UNIVERSITY OF STRATHCLYDE

DEPARTMENT OF MECHANICS OF MATERIALS

RATIONAL CREEP MECHANICS
And engineering applications to high temperature pipework design

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ABSTRACT

The thesis is divided into three sections:
The first section is aimed at providing a rational basis for the behaviour of complex structures deforming in time due to creep under the action of arbitrary loading. Two constitutive models were chosen to be studied - the "internal variable theory" and the well known "time hardening theory". The main content of the approach is the derivation of an equation of evolution for the growth of inelasticity; this equation may be rigorously defined. Resulting from this description are coupled evolution equations for the stress, strain and displacement field. It is the author's contention that these equations are fundamental to understanding the process of creep in a structure and bring out the essential nature of the temporal response. Such an approach allows the basic properties of solution to a creep problem under variable loading to be studied.

The second section applies the ideas from the first section to the study of flexible piping components - in particular the curved pipe; however to begin with, in order to fix ideas the straight pipe is discussed. A novel nonlinear shell model of a curved pipe is derived (to cope with internal pressure). The elastic behaviour of the in-plane bending of this model is studied numerically. Two distinct cases of the creep of a curved pipe, with no internal pressure, are investigated in detail; these analyses are realized from the equation of evolution of stress and allow comparison with simple approximate methods - which turn out to be quite successful.

The third section examines the more complex problem of a high temperature piping system. On a basis of results of

the second section a simple approximate method is suggested in the spirit of well known "flexibility" methods for elasticity. A simple example is worked out to demonstrate the method. The thesis concludes with a view into a possible future for inelastic pipework analysis and creep mechanics in general.

PREFACE

"There was something ominous about the verb 'to creep' which seemed to signify something stealthy, and, indeed ominous it was"

Rear-Admiral Grousaz

The subject of time dependent inelastic material behaviour has furtively crept up to the strongholds of mathematics - quietly enough to have remained virtually unnoticed. Advocates of neo-classical mechanics would contest this - for has not the edifice constructed by the rational school of Clifford Truesdell and Walter Noll and their myriad students reached almost monolithic proportions? In fact the truth is difficult to detect. This is not the place to justify the often obtuse thinking of the rational approach - indeed it has been eloquently defended by Truesdell himself(1,2). It is, without question, necessary - more as a familiar rest than as a point of departure for the fabrication of workable tools. The applications oriented approach of the engineer, with its associated analytical and numerical content, must by its very urgency preclude a ceasation of activity to systematically review achievements and aims - the motives of a rational formulation. Such a review is essential in the opinion of the author, since the inventive urgency of the engineer has taken the study of creep into realms in which it should not belong.

However it is the engineering approach which shall ultimately challenge the rational school and force it to examine itself. In the field of time dependent material behaviour this defial is about to commence, being directed

at the already much criticized "Axioms of Constitutive Theory" (3,4), and, in particular, the so-called "Principle of Determinism" which asserts that at any instant in time, the stress in a body at that instant shall be determined by the history of deformation of the body up to that time. Although intended to eliminate future events, the Principle of Determinism is used to imply a constitutive model with stress expressed in terms of the strain history. Simple observation of the methods and possibilities of large scale material testing and documentation for creep highlights the excessive practical limitations of the principle - essentially restricting any experimental program to short term tensile tests at variable load producing data on strain change in terms of stress. Unless a sound physical theory of material behaviour on the atomistic level is forthcoming, and this does not appear to be remotely accessible, any mathematical constitutive model shall be largely speculative, and the problems involved in developing a general multiaxial constitutive equation would thus seem insurmountable. Nevertheless an alternative view of creep constitutive theory has recently been expressed by Fred Leckie (5). The underlying philosophy is the treatment of the material as a "black box" with an input of stress and an output of strain rate - thus the effect of the black box may be assessed by performing suitable tests, even though the actual mechanism of the box is unknown. A suitable mathematical model of the black box is a form of the so-called internal, or "hidden" state variable theory, which shall be described in the text of this thesis (Chapter 2, Section 2); indeed Leckie has demonstrated how the pertinent parameters of the constitutive equation which results may be determined by

operating the black box. It is the prime aim of this thesis to attempt an initial step in the formulation of a rational mechanics for creep based on this theme of black boxes, rather than the Axioms of Constitutive Theory. The approach shall be systematic in its generalization and clarification of known results, as well as providing some new ones.

The chronological development of the subject matter of this thesis was not exactly as presented here. Throughout the course of study an attempt was made to establish existence and uniqueness of solution for classical material creep models, while the main effort was directed at the analysis of flexible piping components - in particular the curved pipe - in creep using standard techniques based on the "initial strain" algorithms. Attempts to construct more accurate temporal integrations convinced the author that some sort of mathematical formulation of the process of stress redistribution should be possible - elementary manipulations of simple problems (viz. Chapter 4, Section 1) showed this to be so. Moreover the publications of W C Carpenter, Z P Bazant and I C Cormeau for finite element analysis confirmed this for more complex problems; the step to the general "equation of stress redistribution" (Chapter 4, Section 6) followed. A major influence on the presentation of the approach came from a publication of I Babuska and I Hlavacek on existence and uniqueness in nonlinear viscoelasticity. Close examination of the details of their proof revealed points of similarity in approach and the author felt sure that the equation of stress redistribution could be linked to this - some experimentation (Chapter 4, Section 4) established the "equation of inelastic growth" which could be identified

as the principle of Babuska and Hlavacek's proof. A study of these fundamental equations forms the main body of the thesis.

The thesis is incomplete in two respects - firstly existence and uniqueness of solution has not been decisively answered (Chapter 5, Section 1) although the author has gone as far as is possible using current mathematical knowledge. Secondly, it was intended to investigate the creep of a pressurized curved pipe (Chapters 10 and 11) but the results of a numerical study of a nonlinear shell model in elasticity proved so surprising and dissimilar from classical results that it was felt that further theoretical and experimental investigation would be required before the creep problem could be tackled.

Some of the work in which the author has been involved during the course of study does not fit into the prime aims of this thesis, but most has been included in some form in the Appendices. Furthermore the discussion of Part C has been largely based on parts of a report prepared by the author (6) for the USAEC, Fast Reactor Safety Division.

All of the work presented was carried out in the Department of Mechanics of Materials of the University of Strathclyde with support from the Science Research Council under the Engineering Mathematics studentship scheme, although the author wishes to acknowledge the hospitality afforded by A I Smith, Head of Division Z3 and members of his staff at the National Engineering Laboratory, East Kilbride, during the summer of 1974. The use of the computing facilities at the University of Strathclyde and Edinburgh Regional Computing Centre are also gratefully acknowledged. Finally the author wishes to record his

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James T. Boyle

JAMES T BOYLE
GLASGOW,
JULY 1975

Notes:

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CHAPTER 1

INTRODUCTION AND DESCRIPTION

The phenomenon of creep in structural materials, and particularly in metals, is of ever growing importance. The prediction of its long term effect on the behaviour of structures operating at high temperatures has become the theme of numerous studies over the past half century. Diverse techniques have been employed in the resolution of time dependent creep problems and have been proven successful in many applications by comparison with those elementary analytical solutions which are available and with experiment. Yet no mathematical basis for an investigation into the validity and relevance of these methods, which arose out of the engineering demands of the time, is to be had. The engineer's desire to extend familiar techniques of linear elastic analysis to creep analysis has put the emphasis on the inherent nonlinearity of the response rather than the time dependence, treating this aspect as something which should either be avoided at all costs or tackled in the simplest possible manner. In this thesis an alternative standpoint is adhered to rigorously; the time dependence is of underlying importance while the non-linearity is incidental.

The first part of this thesis deals with a more rational approach to creep mechanics. An attempt at an axiomatic foundation is made by formulating compact equations for the time dependent response of a continuum to creep. These equations - called here the " equation of inelastic growth" and the "redistribution equation" - can be identified as orthodox forms in functional, rather than classical, analysis. The premiss here is that this

rational approach isolates the important aspects of the creep behaviour. Throughout two simple, yet versatile, models of the mechanical response of a creeping material are employed - the "internal state variable theory" and the "time hardening theory" - both of which are described in the preliminary review. It is important that the equations of inelastic growth and stress redistribution demonstrate anticipated properties of solution. The most obvious are existence and uniqueness, which are discussed as far as possible using current mathematical knowledge, and asymptotic behaviour of large times, establishing the ultimate state of continuum. Of lesser importance are global statements which are valid for the whole body - variational principles and energy bounding theorems - possessing a certain intrinsic attractiveness and utility. The part concludes with an investigation into the current methods of numerically resolving creep problems and their identification within the rational framework. Many investigators have presumed, quite correctly, that it is out of the question to attempt a complete analysis of a complex, long term creep process, pointing out the need for an approximate method. Although such a generally acceptable technique has not been forthcoming the predominant candidates are examined.

The design of pressure vessels operating in the creep regime can involve a detailed study of the deformations of a vessel either for its own sake or as a basis for constructing a systematic design procedure. However, this detail cannot be expected for complex, large diameter piping networks, common in nuclear and petrochemical installations; anyhow the problem is to produce a flexible system. Indeed not only must a simplification of the creep

response be made but also the involved, interactive geometry must be simplified. A first step in this overall simplification would involve a detailed analysis of the most common components - the straight and the curved pipe. The second part is a study of these components in creep. After a brief discussion of the straight pipe under combined loading, a thin shell model of a curved pipe is described on a basis of definitive deformation assumptions which appear new to the literature. An elliptic cross section curved pipe under the action of in-plane bending and internal pressure is examined numerically in elasticity using both a linear and nonlinear theory. It is found that a discrete model based on finite differences is satisfactory, by comparison with a more exact analysis formed by direct integration of the fundamental equations. Using this model a creep analysis can be performed (as an excellent example of the rational approach postulated in the first part). Two separate problems are considered; the first is the situation wherein a circular cross section curved pipe deforms under the influence of an applied in-plane bending moment, constant in time - it is possible to study the evolution of the rotation of the ends of the pipe, progressing towards a steady state and confirming previous analyses due to J Spence; the second is the situation wherein a constant curvature is imposed on a circular cross section curved pipe and the resultant bending moment relaxes in time. In elasticity these two problems are identical, but in creep, and, indeed, in other forms of inelasticity, they are not. In both cases the rational procedure is used to derive a finite system of initial value problems which can be integrated forward in time by suitable sequential methods. Finally two simple approximate

methods, with potential application to piping network analysis, are considered, these are found to be quite successful.

The third part reviews the current situation in elastic analysis of pipework systems and the state of inelastic methods. A possible design procedure is suggested based on a step wise approach to the overall design. An efficient flexibility type method, similar to that commonly employed in linear elastic analysis, is recommended as a primary method. The general technique for arbitrary two anchor runs is described and the method explained for a simple example.

The thesis concludes with an essay on the future of creep research as viewed by the writer.

The body of this thesis is concerned with the deformation problem; unfortunately considerations of gross material deterioration such as rupture, fatigue and ratchetting cannot be included due to a lack of space. It is hoped that this position shall be rectified in the near future and meaningfully incorporated into the rational approach. As mentioned in the Preface, many of the ideas contained herein need sharpening and strengthening; the achievements and the prospects of this work remain to be investigated.

PART A

RATIONAL CREEP MECHANICS

The mathematical formulation of the creep of solid bodies has progressed to a point where a number of different outlooks can be identified, yet a firm basis has not yet been developed to unify them. It is the purpose of this part to propose a rational approach to creep mechanics such that the nature of time dependent inelastic material response can be isolated and clearly understood, thus hopefully providing such a base.

CHAPTER 2

BRIEF REVIEW OF CREEP CONSTITUTIVE THEORY

No essay on the mathematical foundations of the creep of solid bodies can be complete without an appraisal of the mechanical laws which characterize this form of inelastic deformation.

Of the representations which are available few are concerned with the atomic structure of the material necessarily involving a stochastic and kinetic description; thus most constitutive approximations are of a purely deterministic, phenomenological nature.

Consider a solid body, supported so that rigid body motions are impossible, at time $t = 0^-$ completely enclosed in an unchanging environment. At time $t = 0$ it is released from this environment so that it is free to be subjected to external influences. After a time t the body has developed a displacement $u(t)$, a compatible strain $\varepsilon(t)$ and a stress field $\sigma(t)$. Clearly this deformation shall depend on the properties of the constitutive material; in fact the values of the complete process $u(t)$, $\varepsilon(t)$, $\sigma(t)$ cannot be completely determined until some material relastionship is specified. In general such a relationship shall depend on the entire history of the quantities $\varepsilon(t)$ and $\sigma(t)$, their rates and accelerations, together with a number of "hidden variables" $\alpha_i(t)$, i = 1,2,3.....N. However, it shall be assumed that the process is slow enough for accelerative effects to be neglected; thus it is quasi-static. Further it shall also be supposed that isothermal conditions are prevelant so that thermodynamic effects can similarly be ignored.

Experimental evidence suggests [1-11] that a material with the ability to creep should also exhibit certain qualitative properties. Firstly it should be rate-sensitive, that is, it should discriminate between different rates of change of the external influences; a constitutive equation can represent rate sensitivity if it is variant under a change in time scale. Secondly it should be path dependent in the sense that the values of stress, strain and hidden variables at some instant cannot be solely determined by the value of the external influences at that instant. Thirdly the path and rate dependence should be nonlinear.

A creep constitutive equation may also be chosen to display two further qualitative properties whose inclusion can be open to opinion. The material may be history dependent in the sense of having a past memory. That is, if it has been removed from its environment and subjected to external influences then returned to its original state and environment at some time before $t = 0$, it may subsequently act differently. Thus a constitutive equation exhibits past memory if it is variant under time origin translation. Such behaviour is well known in classical plasticity in the form of shakedown and the Bauschinger effect. It also appears to be manifest in creep under cyclic loading [10]. The material may also be required to show ageing; suppose the body is removed from its original environment at time $t = 0$ but not subjected to external influences until $t = T$ thus having a rest period of duration T; then if the state at time $T + t$ with rest period T is different from the state at time t assuming there to have been no rest period the material is ageing. The constitutive equation should

therefore depend explicitly on time t.

A general set of constitutive equations takes the form

$$\mathcal{H}_i^{t}(t, \sigma(\tau), \dot{\sigma}(\tau), \varepsilon(\tau), \dot{\varepsilon}(\tau), \alpha_j(\tau), \dot{\alpha}_j(\tau)) = 0 \qquad (2.1)$$

$\tau=-\infty$

for i = 1,2,.....N + 1, where the $\mathcal{H}_i$ are a set of operators dependent on the whole past history of deformation. The set of relations (2.1) includes most, if not all, of the classical phenomenological constitutive theories of creep.

2.1 The equation of state

For the theory of the equation of state the only hidden variable is the inelastic viscous creep strain ε_v, since the actual strain ε can be related to the creep strain (for example by decoupling) a single constitutive equation

$$H(\sigma(t), \dot{\sigma}(t), \varepsilon_v(t), \dot{\varepsilon}_v(t)) = 0 \qquad (2.2)$$

is required for some function H. Two particular examples of this are the "strain hardening law" [1-8]

$$\dot{\varepsilon}_v(t) = f(\sigma, \varepsilon_v) \qquad (2.3)$$

and "Odquists primary creep law" [5]

$$\dot{\varepsilon}_v(t) = f(\sigma, \dot{\sigma}) \qquad (2.4)$$

the functions f usually being determined by experiment [8].

This type of constitutive law is rate sensitive and path dependent, but has no memory and is non-ageing. In order to include these effects (2.2) would need to be modified, for example [10,11] as

$$H(t, \sigma(t), \dot{\sigma}(t), \varepsilon_v(t), \dot{\varepsilon}_v(t), p_j(t)) = 0 \qquad (2.5)$$

where $p_j(t)$ is some measure of the past history [6,12-15]. This more general notion of an equation of state for creep was suggested as a possible theory by Yu N Rabotnov [6,12-15] and developed to a high degree of sophistication

by E T Onat [16-20] with the system

$$H_i(\sigma(t), \dot{\sigma}(t), \varepsilon(t), \dot{\varepsilon}(t), \alpha_j(t), \dot{\alpha}_j(t)) = 0 \qquad (2.6)$$

for i = 1,2,.....N+1 for some functions H_i and known as the theory of hidden, or "internal" state variables.

2.2 The internal state variable theory

Using the Principle of Determinism (i.e. the stress in a body is uniquely determined by its past history of deformation) Onat [16] reduced (2.6) to the form

$$\sigma = f(\alpha_j)$$
$$\dot{\alpha}_i = g_i(\dot{\varepsilon}, \alpha_j) \qquad (2.7)$$

which includes the special representation

$$\dot{\varepsilon}_v = f(\sigma, \alpha_j)$$
$$\dot{\alpha}_i = g_i(\sigma, \alpha_j) \qquad (2.8)$$

if one of the internal variables is identified as the inelastic viscous strain.

This theory has achieved a certain amount of popularity amongst those searching for a realistic model of metallic creep [21,22] and has a firm foundation on metallurgical observations often being strongly linked to dislocation mechanics and crystal physics. A favourable aspect of the theory is that it is not exclusive to isothermal metallic creep; it may be suitably generalized to include thermodynamic effects [23] together with finite strain [24,25] and history dependence [26]. Furthermore it can represent geometrical properties such as material stability [27] and viscoplasticity [28-31,32]. Indeed the internal state variable theory has been rationalized into a complete, abstract thermodynamic theory [33] based on fundamental concepts from W Noll's mechanics of simple materials [34]. The majority of these thermodynamic approaches are fairly obscure, but a more accessible treatment has been given by J F Besselling [35-37].

The internal state variable theory (2.7) and (2.8) is rate sensitive and path dependent and can be made ageing if time is explicitly included. As to it being capable of having a past memory is a matter open to question being dependent on the nature of the internal variables. In the opinion of E Krempl [10] the majority of suggested internal variables are not memory parameters, although it seems to the writer that those based on dislocations or the notion of visco-plastic behaviour must implicitly have some form of memory if properly chosen.

2.3 Nonlinear viscoelastic creep

The phenomenon of nonlinear viscoelasticity [38-40] may be postulated as a model of metallic creep by ignoring the internal variables and the past history ($t \leqslant 0$) so that (2.1) becomes

$$\overset{t}{\underset{\tau=0}{\mathcal{H}}}\left(t, \sigma(\tau), \dot{\sigma}(\tau), \varepsilon(\tau), \dot{\varepsilon}(\tau)\right) = 0 \qquad (2.9)$$

Particular examples are provided by Aratrunyan's equation [38]

$$\varepsilon(t) = g(t, \sigma(t)) + \int_0^t f(t, \tau, \sigma(\tau))\, d\tau \qquad (2.10)$$

or Rabotnov's equation [6]

$$g(t, \varepsilon(t)) = f(t, \sigma(t)) + \int_0^t k(t, \tau)\sigma(\tau)\, d\tau \qquad (2.11)$$

or the standard viscoelastic constitutive model based on the hypothesis of fading memory (i.e. prior deformations which took place a long time ago are ignored) and the Frechet derivative of the material [39-42] i.e.

$$\varepsilon(t) = \sum_{K=1}^{\infty} \int_0^t \int_0^t \cdots \int_0^t J_K(t-\tau_1, \cdots, t-\tau_K) \frac{\partial \sigma(\tau_1)}{\partial \tau_1} \cdots \frac{\partial \sigma(\tau_K)}{\partial \tau_K}\, d\tau_1 \cdots d\tau_K \qquad (2.12)$$

or alternatively [43,44]

$$\varepsilon(t) = \int_0^t f(\sigma(\tau), t-\tau) \frac{d\sigma(\tau)}{d\tau}\, d\tau \qquad (2.13)$$

which have equivalent differential operator forms [40].

The theories (2.10) and (2.11) are rate sensitive, path dependent and ageing while (2.12) and (2.13) are nonageing. Further none have any repository for past memory [10] . This can be overcome by introducing a memory parameter $p(t)$ whence [10]

$$\varepsilon(t) = \int_0^t f(\sigma(\tau), t-\tau, p(\tau)) \frac{d\sigma}{d\tau} \, d\tau \qquad (2.14)$$

or an internal variable (for example K Valanis's "endochronic time scale" [45]).

Such nonlinear viscoelastic models of creep are not widely accepted for metals, being more applicable to concrete and polymers.

2.4 A critical review

It is outwith the scope of this essay to critically review these phenomenological creep theories with regard to quantitative physical reality but merely remark on their qualitative properties. The basic problems together with experimental evidence are reviewed in the articles by E Krempl [10,11] . None of the theories are widely satisfactory, being to a great extent hypothetical and untested on a broad basis. Although it is fairly easy to propose abstract generalizations (e.g. (2.1))[46,33] it is extremely difficult to substantiate them for particular materials. In deciding on a satisfactory theory, the ascertainment of specific functions in the constitutive equations and their application to complex structural calculations must be taken into account. On both counts the nonlinear viscoelastic model should be discounted as remotely practical as an accurate physical representation is quite complicated [41] and since numerical methods for nonlinear integro-differential equations are in their infancy crude temporal approximations would be used.

The equation of state theory is in some sense a particular case of the internal variable theory, the mathematical form of which being useful both computationally and theoretically. Furthermore evidence suggests that a small number of state variables may be sufficient to describe the phenomenon [16] and systematic techniques for their identification and functional form in the constitutive equations have been proposed [22] . The conclusion is that an internal state variable theory, with proper choice of state parameters, comes close to an ideal.

Nevertheless there are drawbacks which must be mentioned; much of the information available is for uniaxial situations and generalizations necessary for complex stress analysis in a multiaxial form are scarce, although it is fully expected that this situation shall be remedied by some investigators in the near future. Finally the development of a perfectly realistic model incorporating past memory and ageing poses overwhelming difficulties in practice and shall probably never be achieved; there is a need for a generally acceptable yet simple theory to be studied, both experimentally and theoretically, in detail.

2.5 Time-hardening

So far the most common and simplext theory of creep has not been mentioned. This is the so-called "time-hardening" theory which is closely related to the nonlinear Maxwell model in rheology [1-9] . Its peculiar properties, advantages and disadvantages deserve a small paragraph of their own.

Briefly the theory supposes classically that

$$\dot{\varepsilon}_v = f(t', \sigma) \qquad (2.15)$$

where t' is "elapsed time" from the initial application of

the external influences i.e. $t' = t - t_o$ with t_o the time at which the influences were first applied. The constitutive equation (2.15) is rate sensitive, path dependent but has no memory and is nonageing (it can be made ageing by replacing elapsed time by real time). The model is often criticised on physical grounds, but nevertheless gives excellent agreement for constant stress uniaxial tests and has proven fairly accurate in complex stress situations under constant applied external conditions [7,47,48] through a simple choice of the functional form (2.15) together with selected uniaxial test results [47,48] . It is usually supposed that the model breaks down under variable loading [7] being demonstrated by taking a constitutive equation derived from constant loading tests. The comparison would be more valid if (2.15) was derived on a basis of tests subjected to a loading programme matching in kind the temporal behaviour of the external influences on the body. Apart from the encouraging results of [47] and the work of J J Williams and F A Leckie [50,51] the writer has been unable to find indications in the literature which resolve this. Anyhow the theory has sufficient scope for generalization by introducing, for example, a "parameter of flow" $\gamma(t')$ which depends on a more arbitrary time scale t' (be it elapsed, real or some other) and which could be a repository for past memory, for example "endochronic time" [45] . Then

$$\dot{\varepsilon}_v = f(\gamma(t'), \sigma) \qquad (2.16)$$

Such possibilities would require much development.

However a considerable point in favour of the time hardening theory at present is that a fair multiaxial theory exists (see [1-9] or the excellent review [49] of work on the subject at the National Engineering Laboratories). It

is simple, and can be used with a certain amount of confidence.

CHAPTER 3

A REVIEW OF THE MATHEMATICAL FORMULATION OF THE CREEP OF SOLID BODIES

The fundamental field equations of continuum mechanics comprise the equations of equilibrium for the stresses and the strain-displacement relations which imply the potential compatibility of the strain field. But the deformation of a solid body shall depend on the material of which it is composed and therefore the field relations are insufficient to completely resolve the problem. Constitutive equations and necessary and sufficient initial and boundary conditions should be fixed in order to obtain a solution for the process of stress, strain and displacement. It is at this point that difficulties occur if complex structural problems are to be treated. How can these fundamental relations be manipulated to yield the required stresses and strains, what types of numerical methods are applicable, and, given a numerical solution, how accurate is it? The answers to these questions can only begin to be answered if the properties of the equations of the mathematical formulation of creep are documented and fully understood.

It is the main purpose of this essay to bring together numerous results, theories and views into a unified whole so that the above questions may be resolved to some extent. In this sense the theory is "rationalized". A first step is to review previous work and allocate it its proper place in the historical development.

Many investigators throughout the past half century have tackled structural problems in the creep regime, more or less beginning with the studies of R W Bailey [52] and Folke Odquist (see [5]), and are thoroughly traced in the

standard texts on the subject [1-9] . Nevertheless the first attempts at an overall continuum theory of creep appears to have been made by Soviet authors [53,54], notably L M Kachanov [2].

The majority of studies concerning the properties of a continuum theory of creep focusses on the so-called steady state phase of creep using the time hardening formulation (section 2.5) with no dependence on elapsed time. In this steady state stresses have redistributed to a constant value along with the rate of change of deformation, elastic effects are neglected and small, isothermal deformations are assumed. Early investigators looked for analogies with well known elastic procedures. Many useful studies were provoked by the work of N J Hoff [55,56,57] who pointed out the analogy between the stress field in steady creep and a suitably defined equivalent nonlinear elastic material (a similar observation was also made by L M Kachanov). The implication was that classical elastic solution methods could be utilised for example, all the energy theorems could be adapted. Following from this C R Calladine and D C Drucker established the "theorem of nesting surfaces of constant energy rate" for the Norton n-power law of creep,

$$\dot{\varepsilon}_v = B\,\sigma^n$$

where B and n are material constants, which indicated that that surfaces of constant energy dissipation rate nested monotonically (with n) between the analogous linear elastic surface (n=1) and perfect plastic surface (n=∞), [58,59]. This result was later generalized by W R Hodgkins [60] with the observation that the "actual" value of the exponent n was not the same as the experimentally approximated "idealized" value. Hodgkins was able to show that the

nesting property was retained for the actual and idealized values. The derivation of these results depend on "energy theorems" for steady creep [2,5,54] similar to the well known principles of "minimum total potential energy" and "complementary energy". These rely on the function of stress in the time hardening creep equation being "potential" i.e. there exists a functional W such that

$$\dot{\varepsilon}_v = \frac{\partial W}{\partial \sigma}$$

R Hill [61], showed that the minimum characteristic of the theorems followed from the requirement that W be "convex" i.e. the inequality

$$W(\sigma_1) - W(\sigma_2) \geq \frac{\partial W(\sigma_2)}{\partial \sigma_2}(\sigma_1 - \sigma_2)$$

holds for any two stress fields σ_1 and σ_2 , and was thus also able to establish uniqueness of solution (this being independently proven by J L Sackman [62] and Yu N Rabotnov [6]). The question of existence of a solution was not answered until the work of G Dinca [63,64] based on some investigations of A Langenbah [65]. Dinca established that a "generalized" solution of the first boundary value problem (i.e. velocities specified on the whole boundary of the body) existed in the sense that a minimum of the associated total potential energy functional existed. These variational theorems also allow the energy dissipation rate of a body to be bounded above and below, by suitable manipulations, dependent on the convexity of the creep potential, J B Martin [66,67] was able to bound above displacement at a point (based on similar manipulations in elasticity [68]), on the other hand A C Palmer [69] provided a lower bound. These bounds compared favourably with exact results and the use of the energy theorms [70], however it has been demonstrated

by B Noble [71,72] that the technique is limited in the sense that it relies strongly on the homogeneity of the power law of creep. The "duality" of the energy theorems was properly identified in a functional analytic sense by H Gajewski [73] . In fact dual theorems do not require the existence of a potential in the creep law - there only needs to be a convex "super-potential" whose "sub-differential" is $\dot{\varepsilon}_v$ [74] (these terms shall be explained at a later stage since they entail a certain amount of preparation).

It is by now well known that the linear theory of elasticity typifies a linear elliptic boundary value problem; in a similar manner nonlinear steady creep can be reduced to a nonlinear elliptic boundary value problem. Thus any technique which applies to the theory of (nonlinear) elliptic boundary value problems may be used in the resolution of steady creep problems. The success of Hoff's elastic analogy and the variational methods is partly due to this observation. For more general time dependent creep, with elastic effects included, this approach is not possible, moreover variable loading has to be included also and the steady creep formulation leads to several paradoxes.

In order to account for elastic effects under constant loading D L Marriott and F A Leckie [75] advanced an approximation known as "super position" by simply adding the initial elastic solution to the steady state creep solution found using either time-hardening [76] or strain hardening [77] models. A similar suggestion was implicitly employed in an early work by V I Rozenblium [78]. The significance of superposition from a design point of view was discussed at length by C R Calladine [79], who

was also able to approximate the order of the time scale for the steady state to be achieved, and it was demonstrated by D L Marriott [76,77] that the technique underestimated the actual response, but it was found possible to bound the ultimate error which, in fact, could be exactly computed for a linear creep law [80]. More recently J Spence and J Hult [81] employed a related approximation to superposition in their study of the relaxation of structures subject to constant applied deformations.

The variational energy methods which form an important description of the linear elastic and steady creep behaviour unfortunately cannot be extended in a natural way to the more general situation. Some "reduced" variational theorems have been given in the literature either for the rate problem presuming the total quantities [82,83] or for the actual problem given the solution of the rate problem [2,84,85] ; these shall be discussed later. Bounding energy dissipation and displacement also cannot be achieved in a direct manner; in linear elasticity and steady creep the energy can be bounded in terms of the energy of an identical body, subjected to the same external influences but not necessarily having compatible strains; however if elastic effects are included in the creep formuation the energy can only be bounded in terms of a similar body composed of a different material [86] . Similarly the displacement bounding theorems can only be extended by using comparisons with a different material [87][88].

If variable external influences are applied then the mathematical description of the physical creep process becomes even more involved. Special simplifications which allow the elastic analogy to be retained have been

proposed by W Prager [89] and J Hult [90] for proportional loading. Essentially these simplifications are affected by ignoring the elastic response thus making the approximation rather artificial, although it was important to the development of the superposition method. One of the first important results regarding variable loading, but using a time hardening creep model, was the "theorem of convergent internal stresses" of C O Frederick and P J Armstrong [91]. This theorem stated that if two identical bodies are subjected to the same loading programme but differed in their initial state then their stress patterns would eventually become the same. The main implication of this result is that if a body is subjected to cycles of load then it would ultimately reach a cyclic stress state [92-95]. Recently A R S Ponter and F A Leckie [96] have confirmed a similar result for an internal state variable constitutive model based on the so-called Bailey Orowan theory [97]. In variable loading both the previously mentioned reduced variational theorems [82-85] and bounding principles [93-95] can be cast in a suitable form.

The needs of high temperature engineering forced the genesis of practical methods for the numerical resolution of complex structural problems. The most popular of such is the "method of elastic solutions" or the "method of initial strains" formed from concepts out of thermo-elasticity applied to elasto-plasticity by A A Ilyushin [98] and time dependent creep by H Poritsky and F A Fend [99], A Mendelsohn, M H Hirshberg and S S Manson [100] and P S Kuratov and V I Rozenblium [101]. These "algorithms" are typically based on a "linearity assumption" which presumes that the stress/time curve can be approximated by a series

of straight lines; thus if $\sigma(t)$ and $\dot{\sigma}(t)$ are known the at the next instant $\sigma(t + \Delta t) = \sigma(t) + \Delta t \dot{\sigma}(t)$ and then problem can be resolved in a step-by-step fashion. The algorithms have been improved by a number of investigators removing the linearity assumption; major refinements have been demonstrated by W C Carpenter [102] and Z P Bazant [103]. Perhaps the most important aspect of this "initial strain" method is that it can be employed in the establishment of existence and uniqueness of solution. This has been done for a class of nonlinear viscoelastic constitutive models by I Babuska and I Hlavacek [104] in the sense of Aratrunyan [38] (equation (2.10)). Further existence theorems for a linear creep theory which uses a constitutive equation relating stress in terms of strain have been obtained by I Hlavacek and M Predeleanu [105-108] and independently for a similar nonlinear model by H Babadshanjan and H Gajewski [109]. More recently a series of existence theorems for an internal state variable theory have been produced by J Necas and his co-workers [110-113]. But the crucial observation is that none of these theorems apply to the particular forms of constitutive equations employed for metallic creep, since all depend on a special form of continuity (Lipschitz continuity) and boundedness of the viscous rates.

It is fairly obvious that progress in the mathematical theory of the creep of solid bodies has taken two major paths - the variational and bounding principles on the one hand, and the initial strain approach on the other. It seems to the writer that some uniform viewpoint is called for from which these ideas can be deduced and which can be identified as some standard mathematical structure. A base is required for a <u>systematic</u> treatment. One possible, and

exciting step has been taken by Q S Nguyen [114,115] on identifying viscoplasticity as a "variational evolution inequality" [116] for an internal state variable constitutive model.

In this essay a different course is taken which includes this approach as a change in viewpoint and which, it is felt, brings out the essential nature of creep.

CHAPTER 4

A RATIONAL APPROACH TO CREEP MECHANICS

In the previous Chapter it was remarked that the behaviour of a solution to a creep problem was only really understood on a basis of numerical results and simple examples. It is natural to ask how for these observed properties carry over to the more general case. This, it is felt, can only be successfully answered from a sound, unifying base. It is the aim of this Chapter to present what the writer considered to be such an approach.

4.1 Motivation

Consider rationally the process of time-dependent creep. A body in equilibrium with itself and its environment is suddenly subjected to a series of external influences such that an initial stress and strain pattern is generated. As time goes on the body creeps and this stress and strain pattern evolves from the initial state. Thus, qualitatively what has to be resolved is a problem of evolution, or an initial value problem. Is it not therefore reasonable to expect that this process should be represented by an equation of evolution? Indeed by discretizing in space (via finite elements) W C Carpenter [102] and Z P Bazant [103] confirmed the question. Moreover it is fairly easy to establish an equation of evolution for simple problems. Consider the simplest problem of a beam in bending, examined by W J Goodey [117]. Suppose that the cross section, of area A, has symmetry about one principal axis of bending and that the longitudinal stress σ depends solely on height above the centroidal plane, x, and time, t. Let $\varepsilon = \varepsilon(x,t)$ be the longitudinal strain, and $\kappa = \kappa(t)$ the curvature change, and assume that the

material is of time hardening type. The fundamental field equations to be satisfied are

(a) Equilibrium with external bending moment M,

$$\int_A \sigma(x,t)\, x\, dA = M$$

(b) Strain-displacement

$$\varepsilon(x,t) = \kappa(t)\cdot x$$

(c)Constitutive equation

$$\varepsilon(x,t) = \varepsilon_e(x,t) + \varepsilon_v(x,t)$$

$$\varepsilon_e(x,t) = \frac{1}{E}\sigma(x,t) \qquad \frac{d\varepsilon_v(x,t)}{dt} = B(t)\sigma(x,t)^n$$

$$\varepsilon_v(x,0) = 0$$

where E is Young's modulus and B, n material creep parameters.

For simplicity assume that the applied bending moment M is constant in time. Firstly differentiate all equations with respect to time

$$\int_A \dot{\sigma}(x,t)\, x\, dA = 0 \qquad \dot{\varepsilon}(x,t) = \dot{\kappa}(t).x$$

$$\dot{\varepsilon}(x,t) = \frac{\dot{\sigma}(x,t)}{E} + \dot{\varepsilon}_v(x,t)$$

thus

$$\dot{\sigma}(x,t) = E\left(\dot{\varepsilon}(x,t) - \dot{\varepsilon}_v(x,t)\right)$$

and on substituting into equilibrium and using strain displacement there results

$$\dot{\sigma}(x,t) = E\left\{\frac{x}{I}\int_A \dot{\varepsilon}_v(x,t)\, x\, dA - \dot{\varepsilon}_v(x,t)\right\} \qquad (4.1)$$

where $I = \int_A x^2 dA$. Hence,

$$\frac{d\sigma}{dt}(x,t) = EB(t)\left\{\frac{x}{I}\int_A \sigma(x,t)^n x\, dA - \sigma(x,t)^n\right\} \qquad (4.2)$$

which is an equation of evolution for the stress field taken with the initial condition

$$\sigma(x,0) = \sigma_0(x) \qquad (4.3)$$

where $\sigma_0(x)$ is the initial elastic stress

$$\sigma_0(x) = \frac{Mx}{I}$$

Thus for this simple problem the existence of an equation of evolution (4.2)(4.3) to describe the creep resonse has

been established. Such an equation similarly prevails for other creep laws (for example, consult the work of B Einarrson [118,119] on thick spheres and cylinders using the strain hardening theory (2.3)).

The question is: Does the procedure described above always operate? At first sight it seems impossible, replying on the ability to express $\dot{\sigma}(x,t)$ in terms of $\dot{\varepsilon}_v(x,t)$. However consider the procedure in a more abstract fashion. What in fact has been shown is that

$$\dot{\sigma}(x\,t) = r(\dot{\varepsilon}_v(x\,t)) \qquad (4.4)$$

for some function or "operator" r (which maps a function of x into another function of x) which is obviously linear. Then using the creep constitutive law

$$\frac{d\sigma}{dt} = B(t)r(\sigma^n) \qquad (4.5)$$

which is equation (4.2).

Thus for <u>any</u> problem, even though an explicit expression for r is unavailable, this operator can still be <u>formally</u> defined. And how is it defined? By examining the course by which (4.4) was deduced it is obvious that r may be defined in the following manner: Let ε^* be an initial strain, then $r(\varepsilon^*)$ is the stress resulting from a solution of an equivalent elastic problem on an identical beam constrained to have no applied moment but with an initial strain field ε^*.

In consequence a similar procedure can be applied in principle to the creep of any body, and although constant loads were prescribed it is possible to extend to variable loading.

It is the above simple observation which provides the motivation for the following discussion.

4.2 The continuum creep problem

Consider a body $\mathcal{B}$ occupying a volume $\mathcal{V}$ in three dimensional Euclidean space $\mathbb{E}^3$ bounded by a surface $\mathcal{S}$; points of $\bar{\mathcal{B}} = \mathcal{B} \cup \mathcal{S}$ are described by the triplet $(X_1, X_2, X_3) = \underset{\sim}{X}$. It is supposed that at time $t = 0^-$ the body is in an unstressed and unstrained state in equilibrium with itself and its environment. At time $t = 0$ the state of the body is suddenly changed by the application of external influences in the form of internal body forces within $\mathcal{B}$, prescribed surface tractions on a part of the boundary $\mathcal{S}_T \subset \mathcal{S}$ and prescribed displacements on the remainder $\mathcal{S}_U = \mathcal{S} - \mathcal{S}_T$. The environment is such that the body creeps inelastically in time.

It shall be assumed that isothermal conditions are maintained so that thermal strains can be discounted; furthermore the deformation is infinitessimal and quasi-static so that inertial and accelerative effects can be discounted also. The initial deformation is supposed to be purely elastic so that instantaneous plasticity, for example, is not taken into account. Finally it is assumed that the portions $\mathcal{S}_T$ and $\mathcal{S}_U$ of the bounding surface do not alter in time. None of these suppositions are restrictive to the theory which can be readily extended to cope with their inclusion - they serve only to add notational complications. For the purposes of demonstration an internal state variable model (2.8) of the constitutive material is employed.

Firstly note some definitions [120-122]:
A continuum field process $(\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma})$ for the body $\mathcal{B}$ is a function defined on the real line $\mathbb{R}$ whose values at time $t \in \mathbb{R}$ are ordered triplets $(\underset{\sim}{u}(t), \underset{\sim}{\varepsilon}(t), \underset{\sim}{\sigma}(t))$ such that $\underset{\sim}{u}(t)$ is a vector displacement field with components

$u_i(\underset{\sim}{X},t)$, i = 1,2,3, $\underset{\sim}{X} \in \bar{\mathcal{B}}$ and $\underset{\sim}{\varepsilon}(t), \underset{\sim}{\sigma}(t)$ are second order tensor strain and stress fields with components $\varepsilon_{ij}(\underset{\sim}{X},t)$ $\sigma_{ij}(\underset{\sim}{X},t)$ ij = 1,2,3, $\underset{\sim}{X} \in \bar{\mathcal{B}}$ respectively. Explicit dependence on t shall be frequently dropped.

The triplet $(\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{\varepsilon}}, \hat{\underset{\sim}{\sigma}})$ is an admissable state for $\mathcal{B}$ if $\hat{\underset{\sim}{u}}(\underset{\sim}{X})$ is smooth on $\mathcal{B}$ with $\hat{\underset{\sim}{u}}, \hat{\underset{\sim}{\nabla}}\hat{\underset{\sim}{u}}$ continuous on $\bar{\mathcal{B}}$, where $\hat{\underset{\sim}{\nabla}}$ is the tensor

$$\hat{\underset{\sim}{\nabla}}\hat{\underset{\sim}{u}} = \tfrac{1}{2}(\hat{u}_{i,j} + \hat{u}_{j,i}) \qquad i,j = 1,2,3$$

$\hat{\underset{\sim}{\varepsilon}}(\underset{\sim}{X})$ is a symmetric tensor continuous on $\mathcal{B}$, and $\hat{\underset{\sim}{\sigma}}(\underset{\sim}{X})$ is a symmetric tensor, smooth on $\mathcal{B}$ and such that $\hat{\underset{\sim}{\sigma}}$ and $div\,\hat{\underset{\sim}{\sigma}}$ are continuous on $\bar{\mathcal{B}}$ where div is the vector.

$$div\,\hat{\underset{\sim}{\sigma}} = \hat{\sigma}_{ij,j} \qquad i = 1,2,3$$

with $(\,)_{,i} = \partial(\,)/\partial X_i$ i = 1,2,3 and using the Einstein summation convention. Then $(\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma})$ is an admissable process for $\mathcal{B}$ if its values $(\underset{\sim}{u}(t), \underset{\sim}{\varepsilon}(t), \underset{\sim}{\sigma}(t))$ are admissable states for $\mathcal{B}$.

The inelastic constitutive equation (2.8) for the internal state variable theory generalizes to

$$\begin{aligned} \dot{\underset{\sim}{\varepsilon}}_v(t) &= \underset{\sim}{F}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t)) \\ \dot{\underset{\sim}{\alpha}}(t) &= \underset{\sim}{G}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t)) \end{aligned} \qquad (4.6)$$

with $\underset{\sim}{F}$ and $\underset{\sim}{G}$ nonlinear tensor and vector valued operators respectively, $\underset{\sim}{\alpha}$ being the process of internal change whose value at time t is the vector field $\underset{\sim}{\alpha}(t)$ with components $\alpha_i(\underset{\sim}{X},t)$ i = 1,2,3.....N, $\underset{\sim}{X} \in \bar{\mathcal{B}}$ representing the hidden state variables (there is no loss in generality in assuming this as opposed to adopting vector and tensor fields). The inelastic strain is related to the actual (total) strain through the decoupling identity

$$\underset{\sim}{\varepsilon}(t) = \underset{\sim}{\varepsilon}_e(t) + \underset{\sim}{\varepsilon}_v(t) \qquad (4.7)$$

where $\underset{\sim}{\varepsilon}_e$ is an elastic strain related to stress through the generalized Hooke's law

$$\underset{\sim}{\varepsilon}_e(t) = \underset{\sim}{C}(\underset{\sim}{\sigma}(t)) \qquad (4.8)$$

with $\underset{\sim}{C}$ a linear operator whose components $C_{ijk\ell}(\underset{\sim}{X})$, i,j,k, , = 1,2,3 form the forth order "compliance" tensor (it is possible to require $\underset{\sim}{C}$ to depend on t but for convenience this shall not be done).

Since the initial response is supposed to be purely elastic the constitutive relations are completed by the identity

$$\underset{\sim}{\varepsilon}_v(0) = \underset{\sim}{0} \qquad (4.9)$$

Now suppose that the body is subjected to an internal force process $\bar{\underset{\sim}{b}}$ with values $\bar{\underset{\sim}{b}}(t)$ and components $\bar{b}_i(\underset{\sim}{X}, t)$, i = 1,2,3, $\underset{\sim}{X} \in \mathcal{B}$. Then by a quasi-static creep process (for the interal variable model) for the body $\mathcal{B}$ is meant an admissable process $[\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}]$ for $\mathcal{B}$ whose values are such that the stress field $\underset{\sim}{\sigma}(t)$ is in equilibrium with the body force $\bar{\underset{\sim}{b}}(t)$,

$$\operatorname{div} \underset{\sim}{\sigma}(t) + \bar{\underset{\sim}{b}}(t) = \underset{\sim}{0} \qquad (4.10)$$

while being related to the strain field $\underset{\sim}{\varepsilon}(t)$ through the constitutive relations (4.6)-(4.9) which is compatible with the displacement field through the strain-displacement identity

$$\underset{\sim}{\varepsilon}(t) = \hat{\underset{\sim}{\nabla}} \underset{\sim}{u}(t) \qquad (4.11)$$

Let a surface displacement vector field process $\bar{\underset{\sim}{u}}$ be given on $\mathcal{S}_u$ and a surface traction vector field process $\bar{\underset{\sim}{\tau}}$ be given on $\mathcal{S}_\tau$, then by a solution of the mixed quasi-static creep problem corresponding to boundary data $[\bar{\underset{\sim}{u}}, \bar{\underset{\sim}{\tau}}]$ is meant a quasi-static creep process for $\mathcal{B}$ such that

$$\underset{\sim}{u}(t) = \bar{\underset{\sim}{u}}(t) \qquad (4.12)$$

$$\underset{\sim}{\sigma}(t).\underset{\sim}{n} = \bar{\underset{\sim}{\tau}}(t) \qquad (4.13)$$

where $\underset{\sim}{n}$ is the unit normal vector to the surface $\mathcal{S}_\tau$.

From this the first boundary value problem $\mathcal{S}_\tau \equiv \phi$,

$\mathcal{S}_u \equiv \mathcal{S}$ and the second boundary value problem $\mathcal{S}_u \equiv \phi$,
$\mathcal{S}_T \equiv \mathcal{S}$ may be distinguished.

4.3 Definition of two auxiliary elastic problems

It has been remarked that the creep process may be represented by an equation of evolution using an auxiliary elastic problem. Since the possibility of time varying boundary data needs to be admitted two auxiliary elastic problems shall be defined.

The Equivalent Elastic Problem

The process $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o]$ satisfies the equivalent elastic problem to the (mixed) quasi-static creep problem corresponding to the boundary data $[\bar{\underset{\sim}{u}}, \bar{\underset{\sim}{\tau}}]$ and body process $\bar{\underset{\sim}{b}}$ if its values satisfy

$$
\begin{aligned}
\operatorname{div} \underset{\sim}{\sigma}_o(t) + \bar{\underset{\sim}{b}}(t) &= \underset{\sim}{0} && \text{in } \mathcal{B} \\
\underset{\sim}{\varepsilon}_o(t) &= \hat{\nabla} \underset{\sim}{u}_o(t) && \text{in } \bar{\mathcal{B}} \\
\underset{\sim}{\varepsilon}_o(t) &= \underset{\sim}{C}(\underset{\sim}{\sigma}_o(t)) && \text{in } \bar{\mathcal{B}} \\
\underset{\sim}{u}_o(t) &= \bar{\underset{\sim}{u}}(t) && \text{on } \mathcal{S}_u \\
\underset{\sim}{\sigma}_o(t).\underset{\sim}{n} &= \bar{\underset{\sim}{\tau}}(t) && \text{on } \mathcal{S}_T
\end{aligned}
$$

It is important to notice that the solution of the initial elastic problem is the value of the process $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o]$ at time t = 0, i.e. $(\underset{\sim}{u}_o(0), \underset{\sim}{\varepsilon}_o(0), \underset{\sim}{\sigma}_o(0))$.

The Residual Elastic Problem

Let $\underset{\sim}{\varepsilon}^*$ be a strain process whose values are admissable strains (i.e. symmetric and continuous on $\mathcal{B}$) then $[\underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_R]$ is a solution of the residual elastic problem (to the mixed quasi-static creep problem) for $\underset{\sim}{\varepsilon}^*$ if its values satisfy

$$
\begin{aligned}
\operatorname{div} \underset{\sim}{\sigma}_R(t) &= \underset{\sim}{0} && \text{in } \mathcal{B} \\
\underset{\sim}{\varepsilon}_R(t) &= \hat{\nabla} \underset{\sim}{u}_R(t) && \text{in } \bar{\mathcal{B}} \\
\underset{\sim}{\varepsilon}_R(t) &= \underset{\sim}{C}(\underset{\sim}{\sigma}_R(t)) + \underset{\sim}{\varepsilon}^*(t) && \text{in } \bar{\mathcal{B}} \\
\underset{\sim}{u}_R(t) &= \underset{\sim}{0} && \text{on } \mathcal{S}_u \\
\underset{\sim}{\sigma}_R(t).\underset{\sim}{n} &= \underset{\sim}{0} && \text{on } \mathcal{S}_T
\end{aligned}
$$

Therefore the residual elastic problem is an elastic initial strain problem such that $\mathcal{S}_T$ is free and $\mathcal{S}_U$ is fixed with no body forces. As for the beam in bending problem the residual operator is defined as

$$\underset{\sim}{\sigma}_R(t) = \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*(t))$$

where maps second order tensor into second order tensors; it turns out that this operator has several important properties which shall be discussed later.

4.4 The equations of inelastic growth

In order to represent the creep process by the two auxiliary elastic problems it is necessary to introduce a third auxiliary problem.

Reconsider the simple beam example: integrate equations (4.4) with respect to time so that

$$\sigma(x,t) = \sigma_0(x) + r(\varepsilon_v(x,t))$$

using the initial condition. But from the constitutive relation

$$\frac{d}{dt}\varepsilon_v(x,t) = B(t)\,\sigma(x,t)^n$$

hence

$$\frac{d}{dt}\varepsilon_v(x,t) = B(t)[\sigma_0(x) + r(\varepsilon_v(x,t))]^n$$

which is an evolution equation for the growth of inelastic strain together with the initial condition

$$\varepsilon_v(x,0) = 0$$

This may be naturally extended to the continuum problem as

$$\frac{d}{dt}\underset{\sim}{\varepsilon}_v(t) = \underset{\sim}{F}(\underset{\sim}{\sigma}_0(t) + \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v(t)), \underset{\sim}{\alpha}(t)) \qquad (4.14)$$

$$\frac{d}{dt}\underset{\sim}{\alpha}(t) = \underset{\sim}{G}(\underset{\sim}{\sigma}_0(t) + \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v(t)), \underset{\sim}{\alpha}(t))$$

subject to the initial condition

$$\underset{\sim}{\varepsilon}_v(0) = \underset{\sim}{0} \qquad (4.15)$$

$$\underset{\sim}{\alpha}(0) = \underset{\sim}{\alpha}_0$$

where $\underset{\sim}{\alpha}_0$ is the initial value of the internal variables.

Although it is tolerably straightforward to define these equations of inelastic growth it is not at all clear that these together with the auxiliary elastic problems yield the solution of the quasi-static creep problem. This needs to be proven and is the purpose of the following section.

4.5 Characterisation of the creep problem in terms of the auxiliary problems

Firstly it should be established that the process of inelastic strain indeed complies with the equations of inelastic growth.

Theorem 4.1 : Let $[\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}]$ be a solution of the quasi-static creep problem with $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o]$ a solution of the equivalent elastic problem then $\underset{\sim}{\varepsilon}_v$ must satisfy the equations of inelastic growth.

Proof: It is fairly obvious that $[\underset{\sim}{u}-\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}-\underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}-\underset{\sim}{\sigma}_o]$ satisfies a residual elastic problem for $\underset{\sim}{\varepsilon}_v$. By definition

$$\dot{\underset{\sim}{\varepsilon}} = \underset{\sim}{C}(\dot{\underset{\sim}{\sigma}}) + \underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$$

hence, since $\underset{\sim}{\varepsilon}_o = \underset{\sim}{C}(\underset{\sim}{\sigma}_o)$

$$\dot{\underset{\sim}{\varepsilon}} - \dot{\underset{\sim}{\varepsilon}}_o = \underset{\sim}{C}(\dot{\underset{\sim}{\sigma}} - \dot{\underset{\sim}{\sigma}}_o) + \underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) \qquad (i)$$

But $\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o$ satisfies the residual elastic problem for $\underset{\sim}{\varepsilon}_v$ so

$$\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o = \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v) \qquad (ii)$$

$$\underset{\sim}{\varepsilon} - \underset{\sim}{\varepsilon}_o = \underset{\sim}{C}(\underset{\sim}{\sigma} - \underset{\sim}{\sigma}_o) + \underset{\sim}{\varepsilon}_v \qquad (iii)$$

comparing (i) and (iii)

$$\dot{\underset{\sim}{\varepsilon}}_v = \underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$$

and from (ii)

$$\dot{\underset{\sim}{\varepsilon}}_v = \underset{\sim}{F}(\underset{\sim}{\sigma}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v), \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v), \underset{\sim}{\alpha})$$

since it is readily verified that $\underset{\sim}{\varepsilon}_v(0) = \underset{\sim}{0}$, $\underset{\sim}{\alpha}(0) = \underset{\sim}{\alpha}_o$ the result is proven QED . Secondly it needs to be established that if a solution, $\underset{\sim}{\varepsilon}^*$ say, of the inelastic

growth equations is attainable then a solution of the creep problem can be constructed.

Theorem 4.2 : Let $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o]$ satisfy the equivalent elastic problem and $[\underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_R]$ satisfy the residual elastic problem for $\underset{\sim}{\varepsilon}^*$ where $\underset{\sim}{\varepsilon}^*$ is a solution of the inelastic growth equations. Then $[\underset{\sim}{u}_o + \underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_o + \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R]$ is a solution of the creep problem.

Proof: By definition

$$\underset{\sim}{\varepsilon}_o = \underset{\sim}{C}(\underset{\sim}{\sigma}_o) \qquad \underset{\sim}{\varepsilon}_R = \underset{\sim}{C}(\underset{\sim}{\sigma}_R) + \underset{\sim}{\varepsilon}^* \qquad \underset{\sim}{\sigma}_R = \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*)$$

and

$$\dot{\underset{\sim}{\varepsilon}}^* = \underset{\sim}{F}(\underset{\sim}{\sigma}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*), \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*), \underset{\sim}{\alpha})$$

combining these

$$\dot{\underset{\sim}{\varepsilon}}_R + \dot{\underset{\sim}{\varepsilon}}_o = \underset{\sim}{C}(\dot{\underset{\sim}{\sigma}}_o + \dot{\underset{\sim}{\sigma}}_R) + \underset{\sim}{F}(\underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R, \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R, \underset{\sim}{\alpha})$$

and similarly

$$\begin{aligned} &div(\underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R) + \bar{\underset{\sim}{b}} = \underset{\sim}{0} && \text{in } \mathcal{B} \\ &\underset{\sim}{\varepsilon}_o + \underset{\sim}{\varepsilon}_R = \hat{\nabla}(\underset{\sim}{u}_o + \underset{\sim}{u}_R) && \text{in } \bar{\mathcal{B}} \\ &\underset{\sim}{u}_o + \underset{\sim}{u}_R = \bar{\underset{\sim}{u}} && \text{on } \mathcal{S}_u \\ &(\underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R)\underset{\sim}{n} = \bar{\underset{\sim}{\tau}} && \text{on } \mathcal{S}_T \end{aligned}$$

Hence since $\underset{\sim}{u}_R(o) = \underset{\sim}{0}$, $\underset{\sim}{\varepsilon}_R(o) = \underset{\sim}{\sigma}_R(o) = \underset{\sim}{0}$, $[\underset{\sim}{u}_o + \underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_o + \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R]$ is a solution of the creep problem. QED

This theorem reestablishes the classical representation of a creep process by an equivalent elastic component and a residual component and further more that this residual part is obtainable through the solution of another elastic problem together with an appropriate equation of evolution. This mode of description is useful in rigorously affirming existence and uniqueness; an interesting corollary which is needed in such proofs is the following.

Lemma 4.1 : Let $[\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}]$ be a solution of the quasi-static creep problem with $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o]$ a solution of the

equivalent elastic problem. Then the difference $[\underset{\sim}{u}-\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}-\underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}-\underset{\sim}{\sigma}_o]$ is a solution of the residual elastic problem for $\underset{\sim}{\varepsilon}^*$ if and only if $\underset{\sim}{\varepsilon}^*$ satisfies the equations of inelastic growth.

Proof : The proof follows immediately from the demonstrations of Theorems 4.1 and 4.2

4.6 The stress redistribution equation

In the short examination of the simple beam in bending an equation of evolution for stresses was derived (section 4.2, equation (4.5)). In describing the properties of solution of creep problems, it is useful to generalize this to the continuum problem - this results in what shall be called the "redistribution equations".

$$\frac{d}{dt}\underset{\sim}{\sigma}(t) = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t))) + \dot{\underset{\sim}{\sigma}}_o(t) \qquad (4.16)$$

$$\frac{d}{dt}\underset{\sim}{\alpha}(t) = \underset{\sim}{G}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t))$$

$$\underset{\sim}{\sigma}(0) = \underset{\sim}{\sigma}_o(0) \qquad \underset{\sim}{\alpha}(0) = \underset{\sim}{\alpha}_o \qquad (4.17)$$

being established with Theorems 4.1 and 4.2 by

Theorem 4.3 :

(a) Let $[\underset{\sim}{u}, \underset{\sim}{\varepsilon}, \underset{\sim}{\sigma}]$ be a solution of the creep problem such that $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o]$ is a solution of the equivalent elastic problem, then $\underset{\sim}{\sigma}$ satisfies the redistribution equations.

(b) Let $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o]$ be a solution of the equivalent elastic problem and $[\underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_R]$ satisfy the residual elastic problem for $\underset{\sim}{\varepsilon}^*$ where $\underset{\sim}{\varepsilon}^*$ satisfies the equations of inelastic growth. Then $\underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R$ satisfies the redistribution equations.

Proof:

(a) By definition $\underset{\sim}{\sigma}-\underset{\sim}{\sigma}_o = \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v)$ and from Theorem 4.1 $\underset{\sim}{\varepsilon}_v$ is a solution of the appropriate equations of evolution

$$\dot{\underset{\sim}{\varepsilon}}_v = \underset{\sim}{F}(\underset{\sim}{\sigma}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v), \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v), \underset{\sim}{\alpha})$$

Consequently $\dot{\underset{\sim}{\varepsilon}}_v = \underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$ and

$$\dot{\underset{\sim}{\sigma}} = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})) + \dot{\underset{\sim}{\sigma}}_0 \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$$

also $\underset{\sim}{\sigma}(0) = \underset{\sim}{\sigma}_0(0)$, $\underset{\sim}{\alpha}(0) = \underset{\sim}{\alpha}_0$ and the result is proven.

(b) By definition $\underset{\sim}{\sigma}_R = \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*)$ and

$$\dot{\underset{\sim}{\varepsilon}}^* = \underset{\sim}{F}(\underset{\sim}{\sigma}_0 + \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*), \underset{\sim}{\alpha}) \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}_0 + \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*), \underset{\sim}{\alpha})$$

hence

$$\dot{\underset{\sim}{\sigma}}_R = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}_0 + \underset{\sim}{\sigma}_R, \underset{\sim}{\alpha}))$$

Consequently

$$\dot{\underset{\sim}{\sigma}}_R + \dot{\underset{\sim}{\sigma}}_0 = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}_0 + \underset{\sim}{\sigma}_R, \underset{\sim}{\alpha}))' + \dot{\underset{\sim}{\sigma}}_0 \qquad \dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma}_0 + \underset{\sim}{\sigma}_R, \underset{\sim}{\alpha})$$

but from Theorem 4.2 $\underset{\sim}{\sigma}_0 + \underset{\sim}{\sigma}_R$ satisfies the creep problem so

$$\underset{\sim}{\sigma}_R(0) + \underset{\sim}{\sigma}_0(0) = \underset{\sim}{\sigma}_0(0)$$

and the result is proven. QED

This theorem shows that the stress satisfies the redistribution equation whether the exact solution or the constructed solution of the creep problem is available. Unless this stress is unique it is not possible to construct a solution of the creep problem by resolving the redistribution equations unless additional conditions on the constitutive equations are specified.

This is discussed at a later stage.

4.7 The evolution of deformation

The preceding analysis has successfully established the existence of initial value problems for inelastic strain and stress. Similar formulations can be inferred for the deformation process, coupled with either of these primitive equations.

From Theorem 4.2 $\underset{\sim}{u}(t) = \underset{\sim}{u}_R(t) + \underset{\sim}{u}_0(t)$, $\underset{\sim}{\varepsilon}(t) = \underset{\sim}{\varepsilon}_R(t) + \underset{\sim}{\varepsilon}_0(t)$ and on defining

$$\underset{\sim}{u}_R = \underset{\sim}{R}_u(\underset{\sim}{\varepsilon}^*)$$

$$\underset{\sim}{\varepsilon}_R = \underset{\sim}{R}_\varepsilon(\underset{\sim}{\varepsilon}^*)$$

there results on rearranging and employing Theorems 4.1 and 4.2

$$\frac{d}{dt}\underset{\sim}{u}(t) = \underset{\sim}{R_J}(\underset{\sim}{F}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t))) + \dot{\underset{\sim}{u}}_o(t) \qquad (4.18)$$

$$\frac{d}{dt}\underset{\sim}{\varepsilon}(t) = \underset{\sim}{R_\varepsilon}(\underset{\sim}{F}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t))) + \dot{\underset{\sim}{\varepsilon}}_o(t)$$

$$\underset{\sim}{u}(0) = \underset{\sim}{u}_o(0) \qquad \underset{\sim}{\varepsilon}(0) = \underset{\sim}{\varepsilon}_o(0) \qquad (4.19)$$

which should be integrated along with (4.16) and (4.17) to obtain a solution. Theorems with similar implications to Theorem 4.3 may be deduced.

4.8 Implication

There are two main implications of this rational approach to creep mechanics. Firstly the inelasticity has been isolated and shown to obey an equation of evolution; the creep process may be constructed from this growth law. Secondly, if it can be established that a unique solution exists, the creep process may be alternatively described by the redistribution equations and the equations of evolution of deformation without resorting to the growth law.

This approach is, in a sense, an exact statement of the well-known "initial strain" algorithm (Chapter 3, references [98-104, 122]). Discrete forms of the redistribution equations have been identified by W C Carpenter [102] and Z P Bazant [103] and of the evolution of deformation by O C Zienkiewicz and I C Cormeau [123] by approximating the space variable using the finite element method. The inelastic growth law can in fact be recognized as the basis of the existence proof for non-linear viscoelasticity using a hereditary integral constitutive equation proposed by I Babuska and I Hlavacek [104].

It is the writer's contention that the equations of inelastic growth are fundamental to the creep process and

that all of the familiar results concerning the mathematical formulation of creep and its numerical resolution should be directly related to the properties of these equations or the associated redistribution equations. The purpose therefore of the following Chapter is to substantiate this together with some novel results.

CHAPTER 5

SOME FUNDAMENTAL CONSEQUENCES OF THE EQUATIONS OF EVOLUTION OF CREEP

The rational approach to creep mechanics would be fatuous if it did not possess underlying consequences in the mathematical theory of creep. In particular the aim now is to demonstrate that the equations of evolution of creep can be used to establish well known properties of solution as well as some new ones. This is not to say that such properties cannot be deduced from classical formulations - the thesis here is that the comprehension and significance of these properties can be simplified.

In the opinion of the writer, the three most important and fundamental properties are existence and uniqueness of solution, asymptotic behaviour at large times and global principles. These shall be treated here.

5.1 Existence and uniqueness of solution

Existence theorems, even though they do not exhibit a solution or exhibit it in a useful form, serve several purposes. They indicate what must be known about a given physical situation to insure a solution and preferably a unique one and moreover that a search for a solution would not be attempting the impossible. Further, some of the methodologies of solution presuppose the existence of a particular solution. The problem of existence and uniqueness of solution for time dependent inelastic material behaviour was largely neglected due to the fact that the differential equations arose in physical problems, and it was intuitively clear that these equations had solutions. First steps were taken for linear viscoelastic constitutive models expressing stress in terms of strain [105-107],

[124-126] and later for a nonlinear variation [109,127], although uniqueness is fairly easy to establish in isolation [108,128-131]. A major advance was made by I Babuska and I Hlavacek [104] who were able to prove existence of solution for a class of nonlinear viscoelastic creep materials typical of concrete and polymers using the initial strain method. Laterally J Necas [110-113] used a variational formulation of the initial strain process to study the problem for the internal variable model. Unfortunately these results cannot be easily extended to the type of constitutive equations which occur in metallic creep. It is the purpose here to discuss the possibility of existence and uniqueness of solution for the internal variable model from an investigation of the equations of evolution of creep - in particular the equation of inelastic growth. Of necessity this requires an excursion into the realms of functional analysis and the theory of ordinary differential equations on an abstract space.

5.1.1. Technical Preliminaries

Let the region $\mathcal{V}$ of three dimensional Euclidean space $\mathbb{E}^3$ be a Lipschitz region i.e. $\mathcal{V}$ is bounded and its boundary $\mathcal{S}$ has the following properties:

(a) to each point $\underset{\sim}{X} \in \mathcal{S}$ an open hypersphere $S(\underset{\sim}{X})$ about $\underset{\sim}{X}$ exists such that the intersection $S(\underset{\sim}{X}) \cap \mathcal{S}$ may be described by means of a Lipschitz function, and,

(b) $S(\underset{\sim}{X}) \cap \mathcal{S}$ divides $S(\underset{\sim}{X})$ into exterior and interior parts with respect to $\mathcal{V}$.

Let the following spaces be defined [133]:

$l_2(\mathcal{V})$ is the real Hilbert space of functions square integrable in the Lebesque sense in $\mathcal{V}$ with inner product

$$\langle \phi, \psi \rangle_{l_2} = \int_{\mathcal{V}} \phi(\underset{\sim}{X})\, \psi(\underset{\sim}{X})\, d\underset{\sim}{X} \qquad \phi, \psi \in l_2(\mathcal{V})$$

$[l_2(\mathcal{V})]^k$ is the real Hilbert space of vector functions with each of the k components square integrable in $\mathcal{V}$ with inner product

$$\langle \underset{\sim}{\phi}, \underset{\sim}{\psi} \rangle_{[l_2]^k} = \int_{\mathcal{V}} \underset{\sim}{\phi} \cdot \underset{\sim}{\psi} \, dX \qquad \underset{\sim}{\phi}, \underset{\sim}{\psi} \in [l_2(\mathcal{V})]^k$$

$W_2^{(1)}(\mathcal{V})$ is the real Sobolev space of functions from $l_2(\mathcal{V})$ having all first partial derivatives in the generalized sense (of distributions) also belonging to $l_2(\mathcal{V})$ with inner product

$$\langle \phi, \psi \rangle_{W_2^{(1)}} = \langle \phi, \psi \rangle_{l_2} + \sum_{i=1}^{3} \left\langle \frac{\partial \phi}{\partial X_i}, \frac{\partial \psi}{\partial X_i} \right\rangle_{l_2} \qquad \phi, \psi \in W_2^{(1)}(\mathcal{V})$$

$[W_2^{(1)}(\mathcal{V})]^k$ is the real Sobolev space of k dimensional vector functions with each component in $W_2^{(1)}(\mathcal{V})$ and inner product

$$\langle \underset{\sim}{\phi}, \underset{\sim}{\psi} \rangle_{[W_2^{(1)}]^k} = \sum_{i=1}^{k} \langle \phi_i, \psi_i \rangle_{W_2^{(1)}} \qquad \underset{\sim}{\phi}, \underset{\sim}{\psi} \in [W_2^{(1)}(\mathcal{V})]^k$$

$[\mathring{W}_2^{(1)}(\mathcal{V})]^k$ is the closure in $[W_2^{(1)}(\mathcal{V})]^k$ of the set of all k dimensional vector functions with components having continuous partial derivatives of all orders and compact support in $\mathcal{V}$, in the sense of the norm of $[W_2^{(1)}(\mathcal{V})]^k$, which is a Hilbert space with inner product

$$\langle \underset{\sim}{\phi}, \underset{\sim}{\psi} \rangle_{[\mathring{W}_2^{(1)}]^k} = \sum_{i=1}^{k} \sum_{j=1}^{3} \left\langle \frac{\partial \phi_i}{\partial X_j}, \frac{\partial \psi_i}{\partial X_j} \right\rangle_{l_2} \qquad \underset{\sim}{\phi}, \underset{\sim}{\psi} \in [\mathring{W}_2^{(1)}(\mathcal{V})]^k$$

$[\mathring{W}_2^{(-1)}(\mathcal{V})]^k$ denotes the space of linear continuous functionals f defined on $[\mathring{W}_2^{(1)}(\mathcal{V})]^k$ with the norm

$$\|f\|_{[\mathring{W}_2^{(-1)}]^k} = \sup |f(\underset{\sim}{\phi})| \,, \qquad \|\underset{\sim}{\phi}\|_{[\mathring{W}_2^{(1)}]^k} = 1$$

$L_2(\mathcal{V})$ denotes the Hilbert space of tensor functions $\underset{\sim}{\Phi}(\underset{\sim}{X})$ with components $\phi_{ij}(\underset{\sim}{X}) \in l_2(\mathcal{V})$ $i,j = 1,2,3$ and inner product

$$\langle \underset{\sim}{\Phi}, \underset{\sim}{\Psi} \rangle_{L_2} = \sum_{i,j=1}^{3} \langle \phi_{ij}, \psi_{ij} \rangle_{l_2} \qquad \underset{\sim}{\Phi}, \underset{\sim}{\Psi} \in L_2(\mathcal{V})$$

Y_2 denotes the space of linear continuous mappings $\underset{\sim}{\Im}$ of $L_2(\mathcal{V})$ into $L_2(\mathcal{V})$ with norm

$$\|\underset{\sim}{\Im}\|_{Y_2} = \sup \|\underset{\sim}{\Im}(\underset{\sim}{\Phi})\|_{L_2} \qquad \|\underset{\sim}{\Phi}\|_{L_2} = 1$$

$l_2(\mathcal{J}_o)$ denotes the Banach space of functions ϕ defined almost everywhere on $\mathcal{J}_o \subseteq \mathcal{J}$ (i.e. almost everywhere on two dimensional regions) for which the norm

$$\|\phi\|^2_{l_2(\mathcal{J}_o)} = \int_{\mathcal{J}_o} \phi^2(\underset{\sim}{X})\, d\underset{\sim}{X} \qquad \phi \in l_2(\mathcal{J}_o)$$

is finite.

$[l_2(\mathcal{J}_o)]^K$ denotes the Banach space of vector functions with components in $l_2(\mathcal{J}_o)$ and norm

$$\|\underset{\sim}{\phi}\|^2_{[l_2(\mathcal{J}_o)]^K} = \sum_{i=1}^{K} \|\phi_i\|^2_{l_2(\mathcal{J}_o)} \qquad \underset{\sim}{\phi} \in [l_2(\mathcal{J}_o)]^K$$

Let $I=[0,T]$, $0<T<\infty$ be an interval of the real line $\mathbb{R}$ and let $\mathcal{X}$ be a Banach (or Hilbert) space with norm $\|\cdot\|_{\mathcal{X}}$ then:

$C(I;\mathcal{X})$ is the Banach space of all continuous $\mathcal{X}$-valued functions $\phi(t)$ defined on I with the maximum norm

$$\|\phi\|_C = \max_{t\in I} \|\phi(t)\|_{\mathcal{X}}$$

$C'(I;\mathcal{X})$ is the Banach space of all continuously differentiable $\mathcal{X}$-valued functions on I with the norm

$$\|\phi\|_{C'} = \|\phi\|_C + \|\frac{d\phi}{dt}\|_C$$

thus $C(I;l_2(V))$, $C'(I;l_2(V))$ etc.....are defined.

5.1.2. Existence and uniqueness of the auxiliary elastic problems

Theorems on existence and uniqueness of boundary value problems in elasticity have been demonstrated by a number of authors [134-138]. Here the most advanced enunciation of these theorems given by I Hlavacek and J Necas shall be used [134, 135,139].

The following assumptions shall be made concerning the elastic constitutive operator $\underset{\sim}{C}$:

(a) $\underset{\sim}{C} \in Y_2$ is an invertible linear operator with inverse $\underset{\sim}{C}^{-1} \in Y_2$;

(b) the coefficients $C^{-1}_{ijkl}(\underset{\sim}{X})$ of the operator $\underset{\sim}{C}^{-1}$ are real, bounded, measurable functions in the Lebesque

sense on $\mathcal{V}$;

(c) the coefficients $C^{-1}_{ijkl}(\underset{\sim}{X})$ of the operator $\underset{\sim}{C}^{-1}$ are symmetric in the sense

$$C^{-1}_{ijkl} = C^{-1}_{ijlk} = C^{-1}_{jikl}$$

(d) the coefficients $C^{-1}_{ijkl}(\underset{\sim}{X})$ of the operator $\underset{\sim}{C}^{-1}$ are uniformly positive bounded below i.e.

$$C^{-1}_{ijkl}(X)\,\phi_{ij}\,\phi_{kl} \geq \mu_0 \sum_{i,j=1}^{3} \phi^2_{ij} \qquad \underset{\sim}{\Phi} \in L_2(\mathcal{V})$$

for almost all $\underset{\sim}{X} \in \mathcal{V}+\mathcal{J}$, where $\mu_0 > 0$ is independent of $\underset{\sim}{X}$.

A simple consequence of these assumptions is:

Lemma 5.1: The operator $\underset{\sim}{C}$ is symmetric and positive definite in $L_2(\mathcal{V})$, that is

$$\langle \underset{\sim}{C}(\underset{\sim}{\Phi}), \underset{\sim}{\Psi} \rangle_{L_2} = \langle \underset{\sim}{\Phi}, \underset{\sim}{C}(\underset{\sim}{\Psi}) \rangle_{L_2} \qquad \underset{\sim}{\Phi}, \underset{\sim}{\Psi} \in L_2(\mathcal{V})$$

$$\langle \underset{\sim}{C}(\underset{\sim}{\Phi}), \underset{\sim}{\Phi} \rangle_{L_2} \geq 0, = 0 \text{ iff } \underset{\sim}{\Phi} = \underset{\sim}{0}$$

Furthermore the operators $\underset{\sim}{C}$ and $\underset{\sim}{C}^{-1}$ are linear, continuous operators in $L_2(\mathcal{V})$ and if $\underset{\sim}{\Phi} \in C[I; L_2(\mathcal{V})]$ then $\underset{\sim}{C}^{-1}(\underset{\sim}{\Phi}) \in C[I; L_2(\mathcal{V})]$ and $\underset{\sim}{C}(\underset{\sim}{\Phi}) \in C[I; L_2(\mathcal{V})]$ etc, the proof of which may be found in [136,137,139].

By a weak solution of the equivalent elastic problem corresponding the boundary data $\bar{\underset{\sim}{u}}(t) \in [W_2^{(1)}(\mathcal{V})]^3$, $\bar{\underset{\sim}{t}}(t) \in [l_2(\mathcal{J}_\tau)]^3$ and body force $\bar{\underset{\sim}{b}}(t) \in [l_2(\mathcal{V})]^3$ is meant a vector field $\underset{\sim}{u}_0(t) \in [W_2^{(1)}(\mathcal{V})]^3$ such that

$$\underset{\sim}{u}_0(t) = \bar{\underset{\sim}{u}}(t) \qquad \text{on } \mathcal{J}_u$$

in the sense that $\underset{\sim}{u}_0(t) - \bar{\underset{\sim}{u}}(t) \in [\mathring{W}_2^{(1)}(\mathcal{V})]^3$, and if the components of the stress tensor $\underset{\sim}{\sigma}_0(t)$ are computed according to

$$\underset{\sim}{\sigma}_0(t) = \underset{\sim}{C}^{-1}(\hat{\underset{\sim}{\nabla}}\,\underset{\sim}{u}_0(t))$$

this should satisfy the equations of equilibrium

$$\operatorname{div}\underset{\sim}{\sigma}_0(t) + \bar{\underset{\sim}{b}}(t) = \underset{\sim}{0}$$

and the boundary conditions

$$\underset{\sim}{\sigma}_0(t).\underset{\sim}{n} = \bar{\underset{\sim}{t}}(t)$$

in the weak sense

$$\int_{\mathcal{V}} \hat{\nabla}\cdot\underset{\sim}{v}\cdot\underset{\sim}{\sigma}_o(t)\,d\underset{\sim}{X} = \int_{\mathcal{S}_T} \underset{\sim}{v}\cdot\bar{\underset{\sim}{\tau}}(t)\,d\underset{\sim}{X} + \int_{\mathcal{V}} \underset{\sim}{v}\cdot\bar{\underset{\sim}{b}}(t)\,d\underset{\sim}{X} \qquad (5.1)$$

for each vector function $\underset{\sim}{v}$ with continuous partial derivatives of all orders in $\mathcal{S}+\mathcal{V}$ and with support disjoint with $\mathcal{S}_u$.

Theorem 5.1 : Under the assumptions made on the operator $\underset{\sim}{C}$ if $\underset{\sim}{u}(t)\in[W_2^{(1)}(\mathcal{V})]^3$, $\bar{\underset{\sim}{\tau}}(t)\in[l_2(\mathcal{S}_T)]^3$ and $\underset{\sim}{b}(t)\in[l_2(\mathcal{V})]^3$ then the equivalent elastic problem has precisely one weak solution $\underset{\sim}{u}_o(t)\in[W_2^{(1)}(\mathcal{V})]^3$ with

$$\|\underset{\sim}{u}_o(t)\|_{[W_2^{(1)}]^3} \le K_o^o\left\{\|\bar{\underset{\sim}{u}}(t)\|_{[W_2^{(1)}]^3} + \|\bar{\underset{\sim}{\tau}}(t)\|_{[L_2(\mathcal{S}_T)]^3} + \|\bar{\underset{\sim}{b}}(t)\|_{[l_2]^3}\right\}$$

where K_o^o does not depend on $\underset{\sim}{X}$ or t.

The proof of this theorem can be implied from Theorem 2.2 of [134] using results of [135].

This theorem treats time as a parameter, however, the following result can be inferred for r = 0,1:

Corollary 5.1: Under the assumptions made on the operator $\underset{\sim}{C}$ if $\bar{\underset{\sim}{u}}\in C^r[I;[W_2^{(1)}(\mathcal{V})]^3]$, $\bar{\underset{\sim}{\tau}}\in C^r[I;[l_2(\mathcal{S}_T)]^3]$ and $\bar{\underset{\sim}{b}}\in C^r[I;[l_2(\mathcal{V})]^3]$ then the equivalent elastic problem has precisely one weak solution $\underset{\sim}{u}_o\in C^r[I;[W_2^{(1)}(\mathcal{V})]^3]$ with

$$\|\underset{\sim}{u}_o\|_{C^r} \le K_r^o\left\{\|\bar{\underset{\sim}{u}}\|_{C^r} + \|\bar{\underset{\sim}{\tau}}\|_{C^r} + \|\bar{\underset{\sim}{b}}\|_{C^r}\right\}$$

where K_r^o does not depend on I or $\underset{\sim}{X}$.

By a weak solution of the residual elastic problem corresponding to an initial strain $\underset{\sim}{\varepsilon}^*(t)\in L_2(\mathcal{V})$ is meant a vector field $\underset{\sim}{u}_R(t)\in[W_2^{(1)}(\mathcal{V})]^3$ such that

$$\underset{\sim}{u}_R(t) = \underset{\sim}{0}$$

on $\mathcal{S}_u$ (in the sense of traces) and if the components of stress $\underset{\sim}{\sigma}_R(t)$ are computed according to

$$\underset{\sim}{\sigma}_R(t) = \underset{\sim}{C}^{-1}(\hat{\nabla}\cdot\underset{\sim}{u}_R(t) - \underset{\sim}{\varepsilon}^*(t))$$

they should satisfy the homogeneous equations of equilibrium

$$div\,\underset{\sim}{\sigma}_R(t) = \underset{\sim}{0}$$

are boundary conditions, and similarly

$$\underset{\sim}{\sigma}_R(t).\underset{\sim}{n} = \underset{\sim}{0}$$

on $\mathcal{S}_T$ in the weak sense (c.f (5.1)).

Theorem 5.2: Under the assumptions made on the operator $\underset{\sim}{C}$ if $\underset{\sim}{\varepsilon}^*(t) \in L_2(V)$ then the residual elastic problem has precisely one weak solution $\underset{\sim}{u}_R(t) \in [W_2^{(1)}(V)]^3$ such that

$$\|\underset{\sim}{u}_R(t)\|_{[W_2^{(1)}]^3} \leq K^R \|\underset{\sim}{\varepsilon}^*(t)\|_{L_2}$$

where K^R is independent of t and $\underset{\sim}{X}$.

The proof of this theorem again follows from [134] on noting that the residual elastic problem may be rewritten as an ordinary elastic problem with no initial strain but with a body force and surface traction i.e. let

$\underset{\sim}{\sigma}_R^* = \underset{\sim}{\sigma}_R + \underset{\sim}{\sigma}^*$ where $\underset{\sim}{\sigma}^* = \underset{\sim}{C}^{-1}(\underset{\sim}{\varepsilon}^*)$ then

$$\begin{aligned} div\,\underset{\sim}{\sigma}_R^* + \bar{\underset{\sim}{b}}^* &= \underset{\sim}{0} && \text{in } V \\ \underset{\sim}{\varepsilon}_R = \hat{\nabla}.\underset{\sim}{u}_R &= \underset{\sim}{C}(\underset{\sim}{\sigma}_R^*) && \text{in } V+\mathcal{S} \\ \underset{\sim}{u}_R &= \underset{\sim}{0} && \text{on } \mathcal{S}_U \\ \underset{\sim}{\sigma}_R^*.\underset{\sim}{n} &= \bar{\underset{\sim}{t}}^* && \text{on } \mathcal{S}_T \end{aligned}$$

$$\bar{\underset{\sim}{b}}^* = -div\,\underset{\sim}{\sigma}^* \qquad \bar{\underset{\sim}{t}}^* = \underset{\sim}{\sigma}^*.\underset{\sim}{n}$$

similarly,

Corollary 5.2: Under the assumptions made on the operator $\underset{\sim}{C}$ if $\underset{\sim}{\varepsilon}^* \in C^r[I\,;\,L_2(V)]$ then the residual elastic problem has precisely one weak solution $\underset{\sim}{u}_R \in C^r[I\,;[W_2^{(1)}(V)]^3]$ such that

$$\|u_R\|_{C^r} \leq K_r^R \|\underset{\sim}{\varepsilon}^*\|_{C^r}$$

where K_r^R is independent of I and $\underset{\sim}{X}$.

These two basic theorems on existence and uniqueness of the auxiliary elastic problems hold for the mixed and first boundary value problems but not for the second boundary value problem. Some adjustment is required (viz.

[135] Theorem *II* .2,3).

Lemma 5.2: For the second boundary value problem the existence and uniqueness of solution of the auxiliary elastic problems can be established provided the total equilibrium conditions

$$\int_{\mathcal{V}} \tilde{\underset{\sim}{b}}\, d\underset{\sim}{X} + \int_{\mathcal{S}} \tilde{\underset{\sim}{\tau}}\, d\underset{\sim}{X} = \underset{\sim}{0}$$

$$\int_{\mathcal{V}} (\tilde{\underset{\sim}{b}} \wedge \underset{\sim}{r})\, d\underset{\sim}{X} + \int_{\mathcal{S}} (\tilde{\underset{\sim}{\tau}} \wedge \underset{\sim}{r})\, d\underset{\sim}{X} = \underset{\sim}{0}$$

are satisfied where r is the position vector of $\underset{\sim}{X}$, and the conditions

$$\sum_{i=1}^{6} (p_i(\underset{\sim}{v}))^2 = 0$$

can be imposed for some system p_i , i = 1,2,3,.....6 of linearly independent, linear continuous functionals on $[W_2^{(m)}(\mathcal{V})]^3$.

The functionals p_i may be chosen in many different ways, for example

$$p_i(\underset{\sim}{v}) = \int_{\mathcal{V}^*} v_i\, d\underset{\sim}{X} \quad i=1,2,3 \quad , \quad \int_{\mathcal{V}^*} (\underset{\sim}{r} \wedge \underset{\sim}{v})_{i-3}\, d\underset{\sim}{X} \quad i=4,5,6$$

$$p_i(\underset{\sim}{v}) = \int_{\mathcal{V}^*} v_i\, d\underset{\sim}{X} \quad i=1,2,3 \quad , \quad \int_{\mathcal{V}^*} (\mathrm{rot}\, \underset{\sim}{v})_{i-3}\, d\underset{\sim}{X} \quad i=4,5,6$$

$$p_i(\underset{\sim}{v}) = \int_{\mathcal{S}^*} v_i\, d\underset{\sim}{X} \quad i=1,2,3 \quad , \quad \int_{\mathcal{S}^*} (\underset{\sim}{r} \wedge \underset{\sim}{v})_{i-3}\, d\underset{\sim}{X} \quad i=4,5,6$$

where by $\mathcal{V}^*$ is meant $\mathcal{V} \setminus \mathcal{N}$ where $\mathcal{N}$ is a set of measure zero.

5.1.3. Properties of the residual operator

A study of the equation of inelastic growth necessitates some familiarity with the nature of the residual operator, for example,

Lemma 5.3: The residual operator $\underset{\sim}{R}$ is a linear continuous operator.

Proof: Linearity is obvious by definition. To establish continuity it is necessary and sufficient to demonstrate boundedness [140]. From Theorem 5.2 (dropping dependence on t for convenience)

$$\| \underset{\sim}{u}_R \|_{[W_2^{(1)}]^3} \leq K^R \| \underset{\sim}{\varepsilon}^* \|_{L_2}$$

however it may be shown that

$$\| \underset{\sim}{\varepsilon}_R \|_{L_2} \leq \| \underset{\sim}{u}_R \|_{[W_2^{(1)}]^3}$$

hence

$$\| \underset{\sim}{\varepsilon}_R \|_{L_2} \leq K^R \| \underset{\sim}{\varepsilon}^* \|_{L_2}$$

But $\underset{\sim}{\varepsilon}_R = \underset{\sim}{C}(\underset{\sim}{R}(\underset{\sim}{\varepsilon}^*)) + \underset{\sim}{\varepsilon}^*$ so

$$\| \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*) \|_{L_2} = \| \underset{\sim}{C}^{-1}(\underset{\sim}{\varepsilon}_R - \underset{\sim}{\varepsilon}^*) \|_{L_2} \leq K^C \| \underset{\sim}{\varepsilon}_R - \underset{\sim}{\varepsilon}^* \|_{L_2}$$

since by Lemma 5.1, $\underset{\sim}{C}^{-1}$ is bounded. Thus

$$\| R(\underset{\sim}{\varepsilon}^*) \|_{L_2} \leq K^C (\| \underset{\sim}{\varepsilon}_R \|_{L_2} + \| \underset{\sim}{\varepsilon}^* \|_{L_2}) \leq K^C (K^R + 1) \| \underset{\sim}{\varepsilon}^* \|_{L_2}$$

and hence $\underset{\sim}{R}$ is bounded, $\underset{\sim}{R} \in Y_2$. QED

Lemma 5.4: The residual operator $\underset{\sim}{R}$ is a symmetric, negative operator on $L_2(V)$.

Proof: Let $\underset{\sim}{\Phi}, \underset{\sim}{\Psi} \in L_2(V)$ and $(\underset{\sim}{u}_\phi, \underset{\sim}{\varepsilon}_\phi, \underset{\sim}{\sigma}_\phi)$, $(\underset{\sim}{u}_\psi, \underset{\sim}{\varepsilon}_\psi, \underset{\sim}{\sigma}_\psi)$ be solutions of corresponding residual elastic problems. Then

$$\langle \underset{\sim}{R}(\underset{\sim}{\Phi}), \underset{\sim}{\Psi} \rangle_{L_2} = \langle \underset{\sim}{\sigma}_\phi, \underset{\sim}{\Psi} \rangle_{L_2} = \langle \underset{\sim}{\sigma}_\phi, \underset{\sim}{\varepsilon}_\psi - \underset{\sim}{C}(\underset{\sim}{\sigma}_\psi) \rangle_{L_2}$$

$$= \langle \underset{\sim}{\sigma}_\phi, \underset{\sim}{\varepsilon}_\psi \rangle_{L_2} - \langle \underset{\sim}{\sigma}_\phi, C(\underset{\sim}{\sigma}_\psi) \rangle_{L_2}$$

But $\langle \underset{\sim}{\sigma}_\phi, \underset{\sim}{\varepsilon}_\psi \rangle_{L_2} = 0$ on application of Green's identity (virtual work).

Hence $\langle \underset{\sim}{R}(\underset{\sim}{\Phi}), \underset{\sim}{\Psi} \rangle_{L_2} = - \langle \underset{\sim}{\sigma}_\phi, \underset{\sim}{C}(\underset{\sim}{\sigma}_\psi) \rangle_{L_2}$ and similarly $\langle \underset{\sim}{R}(\underset{\sim}{\Psi}), \underset{\sim}{\Phi} \rangle_{L_2} = - \langle \underset{\sim}{\sigma}_\psi, \underset{\sim}{C}(\underset{\sim}{\sigma}_\phi) \rangle_{L_2}$.

Since $\underset{\sim}{C}$ is symmetric (Lemma 5.1) it follows that $\underset{\sim}{R}$ is symmetric, i.e.

$$\langle \underset{\sim}{R}(\underset{\sim}{\Phi}), \underset{\sim}{\Psi} \rangle_{L_2} = \langle \underset{\sim}{\Phi}, \underset{\sim}{R}(\underset{\sim}{\Psi}) \rangle_{L_2}$$

and further since $\underset{\sim}{C}$ is positive it follows that $\underset{\sim}{R}$ is negative

$$\langle \underset{\sim}{R}(\underset{\sim}{\Phi}), \underset{\sim}{\Phi} \rangle \leq 0$$

However it is not necessarily true that

$$\langle \underset{\sim}{R}(\underset{\sim}{\Phi}), \underset{\sim}{\Phi} \rangle = 0 \quad \text{iff} \quad \underset{\sim}{\Phi} = \underset{\sim}{0}$$ QED

5.1.4 A necessary and sufficient condition for existence and uniqueness of the creep problem

Theorem 5.3: Suppose that the hypothesis of Theorem 5.1 are satisfied then a necessary and sufficient condition for there to exist a unique solution of the creep problem is that the equation of inelastic growth should have a unique solution.

Proof: If there exists a solution of the creep problem there also exists an associated inelastic strain. However, by Theorem 4.1 this inelastic strain satisfies the growth equation; hence the growth equation has a solution. Further if the growth equation had two solutions, two solutions of the creep problem could be constructed via Theorem 4.2 which would contradict uniqueness; the growth equation must therefore have a unique solution.

Alternatively suppose that there exists a solution $\underset{\sim}{\varepsilon}^{*} \in C^{r}[I; L_2(V)]$ of the inelastic growth equation so that there is a solution of the residual elastic problem for $\underset{\sim}{\varepsilon}^{*}$ denoted by $[\underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_R] \in C^{r}[I; [W_2^{(1)}(V)]^3 \times L_2(V) \times L_2(V)]$. There also exists a solution $[\underset{\sim}{u}_o, \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}_o] \in C^{r}[I; [W_2^{(1)}(V)]^3 \times L_2(V) \times L_2(V)]$ of the equivalent elastic problem.

Therefore by Theorem 4.2 there is a solution $[\underset{\sim}{u}_o + \underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_o + \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_o + \underset{\sim}{\sigma}_R]$ which satisfies the creep problem. To establish uniquess suppose there are two processes $[\underset{\sim}{u}', \underset{\sim}{\varepsilon}', \underset{\sim}{\sigma}']$ and $[\underset{\sim}{u}^2, \underset{\sim}{\varepsilon}^2, \underset{\sim}{\sigma}^2]$ which are solutions of the creep problem.

Define $\underset{\sim}{\varepsilon}_j^{*}$, $j = 1,2$ by

$$\dot{\underset{\sim}{\varepsilon}}_j^{*} = \underset{\sim}{F}(\underset{\sim}{\sigma}^j, \underset{\sim}{\alpha}^j) \qquad \dot{\underset{\sim}{\alpha}}^j = \underset{\sim}{G}(\underset{\sim}{\sigma}^j, \underset{\sim}{\alpha}^j)$$
$$\underset{\sim}{\varepsilon}_j^{*}(0) = \underset{\sim}{0} \qquad \underset{\sim}{\alpha}^j(0) = \underset{\sim}{\alpha}_o$$

then the difference $[\underset{\sim}{u}_R^j, \underset{\sim}{\varepsilon}_R^j, \underset{\sim}{\sigma}_R^j] = [\underset{\sim}{u}^j - \underset{\sim}{u}_o, \underset{\sim}{\varepsilon}^j - \underset{\sim}{\varepsilon}_o, \underset{\sim}{\sigma}^j - \underset{\sim}{\sigma}_o]$ $j = 1,2$ is a solution of a residual elastic problem for

$\underset{\sim}{\varepsilon}^*_j$ by Lemma 4.1. But also by Lemma 4.1 $\underset{\sim}{\varepsilon}^*_j$ $j = 1,2$ should satisfy the equation of inelastic growth and it is assumed that this has a unique solution, therefore

$$\underset{\sim}{\varepsilon}^*_1 = \underset{\sim}{\varepsilon}^*_2$$

but from Theorem 5.2

$$\| \underset{\sim}{u}^1_R - \underset{\sim}{u}^2_R \|_{[W^{(1)}_2]^3} \leq K^R_\gamma \| \underset{\sim}{\varepsilon}^*_1 - \underset{\sim}{\varepsilon}^*_2 \|_{L_2} = 0$$

hence

$$\| \underset{\sim}{u}^1 - \underset{\sim}{u}^2 \|_{[W^{(1)}_2]^3} = \| \underset{\sim}{u}^1_R - \underset{\sim}{u}^2_R \|_{[W^{(1)}_2]^3} = 0$$

and so $\underset{\sim}{u}^1 \equiv \underset{\sim}{u}^2$. Similarly $\underset{\sim}{\varepsilon}^1 \equiv \underset{\sim}{\varepsilon}^2$ and $\underset{\sim}{\sigma}^1 \equiv \underset{\sim}{\sigma}^2$ and uniqueness is proven. QED

The implication of this result is that existence and uniqueness of the creep problem can be reduced to a study of existence and uniqueness of solution of the equation of inelastic growth. A discussion of the possibility of this follows.

5.1.5. On the existence and uniqueness of solution of of the equations of inelastic growth

Consider the equations of inelastic growth (4.14) in the compact form

$$\dot{\underset{\sim}{\varepsilon}}^* = \underset{\sim}{f}(t, \underset{\sim}{\varepsilon}^*, \underset{\sim}{\alpha}) \qquad (5.2)$$

$$\dot{\underset{\sim}{\alpha}} = g(t, \underset{\sim}{\varepsilon}^*, \underset{\sim}{\alpha})$$

$$\underset{\sim}{\varepsilon}^*(0) = \underset{\sim}{0} \qquad \underset{\sim}{\alpha}(0) = \underset{\sim}{\alpha}_0 \qquad (5.3)$$

which can be identified as an initial value problem in a Hilbert space. Define the couple $E = (\underset{\sim}{\varepsilon}^*, \underset{\sim}{\alpha})$ as a process with values in the Hilbert space $\mathcal{H}$ given by the product space

$$\mathcal{H} = L_2(V) \times [l_2(V)]^N$$

with the inner product

$$\langle E_1, E_2 \rangle_{\mathcal{H}} = \langle \underset{\sim}{\varepsilon}^*_1, \underset{\sim}{\varepsilon}^*_2 \rangle_{L_2} + \langle \underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2 \rangle_{[l_2]^N}$$

and norm

$$\| E \|^2_{\mathcal{H}} = \langle E, E \rangle_{\mathcal{H}}$$

Then (5.2) can be expressed in the contracted form

$$\frac{dE}{dt} = \mathfrak{I}(t, E) \qquad (5.3)$$

where $\mathfrak{I} : I \times \mathcal{H} \longrightarrow \mathcal{H}$ is defined by

$$\mathfrak{I}(t, E) = (f(t, \xi^*, \alpha), g(t, \xi^*, \alpha))$$

and $E(0) = E_0 = (\underset{\sim}{0}, \alpha_0)$.

As a consequence the ordinary differential equation (5.3) shall be studied. The theory of such equations is embryonic. The classical text on the subject is by the French conglomerate N Bourbaki [141], more recent works merely affirm the results [142-144]. The Bourbaki theorems require continuity of $\mathfrak{I}$ and boundedness together with the Lipschitz condition

$$\| \mathfrak{I}(t, E_1) - \mathfrak{I}(t, E_2) \|_{\mathcal{H}} \leq K \| E_1 - E_2 \|_{\mathcal{H}}$$

for some constant K. This condition has been relaxed by F Browder [145-147] to monotonicity

$$\langle \mathfrak{I}(t, E_1) - \mathfrak{I}(t, E_2), E_1 - E_2 \rangle_{\mathcal{H}} \leq K \| E_1 - E_2 \|^2_{\mathcal{H}}$$

and to a weaker statement, known as accretivity by T Kato [148,149] and H Brezis [150,151]. All of these publications (and an ever growing number not mentioned here) aim at relaxing the genus of abstract space and continuity. Notable however, are the studies of H Murakami [152] and R H Martin Jr [153,154] whose objective is the relaxation of monotonicity and whose course is followed here to an extent.

Definition 5.1: Any function $E(t)$ is said to be a solution of (5.3) if the following conditions are satisfied,

(a) $E \in C[I ; \mathcal{H}]$;

(b) $E(0) = E_0$;

(c) $E(t)$ is strongly differentiable in t for $[0, a)$ and satisfies (5.3) for $[0, a]$.

Definition 5.2: The functional $V \in C[I \times \mathcal{H} \times \mathcal{H}; \mathbb{R}^+]$ is a Lyapunov function if

(a) $V(t, E_1, E_2) > 0$ if $E_1 \neq E_2$, $= 0$ $E_1 = E_2$ $\forall t$;

(b) $V(t, E_1^{(n)}, E_2^{(n)}) \longrightarrow 0$ as $n \to \infty$ implies $E_1^{(n)} - E_2^{(n)} \longrightarrow 0$ $\forall t$ and uniformly in t ;

(c) $V(t, E_1, E_2)$ has continuous (bounded) Gateaux derivatives and for any positive number N

$$\frac{\partial V}{\partial t} \quad , \quad D_1 V \cdot H \quad , \quad D_2 V \cdot H$$

are continuous in E_1 and E_2 uniformly for $(t, E_1, E_2) \in I \times \mathcal{H} \times \mathcal{H}$ and $\| H \|_{\mathcal{H}} \leq N$. Initially a local existence theorem shall be established. The proof of this theorem, which is a generalization of Murakami's result [152], can be found in Appendix 1.

Theorem 5.4: Consider the closed ball $S_b = \{ E \in \mathcal{H} / \| E - E_0 \|_{\mathcal{H}} \leq b \}$ and let the function $\mathfrak{F}$ satisfy the following conditions in S_b:

(a) $\mathfrak{F} \in C[I \times S_b ; \mathcal{H}]$;

(b) $\mathfrak{F}$ maps bounded sets into bounded sets;

(c) for some Lyapunov function V the inequality

$$\frac{\partial V}{\partial t} + D_1 V \cdot \mathfrak{F}(t, E_1) + D_2 V \cdot \mathfrak{F}(t, E_2) \leq \omega(t, V)$$

holds for $(t, E_1, E_2) \in I \times S_b \times S_b$ where $\omega(t, r)$ is a continuous function from $[0, \infty)$ into $\mathbb{R}$ such that $\omega(0,0) = 0$ and the unique solution of

$$\frac{dr}{dt} = \omega(t, r)$$
$$r(0) = 0$$

is trivial.

Then equation (5.3) has a solution on $[0, a]$ a being the number such that $aM \leq b$ where M is such that

$$\| \mathfrak{F}(t, E) \|_{\mathcal{H}} \leq M \qquad (t, E) \in I \times S_b$$

If the condition (c) of the therem can be established for suitable V and ω the existence of solution is ensured

locally for t Uniqueness is ensured by

Theorem 5.5: Under the assumptions of Theorem 5.4 equation (5.3) has a unique solution.

In fact the conditions on $\mathfrak{F}$ can be considerably weakened to demi-continuity [152, 155,156]; uniqueness holds only if condition (c) of Theorem 5.4 is verified. Global existence, for all time, replaces the local result if the conditions of the theorems can be extended to the whole of $\mathcal{H}$.

Consequently Theorem 5.3 ensures existence and uniqueness of a continuous, strongly differentiable solution of the creep problem if the assumptions of Theorem 5.4 are met. Thus the applicability of internal variable constitutive equations needs to be studied. The simplest possible Lyapunov function is

$$V(t,E_1,E_2) = V(E_1,E_2) = \|E_1 - E_2\|^2_{\mathcal{H}}$$

For this the Gateaux derivatives are

$$D_1V\cdot H = \lim_{s\to 0}\frac{1}{s}[V(E_1+sH,E_2) - V(E_1,E_2)] = 2\langle E_1 - E_2, H\rangle_{\mathcal{H}}$$

$$D_2V\cdot H = \lim_{s\to 0}\frac{1}{s}[V(E_1,E_2+sH) - V(E_1,E_2)] = -2\langle E_1 - E_2, H\rangle_{\mathcal{H}}$$

which are linear in H (and thus continuous Frechet derivatives). Condition (c) of Theorem 5.4 becomes

$$\langle \mathfrak{F}(t,E_1) - \mathfrak{F}(t,E_2), E_1 - E_2\rangle_{\mathcal{H}} \leq \omega(t, \|E_1 - E_2\|^2_{\mathcal{H}}) \qquad (5.4)$$

which reduces to Browders monotonicity condition on taking

$$\omega(t,r) = Kr$$

In order to apply the inequality (5.4) to constitutive functions the following is useful:

Lemma 5.5: If in a neighbourhood of $\underset{\sim}{\sigma}_o$ and $\underset{\sim}{\alpha}_o$ the constitutive functions $\underset{\sim}{F}$ and $\underset{\sim}{G}$ of equations (4.6) satisfy the conditions

$$\|\underset{\sim}{F}(\underset{\sim}{\sigma}_1,\underset{\sim}{\alpha}_1) - \underset{\sim}{F}(\underset{\sim}{\sigma}_2,\underset{\sim}{\alpha}_2)\|^2_{L_2} \leq W_1(\|\underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2\|^2_{L_2} + \|\underset{\sim}{\alpha}_1 - \underset{\sim}{\alpha}_2\|^2_{C[t_1]^N})$$

$$\|\underset{\sim}{G}(\underset{\sim}{\sigma}_1,\underset{\sim}{\alpha}_1) - \underset{\sim}{G}(\underset{\sim}{\sigma}_2,\underset{\sim}{\alpha}_2)\|^2_{C[t_2]^N} \leq W_2(\|\underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2\|^2_{L_2} + \|\underset{\sim}{\alpha}_1 - \underset{\sim}{\alpha}_2\|^2_{C[t_1]^N})$$

such that the positive valued scalar functions $W_1(s), W_2(s)$ are monotonic increasing with $s \geq 0$ and the function

$$\sqrt{r\,[\,W_1(\bar{K}r) + W_2(\bar{K}r)\,]}$$

satisfies the conditions imposed on $\omega(t,r)$ for any constant $\bar{K}$, and if $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are bounded, in the sense of mapping bounded sets into bounded sets, then the requirements of Theorem 5.4 are satisfied in the form of (5.4).

The proof of this Lemma is straightforward yet laborious, following simply from standard inequalities for norms and inner products [157], and for this reason shall not be presented here.

The functions W_1 and W_2 can be chosen in a number of ways, for example for ordinary Lipschitz continuity

$$W(s) = Ks \qquad W(s) = Ks$$

and, indeed, if $\underset{\sim}{F}$ and $\underset{\sim}{G}$ are derivable from potentials this is identically true [157].

These results generalize those of J Necas [110 113] as the hypotheses on the constitutive functions admit a broader spectrum in the nonlinear range. Unfortunately a large class of time dependent inelastic material models for metallic creep are <u>not</u> included since often the initial inelastic strain rate is unbounded and thus the conditions of Theorem 5.4 cannot be ensured. If this singularity is due to imposing t = 0 (as in the time hardening model (section 2.5)) then the existence theorem can be re-established (viz. Appendix 1 and the next chapter); however, if it is due to the initial condition $\underset{\sim}{\varepsilon}^* = \underset{\sim}{0}$ complications result. In real analysis this difficulty can usually be overcome by rearranging the equations to remove the singularity (see for example the manipulations for the strain-hardening model by J Hult [4,80] and

B Einarrson [118,119]. Although this is theoretically possible for particular cases of the internal variable theory, it is intractable in the general situation. To the writer's knowledge there is no general (or particular) theory to account for singularities in the equations due to the initial values of the unknown. In this sense the question of existence and uniqueness must remain open.

5.2 Asymptotic behaviour

The existence theorem establishes only local behaviour over a small time imterval. It is often extremely difficult to prove overall global existence for all time. An alternative approach to this question is to investigate the asymptotic behaviour of the body at large times and heed what additional conditions need apply.

Information on the behaviour of creeping bodies at large times is scant and the proofs of such decidedly lacking in rigour. Under constant loading F A Leckie and J B Martin and A R S Ponter [87,88] have demonstrated that the stress field is asymptotic with the steady state phase of constant stress and although they realised the nature of the limiting state they failed to consider how this state was reached. For variable, and in particular for cyclic, loading the only specific result is the "Theorem of Convergent internal stresses" of C O Frederick and P J Armstrong [91, 92-95] which established that for large times the initial behaviour of a body has no effect on the ultimate stress pattern. To begin with this theorem shall be reconsidered.

5.2.1 A constitutive model and the energy norms

The majority of information on asymptotic behaviour applies to the time hardening model of creep. Here the internal variable theory shall be used once more but in

a special, more beneficial, form. Throughout dependence on t shall be dropped.

The constitutive equations (4.6) are rewritten as

$$\begin{aligned} \dot{\underset{\sim}{\varepsilon}}_v &= \underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) \\ -\dot{\underset{\sim}{\beta}} &= \underset{\sim}{G}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) \end{aligned} \qquad (5.5)$$

where

$$\underset{\sim}{\beta} = \underset{\sim}{L}(\underset{\sim}{\alpha})$$

$\underset{\sim}{L}$ being a linear, symmetric and positive definite operator in $\underset{\sim}{L}_2(V)$ (notice that although the same $\underset{\sim}{G}$ has been used notationally it is, in fact, $-\underset{\sim}{L}(\underset{\sim}{G})$ but no confusion should arise if (5.5) is taken on its own merits). It is further assumed, although this is not strictly necessary, that $\underset{\sim}{L}$ is also invertible. In the terminology of Q S Nguyen [158] the material is "standard, generalized".

The norms and inner products in $L_2(V)$ and $[l_2(V)]^N$ are denoted now in the more convenient form

$$\langle \cdot, \cdot \rangle_{L_2} = \langle \cdot, \cdot \rangle$$

$$\langle \cdot, \cdot \rangle_{[l_2]^N} = \{ \cdot, \cdot \}$$

Thus $\underset{\sim}{L}$ has the properties

$$\begin{aligned} &\{\underset{\sim}{L}(\underset{\sim}{\alpha}_1), \underset{\sim}{\alpha}_2\} = \{\underset{\sim}{\alpha}_1, \underset{\sim}{L}(\underset{\sim}{\alpha}_2)\} \\ &\{\underset{\sim}{L}(\underset{\sim}{\alpha}), \underset{\sim}{\alpha}\} \geq 0, = 0 \ \text{iff} \ \underset{\sim}{\alpha} = \underset{\sim}{0} \qquad (5.6) \\ &\{\underset{\sim}{L}(\mu\underset{\sim}{\alpha}_1 + \lambda\underset{\sim}{\alpha}_2), \underset{\sim}{\alpha}_3\} = \mu\{\underset{\sim}{L}(\underset{\sim}{\alpha}_1), \underset{\sim}{\alpha}_3\} + \lambda\{\underset{\sim}{L}(\underset{\sim}{\alpha}_2), \underset{\sim}{\alpha}_3\} \end{aligned}$$

It shall be assumed that the operators $\underset{\sim}{F}$ and $\underset{\sim}{G}$ in (5.5) have the following monotonicity condition

$$\begin{aligned} &\langle \underset{\sim}{F}(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1) - \underset{\sim}{F}(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2), \underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2 \rangle + \{\underset{\sim}{G}(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1) - \underset{\sim}{G}(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2), \underset{\sim}{\alpha}_1 - \underset{\sim}{\alpha}_2\} \\ &\qquad \geq 0, \ = 0 \ \text{iff} \ (\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1) = (\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2) \end{aligned} \qquad (5.7)$$

In order to help the development two further inner products and norms are introduced:

Definition 5.3: Let the bilinear functional $\langle \underset{\sim}{C}(\underset{\sim}{\sigma}_1), \underset{\sim}{\sigma}_2 \rangle$, where $\underset{\sim}{C}$ is the elastic operator, be used to define an inner product

$$[\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2]_e = \langle \underset{\sim}{C}(\underset{\sim}{\sigma}_1), \underset{\sim}{\sigma}_2 \rangle$$

noting the properties of Lemma 5.1, then the elastic energy

norm is

$$\|\underset{\sim}{\sigma}\|_e = \sqrt{[\underset{\sim}{\sigma}, \underset{\sim}{\sigma}]_e}$$

Definition 5.4: Introduce the product space $L_2(\mathcal{V}) \times [l_2(\mathcal{V})]^N = \mathcal{H}_E$ with components $\Sigma = (\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$ where $\underset{\sim}{\sigma} \in L_2(\mathcal{V})$, $\underset{\sim}{\alpha} \in [l_2(\mathcal{V})]^N$ and inner product

$$[\Sigma_1, \Sigma_2]_E = \langle \underset{\sim}{C}(\underset{\sim}{\sigma}_1), \underset{\sim}{\sigma}_2 \rangle + \{\underset{\sim}{L}(\underset{\sim}{\alpha}_1), \underset{\sim}{\alpha}_2\}$$

noting Lemma 5.1 and relations (5.6), then the internal energy norm is

$$\|\Sigma\|_E = \sqrt{[\Sigma, \Sigma]_E}$$

Clearly these norms are related to the physical energies

$$\mathcal{E}(\underset{\sim}{\sigma}) = \frac{1}{2}\|\underset{\sim}{\sigma}\|_e^2 = \text{elastic energy}$$
$$\mathcal{J}(\Sigma) = \frac{1}{2}\|\Sigma\|_E^2 = \text{internal (elastic) energy}$$

and hence their designations.

The following results are pertinent:

Lemma 5.6: The elastic energy norm and the internal energy norms are related by the inequality

$$\|\underset{\sim}{\sigma}\|_e \leq \|\Sigma\|_E$$

Lemma 5.7: The stress field $\hat{\underset{\sim}{Q}} \in L(\mathcal{V})$ is said to be "self-equilibrating" for the mixed problem if

$$\operatorname{div} \hat{\underset{\sim}{Q}} = \underset{\sim}{0} \quad \text{in } \mathcal{V}$$
$$\hat{\underset{\sim}{Q}} \cdot \underset{\sim}{n} = 0 \quad \text{on } \mathcal{S}_T$$

Then if $\underset{\sim}{\varepsilon}^* \in L(\mathcal{V})$

$$[\underset{\sim}{R}(\underset{\sim}{\varepsilon}^*), \hat{\underset{\sim}{Q}}]_e = - \langle \underset{\sim}{\varepsilon}^*, \hat{\underset{\sim}{Q}} \rangle$$

Proof: Let $(\underset{\sim}{u}_R, \underset{\sim}{\varepsilon}_R, \underset{\sim}{\sigma}_R)$ be the solution of the residual elastic problem corresponding to $\underset{\sim}{\varepsilon}^*$ then

$$[\underset{\sim}{R}(\underset{\sim}{\varepsilon}^*), \hat{\underset{\sim}{Q}}]_e = \langle \underset{\sim}{C}(\underset{\sim}{R}(\underset{\sim}{\varepsilon}^*)), \hat{\underset{\sim}{Q}} \rangle = \langle \underset{\sim}{\varepsilon}_R - \underset{\sim}{\varepsilon}^*, \hat{\underset{\sim}{Q}} \rangle$$
$$= \langle \underset{\sim}{\varepsilon}_R, \hat{\underset{\sim}{Q}} \rangle - \langle \underset{\sim}{\varepsilon}^*, \hat{\underset{\sim}{Q}} \rangle$$

But using Green's identity

$$\langle \underset{\sim}{\varepsilon}_R, \hat{\underset{\sim}{Q}} \rangle = \int_{\mathcal{V}} \hat{\underset{\sim}{\nabla}} \cdot \underset{\sim}{u}_R \hat{\underset{\sim}{Q}} \, d\underset{\sim}{X} = - \int_{\mathcal{V}} \operatorname{div} \hat{\underset{\sim}{Q}} \cdot \underset{\sim}{u}_R \, d\underset{\sim}{X} + \int_{\mathcal{S}} \hat{\underset{\sim}{Q}} \cdot \underset{\sim}{n} \cdot \underset{\sim}{u}_R \, d\underset{\sim}{X} = 0$$

by definition. Hence result. QED

5.2.2. The Theorem of Convergent Internal Stresses

Whereas existence and uniqueness were studied using the equation of inelastic growth, asymptotic behaviour is more conveniently handled using the redistribution equations (4.16) which, for equation (5.5) become

$$\begin{aligned} &\frac{d}{dt}\underset{\sim}{\sigma}(t) = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t))) + \dot{\underset{\sim}{\sigma}}_o(t) \\ &-\frac{d}{dt}\underset{\sim}{\beta}(t) = \underset{\sim}{G}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t)) \qquad (5.8) \\ &\underset{\sim}{\sigma}(0) = \underset{\sim}{\sigma}_o(0) \qquad \underset{\sim}{\alpha}(0) = \underset{\sim}{\alpha}_o \end{aligned}$$

The theorem compares two boundary data processes acting on the same body.

For problem 1 let boundary data $(\bar{\underset{\sim}{u}}_1, \bar{\underset{\sim}{t}}_1)$ and body force $\bar{\underset{\sim}{b}}_1$, be applied to the body, the resulting process includes a stress field $\underset{\sim}{\sigma}_1$ and internal variables $\underset{\sim}{\alpha}_1$ satisfying the redistribution equations

$$\begin{aligned} &\dot{\underset{\sim}{\sigma}}_1 = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1)) + \dot{\underset{\sim}{\sigma}}_{o1}, \quad \underset{\sim}{\sigma}_1(0) = \underset{\sim}{\sigma}_{o1}(0) \\ &-\dot{\underset{\sim}{\beta}}_1 = \underset{\sim}{G}(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1), \qquad \underset{\sim}{\alpha}_1(0) = \underset{\sim}{\alpha}_{o1} \end{aligned} \qquad (5.9)$$

For problem 2 let data $(\bar{\underset{\sim}{u}}_2, \bar{\underset{\sim}{t}}_2, \bar{\underset{\sim}{b}}_2)$ apply, then the resultant stress $\underset{\sim}{\sigma}_2$ and hidden variables $\underset{\sim}{\alpha}_2$ satisfy

$$\begin{aligned} &\dot{\underset{\sim}{\sigma}}_2 = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2)) + \dot{\underset{\sim}{\sigma}}_{o2}, \quad \underset{\sim}{\sigma}_2(0) = \underset{\sim}{\sigma}_{o2}(0) \\ &-\dot{\underset{\sim}{\beta}}_2 = \underset{\sim}{G}(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2), \qquad \underset{\sim}{\alpha}_2(0) = \underset{\sim}{\alpha}_{o2} \end{aligned} \qquad (5.10)$$

then,

Theorem 5.6: Suppose that after a finite time $(\bar{\underset{\sim}{u}}_1, \bar{\underset{\sim}{t}}_1, \bar{\underset{\sim}{b}}_1)$ becomes identical to $(\bar{\underset{\sim}{u}}_2, \bar{\underset{\sim}{t}}_2, \bar{\underset{\sim}{b}}_2)$ then $(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1)$ is asymptotically equivalent to $(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2)$ in the internal energy norm.

Proof: From (5.9) and (5.10) with $\Sigma_i = (\underset{\sim}{\sigma}_i, \underset{\sim}{\alpha}_i)$ $i = 1, 2$

$$\left[\frac{d}{dt}(\Sigma_1 - \Sigma_2), \Sigma_1 - \Sigma_2\right]_E = \left[\underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1) - \underset{\sim}{F}(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2)), \underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2\right]_e$$

$$-\left\{\underset{\sim}{G}(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1) - \underset{\sim}{G}(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2), \underset{\sim}{\alpha}_1 - \underset{\sim}{\alpha}_2\right\} + \left[\dot{\underset{\sim}{\sigma}}_{o1} - \dot{\underset{\sim}{\sigma}}_{o2}, \underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2\right]_e$$

After a finite time $\underset{\sim}{\sigma}_{01} = \underset{\sim}{\sigma}_{02}$ and $\underset{\sim}{\sigma}_1 - \sigma_2$ is self equilibrating, hence, using Lemma 5.7

$$\frac{1}{2}\frac{d}{dt}\|\Sigma_1-\Sigma_2\|_E^2 = -\langle \underset{\sim}{F}(\underset{\sim}{\sigma}_1,\underset{\sim}{\alpha}_1)-\underset{\sim}{F}(\underset{\sim}{\sigma}_2,\underset{\sim}{\alpha}_2), \underset{\sim}{\sigma}_1-\underset{\sim}{\sigma}_2\rangle - \{\underset{\sim}{G}(\underset{\sim}{\sigma}_1,\underset{\sim}{\alpha}_1)-\underset{\sim}{G}(\underset{\sim}{\sigma}_2,\underset{\sim}{\alpha}_2), \underset{\sim}{\alpha}_1-\underset{\sim}{\alpha}_2\} \quad (i)$$

and using monotonicity (5.7)

$$\frac{1}{2}\frac{d}{dt}\|\Sigma_1-\Sigma_2\|_E^2 \leq 0$$

Thus $\|\Sigma_1-\Sigma_2\|_E^2$ is monotonically decreasing, but it is also bounded below by zero hence must tend to a constant as $t \to \infty$ and in the limit $d/dt\,\|\Sigma_1-\Sigma_2\|_E^2 = 0$; but from (i) this is true if and only if $\Sigma_1 \equiv \Sigma_2$ by (5.7) and so $\|\Sigma_1-\Sigma_2\|_E^2 \to 0$ as $t \to \infty$ and the result is proven. QED

Frederick and Armstrong proved this result only for different initial states and a time hardening, ideally plastic constitutive model. In fact the theorem may be further extended, in a manner similar to [9].

Theorem 5.7: Suppose that after a finite time $(\underset{\sim}{u}_1, \underset{\sim}{\varepsilon}_1, \underset{\sim}{b}_1)$ becomes identical to $(\underset{\sim}{u}_2, \underset{\sim}{\varepsilon}_2, \underset{\sim}{b}_2)$ except for a constant displacement vector field (i.e. $\underset{\sim}{u}_1(t) = \underset{\sim}{u}_2(t) + \underset{\sim}{u}_\infty$, where $\underset{\sim}{u}_\infty$ is a constant) then $(\underset{\sim}{\sigma}_1, \underset{\sim}{\alpha}_1)$ and $(\underset{\sim}{\sigma}_2, \underset{\sim}{\alpha}_2)$ are asymptotically equivalent in the internal energy norm.

Proof: After a finite time $\underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2$ is self equilibrating and moreover $\dot{\underset{\sim}{\sigma}}_{01} \equiv \dot{\underset{\sim}{\sigma}}_{02}$. Proof proceeds as before. QED

None of the investigators concerned with the maturation of the Theorem of Convergent Internal Stresses recognised the role of the norm. The definition of an appropriate norm in a limiting process is of fundamental importance. It allows the behaviour to be kept strictly under control, and moreover, classical concepts from real analysis can be employed. Unfortunately, the extension to an internal variable model proposed by A R S Ponter and F A Leckie [96] does not use a properly defined norm.

The Theorem is interesting but unhappily does not say anything about the nature of an asymptotic state. It is often quoted as a proof of progression to the steady state under constant loads [9,96] but this is more fancied than rigorous.

5.2.3. The Pure Creep Problem

It is well known that a simple hyperstatic component subjected to a constant load shall eventually reach a steady phase wherein stresses are constant; this steady state can be characterized by the neglect of elastic effects. It also seems that complex structures under constant loading similarly attains a steady phase of constant stress which may be secured on solving a nonlinear boundary value problem. Expanding this definition the "pure creep problem" is postulated here; just as the equivalent (pure) elastic problem ignores inelasticity, the pure creep problem ignores elasticity.

Definition 5.5: The process $[\dot{\underset{\sim}{u}}_c, \dot{\underset{\sim}{\varepsilon}}_c, \underset{\sim}{\sigma}_c]$ satisfies the pure creep problem associated with the (mixed) quasi-static creep problem corresponding to the boundary data $[\bar{\underset{\sim}{u}}, \bar{\underset{\sim}{\tau}}]$ and body process $\bar{\underset{\sim}{b}}$ if its values satisfy:

$$
\begin{aligned}
&\operatorname{div} \underset{\sim}{\sigma}_c(t) + \bar{\underset{\sim}{b}}(t) = \underset{\sim}{0} && \text{in } V \\
&\dot{\underset{\sim}{\varepsilon}}_c(t) = \hat{\nabla} \cdot \dot{\underset{\sim}{u}}_c(t) && \\
&\dot{\underset{\sim}{\varepsilon}}_c(t) = \underset{\sim}{F}(\underset{\sim}{\sigma}_c(t), \underset{\sim}{\alpha}_c(t)) && \text{in } V+J \\
&-\dot{\underset{\sim}{\beta}}_c(t) = \underset{\sim}{G}(\underset{\sim}{\sigma}_c(t), \underset{\sim}{\alpha}_c(t)) && \\
&\dot{\underset{\sim}{u}}_c(t) = \dot{\bar{\underset{\sim}{u}}}(t) && \text{on } J_u \\
&\underset{\sim}{\sigma}_c(t)\underset{\sim}{n} = \bar{\underset{\sim}{\tau}}(t) && \text{on } J_\tau
\end{aligned}
$$

assuming $\bar{\underset{\sim}{u}}$ to be differentiable.

In fact here $\underset{\sim}{\alpha}_c(t)$ appears only as a parameter, whence once $\underset{\sim}{\sigma}_c(t)$ is determined the correspondence

$$\underset{\sim}{\sigma}_c(t) = \underset{\sim}{P}(t, \underset{\sim}{\alpha}_c(t)) \qquad (5.11)$$

is constructed for some nonlinear operator $P: I \times [l_2(V)]^N \to L_2(V)$ Then the ordinary differential equation

$$\frac{d}{dt} \underset{\sim}{\beta}_c(t) = \underset{\sim}{G}(\underset{\sim}{P}(t, \underset{\sim}{\alpha}_c(t)), \underset{\sim}{\alpha}_c(t)) \qquad (5.12)$$

needs to be resolved, subject to some initial condition.

It would be of interest to investigate the properties of the pure creep problem alone - for example, existence and uniqueness of solution - but since the equations constitute a nonlinear elliptic boundary value problem standard techniques ([159-163],[63-65],[144,147,156]) can be employed by imposing sufficient conditions on $\underset{\sim}{F}$. In consequence the ordinary differential equation (5.12) alone need be studied. This topic shall not be pursued here.

The pure creep problem is of significance in the study of asymptotic behaviour; it is an extension of the well known steady state for constant loads, and stationary state for proportional loading [89,90].

5.2.4 Some results on asymptotic behaviour

The principal result is that, for a class of mixed boundary value problems, the actual stress field shall tend to be the same as the pure creep stress field.

Theorem 5.8: For the mixed boundary value problem of quasi-static creep if the applied displacement $\bar{\underset{\sim}{u}}$ becomes constant after a finite time then the actual stress and internal variable fields shall be asymptotically equivalent to the pure creep fields in the internal energy norm if the boundary data is such that

$$\| \dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}_c \|_e \longrightarrow 0 \qquad t \to \infty$$

in such a way that

$$\int_0^t \| \dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}_c \|_e dt$$

remains bounded.

Proof: From the redistribution equation (5.8)

$$\frac{1}{2}\frac{d}{dt}\|\Sigma-\Sigma_c\|_E^2 = [\underline{R}(\underline{F}(\underline{\sigma},\underline{\alpha})),\underline{\sigma}-\underline{\sigma}_c]_e - \{\underline{G}(\underline{\sigma},\underline{\alpha})-\underline{G}(\underline{\sigma}_c,\underline{\alpha}_c),\underline{\alpha}-\underline{\alpha}_c\} + [\dot{\underline{\sigma}}_o-\dot{\underline{\sigma}}_c,\underline{\sigma}-\underline{\sigma}_c]_e$$

But $\underline{\sigma}-\underline{\sigma}_c$ is self equilibrating, hence, from Lemma 5.7

$$\frac{1}{2}\frac{d}{dt}\|\Sigma-\Sigma_c\|_E^2 = -\langle \underline{F}(\underline{\sigma},\underline{\alpha})-\underline{F}(\underline{\sigma}_c,\underline{\alpha}_c),\underline{\sigma}-\underline{\sigma}_c\rangle + \{\underline{G}(\underline{\sigma},\underline{\alpha})-\underline{G}(\underline{\sigma}_c,\underline{\alpha}_c),\underline{\alpha}-\underline{\alpha}_c\} - \langle \dot{\underline{\varepsilon}}_c,\underline{\sigma}-\underline{\sigma}_c\rangle + [\dot{\underline{\sigma}}_o-\dot{\underline{\sigma}}_c,\underline{\sigma}-\underline{\sigma}_c]_e \qquad (i)$$

From the conditions of the theorem, after a finite time, using Green's identity

$$\langle \dot{\underline{\varepsilon}}_c,\underline{\sigma}-\underline{\sigma}_c\rangle = 0$$

since $\underline{\sigma}-\underline{\sigma}_c$ is self equilibrating. Hence using monotonicity (5.7)

$$\frac{1}{2}\frac{d}{dt}\|\Sigma-\Sigma_c\|_E^2 \leq [\dot{\underline{\sigma}}_o-\dot{\underline{\sigma}}_c,\underline{\sigma}-\underline{\sigma}_c]_e \leq \|\dot{\underline{\sigma}}_o-\dot{\underline{\sigma}}_c\|_e\|\Sigma-\Sigma_c\|_E \qquad (ii)$$

with Lemma 5.6 and the Schwartz inequality [157]. From the theory of differential inequalities [155 : Theorem 1.4.1. p15].

$$\|\Sigma-\Sigma_c\|_E \leq \|\Sigma(0)-\Sigma_c(0)\|_E + \int_o^t \|\dot{\underline{\sigma}}_o-\dot{\underline{\sigma}}_c\|_e dt$$

and thus $\|\Sigma-\Sigma_c\|_E$ is bounded i.e. there exists a constant M such that

$$\|\Sigma-\Sigma_c\|_E \leq M$$

whence from (ii)

$$\frac{1}{2}\frac{d}{dt}\|\Sigma-\Sigma_c\|_E^2 \leq \|\dot{\underline{\sigma}}_o-\dot{\underline{\sigma}}_c\|\cdot M \longrightarrow 0 \quad as \quad t\longrightarrow\infty$$

and $\|\Sigma-\Sigma_c\|_E^2 \longrightarrow$ a constant. But from (i) and (5.7)

$$\frac{1}{2}\frac{d}{dt}\|\Sigma-\Sigma_c\|_E^2 - [\dot{\underline{\sigma}}_o-\dot{\underline{\sigma}},\underline{\sigma}-\underline{\sigma}_c]_e = 0 \quad iff \quad \Sigma\equiv\Sigma_c$$

Therefore

$$\|\Sigma-\Sigma_c\|_E^2 \longrightarrow 0 \quad t\longrightarrow\infty$$

and the result is proven. QED

Four important corollaries are:

Corollary 5.3: If the applied boundary data and body forces become constant after a finite time then the stress field is asymptotically equivalent to the steady state stress field in the internal energy norm.

Corollary 5.4: For the second boundary value problem under the conditions of the theorem the stress and internal variable fields are asymptotically equivalent to the pure creep stress and internal variable fields in the internal energy norm.

Corollary 5.5: For the first boundary value problem if the conditions of the Theorem are met then the result of the Theorem remains true.

Corollary 5.6: For all time the inequality

$$\|\Sigma(t)-\Sigma_c(t)\|_E \leq \|\Sigma(0)-\Sigma_c(0)\|_E + \int_0^t \|\dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}_c\|_e \, dt$$

is satisfied.

Theorem 5.8 unfortunately does not include the general case of the mixed boundary value problem with continuously variable displacements for all time. Nonetheless for the first boundary value problem with body force absent - known as the "relaxation problem".

Theorem 5.9: For the relaxation problem if $\|\dot{\underset{\sim}{\sigma}}_o\|_e \rightarrow 0$ so that $\int_0^t \|\dot{\underset{\sim}{\sigma}}_o\| dt$ is bounded then the stress and internal variable fields are asymptotically stable (i.e. tend to zero) in the internal energy norm.

Proof: Identical to that of Theorem 5.8 replacing the pure creep fields by zero. QED

The assumptions concerning the asymptotic behaviour of the equivalent elastic and pure creep problems can be fulfilled in a number of ways. A special situation is where

$$\|\dot{\underset{\sim}{\bar{q}}}\|_{[L_2]^3} \qquad \|\dot{\underset{\sim}{\bar{\varepsilon}}}\|_{[L_2(S_T)]^3} \qquad \|\dot{\underset{\sim}{\bar{b}}}\|_{[L_2]^3}$$

all tend to zero such that their temporal integrals remain bounded. Obviously these quantities should tend to zero like $1/t^n$, $n>1$.

Finally for some structures and materials

$$\|\underline{\varrho}\|_e^2 \geq \mu_0 \|\underline{\varrho}\|^2$$

for a constant $\mu_0 > 0$. Thus by Lemma 5.6

$$\|\Sigma\|_E^2 \geq \mu_0 \|\underline{\underline{\varrho}}\|^2$$

and consequently as $\|\Sigma - \Sigma_c\|_E^2 \longrightarrow 0$, $\|\underline{\varrho} - \underline{\varrho}_c\|^2 \longrightarrow 0$.

5.3 Global properties

Variational formulations of physical laws describe global properties of the whole system rather than its component parts, encompassing in concise statements entire collections of equations and boundary conditions. They often lead to information on bounds to solution and usually provide a natural means of approximation.

5.3.1. Classical variational formulations

The bulk of variational principles for time dependent inelasticity is concerned with linear viscoelasticity [164-175] being of convolutional type which strongly requires linearity of the constitutive equation. In fact as long as the material model is linear in character there will always be a variational formulation [176-180] and [181,182]. Attempts have been made to derive similar statements for nonlinear viscoelasticity together with thermodynamic effects and an internal variable theory [183-185]; these results, however, contain a fundamental error [186] which restricts their applicability especially to linear behaviour [187] and classes of material with small nonlinearities [188].

It has already been mentioned that one of the dominant features of metallic creep is a strong nonlinear

response (Chapter 2); thus principles of convolutional type, common in linear viscoelasticity, cannot be reaffirmed for creep. Nonetheless two "reduced" variational approaches have been suggested. The first is the Wang and Prager theorem [82], which has been extended to anistropy [189], thermo-visco-plasticity [190], and finite deformations [83,191] - The second is based on work of L M Kachanov [2,54,84,85] which can also be extended to finite deformations [182]; the theorems strictly only apply to a time hardening creep theory but may be altered to cope with other models, for example, strain-hardening [193], by combining with the Wang and Prager approach. A useful discussion of these reduced variational theorems can be found in the book by Yu N Rabotnov [6]. They shall now be identified within the rational framework.

But firstly, what is a variational principle?

Consider the operator equation

$$A(U) = F \qquad (5.13)$$

where U is a set of unknowns such that the set of all possible U forms an inner product space $\mathcal{H}$ i.e. there exists a real valued, bilinear function $\langle \cdot, \cdot \rangle$ with the properties:

$$\langle U, V \rangle = \langle V, U \rangle$$

$$\langle U, U \rangle \geq 0 \ , = 0 \ \text{iff} \ U = 0$$

$$\langle U, \mu V + \lambda W \rangle = \mu \langle U, V \rangle + \lambda \langle U, W \rangle$$

where $U, V, W \in \mathcal{H}$. The operator A, which can be differential, integral, algebraic and so on, transforms members of $\mathcal{H}$ into another member of $\mathcal{H}$; it is therefore supposed that $F \in \mathcal{H}$.

A variational formulation of (5.13) is possible if there exists a functional $W(U)$ (mapping $\mathcal{H}$ onto the real line) such that the gradient of $W(U)$, given by the Gateaux derivative

$$DW(U).H = \lim_{s \to 0} \frac{W(U+sH) - W(U)}{s} = \langle \text{grad} W, H \rangle \qquad \forall H \in \mathcal{H}$$

exists and is (5.13) i.e.

$$grad W(U) = A(U) - F$$

Then A is said to be a potential operator.

It is easily shown that if A is a potential operator, then its potential $W(U)$ is given by

$$W(U) = \int_0^1 \langle A(U_0 + s(U-U_0)), U-U_0 \rangle ds - \langle U, F \rangle + W(U_0)$$

where U_0 is arbitrary; if $W(0) = 0$ then

$$W(U) = \int_0^1 \langle A(sU), U \rangle ds - \langle U, F \rangle \qquad (5.14)$$

A necessary and sufficient condition for A to be potential is that its Gateaux derivative

$$DA(U)\cdot H = \lim_{s \to 0} \frac{A(U+sH) - A(U)}{s}$$

be symmetric i.e. $\forall U, H_1, H_2 \in \mathcal{H}$

$$\langle DA(U)\cdot H_1, H_2 \rangle = \langle DA(U)\cdot H_2 \cdot H_1 \rangle$$

Clearly if A is linear and symmetric

$$W(U) = \frac{1}{2} \langle A(U), U \rangle - \langle U, F \rangle \qquad (5.14)'$$

The theory of potential operators and variational principles derives from work of S G Mihlin [136,194] and E Tonti [195] for linear operators extended to the nonlinear case by V Vainberg [196-199] and A Langebah [65,200] and further developed by E Tonti [201-204] and J Necas [159-161]. Not only has it been employed in steady creep [63-65] but also in other branches of nonlinear continuum mechancs [205-207].

The Wang and Prager variational principle is founded on a standard linear elastic problem treating the inelastic strain rate as a determined quantity at some instant. The creep problem is thus written

$$\begin{bmatrix} 0 & div \\ -\hat{\nabla} & \underset{\sim}{C} \end{bmatrix} \begin{bmatrix} \dot{\underset{\sim}{u}} \\ \dot{\underset{\sim}{\sigma}} \end{bmatrix} = \begin{bmatrix} -\dot{\underset{\sim}{b}} \\ -\dot{\underset{\sim}{\varepsilon}}_v \end{bmatrix} \qquad (5.15)$$

and identified with (5.13) by

$$A = \begin{bmatrix} 0 & div \\ -\hat{\nabla} & \underset{\sim}{C} \end{bmatrix} \qquad U = \begin{bmatrix} \dot{\underset{\sim}{u}} \\ \dot{\underset{\sim}{\sigma}} \end{bmatrix} \qquad F = \begin{bmatrix} -\dot{\underset{\sim}{b}} \\ -\dot{\underset{\sim}{\varepsilon}}_v \end{bmatrix}$$

together with the inner product on the space $\mathcal{H}$ of all couples $(\underset{\sim}{u}, \underset{\sim}{\sigma})$

$$< U_1, U_2 > = < \underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2 >_{L_2} + < \underset{\sim}{u}_1, \underset{\sim}{u}_2 >_{[L_2]^3}$$

Then noting that A is linear the symmetry of its Gateaux derivative $DA(U).H = A(H)$ is ensured by the assumed symmetry of the Hooke's law operator $\underset{\sim}{C}$ in L_2 (Lemma 5.1) together with Green's identity

$$< \underset{\sim}{u}, div\, \underset{\sim}{\sigma} >_{[L_2]^3} = < \underset{\sim}{\sigma}, -\hat{\underset{\sim}{\nabla}} \underset{\sim}{u} >_{L_2} + \{boundary\ terms\} \qquad (5.16)$$

Thus the appropriate potential functional is given by (5.14) with the addition of suitable boundary terms. From this functional may be deduced the standard variational formulations of elasticity [205-217] in a systematic manner [218-220] with terms for the inelastic strain. Thus the Wang and Prager principle is a global formulation of the "initial strain method", or from the rational standpoint adopted in this essay <u>it is a variational principle for the equivalent and residual elastic problems</u> in the form of a "modified" residual process $[\underset{\sim}{u}^r, \underset{\sim}{\varepsilon}^r, \underset{\sim}{\sigma}^r]$ where

$$\underset{\sim}{u}^r = \underset{\sim}{u}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*) \qquad \underset{\sim}{\varepsilon}^r = \underset{\sim}{\varepsilon}_o + \underset{\sim}{R}_\varepsilon(\underset{\sim}{\varepsilon}^*) \qquad \underset{\sim}{\sigma}^r = \underset{\sim}{\sigma}_o + \underset{\sim}{R}(\underset{\sim}{\varepsilon}^*)$$

Whereas the Wang and Prager principle is based on a linear elastic formulation, the Kachanov principle is based on a nonlinear pure creep formulation treating the elastic strain rate as a determined quantity and $\underset{\sim}{\alpha}$ as a parameter. Thus the creep problem is written

$$\begin{bmatrix} 0 & div \\ -\hat{\underset{\sim}{\nabla}} & \underset{\sim}{F}(\cdot, \underset{\sim}{\alpha}) \end{bmatrix} \begin{bmatrix} \dot{\underset{\sim}{u}} \\ \dot{\underset{\sim}{\sigma}} \end{bmatrix} = \begin{bmatrix} -\dot{\underset{\sim}{b}} \\ -\dot{\underset{\sim}{\varepsilon}}_e \end{bmatrix} \qquad (5.17)$$

and identified with (5.13) by

$$A = \begin{bmatrix} 0 & div \\ -\hat{\underset{\sim}{\nabla}} & \underset{\sim}{F}(\cdot, \underset{\sim}{\alpha}) \end{bmatrix} \quad U = \begin{bmatrix} \dot{\underset{\sim}{u}} \\ \dot{\underset{\sim}{\sigma}} \end{bmatrix} \quad F = \begin{bmatrix} -\dot{\underset{\sim}{b}} \\ \dot{\underset{\sim}{\varepsilon}}_e \end{bmatrix}$$

with inner product in $\mathcal{H} = [l_2]^3 \times L_2$ as before. The symmetry of the Gateaux derivative of A is ensured if there exists a functional $\omega(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$ such that

$$\dot{\underset{\sim}{\varepsilon}}_v = \frac{\partial}{\partial \underset{\sim}{\sigma}} \omega(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) = \underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$$

noting (5.16). Again the appropriate functional is given by (5.14) with the addition of boundary terms.

In order to discern how this fits into the rational approach it is noted that an operator $\underset{\sim}{S}(t, \underset{\sim}{\alpha}^*, \dot{\underset{\sim}{\varepsilon}}^*)$ can be defined by a solution of the residual pure creep problem $[\dot{\underset{\sim}{u}}_s, \dot{\underset{\sim}{\varepsilon}}_s, \underset{\sim}{\sigma}_s]$:

$$\begin{aligned} \operatorname{div} \underset{\sim}{\sigma}_s(t) + \bar{\underset{\sim}{b}}(t) &= \underset{\sim}{0} && \text{in } V \\ \dot{\underset{\sim}{\varepsilon}}_s(t) = \hat{\underset{\sim}{\nabla}} \cdot \dot{\underset{\sim}{u}}_s(t) &= \underset{\sim}{F}(\underset{\sim}{\sigma}_s(t), \underset{\sim}{\alpha}^*(t)) + \dot{\underset{\sim}{\varepsilon}}^*(t) && \text{in } V+S \\ \dot{\underset{\sim}{u}}_s(t) &= \dot{\bar{\underset{\sim}{u}}}(t) && \text{on } S_v \\ \underset{\sim}{\sigma}_s(t).\underset{\sim}{n} &= \bar{\underset{\sim}{t}}(t) && \text{on } S_T \end{aligned}$$

then

$$\underset{\sim}{\sigma}_s(t) = \underset{\sim}{S}(t, \underset{\sim}{\alpha}^*, \dot{\underset{\sim}{\varepsilon}}^*)$$

and proceeding in a manner similar to the derivation of the equation of inelastic growth the creep problem can be reduced to the study of

$$\begin{aligned} \underset{\sim}{C}^{-1}(\dot{\underset{\sim}{\varepsilon}}^*) &= \underset{\sim}{S}(t, \underset{\sim}{\alpha}^*, \dot{\underset{\sim}{\varepsilon}}^*) \\ \dot{\underset{\sim}{\alpha}}^* &= \underset{\sim}{G}(\underset{\sim}{C}^{-1}(\dot{\underset{\sim}{\varepsilon}}^*), \underset{\sim}{\alpha}^*) \end{aligned} \qquad (5.18)$$

or

$$\begin{aligned} \dot{\underset{\sim}{\sigma}} &= \underset{\sim}{S}(t, \underset{\sim}{\alpha}, \underset{\sim}{C}(\dot{\underset{\sim}{\sigma}})) \\ \dot{\underset{\sim}{\alpha}} &= \underset{\sim}{G}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}) \end{aligned} \qquad (5.19)$$

which can be demonstrated (by considerable manipulation) to be formally the redistribution equation. It is evident that Kachanov's principle is a variational principle for the residual pure creep problem. The writer can see no advantage in Kachanov's principle and associated residual pure creep problem since it implies the resolution of a sequence of nonlinear problems for the operator $\underset{\sim}{S}$. However Rabotnov [6] has shown that if numerical methods are applied the resulting discrete equations from Kachanov's principle are the same as those derived from the Wang and Prager principle. Conceptually the residual operator $\underset{\sim}{R}$ is superior.

5.3.2. Bounding Theorems

It has often been found that the potential functional W has physical significance in continuum mechanics being associated with the energy of the structure. Thus if W is also convex i.e. $\forall U_1, U_2 \in \mathcal{H}$

$$W(U_1) - W(U_2) - \langle \operatorname{grad} W(U_1), U_1 - U_2 \rangle \geq 0$$

it is minimized by the solution of (5.13) and may in consequence be bounded, i.e.

$$W(U) \leq W(U^*)$$

where U minimizes W (and thus solives (5.13)) and U^* is any member of $\mathcal{H}$. In elasticity this is possible by specialization of W to two dual or complementary principles

$$\begin{aligned} W_1(\underset{\sim}{\sigma}) &\leq W_1(\underset{\sim}{\sigma}^*) \\ W_2(\underset{\sim}{u}) &\leq W_2(\underset{\sim}{u}^*) \end{aligned} \qquad (5.18)$$

where $(\underset{\sim}{u}, \underset{\sim}{\sigma})$ is the solution to the creep problem, $\underset{\sim}{u}^*$ is any admissable displacement vector field and $\underset{\sim}{\sigma}^*$ is any statically admissable stress field

$$\begin{aligned} \operatorname{div} \underset{\sim}{\sigma}^* + \underset{\sim}{\tilde{b}} &= \underset{\sim}{0} \quad \text{in } \nu \\ \underset{\sim}{\sigma}^* \cdot \underset{\sim}{n} &= \underset{\sim}{\tilde{\xi}} \quad \text{on } \Gamma_\tau \end{aligned}$$

Equality holds in (5.18) if, and only if $\underset{\sim}{u}^* = \underset{\sim}{u}$ and $\underset{\sim}{\sigma}^* = \underset{\sim}{\sigma}$.

The concept of dual variational principles has been proposed by a number of investigators [221-223] as a systematic approach to the global formulation of equations of mathematical physics. For linear elasticity the principles imbodied in (5.18) are "total potential energy" W_2 and "complementary energy" W_1 ; these can be used to bound the elastic internal energy [224-227] (also the review in [228]).

The bounding theorems for elasticity can be easily extended to pure creep provided W (or ω) is convex [5,70]; an interesting application of the dual principles has been

given by N Anderson and A M Arthurs [229,230].

The significance, therefore, of these bounding theorems is that they may be translated into the rational approach and applied to the equivalent and residual elastic problems, and the residual pure creep problem.

Since no variational statement has yet been found for the quasi-static creep problem it is difficult to directly describe similar bounding theorems. J J Williams and J B Martin [231] have supposed existence of an extremum principle in the limiting state (since it is demonstrated here (section 5.2) that this is often the pure creep problem this is not surprising). Using an entirely novel approach A R S Ponter [86,95,232-236] has shown that the functionals usually associated with total potential and complementary energies (viz. W_1 and W_2) may be bounded. This is achieved by employing a different functional form so that

$$W_1(\underline{\varrho}) \leqslant \bar{W}_1(\underline{\varrho}^*)$$
$$W_2(\underline{\mu}) \leqslant \bar{W}_2(\underline{\mu}^*) \qquad (5.19)$$

where $\bar{W}_1$ and $\bar{W}_2$ are defined relative to an associated elastic material. Ponter [95] has further demonstrated a procedure for bounding deformation and inelastic work, based on (5.19).

The most unfortunate aspect of (5.19) is that even if $\underline{\varrho}^* = \underline{\varrho}$ and $\underline{\mu}^* = \underline{\mu}$ equality may not hold. Moreover the bounds in themselves are of no consequence since they do not necessarily produce approximations to the creep process; the point shall be discussed in more detail at a later stage in Chapters 6 and 7.

For the time being it shall be shown here that the redistribution equations provide a more useful alternative

to the Ponter bounds.

Theorem 5.10 Let $\underset{\sim}{\sigma}^*$ be an arbitrary statically admissable stress field process for the mixed quasi-static creep problem, and $\underset{\sim}{\alpha}^*$ an arbitrary internal field process, such that

$$\underset{\sim}{\sigma}^*(0) = \underset{\sim}{\sigma}_o(0) \qquad \underset{\sim}{\alpha}^*(0) = \underset{\sim}{\alpha}_o$$

then if there exists a functional ω such that for any stress and internal fields

$$\langle \underset{\sim}{F}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t)), \underset{\sim}{\sigma}^*(t) - \underset{\sim}{\sigma}(t) \rangle + \{ \underset{\sim}{G}(\underset{\sim}{\sigma}(t), \underset{\sim}{\alpha}(t)), \underset{\sim}{\alpha}^*(t) - \underset{\sim}{\alpha}(t) \} \leq \omega(\underset{\sim}{\sigma}^*(t), \underset{\sim}{\alpha}^*(t), t)$$

the difference $\|\Sigma - \Sigma^*\|_E^2$ where $\Sigma = (\underset{\sim}{\sigma}, \underset{\sim}{\alpha})$ is the solution of the creep problem, may be bounded above by $U(\Sigma^*(t), t)$ being the solution of the differential equation

$$\frac{1}{2}\frac{dr}{dt} = \omega(\underset{\sim}{\sigma}^*(t), \underset{\sim}{\alpha}^*(t), t) + \bar{\omega}(\underset{\sim}{\sigma}^*(t), \underset{\sim}{\alpha}^*(t), t)\sqrt{r} \qquad (5.20)$$

where $r(0) = 0$, $\bar{\omega} = \|\dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}^*\|_e + \sqrt{\{\dot{\underset{\sim}{\alpha}}^*, \dot{\underset{\sim}{\alpha}}^*\}}$.

Proof: From the redistribution equations (5.8)

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|\Sigma - \Sigma^*\|_E^2 &= -\langle \underset{\sim}{F}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}), \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^* \rangle - \{ \underset{\sim}{G}(\underset{\sim}{\sigma}, \underset{\sim}{\alpha}), \underset{\sim}{\alpha} - \underset{\sim}{\alpha}^* \} \\ &\quad - \{ \dot{\underset{\sim}{\beta}}^*, \underset{\sim}{\alpha} - \underset{\sim}{\alpha}^* \} + [\dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}^*, \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^*]_e \\ &\leq \omega(\underset{\sim}{\sigma}^*, \underset{\sim}{\alpha}^*, t) + [\dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}^*, \underset{\sim}{\sigma} - \underset{\sim}{\sigma}^*]_e - \{ \underset{\sim}{L}(\dot{\underset{\sim}{\alpha}}^*), \underset{\sim}{\alpha} - \underset{\sim}{\alpha}^* \} \\ &\leq \omega(\underset{\sim}{\sigma}^*, \underset{\sim}{\alpha}^*, t) + \bar{\omega}(\underset{\sim}{\sigma}^*, \underset{\sim}{\alpha}^*, t)\|\Sigma - \Sigma^*\|_E\end{aligned}$$

Thus by the theory of differential inequalities [155] the solution of

$$\frac{1}{2}\frac{dv}{dt} \leq \omega + \bar{\omega}\sqrt{v} \quad , \quad v(0) = 0$$

where $v = \|\Sigma - \Sigma\|_E^2$ is less than or equal to the solution of

$$\frac{1}{2}\frac{dr}{dt} = \omega + \bar{\omega}\sqrt{r} \qquad r(0) = 0$$

thus $v \leq r$ and the result is proven. QED

The theorem indicates that for any statically admissable $\underset{\sim}{\sigma}^*$ and admissable $\underset{\sim}{\alpha}^*$.

$$\|\Sigma - \Sigma^*\|_E^2 \leq U(\Sigma^*, t) \qquad (5.21)$$

where U is defined for a given Σ^* through the solution of

an ordinary differential equation (5.20). The possibility therefore exists of minimizing the functional $U(\Sigma^*,t)$ so that if Σ_{min} is the minimal solution and U_{min} the minimum value then

$$\|\Sigma(t)-\Sigma_{min}(t)\| \leq U_{min} \qquad (5\cdot22)$$

Thus if U_{min} is small it is to be expected that $\Sigma(t)$ is close to $\Sigma_{min}(t)$. This procedure therefore not only provides an efficient means of obtaining better trial fields by minimizing a suitable functional but also supplies a measure of its correctness.

5.3.3. Super-Variational Formulations

With the growing recognition that only a very restricted number of problems possess a variational formulation in the classical sense, a number of mathematicians, notably J L Lions [162,163] and F E Browder [147] have re-examined the technique known as the calculus of variations. The existence of a potential form of a given problem is attractive but not the essence of the matter. The bounding theorem for convex W

$$W(U) \leq W(U^*) \qquad (5\cdot23)$$

can often only be realised if $U,U^* \in K$ a convex set, i.e.

$$\forall U_1,U_2 \in K\,.\;\lambda\in[0,1] \qquad \lambda U_1+(1-\lambda)U_2 \in K$$

for example the set of all statically admissable stresses is a convex set. Now (5.23) is equivalent to the condition

$$\langle \mathrm{grad}W(U), U^*-U\rangle \geq 0 \qquad \forall U^*\in K$$

or

$$\langle A(U)-F,\, U^*-U\rangle \geq 0 \qquad \forall U^*\in K$$

which is calleda variational inequality [116,150,162,163, 144,147].

If the indicatrix of a convex set K

$$\psi_K(U) = \begin{cases} +\infty & U \notin K \\ 0 & U \in K \end{cases}$$

is introduced then (5.24) is identical to

$$\langle A(U)-F, U^*-U \rangle + \psi_K(U^*) - \psi_K(U) \geq 0 \qquad \forall U^* \in K \qquad (5.25)$$

which may be further generalized on letting ψ be a <u>proper convex function</u> (i.e. convex, semi-continuous from below not identically equal to $+\infty$ [237]) so that (5.25) becomes

$$\langle A(U)-F, U^*-U \rangle + \psi(U^*) - \psi(U) \geq 0 \qquad \forall U^* \in K \qquad (5.26)$$

The problem is to find a U such that (5.26) is satisfied for all $U^* \in K$. It is possible to define the <u>subdifferential</u> $\partial\psi(U)$ of ψ as the set [238]

$$\partial\psi(U) = \{ \xi \mid \psi(U^*) - \psi(U) - \langle \xi, U^*-U \rangle \geq 0 \}$$

and then (5.26) is equivalent to

$$-(A(U)-F) \in \partial\psi(U) \qquad (5.27)$$

which is an equation involving a many valued function (note that if ψ is differentiable then $\partial\psi(U) = \{\psi'(U)\}$.)

The theory of variational inequalities and many valued equations is being studied in great depth at present. It turns out that in this sense the creep problem has a variational formulation.

Consider the quasi-static creep problem and define the convex set $S(t)$ of statically admissable stresses

$$S(t) = \{ \underset{\sim}{\sigma}(t) \in L_2(V) \mid div\, \underset{\sim}{\sigma}(t) + \bar{\underset{\sim}{b}}(t) = \underset{\sim}{0} \text{ in } V;\ \underset{\sim}{\sigma}(t)\underset{\sim}{n} = \bar{\tau}(t) \text{ on } S_\tau \}$$

and let $\psi_{S(t)}(\underset{\sim}{\sigma})$ be the indicatrix of $S(t)$. On $S(t)$ define the elastic energy inner product (Definition 5.3). Then the subdifferential of $\psi_{S(t)}$ is the set

$$\partial\psi_{S(t)}(\underset{\sim}{\sigma}) = \{ \underset{\sim}{\xi} \in L_2(V) \mid [\underset{\sim}{\xi}, \underset{\sim}{\sigma}^* - \underset{\sim}{\sigma}]_e \leq 0 \quad \forall \underset{\sim}{\sigma}^* \in S(t) \}$$

But from Lemma 5.7 if $\hat{\underset{\sim}{\sigma}}$ is self-equilibrating and $\underset{\sim}{\varepsilon}^* \in L_2(V)$

$$[\underset{\sim}{R}(\underset{\sim}{\varepsilon}^*), \hat{\underset{\sim}{\sigma}}]_e = -\langle \underset{\sim}{\varepsilon}^*, \hat{\underset{\sim}{\sigma}} \rangle$$

thus

$$[\underset{\sim}{R}(\underset{\sim}{\varepsilon}^*) + \underset{\sim}{C}^{-1}(\underset{\sim}{\varepsilon}^*), \hat{\underset{\sim}{\sigma}}]_e = 0$$

and if $\underset{\sim}{\sigma}, \underset{\sim}{\sigma}^* \in S(t)$

$$[\underset{\sim}{R}(\underset{\sim}{\xi}^*) + \underset{\sim}{C}^{-1}(\underset{\sim}{\xi}^*), \underset{\sim}{\sigma}^* - \underset{\sim}{\sigma}]_e = 0$$

and by definition

$$\underset{\sim}{R}(\underset{\sim}{\xi}^*) + \underset{\sim}{C}^{-1}(\underset{\sim}{\xi}^*) \in \partial\psi_{S(t)}(\underset{\sim}{\sigma})$$

indeed if $\underset{\sim}{\sigma}$ is identified with the stress from the creep problem and $\underset{\sim}{\xi}^*$ with $\dot{\underset{\sim}{\varepsilon}}_v$ then from the redistribution equation (4.16)

$$\underset{\sim}{R}(\dot{\underset{\sim}{\varepsilon}}_v) + \underset{\sim}{C}^{-1}(\dot{\underset{\sim}{\varepsilon}}_v) = \dot{\underset{\sim}{\sigma}} - \dot{\underset{\sim}{\sigma}}_o + \underset{\sim}{C}^{-1}(\underset{\sim}{F}(\underset{\sim}{\sigma},\underset{\sim}{\alpha}))$$

and hence there results the <u>multivalued equation of evolution for creep</u>

$$\dot{\underset{\sim}{\sigma}} - \dot{\underset{\sim}{\sigma}}_o + \underset{\sim}{C}^{-1}(\underset{\sim}{F}(\underset{\sim}{\sigma},\underset{\sim}{\alpha})) \in \partial\psi_{S(t)}(\underset{\sim}{\sigma})$$

$$\dot{\underset{\sim}{\alpha}} = \underset{\sim}{G}(\underset{\sim}{\sigma},\underset{\sim}{\alpha}) \qquad (5.28)$$

which is equivalent to the <u>variational inequality of evolution</u> [150,151,239,116,163]

$$[\underset{\sim}{C}^{-1}(\underset{\sim}{F}(\underset{\sim}{\sigma},\underset{\sim}{\alpha})) + \dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}, \underset{\sim}{\sigma}^* - \underset{\sim}{\sigma}]_e + \psi(\underset{\sim}{\sigma}^*) - \psi(\underset{\sim}{\sigma}) \geq 0 \quad \forall \underset{\sim}{\sigma}^* \in S(t) \qquad (5.29)$$

Thus the approach of Q S Nguyen [115,158] in visco-plasticity is regained for creep.

This viewpoint shall not be considered any further here as the inelastic growth equations and the redistribution equations are regarded as more general. However, an interesting application to steady creep has been presented by B Nayrolles [74] (and see [240] for a good explanation of the underlying theory in Nayrolles approach).

With this is concluded the discussion of the properties of the equations of creep. Obviously much more could be said, but it is hoped that a basis for further, better investigations has been established.

CHAPTER 6

SPECIALIZATION TO THE TIME HARDENING THEORY

It has already been remarked that the time hardening theory of creep is the most common in engineering calculations, and the one most widely studied. It is the purpose of this brief chapter to summarize the results of the previous chapters for this simple theory.

The inelastic, viscous constitutive equation is assumed to be of the form

$$\dot{\underset{\sim}{\varepsilon}}_v(t) = \underset{\sim}{F}(\underset{\sim}{\sigma}(t), t) \tag{6.1}$$

so that the equation of inelastic growth is

$$\frac{d}{dt}\underset{\sim}{\varepsilon}_v(t) = \underset{\sim}{F}(\underset{\sim}{\sigma}_o(t) + \underset{\sim}{R}(\underset{\sim}{\varepsilon}_v(t), t)) = \underset{\sim}{f}(\underset{\sim}{\varepsilon}_v(t), t)$$
$$\underset{\sim}{\varepsilon}_v(0) = \underset{\sim}{0} \tag{6.2}$$

and the redistribution equation is

$$\frac{d}{dt}\underset{\sim}{\sigma}(t) = \underset{\sim}{R}(\underset{\sim}{F}(\underset{\sim}{\sigma}(t), t)) + \dot{\underset{\sim}{\sigma}}_o(t)$$
$$\underset{\sim}{\sigma}(0) = \underset{\sim}{\sigma}_o(0) \tag{6.3}$$

with similar simplifications for the evolution of deformation.

6.1 Existence and uniqueness of solution

The notation of Section 5.1 is adopted. The basic result, Theorem 5.3, asserts that the mixed quasi-static creep problem has a solution if and only if the equation of inelastic growth (6.2) has a solution. The following result concerning existence and uniqueness may be found in Appendix 1:

Theorem 6.1: Suppose there exists a set $\mathcal{D} \subset \{(t, \underset{\sim}{\xi}^*) \mid 0 < t \leq a, \underset{\sim}{\xi}^* \in L_2\}$ such that $\mathcal{D}(t) = \{\underset{\sim}{\xi}^* \mid (t, \underset{\sim}{\xi}^*) \in \mathcal{D}\}$ is closed in $L_2(V)$ for each $t \in (0, a]$ and such that the following conditions hold for $\underset{\sim}{f}$

(a) $\underset{\sim}{f} \in C[\mathcal{D}; L_2(V)]$;

(b) $\|\underset{\sim}{f}(\underset{\sim}{\xi}^*, t)\|_{L_2} \leq M(t)$ on $\mathcal{D}$, where M is summable;

(c) for some Lyapunov function V the inequality

$$\frac{\partial V}{\partial t} + D_1 V . \underset{\sim}{f}(\underset{\sim}{\xi}_1^*, t) + D_2 V . \underset{\sim}{f}(\underset{\sim}{\xi}_1^*, t) \leq \omega(t, V)$$

such that the unique solution of $\dot{r} = \omega(t, r)$, $r(0) = 0$ is trivial. Then if for some $\underset{\sim}{\Phi} \in C[\mathcal{D}; L_2(V)]$

$$\sup_{\underset{\sim}{\xi}^* \in \partial\mathcal{D}(t)} V(t, \underset{\sim}{\Phi}(t), \underset{\sim}{\xi}^*(t)) \longrightarrow 0 \qquad t \longrightarrow 0$$

such that (6.2) has a solution starting from $(t_1, \underset{\sim}{\Phi}(t_1))$, $\forall t_1 \in (0, a]$ reaching a in $\mathcal{D}$, then there exists a unique solution of (6.2) with

$$\underset{\sim}{\xi}^*(0) = \underset{\sim}{\Phi}(0) = \underset{\sim}{0}$$

and furthermore a is such that

$$\max_{t \in (0,a]} \int_0^t M(s)ds \leq \sup_{\underset{\sim}{\xi}^* \in \mathcal{D}} \| \underset{\sim}{\xi}^* \|_{L_2}$$

A similar discussion to that accompanying Theorem 5.4 could be given here. However an important case is the Nagumo like Lypschitz condition

$$\| \underset{\sim}{F}(\underset{\sim}{\sigma}_1, t) - \underset{\sim}{F}(\underset{\sim}{\sigma}_2, t) \|_{L_2} \leq g(t) \| \underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2 \|_{L_2} \qquad \forall \underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2 \in L_2(V) \quad (6.4)$$

in some neighbourhood of $\underset{\sim}{\sigma}_0$ where the scalar function $g(t)$ is such that

$$g(t) \longrightarrow \infty \qquad \Omega(t) = \int_0^t g(s)ds \longrightarrow 0 \qquad t \longrightarrow 0$$

Then on choosing

$$\mathcal{D}(t) = \{ \underset{\sim}{\xi}^* \mid \| \underset{\sim}{\xi}^* \|_{L_2} \leq \Omega(t) \}$$

$$\underset{\sim}{\Phi}(t) \equiv \underset{\sim}{0}$$

and

$$V(t, \underset{\sim}{\xi}_1^*, \underset{\sim}{\xi}_2^*) = \| \underset{\sim}{\xi}_1^* - \underset{\sim}{\xi}_2^* \|_{L_2}^2$$

$$\omega = 0$$

it is obvious that conditions (a), (b) and (c) of Theorem 6.1 hold and that (6.2) has a solution starting from $(t_1, \underset{\sim}{0})$, $\forall t_1 \in (0, a)$. Further

$$\sup_{\underset{\sim}{\xi}^* \in \partial\mathcal{D}(t)} V(t, \underset{\sim}{\Phi}, \underset{\sim}{\xi}^*) = \sup_{\| \underset{\sim}{\xi}^* \| \leq \Omega(t)} \| \underset{\sim}{\xi}^* \|_{L_2}^2 \leq \Omega(t) \longrightarrow 0 \qquad t \longrightarrow 0$$

and the theorem is established for the Nagumo condition (6.4). This is true for most real time hardening laws

with, for example

$$g(t) = t^{-1/3}$$

6.2 Asymptotic behaviour

The results on asymptotic behaviour Theorems 5.6, 5.7 and 5.8 can be established for the time hardening theory with the monotonicity condition (5.7) replaced by

$$< \underset{\sim}{F}(\underset{\sim}{\sigma}_1,t) - \underset{\sim}{F}(\underset{\sim}{\sigma}_2,t), \underset{\sim}{\sigma}_1 - \underset{\sim}{\sigma}_2 > \geq 0 \,, = 0 \text{ iff } \underset{\sim}{\sigma}_1 = \underset{\sim}{\sigma}_2 \qquad (6.5)$$

for all t. The limiting process then occurs in the elastic energy norm (Definition 5.3). Obviously the condition $\| \dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}_c \|_e \longrightarrow 0$ shall depend on the functional form of t in $\underset{\sim}{F}$ (6.1).

6.3 Global properties

Clearly the identification of the Wang and Prager principle on the one hand and the Kachanov principle on the other as global statements of the auxiliary elastic and pure creep problems respectively equally applies to time hardening.

In a similar manner to Theorem 5.10 it can be shown that if $\underset{\sim}{\sigma}^*(t)$ is statically admissable, with $\underset{\sim}{\sigma}^*(0) = \underset{\sim}{\sigma}_o(0)$ then the inequality

$$\| \underset{\sim}{\sigma}(t) - \underset{\sim}{\sigma}^*(t) \|_e^2 \leq U(\underset{\sim}{\sigma}^*, t) \qquad (6.6)$$

holds, where U is a solution of the differential equation

$$\frac{1}{2}\frac{du}{dt} = w(\underset{\sim}{\sigma}^*, t) + \| \dot{\underset{\sim}{\sigma}}_o - \dot{\underset{\sim}{\sigma}}^* \|_e \sqrt{u}$$

$$u(0) = 0$$

with w such that [95] $\forall \underset{\sim}{\sigma}_1^*, \underset{\sim}{\sigma}_2^* \in L_2(V)$

$$< \underset{\sim}{F}(\underset{\sim}{\sigma}_2^*, t), \underset{\sim}{\sigma}_1^* - \underset{\sim}{\sigma}_2^* > \leq w(\underset{\sim}{\sigma}_1^*, t)$$

and $\underset{\sim}{\sigma}(t)$ is the solution of the creep problem.

As before $U(\underset{\sim}{\sigma}^*, t)$ may be minimized with respect to $\underset{\sim}{\sigma}^*(t)$ and a measure of the accuracy of the minimal solution (6.6) obtained.

Two interesting special cases of (6.6) are

$$\|\underset{\sim}{\sigma}(t)-\underset{\sim}{\sigma}_o(t)\|_e^2 \le 2\int_o^t \omega(\underset{\sim}{\sigma}_o(t),t)\,dt \qquad (6.7)$$

and

$$\|\underset{\sim}{\sigma}(t)-\underset{\sim}{\sigma}_c(t)\|_e^2 \le U(\underset{\sim}{\sigma}_c(t),t) \qquad (6.8)$$

taking $\underset{\sim}{\sigma}_c(0)=\underset{\sim}{\sigma}_o(0)$, which effectively demonstrate the maximum error on using the pure elastic or the pure creep stress patterns to approximate the actual behaviour.

Indeed Ponter [94] has suggested using the stress field

$$\underset{\sim}{\sigma}^*(t)=\underset{\sim}{\sigma}_o(t)+\underset{\sim}{\varrho} \qquad (6.9)$$

where $\underset{\sim}{\varrho}$ is self-equilibrating and independent of t ; whence from (6.6)

$$\|\underset{\sim}{\sigma}(t)-\underset{\sim}{\sigma}^*(t)\|_e^2 \le 2\int_o^t \omega(\underset{\sim}{\sigma}^*(t),t)\,dt \qquad (6.10)$$

and the right hand side may be minimized with respect to $\underset{\sim}{\varrho}$; a more general approximation is

$$\underset{\sim}{\sigma}^*(t)=\underset{\sim}{\sigma}_o(t)\,\lambda(t)+(1-\lambda(t))\,\underset{\sim}{\sigma}_c(t)+\underset{\sim}{\varrho}(t)$$

for any scalar $\lambda(t)$, $\lambda(0)=1$ and self-equilibrating process $\underset{\sim}{\varrho}(t)$, $\underset{\sim}{\varrho}(0)=\underset{\sim}{\varrho}$. Thus, whereas Ponter only contrives to bound above and below the inelastic energy dissipation using approximations (6.9) and the pure creep solution respectively, here a measure of the accuracy of these stress fields themselves has been derived (6.8) and (6.10). The possibilities of this technique are many, but space does not permit a fuller discussion.

CHAPTER 7

THE NUMERICAL REALIZATION OF CREEP PROBLEMS

The rational scheme is of use not only in an investigation of properties of solution of creep problems but also in the clarification of their numerical realization. It is the intention of this Chapter to contribute a unifying treatment of the numerical solution of time dependent inelastic continuum problems by studying the redistribution equations.

The redistribution equations (4.16) and (4.17) are a continuous system of first order ordinary differential equations in time and in principle can be integrated forward numerically in a sequential manner by any competent method starting from the initial state. But continuity in the space variable makes this impracticable unless some technique is employed to reduce the problem to the numerically plausible finite system of ordinary differential equations. There are two possibilities:
a "direct" method, or a "discrete" method.

7.1 Direct methods

A direct method presumes the approximation

$$\underset{\sim}{\sigma}_n(\underset{\sim}{X},t) = \underset{\sim}{\sigma}^{(s)}(\underset{\sim}{X},t) + \sum_{i=1}^{n} c_i(t)\,\underset{\sim}{\sigma}^{(i)}(\underset{\sim}{X})$$

$$\underset{\sim}{\alpha}_m(\underset{\sim}{X},t) = \sum_{i=1}^{m} \bar{c}_i(t)\,\underset{\sim}{\alpha}^{(i)}(\underset{\sim}{X})$$

such that $\underset{\sim}{\sigma}^{(s)}(\underset{\sim}{X},t)$ is a statically admissable stress field and $\underset{\sim}{\sigma}^{(i)}(\underset{\sim}{X},t)$ are self-equilibrating for the mixed problem, with $\underset{\sim}{\alpha}^{(i)}(\underset{\sim}{X})$ given. The approximation is said to converge in the sense that

$$\underset{\sim}{\sigma}_n(\underset{\sim}{X},t) \longrightarrow \underset{\sim}{\sigma}(\underset{\sim}{X},t) \qquad n \longrightarrow \infty$$

$$\underset{\sim}{\alpha}_m(\underset{\sim}{X},t) \longrightarrow \underset{\sim}{\alpha}(\underset{\sim}{X},t) \qquad m \longrightarrow \infty$$

The coefficients $c_i(t)$ and $\bar{c}_i(t)$ can be determined by "the method of weighted residuals" [241]:

Let $B(\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2)$, $\underset{\sim}{\sigma}_1, \underset{\sim}{\sigma}_2 \in L_2(V)$ and $b(\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2)$, $\underset{\sim}{\alpha}_1, \underset{\sim}{\alpha}_2 \in [l_2(V)]^N$ be bilinear functions so that

$$B(\underset{\sim}{\sigma}^{(i)}, \underset{\sim}{\sigma}^{(j)}) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$b(\underset{\sim}{\alpha}^{(i)}, \underset{\sim}{\alpha}^{(j)}) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

This includes the well known "Galerkin method" [242] as a special case if B and b are taken as the inner products in $L_2(V)$ and $[l_2(V)]^N$ respectively.

Now if it is supposed that the approximation also satisfies the re-distribution equations and that the initial state and equivalent elastic problems have solutions of the form

$$\underset{\sim}{\sigma}_o(\underset{\sim}{X}, t) = \underset{\sim}{\sigma}^{(s)}(\underset{\sim}{X}, t) + \sum_{i=1}^{n} c_i^o(t)\,\underset{\sim}{\sigma}^{(i)}(\underset{\sim}{X})$$

$$\underset{\sim}{\alpha}_o(\underset{\sim}{X}) = \sum_{i=1}^{m} \bar{c}_i^o\, \underset{\sim}{\alpha}^{(i)}(\underset{\sim}{X})$$

then the coefficients c_i and $\bar{c}_i$ must satisfy

$$\dot{c}_i = B\Big(R\big(F(\underset{\sim}{\sigma}^{(s)} + \sum_{i=1}^{n} c_i\,\underset{\sim}{\sigma}^{(i)}, \sum_{i=1}^{m} \bar{c}_i \underset{\sim}{\alpha}^{(i)})\big), \underset{\sim}{\sigma}^{(i)}\Big) + B(\underset{\sim}{\sigma}^{(s)}, \underset{\sim}{\sigma}^{(i)}) + \dot{c}_i^o$$

$$\dot{\bar{c}}_j = b\Big(G(\underset{\sim}{\sigma}^{(s)} + \sum_{i=1}^{n} c_i\,\underset{\sim}{\sigma}^{(i)}, \sum_{i=1}^{m} \bar{c}_i \underset{\sim}{\alpha}^{(i)}), \underset{\sim}{\alpha}^{(j)}\Big) \qquad (7.1)$$

$$c_i(0) = c_i^{(o)}(0) \qquad \bar{c}_j(0) = \bar{c}_j^o \qquad (7.2)$$

for i = 1,2,.....n and j = 1,2,.....m.

The finite system of ordinary differential equations (7.1) with (7.2) could also be derived using one of the reduced variational principles (e.g. for a time hardening law and a Galerkin method [6,243-254]).

The question of convergence cannot be completely resolved at present although limited information is available for linear visco-elasticity [246] and nonlinear viscoelasticity [109]. For the internal variable theory, provided that the monotonicity condition (5.4) holds, it can probably be shown that the Galerkin method converges [145,146]; the details shall not be considered here.

7.2 Discrete methods

The direct method retains continuity in the space variable; on the other hand the discrete method assumes only a finite number of points $\underset{\sim}{X}_1, \underset{\sim}{X}_2, \ldots\ldots \underset{\sim}{X}_n$ within the body, replacing the differential equations in space by their discrete equivalents using for example finite differences (or direct integration, or finite elements) [247], giving approximations $\underset{\sim}{\sigma}_n(\underset{\sim}{X}_i, t)$ and $\underset{\sim}{\alpha}_n(\underset{\sim}{X}_i, t)$ at each $\underset{\sim}{X}_i$, i = 1,2,.....n, together with discrete residual operators $\underset{\sim}{R}_n^{(i)}$ such that $\underset{\sim}{R}_n^{(i)}(\underset{\sim}{\xi}^*) = \underset{\sim}{\sigma}_R(\underset{\sim}{X}_i)$. The method is supposed to converge in the sense

$$\begin{aligned} \underset{\sim}{\sigma}_n(\underset{\sim}{X}_i, t) &\longrightarrow \underset{\sim}{\sigma}(\underset{\sim}{X}_i, t) \\ \underset{\sim}{\alpha}_n(\underset{\sim}{X}_i, t) &\longrightarrow \underset{\sim}{\alpha}(\underset{\sim}{X}_i, t) \end{aligned} \qquad i = 1,2 \ldots n,\ n \to \infty$$

Let

$$\underset{\sim}{\sigma}_n^{(i)}(t) = \underset{\sim}{\sigma}_n(\underset{\sim}{X}_i, t) \qquad \underset{\sim}{\alpha}_n^{(i)}(t) = \underset{\sim}{\alpha}_n(\underset{\sim}{X}_i, t)$$

then the redistribution equations become

$$\begin{aligned} \dot{\underset{\sim}{\sigma}}_n^{(i)}(t) &= \underset{\sim}{R}_n^{(i)}(\underset{\sim}{F}(\underset{\sim}{\sigma}_n^{(i)}(t), \underset{\sim}{\alpha}_n^{(i)}(t))) + \dot{\underset{\sim}{\sigma}}_{on}^{(i)} \\ \dot{\underset{\sim}{\alpha}}_n^{(i)}(t) &= \underset{\sim}{G}(\underset{\sim}{\sigma}_n^{(i)}(t), \underset{\sim}{\alpha}_n^{(i)}(t)) \end{aligned} \qquad (7.3)$$

$$\underset{\sim}{\sigma}_n^{(i)}(0) = \underset{\sim}{\sigma}_{on}^{(i)}(0) \qquad \underset{\sim}{\alpha}_n^{(i)}(0) = \underset{\sim}{\alpha}_o^{(i)}$$

for i = 1,2,.....n.

The problem has thus once more been separated into a finite system of ordinary differential equations so that practical numerical computations are possible. This method shall be employed later in the solution of some real creep problems.

7.3 Review of the numerical realization of creep problems

The attainment of a numerical solution to a creep problem therefore consists of two steps: (a) reduction of the continuous redistribution equations to a finite system, then (b) forward integration of these equations in time.

The latter step can be achieved by any standard method such as Runge-Kutta or predictor-corrector, whatever is the most satisfactory for the demands of the problem at hand (this shall be discussed more fully at a later stage in an Appendix). The most common method (and as it happens, the least efficient) is the "method of elastic solutions" [98-101, 7,248-253] which corresponds to Eulers method [7,99,250]

$$\underset{\sim}{\sigma}(t+\Delta t) = \underset{\sim}{\sigma}(t) + \Delta t\, \dot{\underset{\sim}{\sigma}}(t)$$

$$\underset{\sim}{\alpha}(t+\Delta t) = \underset{\sim}{\alpha}(t) + \Delta t\, \dot{\underset{\sim}{\alpha}}(t)$$

and Picard iteration [100,101,248,249]

$$\underset{\sim}{\sigma}(t+\Delta t) = \underset{\sim}{\sigma}(t) + \Delta \underset{\sim}{\sigma}(t)$$

$$\underset{\sim}{\alpha}(t+\Delta t) = \underset{\sim}{\alpha}(t) + \Delta \underset{\sim}{\alpha}(t)$$

where $\Delta\underset{\sim}{\sigma}$ and $\Delta\underset{\sim}{\alpha}$ are solutions of the algebraic equation

$$\Delta \underset{\sim}{\sigma} = \Delta t\, \underset{\sim}{R}\left(\underset{\sim}{F}(\underset{\sim}{\sigma}+\Delta\underset{\sim}{\sigma}, \underset{\sim}{\alpha}+\Delta\underset{\sim}{\alpha})\right) + \underset{\sim}{\sigma}_o(t+\Delta t) - \underset{\sim}{\sigma}_o(t)$$

$$\Delta \underset{\sim}{\alpha} = \Delta t\, \underset{\sim}{G}(\underset{\sim}{\sigma}+\Delta\underset{\sim}{\sigma}, \underset{\sim}{\alpha}+\Delta\underset{\sim}{\alpha})$$

found by iteration.

Several authors have considered the convergence of this technique (in plasticity [254-256] and viscoelasticity [257-261,109,130] but at the level of their investigations (no discretization) the study reduces to the question of convergence of numerical methods for the sequential integration of initial value problems on an abstract space.

With the rapid development of the "Finite Element Method"(a powerful combination of direct and discrete approximation (sic)) the method of elastic solutions has extensively been employed in the solution of spatially complex structural problems for time-dependent creep [262-270]. The initial publications of J F Besseling [271], G A Greenaum and M F Rubenstein [272], and W Sutherland [273], provided the impetus for a deluge of finite element

creep solutions [274-281]. The majority of these analyses used a time hardening model of creep, but it was found that the method could be extended to include a viscoplastic response [123, 282-284]. Further I C Cormeau, B M Irons and others [285, 286] attempted to resolve the somewhat nebulous question of "stability"; this implies a search for an optimum time step just large enough for the solution to be well behaved. The writer's opinion is that the conclusions of these investigators should not be taken too seriously as they only take into consideration instabilities due to approximations to the temporal behaviour. But since Euler's method and Picard iteration are strictly mathematically always stable in this sense, it is concluded that reported instabilities are due to the exponential growth in parasitic errors due to the spatial approximation. A successful treatment of the problem of stability would therefore need to take into account the mode and reliability of the spatial approximations, as well as the wide variations in the rates of change of stress.

In the context of finite elements the method of elastic solutions was refined by W C Carpenter [102] and Z P Bazant [103] who were able to deduce a discrete form of the redistribution equations (7.3). Similar results were obtained by I C Cormeau [286] and Q S Nguyen and his co-workers [290-291] The conclusion was that more efficient time integration schemes could be employed; their effectiveness has been demonstrated for a number of practical problems [287-289].

7.4 Approximate methods for time dependent creep

By an "approximate method" is meant here a method which avoids the direct temporal integration of the redistribution equations. Two such methods have been derived from the bounding theorems (Section 5.3) and superposition of the pure creep solution onto the equivalent elastic solution, called here "superposition of states".

A R S Ponter [94,95] has demonstrated how the energy dissipated within a structure due to the formation of inelastic strains within some time interval may be bounded above and below by the stress field (6.9) $\underset{\sim}{\sigma}_\rho = \underset{\sim}{\sigma}_o + \underset{\sim}{\rho}$ and the pure creep stress field $\underset{\sim}{\sigma}_c$ respectively, for the time hardening model (also including initial plasticity and thermal strains). These results, and associated theorems, were formulated in a series of publications [94,95,233,292-294] (see also the reviews [295-301]) and have been applied to some simple problems [302-304] with some success. The bounds on the energy were found to be quite close in most cases but, unhappily, no indications of the nature of the stress field could be deduced. It is suggested here that the approach advocated in Sections 5.3 and 6.3 could be employed as a systematic procedure for optimizing an approximate stress field in conjunction with the bounding theorems.

Superposition of states arises out of the initial work of D L Marriott [75-77] who noticed that a good approximation to a constant load (second boundary value problem) situation can be constructed by superinposing the steady state (pure creep) deformation fieldon to the initial elastic deformation. Similar approximations were suggested by other investigators [78,79] and a thorough study has been undertaken [80,305] for a linear, time

hardening creep law. Later J Spence and J Hult [81] showed that a similar approximation was possible in the relaxation situation which again was fairly accurate.

The method of superposition of states is strictly an approximation for the deformation process in the second boundary value problem in that

$$\begin{aligned} \underset{\sim}{u}(t) &\triangleq \underset{\sim}{u}_e(t) + \underset{\sim}{u}_c(t) \\ \underset{\sim}{\varepsilon}(t) &\triangleq \underset{\sim}{\varepsilon}_e(t) + \underset{\sim}{\varepsilon}_c(t) \end{aligned} \qquad (7.6)$$

but can be applied to the first and mixed boundary value problems by transforming into a second boundary value problem i.e. on S_u assumetractions $\tau_u(t)$ are applied and determine their proper values from the condition on displacements. The approximation is best described by an example and this shall be given in a later chapter.

7.5 Discussion

Each of the approximate methods suffer from a conceptual and computational disadvantage. This does not seem to have been appreciated previously. The bounding methods require the storage of the whole history of the equivalent elastic and pure creep problems; superposition of states requires the solution of a sequence on nonlinear boundary value problems. Both methods therefore become hopelessly involved if more realistic material laws and rapid loading are to be included. Then why are they used at all? For whole classes of problems with engineering importance simplifications can be made -for example if partially closed form solutions are available for the equivalent elastic and pure creep problems, this being true of complex piping networks (sic). When these simplifications are plausible it turns out that the approximations are simple, useful and eminently worthy of consideration. A fuller discussion of the aims and

desirabilities of approximate methods for time dependent creep shall be given in the final chapter.

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PART B

THE CREEP BEHAVIOUR OF FLEXIBLE PIPING COMPONENTS

The rational approach to creep mechanics would be inconsequential if it could not be employed in the solution of practical creep problems. An area of particular interest to the writer is the study of piping components and pipework design in the creep regime. It is to this field that that the ideas shall be applied. This part studies the simplest most common components - the straight and curved pipes.

CHAPTER 8

THE STRAIGHT PIPE

The problem of the thin, circular cross section straight pipe, in particular under combined loading in creep, is of considerable practical interest within the field of elevated temperature pipework design. The steady creep of this component under bending alone is straight forward, but when internal pressure and torque are also admitted the analysis becomes more involved; nonetheless results are available due to the investigations of J Spence [306,307] and the writer [308] (see Appendix 2), being extended still further in [309].

If nonsteady behaviour is subsumed the solution is still straight forward but provides an excellent example for the rational approach to creep.

8.1 Flexibility of a straight pipe in creep

In a later chapter the flexibility of a curved pipe shall be studied - in order to facilitate this the response of a straight pipe under bending alone shall be discussed.

Consider a long, thin, constant thickness circular cross section cylindrical shell of mean radius r, thickness $2h$ and length L, as shown in Figure 8.1. It is supposed that $2h/r \ll 1$ so that the radial stress components are negligibly small with the remaining longitudinal stress $\sigma(\phi,t)$ being constant through the thickness.

Let the pipe be loaded by a bending moment M which is kept fixed as the pipe creeps, the problem is to determine the resultant curvature change $\kappa(t)$.

The field equations are those of a simple beam

$$\varepsilon = \kappa r \sin\phi$$

$$M = \int_0^{2\pi} (2hr^2)\,\sigma \sin\phi \, d\phi$$

where ε is the longitudinal strain composed of an elastic and a creep part

$$\varepsilon = \varepsilon_e + \varepsilon_v$$

related to the stress through the constitutive equations

$$\varepsilon_e = \sigma / E$$

$$\frac{d}{dt}\varepsilon_v = g(t)\sigma^n$$

E being Young's modulus, and $g(t), n$ material parameters associated with Norton's power law of creep [1-9]. With reference to (6.3) the redistribution equation can be written down

$$\frac{d}{dt}\sigma = g(t)R(\sigma^n) + \dot{\sigma}_o \qquad (8.1)$$

$$\sigma(0) = \sigma_o(0)$$

as well as the equation of evolution of curvature

$$\frac{d}{dt}\kappa = g(t)R_v(\sigma^n) + \dot{\kappa}_o \qquad (8.2)$$

$$\kappa(0) = \kappa_o(0)$$

where σ_o and κ_o are the stress and curvature from the equivalent elastic problem

$$\varepsilon_o = \kappa_o r \sin\phi$$

$$M = \int_0^{2\pi} (2hr^2)\sigma_o \sin\phi \, d\phi$$

$$\varepsilon_o = \sigma_o / E$$

with solution

$$\sigma_o = \frac{M}{2hr^2}\frac{\sin\phi}{\pi} \qquad \kappa_o = \frac{M}{2hr^3}\cdot\frac{1}{\pi E}$$

For an arbitrary initial strain ε^*, $R(\varepsilon^*)$ and $R_v(\varepsilon^*)$ are the stress and curvature from the residual elastic problem for ε^*:

$$\varepsilon_R = \kappa_R r \sin\phi$$

$$0 = \int_0^{2\pi} (2hr^2)\sigma_R \sin\phi \, d\phi$$

$$\varepsilon_R = \sigma_R / E + \varepsilon^*$$

with solution

$$R(\varepsilon^*) = \sigma_R = E\left\{\frac{\sin\phi}{\pi}\int_0^{2\pi} \varepsilon^* \sin\phi \, d\phi - \varepsilon^*\right\}$$

$$R_v(\varepsilon^*) = \kappa_R = \frac{1}{\pi r}\int_0^{2\pi} \varepsilon^* \sin\phi \, d\phi$$

On adopting the nondimensional time scale

$$\tau = E\sigma_D^{n-1}\int g(t)\,dt$$

where $\sigma_D = M/2h\pi r^2$, equations (8.1) and (8.2) become

$$\frac{dS}{d\tau} = \frac{\sin\phi}{\pi}\int_0^{2\pi} S^n \sin\phi \, d\phi - S^n$$

$$\frac{dK}{d\tau} = \frac{1}{\pi}\int_0^{2\pi} S^n \sin\phi \, d\phi \qquad (8.3)$$

$$S(\phi,0) = \sin\phi \qquad K(0) = 1$$

where $S(\phi,\tau) = \sigma(\phi,t)/\sigma_D$ and $K(\tau) = \kappa(t)/\kappa_o$ remembering that M and therefore σ_o and κ_o are constant in time.

To resolve the continuous initial value problem (8.3) a discrete method is employed: let the interval $[0, 2\pi]$ of the variable ϕ be replaced by the finite mesh ϕ_i, i = i = 1,2.....N, where $0 = \phi_1 < \phi_2 < \phi_3 < \ldots\ldots < \phi_{N-1} < \phi_N = 2\pi$ and

$$\phi_i = i\Delta\phi$$

$$\Delta\phi = 2\pi/(N-1)$$

In this way the integrals in (8.3) may be replaced by a finite sum, using, for example, Simpson's quadrature rule

$$\int_0^{2\pi} f(\phi)\,d\phi = \sum_{i=1}^{N} a_i f(\phi_i)$$

$$\{a_i\}_{i=1}^{N} = \{1 \;\; 2 \;\; 4 \;\; 2 \;\; \ldots \;\; 2 \;\; 4 \;\; 1\}\frac{\Delta\phi}{3}$$

Thus the continuous system (8.3) takes the discrete form

$$\frac{d\underset{\sim}{S}}{d\tau} = \underset{\sim}{F}(\underset{\sim}{S}) \qquad (8.4)$$

where $\underset{\sim}{S} = (S(\phi_1), S(\phi_2) \cdots S(\phi_N), K)^T$ $S(0) = (\sin\phi_1 \cdots \sin\phi_N, 1)$

$$F_j = \frac{\sin\phi_j}{\pi} \sum_{i=1}^{N} a_i S_i^n \sin\phi_i - S_j^n \qquad j=1,2,\cdots N$$

$$F_{N+1} = \frac{1}{\pi} \sum_{i=1}^{N} a_i S_i^n \sin\phi_i$$

System (8.4) comprises a finite number of first order ordinary differential equations which can be resolved numerically, ysing for example a Runge-Kutta method. Results for K with creep indices n = 3.0,5.0 and 7.0 are shown in Figure 8.2

This example has effectively demonstrated the systematic procedure developed within the rational approach.

8.2 The effect of internal pressure on the creep/relaxation of a straight pipe

In high temperature pipework applications the most common situation is relaxation viz. displacements on a system at an anchor are fixed. Once again if bending alone is present the analysis is simple [81] obtainable in closed form. Complications are introduced if a constant internal pressure is imposed on the pipe, so that most simple creep analyses of piping systems ignore the effect of internal pressure even though it is demonstrated in [306-309] that it may have a considerable effect on the steady creep. A simple study of the influence of internal pressure on the creep relaxation of a straight pipe has been presented by the writer [310]. This shall be re-examined from the rational point of view.

The problem is geometrically identical to the previous except that now there are two stress components; circumferential $\sigma_\phi(\phi,t)$ and longitudinal $\sigma_z(\phi,t)$ with associated strains $\varepsilon_\phi(\phi,t)$ and $\varepsilon_z(\phi,t)$. Under an applied moment M_o and internal pressure P the pipe is bent to a curvature κ and an axial elongation δ; κ and δ are then

fixed and the problem is to determine the resultant bending moment $M(t)$ such that initially $M(0) = M_0$ if the pressure is maintained at a constant throughout.

The longitudinal strain

$$\varepsilon_\ell = \delta + \kappa r \sin\phi$$

is decomposed into an elastic and a creep component related to the stress field through the constitutive equations

$$\varepsilon_\ell = \varepsilon_\ell^e + \varepsilon_\ell^v$$

$$\varepsilon_\ell^e = (\sigma_\ell - \nu\sigma_\phi)/E$$

$$\frac{d}{dt}\varepsilon_\ell^v = g(t)[\bar{\sigma}]^{\frac{n-1}{2}}(\sigma_\ell - \tfrac{1}{2}\sigma_\phi) = g(t) f_\ell(\sigma_\ell, \sigma_\phi)$$

where $\bar{\sigma}$ is the effective stress $\bar{\sigma}^2 = \sigma_\phi^2 - \sigma_\phi\sigma_\ell + \sigma_\ell^2$ associated with Odquist's multiaxial generalization of Norton's power law [5]. Since the circumferential stress is isostatic it may be obtained from equilibrium alone

$$\sigma_\phi = Pr/2h$$

The first step in the rational approach is to write down the redistribution equations

$$\frac{d}{dt}\sigma_\ell = g(t) R[f_\ell(\sigma_\ell, \sigma_\phi), f_\phi(\sigma_\ell, \sigma_\phi)] + \dot{\sigma}_{\ell o}$$

$$\sigma_\ell(0) = \sigma_{\ell o}(0) \qquad (8.5)$$

Secondly define the residual operator - which in this case takes the simple form

$$R(\varepsilon_\ell^*, \varepsilon_\phi^*) = -E\varepsilon_\ell^*$$

Thirdly, since $\sigma_{\ell o}$ is constant in time the initial elastic stress alone needs to be taken into consideration

$$\sigma_\ell(0) = Pr/4h + M_0 \sin\phi/2\pi h r^2$$

Consequently on using the time scale

$$\tau = E\sigma_0^{n-1}\int g(t)dt$$

where $\sigma_0 = M_0/2\pi h r^2$, and the new variable

$$S(\phi,\tau) = \frac{1}{\pi\sigma_0}\left(\sigma_\ell(\phi,t) - \frac{Pr}{4h}\right)$$

the redistribution equation (8.5) becomes

$$\frac{d}{d\tau}S + \left(\frac{3}{4}\pi^2\$_0^2 + \pi^2 S^2\right)^{\frac{n-1}{2}} S = 0$$

$$S(\phi,0) = \sin\phi / \pi \qquad (8.6)$$

where $\$_0 = Pr^3/M_0$ is a loading parameter.

The resultant bending moment may now be easily obtained from the external equilibrium condition

$$M(t) = \int_0^{2\pi} (2hr^2) \sin\phi \, \sigma_i \, d\phi$$

which may be written as

$$m(\tau) = \frac{M(t)}{M_0} = \int_0^{2\pi} S(\phi,\tau) \sin\phi \, d\phi \qquad (8.7)$$

Hence $m(\tau)$ is determined on solving the initial value problem (8.6) and using (8.7). This can be achieved using a discrete method [310] to reduce (8.6) to a finite system of initial value problems and (8.7) to a finite sum.

Results for n = 3.0, 5.0 and 7.0 are shown in Figures 8.3, 8.4 and 8.5 for various values of the loading parameter $\$_0$ with the conclusion that even for "small" values of $\$_0$ there is some significant difference from the primary loading case $\$_0 = 0$ where pressure is absent.

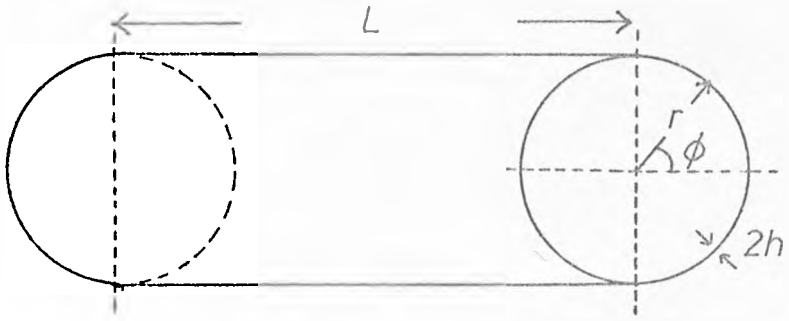


Fig 8·1 Geometry of a straight pipe

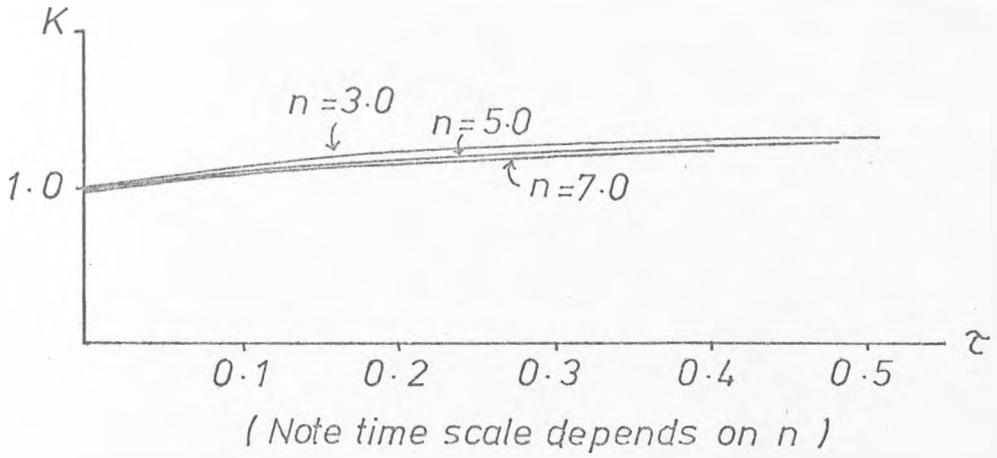


Fig 8·2 Flexibility of a straight pipe in creep

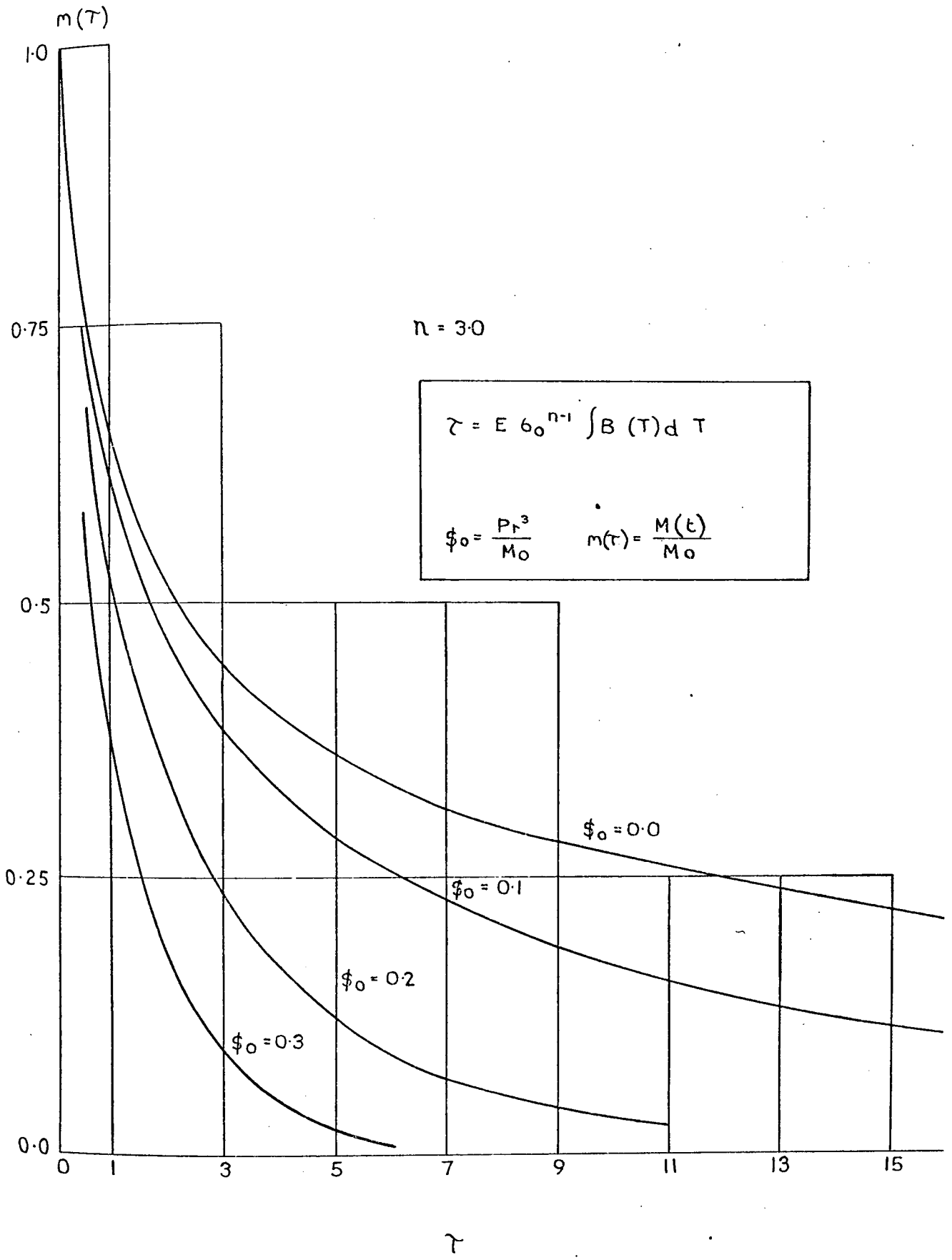

m(τ)
1·0
0·75
0·5
0·25
0·0
n = 3·0
τ = E δ_0^{n-1} ∫B (T) d T
$\$_0 = \frac{P_r^3}{M_0}$
$m(\tau) = \frac{M(t)}{M_0}$
$\$_0$ = 0·0
$\$_0$ = 0·1
$\$_0$ = 0·2
$\$_0$ = 0·3
0
1
3
5
7
9
11
13
15
τ

Fig 8·3

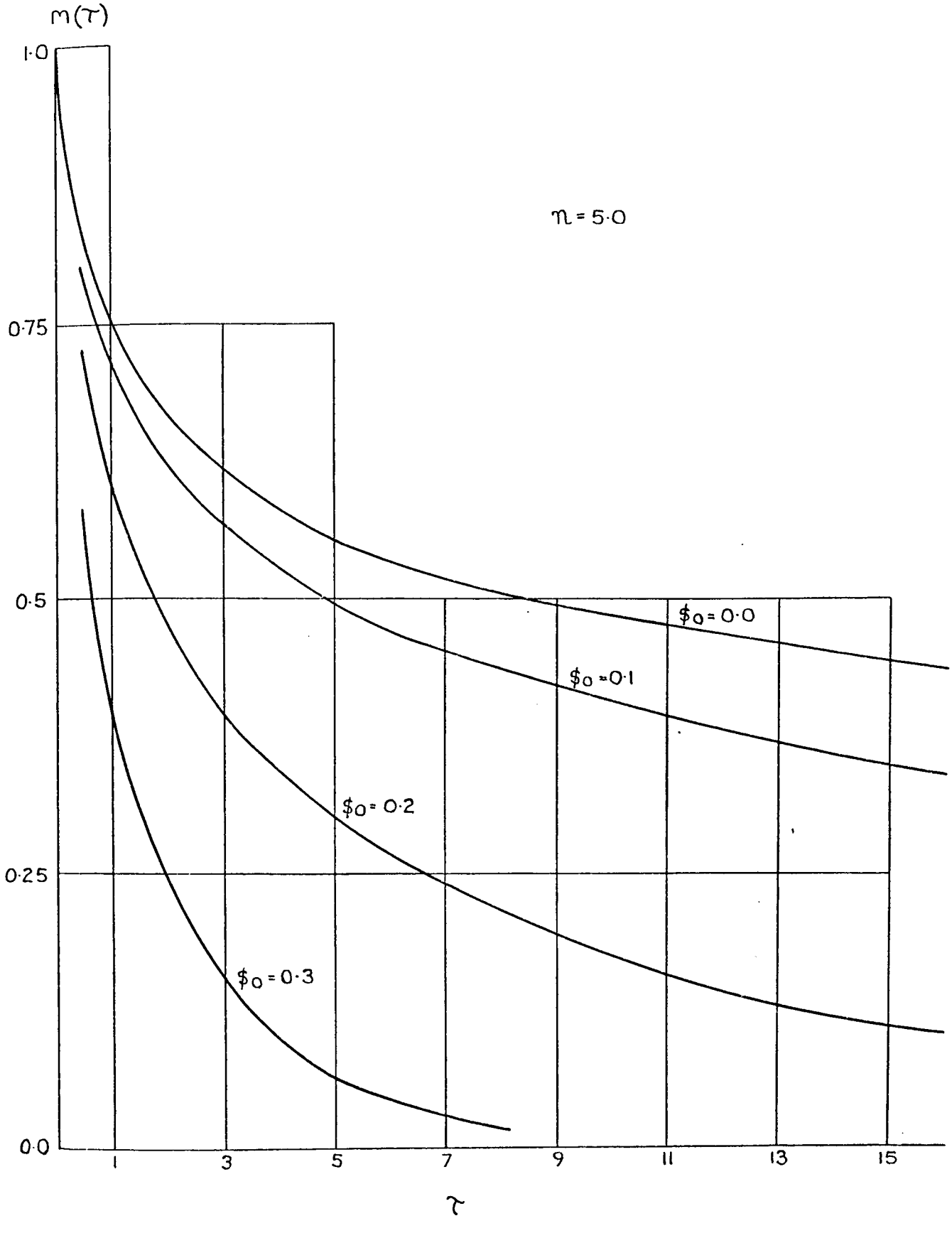


Fig 8·4

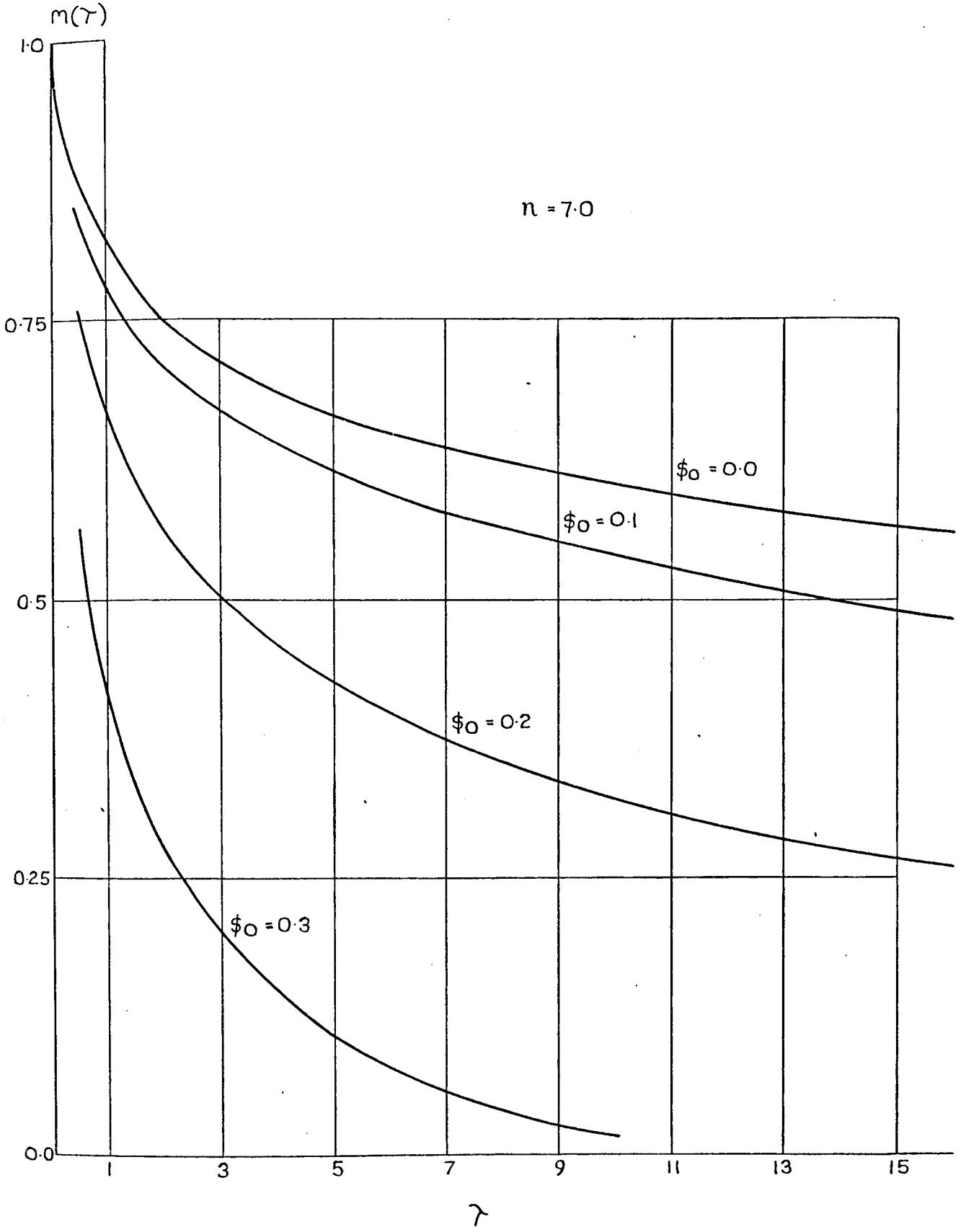


Fig 8·5

CHAPTER 9

THE CURVED PIPE

The curved pipe, or as it is more commonly known the "pipe bend", "piping elbow" or "welding elbow", being the foremost contributor to flexibility in a pipeline system shall be studied for the remainder.

The ultimate aim is to investigate the inelastic behaviour of this component in creep as a further, more involved example of the rational procedure. In order to do this a mathematical model of the geometry and deformation characteristics of the curved pipe needs to be critically assessed. This shall be achieved by first analysing the elastic behaviour.

Historically, Th Von Karman produced the earliest simple mathematical model of the peculiar deformation of the curved pipe, and many investigators have attempted to improve this analysis ever since. A full survey would of necessity repeat the extensive reviews of J Spence [311] and G Findlay [312] and shall therefore not be duplicated here in any depth. Instead the treatment of the bending of a curved pipe as a problem in the theory of thin shells shall be examined.

By a suitable modification of the equations of a shell of revolution E Reissner [313] developed a general model of a thin, elastic, arbitrary cross section smooth curved pipe which contained all the characteristics of the Von Karman theory, although particular equations for a cross section composed of circular arcs had been established and analysed by M Tueda [314]. The circular cross section was analysed in detail by R A Clark [315] and extended to an

elliptic cross section [316] in a later publication. In order to account for the effects of internal pressure S G Crandall and N C Dahl [317] and E Reissner [318,319] suggested a nonlinear variation.

These analyses were produced in the era preceding the digital computer so that the governing equations were simplified and resolved using series solution and an asymptotic technique [320]. Later R Cheng and H J Thailer [321,322] re-examined the equations using an energy method thus combining the classical Von Karman approach with the shell theory analysis. With the advent of high speed computers more exact analyses were performed by J A Blomfield [323], A Robert and C Du Fort [324] and J J Thompson [325].

All of these mathematical models considered the "pure bending" situation of a curved pipe loaded by an in-plane moment, thus assuming that stresses and strains remained rotationally symmetric, eliminating the circumferential angle [326], and in this sense are restricted. In Order to account for non-uniform and asymmetrical loading such as lateral bending and twisting an approximate model has been developed [327-329] on the lines of the Reissner linear theory. These however relied strongly on this linearity of deformation and constitutive equation.

A more general linear elastic shell model which removes the assumptions inherent in the Karman and Reissner theories was first proposed by A Kalnins [330,331], resolving the complicated equations using a technique of direct integration known as the "multisegment method". A similar investigation was undertaken by A T Vasilenko et al [332]. Tne rapid development of the Finite Element

Method has allowed a number of discrete models of a complete curved pipe and attachments to be devised [333-336], holding great potential for the future, but have, unfortunately, not been studied in any detail yet.

It is the purpose here to re-examine the Von Karman-Reissner theory from a more general standpoint to correctly identify the deformation characteristics. To this end a general, nonlinear, thin elastic shell model shall be developed in the next chapter. Firstly, the differential geometry of a curved pipe shall be described.

9.1 Differential geometry of a curved pipe

A smooth curved pipe can be recognized as a thin shell whose mid surface is geometrically equivalent to that sector of a torus formed by rotating a closed curve C, with parametric equations $r(\xi)$ and $l(\xi)$, about an axis through an angle Θ. It may therefore be described by the orthogonal coordinate system (ξ, θ, z) where θ is circumferential angle $-\Theta/2 \leq \theta \leq +\Theta/2$ and z is measured normal to the mid surface, $-h/2 \leq z \leq +h/2$ (Figures 9.1 and 9.2).

The midsurface is described by the vector function $\underset{\sim}{r}(\xi,\theta)$ of the parameters ξ,θ

$$\underset{\sim}{r} = r\cos\theta\,\underset{\sim}{i} + r\sin\theta\,\underset{\sim}{j} + l\,\underset{\sim}{k} \qquad (9.1)$$

with $(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$ a triad of unit vectors forming the basis of a right handed Cartesian coordinate system (x,y,z) with z coincident with l (Figure 9.1).

The tangent vectors at (ξ,θ) are

$$\begin{aligned} \underset{\sim}{r}_{,\xi} &= r'\cos\theta\,\underset{\sim}{i} + r'\sin\theta\,\underset{\sim}{j} + l'\,\underset{\sim}{k} \\ \underset{\sim}{r}_{,\theta} &= -r\sin\theta\,\underset{\sim}{i} + r\cos\theta\,\underset{\sim}{j} \end{aligned} \qquad (9.2)$$

where $(\,)' = \partial(\,)/\partial\xi$. Thus

$$\underset{\sim}{t}_\xi = \frac{\underset{\sim}{r}_{,\xi}}{|\underset{\sim}{r}_{,\xi}|} = \frac{1}{\alpha}\underset{\sim}{r}_{,\xi}$$

$$\underset{\sim}{t}_\theta = \frac{\underset{\sim}{r}_{,\theta}}{|\underset{\sim}{r}_{,\theta}|} = \frac{1}{r}\underset{\sim}{r}_{,\theta}$$

where $\alpha^2 = (r')^2 + (l')^2$, are the unit tangents, the unit normal to the surface at (ξ, θ) is then

$$n = |\underset{\sim}{t}_\xi \wedge \underset{\sim}{t}_\theta| = -\frac{l'}{\alpha}\cos\theta\,\underset{\sim}{i} - \frac{l'}{\alpha}\sin\theta\,\underset{\sim}{j} + \frac{r'}{\alpha}\underset{\sim}{k}$$

The First Fundamental Form is defined as the distance between two points $\underset{\sim}{r}$ and $\underset{\sim}{r} + d\underset{\sim}{r}$ i.e.

$$\begin{aligned} d\underset{\sim}{r}.d\underset{\sim}{r} &= |\underset{\sim}{r}_{,\xi}d\xi + \underset{\sim}{r}_{,\theta}d\theta|^2 \\ &= |\underset{\sim}{r}_{,\xi}|^2 d\xi^2 + 2\underset{\sim}{r}_{,\xi}\cdot\underset{\sim}{r}_{,\theta}d\xi\,d\theta + |\underset{\sim}{r}_{,\theta}|^2 d\theta^2 \\ &= \alpha^2 d\xi^2 + r^2 d\theta^2 \end{aligned}$$

The Second Fundamental Form is defined as

$$-d\underset{\sim}{r}.d\underset{\sim}{n} = -(\underset{\sim}{r}_{,\xi}d\xi + \underset{\sim}{r}_{,\theta}d\theta).(\underset{\sim}{n}_{,\xi}d\xi + \underset{\sim}{n}_{,\theta}d\theta)$$

which reduces to

$$-d\underset{\sim}{r}.d\underset{\sim}{n} = (\underset{\sim}{r}_{,\xi\xi}d\xi^2 + 2\underset{\sim}{r}_{,\xi\theta}d\xi d\theta + \underset{\sim}{r}_{,\theta\theta}d\theta^2).\underset{\sim}{n}$$

on using the orthogonality of $\underset{\sim}{r}_{,\xi}$, $\underset{\sim}{r}_{,\theta}$ and $\underset{\sim}{n}$.

But

$$\begin{aligned} \underset{\sim}{r}_{,\xi\xi} &= r''\cos\theta\,\underset{\sim}{i} + r''\sin\theta\,\underset{\sim}{j} + l''\underset{\sim}{k} \\ \underset{\sim}{r}_{,\theta\xi} &= -r'\sin\theta\,\underset{\sim}{i} + r'\cos\theta\,\underset{\sim}{j} \\ \underset{\sim}{r}_{,\theta\theta} &= -r\cos\theta\,\underset{\sim}{i} - r\sin\theta\,\underset{\sim}{j} \end{aligned} \qquad (9.3)$$

and if ϕ is defined as the tangent to the meriodional curve C (Figure 9.1)

$$r' = \alpha\cos\phi \qquad l' = \alpha\sin\phi$$

and the second fundamental form is

$$-d\underset{\sim}{r}.d\underset{\sim}{n} = \alpha\phi' d\xi^2 + r\sin\phi\,d\theta^2$$

The curvatures are given by the Gauss-Weingarten relations [337,338]

$$\begin{aligned} \underset{\sim}{r}_{,\xi\xi} &= \frac{1}{S_\xi}\underset{\sim}{t}_\theta - \frac{1}{R_\xi}\underset{\sim}{n} & \underset{\sim}{r}_{,\xi\theta} &= \frac{1}{S_\theta}\underset{\sim}{t}_\theta - \frac{1}{R_{\theta\xi}}\underset{\sim}{n} \\ \underset{\sim}{r}_{,\theta\xi} &= -\frac{1}{S_\xi}\underset{\sim}{t}_\xi - \frac{1}{R_{\xi\theta}}\underset{\sim}{n} & \underset{\sim}{r}_{,\theta\theta} &= -\frac{1}{S_\theta}\underset{\sim}{t}_\xi - \frac{1}{R_\theta}\underset{\sim}{n}. \end{aligned} \qquad (9.4)$$

and by suitable manipulation using (9.2) and (9.3)

$$1/S_\xi = 0 \qquad 1/S_\theta = 0$$

$$\frac{1}{R_\xi} = -\frac{\phi'}{\alpha} \qquad \frac{1}{R_{\xi\theta}} = \frac{1}{R_{\theta\xi}} = 0 \qquad \frac{1}{R_\theta} = -\frac{\sin\phi}{r}$$

It is to be noted that the signs of these are opposite to those usually found in differential geometry [337,338] since here the "natural" convention that a radius of curvature is positive (negative) if the unit normal to the surface is directed away from (toward) the centre of curvature, and that C is described by increasing ξ in an anti clockwise sense. These geometrical considerations relate to the undeformed mid surface of the shell; the deformed mid surface still remains to be specified.

9.2 Deformation of a curved pipe

Let the deformed mid surface be represented by the vector function $\underset{\sim}{\varrho}(\xi,\theta)$ and let $\underset{\sim}{T}_\xi$, $\underset{\sim}{T}_\theta$ and $\underset{\sim}{N}$ be an, as yet undefined, mutually perpendicular triad of unit vectors which can be used to describe the deformed shell.

The following fundamental assumptions concerning the deformation of a curved pipe under a uniformly applied in-plane bending moment are made:

Assumption 1: The deformed mid surface can be achieved from the undeformed mid surface by a rotation β about $\underset{\sim}{t}_\theta$ followed by a rotation of the normal to a meridional plane through an angle ω about $\underset{\sim}{k}$.

The former rotation is within the meridional plane while the latter is purely in the plane of the bend; furthermore the deformation described is admissable since these rotations are commutative.

Assumption 2: The deformation of each meridional section of the bend is identical.

Thus β is presumed to be independent of θ.

With these two assumptions a model of the deformation of a curved pipe can be constructed on choosing the triad $(\underset{\sim}{T}_\xi, \underset{\sim}{T}_\theta, \underset{\sim}{N})$ as the product of $(\underset{\sim}{t}_\xi, \underset{\sim}{t}_\theta, \underset{\sim}{n})$ resulting from the two rotations. This is done in two stages; firstly the rotation about $\underset{\sim}{t}_\theta$ through β changes $(\underset{\sim}{t}_\xi, \underset{\sim}{t}_\theta, \underset{\sim}{n})$ into $(\underset{\sim}{\bar{t}}_\xi, \underset{\sim}{\bar{t}}_\theta, \underset{\sim}{\bar{n}})$ related by

$$\begin{bmatrix} \underset{\sim}{\bar{t}}_\xi \\ \underset{\sim}{\bar{t}}_\theta \\ \underset{\sim}{\bar{n}} \end{bmatrix} = \begin{bmatrix} \cos\beta & 0 & -\sin\beta \\ 0 & 1 & 0 \\ \sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} \underset{\sim}{t}_\xi \\ \underset{\sim}{t}_\theta \\ \underset{\sim}{n} \end{bmatrix}$$

Secondly the rotation about $\underset{\sim}{k}$ through ω changes $(\underset{\sim}{\bar{t}}_\xi, \underset{\sim}{\bar{t}}_\theta, \underset{\sim}{\bar{n}})$ into $(\underset{\sim}{T}_\xi, \underset{\sim}{T}_\theta, \underset{\sim}{N})$ related by

$$\begin{bmatrix} \underset{\sim}{T}_\xi \\ \underset{\sim}{T}_\theta \\ \underset{\sim}{N} \end{bmatrix} = \begin{bmatrix} \cos\omega+(1-\cos\omega)\sin^2\bar{\Phi} & \sin\omega\cos\bar{\Phi} & (1-\cos\omega)\cos\bar{\Phi}\sin\bar{\Phi} \\ -\sin\omega\cos\bar{\Phi} & \cos\omega & \sin\bar{\Phi}\sin\omega \\ (1-\cos\omega)\cos\bar{\Phi}\sin\bar{\Phi} & -\sin\omega\sin\bar{\Phi} & \cos\omega+(1-\cos\omega)\cos^2\bar{\Phi} \end{bmatrix} \begin{bmatrix} \underset{\sim}{\bar{t}}_\xi \\ \underset{\sim}{\bar{t}}_\theta \\ \underset{\sim}{\bar{n}} \end{bmatrix}$$

where $\bar{\Phi} = \phi - \beta$.

Thus for the whole transformation (Table 9.1)

$$\begin{bmatrix} \underset{\sim}{T}_\xi \\ \underset{\sim}{T}_\theta \\ \underset{\sim}{N} \end{bmatrix} = \left[\mathbb{T}_t^T\right] \times \begin{bmatrix} \underset{\sim}{t}_\xi \\ \underset{\sim}{t}_\theta \\ \underset{\sim}{n} \end{bmatrix} \qquad (9.5)$$

It can similarly be established that

$$\begin{bmatrix} \underset{\sim}{t}_\xi \\ \underset{\sim}{t}_\theta \\ \underset{\sim}{n} \end{bmatrix} = \left[\mathbb{T}_T^t\right] \times \begin{bmatrix} \underset{\sim}{T}_\xi \\ \underset{\sim}{T}_\theta \\ \underset{\sim}{N} \end{bmatrix} \qquad (9.6)$$

These are the basic transformation laws of the undeformed mid surface $\underset{\sim}{r}$ defined by $(\underset{\sim}{t}_\xi, \underset{\sim}{t}_\theta, \underset{\sim}{n})$ into the deformed mid surface $\underset{\sim}{\varrho}$ defined by $(\underset{\sim}{T}_\xi, \underset{\sim}{T}_\theta, \underset{\sim}{N})$ and vice versa.

The curvatures of the deformed mid surface are given by the Gauss-Weingarten relations (c.f. (9.4))

$$\underset{\sim}{\varrho}_{,\xi\xi} = \frac{1}{s_\xi}\underset{\sim}{T}_\theta - \frac{1}{r_\xi}\underset{\sim}{N} \qquad \underset{\sim}{\varrho}_{,\xi\theta} = \frac{1}{s_\theta}\underset{\sim}{T}_\theta - \frac{1}{r_{\theta\xi}}\underset{\sim}{N}$$

$$\underset{\sim}{\varrho}_{,\theta\xi} = \frac{-1}{s_\xi}\underset{\sim}{T}_\xi - \frac{1}{r_{\xi\theta}}\underset{\sim}{N} \qquad \underset{\sim}{\varrho}_{,\theta\theta} = \frac{-1}{s_\theta}\underset{\sim}{T}_\xi - \frac{1}{r_\theta}\underset{\sim}{N} \qquad (9.7)$$

which can be derived by differentiating (9.5) with respect to ξ and θ (noting Assumption 9.2) using (9.4) then (9.6). The calculation is involved but the end result is

$$1/s_\xi = \cos\bar{\phi}\,\omega'/\alpha \qquad 1/s_\theta = \cos\bar{\phi}\,(1+\dot{\omega})/\alpha$$

$$\frac{1}{r_\xi} = -\frac{\bar{\Phi}'}{\alpha} \qquad \frac{1}{r_{\xi\theta}} = -\sin\bar{\phi}\,\frac{\omega'}{\alpha} \qquad \frac{1}{r_{\theta\xi}} = 0 \qquad \frac{1}{r_\theta} = -\frac{\sin\bar{\Phi}}{r}(1+\dot{\omega}) \tag{9.8}$$

where $(\,)^{\cdot} = \partial(\,)/\partial\theta$.

A point has now been reached where a curved pipe model may be described.

TABLE 9.1

TRANSFORMATION MATRICES

$$[\mathbb{T}_t^T] = \begin{bmatrix} \cos\omega\cos\beta + & \sin\omega\cos\Phi & -\sin\beta\cos\omega + \\ +(1-\cos\omega)\sin\Phi\sin\phi & & +(1-\cos\omega)\sin\Phi\cos\phi \\ -\sin\omega\cos\phi & \cos\omega & \sin\omega\sin\phi \\ \sin\beta\cos\omega + & -\sin\omega\sin\Phi & \cos\omega\cos\beta + \\ +(1-\cos\omega)\cos\Phi\sin\phi & & +(1-\cos\omega)\cos\Phi\cos\phi \end{bmatrix}$$

$$[\mathbb{T}_T^t] = \begin{bmatrix} \cos\omega\cos\beta + & -\sin\omega\cos\phi & \sin\beta\cos\omega + \\ +(1-\cos\omega)\sin\Phi\sin\phi & & +(1-\cos\omega)\cos\Phi\sin\phi \\ \sin\omega\cos\Phi & \cos\omega & -\sin\Phi\sin\omega \\ -\sin\beta\cos\omega + & \sin\omega\sin\phi & \cos\omega\cos\beta + \\ +(1-\cos\omega)\cos\phi\sin\Phi & & +(1-\cos\omega)\cos\Phi\cos\phi \end{bmatrix}$$

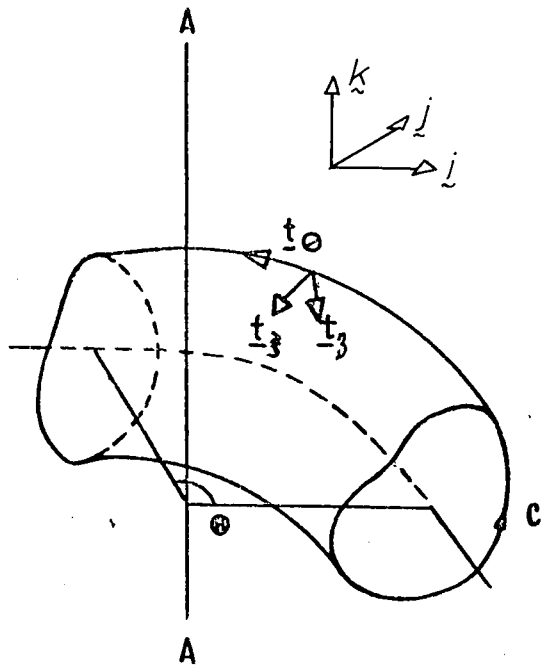


Fig 9·1

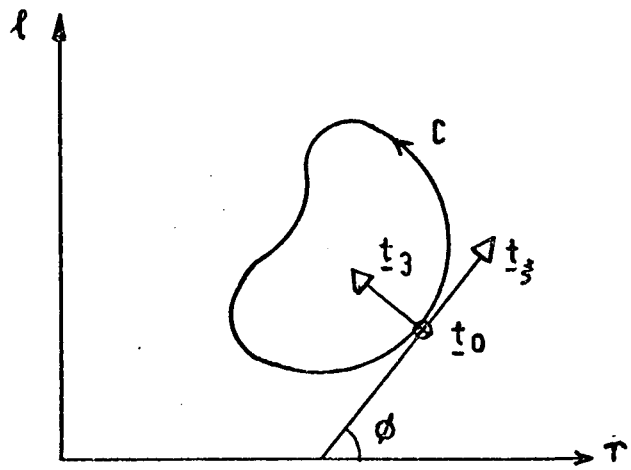


Fig 9·2

CHAPTER 10

A THIN SHELL MODEL OF THE CURVED PIPE

Nonlinear thin shell theory has reached a fair degree of sophistication, yet in practice it is still very much in its infancy.

The "best" linear shell theory was developed simultaneously by J L Sanders [339] and W T Koiter [340] (also [341,342]), although it is only better [343,344] in the sense that its constitutive relations are symmetric. However the most practical theory is that of E Reissner [326]. Both Sanders [345] and Koiter [346] provided general nonlinear extensions of their linear theories being refined respectively by B Budiansky [347] and D A Danielson [348]. A full review of these theories, and others like them with further developments, has been given by P M Naghdi [349] and W T Koiter and J G Simmonds [350]. These shell theories are quite general, requiring only an arbitrary description of the deformed reference surface, but are not particularly practical except in special simple cases. A more functional model was advanced by E Reissner [351-355] in a series of publications, applicable to shells of revolution (see also [356]). The principle of this theory was that the deformed mid surface could be achieved by a single rotation. The idea was perfected for arbitrary shells by J G Simmonds and D A Danielson [357,358], but still was only valid for a single rotation. The next, and obvious, step was taken by E Reissner [359] who postulated a form of the theory for an arbitrary description of the deformed shell. It is this final, simple theory of Reissner which shall be used here. First this theory shall be summarised.

10.1 A nonlinear shell theory

Consider a given surface in three dimensional Euclidean space with equation $\underset{\sim}{r} = \underset{\sim}{r}(\xi_1, \xi_2)$ with (ξ_1, ξ_2) an orthogonal, curvilinear coordinate system and First Fundamental Form

$$d\underset{\sim}{r}.d\underset{\sim}{r} = \alpha_1^2 d\xi_1^2 + \alpha_2^2 d\xi_2^2$$

and Second Fundamental Form

$$-d\underset{\sim}{r}.d\underset{\sim}{n} = \beta_{11} d\xi_1^2 + \beta_{12} d\xi_1 d\xi_2 + \beta_{22} d\xi_2^2$$

The tangent vectors $\underset{\sim}{r}_{,1}$ and $\underset{\sim}{r}_{,2}$ are used to define the unit tangents

$$\underset{\sim}{t}_1 = \underset{\sim}{r}_{,1} / \alpha_1 \qquad \underset{\sim}{t}_2 = \underset{\sim}{r}_{,2} / \alpha_2$$

and unit norm

$$\underset{\sim}{n} = \underset{\sim}{t}_1 \wedge \underset{\sim}{t}_2$$

The surface $\underset{\sim}{r}$ denotes the undeformed reference surface of a thin shell; let a deformed reference surface be represented by $\underset{\sim}{\varrho}$ and as yet undefined, mutually perpendicular triad of unit vectors $\underset{\sim}{T}_i$ and $\underset{\sim}{N}$ chosen in such a way that the difference between it and the deformed tangent $\underset{\sim}{\varrho}_{,i}$ and normal $\underset{\sim}{\varrho}_{,1} \wedge \underset{\sim}{\varrho}_{,2}$ vectors is a measure of the state of strain ϵ_{ij} and γ_i, i,j = 1,2, in the form

$$\underset{\sim}{\varrho}_{,i} = \langle \alpha_i \rangle [(\delta_{ij} + \epsilon_{ij}) \underset{\sim}{T}_j + \gamma_i \underset{\sim}{N}]$$

If the displacement vector $\underset{\sim}{u}$ is defined by

$$\underset{\sim}{u} = \underset{\sim}{\varrho} - \underset{\sim}{r}$$

then the in-plane strain resultants ϵ_{ij} and transverse (shearing) strain resultants γ_i are

$$\epsilon_{ij} = \left(\underset{\sim}{t}_i + \frac{\underset{\sim}{u}_{,i}}{\langle \alpha_i \rangle}\right).\underset{\sim}{T}_j - \delta_{ij}$$

$$\gamma_j = \left(\underset{\sim}{t}_i + \frac{\underset{\sim}{u}_{,i}}{\langle \alpha_i \rangle}\right).\underset{\sim}{N} \qquad (10\text{-}1)$$

The Gauss-Weingarten relations for the undeformed and deformed mid surfaces are

$$\frac{\underset{\sim}{t}_{i,j}}{\langle\alpha_j\rangle} = e_{i\beta}\frac{1}{S_j}\underset{\sim}{t}_{\beta} - \frac{1}{R_{ji}}\underset{\sim}{n}$$
$$\frac{\underset{\sim}{n}_{,i}}{\langle\alpha_i\rangle} = \frac{1}{R_{i\beta}}\underset{\sim}{t}_{\beta} \qquad (10.2)$$

$$\frac{\underset{\sim}{T}_{i,j}}{\langle\alpha_j\rangle} = e_{i\beta}\frac{1}{s_j}\underset{\sim}{T}_{\beta} - \frac{1}{r_{ji}}\underset{\sim}{N}$$
$$\frac{\underset{\sim}{N}_{,i}}{\langle\alpha_i\rangle} = \frac{1}{r_{i\beta}}\underset{\sim}{T}_{\beta} \qquad (10.3)$$

respectively where the S_i and s_i are in-plane radii of curvature and the R_{ij}, r_{ij} are the usual out-of-plane radii of curvature, with e_{ij} the permutation tensor, i,j = 1,2. It can in fact be shown [337,338] that

$$1/S_1 = -\alpha_{1,2}/\alpha_1\alpha_2 \qquad 1/S_2 = \alpha_{2,1}/\alpha_1\alpha_2$$

$$\frac{1}{R_{11}} = -\frac{\beta_{11}}{\alpha_1^2} \qquad \frac{1}{R_{12}} = \frac{1}{R_{21}} = -\frac{\beta_{12}}{\alpha_1\alpha_2} \qquad \frac{1}{R_{22}} = -\frac{\beta_{22}}{\alpha_2^2}$$

The transverse (bending and twisting) strain couples κ_{ij} and the in-plane strain couples λ_i are defined by

$$\kappa_{ij} = 1/r_{ij} - 1/R_{ij}$$
$$\lambda_i = 1/s_i - 1/S_i \qquad (10.4)$$

(it is to be noted that this is actually opposite in sign to that of [359] but both reduce to the same since the definitions of radii of curvature are also opposite in sign to [359]).

The Compatibility Relations are the six independent scalar equations resulting from the identities

$$\underset{\sim}{\varrho}_{,12} = \underset{\sim}{\varrho}_{,21}$$
$$\underset{\sim}{T}_{i,12} = \underset{\sim}{T}_{i,12} \qquad (10.5)$$
$$\underset{\sim}{N}_{,12} = \underset{\sim}{N}_{,12}$$

which may be reduced to simpler forms, using (10.1)-(10.4), (Table 10.1)

Let $\underset{\sim}{N}_i$ and $\underset{\sim}{M}_i$, i = 1,2, be forces and moments per unit undeformed length and $\underset{\sim}{p}$ and $\underset{\sim}{q}$ be applied forces and moments per unit undeformed area. Balance of forces and moments

leads to the Equations of Equilibrium:

$$(\alpha_1 \underset{\sim}{N}_1)_{,1} + (\alpha_2 \underset{\sim}{N}_2)_{,2} + \alpha_1\alpha_2 \underset{\sim}{p} = \underset{\sim}{0}$$
$$(\alpha_2 \underset{\sim}{M}_1)_{,1} + (\alpha_1 \underset{\sim}{M}_2)_{,2} + \underset{\sim}{e}_{,1} \wedge (\alpha_2 \underset{\sim}{N}_1) + \underset{\sim}{e}_{,2} \wedge (\alpha_1 \underset{\sim}{N}_2) + \alpha_1\alpha_2 \underset{\sim}{q} = \underset{\sim}{0} \qquad (10.6)$$

On defining

$$\underset{\sim}{N}_i = N_{ij} \underset{\sim}{T}_j + Q_i \underset{\sim}{N} \qquad \underset{\sim}{M}_i = M_{ij} \underset{\sim}{N} \wedge \underset{\sim}{T}_j + P_i \underset{\sim}{N}$$
$$\underset{\sim}{p} = p_i \underset{\sim}{T}_i + p_N \underset{\sim}{N} \qquad \underset{\sim}{q} = q_i \underset{\sim}{N} \wedge \underset{\sim}{T}_i + q_N \underset{\sim}{N}$$

and using (10.2)-(10.4) the equations (10.6) reduce to six scalar equilibrium equations (Table 10.2).

A more conventional system of equations is obtained on putting

$$P_i = 0 \qquad \gamma_i = 0 \qquad (10.7)$$

so that λ_i and Q_i may be defined in terms of ϵ_{ij}, κ_{ij} and N_{ij} and κ_{ij} [345,346].

10.2 Application to the deformation of a curved pipe

This nonlinear shell theory may now be used to model the deformation of a curved pipe described in Section 9.2.

Obviously the identification

$$\xi_1 = \xi \qquad \xi_2 = 0$$
$$\alpha_1 = \alpha \qquad \alpha_2 = r$$

is made, and the triad $\underset{\sim}{T}_i, \underset{\sim}{N}$ is taken as $\underset{\sim}{T}_\xi, \underset{\sim}{T}_\theta$ and $\underset{\sim}{N}$.

10.2.1. Kinematic Relations

Immediately from the Gauss-Weingarten relations (9.4) and (9.7) the strain couples (10.4) become

$$\kappa_{ij} = \begin{bmatrix} \dfrac{\beta'}{\alpha} & -\sin\Phi\, \dfrac{\omega'}{\alpha} \\ 0 & \dfrac{\sin\phi - \sin\Phi}{r} - \dfrac{\sin\Phi}{r}\,\omega^* \end{bmatrix}$$
$$\lambda_i = \begin{bmatrix} \cos\Phi\, \dfrac{\omega'}{\alpha} \\ \dfrac{\cos\Phi - \cos\phi}{r} + \dfrac{\cos\Phi}{r}\,\omega^* \end{bmatrix} \qquad (10.8)$$

for $i,j = \xi, \theta$.

Furthermore if scalar displacements u_i, w are defined by

$$\underset{\sim}{u} = u_i \underset{\sim}{t}_i + w \underset{\sim}{n}$$

with respect to the undeformed mid surface, then the strain-displacement relations (10.1) are

$$\begin{aligned}
\epsilon_{\xi\xi} &= \cos\omega\cos\beta + (1-\cos\omega)\sin\Phi\sin\phi - 1 \\
&\quad + (\cos\omega\cos\beta + (1-\cos\omega)\sin\Phi\sin\phi)\left(\frac{u_\xi'}{\alpha} + \frac{w}{R_\xi}\right) \\
&\quad + \sin\omega\cos\Phi\,\frac{u_\theta'}{\alpha} \\
&\quad + \left(-\sin\beta\cos\omega + (1-\cos\omega)\sin\Phi\cos\phi\right)\left(\frac{w'}{\alpha} - \frac{u_\xi}{R_\xi}\right) \\
\epsilon_{\xi\theta} &= -\sin\omega\cos\phi \\
&\quad - \sin\omega\cos\phi\left(\frac{u_\xi'}{\alpha} + \frac{w}{R_\xi}\right) \\
&\quad + \cos\omega\,\frac{u_\theta'}{\alpha} \\
&\quad + \sin\omega\sin\phi\left(\frac{w'}{\alpha} - \frac{u_\xi}{R_\xi}\right) \\
\epsilon_{\theta\xi} &= \sin\omega\cos\Phi \\
&\quad + (\cos\omega\cos\beta + (1-\cos\omega)\sin\Phi\sin\phi)\left(\frac{\dot{u}_\xi}{r} - \frac{u_\theta}{S_\theta}\right) \\
&\quad + \sin\omega\cos\Phi\left(\frac{\dot{u}_\theta}{r} + \frac{u_\xi}{S_\theta} + \frac{w}{R_\theta}\right) \\
&\quad + \left(-\sin\beta\cos\omega + (1-\cos\omega)\sin\Phi\cos\phi\right)\left(\frac{\dot{w}}{r} - \frac{u_\theta}{R_\theta}\right) \\
\epsilon_{\theta\theta} &= \cos\omega - 1 \\
&\quad - \sin\omega\cos\phi\left(\frac{\dot{u}_\xi}{r} - \frac{u_\theta}{S_\theta}\right) \\
&\quad + \cos\omega\left(\frac{\dot{u}_\theta}{r} + \frac{u_\xi}{S_\theta} + \frac{w}{R_\theta}\right) \\
&\quad + \sin\omega\sin\phi\left(\frac{\dot{w}}{r} - \frac{u_\theta}{R_\theta}\right) \\
\gamma_\xi &= \sin\beta\cos\omega + (1-\cos\omega)\cos\Phi\sin\phi \\
&\quad + (\sin\beta\cos\omega + (1-\cos\omega)\cos\Phi\sin\phi)\left(\frac{u_\xi'}{\alpha} + \frac{w}{R_\xi}\right) \\
&\quad - \sin\omega\sin\Phi\,\frac{u_\theta'}{\alpha} \\
&\quad + (\cos\beta\cos\omega + (1-\cos\omega)\cos\Phi\cos\phi)\left(\frac{w'}{\alpha} - \frac{u_\xi}{R_\xi}\right) \\
\gamma_\theta &= -\sin\omega\sin\Phi \\
&\quad + (\sin\beta\cos\omega + (1-\cos\omega)\cos\Phi\sin\phi)\left(\frac{\dot{u}_\xi}{r} - \frac{u_\theta}{S_\theta}\right) \\
&\quad - \sin\omega\sin\Phi\left(\frac{\dot{u}_\theta}{r} + \frac{u_\xi}{S_\theta} + \frac{w}{R_\theta}\right) \\
&\quad + (\cos\beta\cos\omega + (1-\cos\omega)\cos\Phi\cos\phi)\left(\frac{\dot{w}}{r} - \frac{u_\theta}{R_\theta}\right)
\end{aligned} \tag{10.9}$$

on using (9.5) and (9.6).

Equations (10.8) and (10.9) represent the fundamental strain-displacement relations for a curved pipe deforming according to Assumptions 9.1 and 9.2. It shall be established that these include the classical curved pipe relations as a special case. Some simplifying assumptions are made.

Assumption 10.1: The displacements u_ξ, u_θ and w and rotation ω are small.

The immediate implication is that second order terms ω^2, ωu etc.....can be neglected, whence (10.9) becomes

$$\epsilon_{ij} = \begin{bmatrix} \cos\beta - 1 & -\omega\cos\phi + \frac{u_\theta'}{\alpha} \\ +\cos\beta\left(\frac{u_\xi'}{\alpha} + \frac{w}{R_\xi}\right) & \\ -\sin\beta\left(\frac{w'}{\alpha} - \frac{u_\xi}{R_\xi}\right) & \\ \omega\cos\phi & \frac{u_\theta^\cdot}{r} + \frac{u_\xi}{S_\theta} + \frac{w}{R_\theta} \\ +\cos\beta\left(\frac{u_\xi^\cdot}{r} - \frac{u_\theta}{S_\theta}\right) & \\ -\sin\beta\left(\frac{w^\cdot}{r} - \frac{u_\theta}{R_\theta}\right) & \end{bmatrix} \qquad (10.10)$$

$$\gamma_i = \begin{bmatrix} \sin\beta + \sin\beta\left(\frac{u_\xi'}{\alpha} + \frac{w}{R_\xi}\right) + \cos\beta\left(\frac{w'}{\alpha} - \frac{u_\xi}{R_\xi}\right) \\ -\omega\sin\phi + \sin\beta\left(\frac{u_\xi^\cdot}{r} - \frac{u_\theta}{S_\theta}\right) + \cos\beta\left(\frac{w^\cdot}{r} - \frac{u_\theta}{R_\theta}\right) \end{bmatrix}$$

for $i, j = \xi, \theta$

Assumption 10.2: This deformation is such that the strains remain rotationally symmetric.

That is (10.10) should be independent of θ thus in the simplest form a necessary condition is

$$u_\xi = u_\xi(\xi) \quad u_\theta = u_\theta^A(\xi) + u_\theta^B(\xi)\theta \quad w = w(\xi) \qquad (10.11)$$

although $\epsilon_{\theta\xi}$ and $\epsilon_{\xi\theta}$ still depend on θ through ω, therefore:

Assumption 10.3: The deformation is such that the bending is pure and shear strains vanish.

Thus $\epsilon_{\xi\theta} = \epsilon_{\theta\xi} = 0$

Now

$$\epsilon_{\theta\xi} = \left(\omega - \frac{u_\theta}{r}\right)\cos\phi = 0 \quad \Rightarrow \quad u_\theta = r\omega \qquad (10.12)$$

and

$$\epsilon_{\xi\theta} = (-\omega r' + u'_\theta)/\alpha = r\omega'/\alpha = 0 \quad \Longrightarrow \quad \omega = \omega(\theta) \qquad (10.13)$$

On comparing (10.11) with (10.12) and (10.13)

$$\omega(\theta) = k\theta \qquad u_\theta^A = 0 \qquad u_\theta^B = kr$$

where k is a constant.

Therefore Assumptions 10.2 and 10.3 imply the standard relations [313]

$$u_\theta = kr\theta \qquad (10.14)$$

as well as the explanation

$$\omega = k\theta \qquad (10.15)$$

A consequence of these is that

$$\kappa_{\xi\theta} = \kappa_{\theta\xi} = 0 \qquad \lambda_\xi = 0 \qquad \gamma_\theta = 0$$

The constant k can be shown geometrically [315] to be the fractional change in the angle of the bend due to deformation, i.e.

$$k = \Delta\textcircled{\scriptsize H} / \textcircled{\scriptsize H}$$

The final reduction is achieved on defining a horizontal and vertical displacement u and v respectively (Figure 10.1) as

$$u_\xi = u\cos\phi + v\sin\phi$$
$$w = -u\sin\phi + v\cos\phi$$

whereupon on substitution into (10.8) and (10.10) the strain displacement relations finally take the form

$$\epsilon_\xi = \cos\beta - 1 + \frac{u'\cos\Phi + v'\sin\Phi}{\alpha}$$
$$\epsilon_\theta = u/r + k$$
$$\gamma_\xi = \sin\beta + \frac{v'\cos\Phi - u'\sin\Phi}{\alpha}$$
$$\gamma_\theta = 0 \qquad (10.16)$$
$$\kappa_\xi = \beta'/\alpha$$
$$\kappa_\theta = \frac{\sin\phi - \sin\Phi}{r} - \frac{\sin\Phi}{r}k$$
$$\lambda_\xi = 0$$
$$\lambda_\theta = \frac{\cos\Phi - \cos\phi}{r} + \frac{\cos\Phi}{r}k$$

where $\epsilon_\xi = \epsilon_{\xi\xi}$ etc.

The conventional form (10.7) is obtained on setting $\gamma_\xi \equiv 0$ so that β may be expressed in terms of u' and v' and also

$$v'/\alpha = (1+\epsilon_\xi)\sin\Phi - \sin\phi \qquad (10.17)$$

These coincide with the forms stated in [317] except for k in the terms in κ_θ and γ_θ which were omitted therein. The compatibility relation of interest is then Table 10.1(ii):

$$(r\epsilon_\theta)' - r'\epsilon_\xi - \alpha r(1+\epsilon_\xi)\lambda_\theta = 0 \qquad (10.18)$$

10.2.2. Static Relations

The equations of equilibrium for a curved pipe are, Table 10.2(i)(iii)(iv)

$$\begin{aligned}
&(rN_\xi)' - r'N_\theta + \alpha r(\kappa_\xi + 1/R_\xi)Q - \alpha r\lambda_\theta N_\theta + \alpha r p_\xi = 0\\
&(rQ)' - \alpha r[N_\xi(\kappa_\xi + 1/R_\xi) + N_\theta(\kappa_\theta + 1/R_\theta)] + \alpha r p_N = 0 \qquad (10.19)\\
&(rM_\xi)' - r'M_\theta - \alpha r(1+\epsilon_\xi)Q - \alpha r\lambda_\theta M_\theta + \alpha r q_\xi = 0
\end{aligned}$$

on writing $Q_\xi = Q$ etc. and defining (10.7), $P_\xi = P_\theta = 0$.

If a horizontal and a vertical stress resultant H and V respectively are defined by (Fig 10·2)

$$\begin{aligned}
N_\xi &= H\cos\Phi + V\sin\Phi\\
Q &= -H\sin\Phi + V\cos\Phi
\end{aligned}$$

and it is assumed that the sole body force is an internal, radial pressure p i.e.

$$p_\xi = 0 \qquad p_N = -p \qquad q_\xi = 0$$

then equations (10.19) are

$$\begin{aligned}
&(rH)' - \alpha(1+k)N_\theta + \alpha r\sin\Phi\, p = 0\\
&(rV)' - \alpha r\cos\Phi\, p = 0 \qquad (10.20)\\
&(rM_\xi)' - r'M_\theta - \alpha r(1+\epsilon_\xi)Q - \alpha r\lambda_\theta M_\theta = 0
\end{aligned}$$

which may be identified as horizontal, vertical and moment equilibrium respectively. Again these are similar to those of [317] with the exception of the term kN_θ in horizontal

equilibrium.

10.2.3. Material Relations

The basic constitutive model which shall be considered is that of an elastic body with initial strains.

Let ε_ξ and ε_θ be principle strains, related to the mid surface strain resultants and couples by

$$\varepsilon_\xi = \epsilon_\xi + \zeta \kappa_\xi \qquad \varepsilon_\theta = \epsilon_\theta + \zeta \kappa_\theta$$

assuming plane sections to remain plane about the mid surface during deformation.

Let σ_ξ and σ_θ be principle stresses related to the strains via Hooke's law for an isotropic, homogeneous material with small strains

$$\begin{aligned} \varepsilon_\xi &= \frac{1}{E}(\sigma_\xi - \nu\sigma_\theta) + \varepsilon_\xi^* \\ \varepsilon_\theta &= \frac{1}{E}(\sigma_\theta - \nu\sigma_\xi) + \varepsilon_\theta^* \end{aligned} \qquad (10.21)$$

where E is Young's modulus, ν Poisson's ratio and $(\varepsilon_\xi^*, \varepsilon_\theta^*)$ an arbitrary initial strain field.

Then on inverting (10.21) and using the definition of stress and moment resultant

$$N_i = \int_{-h/2}^{h/2} \sigma_i \, d\zeta \qquad M_i = \int_{-h/2}^{h/2} \sigma_i \, \zeta \, d\zeta \qquad i = \xi, \theta$$

and assuming that the shell has constant thickness there results

$$\begin{aligned} N_\xi &= \bar{N}_\xi - N_\xi^* & N_\theta &= \bar{N}_\theta - N_\theta^* \\ M_\xi &= \bar{M}_\xi - M_\xi^* & M_\theta &= \bar{M}_\theta - M_\theta^* \\ \epsilon_\xi &= \bar{\epsilon}_\xi + \epsilon_\xi^* & \epsilon_\theta &= \bar{\epsilon}_\theta + \epsilon_\theta^* \\ \kappa_\xi &= \bar{\kappa}_\xi + \kappa_\xi^* & \kappa_\theta &= \bar{\kappa}_\theta + \kappa_\theta^* \end{aligned}$$

where

$$\begin{aligned} \bar{N}_\xi &= C(\epsilon_\xi + \nu\epsilon_\theta) & \bar{N}_\theta &= C(\epsilon_\theta + \nu\epsilon_\xi) \\ \bar{M}_\xi &= D(\kappa_\xi + \nu\kappa_\theta) & \bar{M}_\theta &= D(\kappa_\theta + \nu\kappa_\xi) \\ \bar{\epsilon}_\xi &= B(N_\xi - \nu N_\theta) & \bar{\epsilon}_\theta &= B(N_\theta - \nu N_\xi) \\ \bar{\kappa}_\xi &= M_\xi - \nu M_\theta & \bar{\kappa}_\theta &= M_\theta - \nu M_\xi \end{aligned}$$

$$C = Eh/(1-\nu^2) \qquad D = Eh^3/12(1-\nu^2) \qquad B = 1/Eh$$

and

$$\epsilon_\xi^* = \frac{1}{h}\int_{-h/2}^{h/2} \varepsilon_\xi^* \, d\zeta \qquad \epsilon_\theta^* = \frac{1}{h}\int_{-h/2}^{h/2} \varepsilon_\theta^* \, d\zeta$$

$$\kappa_\xi^* = \frac{1}{h^2}\int_{-h/2}^{h/2} \varepsilon_\xi^* \, \zeta \, d\zeta \qquad \kappa_\theta^* = \frac{1}{h^2}\int_{-h/2}^{h/2} \varepsilon_\theta^* \, \zeta \, d\zeta$$

$$N_\xi^* = C(\epsilon_\xi^* + \nu\epsilon_\theta^*) \qquad N_\theta^* = C(\epsilon_\theta^* + \nu\epsilon_\xi^*)$$

$$M_\xi^* = Ch(\kappa_\xi^* + \nu\kappa_\theta^*) \qquad M_\theta^* = Ch(\kappa_\theta^* + \nu\kappa_\xi^*)$$

It is useful to note that compatibility (10.18) and moment equilibrium $(10.19)_3$ becomes

$$(r\bar{M}_\xi)' - r'\bar{M}_\theta - \alpha r(1+\bar{\epsilon}_\xi)Q - \alpha r\lambda_\theta \bar{M}_\theta = f_1^*$$
$$(r\bar{\epsilon}_\theta)' - r'\bar{\epsilon}_\xi - \alpha r(1+\bar{\epsilon}_\xi)\lambda_\theta = f_2^* \qquad (10.22)$$

where

$$f_1^* = (rM_\xi^*)' - r'M_\theta^* - \lambda_\theta M_\theta^* \alpha r - \alpha r Q\epsilon_\xi^*$$
$$f_2^* = -[(r\epsilon_\theta^*)' - r'\epsilon_\xi^* - \alpha r \lambda_\theta \epsilon_\xi^*]$$

10.2.4. The Bending Equations

Two basic equations are now derived which serve to describe the bending of the curved pipe; they are equivalent to E Reissner's "bending equations"for shells of revolution [351] with additional terms in k. The first is obtained by substitution into (10.22), for barred quantities using (10.16)

$$\beta'' + \frac{(r/\alpha)'}{(r/\alpha)}\beta' - \frac{\nu\alpha}{r}[\cos\Phi(1+k) - \cos\phi]\phi' + \left(\frac{\alpha}{r}\right)^2 \cos\Phi[\sin\Phi(1+k) - \sin\phi](1+k)$$
$$- \alpha^2[-(Hr)\sin\Phi + (Vr)\cos\Phi]/Dr$$
$$-(B/D)(\alpha/r)^2[(Hr)\cos\Phi + (Vr)\sin\Phi - \frac{\nu}{1+k}\left(\frac{r}{\alpha}\right)((rH)' + \alpha r\sin\Phi p)][-(Hr)\sin\Phi + (Vr)\cos\Phi]$$
$$= \alpha f_1^*/Dr \qquad (10.23)$$

The second is obtained by substitution into (10.22), for barred quantities using (10.20),

$$B\{(rH)'' + \frac{(r/\alpha)'}{(r/\alpha)}(rH)' - \nu(1+k)\alpha\frac{\sin\Phi}{r}(rV)'$$
$$- (rH)[(\alpha/r)^2\cos^2\Phi(1+k) - \nu\alpha/r\sin\Phi\cdot\Phi'](1+k)$$
$$- (rV)[(\alpha/r)^2\cos\Phi\sin\Phi(1+k) + \nu\alpha/r\cos\Phi\cdot\Phi'](1+k)$$
$$+ \alpha/r(r^2\sin\Phi)'p + \nu(\alpha/r)^2\cos\Phi(r^2\sin\Phi)p(1+k)\}$$
$$- (1+k)\alpha^2/r(\cos\Phi(1+k) - \cos\phi) = \frac{(1+k)}{(r/\alpha)}f_2^* \qquad (10.24)$$

10.2.5. Additional Equations

The bending equations contain three unknowns $(Hr), (Vr)$ and β; an additional equation is therefore required. This is supplied by vertical equilibrium $(10.20)_2$

$$(rV)' - \alpha r \cos\Phi p = 0 \qquad (10.25)$$

The system (10.23), (10.24) and (10.25) is therefore complete provided sufficient boundary conditions are supplied. In some cases - in particular under the action of internal pressure - such conditions are unavailable in the required form and yet another equation is needed. This is (10.17) which can be rewritten as

$$(1+k)\frac{v'}{\alpha} = (1+k)(\sin\Phi - \sin\phi) + (1+k)\,B/r \sin\Phi[(Hr)\cos\Phi + (Vr)\sin\Phi]$$
$$- B\nu/\alpha\,[(Hr)' + \alpha r \sin\Phi p]\sin\Phi + (1+k) f_3^* \qquad (10.26)$$

where

$$f_3^* = \epsilon_\xi^* \sin\Phi$$

10.2.6. External Moment Equilibrium

For the curved pipe there are two possible problems; the first is the case where an external in-plane bending moment is applied resulting in an end rotation, the second is the case where an end rotation is specified and there is consequently a resultant in-plane moment. In the case of small strains and small deflections these problems are identical, however, in the nonlinear case here they are not. Nevertheless in either case if M is the applied - or resultant - in-plane moment then external equilibrium must be satisfied:

$$M = \oint_C [\alpha(r-a)N_\Theta - l' M_\Theta]\, d\xi \qquad (10.27)$$

where a is the distance of the z-axis from the centre of inertia of the cross sectional curve C.

10.2.7. Evaluation of the Stress Field

Given the quantities (Hr), (Vr) and β it is necessary to evaluate the stress field σ_ξ and σ_θ. It is readily verified that

$$\sigma_\xi = \frac{N_\xi}{h} + \frac{12}{h^3}\frac{E}{1-\nu^2}(\kappa_\xi + \nu\kappa_\theta) + \sigma_\xi^*$$
$$\sigma_\theta = \frac{N_\theta}{h} + \frac{12}{h^3}\frac{E}{1-\nu^2}(\kappa_\theta + \nu\kappa_\xi) + \sigma_\theta^* \qquad (10.28)$$

where

$$\sigma_\xi^* = \frac{N_\xi^*}{h} - \frac{E}{1-\nu^2}(\varepsilon_\xi^* + \nu\varepsilon_\theta^*)$$
$$\sigma_\theta^* = \frac{N_\theta^*}{h} - \frac{E}{1-\nu^2}(\varepsilon_\theta^* + \nu\varepsilon_\xi^*)$$

All the quantities in (10.28) may be directly expressed in terms of $(Hr), (Vr), \beta$.

10.2.8. A linear Theory for no internal pressure

If small rotations as well as small strains are assumed then the bending equations reduce to the equations of the linear theory on neglecting all nonlinear terms [323,326] and internal pressure

$$\beta'' + \frac{(r/\alpha)'}{(r/\alpha)}\beta' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha)'}{(r/\alpha)}\right]\beta + \frac{\alpha\ell'}{rD}(Hr) = \left(-\frac{r'\ell'}{r^2} + \nu\frac{(\ell'/\alpha)'}{(r/\alpha)}\right)k + \frac{\alpha}{Dr}f_1^*$$

$$B\left\{(Hr)'' + \frac{(r/\alpha)'}{(r/\alpha)}(Hr)' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha)'}{(r/\alpha)}\right](Hr)\right\} - \frac{\alpha\ell'}{r}\beta =$$
$$= \alpha\frac{r'}{r}k + \frac{\alpha}{r}f_2^* \qquad (10.29)$$

where

$$f_1^* = (rM_\xi^*)' - r'M_\theta^*$$
$$f_2^* = -[(r\epsilon_\theta^*)' - r'\epsilon_\xi^*]$$

10.3 Discussion

In this chapter a new, consistent thin shell model of the deformation of a smooth, curved pipe has been derived based on two fundamental assumptions together with various simplifications to reduce the equations to a tractable form. This formulation contains the equations presented by S H Crandall and N C Dahl [317] and J A Blomfield [323] as special cases, and in particular the classical linear form [326].

The ad hoc assumptions, Equation (10.14), which is usually made about the displacement field has been reduced to a more physically motivated one, Equation (10.15). It is obvious to ask how accurate the model is? In essence this questions the acceptability of Assumptions 9.1 and 9.2. The usual verification is through an experiment to determine a numerical value for the end rotation for comparison with a numerical study of the equations. The writer would personally prefer a test of the two basic assumptions.

Nonetheless the following chapter is a numerical study of the model and a comparison with previous numerical results for elastic behaviour.

TABLE 10.1

EQUATIONS OF COMPATIBILITY FOR A THIN SHELL

$$\frac{\epsilon_{21,1}}{\alpha_1} - \frac{\epsilon_{11,2}}{\alpha_2} + \frac{\epsilon_{21}+\epsilon_{12}}{S_2} + \frac{\epsilon_{11}-\epsilon_{22}}{S_1} - \gamma_1\left(\frac{1}{R_{21}} + \kappa_{21}\right) + \gamma_2\left(\frac{1}{R_{11}} + \kappa_{11}\right)$$
$$- \lambda_1(1+\epsilon_{22}) - \lambda_2\epsilon_{12} = 0 \qquad (i)$$

$$\frac{\epsilon_{22,1}}{\alpha_1} - \frac{\epsilon_{12,2}}{\alpha_2} + \frac{\epsilon_{12}+\epsilon_{21}}{S_1} + \frac{\epsilon_{22}-\epsilon_{11}}{S_2} + \gamma_2\left(\frac{1}{R_{12}} + \kappa_{12}\right) - \gamma_1\left(\frac{1}{R_{22}} + \kappa_{22}\right)$$
$$- \lambda_2(1+\epsilon_{11}) - \lambda_1\epsilon_{21} = 0 \qquad (ii)$$

$$\frac{\gamma_{2,1}}{\alpha_1} - \frac{\gamma_{1,2}}{\alpha_2} + \frac{\gamma_2}{S_2} + \frac{\gamma_1}{S_1} + \frac{\epsilon_{12}}{R_{22}} - \frac{\epsilon_{21}}{R_{11}} + \frac{\epsilon_{11}-\epsilon_{22}}{R_{12}}$$
$$+ \kappa_{21}(1+\epsilon_{11}) - \kappa_{12}(1+\epsilon_{22}) + \epsilon_{12}\kappa_{22} - \epsilon_{21}\kappa_{11} = 0 \qquad (iii)$$

$$\frac{\kappa_{21,1}}{\alpha_1} - \frac{\kappa_{11,2}}{\alpha_2} + \frac{\kappa_{12}+\kappa_{21}}{S_2} + \frac{\kappa_{11}-\kappa_{22}}{S_1}$$
$$+ \lambda_2\left(\frac{1}{R_{12}} + \kappa_{12}\right) - \lambda_1\left(\frac{1}{R_{22}} + \kappa_{22}\right) = 0 \qquad (iv)$$

$$\frac{\kappa_{22,1}}{\alpha_1} - \frac{\kappa_{12,2}}{\alpha_2} + \frac{\kappa_{21}+\kappa_{12}}{S_1} + \frac{\kappa_{22}-\kappa_{11}}{S_2}$$
$$+ \lambda_1\left(\frac{1}{R_{21}} + \kappa_{21}\right) - \lambda_2\left(\frac{1}{R_{11}} + \kappa_{11}\right) = 0 \qquad (v)$$

$$\frac{\lambda_{2,1}}{\alpha_1} - \frac{\lambda_{1,2}}{\alpha_2} + \frac{\lambda_2}{S_2} + \frac{\lambda_1}{S_1} + \frac{\kappa_{11}}{R_{22}} + \frac{\kappa_{22}}{R_{11}} - \frac{(\kappa_{12}+\kappa_{21})}{R_{12}}$$
$$+ \kappa_{11}\kappa_{22} - \kappa_{12}\kappa_{21} = 0 \qquad (vi)$$

TABLE 10.2

EQUATIONS OF EQUILIBRIUM FOR A THIN SHELL

$$\frac{N_{11,1}}{\alpha_1} + \frac{N_{21,2}}{\alpha_2} - \frac{N_{21}+N_{12}}{S_1} + \frac{N_{11}-N_{22}}{S_2} + Q_1\left(\frac{1}{R_{11}} + \kappa_{11}\right) + Q_2\left(\frac{1}{R_{21}} + \kappa_{21}\right)$$
$$- \lambda_1 N_{12} - \lambda_2 N_{22} + p_1 = 0 \qquad (i)$$

$$\frac{N_{12,1}}{\alpha_1} + \frac{N_{22,2}}{\alpha_2} + \frac{N_{21}+N_{12}}{S_2} - \frac{N_{22}-N_{11}}{S_1} + Q_2\left(\frac{1}{R_{22}} + \kappa_{22}\right) + Q_1\left(\frac{1}{R_{12}} + \kappa_{12}\right)$$
$$+ \lambda_2 N_{21} + \lambda_1 N_{11} + p_2 = 0 \qquad (ii)$$

$$\frac{Q_{1,1}}{\alpha_1} + \frac{Q_{2,2}}{\alpha_2} + \frac{Q_1}{S_2} - \frac{Q_2}{S_1} - N_{11}\left(\frac{1}{R_{11}} + \kappa_{11}\right) - N_{22}\left(\frac{1}{R_{22}} + \kappa_{22}\right)$$
$$- N_{12}\left(\frac{1}{R_{12}} + \kappa_{12}\right) - N_{21}\left(\frac{1}{R_{21}} + \kappa_{21}\right) + p_N = 0 \qquad (iii)$$

$$\frac{M_{11,1}}{\alpha_1} + \frac{M_{21,2}}{\alpha_2} - \frac{M_{12}+M_{21}}{S_1} + \frac{M_{11}-M_{22}}{S_2} + P_1\left(\frac{1}{R_{12}} + \kappa_{12}\right) + P_2\left(\frac{1}{R_{22}} + \kappa_{22}\right)$$
$$- \lambda_1 M_{12} - \lambda_2 M_{22} - (1+\epsilon_{11})Q_1 - \epsilon_{21} Q_2 + \gamma_1 N_{11} + \gamma_2 N_{21} + q_1 = 0 \qquad (iv)$$

$$\frac{M_{12,1}}{\alpha_1} + \frac{M_{22,2}}{\alpha_2} + \frac{M_{12}+M_{21}}{S_2} - \frac{M_{22}-M_{11}}{S_1} - P_1\left(\frac{1}{R_{11}} + \kappa_{11}\right) - P_2\left(\frac{1}{R_{21}} + \kappa_{21}\right)$$
$$+ \lambda_2 M_{21} + \lambda_1 M_{11} - (1+\epsilon_{22})Q_2 - \epsilon_{12} Q_1 + \gamma_2 N_{22} + \gamma_1 N_{12} + q_2 = 0 \qquad (v)$$

$$\frac{P_{1,1}}{\alpha_1} + \frac{P_{2,2}}{\alpha_2} + \frac{P_1}{S_2} - \frac{P_2}{S_1} + M_{12}\left(\frac{1}{R_{11}} + \kappa_{11}\right) - M_{21}\left(\frac{1}{R_{22}} + \kappa_{22}\right)$$
$$+ M_{22}\left(\frac{1}{R_{21}} + \kappa_{21}\right) + M_{11}\left(\frac{1}{R_{12}} + \kappa_{12}\right)$$
$$+ (1+\epsilon_{11})N_{12} - (1+\epsilon_{22})N_{21} + \epsilon_{21} N_{22} - \epsilon_{12} N_{11} + q_N = 0 \qquad (vi)$$

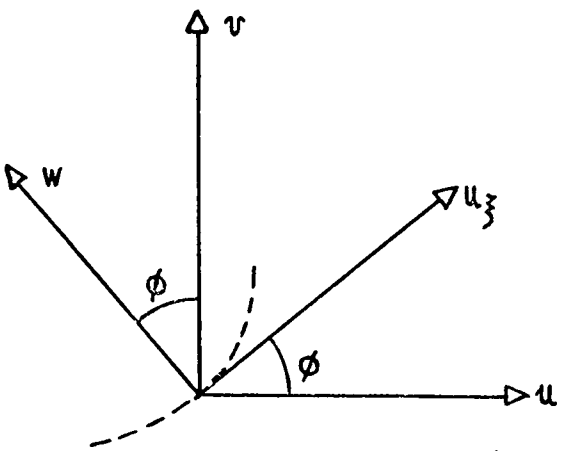


Fig 10·1

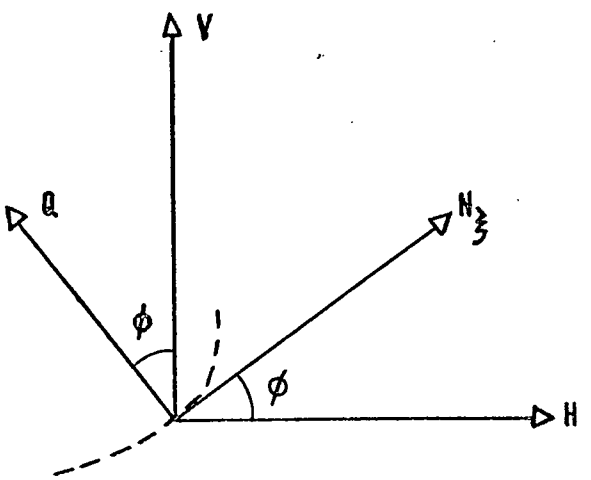


Fig 10·2

CHAPTER 11

NUMERICAL STUDY OF THE CURVED PIPE MODEL IN ELASTICITY

The aim of this Chapter is a numerical study of the elastic response of the (nonlinear) curved pipe model developed in the previous chapter. This shall be done in two stages; firstly a pressurized (elliptic) curved pipe shall be analysed using the nonlinear theory, secondly a circular cross section curved pipe shall be analysed using the linear theory. It is this latter case which shall be employed in the investigation of the creep behaviour, but the former enables the accuracy of this case to be gauged.

11.1 Numerical shell analysis

In general the nonlinear equations of a thin elastic shell may be reduced to a nonlinear elliptic boundary value problem and thus any of the well known methods of solution for these types of equations can be used (see [162,397]) and in particular the finite element method [270]. It is often possible to express the problem in the form of a set of ordinary differential equations; there are in fact only two major techniques for the numerical resolution of this type of problem - the "finite difference method" and "direct integration" via a reduction to an initial value problem [360,361]. Both are discrete methods - the former being a technique for constructing a discrete analogue of the differential equations, the latter being an ingenious contrivance which converts a boundary value problem into an initial value problem to which standard sequential methods can be applied. Both shall be considered here.

In the linear case (subsection 10.2.8.) both methods are straightforward, and their applicability to shell

analysis is well known [362]. The basis of the finite difference method is that a matrix equation can be formed [363-366] for the discrete model and standard solution methods can then be employed. Its main failing is that it can overwhelm the computer capacity and requires much analytical preparation, yet it is well behaved in the sense that its solution is stable. The direct integration [362], or "shooting", method [360,361] on the other hand is fast and easy to use yet it is very often unstable so that the approximate solution becomes unbounded due to the growth of parasitic errors. It is possible to overcome this difficulty by using a method known as "multiple, or parallel, shooting" which is a combination of finite differences and shooting; multiple shooting has been rigorously developed in shell analysis by A Kalnins [367,368] (the "multisegment method") and J C Falkenberg [369]. An alternative, yet less satisfactory, method of avoiding instability is the so-called "field method" [370,371]. A comparison of direct integration with other methods, notably finite elements, can be found in references [372, 373].

In the nonlinear geometric situation the problem is a good deal more involved. The finite difference method as an exercise in matrix analysis may only be possible through the iterative solution of a sequence of associated linearized equations and is consequently an even greater strain on a computer, except for simple problems, nonetheless it has been applied to shell analysis [374]. An alternative realization of the finite difference method, not using matrix analysis, has been suggested [375,376] - the so-called "energy method" - which has several points in common with the finite element method.

For nonlinear equations direct integration comes into its own [366]. There are two different techniques for realizing direct integration, although they are essentially similar [377]. One is known as "nonlinear shooting" [360,378], which is a direct extension of the linear method of converting a boundary value into an initial value problem to the nonlinear situation; the other uses a method of linearization, known as "quasilinearization" [379,380] being a generalization of the well known Newton-Raphson technique to differential equations. The sequence of linear problems which result can then be solved using direct integration (or, indeed, finite differences). A major disadvantage of quasilinearization is a possible lack of convergence, although this may be overcome using a method known as "modified quasilinearization" [381,382]. Nonlinear shooting has been applied to shell analysis by A Kalnins [383] extending his previous multisegment method; however to the writer's knowledge there are no examples of quasilinearization with linear shooting in shell analysis.

Finally, it is to be noted that direct integration, which had previously been thought of as a purely numerical technique, has been put on a rigorous, analytic foundation as a procedure known as "invariant imbedding" [384,385]; so far the invariant imbedding equations for a thin shell have only been formulated in the linear case [386], and no numerical results are available.

So far as the curved pipe is concerned direct integration has been employed in the linear case by J A Blomfield [323], based on Sander's theory [339] and by A Robert and C Du Fort [324] based on Koiter's theory [340].

To the writer's knowledge no true nonlinear curved pipe analysis has been performed. Here the response of a curved pipe to internal pressure and a uniform in-plane bending moment shall be studied numerically using quasilinearization and a linear shooting method called the "method of particular solutions" [380,381]. The properties of the thin shell model shall be discussed and then compared with a discrete model based on finite differences.

11.2 A nondimensionalized, second order theory

It is entirely possible to study the general, nonlinear equations. But it is presumed that large, finite deflections and small strains are physically incompatible. The assumption is therefore made of moderately large deflections [345,354], requiring a second order theory. A second order theory results on expanding $\cos\Phi$ and $\sin\Phi$ in powers of β

$$\cos\Phi = \cos\phi + \beta\sin\phi - \tfrac{1}{2}\beta^2\cos\phi + \cdots\cdots$$
$$\sin\Phi = \sin\phi - \beta\cos\phi - \tfrac{1}{2}\beta^2\sin\phi + \cdots\cdots$$

substituting into the equations of the general theory and retaining terms like β^2, $\beta(Hr)$, βk etc. while ignoring higher order terms like β^3, $\beta^2(Hr)$, $\beta^2 k$ as well as k^2, $k(Hr)$, $(Hr)(Hr)$ etc.

It is convenient to nondimensionalize the equation in the following manner:
Let σ_o be an arbitrary stress and α_o, α_1 arbitrary lengths; define new variables as follows

$$X = \frac{\beta}{\alpha_o\mu_o\varepsilon_o} \qquad Y = \frac{B(Hr)}{\alpha_o\mu_o} \qquad Z = \frac{B(Vr)}{\alpha_o\mu_o} \qquad U = \frac{V}{\alpha_o\varepsilon_o}$$

$$T = \frac{k}{\varepsilon_o} \qquad P_o = B\mu_o p$$

where

$$\mu_o = \sqrt{12(1-\nu^2)}/h \qquad \varepsilon_o = \sigma_o/E$$

Then equations (10.23), (10.24), (10.25) and (10.26) become:

$$X'' + \frac{(r/\alpha)'}{(r/\alpha)}X' - \left[\left(\frac{r'}{r}\right)^2 - \nu\frac{(r'/\alpha)'}{(r/\alpha)}\right]X - \frac{\alpha\mu_o}{r}\left[-Yl' + Zr'\right] - \frac{\Gamma}{\alpha_o\mu_o}\left[-\frac{r'l'}{r^2} + \nu\frac{(l'/\alpha)'}{(r/\alpha)}\right]$$

$$- \varepsilon_1 X\left\{\frac{X}{2}\left[3\frac{r'l'}{r^2} - \nu\frac{(l'/\alpha)'}{(r/\alpha)}\right] + \frac{\alpha\mu_o}{r}(Yr' + Zl') + \frac{\Gamma}{\alpha_o\mu_o}\left[\frac{2(r')^2 - (l')^2}{r^2} - \nu\frac{(r'/\alpha)'}{(r/\alpha)}\right]\right\} = 0$$

$$Y'' + \frac{(r/\alpha)'}{(r/\alpha)}Y' - \nu\frac{l'}{r}Z' - \left[\left(\frac{r'}{r}\right)^2 + \nu\frac{(r'/\alpha)'}{(r/\alpha)}\right]Y - \left[\frac{r'l'}{r^2} + \nu\frac{(l'/\alpha)'}{(r/\alpha)}\right]Z - \frac{\alpha\mu_o}{r}\left[Xl' + r'\frac{\Gamma}{\alpha_o\mu_o}\right]$$

$$+ Q_1\left[(\nu+2)\frac{r'l'}{\alpha_1^2} + \frac{\alpha r}{\alpha_1^2}\left(\frac{l'}{\alpha}\right)'\right]$$

$$- \varepsilon_1 X\left\{-\nu\frac{r'}{r}Z' + \left[2\frac{r'l'}{r^2} + \nu\frac{(l'/\alpha)'}{(r/\alpha)}\right]Y + \left[\frac{(l')^2 - (r')^2}{r^2} - \nu\frac{(r'/\alpha)'}{(r/\alpha)}\right]Z - \frac{\alpha\mu_o}{r}\left[\frac{r'}{2}X - 2l'\frac{\Gamma}{\alpha_o\mu_o}\right]\right.$$

$$\left. + Q_1\left[(\nu+2)\left(\frac{r'}{\alpha}\right)^2 - \nu\left(\frac{l'}{\alpha}\right)^2 + \frac{\alpha r}{\alpha_1^2}\left(\frac{r'}{\alpha}\right)'\right]\right\}$$

$$- \varepsilon_1 X'\left\{\nu\left(Y\frac{l'}{r} - Z\frac{r'}{r}\right) + Q_1\frac{rr'}{\alpha_1^2}\right\} = 0$$

$$Z' = \frac{r}{\alpha_1^2}Q_1(r' + \varepsilon_1 Xl') \qquad (11.1)$$

$$U' = -\mu_o r'X + \frac{l'}{\alpha r}\left[Yr' + Zl'\right] - \nu\frac{l'}{\alpha}Y' - Q_1\nu\frac{r}{\alpha}\left(\frac{l'}{\alpha_1}\right)^2$$

$$+ \varepsilon_1 X\left\{-\alpha_o\mu_o\left[\frac{l'}{\alpha_o}\frac{X}{2} + \frac{r'}{\alpha_o}\frac{\Gamma}{\mu_o}\right] + \frac{Y}{\alpha r}\left[(l')^2 - (r')^2\right] - 2\frac{r'l'}{\alpha r}Z + \nu\frac{r'}{\alpha}Y'\right.$$

$$\left. + 2\nu\left(\frac{r}{\alpha}\right)\left(\frac{r'}{\alpha_1}\right)\left(\frac{l'}{\alpha_1}\right)Q\right\}$$

where

$$\varepsilon_1 = \alpha_o\mu_o\varepsilon_o \qquad Q_1 = \alpha_1^2 P_o/\varepsilon_1$$

Equilibrium of moments (10.27) becomes

$$\frac{M}{\sigma_o h\alpha_o^2} = \oint_C\left\{\frac{(r-a)}{\alpha_o}\left(Y' + \frac{r}{\alpha_1^2}(l' - \varepsilon_1 Xr')Q_1\right) + \frac{(l')^2}{(\alpha_o\mu_o)^2}\frac{\Gamma}{\alpha r}\right.$$

$$\left. - \frac{(l'/\alpha)}{\mu_o\alpha_o}\left[\left(\frac{r'}{r}\right)X + \nu X' + \varepsilon_1 X\left[\frac{X}{2}\frac{l'}{r} + \frac{\Gamma}{\alpha_o\mu_o}\frac{r'}{r}\right]\right]\right\}d\xi \qquad (11.2)$$

and finally these are related to stress via

$$S_\xi = \frac{\sigma_\xi}{\sigma_o} = \frac{N_\xi}{\sigma_o h} + \zeta\frac{h}{1-\nu^2}\left(\frac{\kappa_\xi}{\varepsilon_o} + \nu\frac{\kappa_\theta}{\varepsilon_o}\right)$$

$$S_\theta = \frac{\sigma_\theta}{\sigma_o} = \frac{N_\theta}{\sigma_o h} + \zeta\frac{h}{1-\nu^2}\left(\frac{\kappa_\theta}{\varepsilon_o} + \nu\frac{\kappa_\xi}{\varepsilon_o}\right) \qquad (11.3)$$

where

$$\frac{N_3}{\sigma_0 h} = \left(\frac{\alpha_0}{\alpha}\right)\left[\frac{r'}{r}Y + \frac{l'}{r}Z + \varepsilon_1 X\left(Y\frac{l'}{r} - Z\frac{r'}{r}\right)\right]$$

$$\frac{N}{\sigma_0 h} = \left(\frac{\alpha_0}{\alpha}\right)\left[Y' + \frac{r}{\bar{\alpha}_1^2}\left(l' - \varepsilon_1 X r'\right)Q_1\right] \qquad (11.4)$$

$$\frac{\kappa_3}{\varepsilon_0} = \mu_0\left(\frac{\alpha_0}{\alpha}\right)X'$$

$$\frac{\kappa_\Theta}{\varepsilon_0} = \mu_0\left(\frac{\alpha_0}{\alpha}\right)\left[\frac{r'}{r}X - \frac{l'}{r}\frac{\Gamma}{\alpha_0\mu_0} + \varepsilon_1 X\left[\frac{X}{2}\frac{l'}{r} + \frac{\Gamma}{\alpha_0\mu_0}\frac{r'}{r}\right]\right]$$

and note that

$$\frac{h}{1-\nu^2}\mu_0 = \sqrt{\frac{12}{1-\nu^2}} \qquad \zeta = \frac{z}{h}$$

The nondimensionalized linear theory results on ignoring all terms with ε_1 (and with Q_1 to obtain (10.29)).

11.3 The elliptic cross section curved pipe under internal pressure and in-plane bending

The basic equations shall now be specialized to an elliptic cross section applicable to numerical solution via direct integration.

For an elliptic cross section (Figure 11.1)

$$r = a + b\sin\xi$$

$$l = -c\cos\xi$$

where $0 \le \xi \le 2\pi$ is a parameter describing the ellipse.

Hence immediately

$$\alpha^2 = (r')^2 + (l')^2 = c^2(\sin^2\xi + \eta^2\cos^2\xi)$$

where

$$\eta = \text{ellipticity ratio} = b/c$$

It is convenient to define the geometric parameters

$$\lambda = \text{pipe bend parameter} = ah/bc$$

$$\varrho = \text{radius ratio} = b/a$$

and the associated parameter

$$\mu = \sqrt{12(1-\nu^2)}/h$$

System (11.1) constitutes four equations in five unknowns X, Y, Z, U and Γ and therefore an additional equation is needed; since Γ is known to be constant it follows

$$\frac{d\Gamma}{d\xi} = 0 \qquad (11.5)$$

Consequently there are two second order and three first order differential equations and for the problem to be well-posed there must therefore be 2 x 2 + 3 x 1 = 7 boundary conditions.

Obviously from symmetry

$$X=0 \qquad Y=0 \qquad U=0 \qquad \xi = \pi/2, 3\pi/2 \qquad (11.6)$$

supplying six of the required conditions. Since the problem of an applied bending moment is to be considered the missing condition is supplied by moment equilibrium (11.2)!

The equations may now be transformed into the standard form

$$\frac{d}{d\xi}\underset{\sim}{y} = \underset{\sim}{f}(\xi, \underset{\sim}{y}) \qquad (11.7)$$

on defining the vector $\underset{\sim}{y} = (y_1\ y_2\ y_3\ y_4\ y_5\ y_6\ y_7)$

$$y_1 = X \quad y_2 = Y \quad y_3 = U \quad y_4 = Z \quad y_5 = X' \quad y_6 = Y' \quad y_7 = K$$

where

$$K = \Gamma\rho$$

and the vector valued function $\underset{\sim}{f} = (f_1\ f_2\ f_3\ f_4\ f_5\ f_6\ f_7)$ is given by

$$f_1 = y_5$$

$$f_2 = y_6$$

$$f_3 = (L_{31} + N^*_{31})y_1 + L_{32}y_2 + L_{34}y_4 + L_{36}y_6 + C_3 + y_1[N_{31}y_1 + N_{32}y_2 + N_{34}y_4 + N_{36}y_6 + N_{37}y_7]$$

$$f_4 = N_{41}y_1 + C_4$$

$$f_5 = L_{51}y_1 + L_{52}y_2 + L_{54}y_4 + L_{55}y_5 + L_{57}y_7 + y_1[N_{51}y_1 + N_{52}y_2 + N_{54}y_4 + N_{57}y_7]$$

$$f_6 = (L_{61} + N^*_{61})y_1 + L_{62}y_2 + L_{64}y_4 + N^*_{65}y_5 + L_{66}y_6 + L_{67}y_7 + C_6 + y_1[(N_{61} + N^*_{61})y_1 + N_{62}y_2 + N_{64}y_4 + N_{67}y_7] + y_5[N^*_{62}y_2 + N^*_{64}y_4]$$

$$f_7 = 0$$

with coefficients L_{ij}, N_{ij}, C_i given in Table 11.1

Due to symmetry (11.7) only need be integrated in the range

$$\pi/2 \leq \xi \leq 3\pi/2$$

with boundary conditions (11.6)

$$\begin{aligned} y_1(\pi/2) &= 0 & y_1(3\pi/2) &= 0 \\ y_2(\pi/2) &= 0 & y_2(3\pi/2) &= 0 \\ y_3(\pi/2) &= 0 & y_3(3\pi/2) &= 0 \end{aligned} \qquad (11.8)$$

and end condition (11.2), on integrating by parts (Table 11.1),

$$\int_{\pi/2}^{3\pi/2} \{ m_1 y_1 + m_2 y_2 + y_1 (m_3 y_1 + m_4 y_4) + m_5 y_5 \} d\xi = \gamma \qquad (11.9)$$

where

$$\gamma = \eta J(\eta) m_p/2 - \pi \lambda Q_1/2$$

on taking

$$\sigma_o = \sigma_p = M_p b/I \qquad m_p = M/M_p \qquad \alpha_o = c \qquad \alpha_1 = a$$

with I the second moment of area of C, $I = hcb^2 J(\eta)$, and

$$J(\eta) = \int_o^{2\pi} \sin^2\xi (\sin^2\xi + \eta^2 \cos^2\xi)^{1/2} d\xi$$

tabulated in Table 11.2.

The nondimensionalizing moment M_p is completely arbitrary, being used to keep the numerical values of $\underset{\sim}{y}$ small.

The system of boundary value problems (11.7) and (11.8), (11.9) may now be integrated directly using quasilinearization and the method of particular solutions, described in Appendix 3.

11.3.1. Geometric, loading and material parameters

Three geometrical, η, ρ and λ, and one material, ν, parameters appear in (11.7) (see Table 11.1) along with the additional loading parameters ε_1, Q_1 and m_p.

Since the equations are nonlinear and there are two

loadings there must be two loading parameters (in the linear case this may be reduced to one). These are chosen to be

$$m_p = M/M_p \qquad \psi_p = a^2 P/Ech$$

with $\varepsilon_p = \sigma_p/E$ chosen arbitrarily as some number of the correct order of magnitude, say $\varepsilon_p = 0.001$. Then

$$\varepsilon_1 = \mu\varepsilon_p/\rho \qquad Q_1 = \psi_p/\varepsilon_p$$

The parameter ψ_p, which lies in the range (0,0.05), is equivalent to that used by E C Rodabaugh and H H George [387] (also [388]); the parameter m_p lies in the range (0,1) could be avoided by setting $M_p = M$ then taking $\varepsilon_p = \varepsilon_o$ as the second parameter - however m_p is more useful in studying the effect of variations in the bending moment. It is also possible to choose the parameter

$$Q_p = Pb^3/M = Q_1\rho^3/J(\eta)$$

suggested by Spence [306] instead of ψ_p.

It is useful to think of m_p and ψ_p as a nondimensionalized moment and pressure, with M_p the limit load.

11.3.2. Results and comparison with other theories

In the classical analysis of curved pipes two numbers are of interest - the "flexibility factor" defined as the ratio of the end rotation of the curved pipe to the end rotation of a geometrically similar straight pipe composed of the same material, and the "maximum stress concentration factor" defined as the maximum ratio of stress in the curved pipe to maximum fibre stress in the similar straight pipe. These shall be denoted here for the time being by K_p and $SCFM_p$ respectively. It is fairly obvious on studying the nondimensionalization that if

$$K_p = k \div Ma/EI \qquad (11\cdot10)$$

then

$$K_p = K / m_p$$

and similarly

$$SCFM_p = \max(S_z, S_\theta) / m_p \qquad (11.11)$$

it is of the utmost importance to notice that in the general, nonlinear situation both K_p and $SCFM_p$ depend on applied moment M. In the linear case they are, however, independent of M.

A FORTRAN IV computer program was written for the ERCC IBM 370/158 system to perform quasilinearization with multiple linear shooting. A typical computation took 40 seconds mill time to do five iterations, although three were sufficient for convergence. The purpose here is not to provide extensive documentation of K_p and $SCFM_p$ for a variety of geometrical, material and loading parameters [387,388]. Rather the computer program is used to study the properties of the proposed deformation model under the effect of combined internal pressure and in-plane bending comparing the nonlinear with a linearized theory. A typical geometry, taken from [387] shall be used i.e.

$$\lambda = 0{\cdot}1068 \qquad \rho = 0{\cdot}3273 \qquad \nu = 0{\cdot}285$$

initially assuming the pipe to be of circular cross section so that

$$\eta = 1{\cdot}0$$

Firstly examine the linear case, obtained by ignoring all terms in (11.7) with ε_1 but retaining internal pressure. Iteration in quasilinearization is not required and only one parameter Q_p is needed on equating m_p to unity.

Figure 11.2 shows K_p plotted against Q_p. When $Q_p = 0$ the well known elastic flexibility factor, with zero pressure, is obtained (compare with [323,387]). Clearly as Q_p increases K_p decreases until at some critical pressure

it becomes negative indicating that the bend is opening up with pressure swamping the tendency of the pipe to close under the effect of a bending moment. To the writer's knowledge a graph like 11.2 is not available in the literature and it appears that this is the first time that this property of the linear model with internal pressure has been observed.

Secondly consider the nonlinear equations: Figure 11.3 shows K plotted against m_p for various ψ_p. In the case of no internal pressure as $m_p \to 0, K \to 0$ and the nonlinear results tends to be linear. As pressure is increased the linear result moves downwards (c.f. Figure 11.2) while the nonlinear result moves outward and down. The results clearly indicate that as pressure increases the predictions of the nonlinear theory become increasingly less like the linear theory, although for "small" m_p they tend to be the same. Figure 11.4 shows the effect of pressure on the circumferential stress concentration factor for linear and nonlinear theories.

The writer finds these results totally surprising since they are so dissimilar from classical analyses of a curved pipe under pressure. In fact it was found that no valid comparison could be made and the question which must be asked is, why?

Three important points must be noted:

(a) The model of the curved pipe employed in previous analyses and that presented above are identical in that essentially the same assumptions concerning a thin, curved shell are made.

(b) All previous analyses predict that, no matter what the value of the internal pressure, if there is no applied

in-plane bending there is no resultant end rotation.

(c) There is a linear relationship between end rotation and applied moment (obviously this implies (b)).

It is of the utmost importance to understand the significance of point (a). Although analyses differ - energy methods in [387-389], series solutions in [317] and linear shooting in [323] - the mathematical models of the curved pipe as a thin shell and the essential assumptions about deformation are similar. Both of the classical analyses - Kafka and Dunn's [389] and Crandall and Dahl's [317] - preduct points (b) and (c) which violently disagrees with the results of Figure 11.3. These analyses predict that K shall always be positive no matter what the pressure and bending moment. So why should the analysis given here be so different?

Kafka and Dunn [389] use a linear shell model with a modification to the potential energy to account for the nonlinear change in cross sectional area due to pressure. Careful study indicates that important terms - equivalent to the linear component of the pressure terms in horizontal and vertical equilibrium (10.20) - are omitted for a reason which is not entirely clear. But the results of Figure 11.3 show that these terms are very important.

Crandall and Dahl [317] presume that under pressure alone the end rotation is negligible; Figure 11.3 indicates otherwise. They then linearise by assuming that the only nonlinear term of interest is the second in equation (11.1) and approximating the horizontal and vertical stress resultants by their membrane values.

Blomfield and Turner [323] also retain only one

nonlinear term viz.

$$-\varepsilon_1 \frac{\alpha\mu_o}{r}(Yr'+Zl')X \triangleq -B(\alpha\mu_o)^2 X.N_{\xi m}$$

where $N_{\xi m}$ is the linear membrane solution - inherent in this is the assumption that K is independent of M. For the elliptic cross section this term is

$$-\frac{\mu^2}{\eta}\frac{1+\lambda/2\ \sin\xi}{1+\lambda\ \sin\xi}\cdot(\sin^2\xi+\eta^2\cos^2\xi)^{3/2}\psi_p\cdot X$$

However then, curiously, they proceed to calculate the ratio of moments in a curved to a straight pipe under the same end rotation. For the linear theory with no pressure this is equal to the flexibility factor, but is incorrect if pressure, and nonlinear terms, are included. A program was written to recalculate Blomfield and Turner's analysis for flexibility not ratio of moments; results are shown in Figures 11.3. Obviously the so-called "nonlinear correction" is an improvement but does not account well for the nonlinear behaviour of high pressures.

Despite the misgivings expressed here about the classical analyses some information is available which indicates that their results are numerically acceptable by comparison with experiment [387,389]. It is expected that this is due to a certain amount of luck. But it is also felt that this may be explained by examining the effect of ellipicity. Again this has a profound effect.

Figure 11.5 shows the results of the linearized theory for η = 0.8 and 1.2 for K_p plotted against Q_p. Obviously the effects are startling.

Figures 11.6 and 11.7 show the results of the non-linear and linear theory for η = 0.8 and 1.2 plotting K against m_p for various ψ_p; obviously the non-linear predictions for η = 1.2 differ widely from the linear in

that the linear predicts increasing flexibility with pressure; however it is clear that for a range of ellipticity ratios a positive end rotation is obtained for large pressures.

For all the experimental pressures quoted by Rodabaugh and George [387] and Kafka and Dunn [389], e.g. ψ_p up to 0.6, a positive end rotation is measured. This would be possible if the ellipticity ratio was greater than unity (and moment high enough). This, however, is uncommon, but can occur in certain seam welded tubes; unfortunately Rodabaugh and George [387] do not recount the mode of manufacture of the bend, but Kafka and Dunn [389] do use a seam welded bend. This could partially explain the experimental results. Nonetheless the writer is doubtful about the experiments since an averaged end rotation, accounting for the behaviour of tangent straights, for various moments was quoted, which is inadmissable for a nonlinear geometric response. It is unhappily impossible to recheck in detail the results of the experiments from the viewpoint used herein.

11.3.2. Discussion

It must be stressed once more that the bewildering results of the analysis make it impossible to draw exact conclusions without a lot of further work to substantiate them, both theoretical and experimental.

Primarily the discussion rests on the two deformation assumptions 9.1 and 9.2 and secondarily on assumption 10.2. If the applied in-plane moment is uniform then the latter assumption is acceptable, but the relevance of a uniform moment must be questioned. To the writer's mind this uniformilty and the deformation assumptions are more

acceptable the smaller the angle of the bend, Θ. This is logically substantiated by the analysis.

It is expected that a complete, doubly curved shell analysis of a curved pipe should be undertaken to correctly study the deformation. Indeed experimental evidence could be interpreted as a justification of this approach! However, this cannot be pursued here. The purpose of the present investigation is to develop the creep analysis rather than the elastic one.

11.4 A discrete model of a circular cross section curved pipe under bending alone

In the following chapters an analysis of the creep of a circular cross section curved pipe under bending along shall be engaged upon. The results of the previous section establish that the linear theory with no internal pressure (Section 10.2.8.) is sufficient; thus the linearized forms of equations (11.1) and moment equilibrium (11.2) obtained by ignoring all terms containing ε_1 and Q_1 may be used.

For a circular cross section pipe (Figure 11.8)

$$r = a + b\sin\xi$$
$$l = -b\cos\xi$$

$$\xi \equiv \phi \qquad\qquad \alpha = b$$

and the parameters

$$\lambda = ah/b^2 \qquad \varrho = b/a \qquad \mu = \sqrt{12(1-\nu^2)}\,/h$$

are once again defined.

Symmetry of the cross section and equation $(11.1)_3$ ensures that

$$\dot{Z} \equiv 0$$

with the boundary conditions

$$X = 0 \qquad Y = 0 \qquad \xi = \pi/2, 3\pi/2 \qquad (11.12)$$

so that the bending equations $(11.1)_{1,2}$ alone need be considered in the range $\pi/2 \le \xi \le 3\pi/2$.

If M is the applied bending moment then on choosing

$$\sigma_o = Mb/I \qquad \alpha_o = b$$

where $I = \pi h b^3$, the bending equations become

$$\begin{aligned} X'' + AX' - (A^2 + \nu\bar{A})X + \mu\bar{A}Y/e &= -(A\bar{A} - \nu A)K/\mu \\ Y'' + AY' - (A^2 - \nu\bar{A})Y - \mu\bar{A}X/e &= AK/e \end{aligned} \qquad (11.13)$$

where

$$A(\xi) = \frac{e\cos\xi}{1 + e\sin\xi} \qquad \bar{A}(\xi) = \frac{e\sin\xi}{1 + e\sin\xi}$$

Moment equilibrium becomes, on integrating by parts,

$$1 = \frac{1}{\pi}\int_o^{2\pi} (-Y\cos\xi - \frac{e}{\mu}X(A\sin\xi - \nu\cos\xi))\,d\xi + \frac{2K}{\mu^2}\left(\frac{1}{\sqrt{1-e^2}} - 1\right) \qquad (11.14)$$

and the stress relations (11.3) and (11.4) are

$$\begin{aligned} S_\xi &= AY + \zeta\sqrt{\frac{12}{1-\nu^2}}\,(X + \nu(AX - K\bar{A}/e)) \\ S_\theta &= Y' + \zeta\sqrt{\frac{12}{1-\nu^2}}\,((AX - K\bar{A}/e) + \nu X') \end{aligned} \qquad (11.15)$$

A <u>discrete model</u> of these equations shall now be formulated based on the method of finite differences [247, 360-362, 366].

Let a uniform mesh be placed on the interval $[\pi/2, 3\pi/2]$ described by

$$\begin{aligned} \xi_i &= \pi/2 + i\Delta\xi \qquad i = 0, 1, 2, \cdots M, M+1 \\ \Delta\xi &= \pi/(M+1) \end{aligned}$$

(see Figure 11.9).

Derivatives of a continuous function at "internal" mesh points $\xi_1, \xi_2, \xi_3 \ldots\ldots \xi_M$ are given by the central differences e.g.

$$\left.\frac{dX}{d\xi}\right|_i = \frac{X_{i+1} - X_{i-1}}{2\Delta\xi} \qquad i = 1, 2 \cdots M$$

$$\left.\frac{d^2X}{d\xi^2}\right|_i = \frac{X_{i+1} - 2X_i + X_{i-1}}{2\Delta\xi}$$

and at the end points ξ_o, ξ_{M+1} by

$$\left.\frac{dX}{d\xi}\right|_0 = \frac{-X_2+4X_1-3X_0}{2\Delta\xi}$$
$$\left.\frac{dX}{d\xi}\right|_{M+1} = \frac{X_{M-1}-4X_M+3X_{M+1}}{2\Delta\xi}$$

remembering that from (11.12)

$$X_0=X_{M+1}=Y_0=Y_{M+1}=0 \qquad (11\cdot16)$$

Then the bending equations (11.13) take the discrete form

$$\begin{aligned} a_i X_{i-1} + b_i X_i + \bar{a}_i X_{i+1} + c_i Y_i &= f_i K \\ a_i Y_{i-1} + \bar{b}_i Y_i + \bar{a}_i Y_{i+1} - c_i X_i &= g_i K \end{aligned} \qquad i=1,2\cdots M \qquad (11\cdot17)$$

where

$$a_i = 1-\Delta\xi/2 A_i \qquad \bar{a}_i = 1+\Delta\xi/2 A_i$$
$$b_i = -2-\Delta\xi^2(A_i^2+\nu\bar{A}_i) \qquad \bar{b}_i = -2-\Delta\xi^2(A_i^2-\nu\bar{A}_i)$$
$$c_i = \Delta\xi^2\mu\bar{A}_i/\varrho$$
$$f_i = -(A_i\bar{A}_i-\nu A_i)\Delta\xi^2/\mu \qquad g_i = A_i\Delta\xi^2/\varrho$$

Similarly moment equilibrium becomes

$$\sum_{i=1}^{M}(s_i X_i+\bar{s}_i Y_i) - f_{M+1}K = f^*_{M+1} \qquad (11\cdot18)$$

on replacing the integral by a finite sum using Simpson's rule, where

$$s_i = \begin{cases} -4\varrho(A_i\sin\xi_i-\nu\cos\xi_i)/\mu & i=1,3,\cdots M \\ -2\varrho(A_i\sin\xi_i-\nu\cos\xi_i)/\mu & i=2,4,\cdots M-1 \end{cases}$$

$$s_i = \begin{cases} -4\cos\xi_i & i=1,3,\cdots M \\ -2\cos\xi_i & i=2,4,\cdots M-1 \end{cases}$$

$$f_{M+1} = -\frac{3\pi}{\Delta\xi}\cdot\frac{1}{\mu^2}\left(\frac{1}{\sqrt{1-\varrho^2}}-1\right)$$

$$f^*_{M+1} = \frac{3\pi}{2\Delta\xi}$$

Therefore equations (11.17) and (11.18) represent $2M+1$ equations in $2M+1$ unknowns $X_1, X_2, \ldots, X_M, Y_1, Y_2, \ldots Y_M$ and K, which may be conviently rewritten in the matrix form

$$[A]\underset{\sim}{U} = \underset{\sim}{F} \qquad (11\cdot19)$$

where

$$\underset{\sim}{U} = (X_1\ Y_1\ X_2\ Y_2 \cdots X_M\ Y_M\ K) \quad F = (f_1^*\ g_1^* \qquad f_1^*\ g_1^*\ f_{M+1}^*)$$

$$f_i^* = 0 \qquad g_i^* = 0 \qquad i = 1,2,\cdots M$$

and $[A]$ is given in Table 11.3. Hence immediately

$$\underset{\sim}{U} = [A]^{-1}\underset{\sim}{F} \qquad (11.20)$$

and the problem is solved.

11.4.1. Results and comparison

Due to the nondimensionalization it is fairly obvious that K is the elastic flexibility factor and S_ξ, S_θ are the stress concentration factors, for example

$$K = \Gamma\rho = \frac{k}{\varepsilon_0}\rho = k\frac{EI}{Mb}\cdot\frac{b}{a} = k \div \frac{Ma}{EI}$$

A FORTRAN IV computer program was written for the University of Strathclyde ICL 1904S system to perform the operation (11.20). Results for $\nu = 0.33$ and $\rho = 0.1, 0.3$ are shown in Table 11.4 with $M = 17$ compared with the results of linear shooting. Allowing for the fact that linear shooting supplies a very accurate discrete model, and that the full equations (11.7) are used, it is obvious that the discrete model based on finite differences is acceptable. The value of $M = 17$ was felt to be an optimum (in particular the linear shooting method employed an equivalent $M = 174$).

11.4.2. Calculation of stresses

Let the interval describing the thickness $\zeta : [-1/2, 1/2]$ be replaced by a mesh $\zeta_j, j = 1,2,\ldots..M+1$ so that the matrices $[S_\xi]$ and $[S_\theta]$ represent the principle nondimensional stress fields, whose (i,j)th elements are the values of these quantities of the mesh points (ξ_i, ζ_j). Using (11.3) and (11.4) it is then possible to relate $[S_\xi]$ and $[S_\theta]$ to the solution vector $\underset{\sim}{U}$ from (11.20) thus

$$[S_\xi] = [\mathcal{B}_\xi]\underline{U}\,\underline{1}^T + [\mathcal{C}_\xi]\underline{U}\,\underline{\xi}^T$$
$$[S_\theta] = [\mathcal{B}_\theta]\underline{U}\,\underline{1}^T + [\mathcal{C}_\theta]\underline{U}\,\underline{\xi}^T \qquad [11.21]$$

the matrices $[\mathcal{B}_i]$ and $[\mathcal{C}_i]$ being given in Table 11.3. Values of SCFM are shown in Table 11.4 for $\nu = 0.33$ and $\varrho = 0.1, 0.3$ taking $N = 10$. Again the model is acceptable.

11.5 Summary

Before the creep of a curved pipe is investigated let us summarize exactly what has been achieved so far.

Firstly the deformation of a curved pipe has been stated explicitly; the writer has been unable to find a rigorous stagement of this in the literature.

Secondly a new, nonlinear shell model of a curved pipe has been postulated to cope with a uniform in-plane bending and an internal pressure.

Thirdly the model has been studied numerically in elasticity using a direct integration technique and compared with previous results. The writer expresses a lack of confidence in these previous investigations and emphasizes the need for further work.

Finally a discrete model of the curved pipe based on finite differences has been presented, suitable for inelastic computation. An optimum discretization is established.

TABLE 11.1

COEFFICIENTS OF STANDARD FORM OF EQUATIONS FOR ELLIPTICAL CURVED PIPE

$$L_{31} = -\frac{\mu\eta}{\lambda}\cos\xi \quad , \quad L_{32} = \frac{\lambda\sin\xi}{(1+\lambda\sin\xi)}\cdot\frac{\cos\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}$$

$$L_{34} = \frac{\lambda\sin\xi}{(1+\lambda\sin\xi)}\cdot\frac{1/\eta\ \sin\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}} \quad , \quad L_{36} = \frac{-\nu\sin\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}$$

$$L_{51} = \left(\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\right)^2 + \nu\frac{\lambda\sin\xi}{1+\lambda\sin\xi}\cdot\frac{1}{\sin^2\xi+\eta^2\cos^2\xi}$$

$$L_{52} = -\frac{\mu}{\lambda}\cdot\frac{\lambda\sin\xi}{1+\lambda\sin\xi}\cdot\sqrt{\frac{\sin^2\xi+\eta^2\cos^2\xi}{\eta}}$$

$$L_{54} = \frac{\mu}{\lambda}\sqrt{\sin^2\xi+\eta^2\cos^2\xi}\cdot\frac{\lambda\cos\xi}{1+\lambda\sin\xi}$$

$$L_{55} = \frac{-\lambda\cos\xi}{1+\lambda\sin\xi} - \frac{(1-\eta^2)\sin\xi\cos\xi}{\sin^2\xi+\eta^2\cos^2\xi}$$

$$L_{57} = \frac{1}{\mu\eta}\left(\frac{-\lambda^2\sin\xi\cos\xi}{(1+\lambda\sin\xi)^2} + \nu\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\cdot\frac{\eta^2}{\sin^2\xi+\eta^2\cos^2\xi}\right)$$

$$L_{61} = -L_{52} \quad , \quad L_{62} = \left(\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\right)^2 - \nu\frac{\lambda\sin\xi}{1+\lambda\sin\xi}\cdot\frac{1}{\sin^2\xi+\eta^2\cos^2\xi}$$

$$L_{64} = \frac{1}{\eta}\left(\frac{\lambda^2\sin\xi\cos\xi}{(1+\lambda\sin\xi)^2} + \nu\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\cdot\frac{\eta^2}{\sin^2\xi+\eta^2\cos^2\xi}\right), \quad L_{66} = L_{55}$$

$$L_{67} = \frac{\lambda\cos\xi}{1+\lambda\sin\xi}\cdot\sqrt{\frac{\sin^2\xi+\eta^2\cos^2\xi}{\lambda}}$$

$$N_{31}^{*} = \varepsilon_1\left\{2\nu Q_1(1+\lambda\sin\xi)\frac{\lambda\cos\xi\sin\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}\right\} \quad , \quad N_{31} = \varepsilon_1\left\{\frac{-\mu}{2\lambda}\sin\xi\right\}$$

$$N_{32} = \varepsilon_1\left\{\frac{\lambda}{\eta}\cdot\frac{1}{1+\lambda\sin\xi}\cdot\frac{\sin^2\xi-\eta^2\cos^2\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}\right\}$$

$$N_{34} = \varepsilon_1\left\{\frac{-2\cos\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}\cdot\frac{\lambda\sin\xi}{1+\lambda\sin\xi}\right\}$$

$$N_{37} = \varepsilon_1\left\{-\frac{\eta}{\lambda}\cos\xi\right\} \quad , \quad N_{36} = \varepsilon_1\left\{\frac{\nu\eta}{\lambda}\frac{1+\lambda\sin\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}\right\}$$

$$N_{41} = \varepsilon_1\left\{ Q_1 \frac{\lambda}{\eta} \sin\xi (1+\lambda\sin\xi)\right\}$$

$$N_{51} = \varepsilon_1\left\{ \frac{1}{2\eta}\left(\frac{3\lambda^2\cos\xi\sin\xi}{(1+\lambda\sin\xi)^2} - \nu \frac{\lambda\cos\xi}{1+\lambda\sin\xi} \cdot \frac{\eta^2}{\sin^2\xi+\eta^2\cos^2\xi}\right)\right\}$$

$$N_{52} = \varepsilon_1\left\{ \frac{\mu}{\lambda}\sqrt{\sin^2\xi+\eta^2\cos^2\xi} \cdot \frac{\lambda\cos\xi}{1+\lambda\sin\xi}\right\}$$

$$N_{54} = \varepsilon_1\left\{ \frac{\mu}{\lambda}\sqrt{\sin^2\xi+\eta^2\cos^2\xi} \cdot \frac{\lambda\sin\xi}{1+\lambda\sin\xi}\right\}$$

$$N_{57} = \varepsilon_1\left\{ \frac{1}{\mu}\left(\frac{\lambda^2}{\eta^2} \frac{(2\eta^2\cos^2\xi - \sin^2\xi)}{(1+\lambda\sin\xi)^2} + \nu \frac{\lambda\sin\xi}{1+\lambda\sin\xi} \cdot \frac{1}{\sin^2\xi+\eta^2\cos^2\xi}\right)\right\}$$

$$N_{61}^{*} = \varepsilon_1 Q_1\left\{ \lambda^2\cos^2\xi(\nu+2) - \nu\frac{\lambda^2\sin^2\xi}{\eta^2} - \frac{(1+\lambda\sin\xi)\lambda\sin\xi}{\sin^2\xi+\eta^2\cos^2\xi} + \nu\frac{\lambda^2\sin^2\xi}{\eta^2} - \nu\lambda^2\cos^2\xi\right\}$$

$$N_{61}^{**} = \varepsilon_1 Q_1\left\{ -\nu\frac{\lambda^2}{\eta}\cos\xi\sin\xi\right\} \quad , \quad N_{65}^{*} = \varepsilon_1 Q_1\left\{ (1+\lambda\sin\xi)\lambda\cos\xi\right\}$$

$$N_{61} = \varepsilon_1\left\{ -\frac{\mu}{2\lambda} \cdot \frac{\lambda\cos\xi}{1+\lambda\sin\xi}\sqrt{\sin^2\xi+\eta^2\cos^2\xi}\right\}$$

$$N_{62} = \varepsilon_1\left\{ \frac{2}{\eta} \cdot \frac{\lambda^2\sin\xi\cos\xi}{(1+\lambda\sin\xi)^2} + \nu\frac{\lambda\cos\xi}{1+\lambda\sin\xi} \cdot \frac{\eta}{\sin^2\xi+\eta^2\cos^2\xi}\right\}$$

$$N_{64} = \varepsilon_1\left\{ \frac{\lambda^2}{\eta^2} \frac{(\sin^2\xi - \eta^2\cos^2\xi)}{(1+\lambda\sin\xi)^2} + \nu\frac{\lambda\sin\xi}{1+\lambda\sin\xi} \cdot \frac{1}{\sin^2\xi+\eta^2\cos^2\xi}\right\}$$

$$N_{67} = \varepsilon_1\left\{ \frac{2\sin\xi}{1+\lambda\sin\xi} \cdot \frac{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}{\eta}\right\}$$

$$N_{62}^{*} = \varepsilon_1\left\{ \frac{\nu}{\eta} \cdot \frac{\lambda\sin\xi}{1+\lambda\sin\xi}\right\} \quad , \quad N_{64}^{*} = \varepsilon_1\left\{ -\nu\frac{\lambda\cos\xi}{1+\lambda\sin\xi}\right\}$$

$$C_3 = -Q_1\nu\frac{\lambda}{\eta}\frac{\sin^2\xi(1+\lambda\sin\xi)}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}} \quad , \quad C_4 = Q_1\lambda\cos\xi(1+\lambda\sin\xi)$$

$$C_6 = Q_1\left\{ -2\frac{\lambda^2}{\eta}\cos\xi\sin\xi + \eta\frac{\cos\xi(1+\lambda\sin\xi)}{\sin^2\xi+\eta^2\cos^2\xi}\right\}$$

$$m_1 = -\frac{\lambda}{\mu}\left(\frac{\lambda\sin\xi}{1+\lambda\sin\xi} - \nu\frac{\eta^2}{\sin^2\xi+\eta^2\cos^2\xi}\right)\frac{\cos\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}} - \varepsilon_1 Q_1\eta\sin\xi(1+\lambda\sin\xi)\lambda\cos\xi$$

$$m_2 = -\eta\cos\xi \quad , \quad m_3 = -\frac{\varepsilon_1}{2} \cdot \frac{\lambda\sin\xi}{1+\lambda\sin\xi} \cdot \frac{\lambda}{\mu\eta} \cdot \frac{\sin\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}}$$

$$m_4 = -\varepsilon_1\frac{\sin\xi}{\sqrt{\sin^2\xi+\eta^2\cos^2\xi}} \cdot \frac{\lambda}{\mu^2} \cdot \frac{\lambda\cos\xi}{1+\lambda\sin\xi}$$

TABLE 11.2

THE ELLIPTIC INTEGRAL $J(\eta)$

η	$J(\eta)$
0.6	2.975 457 9
0.8	3.060 954 6
1.0	3.141 592 7 (π)
1.2	3.218 282 0
1.4	3.291 657 5

TABLE 11.3

THE MATRIX $[\mathcal{A}]$, ETC

$$[\mathcal{A}]=\begin{bmatrix}
b_1 & c_1 & \bar{a}_1 & \cdot & \cdot & \cdot & & & & \cdot & \cdot & -f_1 \\
-c_1 & \bar{b}_1 & \cdot & \bar{a}_1 & \cdot & \cdot & & & & \cdot & \cdot & -g_1 \\
a_2 & \cdot & b_2 & c_2 & \bar{a}_2 & \cdot & & & & \cdot & \cdot & -f_2 \\
\cdot & a_2 & -c_2 & \bar{b}_2 & \cdot & \bar{a}_2 & & & & \cdot & \cdot & -g_2 \\
 & & & & \ddots & & & & & & & \vdots \\
 & & & & & \ddots & & & & & & \vdots \\
 & & & & & & \ddots & & & & & \vdots \\
 & & & & & & & a_{M-1} & \cdot & b_{M-1} & c_{M-1} & \bar{a}_{M-1} & \cdot & \vdots \\
 & & & & & & & & a_{M-1} & -c_{M-1} & \bar{b}_{M-1} & \cdot & \bar{a}_{M-1} & \vdots \\
 & & & & & & & & & a_M & \cdot & b_M & c_M & -f_M \\
 & & & & & & & & & \cdot & a_M & -c_M & \bar{b}_M & -g_M \\
\cdot & \cdot & \cdot & \cdot & & & & & & & & & & \\
s_1 & \bar{s}_1 & s_2 & \bar{s}_2 & - & - & - & & & - & - & \bar{s}_{M-1} & s_M & \bar{s}_M & -f_{M+1}
\end{bmatrix}$$

$$\underset{\sim}{1} = (1\ 1\ 1 \cdots 1)^T$$

$$\underset{\sim}{\zeta} = (\zeta_1\ \zeta_2 \cdots \zeta_N\ \zeta_{N+1})^T$$

$$[\mathcal{B}_\xi] = \begin{bmatrix} \cdot & \cdot & \cdot & \cdot & & & \cdot \\ \cdot & A_1 & \cdot & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & A_2 & & & \cdot \\ & & & - - - - & & & \\ & & & & A_{M-1} & \cdot & \cdot \\ & & & & \cdot & \cdot & A_M \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[\mathcal{B}_\theta] = \frac{1}{\Delta\xi} \begin{bmatrix} \cdot & 4/2 & \cdot & -1/2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & 1 & \cdot & \cdot & \cdot \\ \cdot & -1 & \cdot & \cdot & \cdot & 1 & \cdot \\ & & & - - - - - & & \\ \cdot & & & -1 & \cdot & \cdot & 1 \\ \cdot & & & \cdot & \cdot & -1 & \cdot \\ \cdot & & & \cdot & \cdot & 1/2 & \cdot \end{bmatrix}$$

$$[\mathcal{C}_\xi] = \frac{1}{\Delta\xi}\sqrt{\frac{12}{1-\nu^2}} \begin{bmatrix} \cdot & \cdot & 4/2 & \cdot & -1/2 & \cdot & & & & \cdot & -\nu\bar{A}_0^\circ \\ \nu A_1^\circ & \cdot & 1 & \cdot & \cdot & \cdot & & & & \cdot & -\nu\bar{A}_1^\circ \\ -1 & \cdot & \nu A_2^\circ & \cdot & 1 & \cdot & & & & \cdot & -\nu\bar{A}_2^\circ \\ \cdot & \cdot & -1 & \cdot & \nu A_3^\circ & -1 & & & & \cdot & -\nu\bar{A}_3^\circ \\ & & & - - - - - & & & & & & \vdots \\ & & & - - - - - & & & & & & \vdots \\ & & & & & -1 & \cdot & \nu A_{M-1}^\circ & \cdot & 1 & \cdot & -\nu\bar{A}_{M-1}^\circ \\ & & & & & \cdot & \cdot & -1 & \cdot & \nu A_M^\circ & \cdot & -\nu\bar{A}_M^\circ \\ \cdot & \cdot & \cdot & \cdot & & \cdot & \cdot & 1/2 & \cdot & -4/2 & \cdot & -\nu\bar{A}_{M+1}^\circ \end{bmatrix}$$

$$[\mathcal{C}_\theta] = \frac{1}{\Delta\xi}\sqrt{\frac{12}{1-\nu^2}} \begin{bmatrix} \cdot & \cdot & 4/2 & \cdot & -1/2 & \cdot & & & & \cdot & -\bar{A}_0^\circ \\ A_1^\circ & \cdot & \nu & \cdot & \cdot & \cdot & & & & \cdot & -\bar{A}_1^\circ \\ -\nu & \cdot & A_2^\circ & \cdot & \nu & , & & & & \cdot & -\bar{A}_2^\circ \\ \cdot & \cdot & -\nu & \cdot & A_3^\circ & -\nu & & & & \cdot & -\bar{A}_3^\circ \\ & & & - & - & - & - & & & & & \vdots \\ & & & & - & - & - & - & & & & \vdots \\ & & & & & & -\nu & \cdot & A & \cdot & \nu & \cdot & -\bar{A}_{M-1}^\circ \\ & & & & & & \cdot & \cdot & -\nu & \cdot & A & \cdot & -\bar{A}_M^\circ \\ & & & & & & \cdot & \cdot & 1/2 & \cdot & -4/2 & \cdot & -\bar{A}_{M+1}^\circ \\ \cdot & \cdot & \cdot & \cdot & & & & & & & & & \end{bmatrix}$$

$$A_i^\circ = A_i \Delta\xi \qquad \bar{A}_i^\circ = \bar{A}_i / \mu$$

TABLE 11.4

COMPARISON OF FINITE DIFFERENCE DISCRETE MODEL AND DIRECT INTEGRATION

Finite difference model : column 1

Direct integration : column 2

FLEXIBILITY FACTOR

λ	$\varrho = 0.1$		$\varrho = 0.3$	
0.05	32.698	32.692	32.694	32.741
0.1	16.347	16.295	16.345	16.345
0.2	8.181	8.1859	8.1667	8.1695
0.5	3.1022	3.0959	3.1234	3.1115
1.0	1.6301	1.6255	1.6633	1.6445
2.0	1.1632	1.1628	1.1592	1.1531

MAXIMUM SCF

λ	$\varrho = 0.1$		$\varrho = 0.3$	
0.05	12.956	13.939	13.427	14.385
0.1	8.3271	8.7111	8.6811	9.0507
0.2	5.2694	5.4215	5.5314	5.6749
0.5	2.7410	2.7856	2.9141	2.9553
1.0	1.5321	1.5491	1.6886	1.7007
2.0	0.8297	0.8366	0.9593	0.9618

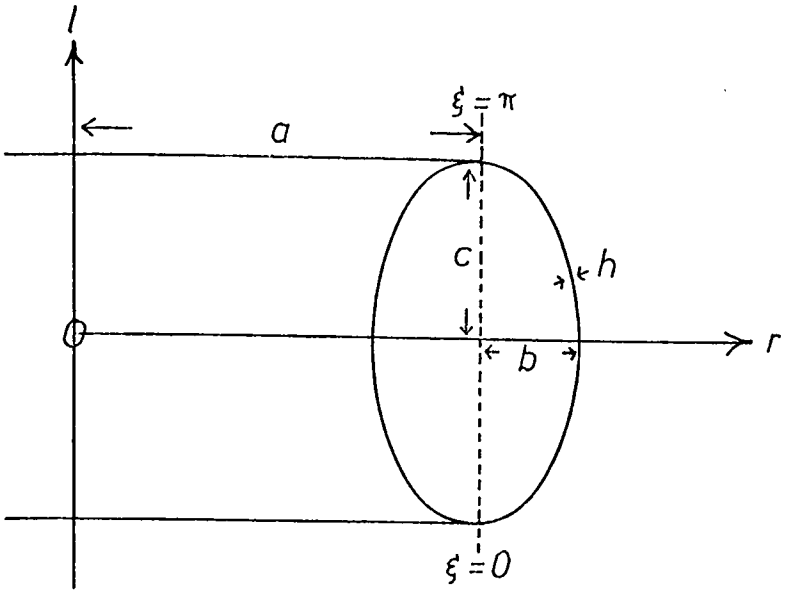


Fig 11·1 Elliptic cross section

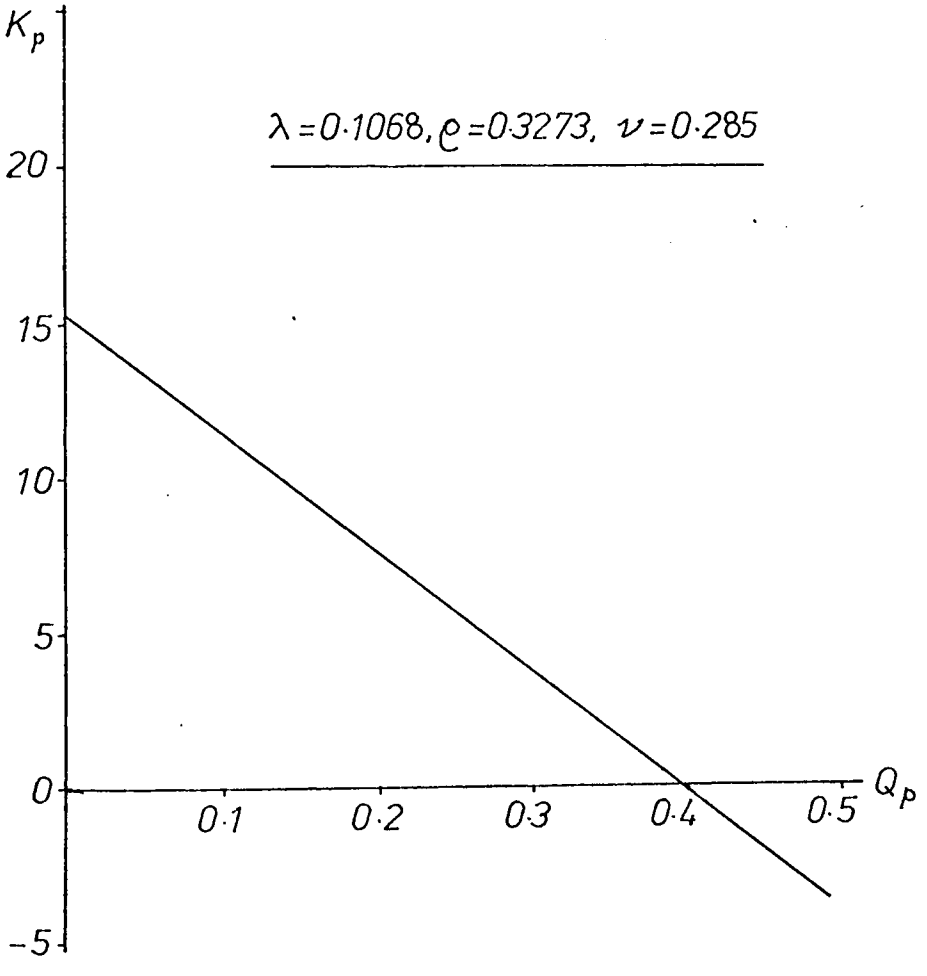


Fig 11·2 Variation of flexibility with pressure : Linear theory

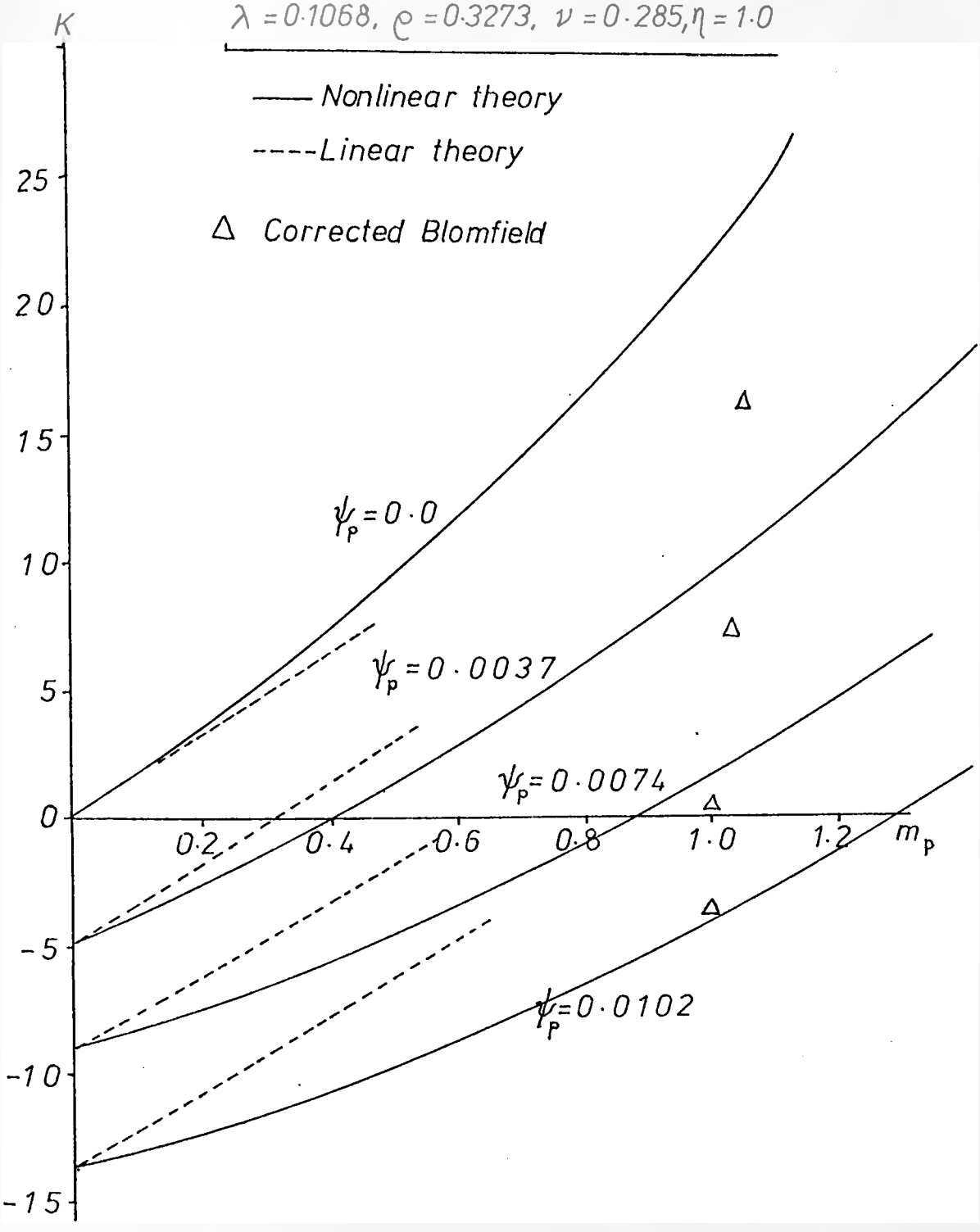
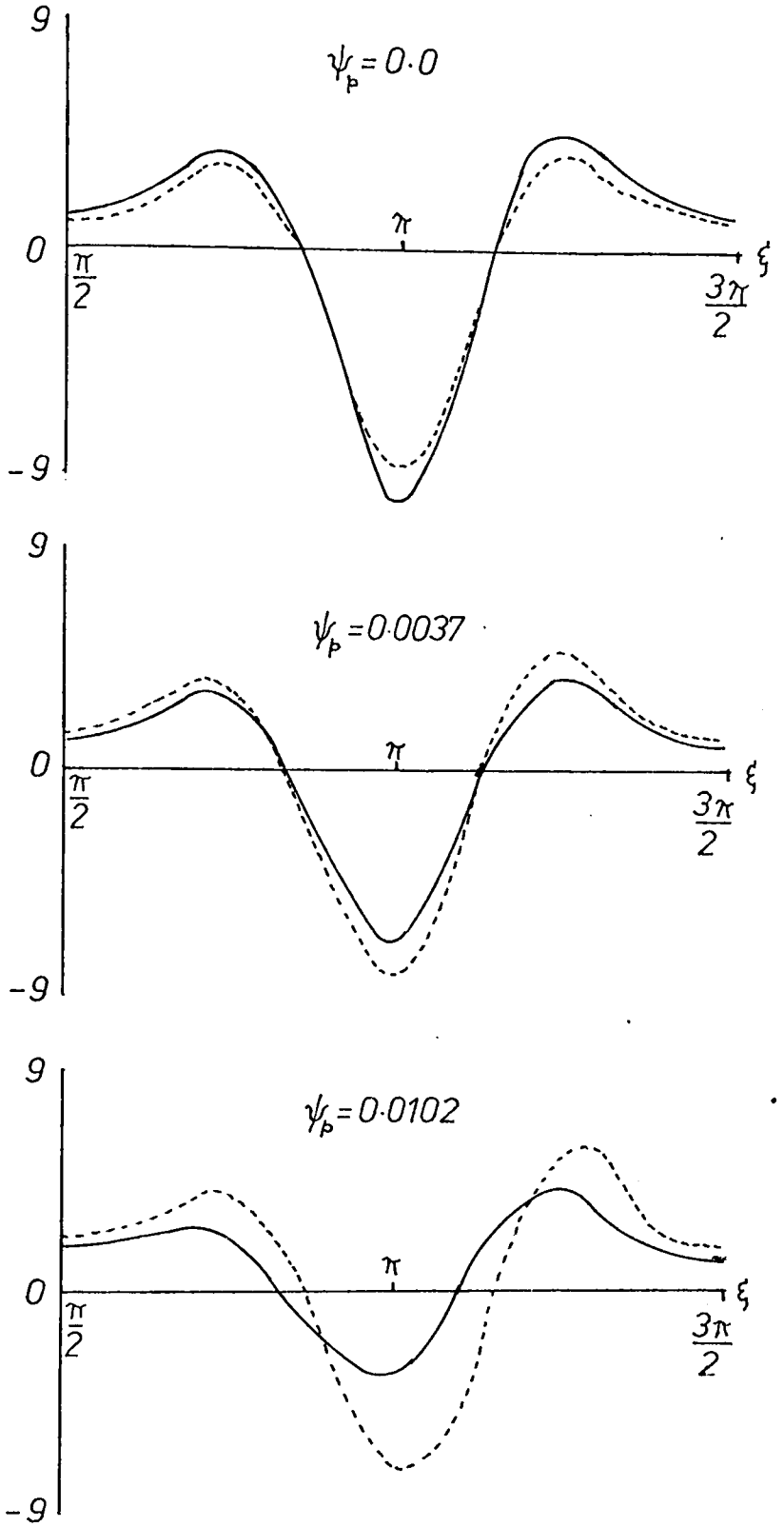


Fig 11·3 Effect of pressure on flexibility versus moment curve

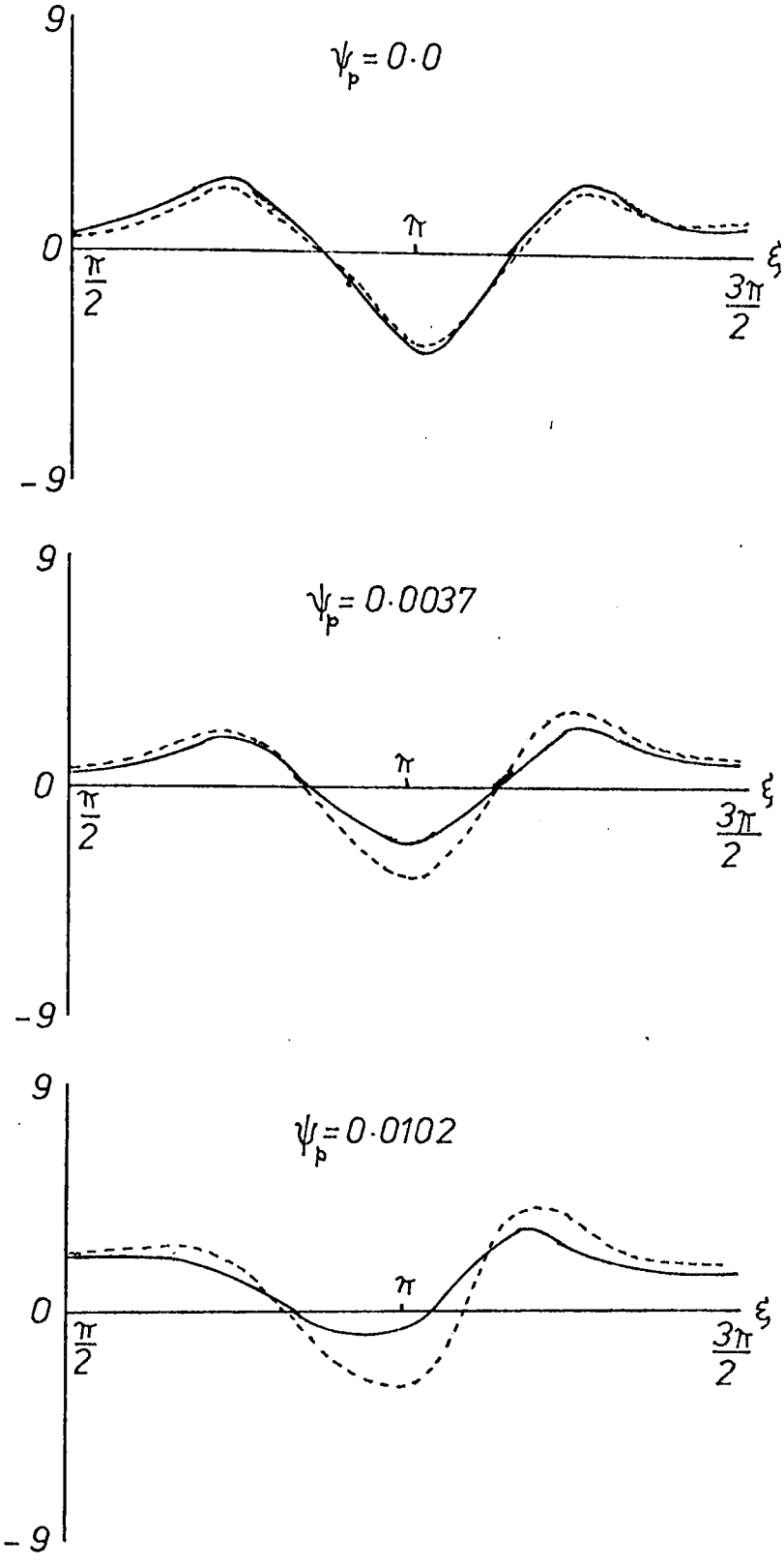


$\lambda = 0.1068,\ \varrho = 0.3273,\ \nu = 0.285,\ \eta = 1.0$

$m_p = 1.0$

—— Nonlinear ------ Linear

Fig 11.4 (a) Effect of pressure on circumferential SCF



$\lambda = 0.1068,\ \rho = 0.3273,\ \nu = 0.285,\ \eta = 1.0$

$m_p = 0.5$

—— Nonlinear ------ Linear

Fig 11.4(b) Effect of pressure on circumferential SCF

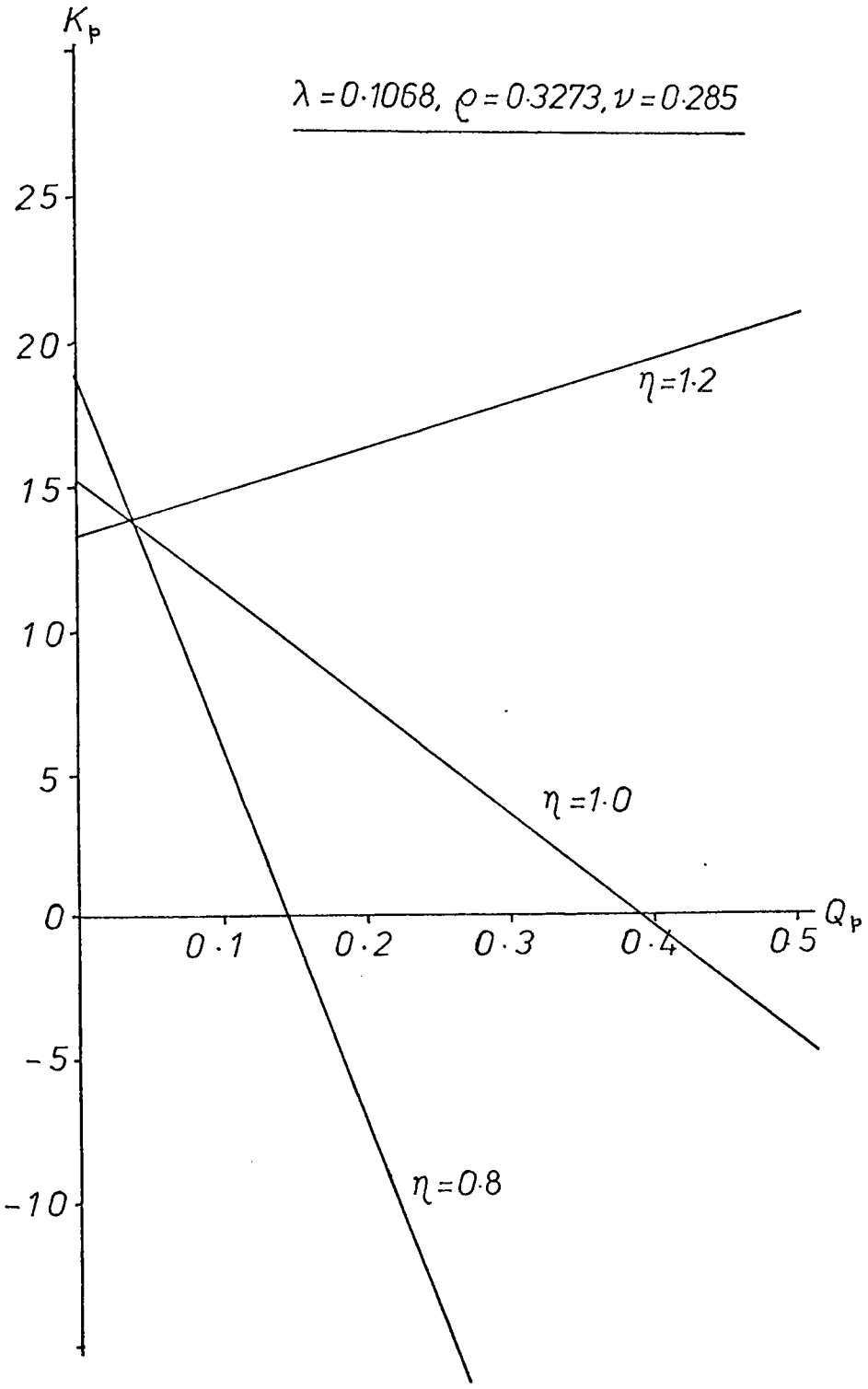


Fig 11·5 Effect of ellipticity ratio on flexibility : Linear theory

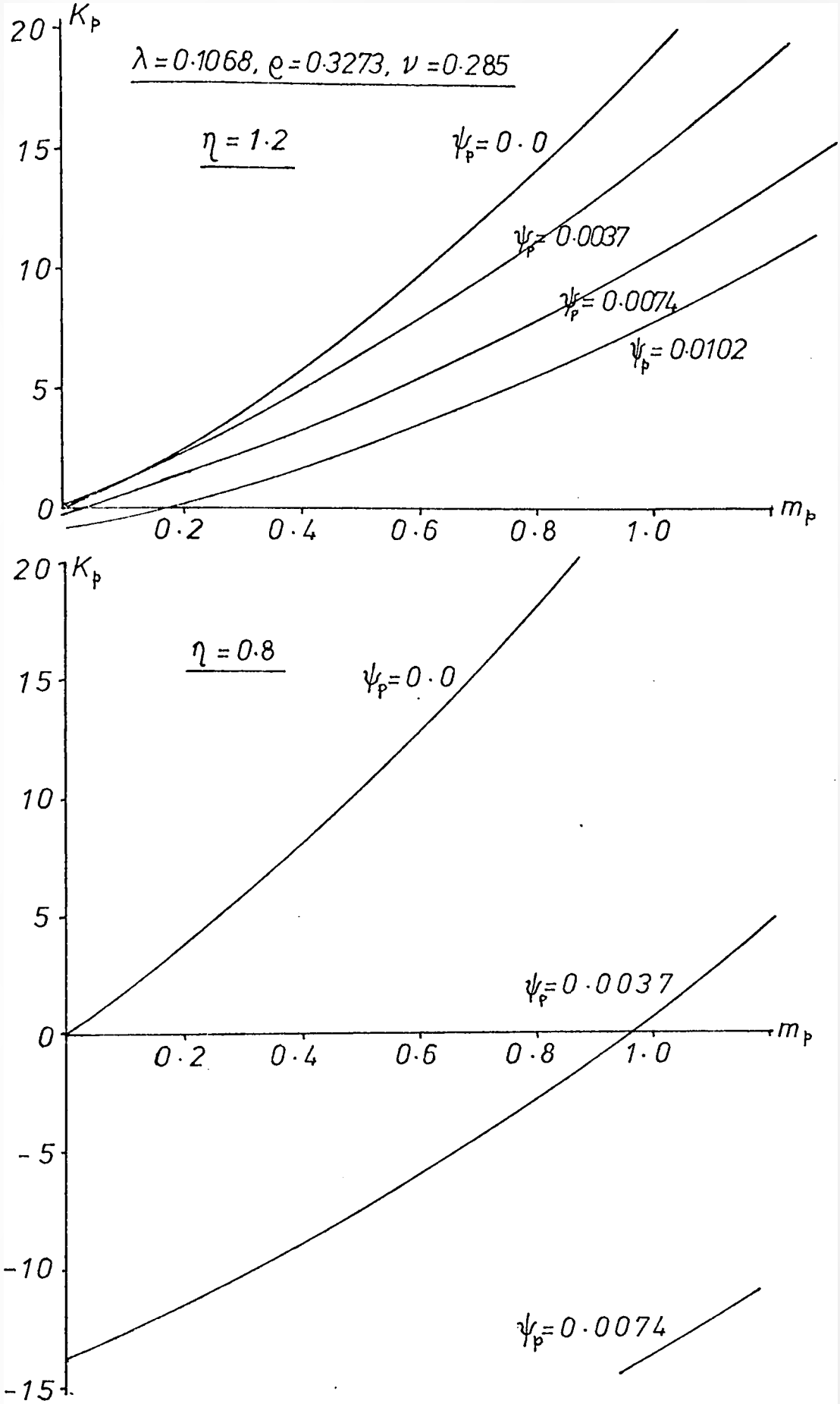
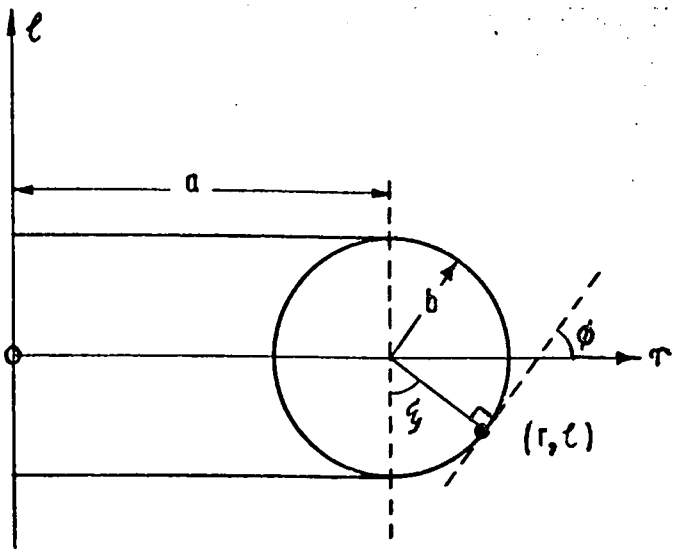


Fig 11.6,11.7 Effect of ellipticity ratio on flexibility: Nonlinear theory



$$r = a + b \sin\xi$$
$$\ell = -b \cos\xi$$

Fig 11.8 GEOMETRY OF A CIRCULAR C/S CURVED PIPE.

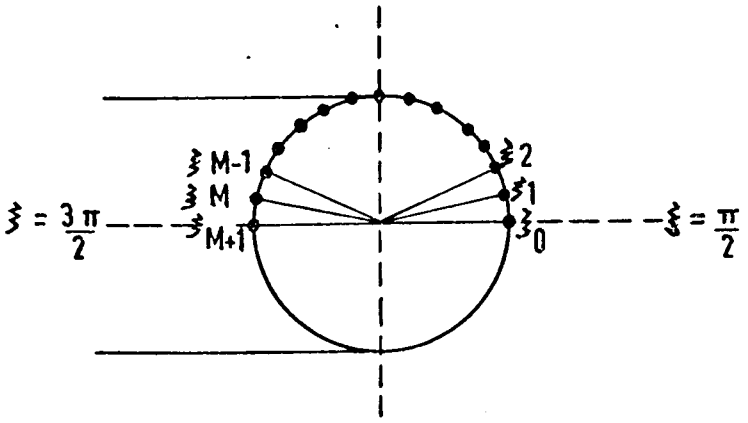


Fig 11.9 DISCRETIZATION OF CROSS SECTION.

CHAPTER 12

INELASTIC SHELL ANALYSIS

The inelastic analysis of shell structures poses some of the most formidable problems in structural analysis. As long as relatively straightforward examples are studied (for example Chapter 8) the difficulties are surmountable. A large number of such simple cases are documented in the standard texts [1-6] and the early works of C R Calladine [390] and S A Patel et al [391,392] on steady creep.

A fairly comprehensive review of the steady creep of shells has been given by W Olsak and A Sawczuk [393-395] and shall therefore not be repeated in any depth here. Recently a general numerical treatment of shells of revolution has been presented by S Murakami [396] based on quasi-linearization and the finite difference method. In fact since the steady analysis of any shell problem is merely a rather complex nonlinear elliptic boundary value problem any suitable method for the resolution of such problems can be employed [397,398]. In elastic shell analysis the problem can be considerably simplified as it is often possible to form a constitutive equation for the stress and moment resultants and midsurface strains (e.g. Section 10.2.3.). No such relationship exists for steady creep. Nonetheless several investigators have proposed approximate relationships [399-401]; most of these are approximations for the creep energy dissipation rate of a shell and should thus only strictly be used in the context of an energy methods. There are standard methods for the resolution of nonlinear variational problems [397,398, 403-405] some of which (notably Kachanov's [403-405]) arose out of attempts to solve steady creep problems. A general energy formulation of arbitrary shells in steady

creep (based on constitutive laws between the shell fields and finite deformation) has recently been given by T Malmberg [406] (see also [414]).

The nonsteady behaviour of shell structures is of much interest to pressure vessel designers. With the appearance of the initial strain algorithms it was natural to attempt to solve shell problems. The general approach for shells of revolution was postulated by T H Lin [249], P S Kuratov and V Levtchenko [407] and R K Penny [7,408,409], and more recently by F A Simonen [410], S Takezono [411] and with improved time integration by S Murakami [412]. These are largely based on a discrete, finite difference model of the shell, although Simonen employs direct integration; another possibility is to use a direct method, in conjunction with the reduced variational principles [413-416]. This also allows the inclusion of nonlinear geometric effects [417,418]. However, the rapid development of the finite element method for inelastic analysis has produced a number of examples for shell structures [262-270].

Despite this activity in inelastic shell analysis the curved pipe did not receive much attention. Indeed in creep so far the steady phase alone has been investigated. Chronologically this was first attempted by L M Kachanov [2,419,85] but independently J Spence [306,420] undertook a more detailed study. Spence defined a "creep flexibility factor" as the ratio of end rotation rate in a curved pipe under bending in steady creep to the end rotation of a similar straight pipe under the same bending in steady creep. He was thus able to obtain lower [421] and upper [422] bounds on this factor for a long radius, circular

cross section curved pipe using the steady creep variational principles, numerically realizing the problem by using a direct method and the technique of functional minimization on an approximate form due to V I Rozenblium [399] and A C Mackenzie [400]. These results showed that in steady creep the flexibility of a curved pipe was greatly increased. This work was later extended by Spence to elliptical cross sections [424-426] and short radius bends [427]. All of these studies took in-plane bending alone into account; recent work by the writer reconsiders the problem and extends to combined torque and out-of-plane bending (see Appendix 4).

The curved pipe has also been studied for other forms of inelasticity, although these are of minor consequence. J Spence and G Findlay [428] were able to utilize elastic and steady creep solutions in the limit analysis of the curved pipe in ideal plasticity; a similar method was used by C R Calladine [429]. The elasto-plastic response of a thin shell model of a curved pipe subjected to a constant end rotation was examined by P V Marcal [430]; an iterative, initial strain method was used and the progression of the plastic region and nature of limit loads discussed. This was later extended by J A Blomfield [431] to cyclic loading for the same problem together with experimental evidence but the analytic problem was that of a curved pipe constrained to a fixed curvature, while the experiments used a cyclically applied bending moment. The plastic collapse of a curved pipe has been studied by R M Mello and D S Griffen [432] using the so-called "reduced shell finite element method" (see later) with curved pipe elements based on Marcal's model [430] and taking attached straights into consideration. A similar

problem was examined by A Hoffman et al [433] using a "complete shell finite element method". All of these applications are important in the documentation of the inelastic behaviour of the component, but cannot be directly employed in piping analysis.

These few references practically exhaust the work done on inelastic curved pipe analysis (except for the simple publication on fracture mechanics from L A James [434]). The purpose of the following chapters is a study of the unsteady creep of the curved pipe. The ultimate aim is to assess the applicability of simple approximate methods with a view to general piping analysis. Nevertheless the numerical studies further contribute to information on inelastic curved pipes.

CHAPTER 13

CREEP OF A CURVED PIPE

It is by now fairly well known that the smooth curved pipe is one of the most important components in any pipework system, being more flexible than an equivalent length of straight pipe. In linear elastic situations this flexibility is measured by the factor:

$$K_e = \frac{\text{end rotation of curved pipe in elasticity}}{\text{end rotation of similar straight in elasticity}}$$

which has been successfully tabulated for a variety of geometries and loading conditions.

In steady creep the measure of flexibility is taken as

$$K_{ss} = \frac{\text{end rotation rate of curved pipe in steady creep}}{\text{end rotation rate of similar straight in steady creep}}$$

However such steady state calculations produce the eventual rate of increase of flexibility alone and it is obvious to ask how much the overall total flexibility is affected if the phase of stress redistribution is included? In this Chapter the problem of calculating variations in flexibility of a curved pipe under nonsteady creep conditions and a constant applied in-plane bending moment is studied. Three different measures of this flexibility are used:

$$K_{c/e} = \frac{\text{end rotation of curved pipe in creep}}{\text{end rotation of similar straight in elasticity}}$$

$$K_{c/c} = \frac{\text{end rotation of curved pipe in creep}}{\text{end rotation of similar straight in creep}}$$

$$K_{c/s} = \frac{\text{end rotation rate of curved pipe in creep}}{\text{end rotation rate of similar straight in steady creep}}$$

all of which are time dependent!

Moreover this constitutes a marvellous example of the use of the rational approach to creep mechanics.

13.1 The creep problem

For simplicity it is supposed that the constitutive model is of time hardening type. In the context here a multiaxial creep relation based on a Norton power law is employed

$$\dot{\varepsilon}_\xi^v = g(t)\,\bar{\sigma}^{n-1}(\sigma_\xi - 1/2\,\sigma_\theta) = g(t)\,f_\xi(\sigma_\xi, \sigma_\theta)$$
$$\dot{\varepsilon}_\theta^v = g(t)\,\bar{\sigma}^{n-1}(\sigma_\theta - 1/2\,\sigma_\xi) = g(t)\,f_\theta(\sigma_\xi, \sigma_\theta) \qquad (13\cdot1)$$

where g(t) and n are material parameters.

The total strain is then a sum of elastic and viscous components,

$$\varepsilon_\xi = \varepsilon_\xi^e + \varepsilon_\xi^v$$
$$\varepsilon_\theta = \varepsilon_\theta^e + \varepsilon_\xi^v$$

such that the elastic strain field (ε_ξ^e, ε_θ^e) is related to stress through Hooke's relation

$$\varepsilon_\xi^e = \frac{1}{E}(\sigma_\xi - \nu\sigma_\theta)$$
$$\varepsilon_\theta^e = \frac{1}{E}(\sigma_\theta - \nu\sigma_\xi)$$

where E is Young's modulus and ν Poisson's ratio.

Consider the following problem:
"Let an in-plane bending moment M be applied uniformly to a curved pipe which deforms elastically resulting in an end rotation γ_o and a stress field ($\sigma_{\xi o}, \sigma_{\theta o}$). The pipe then expands inelastically due to creep resulting in a variation in the end rotation $\gamma(t)$ as time progresses. The problem is to determine $\gamma(t)$ with $\gamma(0) = \gamma_o$. " The first step in the rational approach is to write down the redistribution equations:

$$\frac{d\sigma_\xi}{dt} = g(t)\,R_\xi(f_\xi(\sigma_\xi, \sigma_\theta), f_\theta(\sigma_\xi, \sigma_\theta))$$
$$\frac{d\sigma_\theta}{dt} = g(t)\,R_\theta(f_\xi(\sigma_\xi, \sigma_\theta), f_\theta(\sigma_\xi, \sigma_\theta)) \qquad (13\cdot2)$$

noting that for constant loads the equivalent elastic problem is constant and equal to the initial elastic solution

$$\sigma_z(0) = \sigma_{z0} \qquad \sigma_\theta(0) = \sigma_{\theta 0} \tag{13.3}$$

Then the residual operators $\mathcal{R}_z$ and $\mathcal{R}_\theta$ are defined through the solution of the residual elastic problem:

"Let $(\varepsilon_z^*, \varepsilon_\theta^*)$ be an arbitrary strain field, then $(\mathcal{R}_z(\varepsilon_z^*, \varepsilon_\theta^*), \mathcal{R}_\theta(\varepsilon_z^*, \varepsilon_\theta^*))$ is the stress field resulting from the elastic deformation of a curved pipe constrained to have no resultant in-plane moment but with $(\varepsilon_z^*, \varepsilon_\theta^*)$ as an initial strain field".

The equation of evolution of end rotation is

$$\frac{dk}{dt} = g(t)\,\mathcal{R}_\gamma(f_z(\sigma_z, \sigma_\theta), f_\theta(\sigma_z, \sigma_\theta)) \tag{13.4}$$

with initial condition

$$k(0) = k_0 \tag{13.5}$$

where $k = \gamma / \Theta$ and $\mathcal{R}_\gamma(\varepsilon_z^*, \varepsilon_\theta^*)$ is the fractional end rotation associated with the residual elastic problem.

Thus the stress field and end rotation are determined on integrating (13.2) and (13.4) subject to (13.3) and (13.5).

13.2 Non-dimensionalization

It is convenient to non-dimensionalize the redistribution equations and in particular the time scale. Let σ_0 be an arbitrary stress and $\varepsilon_0 = \sigma_0 / E$ the associated elastic strain. If an alternative time scale

$$\tau = E\,\sigma_0^{n-1} \int g(t)\,dt$$

is adopted then (13.2)-(13.5) become

$$\frac{dS_z}{d\tau} = \mathcal{R}_z(F_z(S_z, S_\theta), F_\theta(S_z, S_\theta)) \tag{13.6}$$

$$\frac{dS_\theta}{d\tau} = \mathcal{R}_\theta(F_z(S_z, S_\theta), F_\theta(S_z, S_\theta))$$

$$S_z(0) = S_{z0} \qquad S_\theta(0) = S_{\theta 0} \tag{13.7}$$

$$\frac{d\Gamma}{d\tau} = \mathcal{R}_\gamma(F_z(S_z, S_\theta), F_\theta(S_z, S_\theta)) \tag{13.8}$$

$$\Gamma(0) = \Gamma_0 \tag{13.9}$$

where

$$S_i = \sigma_i/\sigma_o \qquad R_i = \mathcal{R}_i/\sigma_o \qquad \Gamma = k/\varepsilon_o \qquad R_\gamma = \mathcal{R}_\gamma/\varepsilon_o \qquad F_i = f_i/\varepsilon_o$$

13.3 The circular cross section curved pipe with initial strains

The derivation of the fundamental equations for a circular cross section curved pipe with initial strains is exactly the same as Section 11.4 (viz. equations (11.13) but with additional terms). These are easily obtained from (10.29) with the result that the bending equations are (c.f. (11.12,11.13)):

$$\begin{aligned} X''+AX'-(A^2+\nu\bar{A})X+\mu\bar{A}Y/\varrho &= -(A\bar{A}-\nu A)K/\mu + (A\mathfrak{I}^*_{11}+(\mathfrak{I}^*_{12})') \\ Y''+AY'-(A^2-\nu\bar{A})Y-\mu\bar{A}X/\varrho &= AK/\varrho + (A\mathfrak{I}^*_{21}-(\mathfrak{I}^*_{22})') \end{aligned} \tag{13.10}$$

$$X=0 \quad Y=0 \qquad \xi = \pi/2,\ 3\pi/2 \tag{13.11}$$

where

$$\mathfrak{I}^*_{11} = (1-\nu)\sqrt{\frac{12}{1-\nu^2}}(K^*_\xi - K^*_\theta) \qquad \mathfrak{I}^*_{12} = \sqrt{\frac{12}{1-\nu^2}}(K^*_\xi + \nu K^*_\theta)$$

$$\mathfrak{I}^*_{21} = \mathcal{E}^*_\xi - \mathcal{E}^*_\theta \qquad \mathfrak{I}^*_{22} = \mathcal{E}^*_\theta$$

$$\mathcal{E}^*_i = \epsilon^*_i/\varepsilon_o \qquad K^*_i = \kappa^*_i/\varepsilon_o$$

again with $K = \Gamma\varrho$ and $\sigma_o = Mb/I$, all other quantities being the same.

Similarly moment equilibrium is,

$$m^* = \frac{1}{\pi}\int_o^{2\pi}\left\{-Y\cos\xi - \frac{\varrho}{\mu}X(A\sin\xi - \nu\cos\xi) + \frac{\varrho}{\mu}\sin\xi\,\zeta^*_3\right\}d\xi + \frac{2K}{\mu^2}\left(\frac{1}{\sqrt{1-\varrho^2}}-1\right) \tag{13.12}$$

and the stress relations are

$$\begin{aligned} S_\xi &= AY + \zeta\sqrt{\tfrac{12}{1-\nu^2}}\,(X' + \nu(AX - K\bar{A}/\mu)) + \zeta^*_1 \\ S_\theta &= Y' + \zeta\sqrt{\tfrac{12}{1-\nu^2}}\,((AX - K\bar{A}/\mu) + \nu X') + \zeta^*_2 \end{aligned} \tag{13.13}$$

where

$$\begin{aligned} \zeta^*_1 &= \tfrac{1}{1-\nu^2}[\mathcal{E}^*_\xi + \nu\mathcal{E}^*_\theta - (E^*_\xi + \nu E^*_\theta)] \\ \zeta^*_2 &= \tfrac{1}{1-\nu^2}[\mathcal{E}^*_\theta + \nu\mathcal{E}^*_\xi - (E^*_\theta + \nu E^*_\xi)] \\ \zeta^*_3 &= \sqrt{\tfrac{12}{1-\nu^2}}\,[K^*_\theta + \nu K^*_\xi] \end{aligned}$$

and $E^*_i = \varepsilon^*_i/\varepsilon_o$.

The scalar $m^* = M^*/M$, for some M^* shall be described later.

The discrete model of this problem is obtained in exactly the same way as Section 11.4 on replacing the interval $[\pi/2, 3\pi/2]$ by the mesh

$$\xi_i = \pi/2 + i\Delta\xi \qquad i = 0,1,2,\cdots M, M+1$$
$$\Delta\xi = \pi/(M+1)$$

whence equation (11.9) is obtained

$$[A]\underset{\sim}{U} = \underset{\sim}{F} \tag{13.14}$$

where now

$$f_i^* = [A_i \Im_{111}^* + \frac{\Im_{12\,i+1}^* - \Im_{12\,i-1}^*}{2\Delta\xi}](\Delta\xi)^2$$
$$g_i^* = [A_i \Im_{211}^* - \frac{\Im_{22\,i+1}^* - \Im_{22\,i-1}^*}{2\Delta\xi}](\Delta\xi)^2 \qquad i = 1,2,\cdots M$$
$$f_{M+1}^* = \frac{3}{2}\frac{\pi}{\Delta\xi}m^* - \frac{3}{\Delta\xi}\frac{\rho}{\mu}\int_{\pi/2}^{3\pi/2}\sin\xi\, \zeta_3^* d\xi$$

Thus given $\Im_{ij}^*$ and ζ_i^*, $\underset{\sim}{F}$ is determined and

$$\underset{\sim}{U} = [A]^{-1}\underset{\sim}{F} \tag{13.15}$$

the stresses being obtained from (13.13) and the problem is solved.

13.4 The initial elastic problem

The general curved pipe problem with initial strains can be used to deduce the solutions of the initial and residual elastic problems.

When the moment M is first applied there is an end rotation resulting from a purely elastic response. The appropriate equations are derived from (13.10)-(13.13) on identifying $\Im_{ij}^* = 0$, $\zeta_i^* = 0$. Thus the initial profile, represented by the matrices $[S_{30}]$ and $[S_{00}]$, and K_o (or K_e), may be obtained on solving (13.14) with $m^* = 1$ and

$$\underset{\sim}{F} = (0\ 0\ 0\ \cdots\ 0\ \ 3\pi/2\Delta\xi)$$

which is then identical to section 11.4 and the results are the same.

13.5 The residual elastic problem

Initial strains are superimposed on a curved pipe constrained to have no resultant moment; whence in (13.12) identify $m^* = 0$. If E^*_ξ and E^*_θ are such initial strains then R_ξ and R_θ are the stresses from (13.13) from solving (13.10), and (13.12); R_γ is the resultant fractional end rotation (nondimensionalized).

On replacing the interval $[-1/2, 1/2]$ by the mesh ζ_j, $j = 1,2,.....N,N+1$ such that $[S^*_\xi]$ and $[S^*_\theta]$ represent stresses from the residual elastic problem, then from (13.13) (c.f. (11.21))

$$[S^*_\xi] = [B_\xi]\underset{\sim}{U}\underset{\sim}{1}^T + [C_\xi]\underset{\sim}{U}\underset{\sim}{\zeta}^T + \frac{1}{1-\nu^2}\{[E^*_\xi] + \nu[E^*_\theta]\}([S]-[I])$$

$$[S^*_\theta] = [B_\theta]\underset{\sim}{U}\underset{\sim}{1}^T + [C_\theta]\underset{\sim}{U}\underset{\sim}{\zeta}^T + \frac{1}{1-\nu^2}\{[E^*_\theta] + \nu[E^*_\xi]\}([S]-[I])$$

but from (13.15), $\underset{\sim}{U} = [A]^{-1}\underset{\sim}{F}$ where

$$\underset{\sim}{F} = [D_{11}]\underset{\sim}{\Im}_{11} + [D_{12}]\underset{\sim}{\Im}_{12} + [D_{21}]\underset{\sim}{\Im}_{21} + [D_{22}]\underset{\sim}{\Im}_{22} + [D]\sqrt{\frac{12}{1-\nu^2}}([E^*_\theta] + \nu[E^*_\xi])\underset{\sim}{\bar{d}}$$

$$\underset{\sim}{\Im}_{11} = (1-\nu)\sqrt{\frac{12}{1-\nu^2}}\{[E^*_\xi]-[E^*_\theta]\}\underset{\sim}{\bar{d}} \qquad \underset{\sim}{\Im}_{12} = \sqrt{\frac{12}{1-\nu^2}}\{[E^*_\xi] + \nu[E^*_\theta]\}\underset{\sim}{\bar{d}}$$

$$\underset{\sim}{\Im}_{21} = \{[E^*_\xi]-[E^*_\theta]\}\underset{\sim}{d} \qquad \underset{\sim}{\Im}_{22} = [E^*_\theta]\underset{\sim}{d}$$

all matrices and vectors being given in Table 13.1.

On combining these the discrete form of the residual operators is the finite sum

$$R_\xi(E^*_\xi, E^*_\theta) = \sum_i [R^i_{\xi\xi}][E^*_\xi][\bar{R}^i_{\xi\xi}] + [R^i_{\xi\theta}][E^*_\theta][\bar{R}^i_{\xi\theta}]$$

$$R_\theta(E^*_\xi, E^*_\theta) = \sum_i [R^i_{\theta\xi}][E^*_\theta][\bar{R}^i_{\theta\xi}] + [R^i_{\theta\theta}][E^*_\theta][\bar{R}^i_{\theta\theta}]$$

$$R_\gamma(E^*_\xi, E^*_\theta) = \sum_i (\underset{\sim}{r}^i_\xi)^T[E^*_\xi]\underset{\sim}{\bar{r}}^i_\xi + (\underset{\sim}{r}^i_\theta)^T[E^*_\theta]\underset{\sim}{\bar{r}}^i_\theta$$

where

$$\dim[R^i_{\alpha\beta}] = (M+2)\times(M+2)$$

$$\dim[\bar{R}^i_{\alpha\beta}] = (N+1)\times(N+1)$$

$$\dim \underset{\sim}{r}^i_\alpha = (M+2)\times 1$$

$$\dim \underset{\sim}{\bar{r}}^i_\alpha = (N+1)\times 1$$

13.6 Forward integration of the redistribution equations

With these discrete definitions of the residual operators $\mathbb{R}_\xi$, $\mathbb{R}_\Theta$ and $\mathbb{R}_\gamma$ it is possible to replace the redistribution equations (13.6) and equation of evolution of end rotation (13.8) by their discrete forms (Section 7·2)

$$\frac{d}{d\tau}[S_\xi] = \sum_i [R^i_{\xi\xi}][F_\xi(S_\xi, S_\Theta)][\bar{R}^i_{\xi\xi}] + [R^i_{\xi\Theta}][F_\Theta(S_\xi, S_\Theta)][\bar{R}^i_{\xi\Theta}]$$

$$\frac{d}{d\tau}[S_\Theta] = \sum_i [R^i_{\Theta\xi}][F_\xi(S_\xi, S_\Theta)][\bar{R}^i_{\Theta\xi}] + [R^i_{\Theta\Theta}][F_\Theta(S_\xi, S_\Theta)][\bar{R}^i_{\Theta\Theta}] \qquad (13\cdot16)$$

$$\frac{dK}{d\tau} = \sum_i (\underline{r}^i_\xi)^T [F_\xi(S_\xi, S_\Theta)]\underline{\bar{r}}^i_\xi + (\underline{r}^i_\Theta)^T [F_\Theta(S_\xi, S_\Theta)]\underline{\bar{r}}^i_\Theta \qquad (13\cdot17)$$

$$[S_\xi(0)] = [S_{\xi 0}] \quad [S_\Theta(0)] = [S_{\Theta 0}] \quad K(0) = K_0 \qquad (13\cdot18)$$

where

$$[F_\xi(S_\xi, S_\Theta)]_{(i,j)} = \left\{[S_\xi]^2_{(i,j)} [S_\xi]_{(i,j)}[S_\Theta]_{(i,j)} + [S_\Theta]^2_{(i,j)}\right\}^{\frac{n-1}{2}} \left\{[S_\xi]_{(i,j)} - \tfrac{1}{2}[S_\Theta]_{(i,j)}\right\}$$

etc. The system (13.16) and (13.17) can be integrated forward in time by any suitable sequential process starting from (13.18). The Runge-Kutta-Gill method was found to be efficient and adequate (Appendix 5). Some technique is required to determine the magnitude of the time steps. There are several standard techniques for varying the time interval from step to step, see for example I C Cormeau [286]. Nevertheless the writer has found such methods to be inadequate at present. The problem was overcome here by estimating the order of the length of duration of the redistribution phase.

C R Calladine [79] has demonstrated that the time to stress redistribution to the steady state is of order

$$O(\tau_\infty) = \frac{2\cdot3}{n}\left(\frac{\hat{\sigma}_{ss}}{\sigma_o}\right)^{1-n}$$

In the time scale τ for the Norton power law where $\hat{\sigma}_{ss}$ is a measure of the peak stress level in the pipe in the steady state. Therefore approximately

$$O(\tau_\infty) = \frac{2\cdot3}{n}(\text{SCFM in steady state})^{1-n}$$

The time step was then taken as uniform throughout redistribution as

$$\Delta\tau = \delta \cdot o(\tau_\infty)$$

where δ is some scalar in the range $[0.01, 0.1]$. Values of the maximum SCF were taken from [423].

The integration is terminated when the maximum stress rate reduces to some small value

13.7 Numerical results

A FORTRAN IV computer program was written for the University of Strathclyde ICL 1904S system to perform the forward creep integration using the Runge-Kutta-Gill procedure. The quantities δ and ε were taken as 0.1 and 0.01 respectively with $M = 17$, $N = 10$ as before. In all cases a nominal value of $\nu = 0.33$ was used. A typical run took about 8 minutes mill time.

13.7.1. Flexibility of a creeping curved pipe compared to a similar elastic straight

The nondimensionalizing stress $\sigma_o = Mb/I$ is the maximum fibre stress in an equivalent straight pipe loaded by M and responding elastically. It follows that K is the ratio of end rotation of a creeping curved pipe to this similar straight and is thus a measure of the flexibility of the curved pipe, denoted by $K_{c/e}(\tau)$ obviously $K_{c/e}(0) = K_o = K_e$ the well known elastic flexibility factor. Similarly S_i is the ratio of stress in a creeping curved pipe to maximum fibre stress in the straight pipe and is consequently a stress concentration factor (SCF), whose initial value is the elastic SCF.

Figures 13.1, 13.2 and 13.3 show $K_{c/e}$ plotted against time for values of n = 3.0, 5.0 and 7.0 respectively, and a radius ratio of $\rho = 0.1$.

Figures 13.4, 13.5 and 13.6 show the variation in maximum $SCF_{c/e}$ plotted against time.

13.7.2. Flexibility of a creeping curved pipe compared to a similarly creeping straight

The factor $K_{c/e}$ effectively compares the flexibility of the curved pipe to initial conditions. It is instructive to examine how the flexibility compares to a similarly creeping straight pipe.

Let $\sigma(\xi,t)$ be the stress and $\kappa(t)$ the curvature in a straight pipe. Then with reference to Section 8.1 the nondimensional quantities

$$S(\xi,\tau) = \sigma(\xi,t)/\sigma \qquad K_{st}(\tau) = \kappa(t)/\kappa_o$$

can be obtained.

It follows that

$$K_{c/c}(\tau) = K_{c/e}(\tau) \div K_{st}(\tau)$$

and that

$$SCF_{c/c}(\tau) = SCF_{c/e}(\tau) \div \max_{\xi} S(\xi,\tau)$$

Figures 13.1, 13.2 and 13.3 show $K_{c/c}$ plotted against time.

Figures 13.4, 13.5 and 13.6 show $SCFM_{c/c}$ plotted against time.

13.7.3. Comparison with previous results

The ratio

$$\frac{d}{d\tau}K_{c/e} \div \frac{d}{d\tau}K_{st}$$

show tend at large times to the steady creep flexibility factor K_{ss} [423]. An alternative method of examining this asymptotic behaviour is to introduce the measure $K_{c/s}(\tau)$. It may be shown that

$$K_{c/s}(\tau) = \frac{d}{d\tau}K_{de}(\tau) \times \left(\frac{D_o}{\pi}\right)^n$$

so that as the steady state is approached $K_{c/s}(\tau) \longrightarrow K_{ss}$ which is a constant.

Figures 13.7, 13.8 and 13.9 show variations in $K_{c/s}$ plotted

against time together with upper and lower bound values of K_{ss} from [423]. The upper bound may be considerably reduced by employing a modified analysis [423,459].

13.7.4. Effect of radius ratio

In elasticity it is well known that variations in the radius ratio ϱ do not have a pronounced effect on the flexibility (see Table 11.4). However this is not the case in creep as is seen from Figures 13.7-13.9. It is true, however, that flexibility rates shall be asymptotically virtually independent of ϱ, by performing extended calculations. Variations in the stress concentration factors in elasticity are known to be significant and this is equally true for creep [427].

13.8 Discussion

It is apparent from the results that creep has a significant effect on the flexibility of a curved pipe under the action of a constant applied moment. The important conclusion is that the steady state results of Spence [423] are verified with indications that the lower bound analysis is better for "small" values of the pipe bend parameter , while the upper bound is better for "large" approaching the geometry of a straight pipe.

TABLE 13.1

MATRICES FOR DISCRETE RESIDUAL OPERATORS

$$\underset{\sim}{\bar{d}}^T = \frac{\Delta\xi}{3}\left(\zeta_1 \quad 4\zeta_2 \quad 2\zeta_3 \quad 4\zeta_4 \quad \cdots \quad 2\zeta_{N-1} \quad 4\zeta_N \quad \zeta_{N+1}\right)$$

$$\underset{\sim}{d}^T = \frac{\Delta\xi}{3}\left(1 \quad 4 \quad 2 \quad 4 \quad \cdots \quad 2 \quad 4 \quad 1\right)$$

$$[D_{11}] = \begin{bmatrix} \cdot & A_1 & & & & \cdot \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & A_2 & & & \\ & & & A_{N-1} & \cdot & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & \cdot & A_M & \cdot \\ & & & \cdot & \cdot & \cdot \\ \cdot & & & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[D_{12}] = \frac{1}{2\Delta\xi}\begin{bmatrix} -1 & \cdot & 1 & \cdot & & & \cdot \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & -1 & \cdot & 1 & & & \\ & & & & -1 & \cdot & 1 & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & -1 & \cdot & 1 \\ & & & & \cdot & \cdot & \cdot & \cdot \\ \cdot & & & & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[D_{21}] = \begin{bmatrix} \cdot & \cdot & \cdot & & & \cdot \\ \cdot & A_1 & \cdot & & & \\ \cdot & \cdot & \cdot & & & \\ \cdot & \cdot & A_2 & & & \\ & & & \cdot & A_{M-1} & \cdot \\ & & & \cdot & \cdot & \cdot \\ & & & \cdot & \cdot & A_M \\ \cdot & & & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[D_{22}] = \frac{-1}{2\Delta\xi}\begin{bmatrix} \cdot & \cdot & \cdot & \cdot & & & \\ -1 & \cdot & 1 & \cdot & & & \\ \cdot & \cdot & \cdot & \cdot & & & \\ \cdot & -1 & \cdot & 1 & & & \\ & & & & -1 & \cdot & 1 & \cdot \\ & & & & \cdot & \cdot & \cdot & \cdot \\ & & & & \cdot & -1 & \cdot & 1 \\ & & & & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$[S] = \begin{bmatrix} 1 & 1 & & & 1 \\ 4 & 4 & & & 4 \\ 2 & 2 & & & 2 \\ 4 & 4 & & & 4 \\ \vdots & \vdots & & & \vdots \\ 2 & 2 & & & 2 \\ 4 & 4 & & & 4 \\ 1 & 1 & \cdots & & 1 \end{bmatrix}$$

$$[D] = -\frac{\rho}{\mu}\begin{bmatrix} \cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\ \vdots & & & & & \vdots \\ \cdot & & & & & \cdot \\ \cdot & & & & & \cdot \\ \cdot & 4\sin\xi_1 & 2\sin\xi_2 & \cdots & 4\sin\xi_M & \cdot \end{bmatrix}$$

$$\underset{\sim}{P}^T = -\frac{1}{\pi}\frac{\Delta\xi}{3}\left(4\cos\xi_1 \quad \frac{4\rho}{\mu}\left(A_1\sin\xi_1 - \nu\cos\xi_1\right) \quad 2\cos\xi_2 \quad \frac{2\rho}{\mu}\left(A_2\sin\xi_2 - \nu\cos\xi_2\right) \cdots\right)$$

$$\underset{\sim}{Q}^T = \frac{\rho}{\mu}\left(\sin\xi_0 \quad 4\sin\xi_1 \quad 2\sin\xi_2 \quad 4\sin\xi_3 \quad \cdots \quad \sin\xi_{M+1}\right)$$

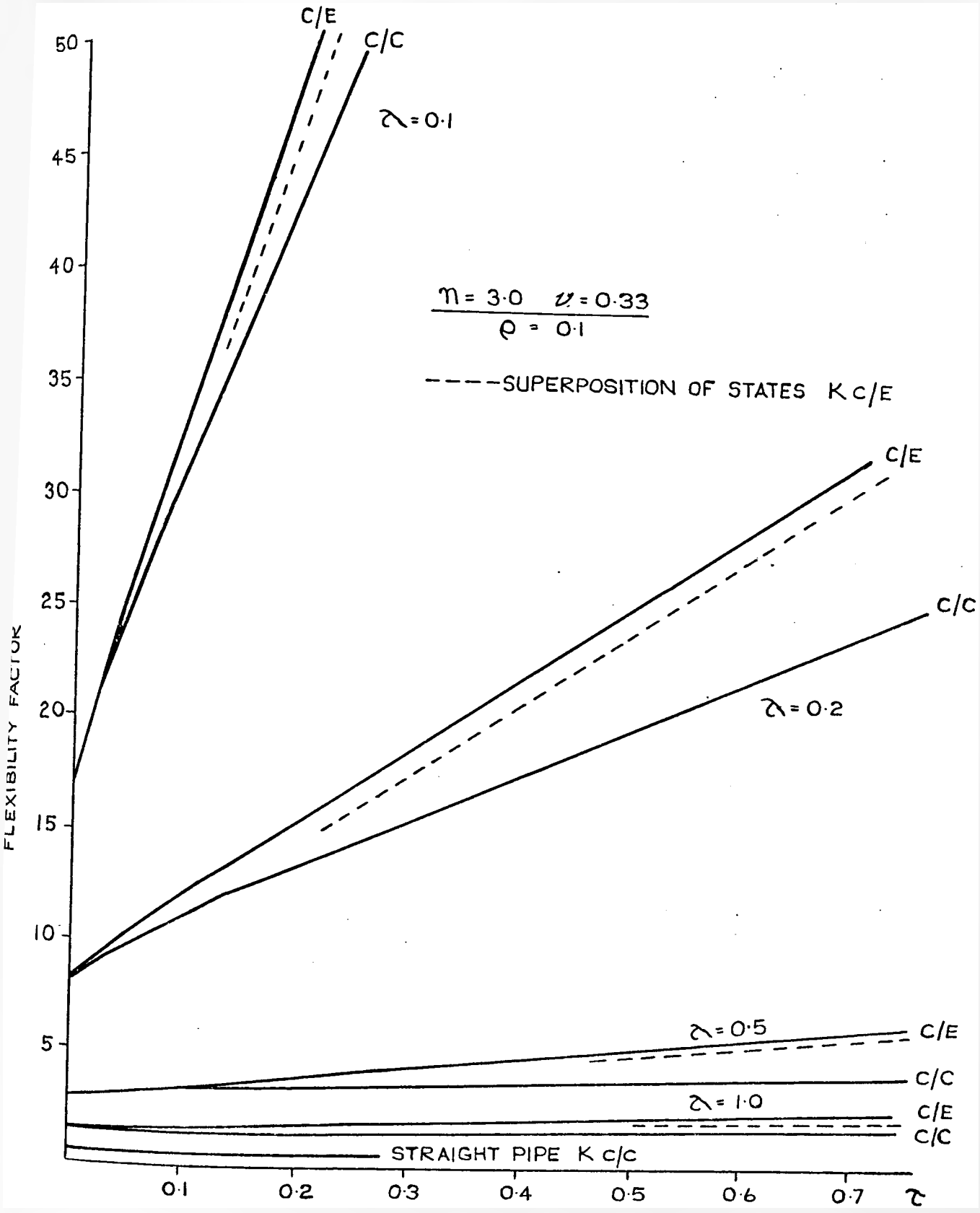


Fig 13.1 VARIATION IN Kc/E AND Kc/c IN TIME

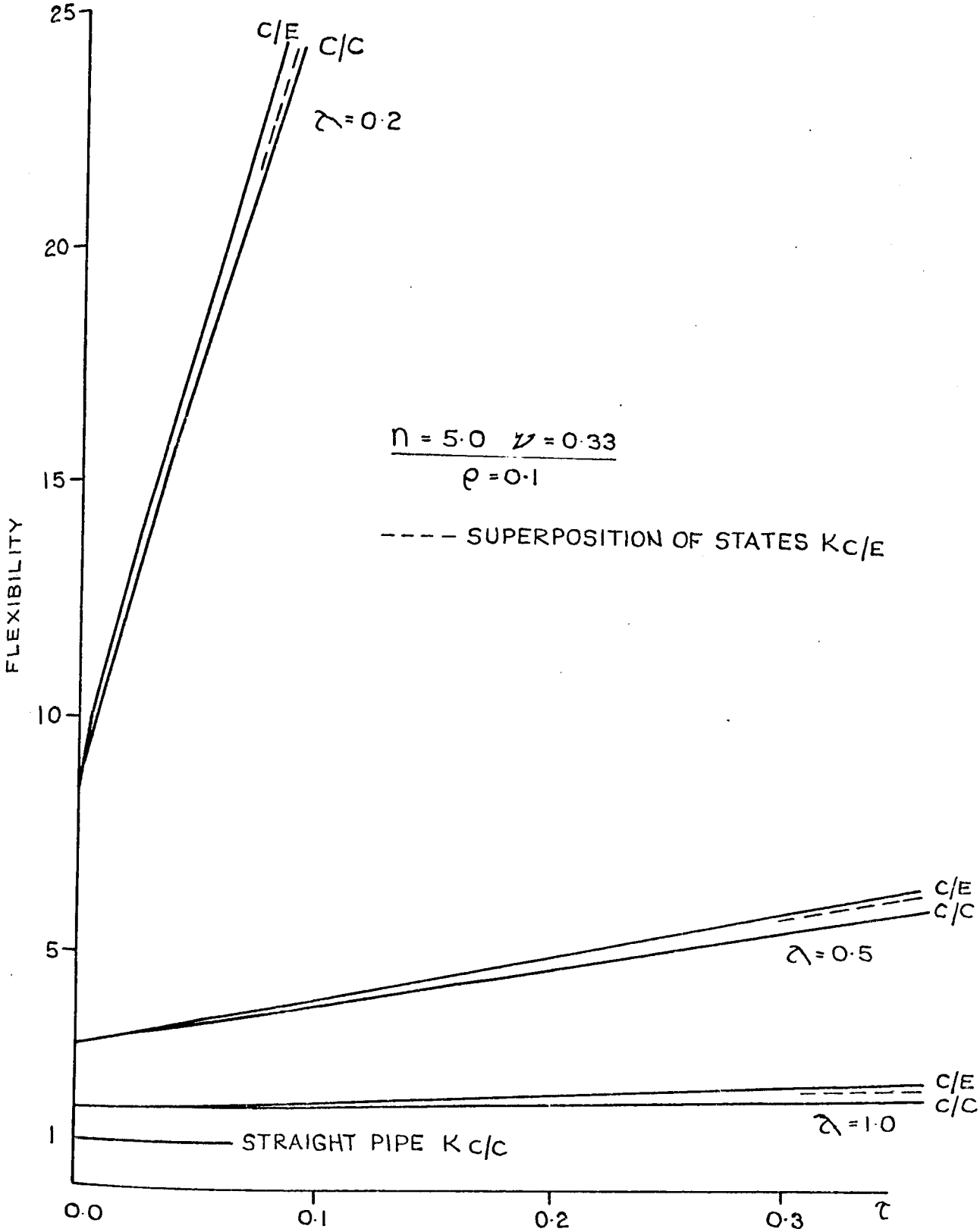


Fig 13·2 VARIATION IN $K_{C/E}$ AND $K_{C/C}$ IN TIME

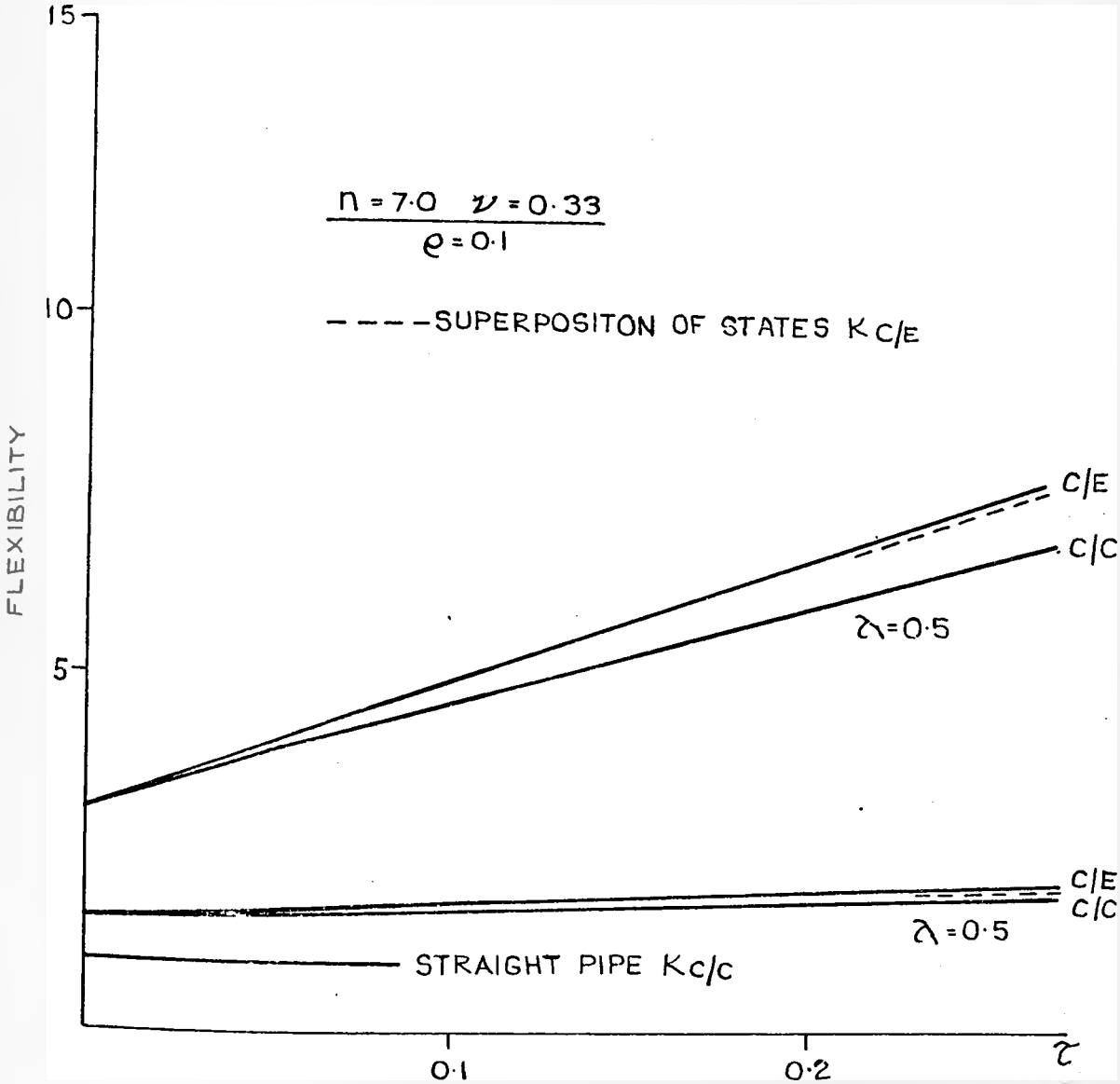


Fig 13·3 VARIATION IN $K_{c/E}$ AND $K_{c/c}$ IN TIME

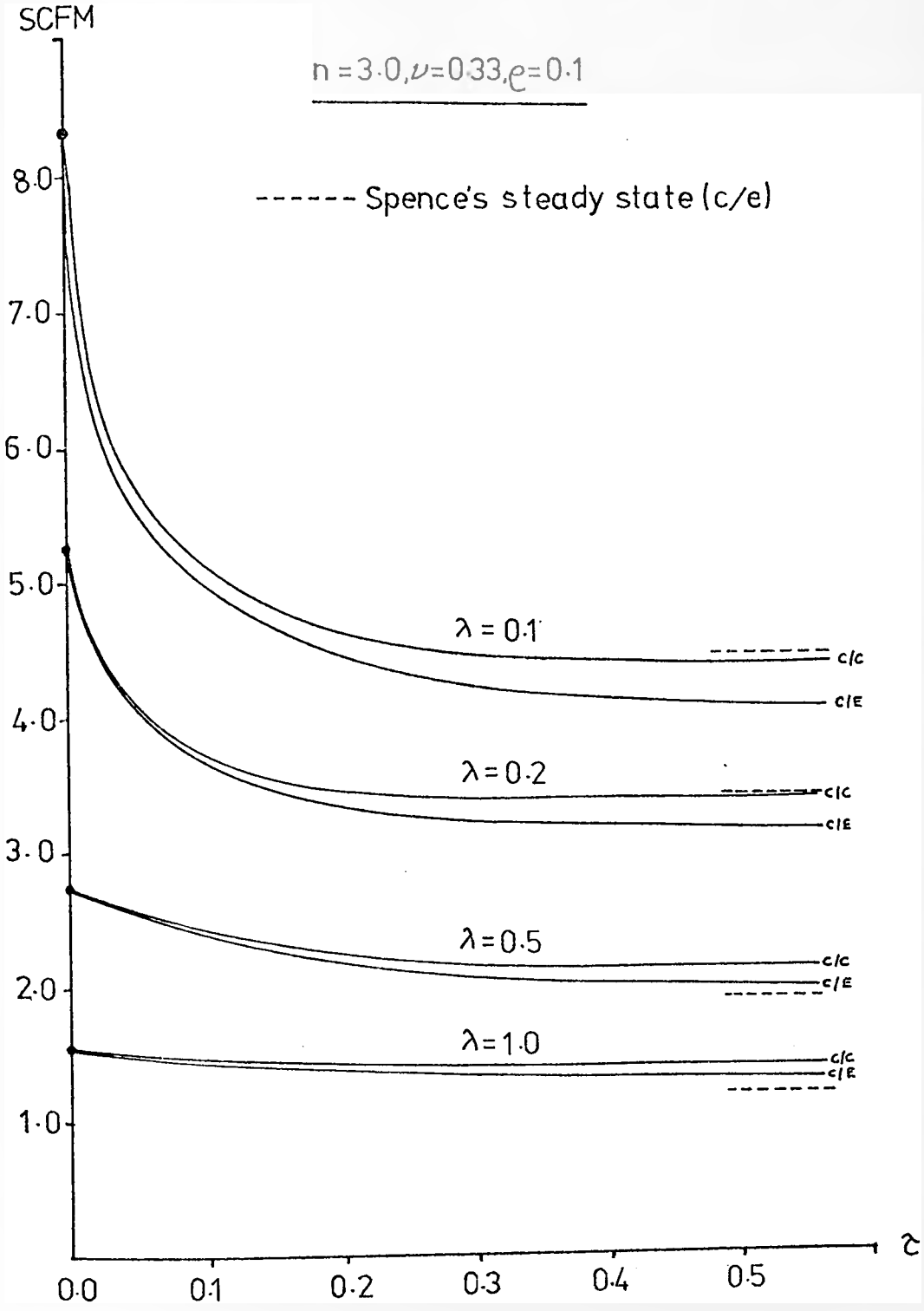


Fig13.4 Redistribution of SCFM

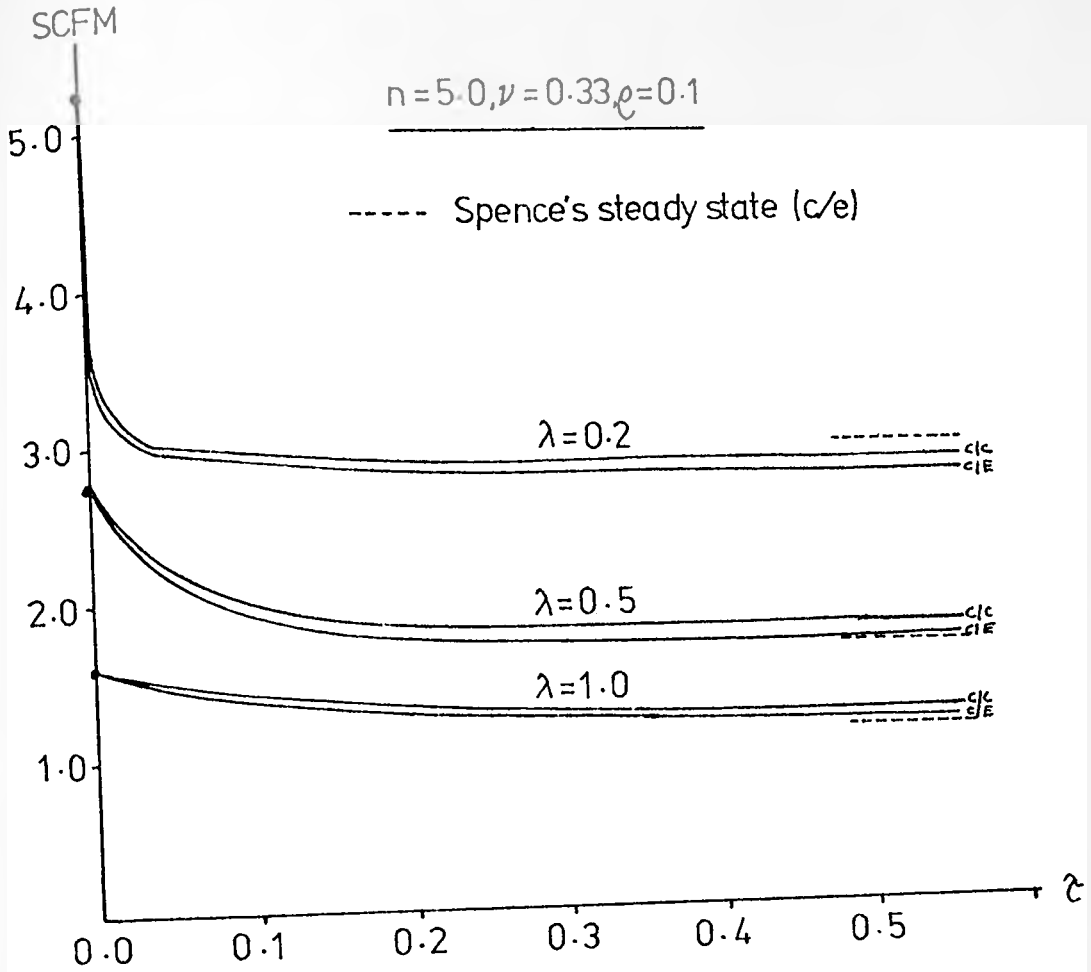


Fig 13·5 Redistribution of SCFM

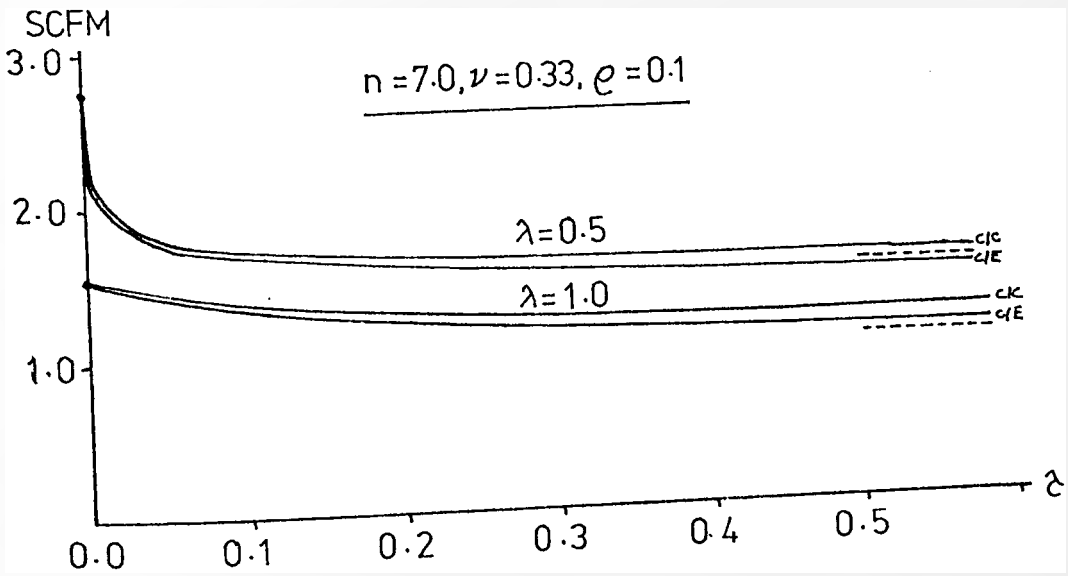


Fig 13·6 Redistribution of SCFM

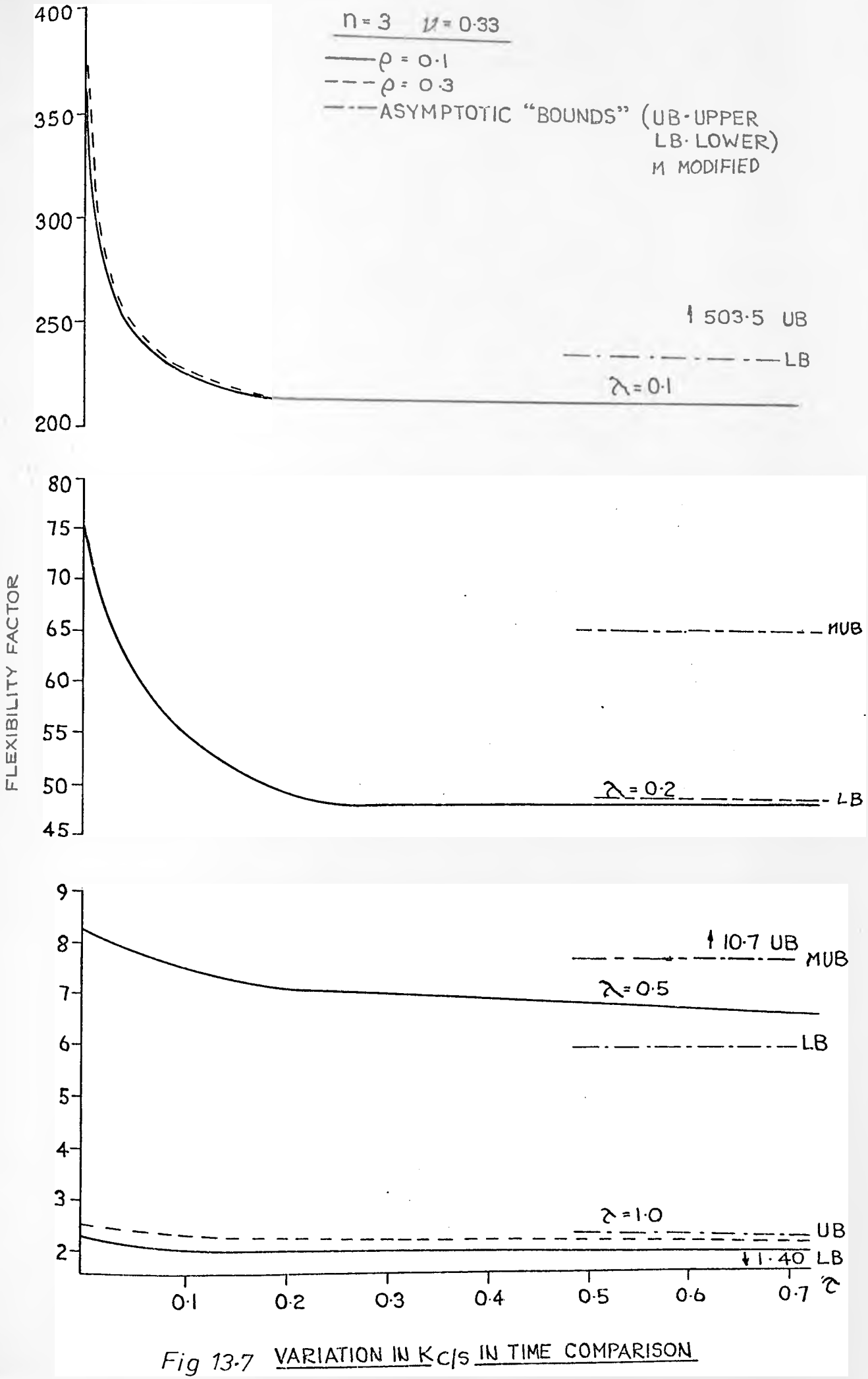


Fig 13·7 VARIATION IN Kc/s IN TIME COMPARISON

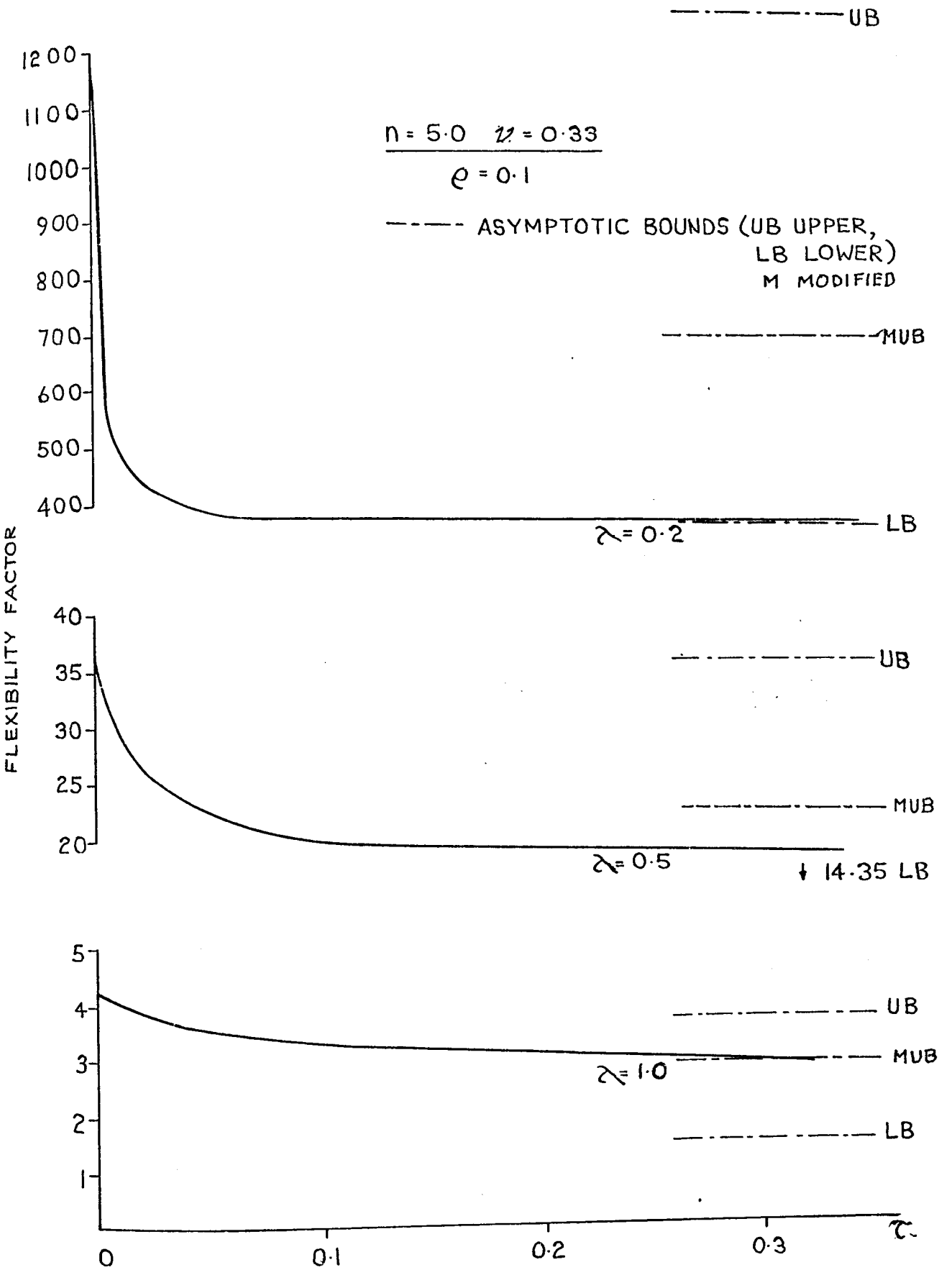


Fig 13·8 VARIATION IN Kc/s IN TIME COMPARISON

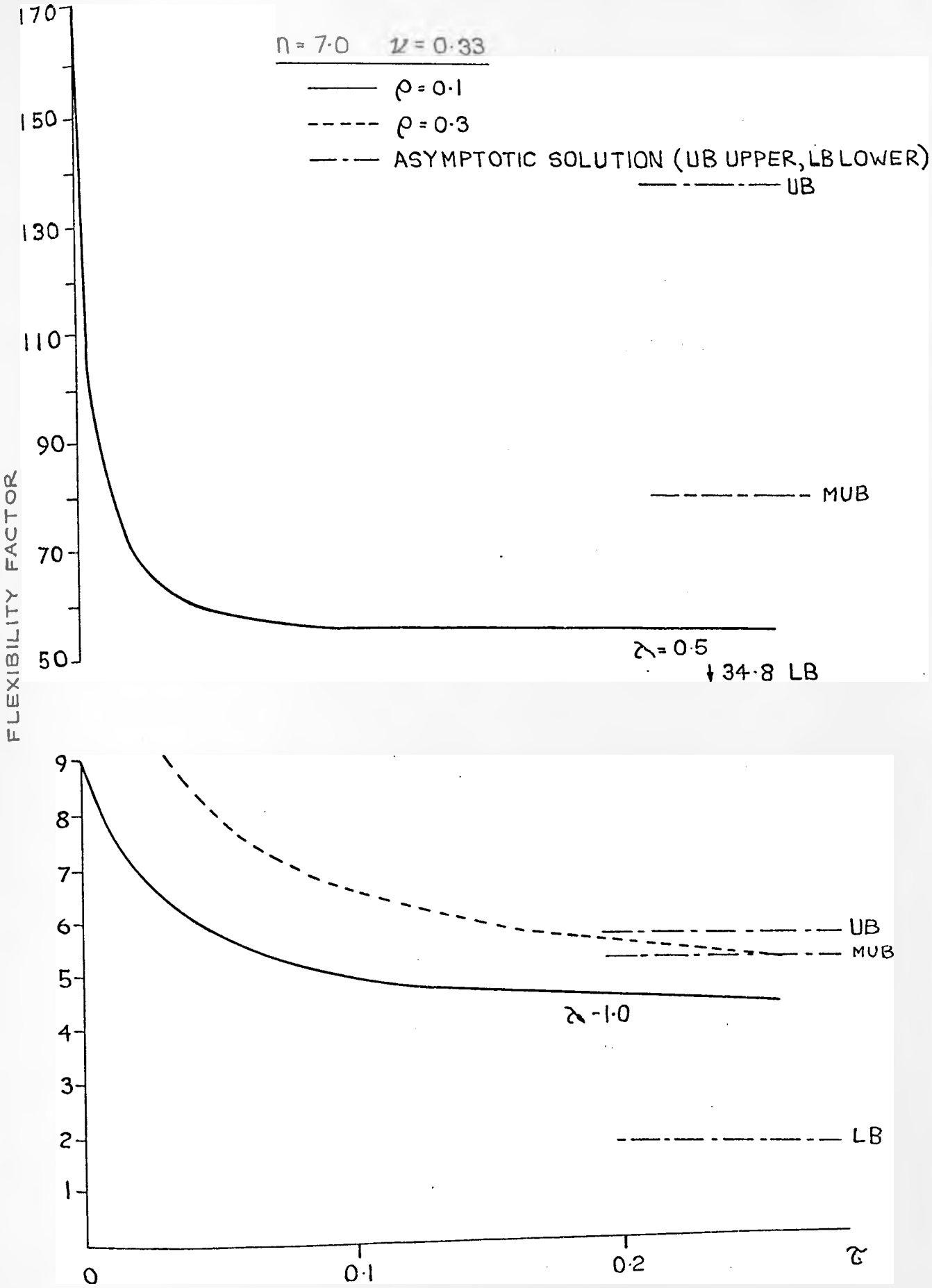


Fig 13.9 VARIATION IN K_C/S IN TIME COMPARISON.

CHAPTER 14

RELAXATION OF A CURVED PIPE

The previous chapter considered the time dependent inelastic response of a curved pipe to a constant applied in-plane bending moment. However the usual situation in high temperature piping analysis is that of relaxation. In the context of the curved pipe alone this means that a constant end rotation is applied resulting in the relaxation of the resultant bending moment.

14.1 The relaxation problem

Again it is presumed that the constitutive material creeps according to a time hardening, Norton power law, equation (13.1).

Consider the following problem:
"Let an in-plane bending moment M_o be applied uniformly to a curved pipe which deforms elastically resulting in an end rotation γ_o and a stress field $(\sigma_{\xi o}, \sigma_{\theta o})$. The ends of the pipe are then fixed, followed by inelastic expansion in the form of creep. The problem is to determine the resultant in-plane moment $M(t)$, where $M(0) = M_o$" .

Immediately the redistribution equations are:

$$\frac{d\sigma_\xi}{dt} = g(t)\,\mathcal{R}_\xi(f_\xi(\sigma_\xi, \sigma_\theta), f_\theta(\sigma_\xi, \sigma_\theta))$$
$$\frac{d\sigma_\theta}{dt} = g(t)\,\mathcal{R}_\theta(f_\xi(\sigma_\xi, \sigma_\theta), f_\theta(\sigma_\xi, \sigma_\theta)) \qquad (14\cdot1)$$

with the initial conditions,

$$\sigma_\xi(0) = \sigma_{\xi o} \qquad \sigma_\theta(0) = \sigma_{\theta o} \qquad (14\cdot2)$$

The residual operators $\mathcal{R}_\xi$ and $\mathcal{R}_\theta$ are defined now through the solution of the residual elastic problem:
"Let $(\varepsilon_\xi^*, \varepsilon_\theta^*)$ be an arbitrary strain field, then $(\mathcal{R}_\xi(\varepsilon_\xi^*, \varepsilon_\theta^*), \mathcal{R}_\theta(\varepsilon_\xi^*, \varepsilon_\theta^*))$ is the stress field resulting

from the elastic deformation of a curved pipe with initial strains ε_ξ^*, ε_θ^*, constrained to have no end rotation".

It is to be noted that $\mathcal{R}_\xi$ and $\mathcal{R}_\theta$ are different from those defined in Chapter 13 for the creep problem.

The relaxation of resultant bending moment is governed by the equation

$$\frac{dM}{dt} = g(t)\mathcal{R}_M(f_\xi(\sigma_\xi, \sigma_\theta), f_\theta(\sigma_\xi, \sigma_\theta)) \qquad (14\cdot3)$$

and initial condition

$$M(0) = M_0 \qquad (14\cdot4)$$

where the operator $\mathcal{R}_M$ is defined such that $\mathcal{R}_M(\varepsilon_\xi^*, \varepsilon_\theta^*)$ is the resultant moment from the residual elastic problem.

Thus the stress field $(\sigma_\xi(t), \sigma_\theta(t))$ and the resultant moment $M(t)$ are obtained on integrating (14.1) and (14.3) from (14.2) and (14.4).

14.2 Nondimensionalization

If σ_o is an arbitrary stress and $\varepsilon_o = \sigma_o / E$ then, on defining the time scale,

$$\tau = E\sigma_o^{n-1}\int g(t)dt$$

there results

$$\frac{dS_\xi}{d\tau} = \mathbb{R}_\xi(F_\xi(S_\xi, S_\theta), F_\theta(S_\xi, S_\theta))$$
$$\frac{dS_\theta}{d\tau} = \mathbb{R}_\theta(F_\xi(S_\xi, S_\theta), F_\theta(S_\xi, S_\theta)) \qquad (14\cdot5)$$

$$S_\xi(0) = S_{\xi 0} \qquad S_\theta(0) = S_{\theta 0} \qquad (14\cdot6)$$

and

$$\frac{dm}{d\tau} = \mathbb{R}_M(F_\xi(S_\xi, S_\theta), F_\theta(S_\xi, S_\theta)) \qquad (14\cdot7)$$

$$m(0) = 1 \qquad (14\cdot8)$$

where

$$S_i = \sigma_i/\sigma_o \qquad F_i = f_i/\varepsilon_o \qquad \mathbb{R}_i = \mathcal{R}_i/\sigma_o \qquad m = M/M_o \qquad \mathbb{R}_M = \mathcal{R}_M/M_o$$

14.3 The initial elastic problem

This is identical to the problem examined in Sections 11.4 and 13.4. It is supposed that the initial stress profile $[S_{s0}]$, $[S_{\theta\theta}]$ corresponding to the discrete model is available.

14.4 The residual elastic problem

In this case the residual elastic problem specifies an end rotation and calculates a resultant moment. The bending equations are thus given by (13.10) and (13.11) with $K \equiv 0$; the resultant moment is calculated from (13.12) (again with $K \equiv 0$). The stress field is obtained from (13.13) in a similar manner.

The matrix equation of the discrete model assumes the reduced form

$$[\hat{\mathcal{A}}]\hat{\underset{\sim}{U}} = \hat{\underset{\sim}{F}} \qquad (14.9)$$

where

$$\hat{\underset{\sim}{U}} = (X_1\ Y_1\ X_2\ Y_2 \cdots X_M\ Y_M)$$

$$\hat{\underset{\sim}{F}} = (f_1^*\ g_1^*\ f_2^*\ g_2^* \cdots f_M^*\ g_M^*)$$

and $[\hat{\mathcal{A}}]$ is a reduction of $[\mathcal{A}]$ (Table 11.3) with the $(M+1)$th row and column removed.

Thus the discrete forms of the residual operators can be obtained as

$$\mathbb{R}_s(E_s^*, E_\theta^*) = \sum_i [\hat{R}^i_{ss}][E_s^*][\hat{\bar{R}}^i_{ss}] + [\hat{R}^i_{s\theta}][E_\theta^*][\hat{\bar{R}}^i_{s\theta}]$$

$$\mathbb{R}_\theta(E_s^*, E_\theta^*) = \sum_i [\hat{R}^i_{\theta s}][E_s^*][\hat{\bar{R}}^i_{\theta s}] + [\hat{R}^i_{\theta\theta}][E_\theta^*][\hat{\bar{R}}_{\theta\theta}]$$

where the matrices $[\hat{R}^i_{\alpha\beta}]$ etc are reduced forms of those in Section 13.5.

From moment equilibrium (13.12)

$$\mathbb{R}_M(E_s^*, E_\theta^*) = \underset{\sim}{P}^T\hat{\underset{\sim}{U}} + \underset{\sim}{Q}^T\left\{\sqrt{\tfrac{12}{1-\nu^2}}([E_s^*] + \nu[E_\theta^*])\right\}\bar{\underset{\sim}{d}}$$

where $\underset{\sim}{P}$ and $\underset{\sim}{Q}$ are given in Table 13.1.

Hence from (14.9) $\hat{\underset{\sim}{U}} = [\hat{\mathcal{A}}]^{-1}\hat{\underset{\sim}{F}}$ and the operator $\mathbb{R}_M$ takes

the form of the finite sum

$$R_M(E^*_\xi, E^*_\theta) = \sum_i (\underset{\sim}{p}^i_\xi)^T [E^*_\xi] \underset{\sim}{\bar{p}}^i_\xi + (\underset{\sim}{p}^i_\theta)^T [E^*_\theta] \underset{\sim}{\bar{p}}^i_\theta$$

14.5 Forward integration of the redistribution equations

It is now possible to construct the discrete forms of the redistribution equations thus

$$\frac{d}{d\tau}[S_\xi] = \sum_i [\hat{R}^i_{\xi\xi}][F_\xi(S_\xi, S_\theta)][\hat{\bar{R}}^i_{\xi\xi}] + [\hat{R}^i_{\xi\theta}][F_\theta(S_\xi, S_\theta)][\hat{\bar{R}}^i_{\xi\theta}]$$

$$\frac{d}{d\tau}[S_\theta] = \sum_i [\hat{R}^i_{\theta\xi}][F_\xi(S_\xi, S_\theta)][\hat{\bar{R}}^i_{\theta\xi}] + [\hat{R}^i_{\theta\theta}][F_\theta(S_\xi, S_\theta)][\hat{\bar{R}}^i_{\theta\theta}] \qquad (14\cdot10)$$

$$[S_\xi(0)] = [S_{\xi 0}] \qquad [S_\theta(0)] = [S_{\theta 0}] \qquad (14\cdot11)$$

$$\frac{dm}{d\tau} = \sum_i (\underset{\sim}{p}^i_\xi)^T [F_\xi(S_\xi, S_\theta)] \underset{\sim}{\bar{p}}^i_\xi + (\underset{\sim}{p}^i_\theta)^T [F_\theta(S_\xi, S_\theta)] \underset{\sim}{\bar{p}}^i_\theta \qquad (14\cdot12)$$

$$m(0) = 1 \qquad (14\cdot13)$$

This finite system of ordinary differential equations may now be integrated forward in time in a step by step manner. Owing to the 'large' times for which this integration should be continued a Runge-Kutta second order method was found sufficient (Appendix 5). Initially a small time step (of the order of 0.001) should be used, continued until the rate of change of moment slows down, whence it is chosen automatically by a method suggested by R K Penny [7,409] i.e. for each time step

$$\Delta\tau = \delta \cdot m(\tau) \div \frac{dm}{d\tau}$$

where δ is a scalar in the range [0.01, 0.1]. The process was terminated after an a priori chosen time.

14.6 Numerical results

A FORTRAN IV computer program was written for the University of Strathclyde ICL 1904S system to perform the numerical integration of the redistribution equations using the previous discrete model (M = 17, N = 10). The scalar parameter δ was taken as 0.05; a typical run took 3 minutes mill time. A nominal value of $\nu = 0.33$ was chosen throughout.

Figures 14.1, 14.2 and 14.3 show $m(\tau)$ plotted against time for values of n = 3.0,5.0 and 7.0 respectively for various λ and $\varrho = 0.1$.

Figure 14.4 shows the effect on bending moment of variations in the radius ratio ϱ in typical cases. It is clear that ϱ does not have a marked effect.

Figures 14.5 and 14.6 show typical stress profiles at various stages of time. It is to be noted that represents the ratio of stress in the bend at various times to the maximum fibre stress in a similar straight pipe under the action of M_o assuming only elastic behaviour. They thus represent measures of SCF.

Figure 14.7 shows a typical variation in maximum SCF in time. The initial points are the corresponding SCFm's.

Obviously many more detailed results than these were generated from the program, but lack of space necessitates their omission.

14.7 The straight pipe

The creep relaxation of a straight pipe constrained to a fixed curvature can be secured from Section 8.2 as

$$m(\tau) = \int_0^{2\pi} \sin^2\xi \left\{1 + (n-1)(\sin\xi)^{n-1}\tau\right\}^{-\frac{1}{n-1}} d\xi$$

in closed form.

Results of such a calculation are shown in Figures 14.1, 14.2 and 14.3. Calculations for large λ indicate that as $\lambda \to \infty$ the results for a curved pipe tend to those of a straight pipe as expected.

14.8 Presentation of results : reference stresses

The so-called Reference Stress Method can be used to present the results of this analysis in a particularly

simple form.

The method was originally introduced for the steady creep of structures to eliminate uncertainty on an assumed value of a material parameter (in this case, for a Norton power law, n) see Appendix 6. The concept was extended to relaxation by J Spence and J Hult [81].

A uniaxial relaxation test started at the stress $\alpha_R \sigma_o$ has the form

$$S(\tau) = \{1 + (n-1)\alpha_R^{n-1}\tau\}^{\frac{-1}{n-1}} \qquad (14.14)$$

in the time scale τ for a power law of creep where $S(\tau)$ is the ratio of actual to initial stress in the test. The possibility exists of expressing the numerical results for load relaxation in a complex structure in the form of (14.14) i.e. in the form of a uniaxial test started at the "reference stress" $\alpha_R \sigma_o$. For a curved pipe this implies finding an α_R such that

$$m(\tau) \doteq \{1 + (n-1)\alpha_R^{n-1}\tau\}^{\frac{-1}{n-1}} \qquad (14.15)$$

In [81] it is suggested that this reference parameter could be the same as that obtained for the steady state of the associated deformation causing relaxation; this, however, shall be discussed in the next chapter in the context of an approximate method. Here values of are obtained by fitting data from the computer program to a curve of the form (14.15) using a least squares minimization technique.

Figure 14.8 shows values of α_R for various λ and n ; when these are back substituted into (14.15) less than 1% error results! The equivalent straight pipe value is also shown.

It is a property of most Reference Stress Methods that the value of α_R for given λ, ρ and ν is virtually

independent of n and thus a typical value can be used for all n. Thus if the relaxation of moment in a curved pipe of given dimensions is required the appropriate reference stress may be computed using Figure 14.8 and the result simulated from a uniaxial relaxation test started at this stress. But if the parameter α_R is to be used to predict the result using (14.15) a knowledge of n is required.

The above results are, in themselves, not of much consequence. Their importance lies in having accessible fairly accurate numerical values for a complex piping component to which simple approximate methods can be compared. It is the purpose of the following chapter to study this.

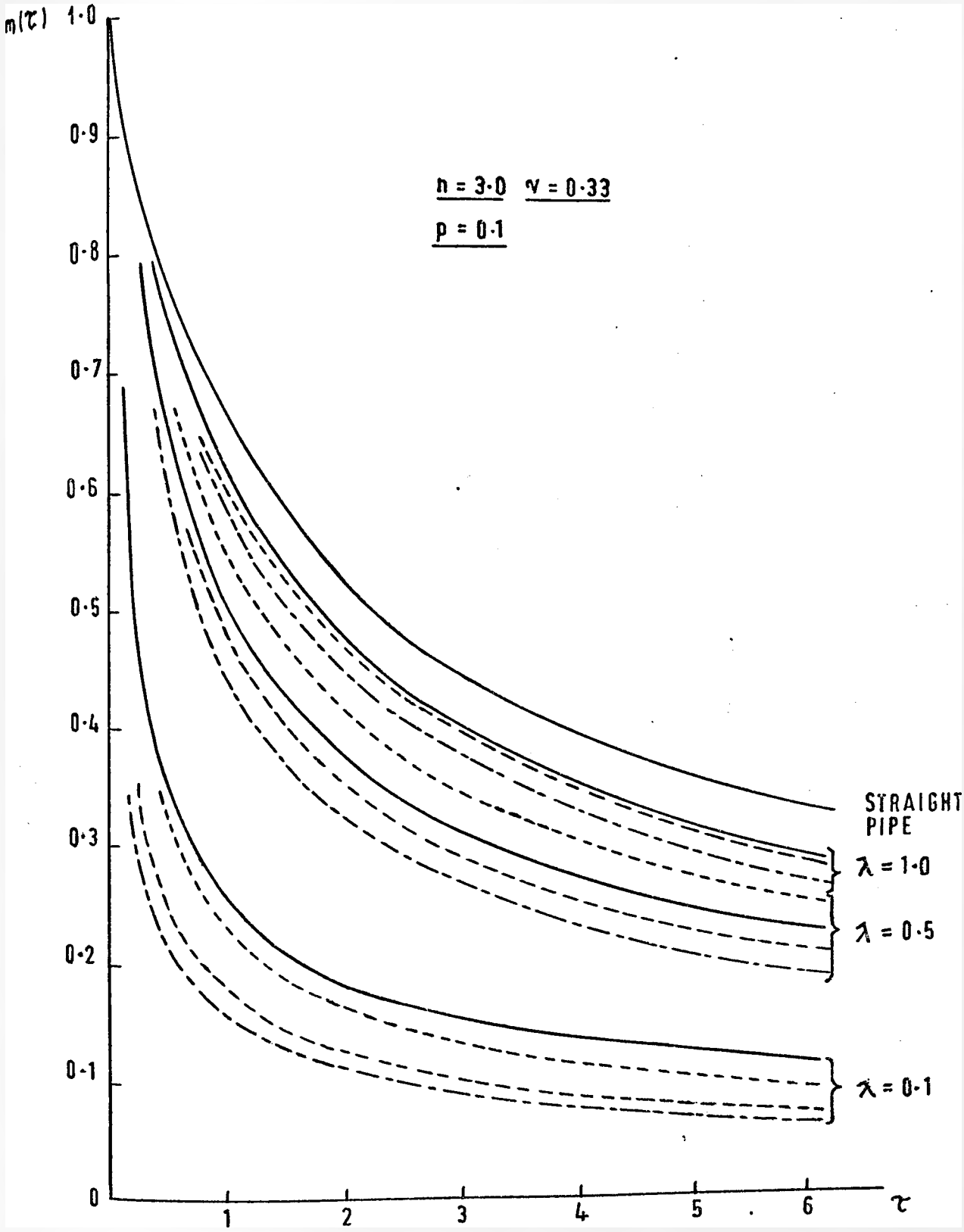


Fig 14·1 RELAXATION OF BENDING MOMENT : n = 3·0

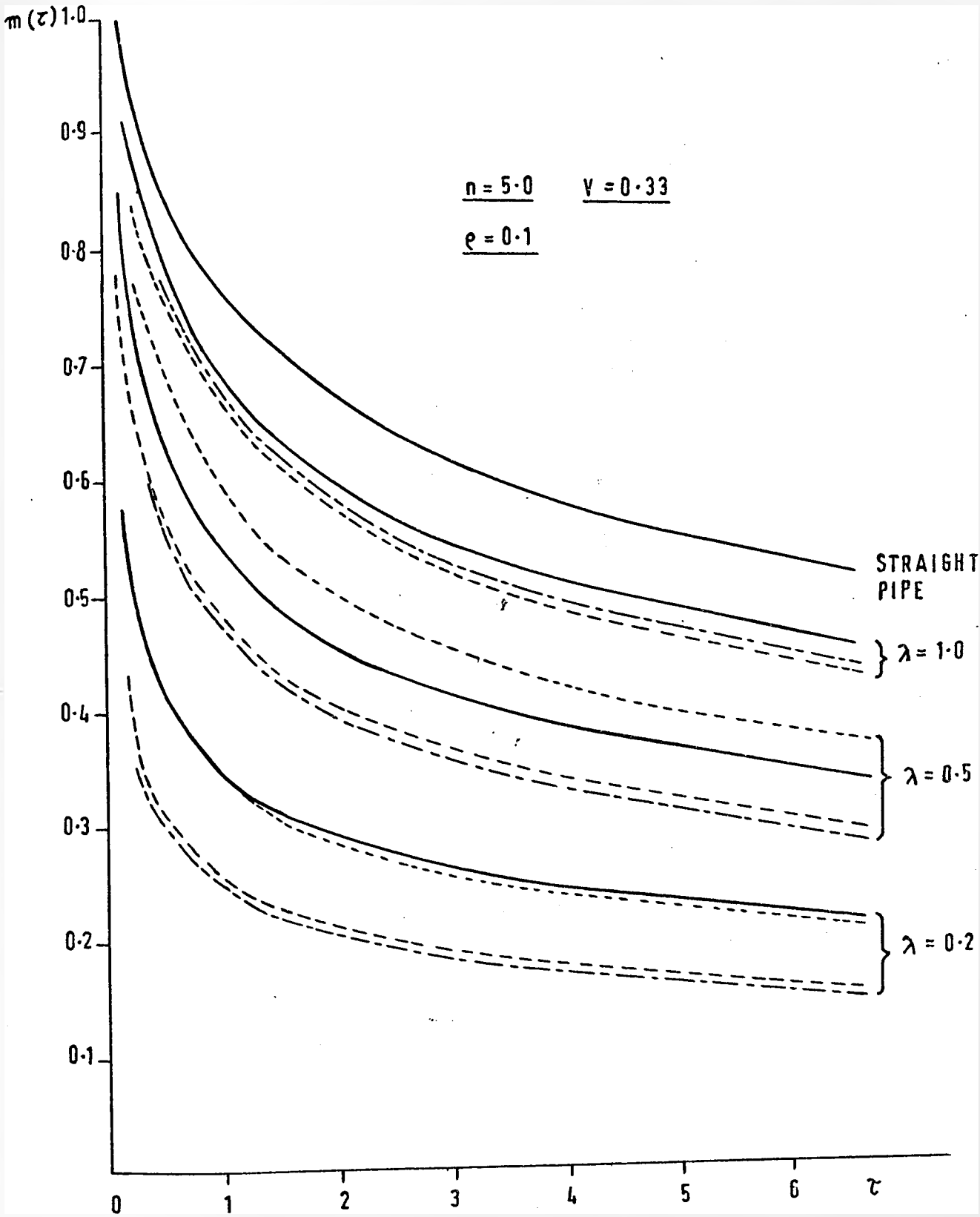


Fig 14·2 : RELAXATION OF BENDING MOMENT n=5·0

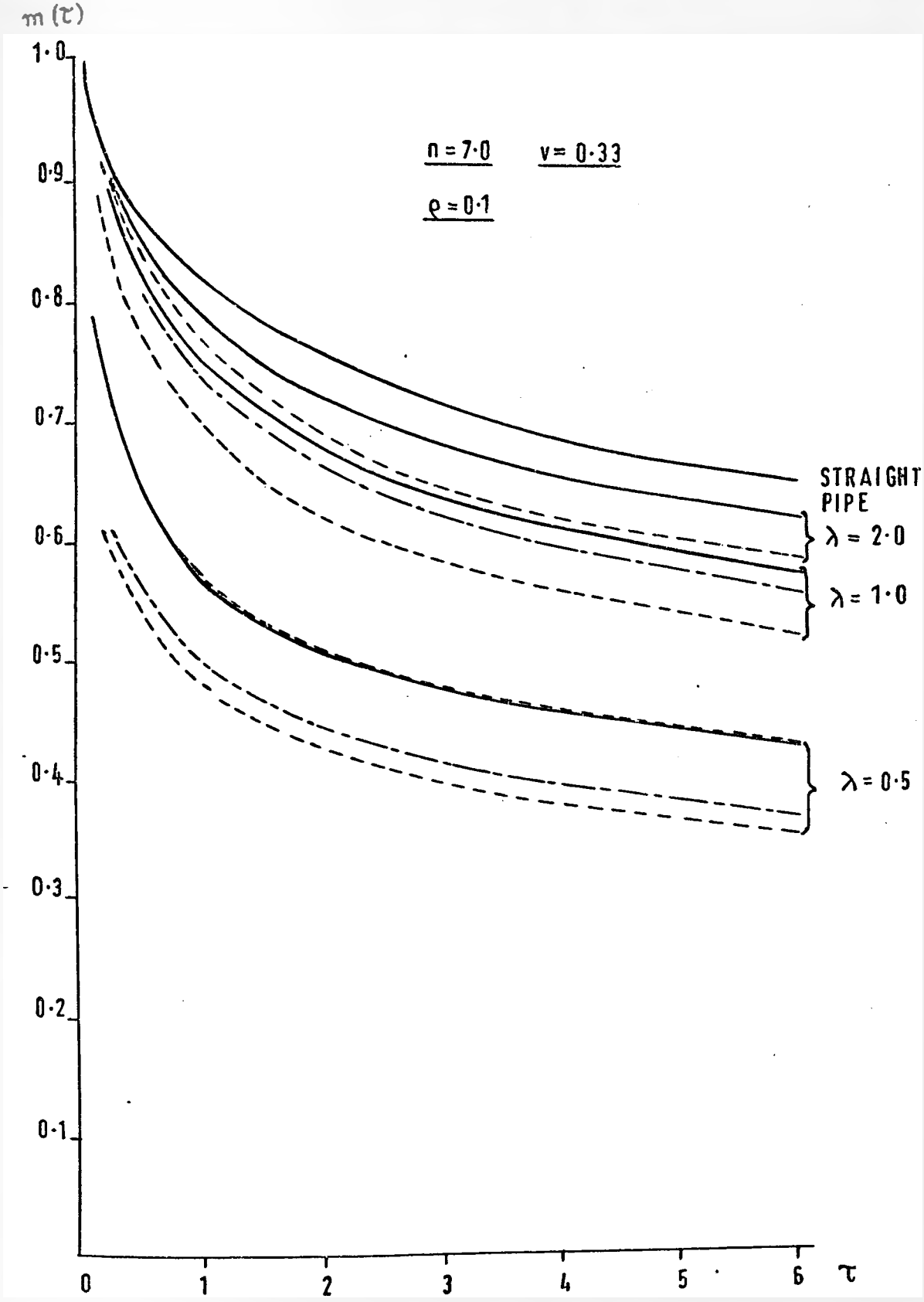


Fig 14·3 RELAXATION OF BENDING MOMENT n=7·0

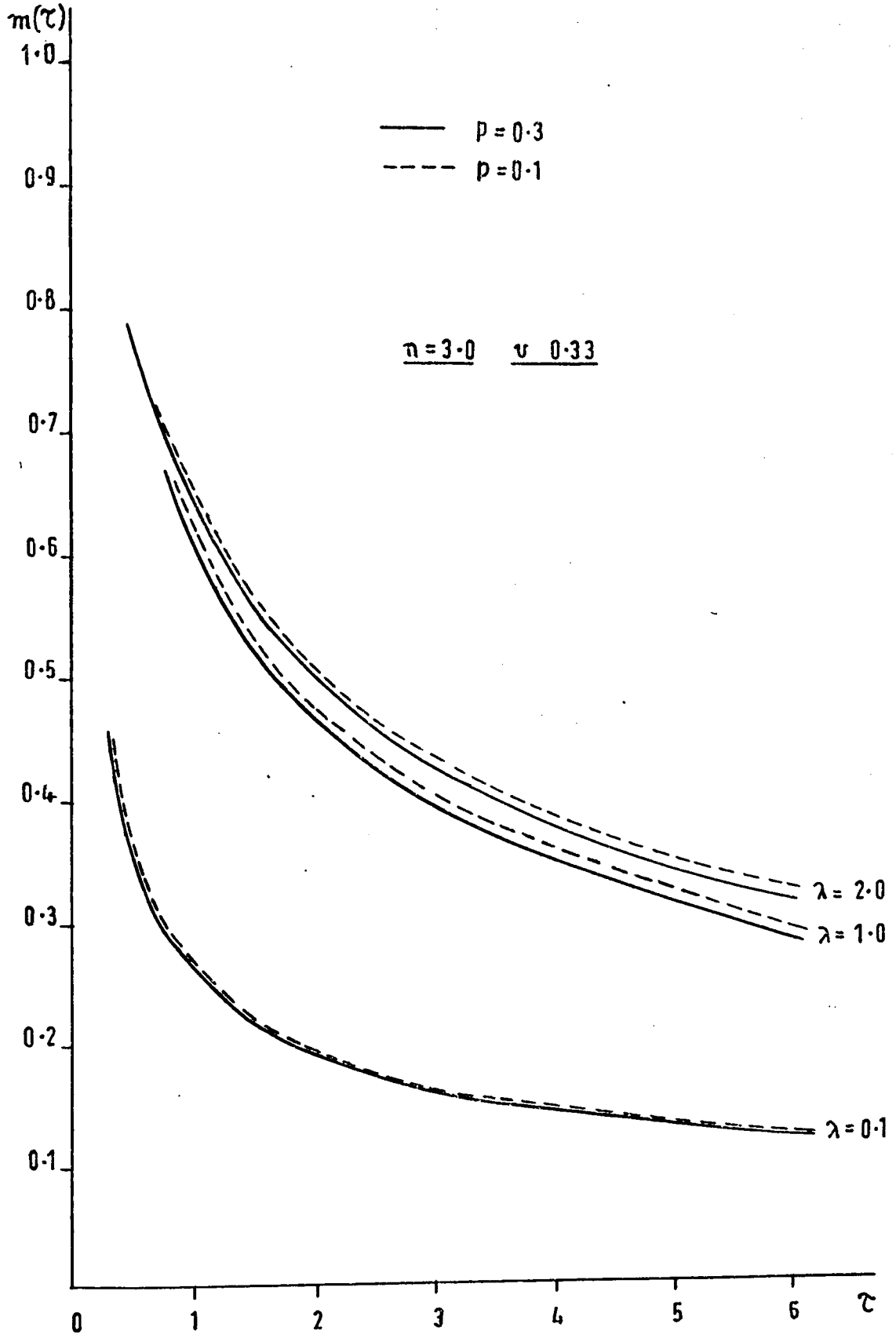


Fig 14·4 : EFFECT OF RADIUS RATIO ℯ

$\pi = 3{\cdot}0$ $\nu = 0{\cdot}33$

$\lambda = 0{\cdot}1$ $\rho = 0{\cdot}1$

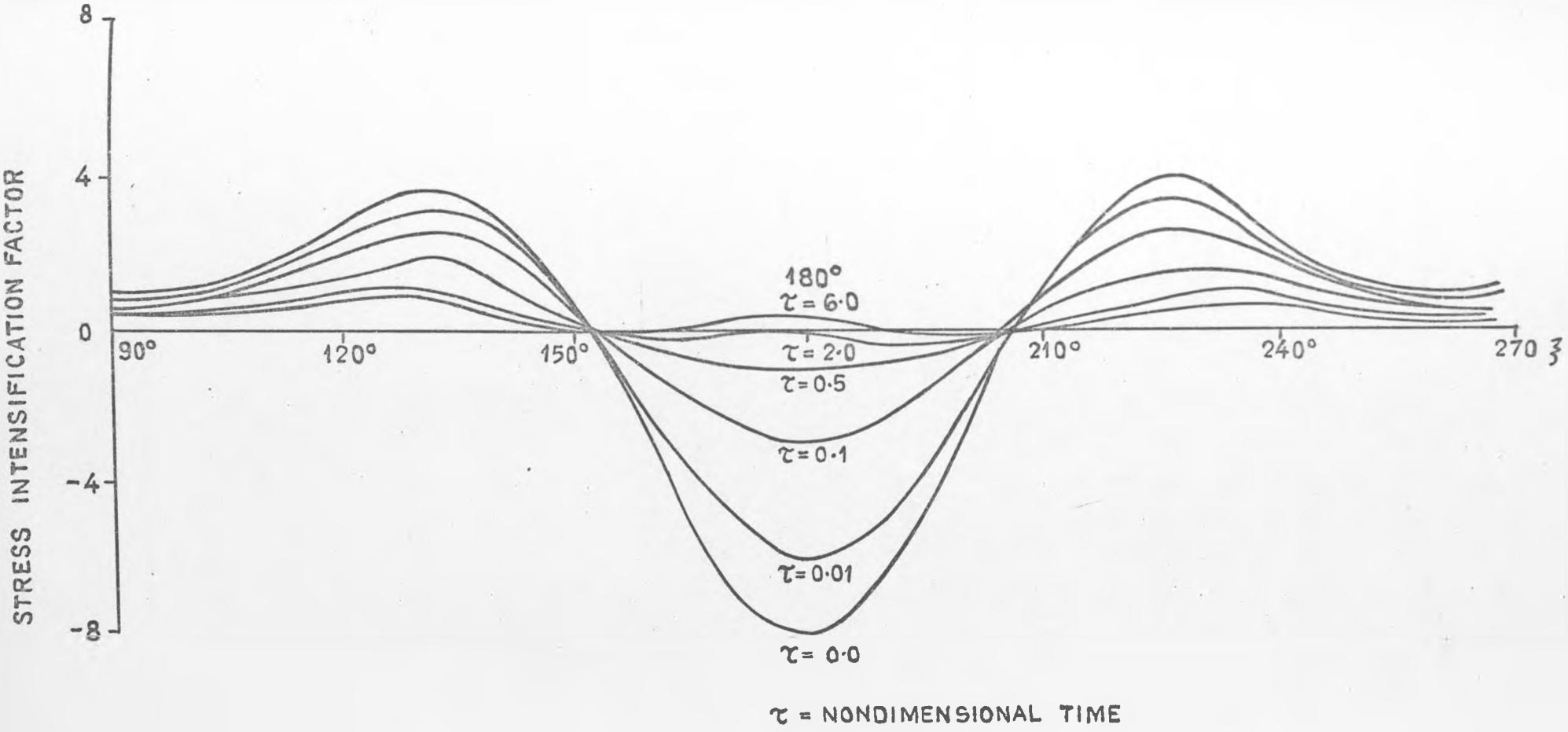


Fig 14·5 TYPICAL VARIATION OF CIRCUMFERENTIAL SCF IN TIME. INSIDE SURFACE

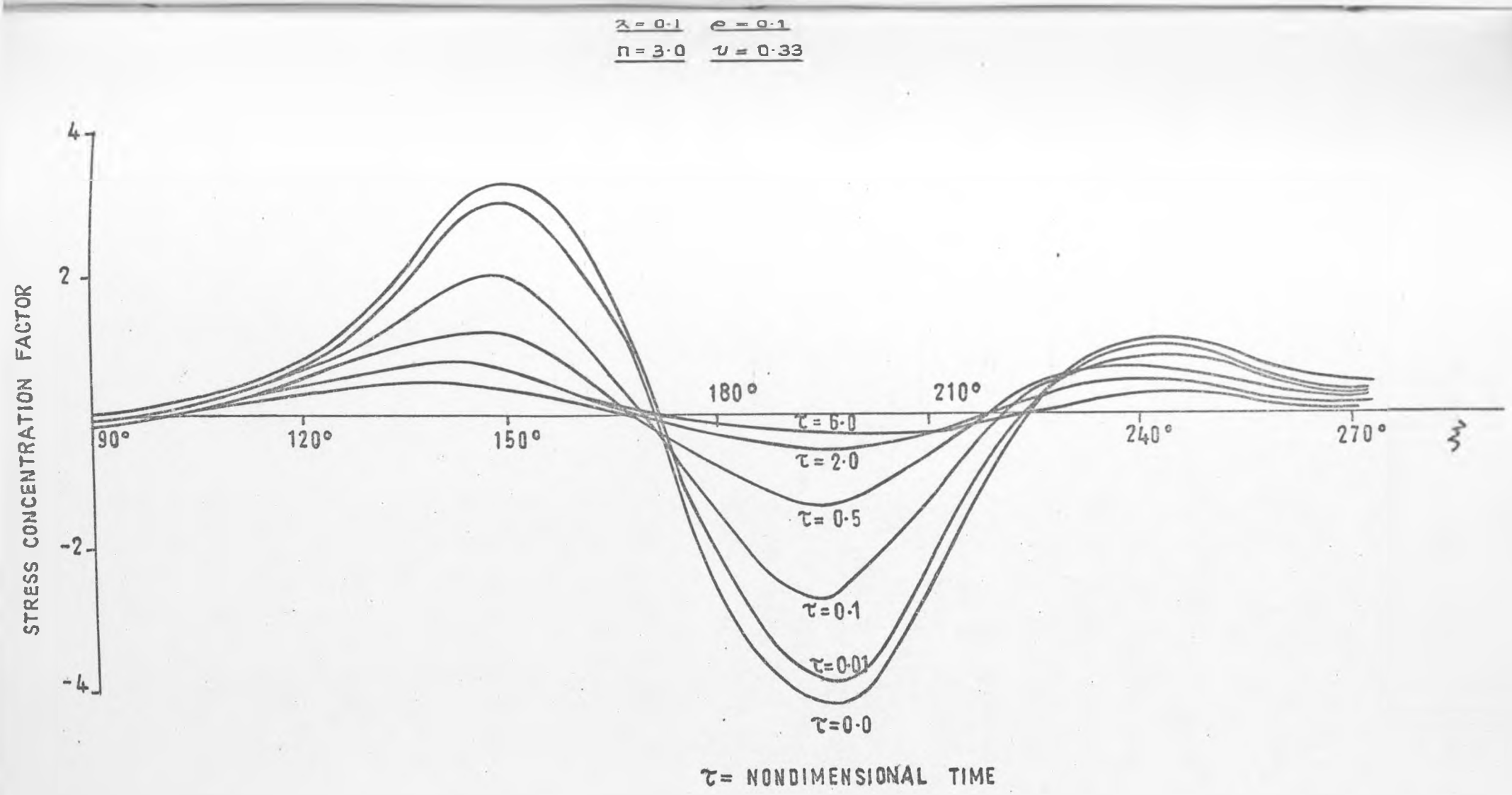


Fig 14·6 TYPICAL VARIATION LONGITUDINAL SCF IN TIME : INSIDE SURFACE

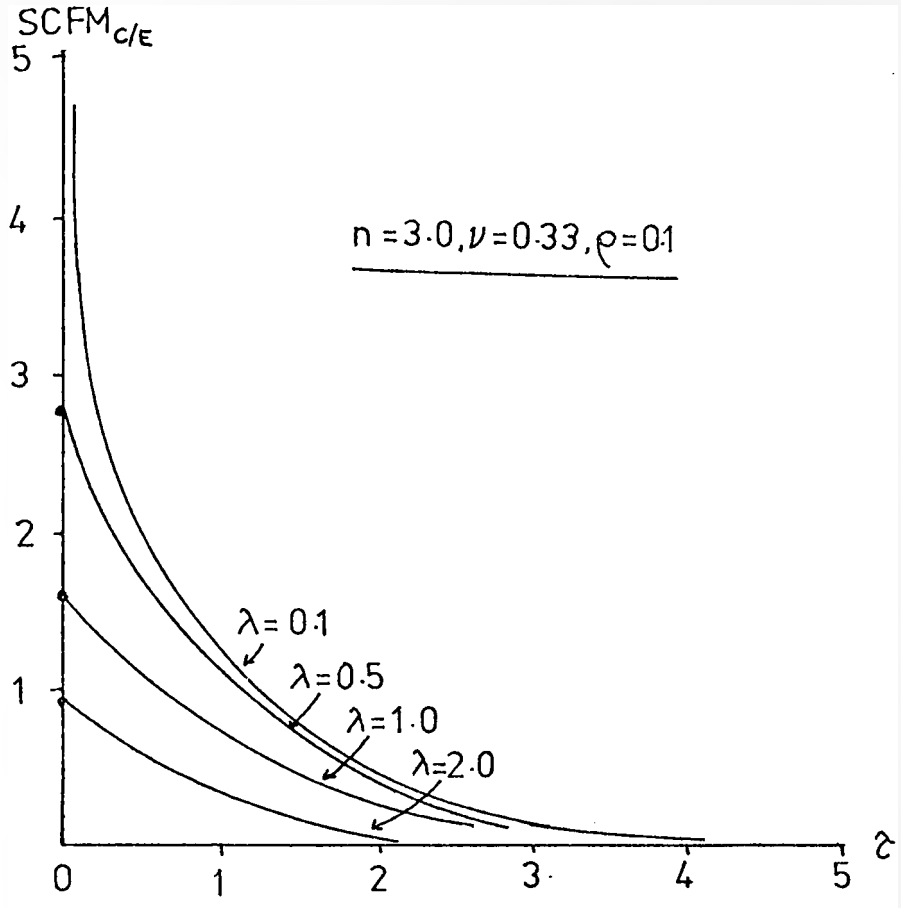


Fig 14·7 Typical relaxation of $SCFM_{c/E}$

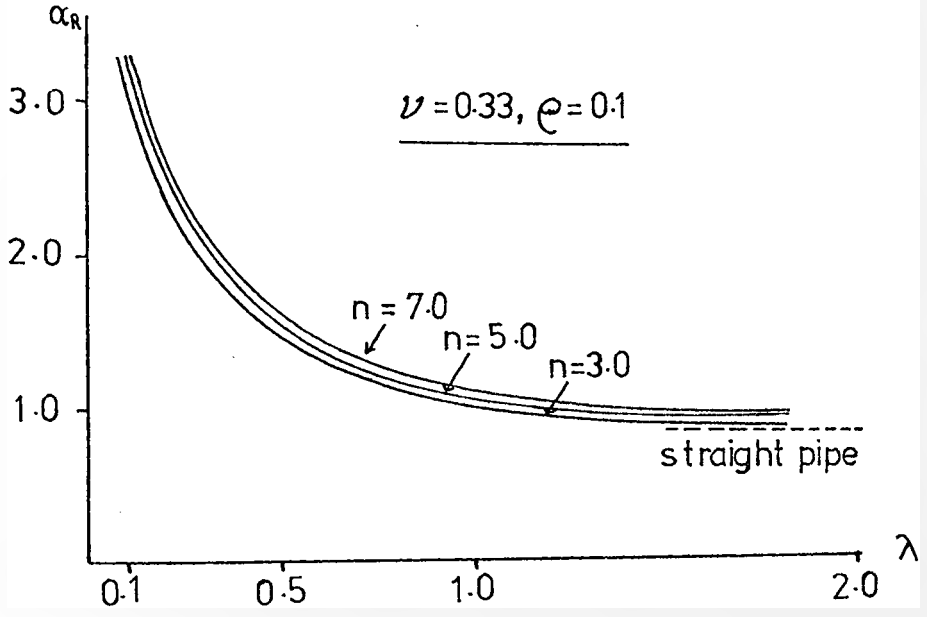


Fig 14·8 Reference parameter α_R

CHAPTER 15

APPROXIMATE METHODS FOR THE CREEP/RELAXATION OF A CURVED PIPE

The analyses which have been performed for the creep and relaxation of a curved pipe, although fairly interesting in themselves, are not in a form which is likely to be of use in complex piping system design. They nevertheless allow the verification of simple approximate methods which do have a certain potential in design. Two important possibilities which have been suggested for time dependent creep shall be examined here; super-position of states, and a simplified direct method due to L M Kachanov.

15.1 Creep of a curved pipe

Consider the problem outlined in Chapter 13.

15.1.1. Superposition of states

This technique approximates the total response of a structure by adding a pure elastic response to a pure creep response (Section 7.4). For constant applied loads (the second boundary value problem) this is identified as the superposition method of D L Marriott and F A Leckie [75]. Then

$$\gamma(t) \doteq \gamma_e + \gamma_c(t)$$

where γ_e is the initial elastic end rotation and $\gamma_c(t)$ is the "pure creep" end rotation. On changing to nondimensional quantities this implies that

$$K_{c/e}(\tau) \doteq K_e + K_c(\tau)$$

But a pure creep analysis in nondimensional time τ under constant applied loads gives a pure creep end rotation rate constant in time i.e.

$$K_{c/e}(\tau) \doteq K_e + \tau \dot{K}_c \qquad (15.1)$$

which is a straight line with initial point K_e and gradient

$\dot{K}_c$. It may be shown (c.f. Section 6.2) that in the time scale τ the pure creep stresses, and thus also the deformation rates, are asymptotically equivalent to the actual stress and deformation rates. Consequently

$$\frac{d}{d\tau} K_{cle}(\tau) \longrightarrow \dot{K}_c \qquad \tau \longrightarrow \infty$$

so that asymptotic values of $dK_{cle}/d\tau$ at large times can be used for $\dot{K}_c$. Results of plotting (15.1) are shown in Figures 13.1, 13.2 and 13.3 for comparison.

It should be noted that alternative values of could be taken from Spence [423] on identifying (Section 13.7.3)

$$\dot{K}_c = K_{ss} \times \left(\frac{\pi}{D_o}\right)^n$$

but this is unnecessary with the improved values produced here.

15.1.2. Kachanov's Method

Kachanov [2,85,243] has proposed a simplified direct method which requires only the initial elastic and asymptotic stress profiles, thus

$$S_{ij}(\tau) = S_{ij0} + c_{ij}(\tau)[S_{ijc} - S_{ij0}]$$
$$S_{\theta}(\tau) = S_{\theta 0} + c_{\theta}(\tau)[S_{\theta c} - S_{\theta 0}] \qquad (15.2)$$

with the conditions

$$c_{ij}(0) = c_{\theta}(0) = 0$$

on noting that in the time scale τ the asymptotic actual stress field is the pure creep stress field, constant in time. In fact Kachanov chooses

$$c_{ij} \equiv c_{\theta}$$

Equations for c_{ij} and c_{θ} can be obtained by the method of weighted residuals, in particular Galerkin's method (Section 7.1) or by using one of the reduced variational principles. Then K_{cle} can be computed from (13.17). An interesting alternative is to use Castigliano's Theorem

for the nonsteady creep of structures (Appendix 7), which implies that, after suitable manipulation

$$\frac{d}{d\tau}K_{c/e} = \frac{1}{2\pi}\int_{-1/2}^{1/2}\int_{0}^{2\pi}\{(S_\xi^2 - S_\xi S_\theta + S_\theta^2) + (S_\xi \dot{S}_\zeta - \nu(S_\xi \dot{S}_\theta + S_\theta \dot{S}_\xi) + S_\theta \dot{S}_\theta)\}\, d\xi\, d\zeta$$

which in fact is another formulation of (13.17) and gives identical results.

Both procedures were programmed for the ICL 1904S. The results computed agreed very well with the computed solution giving less than 1% error in most cases in $\frac{d}{d\tau}K_{c/e}$ They are therefore not graphed.

15.2 Relaxation of a curved pipe

Consider the problem outlined in Chapter 14.

15.2.1. Superposition of states

This was identified for the relaxation problem by J Spence and J Hult [81] although it had unwittingly been used in earlier pipework problems (see next Part). A constant end rotation γ_o is applied to the curved pipe resulting in a bending moment $M(t)$ due to the response to creep. Consider the alternative problem of a curved pipe creeping under the action of the variable load $M(t)$. Similarly to constant load situations it is assumed that the total resultant end rotation is composed of a pure elastic part $\gamma_e(t)$ added to a pure creep part $\gamma_c(t)$ i.e.

$$\gamma_o = \gamma_e(t) + \gamma_c(t) \qquad (15{\cdot}3)$$

For the curved pipe

$$\gamma_e(t) = \left(\frac{M(t)}{\pi h b^2}\right)\frac{K_e}{\varrho}\frac{1}{E} \quad \Theta$$

$$\frac{d}{dt}\gamma_c(t) = \left(\frac{M(t)}{\pi h b^2}\right)^n \frac{\dot{K}_c}{\varrho}\, g(t) \left(\frac{\pi}{D_o}\right)^n \quad \Theta$$

Since γ_o is constant substitution into (15.3) yields the equation for m

$$\frac{dm}{d\tau} + \alpha_s^{n-1} m^n = 0 \qquad (15{\cdot}4)$$

where

$$\alpha_s^{n-1} = \dot{K}_c / K_e \times (\pi / D_o)^n$$

The initial value problem (15.4) with $m(0)=1$ has solution

$$m(\tau) = \{1 + (n-1)\alpha_s^{n-1}\tau\}^{\frac{-1}{n-1}} \qquad (15.5)$$

Results of plotting (15.5) are shown in Figures 14.1, 14.2 and 14.3, for comparison; values of K_e can be taken from Section 13.4 with values of $\dot{K}_c$ chosen in a similar manner to Section 15.1.1, in fact with this choice there is less than 2% error from the computed results and are therefore not shown. It is also interesting to use upper and lower bound values of $\dot{K}_c$ from [423]. Further it is instructive notice that (15.5) is already in the form of a uniaxial relaxation test started at $\alpha_s \sigma_o$ (Section 14.7); values of α_s for the upper and lower bound $\dot{K}_c$ and asymptotic values are shown in Table 15.1 with the least squares fit α_R.

15.2.2. Kachanov's Method

The approximation of L M Kachanov assumed in Section 15.1.2 may also be assumed in this situation observing that the asymptotic pure creep stress field is zero. Then if, simply

$$1 - c_s(\tau) = 1 - c_o(\tau) = p(\tau)$$

the unknown scalar function $p(\tau)$ can be obtained from Galerkin's method, or by using one of the reduced variational principles [81] so that

$$m(\tau) \doteq p(\tau) = \{1 + (n-1)\alpha_K^{n-1}\tau\}^{\frac{-1}{n-1}} \qquad (15.6)$$

where

$$\alpha_K^{n-1} = \frac{\int_{\pi/2}^{3\pi/2}\int_{-1/2}^{1/2} (S_{ij0}^2 - S_{ij0}S_{\infty\infty} + S_{\infty\infty}^2)^{\frac{n+1}{2}} d\xi\, d\zeta}{\int_{\pi/2}^{3\pi/2}\int_{-1/2}^{1/2} (S_{ij0}^2 - 2\nu S_{ij0}S_{\infty\infty} + S_{\infty\infty}^2) d\xi\, d\zeta}$$

Results of calculating (15.6) are shown in Figures 14.1, 14.2 and 14.3 for comparison. Undoubtedly the

approximation underestimates the moment improving with increasing λ.

Once again (15.6) is already in the form of a uniaxial relaxation test started at the stress $\alpha_\kappa \sigma_o$; values of α_κ are shown in Table 15.1.

15.3 Discussion and conclusions

The most remarkable aspect of the above approximate methods, and in particular superposition of states, is the simplicity in derivation and ultimate accuracy of the results. Furthermore they can be easily represented in terms of a reference stress" - recent experimental investigations at the National Engineering Laboratory by J Henderson [435] suggest that this may very closely describe the actual response! The important point is that it is fairly easy to extend the method of superposition of states to more complex piping systems and here the method has been justified for the curved pipe. This point shall be discussed further in the next part.

TABLE 15.1

REFERENCE STRESS PARAMETERS

a. Computed from exact method, α_R

b. Superposition of States using lower bound creep flexibility factors, α_s

c. Superposition of States using upper bound creep flexibility factors, α_s

d. Kachanov's method, α_κ

λ		n		
		3.0	5.0	7.0
0.1	a	3.0	-	-
	b	3.0	-	-
	c	4.4	-	-
	d	4.0	-	-
0.5	a	1.25	1.29	1.31
	b	1.10	1.17	1.19
	c	1.49	1.48	1.49
	d	1.34	1.46	1.56
1.0	a	0.93	0.96	0.97
	b	0.75	0.80	0.82
	c	1.01	0.99	1.05
	d	0.95	1.01	1.05
2.0	a	0.87	0.88	0.90
	b	0.63	0.67	0.69
	c	0.85	0.85	0.84
	d	0.87	0.91	0.94
straight		0.82	0.83	0.84

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PART B

REFERENCES : PART B

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PART C

HIGH TEMPERATURE PIPING SYSTEM ANALYSIS

Until recently piping system design was justified in terms of purely elastic analysis principles. In evaluating the safety and reliability of high temperature systems, inelastic behaviour must be admitted and the possibility of the accumulation of strain in highly stressed locations accounted for. This Part provides a survey of the current situation in elastic analysis and the state of time dependent inelastic methods as applied to pipework. A possible design procedure is suggested based on a step wise approach to the overall design. Based on observations on the behaviour of components obtained in the previous Part from the rational approach an efficient flexibility type method is recommended as a primary method backed up by certain finite element type analyses. Desirable future developments are briefly discussed.

CHAPTER 16

BRIEF REVIEW OF LINEAR ELASTIC TECHNIQUES

The calculation of stresses in a piping system has been the subject of considerable attention and there are several approaches available - graphical, manual and analytical - which focus on the linear elastic behaviour of the system. What follows provides a brief review of the current situation and discusses the applicability of finite element methods.

16.1 Basic methods available

Probably the most well known and comprehensive technique is the so-called "matrix method" [436,437,438] which derives from classical methods [439,440] and is eminently suitable for computer application. Several computer programs are now available which deal adequately and cheaply with the flexibility analysis of many types of pipework systems [441,442,443]. Such matrix applications of flexibility analysis developed simultaneously with similar approaches to complex structures culminating in the well known finite element method.

In the finite element method a complex structure is replaced by a finite number of basic shapes and in principle any of the large finite element program systems, such as EPACA, COPATS, STARS, NASTRAN, ANYSYS, BOFORS or TESS/STAG21/BERSAFE (see [441] or [444] for recent summaries or any of the Conferences on Structural Mechanics in Reactor Technology proceedings) which can use doubly curved thin shell finite elements, could deal with a piping network. Indeed such an approach has been used to analyse some simple configurations and components in [333,334,336,433] for example. This "complete shell finite element method" has

been widely accepted as an important technique of pressure vessel analysis where single components have to be studied. However, in piping analysis the large number of components prohibits its effective use. In order to overcome this difficulty a simpler finite element model has been suggested, called the "reduced shell finite element method" [335,445]. This has been implemented in the MARC program [432,446]. Nevertheless, although the "reduced shell finite element method" enjoys the advantages of the "complete shell finite element method" and the simple "flexibility (matrix) method", it also suffers from their respective disadvantages.

16.2 A comparison of assumptions

The pipe stressing problem is particularly suited to computer solution and the scope of present day computer programs is large. However, the analytical techniques on which they are based are all approximate to a greater or lesser degree.

The "flexibility (matrix) method" is founded on the primary assumption that plane cross sections remain plane after bending, the implication being that a straight pipe can be treated using elementary beam theory and for this reason is often called a "beam" type method. The method secondly assumes that a curved pipe is incorporated as an equivalent straight pipe together with a suitable "flexibility factor". That is, if the flexibility matrices are derived from energy considerations [436-439] the strain (or complementary) energy of a curved pipe under applied loading is equivalent to that of a similar straight pipe under the same loadings multiplied by appropriate "flexibility factors", which, by definition shall depend on material as well as geometric parameters.

The "complete shell finite element method" utilises either an assemblage of small flat plate elements with sufficient degrees of freedom to adequately describe the deformation or curved, three dimensional "isoparametric" shell elements - in particular doubly curved elements for bends, or single curved "shell of revolution" type elements for straights. It is the best mathematical (and usually, physical) model of the complex interactive geometry of a pipe network, and easily the most versatile since no assumptions other than the usual finite element ones are required.

The "reduced shell finite element method" uses either a number of three dimensional simple beam elements or a shell of revolution element together with a number of beam elements to model a straight section of pipe; such elements usually embody a "plane sections remain plane" type assumption similar to the "flexibility method". Bends are treated as a number of so-called "Von Karman Pipe Bend Elements" which are basically "shell of revolution elements" with extra degrees of freedom to cope with in-plane and out-of-plane bending tied together with simple beam elements [335]. This element is a pure bending element, that is it is implicitly assumed that the same bending moment acts circumferentially over the element. This assumption also is similar to the implicit pure bending assumption of "flexibility methods". However the "flexibility method" allows for warping and the nonlinear effects of pressure etc. by suitable redefinition of the flexibility factors, whereas the current "reduced shell finite element method" cannot yet cope (some progress is however, reported in [335]).

16.3 A comparison of elastic techniques: The "Best" method

The basic criterion which it is felt should be met by the "best" method of elastic pipework analysis is that it should be versatile while retaining a simplicity in use and cheapness. The "flexibility method" amply satisfies this criterion in being able to cope with a variety of loading conditions (pressure, deadweight, thermal etc) and with highly involved components, such as elbows, through the use of appropriate "flexibility" factors; it may also be extended to deal with problems of stability [442,443]. The most serious disadvantage of the "flexibility method" is the crude model it uses - nevertheless this model has proven acceptable through experience. A less serious disadvantage is its dependence on the generation of suitable "flexibility" factors, yet the most important factors are readily available.

Finite element techniques and their computer application to pipework are in their infancy. In particular the "complete shell finite element method", while being by far the best and most versatile system model, is singularly difficult to apply and, significantly expensive. The systems which may be analysed are practically limited to overtly simple configurations. In plain words the "complete shell finite element method" cannot yet handle even moderately complex piping systems; however, in fairness this is a criticism of current digital computer development rather than the method itself. It should be noted that this is quite different from the state of current pressure vessel analysis. On the other hand the "reduced shell finite element method" while being an excellent compromise between the best of the "flexibility" and "complete shell" methods also combines the worst of these

methods - the relative crudity of model, and the excessive cost limiting the size of configuration.

In conclusion for elastic behaviour, since adequate flexibility factors are available, the current "best" method is the "flexibility (matrix) method" of pipework analysis. However, in some cases an additional method, such as the "complete shell finite element method" could be justified to calculate stresses and strains locally when local bending is expected, for example in the bends, tee or anchor intersections, etc. after restraining forces have been calculated using the simpler method.

16.4 Summary

Pipework design is in a permanent state of gradual change. While currently the best technique is the "flexibility (matrix) method" it should be remembered that the biggest steps forward in the past few years have been in computer techniques and in basic research into detailed analyses of piping components so that the finite element techniques may still, though not yet, come into their own for the improved analysis of complete pipe networks.

CHAPTER 17

CURRENT INELASTIC PIPEWORK ANALYSIS

Design procedures based on linear elastic flexibility have advanced to a state where a great degree of sophistication can be achieved in computer application. Little consideration has been given to inelasticity except in special cases; indeed none of the existing techniques for such an analysis are yet sufficiently developed. Powerful methods, primarily based on finite elements, for the analysis of vessel-type structures have been developed to a point where virtually any geometry, loading condition and material behaviour may be treated. This is not the case for pipework. A finite element analysis of a large diameter complex primary piping system for high temperature reactor service is simply not feasible economically to the detail and accuracy of a pressure vessel analysis; in any case the aim is to produce a flexible system.

This present Chapter surveys the field of inelastic pipework analysis in broad divisions and is not concerned with specific numerical results. In summary proposals are made fo an ideal, piping oriented design procedure based on several discrete steps each of which may employ different analysis methods pointing out the area which the writer feels should be investigated.

17.1 Simplified system analysis

The early analysis of E L Robinson [447] of simple systems unwittingly used superposition of states (Section 7) and included only straight pipes. Despite this it brought out several points which were important for future developments. W Gorczynski [448,449] followed essentially the same approach. Later J Spence [450,451] included the

effect of bends through the use of his flexibility factors [423], although there is a slight error in the analysis. Recently T Johnston at UKAEA [456] has utilized Spence's approach on a more complex problem. A similar though more general method to that of Spence was suggested by E C Rodabaugh and G H Workman [452]. Unfortunately some fundamentally wrong assumptions were made which invalidated their analysis [453] (in particular decoupling the effects of combined loading). The problems involved are discussed in [451].

Finally Kachanov's method for relaxation has been used on some simple configurations, ignoring bends, in relaxation [457] and rupture [458]. The basis of the method is that all stresses in the system shall exhibit relaxation in the same manner being merely of the same shape as the initial elastic stress field multiplied by a function of time. For this reason alone it is unsuited to more complex piping systems despite its nominal requirements and encouraging results (Chapter 15).

17.2 System analysis using the reduced shell finite element method

An accurate analysis of a complete pipework system in creep using the "complete shell finite element method" is out of the question for both the present and the immediate future. Consequently several investigators have turned to the "reduced shell finite element method" for system analysis in creep - in particular the MARC code [335,432, 445,446]. Although such an approach is fairly accurate and versatile it still suffers from the serious disadvantage of excessive computer time (23 hours in [446]!) and perhaps less serious disadvantage of crude handling of the inelastic analysis through the use of inefficient time

integration via the initial strain algorithm (Section 7)

An alternative simple finite element analysis has been suggested by G H Workman and E C Rodabaugh [454] whose intent was to account for the inelastic behaviour in an "initial strain" sense by solving a succession of elastic flexibility problems. The analysis, however, suffers from two fundamental misconceptions [455] in the decoupling of combined loads and the attempt to relate initial strains in the system to resultant forces. In particular, the authors confused the superposition of state method into exact time integration and muddled the creep response. This is discussed in Appendix 8 together with Rodabough's and Workman's reply from [455].

17.3 Summary : The need for a simplified analysis

Whereas the "complete shell finite element method" is not feasible the "reduced shell" method is a great improvement but still very costly for large system analysis. On the other hand superposition of states allows a simple "flexibility" method in the spirit of classical techniques while still accounting fairly well for the important features of the creep response. The fact and basis of a general "flexibility" method based on superposition of states, with projections for the type of data required shall be discussed in the following chapter.

Nevertheless it is fairly obvious from the foregoing synoptic review that very little quantitative information is provided to aid the designer in the choise of a suitable method for the analysis of piping systems operating in the creep regime. Thus there is a growing demand for an essential evaluation of the possibilities.

The size of modern configurations alone necessitates a

simple technique and appears to eliminate even the "reduced shell finite element method" as a possibility for overall analysis. Furthermore the requirement of projected forcasts for the operating life of involved piping systems over a long period of time conclusively eliminates such methods as a viable proposition for design by computer.

A flexibility type method using simple superposition of states is relatively cheap and simple to apply. The results of Spence and Hult [81] on straight pipes and those of Chapter 15 on the curved pipe significantly increase confidence in the use of such a method (and in this sense the analyses are justified). Thus for a general system, although it may not accurately predict the magnitude of field variables, it nevertheless is expected to highlight the overall trend. For this reason alone it is necessary to choose the method as a likely mode for design.

In conclusion it must be remembered that where the integrity of an important piping system is in question it would seem dangerous, if not foolhardy, to choose any one method over another in the final analysis - accurate auxiliary checks must also be an important consideration.

17.4 Proposals for a possible procedure of inelastic pipework design based on current methods

Bearing the above comments in mind the following IDEAL SITUATION is visualised:

(i) Analyse the complete system using a PRIMARY METHOD - namely the simple flexibility approach with superposition of states. Such a method should be quick, cheap yet retain sufficient accuracy.

(ii) On a basis of the results of (i) isolate and reanalyse a (small) loop of interest using a SECONDARY (BACKUP) METHOD - namely the "reduced shell

finite element method" with exact time integration based on the rational approach to creep proposed herein. Such a method should be capable of accurately modelling the behaviour of the loop.

(iii) On the basis of (i) and (ii) isolate components of interest and reanalyse using TERTIARY (AUXILIARY) METHODS. It is possible at this stage to include more detailed material behaviour such as the possibility of rupture and fracture so that the component studies presented herein and the limit analysis techniques previously mentioned could be used. Such Tertiary methods would involve "complete shell finite element" methods coupled with auxiliary checks using alternative procedures.

This rationalisation of the current situation eliminates the need to analyse whole systems or large loops, with enormous cost, using finite element methods. The cheap, quick flexibility method focuses interest on the important points. The safeguards, as well as the speed and economics of such a design by computer approach to the ideal situation is obvious.

17.5 Recommendations and a future development

Although current knowledge is sufficient in principle to cope with the more usual configurations and behaviour, the requirements of the design procedure for the ideal situation have yet to be realized in total. Several facets remain to be cleared up:

(a) Suitable elastic and creep flexibility factors should be generated to cope with the general case of combined loading and constraints. A review of the current situation was given in Chapter 12.

(b) A good curved pipe finite element needs to be developed

to cope with twisting etc. and the nonlinear effects of internal and external pressure.

(c) A more mathematically sophisticated approach to the exact time integration needs to be incorporated into the finite element codes. This of necessity would be achieved via the rational approach to creep mechanics proposed here.

The above suggested design procedure is based on currently available methods of analysis. The situation would be improved if a simple flexibility type method existed which did not rely on superposition of states. Is such an approach possible? The answer, it seems to the writer, is in the affirmative. It would be supplied by the rational basis of this thesis. Approximations to the creep response are unnecessary; all that are needed are practical approximations to the residual operators appearing in the redistribution and evolution of deformation equations. Since these arise from elastic considerations alone the approximations need only be investigated from a purely linear, time independent elastic standpoint. In particular they may be determined from simple elastic flexibility (beam type) matrix methods....the subject is under consideration.

CHAPTER 18

OUTLINE OF A SIMPLIFIED THEORETICAL BASIS

The analysis of complex piping systems operating at high temperatures in the creep regime is in its infancy. The preceding chapters attempted to identify the factors which must be taken into consideration in the stress analysis as regards applicability and adequacy as criteria for design by computer. A possible design procedure was outlined which could be implemented using existing information coupled with some development work. This relied on a suitable Primary Method based on the approximation of superposition of states. It shall be shown here how this may be tackled by re-examining the classical matrix methods [436-438].

For convenience a single run of pipe, anchored at some point O, with the other end, denoted by A, shall be considered (Figure 18.1). There is no essential difficulty in including multi anchor runs of pipe and distributed load.

Points on the pipe run are described by a fixed Cartesian frame (x,y,z) centred at O or by s the arc length. The displacement $\underset{\sim}{\Delta}$ at an arbitrary point S of the run is given by

$$\underset{\sim}{\Delta} = (\delta_x , \delta_y , \delta_z , \gamma_x , \gamma_y , \gamma_z)^T$$

In the Cartesian frame, where δ is an extension and γ a twist.

Similarly the force $\underset{\sim}{F}$ at S is

$$\underset{\sim}{F} = (F_x , F_y , F_z , M_x , M_y , M_z)^T$$

where F is a load and M moment. However, for a particular point S of the run it is more usual to describe the "rotated loading vector" $\underset{\sim}{F}^r$

$$\underset{\sim}{F}^{r} = (F_T, F_N, F_B, M_T, M_N, M_B)^T$$

given in terms of the tangential unit vector $\underset{\sim}{T}$ at S in the direction of increasing s, the normal unit vector $\underset{\sim}{N}$ at S and the binormal unit vector $\underset{\sim}{B}$ at S, normal to the plane containing S. This rotated loading is related to $\underset{\sim}{F}$ by the identity

$$\underset{\sim}{F}^{r} = [R_I]\underset{\sim}{F} \qquad (18.1)$$

with

$$[R_I] = \begin{bmatrix} [R_{II}] & [0] \\ [0] & [R_{II}] \end{bmatrix} \qquad [R_{II}] = \begin{bmatrix} \cos(T,x) & \cos(T,y) & \cos(T,z) \\ \cos(N,x) & \cos(N,y) & \cos(N,z) \\ \cos(B,x) & \cos(B,y) & \cos(B,z) \end{bmatrix}$$

where

$$\underset{\sim}{T} = \begin{bmatrix} \cos(T,x) \\ \cos(T,y) \\ \cos(T,z) \end{bmatrix} \quad \underset{\sim}{N} = \begin{bmatrix} \cos(N,x) \\ \cos(N,y) \\ \cos(N,z) \end{bmatrix} \quad \underset{\sim}{B} = \begin{bmatrix} \cos(B,x) \\ \cos(B,y) \\ \cos(B,z) \end{bmatrix}$$

with respect to the Cartesian frame, the components being the direction cosines of the respective vectors. The matrix $[R_I]$ is orthogonal

$$[R_I]^T[R_I] = [R_I][R_I]^T = [I]$$

where $[I]$ is the unit matrix.

18.1 Transfer relations

The identity (18.1) allows the transference of the representation of the load from one coordinate system to the other. In the following development it shall be required to transfer loads from one point of the pipe run to another.

Let P,Q be arbitrary points on the run. Let a loading $\underset{\sim}{F}_P$ be applied at P and let $\underset{\sim}{F}_Q$ be the effect of $\underset{\sim}{F}_P$ at Q. From equilibrium alone it may be shown that

$$\underset{\sim}{F}_Q = [T_Q^P]\underset{\sim}{F}_P \qquad (18.2)$$

where

$$[T_Q^P] = \begin{bmatrix} [I] & [0] \\ [t_Q^P] & [I] \end{bmatrix} \qquad [t_Q^P] = [t_Q] - [t_P] \qquad [t_S] = \begin{bmatrix} 0 & z_s & -y_s \\ -z_s & 0 & x_s \\ y_s & -x_s & 0 \end{bmatrix}$$

The matrix $[T_Q^P]$ is called a "transfer matrix" and has the properties

$$[T_o^Q][T_Q^P] = [T_o^P]$$

and is easily invertible

$$[T_Q^P]^{-1} = \begin{bmatrix} [I] & [0] \\ -[t_Q^P] & [I] \end{bmatrix}$$

18.2 The relaxation problem

Consider the following problem:

"Let a system of forces $\underset{\sim}{F}_{A0}$ be applied at the point A of the pipe run which deforms elastically giving a displacement $\underset{\sim}{\Delta}_{A0}$ at A. Let this displacement be fixed. The pipe run then expands inelastically in the form of creep and the forces $F_A(t)$ at A relax in time. The problem is to determine $F_A(t)$ such that $F_A(0) = F_{A0}$."

This shall be achieved via the approximation of superposition of states. The total displacement at A for all time is thus supposed to be composed of a pure elastic part, added to a pure creep part, namely

$$\underset{\sim}{\Delta}_{A0} = \underset{\sim}{\Delta}_A^e + \underset{\sim}{\Delta}_A^c \qquad (18{\cdot}3)$$

But $\underset{\sim}{\Delta}_{A0}$ is fixed, consequently on differentiating with respect to time:

$$\dot{\underset{\sim}{\Delta}}_A^e + \dot{\underset{\sim}{\Delta}}_A^c = \underset{\sim}{0} \qquad (18{\cdot}4)$$

The pure elastic component $\underset{\sim}{\Delta}_A^e$ and the pure creep component $\underset{\sim}{\Delta}_A^c$ shall be determined from Castigliano's Theorem; this states that if P is an arbitrary point of the pipe run and $\underset{\sim}{p}$ is a load applied at this point then the pure elastic and creep displacements at P are

$$\underset{\sim}{\Delta}^{e}_{P} = \lim_{\underset{\sim}{p}\to 0} \frac{D U_o^P}{D \underset{\sim}{p}} \qquad \dot{\underset{\sim}{\Delta}}^{c}_{P} = \lim_{\underset{\sim}{p}\to 0} \frac{D W_o^P}{D \underset{\sim}{p}}$$

where U_o^P is the elastic strain energy associated with the section of the run from O to P, and W_o^P is the creep strain energy rate associated with this section, assuming a time hardening creep law. These are given by

$$U_o^P = \int_o^P \mathcal{U}(\underset{\sim}{F}^r)\, ds$$

$$W_o^P = \int_o^P \mathcal{W}(t, \underset{\sim}{F}^r)\, ds$$

where $\mathcal{U}$ is a quadratic functional of the rotated loading $\underset{\sim}{F}^r$ at S and $\mathcal{W}$ a nonquadratic functional; both are functions of position on the run, s.

18.3 Derivation of equations for relaxation of load

Let a load $\underset{\sim}{p}$ be applied at A; this has an effect at an arbitrary point S of the run given by (18.2)

$$\underset{\sim}{F}_p = [T_s^A]\underset{\sim}{p}$$

Hence the total load at S is $\underset{\sim}{F} + \underset{\sim}{F}_p$ where $\underset{\sim}{F}$ is the actual load at S; the corresponding rotated load is from (18.1)

$$\underset{\sim}{F}_p^r = [R_1](\underset{\sim}{F} + \underset{\sim}{F}_p)$$

Clearly as $\underset{\sim}{p} \to 0$, $\underset{\sim}{F}_p \to 0$ and $\underset{\sim}{F}_p^r \to \underset{\sim}{F}^r$.

Now

$$\frac{D U_o^A}{D \underset{\sim}{p}} = \int_o^A \frac{D \mathcal{U}(\underset{\sim}{F}_p^r)}{D \underset{\sim}{p}}\, ds = \int_o^A \left(\frac{D \underset{\sim}{F}_p^r}{D \underset{\sim}{p}}\right)^T \frac{D \mathcal{U}}{D \underset{\sim}{F}_p^r}\, ds$$

but

$$\frac{D \underset{\sim}{F}_p^r}{D \underset{\sim}{p}} = [R_1][T_s^A]$$

hence from Castigliano

$$\underset{\sim}{\Delta}^{e}_{A} = \int_o^A ([R_1][T_s^A])^T \frac{D \mathcal{U}(\underset{\sim}{F}^r)}{D \underset{\sim}{F}^r}\, ds$$

However $\frac{D \mathcal{U}(\underset{\sim}{F}^r)}{D \underset{\sim}{F}^r} = ([R_1]^{-1})^T \frac{D \mathcal{U}([R_1]\underset{\sim}{F})}{D \underset{\sim}{F}} = [R_1] \frac{D \mathcal{U}([R_1]\underset{\sim}{F})}{D \underset{\sim}{F}}$

and therefore since $[R_1]^T[R] = [I]$, $([R_1][T_s^A])^T = [T_s^A]^T[R_1]^T$

there results

$$\underset{\sim}{\Delta}^{e}_{A} = \int_{o}^{A} [T_s^A]^T \frac{D\,U([R_1]\underset{\sim}{F})}{D\underset{\sim}{F}} ds$$

Similarly

$$\dot{\underset{\sim}{\Delta}}^{c}_{A} = \int_{o}^{A} [T_s^A]^T \frac{D\,W(t,[R_1]\underset{\sim}{F})}{D\underset{\sim}{F}} ds$$

Finally the actual loading at S is related to the applied loading at A, $\underset{\sim}{F}_A$ by (18.2) and therefore

$$\underset{\sim}{\Delta}^{e}_{A} = \int_{o}^{A} \frac{D\,U([R_1][T_s^A]\underset{\sim}{F}_A)}{D\underset{\sim}{F}_A} ds$$

$$\dot{\underset{\sim}{\Delta}}^{c}_{A} = \int_{o}^{A} \frac{D\,W(t,[R_1][T_s^A]\underset{\sim}{F}_A)}{D\underset{\sim}{F}_A} ds \qquad (18.5)$$

It is more convenient to define $\bar{\underset{\sim}{F}}_A$, $\bar{\underset{\sim}{\Delta}}_A$ by

$$\bar{\underset{\sim}{F}}_A = [T_o^A]\underset{\sim}{F}_A$$

$$\bar{\underset{\sim}{\Delta}}_A = ([T_o^A]^{-1})^T \underset{\sim}{\Delta}_A$$

since then from (18.5) on noting that U is a quadratic functional

$$\bar{\underset{\sim}{\Delta}}^{e}_{A} = \int_{o}^{A} \frac{D\,U([R_1][T_o^s]^{-1}\bar{\underset{\sim}{F}}_A)}{D\underset{\sim}{F}_A} ds = [U]\bar{\underset{\sim}{F}}_A$$

$$\dot{\bar{\underset{\sim}{\Delta}}}^{c}_{A} = \int_{o}^{A} \frac{D\,W(t,[R_1][T_o^s]^{-1}\bar{\underset{\sim}{F}}_A)}{D\underset{\sim}{F}_A} ds = \underset{\sim}{W}(\bar{\underset{\sim}{F}}_A) \qquad (18.6)$$

the vector $\bar{\underset{\sim}{F}}_A$ being independent of s, where $[U]$ is a matrix and $\underset{\sim}{W}$ and nonlinear vector valued function.

On substituting (18.6) into (18.4) there results

$$\frac{d}{dt}\bar{\underset{\sim}{F}}_A + [U]^{-1}\underset{\sim}{W}(\bar{\underset{\sim}{F}}_A) = \underset{\sim}{0} \qquad (18.7)$$

which is a vector ordinary differential equation of the first order for relaxation of load and is an initial value problem with the condition

$$\bar{\underset{\sim}{F}}_A(0) = [T_o^A]\underset{\sim}{F}_{Ao} \qquad (18.8)$$

Thus for a given geometry $[U]$ and $\underset{\sim}{W}$ may be specified and (18.7) (18.8) integrated to given $\bar{\underset{\sim}{F}}_A$ and thus $\underset{\sim}{F}_A$.

18.4 The elastic strain energy

The elastic strain energy dissipation functional is usually given by [438]

$$U = \frac{1}{2}\left(\frac{\bar{K}_T F_T^2}{EA} + \frac{\bar{K}_N F_N^2}{GA}\gamma + \frac{\bar{K}_B F_B^2}{GA}\gamma + \frac{K_T M_T^2}{GJ} + \frac{K_N M_N^2}{EI} + \frac{K_B M_B^2}{EI} \right)$$

where

E = modulus of elasticity in tension

G = modulus of elasticity in shear

I = axial moment of inertia of pipe cross section

J = polar moment of inertia

γ = shear distribution factor

K = flexibility factor

all of which depend on s; when $K = 1$ a straight pipe is specified. It is readily shown that the matrix $[U]$ takes the form

$$[U] = \int_0^A \left([k][R_1][T_0^s]^{-1} \right)^T [k][R_1][T_0^s]^{-1} q \, ds$$

where $q = 1/EI$ and is called the "performanee" or "flexibilty" matrix.

The matrix $[k]$ is given by

$$[k] = \begin{bmatrix} [\bar{k}_s] & [0] \\ [0] & [k_s] \end{bmatrix}$$

$$\bar{k}_s = \begin{bmatrix} \bar{k}_T & 0 & 0 \\ 0 & \bar{k}_N & 0 \\ 0 & 0 & \bar{k}_B \end{bmatrix} \qquad k_s = \begin{bmatrix} k_T & 0 & 0 \\ 0 & k_N & 0 \\ 0 & 0 & k_B \end{bmatrix}$$

$$\bar{k}_T^2 = \frac{I}{A}\bar{K}_T \quad k_T^2 = \frac{EI}{GJ}K_T \quad \bar{k}_N^2 = \frac{EI}{GA}\gamma\bar{K}_N \quad k_N^2 = K_N \quad \bar{k}_B^2 = \frac{EI}{GA}\gamma\bar{K}_B \quad k_B^2 = K_B$$

18.5 The creep strain energy rate

The development of a suitable strain energy functional for creep, similar to the elastic case above, is still only in its early stages due to the nonlinearity of the material law.

For in plane bending alone and a power law of creep [423]

$$W(t, \underset{\sim}{F}^r) = \frac{1}{n+1} \dot{K}_c \, g(t) \frac{M_B}{I}^{n+1}$$

where

I_n = nth moment of area of pipe cross section

$\dot{K}_c$ = creep flexibility factor

For combined loading much progress has still to be made; an initial step has been taken in Appendix 4 by the writer to account for out of plane bending and torsion. Nevertheless the whole area needs careful and meaningful development.

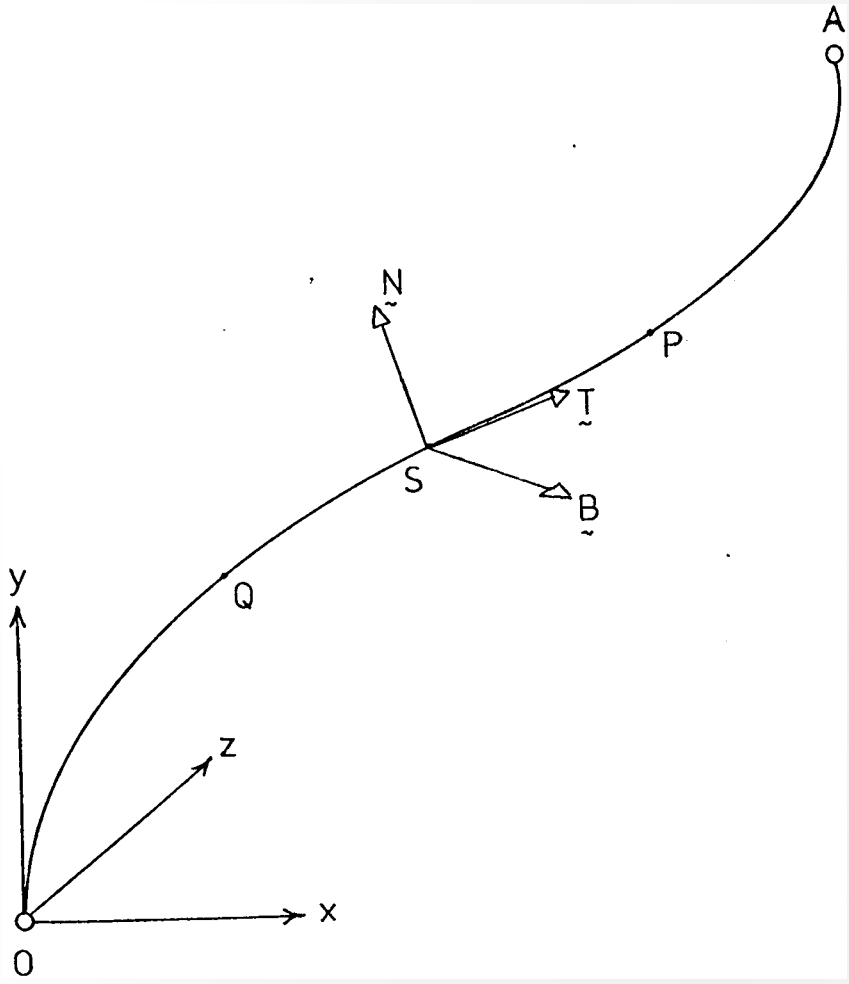


Fig18.1 Two anchor pipe run

CHAPTER 19

AN EXAMPLE

By far the best means of illustrating the simplified method based on superposition of states described in the previous chapter is by way of an example. In this, penultimate chapter the simple loop considered by J Spence [450] shall be reanalysed (and corrected).

Consider the loop shown in Figure 19.1 which lies completely in a two dimensional plane consisting of three 90° bends and three straights. It is assumed that all the pipes are of circular cross section with the same uniform thickness $2h$ and radius r, composed of the same material. Let R be the radius of the bends and l a typical length of the straights (as shown in Figure 19.1) completely describing the geometry of the loop. The point 0 is fixed but free to rotate and it is assumed that there is only one degree of freedom at A (if a Cartesian frame (x,y,z) is placed at 0 with unit vectors $(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$ then A moves solely in the direction of $\underset{\sim}{i}$).

Let F_{A0} be the initial force in the $\underset{\sim}{i}$ direction required to elastically obtain the displacement δ_0 at A. This displacement is fixed and the loop deforms inelastically due to creep. The problem is to determine the mode of relaxation of $F_A(t)$, the resultant force at A.

It is assumed that only in-plane bending is of significance, whence the strain energy densities are

$$U = \frac{1}{2} K_e \frac{M_B^2}{EI}$$

$$W = \frac{1}{n+1} \dot{K}_c \frac{M_B^{n+1}}{I_n} g(t)$$

where

$$I = 2\pi r^3 h \qquad I_n = r\,(2hr^2\pi D_1)^n \qquad D_1 = \frac{1}{\pi}\int_0^{2\pi} \sin\xi^{1+1/n}\, d\xi$$

and K_e is the elastic flexibility factor for the bends and $\dot{K}_c$ the creep flexibility factor [423].

The equation (18.7) for relaxation of load reduces to the single identity

$$\frac{d}{dt}\left\{\frac{d}{d[\bar{F}_A]}\int_0^A U([R_l][T_o^s]^{-1}\bar{F}_A)ds\right\} + \frac{d}{d[\bar{F}_A]}\int_0^A W(t,[R_l][T_o^s]^{-1}\bar{F}_A)ds = 0 \qquad (19.1)$$

where

$$\bar{F}_A = [T_o^A]F_A \qquad F_A = (F_A\ 0\ 0\ 0\ 0\ 0)^T .$$

The matrices $[R_l]$ and $[T_o^s]$ need to be specified for the loop.

Let the sections of the loop be denoted by ① - ⑥ as shown in Figure 19.1. Then in terms of the Cartesian frame $(\underset{\sim}{i},\underset{\sim}{j},\underset{\sim}{k})$ the rotated coordinate axii $(\underset{\sim}{T},\underset{\sim}{N},\underset{\sim}{B})$ is for each section

	①	②	③
$\underset{\sim}{T}$	$(0\ \ 1\ \ 0)$	$(\sin\theta\ \ \cos\theta\ \ 0)$	$(1\ \ 0\ \ 0)$
$\underset{\sim}{N}$	$(-1\ \ 0\ \ 0)$	$(-\cos\theta\ \ \sin\theta\ \ 0)$	$(0\ \ 1\ \ 0)$
$\underset{\sim}{B}$	$(0\ \ 0\ \ 1)$	$(0\ \ 0\ \ 1)$	$(0\ \ 0\ \ 1)$
	④	⑤	⑥
$\underset{\sim}{T}$	$(\cos\phi\ \ -\sin\phi\ \ 0)$	$(0\ \ -1\ \ 0)$	$(\sin\psi\ \ -\cos\psi\ \ 0)$
$\underset{\sim}{N}$	$(\sin\phi\ \ \cos\phi\ \ 0)$	$(1\ \ 0\ \ 0)$	$(\cos\psi\ \ \sin\psi\ \ 0)$
$\underset{\sim}{B}$	$(0\ \ 0\ \ 1)$	$(0\ \ 0\ \ 1)$	$(0\ \ 0\ \ 1)$

and the matrix $[R_{11}]$ is thus given by:

①	②	③
$\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \sin\theta & \cos\theta & 0 \\ -\cos\theta & \sin\theta & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
④	⑤	⑥
$\begin{bmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} \sin\psi & -\cos\psi & 0 \\ \cos\psi & \sin\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$

where the angles θ, ϕ and ψ are as given in Figure 19.1.

Then the matrix

$$[R_i] = \begin{bmatrix} [R_{ii}] & [0] \\ [0] & [R_{ii}] \end{bmatrix}$$

can be defined for each section.

The transfer matrix $[T_o^s]$ is specified by

$$[T_o^s] = \begin{bmatrix} [I] & [0] \\ [t_o^s] & [I] \end{bmatrix} \qquad [T_o^s]^{-1} = \begin{bmatrix} [I] & [0] \\ -[t_o^s] & [I] \end{bmatrix}$$

$$[t_o^s] = [t_o] - [t_s] \qquad [t_o] = [0]$$

$$[t_s] = \begin{bmatrix} 0 & z_s & -y_s \\ -z_s & 0 & x_s \\ y_s & -x_s & 0 \end{bmatrix}$$

The coordinates (x_s, y_s, z_s) for each section are

	x_s	y_s	z_s	
①	0	u	0	$0 \leqslant u \leqslant l+R$
②	$R-R\cos\theta$	$l+R+R\sin\theta$	0	$0 \leqslant \theta \leqslant \pi/2$
③	$R+v$	$l+2R$	0	$0 \leqslant v \leqslant l$
④	$l+R+R\sin\phi$	$l+R+R\cos\phi$	0	$0 \leqslant \phi \leqslant \pi/2$
⑤	$l+2R$	$-w+l+R$	0	$0 \leqslant w \leqslant l$
⑥	$l+3R-R\cos\psi$	$R-R\sin\psi$	0	$0 \leqslant \psi \leqslant \pi/2$

u, v, w being local coordinates for the straight sections and θ, ϕ, ψ local coordinates for the curved sections (Figure 19.1); the point A is specified by $\psi = \pi/2$.

Obviously by multiplication of the matrices

$$\underset{\sim}{\bar{F}}_A = [T_o^A]\underset{\sim}{F}_A = (F_A \; 0 \; 0 \; 0 \; 0 \; 0)^T = \underset{\sim}{F}_A$$

$$[R][T_o^s]^{-1}\underset{\sim}{\bar{F}}_A = F_A(r_{1s} \; r_{2s} \; 0 \; 0 \; 0 \; y_s)^T$$

where

	①	②	③	④	⑤	⑥
r_{1s}	0	$\sin\theta$	1	$\cos\phi$	0	$\sin\psi$
r_{2s}	-1	$-\cos\theta$	0	$\sin\phi$	1	$\cos\psi$

but since U and W depend only on M_B these are not required; also

$$M_B = F_A y_s$$

Hence

$$U([R_1][T_o^s]^{-1}\underset{\sim}{\bar{F}}_A) = \frac{1}{2}K_e\frac{(F_A y_s)^2}{EI}, \quad W(t,[R_1][T_o^s]^{-1}\underset{\sim}{\bar{F}}_A) = \frac{1}{n+1}\frac{(F_A y_s)^{n+1}}{I_o^n}g(t)$$

These may be substituted into (19.1) and on identifying s with the local coordinates there results

$$\frac{dF_A}{dt} + Eg(t)\left(\frac{R}{2hr^2\pi}\right)^{n-1}\left(\frac{W^*}{U^*}\cdot\frac{1}{D_1^n}\right)F_A^n = 0 \qquad (19\text{-}2)$$

where $\mu = l/R$ and

$$U^* = \frac{2(\mu+1)^3-\mu^3}{3} + \mu(\mu+2)^2 + 2K_e\int_0^{\pi/2}(1+\mu+\sin\theta)^2 d\theta + K_e\int_0^{\pi/2}(1-\sin\psi)^2 d\psi$$

$$W^* = \frac{2(\mu+1)^{n+2}-\mu^{n+2}}{n+2} + \mu(\mu+2)^{n+1} + 2\dot{K}_c\int_0^{\pi/2}(1+\mu+\sin\theta)^{n+1} d\theta + \dot{K}_c\int_0^{\pi/2}(1-\sin\psi)^{n+1} d\psi$$

The initial value problem (19.2) has solution

$$\frac{F_A}{F_{AO}} = \left\{1+(n-1)E\int_0^t g(t)dt\ \sigma_R^{n-1}\right\}^{-\frac{1}{n-1}} \qquad (19.3)$$

where

$$\sigma_R = \alpha_R\left(\frac{F_{AO}R}{2hr^2\pi}\right) \qquad \alpha_R = \left(\frac{W^*}{U^*}\cdot\frac{1}{D_i^n}\right)^{-\frac{1}{n-1}}$$

Obviously this answer (19.3) is in the form of a uniaxial relaxation test started at σ_R being a relaxation reference stress.

Figure 19.2 shows values of α_R for various μ against n; Figure 19.3 shows F_A/F_{AO} plotted against nondimensional time $\tau = E\sigma_o^{n-1}\int g(t)dt$, $\sigma_o = F_{AO}R \div 2hr^2\pi$ for a particular geometry taking values of K_e, $\dot{K}_c$ from Chapter 17.

Finally it is to be noted that the results of Robinson [447] are regained on setting $K_e = \dot{K}_c = 0$ and that the results disagree with those of Spence [450], due to an error in his analysis, although not wildly.

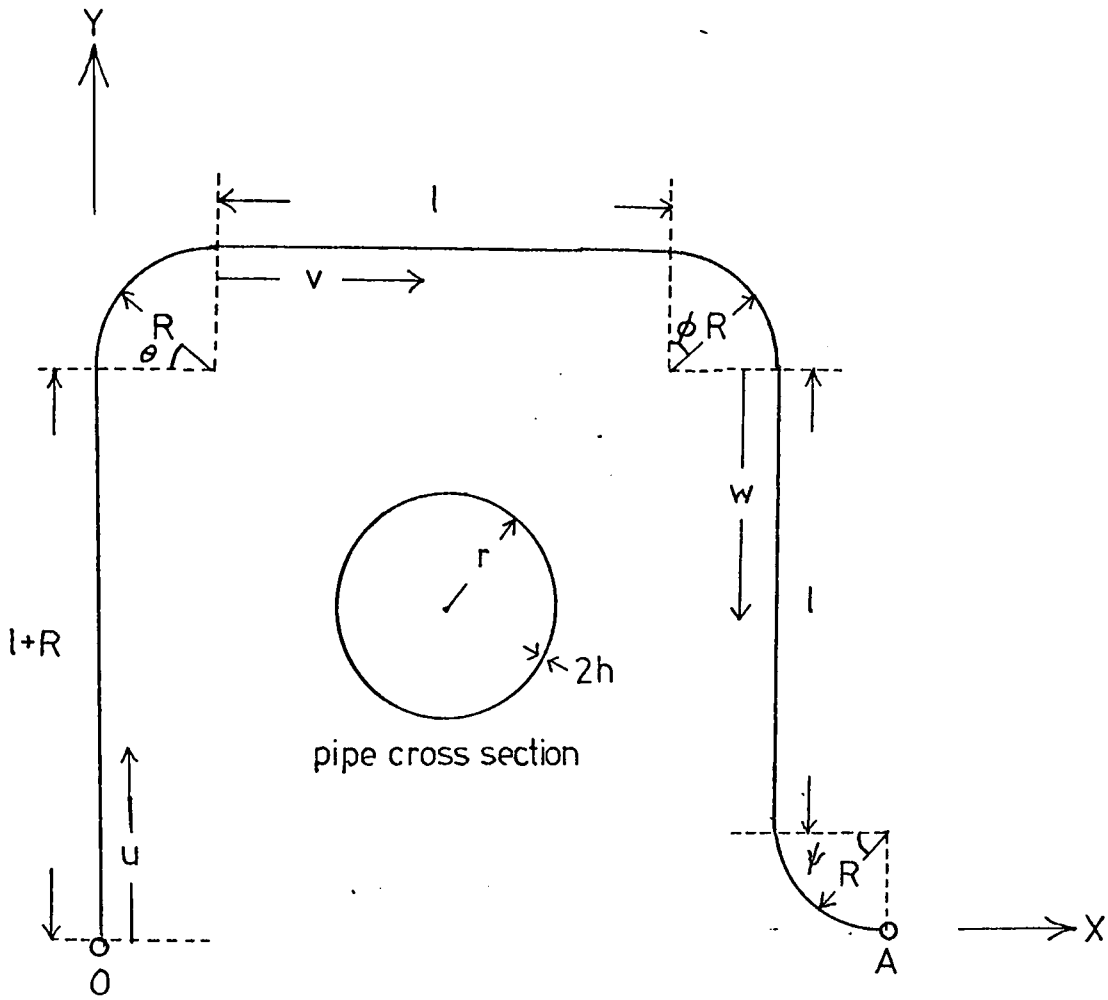


Fig19.1 Example pipe run problem

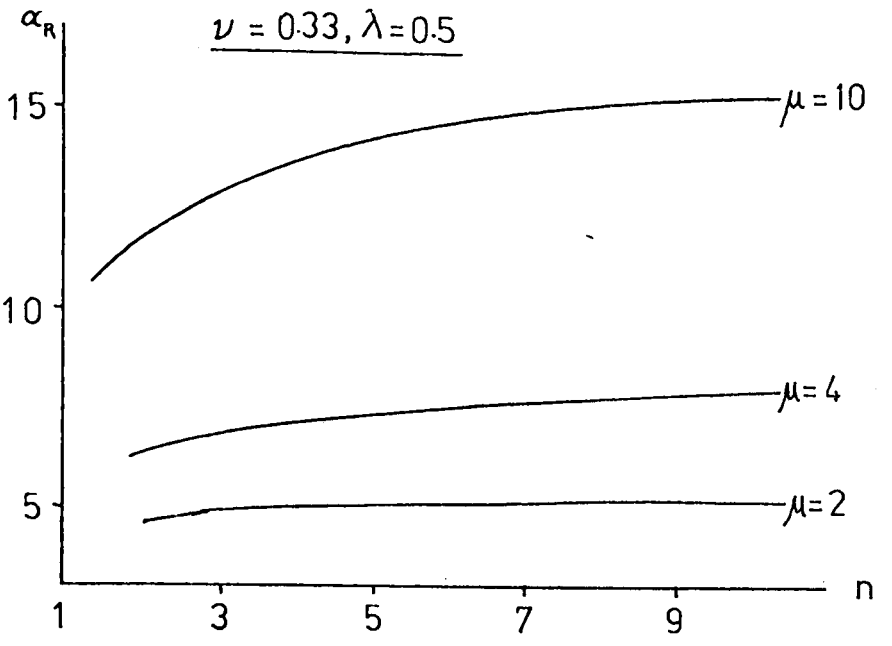


Fig19.2 Reference parameter, α_R

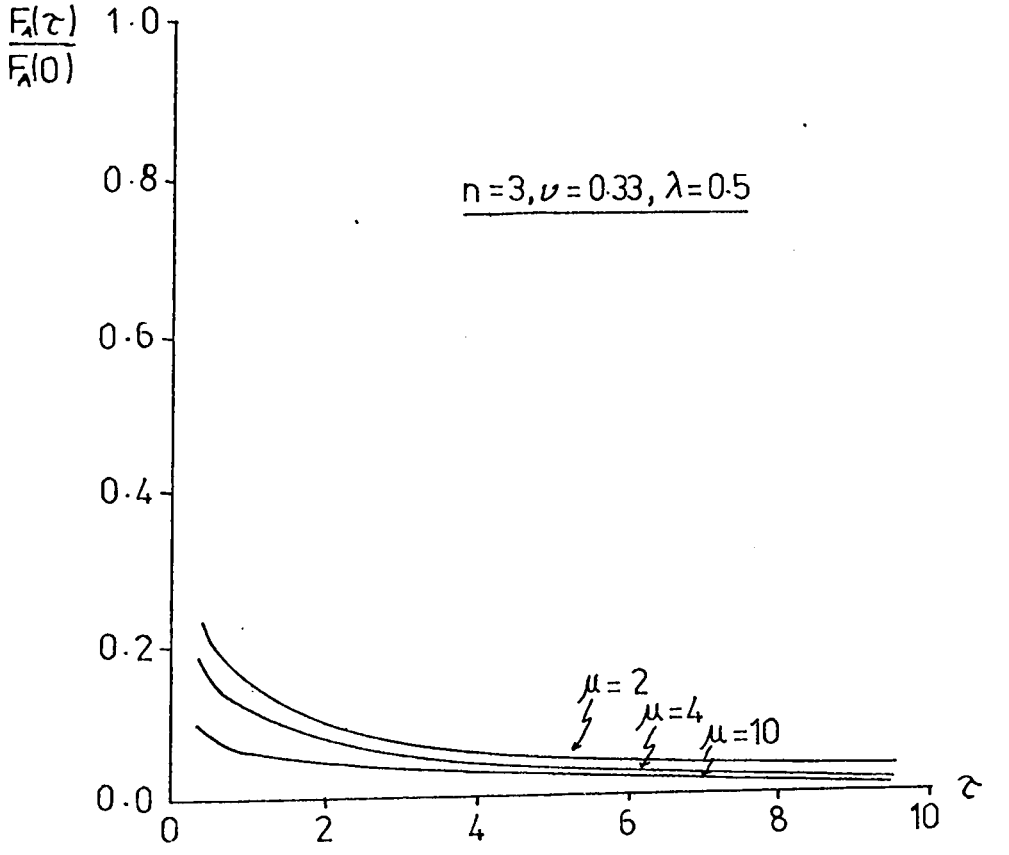


Fig 19.3 Force relaxation

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PART C

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CHAPTER 20

THE FUTURE ?

"No sooner had I completed the course of study at the end of which one would normally be admitted to the degree of Doctor, than I changed my views completely because I found myself beset by so many doubts and errors that I felt that nothing had accrued to me from my efforts at learning but the discovery that I was more ignorant even than I had supposed ....." (1)

Although the writer would perhaps not go so far as Descartes in admitting to a reversal of opinion in the later stages of this work, many adjustments were made in an attempt to follow whatever aspect of creep research was considered to be the most fruitful of the future. Perhaps the ultimate destination of inelastic analysis is not hard to discern - the complete computer simulation of the deformation and deterioration of a complex structure designed on a basis of simple, reliable rules - for it is the path by which this future shall be achieved that is unclear. Maybe relativity shall apply and the nature of the events which shall eventually alter the present course cannot be determined - we shall merely realise that the change has occurred. To the writer there is a feeling that the direction of creep research may be unwise. It appears that there is an emphasis on the nonlinearity of the behaviour at the expense of the time dependence which is admitted only to a secondary rôle and treated often in the basest possible manner. One only needs to refer to most reviews or modern monograph concerned with metallic creep to discover it under the designation "nonlinear material behaviour". This has led to attempts to extend familiar

elastic design procedures and techniques of analysis into the creep regime.

In this thesis creep is considered to be a field worthy of innovation, with temporal behaviour of prime importance. There has been a tendency in the past decade to believe that this aspect of creep is routine through application of any of the initial strain algorithms, being shunned to an extent due to excessive computational costs and the feeling that expensive sledge hammers were being employed in the destruction of rather ill-defined nuts. While the phase is certainly only transient, this view is possessed of an amount of truth in the solution of basic problems of engineering importance. Therefore there have been suggested several simplified approximate methods of analysis, the most popular of which being described in Chapter 7, Section 4. Although confirming research to be the art of the soluble /2/, these methods have met with a certain amount of success as it has often been demonstrated (for example Chapter 15 of this thesis) that they are effective. The question is whether or not the future of creep research lies in the further development of these techniques? When they succeed, and there is no reason to suppose that this will always be the case, such methods cannot be discredited - however taken to their logical conclusion they produce serious disadvantages which far outweigh those of attempting actual temporal integration. It is possible that these methods shall not weaken in the light of future scrutiny, but it is the writer's belief that they are sterile. As an alternative one can only reasonably suggest a point of departure; rather than viewing creep analysis as an application of the theory of nonlinear partial differential equations, it would be

instructive to think of it formulated as an initial value problem. Computationally, above all else, this is to be preferred (3), moreover a different species of analytical technique could be exercised. In keeping with the spirit of this suggestion a rational approach to creep mechanics has been adopted through the derivation of the equations of inelastic growth and stress redistribution, Chapter 4. Retaining these forms would necessitate approximations for the "residual operator", rather than for the actual creep process. It is important to realise that such a study would only be concerned with (linear) elastic material problems! The view proposed here is that future of creep research should not lie in a progression from its current state, instead it should lie in a search for a new basis.

Despite such bemusing speculation, piping systems have to be designed in the present. As far as the curved pipe is concerned its short term future seems very much evident. There is a need for a nonlinear geometric finite element curved shell model to be developed to cope with arbitrary loading and internal pressure. This - together with experimental evidence in elasticity - should indicate those deformation assumptions which are acceptable so that a simpler model can be deduced for use in piping analysis. Only with this as a basis should the inelastic behaviour of a curved pipe be examined. On its own the curved pipe is not of much consequence, being more important for its ability to deform within a complex piping system. As yet there is no completely satisfactory method for designing and verifying such a system for creep although in this thesis (Chapter 18) a rational procedure is suggested based on current information. Nevertheless, is it not possible

that the "residual operator" for a piping system could be formed in a simple manner? The matter is under consideration by the writer at present.

The thesis began with a discussion of creep constitutive theory and it is fitting that it should also end on this point. All of the above remarks are extremely tenuous if suitable material models for creep are not available. The engineer looks to the metallurgist for guidance in interpreting his observations (4); while a lead may possibly soon be taken (5) one should ask if this is entirely necessary. In mathematics the point of view has evolved (6) that one need not know what things are so long as one knows what statements about them one is allowed to make. Applying this premise to constitutive theory the principle of the "black box" is regained.....is this the hope for the future?

Whatever happens, it is only wished that common sense will prevail.

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APPENDICES

APPENDIX 1

AN EXISTENCE AND UNIQUENESS THEOREM FOR AN ORDINARY DIFFERENTIAL EQUATION IN A BANACH SPACE

Let X be a Banach space, and for any $u \in X$ let $\|u\|$ denote the norm of u; let $I = [t_o, t_o + a]$.

Consider the ordinary differential equation

$$\frac{du}{dt} = f(t,u) \qquad (i)$$

where $f(t\ u) : I \times X \longrightarrow X$, then any function $u(t)$ is said to be a solution of (i) if it is a continuous mapping from I into X, strongly differentiable in t for $[t_o, t_o + a]$ and satisfies (i) in $[t_o, t_o + a]$ such that

$$u(t_o) = u_o \qquad (ii)$$

Theorem: Assume that the mapping f satisfies

(a) $f \in C[I \times S_b, X]$, $S_b = \{u \in X / \|u - u_o\| \leq b\}$

(b) f maps bounded sets into bounded sets so that for any $b > 0$ there exists a positive number M such that

$$\|f(t,u)\| \leq M \qquad (t,u) \in I \times S_b$$

Let there exist a functional $V \in C[I \times S_b \times S_b, \mathbb{R}_+]$ which is positive definite, has continuous bounded Gateaux derivatives uniform for $(t, u, v) \in I \times S_b \times S_b$ in the ball $\|w\| \leq M$ such that $V(t, u_n, v_n) \longrightarrow 0$ implies $\|u_n - v_n\| \longrightarrow 0$ for each $t \in I$.

Then if

$$\frac{\partial V}{\partial t} + D_u V \cdot f(t,u) + D_v \cdot f(t,v) \leq \omega(t,V)$$

for any $(t, u, v) \in I \times S_b \times S_b$ where $\omega(t,r)$ is a continuous function from $[0,\infty)$ into $\mathbb{R}$ such that $\omega(0,0) = 0$ and $\psi(t) \equiv 0$, $t \geq 0$ is the unique solution to

$$\dot{r} = \omega(t,r)$$

$$r(0) = 0$$

Then equation (i) has a solution on $[t_o, t_o + \alpha]$, α being the number such that $\alpha M \leq b$, which is unique.

Proof: Let Λ be a subdivision of $[t_0, t_0+\alpha]$, namely

$$t_0 < \hat{t}_1 < \cdots < \hat{t}_n = t_0+\alpha$$

For $\hat{t}_{\kappa-1} \le t \le \hat{t}_\kappa$ define

$$\phi_\Lambda(t) = \phi_\Lambda(\hat{t}_{\kappa-1}) + \int_{t_{\kappa-1}}^{t_\kappa} f_\Lambda(s)\,ds \quad . \quad f_\Lambda(t) = f(t, \phi_\Lambda(\hat{t}_{\kappa-1}))$$

$$\phi_\Lambda(t_0) = u_0$$

then

$$\phi_\Lambda(t) = u_0 + \int_{t_0}^{t} f_\Lambda(s)\,ds \qquad (*)$$

$$\dot{\phi}_\Lambda(t) = f_\Lambda(t) \qquad (**)$$

Let Λ and Π be two such subdivisions,; if t is not a subdivision point of either of these, and

$$\hat{t}_{\kappa-1} < t < \hat{t}_\kappa \qquad t^\Pi_{j-1} < t < t^\Pi_j$$

then from $(**)$

$$\frac{d}{dt}V(t, \phi_\Lambda(t), \phi_\Pi(t)) \le \left(\frac{\partial V}{\partial t} - \frac{\partial \bar{V}}{\partial t}\right) + (D_u V - D_u \bar{V})\cdot f(t, \phi_\Lambda(\hat{t}_{\kappa-1})) + (D_v V - D_v \bar{V})\cdot f(t, \phi_\Pi(t^\Pi_{j-1})) + \omega(t, \bar{V})$$

where $\bar{V} = V(t, \phi_\Lambda(\hat{t}_{\kappa-1}), \phi_\Pi(t^\Pi_{j-1}))$

But by continuity given $\varepsilon > 0$ there exists a $\delta > 0$ such that if

$$\max(\hat{t}_\kappa - \hat{t}_{\kappa-1}) < \delta \qquad \max(t^\Pi_j - t^\Pi_{j-1}) < \delta$$

then

$$\left|\frac{\partial V}{\partial t} - \frac{\partial \bar{V}}{\partial t}\right| < \frac{\varepsilon}{4} \qquad |\omega(t, \bar{V}) - \omega(t, V)| < \frac{\varepsilon}{4}$$

$$|(D_u V - D_u \bar{V})\cdot f(t, \phi_\Lambda(\hat{t}_{\kappa-1}))| < \frac{\varepsilon}{4} \qquad |(D_v V - D_v \bar{V})\cdot f(t, \phi_\Pi(t^\Pi_{j-1}))| < \frac{\varepsilon}{4}$$

and hence

$$\frac{d}{dt}V(t, \phi_\Lambda(t), \phi_\Pi(t)) < \varepsilon + \omega(t, V(t, \phi_\Lambda(t), \phi_\Pi(t)))$$

If $U(t,\varepsilon)$ is the maximal solution of $\dot{U} = \varepsilon + \omega(t,U)$, $U(0)=0$ so that as $\varepsilon \to 0$ $U(t,\varepsilon) \to \psi(t) = 0$, then

$$V(t, \phi_\Lambda(t), \phi_\Pi(t)) \le U(t,\varepsilon)$$

On defining $|\Lambda| = \max(\hat{t}_\kappa - \hat{t}_{\kappa-1})$ then as $|\Lambda|, |\Pi| \to 0$

$$V(t, \phi_\Lambda(t), \phi_\Pi(t)) \to 0 \Rightarrow \|\phi_\Lambda(t) - \phi_\Pi(t)\| \to 0$$

Hence since X is a Banach space there exists a $\phi(t)$ such that

$$\phi_\Lambda(t) \to \phi(t) \qquad \forall\, t \in I$$

Let $t \in [t_o, t_o + \alpha]$ be fixed, then for each subdivision there exists a k such that $t^\wedge_{k-1} \leq t \leq t^\wedge_k$ and from $(*)$

$$\| \phi_\Lambda(t^\wedge_{k-1}) - \phi_\Lambda(t) \| \leq M(t - t^\wedge_{k-1}) \leq M|\Lambda|$$

and thus as $|\Lambda| \to 0$, $\phi_\Lambda(t^\wedge_{k-1}) \to \phi(t)$, consequently from continuity

$$f(t, \phi_\Lambda(t^\wedge_{k-1})) = f_\Lambda(t) \to f(t, \phi(t))$$

and by dominated convergence $\phi(t)$ satisfies

$$\phi(t) = u_o + \int_{t_o}^{t} f(s, \phi(s))\, ds$$

or

$$\frac{d\phi}{dt} = f(t, \phi) \qquad \phi(t_o) = u_o$$

Hence ϕ is a solution of (i)

For any $t \in I$

$$\|\phi(t) - u_o\| \leq \int_{t_o}^{t} \|f(s, \phi(s))\| ds \leq \int_{t_o}^{t_o+\alpha} \|f(s, \phi(s))\| ds \leq \alpha M$$

and if $\phi \in S_b$, α must be such that $\alpha M \leq b$.

Now suppose there are two solutions $u(t)$, $v(t)$ of (i) satisfying the initial condition. Then

$$\frac{dV}{dt}(t,u,v) = \frac{\partial V}{\partial t} + D_u V \cdot \dot{u} + D_v V \cdot \dot{v}$$

$$= \frac{\partial V}{\partial t} + D_u V \cdot f(t,u) + D_v V \cdot f(t,v) \leq \omega(t, V(t,u,v))$$

and

$$V(t, u(t_o), v(t_o)) = V(t, u_o, u_o) = 0$$

Thus $V(t,u,v) \leq \psi(t) \equiv 0$ and $u \equiv v$. QED

It may be established that uniqueness of solution does not require such stringent conditions of the mapping f and the function V, and further that existence only needs a demi-continuous f!

Theorem: Define a set $\mathcal{D} \subset \{(t,u) \mid t_o < t \leq t_o + a,\ u \in X\}$ such that $\mathcal{D}(t) = \{u \mid (t,u) \in \mathcal{D}\}$ is closed in X for each t in $(t_o, t_o + a]$. Assume that the mapping f satisfies

(a) $f(t,u) \in C[\mathcal{D}, X]$;

(b) $\|f(t,u)\| \leq M(t)$ on $\mathcal{D}$ where $M(t)$ is summable;

and let V be a functional defined on $\mathcal{D}$ satisfying the conditions of the previous theorem.

Let $\Phi \in C[\mathcal{D}, X]$ be such that

$$\sup_{u \in \mathcal{D}(t)} V(t, \Phi(t), u) \longrightarrow 0 \qquad t \longrightarrow t_o$$

then if for every t_1, $t_o < t_1 < t_o + a$, (i) has a solution starting from $(t_1, \Phi(t_1))$ reaching $t_o + a$ in $\mathcal{D}$ there is a unique solution of (i) with $u(t_o) = \Phi(t_o) = u_o$.

Proof: Let $\phi(t; t_1)$ be a solution starting from $(t, \Phi(t_1))$ then if $t_o < t_1 < t_2$

$$\frac{dV}{dt}(t, \phi(t;t_1), \phi(t;t_2)) \leq \omega(t, V) \qquad t_2 \leq t \leq t_o + a$$

so that

$$V(t, \phi(t;t_1), \phi(t;t_2)) \leq V(t_2, \phi(t_2;t_1), \phi(t_2;t_2)) = V(t_2, \phi(t_2;t_1), \Phi(t_2))$$

which $\longrightarrow 0$ as $t_2 \longrightarrow t_o$. Hence

$$\phi(t;t_1) \longrightarrow \phi(t;t_2) \qquad t_1, t_2 \longrightarrow t_o$$

and

$$\lim_{t_1 \to t_o} \phi(t;t_1) = \phi(t)$$

exists.

Then $(t, \phi(t)) \in \mathcal{D}$ and $\phi(t)$ is a solution of (i) since for arbitrary $\bar{t} > t_1$

$$\phi(t;t_1) - \phi(\bar{t};t_1) = \int_{\bar{t}}^{t} f(s, \phi(s;t_1))\, ds \qquad t_o < t_1 < t$$

$$\phi(t) - \phi(\bar{t}) = \int_{\bar{t}}^{t} f(s, \phi(s))\, ds$$

and let $\bar{t} \to t_o$.

QED

Uniqueness is obvious.

It should be noticed that a must be chosen such that

$$\max_{t \in (t_o, t_o+a]} \int_{t_o}^{t} M(s)\,ds \leq b$$

where $b = \max_{u \in \mathcal{D}} \|u - u_o\|$.

APPENDIX 2

ON THE EXACT ANALYSIS OF THE STEADY CREEP OF A THIN TUBE UNDER COMBINED LOADING

ROZPRAWY INŻYNIERSKIE • ENGINEERING TRANSACTIONS • 22, 3, 411–420, 1974
Polska Akademia Nauk • Instytut Podstawowych Problemów Techniki

ON THE EXACT ANALYSIS OF THE STEADY CREEP OF A THIN TUBE UNDER COMBINED LOADING

J. BOYLE and J. SPENCE (GLASGOW)

An exact analysis is presented for the creep deformation of a long, thin, constant thickness circular cross-section cylindrical shell under various loadings. Results are presented in the form of a design chart which can be used for several load combinations, and as flexibility factors. Previous approximate analyses are discussed.

NOTATION

M applied bending moment loading,
P applied internal pressure loading,
S $=(Pr/2h)\,(\dot{\varepsilon}_0/\sigma_0)$
$\mathcal{S}$ $=Pr^3/M$,
T applied torsion loading,
W $=T/\pi M$,
W_1 $=(T/4h\pi r^2)\,(\dot{\varepsilon}_0/\sigma_0)$,
β $=(\mathcal{S}^2+W^2)^{1/2}$,
$\dot{\gamma}$ end rotation rate of tube under combined loading,
$\dot{\gamma}_0$ end rotation rate of tube under bending alone;

suffices

Φ circumferential direction,
l longitudinal direction,
Φ_l shear direction,

All other relevant symbols are either defined in the text and fig. 1 or in reference [2].

1. INTRODUCTION

The exact analysis of the steady creep behaviour of a thin walled tube was considered by EDSTAM and HULT [1] using an isothermal steady creep law of the form

$$\frac{\dot{\bar{\varepsilon}}}{\dot{\varepsilon}_0}=\left(\frac{\bar{\sigma}}{\sigma_0}\right)^n. \tag{1.1}$$

The object of their work was to obtain equivalent stress levels under combined pressure and bending loading for comparison with the linear elastic case, the results being presented as a useful design chart. However, their solution was restricted to the case $n=2$. It is the purpose of this investigation to derive a corrected design chart based on a more general analysis. Further it is shown how various loadings, with

bending taken as the primary loading, can be incorporated into the same chart. Finally, results are presented in terms of flexibility factors with reference to the case of bending alone.

2. Stress ratio analysis

Consider a thin tube of circular cross-section under internal pressure P and subjected to a bending moment M and a torque T, as defined in Fig. 1. The radial stress component is negligibly small if $2h/r \ll 1$. The circumferential stress σ_ϕ and the shear stress $\sigma_{\phi l}$ are isostatic and may be determined directly from equilibrium alone

$$\sigma_\phi = \frac{Pr}{2h}, \tag{2.1}$$

$$\sigma_{\phi l} = T/(4\pi h r^2), \tag{2.2}$$

both stresses being taken as constant through the thickness.

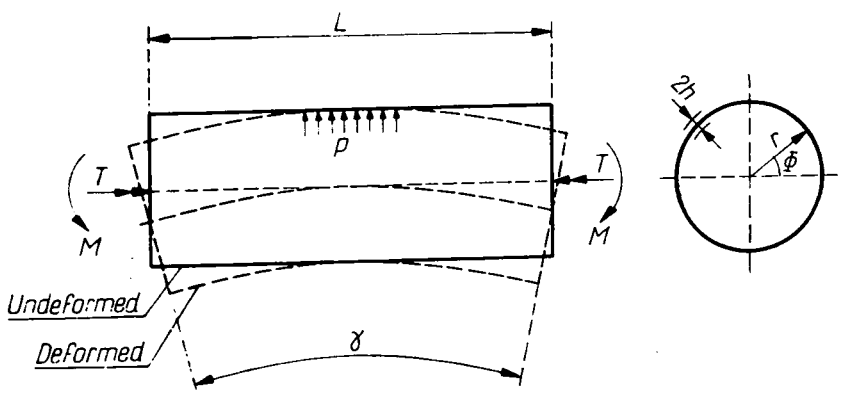


Fig. 1

It is found convenient to write the longitudinal stress

$$\sigma_l = \frac{Pr}{4h} + \rho(\Phi), \tag{2.3}$$

in a manner similar to [1].

This stress field must satisfy equilibrium with the applied moment

$$M = 2hr^2 \int_0^{2\pi} \sigma_l \sin\Phi \, d\Phi,$$

which yields

$$\int_0^{2\pi} R(\Phi) \sin\Phi \, d\Phi = 1, \tag{2.4}$$

on substitution of σ_l from (2.3), where $R(\Phi) = B_1 \rho(\Phi)$ and $B_1 = 2hr^2/M$.

If a pure bending moments alone were acting, the longitudinal strain rate would be

$$\dot{\varepsilon}_l = r\dot{\gamma}\sin(\Phi)/L, \tag{2.5}$$

where $\dot{\gamma}$ is the end rotation rate, related to the curvature rate $\dot{\kappa}$

$$\dot{\kappa} = \frac{\dot{\gamma}}{L}. \tag{2.6}$$

If on the other hand, if pressure and torque alone were acting, the longitudinal strain rate would be zero everywhere. Thus, under combined loading, the longitudinal strain rate would be as given by (2.5).

Further, the constitutive relation (1.1) yields an expression for $\dot{\varepsilon}_l$ in terms of the stresses (2.1), (2.2) and (2.3) [see Appendix II, Eq. (b)], and comparison with (2.5) produces the equation

$$\left(\frac{3}{4}\beta^2 + R^2\right)^{n-1} R^2 = \left(\frac{r\dot{\kappa}(\sigma_0 B_1)^n}{\dot{\varepsilon}_0}\right)\sin^2\Phi, \tag{2.7}$$

where $\beta = (\mathcal{S}^2 + W^2)^{1/2}$.

EDSTAM and HULT [1] have proposed the use of the non-dimensional design parameter

$$\Theta = \frac{\bar{\sigma}^{(n)}}{\bar{\sigma}^{(1)}},$$

known as the "stress ratio", where $\bar{\sigma}^{(n)}$ is the maximum effective stress occurring in the tube, creeping according to the constitutive relation (1.1). Assuming that this occurs when $\Phi = \pi/2$ it follows that

$$\Theta^2 = \frac{3/4 + R^2(\pi/2)/\beta^2}{3/4 + 1/(\pi\beta)^2}, \tag{2.8}$$

and hence that Θ depends on β and n alone.

The suggested design chart plots lines of constant Θ on a diagram with $1/n$ along a horizontal axis, and the quantities β, $1/\beta$ along a vertical axis. Consideration of the moment equilibrium equation (2.4) with the equation (2.7) for $R(\Phi)$ and the relation (2.8) reduces the construction of the chart to the following problem:

Let Θ be given, then for each n solve the algebraic equation

$$F(\bar{\beta}) = \int_0^{2\pi} \bar{R}(\Phi)\sin\Phi\, d\Phi - \bar{\beta} = 0,$$

for $\bar{\beta} = 1/\beta$, where $\bar{R}(\Phi)$ satisfies:

$$\left(\frac{3}{4} + \bar{R}^2(\Phi)\right)^{n-1} \bar{R}^2(\Phi) = \left(\frac{3}{4} + \bar{R}^2(\pi/2)\right)^{n-1} \bar{R}^2(\pi/2)\sin^2\Phi,$$

with

$$\bar{R}^2(\pi/2)=\Theta^2(3/4+\bar{\beta}^2/\pi^2)-3/4,$$

and $\bar{R}(\Phi)=\bar{\beta}\,R(\Phi)$.

The solution of this problem is outlined in Appendix I. The results of such a calculation are given as a design chart in Fig. 2.

3. Comparison with a previous result

For the case of combined pressure and bending, with no applied torque, $W=0$ and $\beta=\delta$. The design chart (Fig. 2) differs significantly from that given in [1] for the same problem. This difference is due to two factors: firstly, the previous analysis [1] was only legitimate for the special case $n=2$, secondly, the authors confined their attention to the calculation of values of Θ for given δ and n (instead of δ being found for given Θ and n) and hence their design chart involved considerable interpolation.

4. Use of the design chart for various loadings

It is important to note that the same design chart can be used for the three loading cases

(a) combined pressure and bending,
(b) combined torsion and bending,
(c) combined pressure, torsion and bending,

if β is simply replaced by $\delta=Pr^3/M$, or $W=T/\pi M$ or $(\delta^2+W^2)^{1/2}$, respectively.

5. Flexibility factor analysis

A further useful design parameter, used extensively in [2 and 3], is the "flexibility factor"

$$K=\frac{\text{end rotation rate of pipe in combined loading}}{\text{end rotation rate of pipe in bending alone}}=\frac{\dot{\gamma}}{\dot{\gamma}_0}.$$

It is easily shown that, with reference to (2.6) and (2.7),

$$K=D_0^n\cdot r\dot{\kappa}(\sigma_0 B_1)^n/\dot{\varepsilon}_0,$$

where $D_0=4\int_0^{\pi/2}(\sin\Phi)^{(1+1/n)}\,d\Phi$.

It is of interest to determine the value of K for a given β and n; since the moment equilibrium equation (2.4) has to be satisfied the problem reduces to the following:

Given β and n, find K such that the algebraic equation

$$G(K)=\int_0^{2\pi}R(\Phi)\sin\Phi\,d\Phi-1=0,$$

is satisfied, with $R(\Phi)$ the solution of the equation

$$\left(\frac{3}{4}\beta^2+R^2(\Phi)\right)^{n-1}R^2(\Phi)=\left(\frac{K\sin\Phi}{D_0^n}\right)^2.$$

The method of solution of this problem is similar to that of the previous stress ratio problem. Flexibility factor results are presented in Fig. 3.

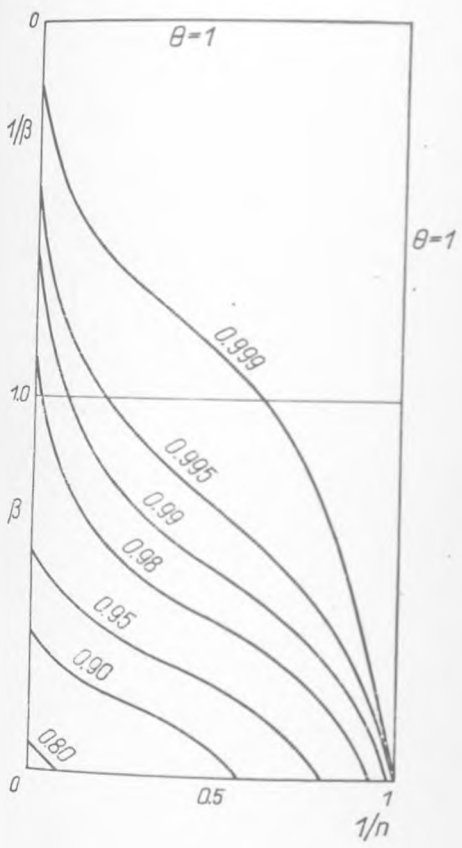


Fig. 2

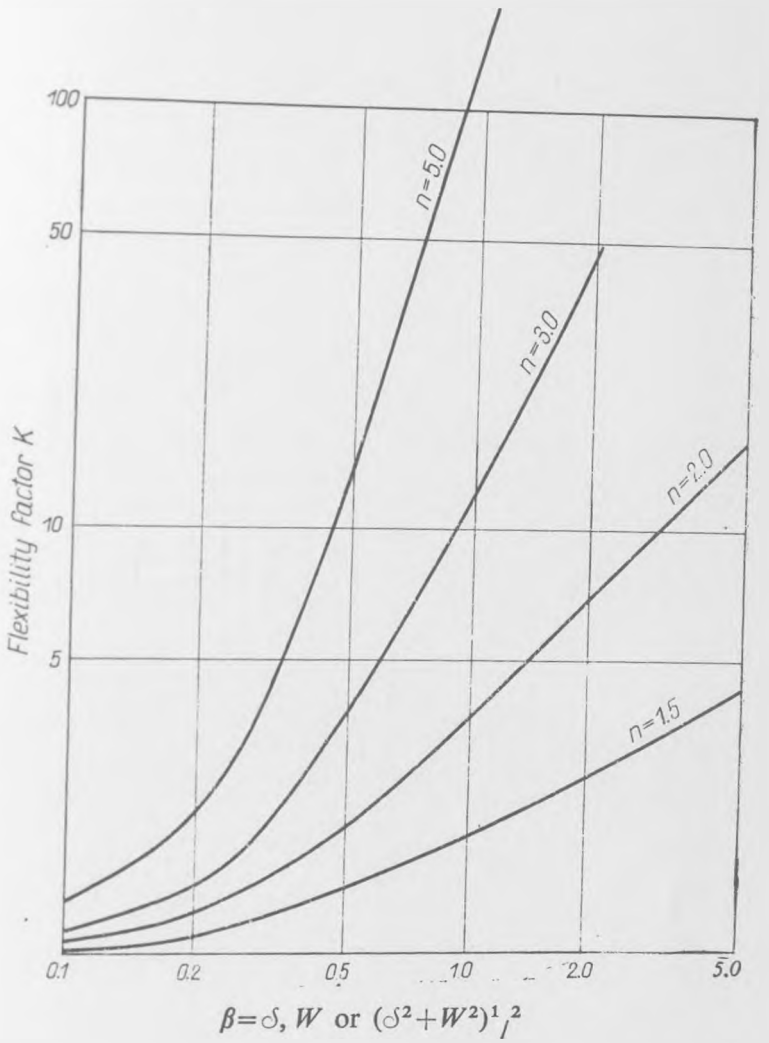


Fig. 3

6. Comparison with previous results

For the case of combined pressure and bending a comparison can be made with the results of [2]. At high δ values the three analyses — "type 1", "type 2" and "exact" — tend to become the same. The assumptions of a "type 2" analysis are not unlike those of the "exact" analysis and consequently the results are fairly close for all δ values. The "type 1" analysis shows a marked disagreement with the "exact" analysis; this shall be discussed further in the next section. For the case of combined loadings a comparison with the results of [3] is possible. The "exact" analysis confirms the simple β replacement scheme outlined in [3] for a "type 2" analysis such that the flexibility graphs can be used for various loadings.

7. Deformation results

It is worth commenting further here on the results of a "type 1" energy analysis as used in [2 and 3]. When the stress field is of the form (2.1), (2.2) and (2.3), the strain-stress relations [Appendix II, Eqs. (a) and (c)] can be written as

$$\dot{\varepsilon}_\phi = \left[\frac{3}{4}(S^2+4W_1^2)+\left(\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0}\rho(\Phi)\right)^2\right]^{(n-1)/2}\frac{3}{4}S-\frac{\dot{\varepsilon}_l}{2},$$

(7.1)

$$\dot{\varepsilon}_{\phi l} = \left[\frac{3}{4}(S^2+4W_1^2)+\left(\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0}\rho(\Phi)\right)^2\right]^{(n-1)/2}\frac{3}{2}W_1,$$

where

$$S=\frac{Pr}{2h}\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0}, \qquad W_1=\frac{T}{4h\pi r^2}\frac{\dot{\varepsilon}_0^{1/n}}{\sigma_0}.$$

However, the assumed strain rate field of a "type 1" energy analysis [3] is

$$\dot{\varepsilon}_\phi=\left[\frac{3}{4}S^2\right]^{(n-1)/2}\frac{3}{4}S-\frac{\dot{\varepsilon}_l}{2}, \qquad \dot{\varepsilon}_{\phi l}=[3W_1^2]^{(n-1)/2}\frac{3}{2}W_1, \tag{7.2}$$

which clearly neglects the effect of the torque T and $\rho(\Phi)$ on the circumferential strain rate, and the effect of the pressure P and $\rho(\Phi)$ on the shear strain rate. This would account for the discrepancies at low values of β between a "type 1" analysis and the "exact" analysis. Nevertheless, for large β values it would seem that assumptions (7.2) as opposed to (7.1) are reasonable approximations.

8. An alternative approach

The deformation results would suggest the following alternative approach to the problem based on assumed strain rates. Suppose that the strain rate field is of the form

$$\dot{\varepsilon}_l=(r\dot{\gamma}\sin\Phi)/L, \qquad \dot{\varepsilon}_\phi=\psi_1(\Phi)-\frac{\dot{\varepsilon}_l}{2}, \qquad \dot{\varepsilon}_{\phi l}=\psi_2(\Phi), \tag{8.1}$$

where ε_l is clearly "exact" and $\psi_1(\Phi)$, $\psi_2(\Phi)$ are functions which depend on the loading parameters.

The applied pressure and torque loadings are known to be isostatic and can be obtained directly from equilibrium as before

$$\sigma_\phi=\frac{Pr}{2h}, \qquad \sigma_{\phi l}=\frac{T}{4\pi hr^2}. \tag{8.2}$$

However, it is possible to express the stresses in terms of the assumed strain rate field (8.1) (Appendix II eqs (d), (f)); if a comparison is made with (8.2) there results

$$\frac{\psi_1}{\frac{3}{4}\left(\frac{Pr}{2h}\right)}=\frac{\psi_2}{\frac{3}{2}\left(\frac{T}{4\pi hr^2}\right)}=\psi(\Phi),$$

say, and

$$\Psi^2\left(\frac{3}{4}\Psi^2\beta^2+\left(\frac{K\sin\Phi}{D_0^n}\right)^2\right)^{1/n-1}=1, \tag{8.3}$$

where

$$B_1\,\Psi(\Phi)=\frac{(B_1\sigma_0)^n}{\dot{\varepsilon}_0}\psi(\Phi).$$

Similarly the moment equilibrium equation can be written as

$$\int_0^{2\pi}\left(\Psi^{-2(n-1)}-\frac{3}{4}\beta^2\right)^{1/2}\sin\Phi\,d\Phi=1. \tag{8.4}$$

In view of Eqs. (8.3) and (8.4) the problem of determining K for a given β and n can thus be written:

Given β and n, find K such that the equation

$$H(K)=\int_0^{2\pi}\left(\Psi^{-2(n-1)}-\frac{3}{4}\beta^2\right)\sin\Phi\,d\Phi-1=0,$$

is satisfied, with $\Psi(\Phi)$ the solution of the equation

$$\Psi^2\left(\frac{3}{4}\Psi^2\beta^2+\left(\frac{K\sin\Phi}{D_0^n}\right)^2\right)^{1/n-1}=1.$$

In fact this problem has exactly the same solution as the previous one for the evaluation of K. This is easily seen if a function $V(\Phi)$ is defined such that

$$\Psi^2(\Phi)=\left(\frac{3}{4}\beta^2+V^2(\Phi)\right)^{n-1},$$

in which case $V(\Phi)\equiv R(\Phi)$ and $H(K)\equiv G(K)$.

Obviously a "type 1" energy analysis could be performed based on assumptions (8.1) and would constitute a better approximation than previously to the above "exact" analysis. Finally it should be noted that the use of the strain rate field (8.1) instead of (7.2) would remove the inconsistencies mentioned in [3] between an α substitution and a β substitution for combined loading cases: that is, the parameter $\alpha=(\mathcal{S}^{2n}+W^{2n})^{1/2n}$ arises out of a "type 1" analysis for the strain rate field (7.2), yet the β substitution given above comes naturally from (8.1) giving agreement for all analyses.

9. Conclusions

An exact analysis has been presented for a thin cylindrical shell in stationary creep under combined loading in terms of two useful design parameters. Results are expected to be more reliable than previously published analyses (i.e., [1 and 2]).

It is shown that the same charts can be used for alternate parameters $\mathcal{S}=\mathrm{Pr}^3/M$, $W=T/\pi M$ or $(\mathcal{S}^2+W^2)^{1/2}$, independently of the method of analysis, which in itself represents a considerable simplification for design purposes as recommended in [3].

Appendix I

It is found necessary to solve an algebraic equation of the form

(a) $$F(x)=\int_a^b f_1(x,t)\,dt+f_2(x)=0,$$

such that the integrand satisfies the algebraic equation

(b) $$g(f_1(x,t))=0.$$

It is possible in some cases to solve (b) analytically. However, in general, this does not seem possible and hence a numerical technique is employed. With an initial approximation $x=x_0$ a first approximation $x=x_1$ is formed

$$x_1=x_0-\frac{F(x_0)}{F'(x_0)},$$

following a Newton/Raphson procedure.

Now $F(x_0)$ requires the evaluation of the integral $\int_a^b f_1(x_0,t)\,dt$; any numerical integration technique (in this case Simpson was used) uses values of the integrand at a number of intermediate points t_j in the closed interval $[a,b]$; such values can be achieved by solving the algebraic equation

$$g(f_1(x_0,t_j))=0$$

for $f_1(x_0,t_j)$ (using, say, a Newton/Raphson technique). Similarly from (a), $F'(x_0)$ requires the evaluation of the integral $\int_a^b \frac{\partial}{\partial x} f_1(x,t)|_{x_0}\,dt$, where for intermediate points t_j in $[a,b]$

$$\frac{\partial}{\partial x} f_1(x,t_j)|_{x_0}=\frac{\partial}{\partial x} g(f_1(x,t_j))|_{x_0}\div\left.\frac{dg}{df_1}\right|_{x_0,t_j}.$$

In the same manner a second approximation $x=x_2$ can be found. The process continues until it has sufficiently converged.

APPENDIX II

The material obeys a Norton type isothermal power law given by (1.1), where

$$\bar{\sigma}^2=\sigma_l^2-\sigma_l\sigma_\phi+\sigma_\phi^2+3\sigma_{\phi l}^2.$$

If it is assumed that it is also incompressible then the effective strain rate is given by

$$\bar{\dot{\varepsilon}}^2=\frac{4}{3}(\dot{\varepsilon}_l^2+\dot{\varepsilon}_l\dot{\varepsilon}_\phi+\dot{\varepsilon}_\phi^2+\dot{\varepsilon}_{\phi l}^2).$$

Then for a von Mises flow rule the strain-stress relations are

(a) $$\frac{\dot{\varepsilon}_\phi}{\dot{\varepsilon}_0}=\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1}\left(\sigma_\phi-\frac{\sigma_l}{2}\right)\Big/\sigma_0,$$

(b) $$\frac{\dot{\varepsilon}_l}{\varepsilon_0}=\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1}\left(\sigma_l-\frac{\sigma_\phi}{2}\right)\Big/\sigma_0,$$

(c) $$\frac{\dot{\varepsilon}_{\phi l}}{\dot{\varepsilon}_0}=\frac{3}{2}\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1}\frac{\sigma_{\phi l}}{\sigma_0};$$

and the stress-strain relations are

(d) $$\frac{\sigma_\phi}{\sigma_0}=\frac{4}{3}\left(\frac{\bar{\dot{\varepsilon}}}{\dot{\varepsilon}_0}\right)^{1/n-1}\left(\dot{\varepsilon}_\phi+\frac{\dot{\varepsilon}_l}{2}\right)\Big/\dot{\varepsilon}_0,$$

(e) $$\frac{\sigma_l}{\sigma_0}=\frac{4}{3}\left(\frac{\bar{\dot{\varepsilon}}}{\dot{\varepsilon}_0}\right)^{1/n-1}\left(\dot{\varepsilon}_l+\frac{\dot{\varepsilon}_\phi}{2}\right)\Big/\dot{\varepsilon}_0,$$

(f) $$\frac{\sigma_{\phi l}}{\sigma_0}=\frac{4}{3}\left(\frac{\bar{\dot{\varepsilon}}}{\dot{\varepsilon}_0}\right)^{1/n-1}\frac{\dot{\varepsilon}_{\phi l}}{2}.$$

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STRESZCZENIE

O ŚCISŁEJ ANALIZIE PROBLEMU USTALONEGO PEŁZANIA RURY CIENKOŚCIENNEJ W ZŁOŻONYM STANIE OBCIĄŻENIA

Przeprowadzono ścisłą analizę procesu pełzania długiej, cienkościennej rury w kształcie walca kołowego poddanej różnym obciążeniom. Wyniki przedstawiono w postaci specjalnego wykresu, z którego można korzystać przy różnych kombinacjach obciążeń, jak również w postaci tabeli współczynników podatności. Przedyskutowano poprzednie rozwiązania przybliżone tego samego problemu.

Резюме

О ТОЧНОМ АНАЛИЗЕ ПРОБЛЕМЫ УСТАНОВИВЩЕЙСЯ ПОЛЗУЧЕСТИ ТОНКОСТЕННОЙ ТРУБЫ В СЛОЖНОМ НАГРУЖЕННОМ СОСТОЯНИИ

Проведен точный анализ процесса ползучести длинной, тонкостенной трубы, в форме кругового цилиндра, подвергнутой разным нагрузкам. Результаты представлены так в виде специальной диаграммы, которой можно пользоваться при разных комбинациях нагрузок, как и в виде таблицы коэффициентов податливости. Обсуждены предущие проближенные решения этой же самой проблемы.

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APPENDIX 3

QUASILINEARIZATION AND LINEAR SHOOTING

The modified Newton-Raphson method for the operator equation

$$P(u) = 0$$

is the iteration [A3.1]:

$$u^{(n+1)} = u^{(n)} - \alpha^{(n)}[P'(u^{(n)})]^{-1}P(u^{(n)}) \qquad n = 0,1,2\cdots$$

where $\alpha^{(n)}$ is a sequence of scalars, $u^{(0)}$ is specified and P' is the Gateaux derivative of P

$$P'(u).h = \lim_{s\to 0}\frac{1}{s}\{P(u+sh)-P(u)\}$$

The modified method of quasilinearization is obtained on rewriting this iteration as a sequence of linearized operator equations,

$$P'(u^{(n)}).u^{(n+1)} = P'(u^{(n)}).u^{(n)} - \alpha^{(n)}P(u^{(n)})$$

which reduces to ordinary quasilinearization on taking each $\alpha^{(n)} = 1$.

A3.1 Boundary value problems

Consider the nonlinear two-point boundary value problem

$$\frac{d}{dx}\underset{\sim}{y} = \underset{\sim}{f}(x,\underset{\sim}{y}) \qquad a \leq x \leq b$$

$$[A]\underset{\sim}{y}(a) = \underset{\sim}{\alpha} \quad , \quad [B]\underset{\sim}{y}(b) = \underset{\sim}{\beta}$$

$$\int_a^b g(x,\underset{\sim}{y})dx = \gamma$$

where

$$\underset{\sim}{y} = (y_1\ y_2\ \cdots\ y_N) \qquad \underset{\sim}{f} = (f_1\ f_2\ \cdots\ f_N)$$

$$\underset{\sim}{\alpha} = (\alpha_1\ \alpha_2\ \cdots\ \alpha_R) \qquad \underset{\sim}{\beta} = (\beta_1\ \beta_2\ \cdots\ \beta_S)$$

$$dim\,[A] = R\times N \qquad dim[B] = S\times N \qquad R+S+1 = N$$

If the identification $u = \underset{\sim}{y}$, $P(\underset{\sim}{y}) = \frac{d\underset{\sim}{y}}{dx} - \underset{\sim}{f}(x,\underset{\sim}{y})$ is made then

$$P'(\underset{\sim}{y}).\underset{\sim}{h} = \frac{d\underset{\sim}{h}}{dx} - \left[\frac{Df}{Dy}\right]\underset{\sim}{h}$$

and the ordinary quasilinearization algorithm is as follows

Let $\underset{\sim}{y}^{(o)}$ be given, then the sequence of solutions $\underset{\sim}{y}^{(n)}$ $n = 1,2,\ldots\ldots$ is obtained on solving the sequence of linear two-point boundary value problem

$$\frac{d}{dx}\underset{\sim}{y}^{(n+1)} = \left[\frac{Df}{D\underset{\sim}{y}^{(n)}}\right]\underset{\sim}{y}^{(n+1)} + \underset{\sim}{f}(x,\underset{\sim}{y}^{(n)}) - \left[\frac{Df}{D\underset{\sim}{y}^{(n)}}\right]\underset{\sim}{y}^{(n)}$$

$$[A]\underset{\sim}{y}^{(n+1)}(a) = \underset{\sim}{\alpha} \qquad [B]\underset{\sim}{y}^{(n+1)}(b) = \underset{\sim}{\beta}$$

$$\int_a^b \left(\frac{Dg}{D\underset{\sim}{y}^{(n)}}\right)^T \underset{\sim}{y}^{(n+1)}\,dx = \gamma + \int_a^b \left\{\left(\frac{Dg}{D\underset{\sim}{y}^{(n)}}\right)^T \underset{\sim}{y}^{(n)} - g(x,y^{(n)})\right\} dx$$

where

$$\frac{Df}{D\underset{\sim}{y}} = \begin{bmatrix} \frac{\partial f_1}{\partial y_1} & \frac{\partial f_1}{\partial y_2} & \cdots & \frac{\partial f_1}{\partial y_N} \\ \vdots & & & \vdots \\ \frac{\partial f_N}{\partial y_1} & \frac{\partial f_N}{\partial y_2} & \cdots & \frac{\partial f_N}{\partial y_N} \end{bmatrix} \qquad \frac{Dg}{D\underset{\sim}{y}} = \left(\frac{\partial g}{\partial y_1} \quad \frac{\partial g}{\partial y_2} \quad \cdots \quad \frac{\partial g}{\partial y_N}\right)^T$$

The process is terminated when some convergence criterion is met. Each of the linear boundary value problems may be solved by a linear shooting technique.

A3.2 The method of particular solutions : Multiple shooting

Consider the linear boundary value problem

$$\frac{d\underset{\sim}{y}}{dx} = [L(x)]\underset{\sim}{y} + \underset{\sim}{l}(x) \qquad a \leq x \leq b$$

$$[A]\underset{\sim}{y}(a) = \underset{\sim}{\alpha} \qquad [B]\underset{\sim}{y}(b) = \underset{\sim}{\beta}$$

$$\int_a^b \underset{\sim}{E}^T \underset{\sim}{y}\,dx = \gamma$$

The interval $S:[a,b]$ is split into M subintervals $S_i:[a_i, b_i]$ where $a_1 = a$, $b_M = b$. Over each subinterval S_i, $\underset{\sim}{y} = \underset{\sim}{y}_i$ where

$$\frac{d\underset{\sim}{y}_i}{dx} = [L(x)]\underset{\sim}{y}_i + \underset{\sim}{l}(x) \qquad a_i \leq x \leq b_i$$

and

$$\underset{\sim}{y}_i = [W_i]\underset{\sim}{c}_i + \underset{\sim}{v}_i$$

where $[W_i]$ and $\underset{\sim}{v}_i$ are solutions of the initial value problems

$$\frac{d}{dx}[W_i] = [L(x)][W_i] \qquad \frac{d\underset{\sim}{v}_i}{dx} = [L(x)]\underset{\sim}{v}_i + \underset{\sim}{l}(x)$$

$$[W_i(a_i)] = [I] \qquad \underset{\sim}{v}_i(a_i) = \underset{\sim}{0}$$

The coefficients $\underset{\sim}{c}_i$ are found from the boundary and end conditions and the continuity condition

$$\underset{\sim}{y}_i(b_i) = \underset{\sim}{y}_{i+1}(a_{i+1}) \qquad i = 1, 2, \cdots, M$$

whence

$$[A][W_1(a)]\underset{\sim}{c}_1 + \underset{\sim}{v}_1(a) = \underset{\sim}{\alpha}$$

$$[B][W_M(b)]\underset{\sim}{c}_M + \underset{\sim}{v}_M(b) = \underset{\sim}{\beta}$$

$$\sum_{i=1}^{M}\left[\left(\int_{a_i}^{b_i} \underset{\sim}{E}^T[W_i]\,dx\right)^T \underset{\sim}{c}_i + \int_{a_i}^{b_i} \underset{\sim}{E}^T \underset{\sim}{v}_i\,dx\right] = \gamma$$

$$[W_i(b_i)]\underset{\sim}{c}_i + \underset{\sim}{v}_i(b_i) = [W_{i+1}(a_{i+1})]\underset{\sim}{c}_{i+1} + \underset{\sim}{v}_{i+1}(a_{i+1}) \qquad i = 1, 2, \cdots M$$

which represents a system of linear simultaneous equations which can be solved for the $\underset{\sim}{c}_i$.

A3.3 Solution of successive initial value problems

Each successive linear two-point boundary value problem requires the solution of a system of initial value problems. These can, in principle, be integrated forward in a step by step manner by any suitable method. Since such methods produce values of $\underset{\sim}{y}^{(n)}$ only at a finite number of mesh points in $[a, b]$ and since the solution for $\underset{\sim}{y}^{(n+1)}$ requires these the method must therefore only use functional evaluations at the mesh points. The predictor-corrector methods [A3.2] are eminently suitable for this, however they must use a suitable technique to start. It is usual to use a Runge-Kutta method but these require functional evaluations at points other than the mesh points. An alternative is to start the predictor-corrector with an implicit 'open' formula which proceeds a number of steps at each stage; for linear problems this method is fairly straightforward and that described by H Wilf [A3.3] has proven acceptable.

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APPENDIX 4

OUT OF PLANE BENDING OF A CURVED PIPE IN PURE CREEP

The flexibility analysis of a high temperature piping network by matrix methods using the "superposition of states" approximation for the creep response requires an expression for the creep complementary energy rate of a bend in terms of resultant loading. If in-plane bending alone is of significance this is fairly easy since the energy may be obtained by simply multiplying the energy of an equivalent straight (under the same loads) by the von-Karman/Spence steady creep flexibility factor. However, for out-of-plane loading and torque this cannot be done directly since an appropriate flexibility factor, which in elasticity is usually taken to be the same as the in-plane factor, is not available and the standard approach to the problem - separation of loads - is unacceptable due to the nonlinearity of the material. A different viewpoint is required.

In order to fix ideas consider a Θ angle bend loaded at one end ($\theta = 0$) with an out of plane moment M and consider a small section of angle $\Delta\theta$ at angle θ;
$0 \leqslant \theta \leqslant \Theta$ as shown in Figure A4.1.

Acting on this section is a torque $T_{\Delta\theta} = M\sin\theta$ and an out of plane moment $M_{\Delta\theta} = M\cos\theta$; it is assumed that over this small section stresses and strains are independent of θ.

Firstly analyse the small section:
An energy method (of Type 1) is used similar to the classical elastic analyses [A4.1-A4.4] but with torque included in a manner suggested by D Q Macdonald [A4.5] . This requires that the total potential energy rate for the section

$$V_{\Delta\Theta} = \int_{\nu_{\Delta\Theta}} U\,d\nu - M_{\Delta\Theta}\dot{\gamma} - T_{\Delta\Theta}\,\dot{\omega} \qquad (i)$$

whence γ is a rotation and ω a twist, is minimized. The rate of dissipation of strain energy per unit volume is given approximately by [A5.5] for a Norton power law of creep as:

$$U = \left(\frac{4}{3}\right)^{\frac{n+1}{2n}} \frac{h}{B}\cdot\frac{n}{n+1}\left[\dot{\epsilon}_\theta^2 + \frac{1}{4}\dot{\epsilon}_{\theta\phi}^2 + \left(\frac{n}{2n+1}\right)^{\frac{2n}{n+1}}\frac{h^2}{4}\dot{\kappa}_\phi^2\right]^{\frac{n+1}{2n}}$$

where the midsurface strains and curvature rates $\dot{\epsilon}_\theta, \dot{\epsilon}_{\theta\phi}$ etc.. can be related to the tangential v and radial w displacement of the pipe wall by

$$\dot{\epsilon}_\theta = \frac{1}{R}\left(\frac{\dot{\gamma} r}{\Delta\theta}\cos\phi + \dot{w}\sin\phi + \dot{v}\cos\phi\right)$$

$$\dot{\epsilon}_\phi = \frac{1}{R}\left(\frac{d\dot{v}}{d\phi} + \dot{w}\right) = 0$$

$$\dot{\epsilon}_{\phi\theta} = B(3)^{\frac{n+1}{2}}\left(\frac{T_{\Delta\theta}}{2hr^2\pi}\right)^n$$

$$\dot{\kappa}_\theta = 0$$

$$\dot{\kappa}_\phi = \frac{1}{r^2}\left(\frac{d^2\dot{w}}{d\phi^2} + \dot{w}\right)$$

$$\dot{\kappa}_{\phi\theta} \doteq 0$$

with the trigonometric series representation

$$\dot{v} = \sum_{p=1}^{\infty} C_p \cos 2p\phi$$

and

$$\dot{w} = \dot{\epsilon}_{\phi\theta}(R\Delta\theta/r)$$

Then (i) can be reduced to

$$\frac{V}{M\dot{\gamma}_o} = \left(\frac{4}{3}\right)^{\frac{n+1}{2n}}\frac{n}{n+1}\cdot\frac{1}{D_o}\int_o^{2\pi} D_2^{\frac{n+1}{2n}}\,d\phi - \Lambda\cos\Theta - 3^{\frac{n+1}{2}}\left(\frac{D_o}{2\pi}\right)^n(\sin\Theta)^{n+1} \qquad (ii)$$

with

$$D_2 = \left\{\Lambda\cos\phi + \sum_p \bar{\beta}_p(2p\sin 2p\phi\sin\phi + \cos 2p\phi\cos\phi)\right\}^2$$

$$+\left(\left(\frac{3}{4}\right)^{\frac{n+1}{2n}}\frac{\sin\Theta}{\pi}D_o\right)^{2n} + \frac{\lambda^2}{4}\left(\frac{n}{2n+1}\right)^{\frac{2n}{n+1}}\left\{\sum_p \bar{\beta}_p(2p - 8p^3)\sin 2p\phi\right\}^2$$

where $\lambda = hR/r^2$ is the pipe bend parameter, $D_o = \int_o^{2\pi}(\sin\phi)^{1+1/n}d\phi$, $\bar{\beta}_p = C_p \div r\dot{\gamma}_o/\Delta\theta$ and $\dot{\gamma}_o$ is the end rotation rate of a similar straight of length $\Delta\theta$ loaded by M so that

$$\Lambda = \dot{\gamma} / \dot{\gamma}_o$$

and is a measure of geometrical flexibility.

The functional (ii) is minimized with respect to the independant variables Λ and $\bar{\beta}_p$. Given the values which achieve this minimum the total strain rates are

$$\dot{\varepsilon}^*_\theta = \dot{\varepsilon}_\theta \div \dot{\gamma}_o r/R\Delta\theta = \Lambda\cos\phi + \sum_p \bar{\beta}_p (2p \sin 2p\phi \sin\phi + \cos 2p\phi \cos\phi)$$

$$\dot{\varepsilon}^*_\phi = \dot{\varepsilon}_\phi \div \dot{\gamma}_o r/R\Delta\theta = \zeta \lambda \sum_p \bar{\beta}_p (2p - 8p^3) \sin 2p\phi$$

$$\dot{\varepsilon}^*_{\theta\phi} = \dot{\varepsilon}_{\theta\phi} \div \dot{\gamma}_o r/R\Delta\theta = 3^{\frac{n+1}{2}} \left(\frac{\sin\theta}{2\pi} D_o \right)^n$$

where $\zeta = z/h$.

The complementary energy of the segment is then

$$W_{\Delta\theta} = \frac{B}{n+1} \int_0^{\Delta\theta} \int_0^{2\pi} \int_{-h/2}^{h/2} \bar{\sigma}^{n+1} R r \, d\theta \, d\phi \, dz$$

which on rearrangement may be expressed as

$$W_{\Delta\theta} = k(\theta) W^\circ_{\Delta\theta} \qquad (iii)$$

where

$$k(\theta) = \frac{1}{D_o} \int_0^{2\pi} \int_{-1/2}^{1/2} \left[\frac{4}{3} \left(\dot{\varepsilon}^{*2}_\phi + \dot{\varepsilon}^*_\phi \dot{\varepsilon}^*_\theta + \dot{\varepsilon}^{*2}_\theta + \frac{1}{4} \dot{\varepsilon}^{*2}_{\phi\theta} \right) \right]^{\frac{n+1}{n}} d\phi \, d\zeta$$

and $W^\circ_{\Delta\theta}$ is the complementary energy of an equivalent straight pipe loaded in bending by M, i.e.

$$W^\circ_{\Delta\theta} = \frac{B}{n+1} \frac{M^{n+1}}{I_n} R\Delta\theta \,, \quad I_n = r(hr^2 D_o)^n$$

secondly consider the whole bend, from (iii)

$$W_{\Delta\theta} = \frac{k(\theta)}{\Theta} W^\circ_\Theta \, \Delta\theta \qquad (iv)$$

where W°_Θ is the complementary energy of an equivalent length of straight pipe, to the whole bend, loaded by M is

$$W^\circ_\Theta = \frac{B}{n+1} \frac{M^{n+1}}{I_n} R\Theta$$

Thus summing (iv) for the whole bend

$$W_\Theta = K(\Theta) W^\circ_\Theta \qquad (v)$$

where

$$K(\Theta) = \frac{1}{\Theta} \int_0^\Theta k(\theta) \, d\theta$$

A FORTRAN IV computer program was written for the University of Strathclyde ICL 1904S using a Hooke-Jeeves [A5.5] minimization routine. Results for n = 1.5, and various λ are shown in Figure A4.2 of the factor $k(\theta)$ and for $K(\Theta)$ in Figure A4.3.

This analysis provides a departure from the standard viewpoint on flexibility factors. It is seen that the natural requirement is an "energy factor", that is the aim is to express the energy of a curved pipe in an analytic form. In elasticity and in in-plane bending alone the energy factor is identical to the flexibility factor; in more complex cases this is not necessarily true. The implication is that research should proceed in the direction of energy factors.....

References:

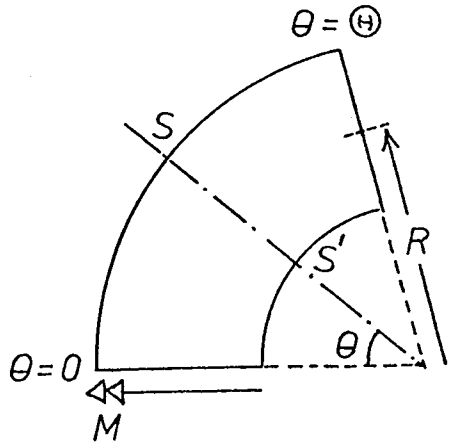
A4.1 I VIGNESS: Elastic properties of curved tubes, Trans ASME 65 (1943) pp105-120

A4.2 E C RODABAUGH, H H GEORGE: Effect of internal pressure on flexibility and stress intensification factors on curved pipes and welding elbows, Trans ASME 74 (1957) pp239-248

A4.3 R T SMITH: Theoretical analysis of the stress in pipe bends subjected to out of plane bending JMES 9 (1967) pp115-123

A4.4 W G DODGE, S E MOORE: Stress indices and flexibility factors for moment loadings on elbows and curved pipes, WRC Bull. No 179 (1972)

A4.5 D Q MACDONALD: Flexibility factors for straight and curved pipes in creep, MSc thesis, University of Strathclyde (1971)



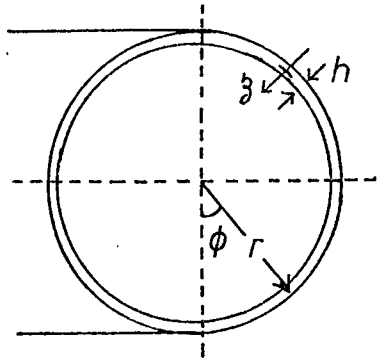
(a) Complete bend

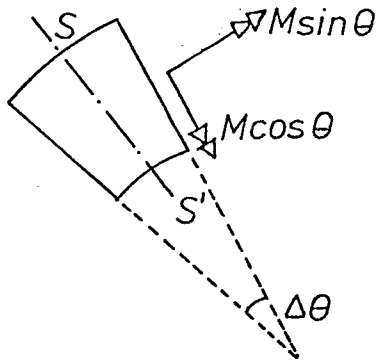
$0 \leq \theta \leq \Theta$

(b) Cross section

$0 \leq \phi \leq 2\pi$

$-h/2 \leq \zeta \leq h/2$





(c) Segment

Creep material law: $\dot{\bar{\varepsilon}} = B\bar{\sigma}^n$

Fig A4.1 Out-of-plane bending of a curved pipe in creep

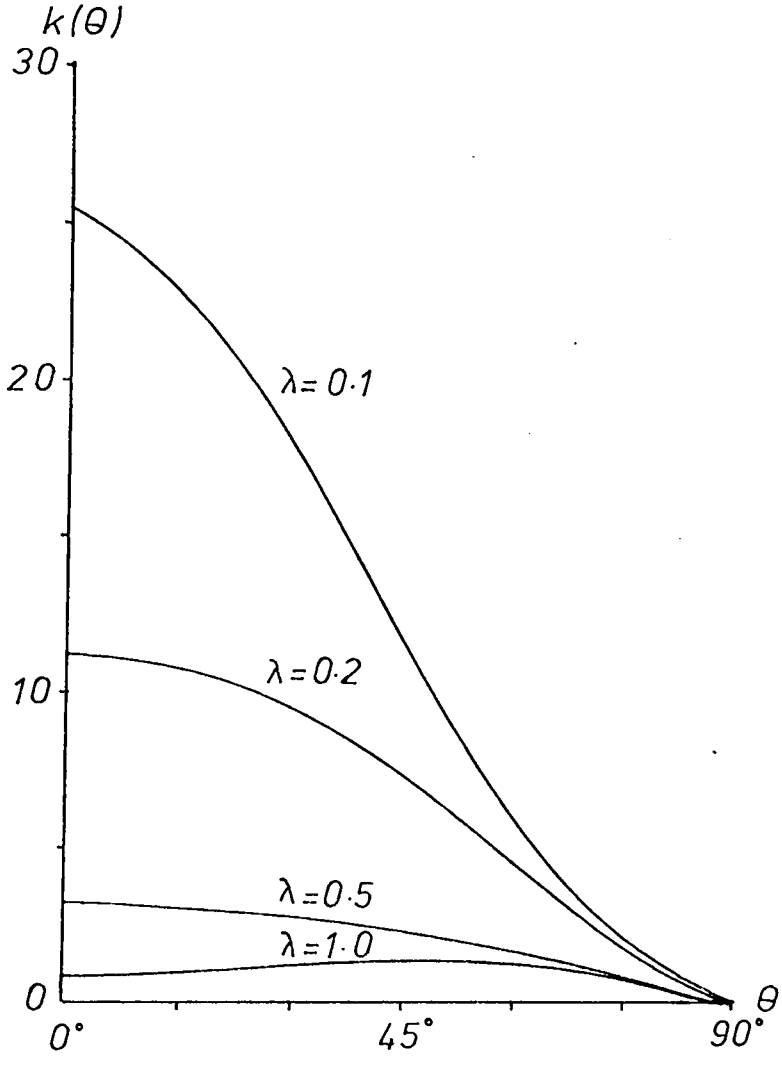
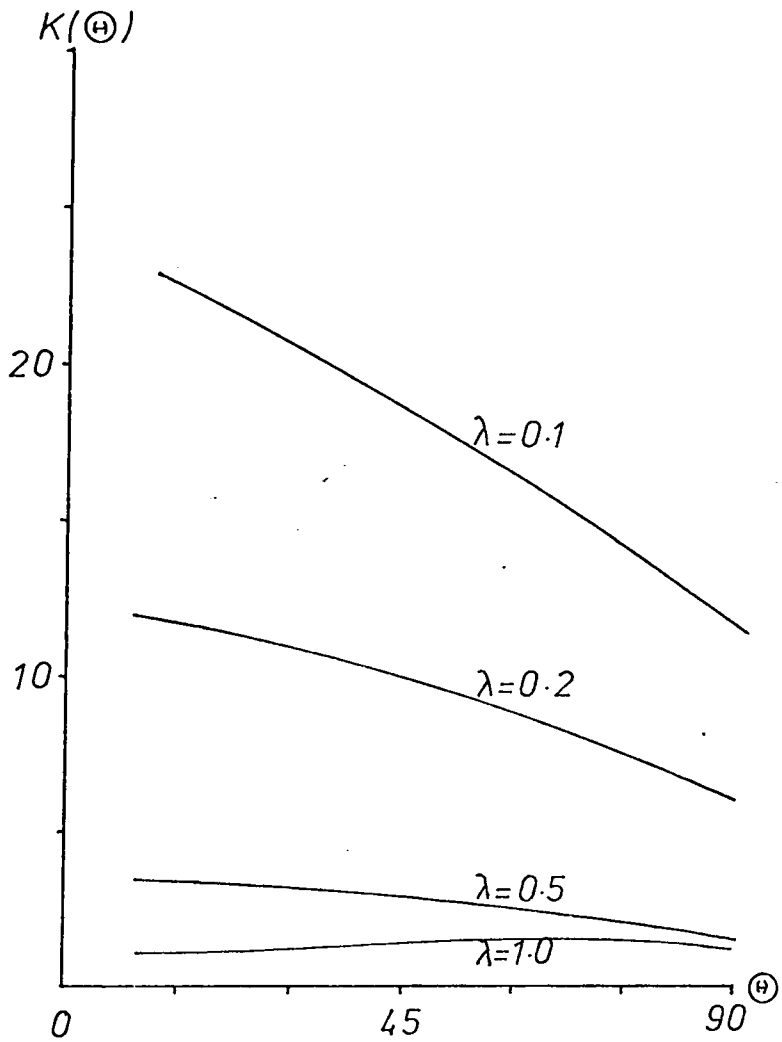


Fig A4·2 The factor $k(\theta)$, $n=1.5$



FigA4·3 The factor K(Θ), n=1·5

APPENDIX 5

NUMERICAL METHODS FOR INITIAL VALUE PROBLEMS WITH PARTICULAR REFERENCE TO CREEP : A CRITICAL REVIEW

For simplicity consider a body creeping according to the time-hardening theory; the ordinary differential equation in a Hilbert space which describes the redistribution of stresses in the body can then be reduced to a finite system of ordinary differential equations

$$\frac{d}{dt}\underset{\sim}{S} = \underset{\sim}{f}(t, \underset{\sim}{S}) \qquad (i)$$

$$\underset{\sim}{S}(0) = \underset{\sim}{S}_0 \qquad (ii)$$

on approximating the spatial problem in a discrete or direct manner. Suitable methods are therefore required for the temporal integration of (i) from (ii).

The choice of a suitable method depends on two parameters

(a) The total computation time, and

(b) The amount of storage required

if it is to be programmed for a digital computer. It is assumed that $\underset{\sim}{f}$ can be stored or recomputed, each time it is used there is a "functional evaluation". Obviously (a) is determined by the number of such functional evaluations. Assuming that a computer program for the integration of (i) occupies P storage locations, while S occupies N storage locations it is clear that (b) is a number of the form $(n + 1) \times N + P$, where the integer n varies from method to method. Furthermore three subsidiary conditions should be met:

(c) The method should be fairly accurate;

(d) It should minimize the effect of parasitic errors to ensure stability;

(e) It would be desirable to be able to vary the step

size, this being the time difference between successively calculated values of $\underset{\sim}{S}$.

The ideal method would be one which minimizes both conditions (a) and (b) while retaining the properties (c), (d) and (e). It is the purpose here to study the applicability of currently available methods.

Some of the most popular methods are described in Table A5.1, being selected as having particular application to the creep problem. These can be divided into two types - Runge-Kutta methods (1)(2)(3)(4)(7) and predictor-corrector methods (5), (6) and (8).

The two most widely used algorithms (1) and (2), used in conjunction with the method of initial strains, have a misleading attractiveness. In particular Euler's method (1) requires a much smaller time step to obtain the same accuracy as either (3) or (4) and in this sense the number of functional evaluations required is consequently much larger. The two fourth order Runge-Kutta methods (3) and (4) are thus more efficient, and since the Gill variation [A5.1, A5.2] requires less storage and minimizes the effect of parasitic error it is highly useful.

It is possible to cut down the computation time while retaining the same accuracy as the fourth order Runge-Kutta method by using the modified Adam's method [A5.3]. This is done at the expense of extra storage and is typical of all predictor-corrector methods. Since storage is often the main problem in creep calculations the predictor-corrector methods are thus wasteful, and furthermore their advantage of small computational times can be eroded if variable step sizes sould be used. Thus, in comparison with the Runge-Kutta-Gill method for creep problems, the predictor-corrector

methods are unsatisfactory. Nevertheless the writer has often found Heun's method (5) to be acceptable for some types of creep problems.

Unfortunately the differential system (i) is often "stiff"; that is some stresses in a structure vary quite slowly, while others vary quite fast. This means that, since the size of time step is determined by the fastest stress variation, a very small step must be used in all the above described methods. This is wasteful in computer processing time. Two techniques have been suggested for overcoming this, due to Treanor [A5.4] and Gear [A5.3], being based on Runge-Kutta and predictor-corrector methods respectively. It is clear from Table A5.1 that a large amount of storage is required in both; further Treanor's algorithm is difficult to extend to systems while Gear's algorithm is inefficient for a varying step size.

What is needed, it is felt, is a Runge-Kutta type method for stiff systems with economical storage and accuracy equivalent to the Gill variation. This would come close to the ideal, but as far as the writer can discern no such suitable method is available.

In general the whole problem of the "best" numerical integration technique is a matter totally open to question. and only wide experience of application to creep problems could come close to answering it with any satisfaction. The least the writer can hope to do is suggest two methods which have been found applicable to particular types of creep problems. For the relaxation problem, wherein the stress field asymptotically vanishes, Heun's method has proven useful, whereas for the forward creep problem, wherein constant loads are applied to a structure, the Runge-Kutta-Gill method is satisfactory.

Both of these methods are given in Table A5.2 for an autonomous system

$$\frac{d}{d\tau}\underset{\sim}{S} = \underset{\sim}{F}(\underset{\sim}{S})$$

resulting after nondimensionalization in time, using the notation $\underset{\sim}{S}_n = \underset{\sim}{S}(\tau_n)$, $\Delta\tau = \tau_{n+1} - \tau_n$.

References:

A5.1 S GILL: A process for step-by-step integration of a differential equation in an automatic digital computing machine, Proc Camb Phil Soc 47 (1951) pp96-108

A5.2 D J FYFE: Economical evaluation of Runge-Kutta formulae Math Comp 20 (1966) pp392-398

A5.3 C W GEAR: Numerical initial value problems in ordinary differential equations, Prentice (1971)

A5.4 C F TREANOR: A method for the numerical integration of coupled first order differential equations with greatly different time constants, Math Comp 20 (1966) pp39-45

	METHOD	ORDER	STORAGE	FUNCTIONAL EVALUATIONS	ULTIMATE STABILITY	ABILITY TO VARY TIME STEP
1	Euler	1	2N + P	1	Poor	Yes
2	Iterated Picard	I	3N + P	I	Bad	Yes
3	Classical Runge-Kutta	4	5N + P	4	Fair	Yes
4	Runge-Kutta-Gill	4	3N + P	4	Good	Yes
5	Heun's	2	4N + P	2	Fair	Yes
6	Adam's	4	7N + P	2	Good	Difficult
7	Treanor	4	9N + P	4	Good	Yes
8	Gear	4	7N + P	4	Good	Difficult

TABLE A5.1

Order = r where error is proportional to $(\Delta t)^{r+1}$, Δt being time step

I = no of iterations

TABLE A5.2

HEUN'S METHOD

Predictor: $\underset{\sim}{S}^{*}_{n+1} = \underset{\sim}{S}_n + \Delta\tau\, \underset{\sim}{F}(\underset{\sim}{S}_n)$

Corrector: $\underset{\sim}{S}_{n+1} = \underset{\sim}{S}_n + \frac{\Delta\tau}{2}\left(\underset{\sim}{F}(\underset{\sim}{S}_n) + \underset{\sim}{F}(\underset{\sim}{S}^{*}_{n+1})\right)$

RUNGE-KUTTA-GILL METHOD

This is achieved in four stages

$$\underset{\sim}{K}^{(1)} = \Delta\tau \underset{\sim}{F}(\underset{\sim}{S}_n)$$
$$\underset{\sim}{S}^{(1)}_{n+1} = \underset{\sim}{S}_n + \tfrac{1}{2}(\underset{\sim}{K}^{(1)} - 2\underset{\sim}{Q}^{(0)})$$
$$\underset{\sim}{Q}^{(1)} = \underset{\sim}{Q}^{(0)} + \tfrac{3}{2}(\underset{\sim}{K}^{(1)} - 2\underset{\sim}{Q}^{(0)}) - \tfrac{1}{2}\underset{\sim}{K}^{(1)}$$

$$\underset{\sim}{K}^{(2)} = \Delta\tau \underset{\sim}{F}(\underset{\sim}{S}^{(1)}_{n+1})$$
$$\underset{\sim}{S}^{(2)}_{n+1} = \underset{\sim}{S}^{(1)}_{n+1} + (1-\sqrt{1/2})(\underset{\sim}{K}^{(2)} - \underset{\sim}{Q}^{(1)})$$
$$\underset{\sim}{Q}^{(2)} = \underset{\sim}{Q}^{(1)} + 3(1-\sqrt{1/2})(\underset{\sim}{K}^{(2)} - \underset{\sim}{Q}^{(1)}) - (1-\sqrt{1/2})\underset{\sim}{K}^{(2)}$$

$$\underset{\sim}{K}^{(3)} = \Delta\tau \underset{\sim}{F}(\underset{\sim}{S}^{(2)}_{n+1})$$
$$\underset{\sim}{S}^{(3)}_{n+1} = \underset{\sim}{S}^{(2)}_{n+1} + (1+\sqrt{1/2})(\underset{\sim}{K}^{(3)} - \underset{\sim}{Q}^{(2)})$$
$$\underset{\sim}{Q}^{(3)} = \underset{\sim}{Q}^{(2)} + 3(1+\sqrt{1/2})(\underset{\sim}{K}^{(3)} - \underset{\sim}{Q}^{(2)}) - (1+\sqrt{1/2})\underset{\sim}{K}^{(3)}$$

$$\underset{\sim}{K}^{(4)} = \Delta\tau \underset{\sim}{F}(\underset{\sim}{S}^{(3)}_{n+1})$$
$$\underset{\sim}{S}^{(4)}_{n+1} = \underset{\sim}{S}^{(3)}_{n+1} + \tfrac{1}{6}(\underset{\sim}{K}^{(4)} - 2\underset{\sim}{Q}^{(3)})$$
$$\underset{\sim}{Q}^{(4)} = \underset{\sim}{Q}^{(3)} + \tfrac{1}{2}(\underset{\sim}{K}^{(4)} - 2\underset{\sim}{Q}^{(3)}) - \tfrac{1}{2}\underset{\sim}{K}^{(4)}$$

with initially $\tau = 0$, $\underset{\sim}{Q}^{(0)} = \underset{\sim}{0}$. It is clear that only three storage locations are required since successive values of $\underset{\sim}{K}$, $\underset{\sim}{S}$ and $\underset{\sim}{Q}$ in the four stages may be overwritten.

APPENDIX 6

THE REFERENCE STRESS METHOD

The following communication relates to the paper "Reference stress for structures obeying the Prandtl and Dorn creep laws" by A JOHNSON, published in 1974, Vol 16, No 5, p298, J Mech Eng Sci.

The author has presented a simple extension of the Reference Stress Method (RSM) to structures obeying creep laws other than Norton's power law. The writer has also obtained some results on a more general reference stress which complement those of the author:

Consider a creep law with two independent material parameters described by

$$\dot{\varepsilon}_c = e\, g(k, \sigma)$$

(using roughly similar notation) and let the deformation rate at some point of interest be

$$\dot{u} = e\, f(\text{geometry}, \text{loading}, k)$$

It is always possible to express the deformation in terms of a uniaxial test via

$$\dot{u} = \Delta(k, \sigma_{ref})\, \dot{\varepsilon}_{ref} \qquad (1)$$

where

$$\dot{\varepsilon}_{ref} = e\, g(k, \sigma_{ref})$$

σ_{ref} being an arbitrary (uniaxial) stress.

The difficulty arises that a true value of k is not available, only an estimate k_p may be found, and thus $\Delta(k, \sigma_{ref})$ in (1) cannot be determined. In order to resolve this let $\Delta(k, \sigma_{ref})$ be expanded in a Taylor series about k_p:

$$\Delta(k, \sigma_{ref}) = \Delta(k_p, \sigma_{ref}) + (k - k_p) \left.\frac{d\Delta}{dk}\right|_{k_p} + o((k - k_p)^2) \qquad (2)$$

if k_p is close enough to its true value for second order terms to be negligible then

$$\Delta(k, \sigma_{ref}) \doteq \Delta(k_p, \sigma_{ref}) + (k - k_p)\left.\frac{d\Delta}{dk}\right|_{k_p}$$

which still contains the unknown k; however σ_{ref} is arbitrary and it may be chosen such that the second term vanishes, i.e.

$$\left.\frac{d\Delta}{dk}\right|_{k_p} = 0 \qquad (3)$$

Consequently, with this choice of Reference Stress $\Delta(k, \sigma_{ref}) \doteq \Delta(k_p, \sigma_{ref})$ and

$$\dot{u} \doteq \Delta(k_p, \sigma_{ref}) \cdot \dot{\varepsilon}_{ref}(k, \sigma_{ref}) \qquad (4)$$

Now from (1)

$$\frac{d\Delta}{dk} = \frac{d}{dk}\left(\frac{\dot{u}}{\dot{\varepsilon}_{ref}}\right) = \frac{\frac{d\dot{u}}{dk}\dot{\varepsilon}_{ref} - \frac{d\dot{\varepsilon}_{ref}}{dk}\dot{u}}{\dot{\varepsilon}_{ref}^2}$$

so (3) is true if and only if

$$\left[\frac{d\,\dot{\varepsilon}_{ref}}{dk} \div \dot{\varepsilon}_{ref}\right]_{k_p} = \left[\frac{d\dot{u}}{dk} \div \dot{u}\right]_{k_p} \qquad (5)$$

where the left hand side is an analytic function of σ_{ref}, $F(\sigma_{ref})$, say; hence

$$\sigma_{ref} = F^{-1}\left[\left.\frac{\frac{d\dot{u}}{dk}}{\dot{u}}\right|_{k_p}\right] \qquad (6)$$

The only remaining problem is the determination of $\left(d\dot{u}/dk\right)_{k_p}$ In the examples used by the author closed form solutions for $\dot{u}(k)$ and $\frac{d\dot{u}}{dk}$ are available. But if $\dot{u}(k)$ is obtained numerically it is inadvisable to use a similar (though linear) method to find $\frac{d\dot{u}}{dk}$ since the amount of problem preparation, etc. increases and some material creep laws may fail to be uniformly differentiable in k for the possible range of stresses - in particular a Norton type power law leads to trouble. A better method is to find $\dot{u}$ for values of k about k_p, construct a difference table and calculate the required derivative using some difference formula (see

e.g. H M Nautical Office "Interpolation and Allied Tables" HMSO, 1956). The significance of this technique can be appreciated on observing that the classical RSM's (e.g. Reference (5) in paper) are merely crude, though often effective, means of approximating the derivative.

In summary σ_{ref} can be found from (6) such that (3) is satisfied identically, and therefore a good approximation to $\dot{u}$ is (4) where k_p is an estimate of the active material parameter k.

When the above method is applied to the problems given by the author, identical results are obtained; in fact it can be shown that this is always true. Nevertheless several points are made clear:

Firstly there is only one equation (3) in one unknown, not two as indicated by the author - the 'scaling factor' $\delta \equiv \Delta(k_p, \sigma_{ref})$ emerges naturally from the derivation; furthermore σ_{ref} is found explicitly from (6), even if $\dot{u}/k$ only is found numerically. Secondly it is clear exactly why (and, if one chooses, under what conditions) a 'small error' is ensured - this method allows a Reference stress to be chosen such that some deformation characteristic may be expressed in terms of calculated quantities and a uniaxial test at this stress, with a fair degree of accuracy.

Two important conclusions can be drawn from this derivation:

(a) The RSM may now be incorporated meaningfully into the body of numerical analysis. The implication being that better error estimates can be made and that by a simple extension higher order estimates of $\dot{u}$ in terms of a number of uniaxial tests, not just one can be found.

(b) There is an obvious generalisation to materials described by more than two parameters, for example

$$\dot{\varepsilon}_c = e\,g(\sigma, k_1, k_2, \cdots k_N)$$

The quantity $\Delta(k) = \dot{u}/e\,U$, where U is a function of a number of uniaxial tests, e.g. a linear combination, is expanded in a Taylor series about estimations $k_{P1}, \ldots\ldots\ldots k_{PN}$

$$\Delta(k) = \Delta(k_{Pj}) + \sum_{i=1}^{N} (k_i - k_{Pi}) \left.\frac{\partial \Delta}{\partial k_i}\right|_{k_{Pj}} + o[(k_i - k_{Pi})^2]$$

and, as before, ignoring second order terms Δ can be approximated by $\Delta(k_{Pj})$ if the function U is such that each term of the first order term vanishes:

$$\left.\frac{\partial \Delta}{\partial k_i}\right|_{k_{Pj}} = 0 \qquad i = 1, 2, \cdots N \qquad (7)$$

The form of the function U and the number of uniaxial tests required must be such that each of (7) has a solution.

This constitutes an improvement over the suggestion of the author, as it is not clear what significance (47) in the paper has - it certainly does not ensure a small error in using estimated material parameters, nor does it admit the smallest possible error since this depends on the estimations k_{Pj}. In fact it is possible that Π is inadmissable as a measurement of error.

This is not the place to develop the ideas contained in (a) and (b) more fully, suffice to say that precise mathematical statements, under certain conditions, can be made (for example in b) with suitable conditions on continuity and differentiability of $\dot{u}$ with respect to k it can be shown that no more than separate uniaxial tests are required); these results seem to indicate that a single

reference stress may be insufficient for all possible cases. However, in agreement with the author, the writer feels that no benefit shall be gained.

Finally it is hoped that this communication contributes to the understanding of the RSM and clarifies the work of the author.

APPENDIX 7

CASTIGLIANO'S THEOREM FOR SMALL INELASTIC DEFORMATIONS

Consider a body occupying a volume V bounded by a surface S deforming inelastically such that the stress field $\underset{\sim}{\sigma}$ is in equilibrium with body forces $\underset{\sim}{b}$ per unit volume in V,

$$div\,\underset{\sim}{\sigma} + \underset{\sim}{b} = \underset{\sim}{0}$$

with surface tractions $\underset{\sim}{\bar{t}}$ on $S_T \subset S$,

$$\underset{\sim}{\sigma}.\underset{\sim}{n} = \underset{\sim}{\bar{t}}$$

where $\underset{\sim}{n}$ is the unit normal vector to S_T, and with a constant point load Q applied to a small surface $S_o \subset S$ in the direction $\underset{\sim}{N}$

$$\int_{S_o} \underset{\sim}{\sigma}.\underset{\sim}{n}\, dS = Q\,\underset{\sim}{N} \qquad (i)$$

Let the corresponding strain rate field $\dot{\underset{\sim}{\varepsilon}}$ be compatible with a displacement rate vector $\dot{u}$ such that the displacement is fixed in time over $S_U \subset S$

$$\dot{\underset{\sim}{u}} = \underset{\sim}{0} \qquad (ii)$$

It is supposed that this problem has a solution, and that this is described by $\underset{\sim}{\sigma}$, $\dot{\underset{\sim}{\varepsilon}}$ and $\dot{\underset{\sim}{u}}$. Further it is assumed that $S_U \cup S_T \cup S_o = S$, $S_U \cap S_T = S_U \cap S_o = S_o \cap S_T = \phi$.

Consider an arbitrary self equilibrating stress field $\underset{\sim}{h}$

$$div\,\underset{\sim}{h} = \underset{\sim}{0} \qquad (iii)$$

$$\underset{\sim}{h}.\underset{\sim}{n} = \underset{\sim}{0} \qquad (iv)$$

and form the functional

$$\int_V \dot{\underset{\sim}{\varepsilon}}.\underset{\sim}{h}\, dV = \int_S \dot{\underset{\sim}{u}}.(\underset{\sim}{h}.\underset{\sim}{n})\, dS$$

using Green's identity together with (i)

$$= \int_{S_T \cup S_o} \dot{\underset{\sim}{u}}.(\underset{\sim}{h}.\underset{\sim}{n})\, dS$$

using (ii)

$$= \int_{S_o} \dot{\underset{\sim}{u}}.(\underset{\sim}{h}.\underset{\sim}{n})\, dS$$

using (iv).

If $\underset{\sim}{h}.\underset{\sim}{n}$ does not change sign over S_o the Mean Value Theorem

for integrals can be used

$$\int_{S_o} \underset{\sim}{u}\cdot(\underset{\sim}{h}\cdot\underset{\sim}{n})\,dS = \dot{\underset{\sim}{q}}\cdot\int_{S_o}\underset{\sim}{h}\cdot\underset{\sim}{n}\,dS$$

where $\dot{\underset{\sim}{q}}$ is the displacement rate vector associated with some point of S_o.

It is supposed that the class of point loads considered is such that S_o is small enough for there to be no sign change in the resultant traction vector $\underset{\sim}{t} = \underset{\sim}{\sigma}\cdot\underset{\sim}{n}$ over it. Then $\underset{\sim}{h} = d\underset{\sim}{\sigma}/dQ$ satisfies the conditions imposed (assuming that $\bar{b}, \bar{t}$ do not depend on Q) and there results using (i)

$$\int_{S_o}\underset{\sim}{h}\cdot\underset{\sim}{n}\,dS = \int_{S_o}\frac{d\underset{\sim}{\sigma}}{dQ}\cdot\underset{\sim}{n}\,dS = \frac{d}{dQ}\int_{S_o}\underset{\sim}{\sigma}\cdot\underset{\sim}{n}\,dS = \underset{\sim}{N}$$

Therefore

$$\dot{q} = \dot{\underset{\sim}{q}}\cdot\underset{\sim}{N} = \int_V \dot{\underset{\sim}{\varepsilon}}\cdot\frac{d\underset{\sim}{\sigma}}{dQ}\,dV$$

where $\dot{q}$ is interpreted as the displacement rate associated with Q. The following theorem can be inferred.

Castigliano's Theorem: Let a body occupying a volume V, bounded by a surface S, be mounted such that rigid body displacements of the entire system are impossible, and a certain external point load Q, constant in time, shall act on the system in addition to distributed loads and displacements, then the displacement rate $\dot{q}$ of the point of application of the force in the direction of the force is given by (v).

Although the theorem has been simply established for a class of point loads it can be extended to include more general force systems and couples [A7.1].

If it is possible to describe the material behaviour in a potential form

$$\dot{\underset{\sim}{\varepsilon}} = \frac{\partial W}{\partial \underset{\sim}{\sigma}}$$

the Castigliano's theorem becomes

$$\dot{q} = \frac{d}{dQ}\int_V W\,dV$$

In nonsteady creep, described by a time hardening law

$$\underset{\sim}{\dot{\varepsilon}} = \underset{\sim}{\dot{\varepsilon}}_e + \underset{\sim}{\dot{\varepsilon}}_v$$

$$\underset{\sim}{\dot{\varepsilon}}_e = \underset{\sim}{C}(\underset{\sim}{\dot{\sigma}})$$

$$\underset{\sim}{\dot{\varepsilon}}_v = \frac{\partial \Lambda}{\partial \underset{\sim}{\sigma}}$$

this is not possible. But in this case the theorem is

$$\dot{q} = \frac{d}{dQ}\int_V \Lambda\,dV + \int_V \underset{\sim}{C}(\underset{\sim}{\dot{\sigma}}).\frac{d\underset{\sim}{\sigma}}{dQ}\,dV$$

which appears to be new to the literature.

Reference:

A7.1 H L LANGHAAR: The principle of complementary energy in nonlinear elasticity theory, J Franklin Inst. 256 (1953) pp255-265

APPENDIX 8

DISCUSSION OF THE ANALYSIS OF G H WORKMAN AND E C RODABAUGH FOR THE CREEP OF PIPING SYSTEMS

Simplified Second Stage Creep/Relaxation Analysis of Moderately Complex Spatially Three-Dimensional Piping Systems[4]

J. BOYLE[5] AND J. SPENCE.[5] The paper is of great interest but, in the opinion of the writers, is dangerous since it infers that a creep/relaxation analysis of a complex, multiload piping system can be tackled in an unusually simple manner. The authors' basic approach is, on the whole, acceptable; however, their handling of the creep problem is open to criticism of a rather fundamental nature. At least two erroneous assumptions are made in their analysis, one stated explicitly which, we fear, cannot be overcome, and one implicit which, fortunately, can be incorporated as a further assumption by stating the problem in a different way.

As a first step in the examination of these assumptions we should like to call attention to several points which require clarification:

(*a*) The general method of analysis is derived from virtual work applied to a force system $\{f_k\}$ which " . . . represents elastic forces acting on the element . . ." However, these "elastic forces" are not defined. Further, the strains are related to the displacements through "geometric considerations" by equations (10)–(13), and the vectors $\{g\}$ and $\{h\}$ may be derived from shape functions ". . . associated with the elastic deflection pattern . . ." Surely these vectors, by definition, should not depend on material behavior, only on geometry? And what is the "elastic deflection pattern"?

(*b*) In order to include the creep effects, a creep/relaxation force vector $\{f^c\}$, defined by equation (16), is used which requires a knowledge of the creep strains in the element; the authors attempt to relate these to the forces acting on the element. Initially it is assumed that the ". . . creep/relaxation strains due to axial, bending and torsional strains can be decoupled . . ." On what grounds can this assumption be made? (We shall return to this point later.)

A further assumption is then made which is not explicitly stated but which, nevertheless, is very important—the authors assume that these decoupled creep strains are compatible with a displacement field, viz., equations (25) and (40). On what grounds can this be justified? In fact, in its proper context, this assumption is an acceptable approximation (see later).

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[4]By G. H. Workman and E. C. Rodabaugh, published in the August, 1974, issue of the JOURNAL OF PRESSURE VESSEL TECHNOLOGY, TRANS. ASME, Vol. 96, Series J, No. 3, p. 184.

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(*c*) The next stage is the development of a "straight pipe element" and a "circularly curved pipe element." That element associated with a straight pipe is a standard beam element wherein the strains are related to the displacements by geometric approximations alone. It is, however, the curved pipe element which is puzzling. The coefficients of the vectors $\{g\}$ and $\{h\}$ given by (65) and (66) obviously depend on the elastic constants by virtue of the elastic flexibility factor k_E and are ". . . those associated with the elastic displacement behavioral patterns of a curved beam formulation . . ." Once again we are forced to comment that $\{g\}$ and $\{h\}$ defined by (10)–(13) should, on the authors' own admission, depend on geometry alone; therefore why was Castigliano's theorem for elasticity used? What is the ". . . elastic displacement behavioral pattern . . ." and furthermore what has a ". . . curved beam formulation . . ." to do with a curved pipe?

A more obvious point is that while the well-known beam element is developed in detail, the more interesting and novel derivation of a curved pipe element is missing.

(*d*) Subsequently the authors use the $\{g\}$ and $\{h\}$ vectors given by (65) and (66) together with (38) and (48), which were derived for a straight pipe, in order to express the creep strains in terms of the forces on the curved pipe element! Surely a step has been omitted?

(*e*) A digital computer program is generated and tested using an example which apparently requires a "creep/relaxation flexibility factor" $k_{C/R}$. But this factor is not mentioned anywhere in the preceding analysis. Is another step missing?

(*f*) Logically one should start the time integration from the initial elastic state. However, the authors start from ". . . an initial time solution obtained by using the creep/relaxation moment of inertia and polar moment of inertia in place of the elastic moment of inertia and polar moment of inertia of the pipe elements . . ." What justification is there for this? And could the authors please clarify what exactly is intended as this step appears to have no logical foundation? Further, " . . . in this way the transition . . . from elastic behavior to time dependent second stage creep/relaxation behavior is not accounted for . . ." needs explanation; in particular are the authors trying to apply concepts from constant stress uniaxial material creep behavior to a body wherein structural stress redistribution occurs due to the inclusion of elastic deformations? What is "second stage creep" in this case?

(*g*) Finally, the authors remark that ". . . the digital computer program produces the elastic solution when the creep/relaxation exponent is unity . . ." This statement requires qualification; its main content is the "elastic analogue" of N. J. Hoff [7][6] which states that the stress patterns in a body creeping steadily (i.e., stresses constant in time) with Norton creep exponent $n = 1$, and elastic effects negligible, and an identical body exhibiting pure, incompressible elastic behavior are identical. However, elastic effects are included (equations (15), (16)) and the body is not creeping steadily. Further, if the pipeline had a pure elastic response incompressibility (Poisson's ratio $= 1/2$) would only be a special case. Consequently, putting $n = 1$ cannot possibly produce the elastic behavior. So what exactly do the authors intend?

In summary, the two main criticisms levelled at this paper are: (1) the effects of combined loading are decoupled, (2) there are inconsistencies arising apparently out of some misconceptions.

No attempt seems to have been made to justify the gross simplification involved in the authors' treatment of combined loading. With a nonlinear constitutive relation (elastic or inelastic) more than one load cannot be treated by adding the effects of the loads taken individually except in rather special, simple cases. To do so is not even an approximation, it is just not correct; the matter has already been pointed out to the authors [1]. To add individual stresses (elastic or creep) and use the relevant constitutive relationship would at least form an approximation. The authors may care to look up the early work of R. W. Bailey or the various contributions of A. E. Johnson. In [2] Finnie and Heller gave approximate results for the increase in bending curvature of a pipe due to internal pressure, while various approximate solutions for the same problem are given in [3]. Torsion is included in [4] and the inherent analytical approximations are removed in [5] (see also [6]). All of these results indicate a potentially large increase in deformation for reasonable load ratios; cases where this would not be true would be where one of the loads dominates. While these references are for forward creep the situation in relaxation is essentially similar, especially with the other approximations made by the authors.

The inconsistencies in the treatment of the creep behavior can only be explained in terms of a single assumption which does not seem to have been appreciated, namely, that the elastic and creep components of the total strains are both compatible with a respective displacement field. This is incorrect. However, it has been demonstrated over the past few years for both forward creep and relaxation, that it is valid to approximate the total deformation of a creeping body by adding a pure elastic part, found by ignoring creep effects, to a pure creep part, found by ignoring elastic effects (see [8, 9, 10 or 11]). The implication of this being that, in a certain sense, the creep component and elastic component of the total strain are compatible with respective displacement, or deformation, fields. Hence the assumption has little basis in physical behavior and is wholly a convenient mathematical approximation. It is therefore only in this context that the assumption may be acceptable.

In the light of the foregoing comments it is indeed surprising that the examples used should give "excellent agreement" with already available work. This agreement needs to be rationalized and put in perspective.

Firstly, consider the cantilever beam examples (i.e., straights only). The simple single load condition obscures our first criticism, and the determinacy of the structure makes the implicit assumption fairly accurate. We must, however, question the inclusion of Figs. 5 and 6, since these merely serve to test the computer program: surely such elementary considerations are superficial?

Secondly, consider the 90-deg bend. Again the computer program was tested by comparison with a program written by Rodabaugh, with excellent agreement. Allowing for the inclusion of the creep flexibility factor $k_{C/R}$ and the initial state from which the time integration proceeds, it appears from examination of both theories that they are equally based on the same assumptions and reduce to the same formulation (this is partly due to bends only being considered and partly due to the derivation of the curved pipe element from elastic theory). Nevertheless, the results of Fig. 7, and also of the main example, must be reconsidered: how may the authors justify a 90-deg bend with flexibility factors $k_E = k_{C/R} = 1$? Also for what geometric parameters and creep index can one have $k_E = 1.0$ with $k_{C/R} = 10.0$? In any case, is the comparison of Fig. 7 meaningful?

In conclusion, although our comments may be rather severe, we would like to say that we have every sympathy with the aims of the paper as we know from experience that the task is formidable. Unfortunately, we believe the problem cannot be simplified in the manner the authors seem to have described.

References

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2 Finnie, I., and Heller, W. R., *Creep of Engineering Materials*, McGraw-Hill, 1959.

[6] Numbers in brackets designate References at end of discussion.

3 Spence, J., *Nuclear Engineering Design*, Vol. 24, 1973, pp. 88–104.
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5 Boyle, J., and Spence, J., *Rozprawy ingynierskie*, (in press).
6 Spence, J., "Creep of Piping," International Mechanical Engineering Conference, Conference Publ. No. 5, Paper No. C48/76, 1974.
7 Hoff, N. J., *Quarterly of Applied Mathematics*, Vol. 12, 1954, pp. 49–55.
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9 Marriott, D. L., *Journal of Strain Analysis*, Vol. 3, 1968, pp. 188–296.
10 Spence, J., and Hult, J., *International Journal of Mechanical Science*, Vol. 15, 1973, pp. 741–755.
11 Rozenblium, V. I., Leningrad University "Investigations on Elasticity and Plasticity," No. 3, 1964, pp. 88–95.

Authors' Closure

The authors would like to thank Professors Boyle and Spence for their indepth review and comments of this paper. Techniques such as outlined in this paper require critical review by a number of investigators before any measure of acceptance or rejection by the general users of such techniques can be obtained.

To repeat and amplify part of the introduction of the paper, the objective of the total effort is to develop a design tool of adequate engineering accuracy which does not require a large computer, excessive running time, or highly complicated input data. Toward this end, certain "simplifying" assumptions had to be made to allow a solution within the above framework. At this time, and therefore not reported in this paper, an attempt is being made to answer the accuracy question both by experimental and analytical means. It should be pointed out that the primary purpose of this technique is to obtain an estimate of the structural response of moderately complex piping systems subjected to inelastic behavior and not the detailed local response of components.

The paper, as noted in the introduction, describes the second step in the total effort. In the final version of this technique initial plasticity behavior is incorporated and the second stage creep/relaxation solution begins from either an initial elastic or elasto-plastic solution.

As in any mathematical formulation certain assumptions have to be made to transform a real problem into one that is amenable to modeling. The most notable of these assumptions are

1 The plasticity behavior can be described by the well-known Ramberg-Osgood [3] formulation.

2 The creep/relaxation behavior can be described by equation (17), thus accounting for classical second-stage creep behavior.

3 The inelastic strains due to axial, bending, and torsional loading can be decoupled and strains due to internal pressure can be neglected.

4 The total strain due to a particular loading component is the sum of the elastic, plastic, creep/relaxation, and thermal strains.

5 Small deflection linear strain theory applies.

6 The uncoupled displacement pattern for each element can be approximated by the corresponding elastic displacement pattern.

7 The structural response of a curved pipe can be characterized by a curved beam formulation with appropriate flexibility factors to account for ovalization of the circular cross section during bending in the elastic, plastic, and creep/relaxation behavior regions.

Once these assumptions have been made, then utilizing the initial strain approach associated with the finite-element matrix-displacement method of analysis, the derivation of the solution is straight forward. The primary purpose of this paper was to present the derivation and give examples to show that the derivation gave solutions to the assumed mathematical problem.

Now let us address the specific comments raised by the discussors.

Comments (A) and (B). The principle of virtual displacements is applied to each element and, therefore, by allowing only external forces to be applied at the nodal points connecting elements, the element experiences only internal forces applied at its boundary. These internal forces are assumed to be completely recoverable by some unloading mechanism and are therefore assumed to be derived through equilibrium considerations from some elastic stress distribution. Thence the term "elastic force system" was utilized.

The strains are related to the displacements through geometric consideration utilizing assumed shape functions for the displacement response of the element. The assumed shape functions used herein were assumed to be the same for all behavior responses of the element, with a modification in the curved pipe element discussed below. These shape functions were assumed to be the same as those determined from the elastic solution. As an example the bending solution for the straight element was assumed to be a cubic function of the axial length coordinate (y), e.g., the first four terms of equations (49) and (51). By following through the logic we find that the unknown coefficients a_i are functions of the end displacements. Therefore, the strain at any point within the element is a function of its position and end displacements and not a function of any material coefficients or cross-sectional response parameters. In the curved pipe element these shape functions are given in equations (65) and (66). Although it appears that these shape functions are functions of material coefficients and cross-sectional response parameters, in reality this is not true. If, for example, the equation for g_i or h_i are expanded using the definitions for the stiffness matrix terms, than these equations, although far more complex, would degenerate to functions of geometric parameters only. The modification utilized in the curved pipe element is that in the use of these equations for the inelastic behavior the elastic flexibility factor is replaced by an inelastic flexibility factor. This inelastic flexibility factor is discussed in detail by the authors [3] in an earlier paper.

The foregoing assumption (3) appears to carry the brunt of comments directed at the basic assumptions associated with the mathematical formulation utilized herein. However, it is this assumption that allows for the computational efficiency of this technique to handle moderately complex piping system. The authors realize that at this time the validity of this assumption to allow for solutions that are within reasonable engineering accuracy for real piping system is not known. Indeed the authors pointed this out in this and other papers and discussions associated with this work. This assumption is one of the main reasons the authors have described this technique as a simplified approach. The authors have undertaken both experimental and analytical means to answer this accuracy question. In ongoing work the authors hope to eliminate this assumption, if necessary, and still maintain a computationally efficient program.

Comments (C) and (E). It is true that "the more interesting and novel derivation of a curved pipe element is missing." This aspect was discussed in detail in reference [3] of the paper. The purpose of the paper was *not* to cover the curved pipe element (which is a broad subject in itself, the literature includes several papers by Prof. Spence), but rather to cover the second stage creep/relaxation analysis of three-dimensional piping systems which, while the method is simplified, still represents a complex analysis procedure.

Comment (D). As reiterated in the introduction to this closure, we do not believe that any step has been omitted within the framework of the listed assumptions.

Comment (F). These questions address the significant problem of initial plasticity of the piping system; an aspect *not* covered by the present paper. As mentioned in the introduction to this closure, subsequent work has lead to inclusion of initial plasticity effects although an analogous simplified approach is used.

Comment (G). The paper apparently did not convey the full implication of using a creep/relaxation exponent equal to unity and failed to mention the (perhaps obvious) auxillary requirement that the value of k/σ_0^n must be equal to $1/E$. The point is that the analysis method gives *no* creep or relaxation with the creep/relaxation exponent equal to unity.

Summary

1 The discussers and authors are in complete agreement that the decoupling of combined loads involves an approximation whose effect cannot, at present, be estimated. The point is made in the paper itself.

2 While there are certainly a number of simplifying assumptions used in the analysis, the authors are not aware of any inconsistency in the analysis. It should be recalled that the paper does *not* address the problem of initial plasticity.

Discussion Following Summary. The discussion following the "Summary" appears to be dividable into four parts:

(*a*) approximations of combined loading
(*b*) compatibility of creep/elastic strain components
(*c*) discussion of examples
(*d*) conclusion.

We will attempt to answer the discussion under these subheadings.

(a) Approximations of Combined Loadings. The discussers cite several references which attempt to solve the problem of combined loadings and indeed, these and other similar published papers give guidance in this area. However, much of the literature addresses relatively simple geometries such as a straight pipe. As mentioned in the paper, the elbow poses formidable problems; even in the elastic regime there can be a complex, nonlinear interaction between moment loading and internal pressure loading.

(b) Compatibility of Creep/Elastic Strain Components. The discussers point here is not clear to the authors. In the relaxation analysis the primary purpose of the analysis method is to determine the rate at which elastic strains are converted into plastic strains; thus determining the load as a function of time as elastic and plastic strains remain compatible.

(c) Discussion of Examples. The discussers point out that the cantilever examples are quite simple and indeed this is true. However, the analysis method of a three-dimensional piping system, with a method that allows for multiple anchors, branches and local restraints, is not simple even in the elastic regime. In the creep/relaxation regime, a further complication exists in the solution techniques for plastic behavior. Accordingly, while the authors would have preferred comparisons with more realistic piping systems, if such could be found in the literature, the check of the computer program on the simplest of models is at least a step in the "de-bugging" process.

The discussors also point out that the basic theory underlying the computer program described in the paper (now called PIRAX2) and the computer program ELLAX is basically the same. This is true, however, PIRAX2 is a relatively complex program and uses different solution techniques than ELLAX. Again, for lack of published data on plastic response of piping systems, the agreement between PIRAX2 and ELLAX was found to be reassuring to the authors, at least as another step in debugging PIRAX2.

The discussors question the k_E and k_{CIR} factors used in Fig. 7; particularly $k_E = k_{CIR} = 1.0$. While Fig. 7 was not intended to represent and real piping system, the choice fo $k_E = k_{CIR} = 1.0$ could represent a real curved pipe. It could, for example, represent a curved pipe with sufficiently large bend radius so that the curved pipe acts like straight pipe in both the elastic and plastic regime.

(d) Conclusions. We agree that the paper is not a complete solution and, indeed, the paper is identified in the introduction as a second step in an ongoing program. Whether the simplifications used will turn out to be useful or not is, in the authors' view, a judgment which cannot be made at this time. In this regard it seems appropriate to briefly discuss the broad aspects of the analytical prediction of the behavior of high-temperature piping systems.

First, as probably apparent to those readers who have been involved in startups and shutdowns (sometimes crash shutdowns) of high-temperature plants and piping systems, the accurate prediction of the loading-temperature-time history itself is a formidable problem. Second, the plastic response of materials used in piping systems is itself quite variable; even the yield strength of material purchased to a given specification may vary by a factor of 2 from 95 percent probability minimum to maximum. Accordingly, even if we had a precise analytical method that method might, in real life, be useful only to the extent of indicating crude (but useful) engineering bounds. Precise analytical methods, in the form of elastic-plastic finite element programs, are becoming available. Unfortunately, the cost of running such programs prohibits their general application to piping systems. It seems, therefore, that a need exists for a simplified inelastic piping system analysis procedure. Obviously, and this is perhaps the main theme of the discussion and closure, the limits of usefulness of such a procedure must be checked out by comparison with test data, precise analytical methods and experience with existing high-temperature piping systems.

APPENDIX 9

REFERENCE ABBREVIATIONS

The following abbreviations are used in the listing of references:

AAM	Advances in Applied Mathematics
AIAA	American Institute for Aeronautics and Astronautics
AMS	American Mathematical Society
Ann.Mat	Annals of Mathematics
Apl. Mat	Aplikace Matematik (Czech)
Arch.Mech	Archives of Mechanics
ARMA	Archives for Rational Mechanics and Analysis
ASCE	American Society of Civil Engineers
-, JEM	-, Journal of Engineering Mechanics
ASR	Applied Scientific Research
BIMA	Bulletin of the Institute of Mathematics and its Applications
CMJ	Czechoslovakian Mathematics Journal
CRAS	Comptes Rendues Academie des Sciences (Paris)
CS	Computers and Structures
Exp.Mech	Experimental Mechanics
IA	Ingenieur Archiv (German)
IJES	International Journal of Engineering Science
IJMS	International Journal of Mechanical Sciences
IJNME	International Journal of Numerical methods in Engineering
IJNLM	International Journal of Nonlinear Mechanics
IJSS	International Journal of Solids and Structures
JAP	Journal of Applied Physics
JAMP	Journal of Applied Mathematics and Physics
JAS	Journal of Aeronautical Sciences
JDE	Journal of Differential Equations

JEM	Journal of Engineering Mathematics
JIMA	Journal of the Institute of Mathematics and its Applications
JMAA	Journal of Mathematical Analysis and applications
JMES	Journal of Mechanical Engineering Science
JMP	Journal of Mathematics and Physics
JMPS	Journal of Mechanics and Physics of Solids
JMPSc	Journal of Mathematics and Physical Sciences
JOTA	Journal of Optimization theory and applications
JRAS	Journal of Royal Aeronautical Society
JSA	Journal of Strain analysis
JSM	Journal of Structural Mechanics
JSME	Japan Society of Mechanical Engineers
Math.Syst.Th.	Mathematical Systems Theory
NED	Nuclear Engineering and design
NM	Numerische Mathematik (German)
ORNL	Oak Ridge National Laboratory
Phy.Rev.	Physics Review
PMM	Prikladnya Mathematika Mechanika (Russian)
QAM	Quarterly of Applied Mathematics
Rheol.Act	Rheologica Acta
RMS	Russian Mathematical Surveys
SAM	Studies in Applied Mathematics
SA Mech	Soviet Applied Mechanics
SIAM	Society for Industrial and applied mathematics
-, JAM	-, Journal of Applied Mechanics
SMD	Soviet Mathematics Doklady
SPD	Soviet Physics Doklady
Trans.ASME	Transactions of the Americal Society of Mechanical Engineers
-, JAM	Journal of Applied Mechanics
-, JBE	Journal of Basic Engineering

-, JEI	Journal of Engineering for industry
-, JPVT	Journal of Pressure Vessel Technology
WRC	Welding Research Council
ZAMM	Zeitschrift fur Angewandte Mathematik und Mechanik