

Analysis, Approximation and Control
of Hybrid Stochastic Systems

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September 19, 2026

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Acknowledgements

First of all, I would like to thank my supervisor, Professor Xuerong Mao, for his guidance, patience and continuous support throughout my PhD. I have learned so much from him over the past four years, and I am grateful for his encouragement and advice at every stage of my research. Beyond academic research, he has also taught me many valuable lessons about personal growth, which I will carry with me long after my PhD. It has been a privilege to work under his supervision.

I owe my deepest thanks to my parents and my sister for their unconditional love and support throughout my PhD. Their encouragement has always given me the confidence to keep going, especially during the difficult periods.

I was fortunate to have my girlfriend with me during the writing of this thesis. Her encouragement and companionship made the final stage of my PhD much easier, and they meant more to me than she knows.

Finally, I would like to thank my two cats, Chouchou and Milk. Although they know nothing about stochastic differential equations, they have been by my side throughout my PhD and have brought plenty of joy during the long days of research and writing.

To everyone mentioned above, and to those not named but no less cherished, thank you for being part of this journey.

Abstract

Hybrid stochastic systems involving memory effects and Markovian switching have been widely used to model a variety of dynamical systems arising in science and engineering. This thesis addresses two fundamental problems in the study of hybrid stochastic systems: the numerical approximation of hybrid stochastic functional differential equations with infinite time delay and the stabilisation of hybrid stochastic differential equations by event-triggered control.

For hybrid stochastic functional differential equations with infinite time delay, the convergence theory of the truncated Euler–Maruyama method is established under a local Lipschitz condition together with a Khasminskii-type condition. The analysis is further extended under a dissipative Khasminskii-type condition, leading to a strong convergence theory over the entire range $q \in [1, p)$. Under suitable additional conditions, convergence rate estimates are also obtained.

For hybrid stochastic differential equations, an event-triggered control framework based on trajectories of discrete-time observations is developed. Both state-feedback and output-feedback controllers are considered. By establishing a connection between the event-triggered controlled system and a corresponding uncertain stochastic system, sufficient conditions guaranteeing exponential mean-square stability are derived.

Taken together, the work presented in this thesis broadens the existing theory for hybrid stochastic systems by extending the convergence theory for numerical methods under weaker assumptions and developing an event-triggered control framework based on discrete-time observations, thereby contributing to the analysis, numerical approximation and control of hybrid stochastic systems.

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Notation

positive	> 0 .
nonpositive	≤ 0 .
negative	< 0 .
nonnegative	≥ 0 .
a.s.	almost surely, or P -almost surely, or with probability 1.
$A := B$	A is defined by B or A is denoted by B .
$A(x) \equiv B(x)$	$A(x)$ and $B(x)$ are identically equal, i.e. $A(x) = B(x)$ for all x .
$\emptyset$	the empty set.
$\mathbf{1}_A$	the indicator function of a set A , i.e. $\mathbf{1}_A(x) = 1$ if $x \in A$ and otherwise 0.
A^c	the complement of A in Ω , i.e. $A^c = \Omega - A$.
$A \subset B$	$A \cap B^c = \emptyset$.
$A \subset B$ a.s.	$P(A \cap B^c) = 0$.
$\sigma(\mathcal{C})$	the σ -algebra generated by $\mathcal{C}$.
$a \vee b$	the maximum of a and b .
$a \wedge b$	the minimum of a and b .
$f : A \rightarrow B$	the mapping f from A to B .
$\mathbb{R} = \mathbb{R}^1$	the real line.
$\mathbb{R}_+$	the set of all non-negative real numbers, i.e. $\mathbb{R}_+ = [0, \infty)$.
$\mathbb{R}_-$	the set of all non-positive real numbers, i.e. $\mathbb{R}_- = (-\infty, 0]$.
$\mathbb{R}^d$	the d -dimensional Euclidean space.
$\mathbb{R}_+^d$	$\{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\}$, i.e. the positive cone.

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$\mathcal{B}^d$	the Borel- σ -algebra on $\mathbb{R}^d$.
$\mathcal{B}$	$:= \mathcal{B}^1$.
$\mathbb{R}^{d \times m}$	the space of real $d \times m$ matrices.
$\mathcal{B}^{d \times m}$	the Borel- σ -algebra on $\mathbb{R}^{d \times m}$.
$\mathbb{C}^d$	the d -dimensional complex space.
$\mathbb{C}^{d \times m}$	the space of complex $d \times m$ matrices.
$ x $	the Euclidean norm of a vector x .
S_h	$:= \{x \in \mathbb{R}^d : x \leq h\}$.
A^T	the transpose of a vector or matrix A .
(x, y)	the scalar product of vectors x and y , i.e. $(x, y) = x^T y$.
$\text{trace}(A)$	the trace of a square matrix $A = (a_{ij})_{d \times d}$, i.e. $\text{trace}(A) = \sum_{i=1}^d a_{ii}$.
$\lambda_{\min}(A)$	the smallest eigenvalue of a matrix A .
$\lambda_{\max}(A)$	the largest eigenvalue of a matrix A .
$\lambda_{\max}^+(A)$	$:= \sup_{x \in \mathbb{R}_+^d, x =1} x^T A x$.
$ A $	$:= \sqrt{\text{trace}(A^T A)}$, i.e. the trace norm of a matrix A .
$\ A\ $	$:= \sup\{ Ax : x = 1\} = \sqrt{\lambda_{\max}(A^T A)}$, i.e. the operator norm of a matrix A .
$A_1 \leq A_2$	for symmetric matrices A_1 and A_2 , means $A_1 - A_2 \leq 0$.
$A_1 < A_2$	for symmetric matrices A_1 and A_2 , means $A_1 - A_2 < 0$.
δ_{ij}	Dirac's delta function, that is $\delta_{ij} = 1$ if $i = j$ and otherwise 0.
$C(D; \mathbb{R}^d)$	the family of continuous $\mathbb{R}^d$ -valued functions defined on D .
$C^m(D; \mathbb{R}^d)$	the family of continuously m -times differentiable $\mathbb{R}^d$ -valued functions defined on D .
$C_0^m(D; \mathbb{R}^d)$	the family of functions in $C^m(D; \mathbb{R}^d)$ with compact support in D .
$C^{2,1}(D \times \mathbb{R}_+; \mathbb{R})$	the family of all real-valued functions $V(x, t)$ defined on $D \times \mathbb{R}_+$ which are continuously twice differentiable in $x \in D$ and once differentiable in $t \in \mathbb{R}_+$.

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∇	$:= \left(\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_d} \right).$
Δ	the Laplace operator, i.e. $\Delta = \sum_{i=1}^d \frac{\partial^2}{\partial x_i^2}.$
V_x	$:= \nabla V = (V_{x_1}, \dots, V_{x_d}) = \left(\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_d} \right).$
V_{xx}	$:= (V_{x_i x_j})_{d \times d} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{d \times d}.$
$\ \xi\ _{L^p}$	$:= (\mathbb{E} \xi ^p)^{1/p}.$
$L^p(\Omega; \mathbb{R}^d)$	the family of $\mathbb{R}^d$ -valued random variables ξ with $\mathbb{E} \xi ^p < \infty.$
$L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^d)$	the family of $\mathbb{R}^d$ -valued $\mathcal{F}_t$ -measurable random variables ξ with $\mathbb{E} \xi ^p < \infty.$
$C([-\tau, 0]; \mathbb{R}^d)$	the space of all continuous $\mathbb{R}^d$ -valued functions φ defined on $[-\tau, 0]$ with a norm $\ \varphi\ = \sup_{-\tau \leq \theta \leq 0} \varphi(\theta) .$
$L^p_{\mathcal{F}}([-\tau, 0]; \mathbb{R}^d)$	the family of all $C([-\tau, 0]; \mathbb{R}^d)$ -valued random variables ϕ such that $\mathbb{E}\ \phi\ ^p < \infty.$
$L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^d)$	the family of all $\mathcal{F}_t$ -measurable $C([-\tau, 0]; \mathbb{R}^d)$ -valued random variables ϕ such that $\mathbb{E}\ \phi\ ^p < \infty.$
$C^b_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^d)$	the family of $\mathcal{F}_t$ -measurable bounded $C([-\tau, 0]; \mathbb{R}^d)$ -valued random variables.
$L^p([a, b]; \mathbb{R}^d)$	the family of Borel measurable functions $h : [a, b] \rightarrow \mathbb{R}^d$ such that $\int_a^b h(t) ^p dt < \infty.$
$\mathcal{L}^p([a, b]; \mathbb{R}^d)$	the family of $\mathbb{R}^d$ -valued $\mathcal{F}_t$ -adapted processes $\{f(t)\}_{a \leq t \leq b}$ such that $\int_a^b f(t) ^p dt < \infty$ a.s.
$\mathcal{M}^p([a, b]; \mathbb{R}^d)$	the family of processes $\{f(t)\}_{a \leq t \leq b}$ in $\mathcal{L}^p([a, b]; \mathbb{R}^d)$ such that $\mathbb{E} \int_a^b f(t) ^p dt < \infty.$
$\mathcal{L}^p(\mathbb{R}_+; \mathbb{R}^d)$	the family of processes $\{f(t)\}_{t \geq 0}$ such that for every $T > 0$, $\{f(t)\}_{0 \leq t \leq T} \in \mathcal{L}^p([0, T]; \mathbb{R}^d).$
$\mathcal{M}^p(\mathbb{R}_+; \mathbb{R}^d)$	the family of processes $\{f(t)\}_{t \geq 0}$ such that for every $T > 0$, $\{f(t)\}_{0 \leq t \leq T} \in \mathcal{M}^p([0, T]; \mathbb{R}^d).$
$\text{sign}(x)$	the sign function, that is $\text{sign}(x) = +1$ if $x \geq 0$ or otherwise $-1.$

Contents

Chapter 1

Introduction

1.1 Background

Many real-world dynamical systems are inevitably influenced by random disturbances arising from external environments, measurement errors, or intrinsic uncertainties. Stochastic differential equations (SDEs) have become one of the fundamental mathematical tools for modelling such systems and have been widely applied in engineering, finance, biology, economics and many other scientific disciplines [22, 24, 35, 45]. Over the past several decades, the mathematical theory of stochastic differential equations has been extensively developed, providing a rigorous foundation for the qualitative analysis and numerical simulation of stochastic dynamical systems. These developments have laid the theoretical foundation for the study of more general stochastic systems involving memory effects and random switching.

In many practical applications, however, the future evolution of a system depends not only on its current state but also on its past history. Such memory effects naturally lead to stochastic functional differential equations (SFDEs), in which the drift and diffusion coefficients depend on the current state together with the past trajectory of the solution [22, 42, 48]. Delay equations constitute an important class of SFDEs and have been widely used to model systems with after-effects and hereditary characteristics. Depending on the length of the delay, these models are commonly divided into finite-delay and infinite-delay systems. In particular, systems with infinite delay allow

the entire past history of the solution to influence its future evolution and provide a more realistic description of long-memory phenomena found in areas such as biology, economics and engineering. Compared with finite-delay models, infinite-delay systems are considerably more difficult to analyse because the associated state space is infinite dimensional [7, 22, 25, 30, 47].

Besides memory effects, many stochastic systems evolve under randomly changing operating environments. Such phenomena are commonly modelled by Markov chains, leading to hybrid stochastic systems with Markovian switching [21, 35]. By incorporating both random switching and infinite memory, one obtains hybrid stochastic functional differential equations with infinite time delay, which provide a flexible mathematical framework for modelling a broad range of practical systems. Throughout this thesis, the systems under consideration can be described by the general form

$$dx(t) = f(x_t, \theta(t), t) dt + g(x_t, \theta(t), t) dB(t), \quad (1.1)$$

where $\{B(t)\}_{t \geq 0}$ is a Brownian motion, $\{\theta(t)\}_{t \geq 0}$ is a finite-state Markov chain, and x_t denotes the history segment of the solution.

The combination of stochastic perturbations, infinite memory and random switching considerably increases the mathematical complexity of these systems. As a result, both their theoretical analysis and practical implementation present significant challenges, motivating further research on numerical methods and feedback control strategies.

1.2 Motivation

For most nonlinear stochastic systems, explicit solutions are generally unavailable, making numerical approximation indispensable for practical computation. Consequently, the numerical analysis of stochastic differential equations and stochastic functional differential equations has been extensively studied over the past several decades. Classical convergence theory for numerical methods is largely developed under global Lipschitz conditions, under which the Euler–Maruyama method and its variants pos-

sess well-established convergence and stability properties [12,35,42,68]. However, many stochastic models encountered in applications exhibit nonlinear behaviours that violate the global Lipschitz condition. This has motivated the development of numerical methods under weaker assumptions, allowing a wider class of stochastic systems to be analysed rigorously.

The numerical approximation of hybrid stochastic functional differential equations with infinite time delay presents additional challenges. Besides the stochastic perturbation, both the infinite-dimensional history space and the randomly switching environments substantially increase the complexity of the analysis. Although recent work has established an effective approximation framework for hybrid stochastic functional differential equations with infinite delay [25], the available convergence theory is still restricted to relatively strong assumptions on the system coefficients. Developing numerical methods and rigorous convergence theory under weaker conditions therefore remains an important problem. This forms the primary motivation for the first part of this thesis.

Besides numerical approximation, the stabilisation of stochastic systems is another fundamental topic in stochastic analysis and control. Feedback control has proved to be an effective approach for stabilising unstable stochastic systems and has been widely studied [33,36,44]. Conventional continuous-time feedback, however, requires continuous observation and communication, which may be difficult or prohibitively expensive in many practical applications. To reduce unnecessary communication while preserving the desired stability properties, event-triggered control has emerged as an attractive alternative and has attracted increasing attention in recent years.

Existing studies on event-triggered control have mainly focused on deterministic systems or stochastic systems under relatively restrictive observation mechanisms. For hybrid stochastic differential equations with Markovian switching, particularly those based on trajectories of discrete-time observations, the available theoretical results remain relatively limited. This motivates the second part of this thesis, which investigates event-triggered stabilisation for hybrid stochastic differential equations under discrete-time state and output observations.

1.3 Main Contributions

For the numerical approximation problem, the convergence theory of the truncated Euler–Maruyama method is established under progressively weaker assumptions. Beginning with the local Lipschitz condition and Khasminskii-type condition, the analysis is further extended to a dissipative Khasminskii-type framework, resulting in the convergence theory over the entire range $q \in [1, p)$. The resulting theory significantly extends the existing convergence results to a substantially broader class of hybrid stochastic functional differential equations with infinite time delay.

For the stabilisation problem, an event-triggered control framework based on trajectories of discrete-time observations is developed for hybrid stochastic differential equations. The framework accommodates both state-feedback and output-feedback controllers, and establishes sufficient conditions for exponential mean-square stability through the analysis of a corresponding uncertain stochastic system. Compared with conventional continuous-time feedback strategies, it provides a communication-efficient approach while maintaining the desired stability properties.

1.4 Organisation of the Thesis

The remainder of this thesis is organised as follows.

Chapter 2 introduces the notation, preliminary results and several auxiliary inequalities that will be used throughout the thesis.

Chapter 3 investigates the numerical approximation of hybrid stochastic functional differential equations with infinite time delay. Under a local Lipschitz condition together with a Khasminskii-type condition, the strong convergence and convergence rate of the truncated Euler–Maruyama method are established for $q \geq 2$.

Chapter 4 extends the numerical analysis developed in Chapter 3 by introducing a dissipative Khasminskii-type condition. The corresponding convergence theory is established for the previously unexplored range $q \in [1, 2)$, yielding a unified convergence framework for the truncated Euler–Maruyama method over the entire range $q \in [1, p)$.

Chapter 5 is devoted to the stabilisation of hybrid stochastic differential equations

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by event-triggered control based on trajectories of discrete-time observations. Both state-feedback and output-feedback controllers are considered, and sufficient conditions for exponential mean-square stability are derived.

Chapter 6 concludes the thesis and discusses several possible directions for future research.

Parts of the work presented in this thesis have appeared in the following publications.

- J. Zhou, H. Xu and X. Mao, “Truncated Euler–Maruyama Method for Hybrid Stochastic Functional Differential Equations with Infinite Time Delay”, [69], *Journal of Computational and Applied Mathematics*, 472 (2026), 116773.
- W. Liu, W. Mao, X. Mao and J. Zhou, “Stabilization of Hybrid Systems by Event-Triggered Control Based on Discrete-Time State Observations” , [31], *SIAM Journal on Control and Optimization*, 63 (2025), No. 5, 3501–3525.

Chapter 2

Preliminaries

2.1 Basic probability theory

Measurable spaces

Probability theory provides a mathematical framework for modelling random phenomena. Let Ω denote the sample space, i.e., the set of all possible outcomes of a random experiment, and let $\omega \in \Omega$ represent a typical outcome.

To assign probabilities in a mathematically consistent manner, one considers a collection $\mathcal{F}$ of subsets of Ω , called events. The collection $\mathcal{F}$ is required to satisfy certain properties, which lead to the notion of a σ -algebra. Specifically, $\mathcal{F}$ must satisfy the following conditions:

1. $\emptyset \in \mathcal{F}$;
2. If $A \in \mathcal{F}$, then $A^c \in \mathcal{F}$;
3. If $\{A_n\}_{n \geq 1} \subseteq \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

A family $\mathcal{F}$ satisfying the above conditions is called a σ -algebra on Ω . The pair $(\Omega, \mathcal{F})$ is referred to as a measurable space, and the elements of $\mathcal{F}$ are called $\mathcal{F}$ measurable sets. Given a collection $\mathcal{C} \subseteq \Omega$, the smallest σ -algebra containing $\mathcal{C}$ is denoted by $\sigma(\mathcal{C})$. Equivalently, $\sigma(\mathcal{C})$ is the intersection of all σ -algebras containing $\mathcal{C}$. The σ -algebra $\sigma(\mathcal{C})$ is said to be generated by $\mathcal{C}$.

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A particularly important example arises when $\Omega = \mathbb{R}^d$. Let $\mathcal{C}$ denote the collection of all open subsets of $\mathbb{R}^d$. The σ -algebra generated by $\mathcal{C}$, $\mathcal{B}(\mathbb{R}^d) = \sigma(\mathcal{C})$ is called the Borel σ -algebra on $\mathbb{R}^d$ and is denoted by $\mathcal{B}^d$. The elements of $\mathcal{B}^d$ are known as Borel sets.

Random variables

Let $(\Omega, \mathcal{F})$ be a measurable space. A function $X : \Omega \rightarrow \mathbb{R}$ is said to be $\mathcal{F}$ -measurable if

$$\{\omega \in \Omega : X(\omega) \leq a\} \in \mathcal{F}, \quad \forall a \in \mathbb{R}.$$

Such a function is called a real-valued random variable. More generally, an $\mathbb{R}^d$ -valued function $X = (X_1, \dots, X_d)^T$ is said to be $\mathcal{F}$ -measurable if each component X_i is $\mathcal{F}$ -measurable. Similarly, a matrix-valued function $X = (X_{ij})_{d \times m}$ is $\mathcal{F}$ -measurable whenever all its entries X_{ij} are $\mathcal{F}$ -measurable. In particular, when the measurable space is $(\mathbb{R}^d, \mathcal{B}^d)$, measurable functions are referred to as Borel measurable functions.

For any set $A \subseteq \Omega$, the indicator function of A is defined by

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

It follows that $\mathbf{1}_A$ is $\mathcal{F}$ -measurable if and only if $A \in \mathcal{F}$.

For a mapping $X : \Omega \rightarrow \mathbb{R}^d$, the σ -algebra generated by X , denoted by $\sigma(X)$, is defined as $\sigma(X) = \sigma(X^{-1}(B) : B \in \mathcal{B}^d)$. Equivalently, $\sigma(X)$ is the smallest σ -algebra on Ω with respect to which X is measurable. In particular, if X is $\mathcal{F}$ -measurable, then $\sigma(X) \subseteq \mathcal{F}$. More generally, for a family of random variables $\{X_i\}_{i \in I}$, the σ -algebra generated by $\{X_i : i \in I\}$ is defined by $\sigma(X_i : i \in I) = \sigma(\bigcup_{i \in I} \sigma(X_i))$. This is the smallest σ -algebra with respect to which every X_i is measurable.

A probability measure on a measurable space $(\Omega, \mathcal{F})$ is a mapping $P : \mathcal{F} \rightarrow [0, 1]$ satisfying $P(\Omega) = 1$, and

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

for every sequence of pairwise disjoint sets $\{A_i\}_{i \geq 1} \subseteq \mathcal{F}$. The triple $(\Omega, \mathcal{F}, P)$ is called

a probability space and forms the fundamental framework for describing random phenomena.

Expectation

Let X be an integrable real-valued random variable defined on $(\Omega, \mathcal{F}, P)$. The expectation of X is defined by

$$\mathbb{E}[X] = \int_{\Omega} X(\omega) dP(\omega),$$

provided that the integral exists. The expectation represents the average or mean value of the random variable with respect to the probability measure P .

More generally, for $p \geq 1$, the quantity $\mathbb{E}[|X|^p]$ is called the p -th moment of X . In particular, if $\mathbb{E}[X^2] < \infty$, the variance of X is defined as

$$\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}[X])^2],$$

which measures the variability of X around its mean.

An $\mathbb{R}^d$ -valued random variable $X = (X_1, \dots, X_d)^T$ induces a probability measure μ_X on $(\mathbb{R}^d, \mathcal{B}^d)$ through $\mu_X(B) = P(X \in B)$, $B \in \mathcal{B}^d$. The measure μ_X is called the distribution (or law) of X . The expectation of X can equivalently be expressed in terms of its distribution as

$$\mathbb{E}[X] = \int_{\mathbb{R}^d} x d\mu_X(x).$$

More generally, if $g : \mathbb{R}^d \rightarrow \mathbb{R}^m$ is a Borel measurable function, then

$$\mathbb{E}[g(X)] = \int_{\mathbb{R}^d} g(x) d\mu_X(x),$$

whenever the integral is well defined. This identity is often referred to as the change-of-variable formula for probability measures.

For $p \in (0, \infty)$, let $L^p = L^p(\Omega; \mathbb{R}^d)$ denote the collection of $\mathbb{R}^d$ -valued random variables satisfying $\mathbb{E}|X|^p < \infty$. In particular, for $X \in L^1$, $|\mathbb{E}[X]| \leq \mathbb{E}[|X|]$. The following inequalities will be used repeatedly throughout this thesis.

(1) Hölder's Inequality. Let $p, q > 1$ be conjugate exponents satisfying $\frac{1}{p} + \frac{1}{q} = 1$.

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Then, for any $X \in L^p$ and $Y \in L^q$,

$$|\mathbb{E}[X^T Y]| \leq (\mathbb{E}|X|^p)^{1/p} (\mathbb{E}|Y|^q)^{1/q}.$$

(2) Minkowski's Inequality. For $p > 1$ and $X, Y \in L^p$,

$$(\mathbb{E}|X + Y|^p)^{1/p} \leq (\mathbb{E}|X|^p)^{1/p} + (\mathbb{E}|Y|^p)^{1/p}.$$

(3) Chebyshev's Inequality. For any $c > 0$ and $X \in L^p$,

$$P(|X| \geq c) \leq \frac{\mathbb{E}|X|^p}{c^p}.$$

As an immediate consequence of Hölder's inequality, $(\mathbb{E}|X|^r)^{1/r} \leq (\mathbb{E}|X|^p)^{1/p}$, $0 < r < p < \infty$.

Conditional expectation

Let $A, B \in \mathcal{F}$ with $P(B) > 0$. The conditional probability of A given B is defined by

$$P(A | B) = \frac{P(A \cap B)}{P(B)}.$$

The notion of conditional probability can be extended to a more general framework through conditional expectation. Let $X \in L^1(\Omega; \mathbb{R})$ and let $\mathcal{G} \subseteq \mathcal{F}$ be a sub- σ -algebra. In general, X need not be $\mathcal{G}$ -measurable. We seek a $\mathcal{G}$ -measurable random variable Y satisfying $\mathbb{E}[\mathbf{1}_G Y] = \mathbb{E}[\mathbf{1}_G X]$, $\forall G \in \mathcal{G}$, or equivalently,

$$\int_G Y(\omega) dP(\omega) = \int_G X(\omega) dP(\omega), \quad \forall G \in \mathcal{G}.$$

By the Radon–Nikodym theorem, such a random variable exists and is unique up to sets of probability zero. It is called the conditional expectation of X with respect to $\mathcal{G}$ and is denoted by $Y = \mathbb{E}[X | \mathcal{G}]$. If $\mathcal{G} = \sigma(Y)$ is the σ -algebra generated by a random variable Y , it is customary to write $\mathbb{E}[X | \mathcal{G}] = \mathbb{E}[X | Y]$.

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As an illustrative example, let $\{A_k\}$ be a partition of Ω satisfying

$$\bigcup_k A_k = \Omega, \quad P(A_k) > 0, \quad A_k \cap A_j = \emptyset, \quad k \neq j,$$

and let $\mathcal{G} = \sigma(\{A_k\})$. Then $\mathbb{E}[X \mid \mathcal{G}]$ is the step function

$$\mathbb{E}[X \mid \mathcal{G}] = \sum_k \frac{\mathbf{1}_{A_k} \mathbb{E}[\mathbf{1}_{A_k} X]}{P(A_k)}.$$

Equivalently, for $\omega \in A_k$,

$$\mathbb{E}[X \mid \mathcal{G}](\omega) = \frac{\mathbb{E}[\mathbf{1}_{A_k} X]}{P(A_k)}.$$

The conditional expectation satisfies $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}]] = \mathbb{E}[X]$ and

$$|\mathbb{E}[X \mid \mathcal{G}]| \leq \mathbb{E}[|X| \mid \mathcal{G}] \quad \text{a.s.}$$

Furthermore, the following properties hold almost surely:

- (1) $\mathcal{G} = \{\emptyset, \Omega\} \Rightarrow \mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$;
- (2) $X \geq 0 \Rightarrow \mathbb{E}[X \mid \mathcal{G}] \geq 0$;
- (3) X is $\mathcal{G}$ -measurable $\Rightarrow \mathbb{E}[X \mid \mathcal{G}] = X$;
- (4) $X = c$ (constant) $\Rightarrow \mathbb{E}[X \mid \mathcal{G}] = c$;
- (5) $\mathbb{E}[aX + bY \mid \mathcal{G}] = a \mathbb{E}[X \mid \mathcal{G}] + b \mathbb{E}[Y \mid \mathcal{G}]$;
- (6) $X \leq Y \Rightarrow \mathbb{E}[X \mid \mathcal{G}] \leq \mathbb{E}[Y \mid \mathcal{G}]$;
- (7) X is $\mathcal{G}$ -measurable $\Rightarrow \mathbb{E}[XY \mid \mathcal{G}] = X \mathbb{E}[Y \mid \mathcal{G}]$;
- (8) if $\sigma(X)$ and $\mathcal{G}$ are independent, then $\mathbb{E}[X \mid \mathcal{G}] = \mathbb{E}[X]$;
- (9) if $\mathcal{G}_1 \subseteq \mathcal{G}_2 \subseteq \mathcal{F}$, then $\mathbb{E}[\mathbb{E}[X \mid \mathcal{G}_2] \mid \mathcal{G}_1] = \mathbb{E}[X \mid \mathcal{G}_1]$.

Finally, for an $\mathbb{R}^d$ -valued random variable $X = (X_1, \dots, X_d)^T$, its conditional expectation with respect to $\mathcal{G}$ is defined componentwise by

$$\mathbb{E}[X | \mathcal{G}] = (\mathbb{E}[X_1 | \mathcal{G}], \dots, \mathbb{E}[X_d | \mathcal{G}])^T.$$

For further background on probability theory, the reader is referred to [59].

2.2 Stochastic processes

Stochastic basis

Let $(\Omega, \mathcal{F}, P)$ be a probability space. A filtration is a family $\{\mathcal{F}_t\}_{t \geq 0}$ of sub- σ -algebras of $\mathcal{F}$ satisfying $\mathcal{F}_t \subseteq \mathcal{F}_s \subseteq \mathcal{F}$, $0 \leq t \leq s$. The filtration is said to be right-continuous if $\mathcal{F}_t = \bigcap_{s > t} \mathcal{F}_s$, $t \geq 0$. A filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfies the usual conditions if the filtration is right-continuous and $\mathcal{F}_0$ contains all P -null sets. Throughout this thesis, unless otherwise specified, all filtrations are assumed to satisfy the usual conditions. We further define $\mathcal{F}_\infty = \bigcup_{t \geq 0} \mathcal{F}_t$.

A family of $\mathbb{R}^d$ -valued random variables $X = \{X_t\}_{t \in I}$ defined on $(\Omega, \mathcal{F}, P)$ is called a stochastic process, where I denotes the index set. In most applications, $I = [0, \infty)$. For each fixed $t \in I$, the mapping $\omega \mapsto X_t(\omega)$ is an $\mathbb{R}^d$ -valued random variable, while for each fixed $\omega \in \Omega$, the mapping $t \mapsto X_t(\omega)$ is called a sample path of the process.

Let $\{X_t\}_{t \geq 0}$ be an $\mathbb{R}^d$ -valued stochastic process. The process X is said to be continuous (respectively, right-continuous or left-continuous) if, for almost every $\omega \in \Omega$, the sample path $t \mapsto X_t(\omega)$ is continuous (respectively, right-continuous or left-continuous) on $[0, \infty)$. It is called càdlàg if its sample paths are right-continuous and admit finite left limits, i.e., $\lim_{s \uparrow t} X_s(\omega)$ exists and is finite for all $t > 0$, almost surely. The process is said to be integrable if $X_t \in L^1(\Omega; \mathbb{R}^d)$ for every $t \geq 0$. Furthermore, X is called $\{\mathcal{F}_t\}$ -adapted if X_t is $\mathcal{F}_t$ -measurable for all $t \geq 0$. A stochastic process X is said to be measurable if the mapping $(t, \omega) \mapsto X_t(\omega)$ from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}^d$ is $\mathcal{B}(\mathbb{R}_+) \otimes \mathcal{F}$ -measurable, where $\mathcal{B}(\mathbb{R}_+)$ denotes the Borel σ -algebra on $\mathbb{R}_+$. Moreover, X is called progressively measurable if, for every $T > 0$, the restriction of $(t, \omega) \mapsto X_t(\omega)$ to $[0, T] \times \Omega$ is $\mathcal{B}([0, T]) \otimes \mathcal{F}_T$ -measurable. Let $\mathcal{O}$ and $\mathcal{P}$ denote the optional

and predictable σ -algebras on $\mathbb{R}_+ \times \Omega$, respectively. The process X is called optional (respectively, predictable) if it is $\mathcal{O}$ -measurable (respectively, $\mathcal{P}$ -measurable). A real-valued stochastic process $A = \{A_t\}_{t \geq 0}$ is called an increasing process if, for almost every $\omega \in \Omega$, $A_s(\omega) \leq A_t(\omega)$, $0 \leq s \leq t$. It is said to be of finite variation if it admits a decomposition $A_t = A_t^+ - A_t^-$, where both A^+ and A^- are increasing processes. Consequently, every process of finite variation has càdlàg sample paths. In particular, every adapted process of finite variation is optional.

Stopping time

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ be a filtered probability space. A random variable $\tau : \Omega \rightarrow [0, \infty]$ is called an $\{\mathcal{F}_t\}$ -stopping time if $\{\omega \in \Omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$, $t \geq 0$. Let τ and ρ be stopping times satisfying $\tau \leq \rho$ a.s. The set $[\tau, \rho] = \{(t, \omega) \in \mathbb{R}_+ \times \Omega : \tau(\omega) \leq t < \rho(\omega)\}$ is called a stochastic interval. Similarly, one may define $[\tau, \rho]$, $] \tau, \rho]$ and $] \tau, \rho[$.

Associated with a stopping time τ is the σ -algebra

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.$$

If τ and ρ are stopping times with $\tau \leq \rho$ a.s., then $\mathcal{F}_\tau \subseteq \mathcal{F}_\rho$.

The following results will be used repeatedly.

Theorem 2.1 (Measurability at a Stopping Time). *Let $X = \{X_t\}_{t \geq 0}$ be a progressively measurable process and let τ be a stopping time. Then $X_\tau \mathbf{1}_{\{\tau < \infty\}}$ is $\mathcal{F}_\tau$ -measurable. In particular, if τ is finite, then X_τ is $\mathcal{F}_\tau$ -measurable.*

Theorem 2.2 (First Exit Time). *Let $X = \{X_t\}_{t \geq 0}$ be an $\mathbb{R}^d$ -valued càdlàg $\{\mathcal{F}_t\}$ -adapted process and let D be an open subset of $\mathbb{R}^d$. Define*

$$\tau = \inf\{t \geq 0 : X_t \notin D\},$$

with the convention $\inf \emptyset = \infty$. Then τ is an $\{\mathcal{F}_t\}$ -stopping time and is called the first exit time from D . Furthermore, if ρ is a stopping time, then

$$\tau_\rho = \inf\{t \geq \rho : X_t \notin D\}$$

is also an $\{\mathcal{F}_t\}$ -stopping time and is referred to as the first exit time from D after ρ .

Martingales

An integrable $\mathbb{R}^d$ -valued adapted process $M = \{M_t\}_{t \geq 0}$ is called a martingale with respect to $\{\mathcal{F}_t\}$ if

$$\mathbb{E}[M_t \mid \mathcal{F}_s] = M_s, \quad 0 \leq s \leq t < \infty.$$

Since the filtration is assumed to satisfy the usual conditions, every martingale admits a càdlàg modification. Hence, throughout this thesis, martingales will always be assumed to be càdlàg.

If $X = \{X_t\}_{t \geq 0}$ is progressively measurable and τ is a stopping time, then the process $X^\tau = \{X_{t \wedge \tau}\}_{t \geq 0}$ is called the stopped process associated with X . The stopped process plays a central role in the theory of martingales and leads to the classical Doob optional stopping theorem.

Theorem 2.3 (Doob stopping theorem). *Let $M = \{M_t\}_{t \geq 0}$ be an $\mathbb{R}^d$ -valued martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$, and let θ and ρ be two finite stopping times. Then*

$$\mathbb{E}[M_\theta \mid \mathcal{F}_\rho] = M_{\theta \wedge \rho}, \quad a.s.$$

In particular, for any stopping time τ ,

$$\mathbb{E}[M_{t \wedge \tau} \mid \mathcal{F}_s] = M_{s \wedge \tau}, \quad 0 \leq s \leq t < \infty.$$

Consequently, the stopped process $M^\tau = \{M_{t \wedge \tau}\}_{t \geq 0}$ is itself a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

A stochastic process $X = \{X_t\}_{t \geq 0}$ is called square-integrable if $\mathbb{E}|X_t|^2 < \infty$, $t \geq 0$. Let $M = \{M_t\}_{t \geq 0}$ be a real-valued square-integrable continuous martingale. Then there exists a unique continuous increasing adapted process $\langle M \rangle = \{\langle M \rangle_t\}_{t \geq 0}$ such that $M_t^2 - \langle M \rangle_t$ is a martingale. The process $\langle M \rangle$ is called the quadratic variation of M . In particular, for any finite stopping time τ , $\mathbb{E}[M_\tau^2] = \mathbb{E}[\langle M \rangle_\tau]$.

For two square-integrable continuous martingales M and N , their joint quadratic

variation is defined by

$$\langle M, N \rangle_t = \frac{1}{2} \left(\langle M + N \rangle_t - \langle M \rangle_t - \langle N \rangle_t \right).$$

Equivalently, $M_t N_t - \langle M, N \rangle_t$ is a martingale.

A continuous adapted process $M = \{M_t\}_{t \geq 0}$ is called a local martingale if there exists an increasing sequence of stopping times $\{\tau_n\}_{n \geq 1}$ with $\tau_n \uparrow \infty$ a.s. such that $\{M_{t \wedge \tau_n}\}_{t \geq 0}$ is a martingale for every n . Every martingale is a local martingale, although the converse is not generally true. For continuous local martingales M and N , the quadratic covariation process $\langle M, N \rangle$ is defined analogously and satisfies that $M_t N_t - \langle M, N \rangle_t$ is a local martingale.

An integrable adapted process $M = \{M_t\}_{t \geq 0}$ is called a supermartingale if $\mathbb{E}[M_t | \mathcal{F}_s] \leq M_s$, $0 \leq s \leq t$, and a submartingale if the above inequality is reversed. Clearly, M is a submartingale if and only if $-M$ is a supermartingale. Moreover, every martingale is both a supermartingale and a submartingale.

The following result provides a strong law of large numbers for continuous local martingales and is frequently used in the asymptotic analysis of stochastic systems.

Theorem 2.4 (Strong Law of Large Numbers). *Let $M = \{M_t\}_{t \geq 0}$ be a real-valued continuous local martingale with $M_0 = 0$. Then*

$$\lim_{t \rightarrow \infty} \langle M \rangle_t = \infty \implies \lim_{t \rightarrow \infty} \frac{M_t}{\langle M \rangle_t} = 0, \quad a.s.$$

Moreover,

$$\limsup_{t \rightarrow \infty} \frac{\langle M \rangle_t}{t} < \infty \implies \lim_{t \rightarrow \infty} \frac{M_t}{t} = 0, \quad a.s.$$

More generally, if $A = \{A_t\}_{t \geq 0}$ is a continuous adapted increasing process such that

$$A_t \rightarrow \infty \quad \text{and} \quad \int_0^\infty \frac{d\langle M \rangle_t}{(1 + A_t)^2} < \infty \quad a.s.,$$

then

$$\lim_{t \rightarrow \infty} \frac{M_t}{A_t} = 0, \quad a.s.$$

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The following inequalities provide useful bounds for the maximal process of a martingale and play an important role in establishing moment estimates and convergence results.

Theorem 2.5 (Doob's Martingale Inequalities). *Let $M = \{M_t\}_{t \geq 0}$ be an $\mathbb{R}^d$ -valued martingale and let $[a, b] \subset \mathbb{R}_+$ be a bounded interval. For $p \geq 1$,*

$$P \left(\sup_{a \leq t \leq b} |M_t| \geq c \right) \leq \frac{\mathbb{E}|M_b|^p}{c^p}, \quad c > 0.$$

Moreover, if $p > 1$, then

$$\mathbb{E} \left[\sup_{a \leq t \leq b} |M_t|^p \right] \leq \left(\frac{p}{p-1} \right)^p \mathbb{E}|M_b|^p.$$

The first inequality controls the tail probability of the running maximum of a martingale, while the second provides an L^p estimate for the maximal process in terms of the terminal value. The results presented in this subsection are standard in martingale theory. For further details and proofs, the reader is referred to [29].

2.3 Brownian motions

This subsection briefly reviews several basic facts about Brownian motion that will be used throughout this thesis. For a comprehensive treatment, the reader is referred to [20].

Brownian motion is one of the fundamental stochastic processes in probability theory and stochastic analysis. It provides a mathematical model for the random motion of particles suspended in a fluid and serves as the basic driving noise in stochastic differential equations. Let $(\Omega, \mathcal{F}, P)$ be a filtered probability space with filtration $\{\mathcal{F}_t\}_{t \geq 0}$. A one-dimensional Brownian motion is a real-valued continuous adapted process $B = \{B_t\}_{t \geq 0}$ satisfying:

- (i) $B_0 = 0$ a.s.;
- (ii) for any $0 \leq s < t < \infty$, $B_t - B_s \sim N(0, t - s)$;

(iii) for any $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$.

If $0 \leq t_0 < t_1 < \dots < t_k$, then the increments $B_{t_i} - B_{t_{i-1}}$, $i = 1, \dots, k$, are mutually independent. Consequently, Brownian motion possesses independent increments. Moreover, the distribution of $B_t - B_s$ depends only on the time difference $t - s$, and hence Brownian motion also has stationary increments.

Throughout this thesis, all stochastic processes are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying the usual conditions. Unless otherwise specified, Brownian motions are assumed to be adapted to this filtration. The following properties of Brownian motion will be useful:

(i) $\{-B_t\}_{t \geq 0}$ is a Brownian motion.

(ii) For any $c > 0$, the scaled process

$$X_t = \frac{B_{ct}}{\sqrt{c}}, \quad t \geq 0,$$

is a Brownian motion.

(iii) B is a continuous square-integrable martingale and

$$\langle B \rangle_t = t, \quad t \geq 0.$$

(iv) For almost every $\omega \in \Omega$, the sample path $t \mapsto B(t, \omega)$ is locally Hölder continuous with exponent δ for every $\delta \in (0, \frac{1}{2})$. Moreover, it is not Hölder continuous with any exponent $\delta > \frac{1}{2}$.

An $\mathbb{R}^d$ -valued stochastic process $B_t = (B_t^1, \dots, B_t^d)^T$, $t \geq 0$, is called a d -dimensional Brownian motion if each component $\{B_t^i\}_{t \geq 0}$, $i = 1, \dots, d$, is a one-dimensional Brownian motion and the processes $\{B_t^1\}_{t \geq 0}, \dots, \{B_t^d\}_{t \geq 0}$ are mutually independent.

2.4 Itô calculus

The Itô stochastic integral, first introduced by Itô [17], provides a rigorous mathematical framework for integration with respect to Brownian motion. Since Brownian

sample paths are almost surely nowhere differentiable, classical Riemann–Stieltjes integration is generally not applicable. We therefore briefly review the construction of the Itô stochastic integral and several of its fundamental properties.

Stochastic integral

Let $(\Omega, \mathcal{F}, P)$ be a complete filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ satisfying the usual conditions, and let $B = \{B_t\}_{t \geq 0}$ be an m -dimensional Brownian motion adapted to the filtration $\{\mathcal{F}_t\}_{t \geq 0}$.

A central object in stochastic analysis is the stochastic integral

$$\int_0^t f(s) dB_s,$$

where $f = \{f(t)\}_{t \geq 0}$ is a $d \times m$ matrix-valued stochastic process. Using the martingale structure of Brownian motion, one can define the integral for a large class of adapted processes. This construction leads to the Itô stochastic integral, which forms the foundation of stochastic calculus and stochastic differential equations.

Definition 2.6. Let $0 \leq a < b < \infty$. Define $\mathcal{M}^2([a, b]; \mathbb{R})$ to be the collection of all real-valued $\{\mathcal{F}_t\}$ -adapted measurable processes $f = \{f(t)\}_{a \leq t \leq b}$ satisfying

$$\|f\|_{\mathcal{M}^2([a, b])}^2 := \mathbb{E} \left[\int_a^b |f(t)|^2 dt \right] < \infty.$$

Two processes $f, \bar{f} \in \mathcal{M}^2([a, b]; \mathbb{R})$ are said to be equivalent if $\|f - \bar{f}\|_{\mathcal{M}^2([a, b])} = 0$. In this case, we identify f and $\bar{f}$ and write $f = \bar{f}$.

Since every process in $\mathcal{M}^2([a, b]; \mathbb{R})$ admits a predictable modification, we shall identify equivalent versions whenever no confusion arises. The Itô integral is first defined for a class of simple adapted processes and then extended to the whole space $\mathcal{M}^2([a, b]; \mathbb{R})$ by a density argument. We therefore introduce the following class of simple processes.

Definition 2.7 (Simple Processes). A process $g = \{g(t)\}_{a \leq t \leq b}$ is called simple if there exist a partition $a = t_0 < t_1 < \dots < t_k = b$ and bounded $\mathcal{F}_{t_i}$ -measurable random

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variables ξ_i , $i = 0, \dots, k-1$, such that

$$g(t) = \xi_0 \mathbf{1}_{[t_0, t_1]}(t) + \sum_{i=1}^{k-1} \xi_i \mathbf{1}_{(t_i, t_{i+1}]}(t). \quad (2.1)$$

The collection of all simple processes is denoted by $\mathcal{M}^0([a, b]; \mathbb{R})$.

Since $\mathcal{M}^0([a, b]; \mathbb{R}) \subset \mathcal{M}^2([a, b]; \mathbb{R})$, we first define the Itô integral on the class of simple processes.

Definition 2.8 (Itô Integral for Simple Processes). *For $g \in \mathcal{M}^0([a, b]; \mathbb{R})$ of the form (2.1), define*

$$\int_a^b g(t) dB_t := \sum_{i=0}^{k-1} \xi_i (B_{t_{i+1}} - B_{t_i}). \quad (2.2)$$

The random variable (2.2) is $\mathcal{F}_b$ -measurable and belongs to $L^2(\Omega)$.

Lemma 2.9 (Itô Isometry for Simple Processes). *For any $g \in \mathcal{M}^0([a, b]; \mathbb{R})$,*

$$\begin{aligned} \mathbb{E} \left[\int_a^b g(t) dB_t \right] &= 0, \\ \mathbb{E} \left[\left| \int_a^b g(t) dB_t \right|^2 \right] &= \mathbb{E} \left[\int_a^b |g(t)|^2 dt \right] \end{aligned}$$

To extend the Itô integral from simple processes to the whole space $\mathcal{M}^2([a, b]; \mathbb{R})$, we require the following density result.

Lemma 2.10 (Density of Simple Processes). *For every $f \in \mathcal{M}^2([a, b]; \mathbb{R})$, there exists a sequence $\{g_n\}_{n \geq 1} \subset \mathcal{M}^0([a, b]; \mathbb{R})$ such that*

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_a^b |f(t) - g_n(t)|^2 dt \right] = 0. \quad (2.3)$$

By the density of simple processes and the Itô isometry, the sequence $\left\{ \int_a^b g_n(t) dB_t \right\}_{n \geq 1}$ is Cauchy in $L^2(\Omega)$. Since $L^2(\Omega)$ is complete, the limit exists and allows us to extend the Itô integral to the whole space $\mathcal{M}^2([a, b]; \mathbb{R})$.

Definition 2.11 (Itô Integral for $\mathcal{M}^2$ -Processes). *Let $f \in \mathcal{M}^2([a, b]; \mathbb{R})$ and let $\{g_n\}_{n \geq 1} \subset$*

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$\mathcal{M}^0([a, b]; \mathbb{R})$ satisfy

$$\lim_{n \rightarrow \infty} \mathbb{E} \left[\int_a^b |f(t) - g_n(t)|^2 dt \right] = 0. \quad (2.4)$$

The Itô integral of f with respect to the Brownian motion B is defined as

$$\int_a^b f(t) dB_t := \lim_{n \rightarrow \infty} \int_a^b g_n(t) dB_t. \quad (2.5)$$

For $f, g \in \mathcal{M}^2([a, b]; \mathbb{R})$ and $\alpha, \beta \in \mathbb{R}$, the Itô integral possesses the following properties:

(i) **Linearity**

$$\int_a^b (\alpha f(t) + \beta g(t)) dB_t = \alpha \int_a^b f(t) dB_t + \beta \int_a^b g(t) dB_t.$$

(ii) **Zero Mean**

$$\mathbb{E} \left[\int_a^b f(t) dB_t \right] = 0.$$

(iii) **Itô Isometry**

$$\mathbb{E} \left| \int_a^b f(t) dB_t \right|^2 = \mathbb{E} \int_a^b |f(t)|^2 dt.$$

(iv) **Continuity** If

$$f_n \rightarrow f \quad \text{in } \mathcal{M}^2([a, b]; \mathbb{R}),$$

then

$$\int_a^b f_n(t) dB_t \longrightarrow \int_a^b f(t) dB_t \quad \text{in } L^2(\Omega).$$

Definition 2.12 (Indefinite Itô Integral). For $f \in \mathcal{M}^2([0, T]; \mathbb{R})$, define

$$I(t) := \int_0^t f(s) dB_s, \quad 0 \leq t \leq T. \quad (2.6)$$

The process $I = \{I(t)\}_{0 \leq t \leq T}$ is called the indefinite Itô integral of f .

The process defined by (2.6) is $\{\mathcal{F}_t\}$ -adapted. We next state its fundamental martingale property.

Theorem 2.13. *Let $f \in \mathcal{M}^2([0, T]; \mathbb{R})$ and define $I(t) = \int_0^t f(s) dB_s$. Then $I = \{I(t)\}_{0 \leq t \leq T}$ is a continuous square-integrable martingale with quadratic variation*

$$\langle I \rangle_t = \int_0^t |f(s)|^2 ds, \quad 0 \leq t \leq T. \quad (2.7)$$

Extensions of the Itô integral

Several useful extensions of the Itô integral are required in the sequel. These include stochastic integration over stopping time intervals and multidimensional Itô integrals, together with their corresponding conditional isometry properties.

Definition 2.14 (Stochastic Integral up to a Stopping Time). *Let $f \in \mathcal{M}^2([0, T]; \mathbb{R})$ and let τ be an $\{\mathcal{F}_t\}$ -stopping time such that $0 \leq \tau \leq T$. Then $\{f(t)\mathbf{1}_{[0, \tau]}(t)\}_{0 \leq t \leq T} \in \mathcal{M}^2([0, T]; \mathbb{R})$, and we define*

$$\int_0^\tau f(s) dB_s := \int_0^T f(s)\mathbf{1}_{[0, \tau]}(s) dB_s.$$

Furthermore, if ρ is another stopping time with $0 \leq \rho \leq \tau$, we define

$$\int_\rho^\tau f(s) dB_s := \int_0^\tau f(s) dB_s - \int_0^\rho f(s) dB_s.$$

The Itô integral over stopping time intervals inherits the fundamental properties of the classical Itô integral. In particular, the following conditional expectation and conditional isometry results are frequently used.

Theorem 2.15 (Conditional Itô Isometry). *Let $f \in \mathcal{M}^2([0, T]; \mathbb{R})$ and let ρ, τ be stopping times satisfying $0 \leq \rho \leq \tau \leq T$. Then*

$$\mathbb{E} \left[\int_\rho^\tau f(s) dB_s \middle| \mathcal{F}_\rho \right] = 0,$$

and

$$\mathbb{E} \left[\left| \int_\rho^\tau f(s) dB_s \right|^2 \middle| \mathcal{F}_\rho \right] = \mathbb{E} \left[\int_\rho^\tau |f(s)|^2 ds \middle| \mathcal{F}_\rho \right].$$

Many stochastic systems are naturally multidimensional. Therefore, it is necessary

to extend the Itô integral from one-dimensional Brownian motion to the multidimensional setting.

Definition 2.16 (Multidimensional Itô Integral). *Let*

$$B = (B^1, \dots, B^m)^T$$

be an m -dimensional Brownian motion and let

$$f = (f_{ij})_{d \times m} \in \mathcal{M}^2([0, T]; \mathbb{R}^{d \times m}).$$

The multidimensional Itô integral is defined componentwise by

$$\int_0^t f(s) dB_s = \left(\sum_{j=1}^m \int_0^t f_{1j}(s) dB_s^j, \dots, \sum_{j=1}^m \int_0^t f_{dj}(s) dB_s^j \right)^T.$$

The multidimensional Itô integral preserves the martingale structure of the one-dimensional case and satisfies an analogous conditional isometry.

Theorem 2.17 (Multidimensional Conditional Itô Isometry). *Let $f \in \mathcal{M}^2([0, T]; \mathbb{R}^{d \times m})$ and let ρ, τ be stopping times satisfying $0 \leq \rho \leq \tau \leq T$. Then*

$$\mathbb{E} \left[\int_{\rho}^{\tau} f(s) dB_s \middle| \mathcal{F}_{\rho} \right] = 0,$$

and

$$\mathbb{E} \left[\left| \int_{\rho}^{\tau} f(s) dB_s \right|^2 \middle| \mathcal{F}_{\rho} \right] = \mathbb{E} \left[\int_{\rho}^{\tau} |f(s)|^2 ds \middle| \mathcal{F}_{\rho} \right].$$

The Itô formula

The Itô formula is one of the central results in stochastic calculus. It can be regarded as the stochastic analogue of the classical chain rule and provides a fundamental tool for studying stochastic differential equations and diffusion processes. We begin with the definition of an Itô process.

Definition 2.18 (One-dimensional Itô Process). *A real-valued continuous adapted pro-*

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cess $X = \{X_t\}_{t \geq 0}$ is called an Itô process if it admits the representation

$$X_t = X_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s, \quad t \geq 0,$$

where $b \in L^1_{\mathcal{F}}(\mathbb{R}_+; \mathbb{R})$ and $\sigma \in L^2_{\mathcal{F}}(\mathbb{R}_+; \mathbb{R})$. Equivalently, X satisfies the stochastic differential equation

$$dX_t = b(t) dt + \sigma(t) dB_t.$$

Theorem 2.19 (One-dimensional Itô Formula). *Let $X = \{X_t\}_{t \geq 0}$ be an Itô process satisfying*

$$dX_t = b(t) dt + \sigma(t) dB_t,$$

and let $V \in C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$. Then

$$V(X_t, t) = V(X_0, 0) + \int_0^t \left(V_t + bV_x + \frac{1}{2}\sigma^2 V_{xx} \right) (X_s, s) ds + \int_0^t \sigma(s) V_x(X_s, s) dB_s.$$

Equivalently,

$$dV(X_t, t) = \left(V_t + bV_x + \frac{1}{2}\sigma^2 V_{xx} \right) (X_t, t) dt + \sigma(t) V_x(X_t, t) dB_t.$$

The above result extends naturally to multidimensional Itô processes.

Definition 2.20 (Multi-dimensional Itô Process). *An $\mathbb{R}^d$ -valued continuous adapted process $X = \{X_t\}_{t \geq 0}$ is called a multidimensional Itô process if*

$$X_t = X_0 + \int_0^t b(s) ds + \int_0^t \sigma(s) dB_s,$$

where $b \in L^1_{\mathcal{F}}(\mathbb{R}_+; \mathbb{R}^d)$ and $\sigma \in L^2_{\mathcal{F}}(\mathbb{R}_+; \mathbb{R}^{d \times m})$. Equivalently,

$$dX_t = b(t) dt + \sigma(t) dB_t.$$

In what follows, V_t , V_x and V_{xx} denote the partial derivative with respect to t , the gradient with respect to x , and the Hessian matrix with respect to x , respectively.

Theorem 2.21 (Multi-dimensional Itô Formula). *Let $X = \{X_t\}_{t \geq 0}$ be a d -dimensional*

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Itô process satisfying

$$dX_t = b(t) dt + \sigma(t) dB_t,$$

and let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Then

$$\begin{aligned} dV(X_t, t) = & \left(V_t + V_x(X_t, t)^T b(t) + \frac{1}{2} \text{trace}(\sigma(t)^T V_{xx}(X_t, t) \sigma(t)) \right) dt \\ & + V_x(X_t, t)^T \sigma(t) dB_t. \end{aligned}$$

As an immediate consequence of the multidimensional Itô formula, we obtain the stochastic integration by parts formula.

Theorem 2.22 (Integration by Parts). *Let $X = \{X_t\}_{t \geq 0}$ and $Y = \{Y_t\}_{t \geq 0}$ be continuous semimartingales. Then*

$$d(X_t Y_t) = X_t dY_t + Y_t dX_t + d\langle X, Y \rangle_t,$$

where $\langle X, Y \rangle$ denotes the quadratic covariation process of X and Y .

Moment inequalities

Moment inequalities are fundamental tools in the study of stochastic differential equations. They provide estimates for stochastic integrals and martingales, and play an important role in proving existence, uniqueness, stability, and convergence results.

Theorem 2.23 (Moment Estimate for Itô Integrals). *Let*

$$g \in \mathcal{M}^p([0, T]; \mathbb{R}^{d \times m}), \quad p \geq 2.$$

Then

$$\mathbb{E} \left| \int_0^T g(s) dB_s \right|^p \leq \left(\frac{p(p-1)}{2} \right)^{p/2} T^{\frac{p}{2}-1} \mathbb{E} \int_0^T |g(s)|^p ds.$$

In particular, when $p = 2$, equality holds and the above estimate reduces to the Itô isometry.

The preceding result estimates the p -th moment of a stochastic integral at a fixed

time. To control the entire sample path, one may combine the above estimate with Doob's martingale inequality.

Theorem 2.24 (Doob-type Maximal Inequality). *Let*

$$g \in \mathcal{M}^p([0, T]; \mathbb{R}^{d \times m}), \quad p \geq 2.$$

Then

$$\mathbb{E} \left(\sup_{0 \leq t \leq T} \left| \int_0^t g(s) dB_s \right|^p \right) \leq \left(\frac{p^3}{2(p-1)} \right)^{p/2} T^{\frac{p}{2}-1} \mathbb{E} \int_0^T |g(s)|^p ds.$$

A more powerful estimate is provided by the Burkholder–Davis–Gundy inequality, which relates the maximal process of a martingale to its quadratic variation.

Theorem 2.25 (Burkholder–Davis–Gundy Inequality). *Let*

$$M_t = \int_0^t g(s) dB_s, \quad A(t) = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, there exist positive constants c_p and C_p , depending only on p , such that

$$c_p \mathbb{E} [A(T)^{p/2}] \leq \mathbb{E} \left[\sup_{0 \leq t \leq T} |M_t|^p \right] \leq C_p \mathbb{E} [A(T)^{p/2}].$$

In particular, one may choose

$$c_p = \begin{cases} (p/2)^p, & 0 < p < 2, \\ 1, & p = 2, \\ (2p)^{-p/2}, & p > 2, \end{cases} \quad C_p = \begin{cases} (32/p)^{p/2}, & 0 < p < 2, \\ 4, & p = 2, \\ [p^{p+1}/(2(p-1)^{p-1})]^{p/2}, & p > 2. \end{cases}$$

2.5 Markov process

Markov property

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a filtered probability space and let $X = \{X_t\}_{t \geq 0}$ be an $\mathbb{R}^d$ -valued $\{\mathcal{F}_t\}$ -adapted stochastic process.

Definition 2.26 (Markov Process). *The process X is called a Markov process if for*

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every $0 \leq s \leq t < \infty$ and every Borel set $A \in \mathcal{B}^d$,

$$\mathbb{P}(X_t \in A \mid \mathcal{F}_s) = \mathbb{P}(X_t \in A \mid X_s). \quad (2.8)$$

Equivalently, for every bounded Borel measurable function $\varphi : \mathbb{R}^d \rightarrow \mathbb{R}$, $\mathbb{E}[\varphi(X_t) \mid \mathcal{F}_s] = \mathbb{E}[\varphi(X_t) \mid X_s]$. The Markov property states that, conditioned on the current state, the future evolution of the process is independent of its past history.

Transition probabilities

For a Markov process X , the transition probability is defined by

$$P(s, x; t, A) = \mathbb{P}(X_t \in A \mid X_s = x), \quad (2.9)$$

where $0 \leq s \leq t < \infty$, $x \in \mathbb{R}^d$, $A \in \mathcal{B}^d$.

The family $\{P(s, x; t, A)\}$ satisfies the following properties:

1. For fixed s, t and x , the mapping $A \mapsto P(s, x; t, A)$ is a probability measure on $\mathcal{B}^d$.
2. For fixed A , the function $x \mapsto P(s, x; t, A)$ is Borel measurable.
3. The Kolmogorov–Chapman equation holds: For all $0 \leq s \leq u \leq t < \infty$,

$$P(s, x; t, A) = \int_{\mathbb{R}^d} P(u, y; t, A) P(s, x; u, dy).$$

In terms of transition probabilities, the Markov property can be written as $\mathbb{P}(X_t \in A \mid \mathcal{F}_s) = P(s, X_s; t, A)$. For any bounded measurable function φ , $\mathbb{E}_{s,x}[\varphi(X_t)] = \int_{\mathbb{R}^d} \varphi(y) P(s, x; t, dy)$.

Continuous-time Markov chains

A continuous-time Markov chain is a Markov process taking values in a countable state space $\mathbb{S}$. Throughout this thesis, we consider homogeneous Continuous-Time Markov Chains, for which

$$\mathbb{P}(X_t = j \mid X_s = i) = \mathbb{P}(X_{t-s} = j \mid X_0 = i), \quad i, j \in \mathbb{S}. \quad (2.10)$$

The transition probabilities are defined by

$$p_{ij}(t) = \mathbb{P}(X_t = j \mid X_0 = i), \quad i, j \in \mathbb{S}, t \geq 0, \quad (2.11)$$

and form the transition probability matrix $P(t) = (p_{ij}(t))_{i,j \in \mathbb{S}}$. The family $\{P(t)\}_{t \geq 0}$ satisfies

$$P(0) = I, \quad P(t+s) = P(t)P(s), \quad \sum_{j \in \mathbb{S}} p_{ij}(t) = 1,$$

where the second identity is the matrix form of the Kolmogorov–Chapman equation. Moreover, $P(t)$ is called *standard* if $\lim_{t \downarrow 0} P(t) = I$. For a homogeneous continuous-time Markov chain with transition probability matrix $P(t) = (p_{ij}(t))_{i,j \in \mathbb{S}}$, the following results characterize the infinitesimal behavior of the chain.

Theorem 2.27. *Let $P(t)$ be a standard transition probability matrix. Then*

$$q_i = \lim_{t \downarrow 0} \frac{1 - p_{ii}(t)}{t} \quad (2.12)$$

exists (possibly infinite) for every $i \in \mathbb{S}$.

A state $i \in \mathbb{S}$ is called *stable* if $q_i < \infty$.

Theorem 2.28. *Let $P(t)$ be a standard transition probability matrix and let $j \in \mathbb{S}$ be a stable state. Then $q_{ij} = p'_{ij}(0)$ exists and is finite for all $i \in \mathbb{S}$.*

Consequently, for sufficiently small $h > 0$,

$$p_{ij}(h) = \begin{cases} 1 + q_{ii}h + o(h), & i = j, \\ q_{ij}h + o(h), & i \neq j. \end{cases} \quad (2.13)$$

For $i \neq j$, the quantity $q_{ij} \geq 0$ is called the transition rate from state i to state j . Since $\sum_{j \in \mathbb{S}} p_{ij}(h) = 1$, it follows that $q_{ii} = -\sum_{j \neq i} q_{ij}$.

The matrix $Q = (q_{ij})_{i,j \in \mathbb{S}}$ is called the generator matrix (or transition rate matrix) of the Markov chain. Moreover,

$$q_{ij} \geq 0, \quad i \neq j, \quad \sum_{j \in \mathbb{S}} q_{ij} = 0.$$

Poisson random measure representation

Let $X = \{X_t\}_{t \geq 0}$ be a finite-state continuous-time Markov chain with state space $\mathbb{S} = \{1, \dots, N\}$ and generator $Q = (q_{ij})_{N \times N}$. For each pair $i \neq j$, construct a collection of disjoint left-closed, right-open intervals $\{\Delta_{ij}\}$ on $\mathbb{R}$ such that $|\Delta_{ij}| = q_{ij}$, where $|\Delta_{ij}|$ denotes the length of the interval. Based on these intervals, define the jump function $h : \mathbb{S} \times \mathbb{R} \rightarrow \mathbb{R}$ by

$$h(i, y) = \sum_{j \neq i} (j - i) \mathbf{1}_{\Delta_{ij}}(y). \quad (2.14)$$

Let $\nu(dt, dy)$ be a Poisson random measure on $[0, \infty) \times \mathbb{R}$ with intensity measure $dt \mu(dy)$, where μ is the Lebesgue measure on $\mathbb{R}$. Then the Markov chain admits the stochastic integral representation

$$dX_t = \int_{\mathbb{R}} h(X_{t-}, y) \nu(dt, dy),$$

with initial condition $X_0 = i_0$. This representation will be useful for the formulation of regime-switching stochastic systems considered later in this thesis.

Generalised Itô formula

Let $B(t)$ be an m -dimensional Brownian motion defined on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$. Let $\theta(t)$ be the finite-state continuous-time Markov chain introduced in the previous subsection, and assume that $\theta(t)$ is independent of $B(t)$.

Consider the regime-switching stochastic differential equation

$$dx(t) = f(x(t), t, \theta(t))dt + g(x(t), t, \theta(t))dB(t),$$

where $f : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \rightarrow \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \rightarrow \mathbb{R}^{d \times m}$.

To extend the classical Itô formula to the paired process $(x(t), \theta(t))$, let

$$V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}).$$

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The infinitesimal generator associated with $(x(t), \theta(t))$ is defined by

$$\begin{aligned} LV(x, t, i) &= V_t(x, t, i) + V_x(x, t, i)f(x, t, i) \\ &\quad + \frac{1}{2} \text{trace} \left(g^T(x, t, i) V_{xx}(x, t, i) g(x, t, i) \right) + \sum_{j=1}^N q_{ij} V(x, t, j). \end{aligned}$$

The following theorem gives the generalized Itô formula for regime-switching diffusion processes.

Theorem 2.29 (Generalised Itô Formula). *Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$. Then, for any $t \geq 0$,*

$$\begin{aligned} V(x(t), t, \theta(t)) &= V(x(0), 0, \theta(0)) + \int_0^t LV(x(s), s, \theta(s)) ds \\ &\quad + \int_0^t V_x(x(s), s, \theta(s)) g(x(s), s, \theta(s)) dB(s) \\ &\quad + \int_0^t \int_{\mathbb{R}} \left[V(x(s), s, \theta(s) + h(\theta(s), l)) - V(x(s), s, \theta(s)) \right] \mu(ds, dl), \end{aligned}$$

where L denotes the infinitesimal generator of the regime-switching diffusion, h is the jump function defined in (2.14), and $\mu(ds, dl)$ is the compensated Poisson random measure associated with the Markov chain $\theta(t)$; the last integral term characterizes the effect of the Markovian switching on the dynamics of $V(x(t), t, \theta(t))$.

As a consequence of the generalized Itô formula, one obtains the following Dynkin formula.

Theorem 2.30 (Dynkin Formula). *Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R}_+)$ and let τ_1, τ_2 be bounded stopping times satisfying*

$$0 \leq \tau_1 \leq \tau_2 \quad a.s.$$

If $V(x(t), t, \theta(t))$ and $LV(x(t), t, \theta(t))$ are bounded on $[\tau_1, \tau_2]$, then

$$\mathbb{E}[V(x(\tau_2), \tau_2, \theta(\tau_2))] = \mathbb{E}[V(x(\tau_1), \tau_1, \theta(\tau_1))] + \mathbb{E} \int_{\tau_1}^{\tau_2} LV(x(s), s, \theta(s)) ds.$$

2.6 Hybrid Stochastic Differential Equations

Let $\theta(t)$ be a finite-state continuous-time Markov chain taking values in $\mathbb{S} = \{1, \dots, N\}$ and independent of the Brownian motion $B(t)$. Consider the Markovian regime-switching stochastic differential equation

$$dx(t) = f(x(t), t, \theta(t))dt + g(x(t), t, \theta(t))dB(t), \quad t \geq 0, \quad (2.15)$$

with initial condition $x(0) = x_0 \in \mathbb{R}^d$, $\theta(0) = i_0 \in \mathbb{S}$.

Since the state process $x(t)$ evolves continuously while the switching process $\theta(t)$ takes values in a finite set, (2.15) is often referred to as a *hybrid* stochastic differential equation.

Equation (2.15) is equivalent to the integral equation

$$x(t) = x_0 + \int_0^t f(x(s), s, \theta(s)) ds + \int_0^t g(x(s), s, \theta(s)) dB(s).$$

Existence and uniqueness

The following assumptions are standard.

[Local Lipschitz Condition] For every $k > 0$, there exists a constant $K_k > 0$ such that

$$|f(x, t, i) - f(y, t, i)|^2 + |g(x, t, i) - g(y, t, i)|^2 \leq K_k |x - y|^2$$

for all $t \geq 0$, $i \in \mathbb{S}$, and $|x| \vee |y| \leq k$.

Theorem 2.31. *Under the local Lipschitz condition, the hybrid SDE (2.15) admits a unique maximal local solution.*

To exclude finite-time explosion, we impose the linear growth condition.

[Linear Growth Condition] There exists a constant $K > 0$ such that

$$|f(x, t, i)|^2 + |g(x, t, i)|^2 \leq K(1 + |x|^2)$$

for all $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$.

Theorem 2.32. *Assume that the local Lipschitz and linear growth conditions hold. Then (2.15) admits a unique global solution.*

Stability concepts

Throughout this thesis, we assume that

$$f(0, t, i) = 0, \quad g(0, t, i) = 0,$$

for all $t \geq 0$ and $i \in \mathbb{S}$. Hence the origin is an equilibrium solution of (2.15).

The following notions of stability will be used.

1. **H_∞ stability:**

$$\int_0^\infty \mathbb{E}|x(t)| \, dt < \infty.$$

2. **p -th moment asymptotic stability:**

$$\lim_{t \rightarrow \infty} \mathbb{E}|x(t)|^p = 0.$$

3. **Almost sure asymptotic stability:**

$$\mathbb{P}\left(\lim_{t \rightarrow \infty} x(t) = 0\right) = 1.$$

4. **p -th moment exponential stability:**

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) \leq -\varepsilon$$

for some $\varepsilon > 0$.

5. **Almost sure exponential stability:**

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log |x(t)| \leq -\varepsilon \quad \text{a.s.}$$

for some $\varepsilon > 0$.

Remark. The above stability notions are related as follows. For $p \geq 1$, p -th moment exponential stability implies p -th moment asymptotic stability. Moreover, p -th moment exponential stability also implies H_∞ stability. By Hölder's inequality,

$$\mathbb{E}|x(t)| \leq (\mathbb{E}|x(t)|^p)^{1/p},$$

and hence the exponential decay of $\mathbb{E}|x(t)|^p$ implies

$$\int_0^\infty \mathbb{E}|x(t)| \, dt < \infty.$$

Similarly, almost sure exponential stability implies almost sure asymptotic stability. The converse implications do not hold in general.

2.7 Useful inequalities

In the remainder of this chapter, we present several auxiliary inequalities that will be used frequently in later developments. For a systematic account of these results, the reader is referred to Chapter 2 of [44].

Young's Inequality. For any $a, b \in \mathbb{R}$ and $\varepsilon > 0$,

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2. \quad (2.16)$$

More generally, if $p, q > 1$ satisfy $\frac{1}{p} + \frac{1}{q} = 1$, then

$$ab \leq \frac{\varepsilon}{p} a^p + \frac{\varepsilon^{-q/p}}{q} b^q, \quad a, b \geq 0. \quad (2.17)$$

Gronwall's Inequality.

Theorem 2.33 (Gronwall Inequality). *Let $u : [0, T] \rightarrow \mathbb{R}_+$ be a continuous function satisfying*

$$u(t) \leq c + \int_0^t v(s)u(s) \, ds, \quad t \in [0, T], \quad (2.18)$$

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where $c \geq 0$ and $v : [0, T] \rightarrow \mathbb{R}_+$ is integrable. Then

$$u(t) \leq c \exp\left(\int_0^t v(s) ds\right), \quad t \in [0, T]. \quad (2.19)$$

Matrix Inequalities. For any matrix $A \in \mathbb{R}^{d \times m}$ and vector $x \in \mathbb{R}^m$,

$$|Ax| \leq \|A\| |x|, \quad (2.20)$$

where $\|A\|$ denotes the operator norm $\|A\| = \sup_{x \neq 0} \frac{|Ax|}{|x|}$.

If $A \in \mathbb{R}^{d \times d}$ is symmetric, then

$$\lambda_{\min}(A)|x|^2 \leq x^T A x \leq \lambda_{\max}(A)|x|^2, \quad x \in \mathbb{R}^d, \quad (2.21)$$

where $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ denote the smallest and largest eigenvalues of A , respectively.

Moreover,

$$\|A\| = \sqrt{\lambda_{\max}(A^T A)}. \quad (2.22)$$

Chapter 3

Truncated Euler-Maruyama Method for Hybrid Stochastic Functional Differential Equations with Infinite Time Delay

3.1 Introduction

Stochastic differential equations have been broadly applied to modelling dynamic systems in the area of science and industry, for example, [22, 24, 35, 45]. As the understanding and research on dynamic systems deepens, the complexity of dynamic systems has increased significantly. In many applications, the future state of a dynamic system is determined not only by its present state but also by its past states, which can be modelled by the time delay (see, e.g., [7, 42, 48]). Long memory is a common phenomenon in many fields, such as economics, geological evolution, DNA sequencing, etc. Furthermore, random switching also plays an important role since discrete events occur inevitably and frequently intersperse within the continuous dynamic system (see, e.g., [21, 41, 44]). Therefore, infinite time delays and random switching have been introduced into stochastic differential equations to build more accurate models for practical problems. Hybrid stochastic functional differential equations with time delay give a

mathematical formulation for such dynamic systems.

Hybrid SFDEs can be divided into two categories according to the types of time delay. The first category is hybrid SFDEs with finite time delay (e.g., [42, 63]); the other is hybrid SFDEs with infinite time delay (e.g., [30, 47]). Hybrid SFDEs with finite time delay have been extensively studied in recent years, particularly with regard to their numerical approximation (see, e.g., [4, 23, 49, 50]). However, developing numerical methods for hybrid SFDEs with infinite time delay remained a challenging problem due to the difficulty of handling the infinite delay term. Recently, Li et al. [25] proposed a new approximation theory to overcome this problem. To illustrate the main idea of this approximation framework, consider the following SFDE with infinite delay:

$$dx(t) = F(x_t)dt + G(x_t)dB(t) \quad (3.1)$$

with infinite time delay segment $x_t = \{x(t+u) : u \leq 0\}$. A precise formulation of hybrid SFDEs with infinite delay will be provided in Section 3.2. To obtain numerical approximations of (3.1), we first approximate it by the following SFDE with finite delay:

$$dx^k(t) = F(\pi_k(x_t^k))dt + G(\pi_k(x_t^k))dB(t), \quad (3.2)$$

where k is a positive integer, π_k is a truncation mapping that reduces the infinite delay to the finite interval $[-k, 0]$, given by

$$\pi_k(\varphi)(u) = \begin{cases} \varphi(u) & \text{if } -k \leq u \leq 0, \\ \varphi(-k) & \text{if } u < -k, \end{cases}$$

for any continuous function φ defined on $(-\infty, 0]$. Subsequently, a numerical method for finite-delay SFDEs can be applied to (3.2), such as the classical Euler-Maruyama (EM) method in [25], that is

$$X_\Delta^k((j+1)\Delta) = X_\Delta^k(j\Delta) + F(\Pi_\Delta^k)\Delta + G(\Pi_\Delta^k)(B((j+1)\Delta) - B(j\Delta)), \quad (3.3)$$

where

$$\Pi_{\Delta}^k = \sum_{h=-kk_1}^{-1} X_{\Delta}^k((j+h)\Delta)\mu([h\Delta, (h+1)\Delta)) + X_{\Delta}^k((j-kk_1)\Delta)\mu((-\infty, -k))$$

with time step $\Delta = \frac{1}{k_1}$ and probability measure μ on $(-\infty, 0]$. The numerical solution X_{Δ}^k serves as an approximation of $x(t)$. Theoretically speaking, the fundamental idea of this method is computing x_t (infinite time delay) by Π_{Δ}^k (finite time delay), where all historical values prior to time $-k$ are replaced by the single value $x(-k)$.

Li et al. [25] provided the first effective numerical approximation framework for hybrid SFDEs with infinite time delay. Unfortunately, it was required that system coefficients should satisfy the global Lipschitz condition therein. However, this condition is so strong that many important SFDEs do not satisfy it such as stochastic functional volatility equations in the following example part. This naturally raises the following question: can numerical approximations for hybrid SFDEs with infinite time delay be developed under weaker assumptions, such as the local Lipschitz condition? As will be shown in this chapter, the answer is affirmative. In fact, we have noticed that in 2015, Mao [37] developed a new explicit method to effectively deal with approximations of stochastic systems under local Lipschitz condition, the so called truncated EM method. Since then, this method has been studied, extended, and applied to stochastic models in different fields by many researchers, for example, [12, 15, 43, 68]. However, the available convergence results for the truncated EM method are largely restricted to stochastic systems with finite delay (e.g. [12, 68]). There is so far no result on numerical solutions of the hybrid SFDEs with infinite time delay under the local Lipschitz condition.

By combining the infinite-delay approximation technique proposed in [25] with the truncated Euler–Maruyama method developed in [37], this chapter develops truncated EM numerical solutions for hybrid SFDEs with infinite time delay. The main contributions of this chapter can be summarised as follows.

- A truncated Euler–Maruyama approximation framework is developed for hybrid stochastic functional differential equations with infinite time delay under the local Lipschitz and Khasminskii-type conditions.

- The strong convergence of the proposed numerical method is established in the general L^q setting ($q \geq 2$), extending previous results that were available only in the L^2 case.
- Convergence rates are obtained under suitable additional conditions, thereby providing a theoretical justification for the efficiency of the proposed numerical scheme.

The remainder of this chapter is organised as follows. In Section 3.2, we introduce the notation, assumptions, and preliminary results required throughout the chapter, and present the truncated Euler-Maruyama scheme for hybrid stochastic functional differential equations with infinite delay. Section 3.3 extends the convergence result of [25] from the L^2 setting to the more general L^q setting under the global Lipschitz condition. These results are subsequently extended to the local Lipschitz framework and used in Section 3.4 to establish convergence of the numerical scheme. Convergence rates are investigated in Section 3.5. Finally, Section 3.6 provides numerical examples and simulations that demonstrate the theoretical results.

3.2 The truncated EM method

Unless otherwise stated, all notation and assumptions introduced in Chapter 2 remain in force throughout this chapter. In addition, several auxiliary notations specific to the analysis in this chapter are introduced below.

Let r be a positive number, and then we define the phase space $\mathcal{C}_r$ with the fading memory by

$$\mathcal{C}_r = \left\{ \varphi \in C(\mathbb{R}_-; \mathbb{R}^d) : \sup_{-\infty < u \leq 0} e^{ru} |\varphi(u)| < \infty \right\}$$

with its norm $\|\varphi\|_r = \sup_{-\infty < u \leq 0} e^{ru} |\varphi(u)|$. It can be seen that $\mathcal{C}_r$ under the norm $\|\cdot\|_r$ forms a Banach space (See [14] for more details). Moreover, denote by $\mathcal{P}_0$ the family of probability measures μ on $\mathbb{R}_-$ and for any positive number $b > 0$, define

$$\mathcal{P}_b = \left\{ \mu \in \mathcal{P}_0 : \int_{-\infty}^0 e^{-bu} \mu(du) < \infty \right\}.$$

Chapter 3. Truncated EM Method for Hybrid SFDEs

Set $\mu^{(b)} := \int_{-\infty}^0 e^{-bu} \mu(du)$ for each $\mu \in \mathcal{P}_b$. Please note that $\mu^{(b)}$ is a positive number but $\mu(\cdot)$ is a measure.

For example, let $\mu = \delta_{-a}$ be the Dirac measure concentrated at $-a$, where $a > 0$. Then

$$\mu^{(b)} = \int_{-\infty}^0 e^{-bu} \delta_{-a}(du) = e^{ab} < \infty.$$

As another example, consider the probability measure μ on $(-\infty, 0]$ with density

$$\frac{d\mu}{du} = \lambda e^{\lambda u}, \quad u \leq 0,$$

where $\lambda > 0$. Then

$$\mu^{(b)} = \lambda \int_{-\infty}^0 e^{(\lambda-b)u} du = \frac{\lambda}{\lambda-b} < \infty, \quad \text{provided that } \lambda > b.$$

Consider a hybrid SFDE with infinite time delay of the form

$$dx(t) = F(x(t), \psi_1(x_t), \theta(t), t)dt + G(x(t), \psi_2(x_t), \theta(t), t)dB(t) \quad (3.4)$$

on $t > 0$ with the initial data

$$x_0 = \xi \in \mathcal{C}_r, \quad \theta(0) = i_0 \in \mathbb{S}, \quad (3.5)$$

where $\theta(t)$ is the finite-state continuous-time Markov chain introduced in Chapter 2, which is assumed to be independent of the Brownian motion $B(t)$.

Here $F : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and $G : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times m}$ are Borel measurable while $\psi_1, \psi_2 : \mathcal{C}_r \rightarrow \mathbb{R}^d$ are defined by

$$\psi_1(\phi) = \int_{-\infty}^0 \phi(u) \mu_1(du), \quad \psi_2(\phi) = \int_{-\infty}^0 \phi(u) \mu_2(du) \quad (3.6)$$

with $\mu_1, \mu_2 \in \mathcal{P}_0$. Moreover, $x(t)$ is an $\mathbb{R}^d$ -valued stochastic process on $t \in (-\infty, \infty)$, while the $x_t = \{x(t+u) : u \in \mathbb{R}_-\}$ can be regarded as a $\mathcal{C}_r$ -valued stochastic process on $t \geq 0$.

If we define $f : \mathcal{C}_r \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and $g : \mathcal{C}_r \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times m}$ by

$$f(\phi, i, t) = F(\phi(0), \psi_1(\phi), i, t), \quad g(\phi, i, t) = G(\phi(0), \psi_2(\phi), i, t) \quad (3.7)$$

respectively, then hybrid SFDE (3.4) becomes the following form

$$dx(t) = f(x_t, \theta(t), t)dt + g(x_t, \theta(t), t)dB(t). \quad (3.8)$$

The main idea of [25] is to approximate the solution of hybrid SFDE (3.4) or (3.8) by the solution of the corresponding hybrid SFDE with a finite time delay. In order to design this hybrid SFDE, we need to define the truncation mapping as follows. For any positive integer k , define the truncation mapping $\pi_k : \mathcal{C}_r \rightarrow \mathcal{C}_r$ by

$$\pi_k(\varphi)(u) = \begin{cases} \varphi(u) & \text{if } -k \leq u \leq 0, \\ \varphi(-k) & \text{if } u < -k. \end{cases}$$

Define the truncation functions $f_k : \mathcal{C}_r \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and $g_k : \mathcal{C}_r \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times m}$ by

$$f_k(\varphi, i, t) = f(\pi_k(\varphi), i, t) \text{ and } g_k(\varphi, i, t) = g(\pi_k(\varphi), i, t).$$

Then we have the corresponding SFDE with finite time delay

$$\begin{aligned} dx^k(t) &= f_k(\pi_k(x_t^k), \theta(t), t)dt + g_k(\pi_k(x_t^k), \theta(t), t)dB(t) \\ &= F(x^k(t), \psi_1(\pi_k(x_t^k)), \theta(t), t)dt + G(x^k(t), \psi_2(\pi_k(x_t^k)), \theta(t), t)dB(t) \end{aligned} \quad (3.9)$$

on $t > 0$ with the initial data $x_0 = \pi_k(\xi)$ and $\theta(0) = i_0$.

In [25], a restrictive global Lipschitz condition is imposed. While in this chapter, we will replace it with the following local Lipschitz condition.

Assumption 3.1. *For each $h > 0$, there is a positive constant K_h such that*

$$|F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)| \vee |G(x, y, i, t) - G(\bar{x}, \bar{y}, i, t)| \leq K_h(|x - \bar{x}| + |y - \bar{y}|) \quad (3.10)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq h$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

But Assumption 3.1 is not able to guarantee the unique existence of global solution of hybrid SFDE (3.4). For this purpose, a Khasminskii-type condition is always wanted.

Assumption 3.2. Let $\mu_1, \mu_2 \in \mathcal{P}_{rp}$. There are nonnegative constants K_1, p such that $p > 2$ and

$$x^T F(x, y, i, t) + \frac{p-1}{2} |G(x, z, i, t)|^2 \leq K_1(1 + |x|^2 + |y|^2 + |z|^2) \quad (3.11)$$

for all $(x, y, z, i, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

It should be highlighted that Assumption 3.2 is not so strong and could be met by many practical systems such as volatility model [3] and population model [43]. Additionally, we need two more assumptions. The first one guarantees the initial data are Hölder continuous with exponent α . the second one requires the system coefficients F and G are Hölder continuous on time t with exponent β . They are necessary even for the classical EM method.

Assumption 3.3. There exist constants $\alpha \geq 1/2$ and $K_2 > 0$ such that the initial function ξ satisfies

$$|\xi(s_1) - \xi(s_2)| \leq K_2 |s_1 - s_2|^\alpha \quad (3.12)$$

for all $s_1, s_2 \in (-\infty, 0]$.

Assumption 3.4. There exists a constant $\beta \geq 1/2$. For each $R > 0$, there is a positive constant K_R such that

$$|F(x, y, i, t_1) - F(x, y, i, t_2)| \vee |G(x, y, i, t_1) - G(x, y, i, t_2)| \leq K_R |t_1 - t_2|^\beta$$

for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq R$, $t_1, t_2 \in \mathbb{R}_+$ and $i \in \mathbb{S}$.

To construct the truncated EM method to hybrid SFDE (3.9), we define the truncated mapping as below.

Chapter 3. Truncated EM Method for Hybrid SFDEs

Step 1. According to the structures of F and G , we first choose a strictly increasing continuous function $\eta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\eta(r) \rightarrow \infty$ as $r \rightarrow \infty$ and

$$\sup_{|x| \vee |y| \leq r, (i,t) \in \mathbb{S} \times \mathbb{R}_+} (|F(x, y, i, t)| \vee |G(x, y, i, t)|) \leq \eta(r), \quad \forall r \geq 1. \quad (3.13)$$

Step 2. Then, we choose a number $\hat{h} \geq 1$ and a strictly decreasing function $h : (0, 1] \rightarrow [\eta(0), \infty)$ such that

$$\lim_{\Delta \rightarrow 0} h(\Delta) = \infty \text{ and } \Delta^{1/4} h(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0, 1]. \quad (3.14)$$

Step 3. For a given step size $\Delta \in (0, 1]$, we define the truncation mapping π_Δ from $\mathbb{R}^d$ to the closed ball $\{x \in \mathbb{R}^d : |x| \leq \eta^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x) = (|x| \wedge \eta^{-1}(h(\Delta))) \frac{x}{|x|}, \quad (3.15)$$

where we set $\frac{x}{|x|} = 0$ when $x = 0$. Here η^{-1} is the inverse function of η and it is clear that η^{-1} is a strictly increasing continuous function from $[\eta(0), \infty)$ to $\mathbb{R}_+$.

Step 4. The truncated functions are defined by for $x, y \in \mathbb{R}^d$

$$F_\Delta(x, y, i, t) = F(\pi_\Delta(x), \pi_\Delta(y), i, t), \quad G_\Delta(x, y, i, t) = G(\pi_\Delta(x), \pi_\Delta(y), i, t). \quad (3.16)$$

It is easy to see that

$$|F_\Delta(x, y, i, t)| \vee |G_\Delta(x, y, i, t)| \leq \eta(\eta^{-1}(h(\Delta))) = h(\Delta), \quad \forall x, y \in \mathbb{R}^d. \quad (3.17)$$

That is, F_Δ and G_Δ are bounded although F and G may not. These truncated functions have the good property of preserving the generalized Khasminskii-type condition as described below.

Lemma 3.5. *Let Assumption 3.2 hold. Then, for every $\Delta \in (0, 1]$, we have*

$$x^T F_\Delta(x, y, i, t) + \frac{p-1}{2} |G_\Delta(x, z, i, t)|^2 \leq \hat{K}_1(1 + |x|^2 + |y|^2 + |z|^2) \quad (3.18)$$

for all $(x, y, z, i, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, where $\hat{K}_1 = 2K_1(1 \vee [1/\eta^{-1}(h(1))])$.

Proof. Fix any $\Delta \in (0, 1]$. For $x \in \mathbb{R}^d$ with $|x| \leq \eta^{-1}(h(\Delta))$ and any $y, z \in \mathbb{R}^d$, we have, by (3.11),

$$\begin{aligned} & x^T F_\Delta(x, y, i, t) + \frac{p-1}{2} |G_\Delta(x, z, i, t)|^2 \\ &= \pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) + \frac{p-1}{2} |G(\pi_\Delta(x), \pi_\Delta(z), i, t)|^2 \\ &\leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) \\ &\leq K_1(1 + |x|^2 + |y|^2 + |z|^2), \end{aligned} \tag{3.19}$$

which implies the desired assertion (3.18). On the other hand, for $x \in \mathbb{R}^d$ with $|x| > \eta^{-1}(h(\Delta))$ and any $y, z \in \mathbb{R}^d$, we have

$$\begin{aligned} & x^T F_\Delta(x, y, i, t) + \frac{p-1}{2} |G_\Delta(x, z, i, t)|^2 \\ &= \pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) + \frac{p-1}{2} |G(\pi_\Delta(x), \pi_\Delta(z), i, t)|^2 \\ &\quad + (x - \pi_\Delta(x))^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) \\ &\leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) + \left(\frac{|x|}{\eta^{-1}(h(\Delta))} - 1 \right) \pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t). \end{aligned} \tag{3.20}$$

But, once again we see from (3.11) that

$$\pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) \leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2). \tag{3.21}$$

Substituting this into (3.20) gives

$$\begin{aligned}
 & x^T F_\Delta(x, y, i, t) + \frac{p-1}{2} |G_\Delta(x, z, i, t)|^2 \\
 & \leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) \\
 & \quad + \left(\frac{|x|}{\eta^{-1}(h(\Delta))} - 1 \right) K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) \\
 & \leq \frac{K_1|x|}{\eta^{-1}(h(\Delta))} (1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) \\
 & \leq K_1(1 \vee [1/\eta^{-1}(h(1))]) |x|(1 + |x| + |y| + |z|) \\
 & \leq 2K_1(1 \vee [1/\eta^{-1}(h(1))]) (1 + |x|^2 + |y|^2 + |z|^2).
 \end{aligned} \tag{3.22}$$

Thus, we have proved that the required condition (3.18) holds for $x \in \mathbb{R}^d$ with $|x| > \eta^{-1}(h(\Delta))$ and any $y, z \in \mathbb{R}^d$. The proof is complete. $\square$

Assumptions 3.1-3.4 could guarantee that SFDE (3.9) with finite time delay has a unique global solution $x^k(t)$. Let us fix a sufficiently large k and apply the truncated EM method to SFDE (3.9) with finite time delay to obtain its numerical solutions. Let k_1 be a positive integer and set the step size $\Delta = 1/k_1$. Let $t_j = j\Delta$ for $j = -kk_1, -kk_1 + 1, \dots, -1, 0, 1, \dots$.

Firstly, we need to define the discrete-time numerical approximations $X_\Delta^k(t_j) \approx x^k(t_j)$ for $j \geq 0$ given the initial data $x_0 = \pi_k(\xi)$ and $\theta(0) = i_0$. From the definition of $\pi_k(\xi)$, we see that $\pi_k(\xi)$ depends only on the values $\xi(u)$ for $u \in [-k, 0]$. Therefore, we set $X_\Delta^k(t_j) = \xi(t_j)$ for $j = -kk_1, -(kk_1 + 1), \dots, -1, 0$ and form $X_\Delta^k(t_{j+1})$ for $j \geq 0$ by

$$X_\Delta^k(t_{j+1}) = X_\Delta^k(t_j) + F_\Delta(X_\Delta^k(t_j), \psi_{1j}, \theta(t_j), t_j)\Delta + G_\Delta(X_\Delta^k(t_j), \psi_{2j}, \theta(t_j), t_j)\Delta B_j, \tag{3.24}$$

where $\Delta B_j = B(t_{j+1}) - B(t_j)$, while

$$\psi_{1j} = \sum_{h=-kk_1}^{-1} X_\Delta^k(t_{j+h})\mu_1([t_h, t_{h+1})) + X_\Delta^k(t_{j-kk_1})\mu_1((-\infty, -k))$$

and ψ_{2j} is defined as ψ_{1j} by replacing μ_1 with μ_2 . We next define the continuous-time

numerical solution

$$X_{\Delta}^k(t) = \sum_{j=0}^{\infty} X_{\Delta}^k(t_j) I_{[t_j, t_{j+1})}(t), \quad t \geq 0. \quad (3.25)$$

Although this is the numerical solution we usually compute in practice, the numerical analysis is carried out via the continuous auxiliary process defined below

$$\bar{X}_{\Delta}^k(t) := \xi(0) + \int_0^t F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) ds + \int_0^t G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s)) dB(s) \quad (3.26)$$

for $t \geq 0$ while set $\bar{X}_{\Delta}^k(t) = \xi(t)$ for $t \in [-k, 0]$, where

$$\tau(t) := t_j, \bar{\theta}(t) := \theta(t_j), \psi_1(t) := \psi_{1j}, \psi_2(t) := \psi_{2j}, t \in [t_j, t_{j+1}).$$

Note that $\bar{X}_{\Delta}^k(t_j) = X_{\Delta}^k(t_j)$ for all j . That is, $\bar{X}_{\Delta}^k(t)$ coincides with the numerical solution $X_{\Delta}^k(t)$ at the grid-points.

Lemma 3.6. *For any $\Delta \in (0, 1]$ and any $\bar{p} \geq 2$, we have*

$$\mathbb{E}|X_{\Delta}^k(t) - \bar{X}_{\Delta}^k(t)|^{\bar{p}} \leq c_{\bar{p}} \Delta^{\bar{p}/2} (h(\Delta))^{\bar{p}}, \quad \forall t \geq 0, \quad (3.27)$$

where $c_{\bar{p}}$ is a positive constant dependent only on $\bar{p}$ and its value may differ between occurrences. Consequently

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|X_{\Delta}^k(t) - \bar{X}_{\Delta}^k(t)|^{\bar{p}} = 0, \quad \forall t \geq 0. \quad (3.28)$$

Proof. Fix $\Delta \in (0, 1]$ arbitrarily. For any $t \geq 0$, there is a unique integer $j \geq 0$ such that $t_j \leq t < t_{j+1}$. By Hölder inequality, Doob martingale inequality and (3.17), we

then derive that

$$\begin{aligned}
 & \mathbb{E}|X_{\Delta}^k(t) - \bar{X}_{\Delta}^k(t)|^{\bar{p}} = \mathbb{E}|\bar{X}_{\Delta}^k(t) - \bar{X}_{\Delta}^k(t_j)|^{\bar{p}} \\
 & \leq c_{\bar{p}} \left(\mathbb{E} \left| \int_{t_j}^t F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) ds \right|^{\bar{p}} + \mathbb{E} \left| \int_{t_j}^t G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s)) dB(s) \right|^{\bar{p}} \right) \\
 & \leq c_{\bar{p}} \left(\Delta^{\bar{p}-1} \mathbb{E} \int_{t_j}^t \left| F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right|^{\bar{p}} ds \right. \\
 & \quad \left. + \Delta^{(\bar{p}-2)/2} \mathbb{E} \int_{t_j}^t \left| G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s)) \right|^{\bar{p}} ds \right) \\
 & \leq c_{\bar{p}} \Delta^{\bar{p}/2} (h(\Delta))^{\bar{p}}
 \end{aligned} \tag{3.29}$$

which is (3.27). Noting from (3.14) that $\Delta^{\bar{p}/2} (h(\Delta))^{\bar{p}} \leq \hat{h}^{\bar{p}} \Delta^{\bar{p}/4}$, we obtain (3.28) from (3.27) immediately. $\square$

Finally, we mention that $X_{\Delta}^k(t_{j+1})$, $X_{\Delta}^k(t)$ and $\bar{X}_{\Delta}^k(t)$ are exactly the required numerical solutions of hybrid SFDE (3.4) with infinite time delay.

3.3 Convergence of modified EM method

Li et al. [25] proposed the modified EM method to approximate hybrid SFDEs with infinite time delay. For the SFDE with finite time delay, they used the classical EM method and obtained strong convergence in L^2 under global Lipschitz condition. These results will be helpful when we study the truncated EM method since we need to construct a space in which SFDE (3.9) satisfies the global Lipschitz condition. Thus in this part, we need to extend their results to L^q -convergence. To distinguish with our systems, we denote by $z^k(t)$ the solution of the following SFDE with finite time delay

$$dz^k(t) = F(z^k(t), \psi_1(\pi_k(z_t^k)), \theta(t), t)dt + G(z^k(t), \psi_2(\pi_k(z_t^k)), \theta(t), t)dB(t), \tag{3.30}$$

which satisfies the global Lipschitz condition.

Assumption 3.7. *There exists a constant $L_1 > 0$ such that*

$$|F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)| \vee |G(x, y, i, t) - G(\bar{x}, \bar{y}, i, t)| \leq L_1(|x - \bar{x}| + |y - \bar{y}|) \tag{3.31}$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.8. *There exists constants $L_2 > 0$ and $\beta \geq 1/2$ such that*

$$|F(x, y, i, t_1) - F(x, y, i, t_2)| \vee |G(x, y, i, t_1) - G(x, y, i, t_2)| \leq L_2(1 + |x| + |y|)|t_1 - t_2|^\beta$$

for all $x, y \in \mathbb{R}^d, t_1, t_2 \in \mathbb{R}_+$ and $i \in \mathbb{S}$.

We denote by $\bar{Z}_\Delta^k$ the continuous auxiliary process of the modified EM method and by Z_Δ^k the continuous-time numerical solution. The modified EM method has numerous good properties established in [25]. We will cite some of them here for the use of this chapter. Let $T > 0$ be any fixed time, and denote by C generic positive real constants dependent on q, T, ξ , etc. but independent of Δ , and its value may be different between occurrences.

Lemma 3.9. *Let Assumption 3.7 holds and $\mu_1, \mu_2 \in \mathcal{P}_r$. Then, for any $q \in [2, p)$*

$$\sup_{0 < \Delta \leq 1} \mathbb{E} \left(\sup_{t \in [0, T]} |\bar{Z}_\Delta^k(t)|^q \right) \leq C. \quad (3.32)$$

Remark. The restriction $q \in [2, p)$ is imposed by the estimates used below. In particular, the condition $q \geq 2$ is needed for the application of Itô's formula to $|\cdot|^q$ and the subsequent estimates involving the factor $|\cdot|^{q-2}$. The condition $q < p$ is required to derive the q -th moment estimates from the available p -th moment bounds using Hölder's and Young's inequalities.

Lemma 3.10. *By virtue of Lemma 3.9,*

$$\sup_{t \in (0, T]} \mathbb{E} |\psi_i(t)|^q \leq C, i = 1, 2.$$

Lemma 3.11. *Let Assumption 3.7 holds and $\mu_1, \mu_2 \in \mathcal{P}_r$. Then, for any $\Delta \in (0, 1]$,*

$$\mathbb{E} |\bar{Z}_\Delta^k(t) - Z_\Delta^k(t)|^q \leq C \Delta^{q/2}, \forall t \in [0, T].$$

Now, we can extend Lemma 5.7 in [25] to the q -th moment of the solution in the

following way.

Lemma 3.12. *Let Assumptions 3.3, 3.7, 3.8 hold and $\mu_1, \mu_2 \in \mathcal{P}_{rp}$. Then for any $\Delta \in (0, 1]$ and $q \in [2, p)$,*

$$\mathbb{E}|z^k(t) - \bar{Z}_\Delta^k(t)|^q \leq C\Delta, \forall t \in [0, T]. \quad (3.33)$$

Proof. Fix $\Delta \in (0, 1]$. It is easy to see that

$$\mathbb{E}|z^k(t) - \bar{Z}_\Delta^k(t)|^q \leq C(I(t) + J(t)), \quad (3.34)$$

where

$$\begin{aligned} I(t) &:= \mathbb{E} \int_0^t |F(z^k(s), \psi_1(\pi_k(z_s^k)), \theta(s), s) - F(Z_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^q ds, \\ J(t) &:= \mathbb{E} \int_0^t |G(z^k(s), \psi_2(\pi_k(z_s^k)), \theta(s), s) - G(Z_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^q ds. \end{aligned}$$

It is clear to see that

$$I(t) \leq C(I_1(t) + I_2(t) + I_3(t)), \quad (3.35)$$

where

$$\begin{aligned} I_1(t) &:= \mathbb{E} \int_0^t |F(z^k(s), \psi_1(\pi_k(z_s^k)), \theta(s), s) - F(Z_\Delta^k(s), \psi_1(s), \theta(s), s)|^q ds, \\ I_2(t) &:= \mathbb{E} \int_0^t |F(Z_\Delta^k(s), \psi_1(s), \theta(s), s) - F(Z_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s)|^q ds, \\ I_3(t) &:= \mathbb{E} \int_0^t |F(Z_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s) - F(Z_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^q ds. \end{aligned}$$

By Assumption 3.7 and Lemma 3.11, we have

$$\begin{aligned} I_1(t) &\leq 2^{q-1} L_1^q \mathbb{E} \int_0^t \left(|z^k(s) - Z_\Delta^k(s)|^q + |\psi_1(\pi_k(z_s^k)) - \psi_1(s)|^q \right) ds \\ &= 2^{q-1} L_1^q \mathbb{E} \int_0^t \left(|z^k(s) - \bar{Z}_\Delta^k(s) + \bar{Z}_\Delta^k(s) - Z_\Delta^k(s)|^q + |\psi_1(\pi_k(z_s^k)) - \psi_1(s)|^q \right) ds \\ &\leq C\Delta^{q/2} + C\mathbb{E} \int_0^t |z^k(s) - \bar{Z}_\Delta^k(s)|^q ds + C\mathbb{E} \int_0^t |\psi_1(\pi_k(z_s^k)) - \psi_1(s)|^q ds. \end{aligned} \quad (3.36)$$

For each $t \in (0, T]$, let $N = \lfloor t/\Delta \rfloor$ and $t_{N+1} = t$. By the Hölder inequality and Assumption 3.3, it can be derived that

$$\begin{aligned}
 & \mathbb{E} \int_0^t |\psi_1(\pi_k(z_s^k)) - \psi_1(s)|^q ds = \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |\psi_1(\pi_k(z_s^k)) - \psi_{1j}|^q ds \\
 &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \left| \int_{-k}^0 z^k(s+u) \mu_1(du) + z^k(s-k) \mu_1((-\infty, -k)) \right. \\
 & \quad \left. - \sum_{h=-kk_1}^{-1} Z_{\Delta}^k(t_{j+h}) \mu_1([t_h, t_{h+1})) - Z_{\Delta}^k(t_j - k) \mu_1((-\infty, -k)) \right|^q ds \\
 &\leq \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \left(\sum_{h=-kk_1}^{-1} \int_{t_h}^{t_{h+1}} |z^k(s+u) - Z_{\Delta}^k(t_{j+h})| \mu_1(du) \right. \\
 & \quad \left. + |z^k(s-k) - Z_{\Delta}^k(t_j - k)| \mu_1((-\infty, -k)) \right)^q ds \\
 &\leq 2^{q-1} \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \left(\sum_{h=-kk_1}^{-1} \int_{t_h}^{t_{h+1}} |z^k(s+u) - Z_{\Delta}^k(t_{j+h})| \mu_1(du) \right)^q ds \\
 & \quad + 2^{q-1} \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} (|\xi(s-k) - \xi(t_j - k)| \mu_1((-\infty, -k)))^q ds \\
 &\leq 2^{q-1} \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \int_{-k}^0 |z^k(s+u) - Z_{\Delta}^k(t_j + u)|^q \mu_1(du) ds + 2^{q-1} T K_2^q \Delta^{q\alpha} \\
 &\leq 4^{q-1} \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \int_{-k}^{-s} |\xi(s+u) - \xi(t_j + u)|^q \mu_1(du) ds \\
 & \quad + 4^{q-1} \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \int_{-s}^0 |z^k(s+u) - Z_{\Delta}^k(t_j + u)|^q \mu_1(du) ds + 2^{q-1} T K_2^q \Delta^{q\alpha} \\
 &= C \Delta^{q\alpha} + C \mathbb{E} \int_0^t \int_{-s}^0 |z^k(s+u) - Z_{\Delta}^k(\tau(s) + u)|^q \mu_1(du) ds. \tag{3.37}
 \end{aligned}$$

It is clear to see that

$$\begin{aligned}
& \mathbb{E} \int_0^t \int_{-s}^0 |z^k(s+u) - Z_\Delta^k(\tau(s)+u)|^q \mu_1(du) ds \\
& \leq 3^{q-1} \mathbb{E} \int_0^t \int_{-s}^0 |z^k(s+u) - \bar{Z}_\Delta^k(s+u)|^q \mu_1(du) ds \\
& \quad + 3^{q-1} \mathbb{E} \int_0^t \int_{-s}^0 |\bar{Z}_\Delta^k(s+u) - \bar{Z}_\Delta^k(\tau(s)+u)|^q \mu_1(du) ds \\
& \quad + 3^{q-1} \mathbb{E} \int_0^t \int_{-s}^0 |\bar{Z}_\Delta^k(\tau(s)+u) - Z_\Delta^k(\tau(s)+u)|^q \mu_1(du) ds.
\end{aligned}$$

By the Assumption 3.7, Lemmas 3.9 and 3.10, we have

$$\begin{aligned}
& \mathbb{E} |\bar{Z}_\Delta^k(t+u) - \bar{Z}_\Delta^k(\tau(t)+u)|^q \\
& \leq 2^{q-1} \left(\mathbb{E} \left| \int_{t_j+u}^{t+u} F(Z_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) ds \right|^q \right. \\
& \quad \left. + \mathbb{E} \left| \int_{t_j+u}^{t+u} G(Z_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s)) dB(s) \right|^q \right) \\
& \leq C \Delta^{(q-2)/2} \int_{t_j+u}^{t+u} \left(1 + \mathbb{E} |Z_\Delta^k(s)|^q + \mathbb{E} |\psi_1(s)|^q + \mathbb{E} |\psi_2(s)|^q \right) ds \\
& \leq C \Delta^{q/2}.
\end{aligned}$$

Applying this and Lemma 3.11, we get

$$\begin{aligned}
& \mathbb{E} \int_0^t \int_{-s}^0 |z^k(s+u) - Z_\Delta^k(\tau(s)+u)|^q \mu_1(du) ds \\
& \leq C \Delta^{q/2} + C \mathbb{E} \int_0^t \int_{-s}^0 |z^k(s+u) - \bar{Z}_\Delta^k(s+u)|^q \mu_1(du) ds \\
& \leq C \Delta^{q/2} + C \mathbb{E} \int_{-\infty}^0 \int_0^t |z^k(s) - \bar{Z}_\Delta^k(s)|^q ds \mu_1(du) \\
& = C \Delta^{q/2} + C \mathbb{E} \int_0^t |z^k(s) - \bar{Z}_\Delta^k(s)|^q ds. \tag{3.38}
\end{aligned}$$

Substituting (3.38) into (3.37) gives

$$\mathbb{E} \int_0^t |\psi_1(\pi_k(z_s^k)) - \psi_1(s)|^q ds \leq C \Delta^{q/2} + C \mathbb{E} \int_0^t |z^k(s) - \bar{Z}_\Delta^k(s)|^q ds. \tag{3.39}$$

Then, substituting the inequality above into (3.36) yields

$$I_1(t) \leq C\Delta^{q/2} + C\mathbb{E} \int_0^t |z^k(s) - \bar{Z}_\Delta^k(s)|^q ds. \quad (3.40)$$

By the following properties of the Markov chain (See [26], Lemma III.2.)

$$\mathbb{P}(r(s) \neq i \text{ for some } s \in [t, t + \nu] | r(t) = i) \leq 1 - e^{-\nu \max_{i \in \mathbb{S}} (-\gamma_{ii})}$$

for any $t \in (0, T]$, $\nu > 0$ and $i \in \mathbb{S}$, we can get, for $s \in [t_j, t_{j+1})$,

$$\mathbb{E}[\mathbb{E}(I_{\{\theta(s) \neq \theta(t_j)\}} | \mathcal{F}_{t_j})] \leq C\Delta.$$

Together with Assumption 3.7, Lemmas 3.9 and 3.10, we can derive that

$$\begin{aligned} I_2(t) &= \mathbb{E} \int_0^t |F(Z^k(s), \psi_1(s), \theta(s), s) - F(Z_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s)|^q ds \\ &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |F(Z_\Delta^k(t_j), \psi_{1j}, \theta(s), t_j) - F(Z_\Delta^k(t_j), \psi_{1j}, \theta(t_j), t_j)|^q ds \\ &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |F(Z_\Delta^k(t_j), \psi_{1j}, \theta(s), t_j) - F(Z_\Delta^k(t_j), \psi_{1j}, \theta(t_j), t_j)|^q I_{\{\theta(s) \neq \theta(t_j)\}} ds \\ &\leq C \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \mathbb{E} \left[\left(1 + |Z_\Delta^k(t_j)|^q + |\psi_{1j}|^q \right) \mathbb{E}(I_{\{\theta(s) \neq \theta(t_j)\}} | \mathcal{F}_{t_j}) \right] ds \\ &\leq C\Delta \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \mathbb{E} \left[\left(1 + |Z_\Delta^k(t_j)|^q + |\psi_{1j}|^q \right) \right] ds \\ &\leq C\Delta. \end{aligned} \quad (3.41)$$

It follows from Assumption 3.8, Lemmas 3.9 and 3.10 that

$$I_3(t) \leq C\mathbb{E} \int_0^t \left(1 + |\bar{Z}_\Delta^k(s)|^q + |\psi_1(s)|^q \right) \Delta^{q\beta} ds \leq C\Delta^{q\beta} \leq C\Delta. \quad (3.42)$$

Substituting (3.40)-(3.42) into (3.35) yields

$$I(t) \leq C\Delta + C\mathbb{E} \int_0^t |z^k(s) - \bar{Z}_\Delta^k(s)|^q ds. \quad (3.43)$$

Similarly, we can also show that

$$J(t) \leq C\Delta + C\mathbb{E} \int_0^t |z^k(s) - \bar{Z}_\Delta^k(s)|^q ds. \quad (3.44)$$

Putting (3.43) and (3.44) into the (3.34), we obtain

$$\mathbb{E}|z^k(t) - \bar{Z}_\Delta^k(t)|^q \leq C\Delta + C \int_0^t \mathbb{E}|z^k(s) - \bar{Z}_\Delta^k(s)|^q ds. \quad (3.45)$$

Applying the well-known Gronwall inequality gives that

$$\mathbb{E}|z^k(t) - \bar{Z}_\Delta^k(t)|^q \leq C\Delta \quad (3.46)$$

as required. The proof is therefore complete. $\square$

Note that due to the special properties of Markov chain $\theta(t)$, here we can only prove that $\mathbb{E}|z^k(t) - \bar{Z}_\Delta^k(t)|^q \leq C\Delta$ instead of $\mathbb{E}|z^k(t) - \bar{Z}_\Delta^k(t)|^q \leq C\Delta^{q/2}$.

3.4 Convergence in L^q

From now on we will fix $T > 0$ arbitrarily. In this section, we will show the truncated EM solution $X_\Delta^k(T)$ can converge to the solution of hybrid SFDE (3.4), namely

$$\lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E}|\bar{X}_\Delta^k(T) - x(T)|^q \right) = 0 \quad \text{and} \quad \lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E}|X_\Delta^k(T) - x(T)|^q \right) = 0.$$

The following lemma gives an upper bound for the p -th moment of $\bar{X}_\Delta^k(t)$.

Lemma 3.13. *Let Assumptions 3.1 and 3.2 hold. Then*

$$\sup_{0 < \Delta \leq 1} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{X}_\Delta^k(t)|^p \right) \leq C. \quad (3.47)$$

Proof. Fix $\Delta \in (0, 1)$ and the initial data ξ arbitrarily. For $R > \|\xi\|$, define the stopping time

$$\rho_R := \inf\{t \geq 0 : |\bar{X}_\Delta^k(t)| \geq R\}.$$

By the Itô formula, for any $0 < t \leq T$, we have

$$\begin{aligned}
 & \sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u \wedge \rho_R)|^p \\
 & \leq |\xi(0)|^p + p \int_0^{t \wedge \rho_R} |\bar{X}_\Delta^k(s)|^{p-2} \left| (\bar{X}_\Delta^k(s))^T F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right. \\
 & \quad \left. + \frac{p-1}{2} |G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right| ds \\
 & \quad + p \sup_{0 \leq u \leq t} \int_0^{u \wedge \rho_R} |\bar{X}_\Delta^k(s)|^{p-2} \left((\bar{X}_\Delta^k(s))^T G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right) dB(s) \\
 & = |\xi(0)|^p + p \int_0^{t \wedge \rho_R} |\bar{X}_\Delta^k(s)|^{p-2} \left| (X_\Delta^k(s))^T F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right. \\
 & \quad \left. + \frac{p-1}{2} |G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right| ds \\
 & \quad + p \int_0^{t \wedge \rho_R} |\bar{X}_\Delta^k(s)|^{p-2} \left| (\bar{X}_\Delta^k(s) - X_\Delta^k(s))^T F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right| ds \\
 & \quad + p \sup_{0 \leq u \leq t} \int_0^{u \wedge \rho_R} |\bar{X}_\Delta^k(s)|^{p-2} \left((\bar{X}_\Delta^k(s))^T G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right) dB(s).
 \end{aligned}$$

Since the process is stopped at ρ_R , all the terms above are finite. By the Young inequality $a^{p-2}b \leq \frac{p-2}{p}a^p + \frac{2}{p}b^{p/2}$ for any $a, b \geq 0$, and Lemma 3.5, we get

$$\begin{aligned}
 & \mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u \wedge \rho_R)|^p \right) \\
 & \leq |\xi(0)|^p + p \mathbb{E} \int_0^{t \wedge \rho_R} |\bar{X}_\Delta^k(s)|^{p-2} \hat{K}_1 (1 + |X_\Delta^k(s)|^2 + |\psi_1(s)|^2 + |\psi_2(s)|^2) ds \\
 & \quad + 2 \mathbb{E} \int_0^{t \wedge \rho_R} |\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^{\frac{p}{2}} |F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^{\frac{p}{2}} ds \\
 & \quad + (p-2) \mathbb{E} \int_0^{t \wedge \rho_R} |\bar{X}_\Delta^k(s)|^p ds \\
 & \quad + p \mathbb{E} \left(\sup_{0 \leq u \leq t} \int_0^{u \wedge \rho_R} |\bar{X}_\Delta^k(s)|^{p-2} \left((\bar{X}_\Delta^k(s))^T G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right) dB(s) \right).
 \end{aligned} \tag{3.48}$$

By the inequalities (3.14), (3.17) and Lemma 3.6, we see that

$$\begin{aligned}
 & \mathbb{E} \int_0^{t \wedge \rho_R} |\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^{\frac{p}{2}} |F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^{\frac{p}{2}} ds \\
 & \leq (h(\Delta))^{\frac{p}{2}} \int_0^t \mathbb{E} |\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^{\frac{p}{2}} ds \\
 & \leq (h(\Delta))^{\frac{p}{2}} c_p \Delta^{\frac{p}{4}} (h(\Delta))^{\frac{p}{2}} T \leq c_p T.
 \end{aligned} \tag{3.49}$$

For each $s \in [0, T]$, there is a unique j such that $s \in [t_j, t_{j+1})$ and $\psi_1(s) = \psi_{1j}$. By the definition of ψ_{1j} , we further derive

$$\begin{aligned}
 |\psi_{1j}| &= \left| \sum_{h=-kk_1}^{-1} X_\Delta^k(t_{j+h}) \mu_1([t_h, t_{h+1})) + X_\Delta^k(t_{j-kk_1}) \mu_1((-\infty, -k)) \right| \\
 &\leq \sum_{h=-kk_1}^{-1} |X_\Delta^k(t_{j+h})| \mu_1([t_h, t_{h+1})) + |X_\Delta^k(t_{j-kk_1})| \mu_1((-\infty, -k)) \\
 &= \sum_{h=-kk_1}^{-1} e^{rt_h} |X_\Delta^k(t_{j+h})| e^{-rt_h} \mu_1([t_h, t_{h+1})) \\
 &\quad + e^{rt_{-kk_1}} |X_\Delta^k(t_{j-kk_1})| e^{-rt_{-kk_1}} \mu_1((-\infty, -k)) \\
 &\leq \left(\sup_{h \leq 0} e^{rt_h} |X_\Delta^k(t_{j+h})| \right) \left(\sum_{h=-kk_1}^{-1} e^{-rt_h} \mu_1([t_h, t_{h+1})) + e^{-rt_{-kk_1}} \mu_1((-\infty, -k)) \right) \\
 &\leq \left(\sup_{h \leq 0} e^{rt_h} |X_\Delta^k(t_{j+h})| \right) \int_{-\infty}^0 e^{r\Delta - ru} \mu_1(du) \\
 &\leq \left(\sup_{h \leq 0} e^{rt_h} |X_\Delta^k(t_{j+h})| \right) e^r \mu_1^{(r)}.
 \end{aligned} \tag{3.50}$$

Note that

$$\begin{aligned}
 \sup_{h \leq 0} e^{rt_h} |X_\Delta^k(t_{j+h})| &= e^{-rt_j} \sup_{h \leq j} e^{rt_h} |X_\Delta^k(t_h)| \\
 &\leq e^{-rt_j} \left(\sup_{h \leq 0} e^{rt_h} |X_\Delta^k(t_h)| + \sup_{0 \leq h \leq j} e^{rt_h} |X_\Delta^k(t_h)| \right) \\
 &\leq C + \sup_{0 \leq h \leq j} |X_\Delta^k(t_h)|.
 \end{aligned} \tag{3.51}$$

Inserting the inequality (3.51) into (3.50) yields

$$|\psi_{1j}| \leq C + \sup_{0 \leq h \leq j} |X_{\Delta}^k(t_h)|.$$

Then, for $s \leq \rho_R$, we have

$$|\psi_1(s)|^p \leq C \left(1 + \sup_{0 \leq u \leq s} |\bar{X}_{\Delta}^k(u \wedge \rho_R)|^p \right). \quad (3.52)$$

Similarly,

$$|\psi_2(s)|^p \leq C \left(1 + \sup_{0 \leq u \leq s} |\bar{X}_{\Delta}^k(u \wedge \rho_R)|^p \right). \quad (3.53)$$

Therefore, we get

$$\begin{aligned} & |\xi(0)|^p + \mathbb{E} \int_0^{t \wedge \rho_R} p |\bar{X}_{\Delta}^k(s)|^{p-2} \hat{K}_1 (1 + |X_{\Delta}^k(s)|^2 + |\psi_1(s)|^2 + |\psi_2(s)|^2) ds \\ & \leq C + C \int_0^t \mathbb{E} \left(\sup_{0 \leq u \leq s} |\bar{X}_{\Delta}^k(u \wedge \rho_R)|^p \right) ds. \end{aligned} \quad (3.54)$$

By the Burkholder-Davis-Gundy inequality and (3.17), we have

$$\begin{aligned} & p \mathbb{E} \left(\sup_{0 \leq u \leq t} \int_0^{u \wedge \rho_R} |\bar{X}_{\Delta}^k(s)|^{p-2} \left((\bar{X}_{\Delta}^k(s))^T G_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right) dB(s) \right) \\ & \leq \sqrt{32p} \mathbb{E} \left(\int_0^{t \wedge \rho_R} |\bar{X}_{\Delta}^k(s)|^{2p-2} |G_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^2 ds \right)^{\frac{1}{2}} \\ & \leq C + \frac{1}{2} \mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_{\Delta}^k(u \wedge \rho_R)|^p \right) + C \mathbb{E} \int_0^{t \wedge \rho_R} |\bar{X}_{\Delta}^k(s)|^p ds. \end{aligned} \quad (3.55)$$

Substituting (3.49), (3.54) and (3.55) into (3.48), and arranging terms gives

$$\begin{aligned} & \mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_{\Delta}^k(u \wedge \rho_R)|^p \right) \\ & \leq C + C \mathbb{E} \int_0^{t \wedge \rho_R} |\bar{X}_{\Delta}^k(s)|^p ds + C \int_0^t \mathbb{E} \left(\sup_{0 \leq u \leq s} |\bar{X}_{\Delta}^k(u \wedge \rho_R)|^p \right) ds \\ & \leq C + C \int_0^t \mathbb{E} \left(\sup_{0 \leq u \leq s} |\bar{X}_{\Delta}^k(u \wedge \rho_R)|^p \right) ds. \end{aligned}$$

Since the process is stopped at ρ_R , the quantity

$$\mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u \wedge \rho_R)|^p \right)$$

is finite. Hence, by the Gronwall inequality,

$$\mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u \wedge \rho_R)|^p \right) \leq C,$$

where C is independent of R and Δ .

Since $\rho_R \rightarrow \infty$ almost surely as $R \rightarrow \infty$, Fatou's lemma gives

$$\mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u)|^p \right) \leq C.$$

Taking $t = T$ and noting that C is independent of Δ , we obtain

$$\sup_{0 < \Delta \leq 1} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{X}_\Delta^k(t)|^p \right) \leq C.$$

The proof is hence complete. □

Lemma 3.14. *By virtue of Lemma 3.13, it follows from (3.52) and (3.53) that*

$$\sup_{t \in [0, T]} \mathbb{E} |\psi_i(t)|^p \leq C, i = 1, 2.$$

The uniform moment estimates established above enable us to derive corresponding stopping-time probability estimates for both the exact solution and the continuous-time truncated Euler–Maruyama approximation. These estimates will be used repeatedly in the convergence analysis presented later in this section. We first establish the following two auxiliary lemmas.

Lemma 3.15. *Let Assumptions 3.1 and 3.2 hold. For any real number $R > \|\xi\|$, define the stopping time*

$$\varphi_R = \inf\{t \geq 0 : |x^k(t)| \geq R\},$$

where throughout this thesis we set $\inf \emptyset = \infty$. Then

$$\mathbb{P}(\varphi_R \leq T) \leq \frac{C}{R^p}. \quad (3.56)$$

(Please note that C here is independent of Δ and R .)

Proof. By the Itô formula and Assumption 3.2, we derive that for $0 < t \leq T$,

$$\begin{aligned} \mathbb{E}|x^k(t \wedge \varphi_R)|^p &\leq |\xi(0)|^p + \mathbb{E} \int_0^{t \wedge \varphi_R} p|x^k(s)|^{p-2} \left((x^k(s))^T F(x^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right. \\ &\quad \left. + \frac{p-1}{2} |G(x^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds \\ &\leq |\xi(0)|^p + \mathbb{E} \int_0^{t \wedge \varphi_R} p|x^k(s)|^{p-2} K_1 (1 + |x^k(s)|^2 + |\psi_1(s)|^2 + |\psi_2(s)|^2) ds. \end{aligned} \quad (3.57)$$

But, by the well-known Young inequality, we compute

$$|x^k(s)|^{p-2} \leq \frac{p-2}{p} |x^k(s)|^p + \frac{2}{p} \leq |x^k(s)|^p + \frac{2}{p}$$

and

$$\begin{aligned} |x^k(s)|^{p-2} |\psi_1(s)|^2 &\leq |x^k(s)|^p + |\psi_1(s)|^p, \\ |x^k(s)|^{p-2} |\psi_2(s)|^2 &\leq |x^k(s)|^p + |\psi_2(s)|^p. \end{aligned}$$

Hence,

$$\begin{aligned} &\mathbb{E}|x^k(t \wedge \varphi_R)|^p \\ &\leq |\xi(0)|^p + pK_1 \mathbb{E} \int_0^{t \wedge \varphi_R} \left(|x^k(s)|^p + \frac{2}{p} \right. \\ &\quad \left. + |x^k(s)|^p + |x^k(s)|^p + |\psi_1(s)|^p + |x^k(s)|^p + |\psi_2(s)|^p \right) ds \\ &\leq C + K_1 \int_0^t \mathbb{E}|x^k(s \wedge \varphi_R)|^p ds \\ &\leq C + K_1 \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x^k(u \wedge \varphi_R)|^p ds. \end{aligned}$$

It is obvious that the sum of the RHS terms is non-decreasing in t , then

$$\sup_{0 \leq u \leq t} \mathbb{E}|x^k(u \wedge \varphi_R)|^p \leq C + K_1 \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|x^k(u \wedge \varphi_R)|^p \right) ds.$$

According to the Gronwall inequality,

$$\sup_{0 \leq u \leq T} \mathbb{E}|x^k(u \wedge \varphi_R)|^p \leq C.$$

In particular, we get

$$\mathbb{E}|x^k(T \wedge \varphi_R)|^p \leq C.$$

This implies, by the Chebyshev inequality,

$$R^p \mathbb{P}(\varphi_R \leq T) \leq C$$

and the assertion (3.56) follows. $\square$

Lemma 3.16. *Let Assumptions 3.1 and 3.2 hold. For any real number $R > \|\xi\|$ and $\Delta \in (0, 1]$, define the stopping time*

$$\rho_{\Delta, R} = \inf\{t \geq 0 : |\bar{X}_{\Delta}^k(t)| \geq R\},$$

Then

$$\mathbb{P}(\rho_{\Delta, R} \leq T) \leq \frac{C}{R^p}. \quad (3.58)$$

(Recall that C is independent of Δ and R .)

Proof. In the same way as that we obtained (3.48), it can be shown that for $0 < t \leq T$,

$$\begin{aligned} \mathbb{E}|\bar{X}_{\Delta}^k(t \wedge \rho_{\Delta, R})|^p &\leq |\xi(0)|^p + (p-2)\mathbb{E} \int_0^{t \wedge \rho_{\Delta, R}} |\bar{X}_{\Delta}^k(s)|^p ds \\ &\quad + \mathbb{E} \int_0^{t \wedge \rho_{\Delta, R}} p|\bar{X}_{\Delta}^k(s)|^{p-2} \hat{K}_1(1 + |X_{\Delta}^k(s)|^2 + |\psi_1(s)|^2 + |\psi_2(s)|^2) ds \\ &\quad + 2\mathbb{E} \int_0^{t \wedge \rho_{\Delta, R}} |\bar{X}_{\Delta}^k(s) - X_{\Delta}^k(s)|^{\frac{p}{2}} |F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^{\frac{p}{2}} ds. \end{aligned} \quad (3.59)$$

Similar to the proofs of Lemmas 3.13 and 3.15, we get

$$\begin{aligned} \mathbb{E}|\bar{X}_\Delta^k(t \wedge \rho_{\Delta,R})|^p &\leq C + C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^p \right) ds \\ &\quad + 2\mathbb{E} \int_0^{t \wedge \rho_{\Delta,R}} |\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^{\frac{p}{2}} |F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^{\frac{p}{2}} ds. \end{aligned} \quad (3.60)$$

By the assertion (3.49), we have

$$\mathbb{E}|\bar{X}_\Delta^k(t \wedge \rho_{\Delta,R})|^p \leq C + C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^p \right) ds.$$

Noting that the sum of RHS terms is increasing in t , hence

$$\sup_{0 \leq u \leq t} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^p \leq C + C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^p \right) ds.$$

By the Gronwall inequality,

$$\sup_{0 \leq u \leq T} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^p \leq C.$$

This implies,

$$R^p \mathbb{P}(\rho_{\Delta,R} \leq T) \leq C$$

and the assertion (3.58) follows. $\square$

Having established the required moment and stopping-time estimates, we are now ready to present the main convergence result.

Theorem 3.17. *Let Assumptions 3.1-3.4 hold. Then, for any $q \in [2, p)$,*

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|\bar{X}_\Delta^k(T) - x^k(T)|^q = 0, \quad \lim_{\Delta \rightarrow 0} \mathbb{E}|X_\Delta^k(T) - x^k(T)|^q = 0. \quad (3.61)$$

Proof. Recalling the definition of φ_R and $\rho_{\Delta,R}$ in the Lemmas 3.15 and 3.16, set

$$\theta_{\Delta,R} = \varphi_R \wedge \rho_{\Delta,R}, \quad e_\Delta(T) = \bar{X}_\Delta^k(T) - x^k(T).$$

It is clear that

$$\mathbb{E}|e_{\Delta}(T)|^q = \mathbb{E}\left(|e_{\Delta}(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) + \mathbb{E}\left(|e_{\Delta}(T)|^q I_{\{\theta_{\Delta,R} \leq T\}}\right). \quad (3.62)$$

By the Young inequality, for an arbitrary positive number δ ,

$$a^q b = (\delta a^p)^{q/p} \left(\frac{b^{p/(p-q)}}{\delta^{q/(p-q)}}\right)^{(p-q)/p} \leq \frac{q\delta}{p} a^p + \frac{p-q}{p\delta^{q/(p-q)}} b^{p/(p-q)}, \forall a, b > 0.$$

Then we get

$$\mathbb{E}\left(|e_{\Delta}(T)|^q I_{\{\theta_{\Delta,R} \leq T\}}\right) \leq \frac{q\delta}{p} \mathbb{E}|e_{\Delta}(T)|^p + \frac{p-q}{p\delta^{q/(p-q)}} \mathbb{P}(\theta_{\Delta,R} \leq T). \quad (3.63)$$

By the Lemma 3.13, we have

$$\mathbb{E}(|e_{\Delta}(T)|^q) \leq C, \quad (3.64)$$

and by the Lemmas 3.15 and 3.16,

$$\mathbb{P}(\theta_{\Delta,R} \leq T) \leq \mathbb{P}(\varphi_R \leq T) + \mathbb{P}(\rho_{\Delta,R} \leq T) \leq \frac{C}{R^p}. \quad (3.65)$$

Substituting (3.64) and (3.65) into (3.63) yields

$$\mathbb{E}\left(|e_{\Delta}(T)|^q I_{\{\theta_{\Delta,R} \leq T\}}\right) \leq \frac{Cq\delta}{p} + \frac{C(p-q)}{p\delta^{q/(p-q)} R^p}. \quad (3.66)$$

Then we have

$$\mathbb{E}|e_{\Delta}(T)|^q \leq \mathbb{E}\left(|e_{\Delta}(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) + \frac{Cq\delta}{p} + \frac{C(p-q)}{p\delta^{q/(p-q)} R^p}. \quad (3.67)$$

Let $\epsilon > 0$ be arbitrary. Now, we choose δ sufficiently small for $\frac{Cq\delta}{p} < \frac{\epsilon}{3}$ and then choose R sufficiently large for $\frac{C(p-q)}{p\delta^{q/(p-q)} R^p} < \frac{\epsilon}{3}$. Then, from (3.67), we see that for this particular choice R ,

$$\mathbb{E}|e_{\Delta}(T)|^q \leq \mathbb{E}\left(|e_{\Delta}(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) + \frac{2\epsilon}{3}. \quad (3.68)$$

If we can prove that

$$\mathbb{E} \left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}} \right) \leq \frac{\epsilon}{3}, \quad (3.69)$$

we will get

$$\lim_{\Delta \rightarrow 0} \mathbb{E} |\bar{X}_\Delta^k(T) - x^k(T)|^q = 0,$$

and then, by the Lemma 3.6, we will also get

$$\lim_{\Delta \rightarrow 0} \mathbb{E} |X_\Delta^k(T) - x^k(T)|^q = 0.$$

In other words, the remaining work is to show (3.69). For this purpose, we define the truncated functions

$$\begin{aligned} F_R(x, y, i, t) &= F \left((|x| \wedge R) \frac{x}{|x|}, (|y| \wedge R) \frac{y}{|y|}, i, t \right), \\ G_R(x, y, i, t) &= G \left((|x| \wedge R) \frac{x}{|x|}, (|y| \wedge R) \frac{y}{|y|}, i, t \right). \end{aligned} \quad (3.70)$$

for $x, y \in \mathbb{R}^d$. Without the loss of any generality, we choose a sufficiently small $\Delta^* \in (0, 1]$ so that $\eta^{-1}(h(\Delta^*)) \geq R$. Then, for all $\Delta \in (0, \Delta^*]$, we have that $F_\Delta(x, y, i, t) = F_R(x, y, i, t)$ and $G_\Delta(x, y, i, t) = G_R(x, y, i, t)$ for $x, y \in \mathbb{R}^d$ with $|x| \wedge |y| \leq R$. Consider the SFDE

$$dz^k(t) = F_R(z^k(t), \psi_1(\pi_k(z^k(t))), \theta(t), t) ds + G_R(z^k(t), \psi_2(\pi_k(z^k(t))), \theta(t), t) dB(t) \quad (3.71)$$

on $t \geq 0$. By Assumption 3.1, it is obvious that both $F_R(x, y, i, t)$ and $G_R(x, y, i, t)$ are global Lipschitz continuous with the Lipschitz constant K_R . So SFDE (3.71) has a unique global solution $z(t)$. It is easy to see that

$$\mathbb{P}\{x^k(t \wedge \varphi_R) = z^k(t \wedge \varphi_R) \text{ for all } 0 \leq t \leq T\} = 1. \quad (3.72)$$

Therefore, for every step size $\Delta \in (0, \Delta^*]$, we can apply the modified EM method to the SFDE (3.71) and we denote by $\bar{Z}_\Delta^k(t)$ the continuous-time continuous EM solution

(continuous auxiliary process). It is again easy to see that

$$\mathbb{P}\{\bar{X}_\Delta^k(t \wedge \varphi_R) = \bar{Z}_\Delta^k(t \wedge \varphi_R) \text{ for all } 0 \leq t \leq T\} = 1. \quad (3.73)$$

By virtue of Lemma 3.12, we have

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |z^k(t) - \bar{Z}_\Delta^k(t)|^q\right) \leq C\Delta.$$

Hence,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |z^k(t \wedge \theta_{\Delta,R}) - \bar{Z}_\Delta^k(t \wedge \theta_{\Delta,R})|^q\right) \leq C\Delta.$$

We then have, by the assertions (3.72) and (3.73),

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |x^k(t \wedge \theta_{\Delta,R}) - \bar{X}_\Delta^k(t \wedge \theta_{\Delta,R})|^q\right) \leq C\Delta,$$

which implies

$$\mathbb{E}\left(|x^k(T \wedge \theta_{\Delta,R}) - \bar{X}_\Delta^k(T \wedge \theta_{\Delta,R})|^q\right) \leq C\Delta.$$

At last

$$\begin{aligned} \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) &= \mathbb{E}\left(|e_\Delta(T \wedge \theta_{\Delta,R})|^q I_{\{\theta_{\Delta,R} > T\}}\right) \\ &\leq \mathbb{E}\left(|x(T \wedge \theta_{\Delta,R}) - x_\Delta(T \wedge \theta_{\Delta,R})|^q\right) \leq C\Delta. \end{aligned} \quad (3.74)$$

This implies (3.61) as required. The proof is hence complete. $\square$

Li et al. [25] proved that the solution of hybrid SFDE (3.9) with finite time delay approximates the solution of hybrid SFDE (3.4) in the q -th moment given k is sufficiently large. We cite this theorem as a lemma here.

Lemma 3.18. *Let Assumptions 3.1 and 3.2 hold. Then, for each $q \in [2, p)$, the solutions of hybrid SFDEs (3.4) and (3.9) have the property that*

$$\lim_{k \rightarrow \infty} \left(\sup_{0 \leq t \leq T} \mathbb{E}|x(t) - x^k(t)|^q \right) = 0, \quad \forall T > 0.$$

Combining Theorem 3.17 and Lemma 3.18, we can get the convergence of the truncated EM solution as below.

Theorem 3.19. *Let Assumptions 3.1 - 3.4 hold. Then, for any $q \in [2, p)$,*

$$\lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E} |\bar{X}_\Delta^k(T) - x(T)|^q \right) = 0, \quad \lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E} |X_\Delta^k(T) - x(T)|^q \right) = 0. \quad (3.75)$$

3.5 Convergence rates

In the last section, we discuss the convergence of the truncated EM solutions in the asymptotic form without the convergence rates. We will consider the convergence rate in this section. Here we need two more assumptions for the convergence rate.

Assumption 3.20. *Assume that there are positive constants α and K_3*

$$\begin{aligned} & (x - \bar{x})^T (F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)) + \frac{1 + \alpha}{2} |G(x, z, i, t) - G(\bar{x}, \bar{z}, i, t)|^2 \\ & \leq K_3 (|x - \bar{x}|^2 + |y - \bar{y}|^2 + |z - \bar{z}|^2) \end{aligned} \quad (3.76)$$

for all $x, y, z, \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.21. *Assume that there is a pair of positive constants γ and K_4 such that*

$$\begin{aligned} & |F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)|^2 \vee |G(x, y, i, t) - G(\bar{x}, \bar{y}, i, t)|^2 \\ & \leq K_4 (|x - \bar{x}|^2 + |y - \bar{y}|^2) (1 + |x|^\gamma + |\bar{x}|^\gamma + |y|^\gamma + |\bar{y}|^\gamma) \end{aligned} \quad (3.77)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Remark. Assumption 3.21 implies Assumption 3.1. For any $h > 0$, if $|x| \vee |\bar{x}| \vee |y| \vee |\bar{y}| \leq h$, then

$$1 + |x|^\gamma + |\bar{x}|^\gamma + |y|^\gamma + |\bar{y}|^\gamma \leq 1 + 4h^\gamma.$$

Thus, the polynomial growth factor in Assumption 3.21 is bounded on any bounded set and can be absorbed into the Lipschitz constant K_h . Hence, Assumption 3.1 follows from Assumption 3.21.

The following lemma shows that the truncated functions F_Δ and G_Δ preserve Assumption 3.21 perfectly.

Lemma 3.22. *Let Assumption 3.21 hold. Then, for every $\Delta \in (0, 1]$, we have*

$$\begin{aligned} & |F_\Delta(x, y, i, t) - F_\Delta(\bar{x}, \bar{y}, i, t)|^2 \vee |G_\Delta(x, y, i, t) - G_\Delta(\bar{x}, \bar{y}, i, t)|^2 \\ & \leq K_4(|x - \bar{x}|^2 + |y - \bar{y}|^2)(1 + |x|^\gamma + |\bar{x}|^\gamma + |y|^\gamma + |\bar{y}|^\gamma) \end{aligned} \quad (3.78)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Proof. For any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$,

$$\begin{aligned} & |F_\Delta(x, y, i, t) - F_\Delta(\bar{x}, \bar{y}, i, t)|^2 \vee |G_\Delta(x, y, i, t) - G_\Delta(\bar{x}, \bar{y}, i, t)|^2 \\ & = |F(\pi_\Delta(x), \pi_\Delta(y), i, t) - F(\pi_\Delta(\bar{x}), \pi_\Delta(\bar{y}), i, t)|^2 \\ & \quad \vee |G(\pi_\Delta(x), \pi_\Delta(y), i, t) - G(\pi_\Delta(\bar{x}), \pi_\Delta(\bar{y}), i, t)|^2 \\ & \leq K_4(|\pi_\Delta(x) - \pi_\Delta(\bar{x})|^2 + |\pi_\Delta(y) - \pi_\Delta(\bar{y})|^2)(1 + |\pi_\Delta(x)|^\gamma \\ & \quad + |\pi_\Delta(\bar{x})|^\gamma + |\pi_\Delta(y)|^\gamma + |\pi_\Delta(\bar{y})|^\gamma). \end{aligned}$$

This implies the assertion by noting that $|\pi_\Delta(x)| \leq |x|$ and $|\pi_\Delta(x) - \pi_\Delta(\bar{x})|^2 \leq |x - \bar{x}|^2$.

The proof is complete. $\square$

The following theorem gives the convergence rate between numerical solution and hybrid SFDE (3.9) with finite time delay.

Theorem 3.23. *Let Assumption 3.2-3.4, 3.20 and 3.21 hold and assume that $p > \gamma + 2$. Then for any $q \in [2, p)$ and $\Delta \in (0, 1]$,*

$$\mathbb{E}|\bar{X}_\Delta^k(T) - x^k(T)|^q \leq C(\Delta \vee \Delta^{q/2}(h(\Delta))^q \vee (\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)}) \quad (3.79)$$

$$\mathbb{E}|X_\Delta^k(T) - x^k(T)|^q \leq C(\Delta \vee \Delta^{q/2}(h(\Delta))^q \vee (\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)}). \quad (3.80)$$

Proof. To make it clearer, this proof is divided into three steps.

Step 1. Fix $\Delta \in (0, 1]$ and the initial data ξ arbitrarily. Let R be any integer such

that $R \geq \|\xi\|$. Recall the definition of the stopping time

$$\theta_{\Delta,R} = \varphi_R \wedge \rho_{\Delta,R} = \inf\{t \geq 0 : |\bar{X}_{\Delta}^k(t)| \wedge |x^k(t)| \geq R\},$$

and the error $e_{\Delta}(t) = \bar{X}_{\Delta}^k(t) - x^k(t)$ for $t \in [-\tau, T]$. We write $\theta_{\Delta,R} = \theta$ for simplicity. It is clear to see that $e_{\Delta}(t) = 0$ for $t \in [-\tau, 0]$. By the Itô formula, we can show that, for $t \in [-\tau, T]$,

$$\begin{aligned} & \mathbb{E}|e_{\Delta}(t \wedge \theta)|^q \\ & \leq \mathbb{E} \int_0^{t \wedge \theta} q|e_{\Delta}(s)|^{q-2} \left(e_{\Delta}^T(s) (F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) \right. \\ & \quad \left. - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \right. \\ & \quad \left. + \frac{q-1}{2} |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds \\ & = \mathbb{E} \int_0^{t \wedge \theta} q|e_{\Delta}(s)|^{q-2} \left(e_{\Delta}^T(s) ([F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) \right. \\ & \quad \left. - F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s)] \right. \\ & \quad \left. + [F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))] \right) \\ & \quad \left. + \frac{q-1}{2} |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s) \right. \\ & \quad \left. + G(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds \\ & \leq \mathbb{E} \int_0^{t \wedge \theta} q|e_{\Delta}(s)|^{q-2} \left(e_{\Delta}^T(s) (F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) \right. \\ & \quad \left. - F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s)) \right. \\ & \quad \left. + \frac{(q-1)(1+\alpha)}{2} |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s)|^2 \right. \\ & \quad \left. + e_{\Delta}^T(s) (F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \right. \\ & \quad \left. + \frac{(q-1)(1+\alpha^{-1})}{2} |G(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds. \end{aligned}$$

Re-arranging the terms gives

$$\mathbb{E}|e_{\Delta}(t \wedge \theta)|^q \leq H_1 + H_2, \quad (3.81)$$

where

$$H_1 := \mathbb{E} \int_0^{t \wedge \theta} q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) - F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)) \right. \\ \left. + \frac{(q-1)(1+\alpha)}{2} |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds$$

and

$$H_2 := \mathbb{E} \int_0^{t \wedge \theta} q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \right. \\ \left. + \frac{(q-1)(1+\alpha^{-1})}{2} |G(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds.$$

Step 2. Then, we estimate H_1 and H_2 . By Assumption 3.20, we get

$$H_1 \leq C \mathbb{E} \int_0^{t \wedge \theta} q |e_\Delta(s)|^{q-2} (|e_\Delta(s)|^2 + |\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 + |\psi_2(\pi_k(x_s^k)) - \psi_2(s)|^2) ds \\ \leq C \mathbb{E} \int_0^{t \wedge \theta} |e_\Delta(s)|^q ds + C \mathbb{E} \int_0^{t \wedge \theta} (|\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^q \vee |\psi_2(\pi_k(x_s^k)) - \psi_2(s)|^q) ds.$$

Similar to the proof of (3.39), for $i = 1, 2$, we have

$$\mathbb{E} \int_0^t |\psi_i(\pi_k(x_s^k)) - \psi_i(s)|^q ds \leq C \Delta^{q/2} + C \mathbb{E} \int_0^t |x^k(s) - \bar{X}_\Delta^k(s)|^q ds. \quad (3.82)$$

Substituting (3.82) into H_1 gives

$$H_1 \leq C \int_0^t \mathbb{E} |e_\Delta(s \wedge \theta)|^q ds + C \Delta^{q/2}. \quad (3.83)$$

To estimate H_2 , we observe that

$$H_2 \leq H_{21} + H_{22} + H_{23} + H_{24}, \quad (3.84)$$

where

$$\begin{aligned}
 H_{21} &:= \mathbb{E} \int_0^T q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)) \right. \\
 &\quad \left. + \frac{(q-1)(1+\alpha^{-1})}{2} |G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds, \\
 H_{22} &:= \mathbb{E} \int_0^T q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s)) \right. \\
 &\quad \left. + \frac{(q-1)(1+\alpha^{-1})}{2} |G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds, \\
 H_{23} &:= \mathbb{E} \int_0^T q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s)) \right. \\
 &\quad \left. + \frac{(q-1)(1+\alpha^{-1})}{2} |G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), s)|^2 \right) ds, \\
 H_{24} &:= \mathbb{E} \int_0^T q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \right. \\
 &\quad \left. + \frac{(q-1)(1+\alpha^{-1})}{2} |G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds,
 \end{aligned}$$

where $t \wedge \theta$ has been replaced by T since $t \wedge \theta \leq T$.

Rearranging H_{21} gives

$$\begin{aligned}
 H_{21} &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + q \mathbb{E} \int_0^T |e_\Delta^T(s)|^{q-2} \left(\frac{1}{2} |F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) \right. \\
 &\quad \left. - F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)|^2 \right. \\
 &\quad \left. + \frac{(q-1)(1+\alpha^{-1})}{2} |G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \mathbb{E} \int_0^T |e_\Delta^T(s)|^{q-2} \left(|F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) \right. \\
 &\quad \left. - F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)|^2 \right. \\
 &\quad \left. \vee |G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds.
 \end{aligned}$$

By the Hölder inequality and Assumption 3.21, we can derive that

$$\begin{aligned}
 & \mathbb{E}|F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)|^2 \\
 &= \mathbb{E}|F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F(\pi_\Delta(\bar{X}_\Delta^k(s)), \pi_\Delta(\psi_1(s)), \theta(s), s)|^2 \\
 &\leq K_4 \mathbb{E}[(|\bar{X}_\Delta^k(s) - \pi_\Delta(\bar{X}_\Delta^k(s))|^2 + |\psi_1(s) - \pi_\Delta(\psi_1(s))|^2)(1 + 2|\bar{X}_\Delta^k(s)|^\gamma + 2|(\psi_1(s))|^\gamma)] \\
 &\leq C \left(\mathbb{E}|\bar{X}_\Delta^k(s) - \pi_\Delta(\bar{X}_\Delta^k(s))|^{2p/(p-\gamma)} + \mathbb{E}|\psi_1(s) - \pi_\Delta(\psi_1(s))|^{2p/(p-\gamma)} \right)^{(p-\gamma)/p} \\
 &\quad \times (1 + \mathbb{E}|\bar{X}_\Delta^k(s)|^p + \mathbb{E}|(\psi_1(s))|^p)^{\gamma/p} \\
 &\leq C \left(\mathbb{E} \left[I_{\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}} |\bar{X}_\Delta^k(s)|^{2p/(p-\gamma)} \right] + \mathbb{E} \left[I_{\{\psi_1(s) > \eta^{-1}(h(\Delta))\}} |\psi_1(s)|^{2p/(p-\gamma)} \right] \right)^{(p-\gamma)/p}.
 \end{aligned}$$

Note that

$$\begin{aligned}
 & \mathbb{E} \left[I_{\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}} |\bar{X}_\Delta^k(s)|^{2p/(p-\gamma)} \right] \\
 &\leq [\mathbb{P}\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}]^{(p-\gamma-2)/(p-\gamma)} [\mathbb{E}|\bar{X}_\Delta^k(s)|^p]^{2/(p-\gamma)} \\
 &\leq C [\mathbb{P}\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}]^{(p-\gamma-2)/(p-\gamma)} \\
 &\leq C \left[\frac{\mathbb{E}\{\bar{X}_\Delta^k(s)\}}{(\eta^{-1}(h(\Delta)))^p} \right]^{(p-\gamma-2)/(p-\gamma)} \leq C(\eta^{-1}(h(\Delta)))^{-p(p-\gamma-2)/(p-\gamma)}.
 \end{aligned}$$

Similarly,

$$\mathbb{E} \left[I_{\{\psi_1(s) > \eta^{-1}(h(\Delta))\}} |\psi_1(s)|^{2p/(p-\gamma)} \right] \leq C(\eta^{-1}(h(\Delta)))^{-p(p-\gamma-2)/(p-\gamma)}.$$

Hence, we get

$$\mathbb{E}|F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)|^2 \leq C(\eta^{-1}(h(\Delta)))^{-(p-\gamma-2)}.$$

Similarly, it can be derived that

$$\mathbb{E}|G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \leq C(\eta^{-1}(h(\Delta)))^{-(p-\gamma-2)}.$$

We then get

$$H_{21} \leq C \int_0^T \mathbb{E}|e_\Delta^T(s)|^q ds + C(\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)}. \quad (3.85)$$

Let us estimate H_{22} now. By Assumption 3.21 and the Young inequality as well as Lemmas 3.6, 3.13 and 3.22, we can show

$$\begin{aligned}
 H_{22} &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds \\
 &\quad + C \mathbb{E} \int_0^T |e_\Delta^T(s)|^{q-2} \left(|F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s)|^2 \right. \\
 &\quad \vee |G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \Big) \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds \\
 &\quad + C \mathbb{E} \int_0^T |e_\Delta^T(s)|^{q-2} \left(|\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^2 (1 + |\bar{X}_\Delta^k(s)|^\gamma + |X_\Delta^k(s)|^\gamma + 2|\psi_1(s)|^\gamma) \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \mathbb{E} \int_0^T (|e_\Delta^T(s)|^q)^{(q-2)/q} \left((|\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^q \right. \\
 &\quad \times (1 + |\bar{X}_\Delta^k(s)|^\gamma + |X_\Delta^k(s)|^\gamma + 2|\psi_1(s)|^\gamma)^{q/2} \Big)^{2/q} ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \int_0^T \mathbb{E} \left(|\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^q \right. \\
 &\quad \times (1 + |\bar{X}_\Delta^k(s)|^{q\gamma/2} + |X_\Delta^k(s)|^{q\gamma/2} + 2|\psi_1(s)|^{q\gamma/2}) \Big) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \Delta^{q/2} (h(\Delta))^q. \tag{3.86}
 \end{aligned}$$

For notational convenience, denote

$$\begin{aligned}
 \Theta_F(s) &:= F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s), \\
 \Theta_G(s) &:= G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), s).
 \end{aligned}$$

Rearranging H_{23} gives

$$\begin{aligned}
 H_{23} &= \mathbb{E} \int_0^T q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (\Theta_F(s)) + \frac{(q-1)(1+\alpha^{-1})}{2} |\Theta_G(s)|^2 \right) ds \\
 &\leq q \mathbb{E} \int_0^T \left(|e_\Delta^T(s)|^q + \frac{1}{2} |\Theta_F(s)|^q + \frac{(q-1)(1+\alpha^{-1})}{2} (|e_\Delta^T(s)|^q + |\Theta_G(s)|^q) \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \mathbb{E} \int_0^T \left(|\Theta_F(s)|^q \vee |\Theta_G(s)|^q \right) ds.
 \end{aligned}$$

By Markov property of $\theta(\cdot)$ and Assumption 3.21 as well as Lemma 3.13, 3.14 and 3.22,

we derive that

$$\begin{aligned}
 & \mathbb{E} \int_0^t |\Theta_F(s)|^q ds \\
 &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |F(X_\Delta^k(t_j), \psi_{1j}, \theta(s), t_j) - F(X_\Delta^k(t_j), \psi_{1j}, \theta(t_j), t_j)|^q ds \\
 &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |F(X_\Delta^k(t_j), \psi_{1j}, \theta(s), t_j) - F(X_\Delta^k(t_j), \psi_{1j}, \theta(t_j), t_j)|^q I_{\{\theta(s) \neq \theta(t_j)\}} ds \\
 &\leq C \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \mathbb{E} \left[\left(1 + |X_\Delta^k(t_j)|^{q(2+\gamma)/2} + |\psi_{1j}|^{q(2+\gamma)/2} \right) \mathbb{E}(I_{\{\theta(s) \neq \theta(t_j)\}} | \mathcal{F}_{t_j}) \right] ds \\
 &\leq C \Delta \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \mathbb{E} \left[\left(1 + |X_\Delta(t_j)|^{q(2+\gamma)/2} + |\psi_{1j}|^{q(2+\gamma)/2} \right) \right] ds \\
 &\leq C \Delta.
 \end{aligned}$$

Substituting above into H_{23} gives

$$H_{23} \leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \Delta. \quad (3.87)$$

For notational convenience, define

$$\begin{aligned}
 \Phi_F(s) &:= F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)), \\
 \Phi_G(s) &:= G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s)).
 \end{aligned}$$

It follows from Assumption 3.4, Lemmas 3.13 and 3.14 that

$$\begin{aligned}
 H_{24} &:= \mathbb{E} \int_0^T q |e_\Delta(s)|^{q-2} \left(e_\Delta^T(s) (\Phi_F(s)) + \frac{(q-1)(1+\alpha^{-1})}{2} |\Phi_G(s)|^2 \right) ds \\
 &\leq q \mathbb{E} \int_0^T \left(|e_\Delta^T(s)|^q + \frac{1}{2} |\Phi_F(s)|^q + \frac{(q-1)(1+\alpha^{-1})}{2} (|e_\Delta^T(s)|^q + |\Phi_G(s)|^q) \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \int_0^T \left(\mathbb{E} |\Phi_F(s)|^q \vee \mathbb{E} |\Phi_G(s)|^q \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \Delta^{q\beta} \leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^q ds + C \Delta. \quad (3.88)
 \end{aligned}$$

Combining (3.85) - (3.88) yields

$$H_2 \leq C \int_0^T \mathbb{E}|e_\Delta^T(s)|^q ds + C(\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)} + C\Delta^{q/2}(h(\Delta))^q + C\Delta. \quad (3.89)$$

Step 3. Insert (3.83) and (3.89) into (3.81), we obtain

$$\mathbb{E}|e_\Delta(t \wedge \theta)|^q \leq C \int_0^T \mathbb{E}|e_\Delta^T(s)|^q ds + C(\Delta \vee \Delta^{q/2}(h(\Delta))^q \vee (\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)}). \quad (3.90)$$

By the Gronwall inequality, we have

$$\mathbb{E}|e_\Delta(T \wedge \theta)|^q \leq C(\Delta \vee \Delta^{q/2}(h(\Delta))^q \vee (\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)}) \quad (3.91)$$

We can get the first assertion (3.79) by letting $R \rightarrow \infty$. The second assertion (3.80) follows the assertion (3.91) and Lemma 3.6. The proof is complete. $\square$

Now, we obtained the convergence rate of $X_\Delta^k(t)$ to $x^k(t)$. We only need the convergence rate of the $x^k(t)$ to $x(t)$ before we get the convergence rate of $X_\Delta^k(t)$ to $x(t)$. Li et al. [25] established the exponential convergence order of $x^k(t)$ to $x(t)$ as follows.

Assumption 3.24. *There are positive numbers $\zeta, \bar{q}, \bar{K}, b_3, b_4$, functions $\bar{V} \in C^{2,1}(\mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, $U \in \mathcal{U}_{0,\zeta}$, as well as two probability measures $\mu_3 \in \mathcal{P}_{b_3}, \mu_4 \in \mathcal{P}_{b_4}$, such that $b_3 > r\bar{q}, b_4 > r\zeta$,*

$$\begin{aligned} \bar{V}(0, i, t) &= 0, \forall (i, t) \in \mathbb{S} \times \mathbb{R}_+, \\ |x|^{\bar{q}} &\leq \bar{V}(x, i, t), \forall (x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+, \end{aligned} \quad (3.92)$$

and

$$\mathcal{L}\bar{V}(\varphi, \phi, i, t) \leq \bar{K} \int_{-\infty}^0 |\varphi(u) - \phi(u)|^{\bar{q}} \mu_3(du) - U(\varphi(0), \phi(0)) + \int_{-\infty}^0 U(\varphi(u), \phi(u)) \mu_4(du), \quad (3.93)$$

for all $(\varphi, \phi, i, t) \in \mathcal{C}_r \times \mathcal{C}_r \times \mathbb{S} \times \mathbb{R}_+$.

Theorem 3.25. *Let Assumptions 3.2, 3.21 and 3.24 hold. Set $\lambda = (b_3 - r\bar{q}) \wedge (b_4 - r\zeta)$.*

Then the solutions of equations (3.4) and (3.9) have the properties that, for all $T > 0$,

$$\lim_{k \rightarrow \infty} \sup \frac{1}{k} \log \left(\mathbb{E} |x(T) - x^k(T)|^{\bar{q}} \right) \leq -\lambda, \quad (3.94)$$

and

$$\lim_{k \rightarrow \infty} \sup \frac{1}{k} \log(|x(T) - x^k(T)|) \leq -\frac{\lambda}{\bar{q}} a.s. \quad (3.95)$$

We recommend readers to [25] for more details about Assumption 3.24 and Theorem 3.25. To apply Theorem 3.25, we need to verify Assumption 3.20 implies Assumption 3.24. Let $\bar{V}(x, i, t) = |x|^{\bar{q}}$ for $\bar{q} \in [2, p)$. From now on, we also let $\delta_0(\cdot)$ be the Dirac measure at 0, which can be regarded as a probability measure on $\mathbb{R}_-$ and it belongs to $\bigcap_{b \geq 1} \mathcal{P}_b$. For $\varphi, \phi \in \mathcal{C}_r$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$, we can derive that

$$\begin{aligned} \mathcal{L}\bar{V}(\varphi, \phi, i, t) &\leq \bar{q}|\varphi(0) - \phi(0)|^{\bar{q}-2} \left((\varphi(0) - \phi(0))^T (f(\varphi, i, t) - f(\phi, i, t)) \right. \\ &\quad \left. + \frac{\bar{q}-1}{2} |g(\varphi, i, t) - g(\phi, i, t)|^2 \right) \\ &= \bar{q}|\varphi(0) - \phi(0)|^{\bar{q}-2} \left((\varphi(0) - \phi(0))^T (F(\varphi(0), \psi_1(\varphi), i, t) - F(\phi(0), \psi_1(\phi), i, t)) \right. \\ &\quad \left. + \frac{\bar{q}-1}{2} |G(\varphi(0), \psi_2(\varphi), i, t) - G(\phi(0), \psi_2(\phi), i, t)|^2 \right) \\ &\leq \bar{q}K_3|\varphi(0) - \phi(0)|^{\bar{q}-2} \left(|\varphi(0) - \phi(0)|^2 + |\psi_1(\varphi) - \psi_1(\phi)|^2 + |\psi_2(\varphi) - \psi_2(\phi)|^2 \right) \\ &\leq 2\bar{q}K_3 \left(|\varphi(0) - \phi(0)|^{\bar{q}} + |\psi_1(\varphi) - \psi_1(\phi)|^{\bar{q}} + |\psi_2(\varphi) - \psi_2(\phi)|^{\bar{q}} \right). \end{aligned}$$

Using $|\psi_1(\phi)|^p \leq \int_{-\infty}^0 |\phi(u)|^p \mu_1(du)$ and $|\psi_2(\phi)|^p \leq \int_{-\infty}^0 |\phi(u)|^p \mu_2(du)$, we obtain

$$\begin{aligned} \mathcal{L}\bar{V}(\varphi, \phi, i, t) &\leq 2\bar{q}K_3 \left(\int_{-\infty}^0 |\varphi(0) - \phi(0)|^{\bar{q}} \delta_0(du) \right. \\ &\quad \left. + \int_{-\infty}^0 |\psi_1(\varphi) - \psi_1(\phi)|^{\bar{q}} \mu_1(du) + \int_{-\infty}^0 |\psi_2(\varphi) - \psi_2(\phi)|^{\bar{q}} \mu_2(du) \right) \\ &\leq 6\bar{q}K_3 \int_{-\infty}^0 |\varphi(0) - \phi(0)|^{\bar{q}} \bar{\mu}_3(du), \end{aligned}$$

where $\bar{\mu}_3(\cdot) = [\delta_0(\cdot) + \mu_1(\cdot) + \mu_2(\cdot)]/3 \in \mathcal{P}_{r\bar{q}}$. This shows that Assumption 3.24 is satisfied with $U(x) = 0$ etc. By virtue of Theorem 3.25, we can conclude that under Assumptions 3.2, 3.20 and 3.21, the solutions of equations (3.4) and (3.9) have the

properties (3.94) and (3.95) with $\lambda = (p - \bar{q})r$ for any $\bar{q} \in [2, p)$.

Corollary 3.26. *Let Assumptions 3.2, 3.20 and 3.21 hold. Then the solution $x^k(t)$ of the truncated SFDE (3.9) approximates the solution $x(t)$ of the given SFDE (3.4) in the sense that*

$$\mathbb{E}|x(t) - x^k(t)|^{\bar{q}} \leq C e^{-(p-\bar{q})rk}, \forall t \in [0, T]. \quad (3.96)$$

Finally, we could give the convergence rate of the truncated EM numerical solution to the given hybrid SFDE (3.4) as follows.

Theorem 3.27. *Suppose that Assumptions 3.2-3.4, 3.20 and 3.21 hold and $\mu_1, \mu_2 \in \mathcal{P}_{rp}$. Then for any $\Delta \in (0, 1], q \in [2, p)$ and any integer $k > T$,*

$$\mathbb{E}|X_{\Delta}^k(t) - x(t)|^q \leq C(\Delta \vee \Delta^{q/2}(h(\Delta))^q \vee (\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)} + e^{-(p-q)rk}) \quad (3.97)$$

for all $t \in [0, T]$.

3.6 Example

In this section, we will discuss an application example and carry out the simulation by MATLAB to illustrate the results. The Black and Scholes formula, proposed in 1973, is one of the most important results of financial model research. The model is widely used by option market participants, although some adjustments are often made. In order to simulate complex financial markets more accurately, assumptions on the model have been relaxed and generalized in many directions, and some variants of the formula have also appeared (see, e.g. [3, 52]). One of these variants is the stochastic functional volatility equation

$$dx(t) = F(x(t), \psi_1(x_t), \theta(t), t)dt + G(x(t), \psi_2(x_t), \theta(t), t)dB(t) \quad (3.98)$$

with the initial data $\xi \in \mathcal{C}_r$ and $\theta(0) = i_0 \in \mathbb{S}$ satisfying Assumption 3.3, where

$$F(x, y, i, t) = a_1 + a_2|y|^{4/3} + a_3x^3, \quad G(x, z, i, t) = a_4|x|^{4/3} + a_5z,$$

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$x, y \in \mathbb{R}$ and $a_1, \dots, a_5$ are all real numbers with $a_3 < 0$. The coefficients F and G are local Lipschitz continuous, i.e. Assumption 3.1 is satisfied. Now, consider the Khasminskii condition. For any $p > 2$, we have

$$xF(x, y, i, t) + \frac{p-1}{2}|G(x, z, i, t)|^2 \leq |a_1||x| + |a_2||x||y|^{4/3} + a_3|x|^4 + (p-1)(a_4^2|x|^{8/3} + a_5^2|z|^2).$$

By the Young inequality, we know that $|x||y|^{4/3} \leq |x|^3 + |y|^2$. Hence, we have

$$\begin{aligned} & xF(x, y, i, t) + \frac{p-1}{2}|G(x, z, i, t)|^2 \\ & \leq |a_1||x| + |a_2||x|^3 + a_4^2(p-1)|x|^{8/3} + a_3|x|^4 + |a_2||y|^2 + a_5^2(p-1)|z|^2 \\ & \leq K_1(1 + |y|^2 + |z|^2), \end{aligned}$$

where $K_1 = (|a_2| + a_5^2(p-1)) \vee \beta_1$ and

$$\beta_1 = \sup_{u \geq 0} [|a_1|u + |a_2|u^3 + a_4^2(p-1)u^{8/3} + a_3u^4] < \infty.$$

We have hence verified that Assumption 3.2 is satisfied for any $p > 2$. Note that, for $x, \bar{x}, y, \bar{y}, z, \bar{z} \in \mathbb{R}$,

$$\begin{aligned} & (x - \bar{x})(F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)) \\ & = (x - \bar{x})[a_2(|y|^{4/3} - |\bar{y}|^{4/3}) + a_3(x^3 - \bar{x}^3)] \\ & \leq \frac{1}{2}a_3|x - \bar{x}|^2(x^2 + \bar{x}^2) + \frac{1}{2}a_2^2(x - \bar{x})^2 + \frac{1}{2}a_2^2(|y|^{4/3} - |\bar{y}|^{4/3})^2 \end{aligned}$$

and

$$|G(x, z, i, t) - G(\bar{x}, \bar{z}, i, t)|^2 = 2a_4^2(|x|^{4/3} - |\bar{x}|^{4/3})^2 + 2a_5^2|z - \bar{z}|^2.$$

By the mean-valued theorem, we see that

$$(|y|^{4/3} - |\bar{y}|^{4/3})^2 \leq \frac{16}{9}|y - \bar{y}|^2(|y|^{1/3} + |\bar{y}|^{1/3})^2 \leq \frac{32}{9}|y - \bar{y}|^2(|y|^{2/3} + |\bar{y}|^{2/3}) \leq \beta_2|y - \bar{y}|^2,$$

where $\beta_2 = \sup_{u \geq 0} \frac{64}{9} u^{2/3} < \infty$. Similarly, we also have

$$(|x|^{4/3} - |\bar{x}|^{4/3})^2 \leq \beta_3 |x - \bar{x}|^2,$$

with $\beta_3 = \sup_{u \geq 0} \frac{64}{9} u^{2/3} < \infty$. Letting $\alpha = 1$ gives

$$\begin{aligned} & (x - \bar{x})(F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)) + \frac{1 + \alpha}{2} |G(x, z, i, t) - G(\bar{x}, \bar{z}, i, t)|^2 \\ & \leq K_3(|x - \bar{x}|^2 + |y - \bar{y}|^2 + |z - \bar{z}|^2), \end{aligned}$$

where $K_3 = a_3 u^2 + \frac{1}{2} a_2^2 (1 + \beta_2) + 2a_4^2 \beta_3 + 2a_5^2 < \infty$. Therefore, it can be concluded that Assumption 3.20 is satisfied. It is easy to show that Assumption 3.21 is satisfied with $\gamma = 4$.

Before applying the truncated EM method, we need to find functions η and h satisfying (3.13) and (3.14). It is easy to see that

$$\sup_{|x| \vee |y| \leq r, (i, t) \in \mathbb{S} \times \mathbb{R}_+} (|F(x, y, i, t)| \vee |G(x, y, i, t)|) \leq \hat{a} r^3, \forall r \geq 1,$$

where $\hat{a} = (|a_1| + |a_2| + |a_3|) \vee (|a_4| + |a_5|)$. We hence choose that $\eta(r) = \hat{a} r^3$ and its inverse function $\eta^{-1}(r) = (r/\hat{a})^{1/3}$ for $r \geq 0$. For $\epsilon \in (0, 1/4]$, we define $h(\Delta) = \Delta^{-\epsilon}$ for $\Delta > 0$. By Theorem 3.27, it can be derived that the truncated EM solutions will converge to the true solution in the sense that

$$\mathbb{E}|X_{\Delta}^k(t) - x(t)|^q \leq C(\Delta \vee \Delta^{q(1/2-\epsilon)} \vee \Delta^{1/3\epsilon q(p-6)} + e^{-(p-q)rk}), \forall \Delta \in (0, 1].$$

Let us discuss a specific example. The coefficients of (3.98) are defined by

$$F(x(t), \psi_1(x_t), \theta(t), t) = \begin{cases} 2 + 3|\psi_1(x_t)|^{4/3} - 3x(t)^3, & i = 1, \\ 3 + |\psi_1(x_t)|^{4/3} - 4x(t)^3, & i = 2, \end{cases}$$

$$G(x(t), \psi_2(x_t), \theta(t), t) = \begin{cases} 2|x(t)|^{4/3} + 2\psi_2(x_t), & i = 1, \\ 3|x(t)|^{4/3} + \psi_2(x_t), & i = 2, \end{cases}$$

where $\psi_1(x_t) = \psi_2(x_t) = \int_{-\infty}^0 x(t+u)\mu(du) \in \mathcal{C}_{1/5}$, the probability measure $\mu(\cdot)$ has its probability density function e^u on $(-\infty, 0]$. $B(t)$ is a scalar Brownian motion and $\theta(t)$ is a Markov chains on the state space $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Let the initial data $\xi(u) = e^u \in \mathcal{C}_{1/5}$ and $\theta(0) = 1$. The corresponding SFDE with finite time delay is given by

$$dx^k(t) = F(x^k(t), \psi_1(\pi_k(x_t)), \theta(t), t)dt + G(x^k(t), \psi_2(\pi_k(x_t)), \theta(t), t)dB(t). \quad (3.99)$$

The coefficients are defined by

$$F(x^k(t), \psi_1(\pi_k(x_t)), \theta(t), t) = \begin{cases} 2 + 3|e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du)|^{4/3} - 3x^k(t)^3, & i = 1, \\ 3 + |e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du)|^{4/3} - 4x^k(t)^3, & i = 2, \end{cases}$$

$$G(x^k(t), \psi_2(\pi_k(x_t)), \theta(t), t) = \begin{cases} 2|x^k(t)|^{4/3} + 2(e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du)), & i = 1, \\ 3|x^k(t)|^{4/3} + e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du), & i = 2. \end{cases}$$

Next, we use MATLAB to perform numerical simulations. We take the truncated EM numerical solution of (3.99) with $\Delta = 2^{-6}$ and $k = 300$ as the exact solution of (3.98). For $k \in \{10, 11, 12, 13, 14, 15\}$, we take the truncated EM numerical solutions of (3.99) with $\Delta = 2^{-6}$ as the exact $x^k(t)$ of (3.99) and plot the mean square approximation error $\mathbb{E}|x(10) - x^k(10)|^2$ for 1000 sample as shown in Figure 3.1a.

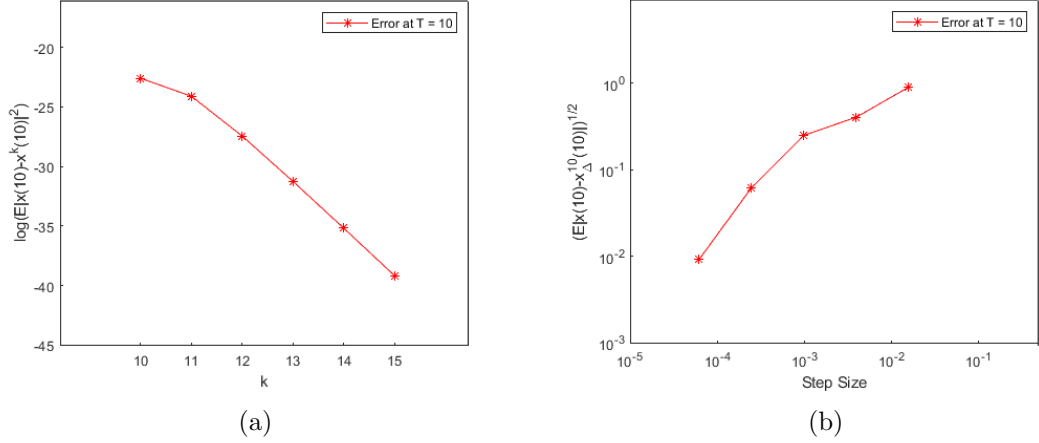


Figure 3.1: The mean square approximation error between the exact solution and the numerical solutions.

To show the relationship between the error of the exact solution and numerical solution and the step size Δ , we set the truncated EM solution of (3.99) with $\Delta = 2^{-18}$ and $k = 300$ as the exact solution of (3.98). Then, we calculate the truncated EM solutions of (3.99) with $\Delta \in \{2^{-6}, 2^{-8}, 2^{-10}, 2^{-12}, 2^{-14}\}$ for 1000 sample and plot the root mean square approximation error between the exact solution and numerical solutions at $T = 10$. Figure 3.1b depicts this result.

3.7 Summary

In this chapter, we developed a truncated Euler–Maruyama method for hybrid stochastic functional differential equations with infinite time delay. The original infinite-delay system was first approximated by a corresponding hybrid SFDE with finite delay, after which the truncated Euler–Maruyama scheme was applied. This approach successfully overcomes both the difficulty caused by the infinite memory and the restrictive global Lipschitz condition required in earlier studies.

Under the local Lipschitz and Khasminskii-type conditions, we established the strong convergence of the numerical solutions. More precisely, for any $T > 0$ and $q \in [2, p)$,

$$\lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E} |\bar{X}_{\Delta}^k(T) - x(T)|^q \right) = 0,$$

and

$$\lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E} |X_{\Delta}^k(T) - x(T)|^q \right) = 0,$$

where p is the parameter appearing in the Khasminskii-type condition.

Furthermore, under additional assumptions, convergence-rate estimates were obtained. In particular,

$$\mathbb{E} |X_{\Delta}^k(t) - x(t)|^q \leq C \left((\Delta \vee \Delta^{q/2} (h(\Delta))^q) \vee (\eta^{-1}(h(\Delta)))^{-q(p-\gamma-2)} + e^{-(p-q)rk} \right), \quad t \in [0, T].$$

These results provide a rigorous theoretical foundation for the numerical approximation of hybrid SFDEs with infinite delay.

The techniques developed in this chapter serve as the basis for the subsequent analysis. In the next chapter, we further relax several assumptions imposed in the present framework and extend the convergence theory to the more challenging case $q \in [1, 2)$. Strong convergence and convergence-rate results will be established under weaker conditions, thereby broadening the applicability of the truncated Euler–Maruyama method.

Chapter 4

Extension of the Truncated Euler–Maruyama Method for Hybrid SFDEs with Infinite Time Delay under Dissipative Khasminskii-Type Conditions

4.1 Introduction

In Chapter 3, a truncated Euler–Maruyama approximation framework was developed for hybrid stochastic functional differential equations with infinite time delay under the local Lipschitz and Khasminskii-type conditions. By combining the infinite-delay truncation technique proposed in [25] with the truncated EM method introduced by Mao [37], the strong convergence and convergence rates of the numerical solutions were established in the L^q sense for $q \geq 2$. These results provided an effective numerical approximation theory for a class of hybrid SFDEs with infinite time delay and significantly extended the applicability of the truncated EM method to systems with infinite memory and Markovian switching.

Although the convergence theory established in Chapter 3 covers a broad class of hybrid SFDEs with infinite delay, the assumptions imposed therein still exclude many important stochastic systems appearing in applications. In particular, the analysis relies on the following Khasminskii-type condition:

$$x^T F(x, y, i, t) + \frac{p-1}{2} |G(x, z, i, t)|^2 \leq K_1 (1 + |x|^2 + |y|^2 + |z|^2), \quad (4.1)$$

where $p > 2$. This condition is sufficient for establishing the moment estimates required in the convergence analysis. However, the Khasminskii-type condition still imposes a restrictive growth constraint on the drift and diffusion coefficients, which excludes many dissipative stochastic systems with super-linear nonlinearities.

In many practical applications, stochastic systems possess a dissipative structure. Typical examples include stochastic models with polynomial nonlinearities, population dynamics, and stochastic volatility systems, where the drift coefficient may exhibit super-linear growth but simultaneously contains a stabilising mechanism that suppresses excessive growth of the solution. Such systems often admit global solutions and possess favourable stability properties despite violating condition (4.1). Therefore, it is desirable to develop a numerical convergence theory under more general dissipative assumptions.

Motivated by this observation, we replace condition (4.1) by the following dissipative Khasminskii-type condition:

$$x^T F(x, y, i, t) + \frac{1}{2} |G(x, z, i, t)|^2 \leq K_1 (1 + |x|^2 + |y|^2 + |z|^2) - c_1 |x|^{\hat{p}} + c_2 |y|^{\hat{p}} + c_3 |z|^{\hat{p}}, \quad (4.2)$$

where $\hat{p} > 2$ and $c_1 \geq c_2 + c_3$.

Compared with (4.1), condition (4.2) permits stronger nonlinear growth in the system coefficients while introducing the dissipative term $-c_1 |x|^{\hat{p}}$, which provides additional control over the solution. The negative coercive term plays a crucial role in preventing excessive growth and enables the derivation of stronger moment estimates for both the exact and numerical solutions. Consequently, the admissible class of hybrid SFDEs with infinite time delay is substantially enlarged.

The dissipative condition (4.2) not only extends the class of stochastic systems that can be analysed, but also provides additional coercive control over the solution compared with the Khasminskii-type condition used in Chapter 3. In particular, the negative term $-c_1|x|^{\hat{p}}$, together with $c_1 \geq c_2 + c_3$, counteracts the potentially super-linear growth of the drift and diffusion coefficients. This coercive structure yields stronger moment estimates for both the exact and numerical solutions, including the uniform second-moment estimate established below. This additional moment information enables us to establish the strong convergence of the truncated Euler–Maruyama method for lower-order moments satisfying

$$1 \leq q < 2.$$

Furthermore, convergence rate estimates are first established for the truncated Euler–Maruyama approximation of the corresponding finite delay system. By combining these estimates with the approximation result of Li et al. [25], convergence rates for the original infinite delay system are obtained, and the results are further extended to all $1 \leq q < 2$ by means of the Hölder inequality. Together with the convergence theory developed in Chapter 3, the truncated Euler–Maruyama method is shown to possess both strong convergence and convergence rates over the entire range $q \in [1, p)$.

Motivated by the above considerations, this chapter develops a convergence theory for the truncated Euler–Maruyama method under dissipative Khasminskii-type conditions. The main contributions of this chapter can be summarised as follows.

- A convergence framework is established for hybrid stochastic functional differential equations with infinite time delay under dissipative Khasminskii-type conditions, substantially extending the applicability of the theory developed in Chapter 3.
- Strong moment estimates are derived under the dissipative structure, allowing stochastic systems with super-linear nonlinearities to be treated within the truncated Euler–Maruyama framework.

- The strong convergence of the truncated Euler–Maruyama method is established in the L^q sense for $1 \leq q < 2$. Moreover, an L^2 convergence rate is derived under the dissipative assumptions and is subsequently extended to all $q \in [1, 2)$.

The remainder of this chapter is organised as follows. In Section 4.2, we present the preliminary results required for the subsequent analysis, including the dissipative assumptions and several auxiliary properties of the truncated Euler–Maruyama approximation. Section 4.3 establishes the strong convergence of the truncated Euler–Maruyama method in the range $1 \leq q < 2$. Convergence rates are investigated in Section 4.4. Finally, Section 4.5 presents numerical examples and simulations that illustrate the theoretical results.

4.2 Preliminary results

Unless otherwise stated, all notation and assumptions introduced in Chapter 2 remain in force throughout this chapter. Moreover, the truncated Euler–Maruyama method developed in Chapter 3 is adopted throughout the subsequent analysis. The principal difference between the present chapter and Chapter 3 lies in the Khasminskii-type condition imposed on the system coefficients. Accordingly, we replace the original Khasminskii-type condition 3.2 by the following dissipative version.

Assumption 4.1. *Let $\mu_1, \mu_2 \in \mathcal{P}_{r\hat{p}}$. There are nonnegative constants $K_1, c_1, c_2, c_3, \hat{p}$ such that $c_1 \geq c_2 + c_3$ and $\hat{p} > 2$, and*

$$x^T F(x, y, i, t) + \frac{1}{2} |G(x, z, i, t)|^2 \leq K_1(1 + |x|^2 + |y|^2 + |z|^2) - c_1 |x|^{\hat{p}} + c_2 |y|^{\hat{p}} + c_3 |z|^{\hat{p}} \quad (4.3)$$

for all $(x, y, z, i, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

Compared with the Khasminskii-type condition employed in Chapter 3, the above assumption introduces an additional dissipative structure through the negative coercive term $-c_1 |x|^{\hat{p}}$. With the condition $c_1 \geq c_2 + c_3$, this provides stronger control over the growth of the solution while allowing the drift and diffusion coefficients to possess stronger nonlinearities.

To carry out the convergence analysis under the above assumption, it is first necessary to verify that the truncated coefficients preserve the dissipative Khasminskii-type condition.

Lemma 4.2. *Let Assumption 4.1 hold. Then, for every $\Delta \in (0, 1]$, we have*

$$\begin{aligned} x^T F_\Delta(x, y, i, t) + \frac{1}{2} |G_\Delta(x, z, i, t)|^2 &\leq \hat{K}_1(1 + |x|^2 + |y|^2 + |z|^2) \\ &\quad - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}} \end{aligned} \quad (4.4)$$

for all $(x, y, z, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $\hat{K}_1 = 2K_1(1 \vee [1/\eta^{-1}(h(1))])$.

Proof. Fix any $\Delta \in (0, 1]$. For $x \in \mathbb{R}^n$ with $|x| \leq \eta^{-1}(h(\Delta))$ and any $y, z \in \mathbb{R}^n$, we have, by (4.3),

$$\begin{aligned} &x^T F_\Delta(x, y, i, t) + \frac{1}{2} |G_\Delta(x, z, i, t)|^2 \\ &= \pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) + \frac{1}{2} |G(\pi_\Delta(x), \pi_\Delta(z), i, t)|^2 \\ &\leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}} \\ &\leq K_1(1 + |x|^2 + |y|^2 + |z|^2) - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}}, \end{aligned} \quad (4.5)$$

which implies the desired assertion (4.4). On the other hand, for $x \in \mathbb{R}^d$ with $|x| > \eta^{-1}(h(\Delta))$ and any $y, z \in \mathbb{R}^n$, we have

$$\begin{aligned} &x^T F_\Delta(x, y, i, t) + \frac{1}{2} |G_\Delta(x, z, i, t)|^2 \\ &= \pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) + \frac{1}{2} |G(\pi_\Delta(x), \pi_\Delta(z), i, t)|^2 \\ &\quad + (x - \pi_\Delta(x))^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) \\ &\leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}} \\ &\quad + \left(\frac{|x|}{\eta^{-1}(h(\Delta))} - 1 \right) \pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t). \end{aligned} \quad (4.6)$$

But we see from (4.3) again that

$$\pi_\Delta(x)^T F(\pi_\Delta(x), \pi_\Delta(y), i, t) \leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2). \quad (4.7)$$

Substituting this into (3.20) gives

$$\begin{aligned}
& x^T F_\Delta(x, y, i, t) + \frac{1}{2} |G_\Delta(x, z, i, t)|^2 \\
& \leq K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}} \\
& + \left(\frac{|x|}{\eta^{-1}(h(\Delta))} - 1 \right) K_1(1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) \\
& \leq \frac{K_1|x|}{\eta^{-1}(h(\Delta))} (1 + |\pi_\Delta(x)|^2 + |\pi_\Delta(y)|^2 + |\pi_\Delta(z)|^2) - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}} \\
& \leq K_1(1 \vee [1/\eta^{-1}(h(1))]) |x|(1 + |x| + |y| + |z|) - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}} \\
& \leq 2K_1(1 \vee [1/\eta^{-1}(h(1))]) (1 + |x|^2 + |y|^2 + |z|^2) - c_1 |\pi_\Delta(x)|^{\hat{p}} + c_2 |\pi_\Delta(y)|^{\hat{p}} + c_3 |\pi_\Delta(z)|^{\hat{p}}.
\end{aligned} \tag{4.8}$$

Thus, we have proved that the required condition (4.4) holds for $x \in \mathbb{R}^d$ with $|x| > \eta^{-1}(h(\Delta))$ and any $y, z \in \mathbb{R}^n$. The proof is complete. $\square$

The above lemma shows that the dissipative Khasminskii-type condition is inherited by the truncated coefficients. Consequently, the analytical framework developed in Chapter 3 can be extended to the present setting. This property plays a fundamental role in establishing the moment estimates, strong convergence and convergence rates presented in the following sections.

4.3 Convergence in L^q for $1 \leq q < 2$

In Chapter 3, the strong convergence of the truncated Euler–Maruyama method was established in the L^q sense for $q \geq 2$. In this section, we extend the convergence theory to the lower-order moment case $1 \leq q < 2$ under the dissipative Khasminskii-type condition 4.1.

The proof follows the general analytical framework developed in Chapter 3. The main difference is that the dissipative structure enables us to establish a uniform L^2 -moment estimate for the truncated numerical solution, from which the desired L^q convergence for $1 \leq q < 2$ can subsequently be obtained.

We first establish a uniform bound for the second moment of the continuous-time numerical solution $\bar{X}_\Delta^k(t)$.

Lemma 4.3. *Let Assumptions 3.1 and 4.1 hold. Then*

$$\sup_{0 < \Delta \leq 1} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{X}_\Delta^k(t)|^2 \right) \leq C. \quad (4.9)$$

Proof. Fix $\Delta \in (0, 1)$ and the initial data ξ arbitrarily. By the Itô formula, it can be derived that for any $0 < t \leq T$,

$$\begin{aligned} \sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u)|^2 &\leq |\xi(0)|^2 + \int_0^t \left(2(\bar{X}_\Delta^k(s))^T F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right. \\ &\quad \left. + |G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds \\ &\quad + 2 \sup_{0 \leq u \leq t} \int_0^u (\bar{X}_\Delta^k(s))^T G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) dB(s) \\ &= |\xi(0)|^2 + \int_0^t \left(2(X_\Delta^k(s))^T F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) \right. \\ &\quad \left. + |G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds \\ &\quad + 2 \int_0^t (\bar{X}_\Delta^k(s) - X_\Delta^k(s))^T F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) ds \\ &\quad + 2 \sup_{0 \leq u \leq t} \int_0^u (\bar{X}_\Delta^k(s))^T G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) dB(s) \end{aligned}$$

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By the Lemma 4.2,

$$\begin{aligned}
& \mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u)|^2 \right) \\
& \leq |\xi(0)|^2 + 2\mathbb{E} \int_0^t \left(\hat{K}_1(1 + |X_\Delta^k(s)|^2 + |\psi_1(s)|^2 + |\psi_2(s)|^2) \right. \\
& \quad \left. - c_1|X_\Delta^k(s)|^{\hat{p}} + c_2|\psi_1(s)|^{\hat{p}} + c_3|\psi_2(s)|^{\hat{p}} \right) ds \\
& \quad + 2\mathbb{E} \int_0^t |\bar{X}_\Delta^k(s) - X_\Delta^k(s)| |F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))| ds \\
& \quad + 2\mathbb{E} \left(\sup_{0 \leq u \leq t} \int_0^u (\bar{X}_\Delta^k(s))^T G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) dB(s) \right). \tag{4.10}
\end{aligned}$$

The following estimates have already been established in the proof of Lemma 3.13 in Chapter 3. Since their derivations are unchanged under the present assumptions, we simply recall them here for completeness.

$$\begin{aligned}
& \mathbb{E} \int_0^t |\bar{X}_\Delta^k(s) - X_\Delta^k(s)| |F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))| ds \\
& \leq h(\Delta) \int_0^t \mathbb{E} |\bar{X}_\Delta^k(s) - X_\Delta^k(s)| ds \leq c_p(h(\Delta))^2 \Delta^{\frac{1}{2}} T \leq c_p T. \tag{4.11}
\end{aligned}$$

$$|\psi_1(s)|^p \leq C \left(1 + \sup_{0 \leq u \leq s} |\bar{X}_\Delta^k(u)|^p \right). \tag{4.12}$$

$$|\psi_2(s)|^p \leq C \left(1 + \sup_{0 \leq u \leq s} |\bar{X}_\Delta^k(u)|^p \right). \tag{4.13}$$

Therefore, we get

$$\begin{aligned}
& |\xi(0)|^2 + 2\mathbb{E} \int_0^t \left(\hat{K}_1(1 + |X_\Delta^k(s)|^2 + |\psi_1(s)|^2 + |\psi_2(s)|^2) \right. \\
& \quad \left. - c_1|X_\Delta^k(s)|^{\hat{p}} + c_2|\psi_1(s)|^{\hat{p}} + c_3|\psi_2(s)|^{\hat{p}} \right) ds \\
& \leq C + C \int_0^t \mathbb{E} \left(\sup_{0 \leq u \leq s} |\bar{X}_\Delta^k(u)|^2 \right) ds. \tag{4.14}
\end{aligned}$$

By the Burkholder-Davis-Gundy inequality and (3.17), we have

$$\begin{aligned}
& 2\mathbb{E} \left(\sup_{0 \leq u \leq t} \int_0^u (\bar{X}_\Delta^k(s))^T G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) dB(s) \right) \\
& \leq 2\sqrt{32}\mathbb{E} \left(\int_0^t |\bar{X}_\Delta^k(s)|^2 |G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^2 ds \right)^{\frac{1}{2}} \\
& \leq C\mathbb{E} \left(\int_0^t |\bar{X}_\Delta^k(s)|^4 ds \right)^{\frac{1}{2}} + C\mathbb{E} \left(\int_0^t |G_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^4 ds \right)^{\frac{1}{2}} \\
& \leq C + \mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u)|^2 \int_0^t C |\bar{X}_\Delta^k(s)|^2 ds \right)^{\frac{1}{2}} \\
& \leq C + \frac{1}{2}\mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u)|^2 \right) + C\mathbb{E} \int_0^t |\bar{X}_\Delta^k(s)|^2 ds. \tag{4.15}
\end{aligned}$$

Substituting (4.11), (4.14) and (4.15) into (4.10), and arranging terms gives

$$\begin{aligned}
\mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u)|^2 \right) & \leq C + C\mathbb{E} \int_0^t |\bar{X}_\Delta^k(s)|^2 ds + C \int_0^t \mathbb{E} \left(\sup_{0 \leq u \leq s} |\bar{X}_\Delta^k(u)|^2 \right) ds \\
& \leq C + C \int_0^t \mathbb{E} \left(\sup_{0 \leq u \leq s} |\bar{X}_\Delta^k(u)|^2 \right) ds.
\end{aligned}$$

Applying the well-known Gronwall inequality yields that

$$\mathbb{E} \left(\sup_{0 \leq u \leq t} |\bar{X}_\Delta^k(u)|^2 \right) \leq C.$$

As this holds for any $t \in (0, T]$, $\Delta \in (0, 1]$ while C is independent of Δ , we get the required assertion

$$\sup_{0 < \Delta \leq 1} \mathbb{E} \left(\sup_{0 \leq t \leq T} |\bar{X}_\Delta^k(t)|^2 \right) \leq C.$$

The proof is hence complete. $\square$

Although the following stopping-time estimates are analogous to those obtained in Chapter 3, their proofs are modified to accommodate the dissipative Khasminskii-type condition introduced in this chapter.

Lemma 4.4. *Let Assumptions 3.1 and 4.1 hold. For any real number $R > \|\xi\|$, define the stopping time*

$$\varphi_R = \inf\{t \geq 0 : |x^k(t)| \geq R\}.$$

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Then

$$\mathbb{P}(\varphi_R \leq T) \leq \frac{C}{R^2}. \quad (4.16)$$

(Please note that C here is independent of Δ and R .)

Proof. By the Itô formula and Assumption 4.1, we derive that for $0 < t \leq T$,

$$\begin{aligned} & \mathbb{E}|x^k(t \wedge \varphi_R)|^2 \\ & \leq |\xi(0)|^2 + 2\mathbb{E} \int_0^{t \wedge \varphi_R} \left((x^k(s))^T F(x^k(s), \psi_1(\pi_k(x^k(s))), \theta(s), s) \right. \\ & \quad \left. + \frac{1}{2} |G(x^k(s), \psi_2(\pi_k(x^k(s))), \theta(s), s)|^2 \right) ds \\ & \leq |\xi(0)|^2 + 2\mathbb{E} \int_0^{t \wedge \varphi_R} \left(K_1(1 + |x^k(s)|^2 + |\psi_1(\pi_k(x^k(s)))|^2 + |\psi_2(\pi_k(x^k(s)))|^2) \right) ds \\ & \quad - 2c_1 \mathbb{E} \int_0^{t \wedge \varphi_R} |x^k(s)|^{\hat{p}} ds + 2c_2 \mathbb{E} \int_0^{t \wedge \varphi_R} |\psi_1(\pi_k(x^k(s)))|^{\hat{p}} ds \\ & \quad + 2c_3 \mathbb{E} \int_0^{t \wedge \varphi_R} |\psi_2(\pi_k(x^k(s)))|^{\hat{p}} ds \\ & \leq |\xi(0)|^2 + 2K_1 T + 2K_1 \mathbb{E} \int_0^{t \wedge \varphi_R} (|x^k(s)|^2 + |\psi_1(\pi_k(x^k(s)))|^2 + |\psi_2(\pi_k(x^k(s)))|^2) ds \\ & \quad - 2c_1 \mathbb{E} \int_0^{t \wedge \varphi_R} |x^k(s)|^{\hat{p}} ds + 2c_2 \mathbb{E} \int_0^{t \wedge \varphi_R} |\psi_1(\pi_k(x^k(s)))|^{\hat{p}} ds \\ & \quad + 2c_3 \mathbb{E} \int_0^{t \wedge \varphi_R} |\psi_2(\pi_k(x^k(s)))|^{\hat{p}} ds. \end{aligned} \quad (4.17)$$

Using the definition of ψ_1 together with Fubini's theorem, we obtain

$$\begin{aligned} & \mathbb{E} \int_0^{t \wedge \varphi_R} |\psi_1(\pi_k(x^k(s)))|^{\hat{p}} ds \\ & = \mathbb{E} \int_0^{t \wedge \varphi_R} \int_{-\infty}^0 |\pi_k(x^k(s+u))|^{\hat{p}} \mu_1(du) ds \\ & = \mathbb{E} \int_0^{t \wedge \varphi_R} \left(\int_{-\infty}^{-s} |\pi_k(x^k(s+u))|^{\hat{p}} \mu_1(du) + \int_{-s}^0 |\pi_k(x^k(s+u))|^{\hat{p}} \mu_1(du) \right) ds \\ & \leq C + \mathbb{E} \int_{-t \wedge \varphi_R}^0 \int_{-u}^{t \wedge \varphi_R} |\pi_k(x^k(s+u))|^{\hat{p}} ds \mu_1(du) \\ & \leq C + \mathbb{E} \int_{-\infty}^0 \int_0^{t \wedge \varphi_R} |\pi_k(x^k(s+u))|^{\hat{p}} ds \mu_1(du) \\ & \leq C + \mathbb{E} \int_0^{t \wedge \varphi_R} |\pi_k(x^k(s))|^{\hat{p}} ds. \end{aligned} \quad (4.18)$$

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Similarly, we have

$$\mathbb{E} \int_0^{t \wedge \varphi_R} |\psi_2(\pi_k(x^k(s)))|^{\hat{p}} ds \leq C + \mathbb{E} \int_0^{t \wedge \varphi_R} |\pi_k(x^k(s))|^{\hat{p}} ds. \quad (4.19)$$

Substituting (4.18) and (4.19) into (4.17) yields

$$\mathbb{E}|x^k(t \wedge \varphi_R)|^2 \leq C + 6K_1 \mathbb{E} \int_0^t |x^k(s \wedge \varphi_R)|^2 ds \leq C + 6K_1 \int_0^t \sup_{0 \leq u \leq s} \mathbb{E}|x^k(u \wedge \varphi_R)|^2 ds.$$

It is obvious that the sum of the RHS terms is non-decreasing in t , then

$$\sup_{0 \leq u \leq t} \mathbb{E}|x^k(u \wedge \varphi_R)|^2 \leq C + 6K_1 \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|x^k(u \wedge \varphi_R)|^2 \right) ds.$$

According to the Gronwall inequality,

$$\sup_{0 \leq u \leq T} \mathbb{E}|x^k(u \wedge \varphi_R)|^2 \leq C.$$

In particular, we get

$$\mathbb{E}|x^k(T \wedge \varphi_R)|^2 \leq C.$$

This implies, by the Chebyshev inequality,

$$R^2 \mathbb{P}(\varphi_R \leq T) \leq C$$

and the assertion (4.16) follows. $\square$

Lemma 4.5. *Let Assumptions 3.1 and 4.1 hold. For any real number $R > \|\xi\|$ and $\Delta \in (0, 1]$, define the stopping time*

$$\rho_{\Delta, R} = \inf\{t \geq 0 : |\bar{X}_{\Delta}^k(t)| \geq R\},$$

Then

$$\mathbb{P}(\rho_{\Delta, R} \leq T) \leq \frac{C}{R^2}. \quad (4.20)$$

(Recall that C is independent of Δ and R .)

Proof. In the same way as that we obtained (4.10), it can be shown that for $0 < t \leq T$,

$$\begin{aligned} \mathbb{E}|\bar{X}_\Delta^k(t \wedge \rho_{\Delta,R})|^2 &\leq |\xi(0)|^2 + 2\mathbb{E} \int_0^{t \wedge \rho_{\Delta,R}} \left(\hat{K}_1(1 + |X_\Delta^k(s)|^2 + |\psi_1(s)|^2 + |\psi_2(s)|^2) \right. \\ &\quad \left. - c_1|X_\Delta^k(s)|^{\hat{p}} + c_2|\psi_1(s)|^{\hat{p}} + c_3|\psi_2(s)|^{\hat{p}} \right) ds \\ &\quad + 2\mathbb{E} \int_0^{t \wedge \rho_{\Delta,R}} |\bar{X}_\Delta^k(s) - X_\Delta^k(s)| |F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))| ds. \end{aligned} \quad (4.21)$$

Similar to the proofs of Lemmas 4.3 and 4.4, we get

$$\begin{aligned} \mathbb{E}|\bar{X}_\Delta^k(t \wedge \rho_{\Delta,R})|^2 &\leq C + C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^2 \right) ds \\ &\quad + 2\mathbb{E} \int_0^{t \wedge \rho_{\Delta,R}} |\bar{X}_\Delta^k(s) - X_\Delta^k(s)| |F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))| ds. \end{aligned} \quad (4.22)$$

By the assertion (4.11), we have

$$\mathbb{E}|\bar{X}_\Delta^k(t \wedge \rho_{\Delta,R})|^2 \leq C + C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^2 \right) ds.$$

Noting that the sum of RHS terms is increasing in t , hence

$$\sup_{0 \leq u \leq t} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^2 \leq C + C \int_0^t \left(\sup_{0 \leq u \leq s} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^2 \right) ds.$$

By the Gronwall inequality,

$$\sup_{0 \leq u \leq T} \mathbb{E}|\bar{X}_\Delta^k(u \wedge \rho_{\Delta,R})|^2 \leq C.$$

This implies,

$$R^2 \mathbb{P}(\rho_{\Delta,R} \leq T) \leq C$$

and the assertion (4.20) follows. $\square$

Having established the corresponding moment and stopping-time estimates under the dissipative Khasminskii-type condition, we now establish the strong convergence of

the truncated Euler–Maruyama method for $1 \leq q < 2$.

Theorem 4.6. *Let Assumptions 3.1, 4.1, 3.3 and 3.4 hold. Then, for any $q \in [1, 2)$,*

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|\bar{X}_\Delta^k(T) - x^k(T)|^q = 0, \quad \lim_{\Delta \rightarrow 0} \mathbb{E}|X_\Delta^k(T) - x^k(T)|^q = 0. \quad (4.23)$$

Proof. Recalling the definition of φ_R and $\rho_{\Delta,R}$ in the Lemmas 4.4 and 4.5, set

$$\theta_{\Delta,R} = \varphi_R \wedge \rho_{\Delta,R}, \quad e_\Delta(T) = \bar{X}_\Delta^k(T) - x^k(T).$$

It is clear that

$$\mathbb{E}|e_\Delta(T)|^q = \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) + \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} \leq T\}}\right). \quad (4.24)$$

By the Young inequality, for an arbitrary positive number δ ,

$$a^q b = (\delta a^2)^{q/2} \left(\frac{b^{2/(2-q)}}{\delta^{q/(2-q)}} \right)^{(2-q)/2} \leq \frac{q\delta}{2} a^2 + \frac{2-q}{2\delta^{q/(2-q)}} b^{2/(2-q)}, \quad \forall a, b > 0.$$

Then we get

$$\mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} \leq T\}}\right) \leq \frac{q\delta}{2} \mathbb{E}|e_\Delta(T)|^2 + \frac{2-q}{2\delta^{q/(2-q)}} \mathbb{P}(\theta_{\Delta,R} \leq T). \quad (4.25)$$

By the Lemma 4.3, we have

$$\mathbb{E}(|e_\Delta(T)|^2) \leq C, \quad (4.26)$$

and by the Lemmas 4.4 and 4.5,

$$\mathbb{P}(\theta_{\Delta,R} \leq T) \leq \mathbb{P}(\varphi_R \leq T) + \mathbb{P}(\rho_{\Delta,R} \leq T) \leq \frac{C}{R^p}. \quad (4.27)$$

Substituting (4.26) and (4.27) into (4.25) yields

$$\mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} \leq T\}}\right) \leq \frac{Cq\delta}{2} + \frac{C(2-q)}{2\delta^{q/(2-q)} R^2}. \quad (4.28)$$

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Then we have

$$\mathbb{E}|e_\Delta(T)|^q \leq \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) + \frac{Cq\delta}{2} + \frac{C(2-q)}{2\delta^{q/(2-q)}R^2}. \quad (4.29)$$

Let $\epsilon > 0$ be arbitrary. Now, we choose δ sufficiently small for $\frac{Cq\delta}{2} < \frac{\epsilon}{3}$ and then choose R sufficiently large for $\frac{C(2-q)}{2\delta^{q/(2-q)}R^2} < \frac{\epsilon}{3}$. Then, from (4.29), we see that for this particular choice R ,

$$\mathbb{E}|e_\Delta(T)|^q \leq \mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) + \frac{2\epsilon}{3}. \quad (4.30)$$

If we can prove that

$$\mathbb{E}\left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}}\right) \leq \frac{\epsilon}{3}, \quad (4.31)$$

we will get

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|\bar{X}_\Delta^k(T) - x^k(T)|^q = 0,$$

and then, by the Lemma 3.6, we will also get

$$\lim_{\Delta \rightarrow 0} \mathbb{E}|X_\Delta^k(T) - x^k(T)|^q = 0.$$

In other words, the remaining work is to show (4.31). For this purpose, we define the truncated functions

$$\begin{aligned} F_R(x, y, i, t) &= F\left((|x| \wedge R) \frac{x}{|x|}, (|y| \wedge R) \frac{y}{|y|}, i, t\right), \\ G_R(x, y, i, t) &= G\left((|x| \wedge R) \frac{x}{|x|}, (|y| \wedge R) \frac{y}{|y|}, i, t\right) \end{aligned} \quad (4.32)$$

for $x, y \in \mathbb{R}^d$. Without the loss of any generality, we choose a sufficiently small $\Delta^* \in (0, 1]$ so that $\eta^{-1}(h(\Delta^*)) \geq R$. Then, for all $\Delta \in (0, \Delta^*]$, we have that $F_\Delta(x, y, i, t) = F_R(x, y, i, t)$ and $G_\Delta(x, y, i, t) = G_R(x, y, i, t)$ for $x, y \in \mathbb{R}^d$ with $|x| \wedge |y| \leq R$. Consider

the SFDE

$$dz^k(t) = F_R(z^k(t), \psi_1(\pi_k(z^k(t))), \theta(t), t)ds + G_R(z^k(t), \psi_2(\pi_k(z^k(t))), \theta(t), t)dB(t) \quad (4.33)$$

on $t \geq 0$. By Assumption 3.1, it is obvious that both $F_R(x, y, i, t)$ and $G_R(x, y, i, t)$ are global Lipschitz continuous with the Lipschitz constant K_R . So SFDE (4.33) has a unique global solution $z(t)$. It is easy to see that

$$\mathbb{P}\{x^k(t \wedge \varphi_R) = z^k(t \wedge \varphi_R) \text{ for all } 0 \leq t \leq T\} = 1. \quad (4.34)$$

Therefore, for every step size $\Delta \in (0, \Delta^*]$, we can apply the modified EM method to the SFDE (4.33) and we denote by $\bar{Z}_\Delta^k(t)$ the continuous-time continuous EM solution (continuous auxiliary process). It is again easy to see that

$$\mathbb{P}\{\bar{X}_\Delta^k(t \wedge \varphi_R) = \bar{Z}_\Delta^k(t \wedge \varphi_R) \text{ for all } 0 \leq t \leq T\} = 1. \quad (4.35)$$

By virtue of Lemma 3.12, we have

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |z^k(t) - \bar{Z}_\Delta^k(t)|^q\right) \leq C\Delta.$$

Hence,

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |z^k(t \wedge \theta_{\Delta, R}) - \bar{Z}_\Delta^k(t \wedge \theta_{\Delta, R})|^q\right) \leq C\Delta.$$

We then have, by the assertions (4.34) and (4.35),

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} |x^k(t \wedge \theta_{\Delta, R}) - \bar{X}_\Delta^k(t \wedge \theta_{\Delta, R})|^q\right) \leq C\Delta,$$

which implies

$$\mathbb{E}\left(|x^k(T \wedge \theta_{\Delta, R}) - \bar{X}_\Delta^k(T \wedge \theta_{\Delta, R})|^q\right) \leq C\Delta.$$

At last

$$\begin{aligned} \mathbb{E} \left(|e_\Delta(T)|^q I_{\{\theta_{\Delta,R} > T\}} \right) &= \mathbb{E} \left(|e_\Delta(T \wedge \theta_{\Delta,R})|^q I_{\{\theta_{\Delta,R} > T\}} \right) \\ &\leq \mathbb{E} (|x(T \wedge \theta_{\Delta,R}) - x_\Delta(T \wedge \theta_{\Delta,R})|^q) \leq C\Delta. \end{aligned} \quad (4.36)$$

This implies (4.23) as required. The proof is hence complete. $\square$

The following approximation result, established by Li et al. [25], remains valid under the dissipative Khasminskii-type condition considered in this chapter. We cite it here as a lemma for completeness.

Lemma 4.7. *Let Assumptions 3.1 and 4.1 hold. Then, for each $q \in [1, 2)$, the solutions of hybrid SFDEs (3.4) and (3.9) have the property that*

$$\lim_{k \rightarrow \infty} \left(\sup_{0 \leq t \leq T} \mathbb{E} |x(t) - x^k(t)|^q \right) = 0, \quad \forall T > 0.$$

Combining Theorem 4.6 with the approximation result in Lemma 4.7, we obtain the following convergence theorem for the truncated Euler–Maruyama solution.

Theorem 4.8. *Let Assumptions 3.1, 4.1, 3.3 and 3.4 hold. Then, for any $q \in [1, 2)$,*

$$\lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E} |\bar{X}_\Delta^k(T) - x(T)|^q \right) = 0, \quad \lim_{k \rightarrow \infty} \left(\lim_{\Delta \rightarrow 0} \mathbb{E} |X_\Delta^k(T) - x(T)|^q \right) = 0. \quad (4.37)$$

4.4 Convergence rates

In the previous section, we established the asymptotic convergence of the truncated Euler–Maruyama solutions without deriving explicit convergence rates. We investigate the convergence rates of the numerical solutions in this section. Here we need two more assumptions for the convergence rate. Before stating these assumptions, we first introduce the following notation. Let $\mathcal{U}$ denote the family of continuous functions $U : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ such that for each $b > 0$, there exists a family of continuous functions $\kappa : \mathbb{R}^d \times \mathbb{R}^d \rightarrow \mathbb{R}_+$ for which

$$U(x, \bar{x}) \leq \kappa(x, \bar{x})|x - \bar{x}|^2 \leq \kappa_b|x - \bar{x}|^2, \quad \forall x, \bar{x} \in \mathbb{R}^d \text{ with } |x| \vee |\bar{x}| \leq b,$$

where $\kappa_b = \sup_{|\psi_1(\pi_k(x_s^k))| \vee |\psi_1(s)| \leq b} \kappa(\pi_k(x_s^k), \psi_1(s))$.

Assumption 4.9. Assume that there are positive constants α and K_3

$$\begin{aligned} & (x - \bar{x})^T (F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)) + \frac{1 + \alpha}{2} |G(x, z, i, t) - G(\bar{x}, \bar{z}, i, t)|^2 \\ & \leq K_3(|x - \bar{x}|^2 + |y - \bar{y}|^2 + |z - \bar{z}|^2) - U(x, \bar{x}) + U(y, \bar{y}) + U(z, \bar{z}) \end{aligned} \quad (4.38)$$

for all $x, y, z, \bar{x}, \bar{y}, \bar{z} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 4.10. Assume that there is a pair of positive constants γ and K_4 such that

$$\begin{aligned} & |F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)|^2 \vee |G(x, y, i, t) - G(\bar{x}, \bar{y}, i, t)|^2 \\ & \leq K_4(|x - \bar{x}|^2 + |y - \bar{y}|^2)(1 + |x|^\gamma + |\bar{x}|^\gamma + |y|^\gamma + |\bar{y}|^\gamma) \end{aligned} \quad (4.39)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

The following lemma shows that the truncated functions F_Δ and G_Δ preserve Assumption 4.10.

Lemma 4.11. Let Assumption 4.10 hold. Then, for every $\Delta \in (0, 1]$, we have

$$\begin{aligned} & |F_\Delta(x, y, i, t) - F_\Delta(\bar{x}, \bar{y}, i, t)|^2 \vee |G_\Delta(x, y, i, t) - G_\Delta(\bar{x}, \bar{y}, i, t)|^2 \\ & \leq K_4(|x - \bar{x}|^2 + |y - \bar{y}|^2)(1 + |x|^\gamma + |\bar{x}|^\gamma + |y|^\gamma + |\bar{y}|^\gamma) \end{aligned} \quad (4.40)$$

for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Proof. For any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and $(i, t) \in \mathbb{S} \times \mathbb{R}_+$,

$$\begin{aligned}
 & |F_\Delta(x, y, i, t) - F_\Delta(\bar{x}, \bar{y}, i, t)|^2 \vee |G_\Delta(x, y, i, t) - G_\Delta(\bar{x}, \bar{y}, i, t)|^2 \\
 &= |F(\pi_\Delta(x), \pi_\Delta(y), i, t) - F(\pi_\Delta(\bar{x}), \pi_\Delta(\bar{y}), i, t)|^2 \\
 &\quad \vee |G(\pi_\Delta(x), \pi_\Delta(y), i, t) - G(\pi_\Delta(\bar{x}), \pi_\Delta(\bar{y}), i, t)|^2 \\
 &\leq K_4(|\pi_\Delta(x) - \pi_\Delta(\bar{x})|^2 + |\pi_\Delta(y) - \pi_\Delta(\bar{y})|^2) \\
 &\quad \times (1 + |\pi_\Delta(x)|^\gamma + |\pi_\Delta(\bar{x})|^\gamma + |\pi_\Delta(y)|^\gamma + |\pi_\Delta(\bar{y})|^\gamma).
 \end{aligned}$$

This implies the assertion by noting that $|\pi_\Delta(x)| \leq |x|$ and $|\pi_\Delta(x) - \pi_\Delta(\bar{x})|^2 \leq |x - \bar{x}|^2$.

The proof is complete. $\square$

The following theorem establishes the convergence rates between the truncated Euler–Maruyama approximation and the hybrid SFDE with finite time delay in the L^2 -sense. This result serves as the key ingredient for deriving the convergence rates for the original hybrid SFDE with infinite time delay.

Theorem 4.12. *Let Assumption 3.1, 4.1, 3.3, 3.4, 4.9 and 4.10 hold and assume that $p > \gamma + 2$. Then for any $\Delta \in (0, 1]$,*

$$\mathbb{E}|\bar{X}_\Delta^k(T) - x^k(T)|^2 \leq C(\Delta \vee \Delta(h(\Delta))^2 \vee (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)}) \quad (4.41)$$

and

$$\mathbb{E}|X_\Delta^k(T) - x^k(T)|^2 \leq C(\Delta \vee \Delta(h(\Delta))^2 \vee (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)}). \quad (4.42)$$

Proof. To make it clearer, this proof is divided into three steps.

Step 1. Fix $\Delta \in (0, 1]$ and the initial data ξ arbitrarily. Let R be any integer such that $R \geq \|\xi\|$. Recall the definition of the stopping time

$$\theta_{\Delta, R} = \varphi_R \wedge \rho_{\Delta, R} = \inf\{t \geq 0 : |\bar{X}_\Delta^k(t)| \wedge |x^k(t)| \geq R\},$$

and the error $e_\Delta(t) = \bar{X}_\Delta^k(t) - x^k(t)$ for $t \in [-\tau, T]$. We write $\theta_{\Delta, R} = \theta$ for simplicity. It is clear to see that $e_\Delta(t) = 0$ for $t \in [-\tau, 0]$. By the Itô formula, we can show that,

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for $t \in [-\tau, T]$,

$$\begin{aligned}
\mathbb{E}|e_\Delta(t \wedge \theta)|^2 &\leq \mathbb{E} \int_0^{t \wedge \theta} 2e_\Delta^T(s) (F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \\
&\quad + |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 ds \\
&= \mathbb{E} \int_0^{t \wedge \theta} 2e_\Delta^T(s) ([F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) - F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)] \\
&\quad + [F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))]) \\
&\quad + |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) \\
&\quad + G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 ds \\
&\leq \mathbb{E} \int_0^{t \wedge \theta} 2 \left(e_\Delta^T(s) (F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) - F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)) \right. \\
&\quad + \frac{1+\alpha}{2} |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \\
&\quad + e_\Delta^T(s) (F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \\
&\quad \left. + \frac{1+\alpha^{-1}}{2} |G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds.
\end{aligned}$$

Re-arranging the terms gives

$$\mathbb{E}|e_\Delta(t \wedge \theta)|^q \leq H_1 + H_2, \quad (4.43)$$

where

$$\begin{aligned}
H_1 &:= \mathbb{E} \int_0^{t \wedge \theta} 2 \left(e_\Delta^T(s) (F(x^k(s), \psi_1(\pi_k(x_s^k)), \theta(s), s) - F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)) \right. \\
&\quad \left. + \frac{1+\alpha}{2} |G(x^k(s), \psi_2(\pi_k(x_s^k)), \theta(s), s) - G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds
\end{aligned}$$

and

$$\begin{aligned}
H_2 &:= \mathbb{E} \int_0^{t \wedge \theta} 2 \left(e_\Delta^T(s) (F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \right. \\
&\quad \left. + \frac{1+\alpha^{-1}}{2} |G(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds.
\end{aligned}$$

Step 2. Then, we estimate H_1 and H_2 . By Assumption 4.9, we get

$$\begin{aligned}
 H_1 &\leq C\mathbb{E} \int_0^{t\wedge\theta} \left(|e_\Delta(s)|^2 + |\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 + |\psi_2(\pi_k(x_s^k)) - \psi_2(s)|^2 \right. \\
 &\quad \left. - U(x^k(t), \bar{X}_\Delta^k(t)) + U(\psi_1(\pi_k(x_s^k)), \psi_1(s)) + U(\psi_2(\pi_k(x_s^k)), \psi_2(s)) \right) ds \\
 &\leq C\mathbb{E} \int_0^{t\wedge\theta} |e_\Delta(s)|^2 ds + C\mathbb{E} \int_0^{t\wedge\theta} (|\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 + |\psi_2(\pi_k(x_s^k)) - \psi_2(s)|^2) ds \\
 &\quad - C\mathbb{E} \int_0^{t\wedge\theta} U(x^k(t), \bar{X}_\Delta^k(t)) ds + C\mathbb{E} \int_0^{t\wedge\theta} U(\psi_1(\pi_k(x_s^k)), \psi_1(s)) ds \\
 &\quad + C\mathbb{E} \int_0^{t\wedge\theta} U(\psi_2(\pi_k(x_s^k)), \psi_2(s)) ds. \tag{4.44}
 \end{aligned}$$

For each $t \in (0, T]$, let $N = \lfloor t/\Delta \rfloor$ and $t_{N+1} = t$. By the Hölder inequality and Assumption 3.3, it can be derived that

$$\begin{aligned}
 &\mathbb{E} \int_0^t |\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 ds \\
 &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 ds \\
 &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \left| \int_{-k}^0 x^k(s+u) \mu_1(du) + x^k(s-k) \mu_1((-\infty, -k)) \right. \\
 &\quad \left. - \sum_{h=-kk_1}^{-1} X_\Delta^k(t_{j+h}) \mu_1([t_h, t_{h+1})) - X_\Delta^k(t_j - k) \mu_1((-\infty, -k)) \right|^2 ds \\
 &\leq \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \left(\sum_{h=-kk_1}^{-1} \int_{t_h}^{t_{h+1}} |x^k(s+u) - X_\Delta^k(t_{j+h})| \mu_1(du) \right. \\
 &\quad \left. + |x^k(s-k) - X_\Delta^k(t_j - k)| \mu_1((-\infty, -k)) \right)^2 ds
 \end{aligned}$$

$$\begin{aligned}
 &\leq 2 \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \left(\sum_{h=-kk_1}^{-1} \int_{t_h}^{t_{h+1}} |x^k(s+u) - X_{\Delta}^k(t_{j+h})| \mu_1(du) \right)^2 ds \\
 &\quad + 2 \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} (|\xi(s-k) - \xi(t_j-k)| \mu_1((-\infty, -k)))^2 ds \\
 &\leq 2 \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \int_{-k}^0 |x^k(s+u) - X_{\Delta}^k(t_j+u)|^2 \mu_1(du) ds + 2TK_2^2 \Delta^{2\alpha} \\
 &\leq 4 \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \int_{-k}^{-s} |\xi(s+u) - \xi(t_j+u)|^2 \mu_1(du) ds \\
 &\quad + 4 \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} \int_{-s}^0 |x^k(s+u) - X_{\Delta}^k(t_j+u)|^2 \mu_1(du) ds + 2TK_2^2 \Delta^{2\alpha} \\
 &= C\Delta^{2\alpha} + C\mathbb{E} \int_0^t \int_{-s}^0 |x^k(s+u) - X_{\Delta}^k(\tau(s)+u)|^2 \mu_1(du) ds. \tag{4.45}
 \end{aligned}$$

It is clear to see that

$$\begin{aligned}
 &\mathbb{E} \int_0^t \int_{-s}^0 |x^k(s+u) - X_{\Delta}^k(\tau(s)+u)|^2 \mu_1(du) ds \\
 &\leq 3\mathbb{E} \int_0^t \int_{-s}^0 |x^k(s+u) - \bar{X}_{\Delta}^k(s+u)|^2 \mu_1(du) ds \\
 &\quad + 3\mathbb{E} \int_0^t \int_{-s}^0 |\bar{X}_{\Delta}^k(s+u) - \bar{X}_{\Delta}^k(\tau(s)+u)|^2 \mu_1(du) ds \\
 &\quad + 3\mathbb{E} \int_0^t \int_{-s}^0 |\bar{X}_{\Delta}^k(\tau(s)+u) - X_{\Delta}^k(\tau(s)+u)|^2 \mu_1(du) ds.
 \end{aligned}$$

By the Assumption 3.1, Lemmas 3.14 and 4.3, we have

$$\begin{aligned}
 \mathbb{E} |\bar{X}_{\Delta}^k(t+u) - \bar{X}_{\Delta}^k(\tau(t)+u)|^2 &\leq 2 \left(\mathbb{E} \left| \int_{t_j+u}^{t+u} F(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s)) ds \right|^2 \right. \\
 &\quad \left. + \mathbb{E} \left| \int_{t_j+u}^{t+u} G(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s)) dB(s) \right|^2 \right) \\
 &\leq C\Delta \int_{t_j+u}^{t+u} \left(1 + \mathbb{E} |X_{\Delta}^k(s)|^2 + \mathbb{E} |\psi_1(s)|^2 + \mathbb{E} |\psi_2(s)|^2 \right) ds \\
 &\leq C\Delta.
 \end{aligned}$$

Applying this and Lemma 3.6, we get

$$\begin{aligned}
& \mathbb{E} \int_0^t \int_{-s}^0 |x^k(s+u) - X_\Delta^k(\tau(s)+u)|^2 \mu_1(du) ds \\
& \leq C\Delta + C\mathbb{E} \int_0^t \int_{-s}^0 |x^k(s+u) - \bar{X}_\Delta^k(s+u)|^2 \mu_1(du) ds \\
& \leq C\Delta + C\mathbb{E} \int_{-\infty}^0 \int_0^t |x^k(s) - \bar{X}_\Delta^k(s)|^2 ds \mu_1(du) \\
& = C\Delta + C\mathbb{E} \int_0^t |x^k(s) - \bar{X}_\Delta^k(s)|^2 ds.
\end{aligned} \tag{4.46}$$

Substituting (4.46) into (4.45) gives

$$\mathbb{E} \int_0^t |\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 ds \leq C\Delta + C\mathbb{E} \int_0^t |x^k(s) - \bar{X}_\Delta^k(s)|^2 ds. \tag{4.47}$$

Then, it can be derived that

$$\begin{aligned}
\mathbb{E} \int_0^{t \wedge \theta} U(\psi_1(\pi_k(x_s^k)), \psi_1(s)) ds & \leq \mathbb{E} \int_0^{t \wedge \theta} \kappa(\pi_k(x_s^k), \psi_1(s)) |\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 ds \\
& \leq \kappa_b \mathbb{E} \int_0^{t \wedge \theta} |\psi_1(\pi_k(x_s^k)) - \psi_1(s)|^2 ds \\
& \leq C\Delta + C\mathbb{E} \int_0^{t \wedge \theta} |x^k(s) - \bar{X}_\Delta^k(s)|^2 ds.
\end{aligned} \tag{4.48}$$

Similarly, we have

$$\mathbb{E} \int_0^{t \wedge \theta} U(\psi_2(\pi_k(x_s^k)), \psi_2(s)) ds \leq C\Delta + C\mathbb{E} \int_0^{t \wedge \theta} |x^k(s) - \bar{X}_\Delta^k(s)|^2 ds \tag{4.49}$$

Substituting (4.47) - (4.49) into (4.44) gives

$$H_1 \leq C \int_0^t \mathbb{E} |e_\Delta(s \wedge \theta)|^2 ds + C\Delta. \tag{4.50}$$

To estimate H_2 , we observe that

$$H_2 \leq H_{21} + H_{22} + H_{23} + H_{24}, \tag{4.51}$$

where

$$\begin{aligned}
 H_{21} &:= \mathbb{E} \int_0^T 2 \left(e_{\Delta}^T(s) (F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s)) \right. \\
 &\quad \left. + \frac{1 + \alpha^{-1}}{2} |G(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s) - G_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds, \\
 H_{22} &:= \mathbb{E} \int_0^T 2 \left(e_{\Delta}^T(s) (F_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \theta(s), s)) \right. \\
 &\quad \left. + \frac{1 + \alpha^{-1}}{2} |G_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds, \\
 H_{23} &:= \mathbb{E} \int_0^T 2 \left(e_{\Delta}^T(s) (F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), s)) \right. \\
 &\quad \left. + \frac{1 + \alpha^{-1}}{2} |G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \theta(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), s)|^2 \right) ds, \\
 H_{24} &:= \mathbb{E} \int_0^T 2 \left(e_{\Delta}^T(s) (F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \right. \\
 &\quad \left. + \frac{1 + \alpha^{-1}}{2} |G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds,
 \end{aligned}$$

where $t \wedge \theta$ has been replaced by T since $t \wedge \theta \leq T$. Rearranging H_{21} gives

$$\begin{aligned}
 H_{21} &\leq C \int_0^T \mathbb{E} |e_{\Delta}^T(s)|^2 ds + \mathbb{E} \int_0^T \left(|F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s)|^2 \right. \\
 &\quad \left. + (1 + \alpha^{-1}) |G(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s) - G_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_{\Delta}^T(s)|^2 ds + C \mathbb{E} \int_0^T \left(|F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s)|^2 \right. \\
 &\quad \left. \vee |G(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s) - G_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_2(s), \theta(s), s)|^2 \right) ds.
 \end{aligned}$$

By the Hölder inequality and Assumption 4.10, we can derive that

$$\begin{aligned}
 &\mathbb{E} |F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F_{\Delta}(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s)|^2 \\
 &= \mathbb{E} |F(\bar{X}_{\Delta}^k(s), \psi_1(s), \theta(s), s) - F(\pi_{\Delta}(\bar{X}_{\Delta}^k(s)), \pi_{\Delta}(\psi_1(s)), \theta(s), s)|^2 \\
 &\leq K_4 \mathbb{E} [(|\bar{X}_{\Delta}^k(s) - \pi_{\Delta}(\bar{X}_{\Delta}^k(s))|^2 + |\psi_1(s) - \pi_{\Delta}(\psi_1(s))|^2) (1 + 2|\bar{X}_{\Delta}^k(s)|^{\gamma} + 2|\psi_1(s)|^{\gamma})]
 \end{aligned}$$

$$\begin{aligned}
&\leq C \left(\mathbb{E} |\bar{X}_\Delta^k(s) - \pi_\Delta(\bar{X}_\Delta^k(s))|^{2p/(p-\gamma)} + \mathbb{E} |\psi_1(s) - \pi_\Delta(\psi_1(s))|^{2p/(p-\gamma)} \right)^{(p-\gamma)/p} \\
&\quad \times (1 + \mathbb{E} |\bar{X}_\Delta^k(s)|^p + \mathbb{E} |\psi_1(s)|^p)^{\gamma/p} \\
&\leq C \left(\mathbb{E} \left[I_{\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}} |\bar{X}_\Delta^k(s)|^{2p/(p-\gamma)} \right] + \mathbb{E} \left[I_{\{\psi_1(s) > \eta^{-1}(h(\Delta))\}} |\psi_1(s)|^{2p/(p-\gamma)} \right] \right)^{(p-\gamma)/p}.
\end{aligned}$$

Note that

$$\begin{aligned}
&\mathbb{E} \left[I_{\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}} |\bar{X}_\Delta^k(s)|^{2p/(p-\gamma)} \right] \\
&\leq [\mathbb{P}\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}]^{(p-\gamma-2)/(p-\gamma)} [\mathbb{E} |\bar{X}_\Delta^k(s)|^p]^{2/(p-\gamma)} \\
&\leq C [\mathbb{P}\{\bar{X}_\Delta^k(s) > \eta^{-1}(h(\Delta))\}]^{(p-\gamma-2)/(p-\gamma)} \\
&\leq C \left[\frac{\mathbb{E}\{\bar{X}_\Delta^k(s)\}}{(\eta^{-1}(h(\Delta)))^p} \right]^{(p-\gamma-2)/(p-\gamma)} \\
&\leq C (\eta^{-1}(h(\Delta)))^{-p(p-\gamma-2)/(p-\gamma)}.
\end{aligned}$$

Similarly,

$$\mathbb{E} \left[I_{\{\psi_1(s) > \eta^{-1}(h(\Delta))\}} |\psi_1(s)|^{2p/(p-\gamma)} \right] \leq C (\eta^{-1}(h(\Delta)))^{-p(p-\gamma-2)/(p-\gamma)}.$$

Hence, we get

$$\mathbb{E} |F(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s)|^2 \leq C (\eta^{-1}(h(\Delta)))^{-(p-\gamma-2)}.$$

Similarly, it can be derived that

$$\mathbb{E} |G(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \leq C (\eta^{-1}(h(\Delta)))^{-(p-\gamma-2)}.$$

We then get

$$H_{21} \leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^2 ds + C (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)}. \quad (4.52)$$

Let us estimate H_{22} now. By Assumption 4.10 and the Young inequality as well as

Lemmas 3.6, 4.3 and 4.11, we can show

$$\begin{aligned}
 H_{22} &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^2 ds + C \mathbb{E} \int_0^T \left(|F_\Delta(\bar{X}_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s)|^2 \right. \\
 &\quad \left. \vee |G_\Delta(\bar{X}_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s)|^2 \right) \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^2 ds \\
 &\quad + C \mathbb{E} \int_0^T \left(|\bar{X}_\Delta^k(s) - X_\Delta^k(s)|^2 (1 + |\bar{X}_\Delta^k(s)|^\gamma + |X_\Delta^k(s)|^\gamma + 2|\psi_1(s)|^\gamma) \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^2 ds + C \Delta (h(\Delta))^2. \tag{4.53}
 \end{aligned}$$

Rearranging H_{23} gives

$$\begin{aligned}
 H_{23} &= \mathbb{E} \int_0^T 2 \left(e_\Delta^T(s) (F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s)) \right. \\
 &\quad \left. + \frac{1 + \alpha^{-1}}{2} |G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), s)|^2 \right) ds \\
 &\leq 2 \mathbb{E} \int_0^T \left(\frac{1}{2} |e_\Delta^T(s)|^2 + \frac{1}{2} |F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s)|^2 \right. \\
 &\quad \left. + \frac{1 + \alpha^{-1}}{2} (|e_\Delta^T(s)|^2 + |G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), s)|^2) \right) ds \\
 &\leq C \int_0^T \mathbb{E} |e_\Delta^T(s)|^2 ds + C \mathbb{E} \int_0^T \left(|F_\Delta(X_\Delta^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s)|^2 \right. \\
 &\quad \left. \vee |G_\Delta(X_\Delta^k(s), \psi_2(s), \theta(s), s) - G_\Delta(X_\Delta^k(s), \psi_2(s), \bar{\theta}(s), s)|^2 \right) ds.
 \end{aligned}$$

By Markov property of $\theta(\cdot)$ and Assumption 4.10 as well as Lemma 4.3, 3.14 and 4.11, we derive that

$$\begin{aligned}
 &\mathbb{E} \int_0^t |F_\Delta(X^k(s), \psi_1(s), \theta(s), s) - F_\Delta(X_\Delta^k(s), \psi_1(s), \bar{\theta}(s), s)|^2 ds \\
 &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |F(X_\Delta^k(t_j), \psi_{1j}, \theta(s), t_j) - F(X_\Delta^k(t_j), \psi_{1j}, \theta(t_j), t_j)|^2 ds \\
 &= \sum_{j=0}^N \mathbb{E} \int_{t_j}^{t_{j+1}} |F(X_\Delta^k(t_j), \psi_{1j}, \theta(s), t_j) - F(X_\Delta^k(t_j), \psi_{1j}, \theta(t_j), t_j)|^2 I_{\{\theta(s) \neq \theta(t_j)\}} ds
 \end{aligned}$$

$$\begin{aligned}
&\leq C \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \mathbb{E} \left[\left(1 + |X_{\Delta}^k(t_j)|^{2+\gamma} + |\psi_{1j}|^{2+\gamma} \right) \mathbb{E}(I_{\{\theta(s) \neq \theta(t_j)\}} | \mathcal{F}_{t_j}) \right] ds \\
&\leq C \Delta \sum_{j=0}^N \int_{t_j}^{t_{j+1}} \mathbb{E} \left[\left(1 + |X_{\Delta}(t_j)|^{2+\gamma} + |\psi_{1j}|^{2+\gamma} \right) \right] ds \\
&\leq C \Delta.
\end{aligned}$$

Substituting above into H_{23} gives

$$H_{23} \leq C \int_0^T \mathbb{E} |e_{\Delta}^T(s)|^2 ds + C \Delta. \quad (4.54)$$

It follows from Assumption 3.4, Lemmas 4.3 and 3.14 that

$$\begin{aligned}
H_{24} &:= \\
&\mathbb{E} \int_0^T 2 \left(e_{\Delta}^T(s) (F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))) \right. \\
&\quad \left. + \frac{1 + \alpha^{-1}}{2} |G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds \\
&\leq 2 \mathbb{E} \int_0^T \left(\frac{1}{2} |e_{\Delta}^T(s)|^2 + \frac{1}{2} |F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^2 \right. \\
&\quad \left. + \frac{1 + \alpha^{-1}}{2} (|e_{\Delta}^T(s)|^2 + |G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2) \right) ds \\
&\leq C \int_0^T \mathbb{E} |e_{\Delta}^T(s)|^2 ds \\
&\quad + C \int_0^T \left(\mathbb{E} |F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), s) - F_{\Delta}(X_{\Delta}^k(s), \psi_1(s), \bar{\theta}(s), \tau(s))|^2 \right. \\
&\quad \left. \vee \mathbb{E} |G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), s) - G_{\Delta}(X_{\Delta}^k(s), \psi_2(s), \bar{\theta}(s), \tau(s))|^2 \right) ds \\
&\leq C \int_0^T \mathbb{E} |e_{\Delta}^T(s)|^2 ds + C \Delta^{2\beta} \\
&\leq C \int_0^T \mathbb{E} |e_{\Delta}^T(s)|^2 ds + C \Delta. \quad (4.55)
\end{aligned}$$

Combining (4.52) - (4.55) yields

$$H_2 \leq C \int_0^T \mathbb{E} |e_{\Delta}^T(s)|^2 ds + C(\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)} + C \Delta (h(\Delta))^2 + C \Delta. \quad (4.56)$$

Step 3. Insert (4.50) and (4.56) into (4.43), we obtain

$$\mathbb{E}|e_\Delta(t \wedge \theta)|^2 \leq C \int_0^T \mathbb{E}|e_\Delta^T(s)|^2 ds + C(\Delta \vee \Delta(h(\Delta))^2 \vee (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)}). \quad (4.57)$$

By the Gronwall inequality, we have

$$\mathbb{E}|e_\Delta(T \wedge \theta)|^2 \leq C(\Delta \vee \Delta(h(\Delta))^2 \vee (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)}) \quad (4.58)$$

We can get the first assertion (4.41) by letting $R \rightarrow \infty$. The second assertion (4.42) follows the assertion (4.58) and Lemma 3.6. The proof is complete. $\square$

To derive the convergence rate for the original hybrid SFDE with infinite time delay, we combine the convergence rate established above with the approximation result of Li et al. [25]. Since Assumption 3.1 is identical to that used in Chapter 3, Assumption 3.24 of Li et al. [25] is satisfied in exactly the same way. Therefore, Theorem 3.25 and the approximation estimate established in Corollary 3.26 remain valid in the present setting.

Corollary 4.13. *Let Assumptions 3.1, 4.1 and 4.9 hold. Then the solution $x^k(t)$ of the truncated SFDE (3.9) approximates the solution $x(t)$ of the given SFDE (3.4) in the sense that*

$$\mathbb{E}|x(t) - x^k(t)|^2 \leq Ce^{-(p-2)rk}, \quad \forall t \in [0, T]. \quad (4.59)$$

Combining Theorem 4.12 with Corollary 4.13, we obtain the convergence rate of the truncated Euler–Maruyama solution to the original hybrid SFDE with infinite time delay.

Theorem 4.14. *Suppose that assumptions of Theorem 4.12 hold and $\mu_1, \mu_2 \in \mathcal{P}_{rp}$. Then for any $\Delta \in (0, 1]$ and any integer $k > T$,*

$$\mathbb{E}|X_\Delta^k(t) - x(t)|^2 \leq C(\Delta \vee \Delta(h(\Delta))^2 \vee (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)} + e^{-(p-2)rk}) \quad (4.60)$$

for all $t \in [0, T]$.

Theorem 4.14 provides the convergence rate in the L^2 sense. By applying the Hölder inequality, the convergence rate can be extended immediately to all lower-order moments.

Corollary 4.15. *Let the assumptions of Theorem 4.14 hold. Then, for any $q \in [1, 2)$, there exists a positive constant C , independent of Δ and k , such that*

$$\mathbb{E} \left| X_{\Delta}^k(t) - x(t) \right|^q \leq C \left((\Delta \vee \Delta(h(\Delta)))^2 \vee (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)} + e^{-(p-2)rk} \right)^{q/2},$$

for all $t \in [0, T]$.

4.5 Example

To highlight the effect of the dissipative Khasminskii condition introduced in this chapter, we reconsider the nonlinear model studied in Chapter 3. Unlike the analysis in Chapter 3, the negative cubic drift is now fully exploited as a dissipative term, allowing the stronger convergence theory established in this chapter to be applied.

4.5.1 Example 1

Consider the stochastic functional volatility equation

$$dx(t) = F(x(t), \psi_1(x_t), \theta(t), t)dt + G(x(t), \psi_2(x_t), \theta(t), t)dB(t) \quad (4.61)$$

with the initial data $\xi \in \mathcal{C}_r$ and $\theta(0) = i_0 \in \mathbb{S}$ satisfying Assumption 3.3, where

$$F(x, y, i, t) = a_1 + a_2|y|^{4/3} - a_3x^3, \quad G(x, z, i, t) = a_4|x|^{4/3} + a_5z,$$

$x, y \in \mathbb{R}$ and $a_1, \dots, a_5$ are all real numbers with $a_3 > 0$. The coefficients F and G are local Lipschitz continuous, i.e. Assumption 3.1 is satisfied. Now, consider the Khasminskii condition. We have

$$xF(x, y, i, t) + \frac{1}{2}|G(x, z, i, t)|^2 \leq |a_1||x| + |a_2||x||y|^{4/3} - a_3|x|^4 + a_4^2|x|^{8/3} + a_5^2|z|^2.$$

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By the Young inequality, we know that $|x||y|^{4/3} \leq |x|^3 + |y|^2$. Hence, we have

$$\begin{aligned} xF(x, y, i, t) + \frac{1}{2}|G(x, z, i, t)|^2 &\leq |a_1||x| + |a_2||x|^3 + a_4^2|x|^{8/3} + |a_2||y|^2 + a_5^2|z|^2 \\ &\leq K_1(1 + |y|^2 + |z|^2) - a_3|x|^4, \end{aligned}$$

where $K_1 = (|a_2| + a_5^2) \vee \beta_1$ and

$$\beta_1 = \sup_{u \geq 0} [|a_1|u + |a_2|u^3 + a_4^2u^{8/3}] < \infty.$$

We have hence verified that Assumption 4.1 is satisfied. Note that, for $x, \bar{x}, y, \bar{y}, z, \bar{z} \in \mathbb{R}$,

$$\begin{aligned} &(x - \bar{x})(F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)) \\ &= (x - \bar{x})[a_2(|y|^{4/3} - |\bar{y}|^{4/3}) - a_3(x^3 - \bar{x}^3)] \\ &\leq -\frac{1}{2}a_3|x - \bar{x}|^2(x^2 + \bar{x}^2) + \frac{1}{2}a_2^2(x - \bar{x})^2 + \frac{1}{2}a_2^2(|y|^{4/3} - |\bar{y}|^{4/3})^2 \end{aligned}$$

and

$$|G(x, z, i, t) - G(\bar{x}, \bar{z}, i, t)|^2 = 2a_4^2(|x|^{4/3} - |\bar{x}|^{4/3})^2 + 2a_5^2|z - \bar{z}|^2.$$

By the mean-valued theorem, we see that

$$\begin{aligned} (|y|^{4/3} - |\bar{y}|^{4/3})^2 &\leq \frac{16}{9}|y - \bar{y}|^2(|y|^{1/3} + |\bar{y}|^{1/3})^2 \\ &\leq \frac{32}{9}|y - \bar{y}|^2(|y|^{2/3} + |\bar{y}|^{2/3}) \\ &\leq \frac{32}{9}|y - \bar{y}|^2\left(\beta_2 + \frac{a_3}{2a_2^2}(|y|^2 + |\bar{y}|^2)\right), \end{aligned}$$

where

$$\beta_2 = \sup_{u \geq 0} \left(2u^{2/3} - \frac{a_3u^2}{a_2^2}\right) < \infty.$$

Similarly, we also have

$$(|x|^{4/3} - |\bar{x}|^{4/3})^2 \leq \frac{32}{9}|x - \bar{x}|^2\left(\beta_3 + \frac{a_3}{8a_4^2}(|x|^2 + |\bar{x}|^2)\right),$$

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with

$$\beta_3 = \sup_{u \geq 0} \left(2u^{2/3} - \frac{a_3 u^2}{8a_4^2} \right) < \infty.$$

Letting $\alpha = 1$ gives

$$\begin{aligned} & (x - \bar{x})(F(x, y, i, t) - F(\bar{x}, \bar{y}, i, t)) + \frac{1 + \alpha}{2} |G(x, z, i, t) - G(\bar{x}, \bar{z}, i, t)|^2 \\ & \leq K_3(|x - \bar{x}|^2 + |y - \bar{y}|^2 + |z - \bar{z}|^2) - U(x, \bar{x}) + U(y, \bar{y}) + U(z, \bar{z}), \end{aligned}$$

where $K_3 = \frac{1}{2}a_2^2(1 + \beta_2) + 2a_4^2\beta_3 + 2a_5^2 < \infty$ and

$$U(x, \bar{x}) = \frac{1}{2}a_3|x - \bar{x}|^2(x^2 + \bar{x}^2).$$

Therefore, it can be concluded that Assumption 4.9 is satisfied. It is easy to show that Assumption 4.10 is satisfied with $\gamma = 4$.

Before applying the truncated EM method, we need to find functions η and h satisfying (3.13) and (3.14). It is easy to see that

$$\sup_{|x| \vee |y| \leq r, (i, t) \in \mathbb{S} \times \mathbb{R}_+} (|F(x, y, i, t)| \vee |G(x, y, i, t)|) \leq \hat{a}r^3, \forall r \geq 1,$$

where $\hat{a} = (|a_1| + |a_2| + |a_3|) \vee (|a_4| + |a_5|)$. We hence choose that $\eta(r) = \hat{a}r^3$ and its inverse function $\eta^{-1}(r) = (r/\hat{a})^{1/3}$ for $r \geq 0$. For $\epsilon \in (0, 1/4]$, we define $h(\Delta) = \Delta^{-\epsilon}$ for $\Delta > 0$. By Theorem 4.14, it can be derived that the truncated EM solutions will converge to the true solution in the sense that

$$\mathbb{E}|X_\Delta^k(t) - x(t)|^2 \leq C(\Delta \vee \Delta^{1-2\epsilon} \vee \Delta^{2/3\epsilon(p-6)} + e^{-(p-q)rk}), \forall \Delta \in (0, 1].$$

Let us discuss a specific example. The coefficients of (4.61) are defined by

$$F(x(t), \psi_1(x_t), \theta(t), t) = \begin{cases} 2 + 3|\psi_1(x_t)|^{4/3} - 3x(t)^3, & i = 1, \\ 3 + |\psi_1(x_t)|^{4/3} - 4x(t)^3, & i = 2, \end{cases}$$

$$G(x(t), \psi_2(x_t), \theta(t), t) = \begin{cases} 2|x(t)|^{4/3} + 2\psi_2(x_t), & i = 1, \\ 3|x(t)|^{4/3} + \psi_2(x_t), & i = 2, \end{cases}$$

where $\psi_1(x_t) = \psi_2(x_t) = \int_{-\infty}^0 x(t+u)\mu(du) \in \mathcal{C}_{1/5}$, the probability measure $\mu(\cdot)$ has its probability density function e^u on $(-\infty, 0]$. $B(t)$ is a scalar Brownian motion and $\theta(t)$ is a Markov chains on the state space $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Let the initial data $\xi(u) = e^u \in \mathcal{C}_{1/5}$ and $\theta(0) = 1$. The corresponding SFDE with finite time delay is given by

$$dx^k(t) = F(x^k(t), \psi_1(\pi_k(x_t)), \theta(t), t)dt + G(x^k(t), \psi_2(\pi_k(x_t)), \theta(t), t)dB(t). \quad (4.62)$$

The coefficients are defined by

$$\begin{aligned} & F(x^k(t), \psi_1(\pi_k(x_t)), \theta(t), t) \\ &= \begin{cases} 2 + 3 \left| e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du) \right|^{4/3} - 3x^k(t)^3, & i = 1, \\ 3 + \left| e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du) \right|^{4/3} - 4x^k(t)^3, & i = 2. \end{cases} \end{aligned}$$

$$\begin{aligned} & G(x^k(t), \psi_2(\pi_k(x_t)), \theta(t), t) \\ &= \begin{cases} 2|x^k(t)|^{4/3} + 2 \left(e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du) \right), & i = 1, \\ 3|x^k(t)|^{4/3} + e^{-k}x(t-k) + \int_{-k}^0 x(t+u)\mu(du), & i = 2. \end{cases} \end{aligned}$$

Next, we use MATLAB to perform numerical simulations. We take the truncated EM numerical solution of (4.62) with $\Delta = 2^{-6}$ and $k = 300$ as the exact solution of (4.61). For $k \in \{10, 11, 12, 13, 14, 15\}$, we take the truncated EM numerical solutions of (4.62) with $\Delta = 2^{-6}$ as the exact $x^k(t)$ of (4.62) and plot the mean square approximation error $\mathbb{E}|x(10) - x^k(10)|^2$ for 1000 sample as shown in Figure 4.1a.

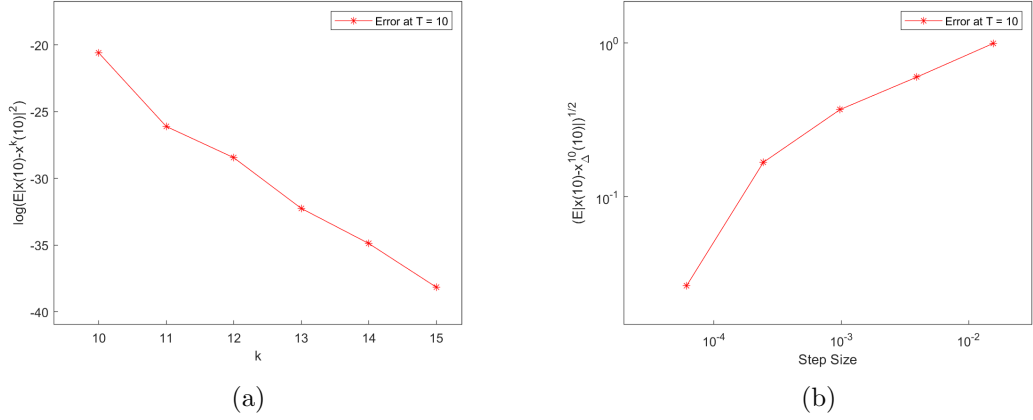


Figure 4.1: The mean square approximation error between the exact solution and the numerical solutions.

To show the relationship between the error of the exact solution and numerical solution and the step size Δ , we set the truncated EM solution of (4.62) with $\Delta = 2^{-18}$ and $k = 200$ as the exact solution of (4.61). Then, we calculate the truncated EM solutions of (4.62) with $\Delta \in \{2^{-6}, 2^{-8}, 2^{-10}, 2^{-12}, 2^{-14}\}$ for 1000 sample and plot the root mean square approximation error between the exact solution and numerical solutions at $T = 10$. Figure 4.1b depicts this result.

4.5.2 Example 2

To further illustrate the difference between the Khasminskii-type conditions in Chapters 3 and 4, consider the scalar stochastic functional differential equation

$$dx(t) = F(x(t), \psi_1(x_t), \theta(t), t)dt + G(x(t), \psi_2(x_t), \theta(t), t)dB(t),$$

where

$$F(x, y, i, t) = -x^3 + y^3, \quad G(x, z, i, t) = 0.$$

The coefficients F and G are locally Lipschitz continuous, and hence Assumption 3.1 is satisfied. Let the initial data satisfy Assumption 3.3 and let the delay measures satisfy $\mu_1, \mu_2 \in \mathcal{P}_{4r}$.

Chapter 4. Extension of the Truncated EM Method

We first verify Assumption 4.1. Since

$$xF(x, y, i, t) + \frac{1}{2}|G(x, z, i, t)|^2 = -|x|^4 + xy^3,$$

by Young's inequality,

$$xy^3 \leq \frac{1}{4}|x|^4 + \frac{3}{4}|y|^4.$$

Therefore,

$$xF(x, y, i, t) + \frac{1}{2}|G(x, z, i, t)|^2 \leq -\frac{3}{4}|x|^4 + \frac{3}{4}|y|^4.$$

Thus, Assumption 4.1 is satisfied by taking

$$\hat{p} = 4, \quad c_1 = \frac{3}{4}, \quad c_2 = \frac{3}{4}, \quad c_3 = 0,$$

and $K_1 = 0$, since $c_1 = c_2 + c_3$.

On the other hand, Assumption 3.2 is not satisfied. Indeed, since $G(x, z, i, t) = 0$, Assumption 3.2 would require that, for some $K_1 > 0$ and $p > 2$,

$$-|x|^4 + xy^3 \leq K_1(1 + |x|^2 + |y|^2 + |z|^2).$$

Taking $x = y/2$ and $z = 0$ gives

$$-|x|^4 + xy^3 = \frac{7}{16}|y|^4.$$

Hence, the above inequality would imply

$$\frac{7}{16}|y|^4 \leq K_1 \left(1 + \frac{5}{4}|y|^2\right),$$

which cannot hold for all $y \in \mathbb{R}$, since the left-hand side has fourth-order growth whereas the right-hand side has only second-order growth. Therefore, Assumption 3.2 is violated.

Consequently, this example satisfies the dissipative Khasminskii-type condition in Chapter 4 but does not satisfy the Khasminskii-type condition in Chapter 3. Numerical

experiments were also conducted for this example, and the results confirm the convergence of the truncated Euler–Maruyama approximations as the step size Δ decreases. Due to space limitations, the corresponding numerical results and convergence plots are not included here.

4.6 Summary

In this chapter, we extended the convergence theory of the truncated Euler–Maruyama method for hybrid stochastic functional differential equations with infinite time delay by replacing the classical Khasminskii-type condition with a more general dissipative Khasminskii-type condition. Compared with the framework developed in Chapter 3, the dissipative assumption permits a broader class of nonlinear stochastic systems with super-linear coefficients while providing stronger control over the moments of both the exact and numerical solutions.

Under the dissipative Khasminskii-type condition, we first established several uniform moment estimates and stopping-time estimates for both the exact solution and the continuous-time truncated Euler–Maruyama approximation. These estimates formed the basis for the subsequent convergence analysis. We then proved the strong convergence of the truncated Euler–Maruyama method in the range $1 \leq q < 2$.

Furthermore, under additional assumptions, convergence-rate estimates were established. In particular, we obtained

$$\mathbb{E}|X_{\Delta}^k(t) - x(t)|^2 \leq C \left((\Delta \vee \Delta(h(\Delta))^2) \vee (\eta^{-1}(h(\Delta)))^{-2(p-\gamma-2)} + e^{-(p-2)rk} \right), \quad t \in [0, T].$$

Finally, by combining the above convergence-rate estimate with the exponential approximation result for the finite-delay system established by Li et al. [25], we derived the convergence rate of the truncated Euler–Maruyama method for the original hybrid stochastic functional differential equation with infinite time delay.

The results obtained in this chapter complement those of Chapter 3 and provide a more comprehensive convergence theory for the truncated Euler–Maruyama method applied to hybrid stochastic functional differential equations with infinite time delay.

Chapter 5

Stabilisation of Hybrid Systems by Event-triggered Control based on Discrete-time State Observations

5.1 Introduction

The previous chapters of this thesis have focused on the numerical analysis of hybrid stochastic functional differential equations with infinite delay. In particular, Chapter 3 established the convergence theory of the truncated Euler–Maruyama method under generalized Khasminskii-type conditions, while Chapter 4 further extended these results by relaxing the underlying Khasminskii-type condition and establishing strong convergence results in the L^q sense for $q \in [1, 2)$. These results establish the theoretical foundation for the numerical approximation of hybrid stochastic functional differential equations with infinite time delay.

Besides numerical approximation, another fundamental aspect in the study of hybrid stochastic systems is the analysis of stability and the design of stabilizing controllers. While numerical methods aim to approximate the trajectories of stochastic systems accurately, control strategies seek to modify the system dynamics so that un-

stable systems become stable while maintaining low implementation cost. Motivated by this complementary perspective, this chapter shifts the focus from numerical analysis to the stabilization problem for hybrid stochastic differential equations with Markovian switching. In particular, we investigate event-triggered feedback control based on discrete-time observations, which significantly reduces the communication and computation burden compared with continuous-time feedback strategies. The results developed in this chapter complement the numerical approximation theory established in the previous chapters, providing a control perspective on the study of hybrid stochastic systems.

Hybrid stochastic differential equations, also known as stochastic differential equations with Markovian switching, constitute an important class of hybrid systems and have been widely used in many branches of science and industry (see, e.g., [33,44,46,58]). Such systems are described by the hybrid stochastic differential equation

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t). \quad (5.1)$$

Here the state $x(t)$ takes values in $\mathbb{R}^d$ and $r(t)$ denotes the current operating mode of the hybrid system, modelled by a Markov chain taking values in a finite space $\mathbb{S} = \{1, 2, \dots, N\}$, $B(t)$ is a Brownian motion and f and g are referred to as the drift and diffusion coefficients, respectively. One of the important issues in the study of hybrid SDEs is the analysis of stability (see, e.g., [19,33,34,41,53–55,67]).

The following stabilisation problem by a feedback control based on discrete-time state observations was initiated in [36]: Assume that the given hybrid SDE (5.1) is not stable and the observations of the state $x(t)$ are only available at discrete times $0, \tau, 2\tau, \dots$, while the mode $r(t)$ is available at any time $t \geq 0$, can we design a feedback control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ so that the controlled system

$$dx(t) = (f(x(t), r(t), t) + u(x(\lfloor t/\tau \rfloor \tau), r(t), t))dt + g(x(t), r(t), t)dB(t) \quad (5.2)$$

becomes stable? Here $\tau > 0$ is a constant which stands for the duration between two consecutive state observations, and $\lfloor t/\tau \rfloor$ is the integer part of t/τ . This is signifi-

cantly different from the stabilisation by a continuous-time (regular) feedback control $u(x(t), r(t), t)$, because the regular feedback control requires the continuous observations of the state $x(t)$ for all $t \geq 0$, while the feedback control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ needs only the discrete observations of the state $x(t)$ at times $0, \tau, 2\tau, \dots$. The latter is clearly more realistic and costs less in practice. This stabilisation problem has been studied by many authors for the past ten years (see, e.g., [38, 40, 51, 61, 62, 66]). It should be pointed out the the controlled system (5.2) is an SDDE with a finite-time delay.

To develop the proposed event-triggered control framework, we first set $t_k = k\tau$ for $k \in \mathbb{N} := \{0, 1, 2, \dots\}$, which are the discrete times when the state observations are made. The feedback control $u(x(\lfloor t/\tau \rfloor \tau), r(t), t)$ in (5.2) can be written as

$$u(x(t_k), r(t), t) \quad \forall t \in [t_k, t_{k+1}), \quad k \in \mathbb{N}. \quad (5.3)$$

That is, the state input $x(t_k)$ in the control is held constant during $[t_k, t_{k+1})$ but is updated to $x(t_{k+1})$ at time t_{k+1} and remains constant during $[t_{k+1}, t_{k+2})$ while the control will be recomputed according to $u(x(t_{k+1}), r(t), t)$. In other words, the state input is updated at every time t_k and the control has to be recomputed accordingly. This naturally raises the question of whether such frequent updates are indeed necessary. Suppose that $x(t_{k+1})$ might be close to $x(t_k)$, we would expect it is unnecessary to update the state input from $x(t_k)$ to $x(t_{k+1})$ at time t_{k+1} . In other words, in this situation, we might still use the control $u(x(t_k), r(t), t)$ for the next duration $[t_{k+1}, t_{k+2})$ to avoid recomputing the new control $u(x(t_{k+1}), r(t), t)$ and hence to reduce the overall control cost.

To develop this idea into a new theory, we first need to describe precisely how $x(t_{k+1})$ is said to be close to $x(t_k)$. We will use the following simple definition in this chapter.

Definition 5.1. Let $\Delta \in (0, 1)$. Let $y = (y_1, \dots, y_d)^T$ and $z = (z_1, \dots, z_d)^T$ be two

vectors in $\mathbb{R}^d$. If

$$|y_i - z_i| \leq \Delta |z_i| \quad \text{for all } i = 1, \dots, d, \quad (5.4)$$

then z is said to be close to y in the order of Δ . We will not mention “in the order of Δ ” any more from now on. We will call Δ the event-triggered parameter.

We next need to form our new controlled system. Let $X_0 = x(0)$, the initial value. The control is $u(X_0, r(t), t)$ for $t \in [0, t_1)$. At time t_1 , we observe the state $x(t_1)$. If $x(t_1)$ is close to X_0 , we will not update the state input in the control; otherwise update it. Hence the control is $u(X_1, r(t), t)$ for $t \in [t_1, t_2)$, where

$$X_1 = \begin{cases} X_0 & \text{if } x(t_1) \text{ is close to } X_0, \\ x(t_1) & \text{otherwise.} \end{cases} \quad (5.5)$$

In general, we define X_k for $k = 1, 2, \dots$ by

$$X_k = \begin{cases} X_{k-1} & \text{if } x(t_k) \text{ is close to } X_{k-1}, \\ x(t_k) & \text{otherwise.} \end{cases} \quad (5.6)$$

Then the control is $u(X_k, r(t), t)$ for $t \in [t_k, t_{k+1})$. Therefore our new controlled system has the following form

$$dx(t) = (f(x(t), r(t), t) + u(X_{\lfloor t/\tau \rfloor}, r(t), t))dt + g(x(t), r(t), t)dB(t) \quad (5.7)$$

on $t \geq 0$.

The idea above is motivated by the popular event-triggered controls. In the field of ordinary differential equations (ODEs), the event-triggered controls have been studied intensively and we only mention [6, 8, 13, 56, 65] among many others. In the field of SDEs, they have received more and more attention recently (see, e.g., [1, 16, 32, 60, 64]). In most of the existing papers, an event-triggered mechanism for SDEs is designed based on a criterion in terms of the mathematical expectation of the system state. Such a

mechanism can only be implemented if a sample of large size of the system state, at any observation time, is available so that the mathematical expectation involved in the event-triggered criterion can be calculated approximately. There are some systems in this category, for example, a stochastic particle system [18]. However, in much more real-world SDE systems, e.g., an SDE model for a stock price, the state observation is just a single-size sample at a time, namely the state observations at discrete times form a trajectory of the solution to the underlying SDE. In such a situation, the event-triggered criterion should be designed based on a trajectory of the state observations at discrete times. To the best of our knowledge, [32] is the only existing work in this direction. There is therefore an urgent need to develop the event-triggered control theory based on a trajectory of the state observations at discrete times.

It is necessary to explain how the results in [32] differ from ours in this chapter. In [32], z is said to be close to y if $|z - y| \leq \Delta|z|$. In their case, there is no form of (5.14) (see Section 2 below) between y and z . We will return to this point in more detail later in Section 2. In particular, the reader will see that the form of (5.14) will play its crucial role in this chapter. Mathematically speaking, our approach is completely different from that in [32]. The novel approach in this chapter is to relate the event-triggered controlled SDE (5.7) to the corresponding uncertain SDE

$$dz(t) = (f(z(t), r(t), t) + u((I_d + \phi(t))z(t), r(t), t))dt + g(z(t), r(t), t)dB(t) \quad (5.8)$$

on $t \geq 0$, where I_d is the $d \times d$ identity matrix and $\phi(t) = \text{diag}(\phi_1(t), \dots, \phi_d(t))$ is an $\mathcal{F}_t$ -adapted process such that its every element $\phi_i(t) \in [-\Delta, \Delta]$ a.s. Please note that the real values of every $\phi_i(t)$ are unknown though they are within $[-\Delta, \Delta]$. In words, $\phi_i(t)$ represents the system uncertainty. We will show that under some regular conditions on the coefficients f, g, u , if the uncertain SDE (5.8) is exponentially stable in mean square, then there is a positive number τ^* such that the event-triggered controlled SDE (5.7) is also exponentially stable in mean square as long as $\tau < \tau^*$.

What we have discussed so far is the case where state $x(t_k)$ is observable at the discrete times $\{t_k\}_{k \in \mathbb{N}}$ and the control is known as the state feedback (see, e.g., [39]). However, there is another important case in practice, where the output $y(t_k) := G_{r(t_k)}x(t_k)$

is observable but state $x(t_k)$ itself is not, in which $G_i \in \mathbb{R}^{l \times d}$ is known for every $i \in \mathbb{S}$ and, in general, $l < d$. In this case, the event-triggered control should be designed based on $y(t_k)$ but not $x(t_k)$. In this chapter we will investigate this case too but [32] did not. This constitutes another important difference between the approach in [32] and that developed in this chapter.

The main contributions of this chapter are summarized as follows:

- We develop an event-triggered control based on a trajectory of the state or output observations at discrete times. This is significantly different from most of the existing papers where an event-triggered mechanism for SDEs is designed based on a criterion in terms of the mathematical expectation of the system state.
- We establish a novel connection between the stability of the event-triggered controlled SDE and that of the corresponding uncertain SDE.
- We design the event-triggered state feedback control for the case where the state is observable while the event-triggered output injection for the case where the output is observable. We explain how both cases differ from each other significantly and hence different mathematical techniques have been developed to study them.

The remainder of this chapter is organised as follows. In Section 5.2, we investigate the stabilisation of hybrid stochastic differential equations by event-triggered state-feedback control and establish the corresponding stability criteria. In Section 5.3, we further study the event-triggered output-feedback case. Section 5.4 presents several examples to illustrate the theoretical results. Finally, Section 5.5 closes this chapter.

5.2 Stabilisation by Event-triggered State Feedback

5.2.1 Additional Notations

Unless otherwise specified, all notation introduced in Chapter 2 will be adopted throughout this chapter. For convenience, we only introduce the additional notation used in the sequel. Let I_d denote the $d \times d$ identity matrix. For symmetric matrices A_1

and $A_2, A_1 \leq A_2$ (respectively, $A_1 < A_2$) means $A_1 - A_2 \leq 0$ (respectively, $A_1 - A_2 < 0$).

If $A \subseteq \Omega$, let $\mathbf{1}_A$ denote its indicator function, namely,

$$\mathbf{1}_A(\omega) = \begin{cases} 1, & \omega \in A, \\ 0, & \omega \notin A. \end{cases}$$

5.2.2 Controlled system

Consider the given unstable d -dimensional hybrid SDE (5.1), where $f : \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^{d \times m}$ are Borel measurable functions. Assume that the state $x(t)$ is observable at the discrete times $\{t_k\}_{k \in \mathbb{N}}$ in all modes or some modes and the Markov chain $r(t)$ is observable at any time (e.g., the Markov chain models the interest rates [9]). Of course, we can consider the situation where the Markov chain $r(t)$ is only observable at the discrete times but we will report the corresponding results elsewhere due to the page limit here. We need to design the control function $u : \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+ \rightarrow \mathbb{R}^d$ so that the controlled SDE (5.7) becomes stable. To make our new theory more understandable, we will only consider the linear control function of the form

$$u(x, i, t) = A_i x \quad \text{with} \quad A_i = F_i G_i, \quad (5.9)$$

where $F_i \in \mathbb{R}^{d \times l}$ and $G_i \in \mathbb{R}^{l \times d}$ for some positive integer l which is less than d in general. If in mode i , state $x(t)$ is not observable, we set $A_i = 0$ (see Example 5.17). This is known as the structure control, which has two cases: (i) state feedback control where F_i 's are known while G_i 's need to be designed; (ii) output injection where G_i 's are known while F_i 's need to be designed (see, e.g., [39]). In this section we consider case (i) but leave case (ii) to the next section. We will explain how to design G_i 's in Section 5.2.5 but in the meanwhile we assume they are known and so are A_i 's. Consequently, our controlled SDE (5.7) becomes

$$dx(t) = (f(x(t), r(t), t) + A_{r(t)} X_{[t/\tau]})dt + g(x(t), r(t), t)dB(t) \quad (5.10)$$

on $t \geq 0$, with initial data $x(0) = x_0 \in \mathbb{R}^d$ and $r(0) = r_0 \in \mathbb{S}$ at time zero. To make this SDE well defined, we impose the following standing hypotheses.

Assumption 5.2. *Assume that the coefficients f and g are locally Lipschitz continuous (see, e.g., [35, 44]). Moreover, they satisfy the following linear growth condition*

$$|f(x, i, t)| \leq K_1|x| \quad \text{and} \quad |g(x, i, t)| \leq K_2|x| \quad (5.11)$$

for all $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, where both K_1 and K_2 are positive numbers.

We claim that under Assumption 5.2, the controlled SDE (5.10) with any initial data $x(0) = x_0 \in \mathbb{R}^d$ and $r(0) = r_0 \in \mathbb{S}$ has a unique global solution $x(t)$ such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \geq 0$. In fact, restricting $t \in [0, t_1]$, we see that the controlled SDE (5.10) becomes

$$dx(t) = (f(x(t), r(t), t) + A_{r(t)}X_0)dt + g(x(t), r(t), t)dB(t), \quad (5.12)$$

where $X_0 = x_0$. It is well known (see, e.g., [44]) that this SDE has a unique solution $x(t)$ on $t \in [0, t_1]$ such that $\sup_{t \in [0, t_1]} \mathbb{E}|x(t)|^2 < \infty$. At time t_1 , $x(t_1)$ is observed and X_1 can be defined by (5.5). For $t \in [t_1, t_2]$, the controlled SDE (5.10) becomes

$$dx(t) = (f(x(t), r(t), t) + A_{r(t)}X_1)dt + g(x(t), r(t), t)dB(t) \quad (5.13)$$

with initial data $x(t_1)$ and $r(t_1)$ at time t_1 . It is well known that this SDE has a unique solution $x(t)$ on $t \in [t_1, t_2]$ such that $\sup_{t \in [t_1, t_2]} \mathbb{E}|x(t)|^2 < \infty$. At time t_2 , $x(t_2)$ is observed and X_2 can be defined by (5.6). Repeating the above arguments for $t \in [t_2, t_3]$ etc. we see what we claimed.

To perform the theoretical analysis, we need to re-write the controlled SDE (5.10) in a modified form. Let $z = (z_1, \dots, z_d)^T$ be close to $y = (y_1, \dots, y_d)^T$. By Definition 5.1, for each $i = 1, \dots, d$, there is a unique number $\sigma_i \in [-\Delta, \Delta]$ such that $y_i = (1 + \sigma_i)z_i$. In fact, $\sigma_i = (y_i - z_i)/z_i$ when $z_i \neq 0$ and we set $\sigma_i = 0$ when $z_i = 0$. Hence

$$y = (I_d + \sigma)z, \quad (5.14)$$

where $\sigma = \text{diag}(\sigma_1, \dots, \sigma_d)$ (and I_d is the $d \times d$ identity matrix as defined before). It is also easy to see

$$|\sigma z| \leq \Delta |z| \quad \text{and} \quad (1 - \Delta)|z| \leq |y| \leq (1 + \Delta)|z|. \quad (5.15)$$

Applying (5.14) to (5.6), we see that for $k = 1, 2, \dots$,

$$X_k = \begin{cases} (I_d + \bar{\Delta}(t_k))x(t_k) & \text{if } x(t_k) \text{ is close to } X_{k-1}, \\ x(t_k) & \text{otherwise,} \end{cases} \quad (5.16)$$

where $\bar{\Delta}(t_k) = \text{diag}(\bar{\Delta}_1(t_k), \dots, \bar{\Delta}_d(t_k))$ with $\bar{\Delta}_i(t_k) \in [-\Delta, \Delta]$ for all $i = 1, \dots, d$, while $\bar{\Delta}(t_k)$ is $\mathcal{F}_{t_k}$ -measurable as it is determined by $x(t_k)$ and X_{k-1} only. We can then further write $X_k = (I_d + \hat{\Delta}(t_k))x(t_k)$, where

$$\hat{\Delta}(t_k) = \begin{cases} \bar{\Delta}(t_k) & \text{if } x(t_k) \text{ is close to } X_{k-1}, \\ 0 & \text{otherwise,} \end{cases} \quad (5.17)$$

which is $\mathcal{F}_{t_k}$ -measurable of course. By (5.15), we also see that $(\bar{\Delta}(t_k))^2 - \Delta^2 I_d \leq 0$,

$$|\hat{\Delta}(t_k)x(t_k)| \leq \Delta |x(t_k)| \quad \text{and} \quad (1 - \Delta)|x(t_k)| \leq |(I_d + \hat{\Delta}(t_k))x(t_k)| \leq (1 + \Delta)|x(t_k)|. \quad (5.18)$$

We set $\hat{\Delta}(t_0) = 0$ so that $\hat{\Delta}(t_k)$ is now defined for all $k \in \mathbb{N}$. Furthermore, we will define $\delta_t = \lfloor t/\tau \rfloor \tau$ for $t \geq 0$. With these notations, the controlled SDE (5.10) can be written in the following modified form

$$dx(t) = (f(x(t), r(t), t) + A_{r(t)}(I_d + \hat{\Delta}(\delta_t))x(\delta_t))dt + g(x(t), r(t), t)dB(t). \quad (5.19)$$

It should be pointed out that this form is convenient when we perform mathematical analysis from next section on by making use of (5.18). However, in practice, we should use (5.10) because once X_k is determined by (5.6), the controlled SDE (5.10) can already be implemented without knowing $\hat{\Delta}(t_k)$.

We will see that the stability of this controlled SDE is related to the robust stability of the following auxiliary uncertain SDE

$$dz(t) = (f(z(t), r(t), t) + A_{r(t)}(I_d + \phi(t))z(t))dt + g(z(t), r(t), t)dB(t) \quad (5.20)$$

on $t \geq 0$, where $\phi(t) = \text{diag}(\phi_1(t), \dots, \phi_d(t))$ is an $\mathcal{F}_t$ -adapted process such that its every element $\phi_i(t) \in [-\Delta, \Delta]$ a.s. Please note that the real values of every $\phi_i(t)$ are unknown though they are within $[-\Delta, \Delta]$. In words, $\phi_i(t)$ represents the system uncertainty.

5.2.3 Lemma

Let us begin this section by stating a key assumption.

Assumption 5.3. *There exist positive numbers β , β_i ($i \in \mathbb{S}$) and symmetric positive definite matrices $Q_i \in \mathbb{R}^{d \times d}$ such that*

$$\begin{aligned} & 2x^T Q_i f(x, i, t) + \text{trace}[g(x, i, t)^T Q_i g(x, i, t)] \\ & + x^T \left(Q_i A_i + A_i^T Q_i + \beta_i Q_i A_i A_i^T Q_i + (\Delta^2 / \beta_i) I_d + \sum_{j=1}^N \gamma_{ij} Q_j \right) x \leq -\beta |x|^2 \end{aligned} \quad (5.21)$$

for all $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

We will explain in detail in Section 5.2.5 how to choose matrices Q_i and free parameters β_i and then design the corresponding matrices G_i 's and hence A_i 's (recalling F_i 's are known) and find the positive number β for (5.21) to hold. In this section, we just assume that Assumption 5.3 holds. By [44, Theorem 5.8 on page 166] (with $V(x, i, t) = x^T Q_i x$ there), we see that Assumption 5.3 guarantees the mean-square exponential stability of the auxiliary uncertain SDE (5.20).

From now on we will fix initial data $x_0 \in \mathbb{R}^d$ and $r_0 \in \mathbb{S}$ arbitrarily and will not

mention them any more unless necessary. Moreover, we will set

$$K_A = \max_{i \in \mathbb{S}} \|A_i\|, \quad K_{QA} = \max_{i \in \mathbb{S}} \|Q_i A_i\|, \quad \lambda_m = \min_{i \in \mathbb{S}} \lambda_{\min}(Q_i), \quad \lambda_M = \max_{i \in \mathbb{S}} \lambda_{\max}(Q_i). \quad (5.22)$$

Before we state our first main theorem, let us present a useful lemma.

Lemma 5.4. *Let Assumption 5.2 hold. If $\tau < 1/(K_A(1 + \Delta))$, then the solution of the controlled SDE (5.10) has the property that*

$$\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) \leq H_1(\tau)\mathbb{E}|x(t)|^2 + H_2(\tau) \int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds \quad (5.23)$$

for all $t \geq 0$, where

$$H_1(\tau) = \frac{\tau[0.5K_1 + K_A(1 + \Delta)] + 0.5K_2\sqrt{\tau}}{1 - \tau K_A(1 + \Delta)} \quad (5.24)$$

and

$$H_2(\tau) = \frac{0.5(K_1 + K_2/\sqrt{\tau})}{1 - \tau K_A(1 + \Delta)}. \quad (5.25)$$

Proof. For $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}_+$, by Assumption 5.2 and property (5.18), it follows from (5.19) that

$$\begin{aligned} \mathbb{E}(|x(t)||x(t) - x(\delta_t)|) &= \mathbb{E}(|x(t)||x(t) - x(t_k)|) \\ &\leq \mathbb{E}\left(|x(t)| \int_{t_k}^t (K_1|x(s)| + K_A(1 + \Delta)|x(t_k)|)ds + |x(t)||M_k(t)|\right), \end{aligned} \quad (5.26)$$

where $M_k(t) = \int_{t_k}^t g(x(s), r(s), s)dB(s)$. But

$$\begin{aligned} &\mathbb{E}\left(|x(t)| \int_{t_k}^t (K_1|x(s)| + K_A(1 + \Delta)|x(t_k)|)ds\right) \\ &\leq \mathbb{E} \int_{t_k}^t \left(K_1|x(t)||x(s)| + K_A(1 + \Delta)|x(t)|[|x(t)| + |x(t) - x(t_k)|]\right)ds \\ &\leq \tau[0.5K_1 + K_A(1 + \Delta)]\mathbb{E}|x(t)|^2 + 0.5K_1\mathbb{E} \int_{t_k}^t |x(s)|^2 ds \\ &\quad + \tau K_A(1 + \Delta)\mathbb{E}(|x(t)||x(t) - x(t_k)|), \end{aligned} \quad (5.27)$$

and

$$\begin{aligned}\mathbb{E}|x(t)||M_k(t)| &\leq 0.5K_2\sqrt{\tau}\mathbb{E}|x(t)|^2 + (0.5/K_2\sqrt{\tau})\mathbb{E}|M_k(t)|^2 \\ &\leq 0.5K_2\sqrt{\tau}\mathbb{E}|x(t)|^2 + (0.5K_2/\sqrt{\tau})\mathbb{E}\int_{t_k}^t |x(s)|^2 ds.\end{aligned}\quad (5.28)$$

Substituting these into (5.26) yields

$$\begin{aligned}&\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) \\ &\leq (\tau[0.5K_1 + K_A(1 + \Delta)] + 0.5K_2\sqrt{\tau})\mathbb{E}|x(t)|^2 + 0.5(K_1 + K_2/\sqrt{\tau})\mathbb{E}\int_{t_k}^t |x(s)|^2 ds \\ &\quad + \tau K_A(1 + \Delta)\mathbb{E}(|x(t)||x(t) - x(t_k)|).\end{aligned}\quad (5.29)$$

This implies the desired assertion (5.23) as $\tau K_A(1 + \Delta) < 1$. $\square$

5.2.4 Theorems

Theorem 5.5. *Let Assumptions 5.2 and 5.3 hold. Let $\tau^* \in (0, (K_A(1 + \Delta))^{-1})$ be the unique root to the following equation of τ*

$$\beta = \frac{2K_{QA}(1 + \Delta)}{1 - \tau K_A(1 + \Delta)} (\tau[K_1 + K_A(1 + \Delta)] + K_2\sqrt{\tau}) \quad (5.30)$$

and it has the explicit form (5.34) below. If $\tau < \tau^$, then the solution of the controlled SDE (5.10) has the properties that*

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \leq -\varepsilon, \quad (5.31)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(|x(t)|) \leq -\frac{\varepsilon}{2} \quad a.s., \quad (5.32)$$

where ε is a unique positive number such that

$$\beta = \varepsilon \lambda_M + 2K_{QA}(1 + \Delta)(H_1(\tau) + \tau e^{\varepsilon\tau} H_2(\tau)), \quad (5.33)$$

in which $H_1(\tau)$ and $H_2(\tau)$ have been defined in Lemma 5.4. That is, the controlled SDE (5.10) is not only exponentially stable in mean square but also almost surely.

Proof. We observe that as τ increases from 0 to $(K_A(1 + \Delta))^{-1}$, the right-hand-side of (5.30) increases strictly and continuously from 0 to positive infinity. Equation (5.30) must therefore have a unique root $\tau^* \in (0, (K_A(1 + \Delta))^{-1})$. It is straightforward to obtain

$$\tau^* = \left(\frac{-b + \sqrt{b^2 + 4a\beta}}{2a} \right)^2, \quad (5.34)$$

where $a = 2K_{QA}(1 + \Delta)[K_1 + K_A(1 + \Delta)] + \beta K_A(1 + \Delta)$ and $b = 2K_2K_{QA}(1 + \Delta)$.

Fix $\tau < \tau^*$ arbitrarily. We then have

$$\beta > \frac{2K_{QA}(1 + \Delta)}{1 - \tau K_A(1 + \Delta)} (\tau[K_1 + K_A(1 + \Delta)] + K_2\sqrt{\tau}). \quad (5.35)$$

Recalling the definitions of $H_1(\tau)$ and $H_2(\tau)$ in Lemma 5.4, we observe that

$$2K_{QA}(1 + \Delta)(H_1(\tau) + \tau H_2(\tau)) = \frac{2K_{QA}(1 + \Delta)}{1 - \tau K_A(1 + \Delta)} (\tau[K_1 + K_A(1 + \Delta)] + K_2\sqrt{\tau}).$$

Making use of (5.35), we see that there is a unique positive number ε for (5.33) to hold.

Applying the generalized Itô formula (see, e.g., [34, 44]) to the Lyapunov function

$$V(x(t), r(t)) := x^T(t)Q_{r(t)}x(t),$$

we obtain

$$dV(x(t), r(t)) = J_1(t)dt + dM(t) \quad (5.36)$$

for $t \geq 0$, where $M(t)$ is a continuous martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this chapter so we do not state it here) and

$$\begin{aligned} J_1(t) := & 2x^T(t)Q_{r(t)}[f(x(t), r(t), t) + A_{r(t)}(I_d + \hat{\Delta}(\delta_t))x(\delta_t)] \\ & + \text{trace}[g^T(x(t), r(t), t)Q_{r(t)}g(x(t), r(t), t)] + \sum_{j=1}^N \gamma_{r(t),j} x^T(t)Q_j x(t). \end{aligned} \quad (5.37)$$

Making use of (5.18), we can derive that

$$\begin{aligned}
 & 2x^T(t)Q_{r(t)}A_{r(t)}(I_d + \hat{\Delta}(\delta_t))x(\delta_t) \\
 &= 2x^T(t)Q_{r(t)}A_{r(t)}(I_d + \hat{\Delta}(\delta_t))(x(t) - [x(t) - x(\delta_t)]) \\
 &\leq x^T(t)\left(Q_{r(t)}A_{r(t)} + A_{r(t)}^TQ_{r(t)} + \beta_{r(t)}Q_{r(t)}A_{r(t)}A_{r(t)}^TQ_{r(t)} + (\Delta^2/\beta_{r(t)})I_d\right)x(t) \\
 &+ 2K_{QA}(1 + \Delta)|x(t)||x(t) - x(\delta_t)|.
 \end{aligned} \tag{5.38}$$

Substituting this into (5.36) and then applying condition (5.21), we get

$$J_1(t) \leq -\beta|x(t)|^2 + 2K_{QA}(1 + \Delta)|x(t)||x(t) - x(\delta_t)|. \tag{5.39}$$

Applying the generalized Itô formula to the Lyapunov function $e^{\varepsilon t}V(x(t), r(t))$, we obtain that

$$e^{\varepsilon t}\mathbb{E}V(x(t), r(t)) = V(x_0, r_0) + \mathbb{E} \int_0^t e^{\varepsilon s} \left(\varepsilon V(x(s), r(s)) + J_1(s) \right) ds \tag{5.40}$$

for $t \geq 0$. By (5.39), Lemma 5.4, etc., we then derive

$$\begin{aligned}
 & \lambda_m e^{\varepsilon t} \mathbb{E}|x(t)|^2 \leq \lambda_M |x_0|^2 \\
 &+ \int_0^t e^{\varepsilon s} \left(-(\beta - \varepsilon \lambda_M) \mathbb{E}|x(s)|^2 + 2K_{QA}(1 + \Delta) \mathbb{E}(|x(s)||x(s) - x(\delta_s)|) \right) ds \\
 &\leq \lambda_M |x_0|^2 - (\beta - \varepsilon \lambda_M - 2K_{QA}(1 + \Delta)H_1(\tau)) \int_0^t e^{\varepsilon s} \mathbb{E}|x(s)|^2 ds \\
 &+ 2K_{QA}(1 + \Delta)H_2(\tau) \int_0^t e^{\varepsilon s} \left(\int_{\delta_s}^s \mathbb{E}|x(v)|^2 dv \right) ds.
 \end{aligned} \tag{5.41}$$

But

$$\begin{aligned}
 & \int_0^t e^{\varepsilon s} \left(\int_{\delta_s}^s \mathbb{E}|x(v)|^2 dv \right) ds \leq \int_0^t e^{\varepsilon s} \left(\int_{(s-\tau) \vee 0}^s \mathbb{E}|x(v)|^2 dv \right) ds \\
 &\leq \int_0^t \mathbb{E}|x(v)|^2 \left(\int_v^{v+\tau} e^{\varepsilon s} ds \right) dv \leq \int_0^t \mathbb{E}|x(v)|^2 \tau e^{\varepsilon(v+\tau)} dv \\
 &= \tau e^{\varepsilon \tau} \int_0^t e^{\varepsilon v} \mathbb{E}|x(v)|^2 dv.
 \end{aligned} \tag{5.42}$$

Substituting this into (5.41) yields

$$\begin{aligned} \lambda_m e^{\varepsilon t} \mathbb{E}|x(t)|^2 &\leq \lambda_M |x_0|^2 \\ &- (\beta - \varepsilon \lambda_M - 2K_{QA}(1 + \Delta)[H_1(\tau) + \tau e^{\varepsilon \tau} H_2(\tau)]) \int_0^t e^{\varepsilon s} \mathbb{E}|x(s)|^2 ds. \end{aligned} \quad (5.43)$$

Recalling (5.33) we obtain

$$\lambda_m e^{\varepsilon t} \mathbb{E}|x(t)|^2 \leq \lambda_M |x_0|^2, \quad (5.44)$$

which implies the required assertion (5.31) immediately.

To prove the second assertion (5.32), we can show easily from (5.19) using the Burkholder-Davis-Gundy inequality (see, e.g., [44, pp.70–76]) etc. that for any $k \in \mathbb{N}_+$,

$$\begin{aligned} &\mathbb{E} \left(\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^2 \right) \\ &\leq 4\mathbb{E}|x(t_k)|^2 + 4\mathbb{E} \int_{t_k}^{t_{k+1}} \left(\tau K_1^2 |x(s)|^2 + \tau K_A^2 (1 + \Delta)^2 |x(t_k)|^2 + 4K_2^2 |x(s)|^2 \right) ds \\ &= 4(1 + \tau K_A^2 (1 + \Delta)^2) \mathbb{E}|x(t_k)|^2 + 4(\tau K_1^2 + 4K_2^2) \int_{t_k}^{t_{k+1}} \mathbb{E}|x(s)|^2 ds. \end{aligned}$$

By (5.44), we obtain

$$\mathbb{E} \left(\sup_{t_k \leq t \leq t_{k+1}} |x(t)|^2 \right) \leq K_3 e^{-\varepsilon t_k},$$

where K_3 is a positive constant independent of k . Let $\bar{\varepsilon} \in (0, \varepsilon)$ be arbitrary. Then

$$\sum_{k=0}^{\infty} \mathbb{P} \left(\sup_{t_k \leq t \leq t_{k+1}} |x(t)| > e^{-0.5\bar{\varepsilon} t_k} \right) \leq \sum_{k=0}^{\infty} K_3 e^{-(\varepsilon - \bar{\varepsilon})t_k} < \infty.$$

The well-known Borel-Cantelli lemma (see, e.g., [44, p.10]) shows that for almost all $\omega \in \Omega$, there is positive integer $k_0 = k_0(\omega)$ such that

$$\sup_{t_k \leq t \leq t_{k+1}} |x(t)| \leq e^{-0.5\bar{\varepsilon} t_k}, \quad k \geq k_0.$$

Hence, for almost all $\omega \in \Omega$,

$$\frac{1}{t} \log(|x(t)|) \leq -\frac{0.5\bar{\varepsilon}\tau k}{\tau(k+1)}, \quad t \in [t_k, t_{k+1}], \quad k \geq k_0.$$

This implies

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(|x(t)|) \leq -0.5\bar{\varepsilon} < 0 \quad a.s.$$

This implies the second assertion (5.32) as $\bar{\varepsilon} \in (0, \varepsilon)$ is arbitrary. The proof is complete.

□

In Section 5.2.5 we will see that the use of Assumption 5.3 requires us to identity the matrices Q_i and parameters β_i wisely so that the corresponding matrices A_i (via G_i and F_i) and the positive number β can be found for (5.21) to hold. On the one side, this gives us flexibility as we have very wide choices of Q_i and β_i . On the other side, more search is required to identity these Q_i and β_i . Let us now present an alternative assumption.

Assumption 5.6. *There exist real numbers b_i ($i \in \mathbb{S}$) such that*

$$\mathcal{A} := -\text{diag}(b_1 + 2\Delta\|A_1\|, \dots, b_N + 2\Delta\|A_N\|) - \Gamma \quad (5.45)$$

is a non-singular M-matrix and

$$2x^T f(x, i, t) + |g(x, i, t)|^2 + x^T (A_i + A_i^T) x \leq b_i |x|^2 \quad (5.46)$$

for all $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

In comparison with Assumption 5.3, there are two advantages of this new one: (i) we only need to design the matrices A_i (or G_i) but not Q_i in application; (ii) it is described in terms of M-matrices which can be verified easily. Of course, we will see that this new assumption is less general than Assumption 5.3. In fact, we now explain that under this assumption, the matrices Q_i and parameter β can be identified

automatically so that Assumption 5.3 is satisfied. Set

$$(q_1, \dots, q_N)^T := \mathcal{A}^{-1}(1, \dots, 1)^T. \quad (5.47)$$

By the theory of M-matrices (see, e.g., [44]), all q_i are positive. Moreover,

$$(b_i + 2\Delta\|A_i\|)q_i + \sum_{j=1}^N \gamma_{ij}q_j = -1, \quad \forall i \in \mathbb{S}. \quad (5.48)$$

For each $i \in \mathbb{S}$, set $Q_i = q_i I_d$. Note that two terms on the left-hand-side of (5.21), namely

$$x^T(\beta_i Q_i A_i A_i^T Q_i + (\Delta^2/\beta_i)I_d)x \leq 2\Delta\|Q_i A_i\||x|^2 = 2q_i\Delta\|A_i\||x|^2$$

by letting $\beta_i = \Delta/\|Q_i A_i\|$ when $A_i \neq 0$, while they can be treated as 0 by letting $\beta_i \rightarrow \infty$ when $A_i = 0$. Alternatively, we observe from (5.38) that these two terms are the upper bound for $2x^T Q_i A_i \hat{\Delta}(\cdot)x$. But we clearly have $2x^T Q_i A_i \hat{\Delta}(\cdot)x \leq 2q_i\Delta\|A_i\||x|^2$ no matter whether $A_i = 0$ or not. Consequently,

the left-hand-side of (5.21)

$$\begin{aligned} &\leq 2q_i x^T f(x, i, t) + q_i |g(x, i, t)|^2 + q_i x^T (A_i + A_i^T)x \\ &\quad + 2q_i\Delta\|A_i\||x|^2 + \sum_{j=1}^N \gamma_{ij}q_j |x|^2 \\ &\leq (b_i + 2\Delta\|A_i\|)q_i |x|^2 + \sum_{j=1}^N \gamma_{ij}q_j |x|^2 \\ &= -|x|^2, \end{aligned}$$

where (5.46) and (5.48) have been used. In other words, we have just shown that Assumption 5.6 implies Assumption 5.3 with $Q_i = q_i I_d$ for $i \in \mathbb{S}$ and $\beta = 1$ while β_i 's are specified above but have no further use. We therefore have the following theorem.

Theorem 5.7. *Let Assumptions 5.2 and 5.6 hold. Let q_i 's be defined by (5.47) and re-set $\beta = 1$, $K_{QA} = \max_{i \in \mathbb{S}} q_i \|A_i\|$, $\lambda_m = \min_{i \in \mathbb{S}} q_i$, $\lambda_M = \max_{i \in \mathbb{S}} q_i$ but leave K_A*

the same as before. Let $\tau^* \in (0, (K_A(1 + \Delta))^{-1})$ be the unique root to equation (5.30). Assume that $\tau < \tau^*$ and let $\varepsilon > 0$ be the unique root to equation (5.33). Then assertions (5.31) and (5.32) of Theorem 5.5 hold.

This theorem is particularly useful when we deal with the situation where the feedback control could only be used for some modes but not every mode. We will illustrate this in detail in Example 5.17.

5.2.5 Design of matrices G_i 's

The implementation of the controlled SDE (5.10) depends on the design of N matrices G_i 's so that we can determine $A_i = F_i G_i$. In this section we will use the technique of linear matrix inequalities (LMIs, see, e.g., [67]) to design G_i 's. Under Assumption 5.2, we will design G_i 's in two steps.

Step 2.5.1. Find N pairs of symmetric matrices Q_i and $\hat{Q}_i$ ($i \in \mathbb{S}$) with $Q_i > 0$ such that

$$2x^T Q_i f(x, i, t) + \text{trace}[g(x, i)^T Q_i g(x, i)] \leq x^T \hat{Q}_i x \quad (5.49)$$

for all $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

There are plenty of choices for these matrices. In fact, for each symmetric positive-definite matrix Q_i , we can let $\hat{Q}_i = \|Q_i\|(2K_1 + K_2^2)I_d$ to make (5.49) hold. However, it is wise to find alternative matrices in order to make use of the given structures of f and g so that Step 2 below can be done more easily. In the case when the given SDE (5.1) is linear, $\hat{Q}_i$'s can be determined explicitly once Q_i 's are chosen. In fact, assume that the coefficients f and g in the given SDE (5.1) have the linear forms

$$f(x, i, t) = U_i x \quad \text{and} \quad g(x, i) = (V_{1i}x, \dots, V_{mi}x) \quad (5.50)$$

for $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$, where $U_i, V_{1i}, \dots, V_{mi} \in \mathbb{R}^{d \times d}$. Then

$$\begin{aligned} & 2x^T Q_i f(x, i, t) + \text{trace}[g(x, i)^T Q_i g(x, i)] \\ &= x^T \left(Q_i U_i + U_i^T Q_i + \sum_{k=1}^m V_{ki}^T Q_i V_{ki} \right) x. \end{aligned} \quad (5.51)$$

We hence have $\hat{Q}_i = Q_i U_i + U_i^T Q_i + \sum_{k=1}^m V_{ki}^T Q_i V_{ki}$.

To explain the second step, we observe that condition (5.21) is now equivalent to

$$\hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + \beta_i Q_i F_i G_i G_i^T F_i^T Q_i + (\Delta^2 / \beta_i) I_d + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in \mathbb{S}. \quad (5.52)$$

Since the matrix on the left-hand side of (5.52) is symmetric, its negative definiteness is equivalent to the existence of some $\beta > 0$ such that the quadratic form is bounded above by $-\beta|x|^2$ for all $x \in \mathbb{R}^d$. Hence, (5.21) and (5.52) are equivalent.

Letting $y_i = 1/\beta_i > 0$, we can re-write the matrix inequalities above as

$$\begin{aligned} & \hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + Q_i F_i G_i (y_i I_d)^{-1} (Q_i F_i G_i)^T \\ &+ y_i \Delta^2 I_d + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in \mathbb{S}. \end{aligned} \quad (5.53)$$

But, by the well-known Schur complements (see, e.g., [44, Theorem 2.8 on page 64]), these are equivalent to the following LMIs of G_i and y_i :

$$\begin{pmatrix} Z_i & Q_i F_i G_i \\ G_i^T F_i^T Q_i & -y_i I_d \end{pmatrix} < 0, \quad i \in \mathbb{S}, \quad (5.54)$$

where $Z_i := \hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + y_i \Delta^2 I_d + \sum_{j=1}^N \gamma_{ij} Q_j$. We hence arrive at our second step.

Step 2.5.2. Find a solution $(G_i, y_i)_{i \in \mathbb{S}}$ of the LMIs (5.54) with $G_i \in \mathbb{R}^{l \times d}$ and $y_i > 0$.

These two steps ensure that Assumption 5.3 is satisfied with $A_i = F_i G_i$ and $\beta_i =$

$1/y_i$ ($i \in \mathbb{S}$) as well as

$$\begin{aligned} \beta = & -\max_{i \in \mathbb{S}} \lambda_{\max} \left(\hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i \right. \\ & \left. + Q_i F_i G_i (y_i I_d)^{-1} (Q_i F_i G_i)^T + y_i \Delta^2 I_d + \sum_{j=1}^N \gamma_{ij} Q_j \right). \end{aligned} \quad (5.55)$$

We hence have the following corollary.

Corollary 5.8. *Under Assumption 5.2, find matrices G_i and positive numbers y_i ($i \in \mathbb{S}$) in two steps as described above. Then Theorem 5.5 holds with β being defined by (5.55) and $A_i = F_i G_i$.*

5.3 Stabilisation by Event-triggered Output Injection

5.3.1 Stabilisation

Let us now return to the given SDE (5.1). Instead assuming the state is observable, we now consider the situation where the output $y(t_k) := G_{r(t_k)} x(t_k)$ is observable at discrete times $\{t_k\}_{k \in \mathbb{N}}$, in which $G_i \in \mathbb{R}^{l \times d}$ is known for every $i \in \mathbb{S}$ (in general, $l < d$). In this situation, the event-triggered control should be designed based on $y(t_k)$ but not $x(t_k)$. More precisely, given we only consider linear control functions in this chapter, we need to design N matrices $F_i \in \mathbb{R}^{d \times l}$ so that the event-triggered controlled SDE

$$dx(t) = (f(x(t), r(t), t) + F_{r(t)} Y_{\lfloor t/\tau \rfloor}) dt + g(x(t), r(t), t) dB(t) \quad (5.56)$$

becomes stable, where $Y_0 = y(0)$ and for $k = 1, 2, \dots$,

$$Y_k = \begin{cases} Y_{k-1} & \text{if } y(t_k) \text{ is close to } Y_{k-1}, \\ y(t_k) & \text{otherwise.} \end{cases} \quad (5.57)$$

The control $F_{r(t)} Y_{\lfloor t/\tau \rfloor}$ is called the event-triggered output injection. We can show in the same way as we did in Section 2 that under Assumption 5.2, the controlled SDE (5.56) with any initial data $x(0) = x_0 \in \mathbb{R}^d$ and $r(0) = r_0 \in \mathbb{S}$ has a unique global

solution $x(t)$ such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \geq 0$.

By Definition 5.1, we can show in the same way as we did in Section 2 that $Y_k = (I_d + \tilde{\Delta}(t_k))G_{r(t_k)}x(t_k)$, where $\tilde{\Delta}(t_k) = \text{diag}(\tilde{\Delta}_1(t_k), \dots, \tilde{\Delta}_d(t_k))$ is $\mathcal{F}_{t_k}$ -measurable and $\tilde{\Delta}_i(t_k) \in [-\Delta, \Delta]$ a.s. for every $i \in \mathbb{S}$. The controlled SDE (5.56) can therefore be written in the modified form

$$dx(t) = (f(x(t), r(t), t) + F_{r(t)}(I_d + \tilde{\Delta}(\delta_t))G_{r(\delta_t)}x(\delta_t))dt + g(x(t), r(t), t)dB(t). \quad (5.58)$$

The stability of this controlled SDE is very much related to the robust stability of the auxiliary uncertain SDE

$$dz(t) = (f(z(t), r(t), t) + F_{r(t)}(I_d + \varphi(t))G_{r(t)}z(t))dt + g(z(t), r(t), t)dB(t), \quad (5.59)$$

where $\varphi(t) = \text{diag}(\varphi_1(t), \dots, \varphi_d(t))$ is an $\mathcal{F}_t$ -adapted process such that its every element $\varphi_i(t) \in [-\Delta, \Delta]$ a.s. We impose the following assumption for the mean square exponential stability of this uncertain SDE (see, e.g., [44]).

Assumption 5.9. *There exist positive numbers β, β_i ($i \in \mathbb{S}$) and symmetric positive definite matrices $Q_i \in \mathbb{R}^{d \times d}$ such that*

$$\begin{aligned} & 2x^T Q_i f(x, i, t) + \text{trace}[g(x, i, t)^T Q_i g(x, i, t)] \\ & + x^T \left(Q_i F_i G_i + G_i^T F_i^T Q_i + \beta_i Q_i F_i F_i^T Q_i + (\Delta^2 / \beta_i) G_i^T G_i + \sum_{j=1}^N \gamma_{ij} Q_j \right) x \leq -\beta |x|^2 \end{aligned} \quad (5.60)$$

for all $(x, i, t) \in \mathbb{R}^d \times \mathbb{S} \times \mathbb{R}_+$.

In what follows, we let λ_m and λ_M be the same as defined by (5.22) and set

$$K_F = \max_{i \in \mathbb{S}} \|F_i\|, \quad K_G = \max_{i \in \mathbb{S}} \|G_i\|, \quad K_{QF} = \max_{i \in \mathbb{S}} \|Q_i F_i\|, \quad D_G = \max_{i, j \in \mathbb{S}} \|G_i - G_j\|^2. \quad (5.61)$$

We will see that the stabilisation by the event-triggered output injection is more com-

plicated than the stabilisation by the event-triggered state feedback. Let us present three lemmas before we form our main result in this section.

Lemma 5.10. *Set $\bar{\gamma} = \max_{i \in \mathbb{S}}(-\gamma_{ii})$. Then*

$$\mathbb{E}(\|G_{r(t)} - G_{r(\delta_t)}\|^2) \leq D_G(1 - e^{-\bar{\gamma}\tau}), \quad \forall t \geq 0. \quad (5.62)$$

Proof. Let $t \geq 0$ be arbitrary and $k = \lfloor t/\tau \rfloor$ so $t_k = \delta_t \leq t < t_{k+1}$. Given $r(t_k) = i$, define the stopping time

$$\eta_i = \inf\{s \geq t_k : r(s) \neq i\}.$$

It is well known (see, e.g., [2]) that $\eta_i - t_k$ has the exponential distribution with parameter $-\gamma_{ii}$. Hence

$$\begin{aligned} \mathbb{P}(r(s) \neq i \text{ for some } s \in [t_k, t_{k+1}] | r(t_k) = i) &= \mathbb{P}(\eta_i - t_k \leq \tau | r(t_k) = i) \\ &= \int_0^\tau (-\gamma_{ii}) e^{\gamma_{ii}s} ds = 1 - e^{\gamma_{ii}\tau} \leq 1 - e^{-\bar{\gamma}\tau}. \end{aligned}$$

Consequently, $\mathbb{P}(r(t) \neq i | r(t_k) = i) \leq 1 - e^{-\bar{\gamma}\tau}$. Therefore

$$\begin{aligned} \mathbb{E}(\|G_{r(t)} - G_{r(\delta_t)}\|^2) &= \sum_{i \in \mathbb{S}} \mathbb{P}(r(t_k) = i) \mathbb{E}(\|G_{r(t)} - G_i\|^2 | r(t_k) = i) \\ &\leq \sum_{i \in \mathbb{S}} \mathbb{P}(r(t_k) = i) D_G \mathbb{P}(r(t) \neq i | r(t_k) = i) \\ &\leq D_G(1 - e^{-\bar{\gamma}\tau}) \end{aligned} \quad (5.63)$$

as required. $\square$

Lemma 5.11. *Let Assumption 5.2 hold. If $\tau < 1/[\sqrt{6}K_F K_G(1 + \Delta)]$, then the solution of the controlled SDE (5.56) has the property that*

$$\mathbb{E}(|x(t) - x(\delta_t)|^2) \leq H_3(\tau) \mathbb{E}|x(t)|^2 + H_4(\tau) \int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds \quad (5.64)$$

for all $t \geq 0$, where

$$H_3(\tau) = \frac{6[\tau K_F K_G(1 + \Delta)]^2}{1 - 6[\tau K_F K_G(1 + \Delta)]^2} \quad \text{and} \quad H_4(\tau) = \frac{3(\tau K_1^2 + K_2^2)}{1 - 6[\tau K_F K_G(1 + \Delta)]^2}. \quad (5.65)$$

Proof. Let $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}_+$. By Assumption 5.2, it follows easily from (5.58) that

$$\begin{aligned} \mathbb{E}(|x(t) - x(\delta_t)|^2) &\leq 3(\tau K_1^2 + K_2^2) \int_{t_k}^t \mathbb{E}|x(s)|^2 ds + 3[\tau K_F K_G(1 + \Delta)]^2 \mathbb{E}|x(t_k)|^2 \\ &\leq 3(\tau K_1^2 + K_2^2) \int_{t_k}^t \mathbb{E}|x(s)|^2 ds + 6[\tau K_F K_G(1 + \Delta)]^2 (\mathbb{E}|x(t)|^2 + \mathbb{E}|x(t) - x(\delta_t)|^2). \end{aligned}$$

This implies the required assertion 5.64 as $6[\tau K_F K_G(1 + \Delta)]^2 < 1$. $\square$

Lemma 5.12. *Let Assumption 5.2 hold. If $\tau < 1/(K_F K_G(1 + \Delta))$, then the solution of the controlled SDE (5.56) has the property that*

$$\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) \leq H_5(\tau) \mathbb{E}|x(t)|^2 + H_6(\tau) \int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds \quad (5.66)$$

for all $t \geq 0$, where

$$H_5(\tau) = \frac{\tau[0.5K_1 + K_F K_G(1 + \Delta)] + 0.5K_2\sqrt{\tau}}{1 - \tau K_F K_G(1 + \Delta)} \quad (5.67)$$

and

$$H_6(\tau) = \frac{0.5(K_1 + K_2/\sqrt{\tau})}{1 - \tau K_F K_G(1 + \Delta)}. \quad (5.68)$$

Proof. For $t \in [t_k, t_{k+1})$ for some $k \in \mathbb{N}_+$, by Assumption 5.2 and property (5.18), it follows from (5.58) that

$$\begin{aligned} \mathbb{E}(|x(t)||x(t) - x(\delta_t)|) &= \mathbb{E}(|x(t)||x(t) - x(t_k)|) \\ &\leq \mathbb{E}\left(|x(t)| \int_{t_k}^t (K_1|x(s)| + K_F K_G(1 + \Delta)|x(t_k)|) ds + |x(t)||M_k(t)|\right), \end{aligned} \quad (5.69)$$

where $M_k(t) = \int_{t_k}^t g(x(s), r(s), s) dB(s)$. We can then prove the lemma in the same way as Lemma 5.4 was proved. $\square$

Theorem 5.13. *Let Assumptions 5.2 and 5.9 hold. Choose a free parameter $\theta \in (0, \beta/[K_{QF}(1 + \Delta)])$ and fix it. Set $\sigma_1 = 2K_{QF}K_G(1 + \Delta)$, $\sigma_2 = \sigma_2(\theta) = K_{QF}(1 + \Delta)\theta$, $\sigma_3 = \sigma_3(\theta) = K_{QF}(1 + \Delta)/\theta$ (so $\beta > \sigma_2$). Let $H_3(\tau), \dots, H_6(\tau)$ be the same as defined in Lemmas 5.11 and 5.12. Set $\bar{\gamma} = \max_{i \in \mathbb{S}}(-\gamma_{ii})$ and define*

$$H_7(\tau) = \beta - \sigma_2 - \sigma_1 H_5(\tau) - 2\sigma_3 D_G(1 - e^{-\bar{\gamma}\tau})[1 + H_3(\tau)], \quad (5.70)$$

$$H_8(\tau) = \sigma_1 H_6(\tau) + 2\sigma_3 D_G(1 - e^{-\bar{\gamma}\tau})H_4(\tau). \quad (5.71)$$

Let $\tau^* \in (0, [\sqrt{6}K_F K_G(1 + \Delta)]^{-1})$ be the unique root to the following equation of τ :

$$H_7(\tau) - \tau H_8(\tau) = 0. \quad (5.72)$$

If $\tau < \tau^*$, then the solution of the controlled SDE (5.56) has the properties that

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^2) \leq -\bar{\varepsilon}, \quad (5.73)$$

and

$$\limsup_{t \rightarrow \infty} \frac{1}{t} \log(|x(t)|) \leq -\frac{\bar{\varepsilon}}{2} \quad a.s., \quad (5.74)$$

where $\bar{\varepsilon} > 0$ is the unique root to the equation

$$H_7(\tau) - \bar{\varepsilon}\lambda_M - \tau e^{\bar{\varepsilon}\tau} H_8(\tau) = 0. \quad (5.75)$$

Proof. Noting that $H_7(\tau) - \tau H_8(\tau)$ is strictly decreasing and continuous for

$$\tau \in \left(0, [\sqrt{6}K_F K_G(1 + \Delta)]^{-1}\right),$$

with

$$\lim_{\tau \rightarrow 0^+} [H_7(\tau) - \tau H_8(\tau)] = \beta - \sigma_2 > 0$$

and

$$\lim_{\tau \uparrow [\sqrt{6}K_F K_G(1 + \Delta)]^{-1}} [H_7(\tau) - \tau H_8(\tau)] = -\infty,$$

there exists a unique

$$\tau^* \in \left(0, [\sqrt{6K_F K_G(1 + \Delta)}]^{-1}\right)$$

satisfying

$$H_7(\tau^*) - \tau^* H_8(\tau^*) = 0.$$

Applying the generalized Itô formula (see, e.g., [34,44]) to $V(x(t), r(t)) := x^T(t)Q_{r(t)}x(t)$, we obtain

$$dV(x(t), r(t)) = J_2(t)dt + dM(t) \quad (5.76)$$

for $t \geq 0$, where $M(t)$ is a continuous martingale with $M(0) = 0$ and

$$\begin{aligned} J_2(t) &:= 2x^T(t)Q_{r(t)}[f(x(t), r(t), t) + F_{r(t)}(I_d + \hat{\Delta}(\delta_t))G_{r(\delta_t)}x(\delta_t)] \\ &+ \text{trace}[g^T(x(t), r(t), t)Q_{r(t)}g(x(t), r(t), t)] + \sum_{j=1}^N \gamma_{r(t),j} x^T(t)Q_j x(t). \end{aligned} \quad (5.77)$$

But

$$\begin{aligned} &2x^T(t)Q_{r(t)}F_{r(t)}(I_d + \hat{\Delta}(\delta_t))G_{r(\delta_t)}x(\delta_t) \\ &= 2x^T(t)Q_{r(t)}F_{r(t)}(I_d + \hat{\Delta}(\delta_t))G_{r(t)}x(t) + J_3(t) \\ &\leq x^T(t) \left(Q_{r(t)}F_{r(t)}G_{r(t)} + G_{r(t)}^T F_{r(t)}^T Q_{r(t)} + \beta_{r(t)} Q_{r(t)}F_{r(t)}F_{r(t)}^T Q_{r(t)} \right. \\ &\quad \left. + (\Delta^2/\beta_{r(t)})G_{r(t)}^T G_{r(t)} \right) x(t) + J_3(t), \end{aligned} \quad (5.78)$$

where

$$J_3(t) = -2x^T(t)Q_{r(t)}F_{r(t)}(I_d + \hat{\Delta}(\delta_t))[G_{r(t)}x(t) - G_{r(\delta_t)}x(\delta_t)].$$

Substituting this into (5.77) and then applying condition (5.60), we get

$$J_2(t) \leq -\beta|x(t)|^2 + J_3(t). \quad (5.79)$$

But

$$\begin{aligned}
|J_3(t)| &\leq 2K_{QF}(1 + \Delta)|x(t)||G_{r(t)}(x(t) - x(\delta_t)) + (G_{r(t)} - G_{r(\delta_t)})x(\delta_t)| \\
&\leq 2K_{QF}K_G(1 + \Delta)|x(t)||x(t) - x(\delta_t)| + 2K_{QF}(1 + \Delta)|x(t)||x(\delta_t)||G_{r(t)} - G_{r(\delta_t)}| \\
&\leq 2K_{QF}K_G(1 + \Delta)|x(t)||x(t) - x(\delta_t)| \\
&\quad + K_{QF}(1 + \Delta)\left(\theta\|x(t)\|^2 + \frac{1}{\theta}\|x(\delta_t)\|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2\right) \\
&= \sigma_1|x(t)||x(t) - x(\delta_t)| + \sigma_2|x(t)|^2 + \sigma_3|x(\delta_t)|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2,
\end{aligned}$$

where $\sigma_1, \sigma_2, \sigma_3$ have been defined in the statement of the theorem. We hence have

$$\begin{aligned}
\mathbb{E}J_2(t) &\leq -(\beta - \sigma_2)\mathbb{E}|x(t)|^2 + \sigma_1\mathbb{E}(|x(t)||x(t) - x(\delta_t)|) \\
&\quad + \sigma_3\mathbb{E}(|x(\delta_t)|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2).
\end{aligned} \tag{5.80}$$

Noting that $x(\delta_t)$ is independent of $G_{r(t)} - G_{r(\delta_t)}$ while applying Lemmas 5.10 and 5.11, we derive that

$$\begin{aligned}
\mathbb{E}(|x(\delta_t)|^2\|G_{r(t)} - G_{r(\delta_t)}\|^2) &= \mathbb{E}|x(\delta_t)|^2\mathbb{E}\|G_{r(t)} - G_{r(\delta_t)}\|^2 \\
&\leq 2(\mathbb{E}|x(t)|^2 + \mathbb{E}|x(t) - x(\delta_t)|^2)\mathbb{E}\|G_{r(t)} - G_{r(\delta_t)}\|^2 \\
&\leq 2D_G(1 - e^{-\bar{\gamma}\tau})\left([1 + H_3(\tau)]\mathbb{E}|x(t)|^2 + H_4(\tau)\int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds\right).
\end{aligned} \tag{5.81}$$

Substituting this into (5.80) and applying Lemma 5.12, we get

$$\mathbb{E}J_2(t) \leq -H_7(\tau)\mathbb{E}|x(t)|^2 + H_8(\tau)\int_{\delta_t}^t \mathbb{E}|x(s)|^2 ds \tag{5.82}$$

for all $t \geq 0$. An application of the generalized Itô formula to $e^{\bar{\varepsilon}t}V(x(t), r(t))$ gives

$$e^{\bar{\varepsilon}t}\mathbb{E}V(x(t), r(t)) = V(x_0, r_0) + \int_0^t e^{\bar{\varepsilon}s}\left(\bar{\varepsilon}\mathbb{E}V(x(s), r(s)) + \mathbb{E}J_2(s)\right)ds. \tag{5.83}$$

This, together with (5.82), implies

$$\begin{aligned} \lambda_m e^{\bar{\varepsilon}t} \mathbb{E}|x(t)|^2 &\leq \lambda_M |x_0|^2 - \int_0^t e^{\bar{\varepsilon}s} [H_7(\tau) - \bar{\varepsilon}\lambda_M] \mathbb{E}|x(s)|^2 ds \\ &\quad + H_8(\tau) \int_0^t e^{\bar{\varepsilon}s} \left(\int_{\delta_s}^s \mathbb{E}|x(v)|^2 dv \right) ds. \end{aligned} \quad (5.84)$$

But, recalling (5.42), we have

$$\int_0^t e^{\bar{\varepsilon}s} \left(\int_{\delta_s}^s \mathbb{E}|x(v)|^2 dv \right) ds \leq \tau e^{\bar{\varepsilon}\tau} \int_0^t e^{\varepsilon v} \mathbb{E}|x(v)|^2 dv. \quad (5.85)$$

Therefore

$$\lambda_m e^{\bar{\varepsilon}t} \mathbb{E}|x(t)|^2 \leq \lambda_M |x_0|^2 - \int_0^t e^{\bar{\varepsilon}s} [H_7(\tau) - \bar{\varepsilon}\lambda_M - \tau e^{\bar{\varepsilon}\tau} H_8(\tau)] \mathbb{E}|x(s)|^2 ds = \lambda_M |x_0|^2, \quad (5.86)$$

where (5.75) has been used. This implies the first assertion (5.73). The second assertion (5.74) can be proved in the same way as Theorem 5.5 was proved. The proof is complete. $\square$

Remark 5.14. In Theorem 5.13, we introduce a free parameter $\theta \in (0, \beta/[K_{QF}(1 + \Delta)])$. We observe that τ^* determined there depends on θ . In application, it is better to have a larger τ^* . At this moment, we do not know how to choose a better θ so that the corresponding τ^* is larger. We will tackle this open problem in the future.

5.3.2 Design of matrices F_i 's

Let us now explain how to design F_i 's in two steps.

Step 3.2.1. The same as Step 2.5.1.

Once this first step is done, we observe that condition (5.60) is now equivalent to

$$\hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + \beta_i Q_i F_i F_i^T Q_i + (\Delta^2 / \beta_i) G_i^T G_i + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in \mathbb{S}. \quad (5.87)$$

By setting $y_i = 1/\beta_i > 0$, the matrix inequalities above becomes

$$\begin{aligned} & \hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + Q_i F_i (y_i I_d)^{-1} (Q_i F_i)^T \\ & + y_i \Delta^2 G_i^T G_i + \sum_{j=1}^N \gamma_{ij} Q_j < 0, \quad i \in \mathbb{S}. \end{aligned} \quad (5.88)$$

By the well-known Schur complements, these are equivalent to the following LMIs of F_i and y_i :

$$\begin{pmatrix} W_i & Q_i F_i \\ F_i^T Q_i & -y_i I_d \end{pmatrix} < 0, \quad i \in \mathbb{S}, \quad (5.89)$$

where $W_i := \hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + y_i \Delta^2 G_i^T G_i + \sum_{j=1}^N \gamma_{ij} Q_j$. We hence arrive at our second step.

Step 3.1.2. Find a solution $(F_i, y_i)_{i \in \mathbb{S}}$ of the LMIs (5.89) with $F_i \in \mathbb{R}^{d \times l}$ and $y_i > 0$.

These two steps ensure that Assumption 5.9 is satisfied with $\beta_i = 1/y_i$ ($i \in \mathbb{S}$) as well as

$$\begin{aligned} \beta = - \max_{i \in \mathbb{S}} \lambda_{\max} & \left(\hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i \right. \\ & \left. + Q_i F_i (y_i I_d)^{-1} (Q_i F_i)^T + y_i \Delta^2 G_i^T G_i + \sum_{j=1}^N \gamma_{ij} Q_j \right). \end{aligned} \quad (5.90)$$

We hence have the following corollary.

Corollary 5.15. *Under Assumption 5.2, find matrices G_i and positive numbers y_i ($i \in \mathbb{S}$) in two steps as described above. Then Theorem 5.13 holds with β being defined by (5.90).*

5.4 Examples

To illustrate our theoretical results, we will discuss a couple of examples.

Example 5.16. Let us consider the linear hybrid SDE

$$dx(t) = U_{r(t)}x(t)dt + V_{r(t)}x(t)dB(t) \quad (5.91)$$

on $t \geq 0$. Here $x(t) = (x_1(t), x_2(t))^T$, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix},$$

and the system matrices are

$$U_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \quad U_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The computer simulation (Figure 4.1) shows this hybrid SDE is not almost surely exponentially stable and hence it is not exponentially stable in mean square either.

Let us now design an event-triggered feedback control to stabilize the system. Let the event-triggered parameter $\Delta = 0.1$. Assume that the controlled SDE has the form

$$dx(t) = (U_{r(t)}x(t) + F_{r(t)}G_{r(t)}X_{\lfloor t/\tau \rfloor})dt + V_{r(t)}x(t)dB(t) \quad (5.92)$$

on $t \geq 0$, which is equivalent to

$$dx(t) = (U_{r(t)}x(t) + F_{r(t)}G_{r(t)}(I_2 + \hat{\Delta}(\delta_t))x(\delta_t))dt + V_{r(t)}x(t)dB(t). \quad (5.93)$$

Assume

$$F_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

but we need to determine G_1 and G_2 in $R^{1 \times 2}$ and then make sure τ is sufficiently

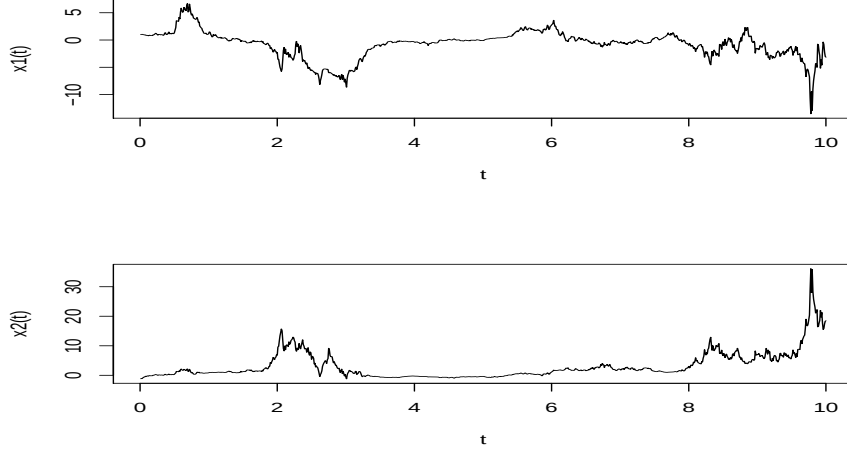


Figure 4.1: The computer simulation of the sample paths of the solution of the SDE (5.91) using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (1, -1)^T$, $r(0) = 1$.

small for the controlled SDE (5.92) to be exponentially stable in mean square and almost surely as well. It is easy to see that Assumption 5.2 holds with $K_1 = 5.236$ and $K_2 = \sqrt{2}$. To apply Corollary 5.8, we apply two steps in Section 4 to determine G_1 and G_2 . In Step 1, we choose $Q_1 = Q_2 = I_2$. Then $\hat{Q}_i = U_i + U_i^T + V_i^T V_i$, namely

$$\hat{Q}_1 = \begin{bmatrix} 4 & 0 \\ 0 & -8 \end{bmatrix}, \quad \hat{Q}_2 = \begin{bmatrix} -8 & 0 \\ 0 & 4 \end{bmatrix}.$$

It is easy to verify that

$$G_1 = (-6, 0), \quad G_2 = (0, -6), \quad y_1 = y_2 = 60$$

form a solution of the LMIs (5.54). By (5.22) and (5.55), we compute $K_A = K_{Q_A} = 6$ and $\beta = 6.8$. Consequently, equation (5.30) becomes

$$6.8 = 13.2(11.836\tau + \sqrt{2\tau})/(1 - 6.6\tau),$$

which has its unique solution $\tau^* = 0.02051$ in $(0, 0.1515152)$. By Corollary 5.8, if we

set G_i as above and make sure that $\tau < 0.02051$, then the controlled SDE (5.92) is exponentially stable in mean square and almost surely as well. We perform a computer simulation (Figure 4.2) with $\tau = 0.01$. It supports the theoretical results clearly.

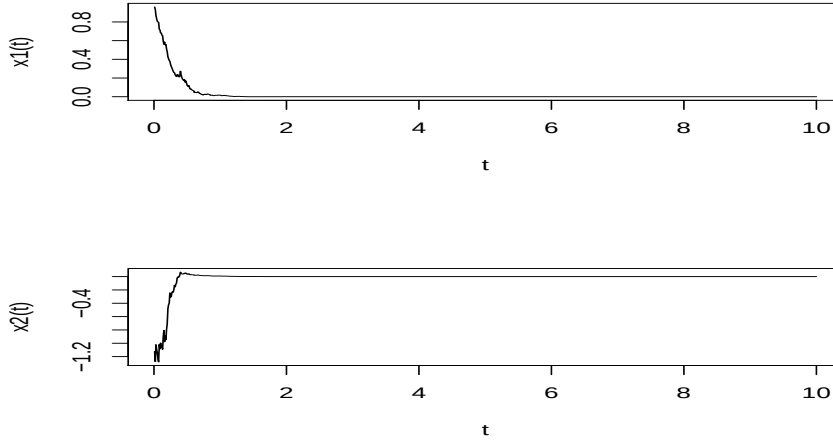


Figure 4.2: The computer simulation of the sample paths of the solution of the controlled SDE (5.92) with $\tau = 0.01$ using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (1, -1)^T$, $r(0) = 1$.

Example 5.17. Let us now apply our theory to the stochastic Hopfield neural networks (see, e.g., [5, 27, 28, 57, 61, 62]). For illustration, it is sufficient to consider the following Hopfield neural network with two neurons

$$dx(t) = [-D_{r(t)}x(t) + U_{r(t)}\varphi(x(t))]dt + V_{r(t)}x(t)dB(t) \quad (5.94)$$

on $t \geq 0$. Here $x(t) = (x_1(t), x_2(t))^T$, $B(t)$ is a scalar Brownian motion, $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 3 & -3 \end{bmatrix},$$

$\varphi(x) = (\tanh(x_1), \tanh(x_2))^T$ and the system matrices are $D_1 = \text{diag}(0.1, 0.2)$, $D_2 =$



Figure 4.3: Hopfield neural network with two neurons. Left: mode 1. Right: mode 2.

$\text{diag}(0.2, 0.1)$,

$$U_1 = \begin{bmatrix} 0.5 & -0.1 \\ 0.1 & 0.5 \end{bmatrix}, \quad U_2 = \begin{bmatrix} 0.5 & 0.1 \\ -0.1 & 0.5 \end{bmatrix},$$

$$V_1 = \begin{bmatrix} -0.1 & 0.1 \\ 0.1 & 0.1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0.1 & -0.1 \\ -0.1 & -0.1 \end{bmatrix}.$$

Matrices U_1 and U_2 consist of the transfer coefficients between two neurons and themselves in mode 1 and 2, respectively, and they are illustrated in Figure 4.3. The computer simulation (Figure 4.4) shows this hybrid SDE is not almost surely exponentially stable and hence it is not exponentially stable in mean square either.

We now design an event-triggered feedback control to stabilize the system. Let the event-triggered parameter $\Delta = 0.1$. Consider the case where the system is observable only in mode 1 but not in mode 2 so we could only use a feedback control in mode 1 (namely we have to set the control function to be 0 in mode 2). Assume the controlled SDE has the form

$$dx(t) = (-D_{r(t)}x(t) + U_{r(t)}\varphi(x(t)) + A_{r(t)}X_{\lfloor t/\tau \rfloor})dt + V_{r(t)}x(t)dB(t) \quad (5.95)$$

on $t \geq 0$, where $A_2 = 0$ and $A_1 = \text{diag}(-2.1, -2)$. It is easy to see that Assumption 5.2 holds with $K_1 = 0.7099$ and $K_2 = 0.1414$. To verify Assumption 5.6, we observe that

$$\begin{aligned} & \text{the left hand side of (5.46) when } i = 1 \\ & \leq -0.2x_1^2 - 0.4x_2^2 + 1.4398|x|^2 - 4.2x_1^2 - 4x_2^2 \leq -2.9602|x|^2, \end{aligned}$$

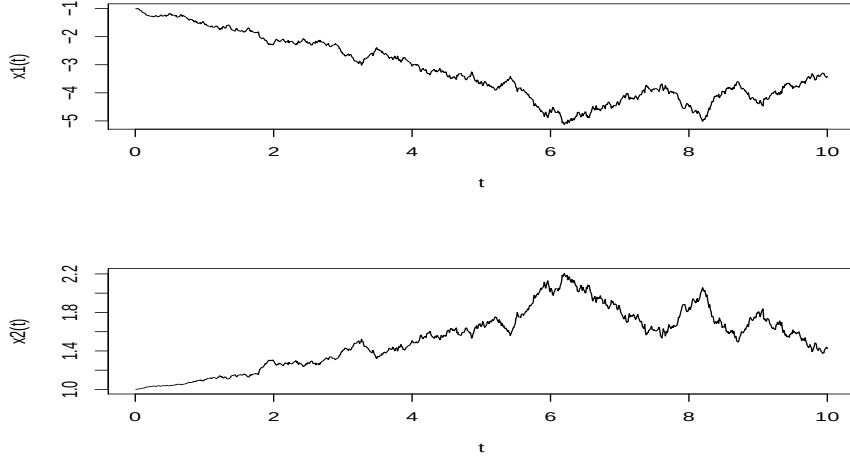


Figure 4.4: The computer simulation of the sample paths of the solution of the SDE (5.94) using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (-1, 1)^T$, $r(0) = 1$.

while

the left hand side of (5.46) when $i = 2$

$$\leq -0.4x_1^2 - 0.2x_2^2 + 1.4398|x|^2 \leq 1.0398|x|^2.$$

We hence have that $b_1 = -2.9602$, $b_1 + 2\Delta\|A_1\| = -2.5402$ while $b_2 = 1.0398$, $b_2 + 2\Delta\|A_2\| = 1.0398$. Consequently, the matrix $\mathcal{A}$ defined by (5.45) becomes

$$\mathcal{A} = \begin{pmatrix} 3.5402 & -1 \\ -3 & 1.0602 \end{pmatrix}$$

which is a non-singular M-matrix. That is, Assumption 5.6 is satisfied. To apply Theorem 5.7, we compute $q_1 = 2.734827$ and $q_2 = 8.681834$ by (5.48). Then $\beta = 1$, $K_A = 2.1$ and $K_{QA} = 5.743137$. Consequently, equation (5.30) becomes

$$1 = 12.6349(3.0199\tau + 0.1414\sqrt{\tau})/(1 - 4.62\tau),$$

which has its unique solution $\tau^* = 0.0178$ in $(0, 0.2164502)$. By Theorem 5.7, if we set

$A_1 = \text{diag}(-2.1, -2)$ and $A_2 = 0$ and make sure that $\tau < 0.0178$, then the controlled SDE (5.95) is exponentially stable in mean square and almost surely as well. We perform a computer simulation (Figure 4.5) with $\tau = 0.01$. It supports the theoretical once again.

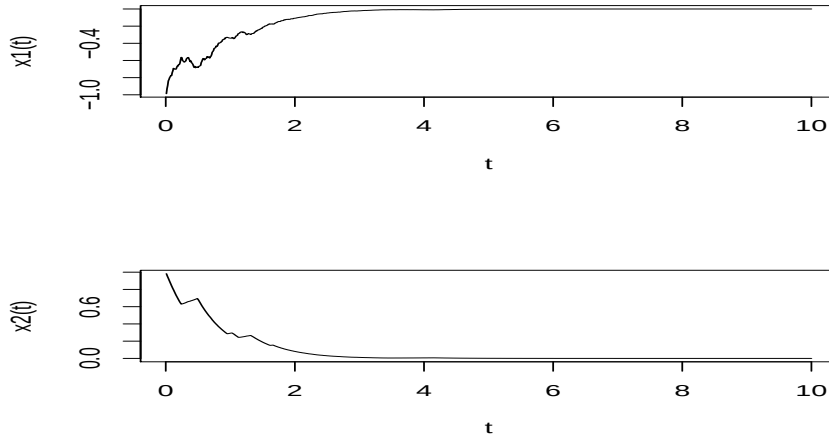


Figure 4.5: The computer simulation of the sample paths of the solution of the controlled SDE (5.95) using the Euler–Maruyama method with step size 10^{-3} and initial data $x(0) = (-1, 1)^T$, $r(0) = 1$.

5.5 Summary

In this chapter, we have investigated the stabilisation problem for hybrid stochastic differential equations by developing an event-triggered control strategy based on trajectories of discrete-time state or output observations. Unlike most existing event-triggered control approaches for stochastic systems, which rely on criteria involving the mathematical expectation of the system state, the proposed strategy is constructed directly from a single trajectory of discrete-time observations, making it more suitable for practical implementation.

Two different observation scenarios have been considered. For systems with observable states, an event-triggered state-feedback control has been developed, while for systems with observable outputs only, an event-triggered output injection strategy

has been established. For both cases, a novel analytical framework has been proposed by relating the stability of the event-triggered controlled system to that of a corresponding uncertain stochastic system. Based on this connection, sufficient conditions guaranteeing exponential stability have been derived.

The theoretical results have been illustrated by numerical examples, demonstrating the effectiveness of the proposed event-triggered control strategies. The stabilisation theory developed in this chapter complements the numerical approximation theory established in the previous chapters and provides a control perspective on the study of hybrid stochastic systems.

Chapter 6

Conclusion and Future Work

6.1 Conclusions

This thesis is concerned with two related aspects of hybrid stochastic systems: the numerical approximation of hybrid stochastic functional differential equations with infinite time delay and the stabilisation of hybrid stochastic differential equations by event-triggered control. Although these two topics address different mathematical problems, they are both motivated by the need to analyse and control hybrid stochastic systems arising in practical applications. Numerical approximation provides reliable computational methods for analysing complex stochastic dynamics, while feedback control aims to modify system behaviour to achieve desired stability properties.

The first part of this thesis considered the numerical approximation of hybrid stochastic functional differential equations with infinite time delay. Infinite memory and random switching substantially increase the mathematical complexity of such systems, making the development of numerical methods considerably more difficult than for stochastic differential equations with finite delay. Existing convergence theory was largely restricted to globally Lipschitz coefficients, limiting its applicability to many nonlinear stochastic models.

To overcome these limitations, Chapter 3 developed a truncated Euler–Maruyama approximation framework under the local Lipschitz and Khasminskii-type conditions. By combining the infinite-delay approximation technique with the truncated Euler–

Maruyama method, strong convergence was established in the general L^q setting for $q \geq 2$, and convergence rate estimates were derived under suitable additional assumptions. These results substantially broadened the applicability of numerical approximation methods for hybrid stochastic functional differential equations with infinite time delay.

Chapter 4 continued this study by replacing the classical Khasminskii-type condition with a dissipative one. Compared with the framework developed in Chapter 3, the new condition permits a wider class of nonlinear stochastic systems with super-linear coefficients while providing stronger moment estimates for both the exact and numerical solutions. This extension enabled the convergence theory to be extended to the previously unexplored range $q \in [1, 2)$, together with corresponding convergence rate estimates. Consequently, the numerical approximation theory established in Chapters 3 and 4 provides a unified convergence framework for the truncated Euler–Maruyama method over the entire range $q \in [1, p)$.

The second part of this thesis shifted the focus from numerical approximation to stochastic stabilisation. Rather than employing continuous-time feedback or event-triggered mechanisms based on mathematical expectations, Chapter 5 developed a new event-triggered control framework constructed directly from trajectories of discrete-time observations. A connection between the event-triggered controlled system and a corresponding uncertain stochastic system was established, allowing stability analysis to be carried out through the uncertain system. Both state-feedback and output-feedback problems were investigated, and sufficient conditions guaranteeing exponential mean-square stability were obtained. These results provide an effective event-triggered control strategy that reduces communication and computational costs while preserving the desired stability properties.

Taken together, the results presented in this thesis establish a theoretical framework for two important aspects of hybrid stochastic systems. The convergence theory developed for the truncated Euler–Maruyama method considerably enlarges the class of hybrid stochastic functional differential equations with infinite time delay for which rigorous numerical analysis can be carried out, while the event-triggered control frame-

work provides an efficient approach to the stabilisation of hybrid stochastic differential equations under discrete-time observations. The methods developed in this thesis may also provide useful techniques for the study of more general hybrid stochastic systems.

6.2 Future work

The work presented in this thesis leaves several questions for further investigation. One possible direction is to extend the numerical approximation framework developed in Chapters 3 and 4 to more general classes of hybrid stochastic functional differential equations. For example, stochastic systems driven by Lévy processes, fractional Brownian motion, or semi-Markov switching arise naturally in many applications and require substantially different analytical techniques. Extending the convergence theory developed in this thesis to such systems remains an interesting direction for future research.

Further improvements may also be made in the design of numerical methods for hybrid stochastic functional differential equations with infinite time delay. The truncated Euler–Maruyama method provides an effective approximation scheme under relatively weak assumptions. Higher-order methods, adaptive step-size strategies, and other efficient numerical algorithms may further improve computational accuracy while preserving strong convergence properties. It remains to be investigated whether the convergence theory established in this thesis can be extended to these numerical methods.

Another interesting direction is to investigate the partially truncated Euler–Maruyama method [10, 11] for hybrid stochastic functional differential equations with infinite time delay. Although the method has been successfully developed for stochastic differential equations, its extension to hybrid SFDEs with infinite time delay remains open. In particular, it would be of interest to establish the corresponding strong convergence theory and investigate whether the numerical solutions preserve the stability and boundedness properties of the underlying systems.

From the control point of view, the event-triggered framework developed in Chapter 5 may be extended in several directions. Possible extensions include hybrid stochastic systems with time delays, communication constraints, or network-induced delays.

Chapter 6. Conclusion and Future Work

Adaptive event-triggered mechanisms and self-triggered control strategies also deserve further investigation, since they may further reduce communication and computational costs while maintaining the required stability properties.

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