On the asymptotic enumeration of monotone grid classes of permutations

PhD Thesis

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The work in this thesis is based on [3] and is the result of the author's work in collaboration with David Bevan.

Abstract

We find a procedure to asymptotically enumerate monotone grid classes of permutations. This is then applied to compute the asymptotic number of permutations in any connected one-corner L-shaped, T-shaped, and X-shaped class. We start by looking at the simplest case of the skinny classes with a single row or column.

Finding the exact enumeration of grid classes is hard, so our goal is to find the asymptotic enumeration. Our strategy consists of enumerating the gridded permutations, finding the asymptotic distribution of points between the cells in a typical large gridded permutation, and determining in detail the ways in which a typical large M-gridded permutation must be structured so that its underlying permutation σ has exactly ℓ distinct M-griddings. We then combine the previous steps to calculate for each $\ell \geq 1$, the asymptotic probability that σ has exactly ℓ distinct M-griddings. Then we deduce the asymptotic enumeration of the number of permutations in the class.

Acknowledgments

First and foremost, I would like to say Alhamduli Allah who inspires me and is always beside me during my study and in all of my life.

I am grateful to my supervisor Dr David Bevan who has always supported me during my PhD journey. I remember the moments when I got new results and I ran to his office interrupting him to tell him about them, and he kept encouraging me. His office door is always open for me and for every other student. I am really happy to have a such smart supportive advisor in my life!

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I cannot forget all my sisters, brothers, my husband, and my sons. They were beside me in all of my moments, the moments of sadness before the happiness!

I am lucky to have such a great supervisor, and such an amazing supportive family!

Contents

Table of notation

Symbol	Meaning
σ	a permutation
$\text{Grid}(M)$	the monotone grid class with gridding matrix M
$\text{Grid}_n(M)$	the set of permutations of length n in $\text{Grid}(M)$
$\text{Grid}^\#(M)$	the gridded class of M-gridded permutations
$\text{gr}(\text{Grid}(M))$	the growth rate of $\text{Grid}(M)$
P_ℓ	$\lim_{n\to\infty} \mathbb{P}\left[\sigma \text{ has } \ell\ M\text{-griddings} : \sigma^\# \in \text{Grid}^\#_n(M)\right]$.
$\theta^\#(n)$	the subexponential term in the asymptotics of a gridded class
$\theta(n)$	the subexponential term in the asymptotics of a grid class
$\sigma^\#$	a gridded permutation
κ_M	the correction factor for $\text{Grid}(M)$
g	the exponential growth rate of the class
Λ	a peak
p	the number of non-corner peaks in a grid class
$\text{Grid}^\#_A(M)$	the set of gridded permutations in $\text{Grid}^\#(M)$ in which the number of points in each cell is specified by matrix A
$\Gamma = (\gamma_{i,j})$	an M-distribution matrix
Γ_M	the unique maximal M-distribution matrix
$\Gamma_{\sigma^\#}$	the M-distribution matrix recording the proportion of the points of $\sigma^\#$ in each cell.

Table of notation continued

Symbol	Meaning
$c+1, r+1$	the number of columns and the number of rows in a connected one-corner class
α	the proportion of points in the corner cell of a connected one-corner class
β	the proportion of points in a row cell of a connected one-corner class
γ	the proportion of points in a column cell of a connected one-corner class
λ	$\frac{\beta}{\alpha+c\beta} = \frac{\gamma}{\alpha+r\gamma}$
q	$\sqrt{(r+c+1)^2 - 4cr}$
τ	a corner type
$\kappa(\tau)$	the correction factor for the corner type τ
τ^{R}	a 90° rotation of corner type τ
C, C_R, C_B	the corner cell, the cell to the right of it, and the cell below it, respectively
x	the letter representing a point in the corner cell
y	the letter representing a point in C_R
z	the letter representing a point in any row cell except C_R

Chapter 1

Introduction

In this chapter we give the main definitions we need. So, in Section 1.1 we define the basics like a permutation and a grid class. In Section 1.2 we briefly consider previous work on grid classes, and in the last section, Section 1.3 we define our goal which is determining the asymptotic enumeration of a monotone grid class.

1.1 Definitions

In this section we define permutations, monotone grid classes, M-griddings, and gridded classes.

1.1.1 Permutations

A *permutation* σ of length n is an arrangement of the numbers $1, 2, ..., n$ for some positive n. Usually, permutations can be considered graphically. In the Euclidean plane a permutation $\sigma = \sigma_1 ... \sigma_n$ can be plotted by the set of ordered pairs (i, σ_i). For example Figure 1.1 shows the plot of the permutation 573614892.

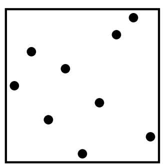


Figure 1.1: The plot of the permutation 573614892

For standard definitions concerning permutations, see [9].

1.1.2 Permutation classes

A permutation τ is said to be *contained* in another permutation σ if σ has a subsequence (not necessarily contiguous) whose terms have the same relative order as τ. From the graphical perspective, σ contains τ if the plot of τ results from erasing zero or more points from the plot of σ and then rescaling the axes appropriately. For example, 573614892 contains 1324 because the subsequence 3649 (among others) is ordered in the same way as 1324. See Figure 1.1.

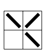
If σ does not contain τ, we say that σ *avoids* τ. For example, 573614892 avoids 1423 since it has no subsequence ordered in the same way as 1423.

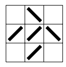
A *permutation class* is a set of permutations closed under the containment relation. That is, $\mathcal{C}$ is a permutation class if and only if $\sigma \in \mathcal{C}$ implies that $\tau \in \mathcal{C}$ for every permutation τ contained in σ. Any permutation class can be defined by the minimal set of permutations that it avoids. For a thorough introduction to permutation classes, see [19].

In this thesis we investigate a special family of permutation classes called monotone grid classes.

1.1.3 Grid(M) and M-gridding

The *monotone grid class* Grid(M) is defined by a matrix M, all of whose entries are in $\{0, 1, -1\}$. This *gridding matrix* specifies the permitted shape for plots of permutations in the class. Each entry of M corresponds to a *cell* in a *gridding* of a permutation. The cells of the matrix M can be displayed easily by sloping lines which create the cell diagram like , , or . With a minor abuse of terminology, we often refer to the matrix entries themselves as cells. For example, Figure 1.2 shows the representation of the same class by a cell diagram and a matrix.

Sometimes, a permutation in a class can be gridded in more than one way. For example, in Figure 1.3 the permutation 843256179 has seven different griddings in the class .



$$\text{Grid}\begin{pmatrix}0 & -1 & 0\\ 1 & 1 & -1\\ 0 & 1 & 0\end{pmatrix}$$

Figure 1.2: The cell diagram and the matrix representation of a class

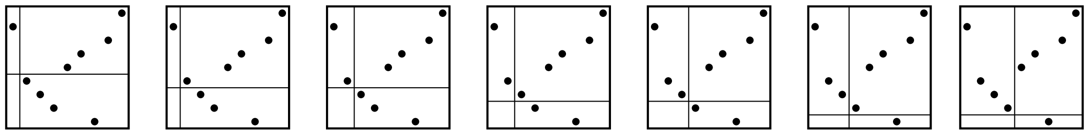


Figure 1.3: The seven -griddings of 843256179

To define an M-gridding formally we index matrices starting from the left lower corner, with the order reversed from what it usually is. For example, $M_{3,1}$ represents the entry located in the third column from left and in the bottom row of M.

Let M be a gridding matrix with s columns and t rows. Suppose σ is a permutation with length n, then an *M-gridding* of σ is a pair of integer sequences, the column dividers $1 = c_1 \leq c_2 \leq c_3 \leq ... \leq c_{s+1} = n+1$, and the row dividers $1 = r_1 \leq r_2 \leq r_3 ... \leq r_{t+1} = n+1$ such that for all $i \in \{1, ..., s\}$ and $j \in \{1, ..., t\}$, the subsequence of σ with indices in $[c_i, c_{i+1})$ and values in $[r_j, r_{j+1})$ is increasing if $M_{i,j} =$ ⧄, decreasing if $M_{i,j} =$ ⧅, and empty if $M_{i,j} =$ □. Note that, the content of a cell may be empty even if it is marked like ⧄ or ⧅. To illustrate, in the rightmost gridding in Figure 1.3, notice that $c_2 = 5$ and $r_2 = 2$. Note that we only index matrices this way in the definition of M-gridding, and not anywhere else. In particular, the normal indexing convention (left-to-right and top-to-bottom) is used for all the matrices in Chapters 3, 4, and 5.

Sometimes, it is more practical to consider the assignment of points to cells induced by a specific M-gridding. In the M-gridding, the column and row *dividers* are vertical and horizontal lines which split the permutation into *blank* cells if the entries look like □, and *non-blank* cells if they look like ⧄ or ⧅.

1.1.4 Grid(M) vs Grid$^{\#}(M)$

The set of permutations that have at least one M-gridding is called the *grid class* Grid(M). We also use $\text{Grid}_n(M)$ for the set of permutations of length n in Grid(M).

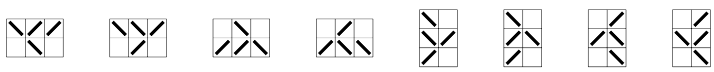
An *M-gridded permutation* is considered to be a permutation itself together with an M-gridding. We call the set of all M-gridded permutations *the gridded class* and it is denoted by Grid$^{\#}(M)$.

The exponential growth rate of Grid(M) is given by the formula:

$$\text{gr}\big(\mathsf{Grid}(M)\big) \;:=\; \lim_{n\to\infty} \big|\mathsf{Grid}_n(M)\big|^{1/n}.$$

This limit is guaranteed to exist as a result of Bevan's Theorem [7], which we discuss in Section 1.3. As there are only $n+1$ possible positions for each row and column divider in the gridding matrix M, for a matrix with dimensions $s \times t$ an upper bound on the number of M-griddings of a permutation of length n is $(n+1)^{s+t-2}$. It is obvious that this is a polynomial in n and hence the exponential growth rate of the gridded class is the same as that of the class itself [18, Proposition 2.1].

Suppose two grid classes $\mathsf{Grid}(M_1)$ and $\mathsf{Grid}(M_2)$ are such that M_2 can be obtained from M_1 by a series of reflections and rotations, then, for each n, the number of n-permutations in the two classes is the same, so they have the same enumeration. The same is true for gridded classes. For example, each of the following eight classes has the same enumeration.



Thus we only need to consider one of the possible orientations of a class when enumerating.

1.1.5 Probability

Much of our analysis is based on probabilistic calculations. In particular, we often consider properties of M-gridded n-permutations drawn uniformly from $\mathsf{Grid}^{\#}(M)$.

If $\mathcal{Q}$ is a property of M-gridded permutations, then we write

$$\mathbb{P}\big[\sigma^\# \text{ satisfies } \mathcal{Q} \;:\; \sigma^\# \in \mathsf{Grid}^\#_n(M)\big]$$

to denote the probability that an n-permutation drawn uniformly from $\mathsf{Grid}^\#(M)$ has the property $\mathcal{Q}$.

The *asymptotic probability* that $\mathcal{Q}$ holds is the following limit, if it exists:

$$\lim_{n\to\infty} \mathbb{P}\big[\sigma^\# \text{ satisfies } \mathcal{Q} \;:\; \sigma^\# \in \mathsf{Grid}^\#_n(M)\big].$$

As usual, given two events A and B, we say that A and B are *independent* if and only if $\mathbb{P}[A \wedge B] = \mathbb{P}[A]\mathbb{P}[B]$. For example, let $\sigma^\#$ be a gridded n-permutation drawn uniformly from $\mathsf{Grid}^\#_n$(⧄⧅⧄⧅). Let A be the event that there is at least one point of $\sigma^\#$ in the first two cells and that the topmost of these points is in the first cell. Similarly, let B be the event that there is at least one point of $\sigma^\#$ in the last two cells and that the topmost of these points is in the last cell. Then events A and B are independent.

Often, we are only interested in independence "in the limit". Given two sequences of events $(A_i)_{i=1}^\infty$ and $(B_i)_{i=1}^\infty$, we say that they are *asymptotically independent* if

$$\lim_{n\to\infty} \frac{\mathbb{P}[A \wedge B]}{\mathbb{P}[A]\mathbb{P}[B]} \;=\; 1.$$

For example, let $\sigma^\#$ be as above. If C is the event that there is at least one point of $\sigma^\#$ in the middle two cells and the bottommost of these points is in the second cell, then it can be shown that A and C are asymptotically independent.

1.1.6 Generating functions

At certain points in the analysis we make use of generating functions. Wilf describes a generating function in this way: "A generating function is a clothesline on which we hang up a sequence of numbers for display" [20]. A generating function is a formal power series. For example, the generating function of the Fibonacci sequence

$0, 1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$ is given by

$$F(z) \;=\; \sum_{n \geqslant 0} F_n z^n \;=\; \frac{z}{1 - z - z^2} \;=\; 0 + z + z^2 + 2z^3 + 3z^4 + 5z^5 + 8z^6 + \ldots$$

This function compactly encodes the Fibonacci recurrence relation, where each term is the sum of the two previous terms. The notation $[z^n]f(z)$ extracts the coefficient of z^n from the generating function $f(z)$. Thus $[z^n]F(z)$ is the nth Fibonacci number F_n.

Given a gridded class $\mathsf{Grid}^\#(M)$, the coefficient of z^n in its generating function $F^\#_M(z)$ is the number of M-gridded permutations of length n:

$$F^\#_M(z) \;=\; \sum_{n \geqslant 0} \left|\mathsf{Grid}^\#_n(M)\right| z^n \;=\; \sum_{\sigma^\# \in \mathsf{Grid}^\#(M)} z^{|\sigma^\#|}.$$

A contribution of $z^{|\sigma^\#|}$ is made for each $\sigma^\#$ in the gridded class.

Sometimes we want to record additional information about the objects we are considering. For this we can use a *bivariate* generating function. For example, let C be a specific cell of gridded class $\mathsf{Grid}^\#(M)$, and let $\mathsf{Grid}^\#_{n,k}(M)$ be the set consisting of those M-gridded n-permutations with k points in C. Then,

$$F^\#_M(z, x) \;=\; \sum_{n \geqslant 0} \left|\mathsf{Grid}^\#_{n,k}(M)\right| z^n x^k$$

is the bivariate generating function for $\mathsf{Grid}^\#(M)$ in which the variable x is used to *mark* the points in C.

Further information concerning generating functions is introduced later when needed.

1.2 Previous work

In this section we summarise the history of grid classes of permutations. We focus on exact enumeration results.

The study of individual grid classes can be traced back to the work of Atkinson [4] and Stankova [16] in the final decade of the 20th century on the skew-merged permutations. However, the study of monotone grid classes was first formally presented by Murphy and Vatter [15]. Murphy and Vatter refer to them as "profile classes" in their work. Huczynska and Vatter [14] were the first to adopt the term grid classes for the monotone grid classes and use the notation Grid(M).

The study of the *exact* enumeration of monotone grid classes is rare as there are few individual classes which have been enumerated, and only two *general* results that have been found. An individual class example is the generating function for the *skew-merged* permutations Grid $\left(\begin{smallmatrix} \searrow & \nearrow \\ \nearrow & \searrow \end{smallmatrix}\right)$ which was determined by Atkinson [4].

The first general result is related to *skinny* grid classes, which are classes with one row. Their gridding matrix is a vector over $\{\nearrow, \searrow\}$. An iterative procedure is presented by Bevan [8], which gives the generating function of any skinny monotone grid class. Moreover, Brignall and Sliačan in [11] generalised Bevan's result to include skinny classes which have a single cell that may be non-monotone.

The second general result covers the *polynomial* grid classes, which are classes with growth rate equal to 1. The monotone grid class Grid(M) is polynomial if M has at most one $\nearrow$ or $\searrow$ in any row or column. An algorithm was described by Homberger and Vatter [13] which can enumerate any permutation class with polynomial enumeration from a structural description of the class. It is a big challange to extend any of the previous results.

A family of permutation classes that are closely related to monotone grid classes are the *geometric grid classes*, introduced and studied in [2] and [6]. Given a gridding matrix M, the geometric grid class $\text{Geom}(M)$ consists of those permutations whose points can be plotted on the sloping lines of the cell diagram of M. Each permutation in $\text{Geom}(M)$ is also a permutation in $\text{Grid}(M)$, so $\text{Geom}(M) \subseteq \text{Grid}(M)$. Indeed, $\text{Geom}(M) = \text{Grid}(M)$ if and only if the cell graph of M is acyclic (see Section 3.3 below for a definition of the cell graph of a gridding matrix).

All the monotone grid classes that are considered in this thesis have acyclic cell graphs, so our results also apply to the corresponding geometric grid classes. However, we do not concern ourselves further with geometric grid classes; see [2] for more on their properties.

1.3 Asymptotic enumeration

In this section we introduce the aim of this thesis which is the asymptotic enumeration of monotone grid classes. We also present our strategy to achieve this aim.

Given the difficulty of exact enumeration we aim to determine the asymptotic enumeration of monotone grid classes instead. Bevan proves in [7] that the exponential *growth rate* of Grid(M) exists and is equal to the square of the spectral radius of a certain graph associated with M.

For a more general version of Bevan's Theorem, a simpler proof was later given by Albert and Vatter in [1]. Thus we know that

$$\left|\mathrm{Grid}_n(M)\right| \;\sim\; \theta(n)\, g^n,$$

where g is the growth rate of the class and $\theta(n)$ is subexponential; that is, we have $\lim_{n\to\infty}\theta(n)^{1/n}=1$. We write $f(n)\sim g(n)$ to denote that $\lim_{n\to\infty} f(n)/g(n)=1$.

Our strategy for the asymptotic enumeration of Grid(M) consists of the following five steps:

1. Find the proportion of points that occur in each cell in a typical large M-gridded permutation.

2. Determine the asymptotic enumeration of the corresponding gridded class:

   $$\left|\mathrm{Grid}^{\#}_n(M)\right| \;\sim\; \theta^{\#}(n)\, g^n,$$

   where g is the exponential growth rate of the class, and $\theta^{\#}(n)$ is subexponential.

3. Determine, for each $\ell \geq 1$, how a typical large M-gridded permutation $\sigma^{\#}$ must be structured so that its underlying permutation σ has exactly ℓ distinct M-griddings.

4. By combining steps 1 and 3, calculate, for each $\ell \geq 1$, the asymptotic probability

$$P_\ell := \lim_{n\to\infty} \mathbb{P}[\sigma \text{ has exactly } \ell \text{ distinct } M\text{-griddings} : \sigma^\# \in \mathrm{Grid}^\#_n(M)].$$

5. Let

$$\kappa_M = \sum_{\ell\geq 1} P_\ell/\ell = \lim_{n\to\infty} \frac{\left|\mathrm{Grid}_n(M)\right|}{\left|\mathrm{Grid}^\#_n(M)\right|}.$$

   Then $\left|\mathrm{Grid}_n(M)\right| \sim \kappa_M\, \theta^\#(n)\, g^n$.

Chapter 2

Skinny Classes

In this chapter we find the asymtotic enumeration of the number of permutations in skinny grid classes. In Section 2.1 we define both peaks and peak points in skinny classes. In Section 2.2 we prove that if $\mathrm{Grid}(M)$ is a k-cell skinny grid class with p peaks, then $\left|\mathrm{Grid}_n(M)\right| \sim 2^{-p}k^n$.

We say that $\mathrm{Grid}(M)$ is *skinny* if M is simply a ⧄ / ⧅ vector. For example, the figure below shows the permutation 472598361 plotted in the skinny class ⧄⧄⧅⧅ .

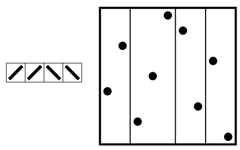


Figure 2.1: The permutation 472598361 plotted in the skinny class ⧄⧄⧅⧅

Although, as mentioned earlier, a method is known which provides the generating function of any skinny class, this would be an extremely inefficient way of finding their asymptotic enumeration.

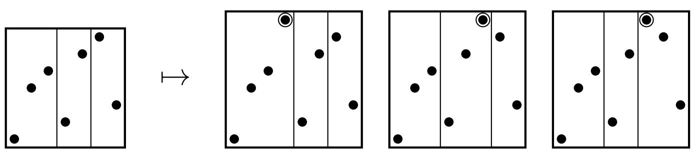


Figure 2.2: The three ways of adding a new maximum point to a gridded permutation in the skinny class $\mathrm{Grid}^{\#}($ ⧄⧄⧅ $)$

For skinny classes, each of the five steps in the asymptotic enumeration analysis is easy.

The start point is to begin with the exact enumeration of the skinny *gridded* classes.

Proposition 2.1. *If* $\mathrm{Grid}(M)$ *is a k-cell skinny grid class, then* $\left|\mathrm{Grid}^{\#}_{n}(M)\right| = k^{n}$.

Proof. Any M-gridded permutation can be uniquely constructed from the empty M-gridded permutation by repeatedly adding a new maximum point. This point may be placed in any one of the k cells. Moreover, there is only one way of adding a maximum to any particular cell, because of the monotonicity constraints. See Figure 2.2 for an illustration. □

2.1 Peaks and peak points

In this section we define both peaks and the peak point in skinny classes.

To analyse how permutations can be gridded requires some additional concepts, so we begin with some basic definitions. First, given a skinny grid class Grid(M), where M is the ⧄ / ⧅ vector $(m_1, ..., m_k)$, a *peak* is formed by a pair (m_i, m_{i+1}) of adjacent cells if $m_i \neq m_{i+1}$. That is, a peak either looks like ⧄⧅ , which we say *points up*, or else looks like or ⧅⧄ , which we say *points down*. For example, Grid(⧄⧅⧅⧄⧅⧄⧅) has five peaks, three pointing up and two pointing down.

Second, suppose we have a skinny grid class $\mathrm{Grid}(M)$ with a peak Λ, and that $\sigma^{\#}$ is an M-gridded permutation with at least two points in each cell, witnessing the orientation of the cell. Then the *peak point* of Λ is the highest of the points of $\sigma^{\#}$ in the two cells of Λ if Λ points up, and is the lowest of the points of $\sigma^{\#}$ in Λ if Λ points down. For example, in the rightmost two gridded permutations in Figure 2.2, the peak point (of the only peak in the class) is circled.

It should be noted that this use of the term "peak" is atypical, differing from its standard use to denote a consecutive 132 or 231 pattern in a permutation. We consider a *peak* to be part of a gridding matrix. In contrast, a *peak point* is the central point of either a peak or valley (in the traditional sense) in a *gridded* permutation. For example, the circled points in Figure 2.3 are the two peak points in a permutation in the skinny grid class ⧄⧄⧅⧄ .

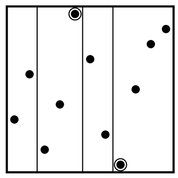


Figure 2.3: The two peak points in a permutation in the skinny class Grid($\boxed{/}\boxed{/}\boxed{\backslash}\boxed{/}$)

We generalise these concepts in Section 4.3 below when taking into account non-skinny classes.

2.2 Dancing and constrained gridded permutations

In this section we start with the definitions of dancing and of a constrained gridded permutation. Then we give two propositions concerning constrained gridded permutations. We then prove that almost all gridded permutations in a skinny class are constrained. Finally, we prove Theorem 2.5 which says that if Grid(M) is a k-cell skinny grid class with p peaks, then $\left|\text{Grid}_n(M)\right| \sim 2^{-p}k^n$ using Propositions 2.1, 2.2, and 2.4.

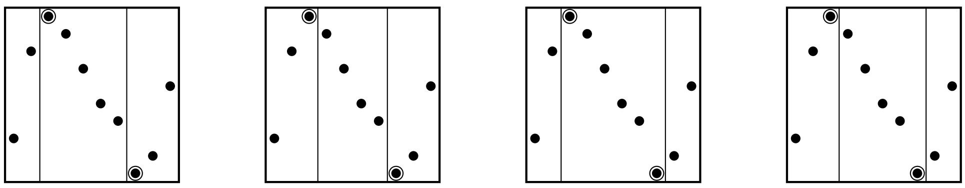


Figure 2.4: The four distinct griddings of a permutation in Grid($\boxed{/}\boxed{\backslash}\boxed{/}$); points which can dance are circled

Suppose $\mathsf{Grid}(M)$ is skinny and $\sigma^\# \in \mathsf{Grid}^\#(M)$. If Q is a peak point of $\sigma^\#$, then Q is immediately adjacent to a column divider. That is, there is no other point between Q and the divider. The movement of this divider to the other side of Q results in another valid M-gridding of σ. We say that Q can *dance* and that this new gridded permutation is the result of Q *dancing*.

We imagine that Q dances from one cell to an adjacent one. (However actually it is the divider rather than the point that moves). For an illustration see Figure 2.4. We generalise the notion of dancing later in Section 4.3, when considering non-skinny classes.

The key behind the approach taken in this thesis is the fact that the gridding possibilities are heavily restricted for most permutations. To formalise this observation, we direct our focus to specific well-behaved gridded permutations in which only peak points can dance. In general, different ways of creating griddings are possible. See the unconstrained gridded permutations in Figure 2.5.

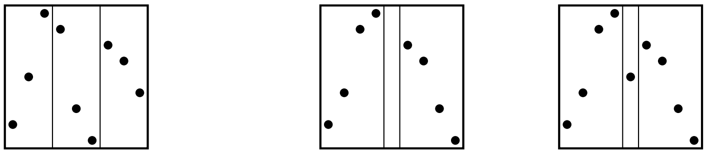
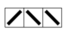


Figure 2.5: One constrained and two non-constrained gridded permutations in $\mathrm{Grid}^{\#}($  $)$

Suppose Grid(M) is skinny and $\sigma^{\#} \in \mathrm{Grid}^{\#}(M)$. We say that $\sigma^{\#}$ is *M-constrained* (or just *constrained*) if

a) every M-gridding of its underlying permutation σ is the result of zero or more peak points of $\sigma^{\#}$ dancing, and

b) in every M-gridding of σ, each cell contains at least two points.

Counting possible griddings becomes easy when dealing with constrained gridded permutations.

Proposition 2.2. *If* Grid*(M) is a skinny grid class with p peaks, and $\sigma^{\#} \in \mathrm{Grid}^{\#}(M)$ is M-constrained, then σ has exactly 2^p distinct M-griddings.*

Proof. Since $\sigma^{\#}$ is constrained, every M-gridding of σ is the result of zero or more peak points of $\sigma^{\#}$ dancing. Since each cell of $\sigma^{\#}$ contains at least two points, $\sigma^{\#}$ has p distinct peak points (one for each peak), which can dance independently. For each peak point of $\sigma^{\#}$, we can choose whether it dances or not, yielding a total of 2^p distinct M-griddings for σ. □

The following proposition gives sufficient conditions for a gridded permutation in a skinny class to be constrained:

Proposition 2.3. *Suppose* Grid*(M) is skinny and* $\sigma^\# \in \text{Grid}^\#(M)$ *is such that each cell contains at least four points and there are no two adjacent cells whose contents together form an increasing or decreasing sequence. Then* $\sigma^\#$ *is M-constrained.*

Proof. The contents of each cell of $\sigma^\#$ consists of an increasing or decreasing sequence of points. However, there is no pair of adjacent cells whose contents together form an increasing or decreasing sequence.

Thus, in any M-gridding of σ, by the monotonicity constraints, there must be a divider between each pair of adjacent cells of $\sigma^\#$ that have the same orientation and also a divider adjacent to each peak point of $\sigma^\#$. So any M-gridding of σ can be formed from $\sigma^\#$ by zero or more of its peak points dancing.

Moreover, in any M-gridding of σ, at most two points from any cell of $\sigma^\#$ (the first and the last) can be gridded in another cell. So each cell of any M-gridding of σ contains at least two points. □

The reason behind our interest in constrained gridded permutations is not only because it is easy to count their griddings, but in addition because almost all gridded permutations are constrained.

Proposition 2.4. *If* Grid*(M) is skinny, then almost all M-gridded permutations are M-constrained:*

$$\lim_{n\to\infty} \mathbb{P}[\sigma^\# \text{ is } M\text{-constrained} : \sigma^\# \in Grid^\#_n(M)] = 1.$$

Proof. Suppose M has k cells. The number of n-point M-gridded permutations with exactly m points in a given cell equals

$$\binom{n}{m}(k-1)^{n-m} < n^m(k-1)^n.$$

Here, $\binom{n}{m}$ is the number of ways of choosing the points in the given cell, and $(k-1)^{n-m}$ is the number of ways of distributing the remaining points. So the total number of n-point M-gridded permutations with fewer than four points in some cell is less than $4kn^3(k-1)^n$, there being four choices for the value of m, and k choices of cell.

Similarly, the number of n-point M-gridded permutations with a given pair of adjacent cells forming an increasing or decreasing sequence of length m is less than

$$(m+1)(k-1)^n < (n+1)(k-1)^n.$$

Here, $(k-1)^n$ is an upper bound on the number of ways of distributing the points if we consider the pair of adjacent cells merged to form a single "super cell", and the factor $m+1$ is the number of choices for the position of the divider that splits the pair of cells. So the total number of n-point M-gridded permutations with two adjacent cells forming an increasing or decreasing sequence is less than $(n+1)^2(k-1)^{n+1}$, there being $n+1$ choices for the value of m, and $k-1$ choices for the pair of cells.

Thus, by Propositions 2.1 and 2.3, the proportion of n-point M-gridded permutations which are not constrained is less than

$$\frac{4kn^3(k-1)^n \;+\; (n+1)^2(k-1)^{n+1}}{k^n} \;<\; 5k(n+1)^3\Big(1-\frac{1}{k}\Big)^n,$$

which converges to zero as n tends to infinity. □

We can then deduce the asymptotic enumeration of skinny grid classes directly from Propositions 2.1, 2.2, and 2.4.

Theorem 2.5. *If* $\mathrm{Grid}(M)$ *is a k-cell skinny grid class with p peaks, then* $\big|Grid_n(M)\big| \sim 2^{-p}k^n$.

Proof. For almost all of the k^n distinct M-gridded permutations, the underlying permutation has exactly 2^p M-griddings. □

Chapter 3

The distribution of points between cells

In this chapter we introduce in Section 3.1 some special types of matrices and use them for recording the number of points or the proportion of points in each cell. We then determine in Section 3.2 the asymptotic number of gridded permutations with a given distribution. In Section 3.3 we consider maximal distributions, with the greatest growth rate. In Section 3.4 we establish that the distribution of points in almost all gridded permutations is close to the maximal distribution.

As we have previously observed, depending only on the number of peaks, almost every large permutation in a given skinny class has the same number of griddings. However, the situation is not the same in non-skinny classes, because the structure of a permutation may affect the number of its griddings.

In this chapter we determine the asymptotic probability of a permutation having a specific number of griddings. To do this, we need to know the proportion of points that occur in each cell in a typical large gridded permutation.

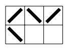
In Bevan's PhD thesis [8, Chapter 6], and in Albert and Vatter [1] much of the required analysis can be found. However, neither reference has all that we require. This chapter uses a combination of the approaches in both of these sources.

3.1 M-admissible and M-distribution matrices

In this section we define some matrices which we use to record the number or proportion of points in each cell.

Given a gridding matrix M, we say that a nonnegative real matrix $A = (a_{i,j})$ of the same dimensions as M is *M-admissible* if $a_{i,j}$ is zero whenever $M_{i,j}$ is blank. We refer to the sum of the entries of such a matrix as its *weight*.

We use *integer* M-admissible matrices to record the number of points in each cell of a gridded permutation. Suppose $A = (a_{i,j})$ is an integer M-admissible matrix of weight n. Then $\mathsf{Grid}^{\#}_{A}(M)$ denotes the subset of $\mathsf{Grid}^{\#}_{n}(M)$ consisting of those M-gridded permutations with $a_{i,j}$ points in cell (i,j), for each (i,j). For an illustration, the gridded permutation in Figure 3.1 below is an element of

$$\mathsf{Grid}^{\#}_{\left(\begin{smallmatrix}2&3&3\\1&0&0\end{smallmatrix}\right)}\Big($$  $$\Big).$$

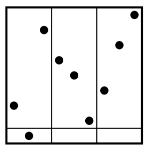
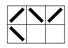


Figure 3.1: A gridding of the permutation 318652479 in $\mathsf{Grid}^{\#}($  $)$,

Thus $\mathsf{Grid}^{\#}_{n}(M)$ can be partitioned into subsets as follows:

$$\mathsf{Grid}^{\#}_{n}(M) = \biguplus_{\|A\|=n} \mathsf{Grid}^{\#}_{A}(M),$$

where the disjoint union is over all integer M-admissible matrices of weight n. The number of gridded permutations in one of these subsets is given by the following product of multinomial coefficients.

Proposition 3.1. *Suppose $A = (a_{i,j})$ is an integer M-admissible matrix with dimensions*

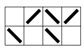
$r \times s$ (r rows and s columns). Then,

$$\left|\mathsf{Grid}^{\#}_{A}(M)\right| = \prod_{i=1}^{r} \binom{\sum_{j=1}^{s} a_{i,j}}{a_{i,1}, a_{i,2}, \ldots, a_{i,s}} \times \prod_{j=1}^{s} \binom{\sum_{i=1}^{r} a_{i,j}}{a_{1,j}, a_{2,j}, \ldots, a_{r,j}}.$$

Proof. The ordering of points (increasing or decreasing) *within* a particular cell is fixed by the corresponding entry of M. However, the interleaving of points in *distinct* cells in the same row or column can be chosen arbitrarily and independently. The multinomial coefficient in the first product counts the number of ways of vertically interleaving the points in the cells in row i. Similarly, the term in the second product counts the number of ways of horizontally interleaving the points in the cells in column j. □

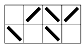
For example, by applying Proposition 3.1 the exact number of gridded permutations in

$$\mathsf{Grid}^{\#}_{\left(\begin{smallmatrix}0&2&3&1\\2&0&1&0\end{smallmatrix}\right)}(\ \ )$$



is:

$$\left|\mathsf{Grid}^{\#}_{\left(\begin{smallmatrix}0&2&3&1\\2&0&1&0\end{smallmatrix}\right)}(\ \ )\right| = \binom{6}{3,2,1}\binom{3}{2}\binom{4}{3} = 720,$$



where the first multinomial coefficient comes from row 1, the second comes from row 2, and the third comes from column 3.

In order to record the *proportion* of points in each cell, we use M-admissible matrices *of weight one*. We call such matrices *M-distribution matrices*. To avoid having to be concerned with rounding the number of points in each cell to an integer, we make use of Baranyai's Rounding Lemma [5] as a consequence of which we have the following result.

Proposition 3.2. *Suppose $\Gamma = (\gamma_{i,j})$ is an M-distribution matrix, and n is any positive integer. Then there exists an integer M-admissible matrix $A = (a_{i,j})$ of weight n such that, for each i, j, we have $|a_{i,j} - n\gamma_{i,j}| < 1$.*

In consideration of this, if $\Gamma = (\gamma_{i,j})$ is an M-distribution matrix and n a positive integer, we use $\mathsf{Grid}^{\#}_{\Gamma n}(M)$ to denote $\mathsf{Grid}^{\#}_{A}(M)$, where A is an integer M-admissible matrix of weight n each of whose entries differs from the corresponding entry of $n\Gamma$ by less than one, the existence of such an A being guaranteed by Proposition 3.2. The exact choice of A is of no consequence to our arguments. Note that in any gridded permutation in $\mathsf{Grid}^{\#}_{\Gamma n}(M)$ the proportion of points in cell (i,j) differs from $\gamma_{i,j}$ by less than $1/n$.

3.2 Asymptotics of gridded permutations with a given distribution

In this section we use Stirling's approximation and Proposition 3.1 to get the asymptotic number of gridded permutations with a given distribution.

Suppose $\tau = \gamma_1 + \gamma_2 + \ldots + \gamma_k$, where each $\gamma_i > 0$. Then Stirling's approximation gives the following asymptotic form for a multinomial coefficient.

$$\binom{\tau n}{\gamma_1 n, \gamma_2 n, \ldots, \gamma_k n} \sim \sqrt{\frac{\tau}{(2\pi)^{k-1}\gamma_1\gamma_2\ldots\gamma_k}}\, n^{-(k-1)/2} \left(\frac{\tau^\tau}{\gamma_1^{\gamma_1}\gamma_2^{\gamma_2}\ldots\gamma_k^{\gamma_k}}\right)^n.$$

Therefore, by Proposition 3.1, the asymptotic enumeration of $\mathsf{Grid}^{\#}_{\Gamma n}(M)$ is as follows.

Proposition 3.3. *If $\Gamma = (\gamma_{i,j})$ is an M-distribution matrix with row sums $\rho_i = \sum_j \gamma_{i,j}$ and column sums $\kappa_j = \sum_i \gamma_{i,j}$, then*

$$\left|\mathsf{Grid}^{\#}_{\Gamma n}(M)\right| \sim C n^\beta g^n,$$

where

$$g = g(\Gamma) := \prod_i \frac{\rho_i^{\rho_i}}{\prod_j \gamma_{i,j}^{\gamma_{i,j}}} \times \prod_j \frac{\kappa_j^{\kappa_j}}{\prod_i \gamma_{i,j}^{\gamma_{i,j}}},$$

and C and β are constants that only depend on Γ.

Proof. Suppose Γ has dimensions $r \times s$, and $A = (a_{ij})$ is as given in the definition of

$\mathsf{Grid}^{\#}_{\Gamma n}(M)$. Then,

$$\begin{aligned}\left|\mathsf{Grid}^{\#}_{\Gamma n}(M)\right| &= \prod_{i=1}^{r}\binom{\sum_{j=1}^{s} a_{i,j}}{a_{i,1}, a_{i,2}, \ldots, a_{i,s}} \times \prod_{j=1}^{s}\binom{\sum_{i=1}^{r} a_{i,j}}{a_{1,j}, a_{2,j}, \ldots, a_{r,j}} \\ &\sim \quad \prod_{i=1}^{r}\sqrt{\frac{\rho_i}{(2\pi)^{s-1}\gamma_{i,1}\gamma_{i,2}\cdots\gamma_{i,s}}}\, n^{-(s-1)/2}\left(\frac{{\rho_i}^{\rho_i}}{\gamma_{i,1}^{\gamma_{i,1}}\gamma_{i,2}^{\gamma_{i,2}}\cdots\gamma_{i,s}^{\gamma_{i,s}}}\right)^{n} \\ &\quad \times \prod_{j=1}^{s}\sqrt{\frac{\kappa_j}{(2\pi)^{r-1}\gamma_{1,j}\gamma_{2,j}\cdots\gamma_{r,j}}}\, n^{-(r-1)/2}\left(\frac{{\kappa_j}^{\kappa_j}}{\gamma_{1,j}^{\gamma_{1,j}}\gamma_{2,j}^{\gamma_{2,j}}\cdots\gamma_{r,j}^{\gamma_{r,j}}}\right)^{n}. \quad \square\end{aligned}$$

Observe that the exponential growth rate of those M-gridded permutations whose points are distributed between the cells in the proportions specified by Γ is $g(\Gamma) = \lim_{n\to\infty}\left|\mathsf{Grid}^{\#}_{\Gamma n}(M)\right|^{1/n}$.

To avoid problems below, we restrict the products in the denominators of the expression for $g(\Gamma)$ to nonzero entries of Γ only.

3.3 Growth rate

In this section we state two results from [1] and [3], concerning maximal M-distribution matrices.

Given a grid class $\mathsf{Grid}(M)$, an M-distribution matrix $\Gamma = (\gamma_{i,j})$ is a *maximal M-distribution matrix* if the growth rate $g(\Gamma)$ is greatest.

The entries of such a matrix must satisfy specific equations that are shown in the following proposition 3.4.

The requirement is to determine which choices of values for the $\gamma_{i,j}$ maximise $g(\Gamma)$, subject to the requirement that $\sum \gamma_{i,j} = 1$. This is a constrained optimisation problem, which can be solved using the method of Lagrange multiplier

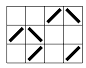
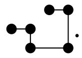
Proposition 3.4 (see [1, Section 5] and [3, Proposition 3.4])**.** *Suppose $\Gamma = (\gamma_{i,j})$ is a maximal M-distribution matrix. Then there exists a constant λ such that, for each nonzero entry $\gamma_{i,j}$ of Γ, we have*

$$\frac{\gamma_{i,j}^2}{\rho_i\kappa_j} = \lambda,$$

where $\rho_i = \sum_j \gamma_{i,j}$ *and* $\kappa_j = \sum_i \gamma_{i,j}$ *are the row and column sums of* Γ.

$\mathsf{Grid}(M)$ may have several maximal M-distribution matrices. Restricting the considered classes allows us to ensure uniqueness.

Given a gridding matrix M, its *cell graph* is the graph whose vertices are its non-blank cells, and in which two vertices are adjacent if they share a row or a column and all the cells between them are blank.

For example, the cell graph of  is .

If the cell graph of a gridding matrix is connected, then we also say that the matrix and the corresponding grid class are *connected*. A connected grid class has a unique maximal distribution matrix.

Proposition 3.5 (see [3, Proposition 3.7])**.** *If* $\mathsf{Grid}(M)$ *is a connected grid class, then it has a unique maximal* M*-distribution matrix* $\Gamma = (\gamma_{i,j})$. *Moreover,* $\gamma_{i,j}$ *is positive if and only if* $M_{i,j}$ *is not blank.*

The proof of Proposition 3.5 makes use of a characterisation by Albert and Vatter in [1] of maximal M-distribution matrices in terms of singular value decompositions, together with an application of the Perron–Frobenius Theorem.

Note that, for a grid class to have a unique maximal M-distribution matrix the connectivity is a sufficient condition. We use Γ_M to denote the unique maximal M-distribution matrix for a connected matrix M.

3.4 The distribution of points in a typical gridded permutation

In this section we state the results from [3] that the distribution of points in almost all gridded permutations in a connected grid class is close to the maximal distribution.

Given any $\sigma^\# \in \mathsf{Grid}_n^\#(M)$, let $\sigma^\#_{(i,j)}$ denote the number of points of $\sigma^\#$ in cell (i,j), and let $\Gamma_{\sigma^\#} = \left(\sigma^\#_{(i,j)}/n\right)$ be the M-distribution matrix recording the proportion of the points of $\sigma^\#$ in each cell.

Theorem 3.6 (see [3, Theorem 3.8])**.** *If* $\mathsf{Grid}(M)$ *is connected and* $\Gamma_M = (\gamma_{i,j})$ *is the unique maximal M-distribution matrix, then for any* $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\big[\max_{i,j} \big|\sigma^{\#}_{(i,j)}/n - \gamma_{i,j}\big| \leqslant \varepsilon \,:\, \sigma^{\#} \in \mathsf{Grid}^{\#}_n(M)\big] \;=\; 1.$$

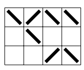
We now know that, if $\mathsf{Grid}(M)$ is connected, then the distribution of points between cells in almost all M-gridded permutations is close to that specified by Γ_M.

Chapter 4

Connected classes with one corner

In this chapter we analyse connected one-corner classes. In Section 4.1, we use Theorem 3.6 to determine the asymptotic distribution of points between cells in connected one-corner classes. In Section 4.2 we then use generating functions to establish the asymptotics of the number of gridded permutations in such a class. In Section 4.3 we generalise our notion of a peak and investigate the ways in which points can dance between the cells in a peak in connected one-corner classes. In Section 4.4 we investigate the ways in which points can dance between the cells in diagonals and tees in connected one-corner classes. In Section 4.5 we generalise the definition of a constrained gridded permutation from skinny classes to M-gridded permutations in connected one-corner classes.

We say that a cell C of a gridding matrix is a *corner cell* or just a *corner*, if there is both another non-blank cell in the same row as C and also another non-blank cell in the same column as C. A cell that is not a corner is a *non-corner* cell.

For example,  has five corners, three in the top row and two in the bottom row.

The focus in the remaining chapters is on connected monotone grid classes that have a single corner. These classes are either L-shaped, T-shaped or X-shaped. See Figure 4.1 for an illustration of each type of class. L-shaped and T-shaped classes come in four different orientations.

The *main row* is the row that contains the corner cell, and the *main column* is the

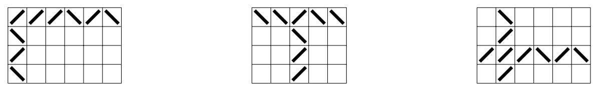


Figure 4.1: An L-shaped class, a T-shaped class, and an X-shaped class

column that contains the corner cell. For conciseness, we say that the *row cells* are the *non-corner* cells in the main row, and *column cells* are the non-corner cells in the main column. Hence each L, T or X-shaped class includes a corner cell, some row cells and some column cells.

During this and the remaining chapters we assume that $\mathsf{Grid}(M)$ is a connected one-corner class with dimensions $(r+1) \times (c+1)$. Hence M has $r+c+1$ non-blank cells: the corner, r column cells, and c row cells. As an example, for the T-shaped class in Figure 4.1, we have $r = 3$ and $c = 4$.

4.1 The distribution of points between cells

In this section, we use Theorem 3.6 to determine the asymptotic distribution of points between cells in connected one-corner classes.

We can determine the asymptotic distribution of points between the cells in a typical M-gridded permutation by using Theorem 3.6 and Proposition 3.4. In a one-corner class the same equations are satisfied for the asymptotic proportion of points in each row cell and thus these proportions are all equal. For the column cells the same holds true. That is because according to 3.4 for each nonzero entry $\gamma_{i,j}$ of Γ, we have have $\frac{\gamma_{i,j}^2}{\rho_i \kappa_j} = \lambda$, where $\rho_i = \sum\limits_j \gamma_{i,j}$ and $\kappa_j = \sum\limits_i \gamma_{i,j}$ are the row and column sums of Γ. So, if we have two nonzero entries γ_1, γ_2 that are in the same row but not the corner cell then this means: $\frac{\gamma_1^2}{\gamma_1 \rho_i} = \frac{\gamma_2^2}{\gamma_2 \rho_i} \implies \frac{\gamma_1}{\rho_i} = \frac{\gamma_2}{\rho_i} \implies \gamma_1 = \gamma_2$. The same holds for columns.

We use α for the proportion of points in the corner cell, β for the proportion in each of the row cells, and γ for the proportion in each of the column cells. So, for illustration, the unique maximal distribution matrix for the T-shaped class in

Figure 4.1 can be presented as

$$\begin{pmatrix} \beta & \beta & \alpha & \beta & \beta \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \\ 0 & 0 & \gamma & 0 & 0 \end{pmatrix}.$$

Then, by Theorem 3.6 and Proposition 3.4, we know that α, β and γ are the unique positive solutions to the equations

$$\alpha + c\beta + r\gamma = 1 \quad \text{and} \quad \frac{\alpha^2}{(\alpha + c\beta)(\alpha + r\gamma)} = \frac{\beta}{\alpha + c\beta} = \frac{\gamma}{\alpha + r\gamma}.$$

Solving these then yields

$$\alpha = \frac{1}{q}, \qquad \beta = \frac{c - r + q - 1}{2cq}, \qquad \gamma = \frac{r - c + q - 1}{2rq}, \tag{4.1}$$

where

$$q = \sqrt{(r + c + 1)^2 - 4cr}. \tag{4.2}$$

Note that in the only other solution to these equations, $\alpha = -1/q < 0$, so there is a unique solution with α, β and γ all positive.

Note that

$$\lambda = \frac{\beta}{\alpha + c\beta} = \frac{\gamma}{\alpha + r\gamma} = \frac{\alpha^2}{(\alpha + c\beta)(\alpha + r\gamma)} = \frac{r + c + 1 - q}{2rc} \tag{4.3}$$

is the common value of the ratios from Proposition 3.4.

For example, if $r = 3$ and $c = 4$ then we have $q = 4$, $\alpha = \frac{1}{4}$, $\beta = \frac{1}{8}$, $\gamma = \frac{1}{12}$ and $\lambda = \frac{1}{6}$.

4.2 The asymptotics of gridded classes

In this section we determine the generating function for a connected one-corner gridded class. We then extract the asymptotic growth of the number of gridded permutations from the generating function.

The exact enumeration of the gridded permutations of skinny monotone grid classes is not difficult, because by Proposition 2.1 we know that for a k-cell skinny grid class there are exactly k^n gridded n-permutations. However, for non-skinny classes, things are not as simple.

For connected one-corner classes, we determine the generating function for the gridded class using a technique which was first described in [8, Chapter 4]. Then using methods from analytic combinatorics we find the asymptotic growth of the number of gridded permutations.

To determine the generating function for the gridded permutations in an L-shaped, T-shaped or X-shaped class we tie together two skinny classes, formed by the main row and the main column that intersect at the corner cell.

The bivariate generating function for the $(c+1)$-cell horizontal skinny gridded class $\mathcal{H}^\#$ formed from the main row is given by the following formula where x marks the points in the corner cell.

$$H^\#(z,x) \;=\; \frac{1}{1-cz-zx}.$$

Similarly, the bivariate generating function for the $(r+1)$-cell vertical skinny gridded class $\mathcal{V}^\#$ formed from the main column, in which y is used to mark the points in the corner cell, is

$$V^\#(z,y) \;=\; \frac{1}{1-rz-zy}.$$

Thus, the set of pairs $(\sigma_{\mathsf{h}}^\#, \sigma_{\mathsf{v}}^\#)$, consisting of an $\mathcal{H}$-gridded permutation $\sigma_{\mathsf{h}}^\#$ and a $\mathcal{V}$-gridded permutation $\sigma_{\mathsf{v}}^\#$, is enumerated by the product of the generating functions of the two skinny classes:

$$P^\#(z,x,y) \;=\; H^\#(z,x)\,V^\#(z,y) \;=\; \frac{1}{(1-cz-zx)(1-rz-zy)}.$$

To count M-gridded permutations, we are only interested in pairs $(\sigma_{\mathsf{h}}^\#, \sigma_{\mathsf{v}}^\#)$ where the number of points of $\sigma_{\mathsf{h}}^\#$ in the corner cell equals the number of points of $\sigma_{\mathsf{v}}^\#$ in the corner cell.

As seen in Figure 4.2, each M-gridded permutation can be broken down into such a pair, and can be combined uniquely to form an M-gridded permutation.

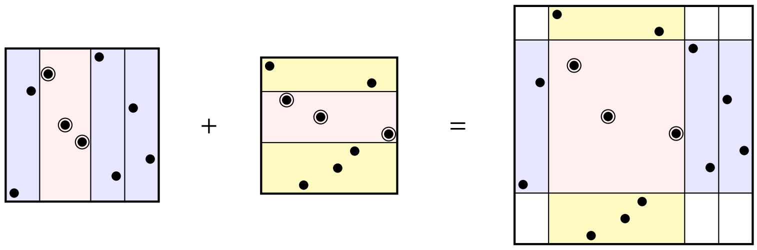
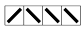
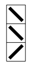
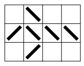


Figure 4.2: Tying together a -gridded permutation and a -gridded permutation to create a -gridded permutation

We extract the x and y terms with the same exponent from the generating function $P^{\#}(z, x, y)$. However, we must avoid double-counting the points in the corner cell. Hence we need

$$\sum_{m \geqslant 0} [x^m y^m] \, P^{\#}(z,\, x/\sqrt{z},\, y/\sqrt{z}).$$

By dividing by $\sqrt{z}$ for both arguments x and y we guarantee that for each point in the corner we decrease the exponent of z by one.

We let $y = x^{-1}$ in order to extract the terms where x and y have the same exponent. This results in a Laurent series in x (a *Laurent series* is a power series in which terms of negative degree are permitted). In this series the difference between the number of points of $\sigma^{\#}_{\mathsf{h}}$ in the corner and the number of points of $\sigma^{\#}_{\mathsf{v}}$ in the corner is recorded in the exponent of x. The constant term is all that we want, when this difference is zero;

$$[x^0] \, P^{\#}(z,\, x/\sqrt{z},\, x^{-1}/\sqrt{z}).$$

Using the result below we extract this constant term.

Proposition 4.1 (Stanley [17, Section 6.3]). *If $f(x) = f(z, x)$ is a Laurent series, then the constant term $[x^0]f(x)$ is given by the sum of the residues*[1] *of $x^{-1}f(x)$ at those poles $\alpha(z)$ of $f(x)$ for which $\lim_{z \to 0} \alpha(z) = 0$. These are known as the* small *poles.*

[1]The residue of $h(x)$ at $x = \alpha$ is the coefficient of $(x - \alpha)^{-1}$ in the Laurent expansion of $h(x)$ around $x = \alpha$. If α is a simple pole, then this is just the value of $yh(y + \alpha)$ at $y = 0$.

For our case we get,

$$x^{-1}P^{\#}(z,\, x/\sqrt{z},\, x^{-1}/\sqrt{z}) \;=\; \frac{1}{(1-cz-x\sqrt{z})(x-rzx-\sqrt{z})}.$$

There are two poles, at $x_1(z) = (1-cz)/\sqrt{z}$, and at $x_2(z) = \sqrt{z}/(1-rz)$.

As $x_2(z)$ is the only small pole, the residue at $x = x_2(z)$ is all that we require. We then find that this implies the generating function for M-gridded permutations in a connected class with one corner is as follows :

$$F^{\#}_M(z) \;=\; \sum_{n\geqslant 0} |\mathsf{Grid}^{\#}_n(M)|\, z^n \;=\; \frac{1}{1-(r+c+1)z+rcz^2}.$$

Therefore, from this generating function and, by using the following standard result we extract the asymptotic growth of the number of gridded permutations.

Proposition 4.2 (see [12, Theorems IV.10 and VI.1]). *Suppose $F(z)$ is the ordinary generating function of a combinatorial class $\mathcal{C}$. Let ρ be the least singularity of $F(z)$ on the positive real axis. If there are no other singularities on the radius of convergence and ρ is a pole of order r, then*

$$|\mathcal{C}_n| \;\sim\; c\rho^{-n}n^{r-1} \qquad \textit{where} \quad c \;=\; \frac{\rho^{-r}}{(r-1)!}\lim_{z\to\rho}(\rho-z)^r F(z).$$

In the case that $F(z)$ is a rational function with a denominator $Q(z)$ of degree d, the greatest root of the polynomial $z^d Q(z^{-1})$ is the exponential growth rate ρ^{-1}. Therefore, the growth rate of $\mathsf{Grid}^{\#}(M)$ is the larger of the two roots of the quadratic equation $z^2-(r+c+1)z+rc = 0$. Hence, by the observation in Section 1.1.4, this is also the growth rate of $\mathsf{Grid}(M)$. Thus,

$$g_M \;=\; \mathrm{gr}(\mathsf{Grid}(M)) \;=\; \tfrac{1}{2}(r+c+1+q),$$

where $q = \sqrt{(r+c+1)^2-4cr}$. Since $z = g_M^{-1}$ is a simple pole of $F^{\#}_M(z)$ the subexponential term is a constant:

$$\theta^{\#}_M(n) \;=\; (r+c+1+q)/2q \;=\; g_M/q.$$

Therefore, the following proposition gives the asymptotic growth of the number of gridded permutations in a connected one-corner class.

Proposition 4.3. *If M is connected with one corner and has dimensions $(r+1) \times (c+1)$, then*

$$\left|\mathsf{Grid}^{\#}_{n}(M)\right| \sim \theta^{\#} g^{n}, \quad \text{where } \theta^{\#} = \frac{r+c+q+1}{2q} \quad \text{and } g = \frac{r+c+q+1}{2},$$

with $q = \sqrt{(r+c+1)^2 - 4rc}$.

For example, if M has dimensions 3×3 ($r = 2$ and $c = 2$), then $\left|\mathsf{Grid}^{\#}_{n}(M)\right| \sim \frac{4}{3} \times 4^n$.

4.3 Dancing: peaks

In this section we generalise our notion of a peak and investigate the ways in which points can dance between the cells in a peak in connected one-corner classes.

To generalise our definition of a peak for skinny classes, two non-blank horizontally or vertically adjacent cells in a gridding matrix form a *peak* if they alternate: that is, if one of them is increasing (⧄) and the other is decreasing (⧅). Therefore, besides peaks that point up ⧄⧅ or point down ⧅⧄, there are also peaks that *point left* (⧄ over ⧅) and *point right* (⧅ over ⧄). For example, in Figure 4.3 there is a peak that points up in the main row, and there is a peak that points right in the main column.

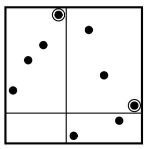


Figure 4.3: A gridding of the permutation 467918523 in Grid$^{\#}$(⧄⧅ / □⧄)

Let $\sigma^{\#}$ be a gridded permutation with at least two points in each cell in a gridded class with a peak Λ that points left or right. Then, the peak point of Λ is the leftmost of the points of $\sigma^{\#}$ in the two cells of Λ if Λ points left, and is the rightmost of the points of $\sigma^{\#}$ in the two cells of Λ if Λ points right. For example, the rightmost circled point in the gridded permutation 467918523 in Figure 4.3 is the peak point of a peak that points right.

To generalise the notation of dancing that we introduced in Section 2.2, we say that a peak point Q can dance if it is the closest point to the row or column divider that separates the two cells of the peak. Another valid gridding results from the movement of this divider to the other side of Q.

Only a single point is involved in the dancing of a peak point which we now call *peak dancing* or *dancing at a peak*.

We say that a peak is a *corner peak* if one of its two cells is a corner cell, and that it is a *non-corner peak* otherwise. For example, the L-shaped class in Figure 4.4 has one corner peak and five non-corner peaks.

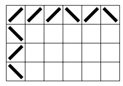


Figure 4.4: An L-shaped class that has one corner peak and five non-corner peaks

4.4 Dancing: diagonals and tees

In this section we investigate the ways in which points can dance between the cells in diagonals and tees in connected one-corner classes.

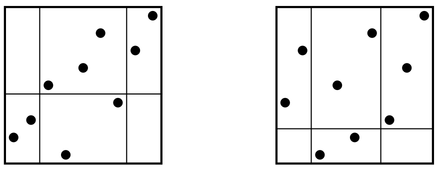
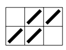
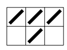
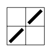
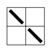


Figure 4.5: Griddings of permutations in Grid# (), and in Grid# ()

In this section we consider shapes found in corners beside peaks. For one-corner classes, in addition to peaks, we also need to take into account two other structures, both of which contain a diagonally adjacent pair of cells oriented either  or . In one-corner classes, these can only occur adjacent to the corner.

If the corner is oriented in the same direction as its two neighbours, then they form a *diagonal*, seen in one of the four rotations of . For example, the class at the left of Figure 4.5 has two diagonals, and the class at the right has only one diagonal.

On the other hand, if the corner is oriented in the opposite direction from its two neighbours, then the three cells form a *tee*: one of the four rotations of . For example, in Figure 4.6 the L-shaped class has a tee.

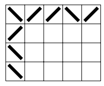


Figure 4.6: An L-shaped class with a tee

In connected one-corner classes unlike the situation for peaks, diagonals and tees only occur at corners. That is because, diagonals can only be formed by two cells which are adjacent to a corner, and tees can only be formed by a corner cell and two of its neighbours.

We know that *peak dancing* or *dancing at a peak* involves only one single point to dance. However, in a diagonal or tee there are new possible ways for points to dance.

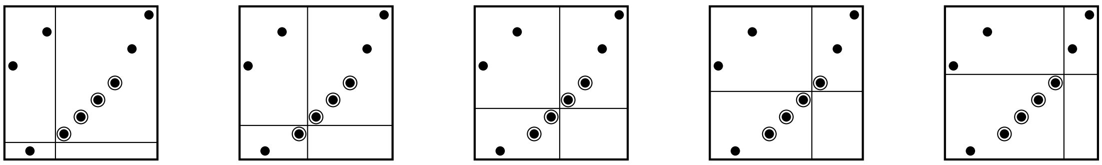


Figure 4.7: The five -griddings of 618234579; the four circled points can dance diagonally

Assume that M is a gridding matrix with a diagonal formed from two cells separated by a row and column divider. Suppose that Q is a point of an M-gridded permutation $\sigma^{\#}$ that is located in one of the two cells of the diagonal. We say that Q can *dance diagonally* if it can jump diagonally to the other cell of the two diagonal cells giving a new valid gridding. As mentioned earlier in Section 2.2 it is the dividers that move, but we imagine that the point dances by jumping. For an example, see Figure 4.7.

Observe that points that lie both horizontally and vertically between Q and the intersection of the dividers are able to dance. This implies a monotone sequence of dancers.

For the tee shape it is similar. Suppose that M is a gridding matrix, with a tee formed from a corner cell and two other cells, C_1 and C_2 say, separated by a row divider and a column divider. Suppose that Q is a point of an M-gridded permutation $\sigma^{\#}$ in one of the three cells of the tee. Q can *dance through the tee* if there is an alternating sequence of one-step moves of the two dividers which leaves one of them at least immediately the other side of Q and results in another valid M-gridding of σ. For an example, see Figure 4.8.

Again, any points that lie between the intersection of the dividers and Q can also dance, yielding a monotone sequence of points that can dance. Note however, that the first (or last) point in this sequence, lying in C_1 say, may only be able to dance into the corner, and not be able to dance through to C_2. For example, the circled point to the right in the gridded permutations in Figure 4.8 cannot dance into the cell below the corner.

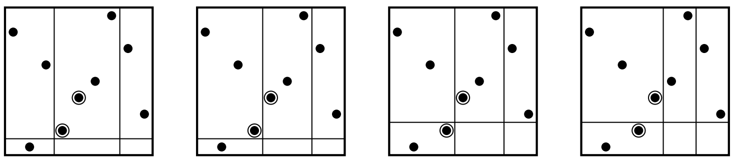
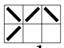


Figure 4.8: The four -griddings of 816245973; the two circled points can dance through the tee.

4.5 Constrained gridded permutations

In this section we give two conditions to generalise the definition of a constrained gridded permutation from skinny classes to M-gridded permutations in connected one-corner classes.

For almost all permutations in the classes we consider, the valid griddings are restricted to those that can be obtained through peak dancing, diagonal dancing and tee dancing. So, generalising the definition for skinny classes, if $\mathsf{Grid}(M)$ is a connected one-corner class we say that an M-gridded permutation $\sigma^{\#}$ is *M-constrained* (or just *constrained*) if

(a) every M-gridding of its underlying permutation σ is the result of zero or more points of $\sigma^{\#}$ dancing at a peak or diagonally or through a tee, and

(b) in every M-gridding of σ, each non-blank cell contains at least two points.

We delay further analysis and a proof that most M-gridded permutations are constrained until Section 5.4, after we discuss the different types of corner.

Chapter 5

Counting griddings

In this chapter we complete our computation of the asymptotic growth for the connected one-corner classes. We start by determining how the structure of a constrained M-gridded permutation $\sigma^{\#}$ affects the number of griddings that its given underlying permutation σ has. This relies on whether the corner cell is oriented in an increasing or decreasing direction, along with the orientation of each of its non-blank neighbouring cells. In L-shaped classes, there are 8 distinct ways to orient the corner cell and the two non-blank cells adjacent to it. In T-shaped classes, there are 16 distinct ways to orient the corner cell and the three non-blank cells adjacent to it. And, in X-shaped classes, there are 32 distinct ways to orient the corner cell and the four non-blank cells adjacent to it. These are all illustrated in Figure 5.1. We call these the *corner types*.

We find the correction factors for each of the eleven corner types of the L, T, and X-shaped connected one-corner classes. In Section 5.1 we examine corner types with peaks. In Section 5.2 we analyse corner types with diagonals. In Section 5.3 we present a detailed examination of corner types with tees. At the end of each section we find the asymptotic number of permutations in a 3×3 class with the specified corner type. In Section 5.4 we conclude by proving that almost all gridded permutations are constrained.

We then combine this analysis of the permutation structure with the asymptotic distribution of points between cells from Section 4.1 to calculate the asymptotic prob-

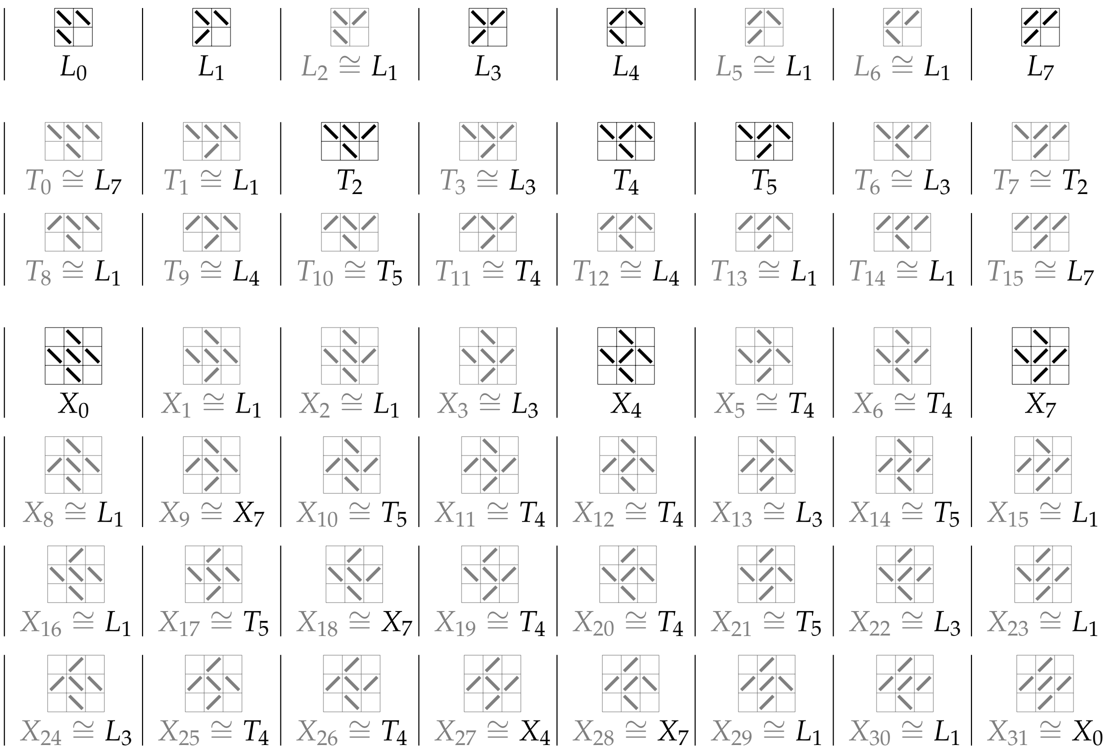


Figure 5.1: The corner types

ability for each $\ell \geqslant 1$. Let

$$P_\ell \;=\; \lim_{n\to\infty} \mathbb{P}\big[\sigma \text{ has exactly } \ell \text{ distinct } M\text{-griddings} \;:\; \sigma^\# \in \mathsf{Grid}^\#_n(M)\big].$$

Next, we let

$$\kappa_M \;=\; \sum_{\ell\geqslant 1} P_\ell/\ell \;=\; \lim_{n\to\infty} \frac{|\mathsf{Grid}_n(M)|}{|\mathsf{Grid}^\#_n(M)|}$$

be the *correction factor* for the class. We then conclude that $|\mathsf{Grid}_n(M)| \sim \kappa_M\, \theta^\#\, g^n$, where $\theta^\#$ and g are as mentioned previously in (step 2);

$$\theta^\# \;=\; \frac{r+c+q+1}{2q}, \quad g \;=\; \frac{r+c+q+1}{2}, \text{ where } q \;=\; \sqrt{(r+c+1)^2 - 4rc}.$$

We make repeated use of the following observation, recalling the common ratio λ from (equation (4.3)) on page 27 for the determination of the probabilities.

Given a gridded class $\mathsf{Grid}^\#(M)$ and a property $\mathcal{Q}$ of M-gridded permutations, the *asymptotic probability* that $\mathcal{Q}$ holds is the following limit, if it exists:

$$\lim_{n\to\infty} \mathbb{P}\big[\sigma^{\#} \text{ satisfies } \mathcal{Q} \,:\, \sigma^{\#} \in \mathsf{Grid}^{\#}_n(M)\big].$$

Observation 5.1. *Suppose* $\mathsf{Grid}(M)$ *is a connected one-corner class with dimensions* $(r+1)\times(c+1)$*, and let* α*,* β *and* γ *be the asymptotic proportion of points of an* M*-gridded permutation in the corner cell, in any row cell, and in any column cell, respectively.*

Then, in an M*-gridded permutation, for each* $k \geqslant 1$*, the asymptotic probability that the kth point from the top (or bottom) in the main row occurs in any given row cell equals* $\lambda = \beta/(\alpha + c\beta)$*. Similarly, the asymptotic probability that the kth point from the left (or right) in the main column occurs in any given column cell also equals* $\lambda = \gamma/(\alpha + r\gamma)$*.*

Moreover, for $j \neq k$*, the events that the jth and kth points from the top or bottom in the main row occur in specific cells are asymptotically independent. And analogously for points in the main column.*

This observation follows from Theorem 3.6, and the fact that the interleaving of points in distinct cells in the same row or column can be chosen arbitrarily and independently.

It is important to notice that we can multiply corresponding correction factors when dancing can occur asymptotically independently in more than one location. This is based on the following arithmetic observation.

Observation 5.2. *Given real sequences* $(P'_i)_{i\geqslant 1}$ *and* $(P''_j)_{j\geqslant 1}$*, let* $\kappa' = \sum_{i\geqslant 1} P'_i/i$ *and* $\kappa'' = \sum_{j\geqslant 1} P''_j/j$*. Suppose* $P_\ell = \sum_{ij=\ell} P'_i\, P''_j$*. Then,*

$$\sum_{\ell\geqslant 1} P_\ell/\ell \;=\; \kappa'\,\kappa''.$$

Proof.

$$\kappa'\,\kappa'' \;=\; \sum_{i\geqslant 1}\frac{P'_i}{i}\sum_{j\geqslant 1}\frac{P''_j}{j} \;=\; \sum_{i,j\geqslant 1}\frac{P'_i\,P''_j}{ij} \;=\; \sum_{\ell\geqslant 1}\frac{1}{\ell}\sum_{ij=\ell}P'_i\,P''_j \;=\; \sum_{\ell\geqslant 1}\frac{P_\ell}{\ell}. \qquad \square$$

Non-corner peaks

Before looking at the corner types, we consider the effect of non-corner peaks. From the definition of the peak point in any constrained M-gridded permutation at each non-corner peak of M there is one point that dances. Therefore, if M has p non-corner peaks, these give a factor of 2^p to the number of possible M-griddings, in a similar way to skinny classes (Proposition 2.2). Hence, the asymptotic enumeration of L, T and X-shaped classes can be given by the result in Theorem 5.3 as almost all M-gridded permutations are M-constrained (see Theorem 5.5 below).

Theorem 5.3. *Suppose* $\mathsf{Grid}(M)$ *is a connected one-corner class with corner type* τ*, and* p *non-corner peaks, then*

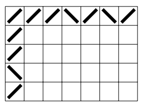
$$\left|\mathsf{Grid}_n(M)\right| \;\sim\; 2^{-p}\,\kappa(\tau)\,\theta^{\#}\,g^n,$$

where $\kappa(\tau)$ *is the correction factor for a gridding matrix with the same corner type and dimensions as M but with no non-corner peaks, and* $\theta^{\#}$ *and* g *are as given by Proposition 4.3.*

Note that $\kappa_M = 2^{-p}\,\kappa(\tau)$ is the correction factor for the class.

Worked example

Before proceeding to calculate the correction factors for each corner type, we briefly show our method with an example, by determining the asymptotic enumeration of the L-shaped class from Figure 4.1:

$M_{\mathsf{L}} \;=$  .

Firstly, from Proposition 4.3, since $r = 4$ and $c = 6$, we have $\left|\mathsf{Grid}^{\#}_n(M_{\mathsf{L}})\right| \sim \frac{8}{5} \times 8^n$.

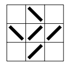
Secondly, M_{L} has six non-corner peaks and has corner type L_7 see table 5.1. Now, $\kappa(L_7) = 1 - \lambda$, and from equation (4.3) we know that $\lambda = \frac{1}{8}$.

Thus, by Theorem 5.3, we have

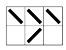
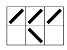
$$\left|\mathsf{Grid}_n(M_{\mathsf{L}})\right| \;\sim\; 2^{-6} \times \left(\tfrac{7}{8}\right) \times \tfrac{8}{5} \times 8^n \;=\; \tfrac{7}{320} \times 8^n.$$

Correction factors for corner types

We find the correction factor by calculating $\kappa(\tau)$ for each corner type τ. All of our results are summarised in Table 5.1.

We determine the subscripts used for corner type names by reading the non-blank cells in normal reading order (left-to-right and top-to-bottom) and creating a binary number by treating ⧄ as 1 and ⧅ as 0. For example, X_7 is  since $7 = 00111_2$.

Two corner types, τ_1 and τ_2, are *equivalent* (denoted $\tau_1 \cong \tau_2$ in Figure 5.1) if τ_2 can be obtained from τ_1 by rotation or reflection and/or by the addition or deletion of non-blank cells without creating or removing any peaks, diagonals or tees. For example, the following corner types are equivalent:

 $\cong$  $\cong$  $\cong$  $\cong$  .

L_1 T_1 T_{14} L_6 L_5

Specifically, $L_1 \cong T_1$ by the addition of a decreasing cell at the left, $T_1 \cong T_{14}$ by reflection about a vertical axis, $T_{14} \cong L_6$ by the deletion of an increasing cell from the left, and $L_6 \cong L_5$ by reflection about a diagonal axis. The analysis of two equivalent corner types is the same. As seen from Figure 5.1, there are eleven inequivalent corner types to consider. Of equivalent corner types, we choose the one with the least subscript as the representative.

In the analysis of a corner type, what is important is the number of peaks, diagonals and tees that make it up, and how these are combined. Note that both L_4 and T_5 consist of two peaks, but their correction factors differ.

From now on, we use dots as an indicator of where there may be additional row and column cells if they don't create additional peaks, diagonals or tees.

We begin with L_0 the simplest of the corner types.

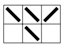
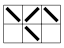
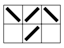
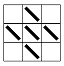
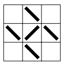
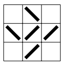
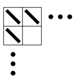
τ	Cell diagram of the corner type	P	D	T	$\kappa(\tau)$
L_0		0	0	0	1
L_1		1	0	0	$\frac{1}{2}\left(1+\frac{c\alpha\lambda}{\alpha+\gamma}\right)$
L_3		0	0	1	$\frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)}$
L_4		2	0	0	$\frac{1}{4}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)\left(1+\frac{c\alpha\lambda}{\alpha+\gamma}\right)$
L_7		0	1	0	$1-\lambda$
T_2		1	1	0	$\frac{1}{2}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)(1-\lambda)$
T_4		1	0	1	$\frac{1}{2}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)\frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)}$
T_5		2	0	0	$\frac{1}{4}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)^2$
X_0		0	2	0	$(1-\lambda)^2$
X_4		0	0	2	$\left(\frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)}\right)^2$
X_7		2	1	0	$\frac{1}{4}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)\left(1+\frac{c\alpha\lambda}{\alpha+\gamma}\right)(1-\lambda)$

Table 5.1: The correction factors and the number of peaks (P), diagonals (D) and tees (T) for each of the eleven corner types.

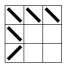
Corner type L_0



In this corner type there are no peaks, diagonals or tees. This corner type cannot occur in a T-shaped or X-shaped class 5.1. There is no possibility for dancing in a constrained gridded permutation in this class, therefore the underlying permutation has a single gridding. So, $P_1 = 1$, and $P_\ell = 0$ if $\ell > 1$. Hence $\kappa(L_0) = 1$.

Worked example

We now briefly apply our method to an example, by determining the asymptotic enumeration of the following class with corner type L_0 with a 3×3 gridding matrix:

$$M_{L_0} = $$  .

Firstly, from Proposition 4.3 we have: $\left|\mathsf{Grid}^{\#}_n(M_{L_0})\right| \sim \theta^{\#} g^n$, where $\theta^{\#} = \frac{r+c+q+1}{2q}$ and $g = \frac{r+c+q+1}{2}$, with $q = \sqrt{(r+c+1)^2 - 4rc}$. Since $r = 2$ and $c = 2$, then we have $q = 3$, so $\left|\mathsf{Grid}^{\#}_n(L_0)\right| \sim \frac{4}{3} \times 4^n$.

Secondly, M_{L_0} has only one non-corner peak. We know that $\kappa(L_0) = 1$.

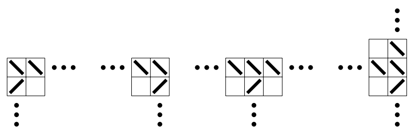
Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{L_0})\right| \sim 2^{-p}\,\kappa(\tau)\,\theta^{\#}\,g^n = 2^{-1} \times 1 \times \tfrac{4}{3} \times 4^n = \tfrac{2}{3} \times 4^n.$$

5.1 Corners with peaks

In this section we examine corner types with peaks. We find the correction factor for corner types L_1, L_4, and T_5. We then find the asymptotic number of permutations in a 3×3 class with the specified corner type.

Corner type L_1



In this corner type there is one peak. This corner type cannot occur in an X-shaped class. From the figures, we can see that the peak may be orientated in different ways with respect to the other cells adjacent to the corner. However, this does not change the analysis, so this is considered to be a single corner type.

Given a constrained gridded permutation, let Q be the peak point, and let R be the lowest point in any of the row cells. The only point that may be able to dance is the peak point Q. It can dance if it is the closest point to the row divider. So, there are two cases in which Q can dance: if it is below R, either in the corner cell or in the cell below the corner. On the other hand Q cannot dance when it is above R in the corner cell. Since R controls whether Q can dance or not, we call it the *controller*. Figure 5.2 shows an illustration of the three cases.

Since the peak point, Q, is either in the corner or in the cell immediately below the corner, the asymptotic probability that Q is in the corner equals $\alpha/(\alpha+\gamma)$. Fur-

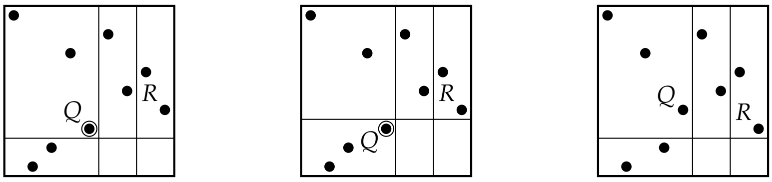
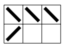


Figure 5.2: Three -gridded permutations; the peak point is circled if it can dance

thermore, the asymptotic probability that the lowest point in the main row is not in the corner equals $c\beta/(\alpha + c\beta)$. These events are asymptotically independent, so

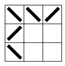
$$\mathbb{P}\big[Q \text{ can't dance}\big] \;\sim\; P_1 \;=\; \frac{\alpha}{\alpha+\gamma} \times \frac{c\beta}{\alpha + c\beta} \;=\; \frac{c\alpha\lambda}{\alpha+\gamma} \qquad \text{and} \qquad P_2 \;=\; 1 - P_1.$$

So,

$$\kappa(L_1) \;=\; P_1 + \tfrac{1}{2}P_2 \;=\; \frac{1}{2}\left(1 + \frac{c\alpha\lambda}{\alpha+\gamma}\right) \;=\; \frac{2r}{3r-c+q-1}.$$

Worked example

We briefly apply our method to an example, by determining the asymptotic enumeration of the following class:

$$M_{L_1} \;=\;$$ .

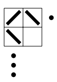
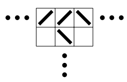
Since for a connected one-corner class the number of gridded permutations depends only on the number of columns and rows, then for M_{L_1} we have $\left|\mathsf{Grid}^{\#}_n(M_{L_1})\right| \sim \frac{4}{3} \times 4^n$.

M_{L_1} has two non-corner peaks. We know that $\kappa(L_1) = \frac{2r}{3r-c+q-1} = \frac{2}{3}$ for a 3×3 matrix.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{L_1})\right| \;\sim\; 2^{-2} \times \tfrac{2}{3} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{2}{9} \times 4^n.$$

Corner type L_4

In this corner type there are two peaks. This corner type cannot occur in an X-shaped class. Given a constrained gridded permutation, let Q_1 be the peak point at the left and Q_2 be the peak point at the top. For each peak, the analysis is the same as for L_1, so it gives

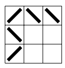
$$\mathbb{P}\big[Q_1 \text{ can't dance}\big] \;\sim\; \frac{c\alpha\lambda}{\alpha+\gamma}, \qquad \mathbb{P}\big[Q_2 \text{ can't dance}\big] \;\sim\; \frac{r\alpha\lambda}{\alpha+\beta}.$$

Whether Q_1 can dance or not depends on the points adjacent to the row divider below the corner. Similarly, whether Q_2 can dance or not depends on the points adjacent to the column divider to the right of the corner. Since these events are asymptotically independent, by Observation 5.2, we have

$$\begin{aligned}\kappa(L_4) \;=\; \kappa(L_1)\,\kappa(L_1^{\mathsf{R}}) \;&=\; \frac{1}{4}\left(1+\frac{c\alpha\lambda}{\alpha+\gamma}\right)\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right) \\ &=\; \frac{4rc}{(3r-c+q-1)(3c-r+q-1)}.\end{aligned}$$

Worked example

We briefly apply our method to an example, by determining the asymptotic enumeration of the following class:

M_{L_4} = .

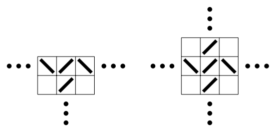
Since for a connected one-corner class the number of gridded permutations depends only on the number of columns and rows, then for M_{L_4} we have $\big|\mathsf{Grid}^{\#}_n(M_{L_4})\big| \sim \frac{4}{3} \times 4^n$.

M_{L_4} has only one non-corner peak. We know for a 3×3 matrix that $\kappa(L_4) = \frac{4rc}{(3r-c+q-1)(3c-r+q-1)} = \frac{4}{9}$.

Thus, by Theorem 5.3, we have

$$\big|\mathsf{Grid}_n(M_{L_4})\big| \;\sim\; 2^{-1} \times \tfrac{4}{9} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{8}{27} \times 4^n.$$

Corner type T_5



In this corner type there are also two peaks. This corner type cannot occur in an L-shaped class. Given a constrained gridded permutation, let Q_1 and Q_2 be the two peak points. Each peak takes the same analysis as for L_1 which implies

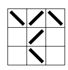
$$\mathbb{P}\big[Q_1 \text{ cannot dance}\big] \;\sim\; \mathbb{P}\big[Q_2 \text{ cannot dance}\big] \;\sim\; \frac{r\alpha\lambda}{\alpha+\beta}.$$

Whether one of the peak points can dance or not depends on the points adjacent to the column divider to the left of the corner, whereas whether the other peak point can dance or not depends on the points adjacent to the column divider to the right of the corner. So, these events are asymptotically independent. Therefore, by Observation 5.2, we have

$$\kappa(T_5) \;=\; \kappa(L_1^{\mathsf{R}})^2 \;=\; \frac{1}{4}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)^2 \;=\; \frac{4c^2}{(3c-r+q-1)^2}.$$

Worked example

We determine the asymptotic enumeration of the following class:

$$M_{T_5} \;=\;$$  .

As before, $\left|\mathsf{Grid}_n^{\#}(M_{T_5})\right| \sim \frac{4}{3} \times 4^n$. M_{T_5} has only one non-corner peak. We know for a 3×3 matrix that $\kappa(T_5) = \frac{4c^2}{(3c-r+q-1)^2} = \frac{4}{9}$.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{T_5})\right| \;\sim\; 2^{-1} \times \tfrac{4}{9} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{8}{27} \times 4^n.$$

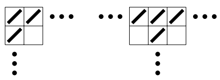
Note that, if $r = c$ then $\kappa(L_4) = \kappa(T_5)$. This is because both corner types have two independent corner peaks. If $r = c$, their different orientations do not affect the final result specifically, if $r = c$ then

$$\kappa(L_4) \;=\; \kappa(T_5) = \frac{4c^2}{(2c+q-1)^2}.$$

5.2 Corners with diagonals

In this section we present a detailed examination of each corner type with diagonals. We find the correction factor for corner types L_7, T_2, X_0 and X_7. We then find the asymptotic number of permutations in a 3×3 class with the specified corner type.

Corner type L_7



In this corner type there is one diagonal. This corner type cannot occur in an X-shaped class. Let us call the cell immediately to the right of the corner C_{R}, and call the cell immediately below the corner C_{B}.

Given a constrained gridded permutation, let Q_1 be the lowest point that is in the main row but is not in C_{R}. Notice that Q_1 may be in the corner cell. Let k_1 be the number of points in C_{R} that lie below Q_1. Similarly, let Q_2 be the rightmost point that is in the main column but is not in C_{B} (Q_2 could also be in the corner cell), and let k_2 be the number of points in C_{B} that lie to the right of Q_2. These k_1 and k_2 points can dance diagonally. They give $k_1 + k_2 + 1$ distinct griddings of the underlying permutation. Note that, each of k_1 and k_2 may be zero. Any point above Q_1 or to the left of Q_2 cannot dance. We call Q_1 and Q_2 the controllers as they control which points can dance. For an illustration, see the following Figure 5.3.

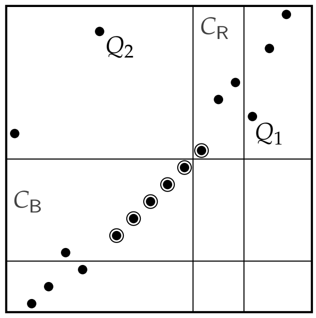
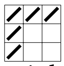


Figure 5.3: A -gridded permutation; the six circled points, below Q_1 and to the right of Q_2, can dance; $k_1 = 1$ and $k_2 = 5$

Applying Observation 5.1, for each $i \geqslant 0$, we have

$$\mathbb{P}\big[k_1 = i\big] \;\sim\; \lambda^i(1-\lambda) \qquad \text{and also} \qquad \mathbb{P}\big[k_2 = i\big] \;\sim\; \lambda^i(1-\lambda).$$

Since the value of k_1 is asymptotically independent of the value of k_2, for each $\ell \geqslant 1$, the asymptotic probability of having exactly ℓ griddings is

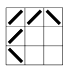
$$\mathbb{P}\big[k_1+k_2+1=\ell\big] \;=\; \sum_{k_1=0}^{\ell-1} \mathbb{P}\big[k_1=i\big]\mathbb{P}\big[k_2=\ell-1-i\big] \;\sim\; \ell\,\lambda^{\ell-1}(1-\lambda)^2.$$

Hence,

$$\kappa(L_7) \;=\; \sum_{\ell=1}^{\infty} \lambda^{\ell-1}(1-\lambda)^2 \;=\; 1-\lambda \;=\; 1-\frac{r+c+1-q}{2rc}.$$

Worked example

We determine the asymptotic enumeration of the following class:

$$M_{L_7} \;=\;$$ .

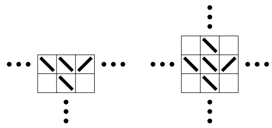
As before, $\left|\mathsf{Grid}_n^{\#}(M_{L_7})\right| \sim \frac{4}{3} \times 4^n$. M_{L_7} has two non-corner peaks. We know for a 3×3 matrix that $\kappa(L_7) = 1 - \frac{r+c+1-q}{2rc} = \frac{3}{4}$.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{L_7})\right| \;\sim\; 2^{-2} \times \tfrac{3}{4} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{1}{4} \times 4^n.$$

Note that, for an L-shaped or T-shaped class with a 3×3 matrix, $\kappa(L_7)$ is equal to the inverse of the subexponential term $\theta^{\#}$. So, for this special case the asymptotic number of permutations in this class is simply $2^{-p} g^n$, where p is the number of non-corner peaks, and g is the growth rate.

Corner type T_2



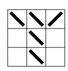
In this corner type there is one peak and one diagonal. This corner type cannot occur in an L-shaped class.

The peak dancing depends on the points that are in any cell adjacent to the column divider to the right of the corner, while the diagonal dancing depends on the points that are adjacent to the column divider to the left of the corner and points that are adjacent to the row divider below the corner. These sources of dancing are asymptotically independent. Therefore, from the L_1 and L_7 analysis, and using Observation 5.2, we get

$$\kappa(T_2) \;=\; \kappa(L_1^{\mathsf{R}})\,\kappa(L_7) \;=\; \frac{1}{2}\left(1 + \frac{r\alpha\lambda}{\alpha+\beta}\right)(1-\lambda).$$

Worked example

We determine the asymptotic enumeration of the following class:

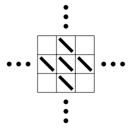
$$M_{T_2} \;=$$  .

As before, $\left|\mathsf{Grid}_n^{\#}(M_{T_2})\right| \sim \frac{4}{3} \times 4^n$. M_{T_2} has zero non-corner peaks. From equations 4.1, 4.2, and 4.3 we find $q = 3, \alpha = \frac{1}{3}, \beta = \gamma = \frac{1}{6}$, and $\lambda = \frac{\alpha^2}{(\alpha+c\beta)(\alpha+r\gamma)} = \frac{r+c+1-q}{2rc} = \frac{1}{4}$. We know for a 3×3 matrix that $\kappa(T_2) = \frac{1}{2}\left(1 + \frac{r\alpha\lambda}{\alpha+\beta}\right)(1-\lambda) = \frac{1}{2}$.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{T_2})\right| \;\sim\; 2^0 \times \tfrac{1}{2} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{2}{3} \times 4^n.$$

Corner type X_0



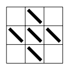
In this corner type there are two diagonals. This corner type cannot occur in an L-shaped or T-shaped class.

Dancing here comes from the two diagonals. For one of them dancing depends on the points adjacent to the row divider below the corner and points adjacent to the column divider to the left of the corner. Dancing on the other diagonal depends on the points adjacent to the row divider above the corner and points adjacent to the column divider to the right of the corner. Thus, these two kinds of diagonal dancing are asymptotically independent. So, by using the analysis for L_7, and Observation 5.2, we have

$$\kappa(X_0) \;=\; \kappa(L_7)^2 \;=\; (1-\lambda)^2.$$

Worked example

We determine the asymptotic enumeration of the following class:

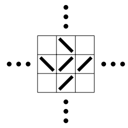
$$M_{X_0} \;=$$  .

As before, $\left|\mathsf{Grid}^{\#}_n(M_{X_0})\right| \sim \frac{4}{3} \times 4^n$. M_{X_0} has zero non-corner peaks. We know for a 3×3 matrix that $\kappa(X_0) = (1-\lambda)^2 = \frac{9}{16}$.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{X_0})\right| \;\sim\; 2^0 \times \tfrac{9}{16} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{3}{4} \times 4^n.$$

Corner type X_7



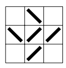
In this corner type there are two peaks and one diagonal. This corner type cannot occur in an L-shaped or T-shaped class.

Dancing here comes from the two peaks and the diagonal. Dancing through one of the peaks depends on points adjacent to the row divider above the corner and points adjacent to the column divider to the left of the corner. Dancing on the diagonal depends on the points adjacent to the row divider below the corner and points adjacent to the column divider to the right of the corner. Thus these types of dancing are asymptotically independent of each other. So, using the analysis for L_4 and L_7, with Observation 5.2, we have

$$\kappa(X_7) \;=\; \kappa(L_4)\,\kappa(L_7) \;=\; \frac{1}{4}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)\left(1+\frac{c\alpha\lambda}{\alpha+\gamma}\right)(1-\lambda).$$

Worked example

We determine the asymptotic enumeration of the following class:

$$M_{X_7} \;=\;$$  .

As before, $\left|\mathsf{Grid}^{\#}_n(M_{X_7})\right| \sim \frac{4}{3}\times 4^n$. M_{X_7} has zero non-corner peaks. We know for a 3×3 matrix that $\kappa(X_7)=\kappa(L_4)\,\kappa(L_7)=\frac{1}{3}$

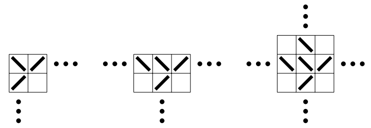
Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{X_7})\right| \;\sim\; 2^0\times\tfrac{1}{3}\times\tfrac{4}{3}\times 4^n \;=\; \tfrac{4}{9}\times 4^n.$$

5.3 Corners with tees

In this section we present a detailed examination of corner types with tees. We find the correction factor for corner types L_3, T_4, and X_4. We then find the asymptotic number of permutations in a 3×3 class with the specified corner type.

Corner type L_3



In this corner type there is one tee. This corner type is the only type of corner that can occur in L, T and X-shaped classes.

This corner type is more complex than any of the types discussed before. That is because the analysis needs more details and points to consider. We start by analysing the main row by reading its points from the bottom. We also do the same for the main column by reading its points from the right. Let C denote the corner cell, and let C_{R} and C_{B} be the cell immediately to the right of C, and the cell immediately below C, respectively.

5.3.1 Sequences of points

Let us consider the main row, and represent the points in this row by words over the alphabet $\{\mathsf{x}, \mathsf{y}, \mathsf{z}\}$. We start reading from the bottom. Let us use x to represent a point in the corner cell C, y to represent a point in C_{R}, and z to represent a point in any other cell of the main row. To identify any specific points we use subscripts. For example, suppose Q is a point in the cell C_{R} then y_Q is the representation of the occurrence of point Q in C_{R}. For example, the sequence of points in the main row of the gridded permutation at the right of Figure 5.4 on page 53 is represented by the word $\mathsf{x}_Q\,\mathsf{y}\mathsf{y}\,\mathsf{z}_S\,\mathsf{z}\,\mathsf{y}_U\,\mathsf{y}\,\mathsf{x}_R\,\mathsf{x}\,\mathsf{y}\,\mathsf{z}\,\mathsf{z}$.

We use simple *regular expressions* for denoting specific sets giving the possible ordering of the initial points. We use a^* to denote a sequence of zero or more copies of letter a in the regular expressions. For example, $\mathsf{z}^*\mathsf{x}_Q$ consists of arrangements of points in the main row where Q is the lowest point in the corner cell (x_Q being the first occurrence of x) and Q is below any points in C_{R} (there being no y in the word). The occurrence of any points below Q can only be elsewhere and is represented by z^*. Note that there is permission for any ordering of points after the initial points specified by the regular expression.

Using Observation 5.1, associated with x, y and z we have the following asymptotic probabilities:

$$p_{\mathsf{x}} \;=\; 1 - c\lambda, \qquad p_{\mathsf{y}} \;=\; \lambda, \qquad p_{\mathsf{z}} \;=\; (c-1)\lambda.$$

We know that $\lambda = \frac{\beta}{\alpha + c\beta}$ is the probability that a point is in C_{R} so $p_{\mathsf{y}} = \lambda$. Then, for the corner cell, because we have $c+1$ columns, $p_{\mathsf{x}} = 1 - c\lambda$. As the probabilities sum up to 1, then we have $p_{\mathsf{z}} = (c-1)\lambda$, which is the probability of a point being in a cell represented by z.

By symmetry for the location of points in the main column, when read from the right, the same approach implies the following asymptotic probabilities:

$$q_{\mathsf{x}} \;=\; 1 - r\lambda, \qquad q_{\mathsf{y}} \;=\; \lambda, \qquad q_{\mathsf{z}} \;=\; (r-1)\lambda.$$

To find the asymptotic probability that the points in the main row are in (the set represented by) some regular expression we take the product after replacing each x, y, z, x^*, y^* and z^* by by p_{x}, p_{y}, p_{z}, $1/(1-p_{\mathsf{x}})$, $1/(1-p_{\mathsf{y}})$ and $1/(1-p_{\mathsf{z}})$, respectively. For example, the asymptotic probability that the arrangement of points in the main row is in $\mathsf{x}_Q\mathsf{y}^*$ equals $p_{\mathsf{x}}/(1-p_{\mathsf{y}})$.

5.3.2 The number of griddings

We recall from the introduction to tee dancing from page 34, that there is a monotone sequence of points which can dance. Except for the end points, the first and last points of the sequence these can always dance between C_{R} and C_{B} through C . However, the end points may only be able to dance between C_{B} and C or between C_{R} and C, but not from C_{R} to C_{B}.

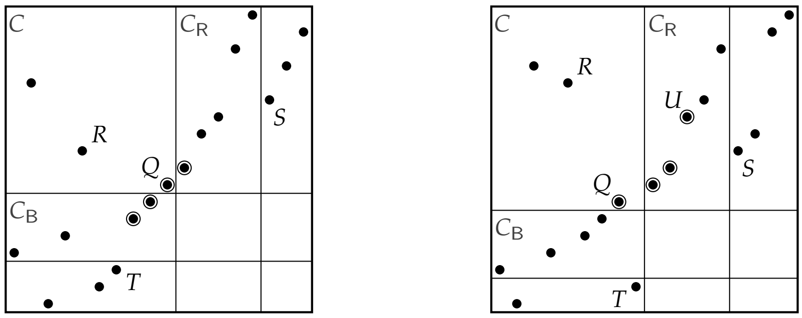
As a result we find that each point which can dance between C_{R} and C_{B} contributes two to the number of griddings. However, a point that can only dance between C_{B} and C or between C_{R} and C contributes only one additional gridding. For the analysis below, we consider the contribution to the number of griddings from points in the main row to be m_1, and we consider the contribution to the number of griddings from points in the main column to be m_2. We must include the original gridding, so we get $m_1 + m_2 + 1$ for the total number of distinct griddings.

5.3.3 Labelled points

Given a constrained gridded permutation, let Q be the lowest point in the corner cell C. The analysis here depends on whether Q is the peak point of each of the two peaks that come from the tee or not.

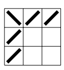
If $c > 1$ (which means there are more than two columns), then let S be the lowest point in the main row which is in any cell other than C or C_R. Similarly, if $r > 1$, then let T be the rightmost point in the main column that is in any other cell than C or C_R.

There are three cases to consider. First, Q is the peak point of both of the tee peaks. Second, Q is not the peak point of either of the tee peaks. Finally, in the third case, Q is the peak point of only one of the two tee peaks.



1a ($k_1 = 1$) and **1a** ($k_2 = 2$)
9 griddings

1b ($k_1 = 2$) and **1c** ($k_2 = 0$)
7 griddings

Figure 5.4: L_3 Case 1: -gridded permutations; the circled points can dance.

Case 1: Q is the peak point of both peaks that form the tee

In Case 1, from our assumption above that Q is the peak point for both peaks, then Q is lower than every point in C_R and to the right of every point in C_B.

If S is above Q, then Q is adjacent to the row divider below C and can dance vertically into C_B, contributing one to the number of griddings. Similarly, if T is to the left of Q, then Q is adjacent to the column divider to the right of C and can dance horizontally into C_{R}, again contributing one to the number of griddings.

Let R be the second lowest point in the corner cell C. That is, R is the point immediately above Q in the corner cell. Points R, S and T are the controllers, because they control which points can dance.

Let k_1 be the number of points in C_{R} below both R and S (if $c > 1$). These k_1 points are above Q, and thus can all dance through the tee to C_{B}, each contributing 2 to the number of griddings. Similarly, let k_2 be the number of points in C_{B} to the right of both R and T (if $r > 1$). These k_2 points are to the right of Q, and can also all dance through to C_{R}, again each contributing 2 to the number of griddings. Note that k_1 and k_2 may be zero. See Figure 5.4 for two gridded permutations satisfying the conditions of Case 1.

In Case 1, the possible ordering of the points in the main row, reading from the bottom, is given by the regular expression $\mathsf{z}^*\mathsf{x}_Q$, the only points below Q (if any) being in cells other than C and C_{R}. So, the asymptotic probability that the points in the main row of a gridded permutation satisfy the conditions of Case 1 is given by

$$p_1 \;:=\; \mathbb{P}\big[\text{Case } 1\big] \;=\; \frac{p_{\mathsf{x}}}{1-p_{\mathsf{z}}} \;=\; \frac{1-c\lambda}{1-(c-1)\lambda}.$$

An analogous result applies for the main column.

For both the main row and the main column, we have three subcases. We analyse the main row, reading its points from the bottom. The analysis of the main column is analogous (reading its points from the right). The three subcases are as follows:

1a. S, if it exists, is above Q, which is the lowest point in the main row. S (if it exists) may be above or below R. There is no point in C_{R} that is above S and below R. See the left of Figure 5.4 for examples in both the main row and the main column. The analysis of Case **1a** is as follows:

Q can dance vertically into C_{B}. Hence k_1 points can dance from C_{R} to C_{B} via C. Thus $m_1 = 2k_1 + 1$.

The possible ordering of the points in the main row, reading from the bottom, is given by the regular expression $\mathsf{x}_Q\mathsf{y}^*\mathsf{z}^*\mathsf{x}_R$. So, for each $i \geqslant 0$, we have

$$p_{\mathbf{1a}}(i) \;:=\; \mathbb{P}\big[\text{Case } \mathbf{1a} \text{ and } k_1 = i\big] \;=\; \frac{p_{\mathsf{x}}^2\, p_{\mathsf{y}}^i}{1-p_{\mathsf{z}}} \;=\; \frac{(1-c\lambda)^2\, \lambda^i}{1-(c-1)\lambda}.$$

1b. S is above Q and below R, and at least one point in C_{R} is above S and below R. Again, Q is the lowest point in the main row. Let U be the lowest of these points (the points that are in C_{R} located above S and below R). Note that U is not one of the k_1 points in C_{R} below both R and S. Other points not in C or C_{R} may lie below U. See the right of Figure 5.4 for an example.

Q can dance vertically into C_{B}, contributing one to the number of griddings. And U can dance horizontally into the corner cell, but not from the corner into C_{B}, so this contributes another gridding. Thus $m_1 = 2k_1 + 2$.

The possible ordering of the points in the main row, reading from the bottom, is given by the regular expression $\mathsf{x}_Q\mathsf{y}^*\mathsf{z}_S\mathsf{z}^*\mathsf{y}_U$. So, for each $i \geqslant 0$, we have

$$p_{\mathbf{1b}}(i) \;:=\; \mathbb{P}\big[\text{Case } \mathbf{1b} \text{ and } k_1 = i\big] \;=\; \frac{p_{\mathsf{x}}\, p_{\mathsf{y}}^{i+1}\, p_{\mathsf{z}}}{1-p_{\mathsf{z}}} \;=\; \frac{(1-c\lambda)\, \lambda^{i+1}\, (c-1)\lambda}{1-(c-1)\lambda}.$$

1c. S is below Q. So Q cannot dance vertically into C_{B}, and $k_1 = 0$, since every point in C_{R} is above Q and hence also above S. Thus $m_1 = 0$. See the right of Figure 5.4 for an example of this in the main *column*, in which the controller T is to the right of Q, so Q can't dance horizontally into C_{R}, and $m_2 = k_2 = 0$.

The possible ordering of the points in the main row, reading from the bottom, is given by the regular expression $\mathsf{z}_S\mathsf{z}^*\mathsf{x}_Q$. Thus,

$$p_{\mathbf{1c}} \;:=\; \mathbb{P}\big[\text{Case } \mathbf{1c} \text{ and } k_1 = 0\big] \;=\; \frac{p_{\mathsf{x}}\, p_{\mathsf{z}}}{1-p_{\mathsf{z}}} \;=\; \frac{(1-c\lambda)\,(c-1)\lambda}{1-(c-1)\lambda}.$$

Table 5.2 gives the total number of griddings for each possibility in Case 1.

		Main row **1a** $m_1 = 2k_1 + 1$	Main row **1b** $m_1 = 2k_1 + 2$	Main row **1c** $m_1 = 0$
Main column **1a**	$m_2 = 2k_2 + 1$	$2k_1 + 2k_2 + 3$	$2k_1 + 2k_2 + 4$	$2k_2 + 2$
Main column **1b**	$m_2 = 2k_2 + 2$	$2k_1 + 2k_2 + 4$	$2k_1 + 2k_2 + 5$	$2k_2 + 3$
Main column **1c**	$m_2 = 0$	$2k_1 + 2$	$2k_1 + 3$	1

Table 5.2: The number of griddings for each combination of subcases in Case 1

We now calculate, for each $\ell \geqslant 1$, the probability $P_{\mathbf{1}}(\ell) := \mathbb{P}\big[\text{Case } \mathbf{1} \text{ and } \ell \text{ griddings}\big]$.

Let $q_{\mathbf{1a}}(i)$, $q_{\mathbf{1b}}(i)$ and $q_{\mathbf{1c}}$ be the subcase probabilities for the main column, formed from $p_{\mathbf{1a}}(i)$, $p_{\mathbf{1b}}(i)$ and $p_{\mathbf{1c}}$ by replacing $p_\times$, p_y, p_z with $q_\times$, q_y, q_z, respectively.

Then, from Table 5.2, we have

$$
\begin{aligned}
P_{\mathbf{1}}(1) &= p_{\mathbf{1c}}\, q_{\mathbf{1c}}, \\
P_{\mathbf{1}}(2) &= p_{\mathbf{1a}}(0)\, q_{\mathbf{1c}} \;+\; p_{\mathbf{1c}}\, q_{\mathbf{1a}}(0), \\
P_{\mathbf{1}}(3) &= p_{\mathbf{1a}}(0)\, q_{\mathbf{1a}}(0) \;+\; p_{\mathbf{1b}}(0)\, q_{\mathbf{1c}} \;+\; p_{\mathbf{1c}}\, q_{\mathbf{1b}}(0), \\
P_{\mathbf{1}}(\ell) &= \quad p_{\mathbf{1a}}\big(\tfrac{\ell-2}{2}\big)\, q_{\mathbf{1c}} \;+\; p_{\mathbf{1c}}\, q_{\mathbf{1a}}\big(\tfrac{\ell-2}{2}\big) \\
&\quad + \sum_{i=0}^{(\ell-4)/2} \Big(p_{\mathbf{1a}}(i)\, q_{\mathbf{1b}}\big(\tfrac{\ell-4}{2} - i\big) \;+\; p_{\mathbf{1b}}\big(\tfrac{\ell-4}{2} - i\big)\, q_{\mathbf{1a}}(i) \Big), \quad \ell \geqslant 4, \text{ even}, \\
P_{\mathbf{1}}(\ell) &= \quad p_{\mathbf{1b}}\big(\tfrac{\ell-3}{2}\big)\, q_{\mathbf{1c}} + p_{\mathbf{1c}}\, q_{\mathbf{1b}}\big(\tfrac{\ell-3}{2}\big) \\
&\quad + \sum_{i=0}^{(\ell-3)/2} p_{\mathbf{1a}}(i)\, q_{\mathbf{1a}}\big(\tfrac{\ell-3}{2} - i\big) \;+\; \sum_{i=0}^{(\ell-5)/2} p_{\mathbf{1b}}(i)\, q_{\mathbf{1b}}\big(\tfrac{\ell-5}{2} - i\big), \quad \ell \geqslant 5, \text{ odd}.
\end{aligned}
$$

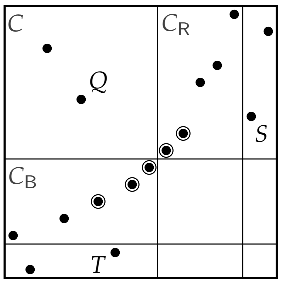
After simplification, facilitated by using a computer algebra system [21] to evaluate the sums and confirm that the probabilities for small values of ℓ are not anomalous yields,

$$
P_{\mathbf{1}}(\ell) \;=\; \begin{cases}
\dfrac{\lambda^{(\ell-3)/2}\, p_\times q_\times \big((\ell-1)\, p_\times q_\times + (\ell+1)\, \lambda\, p_z q_z\big)}{2\,(1-p_z)\,(1-q_z)}, & \ell \geqslant 1, \text{ odd}, \\[2ex]
\dfrac{\ell\, \lambda^{(\ell-2)/2}\, p_\times q_\times\, (p_\times q_z + p_z q_\times)}{2\,(1-p_z)\,(1-q_z)}, & \ell \geqslant 2, \text{ even}.
\end{cases}
$$

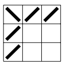
Case 2: Q is not the peak point of either of the peaks that form the tee

In Case 2, there is a point in C_R below Q and a point in C_B to the right of Q. So Q can't dance into either C_B or C_R. Points Q, S and T are the controllers, controlling which points can dance.

Let k_1 be the number of points in C_R below both Q and S (if $c > 1$). These k_1 points can all dance through the tee to C_B, each contributing 2 to the number of griddings. Similarly, let k_2 be the number of points in C_B to the right of both Q and T (if $r > 1$). These k_2 points can also all dance through to C_R, again each contributing 2 to the number of griddings. See Figure 5.5 for a gridded permutation satisfying the conditions of Case 2. Again, k_1 and k_2 may be zero.



2a ($k_1 = 2$) and **2b** ($k_2 = 2$)
10 griddings

Figure 5.5: L_3 Case 2: a  -gridded permutation; the circled points can dance

For both the main row and the main column, we have two subcases. We analyse the main row, the analysis of the main column being analogous.

2a. There is no point in C_R above S and below Q. S may be above or below Q, or S may not exist. Since there is a point in C_R below Q (by the definition of Case 2), in this subcase (only) we know that $k_1 > 0$.

The k_1 points in C_R below both Q and S (if it exists) can all dance through to C_B. Thus $m_1 = 2k_1$.

The possible ordering of the points in the main row, reading from the bottom, is given by the regular expression $\mathsf{y}\mathsf{y}^*\mathsf{z}^*\mathsf{x}_Q$. So, for each $i \geqslant 1$, we have

$$p_{\mathbf{2a}}(i) \;:=\; \mathbb{P}\big[\text{Case } \mathbf{2a} \text{ and } k_1 = i\big] \;=\; \frac{p_{\mathsf{x}}\, p_{\mathsf{y}}^{i}}{1-p_{\mathsf{z}}} \;=\; \frac{(1-c\lambda)\,\lambda^{i}}{1-(c-1)\lambda}.$$

2b. S is below Q with at least one point in C_{R} above S and below Q. Let U be the lowest of these points. Note that U is not one of the k_1 points in C_{R} that are below both Q and S. Other points not in C or C_{R} may lie below U.

U can dance horizontally into the corner cell, but not from the corner into C_{B}. Thus $m_1 = 2k_1 + 1$.

The possible ordering of the points in the main row, reading from the bottom, is given by the regular expression $\mathsf{y}^*\mathsf{z}_S\mathsf{z}^*\mathsf{y}_U$. So, for each $i \geqslant 0$, we have

$$p_{\mathbf{2b}}(i) \;:=\; \mathbb{P}\big[\text{Case } \mathbf{2b} \text{ and } k_1 = i\big] \;=\; \frac{p_{\mathsf{y}}^{i+1}\, p_{\mathsf{z}}}{1-p_{\mathsf{z}}} \;=\; \frac{\lambda^{i+1}\,(c-1)\lambda}{1-(c-1)\lambda}.$$

Table 5.3 gives the total number of griddings for each possibility in Case 2.

		Main row **2a** $m_1 = 2k_1$	Main row **2b** $m_1 = 2k_1 + 1$
Main column **2a**	$m_2 = 2k_2$	$2k_1 + 2k_2 + 1$	$2k_1 + 2k_2 + 2$
Main column **2b**	$m_2 = 2k_2 + 1$	$2k_1 + 2k_2 + 2$	$2k_1 + 2k_2 + 3$

Table 5.3: The number of griddings for each combination of subcases in Case 2

We now calculate, for each $\ell \geqslant 1$, the probability $P_{\mathbf{2}}(\ell) = \mathbb{P}\big[\text{Case } \mathbf{2} \text{ and } \ell \text{ griddings}\big]$.

Let $q_{\mathbf{2a}}(i)$ and $q_{\mathbf{2b}}(i)$ be the subcase probabilities for the main column, formed from $p_{\mathbf{2a}}(i)$ and $p_{\mathbf{2b}}(i)$ by replacing p_{x}, p_{y}, p_{z} with q_{x}, q_{y}, q_{z}, respectively.

Then, from Table 5.3, we have

$$
\begin{aligned}
P_2(3) &= p_{2b}(0)\, q_{2b}(0), \\
P_2(\ell) &= \sum_{i=1}^{(\ell-2)/2} \Big(p_{2a}(i)\, q_{2b}\big(\tfrac{\ell-2}{2} - i\big) + p_{2b}\big(\tfrac{\ell-2}{2} - i\big)\, q_{2a}(i) \Big), \quad \ell \geqslant 4, \text{ even}, \\
P_2(\ell) & \sum_{i=1}^{(\ell-1)/2-1} p_{2a}(i)\, q_{2a}\big(\tfrac{\ell-1}{2} - i\big) + \sum_{i=0}^{(\ell-3)/2} p_{2b}(i)\, q_{2b}\big(\tfrac{\ell-3}{2} - i\big), \quad \ell \geqslant 5, \text{ odd}.
\end{aligned}
$$

After simplification, this yields

$$
P_2(\ell) = \begin{cases} 0, & \ell = 1, \\ \dfrac{(\ell-2)\,\lambda^{\ell/2}\,(p_x q_z + p_z q_x)}{2\,(1-p_z)\,(1-q_z)}, & \ell \geqslant 2, \text{ even}, \\ \dfrac{\lambda^{(\ell-1)/2}\,\big((\ell-3)\, p_x q_x + (\ell-1)\,\lambda\, p_z q_z\big)}{2\,(1-p_z)\,(1-q_z)}, & \ell \geqslant 3, \text{ odd}. \end{cases}
$$

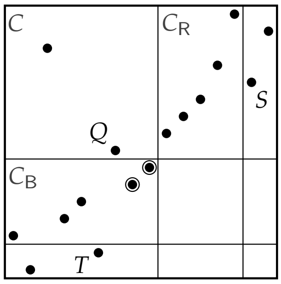
Case 3: Q is the peak point of just one of the two peaks that form the tee

Case 3 combines Case 1 for the main row and Case 2 for the main column, or vice versa.

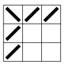
Suppose the main row satisfies Case 1 and the main column satisfies Case 2, as in Figure 5.6. Then Q cannot dance into either C_R or C_B because of the points to its left in C_B. Neither can the points in C_R dance, because Q is below them. So $m_1 = 0$. On the other hand, the Case 2 analysis of the main column is still valid, the k_2 points in C_B to the right of Q being able to dance through to C_R.

The situation is analogous if the main row satisfies Case 2 and the main column satisfies Case 1. Table 5.4 gives the total number of griddings for each possibility in Case 3.

Thus, for each $\ell \geqslant 1$, the probability $P_3(\ell) = \mathbb{P}\big[\text{Case } \mathbf{3} \text{ and } \ell \text{ griddings}\big]$ is given



1 ($k_1 = 0$) and **2a** ($k_2 = 2$)
5 griddings

Figure 5.6: L_3 Case 3: a -gridded permutation; the circled points can dance

		Main row **1** $m_1 = 0$	Main row **2a** $m_1 = 2k_1$	Main row **2b** $m_1 = 2k_1 + 1$
Main column **1**	$m_2 = 0$		$2k_1 + 1$	$2k_1 + 2$
Main column **2a**	$m_2 = 2k_2$	$2k_2 + 1$		
Main column **2b**	$m_2 = 2k_2 + 1$	$2k_2 + 2$		

Table 5.4: The number of griddings for each combination of subcases in Case 3

by

$$P_{\mathbf{3}}(\ell) \;=\; \begin{cases} 0, & \ell = 1, \\ p_{\mathbf{2b}}(\tfrac{\ell-2}{2})\, q_{\mathbf{1}} \;+\; p_{\mathbf{1}}\, q_{\mathbf{2b}}(\tfrac{\ell-2}{2}) \;=\; \dfrac{\lambda^{\ell/2}\,(p_\times q_z + p_z q_\times)}{(1-p_z)\,(1-q_z)}, & \ell \geqslant 2, \text{ even}, \\ p_{\mathbf{2a}}(\tfrac{\ell-1}{2})\, q_{\mathbf{1}} \;+\; p_{\mathbf{1}}\, q_{\mathbf{2a}}(\tfrac{\ell-1}{2}) \;=\; \dfrac{2\,\lambda^{(\ell-1)/2}\, p_\times q_\times}{(1-p_z)\,(1-q_z)}, & \ell \geqslant 3, \text{ odd}, \end{cases}$$

where $q_{\mathbf{1}}$ is the probability that the main column satisfies Case 1, formed from $p_{\mathbf{1}}$ by replacing $p_\times$ and p_z with $q_\times$ and q_z, respectively.

We can now combine the three cases. The asymptotic probability that the underlying permutation of a gridded permutation has exactly ℓ griddings is

$$P_\ell \;=\; P_{\mathbf{1}}(\ell) + P_{\mathbf{2}}(\ell) + P_{\mathbf{3}}(\ell).$$

After considerable simplification, facilitated by using a computer algebra system, we have

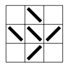
$$P_\ell \;=\; \begin{cases} \dfrac{p_\mathsf{x}\, p_\mathsf{z}\, q_\mathsf{x}\, q_\mathsf{z}}{(1-p_\mathsf{z})\,(1-q_\mathsf{z})}, & \ell = 1, \\[2ex] \dfrac{\ell\, \lambda^{(\ell-2)/2}\,(\lambda + p_\mathsf{x}\, q_\mathsf{x})\,(p_\mathsf{x}\, q_\mathsf{z} + p_\mathsf{z}\, q_\mathsf{x})}{2\,(1-p_\mathsf{z})\,(1-q_\mathsf{z})}, & \ell \geqslant 2, \text{ even}, \\[2ex] \dfrac{\ell\, \lambda^{(\ell-3)/2}\,(\lambda + p_\mathsf{x}\, q_\mathsf{x})\,(p_\mathsf{x}\, q_\mathsf{x} + \lambda\, p_\mathsf{z}\, q_\mathsf{z})}{2\,(1-p_\mathsf{z})\,(1-q_\mathsf{z})}, & \ell \geqslant 3, \text{ odd}. \end{cases}$$

Finally, after further simplification, this gives us the correction factor for the L_3 corner type:

$$\kappa(L_3) \;=\; \sum_{\ell \geqslant 1} P_\ell/\ell \;=\; \frac{\lambda\,(1-\lambda)}{(1-p_\mathsf{z})\,(1-q_\mathsf{z})} \;=\; \frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)}.$$

Worked example

We determine the asymptotic enumeration of the following class:

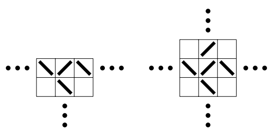
M_{L_3} =  .

As before, $\left|\mathsf{Grid}^\#_n(M_{L_3})\right| \sim \frac{4}{3} \times 4^n$. M_{L_3} has zero non-corner peaks. We know for a 3×3 matrix that $\kappa(L_3) = \frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)} = \frac{1}{3}$.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{L_3})\right| \;\sim\; 2^0 \times \tfrac{1}{3} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{4}{9} \times 4^n.$$

Corner type T_4



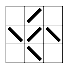
In this corner type there is one peak and one tee. This corner type cannot occur in an L-shaped class.

Dancing at the peak depends on the points adjacent to the column divider to the right of the corner, whereas tee dancing depends on the points adjacent to the column divider to the left of the corner and points adjacent to the row divider below the corner. So these are asymptotically independent. Thus, from the analysis for L_1 and L_3, and by Observation 5.2, we have

$$\kappa(T_4) \;=\; \kappa(L_1^{\mathsf{R}})\,\kappa(L_3) \;=\; \frac{1}{2}\left(1+\frac{r\alpha\lambda}{\alpha+\beta}\right)\frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)}.$$

Worked example

We determine the asymptotic enumeration of the following class:

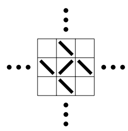
M_{T_4} = .

As before, $\left|\mathsf{Grid}_n^{\#}(M_{T_4})\right| \sim \frac{4}{3}\times 4^n$. M_{T_4} has zero non-corner peaks. We know for a 3×3 matrix that $\kappa(T_4)=\kappa(L_1^{\mathsf{R}})\,\kappa(L_3)=\frac{2}{3}\times\frac{1}{3}=\frac{2}{9}$.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{T_4})\right| \;\sim\; 2^0\times\tfrac{2}{9}\times\tfrac{4}{3}\times 4^n \;=\; \tfrac{8}{27}\times 4^n.$$

Corner type X_4



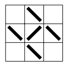
In this corner type there are two tees. This corner type cannot occur in an L-shaped or T-shaped class.

Dancing at one of the tees depends on the points adjacent to the column divider to the right of the corner and the row divider above the corner, whereas dancing at the other tee depends on the points adjacent to the column divider to the left of the corner and points adjacent to the row divider below the corner. So these are asymptotically independent. Thus, from the analysis for L_3, and by Observation 5.2, we have

$$\kappa(X_4) \;=\; \kappa(L_3)^2 \;=\; \left(\frac{\lambda\,(1-\lambda)}{(1-(c-1)\lambda)\,(1-(r-1)\lambda)}\right)^2.$$

Worked example

We determine the asymptotic enumeration of the following class:

$$M_{X_4} \;=\;$$  .

As before, $\left|\mathsf{Grid}^{\#}_n(M_{X_4})\right| \sim \frac{4}{3} \times 4^n$. M_{X_4} has zero non-corner peaks. We know for a 3×3 matrix that $\kappa(X_4) = (\kappa(L_3))^2 = \frac{1}{9}$.

Thus, by Theorem 5.3, we have

$$\left|\mathsf{Grid}_n(M_{T_4})\right| \;\sim\; 2^0 \times \tfrac{1}{9} \times \tfrac{4}{3} \times 4^n \;=\; \tfrac{4}{27} \times 4^n.$$

5.4 Constrained gridded permutations

In this section we conclude by proving that almost all gridded permutations are constrained.

Recall that, if $\mathsf{Grid}(M)$ is a connected one-corner class, then an M-gridded permutation $\sigma^{\#}$ is M-constrained if

(a) every M-gridding of its underlying permutation σ is the result of zero or more points of $\sigma^{\#}$ dancing at a peak or diagonally or through a tee, and

(b) in every M-gridding of σ, each non-blank cell contains at least two points.

Suppose a gridded permutation satisfies Part (a) of this definition. Then, in order to satisfy Part (b), it is sufficient that, in each non-blank cell, there are at least two points that cannot dance, since these points will be in the same cell in all griddings of the underlying permutation.

Non-corner peak points can always dance. However, the set of points which can dance at the corner is determined by the position of the controllers.

For peak dancing (L_1) and diagonal dancing (L_7), each controller is an *extremal* (lowest, highest, leftmost or rightmost) point in one of the cells. For diagonal dancing, the points which can dance are those that are in one of the cells adjacent to the corner and lie between the relevant controller and cell divider.

For tee dancing (L_3), each controller is either an extremal point in a cell, or else is the second lowest or second highest point in the corner cell. As with diagonal dancing, points between a controller and the corresponding cell divider can dance. However, there may also be a single point that is not between the controller and cell divider that can dance. In addition, one or both of the extremal points in the corner cell may also be able to dance.

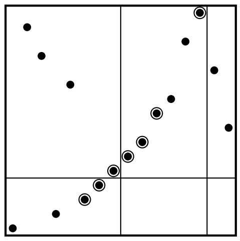
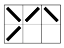


Figure 5.7: A constrained -gridded permutation; the circled points can dance

See Figure 5.7 for an example of a gridded permutation in a class with L_3 corner type and a non-corner peak: in each non-blank cell there are at least two points that can't dance. Note that we may need four points in a cell above a controller in the main row to guarantee two points that cannot dance, since one point just above the controller may be able to dance, and the highest point may be a peak point that can dance.

In this context, we make the following definition. Given a gridded permutation, if C_1 and C_2 are two distinct cells in the same row, then they are *interlocked* if

- C_1 contains at least four points above the second lowest point in C_2,
- C_2 contains at least four points above the second lowest point in C_1,
- C_1 contains at least four points below the second highest point in C_2, and
- C_2 contains at least four points below the second highest point in C_1.

Similarly, if C_1 and C_2 are two distinct cells in the same column, then they are interlocked if

- C_1 contains at least four points to the right of the second leftmost point in C_2,
- C_2 contains at least four points to the right of the second leftmost point in C_1,

- C_1 contains at least four points to the left of the second rightmost point in C_2, and
- C_2 contains at least four points to the left of the second rightmost point in C_1.

Note that if two adjacent cells are interlocked, then their contents together do not form an increasing or decreasing sequence.

The following proposition gives sufficient conditions for a gridded permutation in a connected one-corner class to be constrained.

Proposition 5.4. *Suppose* $\mathsf{Grid}(M)$ *is a connected one-corner class and* $\sigma^\# \in \mathsf{Grid}^\#(M)$ *is such that each pair of cells in the main row is interlocked and each pair of cells in the main column is interlocked. Then* $\sigma^\#$ *is* M*-constrained.*

Proof. We use an argument similar to that employed in the proof of Proposition 2.3 for skinny classes. The contents of each cell of $\sigma^\#$ consists of an increasing or decreasing sequence of points. However, there is no pair of adjacent cells whose contents together form an increasing or decreasing sequence, since each pair of cells in the main row and each pair of cells in the main column is interlocked.

Thus, in any M-gridding of σ, by the monotonicity constraints, there must be a divider between each pair of adjacent non-corner cells of $\sigma^\#$ that have the same orientation and also a divider adjacent to each non-corner peak point of $\sigma^\#$. So the only dancing possible across these dividers is by non-corner peak points.

What about the dividers adjacent to the corner cell? For this, we require a case analysis of each corner type. But this is exactly what is presented in Sections 5.1 to 5.3. The definition of interlocking was chosen precisely so that the interlocking of cells in $\sigma^\#$ guarantees the existence of the controllers and that in each non-blank cell there are at least two points that can't dance.

By the interlocking of cells in $\sigma^\#$, every cell contains at least four points. Each controller is either the first or second point in a cell, when considered in the appropriate direction. Thus, in each case, the points which control the dancing at the corner are present, and the only possibilities for dancing are those described above. Specifically,

- If there is a corner peak (Section 5.1), then the controller is the first point in its cell, and only the peak point (the first point in its cell) may be able to dance.
- If there is a diagonal (Section 5.2), then the only points that can dance are those that precede one of the two controllers, each of which is the first point in its cell.
- If there is a tee (Section 5.3), then each controller is either the first or second point in its cell, and the only points that can dance either precede a controller or immediately follow one.

Thus the possible griddings are restricted to those that result from zero or more points of $\sigma^\#$ dancing at a peak or diagonally or through a tee

The only points in a cell that may be able to dance are those that precede a controller and the one that immediately follows it. The interlocking of cells in $\sigma^\#$ guarantees that there are at least four points in each cell following any controller, of which at most two (the first and last) can dance. Thus each cell of any M-gridding of σ contains at least two points. □

Finally, we prove that almost all gridded permutations in a connected one-corner class are constrained.

Proposition 5.5. *If* $\mathsf{Grid}(M)$ *is a connected one-corner class, then almost all M-gridded permutations are M-constrained:*

$$\lim_{n\to\infty} \mathbb{P}\big[\sigma^\# \text{ is } M\text{-constrained} \;:\; \sigma^\# \in \mathsf{Grid}^\#_n(M)\big] \;=\; 1.$$

Proof. It is sufficient to prove that almost all M-gridded permutations satisfy the conditions of Proposition 5.4. By Theorem 3.6, we know that the number of points in each non-blank cell of almost all n-point M-gridded permutations grows with n.

Suppose C_1 and C_2 are two cells in the same row, containing αn and βn points respectively. Then the probability that C_1 contains exactly k points above the second lowest point in C_2 equals

$$(\alpha n - k)\binom{\beta n + k - 2}{k} \Big/ \binom{(\alpha+\beta)n}{\alpha n}.$$

For fixed k, this tends to zero as n grows, since the numerator is polynomial in n, whereas the denominator grows exponentially.

Thus, given any pair of cells in same row or column, the probability that they are interlinked in an n-point M-gridded permutation converges to 1 as n tends to infinity. □

Chapter 6

Possible future work

In this chapter, we look at how our approach can be extended. So, beyond connected one-corner classes we suggest some classes that have not been enumerated yet, such as the connected two (or more) corner classes (Section 6.1), and disconnected classes (Section 6.2).

First, for connected classes, we can establish the asymptotic distribution of points between the cells in a typical gridded permutation by using the method presented in Section 3. Second, for any connected class with a cell graph that is either acyclic or unicyclic, the generating function approach in Section 4.2 can be extended to give the asymptotic number of gridded permutations. Further details can be found in [8, Theorems 4.3 and 4.5].

6.1 Connected classes

Extending our approach beyond L, T and X-shaped classes we first consider connected classes without adjacent corners such as some connected 2-corner classes, and some zigzag classes. Then, we consider connected classes with adjacent corners such as semi-skinny classes, and some other zig zag classes. Finally, we consider connected classes with more than two corners including cyclic classes.

6.1.1 Connected classes without adjacent corners

Connected classes without adjacent corners such as some two connected corner classes, and some zigzag classes do not support any new possibilities for dancing, and are thus directly amenable to the analysis presented above.

2-corner classes

One family of connected classes without adjacent corners are those with two corners. For example, in Figure 6.1 below there are three connected 2-corner classes with no adjacent corner cells.

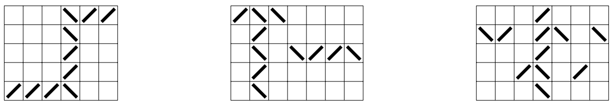


Figure 6.1: Connected 2-corner classes with no adjacent corners

Zigzag classes without adjacent corners

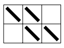
A *zigzag* class is a class in which the corner orientations are restricted to northeast and southwest (or northwest and southeast). The class at the left of Figure 6.1 above is a 2-corner zigzag class without adjacent corners. Zigzag classes can have more than two corners. For example in Figure 6.2 there are two 4-corner zig zag classes with no adjacent corners.

Figure 6.2: Four-corner zig zag classes with no adjacent corners

6.1.2 Connected classes with adjacent corners

Connected classes with corners in adjacent cells, such as the *semi-skinny* (two row) classes, and other zig zag classes, may have non-corner diagonals. Dancing at non-corner diagonals interacts with the dancing at the corners, so additional analysis would be required.

Semi skinny classes

Semi-skinny classes are those with two rows (or columns). Semi-skinny classes may exhibit diagonally adjacent pairs of non-blank cells which are not at a corner. We call these *non-corner diagonals*. Another possibility is the four-cell double-corner  with two diagonals. For examples of non-corner diagonals and a two-diagonal double-corners, see Figure 6.3 below. Further analysis is required of the possibilities for dancing in these structures.

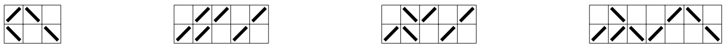
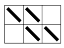
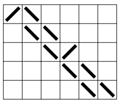


Figure 6.3: Semi-skinny classes, with two adjacent corners showing non-corner diagonals and a 2-diagonal double corner

Note that we are only considering two-corner classes here, so we are not allowing cycles.

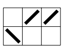
Zigzag classes with adjacent corners

The previous semi-skinny class with the four-cell double-corner  with two diagonals is also a zigzag class. In zigzag classes we may also have longer diagonals such as in . The possibilities of dancing in such classes needs more analysis.

6.1.3 Connected classes with more than two corners

With three or more corners, the general situation quickly gets more complicated. With four or more corners, the cell graph may be unicyclic such as . We would require additional analysis here. See [10] for a detailed investigation of the structural complexity of unicyclic classes.

6.2 Disconnected classes

Generally, *disconnected* classes are not directly amenable to our approach. The reason is that typical large gridded permutations in such classes may not be constrained. For example, in , the asymptotic expected number of points in the non-blank cell at the lower left is finite. Furthermore, the asymptotic probability of that cell being empty or containing just a single point is positive. Specifically, in this class there are 2^{n-k} gridded n-permutations with k points in the lower left cell (for $k = 0, \ldots, n$), and therefore $2^{n+1} - 1$ gridded n-permutations overall. So the asymptotic probability of there being k points in the lower left cell is $2^{-(k+1)}$, and the expected number of points in the lower left cell asymptotically equals 1. We need additional analysis to handle this sort of situation.

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