Stabilization of Stochastic Differential Equations by Feedback Controls Based on Discrete-time Observations

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Abstract

Traditionally, to stabilize an unstable continuous-time stochastic differential equation (SDE) by feedback control, continuous-time observations of the system state are required. This is obviously expensive and unrealistic, so recently Mao discretized the observations. This thesis is to investigate the stabilization problem of continuous-time differential equation systems by deterministic and stochastic feedback controls based on discrete-time observations. This problem includes determining the conditions for original system and controller, and calculating the upper bound of the observation interval, namely the minimum of the observation frequency.

The SDEs discussed in this thesis are all in the Itô sense. The main mathematical fundamentals used are Itôs formula, Lyapunov's second method and inequalities. The problem was investigated under Lipschitz continuity and linear growth condition.

Firstly, I investigated the hybrid SDEs, which is also known as stochastic differential equations with Markovian switching. Using discrete-time observations of system state and mode, we can achieve pth moment stabilization in the sense of asymptotic and exponential stability for $p > 1$. Our new theory expands from the second moment to pth moment and reduces the observation frequency.

Secondly, I used stochastic feedback control, which is based on Brownian motion, to stabilize non-autonomous linear scalar ODEs as well as nonlinear multidimensional hybrid SDEs. Almost sure exponential stabilization is discussed. The new established theory expands the scope of applicable original unstable systems from autonomous ODEs to non-autonomous ODEs and hybrid SDEs.

Thirdly, by making full use of the time-varying system property, I used the time-varying observation intervals instead of a constant as before. Non-autonomous periodic

SDEs and hybrid SDEs are investigated. Many stabilities are discussed, including asymptotic and exponential stabilities in pth moment for $p > 1$ and almost surely. My new established theory not only reduces the observation frequencies, but also offers flexibility on the setting of observations.

Acknowledgements

When I applied for this PhD course, I thought it's statistics. I've always been keen on applied research that will have an impact in practice. Although my PhD study is more like theoretical mathematics than what I wanted, I'm proud of my work and I hope the meaning of my three-year research does not only lie in paper publication and the doctorate degree. Before making acknowledgement, I hope the readers enjoy my thesis and find what I'm proud of has some value.

My acknowledgement goes to my parents, the University and my first supervisor Professor Xuerong Mao. I don't express more gratitude to my parents, as the Chinese saying goes, "If somebody gives you or helps you too much that you cannot thank him or her enough, then you don't have to say it out." I thank the University of Strathclyde for the studentship, which is not only financial support but also an honour awarded to me. When I needed help in mathematics or got lost in career development, I went to Professor Mao, who was always there to provide support and guidance. As a stressed student with high expectation but low confidence, I appreciate his encouragement, patience and offering me flexibility and space to fulfill my potential.

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Notation

$a.s.$	:	almost surely, or with probability 1.
$A := B$	:	A is defined by B or B is denoted by A.
$\emptyset$	:	the empty set.
I_A	:	the indicator function of a set A, i.e. $I_A(x) = 1$ if x A or otherwise 0.
A^c	:	the complement of A in Ω, i.e. $A^c = \Omega - A$.
$A \subset B$	:	$A \cap B^c = \emptyset$.
$\sigma(\mathcal{C})$	:	The σ-algebra generated by $\mathcal{C}$.
$a \vee b$	:	the maximum of a and b.
$a \wedge b$	:	the minimum of a and b.
$f : A \to B$	:	the mapping f from A to B.
$\mathbb{R} = \mathbb{R}^1$	:	the real line.
$\mathbb{R}_+$	:	the set of all nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, \infty)$.
$\mathbb{R}^n$	:	the n-dimensional Euclidean space.
$\mathbb{R}^d_+$	:	$= \{x \in \mathbb{R}^d, x_i > 0, 1 \le i \le d\}$, i.e. the positive cone.
$\mathcal{B}^n$	:	the Borel-σ-algebra on $\mathbb{R}^n$.
$\mathcal{B}$	:	$= \mathcal{B}^1$.
$\|x\|$	:	the Euclidean norm of a vector x.
A^T	:	the transpose of a vector of matrix A.
$\text{trace}(A)$	:	the trace of a square matrix $A = (a_{ij})_{n \times n}$, i.e., $\text{trace}(A) = \Sigma_{1 \le i \le n} a_{ii}$.
$\lambda_{min}(A)$	:	The smallest eigenvalue of a matrix A.
$\lambda_{max}(A)$	:	The largest eigenvalue of a matrix A.
$A = \text{diag}(a_1, \cdots, a_n)$	:	A is an $n \times n$ diagonal matrix with $A_{ii} = a_i$ for $1 \le i \le n$.
$\|A\|$	:	the trace norm of a matrix A, i.e., $\|A\| = \sqrt{\text{trace}(A^T A)}$.
$\\|A\\|$	:	the operator norm of a matrix A, i.e., $\\|A\\| = sup\{\|Ax\| : \|x\| = 1\} = \sqrt{\lambda_{max}(A^T A)}$.

∇	:	$= \big(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\big)$.
V_x	:	the first order partial derivative of V with respect to x, i.e., $V_x = \nabla V = \big(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}\big)$.
V_{xx}	:	the second order partial derivative of V with respect to x, i.e., $V_{xx} = \big(\frac{\partial^2 V}{\partial x_i \partial x_j}\big)_{n\times n}$.
$\\|\xi\\|_{L^p}$	:	$= (\mathbb{E}\|\xi\|^p)^{1/p}$.
$C(D;\mathbb{R}^n)$	:	the family of continuous $\mathbb{R}^n$-valued functions defined on D.
$C^{2,1}(\mathbb{R}\times\mathbb{R}_+;\mathbb{R})$	:	the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}\times\mathbb{R}_+$ which are continuously twice differentiable in $x\in\mathbb{R}$ and once differentiable in $t\in\mathbb{R}_+$.
$L^p(\Omega;\mathbb{R}^n)$	:	the family of $\mathbb{R}^n$-valued random variables ξ with $\mathbb{E}\|\xi\|^p < \infty$.
$L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^n)$	:	the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-measurable random variables ξ with $\mathbb{E}\|\xi\|^p < \infty$.
$L^p([a,b];\mathbb{R}^n)$	:	the family of Borel measuable functions $h:[a,b]\to\mathbb{R}^n$ such that $\int_a^b \|h(t)\|^p dt < \infty$.
$\mathcal{L}^p([a,b];\mathbb{R}^n)$	:	the family of $\mathbb{R}^n$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b \|f(t)\|^p dt < \infty$ *a.s.*.
$\mathcal{M}^p([a,b];\mathbb{R}^n)$	:	the family of processes $\{f(t)\}_{a\le t\le b}$ in $\mathcal{L}^p([a,b];\mathbb{R}^n)$ such that $\mathbb{E}\int_a^b \|f(t)\|^p dt < \infty$.
$\mathcal{L}^p(\mathbb{R}_+;\mathbb{R}^n)$	:	the family of processes $\{f(t)\}_{t\ge 0}$ such that for every $T>0$, $\{f(t)\}_{0\le t\le T}\in\mathcal{L}^p([0,T];\mathbb{R}^n)$.
$\mathcal{M}^p(\mathbb{R}_+;\mathbb{R}^n)$	:	the family of processes $\{f(t)\}_{t\ge 0}$ such that for every $T>0$, $\{f(t)\}_{0\le t\le T}\in\mathcal{M}^p([0,T];\mathbb{R}^n)$.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

1.1 Background of SDEs

Let's start with the birth of stochastic differential equations (SDEs). After more than 70 years of Robert Brown's discovery of particles' random motion in a fluid, Louis Bachelier, Albert Einstein and Marian Smoluchowski used mathematical models to describe the well known phenomenon Brownian motion at the start of 1900s ([1, 2, 3]). Bacheliers Brownian motion arose as a model of fluctuations in stock prices with the 'lack of memory' property, which now known as the Markov property, and both Bachelier and Einstein derived connection between Brownian motion and the heat equation ([4]).

In 1923, Norbert Wiener introduced a rigorous mathematical definition of Brownian motion, so standard Brownian motion is also called the Wiener process ([4, 5]). However, due to the non-differentiability of the Brownian motion paths proved by Wiener, the stochastic integral with respect to a Brownian motion cannot be defined in the ordinary way.

This problem was solved by Kiyosi Itô. Itô gave a mathematical definition of stochastic integral in 1944, which is known as the Itô integral ([6]). Roughly speaking, an 'ordinary integral' of the derivative of a function equals to the function value at the upper limit subtracts its value at the lower limit. However, a stochastic integral of the derivative with respect to a Brownian motion equals to another term subtracted from it. That term is half of the 'ordinary integral' of the second order derivative. This is the difference between ordinary and Itô's integrals ([6]). In 1951, he proposed one of the most powerful tools for stochastic calculus: 'Itô's formula' ([7]). Later Ruslan Stratonovich invented the Stratonovich stochastic integral as an alternative to Itô's integral especially in physics. Since Itô's integral is a more often choice in applied mathematics, this thesis focuses on Itô-type stochastic differential equations. It has the general form:

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \tag{1.1}$$

where $B(t)$ is a Brownian motion and the formal definitions are given in Chapter 2. In (1.1), f and g are system coefficients. f is called the drift and g is sometimes called the diffusion part of the system.

Moreover, many other scientists contributed to the early development of stochastic process. Andrey Kolmogorov defined conditional probabilities as random variables, laid the modern axiomatic foundations of probability theory with his famous book "Foundations of the Theory of Probability" in 1933. According to Davis [4], Paul Lévy formalised the concept of a martingale in 1934; Markov property of Brownian motion was formulated by Joseph Doob in the 1940s and established by Hunt in 1956; influenced by Bachelier, who considered the price process as a martingale and investigated the trajectories of stochastic process, Joseph Doob developed the Martingale theory in his 1953 book "Stochastic Processes" and made martingales a powerful tool in both probability and analysis.

As a type of important mathematical model, stochastic differential equation has been widely used in many areas such as physics, engineering, finance, economics, population, epidemiology, etc., and studied by scientists, engineers and economists all over the world.

Compared to deterministic models, stochastic models incorporate influence of external environmental disturbances and hence are more realistic. As a result, many classic models and theories in various areas have been generalized to stochastic versions. For example, the LotkaVolterra model of predator-prey systems, Gompertz models, oscillators and many other ODE systems have been generalized to SDEs ([8, 9, 10]). Stochastic versions of center manifold theory, Thiele's differential equation and LaSalle's theorem were all studied for SDEs ([11, 12, 13]). The Euler method for ODEs was also generalized to a classic approximation method for SDEs: the Euler-Maruyama method, named after Leonhard Euler and Gisiro Maruyama.

Phase locked loop (PLL) is a control system widely employed in radio, telecommunications, computers and other electronic applications and it can be modelled as SDEs ([14, 15, 16]). According to [16], such system can be used to generate, stabilize, modulate, demodulate, filter or recover a signal from a 'noisy' communications channel where data have been interrupted.

A good example of SDE's applications in finance is the well-known Nobel prize winning work for options pricing - the Black-Scholes model, by Fischer Black, Myron Scholes and Robert Merton. The price of the underlying asset in the Black-Scholes model follows a geometric Brownian motion, which guarantees the price is non-negative. Apart from this, many other SDEs are used in finance. For example, mean reverting process such as the Ornstein-Uhlenbeck process is often used to model interest rates and exchange rates; the Cox-Ingersoll-Ross (CIR) model is used to describe interest rate movement.

Moreover, an important application of SDEs in mathematics is the Feynman-Kac formula, named after Richard Feynman and Mark Kac. That allows us to write the solution of a partial differential equation (PDE) as a stochastic process with SDE of Itô sense involved. The formula builds a link between SDEs and PDEs. So on one hand, PDEs can be studied through probabilistic approach; and on the other hand, expectations of stochastic processes can be computed with deterministic methods ([9]).

Apart from the Brownian motion, other random elements are also incorporated into SDEs. For example, Poisson-driven SDEs have applications in physics, population dynamics and engineering ([17]-[23]). Another classic type of stochastic processes is the Markov chains, proposed by A. A. Markov (Andrei Andreyevich Markov) in his famous 1906 paper ([24]). Markov chain has been widely used in many fields such as population dynamics, chemical reactions, DNA analysis, speech recognition, gambling and economics; it also forms one of the bases of quantum mechanics, according to

Gagniuc [25]. This thesis will discuss SDEs with Markovian switching, we also call it hybrid SDEs.

Several decades ago, people found that some events can cause an abrupt change in the dynamic behaviour of some time-driven systems. For example, component failures, changes in the interconnections of subsystems or external environmental disturbances could lead to sudden change in system structure or parameter ([26]). Consequently the hybrid system arose as the combination of a time-driven system and an event-driven system, where the time-driven system can be modelled by continuous-time differential equations and the event-driven system is described by system mode ([27]).

Since multiple-mode systems fit many situations in practice, hybrid system has drawn considerable attentions in the past decades. It has been used to model national economy and electric power systems; it's also used in financial engineering, wireless communications, substations, powertrain systems, autonomous vehicles and hybrid electric vehicle energy management; hybrid system is also considered as a convenient mathematical framework for multiple target tracking, fault tolerant control, aircraft control, (see e.g. [27]-[30]) etc.

Markov models are good at modelling fault. Fault detection and diagnosis by models based on Markov chains or hidden Markov chains are studied by many researchers (see e.g. [31]-[34]). Roughly speaking, a hybrid SDE is a combination of an SDE and a Markov Chain with the general form

$$dx(t) = f(x(t), r(t))dt + g(x(t), r(t))dB(t), \tag{1.2}$$

where $B(t)$ is a Brownian motion, $r(t)$ is a Markov chain representing the system mode. The formal definitions are given in Chapter 2. An advantage of the hybrid SDE over the regular SDE is that it's good at modelling systems that may experience abrupt changes in structure and parameter because of its Markovian structure. For example, the Markov chain jumps from one state to another when a fault is estimated to happen, then the coefficients f and g change as if the system switches from one mode to another. Song et al. [35] mentioned that recently the Black-Scholes SDE model had been generalised to a linear hybrid SDE of the form

$$dx(t) = \mu(r(t))x(t)dt + \sigma(r(t))x(t)dB(t).$$

1.2 Stabilization and feedback control

Some dynamic systems are important or special and we cannot let them evolve freely without any restriction. Most engineering systems require some attributes (e.g., temperature or voltage) to maintain at some certain levels for normal operations. Governments intervene in markets to overcome market failure, employ monetary and fiscal policy to promote economic growth and control inflation. To preserve the ecosystem, endangered species have been protected by law (e.g., the Endangered Species Act) to prevent extinction. Vehicles, aircraft and artificial satellites all need a good control of velocity, i.e., speed and direction, to work normally and avoid collision. So control is obviously necessary and control theory, which is to ensure the desired outcome through a control action, has extensive applications.

Fundamentally, there are two types of control loops: open loop control, also called feed forward control, and closed loop control, also called feedback control. "Feedback and feed forward are two types of control schemes for systems that react automatically to changing environmental dynamics," stated in [36].

Feedback control is a reactive control. It measures the system state, then the controller generates a control signal by comparing the actual and desired system state. Specifically, the block diagram of a feedback control system is shown in Figure 1.1 as a basic example. A desired state value is given as the reference input. A sensor measures the system output. The error signal is the difference between the desired and observed values. Then the controller generates a control signal by a control law with the error signal. Even if the plant is affected by some unknown disturbances, the output is measured and controller generates corrective action to compensate for it. In other words, feedback control enables the system to adjust automatically without understanding or estimating the influence of disturbances. Therefore, the system can self-correct regardless of the source and type of disturbances ([37]). This lead to advantages of feedback control: it's robust for condition changes and it reduces the system's sensitivity to external disturbances. In contrast, feedforward control is a proactive control and it's based on prediction rather than output observations. It predicts the influence of detected environment change and takes corrective actions before the error occurs. Since Brownian motion is unpredictable, we use feedback control instead of feedforward control for stabilization of SDEs in this thesis.

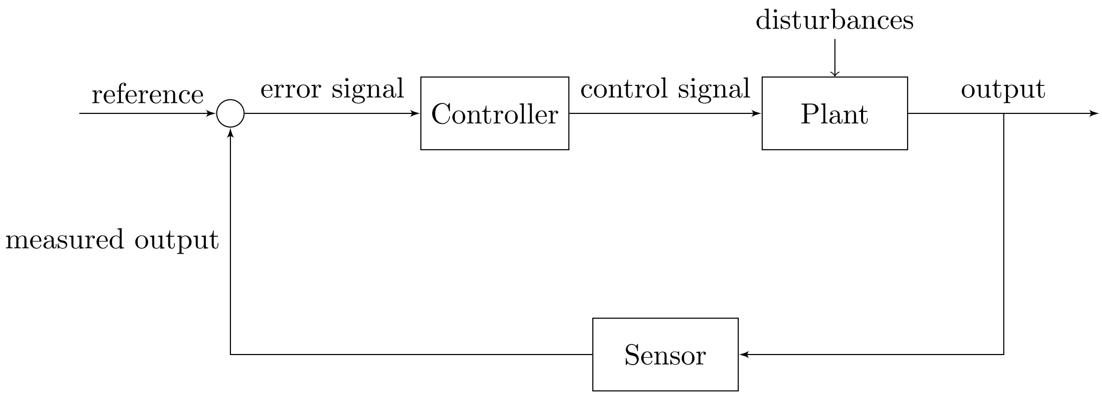

Figure 1.1: Block diagram of feedback control.

The desired behaviour of a system investigated in this thesis is stability. Stability of a process is the ability to resist a priori unknown, small disturbances, according to Kolmanovskii and Myshkis [38]. Stability has many different specific definitions. Roughly speaking, for my research work in this thesis, I refer to it as convergence of the system state to the origin, which is the unique equilibrium point of our SDEs. The formal definitions are stated in Chapter 2.

An important theory of stability analysis is the Lyapunov stability theory, named after the Russian mathematician Aleksandr Mikhailovich Lyapunov. In his 1892 doctoral dissertation ([39]), Lyapunov's second method (also known as Lyapunov's direct method) was proposed, but the magnitude of this great stability theory was not realized until several decades later. Lyapunov's second method provides a sufficient condition to determine the stability of a system governed by differential equation without explicitly calculating the solution. The basic idea of Lyapunov's second method comes from physical systems: if a system is spending energy and the system energy is nonincreasing, then the system must end up at an equilibrium point eventually. From this idea of "measure of energy" ([40]), Lyapunov function was born to be nonnegative, continuous and nonincreasing. An limitation of Lyapunov's second method is that it only provides sufficient condition. If a Lyapunov function satisfying corresponding conditions has not been found, then that theory cannot determine the stability.

Lyapunov stability theory was initially proposed for ODEs and was later developed for SDEs. Khas'minskii said in [41]: "in practical applications one may often assume that the 'noise' has a 'short memory interval'. The natural limiting case of such noise is of course white noise. Thus it is very important to study the stability of solutions of Itô equations since this is equivalent to the study of stability of systems perturbed by white noise."

In 1965, Bucy [42] introduced the concept of "stability of a random solution" in probability and almost sure. In 1967, Khas'minskii investigated necessary and sufficient conditions for the asymptotic stability of linear SDEs [43]. Let's roughly compare the Lyapunov stability theory for ODEs and SDEs in a non-rigorous way. Lyapunov's second method said that for an ODE with the form

$$\dot{x}(t) = f(x(t), t),$$

if there is a positive-definite function $V(x, t)$ such that

$$\dot{V}(x, t) := V_t(x, t) + V_x(x, t)f(x, t)$$

is negative semidefinite, then the system is stable. Here $V(x, t)$ is called the Lyapunov function. For an SDE with the form

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t),$$

if there is a positive-definite function $V(x, t)$ such that

$$\mathcal{L}V(x, t) := V_t(x, t) + V_x(x, t)f(x, t) + \frac{1}{2}\text{trace}[g^T(x, t)V_{xx}(x, t)g(x, t)]$$

is negative semidefinite, then the system is stable. The difference is that $\dot{V}(x, t)$ is replaced with $\mathcal{L}V(x, t)$. More specifically, stochastic system has an additional term $\frac{1}{2}\text{trace}[g^T(x, t)V_{xx}(x, t)g(x, t)]$ compared to the deterministic system. This means the size of noise can affect the stability of the stochastic system.

Apart from Lyapunov theory, LaSalle's theorem (also called LaSalle's invariance principle) is another important criterion for asymptotic stability. To establish asymptotic stability, Lyapunov theory requires $\dot{V}(x, t)$ to be negative definite, while LaSalle's theorem only requires $\dot{V}(x,t)$ to be negative semidefinite.

In 1967, Khas'minskii opened a new chapter in stochastic stability theory by giving a necessary and sufficient criterion for asymptotic stability of linear SDEs [44]. In 1969, Khas'minskii presented a theory of stability for SDEs, some of which are generalizations of Lyapunovs second method to stochastic systems, including stability in probability and almost sure, moment stability, asymptotic and exponential stabilities [41]. Roughly speaking, asymptotic stability means convergence to the equilibrium point 0; exponential stability means exponentially fast convergence; p-th moment stability means it's $\mathbb{E}|x(t)|^p$ that converges; probability and almost sure stabilities mean it's $x(t)$ that converges in probability and almost sure respectively. In 1981, Curtain gave sufficient conditions for pathwise asymptotic stability and exponential stability of stochastic partial differential equations [45]. In 1986, Arnold, Oeljeklaus and Pardoux studied more systematically the almost sure and moment stability for linear SDEs [46], said by Mao [47], which discusses the mean square eventually uniformly asymptotic stability for nonlinear SDEs in 1990. In 1987, Brusin and Ugrinovskii provided sufficient conditions for mean square asymptotic and exponential stabilities of SDEs by Lyapunov functional. Mean square asymptotic stability means $\mathbb{E}|x(t)|^2$ converges to 0 as t goes to infinity. Mean square exponential stability means $\mathbb{E}|x(t)|^2$ converges exponentially fast. Stability of stochastic functional differential equations was studied (see e.g. [48, 38, 49]). In 1996, Razumikhin-type theorems for the ordinary differential delay equations was generalized to stochastic functional differential equations (SFDEs) [50]. In 1999, Mao [13] presented stochastic versions of the LaSalle theorem, providing another method and leading more study on asymptotic stability analysis of SDDEs and SFDEs (see e.g. [51]-[54]). In 2007, Mao [55] discussed exponential and asymptotic stabilities for linear and nonlinear SDDEs. Stability of stochastic neural networks was also discussed by Huang et al. [56, 57]), etc. Stability analysis for SDEs with uncertainties was also studied, for example, Gao et al. [58] provided criteria for robust mean square asymptotic stability. Moreover, stabilities of SDEs have also been studied by many other researchers such as [23], [59]-[63].

Single-mode SDE system is not enough, many practical systems have several modes or subsystems. Therefore, hybrid system has attracted increasing attention in past decades. Now let us briefly review the development of stability analysis for hybrid SDEs.

In 1996, Basak et al. ([64]) studied the hybrid ODE (also called linear jump system) with $r(t)$ representing the Markov chain

$$dx(t) = A(r(t))x(t)dt \tag{1.3}$$

disturbed by Gaussian type noise with the following semi-linear SDE form

$$dx(t) = A(r(t))x(t)dt + \sigma(x(t), r(t))dB(t). \tag{1.4}$$

They investigated conditions under which the stability of the linear jump system (1.3) is remained in the perturbed one (1.4). They also showed that under certain conditions, the unstable linear jump system (1.3) can be stabilized by noisy perturbation. For the hybrid system (1.4), they also introduced the notion of stability in distribution for the time-homogeneous Markov process $(x(t), r(t))$. One interesting point discovered by Basak et al. [64] is that: even though all the subsystems in different modes are stable, the hybrid system may not be stable; and on the other hand, the hybrid system may be stable even if all the subsystems are not stable. The influence of Markovian switching on the system stability attracted many researchers to study the stability for hybrid SDEs.

In 1999, Mao [65] discussed the exponential stability in moment and almost sure for general nonlinear hybrid SDEs. In 2000, Mao, Matasov and Piunovskiy [66] investigated stability of hybrid SDDEs (SDDEs with Markovian switching), including moment and almost sure exponentially stability, with linear or nonlinear delay. Later, asymptotic stability in distribution of hybrid SDEs, exponential and asymptotic stabilities in moment and almost sure for hybrid SDDEs were studied (see e.g.[67, 55]). In 2007, Yin etal [68] provided sufficient conditions for almost sure exponential stability and almost sure exponential instability for linear and nonlinear hybrid SDEs. Their study reveals more about the mixed influence of system coefficients and Markov chain. For nonlinear systems, their proposed conditions are combinations of the stationary distribution or transition rate of the Markov chain and the growth rates of system coefficients under each system mode. Stability of hybrid SDDEs were also studied by other researchers in recent years (see e.g. [69, 70, 71]).

Compared to $\mathcal{L}V(x(t),t)$ for a single-mode SDE, $\mathcal{L}V(x(t),r(t),t)$ for a hybrid SDE has an additional term involving transition rate of the Markov chain. Let $r(t)=i$ and denote the transition rate from state i to k by γ_{ik}, then (see e.g. [65, 67])

$$\begin{aligned}\mathcal{L}V(x,i,t) :=& V_t(x,i,t) + V_x(x,i,t)f(x,i,t) + \frac{1}{2}\text{trace}[g^T(x,i,t)V_{xx}(x,i,t)g(x,i,t)]\\ &+ \sum_{k=1}^{N}\gamma_{ik}V(x,k,t).\end{aligned}$$

With knowledge of the system stability, why do we want to stabilize the system by a control? Suppose we want a variable of interest in a dynamic system to stay at a certain level. Let a new variable equal to the target value subtracted from the interested variable. Then by stabilizing the system in terms of the new variable, our interested variable converges to the target value. Stabilization problem has been studied for decades.

In 1990, control problem of linear jump systems was studied by Ji, Chizeck and Mariton ([72, 73]). Ji and Chizeck proposed definitions of stochastic stabilizability and stochastic controllability for continuous-time Markovian jump linear systems [72]. In 1995, Chen and Francis discussed stabilization problem of discrete-time feedback systems by observer-based controllers [74]. In 2002, exponential stabilization of non-holonomic dynamic systems by time-varying control was studied in [75]. Later, adaptive stabilization of uncertain systems was studied in [76]. In 2005, Allwright, Astolfi and Wong discussed asymptotic stabilization of continuous-time linear systems by periodic piecewise constant output feedback and gave a necessary and sufficient condition [77]. In 2007, Xu et al. [78] investigated robust exponential stabilization for Markovian jump systems with delay. Moreover, stabilization of uncertain SDEs by robust control also attracted attention. Different types of control are discussed for this problem (see e.g. [79, 80]). In addition, adaptive control has also been studied for uncertain stochastic systems by many authors (see e.g. [81]-[85]).

Apart from [64], most of above stabilization research uses deterministic controller. Many other researchers study how to control the stability of a system through stochastic disturbances. In 1985, Scheutzow showed that some type of diffusion can be stabilized by noise [86]. Stabilization and destabilization of ODEs by noise have been studied by many authors. In 1993, Scheutzow provided two ODE dynamical systems in the plane, one was stabilised and the other one was destabilised by white noise [87]. Specifically, [87] mentioned that: one ODE which explodes in finite time for every initial condition becomes stable by adding white noise in the sense that the system becomes nonexplosive and even positive recurrent; on the other hand, the globally asymptotically stable ODE system becomes explosive when it is perturbed by additive white noise. Later, Mao showed that any multidimensional nonlinear ODEs can be stabilised or destabilised by Brownian motion under some conditions [88]. In 2007, Yin et al. [68] investigated stabilization and destabilization of hybrid ODEs through stability and instability of hybrid SDEs. They discussed almost sure exponential stability and instability. In 2015, Li and Liu established almost sure stabilization criteria for nonlinear SDEs by adding feedback controls on both drift and diffusion part [89]. In 2016, Li and Liu investigated stabilization and destabilization of ODE systems by adding time-varying noise as diffusion part of the controlled SDE systems [90]. An important advantage of stochastic feedback controls over the deterministic ones is that it can make the controlled system almost surely exponentially stable without changing the orginal state mean. This is in accordance with the concept called volatility-stabilized market in mathematical finance (see e.g. [91, 92, 93]). In ecosystem, the stability of SDE models also reveal that the environmental noise might cause a population extinct (see e.g. Chapter 11 in [9]), said in [91].

1.3 Motivation and research background

Usually, stabilization of continuous-time SDEs requires continuous-time observations of the system (see e.g. [79, 80, 68]). However, this would lead to high cost and sometimes it's unrealistic as the observations are often of discrete time in practice. Therefore, in 2013, Mao [94] initiated the study of stabilization of continuous-time SDEs by feedback controls based on discrete-time observations.

Stabilization problem by feedback control based on the discrete-time state observations for the deterministic differential equations has been studied since decades ago. For example, in 1988, Hagiwara and Araki studied stabilizability of continuous-time ODE systems by periodically time-varying controller that detects the plant output only once during a time interval called "a frame period" [95]. Many other papers discussed controller that detects several times during a frame period.

Consider an unstable continuous-time hybrid SDE in the Itô sense

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t), \tag{1.5}$$

on $t \geq 0$, where $x(t) \in \mathbb{R}^n$ is the system state, $B(t) = (B_1(t), \cdots, B_m(t))^T$ is an m-dimensional Brownian motion and $r(t)$ is a Markov chain which represents the system mode. When stabilizing the system with a feedback control, a traditional (or regular) choice form is $u(x(t), r(t), t)$ based on continuous-time observations of state $x(t)$, and the controlled system has the form

$$dx(t) = \big(f(x(t), r(t), t) + u(x(t), r(t), t)\big)dt + g(x(t), r(t), t)dB(t). \tag{1.6}$$

" it is more realistic and costs less in practice if the state is only observed at discrete times, say $0, \tau, 2\tau, \cdots$", said in [94], "the feedback control should be designed based on these discrete-time observations, namely the feedback control should be of the form $u(x([t/\tau]\tau), r(t), t)$ where $[t/\tau]$ is the integer part of t/τ."

That is, Mao [94] discretized the state observations by setting a constant observation interval. Observation interval is the time length between two consecutive observations. If we denote the observation interval by τ, then x is observed at time points $\delta_t = k\tau$ for $t \in [k\tau, (k+1)\tau)$, where $k = 0, 1, 2, \cdots$. δ_t can also be seen as a step function of time t. For convenience of notation, let $\delta_t = [t/\tau]\tau$ where $[t/\tau]$ denotes the integer part of t/τ. So by choosing a positive number τ representing the observation interval, the controller has the form $u(x([t/\tau]\tau), r(t), t)$ and the consequent controlled system becomes

$$dx(t) = \big(f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t)\big)dt + g(x(t), r(t), t)dB(t). \qquad (1.7)$$

The new form of controller needs state observations at time points $0, \tau, 2\tau, 3\tau, \cdots$, which is more realistic and costs less than the traditional control based on continuous observations. In [94], the original system starts from initial time t_0. In that case, x is observed at $t_0, t_0 + \tau, t_0 + 2\tau, \cdots$. Letting $\delta_t = t_0 + [t/\tau]\tau$ enables us to write the controlled system in a simpler way,

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(t), t)\big)dt + g(x(t), r(t), t)dB(t). \qquad (1.8)$$

In [94], the system coefficients f, g and controller function u are all assumed to be globally Lipschitz continuous on x and vanish when $x = 0$. This implies that f, g and u all satisfy linear growth condition on x. Under this framework, [94] shows that, if the traditionally controlled SDE (1.6) is mean square exponentially stable, then so is the new controlled system (1.8), as long as the observation interval τ is sufficiently small.

The existence and uniqueness of the solution is guaranteed by the fact that (1.8) is an SDDE with a bounded variable delay, under the framework of Lipschitz continuity and linear growth. Specifically, we can write $\delta_t = t - \zeta$ where $\zeta = t - (t_0 + [t/\tau]\tau) \in [0, \tau)$. In system (1.8), the pair $(x(t), r(t))$ has Markov property at discrete times δ_t. That is, the evolution of $(x(t), r(t))$ after a time point does not depend on its history before that time point. The new theorem in [94] is mainly derived through the Markov property of the pair and by comparing the new controlled system (1.8) with an auxiliary system of the form (1.6). The auxiliary continuous-time observed system is assumed to be mean square exponentially stable in [94]. The difference between the two systems (the continuous-time observed system and discrete-time observed system) is bounded over each observation interval, with upper bound positively related to the length of observation interval. Hence as long as the observation interval is short enough, the new controlled system can achieve mean square exponentially stable, although with a different rate. An upper bound for observation interval was also derived in [94]. Since comparing two systems is a very general proof method, and existence of Lyapunov function is not assumed, some properties of the system are not fully used. As a result, the upper bound is very small, but luckily it's still large enough for computer simulation.

In 2014, Mao et al. [96] investigated mean square exponential stabilization for both linear and nonlinear hybrid SDEs. They used linear controller and the linear controlled system has the form

$$dx(t) = [A_{r(t)}x(t) + D_{r(t)}x(\delta_t)]dt + \sum_{k=1}^{m} B_{r(t)}x(t)dB_k(t) \tag{1.9}$$

and the semilinear controlled system has the form

$$dx(t) = [f(x(t), r(t), t) + F_{r(t)}G_{r(t)}x(\delta_t)]dt + g(x(t), r(t), t)dB(t), \tag{1.10}$$

where $A_{r(t)}$, $B_{r(t)}$, $D_{r(t)}$, $F_{r(t)}$ and $G_{r(t)}$ are all matrices and $D_{r(t)}x(\delta_t)$ as well as $F_{r(t)}G_{r(t)}x(\delta_t)$ are controllers. For the nonlinear system (1.10), they relaxed the Lipschitz continuity condition of system coefficients from globally in [94] to locally.

Without the auxiliary traditionally controlled system, they analyzed the new controlled system directly by Lyapunov method. They added assumptions of Lyapunov functions by assuming existence of symmetric positive-definite matrices, which was used as the core part of the quadratic-form Lyapunov function. Namely, Lyapunov function $V(x(t), r(t), t) = x^T(t)Q_{r(t)}x(t)$ for symmetric positive-definite matrices $\{Q_{r(t)}\}_{r(t)\in\mathbb{S}}$. Mao et al. [96] wrote the stability condition that $\mathcal{L}V(x(t), r(t), t)$ needs to be negative-definite as matrix inequalities.

The paper [96] didn't connect the discrete-time observed system with the continuous-time observed system, but since the state observations were discretized, there must be a connection between the state values in discrete and continuous time, and we have to make use of it to analyse the new system. Therefore Mao et al. [96] calculated $\mathbb{E}|x(t) - x(\delta_t)|^2$. We also need to connect the difference $\mathbb{E}|x(t) - x(\delta_t)|^2$ with some quantity we know or easy to handle. This is usually either $x(t)$ or $x(\delta_t)$ with $x(t)$ more often. They showed that the difference $\mathbb{E}|x(t)-x(\delta_t)|^2$ cannot be greater than the product of $\mathbb{E}|x(t)|^2$ and a positive number, which is positively related to the observation interval.

Two frequently used tricks for mean square exponential stabilization are: 1) setting the Lyapunov function as quadratic form of x, e.g., $V = x^T Qx$ where Q is a symmetric positive-definite matrix; and 2) applying the Itô formula to the product of the exponential of time and quadratic form of x, e.g., $e^{\theta t}x^TQx$. By using these tricks and connecting $x(\delta_t)$ with $x(t)$ as explained above, [96] dramatically improved the upper bound on observation interval. For the same example, the observation interval was improved from 0.0000308 in [94] to 0.0046 in [96]. The analysis technique used in [96] is usually called LMI (Linear Matrix Inequality) and is one of the most popular methods for control problem.

In 2015, You et al. [97] studied stabilization problem for nonlinear hybrid SDEs with the controlled system of the same form as (1.8). Regarding to the Lipschitz continuity condition, coefficients f and g were assumed to be locally Lipschitz continuous on x and controller u was assumed to have globally Lipschitz continuity. Compared to [94] and [96], [97] established stabilization criteria using a more sophisticated method - constructing Lyapunov functional. The Lyapunov functional is the sum of a Lyapunov function and a double integral:

$$\begin{aligned} &U(\hat{x}_t, \hat{r}_t, t) = V(x(t), r(t), t) \\ &+ \theta \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau |f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dv ds, \end{aligned} \tag{1.11}$$

where $\hat{x}_t$ denotes a segment of x over $[t - 2\tau, t]$ and so does $\hat{r}_t$.

This more complicated method has two advantages for analysis. Firstly, assumption was made on the Lyapunov function's partial derivatives and their combinations with system coefficients as well as transition rates of the Markov chain. This assumption is more specific about the property of the Lyapunov function than [96], so it has no requirement on the form of Lyapunov function. Apart from quadratic, more general forms can be used to establish stabilization, compared to [96]. Secondly, the double integral in (1.11) can cancel out the term $\mathbb{E}|x(t) - x(\delta_t)|^2$ through parameter setting. In other words, instead of finding an upper bound of $\mathbb{E}|x(t) - x(\delta_t)|^2$ and restricting it as in [96], this term was ingeniously cancelled out and had little influence on remaining analysis in [97]. This is one of the reasons why [97] can get larger upper bound of observation interval.

The analysis idea in [97] is more complicated but ingenious than that in [96]. In [96], exponential stability was derived from assumptions of Lyapunov functions only, which is more straightforward. However in [97], exponential stability was built on Lyapunov functional and other weaker stabilities. You et al. started with a relatively weak stability - mean square H_∞ stabilization, by the above two advantages. It was proved that the controlled system satisfied

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty. \tag{1.12}$$

Based on the H_∞ stability and other system properties, [97] proved that the controlled system was mean square asymptotic stable. Then based on these two stabilities, almost sure asymptotic stability was achieved. With an additional assumption about the range of Lyapunov function, and making use of the Lypunov functional, [97] built mean square exponential stability for the controlled system on mean square H_∞ stability and mean square asymptotic stability. From that, almost sure exponential stability was implied.

Constructing the Lyapunov functional gives two advantages for the results. One is that it brings more results as more stabilities were built. The other one is reduction of observation frequency. The upper bound on observation interval was improved from 0.0046 to 0.0074 compared to [96] for the same numerical example. As quadratic-form Lyapunov functions are used frequently, said in [97], corollaries and alternative assumptions were given in this case.

Moreover, the stabilization problem by feedback control based on discrete-time observations was studied by many researchers. In 2015, You et al. [98] investigated robust stabilization problem for hybrid uncertain stochastic systems. They established the robust mean square exponential stability for the controlled linear hybrid SDEs with norm bounded uncertainties. The drift and diffusion part of the system have different uncertainties. Controller appears in both drift and diffusion part. Only the uncertainties depend on time explicitly. In 2016, Qiu et al. [99] investigated mean square exponential stabilization of linear and nonlinear hybrid SDEs. Apart from the observation interval, they also considered a constant time delay for observations, which may be caused by data transmission. Similar to [96], both [98] and [99] used the Linear Matrix Inequality technique for analysis.

I noticed an obvious limitation of the research on this problem - stabilization of SDEs by feedback controls based on discrete-time observations. All the research focused only on the mean square exponential stabilization except [99], which still only discussed the mean square stabilities in terms of the moment order.

Mean square stability is only a special case of moment stability when moment order $p = 2$. Stabilities in terms of a more general moment order needed to be discussed. In terms of the moment order, mean square stability is the simplest case. When $p = 2$, in the calculation of $\mathcal{L}V(\cdot)$ or Lyapunov functional, terms appearing have the simplest coefficients and some have coefficients 0. In terms of asymptotic and exponential stabilities, higher moment stability is stronger than lower moment stability, so the former can imply the latter.

On one hand, higher moment is important and frequently used. For example, skewness ($p = 3$) and kurtosis ($p = 4$) are basic and important measures in statistics and frequently used in finance; study in digital image process can use moment order $p = 50$ (see e.g. [100]-[103]). On the other hand, some problems only need lower moment for $p < 2$ and mean square stability is unnecessarily too strong. For example, to make the controlled system almost surely exponentially stable, You et al. [99] had to achieve mean square exponential stabilization. By choosing a smaller p, I can make it through lower moment exponential stabilization. In other words, I can make the system almost surely exponentially stable by weaker conditions than what [97] required. Therefore, it's very necessary to investigate pth moment stabilization for a wide range of p, but no results about this problem had been reported. This motivated me to investigate what was stated in Chapter 3 - the pth moment stabilization for $p \in (1, \infty)$. My new theory enables researchers to choose p flexibly according to their needs from the wide range $(1, \infty)$. I submitted the results to the peer-reviewed journal "Stochastic Analysis and

Applications" in November 2016, which has been published as [104].

Later, the observations of system mode, i.e. the Markov chain have also been discretized. That is, both system state x and system mode r are observed at time points $0, \tau, 2\tau, 3\tau, \cdots$. The controller has the form $u(x(\delta_t), r(\delta_t), t)$ and the controlled system is

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), r(t), t)dB(t). \tag{1.13}$$

Discrete-time observations of the Markov chain are analyzed through its property that, the time until the process jumps follows the exponential distribution with rate equivalent to the transition rate of current state. In 2016, Song et al. [35] mentioned that "it could often be the case where the mode is not obvious and it costs to identify the current mode of the system." To reduce the cost control, they used discrete-time observations of both system state and system mode to control the unstable hybrid SDEs of a very general nonlinear form. The general idea is similar to that of Mao's paper in 2013 [94]. They assumed that the auxiliary traditionally controlled system, which is based on continuous-time observations with the form (1.6), was mean square exponential stable. Then they compared the auxiliary controlled system and the new controlled system with form (1.13), by properties of Markov chain and the Markov property of the pair $(x(t), r(t))$ at discrete times δ_t. Finally mean square exponential stability for the new controlled system was established.

Apart from stabilization by deterministic feedback controls, which appear on the drift, stabilization by stochastic feedback controls based on discrete-time observations was studied by Mao [91] in 2016. Mao showed that under some conditions, the scalar linear autonomous [1] ODE $dx(t) = \alpha x(t)dt$ and the multidimensional nonlinear ODE $dx(t) = f(x(t))dt$ starting from $t_0 = 0$ can be stabilized in the sense of that the controlled systems

$$dx(t) = \alpha x(t)dt + \sigma x(\delta_t)dB(t) \tag{1.14}$$

[1] The system is said to be autonomous if the coefficients do not depend on time explicitly. Here the coefficients depend on t through x.

and

$$dx(t) = f(x(t))dt + Ax(\delta_t)dB(t) \tag{1.15}$$

become almost surely exponentially stable. Almost sure exponential stability was derived from the pth moment exponential stability through the Borel-Cantelli lemma under some conditions. The main analysis technique for the scalar linear case is very special. It's to make $\mathbb{E}|x((k+1)\tau)|$ equal to or less than $\mathbb{E}|x(k\tau)|$ times a positive number less than 1. This method makes use of the variation-of-constants formula at each observation interval $[k\tau, (k+1)\tau]$ and the moment properties of the standard normal distribution. It's not easy to apply this special analysis technique to multidimensional case. The more general nonlinear multidimensional case was investigated through comparing two controlled systems - (1.15) and the continuous-time observations based auxiliary system, similar to [94]. The observation interval upper bounds derived in [91] are too small to give any numerical example, so it can only be said that the upper bounds exist.

Inspired by Mao [91], I investigated the stabilization problem by stochastic feedback controls based on discrete-time observations for more complex models.

The most popular methods used in stabilization problem for deterministic feedback controls are usually based on constructing Lyapunov function or functional. However, these methods cannot be applied to stochastic feedback control. Particularly this means the Linear Matrix Inequality method cannot be used here. This is probably an important reason why there was no more published improvement or development after [91]. Recall that stabilization by noise has an advantage that it would not change the original state mean $\mathbb{E}x(t)$. So I'm still interested in this problem regardless of its difficulty and complexity. Mao [91] only discussed stabilization of autonomous ODEs, which is not enough. Systems of more complex forms should also be studied, for example, non-autonomous ODEs and multidimensional hybrid SDEs. Non-autonomous ODEs have been extensively applied in science and engineering over the past decades. For example, time-varying complex dynamical network model (see e.g. [105, 106]), linear time-varying (LTV) plants (see e.g. [81, 107]) and LTV multiple-inputmultiple-output (MIMO) state-space systems (see e.g. [108]). Motivated by above, I investigated stabilization of scalar non-autonomous ODEs and multidimensional hybrid SDEs, by stochastic feedback control based on discrete-time observations. For hybrid systems, observations for both system state and mode are in discrete time. The results were stated in Chapter 4. It was also submitted to the journal "Stochastic Analysis and Applications" in May 2017 and published as [109].

Similarly to the development from comparison of two controlled systems in [94] to the Linear Matrix Inequality analysis technique in [96], in 2017, Li et al. used Linear Matrix Inequality to established stabilization criteria using discrete-time state and mode observations. In [110], Li et al. discussed mean square exponential stabilization for both linear and nonlinear hybrid SDEs. The controlled linear system has form (1.9) and the nonlinear system uses linear controller. In [111], Li et al. investigated robust stabilization of hybrid uncertain stochastic systems. The drift and diffusion part of the system have different uncertainties. Only the uncertainties depend on time explicitly. They showed that the controlled system with norm bounded uncertainties is robustly exponentially stable in mean square. The main difference between [111] and [98] is that the controller only works on the drift and observations of mode are in discrete-time in [111].

Moreover in 2017, Zhu and Zhang [112] also discussed the pth moment stabilization problem, but for pth moment exponential stability where $p > 2$ only. The key difference between my paper [104] and theirs [112] is that, they considered the constant time delay for observations, similarly to [99]. They extended the exponential stability from mean square to higher moment. As for system observations, they used feedback controls based on discrete-time state observations to stabilize hybrid SDEs.

I noticed that all the research on this topic (stabilization by feedback controls based on discrete-time observations), as far as I know, used a constant observation interval with no flexibility on observation frequency. None of the papers mentioned above made use of the time-varying property for non-autonomous systems. Although the systems have the form

$$dx = [f(x, r, t) + u(x, r, t)]dt + g(x, r, t)dB,$$

since most calculation and analysis were did to the Lipschitz constants of f, g and u

instead of to f, g and u directly, the effect of time t was neglected. Their results would not be much different if the system is autonomous:

$$dx = [f(x,r) + u(x,r)]dt + g(x,r)dB.$$

If we consider the time-varying property of the coefficients and controller into our analysis of the non-autonomous system, if we use a time-varying function with upper bound equivalent to the Lipschitz constant, then the parameters which were previously constants would become time-varying functions, and the observation interval would no longer be a constant but depend on time as well. Consequently, instead of a constant observation interval, we will have an observation interval sequence $\{\tau_j\}_{j\geq 1}$. That is, the system is observed at time points $0, \tau_1, \tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3, \cdots$.

Although the discrete-time observation-based feedback control is already more realistic and costs less than the traditional one, it still fails to make use of the time-varying property of the non-autonomous system. If the controlled system is non-autonomous (i.e., f or g or u depends on time explicitly), then the time-varying observation frequencies make more sense than the constant one. Intuitively, when the system state or mode change rapidly, we should observe them very frequently, and vice versa, low-frequency observation is allowed when system changes slowly. However, previous research has to require the highest observation frequency for all time. Making use of the time-varying property and using time-varying observation frequencies can reduce the cost of control. Especially if the observation is not free, for example, some systems are observed by human, monitor or sensor. Wide observation interval may indicate holiday for observer or we can turn off the monitor or sensor to save power.

A particular interest for a time-varying coefficient is its periodicity because periodic phenomena are all around us. Satellite orbit, working days per week, seasons or months in a year, wave vibration, etc., are all periodic. Furthermore, stochastic models involving periodicity have been studied by many authors (see e.g. [113]-[121]) due to their wide applications in many areas. If the system coefficients and controller are all periodic, then it makes sense to use periodic observations.

Motivated by above discussion, I investigated how to stabilize a given non-autonomous unstable SDE or hybrid SDE with periodic coefficients by a periodic feedback control based on periodic discrete-time observations. The results stated in Chapter 5 has been submitted to the journal "IEEE Transactions on Automatic Control" as [122], which is still under review.

1.4 Scope and structure of this thesis

In Chapter 1, I briefly introduces development of SDEs and stability theory, why we want to make the system stable and why we use feedback control, the development of study on a new type of feedback control (based on discrete-time observations) in recent years and my research motivation.

To make this thesis self-contained, Chapter 2 is preliminaries for readers. Chapter 2 includes some useful definitions, theorems about SDEs and stability theory, and some useful inequalities.

In Chapter 3, I investigate the pth moment stabilization of hybrid SDEs. Stabilities discussed include pth moment H_∞ stability for $p > 1$, pth moment asymptotic stability for $p \geq 2$, pth moment exponential stability for $p > 1$ and almost sure exponential stability.

The key technique used is constructing Lyapunov functional. Observation of the system state was discretized firstly and then observation of the system mode was discretized as well. Rigorous proofs are followed by numerical examples for illustration of the new established theory. The main contributions of Chapter 3 are: 1) developing the criterion on asymptotic stabilization from mean square ($p = 2$) to pth moment for all $p \geq 2$, developing the criterion on H_∞ stabilization and exponential stabilization from mean square to pth moment for all $p > 1$; and 2) improving the upper bound of observation interval, namely reducing the observation frequency and hence reducing the cost of control.

In Chapter 4, I use stochastic feedback control, which is based on Brownian motions and discrete-time observations, to stabilize non-autonomous linear scalar ODEs and nonlinear multidimensional hybrid SDEs. The analysis technique is as follows. By comparing the traditionally and new controlled systems, I prove that the new controlled system is pth moment ($0 < p < 1$) exponentially stable, then use it as a steppingstone, I prove the almost sure exponential stability. The main contribution of this chapter is expanding the scope of applicable original unstable systems from autonomous ODEs to non-autonomous ODEs and hybrid SDEs.

Chapter 5 states my proudest work. In Chapter 5, I studied stabilization problem for non-autonomous periodic SDEs as well as hybrid SDEs. The feedback control is based on the innovative periodic discrete-time observations. By making use of Lyapunov functions and inequalities, I prove that the controlled system can achieve many stabilities, including pth moment H_∞ stability and exponential stability for $p > 1$, pth moment asymptotic stability for $p \geq 2$, almost sure asymptotic and exponential stabilities. Compared to existing results, my new established theory not only reduces the cost of control by reducing observation frequency dramatically, but also offers flexibility on the setting of observations - we can choose when to observe more frequently or less frequently. Numerical examples are given to compare the observation frequencies under others' existing theory and my new theory. My highest frequency is still lower than the constant frequency required by existing theory, and if the time unit is a year, then an example shows observers can have long holiday under my new theory.

Finally Chapter 6 concludes this thesis, summarizes contributions of this thesis and proposes some potential improvement research work for the future.

In this thesis, for technical reason, the nonlinear original system generally needs to satisfy locally Lipschitz condition and linear growth condition; and the controller generally needs to satisfy globally Lipschitz condition and vanish when the system reaches the equilibrium point (i.e., the origin). As a result, the existence and uniqueness of the solution to the controlled system is guaranteed. As far as the author knows, the stabilization problem of SDEs by either deterministic or stochastic feedback control based on discrete-time observations is still investigated under the linear growth condition. Technical difficulties make it a challenge to relax the linear growth condition.

From convergence's speed point of view, this thesis mainly discusses asymptotic stability and exponential stability; from probability's point of view, the stochastic stabilities discussed in this thesis include moment stability and almost sure stability (also known as almost everywhere or pathwise stability). Other stabilities such as polynomial stability, stability in probability and distribution are not involved. From equation type's point of view, this thesis focuses on the regular stochastic differential equations. Stochastic functional differential equations, stochastic equations of neutral type and backward stochastic differential equations are not involved.

Chapter 2

Preliminary

2.1 Basic notations of probability theory and stochastic processes

In this section, let us briefly review some basic notations of probability theory and stochastic processes. Readers are referred to [9, 26] for more details.

Probability theory deals with mathematical models of trials whose outcomes depend on chance. All the possible outcomes form a set Ω, with typical element $\omega \in \Omega$. The observable or interesting events contained in Ω form a family $\mathcal{F}$. A family $\mathcal{F}$ is called a σ-algebra if it has the following properties:

(i) $\emptyset \in \mathcal{F}$, where $\emptyset$ denotes the empty set;

(ii) $E \in \mathcal{F} \Rightarrow E^c \in \mathcal{F}$, where E^c is the complement of E in Ω;

(iii) $\{E_i\}_{i\geq 1} \subset \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} E_i \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is a measurable space and the elements of $\mathcal{F}$ are $\mathcal{F}$-measurable sets.

A function X: $\Omega \to \mathbb{R}$ is $\mathcal{F}$-measurable if

$$\{\omega : X(\omega) \leq y\} \in \mathcal{F}, \text{ for all } y \in \mathbb{R}.$$

The function X is called a real-valued $\mathcal{F}$-measurable random variable. An $\mathbb{R}^n$-valued function is $\mathcal{F}$-measurable if all its elements are $\mathcal{F}$-measurable. When the measurable space is $(\mathbb{R}^n, \mathcal{B}^n)$, a $\mathcal{B}^n$-measurable function is call a Borel measurable function. The system coefficients in this thesis are all Borel measurable functions.

A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P}$: $\mathcal{F} \to [0, 1]$

such that

(i) $\mathbb{P}(\Omega) = 1$;

(ii) for any disjoint sequence $\{E_i\}_{i\geq 1} \subset \mathcal{F}$, $\mathbb{P}(\cup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} \mathbb{P}(E_i)$. Then the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let

$$\bar{\mathcal{F}} = \{A \subset \Omega : \exists B, C \in \mathcal{F} \text{ such that } B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}.$$

If $\mathcal{F} = \bar{\mathcal{F}}$, then the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete.

For a real-valued random variable integrable with respect to the probability measure $\mathbb{P}$, the expectation and variance of X are $\mathbb{E}X = \int_{\Omega} X(\omega) d\mathbb{P}(w)$ and $Var(X) = \mathbb{E}(X - \mathbb{E}X)^2$ respectively. The pth moment of X is $\mathbb{E}|X|^p (p > 0)$. A family of k random variables $\{X_i\}_{1\leq i\leq k}$ are independent if the σ-algebras generated by them - $\{\sigma(X_i)\}_{1\leq i\leq k}$ are independent.

Now we give two concepts of convergence for $\mathbb{R}^n$-valued random variables X and $\{X_i\}_{i\geq 1}$ ([9, Section 1.2]):

(i) The sequence $\{X_i\}_{i\geq 1}$ is said to converge to X almost surely, or almost everywhere, or with probability 1, if

$$\mathbb{P}\big(\omega \in \Omega : \lim_{i\to\infty} X_i(\omega) = X(\omega)\big) = 1.$$

(ii) If for every $\varepsilon > 0$, $\mathbb{P}\big(\omega : |X_i(\omega) - X(\omega)| > \varepsilon\big) \to 0$ as $i \to \infty$, then $\{X_i\}_{i\geq 1}$ is said to converge to X stochastically or in probability.

(iii) The sequence $\{X_i\}_{i\geq 1}$ is said to converge to X in pth moment or in L^p, if $X_i, X \in L^p$ and $E|X_k - X|^p \to 0$.

Here are some important convergence theorems.

Theorem 2.1.1 ***(Monotonic convergence theorem)*** *If $\{X_i\}$ is an increasing sequence of nonnegative random variables, then*

$$\lim_{i\to\infty} \mathbb{E}X_i = \mathbb{E}(\lim_{i\to\infty} X_i).$$

Theorem 2.1.2 ***(Dominated convergence theorem)*** *Let $p \geq 1$, $\{X_i\} \subset L^p(\Omega; \mathbb{R}^n)$ and $Y \in L^p(\Omega; \mathbb{R})$. Assume $|X_i| \leq Y$ a.s. and $\{X_i\}$ converges to X in probability.*

Then $X \in L^p(\Omega;\mathbb{R})$*,* $\{X_i\}$ *converges to* X *in* L^p*, and*

$$\lim_{i\to\infty} \mathbb{E}X_i = \mathbb{E}(X).$$

Then we state the well-known Borel-Cantelli lemma.

Lemma 2.1.3 ***(Borel-Cantelli's lemma)***
Let $\{E_k\}_{k\geq 1} \subset \mathcal{F}$ *and* $\limsup_{k\to\infty} E_k = \cap_{i=1}^{\infty} \cup_{k=i}^{\infty} E_k$*.*
(1) If $\sum_{k=1}^{\infty} \mathbb{P}(E_k) < \infty$*, then*

$$\mathbb{P}(\limsup_{k\to\infty} E_k) = 0.$$

(2) If the sequence $\{E_k\}_{k\geq 1}$ *is independent and* $\sum_{k=1}^{\infty} \mathbb{P}(E_k) = \infty$*, then*

$$\mathbb{P}(\limsup_{k\to\infty} E_k) = 1.$$

Here are some useful properties of conditional expectation. For $X \in L^1(\Omega;\mathbb{R})$ and $\mathcal{G} \subset \mathcal{F}$:
(a) $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$;
(b) X is $\mathcal{G}$-measurable $\Rightarrow \mathbb{E}(X|\mathcal{G}) = X$;
(c) X is $\mathcal{G}$-measurable $\Rightarrow \mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$.

Now let's review some important concepts about stochastic processes.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration is a family $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$ (i.e. $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leq t < s < \infty$). The filtration is right continuous if $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geq 0$. Unless otherwise stated, $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ used in this thesis is a complete probability space with the increasing and right-continuous filtration $\{\mathcal{F}_t\}_{t\geq 0}$ where $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

A family $\{X_t\}_{t\in I}$ of $\mathbb{R}^n$-valued random variables is called a stochastic process with state space $\mathbb{R}^n$. A random variable ϱ: $\Omega \to [0,\infty]$ is a stopping time if $\{\omega : \varrho(\omega) \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. An $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted integrable process $\{M_t\}_{t\geq 0}$ is a martingale if

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \quad a.s. \qquad \text{for } all \quad 0 \leq s < t < \infty.$$

A real-valued square-integrable continuous martingale $M = \{M_t\}_{t\geq 0}$ has a quadratic variation $\{\langle M, M\rangle\}_{t\geq 0}$. A right-continuous adapted process $M = \{M_t\}_{t\geq 0}$ is a local martingale if there is a nondecreasing sequence $\{\varrho_k\}_{k\geq 1}$ of stopping times with $\varrho_k \to \infty$ *a.s.* such that every $\{M_{\varrho_k\wedge t} - M_0\}_{t\geq 0}$ is a martingale. Every martingale is a local martingale, but the converse is not true (see e.g. [9, 26]).

Here are some important and useful theorems on martingale and local martingale. We refer the readers to [9, Section 1.3].

Theorem 2.1.4 ***(Strong law of large numbers)*** *Let $M = \{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty} \langle M, M\rangle_t = \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \quad a.s.$$

and also

$$\limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.$$

More generally, if $Y = \{Y_t\}_{t\geq 0}$ is a continuous adapted increasing process such that

$$\lim_{t\to\infty} Y_t = \infty \quad and \quad \int_0^\infty \frac{d\langle M, M\rangle_t}{(1 + Y_t)^2} < \infty \quad a.s.$$

then

$$\lim_{t\to\infty} \frac{M_t}{Y_t} = 0 \quad a.s.$$

Theorem 2.1.5 ***(Doob's martingale inequalities)*** *Let $\{M_t\}_{t\geq 0}$ be an $\mathbb{R}^n$-valued martingale and let $[a, b]$ be a bounded interval in $\mathbb{R}_+$. Assume $M_t \in L^p(\Omega; \mathbb{R}^n)$.*

(i) If $p \geq 1$, then for all $z > 0$,

$$\mathbb{P}\big(\omega : \sup_{a\leq t\leq b} |M_t(\omega)| \geq z\big) \leq \frac{\mathbb{E}|M_b|^p}{z^p}.$$

(ii) If $p > 1$, then

$$\mathbb{E}(\sup_{a\leq t\leq b} |M_t|^p) \leq (\frac{p}{p-1})^p \mathbb{E}|M_b|^p.$$

Theorem 2.1.6 *Let $\{A_t\}_{t\geq 0}$ and $\{U_t\}_{t\geq 0}$ be two continuous adapted increasing processes with $A_0 = U_0 = 0$ a.s. Let $\{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale* *with $M_0 = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0$-measurable random variable and*

$$X_t = \xi + A_t - U_t + M_t \qquad \textit{for } t \geq 0.$$

If X_t is nonnegative, then

$$\{\lim_{t\to\infty} A_t < \infty\} \subset \{\lim_{t\to\infty} X_t \textit{ exists and is finite}\} \cap \{\lim_{t\to\infty} U_t < \infty\} \quad a.s.$$

where $B \subset D$ a.s.. If $\lim_{t\to\infty} A_t < \infty$ a.s., then for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} X_t(\omega) \textit{ exists and is finite, and } \lim_{t\to\infty} U_t(\omega) < \infty.$$

2.2 Brownian motions and stochastic integrals

After reviewing some basic notations and theorems of probability theory and stochastic processes, let us look at the foundation of SDEs - Brownian motions and stochastic integrals. Readers are referred to [9, Section 1.4, 1.5] for more details.

Definition 2.2.1 *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A standard one-dimensional Brownian motion is a real-valued continuous $\mathcal{F}_t$-adapted process $\{B_t\}_{t\geq 0}$ with the following properties:*
(i) $B(0) = 0$ a.s.;
(ii) for $0 \leq s < t < \infty$, the increment $B(t) - B(s) \sim N(0, t-s)$, i.e. normally distributed with mean zero and variance $t - s$;
(iii) for $0 \leq s < t < \infty$, the increment $B(t) - B(s)$ is independent of $\mathcal{F}_s$.

The standard one-dimensional Brownian motion $\{B_t\}_{t\geq 0}$ has the following properties:

(a) $\{B_t\}_{t\geq 0}$ is a continuous square-integrable martingale, its quadratic variation $\langle B, B\rangle_t = t$ for all $t \geq 0$ and the strong law of large numbers indicates

$$\lim_{t\to\infty} \frac{B(t)}{t} = 0 \quad a.s.$$

(b) The sample path $B(t,\omega)$ is nowhere differentiable for almost all $\omega \in \Omega$.

Definition 2.2.2 *A m-dimensional process $\{B(t) = (B_1(t), \cdots, B_m(t))^T\}_{t\geq 0}$ is an m-dimensional Brownian motion if all elements $\{B_i(t)\}'s$ are independent one-dimensional Brownian motion.*

In this thesis, the one-dimensional Brownian motion is standard as stated above and is defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, which was defined in Section 2.1. In this thesis, we denote the m-dimensional Brownian motion by $B(t) = (B_1(t), \cdots, B_m(t))^T$.

Following is the definition of Itô's stochastic integral.

Let $g = \{g(t)\}_{a\leq t\leq b}$ be a simple (or step) process, that is, there is a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$, and bounded random variables ξ_i, $0 \leq i \leq k-1$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable and

$$g(t) = \xi_0 I_{[t_0,t_1]}(t) + \sum_{i=1}^{k-1} \xi_i I_{(t_i,t_{i+1}]}(t).$$

Definition 2.2.3 ***(Part 1 of the definition of Itô's integral)*** *For a simple process g defined as above, the stochastic integral of g with respect to the Brownian motion $B(t)$ is*

$$\int_a^b g(t)dB(t) = \sum_{i=0}^{k-1} \xi_i [B(t_{i+1}) - B(t_i)].$$

Lemma 2.2.4 *A process g defined as above has the following properties*

$$\mathbb{E}\int_a^b g(t)dB(t) = 0 \text{ and } \mathbb{E}\Big|\int_a^b g(t)dB(t)\Big|^2 = \mathbb{E}\int_a^b |g(t)|^2 dB(t).$$

Definition 2.2.5 ***(Part 2 of the definition of Itô's integral)*** *Let $f \in \mathcal{M}^2([a,b];\mathbb{R})$. The Itô integral of f with respect to $B(t)$ is*

$$\int_a^b f(t)dB(t) = \lim_{k\to\infty} \int_a^b g_k(t)dB(t) \quad \text{in } L^2(\Omega;\mathbb{R}),$$

where $\{g_k\}$ is a sequence of simple process such that

$$\lim_{k\to\infty} \mathbb{E}\int_a^b |f(t) - g_k(t)|^2 dt = 0.$$

Following are some properties of Itô's stochastic integral.

Let $f, g \in \mathcal{M}^2([a,b];\mathbb{R})$, and let ϱ_1, ϱ_2 be two stopping times such that $a \le \varrho_1 \le \varrho_2 \le b$, then

(a) $\int_a^b f(t)dB(t)$ is $\mathcal{F}_b$-measurable;

(b) $\mathbb{E}\int_a^b f(t)dB(t) = \mathbb{E}\int_{\varrho_1}^{\varrho_2} f(t)dB(t) = \mathbb{E}(\int_a^b f(t)dB(t)|\mathcal{F}_a) = 0$;

(c) $\mathbb{E}|\int_a^b f(t)dB(t)|^2 = \mathbb{E}\int_a^b |f(t)|^2 dt$;

(d) $\mathbb{E}|\int_{\varrho_1}^{\varrho_2} f(t)dB(t)|^2 = \mathbb{E}\int_{\varrho_1}^{\varrho_2} |f(t)|^2 dt$;

(e) $\mathbb{E}(|\int_a^b f(t)dB(t)|^2|\mathcal{F}_a) = \int_a^b \mathbb{E}(|f(t)|^2|\mathcal{F}_a)dt$.

Itô's integral under some conditions can be a martingale.

Theorem 2.2.6 *Let $f \in \mathcal{M}^2([0,T];\mathbb{R})$. Then Itô's integral $M = \int_0^t f(s)dB(s)$ is a square-integrable continuous martingale and its quadratic variation $\langle M, M\rangle_t = \int_0^t |f(s)|^2 ds$ for any $0 \le t \le T$.*

Furthermore, if $f \in \mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{n\times m})$, (i.e. the value of f is $n \times m$ real matrix) and $B(t)$ is an m-dimensional Brownian motion, then the Itô integral $\int_0^t f(s)dB(s)$ for $t \ge 0$ is an $\mathbb{R}^n$-valued continuous square-integrable martingale.

2.3 SDEs, SDDEs, Markov processes and hybrid SDEs

This section is to review some important concepts of SDEs, SDDEs, Markov processes and hybrid SDEs. Firstly we introduce the formal definitions and some important properties of SDEs and SDDEs. Then we introduce stochastic differential equations with Markovian switching (hybrid SDEs), before which we review some concepts of Markov processes including Markov chains and Markov property. Finally we review the connections between SDEs and Markov processes. Readers are referred to [9, 26] for more details.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge 0}, \mathbb{P})$ be a probability space defined in Section 2.1 and let $B(t) = (B_1(t), \cdots, B_m(t))^T$ be an m-dimensional Brownian motion defined on it. Let $0 \le t_0 < T < \infty$. Let $x_0 \in L^2(\Omega;\mathbb{R}^n)$ be $\mathcal{F}_{t_0}$-measurable, i.e. $\mathbb{E}|x_0|^2 < \infty$. Let $f : \mathbb{R}^n \times [t_0, T] \to$

$\mathbb{R}^n$ and $g : \mathbb{R}^n \times [t_0, T] \to \mathbb{R}^{n\times m}$ be both Borel measurable functions. Then the n-dimensional stochastic differential equations of Itô type has the general form

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \quad \text{on } t_0 \le t \le T \tag{2.1}$$

with initial value $x(t_0) = x_0$. This is equivalent to

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s) \quad \text{on } t_0 \le t \le T. \tag{2.2}$$

Definition 2.3.1 *An $\mathbb{R}^n$-valued stochastic process $\{x(t)\}_{t_0\le t\le T}$ is called a solution of equation (2.1) if it has the following properties:*

(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;

(ii) $\{f(x(t), t)\}_{t_0\le t\le T} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^n)$ and $\{g(x(t), t)\}_{t_0\le t\le T} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{n\times m})$;

(iii) equation (2.1) holds for $\forall t \in [t_0, T]$ with probability 1.

A solution $\{x(t)\}$ is unique if $\mathbb{P}\big(x(t) = \bar{x}(t) \text{ for } \forall t \in [t_0, T]\big) = 1$.

In the following we give the definitions of Lipschitz condition and linear growth condition, which guaranteed the existence and uniqueness of the system solution.

(i) **Global Lipschitz condition:** There is a positive constant K such that for all $x, y \in \mathbb{R}^n$ and $t \in [t_0, T]$

$$|f(x,t) - f(y,t)|^2 \bigvee |g(x,t) - g(y,t)|^2 \le K|x - y|^2. \tag{2.3}$$

(ii) **Local Lipschitz condition:** For every integer $n \ge 1$, there exists a positive constant K_n such that, for all $t \in [t_0, T]$ and $x, y \in \mathbb{R}^n$ with $|x| \wedge |y| \le n$,

$$|f(x,t) - f(y,t)|^2 \bigvee |g(x,t) - g(y,t)|^2 \le K_n|x - y|^2. \tag{2.4}$$

(iii) **Linear growth condition:** There is a positive constant $\bar{K}$ such that for all $x, y \in \mathbb{R}^n$ and $t \in [t_0, T]$

$$|f(x,t)|^2 \bigvee |g(x,t)|^2 \le \bar{K}(1 + |x|^2). \tag{2.5}$$

Theorem 2.3.2 *If equation (2.1) satisfies the global Lipschitz condition and Linear growth condition, or if equation (2.1) satisfies the local Lipschitz condition and Linear growth condition, then (2.1) has a unique solution $x(t)$ and $x(t) \in \mathcal{M}^2([t_0, T]; \mathbb{R}^n)$.*

In this thesis the controlled single-mode SDE system has the form

$$dx(t) = \big(f(x(t), t) + u(x(\delta_t), t)\big)dt + g(x(t), t)dB(t) \tag{2.6}$$

where $\delta_t = t_0 + [t/\tau]\tau$ and $[t/\tau]$ is the integer part of t/τ. So $\delta_t \in (t - \tau, t]$ and (2.6) can be written as a stochastic differential delay equation

$$dx(t) = \big(f(x(t), t) + u(x(t - \zeta(t)), t)\big)dt + g(x(t), t)dw(t),$$

where $\zeta(t) = t - t_0 - [t/\tau]\tau$. If (2.6) satisfies the Local Lipschitz condition and linear growth condition for SDDEs, which is similar as stated above, then (2.6) has a unique solution $x(t)$ such that $\mathbb{E}|x(t)|^p < \infty$ for all $t \geq t_0$ and $p > 0$. We refer to Section 5.3 and 5.4 of [9] for details.

Then let us review the Markov processes and its connections with SDEs.

An $\mathbb{R}^n$-valued $\{\mathcal{F}_t\}$-adapted process $X = \{X_t\}_{t \geq 0}$ is a Markov process if the following *Markov property* holds:

$$\mathbb{P}(X(t) \in A | \mathcal{F}_s) = \mathbb{P}(X(t) \in A | X(s)) \text{ for all } 0 \leq s \leq t < \infty \text{ and } A \in \mathcal{B}^n.$$

Theorem 2.3.3 *If equation (2.1) satisfies the local Lipschitz condition and Linear growth condition, then its solution $x(t)$ is a Markov process.*

Before introducing hybrid SDEs, we review the definition of Markov chains first.

A stochastic process $X = \{X_t\}_{t \geq 0}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with values in state space $\mathbb{S}$, which is a countable set, is a continuous-time Markov chain if for any finite set of time points $t_i's$ and corresponding states in $\mathbb{S}$ such that $\mathbb{P}\big(X(t_n) = i, X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1\big) > 0$, the following holds:

$$\mathbb{P}\big(X(t_{n+1}) = j | X(t_n) = i, X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1\big) = \mathbb{P}\big(X(t_{n+1}) = j | X(t_n) = i\big).$$

In this thesis, unless otherwise stated, let $r(t)$ for $t \geq 0$ be a right-continuous Markov chain on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$, which was defined in Section 2.1, taking values in a finite state space $\mathbb{S} = \{1, 2, \cdots, N\}$ with generator matrix $\Gamma = (\gamma_{ij})_{N\times N}$, whose elements γ_{ij} are the transition rates from state i to j for $i \neq j$ and $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. The generator $\Gamma = (\gamma_{ij})_{N\times N}$ is given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$.

In this thesis, define $\overline{\gamma} := \max_{i\in\mathbb{S}}(-\gamma_{ii})$. We present a useful lemma here (see Lemma 1 in [35]).

Lemma 2.3.4 *For any $t \geq t_0$, $v > 0$ and $i \in \mathbb{S}$,*

$$\mathbb{P}\Big(r(s) \neq i \text{ for some } s \in [t, t+v] \Big| r(t) = i\Big) \leq 1 - e^{-\overline{\gamma}v}. \tag{2.7}$$

Generally, a stochastic differential equation with Markovian switching (hybrid SDEs) has the form

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t), \quad \text{on } t_0 \leq t \leq T \tag{2.8}$$

with initial data $x(t_0) = x_0 \in \mathcal{L}^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$ and $r(t_0) = r_0 \in \mathbb{S}$ is $\mathcal{F}_{t_0}$-measurable. System coefficients are all Borel measurable functions

$$f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $B(\cdot)$.

Definition of the solution to the hybrid SDE (2.8) is similar to Definition 2.3.1, with $f(x(t), t)$ and $g(x(t), t)$ replaced by $f(x(t), r(t), t)$ and $g(x(t), r(t), t)$ respectively.

The Lipschitz condition and linear growth condition for hybrid SDEs are similar to those for single-mode SDEs stated above. With $f(x, t)$ replaced by $f(x, r, t)$, $g(x, t)$

replaced by $g(x, r, t)$, and $x, y, t \in \mathbb{R}^n \times [t_0, T]$ replaced by $x, y, i, t \in \mathbb{R}^n \times \mathbb{S} \times [t_0, T]$, either global or local Lipschitz condition in addition to the linear growth condition guarantee that (2.8) has a unique solution $x(t)$ and $x(t) \in \mathcal{M}^2([t_0, T]; \mathbb{R}^n)$.

Theorem 2.3.5 *If equation (2.8) satisfies the local Lipschitz condition and Linear growth condition, then the pair $(x(t), r(t))$ is a Markov process.*

2.4 Itô's formula

In this section, we present Ito's formula for single-mode SDEs and the generalized Ito formula for hybrid SDEs.

Let $C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$ denote the family of all real-valued functions $V(x, t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t. For $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, its partial derivatives

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = (\frac{\partial V}{\partial x_1}, \dots, \frac{\partial V}{\partial x_n}), \quad V_{xx} = \big(\frac{\partial^2 V}{\partial x_i \partial x_j}\big)_{n \times n} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_n} \\ \vdots & & \vdots \\ \frac{\partial^2 V}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_n \partial x_n} \end{pmatrix}.$$

Following is the well-known Itô's formula.

Theorem 2.4.1 ***(The multi-dimensional Itô formula)*** *Let $x(t)$ be an n-dimensional Itô process on $t \geq 0$ governed by*

$$dx(t) = f(t)dt + g(t)dB(t),$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^n)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{n \times m})$. Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t), t)$ is also an Itô process governed by

$$\begin{aligned} dV(x(t), t) = &\big[V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}\mathrm{trace}\big(g^T(t)V_{xx}(x(t), t)g(t)\big)\big]dt \\ &+ V_x(x(t), t)g(t)dB(t) \quad a.s. \end{aligned} \tag{2.9}$$

For the one-dimensional Itô formula where $x \in \mathbb{R}$, (2.9) has $\mathrm{trace}\big(g^T(t)V_{xx}(x(t), t)g(t)\big)$

replaced by $V_{xx}(x(t),t)g^2(t)$.

Let us introduce a multiplication table:

$$dtdt = 0, \quad dB_i dt = 0,$$
$$dB_i dB_i = dt, \quad dB_i dB_j = 0 \quad \text{for} \quad i \neq j.$$

According to [123], a continuous-time Markov chain can be represented as a stochastic integral with respect to a Poisson random measure. Then for hybrid SDE, we have the following generalized Ito formula.

For (2.8), let $V(x,i,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R})$, the family of all real-valued functions defined on $\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$ which are continuously twice differentiable in x and once in t. Define an operator LV from $\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$ to $\mathbb{R}$ by

$$\begin{aligned} LV(x,i,t) &= V_t(x,i,t) + V_x(x,i,t)f(x,i,t) \\ &+ \frac{1}{2}[trace(g^T(x,i,t)V_{xx}(x,i,t)g(x,i,t)], + \sum_{j=1}^{N} \gamma_{ij} V(x,j,t). \end{aligned} \tag{2.10}$$

Theorem 2.4.2 ***(The generalized Itô Formula)*** *If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R})$, then for any $t \geq 0$*

$$\begin{aligned} &V(x(t),r(t),t) \\ =&V(x(0),r(0),0) + \int_0^t LV(x(s),r(s),s)ds \\ &+ \int_0^t V_x(x(s),r(s),s)g(x(s),r(s),s)dB(s) \\ &+ \int_0^t \int_{\mathbb{R}} (V(x(s),r_0 + h(r(s),l),s) - V(x(s),r(s),s))\mu(ds,dl), \end{aligned} \tag{2.11}$$

where $\mu(ds,dl) = v(ds,dl) - \mu(dl)ds$ is a martingale random measure.

We refer the definition of function h to page 47 of [26].

Taking expectation on both sides of (2.11) gives the following useful lemma.

Lemma 2.4.3 *Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R})$ and ϱ_1, ϱ_2 be bounded stopping times such* *that* $0 \le \varrho_1 \le \varrho_2$ *a.s. If* $V(x(t), r(t), t)$ *and* $LV(x(t), r(t), t)$ *are bounded on* $t \in [\rho_1, \rho_2]$ *with probability* 1*, then*

$$\mathbb{E}V(x(\varrho_2), r(\varrho_2), \varrho_2) = \mathbb{E}V(x(\varrho_1), r(\varrho_1), \varrho_1) + \mathbb{E}\int_{\varrho_1}^{\varrho_2} LV(x(s), r(s), s)ds. \qquad (2.12)$$

2.5 Stability

In this section, we present definitions of several frequently used stabilities and then introduce some theorems on stability.

For an n-dimensional SDE (e.g. of the form (2.1)) or hybrid SDE (e.g. of the form (2.8)), following is several stabilities discussed in this thesis. In this thesis, $p > 0$ is the moment order.

Readers are referred to [9, 26] for more details.

H_∞ **stability**

The system is H_∞-stable in $L^p(\Omega \times \mathbb{R}_+; \mathbb{R}^n)$ (also known as $L^p(\Omega \times \mathbb{R}_+; \mathbb{R}^n)$-stable) if

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$ if it's a hybrid SDE.

Asymptotic stability

The system is asymptotically stable in pth moment if

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0$$

for any initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$ if it's a hybrid SDE.

When $p = 2$, it's called mean square asymptotic stability.

The system is almost surely asymptotically stable if

$$\lim_{t\to\infty} x(t) = 0 \quad a.s.$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$ if it's a hybrid SDE.

Exponential stability

The system is exponentially stable in pth moment if

(1) $\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) < 0;$

or equivalently

(2) there is a pair of positive constants λ and C such that

$$\mathbb{E}|x(t)|^p \le C|x_0|^p e^{-\eta(t-t_0)};$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$ if it's a hybrid SDE.

When $p = 2$, it's called mean square exponential stability.

Noting that $\mathbb{E}|x(t)|^p \le \left(\mathbb{E}|x(t)|^{\hat{p}}\right)^{p/\hat{p}}$ for $0 < p < \hat{p}$, we can see that the $\hat{p}$th moment exponential stability implies the pth moment exponential stability.

The system is almost surely exponentially stable if

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) < 0 \quad a.s.$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$ if it's a hybrid SDE.

The following lemma is [9, Lemma 3.2 on page 120]. It indicates that the SDE solution x cannot reach the origin almost surely if it does not start from origin.

Lemma 2.5.1 *For all $x_0 \ne 0$ in $\mathbb{R}^n$*

$$P\big(x(t) \ne 0 \ \text{ on } \ t \ge t_0\big) = 1.$$

Hybrid SDEs have similar lemma ([26, Lemma 5.1 on page 164]).

Assumption 2.5.2 *Assume that for each $n = 1, 2, \cdots$, there is a $K_n > 0$ such that*

$$|f(x,i,t)| \vee |g(x,i,t)| \le K_n|x|$$

for all $0 \le t \le n$, system mode $i \in \mathbb{S}$ and system state $x \in \mathbb{R}^n$ with $|x| \le n$.

Lemma 2.5.3 *Let Assumption 2.5.2 hold. Then*

$$P\big(x(t) \neq 0 \ \ on \ \ t \geq t_0\big) = 1$$

for all $x_0 \neq 0$ *in* $\mathbb{R}^n$, $t_0 \in \mathbb{R}_+$ *and* $i \in \mathbb{S}$.

The following theorems give conditions under which the moment exponential stability can imply the almost sure exponential stability.

Theorem 2.5.4 *For an* n*-dimensional SDE (e.g. of the form (2.1)), if there is a positive constant* K *such that*

$$x^T f(x,t) \vee |g(x,t)|^2 \leq K|x|^2 \quad for \ all \ \ (x,t) \in \mathbb{R}^n \times [t_0, \infty).$$

Then the pth moment exponential stability (for $p > 0$*) implies the almost sure exponential stability ([9, Lemma 4.2 on page 128]).*

For the n-dimensional stochastic functional differential equation

$$dx(t) = f(x_t,t)dt + g(x_t,t)dB(t) \quad \text{on} \ \ t_0 \leq t \leq T$$

where $x_t = \{x(t+s) : -\tau \leq s \leq 0\}$, we have the following theorem.

Theorem 2.5.5 *Let* $p \geq 1$*. Assume there is a positive constant* K *such that for every solution* $x(t)$ *of the SFDE stated above and for all* $t \geq t_0$*,*

$$\mathbb{E}\big(|f(x_t,t)|^p + |g(x_t,t)|^p\big) \leq K \ \mathbb{E}\big|\sup_{-\tau \leq s \leq 0} x(t+s)\big|^p. \tag{2.13}$$

Then

$$\mathbb{E}|x(t)|^p \leq Ce^{-\eta(t-t_0)} \quad for \ all \ \ t \geq t_0 \tag{2.14}$$

implies

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \leq -\frac{\eta}{p} \quad a.s. \tag{2.15}$$

We refer the details to [9, Theorem 6.2 on page 175]. Hybrid SFDEs of the form $dx(t) = f(x_t, r(t), t)dt + g(x_t, r(t), t)dB(t)$ have similar theorem.

Theorem 2.5.6 *Let $p \geq 1$. Assume there is a positive constant K such that for every solution $x(t)$ of the hybrid SFDE, for all $t \geq t_0$ and $i \in \mathbb{S}$,*

$$\mathbb{E}\big(|f(x_t, i, t)|^p + |g(x_t, i, t)|^p\big) \leq K \ \mathbb{E}\big|\sup_{-\tau \leq s \leq 0} x(t+s)\big|^p. \tag{2.16}$$

Then (2.14) can imply (2.15) for hybrid SFDEs.

We refer the details to [26, Theorem 8.8 on page 309].

Now we introduce the definition of a nonsingular M-matrix and some relevant theorems for stability of hybrid SDEs used in Chapter 4.

Definition 2.5.7 *A square matrix $A = (a_{ij})_{n\times n}$ is called a nonsingular M-matrix if A can be expressed as $A = sI - G$, where the matrix G has all elements non-negative, s is a real number larger than the spectral radius of G and I is the $n \times n$ identity matrix.*

Theorem 2.5.8 *If $A = (a_{ij})_{n\times n}$ with $a_{ij} \leq 0$ for $i \neq j$, then the following statements are equivalent:*
(1) A is a nonsingular M-matrix.
(2) Every real eigenvalue of A is positive.
(3) There is an n-dimensional real vector x whose elements are all positive such that Ax has all elements positive.

We present the Theorem 5.12 on page 172 of [26] below.

Theorem 2.5.9 *Assume that there are constants $K > 0$, $\alpha_i \in \mathbb{R}$, $\sigma_i \geq 0$ and $\rho_i \geq 0$ ($i \in \mathbb{S}$) such that*

$$|f(x,i,t)| \leq K|x|, \quad x^T f(x,i,t) \leq \alpha_i |x|^2,$$
$$|g(x,i,t)| \leq \rho_i |x|, \quad |x^T g(x,i,t)| \geq \sigma_i |x|^2,$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$. For $0 < p < 2$, define an $N \times N$ matrix

$$\mathcal{A}(p) := \mathrm{diag}(\theta_1(p), \cdots, \theta_N(p)) - \Gamma,$$

where

$$\theta_i(p) = \frac{p}{2}\Big[(2-p)\sigma_i^2 - \rho_i^2\Big] - p\alpha_i \quad \textit{for} \quad 1 \le i \le N$$

and $\Gamma = (\gamma_{ij})_{N\times N}$ *is the generator matrix of Markov chain* $r(t)$*. If* $\mathcal{A}(p)$ *is a nonsingular M-matrix, then the hybrid SDE (e.g. of the form (2.8)) is exponentially stable in pth moment and also almost surely.*

We present the Theorem 5.8 on page 166 of [26] below.

Theorem 2.5.10 *Let Assumption 2.5.2 hold. Let* p, λ, c_1, c_2 *be positive numbers. If there is a function* $V(x,i,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ *such that*

$$c_1|x|^p \le V(x,i,t) \le c_2|x|^p$$

and

$$LV(x,i,t) \le -\lambda|x|^p$$

where $LV(x,i,t)$ *was defined in (2.10), for all* $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$*. Then*

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) \le -\frac{\lambda}{c_2}$$

for all $(x_0, r_0, t_0) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$*.*

Barbalat's lemma (see e.g. [124, page 123]) was used in the proof, so we present it here.

Lemma 2.5.11 *If the differentiable function* $f(t)$ *has a finite limit as* $t \to \infty$*, and if its derivative* $\dot{f}$ *is uniformly continuous, then* $\dot{f} \to 0$ *as* $t \to \infty$*.*

2.6 Frequently used inequalities

In this section, I present some inequalities (see e.g. [9, 26]) used in this thesis.

The Young inequality

$$|a|^\beta |b|^{1-\beta} \le \beta|a| + (1-\beta)|b|, \quad \forall a, b \in \mathbb{R} \text{ and } \forall \beta \in [0,1].$$

Then we can easily derive that

$$|a|^{\beta}|b|^{1-\beta} \le \varepsilon\beta|a| + \varepsilon^{-\beta/(1-\beta)}(1-\beta)|b|, \quad \forall \varepsilon > 0.$$

For $x_i \in \mathbb{R}$, we have

$$|\sum_{i=1}^{k} x_i|^p \le k^{p-1}\sum_{i=1}^{k}|x_i|^p \quad \text{for } p \ge 1 \text{ and } |\sum_{i=1}^{k} x_i|^p \le k^p \sum_{i=1}^{k}|x_i|^p \quad \text{for } p \in (0,1). \tag{2.17}$$

Another useful inequality is

$$|a^p - b^p| \le p|a-b|(a^{p-1} + b^{p-1}) \quad \text{for } \forall a, b \ge 0 \text{ and } p \ge 1.$$

Following inequalities for random variables are useful:

(i) **Hölder's inequality**

$$|\mathbb{E}(X^T Y)| \le (\mathbb{E}|X|^p)^{1/p}(\mathbb{E}|Y|^q)^{1/q}$$

if $p > 1$, $1/p + 1/q = 1$, $X \in L^p$, $Y \in L^q$. This implies $\big(\mathbb{E}|X|^a\big)^{1/a} \le \big(\mathbb{E}|X|^b\big)^{1/b}$ for $0 < a < b < \infty$ and $X \in L^p$. Especially when $0 < p < 2$,

$$\mathbb{E}|X|^p \le \big(\mathbb{E}|X|^2\big)^{p/2},$$

which can also be derived from Jensen's inequality.

(ii) **Chebyshev's inequality**

$$\mathbb{P}\big(\omega : |X(\omega)| \ge c\big) \le c^{-p}\mathbb{E}|X|^p$$

for $c > 0$, $p > 0$, $X \in L^p$.

We refer the readers to [9, Section 1.7, 1.8] for the following frequently used inequalities.

Theorem 2.6.1 ***(Gronwall's inequality)*** *Let $T > 0$ and $c \ge 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0,T]$, and let $v(\cdot)$ be a nonnegative inte-*

grable function on $[0,T]$. *If*

$$u(t) \le c + \int_0^t v(s)u(s)ds \quad \textit{for all } 0 \le t \le T,$$

then

$$u(t) \le c\exp(\int_0^t v(s)ds) \quad \textit{for all } 0 \le t \le T.$$

In this thesis, we write the exponential function as $e^{(\cdot)}$ or sometimes $\exp(\cdot)$.

Following are moment inequalities for Itô's integral.

Theorem 2.6.2 *For* $p \ge 2$, *let* $g \in \mathcal{M}^2([0,T];\mathbb{R}^{n\kappa m})$, *then*

$$\mathbb{E}|\int_0^T g(s)dB(s)|^p \le [\frac{p(p-1)}{2}]^{p/2} \; T^{p/2-1} \; \mathbb{E}\int_0^T |g(s)|^p ds.$$

When $p = 2$, *it's an equality.*

Theorem 2.6.3 ***(The Burkholder-Davis-Gundy inequality)***
Let $g \in \mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{n\kappa m})$. *For* $t \ge 0$, *define*

$$x(t) = \int_0^t g(s)dB(s) \;\; \textit{and} \;\; A(t) = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, *there are positive constants* c_p, C_p *depending on* p *such that*

$$c_p\mathbb{E}|A(t)|^{p/2} \le \mathbb{E}\Big(\sup_{0\le s\le t}|x(t)|^p\Big) \le C_p\mathbb{E}|A(t)|^{p/2}$$

for all $t \ge 0$. *In particular, one may take*

$$c_p = \big(\frac{p}{2}\big)^p \;\; \textit{and} \;\; C_p = \big(\frac{32}{p}\big)^{p/2} \quad \textit{for} \quad 0 < p < 2;$$

$$c_p = 1 \;\; \textit{and} \;\; C_p = 4 \quad \textit{for} \quad p = 2;$$

$$c_p = \big(2p\big)^{-p/2} \;\; \textit{and} \;\; C_p = [\frac{p^{p+1}}{2(p-1)^{p-1}}]^{p/2} \quad \textit{for} \quad p > 2.$$

Chapter 3

On Pth Moment Stabilization of Hybrid SDEs

As discussed in Section 1.3, mean square stability is not enough and it's necessary to investigate the pth moment stabilization for a wide range of p. This chapter is devoted to pth moment stabilization of hybrid SDEs by feedback control based on discrete-time observations, in the sense that the controlled system becomes H_∞ in pth moment stable for $p > 1$, asymptotic stable in pth moment for $p \geq 2$, exponentially stable in pth moment and almost surely exponentially stable.

The results in this chapter have been published in "Stochastic Analysis and Applications" in 2017 as [104] [1].

This chapter is organised as follows. Section 3.1 states our stabilization problem, proposes assumptions and defines the Lyapunov functional that will be used for analysis. Section 3.2 and 3.3 analyze pth moment asymptotic and exponential stabilization respectively. Then Section 3.4 gives both linear and nonlinear examples for illustration. In Sections 3.2-3.4, the controller is based on discrete-time state observations and continuous-time mode observations. In Section 3.5, mode observations are also discretized. The conclusion is stated in Section 3.6.

[1] Dong, R and Mao, X. 2017. On Pth Moment Stabilization of Hybrid Systems by Discrete-time Feedback Control. *Stochastic Analysis and Applications*, 35, pp. 803–822.

3.1 Stabilization problem

Consider an n-dimensional hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{3.1}$$

on $t \geq 0$, with initial values $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in \mathbb{S}$. Here

$$f, u : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}$$

are all Borel measurable functions. The given system may not be stable and our aim is to design a feedback control $u(x(\delta_t), r(t), t)$ so that the controlled hybrid SDE

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(t), t)\big)dt + g(x(t), r(t), t)dB(t) \tag{3.2}$$

becomes stable, where

$$\delta_t = [t/\tau]\tau \tag{3.3}$$

for $\tau > 0$.

So our controller $u(x(\delta_t), r(t), t)$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$. Now we impose the following standing hypotheses.

Assumption 3.1.1 *Assume that the coefficients f and g are all locally Lipschitz continuous We also assume that they satisfy the following linear growth conditions*

$$|f(x, i, t)| \leq K_1|x| \quad \textit{and} \quad |g(x, i, t)| \leq K_2|x| \tag{3.4}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where K_1 and K_2 are both positive numbers.

Obviously, (3.4) implies that

$$f(0, i, t) = 0, \qquad g(0, i, t) = 0 \tag{3.5}$$

for all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 3.1.2 *Assume the controller function u are globally Lipschitz continuous, i.e., there exists a positive constant K_3 such that*

$$|u(x,i,t) - u(y,i,t)| \le K_3|x-y| \tag{3.6}$$

for all $(x,y,i,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$. We also assume that

$$u(0,i,t) = 0 \tag{3.7}$$

for all $(i,t) \in \mathbb{S} \times \mathbb{R}_+$.

We can easily see that Assumption 3.1.2 implies the following linear growth condition on the controller function

$$|u(x,i,t)| \le K_3|x| \tag{3.8}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Discrete-time observations bring a delay term into the system. Specifically,

$$x(\delta_t) = x(t-\zeta(t)) \text{ where } \zeta(t) = t-[t/\tau]\tau = t-k\tau \quad \text{for } t \in [k\tau,(k+1)\tau), k = 0,1,2,\cdots.$$

As discussed in section 2.3, the controlled system (3.2) can be written as a hybrid SDDE. Then under Assumptions 3.1.1 and 3.1.2, system (3.2) has a unique solution $x(t)$ such that $\mathbb{E}|x(t)|^p < \infty$ for all $t \ge 0$ and $p > 1$ (see e.g. Corollary 7.15 in [26]).

Readers may wonder why we do not apply the existing stability criteria for SDDEs or hybrid SDDEs directly to our controlled system. There are two reasons. One is that the existing stability criteria for delay systems usually rely on assumptions involving the delay term, for example $V(x(t), x(\delta_t))$ or $V(x(\delta_t), \delta_t)$. However, before we know the upper bound of the observation interval τ, δ_t is unknown and so conditions involving δ_t cannot be verified. The other reason is that usually the bounded variable delay is required to be differentiable with derivative less than one (see e.g. [69, 97] and [26, p.285]). However, our delay $\zeta(t)$ has derivative 1 when $t \in ((k-1)\tau, k\tau)$ and is not differentiable when $t = k\tau$, $k = 1, 2, 3 \cdots$. Therefore, we need to establish new stability criteria for the controlled system.

For stabilization purpose related to the controlled system (3.2), we introduce the following Lyapunov function operator and Lyapunov functional.

Let $V(x, i, t)$ be a Lyapunov function and we require $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, i.e., the family of non-negative functions $V(x, i, t)$ is defined on $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$ which are continuously twice differentiable in x and once in t. Then define an operator $\mathcal{L}V : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(x, i, t) = {} & V_t(x, i, t) + V_x(x, i, t)[f(x, i, t) + u(x, i, t)] \\ & + \frac{1}{2}\text{trace}[g^T(x, i, t)V_{xx}(x, i, t)g(x, i, t)] \\ & + \sum_{k=1}^{N} \gamma_{ik} V(x, k, t), \end{aligned} \tag{3.9}$$

where V_t, V_x and V_{xx} is the first order partial derivative with respect to t, x and the second order partial derivative with respect to x respectively.

Now we define a Lyapunov functional for a fixed moment order $p > 1$ by

$$\hat{V}(x_t, r_t, t) = \theta\tau^{\frac{p-2}{2}} \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau^{\frac{p}{2}} |f(x(z), r(z), z) + u(x(\delta_z), r(z), z)|^p + \rho |g(x(z), r(z), z)|^p\Big] dz ds \tag{3.10}$$

for $t \geq 0$, where $x_t := \{x(t + s) : -2\tau \leq s \leq 0\}$, $r_t := \{r(t + s) : -\tau \leq s \leq 0\}$, θ is a positive number to be determined and

$$\rho = \begin{cases} (\frac{32}{p})^{\frac{p}{2}} & \text{for } p \in (1, 2), \\ [\frac{p(p-1)}{2}]^{\frac{p}{2}} & \text{for } p \geq 2. \end{cases} \tag{3.11}$$

For the definition of x_t, we require $s \in [-2\tau, 0]$, because $z - \tau < \delta_z \leq z$ in (3.10). At the starting point $z = s = t - \tau$, we have $t - 2\tau < \delta_z \leq t - \tau$.

For the functional to be well defined over $0 \leq t < 2\tau$, we define

$$x(s) = x_0, \;\; r(s) = r_0, \;\; f(x, i, s) = f(x, i, 0), \;\; u(x, i, s) = u(x, i, 0), \;\; g(x, i, s) = g(x, i, 0)$$

for all $(x, i, s) \in \mathbb{R}^n \times \mathbb{S} \times [-2\tau, 0)$.

In addition, we need to construct another functional by

$$U(x_t, r_t, t) = V(x(t), r(t), t) + \hat{V}(x_t, r_t, t). \tag{3.12}$$

Let's impose an assumption on the Lyapunov function.

Assumption 3.1.3 *Assume that there is a function $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$ and two positive numbers l, λ such that*

$$\mathcal{L}V(x, i, t) + l|V_x(x, i, t)|^{\frac{p}{p-1}} \leq -\lambda |x|^p \tag{3.13}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

3.2 Asymptotic stabilization

3.2.1 H_∞ stability

Theorem 3.2.1 *Fix the moment order $p > 1$. Let Assumptions 3.1.1, 3.1.2 and 3.1.3 hold. If $\tau > 0$ is sufficiently small for*

$$\begin{aligned} \lambda > & \frac{[2(p-1)]^{p-1} K_3^p}{p^p l^{p-1}(1 - 8^{p-1}\tau^p K_3^p)} \tau^{\frac{p}{2}} \left[2^{p-1}\tau^{\frac{p}{2}} K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}} K_3^p \right] \\ & \textit{and} \quad \tau \leq 8^{-\frac{p-1}{p}} / K_3 \end{aligned} \tag{3.14}$$

then the controlled system (3.2) is H_∞-stable in $L^p(\Omega \times \mathbb{R}_+; \mathbb{R}^n)$ (also known as $L^p(\Omega \times \mathbb{R}_+; \mathbb{R}^n)$-stable) in the sense

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty \tag{3.15}$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$.

Proof. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. Let

$$\Phi(x_t, r_t, t) = \theta \tau^{\frac{p-2}{2}} \int_{t-\tau}^{t} \left[\tau^{\frac{p}{2}} |f(x(s), r(s), s) + u(x(\delta_s), r(s), s)|^p + \rho |g(x(s), r(s), s)|^p \right] ds. \tag{3.16}$$

Notice that the integrand in (3.10) is right-continuous in t, then we can use the Leibniz integral rule to calculate the partial derivative of $\hat{V}(x_t, r_t, t)$ with respect to t.

$$\hat{V}_t(x_t, r_t, t) = \theta\tau^{\frac{p}{2}}\left[\tau^{\frac{p}{2}}|f(x(t), r(t), t) + u(x(\delta_t), r(t), t)|^p + \rho|g(x(t), r(t), t)|^p\right] - \Phi(x_t, r_t, t).$$

We apply the generalized Itô formula to $U(x_t, r_t, t)$ and obtain that

$$dU(x_t, r_t, t) = LU(x_t, r_t, t)dt + dM(t)$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ and

$$\begin{aligned} & LU(x_t, r_t, t) \\ =& V_t(x(t), r(t), t) + V_x(x(t), r(t), t)[f(x(t), r(t), t) + u(x(\delta_t), r(t), t)] \\ & + \frac{1}{2}\text{trace}[g^T(x(t), r(t), t)V_{xx}(x(t), r(t), t)g(x(t), r(t), t)] \\ & + \sum_{j=1}^{N} \gamma_{r(t),j} V(x(t), j, t) + \hat{V}_t(x_t, r_t, t). \end{aligned} \tag{3.17}$$

Replacing some terms with the operator defined in (3.9), we have

$$\begin{aligned} & LU(x_t, r_t, t) \\ =& \mathcal{L}V(x(t), r(t), t) - V_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(t), t)] \\ & + \theta\tau^{\frac{p}{2}}\left[\tau^{\frac{p}{2}}|f(x(t), r(t), t) + u(x(\delta_t), r(t), t)|^p + \rho|g(x(t), r(t), t)|^p\right] - \Phi(x_t, r_t, t). \end{aligned} \tag{3.18}$$

By the Young inequality (presented in section 2.6 or see e.g. [26, page 52]) and

Assumption 3.1.2, we can derive that

$$\begin{aligned}
& -V_x(x(t),r(t),t)[u(x(t),r(t),t)-u(x(\delta_t),r(t),t)] \\
\le & |V_x(x(t),r(t),t)||u(x(t),r(t),t)-u(x(\delta_t),r(t),t)| \\
\le & \Big[\varepsilon|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}}\Big[\varepsilon^{1-p}|u(x(t),r(t),t)-u(x(\delta_t),r(t),t)|^p\Big]^{\frac{1}{p}} \\
\le & \frac{p-1}{p}\varepsilon|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}+\frac{1}{p}\varepsilon^{1-p}|u(x(t),r(t),t)-u(x(\delta_t),r(t),t)|^p \\
\le & l|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}+\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p|x(t)-x(\delta_t)|^p, \qquad (3.19)
\end{aligned}$$

where $l=\frac{p-1}{p}\varepsilon$ for $\forall\varepsilon>0$. Moreover, by Assumptions 3.1.1, 3.1.2 and the elementary inequality $|a+b|^p\le 2^{p-1}(|a|^p+|b|^p)$ for $\forall a,b\in\mathbb{R}$, we have

$$\begin{aligned}
& |f(x(t),r(t),t)+u(x(\delta_t),r(t),t)|^p \\
\le & 2^{p-1}\Big[K_1^p|x(t)|^p+K_3^p|x(\delta_t)|^p\Big] \\
\le & 2^{p-1}(K_1^p+2^{p-1}K_3^p)|x(t)|^p+4^{p-1}K_3^p|x(t)-x(\delta_t)|^p.
\end{aligned} \qquad (3.20)$$

Substituting (3.19) and (3.20) into (3.18) yields that

$$\begin{aligned}
LU(x_t,r_t,t)\le & \mathcal{L}V(x(t),r(t),t)+l|V_x(x(t),r(t),t)|^{\frac{p}{p-1}} \\
& +\theta\tau^{\frac{p}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p+\rho K_2^p+4^{p-1}\tau^{\frac{p}{2}}K_3^p\Big]|x(t)|^p \\
& +\Big(4^{p-1}\theta\tau^p+\frac{1}{p}(\frac{p-1}{pl})^{p-1}\Big)K_3^p|x(t)-x(\delta_t)|^p-\Phi(x_t,r_t,t). \qquad (3.21)
\end{aligned}$$

Then Assumption 3.1.3 implies that

$$LU(x_t,r_t,t)\le -\beta|x(t)|^p+\Big(4^{p-1}\theta\tau^p+\frac{1}{p}(\frac{p-1}{pl})^{p-1}\Big)K_3^p|x(t)-x(\delta_t)|^p-\Phi(x_t,r_t,t). \qquad (3.22)$$

where

$$\beta=\beta(\theta,\tau):=\lambda-\theta\tau^{\frac{p}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p+\rho K_2^p+4^{p-1}\tau^{\frac{p}{2}}K_3^p\Big]. \qquad (3.23)$$

Furthermore, it's easy to see from the definition of hybrid SDEs solutions and the elementary inequality in (2.17) that

$$\begin{aligned}&|x(t)-x(\delta_t)|^p\\ \leq& 2^{p-1}\Big(\Big|\int_{\delta_t}^t [f(x(s),r(s),s)+u(x(\delta_s),r(s),s)]ds\Big|^p+\Big|\int_{\delta_t}^t g(x(s),r(s),s)dw(s)\Big|^p\Big).\end{aligned} \tag{3.24}$$

Since $t-\delta_t\leq\tau$ for all $t\geq 0$, Hölder's inequality implies that

$$\Big|\int_{\delta_t}^t [f(x(s),r(s),s)+u(x(\delta_s),r(s),s)]ds\Big|^p\leq \tau^{p-1}\int_{\delta_t}^t |f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^p ds. \tag{3.25}$$

For $p\in(1,2)$, we use the Burkholder-Davis-Gundy inequality (presented in section 2.6 or see e.g. [9, page 40]) and Hölder's inequality to obtain that

$$\begin{aligned}&\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),r(s),s)dw(s)\Big|^p\leq \mathbb{E}\Big(\sup_{\delta_t\leq z\leq t}\Big|\int_{\delta_t}^z g(x(v),r(v),v)dw(v)\Big|^p\Big)\\ \leq&(\frac{32}{p})^{\frac{p}{2}}\mathbb{E}\Big[\int_{\delta_t}^t |g(x(s),r(s),s)|^2 ds\Big]^{\frac{p}{2}}\leq(\frac{32}{p})^{\frac{p}{2}}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t |g(x(s),r(s),s)|^p ds.\end{aligned} \tag{3.26}$$

For $p\geq 2$, we use Theorem 2.6.2 (or see [9, Theorem 7.1 on page 39]) to obtain that

$$\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),r(s),s)dw(s)\Big|^p\leq[\frac{p(p-1)}{2}]^{\frac{p}{2}}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t |g(x(s),r(s),s)|^p ds. \tag{3.27}$$

Substituting (3.25), (3.26), (3.27) and (3.11) into (3.24) yields

$$\begin{aligned}&\mathbb{E}|x(t)-x(\delta_t)|^p\\ \leq& 2^{p-1}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t\Big[\tau^{\frac{p}{2}}|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^p+\rho|g(x(s),r(s),s)|^p\Big]ds.\end{aligned} \tag{3.28}$$

Substitute (3.16) and (3.28) into (3.22). Then taking expectations on both sides gives

$$\begin{aligned}
&\mathbb{E}(LU(x_t, r_t, t)) \\
\leq &-\beta\mathbb{E}|x(t)|^p + 2^{p-1}\tau^{\frac{p-2}{2}}\Big(4^{p-1}\theta\tau^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}\Big)K_3^p \\
&\times \mathbb{E}\int_{\delta_t}^t \Big[\tau^{\frac{p}{2}}|f(x(s), r(s), s) + u(x(\delta_s), r(s), s)|^p + \rho|g(x(s), r(s), s)|^p\Big]ds \\
&- \theta\tau^{\frac{p-2}{2}}\mathbb{E}\int_{t-\tau}^t \Big[\tau^{\frac{p}{2}}|f(x(s), r(s), s) + u(x(\delta_s), r(s), s)|^p + \rho|g(x(s), r(s), s)|^p\Big]ds.
\end{aligned} \tag{3.29}$$

Let us now choose

$$8^{p-1}\tau^p K_3^p < 1 \quad \text{and} \quad \theta = \frac{[2(p-1)]^{p-1}}{p^p(1-8^{p-1}\tau^p K_3^p)}l^{1-p}K_3^p. \tag{3.30}$$

Then

$$2^{p-1}\tau^{\frac{p-2}{2}}[4^{p-1}\theta\tau^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}]K_3^p = \theta\tau^{\frac{p-2}{2}} \tag{3.31}$$

Noting that $t - \tau \leq \delta_t$ and combining (3.31) with (3.29), we get

$$\mathbb{E}(LU(x_t, r_t, t)) \leq -\beta\mathbb{E}|x(t)|^p. \tag{3.32}$$

By condition (3.14), we have $\beta > 0$.

Moreover, we know from the generalized Itô formula that

$$\mathbb{E}U(x_t, r_t, t) = U(x_0, r_0, 0) + \mathbb{E}\int_0^t LU(x_s, r_s, s)ds, \qquad \text{for } t \geq 0. \tag{3.33}$$

Denote $U(x_0, r_0, 0)$ by C_0 for simplicity, then

$$C_0 = V(x_0, r_0, 0) + 0.5\theta\tau^{\frac{p+2}{2}}\Big[\tau^{\frac{p}{2}}|f(x_0, r_0, 0) + u(x_0, r_0, 0)|^p + \rho|g(x_0, r_0, 0)|^p\Big]. \tag{3.34}$$

Clearly, C_0 is a positive number. Consequently, substituting (3.32) into (3.33) and by the Fubini theorem, we obtain that

$$0 \le \mathbb{E}U(x_t, r_t, t) \le C_0 - \beta \int_0^t \mathbb{E}|x(s)|^p ds, \tag{3.35}$$

for $t \ge 0$. Hence

$$\int_0^\infty \mathbb{E}|x(s)|^p ds \le C_0/\beta,$$

which implies the desired assertion (3.15). The proof is complete. □

Denote the right-hand-side of the first inequality in (3.14) by L_τ, i.e.

$$L_\tau = \frac{[2(p-1)]^{p-1} K_3^p}{p^p l^{p-1}(1 - 8^{p-1}\tau^p K_3^p)} \tau^{\frac{p}{2}} \Big[2^{p-1}\tau^{\frac{p}{2}} K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}} K_3^p \Big].$$

Condition (3.14) requires $\lambda > L_\tau$. When $\tau = 0$, $L_\tau = 0$ and then (3.14) holds. L_τ is obviously continuous on τ and increases with τ. Hence, for any fixed values (within the ranges stated above) of the parameters p, λ, l, K_1, K_2 and K_3, there is a unique positive number τ^* such that $\lambda = L_\tau$. Setting $\tau \in (0, \tau^*)$ guarantees $\lambda > L_\tau$. Setting $0 < \tau < \min(\tau^*, 8^{-\frac{p-1}{p}}/K_3)$ guarantees that condition (3.14) is satisfied. Besides, the derivative of L_τ with respect to τ, K_1, K_2, K_3 are all nonnegative.

Compared to [97] which studied mean square stabilization, my theory requires weaker conditions on the observation interval τ. You et al. [97] required τ to satisfy

$$l\lambda > \tau K_3^2[2\tau(K_1^2 + 2K_3^2) + K_2^2] \quad \text{and} \quad \tau \le \frac{1}{4K_3}. \tag{3.36}$$

When $p = 2$, condition (3.14) becomes

$$2l(1 - 8K_3^2\tau^2)\lambda > \tau K_3^2[2\tau(K_1^2 + 2K_3^2) + K_2^2] \quad \text{and} \quad \tau \le \frac{1}{2\sqrt{2}K_3}.$$

On one hand, $\tau \le \frac{1}{2\sqrt{2}K_3}$ is weaker than $\tau \le \frac{1}{4K_3}$. On the other hand, when $\tau \le \frac{1}{4K_3}$, $2(1-8K_3^2\tau^2) > 1$, i.e. we allow the right-hand-side $\tau K_3^2[2\tau(K_1^2+2K_3^2)+K_2^2]$ larger than [97]. Therefore, we allow for larger observation interval for mean square stabilization than [97]. This will be shown later in Example 3.4.2.

3.2.2 Asymptotic stability

Theorem 3.2.2 *Let the moment order $p \geq 2$. Under the same assumptions of Theorem 3.2.1, the solution of the controlled system (3.2) satisfies*

$$\lim_{t \to \infty} \mathbb{E}|x(t)|^p = 0$$

for any initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. In other words, the controlled system (3.2) is asymptotically stable in pth moment.

Proof. Again, fix any $x_0 \in \mathbb{R}^n$, $r_0 \in \mathbb{S}$. We know from the Itô formula that for $t \geq 0$,

$$\begin{aligned} \mathbb{E}(|x(t)|^p) &= |x_0|^p + \mathbb{E}\int_0^t \Big(p|x(s)|^{p-2}x^T(s)[f(x(s),r(s),s) + u(x(\delta_s),r(s),s)]\Big)ds \\ &+ \mathbb{E}\int_0^t \Big(\frac{p}{2}|x(s)|^{p-2}|g(x(s),r(s),s)|^2 + \frac{p(p-2)}{2}|x(s)|^{p-4}|x^T(s)g(x(s),r(s),s)|^2\Big)ds. \end{aligned} \tag{3.37}$$

Since $x^T y \leq |x||y|$ and $|x^T g| \leq |x||g|$ for $\forall x, y \in \mathbb{R}^n, g \in \mathbb{R}^{n\times m}$, we have

$$\begin{aligned} \mathbb{E}(|x(t)|^p) &\leq |x_0|^p + \int_0^t p\mathbb{E}\Big[|x(s)|^{p-1}(|f(x(s),r(s),s)| + |u(x(\delta_s),r(\delta_s),s)|)\Big]ds \\ &+ \int_0^t \Big(\frac{p}{2}\mathbb{E}\Big[|x(s)|^{p-2}|g(x(s),r(s),s)|^2\Big] + \frac{p(p-2)}{2}\mathbb{E}[|x(s)|^{p-2}|g(x(s),r(s),s)|^2]\Big)ds \end{aligned} \tag{3.38}$$

for $p \geq 2$. Then Assumptions 3.1.1 and 3.1.2 imply

$$\mathbb{E}(|x(t)|^p) \leq |x_0|^p + \int_0^t \Big(pK_1\mathbb{E}|x(s)|^p + pK_3\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big] + \frac{p(p-1)}{2}K_2^2\mathbb{E}|x(s)|^p\Big)ds. \tag{3.39}$$

Moreover, the Young inequality and the elementary inequality in (2.17) ($|a+b|^p \leq$

$2^{p-1}(|a|^p + |b|^p)$ for $\forall a, b \in R$) imply that

$$
\begin{aligned}
|x(s)|^{p-1}|x(\delta_s)| &\leq \Big[\frac{p-1}{p}||x(s)| + \frac{1}{p}|x(\delta_s)|\Big]^p \\
&\leq \frac{2^{p-1}}{p^p}\Big[(p-1)^p|x(s)|^p + |x(\delta_s)|^p\Big] \\
&\leq \frac{2^{p-1}}{p^p}\Big[((p-1)^p + 2^{p-1})|x(s)|^p + 2^{p-1}|x(s) - x(\delta_s)|^p\Big].
\end{aligned}
\tag{3.40}
$$

Substituting this into (3.39) gives

$$
\mathbb{E}(|x(t)|^p) \leq |x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds + C\int_0^t \mathbb{E}|x(s) - x(\delta_s)|^p ds, \tag{3.41}
$$

where, here and in the remaining part of the thesis, C's denote positive constants that may change from line to line.

Note that for any $s \geq 0$, there is a unique integer $v \geq 0$ for $s \in [v\tau, (v+1)\tau)$, and $\delta_z = v\tau$ for any $z \in [v\tau, s]$.

Recall (3.28) as well as the Assumptions 3.1.1 and 3.1.2, we derive that

$$
\begin{aligned}
&\mathbb{E}|x(s) - x(\delta_s)|^p = \mathbb{E}|x(s) - x(v\tau)|^p \\
\leq& 2^{p-1}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{v\tau}^s \tau^{\frac{p}{2}}|f(x(z), r(z), z) + u(x(\delta_z), r(z), z)|^p + \rho|g(x(z), r(z), z)|^p dz \\
\leq& 2^{p-1}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{v\tau}^s 2^{p-1}\tau^{\frac{p}{2}}\Big[K_1^p|x(z)|^p + K_3^p|x(v\tau)|^p\Big] + \rho K_2^p|x(z)|^p dz \\
\leq& 2^{p-1}\tau^{\frac{p-2}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p\Big]\int_{v\tau}^s \mathbb{E}|x(z)|^p dz + 4^{p-1}\tau^p K_3^p \mathbb{E}|x(v\tau)|^p \\
\leq& 2^{p-1}\tau^{\frac{p-2}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p\Big]\int_{v\tau}^s \mathbb{E}|x(z)|^p dz + 8^{p-1}\tau^p K_3^p\Big(\mathbb{E}|x(s)|^p + \mathbb{E}|x(s) - x(v\tau)|^p\Big).
\end{aligned}
\tag{3.42}
$$

Note that the condition (3.14) implies $8^{p-1}\tau^p K_3^p < 1$, then we can rearrange it and obtain that

$$\mathbb{E}|x(s)-x(\delta_s)|^p \le \frac{2^{p-1}\tau^{\frac{p-2}{2}}\left[2^{p-1}\tau^{\frac{p}{2}}K_1^p+\rho K_2^p\right]}{1-8^{p-1}\tau^p K_3^p}\int_{\delta_s}^{s}\mathbb{E}|x(z)|^p dz+\frac{8^{p-1}\tau^p K_3^p}{1-8^{p-1}\tau^p K_3^p}\mathbb{E}|x(s)|^p. \tag{3.43}$$

Substituting this into (3.41) yields

$$\mathbb{E}|x(t)|^p \le |x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds + C\int_0^t\int_{\delta_s}^{s}\mathbb{E}|x(z)|^p dzds. \tag{3.44}$$

Besides, it's easy to show that for a non-negative function $F(t)$,

$$\begin{aligned}&\int_0^t\int_{\delta_s}^{s} F(z)dzds \le \int_0^t\int_{s-\tau}^{s} F(z)dzds\\ &\le \int_{-\tau}^{t} F(z)\int_{z}^{z+\tau} dsdz \le \tau\int_{-\tau}^{t} F(z)dz.\end{aligned}$$

Applying this to $\mathbb{E}|x(z)|^p$ gives

$$\int_0^t\int_{\delta_s}^{s}\mathbb{E}|x(z)|^p dzds \le \tau\int_{-\tau}^{t}\mathbb{E}|x(z)|^p dz \le \tau^2|x_0|^p + \tau\int_0^t \mathbb{E}|x(z)|^p dz,$$

then we can rewrite (3.44) as

$$\mathbb{E}|x(t)|^p \le C|x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds. \tag{3.45}$$

So by Theorem 3.2.1, we have

$$\mathbb{E}|x(t)|^p \le C \quad \forall t \ge 0. \tag{3.46}$$

Furthermore, it's easy to see from the Itô formula that

$$\begin{aligned}
&\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \\
=&\mathbb{E}\int_{t_1}^{t_2} \Big(p|x(t)|^{p-2}x^T(t)[f(x(t),r(t),t) + u(x(\delta_t),r(t),t)]\Big)dt \\
&+\mathbb{E}\int_{t_1}^{t_2} \Big(\frac{p}{2}|x(t)|^{p-2}|g(x(t),r(t),t)|^2 + \frac{p(p-2)}{2}|x(t)|^{p-4}|x^T(t)g(x(t),r(t),t)|^2\Big)dt.
\end{aligned}$$

After similar calculations to (3.37) and(3.39), we derive that

$$\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p \le \int_{t_1}^{t_2} \Big(pK_1\mathbb{E}|x(t)|^p + pK_3\mathbb{E}\Big[|x(t)|^{p-1}|x(\delta_t)|\Big] + \frac{p(p-1)}{2}K_2^2\mathbb{E}|x(t)|^p\Big)dt.$$

Then by (3.46), we get that for any $0 \le t_1 < t_2 < \infty$,

$$\Big|\mathbb{E}|x(t_2)|^p - \mathbb{E}|x(t_1)|^p\Big| \le C(t_2 - t_1).$$

According to Barbalat's lemma [2] (presented in Lemma 2.5.11 or see [124, page 123]), combining this uniform continuity with Theorem 3.2.1 yields that $\lim_{t\to\infty}\mathbb{E}|x(t)|^p = 0$. The proof is complete. □

3.3 Exponential stabilization

In last section, we discussed the asymptotic stabilization and proved that eventually (as $t \to \infty$), $\mathbb{E}|x(t)|^p$ goes to 0, but we don't know its speed. To explore the rate at which the solution tends to zero, let us discuss the exponential stabilization in this section. We need to impose the following condition.

Assumption 3.3.1 *Assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^p \le V(x,i,t) \le c_2|x|^p \tag{3.47}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

[2]Consider $\int_0^t \mathbb{E}|x(s)|^p ds$ to be the 'differentiable function f(t)' in Lemma 2.5.11, consider $\mathbb{E}|x(t)|^p$ as 'its derivative $\dot{f}$' in Lemma 2.5.11.

Theorem 3.3.2 *Fix the moment order $p > 1$. Let Assumptions 3.1.1, 3.1.2, 3.1.3 and 3.3.1 hold. Choose $\tau > 0$ sufficiently small for (3.14) to hold. Then the solution of the controlled system (3.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) \le -\eta \tag{3.48}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\eta}{p} \quad a.s. \tag{3.49}$$

for any initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$, where $\eta > 0$ is the unique root to the following equation

$$2\tau\eta e^{2\tau\eta}(H_1 + \tau H_2) + \eta c_2 = \beta, \tag{3.50}$$

in which

$$\begin{aligned} H_1 &= \theta\tau^{\frac{p}{2}}\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p + \frac{32^{p-1}\tau^{\frac{3p}{2}}K_3^{2p}}{1-8^{p-1}\tau^p K_3^p}\Big] \\ \text{and} \qquad H_2 &= \frac{8^{p-1}\theta\tau^{\frac{3p-2}{2}}K_3^p\Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p\Big]}{1-8^{p-1}\tau^p K_3^p}. \end{aligned} \tag{3.51}$$

Remark 3.3.3 *When $\tau = 0$, obviously $H_1 = H_2 = 0$. For any $\tau > 0$, we have $H_1 > 0$ and $H_2 > 0$. The left hand side of (3.50) is continuous on η. For a fixed $\tau > 0$, when $\eta = 0$, the left hand side of (3.50) is 0 when $\eta = 0$ and its derivative on η is positive. Therefore for any given positive number β, equation (3.50) has a unique root.* [3]

Proof. It's easy to see from the generalized Itô formula that

$$\mathbb{E}\big[e^{\eta t}U(x_t, r_t, t)\big] = U(x_0, r_0, 0) + \mathbb{E}\int_0^t e^{\eta s}[\eta U(x_s, r_s, s) + LU(x_s, r_s, s)]ds \tag{3.52}$$

[3] We can calculate η in this way: The system coefficients and controller determine K_1, K_2 and K_3; moment order determines p and ρ; set parameters c_2 and λ according to corresponding conditions, choose $\tau > 0$ sufficiently small for (3.14) to hold; then calculate β, H_1 and H_2; finally η can be found by solving equation (3.50) and actually finding $\eta > 0$ such that $2\tau\eta e^{2\tau\eta}(H_1 + \tau H_2) + \eta c_2 \le \beta$ is sufficient for (3.48) and (3.49) to hold.

for $t \geq 0$. By (3.47) and (3.12), we have

$$c_1 e^{\eta t}\mathbb{E}|x(t)|^p \leq e^{\eta t}\mathbb{E}V(x(t), r(t), t) \leq e^{\eta t}\mathbb{E}U(x_t, r_t, t)$$

Then combining (3.52), (3.32) and (3.34) gives

$$c_1 e^{\eta t}\mathbb{E}|x(t)|^p \leq C_0 + \int_0^t e^{\eta s}[\eta \mathbb{E}U(x_s, r_s, s) - \beta \mathbb{E}|x(s)|^p]ds. \tag{3.53}$$

Moreover, substutiting (3.10) and (3.47) into (3.12) gives

$$\mathbb{E}U(x_s, r_s, s) \leq c_2 \mathbb{E}|x(s)|^p + \mathbb{E}\hat{V}(x_s, r_s, s). \tag{3.54}$$

Since for a function $F(v)$, we have

$$\begin{aligned}\int_{s-\tau}^s \int_z^s F(v)dvdz &= \int_{s-\tau}^s \int_{s-\tau}^v F(v)dzdv = \int_{s-\tau}^s F(v)\int_{s-\tau}^v dzdv \\ &= \int_{s-\tau}^s F(v)(v-s+\tau)dv < \tau \int_{s-\tau}^s F(v)dv.\end{aligned}$$

Applying this to $\mathbb{E}\hat{V}(x_s, r_s, s)$ yields that

$$\begin{aligned}&\mathbb{E}\hat{V}(x_s, r_s, s) \\ \leq& \theta \tau^{\frac{p}{2}} \mathbb{E}\int_{s-\tau}^s \Big[\tau^{\frac{p}{2}}|f(x(v), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^p + \rho |g(x(v), r(v), v)|^p\Big]dv \\ \leq& \theta \tau^{\frac{p}{2}} \int_{s-\tau}^s \Big[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p\Big]\mathbb{E}|x(v)|^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p \mathbb{E}|x(v) - x(\delta_v)|^p dv.\end{aligned}$$

To make $\delta_v > 0$, we need $v \geq \tau$ and so $s \geq 2\tau$. Note that (3.42) and (3.43) both hold for any $p > 1$. Then we have

$$\mathbb{E}\hat{V}(x_s, r_s, s) \leq H_1 \int_{s-\tau}^s \mathbb{E}|x(v)|^p dv + H_2 \int_{s-\tau}^s \int_{\delta_v}^v \mathbb{E}|x(y)|^p dydv. \tag{3.55}$$

where both H_1 and H_2 have been defined by (3.51).

Since for a non-negative function $F(y)$,

$$\int_{s-\tau}^{s}\int_{\delta_v}^{v} F(y)dydv \leq \int_{s-\tau}^{s}\int_{v-\tau}^{v} F(y)dydv$$
$$< \int_{s-2\tau}^{s}\int_{s-\tau}^{s} F(y)dvdy = \tau\int_{s-2\tau}^{s} F(y)dy.$$

Thus, $\int_{s-\tau}^{s}\int_{\delta_v}^{v}\mathbb{E}|x(y)|^p dydv \leq \tau\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dy$. Hence we have

$$\mathbb{E}(\hat{V}(x_s, r_s, s)) \leq (H_1 + \tau H_2)\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dy. \tag{3.56}$$

Combining (3.53), (3.54) and (3.56), we obtain that

$$\begin{aligned} &c_1 e^{\eta t}\mathbb{E}|x(t)|^p \\ \leq &C_0 - (\beta - \eta c_2)\int_0^t e^{\eta s}\mathbb{E}|x(s)|^p ds + \eta(H_1 + \tau H_2)\int_0^t e^{\eta s}\Big(\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dy\Big)ds \end{aligned} \tag{3.57}$$

for $\forall t \geq 2\tau$. Obviously,

$$\int_0^{2\tau} e^{\eta s}\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dyds \leq \int_{-2\tau}^{2\tau}\int_0^{2\tau} e^{\eta s}\mathbb{E}|x(y)|^p dsdy = \frac{e^{2\tau\eta - 1}}{\eta}\int_{-2\tau}^{2\tau}\mathbb{E}|x(y)|^p dy. \tag{3.58}$$

Besides, it can be easily seen that

$$\begin{aligned} \int_{2\tau}^{t} e^{\eta s}\Big(\int_{s-2\tau}^{s}\mathbb{E}|x(y)|^p dy\Big)ds &\leq \int_0^t \mathbb{E}|x(y)|^p\Big(\int_y^{y+2\tau} e^{\eta s}ds\Big)dy \\ &\leq 2\tau e^{2\tau\eta}\int_0^t e^{\eta y}\mathbb{E}|x(y)|^p dy. \end{aligned} \tag{3.59}$$

Substituting (3.58) and (3.59) into (3.57) gives

$$c_1 e^{\eta t}\mathbb{E}|x(t)|^p \leq C + \Big(2\tau\eta e^{2\tau\eta}(H_1 + \tau H_2) + \eta c_2 - \beta\Big)\int_0^t e^{\eta s}\mathbb{E}|x(s)|^p ds. \tag{3.60}$$

The condition (3.50) implies that for $\forall t \geq 2\tau$,

$$c_1 e^{\eta t}\mathbb{E}|x(t)|^p \leq C. \tag{3.61}$$

Hence we obtain the assertion (3.48). According to Theorem 2.5.6 (or see [26, Theorem 8.8 on page 309]), which shows that pth moment exponential stability implies the almost sure exponential stability under the linear growth condition through definition of the SDE solutions, the Hölder's inequality, the Burkholder-Davis-Gundy inequality and Borel-Cantelli lemma, we finally obtain the assertion (3.49). The proof is complete. □

3.3.1 Corollary

In practice, a common choice of Lyapunov functions is quadratic functions, for example, $V(x(t), r(t), t) = (x^T(t)Q_{r(t)}x(t))^{\frac{p}{2}}$ where $Q_{r(t)}$ are positive-definite symmetric $n \times n$ matrices for $p \geq 2$. So we propose the following corollary to state how to use this kind of Lyapunov functions to help exponentially stabilize an unstable hybrid system.

Since $V_x(x, i, t) = p(x^T Q_i x)^{\frac{p}{2}-1} x^T Q_i$, then we have

$$|V_x(x, i, t)| \leq p\lambda_{\max}^{\frac{p}{2}-1}(Q_i)\|Q_i\||x|^{p-1} = p\lambda_{\max}^{\frac{p}{2}}(Q_i)|x|^{p-1}.$$

So we only need to require $\mathcal{L}V(x, i, t) \leq -b|x|^p$ for $b > 0$ to satisfy Assumption 3.1.3. This leads to the following alternative assumption. Moreover, Assumption 3.3.1 holds with $c_1 = \min_{i\in\mathbb{S}} \lambda_{\min}^{\frac{p}{2}}(Q_i)$ and $c_2 = \max_{i\in\mathbb{S}} \lambda_{\max}^{\frac{p}{2}}(Q_i)$. By calculating the derivatives $V_t(x, i, t) = 0$ and $V_{xx}(x, i, t) = p(p-2)[x^T Q_i x]^{\frac{p}{2}-2} Q_i x x^T Q_i + p[x^T Q_i x]^{\frac{p}{2}-1} Q_i$, we can easily obtain $\mathcal{L}V(x, i, t)$, which is the left-hand-side of (3.62) below.

Assumption 3.3.4 *Assume that there exist positive-definite symmetric matrices $Q_i \in \mathbb{R}^{n\times n}$ $(i \in \mathbb{S})$ and a constant $b > 0$ such that*

$$\begin{aligned} &p(x^T Q_i x)^{\frac{p}{2}-1}\Big(x^T Q_i[f(x, i, t) + u(x, i, t)] + \frac{1}{2}\text{trace}[g^T(x, i, t)Q_i g(x, i, t)]\Big) \\ &+ p(\frac{p}{2} - 1)[x^T Q_i x]^{\frac{p}{2}-2}|g^T Q_i x|^2 + \sum_{j=1}^{N} \gamma_{ij}[x^T Q_j x]^{\frac{p}{2}} \leq -b|x|^p, \end{aligned} \tag{3.62}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Corollary 3.3.5 *Fix moment order $p \geq 2$. Let Assumptions 3.1.1, 3.1.2 and 3.3.4 hold. Set*

$$c_2 = \max_{i \in \mathbb{S}} \lambda_{\max}^{\frac{p}{2}}(Q_i) \quad \text{and} \quad d = (pc_2)^{\frac{p}{p-1}}.$$

Choose $l \in (0, b/d)$ to maximize $bl^{p-1} - dl^p$. Then set $\lambda = b - ld$. Let $\tau > 0$ be sufficiently small for (3.14) to hold. Then (3.48) holds, i.e., the controlled system (3.2) is pth moment exponentially stable.

The reason to require $l < b/d$ is because, condition (3.13) in Assumption 3.1.3 is equivalent to $-b + ld \leq -\lambda < 0$. The reason to maximize $bl^{p-1} - dl^p$ is because, substituting $\lambda = b - ld$ into (3.14) yields that large $bl^{p-1} - dl^p$ allows for relatively large τ.

It can be seen that the condition (5.4) in [97] is a special case of (3.62) when $p = 2$. Besides, You et al. [97] didn't discuss how to choose l. My parameter settings can give a better (larger) observation interval, which is shown in Example 3.4.2 below.

Unlike the mean square case ($p = 2$), a more general range of moment order brings more complexity and difficulty to the stabilization problem. Firstly, I need to use more general inequalities. Secondly, some terms would have more complex coefficients. If $p = 2$, in Assumption 3.3.4 condition (3.62), the term $(x^T Q_i x)^{\frac{p}{2}-1}$ and $p(\frac{p}{2} - 1)[x^T Q_i x]^{\frac{p}{2}-2}|g^T Q_i x|^2$ would be simply 1 and 0 respectively.

3.4 Examples

Now we illustrate our theory with two examples.

Example 3.4.1 Now we consider a nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{3.63}$$

on $t \geq 0$. Here

$$x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix};$$

$B(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator matrix

$$\Gamma = \begin{bmatrix} -4 & 4 \\ 1 & -1 \end{bmatrix};$$

and the coefficients are

$$f(x(t), 1, t) = \begin{bmatrix} x_2(t)\cos(x_1(t)) \\ x_1(t)\sin(x_2(t)) \end{bmatrix}, \quad f(x(t), 2, t) = \begin{bmatrix} x_2(t)\sin(x_1(t)) \\ x_1(t)\cos(x_2(t)) \end{bmatrix},$$

$$g(x(t), 1, t) = \begin{bmatrix} 0.2\sqrt{3x_1^2(t) + x_2^2(t)} \\ 0.2\sqrt{x_1^2(t) + 3x_2^2(t)} \end{bmatrix}, \quad g(x(t), 2, t) = \begin{bmatrix} 0.1 & -0.1 \\ -0.2 & 0.4 \end{bmatrix} x(t).$$

Figure 3.1 below shows simulated paths and obviously this system is not stable in the sense of 3rd moment exponential stability.

Note that this system satisfies the Assumption 3.1.1 with $K_1 = 1$ and $K_2 = 0.4671$. We will design a feedback control of the form $u(x, i, t) = A_i(x)x$ and find the observation interval τ to make the controlled system

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(t), t)\big)dt + g(x(t), r(t), t)dB(t), \tag{3.64}$$

become 3rd moment exponentially stable. In the controller, $A_i(x) : \mathbb{R}^2 \to \mathbb{R}^{2\times 2}$ and Assumption 3.1.2 will hold with $K_3 = \max_{i\in\mathbb{S}, x\in\mathbb{R}^2} \|A_i(x)\|$.

Now we can start designing $A_i(x)$ by choosing our auxiliary Lyapunov functions. We choose Lyapunov functions of the form $V(x, i, t) = (x^T Q_i x)^{1.5}$ where $Q_i = q_i I$ (I is the 2×2 identity matrix), so Corollary 3.3.5 can be applied.

Let $V(x, i, t) = q_i^{1.5}|x|^3$ where $q_1 = 2, q_2 = 1$. Then the left-hand-side of (3.62)

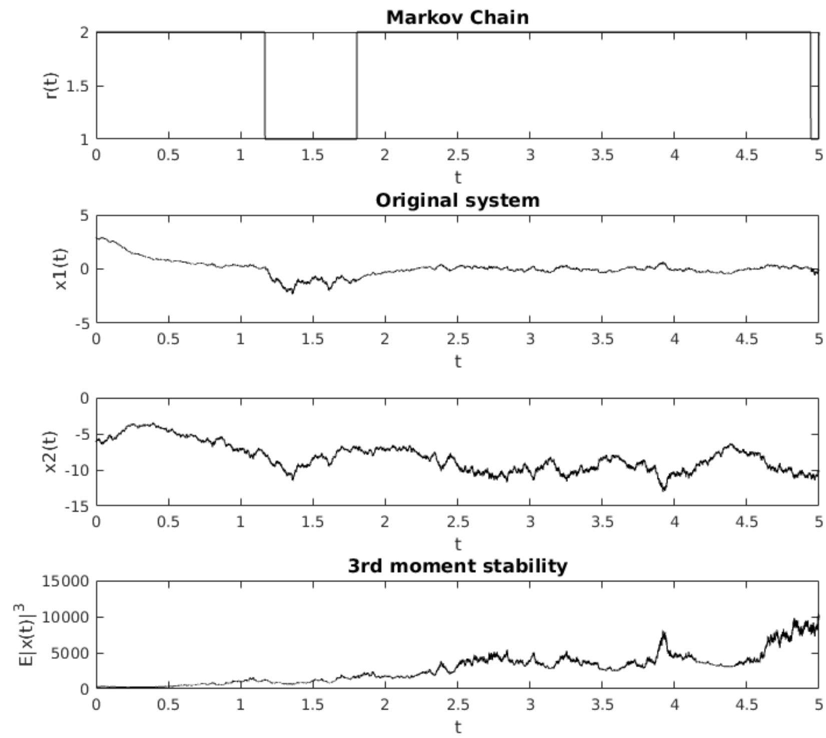

Figure 3.1: Simulation of system (3.63) with random initial values using the Euler-Maruyama method with step size $1e-6$. The upper three plots show one path of system mode and state. The bottom plot is the sample mean of $|x(t)|^3$ from 2000 paths.

becomes

$$
\begin{aligned}
&3q_i^{0.5}|x|[q_i x^T(f(x,i,t)+u(x,i,t))+0.5q_i|g(x,i,t)|^2]+1.5q_i|x|^{-1}|g^Tx|^2+\sum_{j=1}^N \gamma_{ij}q_j^{1.5}|x|^3\\
\le&1.5q_i^{1.5}|x|[(2K_1+K_2^2)|x|^2+2x^TA_i(x)x]+1.5q_iK_2^2|x|^{1.5}+\sum_{j=1}^N \gamma_{ij}q_j^{1.5}|x|^3\\
\le&|x|x^T\tilde{Q}_ix\le\lambda_{\max}(\tilde{Q}_i)|x|^3
\end{aligned}
\tag{3.65}
$$

for all $i\in\mathbb{S}$, where

$$
\tilde{Q}_i=1.5q_i^{1.5}(2K_1+K_2^2)I+1.5q_i^{1.5}(A_i(x)+A_i^T(x))+1.5q_iK_2^2I+\sum_{j=1}^N \gamma_{ij}q_j^{1.5}I.
$$

Substituting the constant coefficients gives

$$\tilde{Q}_1 = 2.7517I + 4.2426(A_1(x) + A_1^T(x))$$
$$\text{and} \quad \tilde{Q}_2 = 5.4829I + 1.5(A_2(x) + A_2^T(x)).$$

Thus, we need to design $A_i(x)$ such that $\tilde{Q}_i$ is negative-definite for $i \in \mathbb{S}$. Of course there are many choices of $A_i(x)$, here we use

$$A_1(x(t)) = \begin{bmatrix} 0.5\sin(x_1(t)) - 1 & -1 \\ 1 & 0.5\cos(x_2(t)) - 1 \end{bmatrix}$$

$$\text{and} \quad A_2(x(t)) = \begin{bmatrix} -2.3 & 0.2\cos(x_1(t)x_2(t)) \\ -0.2\cos(x_1(t)x_2(t)) & -2.3 \end{bmatrix}.$$

Substituting the coefficient matrices gives $\lambda_{\max}(\tilde{Q}_1) = -1.491$ and $\lambda_{\max}(\tilde{Q}_2) = -1.417$. That is, Assumption 3.3.4 holds with $b = 1.417$. Assumption 3.1.2 holds with $K_3 = 2.309$. Then we calculate parameters in Corollary 3.3.5 and get $c_1 = 1$, $c_2 = 2.828$ and $d = 24.7116$. To obtain a relatively large observation interval τ, we choose $l = 0.0382$. This gives $\lambda = 0.473$. Then $\tau \le 0.003$ satisfies condition (3.68) and $\beta = 0.0109 > 0$. Corollary 3.3.5 indicates that the controlled system (3.64) with feedback control defined as above and $\tau \le 0.003$ is 3rd moment exponentially stable, which is indeed in accordance with the Figure 3.2.

Example 3.4.2 We can use a larger observe interval than the Example 6.1 in [97] to achieve the mean square exponential stabilization for the same original system and controller.

As stated in Example 6.1 in [97], the original system is not mean square exponential stable. Figure 3.3 shows my simulated paths and the results agree with it.

We use the same controller, same parameters except l and λ, same Lyapunov functions as stated in [97]. That is, $K_1 = 5.236$, $K_2 = \sqrt{2}$, $K_3 = 10$; $c_1 = c_2 = 1$, $b = 8$, $d = 4$; $Q_1 = Q_2 = I$ (the 2×2 identity matrix).

By Corollary 3.3.5, we choose $l \in (0, b/d)$ to maximize $bl - dl^2$. So let $l = 1$. Then

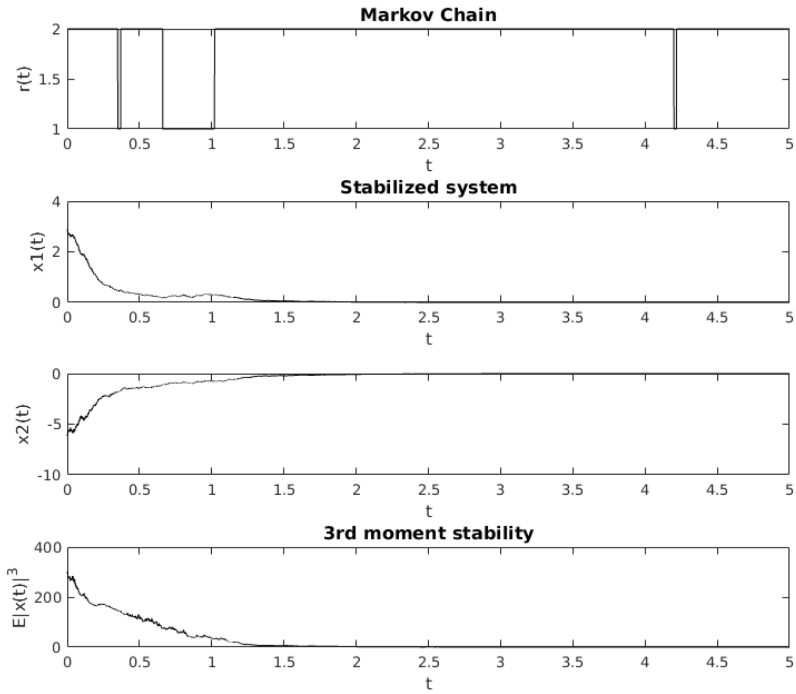

Figure 3.2: Simulation of the controlled system (3.64) with observation interval $\tau = 0.003$ and random initial values using the Euler-Maruyama method with step size $1e{-}6$. The upper three plots show one path of system mode and state. The bottom plot is the sample mean of $|x(t)|^3$ from 2000 paths.

this gives $\lambda = 4$. Finally $\tau \leq 0.0088$ can satisfy condition (3.68). You et al. [97] required $\tau \leq 0.0074$. By Corollary 3.3.5, the controlled system with $\tau \leq 0.0088$ is mean square exponentially stable, which was in accordance with the simulation results in Figure 3.4.

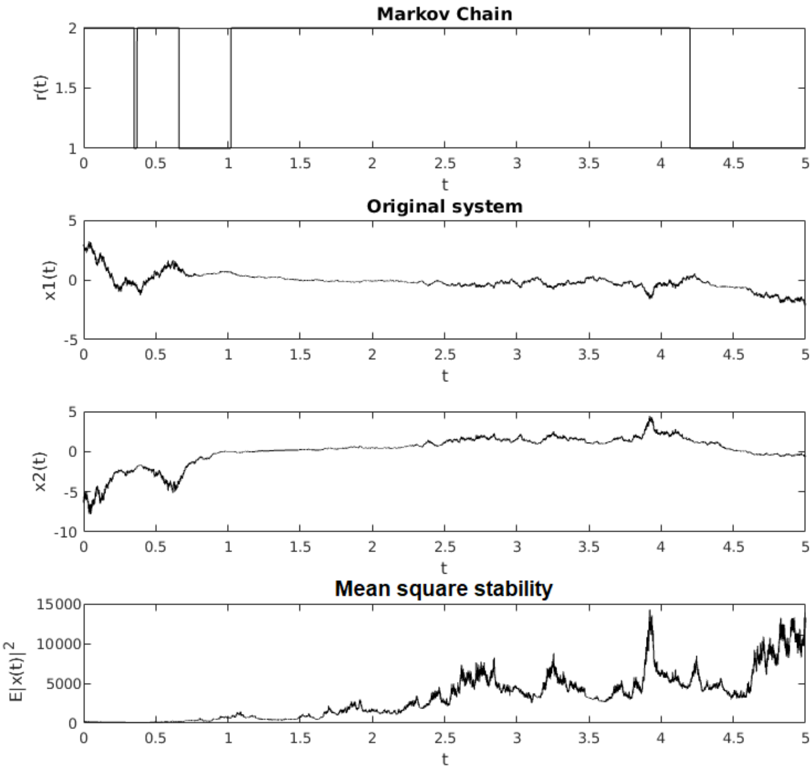

Figure 3.3: Simulation of the original system with random initial values using the Euler-Maruyama method with step size $1e-6$. The upper three plots show one path of system mode and state. The bottom plot is the sample mean of $|x(t)|^2$ from 2000 paths.

Larger observation interval means less frequent observations and less cost of control. This is one of the advantages of my results over the existing best result. The improvement of the upper bound of observation interval is because of the weaker condition (3.68) and the better parameter settings.

3.5 Discretization of mode observation

In previous sections, we discussed the pth moment stabilization with controller $u(x(\delta_t), r(t), t)$. However, it's more practical and cost-effective to observe the system mode in discrete time as well. So in this section, we discuss the stabilization problem with controller $u(x(\delta_t), r(\delta_t), t)$ is based on observations of both system state and mode

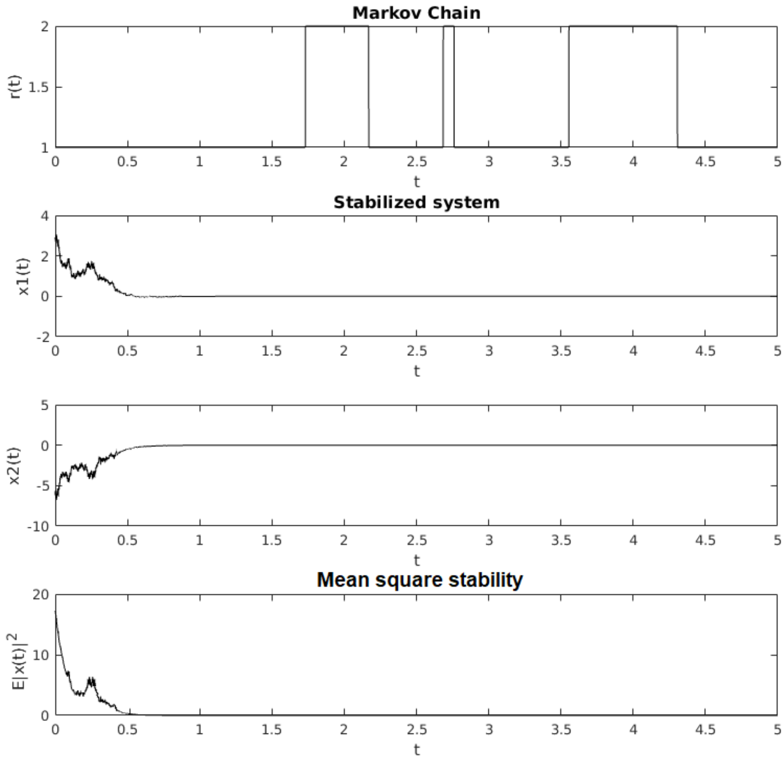

Figure 3.4: Simulation of the controlled system with observation interval $\tau = 0.0088$ and random initial values using the Euler-Maruyama method with step size $1e-6$. The upper three plots show one path of system mode and state. The bottom plot is the sample mean of $|x(t)|^2$ from 2000 paths.

at time points $0, \tau, 2\tau, 3\tau, \cdots$. Consequently, the controlled system has the form

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), r(t), t)dB(t). \tag{3.66}$$

Generally speaking, the conclusions in previous theorems and corollary still hold here with a smaller observation interval.

We define a similar Lyapunov functional for a fixed moment order $p > 1$ by

$$\hat{V}(x_t, r_t, t) = \theta\tau^{\frac{p-2}{2}} \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau^{\frac{p}{2}} |f(x(z), r(z), z) + u(x(\delta_z), r(\delta_z), z)|^p + \rho |g(x(z), r(z), z)|^p\Big] dz ds \tag{3.67}$$

for $t \geq 0$, where $x_t := \{x(t+s) : -2\tau \leq s \leq 0\}$ and $r_t := \{r(t+s) : -2\tau \leq s \leq 0\}$.

Define $\overline{\gamma} := \max_{i\in\mathbb{S}}(-\gamma_{ii})$.

Theorem 3.5.1 *Fix the moment order $p > 1$. Let Assumptions 3.1.1, 3.1.2 and 3.1.3 hold. Choose $\tau > 0$ sufficiently small for*

$$\lambda > \frac{[4(p-1)]^{p-1}[2^{2p-1}(1-e^{-\overline{\gamma}\tau})+1]K_3^p}{p^p l^{p-1}(1-8^{p-1}\tau^p K_3^p)}\tau^{\frac{p}{2}}\left[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p\right]$$
$$+\frac{2^{3p-2}}{p}(\frac{p-1}{pl})^{p-1}K_3^p(1-e^{-\overline{\gamma}\tau}) \qquad \text{and} \qquad \tau \le 8^{-\frac{p-1}{p}}/K_3, \tag{3.68}$$

then the controlled system (3.66) satisfies

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty \tag{3.69}$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$.

Denote the right-hand-side of the first inequality in (3.68) by $\hat{L}_\tau$. Similarly to the discussion in section 2.2.1, for any fixed values (within the ranges stated above) of the parameters $p, \lambda, l, K_1, K_2, K_3$ and $\overline{\gamma}$, there is a unique positive number τ^* such that $\lambda = \hat{L}_\tau$. So it's guaranteed that condition (3.68) can be satisfied.

Let's compare the difference between condition (3.14) and (3.68). Recall $L_\tau = \frac{[2(p-1)]^{p-1}K_3^p}{p^p l^{p-1}(1-8^{p-1}\tau^p K_3^p)}\tau^{\frac{p}{2}}\left[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p\right]$. Therefore

$$\hat{L}_\tau = [2^{3p-2}(1-e^{-\overline{\gamma}\tau})+1]L_\tau + \frac{2^{3p-2}}{p}(\frac{p-1}{pl})^{p-1}K_3^p(1-e^{-\overline{\gamma}\tau}) > L_\tau \quad \text{for } \tau > 0.$$

That is, condition $\lambda > \hat{L}_\tau$ requires a smaller τ than condition $\lambda > L_\tau$. In other words, discretization of the mode observation increases the observation frequency, which makes sense. Moreover, we notice that the larger $\overline{\gamma}$ is, the smaller τ has to be. This means, large maximum jump rate, or frequent mode switching requires frequent system observations.

Theorem 3.5.1 can be proved in similar way as Theorem 3.2.1. The key difference is that after replacing $u(x(\delta_t), r(t), t)$ with $u(x(\delta_t), r(\delta_t), t)$ in (3.19), we need to calculate $\mathbb{E}|u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^p$. To avoid redundancy, I only show the key difference below and put the complete proof in Appendix A.1.

Key difference:

By the Young inequality and Assumption 3.1.2, we can derive that

$$\begin{aligned}
&- V_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)] \\
\le& \Big[\varepsilon |V_x(x(t), r(t), t)|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} \Big[\varepsilon^{1-p} |u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^p\Big]^{\frac{1}{p}} \\
\le& l |V_x(x(t), r(t), t)|^{\frac{p}{p-1}} + \frac{1}{p}(\frac{p-1}{pl})^{p-1} |u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^p,
\end{aligned} \tag{3.70}$$

where $l = \frac{p-1}{p}\varepsilon$ for $\forall \varepsilon > 0$.

Since

$$\begin{aligned}
&u(x(\delta_t), r(\delta_t), t) - u(x(t), r(t), t) \\
=& u(x(\delta_t), r(\delta_t), t) - u(x(\delta_t), r(t), t) \\
&+ u(x(\delta_t), r(t), t) - u(x(t), r(t), t),
\end{aligned}$$

and the elementary inequality $|a + b|^p \le 2^{p-1}(|a|^p + |b|^p)$ for $a, b \in \mathbb{R}$, $p > 1$, we can obtain that

$$\begin{aligned}
&\mathbb{E}|u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^p \\
\le& 2^{p-1}\mathbb{E}|u(x(\delta_t), r(\delta_t), t) - u(x(\delta_t), r(t), t)|^p + 2^{p-1}\mathbb{E}|u(x(\delta_t), r(t), t) - u(x(t), r(t), t)|^p \\
\le& 2^{p-1}\mathbb{E}|u(x(\delta_t), r(\delta_t), t) - u(x(\delta_t), r(t), t)|^p + 2^{p-1}K_3^p\mathbb{E}|x(\delta_t) - x(t)|^p.
\end{aligned} \tag{3.71}$$

According to Lemma 2.3.4 (or see Lemma 1 in [35]), for any $t \ge t_0$, $v > 0$ and $i \in \mathbb{S}$,

$$\mathbb{P}(r(s) \ne i \text{ for some } s \in [t, t+v] \Big| r(t) = i) \le 1 - e^{-\overline{\gamma} v}. \tag{3.72}$$

Then by Assumption 3.1.2, we have

$$\begin{aligned}
&\mathbb{E}|u(x(\delta_t),r(\delta_t),t)-u(x(\delta_t),r(t),t)|^p\\
=&\mathbb{E}\Big[\mathbb{E}|u(x(\delta_t),r(\delta_t),t)-u(x(\delta_t),r(t),t)|^p\Big|\mathcal{F}_{\delta_t})\Big]\\
\le&\mathbb{E}\Big[2^pK_3^p|x(\delta_t)|^p\mathbb{E}\Big(I_{\{r(s)\neq r_k\}}|\mathcal{F}_{\delta_t}\Big)\Big]\\
\le&\mathbb{E}\Big[2^pK_3^p|x(\delta_t)|^p(1-e^{-\overline{\gamma}\tau})\Big]\\
\le&2^{2p-1}K_3^p(1-e^{-\overline{\gamma}\tau})[\mathbb{E}|x(t)|^p+\mathbb{E}|x(\delta_t)-x(t)|^p]
\end{aligned}\tag{3.73}$$

Substituting (A.7) into (A.5) gives

$$\begin{aligned}
&\mathbb{E}|u(x(t),r(t),t)-u(x(\delta_t),r(\delta_t),t)|^p\\
\le&2^{3p-2}K_3^p(1-e^{-\overline{\gamma}\tau})\mathbb{E}|x(t)|^p+[2^{3p-2}K_3^p(1-e^{-\overline{\gamma}\tau})+2^{p-1}K_3^p]\mathbb{E}|x(\delta_t)-x(t)|^p.
\end{aligned}\tag{3.74}$$

Moreover, by Assumptions 3.1.1, 3.1.2 and the elementary inequality $|a+b|^p\le 2^{p-1}(|a|^p+|b|^p)$ for $\forall a,b\in\mathbb{R}$, we have

$$\begin{aligned}
&|f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)|^p\\
\le&2^{p-1}\Big[K_1^p|x(t)|^p+K_3^p|x(\delta_t)|^p\Big]\\
\le&2^{p-1}(K_1^p+2^{p-1}K_3^p)|x(t)|^p+4^{p-1}K_3^p|x(t)-x(\delta_t)|^p.
\end{aligned}\tag{3.75}$$

Substitute (A.4) and (A.9) into the calculation of $LU(x_t,r_t,t)$ which is similar to (3.18). Taking the mean and by (A.8), we have

$$\begin{aligned}
&\mathbb{E}LU(x_t,r_t,t)\\
\le&\mathbb{E}\Big[\mathcal{L}V(x(t),r(t),t)+l|V_x(x(t),r(t),t)|^{\frac{p}{p-1}}\Big]\\
&+\Big[\theta\tau^{\frac{p}{2}}(2^{p-1}\tau^{\frac{p}{2}}K_1^p+\rho K_2^p+4^{p-1}\tau^{\frac{p}{2}}K_3^p)+\frac{2^{3p-2}}{p}(\frac{p-1}{pl})^{p-1}K_3^p(1-e^{-\overline{\gamma}\tau})\Big]\mathbb{E}|x(t)|^p\\
&+\Big[4^{p-1}\theta\tau^p+\frac{1}{p}(\frac{p-1}{pl})^{p-1}[2^{3p-2}(1-e^{-\overline{\gamma}\tau})+2^{p-1}]\Big]K_3^p\mathbb{E}|x(t)-x(\delta_t)|^p-\mathbb{E}\Phi(x_t,r_t,t).
\end{aligned}\tag{3.76}$$

Then Assumption 3.1.3 implies that

$$\begin{aligned}\mathbb{E}LU(x_t,r_t,t) &\le -\beta\mathbb{E}|x(t)|^p - \mathbb{E}\Phi(x_t,r_t,t)\\ &\quad + \Big[4^{p-1}\theta\tau^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}[2^{3p-2}(1-e^{-\overline{\gamma}\tau}) + 2^{p-1}]\Big]K_3^p\mathbb{E}|x(t)-x(\delta_t)|^p,\end{aligned} \tag{3.77}$$

where

$$\beta = \beta(\theta,\tau) := \lambda - \theta\tau^{\frac{p}{2}}[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p] - \frac{2^{3p-2}}{p}(\frac{p-1}{pl})^{p-1}K_3^p(1-e^{-\overline{\gamma}\tau}). \tag{3.78}$$

If we choose

$$8^{p-1}\tau^p K_3^p < 1 \quad \text{and} \quad \theta = \frac{[4(p-1)]^{p-1}}{p^p(1-8^{p-1}\tau^p K_3^p)}l^{1-p}[2^{2p-1}(1-e^{-\overline{\gamma}\tau})+1]K_3^p. \tag{3.79}$$

then we can obtain

$$\mathbb{E}(LU(x_t,r_t,t)) \le -\beta\mathbb{E}|x(t)|^p. \tag{3.80}$$

□

Theorem 3.5.2 *Fix the moment order $p \ge 2$. Under the same assumptions of Theorem 3.5.1, the solution of the controlled system (3.66) satisfies*

$$\lim_{t\to\infty}\mathbb{E}|x(t)|^p = 0$$

for any initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$.

Theorem 3.5.3 *Fix the moment order $p > 1$. Let Assumptions 3.1.1, 3.1.2, 3.1.3 and 3.3.1 hold. Choose $\tau > 0$ sufficiently small for (3.68) to hold. Then the solution of the controlled system (3.66) satisfies (3.48) and (3.49) for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$.*

Theorems 3.5.2 and 3.5.3 can be proved in the same way as Theorems 3.2.2 and 3.3.2 respectively, the only difference is replacing $u(x(\delta_t), r(t), t)$ with $u(x(\delta_t), r(\delta_t), t)$.

Corollary 3.5.4 *Fix the moment order $p \geq 2$. Let Assumptions 3.1.1, 3.1.2 and 3.3.4 hold for $p \geq 2$. Set parameters the same way as stated in Corollary 3.3.5. Let $\tau > 0$ be sufficiently small for (3.68) to hold. Then the controlled system (3.66) is pth moment exponentially stable.*

Now we discretize the observation of system mode $r(t)$ for the two examples in Section 3.4. For Example 3.4.1, choose the Lyapunov functions and parameters the same as above, substitute $\overline{\gamma} = 4$ into (3.68). Then the controlled system

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), r(t), t)dB(t), \qquad (3.81)$$

can achieve the 3rd moment exponential stability provided that the observation interval $\tau \leq 7.39e-7$. It's dramatically decreased from 0.003. Similarly, for Example 3.4.2, discretizing the mode observation reduces the observation interval from 0.0088 to 0.00365 for mean square exponential stabilization. Obviously, the decrease of observation interval for Example 3.4.1 is more dramatic than that for Example 3.4.2. An important reason is because Example 3.4.1 has larger maximum jump rate of the Markov chain than Example 3.4.2, in which $\overline{\gamma} = 1$.

3.6 Conclusion

In this chapter we have discussed the pth moment stabilization of hybrid stochastic differential equations by feedback controls based on discrete-time observations for $p > 1$. We firstly use $u(x(\delta_t), r(t), t)$ which needs discrete-time state observations and continuous-time mode observations. Later we use a more practical controller of the form $u(x(\delta_t), r(\delta_t), t)$ which requires higher observation frequency than the first controller. Many stabilities were investigated, including pth moment H_∞ stability for $p > 1$, pth moment asymptotic stability for $p \geq 2$, pth moment exponential stability for $p > 1$ and almost sure exponential stability. This chapter has two main contributions:

- developing the criterion on asymptotic stabilization from mean square ($p = 2$) to pth moment for all $p \geq 2$, developing the criterion on H_∞ stabilization and exponential stabilization from mean square to pth moment for all $p > 1$;

- reducing the observation frequency and hence reducing the cost of control.

Chapter 4

Almost Sure Exponential Stabilization by Stochastic Feedback Control

As discussed in Section 1.3, stabilization by noise based on discrete-time observations was only discussed by [91] for autonomous ODEs, which is not enough, and it's necessary to discuss this problem for more complex systems. This chapter investigates how to stabilize a given unstable linear non-autonomous ODE by controller $\sigma(t)x(\delta_t)dB(t)$, and how to stabilize an unstable nonlinear hybrid SDE by controller $G(r(\delta_t))x(\delta_t)dB(t)$, in the sense the controlled stochastic system is pth moment ($0 < p < 1$) and almost surely exponentially stable. The results in this chapter have been published in "Stochastic Analysis and Applications" in 2018 as [109] [1].

This chapter is organised as follows. Section 4.1 discusses stabilization of linear scalar ODEs and ends with expansion of the established theory to some nonlinear multidimensional ODEs. Section 4.2 discusses stabilization of nonlinear hybrid SDEs and ends with a corollary for linear hybrid ODEs. The conclusion is given in Section 4.3.

Following explains some special notations in this chapter.

[1]Dong, R. 2018. Almost sure exponential stabilization by stochastic feedback control based on discrete-time observations. *Stochastic Analysis and Applications* pp.1-23.

For clarity, sometimes I will write exponential function as $e^{(\cdot)}$ as $\exp(\cdot)$.
To emphasize the role of the initial data, sometimes I denote the solution x by $x(t; x_0, t_0)$ in Section 4.1, and in Section 4.2, sometimes I denote the solution x by $x(t; x_0, r_0, t_0)$ and the Markov chain r by $r(t; r_0, t_0)$.
Let the initial value of the system state $x_0 \in L^p_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R})$. This means x_0 is a $\mathbb{R}$-valued $\mathcal{F}_{t_0}$ measurable random variable such that $\mathbb{E}|x_0|^p < \infty$.
Let the initial value of the Markov chain $r_0 \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$. This means r_0 is $\mathcal{F}_{t_0}$ measurable from state space $\mathbb{S}$.

4.1 Scalar linear ODE

4.1.1 Stabilization problem and main result

Given an unstable linear ODE

$$\dot{x}(t) = \alpha(t)x(t) \tag{4.1}$$

on $t \geq t_0(\geq 0)$ with $x_0 = x(t_0) \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R})$, [2] where $\alpha : \mathbb{R}_+ \to \mathbb{R}$ is bounded. We can stabilize it by noise based on the m-dimensional Brownian motion $B(t) = (B_1(t), \cdots, B_m(t))^T$. The controlled SDE has the form

$$dx(t) = \alpha(t)x(t)dt + \sum_{i=1}^{m} \sigma_i(t)x(\delta_t)dB_i(t), \tag{4.2}$$

where $\sigma_i : \mathbb{R}_+ \to \mathbb{R}$ for $1 \leq i \leq m$ are bounded and

$$\delta_t = t_0 + \left[\frac{t - t_0}{\tau}\right]\tau \quad \text{for } \tau > 0, \tag{4.3}$$

in which $\left[\frac{t-t_0}{\tau}\right]$ is the integer part of $\frac{t-t_0}{\tau}$.

Since $\alpha(t)$ and $\sigma_i(t)$ for $1 \leq i \leq m$ are all bounded, the controlled system (4.2) satisfies the global Lipschitz condition and linear growth condition (see Section 2.3). So it has a unique solution $x(t)$ such that $\mathbb{E}|x(t)|^p < \infty$ for all $t \geq t_0$ and $p > 0$ (see

[2] I require $\mathbb{E}|x_0|^2 < \infty$ because the term $\mathbb{E}|x_0|^2$ will be used in the proof. Note that $L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}) \subset L^p_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R})$ for $0 < p < 1$ as condition $\mathbb{E}|x_0|^2 < \infty$ is stronger than $\mathbb{E}|x_0|^p < \infty$.

e.g. [26]).

Let $\hat{\alpha}$ be the upper bound of α and $\overline{\sigma_i}$ be the bound of $|\sigma_i|$ (the lower bound of α is not used so we do not state it). That is,

$$\alpha(t) \le \hat{\alpha} \quad \text{and} \quad |\sigma_i(t)| \le \overline{\sigma_i} \quad \text{for } \forall t \ge t_0. \tag{4.4}$$

Obviously $\hat{\alpha} > 0$, because otherwise, the original system (4.1) would converge to 0 and hence is already stable.

Assumption 4.1.1 *There is a positive constant $p \in (0,1)$ such that*

$$\limsup_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} \Big(\alpha(s) - \frac{1-p}{2}\sum_{i=1}^{m} \sigma_i^2(s)\Big) ds < 0. \tag{4.5}$$

Remark 4.1.2 *The condition (4.5) means that the upper limit of the time average of $\alpha(t) - \frac{1-p}{2}\sum_{i=1}^{m}\sigma_i^2(t)$ is negative, which is weaker than*

$$\alpha(t) - \frac{1-p}{2}\sum_{i=1}^{m} \sigma_i^2(t) < 0 \quad \textit{for } \forall t \ge t_0,$$

because $\alpha(t) - \frac{1-p}{2}\sum_{i=1}^{m}\sigma_i^2(t)$ is allowed to be positive at some time points.

Remark 4.1.3 *Assumption 4.1.1 implies that there is a constant $z > 0$ such that*

$$\limsup_{t\to\infty} \frac{1}{t-t_0} \int_{t_0}^{t} \Big(\alpha(s) - \frac{1-p}{2}\sum_{i=1}^{m} \sigma_i^2(s)\Big) ds = -z. \tag{4.6}$$

That is, we can find a pair of $T > t_0$ and $\epsilon \in (0, z/2)$ such that for $\forall t > T$,

$$\frac{1}{t-t_0} \int_{t_0}^{t} \Big(\alpha(s) - \frac{1-p}{2}\sum_{i=1}^{m} \sigma_i^2(s)\Big) ds \le -z + \epsilon. \tag{4.7}$$

Theorem 4.1.4 *Let Assumption 4.1.1 hold. Choose a constant $\rho \in (0,1)$. Then for any $t_0 \ge 0$ and $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R})$, the solution of equation (4.2) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) < 0 \tag{4.8}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|)<0 \quad a.s. \tag{4.9}$$

provided $\tau\in(0,\bar{\tau})$*, where* $\bar{\tau}>0$ *is the unique root to*

$$\hat{H}(\tau)=1-\rho \tag{4.10}$$

where

$$\hat{H}(\tau)=2^pK(\tau)\Big[\exp\Big([\tau+\log(2^pM/\rho)/\chi](4\hat{\alpha}+3m\sum_{i=1}^{m}\overline{\sigma_i}^2)\Big)-1\Big]^{\frac{p}{2}},$$

in which

$$K(\tau)=\Big[\frac{2m\sum_{i=1}^{m}\overline{\sigma_i}^2[2\tau(\tau\hat{\alpha}^2+m\sum_{i=1}^{m}\overline{\sigma_i}^2)]}{2\hat{\alpha}+m\sum_{i=1}^{m}\overline{\sigma_i}^2}\Big]^{\frac{p}{2}}, \tag{4.11}$$

$$M=\max\Big\{\exp\Big(-p\int_{t_0}^{\hat{T}}[\alpha(s)-\frac{1-p}{2}\sum_{i=1}^{m}\sigma_i^2(s)]ds-\chi(\hat{T}-t_0)\Big),1\Big\}, \tag{4.12}$$

$$\chi=-\frac{p}{2}\limsup_{t\to\infty}\frac{1}{t-t_0}\int_{t_0}^{t}\Big(\alpha(s)-\frac{1-p}{2}\sum_{i=1}^{m}\sigma_i^2(s)\Big)ds>0, \tag{4.13}$$

$$\hat{T}=t_0+\Big[\frac{T-t_0}{\tau}\Big]\tau+\tau, \tag{4.14}$$

in which $[\cdot]$ *means its integer part, and* T *is as discussed in Remark 4.1.3.*

4.1.2 Proof

To prove the new theory, we introduce an auxiliary traditionally controlled system $y(t)$, which is the solution to the SDE

$$dy(t)=\alpha(t)y(t)dt+\sum_{i=1}^{m}\sigma_i(t)y(t)dB_i(t) \tag{4.15}$$

with the same inital data t_0 and $y_0=x_0$ as in the discrete-time controlled system (4.2).

Lemma 4.1.5 *Let Assumption 4.1.1 hold. Then for any $t_0 \geq 0$, $x_0 \in L^p_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R})$,*

$$\mathbb{E}|y(t)|^p \leq M\mathbb{E}|y(\hat{T})|^p e^{-\chi(t-\hat{T})} \quad \text{for } \forall t > \hat{T}, \tag{4.16}$$

where $M, \hat{T}$ and χ have been defined above.

Proof. The system will remain at 0 if $x_0 = 0$. So we only need to consider when $x_0 \neq 0$. Lemma 2.5.1 (or see [9] p120) indicates that for any $y_0 \neq 0$, $y(t) \neq 0$ for all $t \geq 0$ almost surely.

Apply the Itô formula to (4.15),

$$\log y(t) = \log y_0 + \int_{t_0}^{t} [\alpha(s) - \frac{1}{2}\sum_{i=1}^{m} \sigma_i^2(s)]ds + \sum_{i=1}^{m}\int_{t_0}^{t} \sigma_i(s)dB_i(s).$$

That is,

$$y(t) = y_0 \exp\Big(\int_{t_0}^{t} [\alpha(s) - \frac{1}{2}\sum_{i=1}^{m} \sigma_i^2(s)]ds + \sum_{i=1}^{m}\int_{t_0}^{t} \sigma_i(s)dB_i(s)\Big). \tag{4.17}$$

Then

$$\begin{aligned}\mathbb{E}|y(t)|^p &= \mathbb{E}\Big[|y_0|^p \exp\Big(p\int_{t_0}^{t} [\alpha(s) - \frac{1}{2}\sum_{i=1}^{m} \sigma_i^2(s)]ds + p\sum_{i=1}^{m}\int_{t_0}^{t} \sigma_i(s)dB_i(s)\Big)\Big] \\ &= \mathbb{E}(|y_0|^p) \exp\Big(p\int_{t_0}^{t} [\alpha(s) - \frac{1}{2}\sum_{i=1}^{m} \sigma_i^2(s)]ds\Big)\mathbb{E}(e^{\xi}), \end{aligned} \tag{4.18}$$

where $\xi = p\sum_{i=1}^{m}\int_{t_0}^{t} \sigma_i(s)dB_i(s)$ is independent of y_0 and it's a normal random variable with mean 0 and variance

$$Var(\xi) = p^2 \sum_{i=1}^{m}\int_{t_0}^{t} \sigma_i^2(s)ds.$$

By the moment-generating function of ξ, we have that

$$\mathbb{E}(e^{\xi}) = \exp\Big(\frac{p^2}{2}\sum_{i=1}^{m}\int_{t_0}^{t} \sigma_i^2(s)ds\Big). \tag{4.19}$$

Substituting into (4.18) gives

$$\mathbb{E}|y(t)|^p = \mathbb{E}|y_0|^p \exp\Big(p\int_{t_0}^t [\alpha(s) - \frac{1-p}{2}\sum_{i=1}^m \sigma_i^2(s)]ds\Big). \tag{4.20}$$

Note that $y(t)$ is a Markov process (see Theorem 2.3.3 or [9] Section 2.9). This means for any $t^* > t_0$ and $t > t^*$, $y(t)$ can be seen as the solution of (4.15) starting from $y(t^*)$ at $t = t^*$, i.e., $y(t; y_0, t_0) = y(t; y(t^*), t^*)$. So we can write

$$\mathbb{E}|y(t)|^p = \mathbb{E}|y(\hat{T})|^p \exp\Big(p\int_{\hat{T}}^t [\alpha(s) - \frac{1-p}{2}\sum_{i=1}^m \sigma_i^2(s)]ds\Big), \tag{4.21}$$

where $\hat{T}$ has been defined in (4.14).

As discussed in Remark 4.1.3, there is a pair of $T > t_0$ and $\epsilon \in (0, z/2)$ such that for $\forall t > T$,

$$\int_{t_0}^t \Big(\alpha(s) - \frac{1-p}{2}\sum_{i=1}^m \sigma_i^2(s)\Big)ds \le -\frac{1}{2}z(t-t_0). \tag{4.22}$$

Then we can easily derive that

$$\int_{\hat{T}}^t \Big(\alpha(s) - \frac{1-p}{2}\sum_{i=1}^m \sigma_i^2(s)\Big)ds \le -\int_{t_0}^{\hat{T}} \Big(\alpha(s) - \frac{1-p}{2}\sum_{i=1}^m \sigma_i^2(s)\Big)ds - \frac{z}{2}(t-\hat{T}) - \frac{z}{2}(\hat{T}-t_0). \tag{4.23}$$

Substituting this into (4.21) gives

$$\mathbb{E}|y(t)|^p \le \exp\Big(-p\int_{t_0}^{\hat{T}} [\alpha(s) - \frac{1-p}{2}\sum_{i=1}^m \sigma_i^2(s)]ds - \frac{p}{2}z(\hat{T}-t_0)\Big)\mathbb{E}|y(\hat{T})|^p e^{-\frac{p}{2}z(t-\hat{T})}$$

for $\forall t > \hat{T}$. Noticing that $\chi = \frac{p}{2}z$, we obtain assertion (4.16). □

Lemma 4.1.6 *For any $p \in (0,1)$, $t_0 \ge 0$ and $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R})$,*

$$\mathbb{E}|x(t)|^p \le \mathbb{E}|x_0|^p \exp[(p(\hat{\alpha} + 0.5\sum_{i=1}^m \overline{\sigma}_i^2)(t-t_0)]. \tag{4.24}$$

Proof. Apply the Itô formula to (4.2),

$$\mathbb{E}[x^2(t)] = \mathbb{E}x_0^2 + \mathbb{E}\int_{t_0}^{t}\Big(2x(s)\alpha(s)x(s) + [\sum_{i=1}^{m}\sigma_i^2(s)x(\delta_s)]^2\Big)ds$$

$$\leq \mathbb{E}x_0^2 + \mathbb{E}\int_{t_0}^{t}\Big(2\hat{\alpha}x^2(s) + \sum_{i=1}^{m}\overline{\sigma}_i^2 x^2(\delta_s)\Big)ds$$

$$\leq \mathbb{E}x_0^2 + (2\hat{\alpha} + \sum_{i=1}^{m}\overline{\sigma}_i^2)\int_{t_0}^{t}\sup_{t_0\leq z\leq s}\mathbb{E}[x^2(z)]ds$$

Since the last term is nondecreasing function of t, and by the Fubini theory,

$$\sup_{t_0\leq z\leq t}\mathbb{E}[x^2(z)] \leq \mathbb{E}x_0^2 + (2\hat{\alpha} + \sum_{i=1}^{m}\overline{\sigma}_i^2)\int_{t_0}^{t}\sup_{t_0\leq z\leq s}\mathbb{E}[x^2(z)]ds.$$

Then the Gronwall inequality implies

$$\sup_{t_0\leq z\leq t}\mathbb{E}[x^2(z)] \leq \mathbb{E}x_0^2 \ \exp[(2\hat{\alpha} + \sum_{i=1}^{m}\overline{\sigma}_i^2)(t-t_0)].$$

Hence

$$\mathbb{E}[x^2(t)] \leq \mathbb{E}x_0^2 \ \exp[(2\hat{\alpha} + \sum_{i=1}^{m}\overline{\sigma}_i^2)(t-t_0)].$$

Finally Hölder's inequality implies assertation (4.24). □

Lemma 4.1.7 *For any $t_0 \geq 0$, $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega,\mathbb{R})$ and $p \in (0,1)$, we have*

$$\mathbb{E}|y(t)-x(t)|^p \leq \mathbb{E}|x_0|^p K(\tau)\Big[\exp\Big((4\hat{\alpha} + 3m\sum_{i=1}^{m}\overline{\sigma_i}^2)(t-t_0)\Big) - 1\Big]^{\frac{p}{2}}, \tag{4.25}$$

where $K(\tau)$ has been defined in (4.11).

Proof. We have

$$d(x(t)-y(t)) = \alpha(t)[x(t)-y(t)]dt + \sum_{i=1}^{m}\sigma_i^2(t)[x(\delta_t)-y(t)]dB(t).$$

By the Itô formula, we can derive that

$$\mathbb{E}[x(t)-y(t)]^2 = \mathbb{E}\int_{t_0}^{t}\Big(2[x(s)-y(s)]\alpha(s)[x(s)-y(s)] + \Big|\sum_{i=1}^{m}\sigma_i^2(s)[x(\delta_s)-y(s)]\Big|^2\Big)ds.$$

Then by (4.4), we obtain that

$$\begin{aligned}
&\mathbb{E}[x(t)-y(t)]^2\\
\le& 2\hat{\alpha}\int_{t_0}^{t}\mathbb{E}[x(s)-y(s)]^2ds + m\sum_{i=1}^{m}\overline{\sigma_i}^2\int_{t_0}^{t}\mathbb{E}[x(\delta_s)-y(s)]^2ds\\
\le& 2(\hat{\alpha}+m\sum_{i=1}^{m}\overline{\sigma_i}^2)\int_{t_0}^{t}\mathbb{E}[x(s)-y(s)]^2ds + 2m\sum_{i=1}^{m}\overline{\sigma_i}^2\int_{t_0}^{t}\mathbb{E}[x(\delta_s)-x(s)]^2ds.
\end{aligned}$$

Then applying the Gronwall inequality gives

$$\mathbb{E}[x(t)-y(t)]^2 \le 2m\sum_{i=1}^{m}\overline{\sigma_i}^2\exp\Big(2(\hat{\alpha}+m\sum_{i=1}^{m}\overline{\sigma_i}^2)(t-t_0)\Big)\int_{t_0}^{t}\mathbb{E}[x(\delta_s)-x(s)]^2ds. \quad (4.26)$$

By the Itô formula,

$$\begin{aligned}
\mathbb{E}[x^2(t)] =& \mathbb{E}x_0^2 + \mathbb{E}\int_{t_0}^{t}2x^T(s)\alpha(s)x(s) + \Big|\sum_{i=1}^{m}\sigma_i^2(s)x(\delta_s)\Big|^2ds\\
\le& \mathbb{E}x_0^2 + 2\hat{\alpha}\int_{t_0}^{t}\mathbb{E}[x^2(s)]ds + m\sum_{i=1}^{m}\overline{\sigma_i}^2\int_{t_0}^{t}\mathbb{E}[x^2(\delta_s)]ds.
\end{aligned}$$

Then by applying the Gronwall inequality on the supremum, we can derive that

$$\begin{aligned}
&\sup_{t_0\le z\le t}\mathbb{E}[x^2(z)]\\
\le& \mathbb{E}x_0^2 + (2\hat{\alpha}+m\sum_{i=1}^{m}\overline{\sigma_i}^2)\int_{t_0}\sup_{t_0\le z\le s}\mathbb{E}[x^2(z)]ds\\
\le& \mathbb{E}x_0^2\ \exp\Big((2\hat{\alpha}+m\sum_{i=1}^{m}\overline{\sigma_i}^2)(t-t_0)\Big).
\end{aligned}$$

Therefore

$$\mathbb{E}[x^2(t)] \le \sup_{t_0\le z\le t}\mathbb{E}[x^2(z)] \le \mathbb{E}x_0^2\ \exp\Big((2\hat{\alpha}+m\sum_{i=1}^{m}\overline{\sigma_i}^2)(t-t_0)\Big). \quad (4.27)$$

Now we calculate the term $\mathbb{E}[x(s)-x(\delta_s)]^2$ in (4.26) by the Itô formula and the elementary inequality $|\sum_{i=1}^m a_i|^2 \le m\sum_{i=1}^m |a_i|^2$ for $\forall a_i \in \mathbb{R}$.

$$
\begin{aligned}
&\mathbb{E}[x(t)-x(\delta_t)]^2\\
\le&2\mathbb{E}\int_{\delta_t}^t \tau\alpha^2(s)x^2(s) + \Big|\sum_{i=1}^m \sigma_i^2(s)x(\delta_s)\Big|^2 ds\\
\le&2\tau\hat{\alpha}^2\int_{\delta_t}^t \mathbb{E}[x^2(s)]ds + 2m\sum_{i=1}^m \overline{\sigma_i}^2\int_{\delta_t}^t \mathbb{E}[x^2(\delta_s)]ds\\
\le&2\tau(\tau\hat{\alpha}^2 + m\sum_{i=1}^m \overline{\sigma_i}^2)\mathbb{E}x_0^2 \ \exp\Big((2\hat{\alpha} + m\sum_{i=1}^m \overline{\sigma_i}^2)(t-t_0)\Big).
\end{aligned}
$$

Substituing this into (4.26) gives

$$
\begin{aligned}
&\mathbb{E}[x(t)-y(t)]^2\\
\le&2m\sum_{i=1}^m \overline{\sigma_i}^2 \exp\Big(2(\hat{\alpha} + m\sum_{i=1}^m \overline{\sigma_i}^2)(t-t_0)\Big)[2\tau(\tau\hat{\alpha}^2 + m\sum_{i=1}^m \overline{\sigma_i}^2)]\\
&\times \mathbb{E}x_0^2\int_{t_0}^t \exp\Big((2\hat{\alpha} + m\sum_{i=1}^m \overline{\sigma_i}^2)(s-t_0)\Big)ds\\
\le&\frac{2m\sum_{i=1}^m \overline{\sigma_i}^2[2\tau(\tau\overline{\alpha}^2 + m\sum_{i=1}^m \overline{\sigma_i}^2)]}{2\hat{\alpha} + m\sum_{i=1}^m \overline{\sigma_i}^2}\mathbb{E}x_0^2\\
&\times\Big[\exp\Big((4\overline{\alpha} + 3m\sum_{i=1}^m \overline{\sigma_i}^2)(t-t_0)\Big) - \exp\Big(2(\hat{\alpha} + m\sum_{i=1}^m \overline{\sigma_i}^2)(t-t_0)\Big)\Big]\\
\le&\frac{2m\sum_{i=1}^m \overline{\sigma_i}^2[2\tau(\tau\hat{\alpha}^2 + m\sum_{i=1}^m \overline{\sigma_i}^2)]}{2\hat{\alpha} + m\sum_{i=1}^m \overline{\sigma_i}^2}\mathbb{E}x_0^2\Big[\exp\Big((4\hat{\alpha} + 3m\sum_{i=1}^m \overline{\sigma_i}^2)(t-t_0)\Big) - 1\Big].
\end{aligned}
\tag{4.28}
$$

Finally Hölder's inequality implies assertation (4.25). □

Remark 4.1.8 *We observe that $\hat{H}(\tau)$ is a continuously increasing function of τ for $\tau \ge 0$. $\hat{H} = 0$ when $\tau = 0$ (as $K(0) = 0$) and $\hat{H} \to \infty$ as $\tau \to \infty$. So the equation (4.10) must have a unique root $\bar{\tau} > 0$ for any $\rho \in (0,1)$. Moreover, for the fixed ρ, $\hat{H}(\tau) < 1-\rho$ for any $\tau \in (0,\bar{\tau})$.*

To simplify the notation, we write $t_k = t_0 + k\tau$, $y(t_k) = y_k$ and $x(t_k) = x_k$ for

$\forall k = 0, 1, 2, \cdots$.

Remark 4.1.9 *At time points t_k, there is no time delay, so $x(t)$ has the Markov property at these discrete time points:*

$$x(t; x_0, t_0) = x(t; x_k, t_k) \quad \textit{for } \forall t > t_k.$$

That is, for any $t > t_k$, $x(t)$ can be seen as the solution of (4.2) starting from x_k at initial time t_k.

Proof of Theorem 4.1.4: We divide the proof into three steps.
Step 1. Let's fix $\tau \in (0, \bar{\tau})$ and $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R})$ arbitrarily.
Since $2^p M/\rho > 1$, we can choose a positive integer ν such that

$$\frac{\log(2^p M/\rho)}{\chi\tau} \le \nu \le 1 + \frac{\log(2^p M/\rho)}{\chi\tau}. \tag{4.29}$$

The left part of it imples that

$$2^p M e^{-\chi\nu\tau} \le \rho. \tag{4.30}$$

Let $y(\hat{T} + \nu\tau) = y(\hat{T} + \nu\tau; x(\hat{T}), \hat{T})$ and recall Lemma 4.1.5, then we have

$$\mathbb{E}|y(\hat{T} + \nu\tau)|^p \le M\mathbb{E}|x(\hat{T})|^p e^{-\chi\nu\tau}. \tag{4.31}$$

By the elementary inequality $|a+b|^p \le 2^p(|a|^p + |b|^p)$ for any $a, b \in \mathbb{R}$ and $p \in (0, 1)$, we have

$$\mathbb{E}|x(\hat{T} + \nu\tau)|^p \le 2^p\mathbb{E}|y(\hat{T} + \nu\tau)|^p + 2^p\mathbb{E}|y(\hat{T} + \nu\tau) - x(\hat{T} + \nu\tau)|^p. \tag{4.32}$$

It follows from (4.31) and (4.30) that

$$2^p\mathbb{E}|y(\hat{T} + \nu\tau)|^p \le \rho\mathbb{E}|x(\hat{T})|^p. \tag{4.33}$$

Let $x(\hat{T}+\nu\tau) = x(\hat{T}+\nu\tau; x(\hat{T}), \hat{T})$ and recall Lemma 4.1.7, then we have

$$\mathbb{E}|y(\hat{T}+\nu\tau) - x(\hat{T}+\nu\tau)|^p \leq \mathbb{E}|x(\hat{T})|^p K(\tau)\Big[\exp\Big(\nu\tau(4\hat{\alpha} + 3m\sum_{i=1}^{m}\overline{\sigma_i}^2)\Big) - 1\Big]^{\frac{p}{2}}. \quad (4.34)$$

Substituting (4.33) and (4.34) into (4.32) gives

$$\mathbb{E}|x(\hat{T}+\nu\tau)|^p \leq \mathbb{E}|x(\hat{T})|^p\Big(\rho + 2^p K(\tau)\Big[\exp\Big(\nu\tau(4\hat{\alpha} + 3m\sum_{i=1}^{m}\overline{\sigma_i}^2)\Big) - 1\Big]^{\frac{p}{2}}\Big). \quad (4.35)$$

The second inequality of (4.29) implies $\nu\tau \leq \tau + \log(2^p M/\rho)/\chi$. It follows from $\hat{H}(\tau) < 1-\rho$ that

$$\begin{aligned} &\rho + 2^p K(\tau)\Big[\exp\Big(\nu\tau(4\hat{\alpha} + 3m\sum_{i=1}^{m}\overline{\sigma_i}^2)\Big) - 1\Big]^{\frac{p}{2}} \\ \leq&\rho + \hat{H}(\tau) \\ <&1. \end{aligned}$$

We may therefore write

$$\rho + \hat{H}(\tau) = e^{-\lambda\nu\tau}$$

for some $\lambda > 0$. It then follows from (4.35) that

$$\mathbb{E}|x(\hat{T}+\nu\tau)|^p \leq \mathbb{E}|x(\hat{T})|^p e^{-\lambda\nu\tau}. \quad (4.36)$$

Step 2. Since $x(t)$ has Markov property at times $k\tau$ for $k = 0, 1, 2\cdots$, we can repeat (4.31)-(4.36) by letting $y(\hat{T}+(i+1)\nu\tau) = y(\hat{T}+(i+1)\nu\tau; x(\hat{T}+i\nu\tau), \hat{T}+i\nu\tau)$ and $x(\hat{T}+(i+1)\nu\tau) = x(\hat{T}+(i+1)\nu\tau; x(\hat{T}+i\nu\tau), \hat{T}+i\nu\tau)$. Finally we could obtain that

$$\mathbb{E}|x(\hat{T}+i\nu\tau)|^p \leq \mathbb{E}|x(\hat{T})|^p e^{-i\lambda\nu\tau} \quad \text{for } i = 0, 1, 2\cdots.$$

Then by Lemma 4.1.6 and the Markov property,

$$\mathbb{E}|x(\hat{T}+i\nu\tau)|^p \leq \mathbb{E}|x_0|^p C_1 e^{-i\lambda\nu\tau}$$

where $C_1 = \exp\Big((p(\hat{\alpha} + 0.5\sum_{i=1}^m \overline{\sigma}_i^2)([T/\tau]+1)\tau\Big)$, for $i = 0, 1, 2\cdots$. Since for any $t \geq \hat{T}$, there is a unique $i \geq 0$ such that $\hat{T} + i\nu\tau \leq t < \hat{T} + (i+1)\nu\tau$, we can derive that

$$\begin{aligned}\mathbb{E}|x(t)|^p &\leq \mathbb{E}|x_0|^p C_1 \exp\Big[p(\hat{\alpha} + 0.5\sum_{i=1}^m \overline{\sigma}_i^2)(t - \hat{T} - i\nu\tau)\Big] e^{-i\lambda\nu\tau} \\ &\leq \mathbb{E}|x_0|^p C_1 \exp\Big[p\nu\tau(\hat{\alpha} + 0.5\sum_{i=1}^m \overline{\sigma}_i^2)\Big] e^{-i\lambda\nu\tau} \\ &\leq C_2 \mathbb{E}|x_0|^p e^{-\lambda t}, \end{aligned} \tag{4.37}$$

where

$$C_2 = C_1 \exp[p\nu\tau(\hat{\alpha} + 0.5\sum_{i=1}^m \overline{\sigma}_i^2)] e^{\lambda\hat{T}} = \exp\Big(p([T/\tau] + 1 + \nu)\tau(\hat{\alpha} + 0.5\sum_{i=1}^m \overline{\sigma}_i^2) + \lambda\hat{T}\Big).$$

So far we have proved (4.8). Next we will prove the almost sure exponential stability of (4.2).

Step 3.

By the definition of solutions of SDEs [3], the inequality in (2.17), Hölder's inequality and the Burkholder-Davis-Gundy inequality, we have that

$$\mathbb{E}\Big(\sup_{t_0 \leq s \leq t} |x(s)|^2\Big) \leq (m+2)\Big[\mathbb{E}x_0^2 + [\hat{\alpha}^2(t - t_0) + 4\sum_{i=1}^m \overline{\sigma}_i^2] \int_{t_0}^t \mathbb{E}\Big(\sup_{t_0 \leq z \leq s} |x(z)|^2\Big) ds\Big].$$

Using Gronwall's inequality and Hölder's inequality, we obtain

$$\mathbb{E}\Big(\sup_{t_0 \leq s \leq t} |x(s)|^p\Big) \leq (m+2)^{\frac{p}{2}} \mathbb{E}|x_0|^p \exp\Big(\frac{p}{2}(m+2)(t - t_0)[\hat{\alpha}^2(t - t_0) + 4\sum_{i=1}^m \overline{\sigma}_i^2]\Big).$$

Let $\hat{t}_k = \hat{T} + k\nu\tau$ for $k = 0, 1, 2, \cdots$. Then by Remark 4.1.9 and (4.37), we have

$$\begin{aligned}\mathbb{E}\Big(\sup_{\hat{t}_k \leq t \leq \hat{t}_{k+1}} |x(t)|^p\Big) &\leq (m+2)^{\frac{p}{2}} \exp\Big(\frac{p}{2}(m+2)\nu\tau[\hat{\alpha}^2\nu\tau + 4\sum_{i=1}^m \overline{\sigma}_i^2]\Big)\mathbb{E}|x(\hat{t}_k)|^p \\ &\leq C_3 \mathbb{E}|x_0|^p e^{-k\lambda\nu\tau},\end{aligned}$$

[3] It was stated in Definition 2.3.1, particularly we use (iii) here.

where $C_3 = (m+2)^{\frac{p}{2}} C_2 \exp\Big(-\lambda\hat{T} + \frac{p}{2}(m+2)\nu\tau[\hat{\alpha}^2\nu\tau + 4\sum_{i=1}^m \overline{\sigma}_i^2]\Big)$.

Then the Chebyshev inequality implies that for all $k = 0, 1, 2, \cdots$,

$$\mathbb{P}\Big(\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t)|^p \ge e^{-0.5k\lambda\nu\tau}\Big) \le C_3\mathbb{E}|x_0|^p e^{-0.5k\lambda\nu\tau}.$$

Define a sequence of events

$$\{A_k\}_{k\ge 0} := \{\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t)|^p \ge e^{-0.5k\lambda\nu\tau}\}.$$

Then

$$\begin{aligned}\sum_{k=0}^{\infty}\mathbb{P}(A_k) &= \sum_{k=0}^{\infty}\mathbb{P}\Big(\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t)|^p \ge e^{-0.5k\lambda\nu\tau}\Big)\\ &\le C_3\mathbb{E}|x_0|^p \sum_{k=0}^{\infty} e^{-0.5k\lambda\nu\tau} < \infty.\end{aligned}$$

By the Borel-Cantelli lemma, $\mathbb{P}\Big(\limsup_{k\to\infty} A_k\Big) = 0$.
Since $\Big(\limsup_{k\to\infty} A_k\Big)^c = \Big(\liminf_{k\to\infty} A_k^c\Big)$, we get

$$\mathbb{P}\Big(\liminf_{k\to\infty} A_k^c\Big) = \mathbb{P}\Big(\liminf_{k\to\infty}\Big(\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t)|^p < e^{-0.5k\lambda\nu\tau}\Big)\Big) = 1.$$

By the definition of limit inferior,

$$\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t)|^p < e^{-0.5k\lambda\nu\tau}$$

holds for all except finitely many k.
That is, for almost all $\omega \in \Omega$, there is an integer $k_0 = k_0(\omega)$ such that

$$\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t,\omega)|^p < e^{-0.5k\lambda\nu\tau}, \quad \forall k \ge k_0(\omega).$$

Therefore, for $\hat{t}_k \le t \le \hat{t}_{k+1}$ and $k \ge k_0$,

$$\frac{1}{t}\log(|x(t,\omega)|) \le \frac{1}{t}\log\Big(\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t,\omega)|\Big) < -\frac{k\lambda\nu\tau}{2p(k+1)\nu\tau}.$$

As $t \to \infty$, $k \to \infty$ and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t,\omega)|) \le -\frac{\lambda}{2p}$$

for almost all $\omega \in \Omega$.

Hence we complete the proof of assertion (4.9). □

Now let us discuss how to calculate the observation interval τ. Since our aim is to find $\tau \in (0,\bar{\tau})$, namely, to find τ such that $\hat{H}(\tau) < 1-\rho$, we can require that a number larger than $\hat{H}(\tau)$ is still less than $1-\rho$. This can be done by replacing M with a larger number $\tilde{M}$ defined below. So we can find a positive observation interval by the following four steps.

Firstly, determine the values of parameters $\hat{\alpha}$, $\overline{\sigma_i}$, m by the original ODE system and controller. Notice that system coefficient $\sigma_i(t)'s$ and moment order p are determined through Assumption 4.1.1. Then $K(\tau)$ defined in (4.11) is determined.

Secondly, calculate z defined in (4.6), then calculate T and χ through Remark 4.1.3 and (4.13) respectively.

Thirdly, set an upper bound for τ. Here we only need the upper bound τ^* to be an arbitrary rough guess, say, $\tau^* = 0.1$. Noticing that $T < \hat{T} \le T + \tau^*$, we can define

$$\tilde{M} := \max\Big\{\exp\Big(\theta - \chi(T - t_0)\Big), 1\Big\},$$

where

$$\theta = \max_{T \le t \le T+\tau^*}\Big(-p\int_{t_0}^{t}[\alpha(s) - \frac{1-p}{2}\sum_{i=1}^{m}\sigma_i^2(s)]ds\Big).$$

Fourthly, choose $\rho \in (0,1)$ and then find solution $\tau > 0$ for the equation

$$2^p K(\tau)\Big[\exp\Big([\tau + \log(2^p\tilde{M}/\rho)/\chi](4\hat{\alpha} + 3m\sum_{i=1}^{m}\overline{\sigma_i}^2)\Big) - 1\Big]^{\frac{p}{2}} = 1-\rho.$$

The reason why there is a positive solution is similar to the discussion in Remark 4.1.8. If the finally calculated τ is larger than its upper bound τ^*, then either let $\tau = \tau^*$ or try a larger τ^* and calculate again.

4.1.3 Corollary

Theorem 4.1.4 can also be applied to nonlinear system. Given an unstable n-dimensional nonlinear ODE

$$\dot{x}(t) = f(x(t), t) \tag{4.38}$$

on $t \geq t_0 (\geq 0)$ with $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$, we can stabilize it by stochastic feedback control $\hat{G}(t)x(t)dB(t)$ based on a scalar Brownian motion. The controlled SDE has the form

$$dx(t) = f(x(t), t)dt + \hat{G}(t)x(\delta_t)dB(t), \tag{4.39}$$

where $\hat{G} : \mathbb{R}_+ \to \mathbb{R}^{n\times n}$ is a bounded and

$$\delta_t = t_0 + \left[\frac{t - t_0}{\tau}\right]\tau \quad \text{for } \tau > 0. \tag{4.40}$$

Assumption 4.1.10 *Assume that there is a positive constant α such that*

$$|f(x,t) - f(y,t)| \leq \alpha|x - y| \quad \text{and} \quad |f(0,t)| = 0$$

for all $(x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$.

In other words, f satisfies the global Lipschitz condition and the linear growth condition $|f(x,t)| \leq \alpha|x|$. Let $\sigma \geq \|\hat{G}(t)\|$ for $\forall t \geq t_0$.

The auxiliary traditionally controlled system with the same initial data as (4.39) has the form

$$dy(t) = f(y(t), t)dt + \hat{G}(t)y(t)dB(t). \tag{4.41}$$

Assumption 4.1.11 *Assume there are positive constants χ and $p \in (0, 1)$ such that* *the solution of (4.41) satisfies*

$$\mathbb{E}|y(t)|^p \le \mathbb{E}|x_0|^p e^{-\chi(t-t_0)} \tag{4.42}$$

for any $t > t_0 \ge 0$ *and* $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$.

Corollary 4.1.12 *Let Assumptions 4.1.10 and 4.1.11 hold. Choose a constant* $\rho \in (0,1)$. *Then for any* $t_0 \ge 0$ *and* $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$, *the solution of equation (4.39) is pth moment exponentially stable and almost sure exponentially stable in the sense that (4.8) and (4.9) hold, as long as* $\tau \in (0, \bar{\tau})$, *where* $\bar{\tau} > 0$ *is the unique root to*

$$\hat{H}(\tau) = 1 - \rho \tag{4.43}$$

where

$$\hat{H}(\tau) = 2^p \Big[\frac{2\sigma^2[2\tau(\tau\alpha^2 + \sigma^2)]}{2\alpha + \sigma^2}\Big]^{\frac{p}{2}} \Big[\exp\Big([\tau + \log(2^p/\rho)/\chi](4\alpha + 3\sigma^2)\Big) - 1\Big]^{\frac{p}{2}}.$$

This corollary can be proved in a similar and simpler way as Theorem 4.1.4. Specifically, Lemma 4.1.5 is now guaranteed by Assumption 4.1.11;
respectively replace the parameters

$$\hat{\alpha}, \quad m, \quad \sum_{i=1}^m \overline{\sigma}_i^2, \quad \hat{T} \text{ and } M,$$

which are defined in Sections 4.1.1 and 4.1.2, by

$$\alpha, \quad 1, \quad \sigma^2, \quad t_0 \text{ and } 1,$$

where α and σ^2 are defined above in this section;
assertions (4.24) and (4.25) in Lemmas 4.1.6 and 4.1.7 now change to be

$$\mathbb{E}|x(t)|^p \le \mathbb{E}|x_0|^p \exp[p(\alpha + 0.5\sigma^2)(t - t_0)] \tag{4.44}$$

and

$$\mathbb{E}|y(t)-x(t)|^p \le \mathbb{E}|x_0|^p K(\tau)\Big[\exp\Big((4\alpha+3\sigma^2)(t-t_0)\Big)-1\Big]^{\frac{p}{2}} \tag{4.45}$$

respectively for any $p \in (0,1)$, $t_0 \ge 0$ and $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$.

4.2 Multidimensional SDE with Markovian switching

4.2.1 Stabilization problem and main result

Consider a nonlinear n-dimensional unstable hybrid SDE in the Itô sense

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{4.46}$$

on $t \ge t_0(\ge 0)$ with $x_0 = x(t_0) \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ and $r_0 = r(t_0) \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$. Its corresponding stochastically controlled SDE is

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) + u(x(\delta_t), r(\delta_t), t)d\tilde{B}(t). \tag{4.47}$$

Here

$$f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n, \qquad g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m} \quad \text{and} \quad u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times \tilde{m}};$$

$$\delta_t = t_0 + \Big[\frac{t-t_0}{\tau}\Big]\tau \quad \text{for } \tau > 0 \tag{4.48}$$

where $\Big[\frac{t-t_0}{\tau}\Big]$ is the integer part of $\frac{t-t_0}{\tau}$;
$B(t)$, $\tilde{B}(t)$ are independent m, $\tilde{m}$-dimensional Brownian motions.

Define $G \in \mathbb{R}^{m+\tilde{m}}$ by combining $g \in \mathbb{R}^m$ and $u \in \mathbb{R}^{\tilde{m}}$.

Let $G = (g,\ u) : \mathbb{R}^n_0 \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times(m+\tilde{m})}$.

That is, $G(x(t), x(\delta_t), r(t), r(\delta_t), t) = (g(x(t), r(t), t),\ u(x(\delta_t), r(\delta_t), t))$.
Define Brownian motion $W(t)$ by combining $B(t)$ and $\tilde{B}(t)$. That is,

$$W(t) = (B_1(t), \cdots, B_m(t), \tilde{B}_1(t), \cdots, \tilde{B}_{\tilde{m}}(t))^T.$$

Then $|G|^2 = |g|^2 + |u|^2$, $|x^T G|^2 = |x^T g|^2 + |x^T u|^2$ and $W(t)$ is an $(m+\tilde{m})$-dimensional Brownian motion. So the equation (4.47) can be written as

$$dx(t) = f(x(t), r(t), t)dt + G(x(t), x(\delta_t), r(t), r(\delta_t), t)dW(t). \tag{4.49}$$

Assumption 4.2.1 *Assume that the coefficients f, g and controller u are all globally Lipschitz continuous. That is, there exist positive constants $\alpha_i, \beta_i, \kappa_i$ for $i \in \mathbb{S}$ such that*

$$|f(x,i,t) - f(y,i,t)| \le \alpha_i |x-y| \tag{4.50}$$

$$|g(x,i,t) - g(y,i,t)| \le \beta_i |x-y| \tag{4.51}$$

$$|u(x,i,t) - u(y,i,t)| \le \kappa_i |x-y| \tag{4.52}$$

for all $(x,y,i,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$. We also assume that

$$|f(0,i,t)| = 0, \quad |g(0,i,t)| = 0 \quad \textit{and} \quad |u(0,i,t)| = 0. \tag{4.53}$$

for all $(i,t) \in \mathbb{S} \times \mathbb{R}_+$.

Note that Assumption 4.2.1 implies the following linear growth condition

$$|f(x,i,t)| \le \alpha_i |x| \tag{4.54}$$

$$|g(x,i,t)| \le \beta_i |x| \tag{4.55}$$

$$|u(x,i,t)| \le \kappa_i |x|. \tag{4.56}$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.
Define three positive constants:

$$\overline{\alpha} = \max_{i \in \mathbb{S}} \alpha_i, \quad \overline{\beta} = \max_{i \in \mathbb{S}} \beta_i, \quad \overline{\kappa} = \max_{i \in \mathbb{S}} \kappa_i.$$

Note that the controlled system (4.47) is in fact a stochastic differential delay equation (SDDE) with a bounded variable delay. If we define the bounded variable delay

$\zeta : [0,\infty) \to [0,\tau]$ by

$$\zeta(t) = t - k\tau \quad \text{for } k\tau \le t < (k+1)\tau,\ k = 0, 1, 2, \cdots,$$

then (4.47) can be written as

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) + u(x(t-\zeta(t)), r(t-\zeta(t)), t)d\tilde{B}(t). \quad (4.57)$$

Again, since the Lipschitz condition and linear growth condition are satisfied, (4.47) has a unique solution $x(t)$ such that $\mathbb{E}|x(t)|^p < \infty$ for all $t \ge t_0$ and $p > 0$ (see e.g. [26]).

Assumption 4.2.2 *There are positive constants ρ_i ($i \in \mathbb{S}$) such that*

$$|x^T g(x,i,t)|^2 + |x^T u(x,i,t)|^2 \ge \rho_i |x|^4 \quad (4.58)$$

for all $(x,i,t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

For each $p \in (0,1)$, define an $N \times N$ matrix

$$\mathcal{A}(p) := \text{diag}(\theta_1(p), \cdots, \theta_N(p)) - \Gamma, \quad (4.59)$$

where

$$\theta_i(p) = \frac{p}{2}\Big[(2-p)\rho_i - \beta_i^2 - \kappa_i^2\Big] - p\alpha_i$$

for $1 \le i \le N$ and $\Gamma = (\gamma_{ij})_{N\times N}$ is the generator matrix of the Markov chain $r(t)$.

Assumption 4.2.3 *There is a constant $p \in (0,1)$ such that $\mathcal{A}(p)$ is a nonsingular M-matrix.*

Remark 4.2.4 *According to Theorem 2.5.8 (or see Theorem 2.10(9) on page 68 of [26]), Assumption 4.2.3 implies that, there exists a vector $\varphi = (\varphi_1, \cdots, \varphi_N)^T \in \mathbb{R}_+^N$ such that*

$$\mathcal{A}(p)\varphi \in \mathbb{R}_+^N.$$

That is, if we denote $\mathcal{A}(p)\varphi$ by $\bar{\varphi}$, i.e.,

$$\mathcal{A}(p)\varphi = \bar{\varphi} = (\bar{\varphi}_1, \cdots, \bar{\varphi}_N)^T,$$

then

$$\bar{\varphi}_i = \theta_i(p)\varphi_i - \sum_{j=1}^{N} \gamma_{ij}\varphi_j > 0 \quad \textit{for } 1 \le i \le N.$$

Theorem 4.2.5 *Let Assumptions 4.2.1, 4.2.2 and 4.2.3 hold. Choose $\varepsilon \in (0,1)$. Then for any initial values $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ and $r_0 \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$, the solution of (4.47) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t)|^p) < 0 \tag{4.60}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) < 0 \qquad a.s. \tag{4.61}$$

provided $\tau \in (0, \bar{\tau})$, where $\bar{\tau} > 0$ is the unique root to

$$\hat{H}(\tau) = 1 - \varepsilon \tag{4.62}$$

where

$$\begin{aligned} \hat{H}(\tau) =& 2^p H^{\frac{p}{2}}(\tau)\Big[\exp\Big((2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)[\tau + \log(2^p K/\varepsilon)/\chi]\Big) - 1\Big]^{\frac{p}{2}} \\ & \times \exp\Big[p(\overline{\alpha} + 0.5\overline{\beta}^2 + 1.5\overline{\kappa}^2)[\tau + \log(2^p K/\varepsilon)/\chi]\Big], \end{aligned} \tag{4.63}$$

in which

$$H(\tau) = \frac{24\overline{\kappa}^2\tau(\tau\overline{\alpha}^2 + \overline{\beta}^2 + \overline{\kappa}^2)\exp[4\overline{\alpha}^2\tau^2 + 16\tau(\overline{\beta}^2 + \overline{\kappa}^2)] + 12\overline{\kappa}^2(1 - e^{-\overline{\gamma}\tau})}{2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2}. \tag{4.64}$$

and

$$K = \frac{\max_{i\in\mathbb{S}}\varphi_i}{\min_{i\in\mathbb{S}}\varphi_i} \ge 1, \quad \chi = \frac{\min_{1\le i\le N}\bar{\varphi}_i}{\max_{1\le i\le N}\varphi_i} > 0, \quad \overline{\gamma} = \max_{i\in\mathbb{S}}(-\gamma_{ii}). \tag{4.65}$$

For simplicity of the notations, let us write $t_k = t_0 + k\tau$, $y(t_k) = y_k$, $x(t_k) = x_k$,

$r(t_k) = r_k$ for $\forall k = 0, 1, 2, \cdots$.

Remark 4.2.6 *Under Assumption 4.2.1, the pair $(x(t), r(t))$ is a Markov process and has Markov property at discrete time points t_k (see [26] Theorem 3.27 on page 104):*

$$\Big(x(t; x_0, r_0, t_0),\ \ r(t; r_0, t_0)\Big) = \Big(x(t; x_k, r_k, t_k),\ \ r(t; r_k, t_k)\Big) \quad \textit{for } \forall t > t_k.$$

That is, for any $t > t_k$, $x(t)$ can be seen as the solution of (4.47) starting from t_k with initial data x_k and r_k.

4.2.2 Proof

To prove the new theory, we introduce an auxiliary traditionally controlled system $y(t)$, which is the solution to the SDE

$$dy(t) = f(y(t), r(t), t)dt + g(y(t), r(t), t)dB(t) + u(y(t), r(t), t)d\tilde{B}(t) \tag{4.66}$$

with the same inital data t_0, $y_0 = x_0$ and r_0 as in the discrete-time controlled system (4.47). Recall $G = (g,\ u)$, then $G(y(t), r(t), t) = (g(y(t), r(t), t),\ u(y(t), r(t), t))$. So (4.66) can be written as

$$dy(t) = f(y(t), r(t), t)dt + G(y(t), r(t), t)dW(t). \tag{4.67}$$

Lemma 4.2.7 *Let Assumptions 4.2.1, 4.2.2 and 4.2.3 hold. Then for any $t_0 \geq 0$, $x_0 \in L^p_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ and $r_0 \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$, the trivial solution of (4.66) satisfies*

$$\mathbb{E}|y(t)|^p \leq K\mathbb{E}|y_0|^p e^{-\chi(t-t_0)}, \quad \forall t \geq t_0 \tag{4.68}$$

where K and χ have been defined in (4.65).

Proof. By Assumption 4.2.1, the system (4.66) will remain at 0 if $x_0 = 0$. So (4.68) holds. Otherwise, we can combine Theorems 2.5.9 and 2.5.10.

Let $V(y, i, t) = \varphi_i |y|^p$ from $\mathbb{R}^n_0 \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}_+$ and define an operator $\mathcal{L}V : \mathbb{R}^n \times \mathbb{S} \times$

$\mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(y,i,t) =& V_t(y,i,t) + V_y(y,i,t)f(y,i,t) + \frac{1}{2}\text{trace}[G^T(y,i,t)V_{yy}(y,i,t)G(y,i,t)] \\ &+ \sum_{k=1}^{N}\gamma_{ik}V(y,k,t).\end{aligned}$$

Then by Assumptions 4.2.1, 4.2.2, 4.2.3 and Remark 4.2.4, we have that for any $t \geq t_0$,

$$\begin{aligned}\mathcal{L}V(y,i,t) &= p\varphi_i|y|^{p-2}y^Tf(y,i,t) + \frac{1}{2}p\varphi_i|y|^{p-2}|G(y,i,t)|^2 \\ &\quad - \frac{p(2-p)}{2}\varphi_i|y|^{p-4}|y^TG(y,i,t)|^2 + \sum_{k=1}^{N}\gamma_{ik}\varphi_k|y|^p \\ &\leq -\bar{\varphi}_i|y|^p \leq -\min_{1\leq i\leq N}\bar{\varphi}_i|y|^p. \end{aligned} \tag{4.69}$$

For each integer $k \geq 1$, define a stopping time $\varrho_k = \inf\{t \geq t_0 : |y(t)| \geq k\}$.
Since

$$\min_{1\leq i\leq N}\varphi_i|y|^p \leq V(y,i,t) \leq \max_{1\leq i\leq N}\varphi_i|y|^p$$

for all $(y,i,t) \in \mathbb{R}_0^n \times \mathbb{S} \times \mathbb{R}_+$ and $t_0 \leq t \leq \varrho_k$ indicates $0 < |y(t)| \leq k$. We can apply the generalized Itô formula to derive that

$$\begin{aligned}&\mathbb{E}\Big[e^{\chi(t\wedge\varrho_k)}V(y(t\wedge\varrho_k), r(t\wedge\varrho_k), t\wedge\varrho_k)\Big] \\ =& e^{\chi t_0}\mathbb{E}V(y_0,r_0,t_0) + \mathbb{E}\int_{t_0}^{t\wedge\varrho_k} e^{\chi s}\Big[\chi V(y(s),r(s),s) + \mathcal{L}V(y(s),r(s),s)\Big]ds \\ \leq& e^{\chi t_0}\max_{1\leq i\leq N}\varphi_i\mathbb{E}|y_0|^p + \mathbb{E}\int_{t_0}^{t\wedge\varrho_k} e^{\chi s}\Big[\chi\max_{1\leq i\leq N}\varphi_i|y(s)|^p - \min_{1\leq i\leq N}\bar{\varphi}_i|y(s)|^p\Big]ds \\ \leq& e^{\chi t_0}\max_{1\leq i\leq N}\varphi_i\mathbb{E}|y_0|^p\end{aligned}$$

by (4.69). Futhermore,

$$\begin{aligned}\min_{1\leq i\leq N}\varphi_i\mathbb{E}\Big[e^{\chi(t\wedge\varrho_k)}|y(t\wedge\varrho_k)|^p\Big] \leq& \mathbb{E}\Big[e^{\chi(t\wedge\varrho_k)}V(y(t\wedge\varrho_k), r(t\wedge\varrho_k), t\wedge\varrho_k)\Big] \\ \leq& e^{\chi t_0}\max_{1\leq i\leq N}\varphi_i\mathbb{E}|y_0|^p.\end{aligned} \tag{4.70}$$

Define

$$Y_k := e^{\chi(t\wedge\varrho_k)}|y(t\wedge\varrho_k)|^p \quad \text{for } k \geq 1.$$

By the definition of ϱ_k, Y_k is increasing.

Then the Monotonic convergence theorem indicates that

$$\lim_{k\to\infty} \mathbb{E}Y_k = \mathbb{E}\lim_{k\to\infty} Y_k.$$

As $k \to \infty$, $\varrho_k \to \infty$ almost surely, then

$$\lim_{k\to\infty} Y_k = e^{\chi(t)}|y(t)|^p.$$

Letting $k \to \infty$ in (4.70) gives

$$\mathbb{E}|y(t)|^p \leq K\mathbb{E}|y_0|^p e^{-\chi(t-t_0)} \quad \text{for } t \geq t_0.$$

The proof is complete. □

Lemma 4.2.8 *Let Assumption 4.2.1 hold. Then*

$$\begin{aligned} &\mathbb{E}|x(t)|^p \leq \mathbb{E}|x_0|^p \exp\Big(p(\overline{\alpha} + 0.5\overline{\beta}^2 + 0.5\overline{\kappa}^2)(t-t_0)\Big) \\ \textit{and}\quad &\mathbb{E}|x(t)|^2 \leq \mathbb{E}|x_0|^2 \exp\Big((2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(t-t_0)\Big) \end{aligned} \tag{4.71}$$

for any $p \in (0,1)$, $t_0 \geq 0$, $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ *and* $r_0 \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$.

Proof. By the generalized Itô formula and Assumption 4.2.1,

$$\begin{aligned} &\mathbb{E}|x(t)|^2 \\ =&\mathbb{E}|x_0|^2 + \mathbb{E}\int_{t_0}^t \Big(2x^T(s)f(x(s),r(s),s) + |g(x(s),r(s),s)|^2 + |u(x(\delta_s),r(\delta_s),s)|^2\Big)ds \\ \leq&\mathbb{E}|x_0|^2 + \mathbb{E}\int_{t_0}^t \Big(2x^T(s)f(x(s),r(s),s) + |g(x(s),r(s),s)|^2 + |u(x(\delta_s),r(\delta_s),s)|^2\Big)ds \\ \leq&\mathbb{E}|x_0|^2 + (2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)\int_{t_0}^t \sup_{t_0\leq z\leq s}|x(z)|^2 ds \end{aligned}$$

Since the last term is nondecreasing function of t and by the Fubini theory,

$$\sup_{t_0 \le z \le t} \mathbb{E}|x(z)|^2 \le \mathbb{E}|x_0|^2 + (2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2) \int_{t_0}^{t} \sup_{t_0 \le z \le s} \mathbb{E}|x(z)|^2 ds.$$

Then the Gronwall inequality implies

$$\sup_{t_0 \le z \le t} \mathbb{E}|x(z)|^2 \le \mathbb{E}|x_0|^2 \exp[(2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(t - t_0)].$$

Finally Hölder's inequality gives the first inequality of (4.71). The proof is complete. □

Lemma 4.2.9 *Let Assumption 4.2.1 hold. Then*

$$\begin{aligned} &\mathbb{E}\Big(\sup_{t_0 \le s \le t} |x(s)|^p\Big) \le 2^p \mathbb{E}|x_0|^p \exp\Big(2p(t-t_0)[\overline{\alpha}^2(t-t_0) + 4(\overline{\beta}^2 + \overline{\kappa}^2)]\Big) \\ \text{and} \quad &\mathbb{E}\Big(\sup_{t_0 \le s \le t} |x(s)|^2\Big) \le 4\mathbb{E}|x_0|^2 \exp\Big(4(t-t_0)[\overline{\alpha}^2(t-t_0) + 4(\overline{\beta}^2 + \overline{\kappa}^2)]\Big). \end{aligned} \tag{4.72}$$

for any $p \in (0,1)$*,* $t_0 \ge 0$*,* $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ *and* $r_0 \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$*.*

Proof. By the definition of solutions of hybrid SDEs, the inequality in (2.17), Assumption 4.2.1, Hölder's inequality and the Burkholder-Davis-Gundy inequality, we have that

$$\begin{aligned} &\mathbb{E}\Big(\sup_{t_0 \le s \le t} |x(s)|^2\Big) \\ \le& 4\mathbb{E}|x_0|^2 + 4\mathbb{E}\Big(\sup_{t_0 \le s \le t} \Big|\int_{t_0}^{s} f(x(z), r(z), z) dz\Big|^2\Big) \\ &+ 4\mathbb{E}\Big(\sup_{t_0 \le s \le t} \Big|\int_{t_0}^{s} g(x(z), r(z), z) dB(z)\Big|^2\Big) + 4\mathbb{E}\Big(\sup_{t_0 \le s \le t} \Big|\int_{t_0}^{s} u(x(\delta_z), r(\delta_z), z) d\tilde{B}(z)\Big|^2\Big) \\ \le& 4\mathbb{E}|x_0|^2 + 4(t-t_0)\int_{t_0}^{t} \overline{\alpha}^2 \mathbb{E}\Big(\sup_{t_0 \le z \le s} |x(z)|^2\Big) ds + 16(\overline{\beta}^2 + \overline{\kappa}^2)\int_{t_0}^{t} \mathbb{E}\Big(\sup_{t_0 \le z \le s} |x(z)|^2\Big) ds. \end{aligned}$$

Then the Gronwall inequality implies

$$\mathbb{E}\Big(\sup_{t_0 \le s \le t} |x(s)|^2\Big) \le 4\mathbb{E}|x_0|^2 \exp\Big(4(t-t_0)[\overline{\alpha}^2(t-t_0) + 4(\overline{\beta}^2 + \overline{\kappa}^2)]\Big).$$

Finally Hölder's inequality gives the first inequality of (4.72). The proof is complete. □

Lemma 4.2.10 *Let Assumptions 4.2.1 and 4.2.2 hold. Then*

$$\begin{aligned}
&\mathbb{E}|x(t)-y(t)|^p \\
\leq &H^{\frac{p}{2}}(\tau)\mathbb{E}|x_0|^p\Big[\exp[(2\overline{\alpha}+\overline{\beta}^2+\overline{\kappa}^2)(t-t_0)]-1\Big]^{\frac{p}{2}}\exp[p(\overline{\alpha}+0.5\overline{\beta}^2+1.5\overline{\kappa}^2)(t-t_0)]
\end{aligned}$$

for any $p\in(0,1)$, $t_0\geq 0$, $x_0\in L^2_{\mathcal{F}_{t_0}}(\Omega,\mathbb{R}^n)$ *and* $r_0\in M_{\mathcal{F}_{t_0}}(\mathbb{S})$, *where* $H(\tau)$ *has been defined in (4.64).*

Proof. For any fixed $x_0\in L^p_{\mathcal{F}_{t_0}}(\Omega,\mathbb{R}^n)$, we have

$$\begin{aligned}
d(x(t)-y(t)) &= [f(x(t),r(t),t)-f(y(t),r(t),t)]dt \\
&+[g(x(t),r(t),t)-g(y(t),r(t),t)]dB(t)+[u(x(\delta_t),r(\delta_t),t)-u(y(t),r(t),t)]d\tilde{B}(t).
\end{aligned}$$

By the generalized Itô formula and Assumption 4.2.1,

$$\begin{aligned}
&\mathbb{E}|x(t)-y(t)|^2 \\
\leq &\mathbb{E}\int_{t_0}^t\Big(2[x(s)-y(s)]^T[f(x(s),r(s),s)-f(y(s),r(s),s)] \\
&+|g(x(s),r(s),s)-g(y(s),r(s),s)|^2+|u(x(\delta_s),r(\delta_s),s)-u(y(s),r(s),s)|^2\Big)ds \\
\leq &(2\overline{\alpha}+\overline{\beta}^2)\int_{t_0}^t\mathbb{E}|x(s)-y(s)|^2ds+J(t), \qquad (4.73)
\end{aligned}$$

where

$$J(t)=\mathbb{E}\int_{t_0}^t|u(x(\delta_s),r(\delta_s),s)-u(y(s),r(s),s)|^2]ds.$$

Since

$$\begin{aligned} u(x(\delta_s), r(\delta_s), s) - u(y(s), r(s), s) =& u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s) \\ &+ u(x(\delta_s), r(s), s) - u(x(s), r(s), s) \\ &+ u(x(s), r(s), s) - u(y(s), r(s), s). \end{aligned}$$

Using the elementary inequality $|a+b+c|^2 \le 3|a|^2 + 3|b|^2 + 3|c|^2$ for $a, b, c \in \mathbb{R}$, we can rewrite $J(t)$ as

$$\begin{aligned} J(t) \le 3\mathbb{E} \int_{t_0}^{t} \Big(&|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^2 \\ &+ |u(x(\delta_s), r(s), s) - u(x(s), r(s), s)|^2 \\ &+ |u(x(s), r(s), s) - u(y(s), r(s), s)|^2 \Big) ds. \end{aligned}$$

Then by Assumption 4.2.1 and the Fubini theory,

$$\begin{aligned} J(t) \le& 3\mathbb{E} \int_{t_0}^{t} \Big(|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^2 \\ &+ \kappa_{r(s)}^2 |x(s) - x(\delta_s)|^2 + \kappa_{r(s)}^2 |x(s) - y(s)|^2 \Big) ds \\ \le& 3 \int_{t_0}^{t} \mathbb{E}|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^2 ds \\ &+ 3\overline{\kappa}^2 \int_{t_0}^{t} \mathbb{E}|x(s) - x(\delta_s)|^2 + \mathbb{E}|x(s) - y(s)|^2 ds. \end{aligned}$$

Substituting into (4.73) gives

$$\begin{aligned} &\mathbb{E}|x(t) - y(t)|^2 \\ \le& (2\overline{\alpha} + \overline{\beta}^2 + 3\overline{\kappa}^2) \int_{t_0}^{t} \mathbb{E}|x(s) - y(s)|^2 ds + 3\overline{\kappa}^2 \int_{t_0}^{t} \mathbb{E}|x(s) - x(\delta_s)|^2 ds \\ &+ 3 \int_{t_0}^{t} \mathbb{E}|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^2 ds, \end{aligned} \tag{4.74}$$

Notice that $\delta_t \le t < \delta_t + \tau$. By Assumption 4.2.1, Lemma 4.2.9, Remark 4.2.6 and

Lemma 4.2.8, we can derive that

$$
\begin{aligned}
&\mathbb{E}|x(t)-x(\delta_t)|^2\\
\le& 2\mathbb{E}\int_{\delta_t}^{t}\Big(\tau|f(x(s),r(s),s)|^2+|g(x(s),r(s),s)|^2+|u(x(\delta_s),r(\delta_s),s)|^2\Big)ds\\
\le& 2\mathbb{E}\int_{\delta_t}^{t}\Big(\tau\alpha_{r(s)}^2|x(s)|^2+\beta_{r(s)}^2|x(s)|^2+\kappa_{r(\delta_s)}^2|x(\delta_s)|^2\Big)ds\\
\le& 2(\tau\overline{\alpha}^2+\overline{\beta}^2+\overline{\kappa}^2)\int_{\delta_t}^{t}\mathbb{E}\Big(\sup_{\delta_t\le z\le s}|x(z)|^2\Big)ds\\
\le& 8\tau(\tau\overline{\alpha}^2+\overline{\beta}^2+\overline{\kappa}^2)\mathbb{E}|x(\delta_t)|^2\exp[4\overline{\alpha}^2\tau^2+16\tau(\overline{\beta}^2+\overline{\kappa}^2)]\\
\le& 8\tau(\tau\overline{\alpha}^2+\overline{\beta}^2+\overline{\kappa}^2)\mathbb{E}|x_0|^2\exp[(2\overline{\alpha}+\overline{\beta}^2+\overline{\kappa}^2)(\delta_t-t_0)]\exp[4\overline{\alpha}^2\tau^2+16\tau(\overline{\beta}^2+\overline{\kappa}^2)]\\
\le& 8\tau(\tau\overline{\alpha}^2+\overline{\beta}^2+\overline{\kappa}^2)\exp[4\overline{\alpha}^2\tau^2+16\tau(\overline{\beta}^2+\overline{\kappa}^2)]\mathbb{E}|x_0|^2\exp[(2\overline{\alpha}+\overline{\beta}^2+\overline{\kappa}^2)(t-t_0)].
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_{t_0}^{t}\mathbb{E}|x(s)-x(\delta_s)|^2ds\le&\frac{8\tau(\tau\overline{\alpha}^2+\overline{\beta}^2+\overline{\kappa}^2)}{2\overline{\alpha}+\overline{\beta}^2+\overline{\kappa}^2}\exp[4\overline{\alpha}^2\tau^2+16\tau(\overline{\beta}^2+\overline{\kappa}^2)]\\
&\times\mathbb{E}|x_0|^2\Big[\exp[(2\overline{\alpha}+\overline{\beta}^2+\overline{\kappa}^2)(t-t_0)]-1\Big].
\end{aligned}
\tag{4.75}
$$

Let N_t be the integer part of $(t-t_0)/\tau$, i,e, number of observations until time t. Recall $t_k=t_0+k\tau$, $x(t_k)=x_k$, $r(t_k)=r_k$ for $k=0,1,2,\cdots$. Then we can write

$$
\begin{aligned}
&\int_{t_0}^{t}\mathbb{E}|u(x(\delta_s),r(\delta_s),s)-u(x(\delta_s),r(s),s)|^2ds\\
=&\sum_{k=0}^{N_t}\int_{t_k}^{t\wedge t_{k+1}}\mathbb{E}|u(x_k,r_k,s)-u(x_k,r(s),s)|^2ds.
\end{aligned}
\tag{4.76}
$$

According to Lemma 2.3.4, for any $t\ge t_0$, $v>0$ and $i\in\mathbb{S}$,

$$
\mathbb{P}(r(s)\neq i \text{ for some } s\in[t,t+v]\Big|r(t)=i)\le 1-e^{-\overline{\gamma}v},
\tag{4.77}
$$

where $\overline{\gamma}$ has been defined in (4.65) . Then by Assumption 4.2.1, for $t_k \le s \le t \wedge t_{k+1}$,

$$\begin{aligned}
&\mathbb{E}|u(x_k, r_k, s) - u(x_k, r(s), s)|^2 \\
=&\mathbb{E}\Big[\mathbb{E}|u(x_k, r_k, s) - u(x_k, r(s), s)|^2 |\mathcal{F}_{t_k})\Big] \\
\le&\mathbb{E}\Big[4\overline{\kappa}^2 |x_k|^2 \mathbb{E}\Big(I_{\{r(s)\neq r_k\}}|\mathcal{F}_{t_k}\Big)\Big] \\
\le&\mathbb{E}\Big[4\overline{\kappa}^2 |x_k|^2 (1 - e^{-\overline{\gamma}\tau})\Big] = 4\overline{\kappa}^2 (1 - e^{-\overline{\gamma}\tau})\mathbb{E}|x_k|^2.
\end{aligned} \tag{4.78}$$

Substituting (4.78) into (4.76) and using Lemma 4.2.8, we obtain that

$$\begin{aligned}
&\int_{t_0}^{t} \mathbb{E}|u(x(\delta_s), r(\delta_s), s) - u(x(\delta_s), r(s), s)|^2 ds \\
\le&4\overline{\kappa}^2 (1 - e^{-\overline{\gamma}\tau}) \sum_{k=0}^{N_t} \int_{t_k}^{t\wedge t_{k+1}} \mathbb{E}|x_k|^2 ds \\
\le&4\overline{\kappa}^2 (1 - e^{-\overline{\gamma}\tau}) \sum_{k=0}^{N_t} \int_{t_k}^{t\wedge t_{k+1}} \mathbb{E}|x_0|^2 \exp[(2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(t_k - t_0)] ds \\
\le&4\overline{\kappa}^2 (1 - e^{-\overline{\gamma}\tau}) \sum_{k=0}^{N_t} \int_{t_k}^{t\wedge t_{k+1}} \mathbb{E}|x_0|^2 \exp[(2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(s - t_0)] ds \\
\le&4\overline{\kappa}^2 (1 - e^{-\overline{\gamma}\tau}) \mathbb{E}|x_0|^2 \int_{t_0}^{t} \exp[(2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(s - t_0)] ds \\
\le&\frac{4\overline{\kappa}^2 (1 - e^{-\overline{\gamma}\tau})}{2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2} \mathbb{E}|x_0|^2 \Big[\exp[(2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(t - t_0)] - 1\Big].
\end{aligned} \tag{4.79}$$

Substituting (4.79) and (4.75) into (4.74) gives

$$\begin{aligned}
&\mathbb{E}|x(t) - y(t)|^2 \\
\le&(2\overline{\alpha} + \overline{\beta}^2 + 3\overline{\kappa}^2) \int_{t_0}^{t} \mathbb{E}|x(s) - y(s)|^2 ds + H(\tau)\mathbb{E}|x_0|^2 \Big[\exp[(2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(t - t_0)] - 1\Big].
\end{aligned}$$

Then the Gronwall inequality implies

$$\mathbb{E}|x(t) - y(t)|^2 \le H(\tau)\mathbb{E}|x_0|^2 \Big[\exp[(2\overline{\alpha} + \overline{\beta}^2 + \overline{\kappa}^2)(t - t_0)] - 1\Big] \exp[(2\overline{\alpha} + \overline{\beta}^2 + 3\overline{\kappa}^2)(t - t_0)].$$

Finally the Hölder inequality indicates the desired assertion. $\square$

Similarly to Remarks 4.1.8 and 4.1.9 in Section 4.1, we have two remarks to explain the existence of a positive observation interval and the Markov property of the controlled system.

Remark 4.2.11 *$\hat{H}(\tau)$ in (4.62) is a continuously increasing function of τ for $\tau \geq 0$. $\hat{H} = 0$ when $\tau = 0$ (as $H(0) = 0$) and $\hat{H} \to \infty$ as $\tau \to \infty$. So equation (4.62) must have a unique root $\bar{\tau} > 0$, and for any $\tau \in (0, \bar{\tau})$, we have $\hat{H}(\tau) < 1 - \varepsilon$.*

Proof of Theorem 4.2.5: Fix $\tau \in (0, \bar{\tau})$ and $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ arbitrarily.
The definition in (4.65) indicates $K \geq 1$, then $2^p K/\varepsilon > 1$. So we can choose a positive integer ν such that

$$\frac{\log(2^p K/\varepsilon)}{\chi\tau} \leq \nu \leq 1 + \frac{\log(2^p K/\varepsilon)}{\chi\tau}. \tag{4.80}$$

The left part of it imples

$$2^p K e^{-\chi\nu\tau} \leq \varepsilon. \tag{4.81}$$

By Remark 4.2.6, we consider $x(t)$ as the solution of (4.47) starting from $t_0 + i\nu\tau$ for $i = 0, 1, 2, \cdots$. Recall the notations $t_k = t_0 + k\tau$, $y(t_k) = y_k$, $x(t_k) = x_k$, $r(t_k) = r_k$ for $\forall k = 0, 1, 2, \cdots$. Let $y_{(i+1)\nu} = y(t_{(i+1)\nu}; x_{i\nu}, r_{i\nu}, t_{i\nu})$ and use Lemma 4.2.7, then we have

$$\mathbb{E}|y_{(i+1)\nu}|^p \leq K\mathbb{E}|x_{i\nu}|^p e^{-\chi\nu\tau}. \tag{4.82}$$

Since the elementary inequality $(a+b)^p \leq 2^p(a^p + b^p)$ holds for any $a, b \in \mathbb{R}$,

$$\mathbb{E}|x_{(i+1)\nu}|^p \leq 2^p\mathbb{E}|y_{(i+1)\nu}|^p + 2^p\mathbb{E}|x_{(i+1)\nu} - y_{(i+1)\nu}|^p. \tag{4.83}$$

Combining (4.82) and (4.81) gives

$$2^p\mathbb{E}|y_{(i+1)\nu}|^p \leq \varepsilon\mathbb{E}|x_{i\nu}|^p.$$

Let $x_{(i+1)\nu} = x(t_{(i+1)\nu}; x_{i\nu}, r_{i\nu}, t_{i\nu})$ and recall Lemma 4.2.10. Since the second inequality in (4.80) indicates that

$$\nu\tau \leq \tau + \log(2^p K/\varepsilon)/\chi,$$

we can obtain

$$\begin{aligned}
&2^p\mathbb{E}|x_{(i+1)\nu}-y_{(i+1)\nu}|^p\\
\le&2^pH^{\frac{p}{2}}(\tau)\mathbb{E}|x_{i\nu}|^p\Big[\exp[(2\overline{\alpha}+\overline{\beta}^2+\overline{\kappa}^2)\nu\tau]-1\Big]^{\frac{p}{2}}\exp[p(\overline{\alpha}+0.5\overline{\beta}^2+1.5\overline{\kappa}^2)\nu\tau]\\
\le&\hat{H}(\tau)\mathbb{E}|x_{i\nu}|^p,
\end{aligned}$$

where $\hat{H}(\tau)$ has been defined in (4.63).

Substituting this into (4.83) yields

$$\mathbb{E}|x_{(i+1)\nu}|^p\le\mathbb{E}|x_{i\nu}|^p[\varepsilon+\hat{H}(\tau)]. \tag{4.84}$$

Since $\hat{H}(\tau)<1-\varepsilon$, there is $\lambda>0$ such that

$$\varepsilon+\hat{H}(\tau)=e^{-\lambda\nu\tau}.$$

Therefore we can write

$$\mathbb{E}|x_{(i+1)\nu}|^p\le\mathbb{E}|x_{i\nu}|^pe^{-\lambda\nu\tau}. \tag{4.85}$$

Repeating this procedure,

$$\mathbb{E}|x_{i\nu}|^p\le\mathbb{E}|x_0|^pe^{-i\lambda\nu\tau},\quad\forall i=0,1,2,\cdots. \tag{4.86}$$

For any $t\ge0$, there is a unique $i\ge0$ such that $t_0+i\nu\tau\le t<t_0+(i+1)\nu\tau$. Again, use the Markov property of the pair $(x(t),r(t))$ and consider $x(t)$ as the solution of (4.47) starting from $t_0+i\nu\tau$ for $i=0,1,2,\cdots$. By Lemma 4.2.8 and (4.86),

$$\begin{aligned}
\mathbb{E}|x(t)|^p&\le\mathbb{E}|x_{i\nu}|^p\exp[p(t-t_0-i\nu\tau)(\overline{\alpha}+0.5\overline{\beta}^2+0.5\overline{\kappa}^2)]\\
&\le\mathbb{E}|x_{i\nu}|^p\exp[p\nu\tau(\overline{\alpha}+0.5\overline{\beta}^2+0.5\overline{\kappa}^2)]\\
&\le\mathbb{E}|x_0|^pe^{-i\lambda\nu\tau}\exp[p\nu\tau(\overline{\alpha}+0.5\overline{\beta}^2+0.5\overline{\kappa}^2)]\\
&\le\mathbb{E}|x_0|^pe^{-\lambda t}\exp[\lambda\nu\tau+p\nu\tau(\overline{\alpha}+0.5\overline{\beta}^2+0.5\overline{\kappa}^2)]\\
&\le\mathbb{E}|x_0|^pMe^{-\lambda(t-t_0)},
\end{aligned} \tag{4.87}$$

where $M = \exp[\lambda\nu\tau + p\nu\tau(\overline{\alpha} + 0.5\overline{\beta}^2 + 0.5\overline{\kappa}^2)]$. So far we have proved assertion (4.60).

By Lemma 4.2.9, Remark 4.2.6 and (4.87), we have

$$\mathbb{E}\Big(\sup_{\hat{t}_k \le t \le \hat{t}_{k+1}} |x(t)|^p\Big) \le 2^p \mathbb{E}|x(\hat{t}_k)|^p \exp\Big(2p\nu\tau[\overline{\alpha}^2\nu\tau + 4(\overline{\beta}^2 + \overline{\kappa}^2)]\Big) \le C\mathbb{E}|x_0|^p e^{-k\lambda\nu\tau}, \tag{4.88}$$

where $C = 2^p M \exp\Big(2p\nu\tau[\overline{\alpha}^2\nu\tau + 4(\overline{\beta}^2 + \overline{\kappa}^2)]\Big)$ and $\hat{t}_k = t_0 + k\nu\tau$ for $k = 0, 1, 2, \cdots$. Then we can prove the almost sure exponential stability in a similar way as in Section 3, namely by Chebyshev's inequality and the Borel-Cantelli lemma. □

4.2.3 Corollary

Theorem 4.2.5 can also be applied to linear hybrid system. Consider an unstable linear n-dimensional ODE with Markovian Switching

$$\dot{x}(t) = A_{r(t)}x(t) \tag{4.89}$$

on $t \ge t_0(\ge 0)$ with $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ and $r_0 \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$, where $A_i \in \mathbb{R}^{n\times n}$ for $i \in \mathbb{S}$.

We can stabilize it by stochastic feedback control $G(r(t))x(t)dB(t)$ based on a scalar Brownian motion independent of $B(t)$. The controlled SDE has the form

$$dx(t) = A_{r(t)}x(t)dt + G_{r(\delta_t)}x(\delta_t)dB(t), \tag{4.90}$$

where $G_i \in \mathbb{R}^{n\times n}$ for $i \in \mathbb{S}$ and

$$\delta_t = t_0 + \Big[\frac{t - t_0}{\tau}\Big]\tau \quad \text{for } \tau > 0. \tag{4.91}$$

Let $\bar{A} = \max_{i\in\mathbb{S}} \|A_i\|$ and $\bar{G} = \max_{i\in\mathbb{S}} \|G_i\|$. By replacing $\overline{\alpha}$, $\overline{\beta}$ and $\overline{\kappa}$ with $\bar{A}$, 0 and $\bar{G}$ respectively, we have the following corollary.

Corollary 4.2.12 *Assume that there is a constant $p \in (0, 1)$ such that $\mathcal{A}(p)$ defined in*

(4.59) with

$$\theta_i(p) = \frac{p}{2}\Big[(2-p)\lambda_{\max}^2(G_i) - \|G_i\|^2\Big] - p\|A_i\| \quad \textit{for } 1 \le i \le N$$

is a nonsingular M-matrix. Then we can find a vector $\varphi \in \mathbb{R}_+^N$ *such that* $\mathcal{A}(p)\varphi \in \mathbb{R}_+^N$. *Choose* $\varepsilon \in (0,1)$. *Then the controlled system (4.90) is exponentially stable in pth moment and almost surely in the sense that (4.60) and (4.61) hold for any initial values* $x_0 \in L^2_{\mathcal{F}_{t_0}}(\Omega, \mathbb{R}^n)$ *and* $r_0 \in M_{\mathcal{F}_{t_0}}(\mathbb{S})$, *provided* $\tau \in (0, \bar{\tau})$, *where* $\bar{\tau} > 0$ *is the unique root to*

$$\hat{H}(\tau) = 1 - \varepsilon \tag{4.92}$$

where

$$\begin{aligned}\hat{H}(\tau) = & 2^p\Big[\frac{6\bar{G}^2\tau(\bar{A}^2\tau + \bar{G}^2) + 12\bar{G}^2(1 - e^{-\bar{\gamma}\tau})}{2\bar{A} + \bar{G}^2}\Big]^{\frac{p}{2}} \\ & \times \Big[\exp\Big((2\bar{A} + \bar{G}^2)[\tau + \log(2^p K/\varepsilon)/\chi]\Big) - 1\Big]^{\frac{p}{2}} \\ & \times \exp\Big[p(\bar{A} + 1.5\bar{G}^2)[\tau + \log(2^p K/\varepsilon)/\chi]\Big],\end{aligned}$$

in which K, χ *and* $\bar{\varphi}_i (1 \le i \le N)$ *all have the same forms as defined in Section 4.2.1.*[4]

4.3 Discussion and Conclusion

Similar to deterministic feedback controls which are added to the drift, a stochastic feedback control uses observations of the system state x and mode r at discrete times $0, \tau, 2\tau, \cdots$. The difference for implementation is that: at each time step, a deterministic feedback control added $u(x(\delta_t), r(\delta_t), t)dt$ to the system, a stochastic feedback control added $u(x(\delta_t), r(\delta_t), t)d\tilde{B}(t)$ to the system, where $\tilde{B}(t)$ is independent from the Brownian motion of the original stochastic system.

In this chapter we have discussed the stabilization of continuous-time non-autonomous

[4]

$$K = \frac{\max_{i\in\mathbb{S}} \varphi_i}{\min_{i\in\mathbb{S}} \varphi_i}, \quad \chi = \frac{\min_{1\le i\le N} \bar{\varphi}_i}{\max_{1\le i\le N} \varphi_i}, \quad \bar{\varphi}_i = \theta_i(p)\varphi_i - \sum_{j=1}^N \gamma_{ij}\varphi_j \quad \text{for } 1 \le i \le N. \tag{4.93}$$

ODEs as well as hybrid SDEs by stochastic feedback control based on Brownian motions and discrete-time state and mode observations, in the sense of pth moment exponential stability for $p \in (0, 1)$ and almost sure exponential stability.

The main contribution of this chapter is expanding the scope of applicable original unstable systems, from autonomous ODEs to non-autonomous ODEs and hybrid SDEs. Due to the complexity of stabilization problem by a stochastic feedback control, we have to use very general methods - by comparing the our new controlled system based on discrete-time observations and the traditional auxiliary controlled controlled system based continuous-time observations. Consequently, the observation interval is too small to give any numerical example by computer simulation.

Chapter 5

Stabilization of Continuous-time Periodic Stochastic Systems

Since 2013, study on stabilization of SDEs using observations at times $0, \tau, 2\tau, 3\tau, \cdots$ has been developed by many researchers. However, if the coefficients of original system and controller change with time explicitly periodically, then the periodic time-varying observation frequencies make more sense than the observations with constant interval. This chapter is devoted to stabilization problem using this new way of observation. This chapter investigates how to stabilize a given non-autonomous periodic unstable SDE or hybrid SDE by a periodic feedback control based on periodic discrete-time observations to make the controlled system become pth moment H_∞-stable and exponentially stable for $p > 1$, pth moment asymptotically stable for $p \geq 2$, almost surely asymptotically stable and exponentially stable. Compared to existing results, the new established theory not only reduces the cost of control by reducing observation frequency, but also offers flexibility on the setting of observations to some extent. The results stated in this chapter was submitted as [122] [1], which is under review of the journal "IEEE Transactions on Automatic Control".

This chapter is organised as follows. We discuss hybrid SDEs before the single-mode SDE systems. Section 5.1 introduces the stabilization problem, proposes definitions of

[1]Dong, R. and Mao, X. 2018. Stabilization of Continuous-time Periodic Stochastic Systems by Feedback Control Based on Discrete-time Periodic Observations. Manuscript submitted for publication.

the observation interval sequence and assumptions. Section 5.2 and Section 5.3 discuss asymptotic and exponential stabilization respectively for hybrid SDEs. Results for single-mode SDEs is shown in Section 5.4. Before the conclusion in Section 5.6, two numerical examples are given in Section 5.5.

5.1 Notation

Given an n-dimensional periodic unstable hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{5.1}$$

on $t \geq 0$, we want to add a controller u in the drift part for stabilization.

If system (5.1) is autonomous, namely the coefficients $f(x(t), r(t))$ and $g(x(t), r(t))$ does not depend on time explicitly, then it makes sense to design such a controller (i.e. $u(x(\delta_t), r(\delta_t))$ does not depend on time explicitly). If system (5.1) is non-autonomous and coefficients $f(x(t), r(t), t)$ and $g(x(t), r(t), t)$ are both periodic with respect to time t, then it makes sense design such a controller (i.e. $u(x(\delta_t), r(\delta_t), t)$ is periodic with respect to time). For these types of systems and controllers, it's feasible to set the observation intervals to be periodic, including time-varying and constant. This chapter will put emphasis on time-varying periodic observations for non-autonomous systems, as the observations with constant interval for autonomous systems has been covered by Chapter 3.

As we know, the observation interval is the length of time between two observations. Define our periodic observation interval sequence to be $\{\tau_j\}_{j\geq 1}$ such that

$$\tau_{kM+i} = \tau_i$$

for a positive integer M, $\forall k = 0, 1, 2, \cdots$ and $i = 1, 2, \cdots, M$.

This means the system is observed at time points $0, \tau_1, \tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3, \cdots$. Then for any $t \geq 0$, there is a positive integer k such that

$$\sum_{j=1}^{k} \tau_j \leq t < \sum_{j=1}^{k+1} \tau_j.$$

Then similarly as before, the step function of time δ_t, which represents the observation time point, is now defined as

$$\delta_t := \sum_{j=1}^{k} \tau_j. \tag{5.2}$$

We will discuss how to calculate the observation intervals later. However, to avoid readers' confusion, I sketch the general idea now. Roughly speaking: the length of one period of $\{\tau_j\}_{j\geq 1}$ will be determined by the original system and controller; we will firstly divide the length of one period into several subintervals, then calculate τ_j on each subinterval, and finally set the infimum on each subinterval or a smaller value as the constant observation interval over the corresponding subinterval.

In this chapter, I will use several symbols to denote the observation intervals for analysis.

For $\forall t \geq 0$, we can find a positive integer k such that $\sum_{j=1}^{k} \tau_j \leq t < \sum_{j=1}^{k+1} \tau_j$. Let $\kappa_t := \tau_{k+1}$ for any $t \in [\sum_{j=1}^{k} \tau_j, \sum_{j=1}^{k+1} \tau_j)$. That is, $\delta_t \leq t < \delta_t + \kappa_t$.
For example, when $t \in [0, \tau_1)$, $\delta_t = 0$ and $\kappa_t = \tau_1$;
when $t \in [\tau_1, \tau_1 + \tau_2)$, $\delta_t = \tau_1$ and $\kappa_t = \tau_2$;
when $t \in [\tau_1 + \tau_2, \tau_1 + \tau_2 + \tau_3)$, $\delta_t = \tau_1 + \tau_2$ and $\kappa_t = \tau_3$; $\cdots$.
Obviously κ_t is actually a periodic step function of time.

Similarly to the notation 'δ_t' and 'κ_t', for a step function of time $\hat{K}$, I will use $\hat{K}_t$ instead of $\hat{K}(t)$ for simplicity and consistency.

5.2 Stabilization problem

Consider an n-dimensional periodic hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{5.3}$$

on $t \geq 0$, with initial values $x(0) = x_0 \in \mathbb{R}^n$ and $r(0) = r_0 \in \mathbb{S}$. Here

$$f : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}.$$

The given system may not be stable and our aim is to design a feedback control $u : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}^n$ for stabilization.

The controlled system corresponding to (5.3) has the form

$$dx(t) =[f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)]dt + g(x(t), r(t), t)dB(t). \tag{5.4}$$

Assumption 5.2.1 *Assume that $f(x, i, t)$, $g(x, i, t)$ and $u(x, i, t)$ are all periodic with respect to time t. Assume f, g, u and κ_t have a common period T.*

The assumption that T is a period of κ_t means $\kappa_t = \kappa_{t+kT}$ for $k = 0, 1, 2, \cdots$ and $\sum_{j=1}^{M} \tau_j = T$.

Assumption 5.2.2 *Assume that the coefficients $f(x, i, t)$ and $g(x, i, t)$ are both locally Lipschitz continuous on x (see e.g. [26]), and they both satisfy the following linear growth condition*

$$|f(x, i, t)| \leq K_1(t)|x| \quad \text{and} \quad |g(x, i, t)| \leq K_2(t)|x| \tag{5.5}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$, where $K_1(t)$ and $K_2(t)$ are periodic bounded non-negative functions with period T.

Note (5.5) implies that

$$f(0, i, t) = 0 \quad \text{and} \quad g(0, i, t) = 0 \tag{5.6}$$

for all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$.

Assumption 5.2.3 *Assume the controller function $u(x, i, t)$ is globally Lipschitz continuous on x and satisfies the following linear growth condition*

$$|u(x, i, t) - u(y, i, t)| \leq K_3(t)|x - y| \tag{5.7}$$

for all $(x, y, i, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$*, where* $K_3(t)$ *is a periodic bounded non-negative function with period* T*. Moreover, we also assume*

$$u(0, i, t) = 0 \tag{5.8}$$

for all $(i, t) \in \mathbb{S} \times \mathbb{R}_+$*.*

Assumption 5.2.3 implies that the controller satisfies the following linear growth condition

$$|u(x, i, t)| \leq K_3(t)|x| \tag{5.9}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Denote by $\overline{K_1}, \overline{K_2}$ and $\overline{K_3}$ the upper bound of $K_1(t), K_2(t)$ and $K_3(t)$ respectively, i.e.,

$$K_1(t) \leq \overline{K_1}, \quad K_2(t) \leq \overline{K_2} \quad \text{and} \quad K_3(t) \leq \overline{K_3}.$$

Denote the largest observation interval $\max_{j\geq 1} \tau_j$ by $\tau_{\max}$. For stabilization purpose, we define the following initial values

$$\begin{aligned} &x(s) = x_0, \;\; r(s) = r_0, \;\; f(x, i, s) = f(x, i, 0), \\ &u(x, i, s) = u(x, i, 0) \;\; \text{and} \;\; g(x, i, s) = g(x, i, 0) \end{aligned}$$

for all $(x, i, s) \in \mathbb{R}^n \times \mathbb{S} \times [-\tau_{\max}, 0)$.

Let $V(x, i, t)$ be a Lyapunov function periodic with respect to t, and we require $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$. Then define an operator $\mathcal{L}V : \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\begin{aligned} \mathcal{L}V(x, i, t) =& V_t(x, i, t) + V_x(x, i, t)[f(x, i, t) + u(x, i, t)] \\ &+ \frac{1}{2}\text{trace}[g^T(x, i, t)V_{xx}(x, i, t)g(x, i, t)] + \sum_{k=1}^{N} \gamma_{ik} V(x, k, t). \end{aligned} \tag{5.10}$$

For simplicity, we denote $V(x(0), r(0), 0)$ by V_0. We impose an assumption on the Lyapunov function.

Assumption 5.2.4 *Assume that there is a Lyapunov function $V(x,i,t)$ periodic with respect to t with period T, constants $l>0, p>1$ and a periodic function $\lambda(t)$ with* $\inf_{t\geq 0}\lambda(t)>0$ *such that*

$$\mathcal{L}V(x,i,t)+l|V_x(x,i,t)|^{\frac{p}{p-1}}\leq -\lambda(t)|x|^p \tag{5.11}$$

for all $(x,i,t)\in\mathbb{R}^n\times\mathbb{S}\times\mathbb{R}_+$.

It can be seen that T is a period of λ. Let $\underline{\lambda}=\inf_{0\leq t<T}\lambda(t)$.

Let us divide $[0,T]$ into $Z-1$ subintervals, where $Z\geq 2$ is an arbitrary integer, by choosing a partition $\{T_j\}_{1\leq j\leq Z}$ with $T_1=0$ and $T_Z=T$. Then we define the following three step functions on $t\geq 0$ with periodic T:

$$\begin{aligned}
\hat{K}_{1t} &= \sup_{T_j\leq s\leq T_{j+1}} K_1(t) \quad \text{for } T_j\leq t<T_{j+1},\\
\hat{K}_{2t} &= \sup_{T_j\leq t\leq T_{j+1}} K_2(t) \quad \text{for } T_j\leq t<T_{j+1},\\
\hat{K}_{3t} &= \sup_{T_j\leq t\leq T_{j+1}} K_3(t) \quad \text{for } T_j\leq t<T_{j+1},
\end{aligned} \tag{5.12}$$

where $j=1,\cdots,Z-1$.

Define a periodic function

$$\begin{aligned}
\beta(t) :&= \beta(\kappa_t,t)=\lambda(t)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}\\
&\times\Big(K_3^p(t)2^{3p-2}(1-e^{-\overline{\gamma}\kappa_t})+\frac{2^{p-1}\kappa_t^{\frac{p}{2}}\hat{K}_{3t}^p}{1-8^{p-1}\kappa_t^p\hat{K}_{3t}^p}[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_t})+2^{p-1}]\\
&\quad\times[2^{p-1}\kappa_t^{\frac{p}{2}}K_1^p(t)+\rho K_2^p(t)+4^{p-1}\kappa_t^{\frac{p}{2}}K_3^p(t)]\Big).
\end{aligned} \tag{5.13}$$

It can be seen that T is a period of $\beta(t)$.

Define two positive numbers depending on the moment order p:

$$\rho=\begin{cases}(\frac{32}{p})^{\frac{p}{2}} & \text{for } p\in(1,2),\\ [\frac{p(p-1)}{2}]^{\frac{p}{2}} & \text{for } p\geq 2.\end{cases}$$

and

$$\nu = \begin{cases} (\frac{32}{p})^{\frac{p}{2}} & \text{for } p \in (1,2), \\ 4 & \text{for } p = 2, \\ \left(\frac{p^{p+1}}{2(p-1)^{p-1}}\right)^{\frac{p}{2}} & \text{for } p > 2. \end{cases}$$

5.3 Asymptotic Stabilization

5.3.1 Moment H_∞ stability

Theorem 5.3.1 *Fix the moment order $p > 1$. Let Assumptions 5.2.1, 5.2.2, 5.2.3 and 5.2.4 hold. Divide $[0, T]$ into $Z-1$ subintervals with $T_1 = 0$ and $T_Z = T$. Choose $\kappa_t > 0$ sufficiently small such that $\kappa_t \le T_{j+1} - T_j$ for $t \in [T_j, T_{j+1})$* [2] *where $j = 1, 2, \cdots, Z-1$ and*

$$\inf_{0 \le t < T} \beta(t) > 0 \quad \text{and} \quad \sup_{0 \le t < T} (\kappa_t \hat{K}_{3t}) < 8^{-\frac{p-1}{p}}, \tag{5.14}$$

where $\beta(t)$ has been defined in (5.13). Then the controlled system (5.4) is H_∞-stable in $L^p(\Omega \times \mathbb{R}_+; \mathbb{R}^n)$ in the sense

$$\int_0^\infty \mathbb{E}|x(s)|^p ds < \infty \tag{5.15}$$

for all initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$.

We will explain why such an observation interval sequence exists and how to calculate it step by step after the proof.

Proof. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. Applying the generalized Itô formula to $V(x(t), r(t), t)$ gives

$$dV(x(t), r(t), t) = LV(x(t), r(t), t)dt + dM(t)$$

[2] In other words, the length of the subinterval cannot be shorter than the observation interval.

for $t \geq 0$, where $M(s)$ is a continuous local martingale with $M(0) = 0$ and

$$\begin{aligned}
&LV(x(t), r(t), t) \\
=&V_t(x(t), r(t), t) + V_x(x(t), r(t), t)[f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)] \\
&+ \frac{1}{2}\text{trace}[g^T(x(t), r(t), t)V_{xx}(x(t), r(t), t)g(x(t), r(t), t)] + \sum_{k=1}^{N} \gamma_{ik} V(x, k, t). \quad (5.16)
\end{aligned}$$

Since $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+; \mathbb{R}_+)$, we can use the generalized Itô formula and get

$$\mathbb{E}V(x(t), r(t), t) = V_0 + \int_0^t \mathbb{E}LV(x(s), r(s), s)ds. \quad (5.17)$$

We can rewrite $LV(x(s), r(s), s)$ by the operator as

$$\begin{aligned}
&LV(x(s), r(s), s) \\
=&\mathcal{L}V(x(s), r(s), s) - V_x(x(s), r(s), s)[u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)]. \quad (5.18)
\end{aligned}$$

By the Young inequality, we can derive that

$$\begin{aligned}
&- V_x(x(s), r(s), s)[u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)] \\
\leq&\Big[\varepsilon|V_x(x(s), r(s), s)|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}} \\
&\times \Big[\varepsilon^{1-p}|u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)|^p\Big]^{\frac{1}{p}} \\
\leq&l|V_x(x(s), r(s), s)|^{\frac{p}{p-1}} \\
&+ \frac{1}{p}(\frac{p-1}{pl})^{p-1}|u(x(s), r(s), s) - u(x(\delta_s), r(\delta_s), s)|^p, \quad (5.19)
\end{aligned}$$

where $l = \frac{p-1}{p}\varepsilon$ for $\forall\varepsilon > 0$.

Using the elementary inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $a, b \in \mathbb{R}$ and $p > 1$, we have

$$\begin{aligned}&\mathbb{E}|u(x(s),r(s),s)-u(x(\delta_s),r(\delta_s),s)|^p\\ \le&2^{p-1}\mathbb{E}|u(x(\delta_s),r(\delta_s),s)-u(x(\delta_s),r(s),s)|^p\\ &+2^{p-1}\mathbb{E}|u(x(\delta_s),r(s),s)-u(x(s),r(s),s)|^p.\end{aligned}\tag{5.20}$$

According to Lemma 2.3.4 (or see Lemma 1 in [35]) and Assumption 5.2.3, we have

$$\begin{aligned}&\mathbb{E}|u(x(\delta_s),r(\delta_s),s)-u(x(\delta_s),r(s),s)|^p\\ =&\mathbb{E}\Big[\mathbb{E}|u(x(\delta_s),r(\delta_s),s)-u(x(\delta_s),r(s),s)|^p\Big|\mathcal{F}_{\delta_s})\Big]\\ \le&\mathbb{E}\Big[2^pK_3^p(s)|x(\delta_s)|^p\mathbb{E}\Big(I_{\{r(s)\ne r_k\}}|\mathcal{F}_{\delta_s}\Big)\Big]\\ \le&2^{2p-1}K_3^p(s)(1-e^{-\overline{\gamma}\kappa_s})[\mathbb{E}|x(s)|^p+\mathbb{E}|x(\delta_s)-x(s)|^p].\end{aligned}\tag{5.21}$$

Substituting (5.21) into (5.20) gives

$$\begin{aligned}&\mathbb{E}|u(x(s),r(s),s)-u(x(\delta_s),r(\delta_s),s)|^p\\ \le&2^{3p-2}K_3^p(s)(1-e^{-\overline{\gamma}\kappa_s})\mathbb{E}|x(s)|^p\\ &+[2^{3p-2}K_3^p(s)(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}K_3^p(s)]\mathbb{E}|x(\delta_s)-x(s)|^p.\end{aligned}\tag{5.22}$$

Substitute (5.22) into (5.19). Then substitute the result into (5.18). By Assumption 5.2.4, we obtain that

$$\begin{aligned}&\mathbb{E}LV(x(s),r(s),s)\\ \le&-[\lambda(s)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})]\mathbb{E}|x(s)|^p\\ &+\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\mathbb{E}|x(\delta_s)-x(s)|^p.\end{aligned}\tag{5.23}$$

Note that $t-\delta_t\le\kappa_t$ for all $t\ge 0$. By the definition of hybrid SDEs solutions, the elementary inequality in (2.17), Hölder's inequality, the Burkholder-Davis-Gundy inequality and Theorem 2.6.2, we obtain that

$$
\begin{aligned}
&\mathbb{E}|x(t)-x(\delta_t)|^p \\
\leq & 2^{p-1}\kappa_t^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^{t}\Big[\kappa_t^{\frac{p}{2}}|f(x(s),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^p+\rho|g(x(s),r(s),s)|^p\Big]ds.
\end{aligned} \tag{5.24}
$$

By Assumptions 5.2.2 and 5.2.3, we have that for any $s\in[\delta_s,\delta_s+\kappa_s)$,

$$
\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p \\
\leq & 2^{p-1}\kappa_s^{\frac{p-2}{2}}\int_{\delta_s}^{s}[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)]\mathbb{E}|x(z)|^p dz \\
&+8^{p-1}\kappa_s^{p-1}\int_{\delta_s}^{s}K_3^p(z)dz[\mathbb{E}|x(s)-x(\delta_s)|^p+\mathbb{E}|x(s)|^p].
\end{aligned}
$$

Since condition (5.14) guarantees $8^{p-1}\kappa_s^p\hat{K}_{3s}^p<1$, we can rearrange it and get

$$
\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p \\
\leq & \frac{8^{p-1}\kappa_s^p\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\mathbb{E}|x(s)|^p+\frac{2^{p-1}\kappa_s^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\int_{\delta_s}^{s}\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)\Big]\mathbb{E}|x(z)|^p dz.
\end{aligned} \tag{5.25}
$$

Recall $\tau_{\max}=\max_{j\geq 1}\tau_j$. Let $x(s)=x_0$, $r(s)=r_0$, $K_1(s)=K_1(0)$, $K_2(s)=K_2(0)$ and $K_3(s)=K_3(0)$ for all $(x,i,s)\in\mathbb{R}^n\times\mathbb{S}\times[-\tau_{\max},0)$. In addition, note that for $\forall z\in[\delta_s,s]$, we have $\kappa_z=\kappa_s$ and $K_3(s)\leq\hat{K}_{3s}=\hat{K}_{3z}$. Since $s-\kappa_s<\delta_s$, it's easy to show that for a non-negative bounded function $F(t)$,

$$
\begin{aligned}
&\int_0^t\int_{\delta_s}^{s}F(z)dzds\leq\int_0^t\int_{s-\kappa_s}^{s}F(z)dzds \\
\leq&\int_{-\kappa_z}^{t}F(z)\int_z^{z+\kappa_z}dsdz\leq\int_{-\kappa_s}^{t}\kappa_zF(z)dz\leq C+\int_0^t\kappa_zF(z)dz.
\end{aligned} \tag{5.26}
$$

Then

$$
\begin{aligned}
&\int_0^t K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\frac{2^{p-1}\kappa_s^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_s^p\hat{K}_{3z}^p}\\
&\times\int_{\delta_s}^s[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)]\mathbb{E}|x(z)|^p dzds\\
\leq&\int_0^t\int_{\delta_s}^s\hat{K}_{3z}^p[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_z})+2^{p-1}]\frac{2^{p-1}\kappa_z^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_z^p\hat{K}_{3z}^p}\\
&\times\Big[2^{p-1}\kappa_z^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)\Big]\mathbb{E}|x(z)|^p dzds\\
\leq&C+\int_0^t\frac{2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\Big[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}\Big]\\
&\times\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(s)+\rho K_2^p(s)\Big]\mathbb{E}|x(s)|^p ds.
\end{aligned}
$$

Recall that C's denote positive constants that may change from line to line.

So

$$
\begin{aligned}
&\int_0^t K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\mathbb{E}|x(s)-x(\delta_s)|^p ds\\
\leq&C+\int_0^t\frac{2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\Big[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}\Big]\\
&\times\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(s)+\rho K_2^p(s)+4^{p-1}\kappa_s^{\frac{p}{2}}K_3^p(s)\Big]\mathbb{E}|x(s)|^p ds.
\end{aligned}
\tag{5.27}
$$

Substitute (5.27) into (5.23), then substitute the result into (5.18). By (5.13), we have

$$
\begin{aligned}
&\mathbb{E}V(x(t),r(t),t)\\
=&V_0+\int_0^t\mathbb{E}LV(x(s),r(s),s)ds\\
\leq&C-\int_0^t\Big[\lambda(s)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})\Big]\mathbb{E}|x(s)|^p ds\\
&+\int_0^t\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\mathbb{E}|x(s)-x(\delta_s)|^p ds\\
\leq&C-\int_0^t\beta(s)\mathbb{E}|x(s)|^p ds.
\end{aligned}
$$

By definition of V, we have that for $\forall t \geq 0$,

$$0 \leq \mathbb{E}V(x(t), r(t), t) \leq C - \int_0^t \beta(s)\mathbb{E}|x(s)|^p ds.$$

Then $\int_0^t \beta(t)\mathbb{E}|x(t)|^p dt \leq C$. Let $\underline{\beta} = \inf_{0 \leq t < T} \beta(t) (> 0)$. Then we have

$$\underline{\beta} \int_0^\infty \mathbb{E}|x(s)|^p ds \leq \int_0^\infty \beta(t)\mathbb{E}|x(t)|^p dt \leq C.$$

Hence we obtain assertion (5.15). □

We use the same observation interval in one subinterval. Observation interval sequence can be calculated by computer in three steps: [3]

1) The first step is to divide $[0, T]$ into $Z - 1$ subintervals and we propose two ways to do it.

One is simple even division. That is, all $Z - 1$ subintervals have the same length and $T_j = \frac{j-1}{Z-1}T$ for $1 \leq j \leq Z$.

The other way is by an auxiliary function $\tilde{\tau_a}(t)$, which satisfies

$$\begin{aligned} 0 \leq & \lambda(t) - \frac{1}{p}(\frac{p-1}{pl})^{p-1}\Big(K_3^p(t)2^{3p-2}(1 - e^{-\overline{\gamma}\tilde{\tau_a}(t)}) \\ & + \frac{2^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_3^p(t)}{1 - 8^{p-1}\tilde{\tau_a}^p(t)K_3^p(t)}[2^{3p-2}(1 - e^{-\overline{\gamma}\tilde{\tau_a}(t)}) + 2^{p-1}] \\ & \times [2^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_1^p(t) + \rho K_2^p(t) + 4^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_3^p(t)]\Big). \end{aligned} \tag{5.28}$$

We want to set the $\tilde{\tau_a}$ to make the right-hand-side of (5.28) as closer to 0 as possible. Then divide $[0, T]$ into $Z - 1$ subintervals according to the shape of $\tilde{\tau_a}(t)$. We want the supremum and the infimum of $\tilde{\tau_a}(t)$ in each subinterval are relatively close, i.e., the difference is not very big. This means, if $\tilde{\tau_a}(t)$ changes slowly over a time interval, then we can set a wide subinterval in this time interval; otherwise if $\tilde{\tau_a}(t)$ changes rapidly over a time interval, then we need to set several narrow subintervals in this

[3] This is only a sketch of one method to calculate the observation interval sequence, obviously there are other ways to guarantee the observation interval sequence satisfies the conditions in Theorem 5.3.1. Since the main research problem is on stabilization and the time is limited, we don't discuss the details about numerical methods here.

time interval.

Note that neither too narrow nor too wide subinterval is a good choice.

2) For the jth subinterval, (i.e., for $t \in [T_j, T_{j+1})$), find a function $\tilde{\tau_j}(t) \in (0, \frac{1}{8^{\frac{p-1}{p}} \hat{K}_{3t}})$ with $\inf_{t\in[T_j,T_{j+1})} \tilde{\tau_j}(t) > 0$ such that

$$\inf_{t\in[T_j,T_{j+1})} \beta(\tilde{\tau_j}(t), t) > 0, \tag{5.29}$$

where β has been defined in (5.13).

Find $\tilde{\tau_j}(t)$ for all $1 \leq j \leq Z - 1$.

3) For the jth subinterval where $1 \leq j \leq Z - 1$, choose a positive integer N_j such that

$$\frac{T_{j+1} - T_j}{N_j} < \inf_{t\in[T_j,T_{j+1})} \tilde{\tau_j}(t).$$

Then let

$$\underline{\kappa}_j = \frac{T_{j+1} - T_j}{N_j}.$$

So the observation interval is $\underline{\kappa}_j$ and we observe N_j times on the jth subinterval. In other words, the system is observed at $t = T_j, T_j + \underline{\kappa}_j, T_j + 2\underline{\kappa}_j, \cdots, T_j + N_j\underline{\kappa}_j$, where $T_j + N_j\underline{\kappa}_j = T_{j+1}$.

Find N_j and $\underline{\kappa}_j$ for all $1 \leq j \leq Z - 1$.

Consequently, our observation interval sequence for one period $[0, T)$ is:

$\tau_1 = \underline{\kappa}_1, \cdots, \tau_{N_1} = \underline{\kappa}_1,$

$\tau_{N_1+1} = \underline{\kappa}_2, \cdots, \tau_{N_1+N_2} = \underline{\kappa}_2,$

$\vdots$

$\tau_{N_1+\cdots+N_{Z-2}+1} = \underline{\kappa}_{Z-1}, \cdots, \tau_{N_1+\cdots+N_{Z-1}} = \underline{\kappa}_{Z-1}.$

Besides, we always observe once at $t = kT$ where $k = 0, 1, 2, \cdots$.

Now let me explain why we can find a positive sequence satisfying condition (5.14) and analyse the correlation between the observation interval sequence and system coefficients and mode switching rate.

When observation interval $\kappa_t = 0$, $\beta(t) = \lambda(t)$. When $\kappa_t < \frac{1}{8^{\frac{p-1}{p}} \hat{K}_{3t}}$, κ_t, $K_1(t)$, $K_2(t)$, $K_3(t)$ and $\overline{\gamma}$ are all negative related to $\beta(t)$. Increase of κ_t leads to decrease of

$\beta(t)$. To guarantee $\inf_{0\leq t<T}\beta(t)>0$, large $K_1(t)$, large $K_2(t)$ and large $K_3(t)$ would lead to small κ_t.

Through calculation and comparison between Theorems 3.5.1 and 5.3.1, we can find that, if we require β and the observation interval to be both constants like in Chapter 3, replacing $\lambda(t)$, $K_1(t)$, $K_2(t)$, $K_3(t)$ and $\hat{K}_{3t}$ in (5.13) with $\underline{\lambda}$, $\overline{K_1}, \overline{K_2}$, $\overline{K_3}$ and $\overline{K_3}$ respectively would give the same condition as (3.68) in Theorem 3.5.1. Therefore, (5.14) in Theorem 5.3.1 is actually a weaker condition than (3.68) in Theorem 3.5.1, as a result of consideration of the time-varying property. In addition, the observation interval in Theorem 5.3.1 can be larger than that in Theorem 3.5.1, expect that:
(1) $\lambda(t)$ reaches the minimum and $K_1(t)$, $K_2(t)$, $K_3(t)$ all reach their maximums at the same time point, say t^*, if the four functions are all continuous;
(2) otherwise, there is a t^* such that, as t goes to t^* from left or right or both sides, $\lambda(t)$ goes to $\inf_{0\leq t<T}\lambda(t)$ and $K_1(t)$, $K_2(t)$, $K_3(t)$ go to $\overline{K_1}, \overline{K_2}$, $\overline{K_3}$ respectively.
When either (1) or (2) happens, the observation interval κ_t for the subinterval which includes t^* need to meet the same condition as (3.68), and hence is the same as the observation interval calculated in Theorem 3.5.1.
Therefore, not only a positive sequence satisfying (5.14) exists, but also the smallest observation interval required in Theorem 5.3.1 is not smaller than the constant observation interval required in Theorem 3.5.1, as long as the subinterval of $[0,T]$ is not too short to restrict the observation interval.

Define

$$f_a(y) := \frac{2^{p-1}a^{\frac{p}{2}}y^p}{1-8^{p-1}a^py^p} \quad \text{for } a>0 \quad \text{on} \quad 0\leq y<\frac{1}{8^{\frac{p-1}{p}}a}.$$

Notice that f_a would be smaller if y is smaller [4]. So replacing $\hat{K}_{3t}$ in (5.13) with $K_3(t)$ in (5.28) allows $\tilde{\tau_a}(t)$ larger than the observation interval. Specifically,

$$0<\min_{1\leq j\leq Z-1}\underline{\kappa_j}\leq \min_{1\leq j\leq Z-1}\inf_{t\in[T_j,T_{j+1})}\tilde{\tau_j}(t)\leq \inf_{0\leq t<T}\tilde{\tau_a}(t).$$

Under the condition (5.14), large $K_1(t), K_2(t)$ and $K_3(t)$ would lead to small κ_t.

[4] $\frac{df_a}{dy}(y)=\frac{2^{p-1}p\tilde{\tau_a}^{\frac{p}{2}}y^{p-1}}{(1-8^{p-1}\tilde{\tau_a}^py^p)^2}\geq 0.$

Notice that large values of $K_1(t), K_2(t)$ and $K_3(t)$ indicate large values of coefficients, which imply rapid change of the solution. So this means when x changes fast, observations need to be more frequently. Similarly, large $\overline{\gamma}$ leads to small κ_t by (5.14). This means if the system mode switches rapidly, then observations need to be very frequently. These make sense in practice and agree with our intuition.

5.3.2 Moment and almost sure asymptotic stability

Theorem 5.3.2 *Fix the moment order $p \geq 2$. Under the same assumptions and conditions of Theorem 5.3.1, the solution of the controlled system (5.4) satisfies*

$$\lim_{t\to\infty} \mathbb{E}|x(t)|^p = 0 \tag{5.30}$$

and

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{5.31}$$

for any initial data $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. In other words, the controlled system (5.4) is asymptotically stable in pth moment and almost surely.

Assertion (5.30) can be proved in the same way as Theorem 3.2.2 and assertion (5.31) can be proved in the same way as Theorem 3.4 of [97]. For clarity, I put the complete proof is in Appendix A.2.

5.4 Exponential Stabilization

In Section 5.2, we discussed asymptotic stabilization and proved the convergence of $\mathbb{E}|x(t)|^p$ to 0. Now let's investigate its exponential stabilization.

Assumption 5.4.1 *Fix the moment order $p > 1$. Assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^p \leq V(x, i, t) \leq c_2|x|^p \tag{5.32}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times \mathbb{R}_+$.

Theorem 5.4.2 *Let Assumptions 5.2.1, 5.2.2, 5.2.3, 5.2.4 and 5.4.1 hold. Divide* $[0,T]$ *into* $Z-1$ *subintervals with* $T_1 = 0$ *and* $T_Z = T$. *Choose* $\kappa_t > 0$ *sufficiently small such that* $\kappa_t \le T_{j+1} - T_j$ *for* $t \in [T_j, T_{j+1})$ *where* $j = 1, 2, \cdots, Z-1$ *and the following two conditions hold:*

1) for $\forall t \in [0,T)$,

either

$$\begin{aligned}\varphi_t := \varphi(\kappa_t, t) = 8^{p-1}\kappa_t^p \hat{K}_{3t}^p + 16^{p-1}\kappa_t^{\frac{p}{2}}(1+\kappa_t^p\hat{K}_{3t}^p)(2^{p-1}\kappa_t^{\frac{p}{2}}\hat{K}_{1t}^p + \rho\hat{K}_{2t}^p) \\ \times \exp(4^{p-1}\kappa_t^p\hat{K}_{1t}^p + 4^{p-1}\kappa_t^{\frac{p}{2}}\nu\hat{K}_{2t}^p) \\ < 1, \end{aligned} \tag{5.33}$$

or

$$\begin{aligned}\varphi_t := \varphi(\kappa_t, t) = 8^{p-1}\kappa_t^p \hat{K}_{3t}^p + \frac{16^{p-1}\kappa_t^{\frac{p}{2}}(1+\kappa_t^p\hat{K}_{3t}^p)(2^{p-1}\kappa_t^{\frac{p}{2}}\hat{K}_{1}^p + \rho\hat{K}_{2}^p)}{1-4^{p-1}\kappa_t^{\frac{p}{2}}(\kappa_t^{\frac{p}{2}}\hat{K}_{1t}^p + \nu\hat{K}_{2t}^p)} \\ < 1 \ \ \text{and} \ \ 4^{p-1}\kappa_t^{\frac{p}{2}}(\kappa_t^{\frac{p}{2}}\hat{K}_{1t}^p + \nu\hat{K}_{2t}^p) < 1; \end{aligned} \tag{5.34}$$

2)

$$\int_0^T \tilde{\beta}(t)dt > 0, \tag{5.35}$$

where

$$\tilde{\beta}(t) := \tilde{\beta}(\kappa_t, t) = \frac{\lambda(t)}{c_2} - \frac{1}{c_2 p(1-\varphi_t)}(\frac{p-1}{pl})^{p-1}K_3^p(t)\Big[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_t}) + 2^{p-1}\varphi_t\Big]. \tag{5.36}$$

Then the solution of the controlled system (5.4) satisfies

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^p) \le -\frac{v}{T} \tag{5.37}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t)|) \le -\frac{v}{pT} \quad a.s. \tag{5.38}$$

for all initial data $x_0 \in \mathbb{R}^n$ *and* $r_0 \in \mathbb{S}$*, where*

$$v = \int_0^T \tilde{\beta}(s)ds.$$

Remark 5.4.3 *Notice that* T *is a period of* φ_t*, then* T *is also a period of* $\tilde{\beta}(t)$*. For* φ_t *defined in either (5.33) or (5.34) and for a fixed time point* t*, we have the following discussion. When* $\kappa_t = 0$*,* $\varphi_t = 0$*, then* $\tilde{\beta}(t) = \lambda(t)/c_2 > 0$*. If* κ_t *increases, both* φ_t *and* $\frac{\varphi_t}{1-\varphi_t}$ *increases, then* $\tilde{\beta}(t)$ *will decrease. So it's possible to find* $\kappa_t > 0$ *for* $0 \leq t < T$ *such that* $\int_0^T \tilde{\beta}(t)dt > 0$*.*

We will explain how to find such an observation interval sequence after the proof. We divide the proof into three steps.

Proof.

Step 1. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$.

Let

$$\hat{V}(x(t), r(t), t) = e^{\int_0^t \tilde{\beta}(s)ds} V(x(t), r(t), t).$$

We can obtain from the generalized Itô formula that

$$\begin{aligned}
&\mathbb{E}\hat{V}(x(t), r(t), t) \\
=&\mathbb{E}V_0 + \mathbb{E}\int_0^t L\hat{V}(x(s), r(s), s)ds \\
\leq&\mathbb{E}V_0 + \int_0^t e^{\int_0^s \tilde{\beta}(z)dz}[\mathbb{E}LV(x(s), r(s), s) + \tilde{\beta}(s)\mathbb{E}V(x(s), r(s), s)]ds, \qquad (5.39)
\end{aligned}$$

where $LV(x(s), r(s), s)$ has been defined in (5.16).

By (5.24), Assumptions 5.2.2 and 5.2.3, we have that for any $s \in [\delta_s, \delta_s + \kappa_s)$,

$$\begin{aligned}
&\mathbb{E}|x(s) - x(\delta_s)|^p \\
\le& 4^{p-1}\kappa_s^{p-1} \int_{\delta_s}^{s} K_3^p(z) dz \mathbb{E}|x(\delta_s)|^p \\
&+ 2^{p-1}\kappa_s^{\frac{p-2}{2}} \mathbb{E} \int_{\delta_s}^{s} [2^{p-1}\kappa_s^{\frac{p}{2}} K_1^p(z) + \rho K_2^p(z)] |x(z)|^p dz \\
\le& 4^{p-1}\kappa_s^p \hat{K}_{3s}^p \mathbb{E}|x(\delta_s)|^p \\
&+ 2^{p-1}\kappa_s^{\frac{p}{2}} [2^{p-1}\kappa_s^{\frac{p}{2}} \hat{K}_{1s}^p + \rho \hat{K}_{2s}^p] \mathbb{E}\Big(\sup_{\delta_s \le t \le s} |x(t)|^p \Big).
\end{aligned} \tag{5.40}$$

Step 2. We will prove that under either condition (5.33) or (5.34), we have

$$\mathbb{E}|x(s) - x(\delta_s)|^p \le \frac{\varphi_s}{1 - \varphi_s} \mathbb{E}|x(s)|^p, \tag{5.41}$$

for the corresponding φ_s.

Firstly, we prove it using condition (5.33).

By the definition of solutions of hybrid SDEs, the inequality in (2.17), Hölder's inequality and the Burkholder-Davis-Gundy inequality, we have that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t f(x(z),r(z),z)dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t u(x(\delta_z),r(\delta_z),z)]dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t g(x(z),r(z),z)dB(z)\Big|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p+(4\kappa_s)^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t[K_1^p(z)|x(z)|^p+K_3^p(z)|x(\delta_s)|^p]dz\Big)\\
&+4^{p-1}\kappa_s^{\frac{p-2}{2}}\nu\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t K_2^p(z)|x(z)|^pdz\Big)\\
\le&\Big[4^{p-1}+(4\kappa_s)^{p-1}\int_{\delta_s}^s K_3^p(z)dz\Big]\mathbb{E}|x(\delta_s)|^p\\
&+\Big[(4\kappa_s)^{p-1}\hat{K}_{1s}^p+4^{p-1}\kappa_s^{\frac{p-2}{2}}\nu\hat{K}_{2s}^p\Big]\int_{\delta_s}^s\mathbb{E}\Big(\sup_{\delta_s\le z\le t}|x(z)|^p\Big)dt
\end{aligned}
$$

Then the Gronwall inequality implies

$$
\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big)\le\Big[4^{p-1}+(4\kappa_s)^{p-1}\int_{\delta_s}^s K_3^p(z)dz\Big]\mathbb{E}|x(\delta_s)|^p\exp(4^{p-1}\kappa_s^p\hat{K}_{1s}^p+4^{p-1}\kappa_s^{\frac{p}{2}}\nu\hat{K}_{2s}^p).
\tag{5.42}
$$

Substituting this into (5.40) gives

$$
\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p\\
\le&4^{p-1}\kappa_s^{\frac{p}{2}}\Big[\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p+2^{p-1}(1+\kappa_s^p\hat{K}_{3s}^p)(2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p+\rho\hat{K}_{2s}^p)\\
&\times\exp(4^{p-1}\kappa_s^p\hat{K}_{1s}^p+4^{p-1}\kappa_s^{\frac{p}{2}}\nu\hat{K}_{2s}^p)\Big]\mathbb{E}|x(\delta_s)|^p.
\end{aligned}
$$

Noticing that

$$
\mathbb{E}|x(\delta_s)|^p\le 2^{p-1}\mathbb{E}|x(s)|^p+2^{p-1}\mathbb{E}|x(s)-x(\delta_s)|^p
$$

for all $p > 1$, we have

$$\mathbb{E}|x(s) - x(\delta_s)|^p \leq \varphi_s[\mathbb{E}|x(s)|^p + \mathbb{E}|x(s) - x(\delta_s)|^p],$$

where φ_s was been defined in (5.33). Rearranging it gives (5.41).

Alternatively, we prove it under condition (5.34).

By the definition of solutions of hybrid SDEs, the inequality in (2.17) and the Burkholder-Davis-Gundy inequality, we have that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta_s\leq t\leq s}|x(t)|^p\Big)\\
\leq &4^{p-1}\mathbb{E}|x(\delta_s)|^p + 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\leq t\leq s}\Big|\int_{\delta_s}^t f(x(z),r(z),z)dz\Big|^p\Big)\\
&+ 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\leq t\leq s}\Big|\int_{\delta_s}^t u(x(\delta_z),r(\delta_z),z)]dz\Big|^p\Big)\\
&+ 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\leq t\leq s}\Big|\int_{\delta_s}^t g(x(z),r(z),z)dB(z)\Big|^p\Big)\\
\leq &4^{p-1}\mathbb{E}|x(\delta_s)|^p + (4\kappa_s)^{p-1}\mathbb{E}\Big(\sup_{\delta_s\leq t\leq s}\int_{\delta_s}^t [K_1^p(z)|x(z)|^p + K_3^p(z)|x(\delta_s)|^p]dz\Big)\\
&+ 4^{p-1}\kappa_s^{\frac{p-2}{2}}\nu\mathbb{E}\Big(\sup_{\delta_s\leq t\leq s}\int_{\delta_s}^t K_2^p(z)|x(z)|^p dz\Big)\\
\leq &4^{p-1}(1+\kappa_s^p\hat{K}_{3s}^p)\mathbb{E}|x(\delta_s)|^p + 4^{p-1}\kappa_s^{\frac{p}{2}}(\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p + \nu\hat{K}_{2s}^p)\mathbb{E}\Big(\sup_{\delta_s\leq t\leq s}|x(t)|^p\Big).
\end{aligned} \tag{5.43}$$

The condition in (5.34) requires that $4^{p-1}\kappa_t^{\frac{p}{2}}(\kappa_t^{\frac{p}{2}}\hat{K}_{1t}^p + \nu\hat{K}_{2t}^p) < 1$. So we can rearrange (5.43) and get

$$\mathbb{E}\Big(\sup_{\delta_s\leq z\leq s}|x(z)|^p\Big) \leq \frac{4^{p-1}(1+\kappa_s^p\hat{K}_{3s}^p)}{1 - 4^{p-1}\kappa_s^{\frac{p}{2}}(\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p + \nu\hat{K}_{2s}^p)}\mathbb{E}|x(\delta_s)|^p. \tag{5.44}$$

Substituting this into (5.40) gives

$$\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p \\
\leq&\Big(4^{p-1}\kappa_s^p\hat{K}_{3s}^p+\frac{8^{p-1}\kappa_s^{\frac{p}{2}}(2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_1^p+\rho\hat{K}_2^p)(1+\kappa_s^p\hat{K}_{3s}^p)}{1-4^{p-1}\kappa_s^{\frac{p}{2}}(\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p+\nu\hat{K}_{2s}^p)}\Big)\mathbb{E}|x(\delta_s)|^p \\
\leq&\varphi_s(\mathbb{E}|x(s)|^p+\mathbb{E}|x(s)-x(\delta_s)|^p), \qquad (5.45)
\end{aligned}$$

where φ_s has been defined in (5.34).

Since condition (5.34) requires $\varphi_t<1$ for all $t>0$, we can rearrange (5.45) and obtain (5.41).

Step 3. Substitute (5.41) into (5.23). Then by (5.36), we have

$$\begin{aligned}
&\mathbb{E}LV(x(s),r(s),s) \\
\leq&-[\lambda(s)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})]\mathbb{E}|x(s)|^p \\
&+\frac{1}{p}(\frac{p-1}{pl})^{p-1}\frac{\varphi_s}{1-\varphi_s}K_3^p(s)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa_s})+2^{p-1}]\mathbb{E}|x(s)|^p \\
\leq&-c_2\tilde{\beta}(s)\mathbb{E}|x(s)|^p. \qquad (5.46)
\end{aligned}$$

Substitute (5.46) into (5.39). Then by Assumption 5.4.1, we have

$$\begin{aligned}
&\mathbb{E}\hat{V}(x(t),r(t),t) \\
\leq&\mathbb{E}V_0+\int_0^t e^{\int_0^s\tilde{\beta}(z)dz}[\mathbb{E}LV(x(s),r(s),s)+c_2\tilde{\beta}(s)\mathbb{E}|x(s)|^p]ds \\
\leq&\mathbb{E}V_0. \qquad (5.47)
\end{aligned}$$

Assumption 5.4.1 indicates that

$$c_1e^{\int_0^t\tilde{\beta}(s)ds}\mathbb{E}|x(t)|^p\leq\mathbb{E}\hat{V}(x(t),r(t),t)\leq\mathbb{E}V_0.$$

Then

$$\mathbb{E}|x(t)|^p\leq Ce^{-\int_0^t\tilde{\beta}(s)ds}.$$

So we have

$$\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|x(t)|^p) \le \limsup_{t\to\infty}\frac{-1}{t}\int_0^t \tilde{\beta}(s)ds = -\frac{v}{T}.$$

Hence we have obtained assertion (5.37).

Let $\epsilon \in (0, \frac{v}{2T})$ be arbitrary. Then (5.37) implies that there exists a constant $C > 0$ such that

$$\mathbb{E}|x(t)|^p \le Ce^{-(v/T-\epsilon)t} \qquad \text{for } \forall t \ge 0. \tag{5.48}$$

Notice that $4^{p-1}(1+\kappa_t^p \hat{K}_{3t}^p)[1-4^{p-1}\kappa_t^{\frac{p}{2}}(\kappa_t^{\frac{p}{2}}\hat{K}_{1t}^p + \nu\hat{K}_{2t}^p)]^{-1}$ in (5.44) is bounded. It follows from (5.42) and (5.44) that

$$\mathbb{E}\Big(\sup_{\delta_t \le s \le \delta_t+\kappa_t}|x(s)|^p\Big) \le C\mathbb{E}|x(\delta_t)|^p \le Ce^{-(v/T-\epsilon)\delta_t} \qquad \text{for } \forall t \ge 0. \tag{5.49}$$

Then use the Chebyshev inequality stated in Section 2.6 with $c = \exp(\frac{2\epsilon T - v}{pT})$, we have

$$\mathbb{P}\Big(\sup_{\delta_t \le s \le \delta_t+\kappa_t}|x(s)| \ge \exp[\frac{\delta_t}{p}(2\epsilon - \frac{v}{T})]\Big) \le Ce^{\epsilon\delta_t}.$$

The Borel-Cantelli lemma indicates that, there is a $t^* = t^*(\omega) > 0$ for almost all $\omega \in \Omega$ such that

$$\sup_{\delta_t \le s \le \delta_t+\kappa_t}|x(s)| < \exp[\frac{\delta_t}{p}(2\epsilon - \frac{v}{T})] \qquad \text{for } \forall t \ge t^*.$$

So

$$\log\frac{1}{t}(|x(t)|) < -(\frac{v}{T} - 2\epsilon)\frac{\delta_t}{pt}.$$

As $t \to \infty$,

$$\limsup_{t\to\infty}\frac{1}{t}\log(|x(t,\omega)|) \le -\frac{1}{p}(\frac{v}{T} - 2\epsilon) \;\; a.s.$$

Letting $\epsilon \to 0$ gives assertion (5.38). The proof is complete. □

Now we discuss how to divide $[0, T]$ and how to calculate the observation interval sequence. Similarly to asymptotic stabilization, we can either use even division or divide according to the shape of an auxiliary function; neither too narrow nor too wide subinterval is a good choice for partition; and again we use the same observation interval in one subinterval of $[0, T]$.

The difference is that here $\tilde{\beta}$ can be negative at some time points, we only need to guarantee that its integral over $[0, T]$ is positive. This gives flexibility on the setting of κ_t. We can choose to increase the shortest observation interval to avoid high frequency observations, or choose to make the large observation intervals even larger. This is illustrated in Example 5.6.2.

Observation interval sequence can be calculated by computer by the following four steps: [5]

1) Choose to satisfy condition (5.33) or (5.34).

Suppose we choose condition (5.33). Firstly, find a positive number $\overline{\kappa}$ such that

$$8^{p-1}\overline{\kappa}^p\overline{K_3}^p + 16^{p-1}\overline{\kappa}^{\frac{p}{2}}(1 + \overline{\kappa}^p\overline{K_3}^p)(2^{p-1}\overline{\kappa}^{\frac{p}{2}}\overline{K_1}^p + \rho\overline{K_2}^p)\exp(4^{p-1}\overline{\kappa}^p\overline{K_1}^p + 4^{p-1}\overline{\kappa}^{\frac{p}{2}}\nu\overline{K_2}^p) \le 1. \tag{5.50}$$

Secondly, let κ be a positive number to be determined. Define

$$\begin{aligned}\tilde{\varphi}(t) =& 8^{p-1}\kappa^p K_3^p(t) + 16^{p-1}\kappa^{\frac{p}{2}}[1 + \kappa^p K_3^p(t)][2^{p-1}\kappa^{\frac{p}{2}}K_1^p(t) + \rho K_2^p(t)] \\ &\times \exp[4^{p-1}\kappa^p K_1^p(t) + 4^{p-1}\kappa^{\frac{p}{2}}\nu K_2^p(t)]\end{aligned}$$

and then

$$\tilde{\beta}_a(t) = \frac{\lambda(t)}{c_2} - [c_2 p(1 - \tilde{\varphi}(t))]^{-1}(\frac{p-1}{pl})^{p-1}K_3^p(t)[2^{3p-2}(1 - e^{-\overline{\gamma}\kappa}) + 2^{p-1}\tilde{\varphi}(t)].$$

Alternatively, suppose we choose (5.34). Firstly, find a positive number $\overline{\kappa}$ such that

$$4^{p-1}\overline{\kappa}^{\frac{p}{2}}(\overline{\kappa}^{\frac{p}{2}}\overline{K_1}^p + \nu\overline{K_2}^p) < 1$$

[5] Again, this is only a sketch of one method to calculate the observation interval sequence. Due to research subject and time limit, we don't discuss the details about numerical methods.

and

$$8^{p-1}\overline{\kappa}^p\overline{K_3}^p + \frac{16^{p-1}\overline{\kappa}^{\frac{p}{2}}(1+\overline{\kappa}^p\overline{K_3}^p)(2^{p-1}\overline{\kappa}^{\frac{p}{2}}\overline{K_1}^p + \rho\overline{K_2}^p)}{1-4^{p-1}\overline{\kappa}^{\frac{p}{2}}(\overline{\kappa}^{\frac{p}{2}}\overline{K_1}^p + \nu\overline{K_2}^p)} < 1.$$

Secondly, let κ be a positive number to be determined. Define

$$\tilde{\varphi}(t) = 8^{p-1}\kappa^p K_3^p(t) + \frac{16^{p-1}\kappa^{\frac{p}{2}}[1+\kappa^p K_3^p(t)][2^{p-1}\kappa^{\frac{p}{2}}K_1^p(t) + \rho K_2^p(t)]}{1-4^{p-1}\kappa^{\frac{p}{2}}[\kappa^{\frac{p}{2}}K_1^p(t) + \nu K_2^p(t)]}$$

and

$$\tilde{\beta}_a(t) = \frac{\lambda(t)}{c_2} - [c_2 p(1-\tilde{\varphi}(t))]^{-1}(\frac{p-1}{pl})^{p-1}K_3^p(t)[2^{3p-2}(1-e^{-\overline{\gamma}\kappa}) + 2^{p-1}\tilde{\varphi}(t)].$$

For choice of either (5.33) or (5.34), using corresponding definitions above, choose a positive number $\kappa < \overline{\kappa}$ such that $\int_0^T \tilde{\beta}_a(t)dt > 0$.

2) The second step is to divide $[0,T]$ into $Z-1$ subintervals. Similarly as discussed in Section 5.2.1 for asymptotic stabilization, we can simply use even division or divide according to the shape of $\tilde{\beta}_a(t)$, in which case we want the supremum and the infimum of $\tilde{\beta}_a(t)$ in each subinterval are relatively close.

Then set a sequence of $Z-1$ numbers $\{\underline{\tilde{\beta}}_j\}_{1\le j\le Z-1}$ such that

$$\underline{\tilde{\beta}}_j \le \inf_{t\in[T_j,T_{j+1})}\tilde{\beta}_a(t) \quad \text{and} \quad \sum_{j=1}^{Z-1}\underline{\tilde{\beta}}_j(T_{j+1}-T_j) \ge 0.$$

3) Find a function $\tilde{\tau}(t)$ with $\inf_{t\in[0,T)}\tilde{\tau}(t) > 0$ such that

$$\tilde{\beta}(\tilde{\tau}(t),t) \ge \underline{\tilde{\beta}}_j \quad \text{for } j = 1,2,\cdots,Z-1. \tag{5.51}$$

We want to set $\tilde{\tau}(t)$ to make the two sides of (5.51) as closer as possible.

Then let $\tilde{\tau}_j = \inf_{t\in[T_j,T_{j+1})}\tilde{\tau}(t)$, i.e. the infimum of $\tilde{\tau}$ over the jth subinterval, for $j = 1,\cdots,Z-1$.

4) This step is similar to the step 3 in Section 5.2.1 for asymptotic stabilization. For the jth subinterval, choose a positive integer N_j such that $\frac{T_{j+1}-T_j}{N_j} < \tilde{\tau}_j$, then let

$\underline{\kappa}_j = \frac{T_{j+1}-T_j}{N_j}$.

Note that on the jth subinterval, $\underline{\kappa}_j$ also needs to satisfy either condition (5.33) or (5.34), which has been determined in Step 1 and can be guaranteed if $\max_{j=1}^{Z-1} \underline{\kappa}_j < \overline{\kappa}$. Find N_j and $\underline{\kappa}_j$ for all $1 \le j \le Z-1$. Finally the observation interval is $\underline{\kappa}_j$ and we observe the system N_j times over the jth subinterval.

Notice $\underline{\kappa}_j < \tilde{\tau}_j \le \tilde{\tau}(t)$ for $t \in [T_j, T_{j+1})$, $j = 1, \cdots, Z-1$ and $\tilde{\beta}(\kappa_t, t)$ defined in (5.36) is negatively related to κ_t. Then we have

$$\begin{aligned}\int_0^T \tilde{\beta}(\kappa_t, t)dt &= \sum_{j=1}^{Z-1} \int_{T_j}^{T_{j+1}} \tilde{\beta}(\underline{\kappa}_j, t)dt \\ &> \sum_{j=1}^{Z-1} \int_{T_j}^{T_{j+1}} \tilde{\beta}(\tilde{\tau}_j, t)dt \ge \sum_{j=1}^{Z-1} \int_{T_j}^{T_{j+1}} \tilde{\beta}(\tilde{\tau}(t), t)dt = \sum_{j=1}^{Z-1} \underline{\tilde{\beta}}_j (T_{j+1} - T_j) \ge 0.\end{aligned}$$

So condition (5.35) can be guaranteed if we follow the above four steps.

The 4-step procedure is only one way to guarantee that all the requirements in Theorem 5.4.2 on observation intervals are met.

(5.35) is a condition on the integral over one period instead of on every time point. This gives flexibility for calculation of observation intervals. The flexibility comes from settings of the partition of $[0, T]$ and $\{\underline{\tilde{\beta}}_j\}_{1 \le j \le Z-1}$. By adjusting the partition of $[0, T]$ and $\underline{\tilde{\beta}}_j$'s for some $j \in [1, Z-1]$, we can change the $\min_{j \ge 1} \tau_j$ or the $\max_{j \ge 1} \tau_j$, or we can choose the observation interval for a specific subinterval of $[0, T]$.

$\tilde{\beta}(t)$ is negative related to φ_t, K_3, $\overline{\gamma}$ and κ_t. φ_t defined in either (5.33) or (5.34) is positive related to K_1, K_2, K_3 and κ_t. So when K_1, K_2, K_3 or κ_t increases, $\tilde{\beta}(t)$ will decrease. Therefore, large $K_1(t), K_2(t)$, $K_3(t)$ and $\overline{\gamma}$ tend to yield small κ_t. However, it's not always necessary here, as long as the negative values of $\tilde{\beta}(t)$ at some time points can be compensated by its positive values at other time points and its integral over $[0, T]$ is positive.

For exponential stabilization, the observation intervals required in Theorem 5.4.2 can be larger than the constant observation interval required in Theorem 3.3.2. As an extreme example, let the periodic system coefficients $f(x, i, t) = g(x, i, t) = 0$ for a time interval, say $[t_1, t_2]$. Then this interval can be set as one subinterval of one period $[0, T]$. As $\hat{K}_{1t} = \hat{K}_{2t} = 0$ for $t \in [t_1, t_2]$ and there is no lower bound requirement on $\tilde{\beta}(t)$ for $t \in [t_1, t_2]$ (the requirement is only on its integral), then the only requirement for the observation interval for this subinterval (i.e. κ_t for $t \in [t_1, t_2]$) is simply $8^{p-1}\kappa_t^p \hat{K}_{3t}^p < 1$. Moreover, we can stop controlling as well as observing when $t \in [t_1, t_2]$. That is, $u(x, i, t) = 0$ for $t \in [t_1, t_2]$ and the observation interval is $t_2 - t_1$. The controlled system stops changing and behaves like 'frozen' in this interval.

5.4.1 Corollary

Similarly to Section 3.3.1, we propose a corollary to state how to use quadratic form of Lyapunov functions to stabilize an unstable hybrid system.

Assumption 5.4.4 *Assume that there exist positive-definite symmetric matrices $Q_i \in \mathbb{R}^{n\times n}$ ($i \in \mathbb{S}$) and a positive periodic function $b(t)$ with $\inf_{0\le t<T} b(t) > 0$ such that*

$$\begin{aligned} &p(x^TQ_ix)^{\frac{p}{2}-1}\Big(x^TQ_i[f(x,i,t)+u(x,i,t)] + \frac{1}{2}\text{trace}[g^T(x,i,t)Q_ig(x,i,t)]\Big) \\ &\quad + p(\frac{p}{2}-1)[x^TQ_ix]^{\frac{p}{2}-2}|g^TQ_ix|^2 + \sum_{j=1}^N \gamma_{ij}[x^TQ_jx]^{\frac{p}{2}} \\ &\le -b(t)|x|^p, \end{aligned} \tag{5.52}$$

for all $(x, i, t) \in \mathbb{R}^n \times \mathbb{S} \times [0, T)$.

We can see that T is a period of $b(t)$. Let $\underline{b} = \inf_{0\le t<T} b(t)$.

Corollary 5.4.5 *If Assumption 5.2.4 are replaced by Assumption 5.4.4, then Theorems 5.3.1 and 5.3.2 still hold for $p \ge 2$ with $\lambda(t) = b(t) - ld$ where $d = [p \max_{i\in\mathbb{S}} \lambda_{\max}^{\frac{p}{2}}(Q_i)]^{\frac{p}{p-1}}$ and $l < \underline{b}/d$.*

Corollary 5.4.6 *If Assumptions 5.2.4 and 5.4.1 are replaced by Assumption 5.4.4, then Theorem 5.4.2 still holds for $p \ge 2$ with $c_2 = \max_{i\in\mathbb{S}} \lambda_{\max}^{\frac{p}{2}}(Q_i)$, $\lambda(t) = b(t) - ld$ where $d = (pc_2)^{\frac{p}{p-1}}$ and $l < \underline{b}/d$.*

Similarly as in Chapter 3, to obtain relatively large observation intervals, we can choose l to maximize $\underline{b}l^{p-1} - dl^p$.

5.5 Single-mode SDEs

The new established stabilization theory for periodic single-mode SDEs by feedback control based on discrete-time periodic observations is similar to that for hybrid SDEs stated in previous sections.

Consider an n-dimensional periodic SDE

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \tag{5.53}$$

on $t \geq 0$, with initial value $x(0) = x_0 \in \mathbb{R}^n$. Here

$$f : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n \quad \text{and} \quad g : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^{n\times m}.$$

If the given system is not stable, then we can design a feedback control $u : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}^n$ to make the controlled system

$$dx(t) = \big(f(x(t), t) + u(x(\delta_t), t)\big)dt + g(x(t), t)dB(t) \tag{5.54}$$

stable.

Assumption 5.5.1 *Assume that $f(x, t)$, $g(x, t)$ and $u(x, t)$ are all periodic with respect to time t. Assume f, g, u and κ_t have a common period T.*

Assumption 5.5.2 *Assume that the coefficients $f(x, t)$ and $g(x, t)$ are both locally Lipschitz continuous on x. We also assume that $f(x, t)$ and $g(x, t)$ both satisfy the following linear growth conditions on x*

$$|f(x, t)| \leq K_1(t)|x| \quad \textit{and} \quad |g(x, t)| \leq K_2(t)|x| \tag{5.55}$$

for all $(x, t) \in \mathbb{R}^n \times \mathbb{R}_+$, where $K_1(t)$ and $K_2(t)$ are periodic bounded non-negative functions with period T.

Assumption 5.5.3 *Assume that the controller function $u(x, t)$ is globally Lipschitz* *continuous on x and also satisfies the following linear growth conditions on x*

$$|u(x,t) - u(y,t)| \leq K_3(t)|x-y| \tag{5.56}$$

for all $(x,y,t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}_+$, where $K_3(t)$ is a periodic bounded non-negative function with period T. We also assume that

$$u(0,t) = 0 \tag{5.57}$$

for all $t \in \mathbb{R}_+$.

Let $V(x(t),t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R}_+)$ be a Lyapunov function periodic with respect to t. Then define an operator $\mathcal{L}V : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\mathcal{L}V(x,t) = V_t(x,t) + V_x(x,t)[f(x,t) + u(x,t)] + \frac{1}{2}\text{trace}[g^T(x,t)V_{xx}(x,t)g(x,t)]. \tag{5.58}$$

Assumption 5.5.4 *Assume that there is a Lyapunov function $V(x,t)$, which is periodic with respect to time t with period T, constants $l > 0, p > 1$ and a periodic function $\lambda(t)$ with $\inf_{t\geq 0} \lambda(t) > 0$ such that*

$$\mathcal{L}V(x,t) + l|V_x(x,t)|^{\frac{p}{p-1}} \leq -\lambda(t)|x|^p \tag{5.59}$$

for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}_+$.

Divide $[0,T]$ into $Z-1$ subintervals, where $Z \geq 2$ is an arbitrary integer, by choosing a partition $\{T_j\}_{1\leq j \leq Z}$ with $T_1 = 0$ and $T_Z = T$. Then let $\hat{K}_{1t}$, $\hat{K}_{2t}$ and $\hat{K}_{3t}$ be the same as defined in (5.12).

Theorem 5.5.5 *Fix the moment order $p > 1$. Let Assumptions 5.5.1, 5.5.2, 5.5.3 and 5.5.4 hold. Divide $[0,T]$ into $Z-1$ subintervals with $T_1 = 0$ and $T_Z = T$. Choose $\kappa_t > 0$ sufficiently small such that $\kappa_t \leq T_{j+1} - T_j$ for $t \in [T_j, T_{j+1})$ where $j = 1, 2, \cdots, Z-1$*

and

$$\inf_{0\le t<T}\beta(t)>0 \quad and \quad \sup_{t\ge 0}(\kappa_t\hat{K}_{3t})<8^{-\frac{p-1}{p}}, \tag{5.60}$$

where

$$\beta(t):=\beta(\kappa_t,t)=\lambda(t)-\frac{[2(p-1)]^{p-1}\hat{K}_{3s}^p}{p^p(1-8^{p-1}\kappa_t^p\hat{K}_{3t}^p)}l^{1-p}\kappa_t^{\frac{p}{2}} \\ \times\Big[2^{p-1}\kappa_t^{\frac{p}{2}}K_1^p(t)+\rho K_2^p(t)+4^{p-1}\kappa_t^{\frac{p}{2}}K_3^p(t)\Big]. \tag{5.61}$$

Then the controlled system (5.54) satisfies (5.15) for all initial data $x_0\in\mathbb{R}^n$.

Theorem 5.5.5 can be proved in the same way as Theorem 5.3.1 and the complete proof is in Appendix A.3.

Theorem 5.5.6 *Fix the moment order $p\ge 2$. Under the same assumptions of Theorem 5.5.5, the solution of the controlled system (5.54) satisfies (5.30) and (5.31) for any initial data $x_0\in\mathbb{R}^n$. In other words, the controlled system (5.54) is asymptotically stable in pth moment and almost surely.*

Theorem 5.5.6 can be proved in the same way as Theorem 5.3.2.

Assumption 5.5.7 *Assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^p\le V(x,t)\le c_2|x|^p \tag{5.62}$$

for all $(x,t)\in\mathbb{R}^n\times\mathbb{R}_+$.

Theorem 5.5.8 *Fix the moment order $p>1$. Let Assumptions 5.5.1, 5.5.2, 5.5.3, 5.5.4 and 5.5.7 hold. Divide $[0,T]$ into $Z-1$ subintervals with $T_1=0$ and $T_Z=T$. Choose $\kappa_t>0$ sufficiently small such that $\kappa_t\le T_{j+1}-T_j$ for $t\in[T_j,T_{j+1})$ where $j=1,2,\cdots,Z-1$ and two conditions hold. The 1st condition is the same as stated in Theorem 5.4.2, i.e., (5.33) or (5.34). The 2nd condition is*

$$\int_0^T\tilde{\beta}(t)dt>0, \tag{5.63}$$

where

$$\tilde{\beta}(t) := \tilde{\beta}(\kappa_t, t) = \frac{\lambda(t)}{c_2} - \frac{1}{c_2 p}(\frac{p-1}{pl})^{p-1}\frac{\varphi_t}{1-\varphi_t}K_3^p(t). \tag{5.64}$$

Then the solution of controlled system (5.54) satisfies (5.37) and (5.38) for all initial data $x_0 \in \mathbb{R}^n$. In other words, the controlled system (5.54) is exponentially stable in pth moment and almost surely.

Theorem 5.5.8 can be proved in the same way as Theorem 5.4.2 and the complete proof is in Appendix A.3.

Calculation of the observation interval sequence for stabilization of single-mode SDE system is similar to that of hybrid SDEs. The difference is definition of $\beta(t)$ for asymptotic stabilization and definition of $\tilde{\beta}(t)$ for exponential stabilization.

For Lyapunov functions of the frequently used form $V(x(t), t) = (x^T(t)Qx(t))^{\frac{p}{2}}$ where Q is a positive-definite $n \times n$ matrices, we have the following alternative assumption and corollaries.

Assumption 5.5.9 *Assume that there exist positive-definite symmetric matrix $Q \in \mathbb{R}^{n\times n}$ and a positive periodic function $b(t)$ with $\inf_{0\le t<T} b(t) > 0$ such that*

$$\begin{aligned} &p(x^TQx)^{\frac{p}{2}-1}\Big(x^TQ[f(x,t)+u(x,t)] + \frac{1}{2}\text{trace}[g^T(x,t)Qg(x,t)]\Big) \\ &\quad + p(\frac{p}{2}-1)[x^TQx]^{\frac{p}{2}-2}|g^TQx|^2 \le -b(t)|x|^p, \end{aligned} \tag{5.65}$$

for all $(x,t) \in \mathbb{R}^n \times [0,T)$.

Corollary 5.5.10 *If Assumption 5.5.4 are replaced by Assumption 5.5.9, then Theorems 5.5.5 and 5.5.6 still hold for $p \ge 2$, with $\lambda(t) = b(t) - ld$ where $d = [p\lambda_{\max}^{\frac{p}{2}}(Q)]^{\frac{p}{p-1}}$, $l < \underline{b}/d$ and $\underline{b} = \inf_{0\le t<T} b(t)$.*

Corollary 5.5.11 *If Assumptions 5.5.4 and 5.5.7 are replaced by Assumption 5.5.9, then Theorem 5.5.8 still holds for $p \ge 2$ with $c_2 = \lambda_{\max}^{\frac{p}{2}}(Q)$, $\lambda(t) = b(t) - ld$ where $d = (pc_2)^{\frac{p}{p-1}}$ and $l < \underline{b}/d$.*

Similarly as before, to obtain relatively large observation intervals, we can choose l to maximize $\underline{b}l^{p-1} - dl^p$.

5.6 Examples

Now we illustrate our theory with two examples.

Example 5.6.1 Now we consider a 2-dimensional SDE

$$dx(t) = F(x(t), t)x(t)dt + G(t)x(t)dB(t) \tag{5.66}$$

on $t \geq 0$, where $x(t) = (x_1(t), x_2(t))^T$ and $B(t)$ is a scalar Brownian motion. Here the coefficients are

$$F(x,t) = [1.5 + \cos(\frac{\pi}{6}t)] \begin{bmatrix} 0 & \sin(x_1) \\ \cos(x_2) & 0 \end{bmatrix}$$

and

$$G(t) = [1 + \sin(\frac{\pi}{6}t - 2.8)] \begin{bmatrix} 0.5 & -0.5 \\ -0.5 & 0.5 \end{bmatrix}.$$

The upper plot in Figure 5.1 shows that the original system (5.66) is neither mean square asymptotically stable nor mean square exponentially stable.

Coefficients $f(x,t) = F(x(t),t)x(t)$ and $g(x,t) = G(t)x(t)$ are time-periodic with common period 12. Assumption 5.5.2 holds with $K_1(t) = 1.5 + \cos(\frac{\pi}{6}t)$ and $K_2(t) = 1+\sin(\frac{\pi}{6}t-2.8)$. Then we can design a feedback control $u(x,t)$ and calculate observation intervals to make the controlled system

$$dx(t) = [F(x(t), t)x(t) + u(x(\delta_t), t)]dt + G(t)x(t)dB(t) \tag{5.67}$$

mean square asymptotically stable and furthermore, mean square exponentially stable.

Let $u(x,t) = A(x,t)x$ where $A : \mathbb{R}^2 \times \mathbb{R}_+ \to \mathbb{R}^{2\times 2}$ with bounded norm. Then assumption 5.5.3 holds with $K_3(t) = \max_{x \in R^2} \|A(x,t)\|$. Let $V(x,t) = x^T Qx$ where $Q = I$, the 2×2 identity matrix, then Corollaries 5.5.10 and 5.5.11 can be applied.

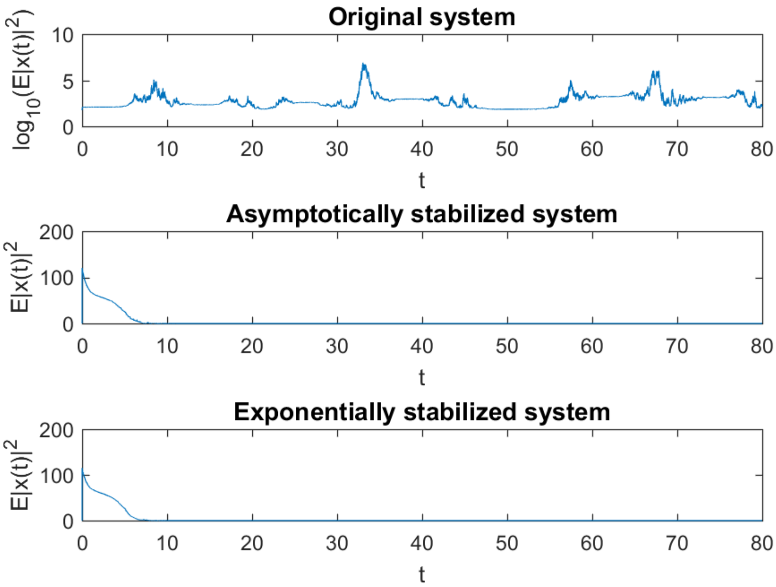

Figure 5.1: Sample averages of $|x|^2$ from 500 simulated paths for original and controlled systems by the Euler-Maruyama method with step size $1e-5$ and random initial values. Upper plot is for original system (5.66); middle and bottom plots are controlled system (5.67) for mean square asymptotically and exponentially stabilization respectively with corresponding observation intervals.

The left-hand-side of (5.65) is

$$\begin{aligned} &2[x^T(f(x,t)+u(x,t))+\frac{1}{2}g^T(x,t)g(x,t)] \\ =&2x^T(F(x,t)+A(x,t))x+x^TG^T(t)G(t)x \\ \leq&x^T\tilde{Q}x\leq\lambda_{\max}(\tilde{Q})|x|^2, \end{aligned} \tag{5.68}$$

where

$$\tilde{Q}=F(x,t)+F^T(x,t)+A(x,t)+A^T(x,t)+G^T(t)G(t).$$

To satisfy Assumption 5.5.9, we design $A(x,t)$ to make $\tilde{Q}$ negative definite.

Let

$$A(x,t)=\begin{bmatrix} B_1(x,t) & B_2(x,t) \\ B_2(x,t) & B_1(x,t)+0.1\sin(\frac{\pi}{6}t) \end{bmatrix},$$

where $B_1(x,t) = -0.25K_2^2(t) - 0.5$ and $B_2(x,t) = -0.25K_1(t)K_2^2(t)[\sin(x_1) + \cos(x_2)]$. Then

$$\tilde{Q} = \begin{bmatrix} -1 & 0 \\ 0 & -1 + 0.2\sin(\frac{\pi}{6}t) \end{bmatrix}.$$

So $b(t) = -\lambda_{\max}(\tilde{Q}) = \min\{1 - 0.2\sin(\frac{\pi}{6}t), 1\}$, $\underline{b} = 0.8$ and $d = 4$.

Choose $l = 0.1$, then $\lambda(t) = b(t) - 0.4$. Obviously, $T = 12$ is a common period of f, g, u, b and λ. $K_1(t)$, $K_2(t)$, $K_3(t)$ and $\lambda(t)$ are shown in Figure 5.2.

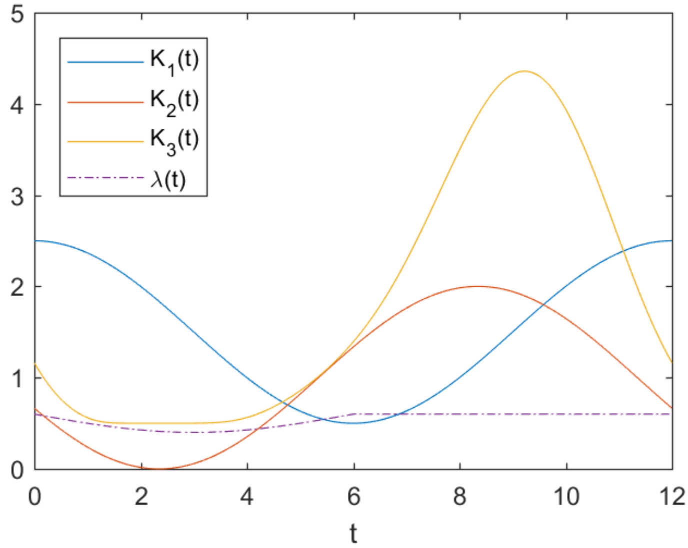

Figure 5.2: Plot of parameters.

Now we calculate $\{\tau_j\}_{j\geq 1}$ for mean square asymptotic stabilization. The positive function $\tilde{\tau_a}(t)$ is calculated by computer as the unique positive root to equation

$$\lambda(t) = \frac{[2(p-1)]^{p-1}}{p^p(1 - 8^{p-1}\tilde{\tau_a}^p(t)K_3^p(t))} l^{1-p} K_3^p(t)\tilde{\tau_a}^{\frac{p}{2}}(t) \times \left[2^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_1^p(t) + \rho K_2^p(t) + 4^{p-1}\tilde{\tau_a}^{\frac{p}{2}}(t)K_3^p(t)\right]$$

and it's shown in the left plot of Figure 5.3. According to the shape of $\tilde{\tau_a}(t)$, we divide $[0, 12]$ into 10 subintervals. The calculated observation interval for each subinterval can be seen in the left plot of Figure 5.3. Table 5.1 clearly shows the partition, the observation interval and observation times for each subinterval.

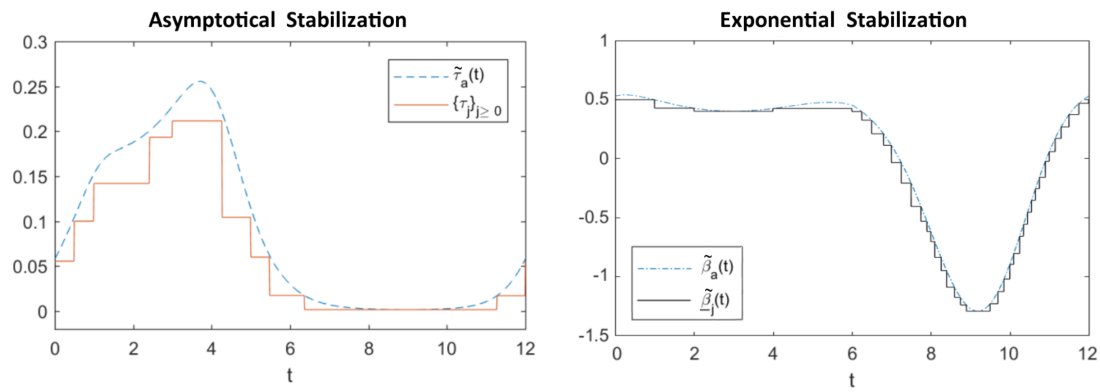

Figure 5.3: Auxiliary functions for stabilization. The left plot shows auxiliary function in blue and observation intervals in orange for asymptotic stabilization; the right plot shows the auxiliary function in blue and its infimum on each subinterval in black for exponential stabilization.

Table 5.1: Partition of the common period, observation interval and observation times in each subinterval.

Subinterval	Observation interval	Observation times
[0, 0.5)	0.05556	9
[0.5, 1)	0.1	5
[1, 2.42)	0.142	10
[2.42, 3)	0.19333	3
[3, 4.27)	0.21167	6
[4.27, 5)	0.10429	7
[5, 5.48)	0.06	8
[5.48, 6.37)	0.01745	51
[6.37, 11.28)	0.00164	2988
[11.28, 12)	0.01714	42

Table 5.1 shows that on the first subinterval $[0, 0.5)$, the system needs to be observed once every 0.05556 time units for 9 times; the 6th row means that when $t \in [3 + 12k, 4.27 + 12k)$ for $k = 0, 1, 2, \cdots$, the system needs to be observed once every 0.21167 time units for 6 times.

This result yields $\inf_{0 \le t < T} \beta(t) = 1.98e - 06 > 0$. Therefore, all the conditions on observation intervals have been satisfied. By Corollary 5.5.10, the controlled system (5.67) with $\{\tau_j\}_{j \ge 1}$ presented in Table 5.1 is mean square and almost surely asymptotically stable. The sample moment of $|x|^2$ in the middle plot of Figure 5.1 agrees with it.

Now let's calculate observation intervals for mean square exponential stabilization using condition (5.33). We start by calculating $\tilde{\beta}_a(t)$ with $\kappa = 0.0012$. Specifically,

$$\begin{aligned}\tilde{\beta}_a(t) &= \frac{\lambda(t)}{c_2} - \frac{1}{c_2 p}(\frac{p-1}{pl})^{p-1}\frac{\tilde{\varphi}(t)}{1-\tilde{\varphi}(t)}K_3^p(t) \\ &= \lambda(t) - \frac{\tilde{\varphi}(t)}{4l[1-\tilde{\varphi}(t)]}K_3^2(t)\end{aligned}$$

where [6]

$$\begin{aligned}\tilde{\varphi}(t) =& 8^{p-1}\kappa^p K_3^p(t) + 16^{p-1}\kappa^{\frac{p}{2}}[1+\kappa^p K_3^p(t)][2^{p-1}\kappa^{\frac{p}{2}}K_1^p(t) + \rho K_2^p(t)] \\ & \times \exp[4^{p-1}\kappa^p K_1^p(t) + 4^{p-1}\kappa^{\frac{p}{2}}\nu K_2^p(t)] \\ =& 8\kappa^2 K_3^2(t) + 16\kappa[1+\kappa^2 K_3^2(t)][2\kappa K_1^2(t) + K_2^2(t)]\exp[4\kappa^2 K_1^2(t) + 16\kappa K_2^2(t)].\end{aligned}$$

We get $\int_0^{12}\tilde{\beta}_a(t)dt > 0$.

Then according to the shape of $\tilde{\beta}_a(t)$, which is plotted in the right plot of Figure 5.3, we divide $[0,12]$ into 38 subintervals. Since $\sum_{j=1}^{38}\inf_{T_j\le t\le T_{j+1}}\tilde{\beta}_a(t)(T_{j+1}-T_j) > 0$, we simply set $\underline{\tilde{\beta}}_j = \inf_{T_j\le t\le T_{j+1}}\tilde{\beta}_a(t)$ for $1 \le j \le 38$. This is shown in the right plot of Figure 5.3.

Our partition and setting of $\{\underline{\tilde{\beta}}_j\}_{j\ge1}$ may be not the best choice. Wide subintervals and large $\underline{\tilde{\beta}}_j$'s can make observation intervals small. After adjusting, we divide $[0,12]$ into 44 subintervals. We found the observation intervals for $t \in [6,12)$ are either 0.0011 or 0.0012. So we merge them and set the observation interval for $t \in [6,12)$ to be 0.001. Consequently, we have 12 subintervals for observations in total and the results are presented in Table 5.2.

The results yield $\max_{0\le t<12}\varphi_t = 0.1872 < 1$ and $\int_0^{12}\tilde{\beta}(t)dt = 0.5373 > 0$. So all conditions on observation intervals are satisfied. By Corollary 5.5.11, the controlled system (5.67) with observation intervals presented in Table 5.2 is mean square and almost surely exponentially stable, which is in accordance with the bottom plot in Figure 5.1.

[6] $c_2 = 1$ when $Q = I$, $\rho = 1$ and $\nu = 4$ when $p = 2$.

Table 5.2: Partition of the common period, observation interval and observation times in each subinterval.

Subinterval	Observation interval	Observation times
[0, 1)	0.0016	625
[1, 1.3)	0.01765	17
[1.3, 1.5)	0.025	8
[1.5,2)	0.03333	15
[2,2.7)	0.06364	11
[2.7, 3.1)	0.05714	7
[3.1, 3.5)	0.05	8
[3.5, 4)	0.03571	14
[4, 5)	0.0024	417
[5, 6)	0.0014	715
[6, 7.9)	0.0012	1584
[7.9, 12)	0.001	4100

Table 5.3: Comparison between our time-varying observation intervals and the constant observation interval.

asymp.min	asymp.max	exp.min	exp.max	constant τ
0.00164	0.21167	0.001	0.0636	8.86e-04

The first four columns of Table 5.3 summarize our shortest and widest observation intervals for mean square asymptotic and exponential stabilization. The largest observation intervals are around 129 and 63 times of the shortest ones for asymptotic and exponential stabilization respectively. The last column of Table 5.3 is the constant observation interval given by existing theory, i.e., using constant parameters as in Chapter 1. Obviously, the new theory gives much better results than existing one. Our smallest observation intervals for both asymptotic and exponential stabilization are still larger than $8.86e-04$. This is because $K_1(t)$, $K_2(t)$ and $K_3(t)$ reach the maximum values and $\lambda(t)$ reach the minimum value at different time points, which can be seen from Figure 5.2. Condition (5.60) and definition of $\beta(t)$ in (5.61) indicate that, the observation interval κ_t is negatively related to $K_1(t)$, $K_2(t)$, $K_3(t)$ and positively related to $\lambda(t)$.

If the time unit is two hours, that is, period T is one day, then no matter for asymptotic or exponential stabilization in mean square, existing theory requires observations once every 6 seconds for all time. While the new results (using time-varying parameters in this chapter) enable the lowest observation frequency to be once every 25min 24s lasting 2h 32min 24s, for asymptotic stabilization. If observation is carried out by a monitor, then it can be switched off during that interval. For exponential stabilization, the new results require observation frequencies between once every 7min 38s and once every 7s.

If the time unit is 30 days, that is, period T is around one year, then existing theory requires observations to be once every 38min 16s for all time, to achieve the mean square asymptotic and exponential stabilization. However, the new results require the lowest observation frequencies to be: once every 6 day 8h 24min lasting around 38 days for asymptotic stabilization, and once every 1 day 21h 47min 31s lasting 21 days for exponential stabilization. If observation is carried out by employees, then this means holidays for them and lower observation frequencies also indicate fewer employees are needed.

Example 5.6.2 Now we consider a 2-dimensional hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dB(t) \tag{5.69}$$

on $t \geq 0$, where $B(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with the generator matrix

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

The subsystem for mode 1 here is the same as system (5.66) and the subsystem for mode 2 has coefficients

$$f(x, 2, t) = [1.5 + \sin(\frac{\pi}{6}t)] \begin{bmatrix} \sin(x_2) \\ \cos(x_1) \end{bmatrix} x$$

and

$$g(x, 2, t) = \frac{1}{2\sqrt{2}}[1 + \cos(\frac{\pi}{6}t + 2.8)] \begin{bmatrix} \sqrt{3x_1^2 + x_2^2} \\ \sqrt{x_1^2 + 3x_2^2} \end{bmatrix}.$$

Since the original system (5.69) is not mean square exponentially stable, which can be seen from the upper plot in Figure 5.4, we design a feedback control $u(x, i, t)$ and calculate an observation interval sequence to make the controlled system

$$dx(t) = \big[f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big]dt + g(x(t), r(t), t)dB(t) \tag{5.70}$$

mean square exponentially stable.

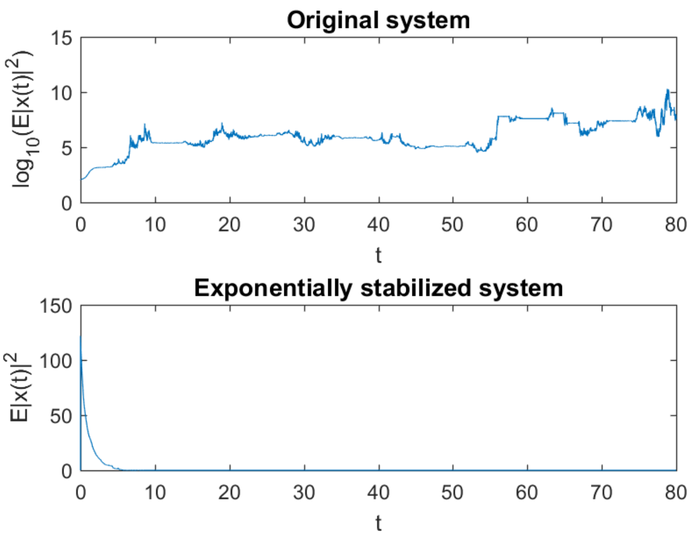

Figure 5.4: Sample averages of $|x(t)|^2$ from 500 simulated paths for original and controlled systems by the Euler-Maruyama method with step size $1e-5$ and random initial values. Upper plot is for original system (5.69); lower plot is for controlled system (5.70) with calculated observation intervals.

Set the controller for mode 1 the same as the controller $u(x, t)$ in Example 5.6.1. Let $u(x, 2, t) = A(x, 2, t)x$ where

$$A(x, 2, t) = \begin{bmatrix} B_3(t)\sin(x_2) - 1.4 & 0 \\ 0 & B_3(t)\cos(x_1) - 1.4 \end{bmatrix},$$

in which $B_3(t) = -[1.5 + \sin(\frac{\pi}{6}t)]$.

After verifying the assumptions and setting parameters, we calculate the observation intervals under condition (5.33) and the results are presented as “original setting” in Figure 5.5.

We divide $[0, 12]$ into 20 subintervals. The largest observation interval is $4.12e-04$ and the smallest one is $2.79e-04$. Corollary 5.4.6 indicates that the controlled system (5.70) with the calculated observation intervals is mean square exponentially stable. This is in accordance with the lower plot in Figure 5.4.

Alternatively, if we use condition (5.34) instead, then the largest and smallest observation intervals are 0.0012 and 0.00037 respectively. Moreover, existing theory (Section 3.5) yields the constant observation interval $\tau \leq 2.62e-4$. Clearly, both conditions (5.33) and (5.34) give better results than this.

Furthermore, condition (5.35) as an integral gives us flexibility to set observation intervals $\{\tau_j\}_{j\geq 1}$ for different subintervals. On one hand, we can make the largest observation interval even larger. If observation is carried out by people, then large observation interval may be considered as holiday. There are two ways to make it. One is by dividing subintervals with large τ_j's into several shorter intervals all with the same $\underline{\tilde{\beta}}_j$ as before. This will not affect τ_j's in other subintervals. The result is shown as a blue line in Figure 5.5. $[0, 12]$ is divided into 26 subintervals. Over time $[0, 0.1)$, the system can be observed once every $7.46e-4$ time units. The other way is to reduce the $\underline{\tilde{\beta}}_j$'s on those subintervals with large τ_j's. However, this would reduce some other observation intervals. This is because, to satisfy $\sum_{j=1}^{20} \underline{\tilde{\beta}}_j (T_{j+1} - T_j) \geq 0$ (required in step 2 of calculating observation interval sequence in Section 5.3), $\underline{\tilde{\beta}}_j$'s on other subintervals have to be increased. On the jth subinterval (for $1 \leq j \leq 20$), a larger $\underline{\tilde{\beta}}_j$ could lead to smaller $\tilde{\tau}(t)$ and then smaller $\tilde{\tau}_j$ (required in step 3); this would lead to larger N_j and finally smaller $\underline{\kappa}_j$ (required by step 4). On the other hand, the flexibility brought by the integral condition enables us to increase the short observation intervals. This will reduce the large τ_j's and the result is shown in Figure 5.5 as a red line. One period $[0, 12]$ is divided into 24 subintervals and $\min_{j\geq 1} \tau_j = 3.19e-4$.

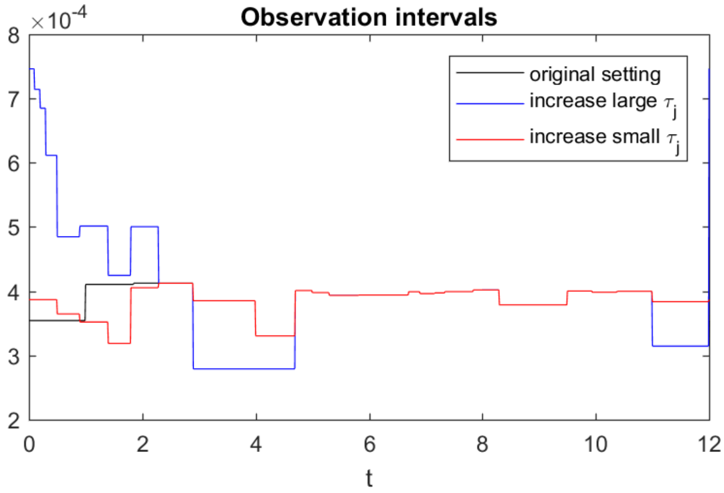

Figure 5.5: Three settings of observation intervals. The black line shows original setting. The blue and red lines show settings to increase the large and small observation intervals respectively.

5.7 Discussion and Conclusion

Now let me explain the connections between the existing theory using constant parameters and my new results using time-varying parameters. Without taking time change into account, existing results simply used the theoretically worst situation to calculation the constant observation interval. The theoretically worst situation means that, $\lambda(t)$ reaches the minimum and $K_1(t)$, $K_2(t)$, $K_3(t)$ reach the maxima all at the same time point, because these make the condition $\inf_{0\le t<T}\beta(t) > 0$ in (5.14) become strongest and hence yield a minimum observation interval. Existing results [35, 96, 97, 104, 110, 111] used the Lyapunov constants of coefficients and controller. They denoted these constants as K_1, K_2 and K_3, which are $\overline{K_1}$, $\overline{K_2}$ and $\overline{K_3}$ in this chapter. Existing results cited above and my Chapter 3 all have to use the shortest observation interval for all times. Even the shortest observation interval derived by the new results in this chapter is still larger than what was used before, let alone over some time intervals, the new results allow for larger observation intervals, which have been clearly shown in Examples 5.6.1 and 5.6.2. So this chapter has dramatically improved the observation intervals.

Example 5.6.2 has illustrated that the integral condition (5.35) enables us to increase those large or short observation intervals. Apart from increasing some specific observation intervals, we can also set a specific value of observation interval for a specific subinterval of $[0, T]$. By adjusting the observation intervals on other subintervals according to the 4-step procedure discussed in Section 5.3, we can find a sequence $\{\tau_j\}_{j\geq 1}$ to meet the corresponding requirements for stabilization purpose. Therefore, the new results in this chapter offers the flexibility and convenience for industry to set observation frequencies according to their needs.

In this chapter we have discussed the stabilization of periodic continuous-time SDEs as well as hybrid SDEs by feedback control based on discrete-time periodic observations. The stabilities analysed include pth moment H_∞ stability and exponential stability for $p > 1$, pth moment asymptotic stability for $p \geq 2$, almost sure asymptotic and exponential stabilities.

This chapter has three main contributions:

- considering the time-varying property of the system into the stability analysis and using time-varying observation frequencies for this stabilization topic;
- reducing the cost of control by reducing the observation frequencies;
- allowing to set observation frequencies flexibly to some extent.

Chapter 6

Conclusion

Summary

This thesis has discussed stabilization problem for stochastic differential equations with or without Markovian switching, by feedback controls based on discrete-time observations.

Firstly, in Chapter 3, we discussed the pth moment stabilization of hybrid SDEs for $p > 1$. We have obtained criteria on pth moment H_∞ stabilization and exponential stabilization for $p > 1$, pth moment asymptotic stabilization for $p \geq 2$ and almost sure exponential stabilization.

Then in Chapter 4, we discussed the stabilization of continuous-time non-autonomous ODEs as well as hybrid SDEs by stochastic feedback control based on Brownian motions. We have obtained criteria on pth moment exponential stabilization for $p \in (0, 1)$ and almost sure exponential stabilization.

Finally in Chapter 5, we discussed the stabilization of periodic SDEs including hybrid SDEs using periodic observation frequencies. We have obtained criteria on pth moment H_∞ stabilization and exponential stabilization for $p > 1$, pth moment asymptotic stabilization for $p \geq 2$, almost sure asymptotic stabilization and almost sure exponential stabilization.

Contribution

The main contributions of this thesis are:

- For stabilization by deterministic feedback controls, when most researches focused on mean square stabilization, this thesis established theory on pth moment stabilization for $p > 1$.

- For stabilization by stochastic feedback controls, when only autonomous ODEs as the unstable original systems were studied, this thesis established theory on stabilization of non-autonomous ODEs and hybrid SDEs.

- Regarding to the research topic "stabilization of SDEs by feedback controls based on discrete-time observations", no other existing research, which considers the influence of time into the control problems, has been reported (as far as the author knows). Other existing researches only use a relatively short constant observation interval for the whole control period.
  For stabilization of periodic systems with time-varying coefficients by deterministic feedback controls, this thesis:
  (1) made use of the time-varying periodic property;
  (2) proposed time-varying periodic observation frequencies (as far as the author knows);
  (3) allowed the industry to set observation frequencies flexibly according to their needs.

- For stabilization by deterministic feedback controls, this thesis reduced the observation frequency compared to existing results and hence reduced the cost of control.

This thesis contributes two publications ([104, 109]) in the peer-reviewed journal "Stochastic Analysis and Applications" and one submitted paper [122] which is under review of the top journal "IEEE Transactions on Automatic Control".

Future Research

There are many directions for future research.
Firstly, I can investigate the stabilization of SDDEs or SFDEs by feedback controls based on discrete-time observations.

Secondly, I can use quantized control. That is, if the control signals derived from the (discrete-time) observations are in a narrow range, then the control signals added to the system are a pre-setted value from the range instead of the derived values. So the control signals will remain the same until the derived values change much and fall in another range. This may be useful when frequently changing control signals are expensive.

Thirdly, the linear growth condition may be too strong as lots of systems in practice do not satisfy it. It's a challenge to relax this condition for stabilization based on discrete-time observations.

In addition, I can try to figure out a way to apply the Lyapunov functions directly to the controlled system using discrete-time observations. Due to this technical difficulty, we now have to use a very general method - by comparing two controlled systems, and as a result, the observation interval is very small.

Appendix A

Appendix for Complete Proof

A.1 Proof for section 3.5

Following is the proof of Theorem 3.5.1.

Proof. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. Let

$$\Phi(x_t, r_t, t) = \theta\tau^{\frac{p-2}{2}} \int_{t-\tau}^{t} \Big[\tau^{\frac{p}{2}} |f(x(s), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^p + \rho |g(x(s), r(s), s)|^p\Big] ds. \tag{A.1}$$

Notice that the integrand in (3.67) is right-continuous in t, then we can use the Leibniz integral ruleto calculate the derivative of $\hat{V}(x_t, r_t, t)$ with respect to t.

$$\hat{V}_t(x_t, r_t, t) = \theta\tau^{\frac{p}{2}} \Big[\tau^{\frac{p}{2}} |f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^p + \rho |g(x(t), r(t), t)|^p\Big] - \Phi(x_t, r_t, t).$$

We apply the generalized Itô formula to $U(x_t, r_t, t)$ and obtain that

$$dU(x_t, r_t, t) = LU(x_t, r_t, t)dt + dM(t)$$

for $t \geq 0$, where $M(t)$ is a continuous local martingale with $M(0) = 0$ and

$$\begin{aligned}
&LU(x_t, r_t, t) \\
=&V_t(x(t), r(t), t) + V_x(x(t), r(t), t)[f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)] \\
&+ \tfrac{1}{2}\text{trace}[g^T(x(t), r(t), t)V_{xx}(x(t), r(t), t)g(x(t), r(t), t)] \\
&+ \sum_{j=1}^{N} \gamma_{r(t),j} V(x(t), j, t) + \hat{V}_t(x_t, r_t, t).
\end{aligned} \tag{A.2}$$

Replacing some terms with the operator defined in (3.9), we have

$$\begin{aligned}
&LU(x_t, r_t, t) \\
=&\mathcal{L}V(x(t), r(t), t) - V_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)] \\
&+ \theta\tau^{\frac{p}{2}}\left[\tau^{\frac{p}{2}}|f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^p + \rho|g(x(t), r(t), t)|^p\right] - \Phi(x_t, r_t, t).
\end{aligned} \tag{A.3}$$

By the Young inequality and Assumption 3.1.2, we can derive that

$$\begin{aligned}
&- V_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)] \\
\leq&|V_x(x(t), r(t), t)||u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)| \\
\leq&\left[\varepsilon|V_x(x(t), r(t), t)|^{\frac{p}{p-1}}\right]^{\frac{p-1}{p}}\left[\varepsilon^{1-p}|u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^p\right]^{\frac{1}{p}} \\
\leq&\frac{p-1}{p}\varepsilon|V_x(x(t), r(t), t)|^{\frac{p}{p-1}} + \frac{1}{p}\varepsilon^{1-p}|u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^p \\
\leq&l|V_x(x(t), r(t), t)|^{\frac{p}{p-1}} + \frac{1}{p}(\frac{p-1}{pl})^{p-1}|u(x(t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^p,
\end{aligned} \tag{A.4}$$

where $l = \frac{p-1}{p}\varepsilon$ for $\forall\varepsilon > 0$.

Since

$$\begin{aligned}
u(x(\delta_t), r(\delta_t), t) - u(x(t), r(t), t) =&u(x(\delta_t), r(\delta_t), t) - u(x(\delta_t), r(t), t) \\
&+ u(x(\delta_t), r(t), t) - u(x(t), r(t), t).
\end{aligned}$$

Using the elementary inequality $|a + b|^p \leq 2^{p-1}(|a|^p + |b|^p)$ for $a, b \in \mathbb{R}$ and $p > 1$, we have

$$\begin{aligned}
&\mathbb{E}|u(x(t),r(t),t)-u(x(\delta_t),r(\delta_t),t)|^p\\
\leq&2^{p-1}\mathbb{E}|u(x(\delta_t),r(\delta_t),t)-u(x(\delta_t),r(t),t)|^p+2^{p-1}\mathbb{E}|u(x(\delta_t),r(t),t)-u(x(t),r(t),t)|^p\\
\leq&2^{p-1}\mathbb{E}|u(x(\delta_t),r(\delta_t),t)-u(x(\delta_t),r(t),t)|^p+2^{p-1}K_3^p\mathbb{E}|x(\delta_t)-x(t)|^p. \qquad \text{(A.5)}
\end{aligned}$$

According to Lemma 1 in [35], for any $t \geq t_0$, $v > 0$ and $i \in \mathbb{S}$,

$$\mathbb{P}(r(s)\neq i \text{ for some } s\in[t,t+v]\Big|r(t)=i)\leq 1-e^{-\overline{\gamma}v}. \qquad \text{(A.6)}$$

Then by Assumption 3.1.2, we have

$$\begin{aligned}
&\mathbb{E}|u(x(\delta_t),r(\delta_t),t)-u(x(\delta_t),r(t),t)|^p\\
=&\mathbb{E}\Big[\mathbb{E}|u(x(\delta_t),r(\delta_t),t)-u(x(\delta_t),r(t),t)|^p\Big|\mathcal{F}_{\delta_t})\Big]\\
\leq&\mathbb{E}\Big[2^pK_3^p|x(\delta_t)|^p\mathbb{E}\Big(I_{\{r(s)\neq r_k\}}|\mathcal{F}_{\delta_t}\Big)\Big]\\
\leq&\mathbb{E}\Big[2^pK_3^p|x(\delta_t)|^p(1-e^{-\overline{\gamma}\tau})\Big]\\
\leq&2^{2p-1}K_3^p(1-e^{-\overline{\gamma}\tau})[\mathbb{E}|x(t)|^p+\mathbb{E}|x(\delta_t)-x(t)|^p] \qquad \text{(A.7)}
\end{aligned}$$

Substituting (A.7) into (A.5) gives

$$\begin{aligned}
&\mathbb{E}|u(x(t),r(t),t)-u(x(\delta_t),r(\delta_t),t)|^p\\
\leq&2^{3p-2}K_3^p(1-e^{-\overline{\gamma}\tau})\mathbb{E}|x(t)|^p+[2^{3p-2}K_3^p(1-e^{-\overline{\gamma}\tau})+2^{p-1}K_3^p]\mathbb{E}|x(\delta_t)-x(t)|^p. \qquad \text{(A.8)}
\end{aligned}$$

Moreover, by Assumptions 3.1.1, 3.1.2 and the elementary inequality $|a+b|^p \leq 2^{p-1}(|a|^p+|b|^p)$ for $\forall a,b\in\mathbb{R}$, we have

$$\begin{aligned}
&|f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)|^p\\
\leq&2^{p-1}\Big[K_1^p|x(t)|^p+K_3^p|x(\delta_t)|^p\Big]\\
\leq&2^{p-1}(K_1^p+2^{p-1}K_3^p)|x(t)|^p+4^{p-1}K_3^p|x(t)-x(\delta_t)|^p.
\end{aligned} \qquad \text{(A.9)}$$

Substitute (A.4) and (A.9) into (A.3). Taking the mean and by (A.8), we have

$$
\begin{aligned}
&\mathbb{E}LU(x_t, r_t, t)\\
\leq&\mathbb{E}\Big[\mathcal{L}V(x(t), r(t), t) + l|V_x(x(t), r(t), t)|^{\frac{p}{p-1}}\Big]\\
&+\Big[\theta\tau^{\frac{p}{2}}(2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p) + \frac{2^{3p-2}}{p}(\frac{p-1}{pl})^{p-1}K_3^p(1-e^{-\overline{\gamma}\tau})\Big]\mathbb{E}|x(t)|^p\\
&+\Big[4^{p-1}\theta\tau^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}[2^{3p-2}(1-e^{-\overline{\gamma}\tau}) + 2^{p-1}]\Big]K_3^p\mathbb{E}|x(t)-x(\delta_t)|^p - \mathbb{E}\Phi(x_t, r_t, t).
\end{aligned}
\tag{A.10}
$$

Then Assumption 3.1.3 implies that

$$
\begin{aligned}
\mathbb{E}LU(x_t,r_t,t) \leq& -\beta\mathbb{E}|x(t)|^p - \mathbb{E}\Phi(x_t, r_t, t)\\
&+\Big[4^{p-1}\theta\tau^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}[2^{3p-2}(1-e^{-\overline{\gamma}\tau}) + 2^{p-1}]\Big]K_3^p\mathbb{E}|x(t)-x(\delta_t)|^p,
\end{aligned}
\tag{A.11}
$$

where

$$
\beta = \beta(\theta,\tau) := \lambda - \theta\tau^{\frac{p}{2}}[2^{p-1}\tau^{\frac{p}{2}}K_1^p + \rho K_2^p + 4^{p-1}\tau^{\frac{p}{2}}K_3^p] - \frac{2^{3p-2}}{p}(\frac{p-1}{pl})^{p-1}K_3^p(1-e^{-\overline{\gamma}\tau}).
\tag{A.12}
$$

Furthermore, it's easy to see from the elementary inequality in (2.17) that

$$
\begin{aligned}
&|x(t)-x(\delta_t)|^p\\
\leq&2^{p-1}\Big(\Big|\int_{\delta_t}^t[f(x(s), r(s), s) + u(x(\delta_s), r(\delta_s), s)]ds\Big|^p + \Big|\int_{\delta_t}^t g(x(s), r(s), s)dB(s)\Big|^p\Big).
\end{aligned}
\tag{A.13}
$$

Since $t-\delta_t \leq \tau$ for all $t \geq 0$, Hölder's inequality indicates that

$$
\Big|\int_{\delta_t}^t[f(x(s), r(s), s)+u(x(\delta_s), r(\delta_s), s)]ds\Big|^p \leq \tau^{p-1}\int_{\delta_t}^t|f(x(s), r(s), s)+u(x(\delta_s), r(\delta_s), s)|^p ds.
\tag{A.14}
$$

For $p \in (1, 2)$, we use the Burkholder-Davis-Gundy inequality and Hölder's inequality to obtain that

$$\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),r(s),s)dB(s)\Big|^p \le \mathbb{E}\Big(\sup_{\delta_t\le z\le t}\Big|\int_{\delta_t}^z g(x(v),r(v),v)dB(v)\Big|^p\Big)$$
$$\le (\frac{32}{p})^{\frac{p}{2}}\mathbb{E}\Big[\int_{\delta_t}^t |g(x(s),r(s),s)|^2 ds\Big]^{\frac{p}{2}} \le (\frac{32}{p})^{\frac{p}{2}}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t |g(x(s),r(s),s)|^p ds. \quad \text{(A.15)}$$

For $p \ge 2$, we use Theorem 2.6.2 (or see [9, Theorem 7.1 on page 39]) to obtain that

$$\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),r(s),s)dB(s)\Big|^p \le [\frac{p(p-1)}{2}]^{\frac{p}{2}}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t |g(x(s),r(s),s)|^p ds. \quad \text{(A.16)}$$

Substituting (A.14), (A.15), (A.16) and (3.11) into (A.13) yields

$$\mathbb{E}|x(t)-x(\delta_t)|^p$$
$$\le 2^{p-1}\tau^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t \Big[\tau^{\frac{p}{2}}|f(x(s),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^p + \rho|g(x(s),r(s),s)|^p\Big]ds. \quad \text{(A.17)}$$

Let us now choose

$$\tau \le 8^{-\frac{p-1}{p}}/K_3 \quad \text{and} \quad \theta = \frac{[4(p-1)]^{p-1}}{p^p(1-8^{p-1}\tau^p K_3^p)} l^{1-p}[2^{2p-1}(1-e^{-\overline{\gamma}\tau})+1]K_3^p. \quad \text{(A.18)}$$

Then

$$2^{p-1}\tau^{\frac{p-2}{2}}\Big(4^{p-1}\theta\tau^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}[2^{3p-2}(1-e^{-\overline{\gamma}\tau})+2^{p-1}]\Big)K_3^p \le \theta\tau^{\frac{p-2}{2}} \quad \text{(A.19)}$$

Combining (A.1), (A.11), (A.17) and (A.19) yields

$$\mathbb{E}(LU(x_t,r_t,t)) \le -\beta\mathbb{E}|x(t)|^p, \quad \text{(A.20)}$$

and by condition (3.68) we have $\beta > 0$.

Moreover, we know from the generalized Itô formula that

$$\mathbb{E}U(x_t,r_t,t) = U(x_0,r_0,0) + \mathbb{E}\int_0^t LU(x_s,r_s,s)ds, \qquad \text{for } t \ge 0. \quad \text{(A.21)}$$

Denote $U(x_0, r_0, 0)$ by C_0 for simplicity, then

$$C_0 = V(x_0, r_0, 0) + 0.5\theta\tau^{\frac{p+2}{2}}\Big[\tau^{\frac{p}{2}}|f(x_0, r_0, 0) + u(x_0, r_0, 0)|^p + \rho|g(x_0, r_0, 0)|^p\Big]. \quad \text{(A.22)}$$

Clearly, C_0 is a positive number. Consequently, substituting (A.20) into (A.21) and by the Fubini theorem, we obtain that

$$0 \leq \mathbb{E}U(x_t, r_t, t) \leq C_0 - \beta\int_0^t \mathbb{E}|x(s)|^p ds, \quad \text{(A.23)}$$

for $t \geq 0$. Hence

$$\int_0^\infty \mathbb{E}|x(s)|^p ds \leq C_0/\beta,$$

which implies the desired assertion (3.69). The proof is complete. □

A.2 Proof for section 5.3

Following is the proof of Theorem 5.3.2.

Proof. Let us prove assertion (5.30) first. Fix any $x_0 \in \mathbb{R}^n$ and $r_0 \in \mathbb{S}$. By the Itô formula, Assumptions 5.2.2 and 5.2.3, we have that for any $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}(|x(t)|^p)\\
=&|x_0|^p + \mathbb{E}\int_0^t \Big(p|x(s)|^{p-2}x^T(s)[f(x(s), r(s), s) + u(x(\delta_s), r(\delta_s), s)]\\
&\quad + \frac{p}{2}|x(s)|^{p-2}|g(x(s), r(s), s)|^2 + \frac{p(p-2)}{2}|x(s)|^{p-4}|x^T(s)g(x(s), r(s), s)|^2\Big)ds\\
\leq&|x_0|^p + \int_0^t \Big(pK_1(s)\mathbb{E}|x(s)|^p + pK_3(s)\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big] + \pi K_2^2(s)\mathbb{E}|x(s)|^p\Big)ds\\
\leq&|x_0|^p + \int_0^t \Big(p\overline{K_1}\mathbb{E}|x(s)|^p + p\overline{K_3}\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big] + \pi\overline{K_2}^2\mathbb{E}|x(s)|^p\Big)ds, \quad \text{(A.24)}
\end{aligned}$$

where $\pi = \frac{p(p-1)}{2}$.

By the Young inequality and the elementary inequality in (2.17) we get

$$\begin{aligned}
&|x(s)|^{p-1}|x(\delta_s)| \\
\le&\Big[\frac{p-1}{p}||x(s)|+\frac{1}{p}|x(\delta_s)|\Big]^p \\
\le&\frac{2^{p-1}}{p^p}\Big[[(p-1)^p+2^{p-1}]|x(s)|^p+2^{p-1}|x(s)-x(\delta_s)|^p\Big]. \qquad (A.25)
\end{aligned}$$

Substituting this into (A.24) gives

$$\mathbb{E}(|x(t)|^p) \le |x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds + C\int_0^t \mathbb{E}|x(s)-x(\delta_s)|^p ds. \qquad (A.26)$$

Recall that C's denote positive constants that may change from line to line.

Denote $\sup_{0\le t<T}(\kappa_t \hat{K}_{3t})$ by H, then (5.14) guarantees $8^{\frac{p-1}{p}}H < 1$. By (5.25) and (5.26), we have

$$\begin{aligned}
&\int_0^t \mathbb{E}|x(s)-x(\delta_s)|^p ds \\
\le&\frac{8^{p-1}\tau_{\max}^{p}\overline{K_3}^p}{1-8^{p-1}H^p}\int_0^t \mathbb{E}|x(s)|^p ds + \frac{2^{p-1}\tau_{\max}^{\frac{p-2}{2}}(2^{p-1}\tau_{\max}^{\frac{p}{2}}\overline{K_1}^p+\rho\overline{K_2}^p)}{1-8^{p-1}H^p}\int_0^t\int_{\delta_s}^s \mathbb{E}|x(z)|^p dz ds \\
\le&C+\frac{2^{p-1}\tau_{\max}^{\frac{p}{2}}}{1-8^{p-1}H^p}[2^{p-1}\tau_{\max}^{\frac{p}{2}}\overline{K_1}^p+\rho\overline{K_2}^p+4^{p-1}\tau_{\max}^{\frac{p}{2}}\overline{K_3}^p]\int_0^t \mathbb{E}|x(s)|^p ds. \qquad (A.27)
\end{aligned}$$

Substituting this into (A.26) yields

$$\mathbb{E}|x(t)|^p \le C+|x_0|^p + C\int_0^t \mathbb{E}|x(s)|^p ds. \qquad (A.28)$$

So by Theorem 5.3.1, we have

$$\mathbb{E}|x(t)|^p \le C \quad \forall t\ge 0. \qquad (A.29)$$

In addition, it follows from the Itô formula that

$$\begin{aligned}&\mathbb{E}|x(t_2)|^p-\mathbb{E}|x(t_1)|^p\\=&\mathbb{E}\int_{t_1}^{t_2}\Big(p|x(t)|^{p-2}x^T(t)[f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)]\\&\qquad+\frac{p}{2}|x(t)|^{p-2}|g(x(t),r(t),t)|^2+\frac{p(p-2)}{2}|x(t)|^{p-4}|x^T(t)g(x(t),r(t),t)|^2\Big)dt.\end{aligned}$$

Then we can easily derive that

$$\mathbb{E}|x(t_2)|^p-\mathbb{E}|x(t_1)|^p\le\int_{t_1}^{t_2}\Big(p\overline{K_1}\mathbb{E}|x(t)|^p+p\overline{K_3}\mathbb{E}\Big[|x(t)|^{p-1}|x(\delta_t)|\Big]+\pi\overline{K_2}^2\mathbb{E}|x(t)|^p\Big)dt.$$

Then by (A.25), (A.27) and (A.29), we get that for any $0\le t_1<t_2<\infty$,

$$\Big|\mathbb{E}|x(t_2)|^p-\mathbb{E}|x(t_1)|^p\Big|\le C\int_{t_1}^{t_2}\mathbb{E}|x(t)|^pdt\le C(t_2-t_1).$$

Finally, according to Barbalat's lemma (see e.g. [124, page 123]), combining this uniform continuity with Theorem 5.3.1 yields that $\lim_{t\to\infty}\mathbb{E}|x(t)|^p=0$.

Now let us prove assertion (5.31) and divide the proof into three steps.

Step 1. Fix $x_0\in\mathbb{R}^n$ and $r_0\in\mathbb{S}$.

The conclusion (5.15) in Theorem 5.3.1 indicates

$$\int_0^\infty|x(t)|^pdt<\infty\quad a.s.$$

Therefore we must have

$$\liminf_{t\to\infty}|x(t)|=0\quad a.s.\tag{A.30}$$

We claim that

$$\lim_{t\to\infty}|x(t)|=0\quad a.s.\tag{A.31}$$

Otherwise, we must have

$$\mathbb{P}\Big(\limsup_{t\to\infty}|x(t)|>0\Big)>0.$$

Consequently, for event $\Omega_1:=\Big\{\limsup_{t\to\infty}|x(t)|>2\varepsilon\Big\}$, we can find a number $\varepsilon>0$

such that

$$\mathbb{P}(\Omega_1) \geq 3\varepsilon. \tag{A.32}$$

Step 2. Let $h > |x_0|$ and define a stopping time

$$\tau_h = \inf\{t \geq 0 : |x(t)| \geq h\},$$

where throughout this thesis we set $\inf \emptyset = \infty$.

Notice that

$$\mathbb{E}|x(t \wedge \tau_h)|^p = h^p \mathbb{P}(\tau_h \leq t) + \mathbb{E}\Big(|x(t)|^p \Big| t < \tau_h\Big). \tag{A.33}$$

By (A.24), we have that for any $t \geq 0$,

$$\begin{aligned}
&\mathbb{E}(|x(t \wedge \tau_h)|^p) \\
\leq &|x_0|^p + \int_0^{t\wedge\tau_h} \Big(p\overline{K_1}\mathbb{E}|x(s)|^p + p\overline{K_3}\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big] + \pi\overline{K_2}^2\mathbb{E}|x(s)|^p\Big)ds \\
\leq &|x_0|^p + \int_0^{t} \Big(p\overline{K_1}\mathbb{E}|x(s)|^p + p\overline{K_3}\mathbb{E}\Big[|x(s)|^{p-1}|x(\delta_s)|\Big] + \pi\overline{K_2}^2\mathbb{E}|x(s)|^p\Big)ds.
\end{aligned}$$

Then by (A.25), (A.26), (A.27) and (A.28), It follows from Theorem 5.3.1 that as $t \to \infty$ and $h \to \infty$, we still have

$$\mathbb{E}|x(t \wedge \tau_h)|^p \leq C.$$

Then by (A.33), we have

$$h^p \mathbb{P}(\tau_h < \infty) \leq C.$$

Choose h sufficiently large so that $\mathbb{P}(\tau_h < \infty) \leq \frac{C}{h^p} \leq \varepsilon$.

Let $\Omega_2 = \{|x(t)| < h \text{ for all } 0 \leq t < \infty\}$. Then

$$\mathbb{P}(\Omega_2) \geq 1 - \varepsilon. \tag{A.34}$$

It follows from (A.32) and (A.34) that

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq 2\varepsilon. \tag{A.35}$$

Step 3. Define a sequence of stopping times:

$$\begin{aligned}
\alpha_1 &= \inf\{t \geq 0 : |x(t)|^p \geq 2\varepsilon\},\\
\alpha_{2k} &= \inf\{t \geq \alpha_{2k-1} : |x(t)|^p \leq \varepsilon\}, \quad k = 1, 2, \cdots,\\
\alpha_{2k+1} &= \inf\{t \geq \alpha_{2k} : |x(t)|^p \geq 2\varepsilon\}, \quad k = 1, 2, \cdots.
\end{aligned}$$

Equation (A.30) implies that $\alpha_{2k} < \infty$ whenever $\alpha_{2k-1} < \infty$ for $k = 1, 2, \cdots$. By definition of Ω_1 and Ω_2, we have

$$\tau_h(\omega) = \infty \text{ and } \alpha_k(\omega) < \infty \text{ for all } k \geq 1 \text{ and } \omega \in \Omega_1 \cap \Omega_2. \tag{A.36}$$

By definitions of α_{2k-1} and α_{2k}, we have

$$|x(t)|^p \geq \varepsilon \quad \text{for} \quad \alpha_{2k-1} \leq t \leq \alpha_{2k}.$$

Hence by Theorem 5.3.1, we can derive that

$$\begin{aligned}
\infty > \mathbb{E}\int_0^\infty |x(t)|^p dt &\geq \sum_{i=1}^{\infty} \mathbb{E}\Big(I_{\{\alpha_{2k-1}<\infty, \tau_h=\infty\}} \int_{\alpha_{2k-1}}^{\alpha_{2k}} |x(t)|^p dt\Big)\\
&\geq \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big(I_{\{\alpha_{2k-1}<\infty, \tau_h=\infty\}} [\alpha_{2k} - \alpha_{2k-1}]\Big).
\end{aligned} \tag{A.37}$$

Let $F(t) = f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)$ and $G(t) = g(x(t), r(t), t)$ for $t \geq 0$. By Assumptions 5.2.2 and 5.2.3, there is a $K_h > 0$ for any $h > 0$ such that

$$|F(t)|^p \vee |G(t)|^p \leq K_h$$

for all $t \geq 0$ and $|x(t)| \vee |x(\delta_t)| \leq h$.

Similarly to (5.24), we can obtain from the elementary inequality in (2.17), the

Hölder inequality and the Burkholder-Davis-Gundy inequality that for any $T > 0$,

$$
\begin{aligned}
&\mathbb{E}\Big(I_A \sup_{0\le t\le T}\Big|x(\tau_h\wedge(\alpha_{2k-1}+t))-x(\tau_h\wedge\alpha_{2k-1})\Big|^p\Big)\\
\le&2^{p-1}\mathbb{E}\Big(I_A \sup_{0\le t\le T}\Big|\int_{\tau_h\wedge\alpha_{2k-1}}^{\tau_h\wedge(\alpha_{2k-1}+t)} F(s)ds\Big|^p\Big)\\
&+2^{p-1}\mathbb{E}\Big(I_A \sup_{0\le t\le T}\Big|\int_{\tau_h\wedge\alpha_{2k-1}}^{\tau_h\wedge(\alpha_{2k-1}+t)} G(s)dB(s)\Big|^p\Big)\\
\le&2^{p-1}T^{p-1}\mathbb{E}\Big(I_A\int_{\tau_h\wedge\alpha_{2k-1}}^{\tau_h\wedge(\alpha_{2k-1}+T)}|F(s)|^pds\Big)\\
&+2^{p-1}T^{\frac{p-2}{2}}\nu\mathbb{E}\Big(I_A\int_{\tau_h\wedge\alpha_{2k-1}}^{\tau_h\wedge(\alpha_{2k-1}+T)}|G(s)|^pds\Big)\\
\le&2^{p-1}K_hT^{\frac{p}{2}}(T^{\frac{p}{2}}+\nu),
\end{aligned}
\tag{A.38}
$$

where $A = \{\tau_h \wedge \alpha_{2k-1} < \infty\}$.

Use the elementary inequality $|a^p - b^p| \le p|a-b|(a^{p-1} + b^{p-1})$ for $\forall a, b \ge 0$ and $p \ge 1$ (presented in section 2.6 or see [26, page 53]). Note that $\Big||x| - |y|\Big| \le |x - y|$ for any $x, y \in \mathbb{R}^n$. Let

$$\theta = \frac{\varepsilon}{2ph^{p-1}},$$

then we have

$$\Big||x|^p - |y|^p\Big| < \varepsilon \text{ whenever } |x-y| < \theta,\ |x| \vee |y| \le h. \tag{A.39}$$

Choose T sufficiently small for

$$\frac{2^{p-1}K_hT^{\frac{p}{2}}(T^{\frac{p}{2}}+\nu)}{\theta^p} < \varepsilon.$$

By Chebyshev's inequality and (A.38), we have

$$
\begin{aligned}
&\mathbb{P}\Big(\{\tau_h\wedge\alpha_{2k-1}<\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\tau_h\wedge(\alpha_{2k-1}+t))-x(\tau_h\wedge\alpha_{2k-1})|\ge\theta\Big\}\Big)\\
\le&\frac{2^{p-1}K_hT^{\frac{p}{2}}}{\theta^p}(T^{\frac{p}{2}}+\nu)<\varepsilon.
\end{aligned}
$$

Moreover, we have

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\alpha_{2k-1}+t)-x(\alpha_{2k-1})|\ge\theta\Big\}\Big)\\
=&\mathbb{P}\Big(\{\tau_h\wedge\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\tau_h\wedge(\alpha_{2k-1}+t))-x(\tau_h\wedge\alpha_{2k-1})|\ge\theta\Big\}\Big)\\
\le&\mathbb{P}\Big(\{\tau_h\wedge\alpha_{2k-1}<\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\tau_h\wedge(\alpha_{2k-1}+t))-x(\tau_h\wedge\alpha_{2k-1})|\ge\theta\Big\}\Big)\\
\le&\varepsilon. \qquad\qquad (A.40)
\end{aligned}$$

It can be seen from (A.36) and (A.35) that

$$\mathbb{P}(\{\alpha_{2k-1}<\infty,\tau_h=\infty\})\ge\mathbb{P}(\Omega_1\cap\Omega_2)\ge 2\varepsilon.$$

Combine this with (A.40), we then obtain

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\alpha_{2k-1}+t)-x(\alpha_{2k-1})|<\theta\Big\}\Big)\\
=&\mathbb{P}(\{\alpha_{2k-1}<\infty,\tau_h=\infty\})\\
&-\mathbb{P}\Big(\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\alpha_{2k-1}+t)-x(\alpha_{2k-1})|\ge\theta\Big\}\Big)\\
\ge&2\varepsilon-\varepsilon=\varepsilon.
\end{aligned}$$

Let

$$\tilde{\Omega}_k=\Big\{\sup_{0\le t\le T}||x(\alpha_{2k-1}+t)|^p-|x(\alpha_{2k-1})|^p|<\varepsilon\Big\}.$$

Then (A.39) implies that

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\tilde{\Omega}_k\Big)\\
\ge&\mathbb{P}\Big(\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\Big\{\sup_{0\le t\le T}|x(\alpha_{2k-1}+t)-x(\alpha_{2k-1})|<\theta\Big\}\Big)\\
\ge&\varepsilon. \qquad\qquad (A.41)
\end{aligned}$$

It follows from the definition of α_k that

$$\alpha_{2k}(\omega) - \alpha_{2k-1}(\omega) \geq T \quad \text{if } \omega \in \{\alpha_{2k-1} < \infty, \tau_h = \infty\} \cap \tilde{\Omega}_k.$$

Combine this with (A.37) and (A.41), we derive that

$$\begin{aligned}
\infty > \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big(I_{\{\alpha_{2k-1}<\infty,\tau_h=\infty\}}[\alpha_{2k} - \alpha_{2k-1}]\Big) \\
\geq \varepsilon \sum_{i=1}^{\infty} \mathbb{E}\Big(I_{\{\alpha_{2k-1}<\infty,\tau_h=\infty\}\cap\tilde{\Omega}_k}[\alpha_{2k} - \alpha_{2k-1}]\Big) \\
\geq \varepsilon T \sum_{i=1}^{\infty} \mathbb{P}\Big(\{\alpha_{2k-1} < \infty, \tau_h = \infty\} \cap \tilde{\Omega}_k\Big) \\
\geq \varepsilon T \sum_{i=1}^{\infty} \varepsilon = \infty,
\end{aligned} \tag{A.42}$$

which is a contradiction. Hence, (A.31) must hold.

The proof is complete. □

A.3 Proof for section 5.5

Following is the proof of Theorem 5.5.5.

Proof. Fix any $x_0 \in \mathbb{R}^n$. By the generalized Itô formula,

$$\begin{aligned}
V(x(t), t) =& V(x(0), 0) + \int_0^t LV(x(s), s)ds \\
&+ \int_0^t V_x(x(s), s)g(x(s), s)dw(s),
\end{aligned}$$

where

$$\begin{aligned}
LV(x(s), s) =& V_s(x(s), s) + V_x(x(s), s)[f(x(s), s) + u(x(\delta_s), s)] \\
&+ \frac{1}{2}\text{trace}[g^T(x(s), s)V_{xx}(x(s), s)g(x(s), s)] \\
=& \mathcal{L}V(x(s), s) - V_x(x(s), s)[u(x(s), s) - u(x(\delta_s), s)].
\end{aligned} \tag{A.43}$$

By the Young inequality and Assumption 5.5.3, we can derive that

$$
\begin{aligned}
&-V_x(x(s),s)[u(x(s),s)-u(x(\delta_s),s)] \\
\leq&|V_x(x(s),s)||u(x(s),s)-u(x(\delta_s),s)| \\
\leq&\Big[\varepsilon|V_x(x(s),s)|^{\frac{p}{p-1}}\Big]^{\frac{p-1}{p}}\Big[\varepsilon^{1-p}|u(x(s),s)-u(x(\delta_s),s)|^p\Big]^{\frac{1}{p}} \\
\leq&\frac{p-1}{p}\varepsilon|V_x(x(s),s)|^{\frac{p}{p-1}}+\frac{1}{p}\varepsilon^{1-p}|u(x(s),s)-u(x(\delta_s),s)|^p \\
\leq&l|V_x(x(s),s)|^{\frac{p}{p-1}}+\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)|x(s)-x(\delta_s)|^p,
\end{aligned}
\tag{A.44}
$$

where $l=\frac{p-1}{p}\varepsilon$ for $\forall\varepsilon>0$.

Then by Assumption 5.5.4, we have that

$$
\begin{aligned}
LV(x(s),s)\leq&\mathcal{L}V(x(s),s)+l|V_x(x(s),s)|^{\frac{p}{p-1}}+\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)|x(s)-x(\delta_s)|^p \\
\leq&-\lambda(s)|x(s)|^p+\frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)|x(s)-x(\delta_s)|^p.
\end{aligned}
\tag{A.45}
$$

It's easy to see from the elementary inequality in (2.17) that

$$
|x(t)-x(\delta_t)|^p \tag{A.46}
$$

$$
\leq 2^{p-1}\Big(\Big|\int_{\delta_t}^t[f(x(s),s)+u(x(\delta_s),s)]ds\Big|^p+\Big|\int_{\delta_t}^t g(x(s),s)dw(s)\Big|^p\Big). \tag{A.47}
$$

Since $t-\delta_t\leq\kappa_t$ for all $t\geq 0$, Hölder's inequality indicates that

$$
\Big|\int_{\delta_t}^t[f(x(s),s)+u(x(\delta_s),s)]ds\Big|^p\leq\kappa_t^{p-1}\int_{\delta_t}^t|f(x(s),s)+u(x(\delta_s),s)|^p ds. \tag{A.48}
$$

For $p\in(1,2)$, we use the Burkholder-Davis-Gundy inequality and Hölder's inequality to obtain that

$$
\begin{aligned}
&\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),s)dw(s)\Big|^p\leq\mathbb{E}\Big(\sup_{\delta_t\leq z\leq t}\Big|\int_{\delta_t}^z g(x(z),z)dw(z)\Big|^p\Big) \\
\leq&(\frac{32}{p})^{\frac{p}{2}}\mathbb{E}\Big[\int_{\delta_t}^t|g(x(s),s)|^2ds\Big]^{\frac{p}{2}}\leq(\frac{32}{p})^{\frac{p}{2}}\kappa_t^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t|g(x(s),s)|^p ds.
\end{aligned}
\tag{A.49}
$$

For $p \geq 2$, we use Theorem 2.6.2 to obtain that

$$\mathbb{E}\Big|\int_{\delta_t}^t g(x(s),s)dw(s)\Big|^p \leq [\frac{p(p-1)}{2}]^{\frac{p}{2}}\kappa_t^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t |g(x(s),s)|^p ds. \tag{A.50}$$

Substituting (A.48), (A.49), (A.50) into (A.46) yields

$$\begin{aligned}
&\mathbb{E}|x(t)-x(\delta_t)|^p \\
\leq & 2^{p-1}\kappa_t^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_t}^t \Big[\kappa_t^{\frac{p}{2}}|f(x(s),s)+u(x(\delta_s),s)|^p + \rho|g(x(s),s)|^p\Big]ds.
\end{aligned} \tag{A.51}$$

By the Assumptions 5.5.2 and 5.5.3, we have that for any $s \in [\delta_s, \delta_s+\kappa_s)$,

$$\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p \\
\leq & 2^{p-1}\kappa_s^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_s}^s \kappa_s^{\frac{p}{2}}|f(x(z),z)+u(x(\delta_s),z)|^p + \rho|g(x(z),z)|^p dz \\
\leq & 2^{p-1}\kappa_s^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_s}^s 2^{p-1}\kappa_s^{\frac{p}{2}}\Big[K_1^p(z)|x(z)|^p + K_3^p(z)|x(\delta_s)|^p\Big] + \rho K_2^p(z)|x(z)|^p dz \\
\leq & 2^{p-1}\kappa_s^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_s}^s 2^{p-1}\kappa_s^{\frac{p}{2}}\Big[K_1^p(z)|x(z)|^p + 2^{p-1}K_3^p(z)[|x(s)-x(\delta_s)|^p + |x(s)|^p]\Big] \\
& \quad + \rho K_2^p(z)|x(z)|^p dz \\
\leq & 2^{p-1}\kappa_s^{\frac{p-2}{2}}\Big[\int_{\delta_s}^s [2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z) + \rho K_2^p(z)]\mathbb{E}|x(z)|^p dz \\
& \quad + 4^{p-1}\kappa_s^{\frac{p}{2}}\int_{\delta_s}^s K_3^p(z)dz[\mathbb{E}|x(s)-x(\delta_s)|^p + \mathbb{E}|x(s)|^p]\Big]
\end{aligned}$$

Note that the condition (5.60) implies $8^{p-1}\kappa_s^p\hat{K}_{3s}^p < 1$, then we rearrange it and obtain

$$\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p \\
\leq & \frac{2^{p-1}\kappa_s^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\int_{\delta_s}^s \Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z) + \rho K_2^p(z)\Big]\mathbb{E}|x(z)|^p dz \\
& + \frac{8^{p-1}\kappa_s^p\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\mathbb{E}|x(s)|^p.
\end{aligned} \tag{A.52}$$

Note that for $\forall z \in [\delta_s, s]$, we have $\kappa_z = \kappa_s$ and $K_3(s) \leq \hat{K}_{3s} = \hat{K}_{3z}$. Since

$s-\kappa_s<\delta_s$, it's easy to show that for a non-negative bounded function $F(t)$,

$$\begin{aligned}&\int_0^t\int_{\delta_s}^s F(z)dzds \le \int_0^t\int_{s-\kappa_s}^s F(z)dzds\\ &\le \int_{-\kappa_z}^t F(z)\int_z^{z+\kappa_z} dsdz \le \int_{-\kappa_s}^t \kappa_z F(z)dz \le C+\int_0^t \kappa_z F(z)dz. \qquad \text{(A.53)}\end{aligned}$$

Let $x(s)=x_0$, $K_1(s)=K_1(0)$, $K_2(s)=K_2(0)$ and $K_3(s)=K_3(0)$ for all $(x,i,s)\in R^n\times S\times[-\tau_{\max},0)$. Then

$$\begin{aligned}&\int_0^t K_3^p(s)\int_{\delta_s}^s \frac{2^{p-1}\kappa_s^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_s^p\hat{K}_{3z}^p}\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)\Big]\mathbb{E}|x(z)|^p dzds\\ \le&\int_0^t\int_{\delta_s}^s \hat{K}_{3z}^p\frac{2^{p-1}\kappa_z^{\frac{p-2}{2}}}{1-8^{p-1}\kappa_z^p\hat{K}_{3z}^p}\Big[2^{p-1}\kappa_z^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)\Big]\mathbb{E}|x(z)|^p dzds\\ \le& C+\int_0^t \frac{2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(s)+\rho K_2^p(s)\Big]\mathbb{E}|x(s)|^p ds.\end{aligned}$$

Recall that C's denote positive constants that may change from line to line.

So

$$\begin{aligned}&\int_0^t K_3^p(s)\mathbb{E}|x(s)-x(\delta_s)|^p ds\\ \le& C+\int_0^t \frac{2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\Big[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(s)+\rho K_2^p(s)+4^{p-1}\kappa_s^{\frac{p}{2}}K_3^p(s)\Big]\mathbb{E}|x(s)|^p ds. \qquad \text{(A.54)}\end{aligned}$$

It's easy to obtain that

$$\begin{aligned}&\mathbb{E}V(x(t),t)=V(x(0),0)+\int_0^t \mathbb{E}LV(x(s),s)ds\\ \le& C-\int_0^t \Big[\lambda(s)-\frac{1}{p}(\frac{p-1}{pl})^{p-1}\frac{2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p}{1-8^{p-1}\kappa_s^p\hat{K}_{3s}^p}\\ &\times[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(s)+\rho K_2^p(s)+4^{p-1}\kappa_s^{\frac{p}{2}}K_3^p(s)]\Big]\mathbb{E}|x(s)|^p ds.\end{aligned}$$

By (5.60) we have that for $\forall t\ge 0$,

$$0<\mathbb{E}V(x(t),t)\le C-\int_0^t \beta(z)\mathbb{E}|x(z)|^p dz.$$

Recall $\inf_{0\le t<T}\beta(t)>0$. Let $\underline{\beta}=\inf_{0\le t<T}\beta(t)$ and we have

$$0<\underline{\beta}\int_0^\infty \mathbb{E}|x(s)|^p ds\le\int_0^\infty \beta(t)\mathbb{E}|x(t)|^p dt\le C.$$

Hence we obtain assertion (5.15). □

Following is the proof of Theorem 5.5.8.

Proof Fix $x_0\in\mathbb{R}^n$. By the Assumptions 5.5.2 and 5.5.3, we have that for any $s\in[\delta_s,\delta_s+\kappa_s)$,

$$\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p\\
\le&2^{p-1}\kappa_s^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_s}^s\kappa_s^{\frac{p}{2}}|f(x(z),z)+u(x(\delta_s),z)|^p+\rho|g(x(z),z)|^p dz\\
\le&2^{p-1}\kappa_s^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_s}^s 2^{p-1}\kappa_s^{\frac{p}{2}}\Big[K_1^p(z)|x(z)|^p+K_3^p(z)|x(\delta_s)|^p\Big]+\rho K_2^p(z)|x(z)|^p dz\\
\le&4^{p-1}\kappa_s^{p-1}\int_{\delta_s}^s K_3^p(z)dz\mathbb{E}|x(\delta_s)|^p+2^{p-1}\kappa_s^{\frac{p-2}{2}}\mathbb{E}\int_{\delta_s}^s[2^{p-1}\kappa_s^{\frac{p}{2}}K_1^p(z)+\rho K_2^p(z)]|x(z)|^p dz\\
\le&4^{p-1}\kappa_s^p\hat{K}_{3s}^p\mathbb{E}|x(\delta_s)|^p+2^{p-1}\kappa_s^{\frac{p}{2}}[2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p+\rho\hat{K}_{2s}^p]\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big).
\end{aligned}\tag{A.55}$$

Now we prove that if condition (5.33) is satisfied, then

$$\mathbb{E}|x(s)-x(\delta_s)|^p\le\frac{\varphi_s}{1-\varphi_s}\mathbb{E}|x(s)|^p.\tag{A.56}$$

By the definition of solutions of hybrid SDEs, the inequality in (2.17), Hölder's inequality and the Burkholder-Davis-Gundy inequality, we have that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t f(x(z),z)dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t u(x(\delta_z),z)]dz\Big|^p\Big)\\
&+4^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\Big|\int_{\delta_s}^t g(x(z),z)dB(z)\Big|^p\Big)\\
\le&4^{p-1}\mathbb{E}|x(\delta_s)|^p+(4\kappa_s)^{p-1}\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t[K_1^p(z)|x(z)|^p+K_3^p(z)|x(\delta_s)|^p]dz\Big)\\
&+4^{p-1}\kappa_s^{\frac{p-2}{2}}\nu\mathbb{E}\Big(\sup_{\delta_s\le t\le s}\int_{\delta_s}^t K_2^p(z)|x(z)|^pdz\Big)\\
\le&\Big[4^{p-1}+(4\kappa_s)^{p-1}\int_{\delta_s}^s K_3^p(z)dz\Big]\mathbb{E}|x(\delta_s)|^p\\
&+\Big[(4\kappa_s)^{p-1}\hat{K}_{1s}^p+4^{p-1}\kappa_s^{\frac{p-2}{2}}\nu\hat{K}_{2s}^p\Big]\int_{\delta_s}^s\mathbb{E}\Big(\sup_{\delta_s\le z\le t}|x(z)|^p\Big)dt
\end{aligned}$$

Then the Gronwall inequality implies

$$\mathbb{E}\Big(\sup_{\delta_s\le t\le s}|x(t)|^p\Big)\le\Big[4^{p-1}+(4\kappa_s)^{p-1}\int_{\delta_s}^s K_3^p(z)dz\Big]\mathbb{E}|x(\delta_s)|^p\exp(4^{p-1}\kappa_s^p\hat{K}_{1s}^p+4^{p-1}\kappa_s^{\frac{p}{2}}\nu\hat{K}_{2s}^p).$$

Substituting this into (A.55) and noticing that

$$\mathbb{E}|x(\delta_s)|^p\le 2^{p-1}\mathbb{E}|x(s)|^p+2^{p-1}\mathbb{E}|x(s)-x(\delta_s)|^p$$

for all $p>1$, we have

$$\begin{aligned}
&\mathbb{E}|x(s)-x(\delta_s)|^p\\
\le&4^{p-1}\kappa_s^{\frac{p}{2}}\Big[\kappa_s^{\frac{p}{2}}\hat{K}_{3s}^p+2^{p-1}(1+\kappa_s^p\hat{K}_{3s}^p)(2^{p-1}\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p+\rho\hat{K}_{2s}^p)\\
&\times\exp(4^{p-1}\kappa_s^p\hat{K}_{1s}^p+4^{p-1}\kappa_s^{\frac{p}{2}}\nu\hat{K}_{2s}^p)\Big]\mathbb{E}|x(\delta_s)|^p.
\end{aligned}$$

By (5.33), we can rearrange it and get (A.56).

Alternatively, we show that if condition (5.34) is satisfied, then (A.56) holds.

By the definition of solutions of hybrid SDEs, the inequality in (2.17) and the Burkholder-Davis-Gundy inequality, we have that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{\delta_s \le t \le s} |x(t)|^p\Big) \\
\le& 4^{p-1}\mathbb{E}|x(\delta_s)|^p + 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s \le t \le s}\Big|\int_{\delta_s}^t f(x(z),z)dz\Big|^p\Big) \\
&+ 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s \le t \le s}\Big|\int_{\delta_s}^t u(x(\delta_z),z)]dz\Big|^p\Big) \\
&+ 4^{p-1}\mathbb{E}\Big(\sup_{\delta_s \le t \le s}\Big|\int_{\delta_s}^t g(x(z),z)dB(z)\Big|^p\Big) \\
\le& 4^{p-1}\mathbb{E}|x(\delta_s)|^p + (4\kappa_s)^{p-1}\mathbb{E}\Big(\sup_{\delta_s \le t \le s}\int_{\delta_s}^t [K_1^p(z)|x(z)|^p + K_3^p(z)|x(\delta_s)|^p]dz\Big) \\
&+ 4^{p-1}\kappa_s^{\frac{p-2}{2}}\nu\mathbb{E}\Big(\sup_{\delta_s \le t \le s}\int_{\delta_s}^t K_2^p(z)|x(z)|^p dz\Big) \\
\le& 4^{p-1}(1+\kappa_s^p\hat{K}_{3s}^p)\mathbb{E}|x(\delta_s)|^p + 4^{p-1}\kappa_s^{\frac{p}{2}}(\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p + \nu\hat{K}_{2s}^p)\mathbb{E}\Big(\sup_{\delta_s \le t \le s}|x(t)|^p\Big).
\end{aligned}$$

By (5.34), we can rearrange it and get

$$\mathbb{E}\Big(\sup_{\delta_s \le z \le s}|x(z)|^p\Big) \le \frac{4^{p-1}(1+\kappa_s^p\hat{K}_{3s}^p)}{1-4^{p-1}\kappa_s^{\frac{p}{2}}(\kappa_s^{\frac{p}{2}}\hat{K}_{1s}^p+\nu\hat{K}_{2s}^p)}\mathbb{E}|x(\delta_s)|^p.$$

Substituting into (A.55) and rearranging yield (A.56).

Recall (A.45):

$$LV(x(s),s) \le -\lambda(s)|x(s)|^p + \frac{1}{p}(\frac{p-1}{pl})^{p-1}K_3^p(s)|x(s)-x(\delta_s)|^p.$$

Combining (A.45), (A.56) and (5.64), we obtain that

$$\mathbb{E}LV(x(s),s) \le -c_2\tilde{\beta}(s)\mathbb{E}|x(s)|^p. \tag{A.57}$$

Let $\hat{V}(x(t),t) = e^{\int_0^t \tilde{\beta}(s)ds}V(x(t),t)$. We can obtain from the Itô formula, Assumption 5.5.7 and (A.57) that

$$
\begin{aligned}
\mathbb{E}\hat{V}(x(t),t) &= \mathbb{E}V(x(0),0) + \mathbb{E}\int_0^t L\hat{V}(x(s),s)ds \\
&\le \mathbb{E}V(x(0),0) + \int_0^t e^{\int_0^s \tilde{\beta}(z)dz}[\mathbb{E}LV(x(s),s) + \tilde{\beta}(s)\mathbb{E}V(x(s),s)]ds \\
&\le \mathbb{E}V(x(0),0). \qquad (A.58)
\end{aligned}
$$

By Assumption 5.5.7 and (A.58), we have

$$c_1 e^{\int_0^t \tilde{\beta}(s)ds}\mathbb{E}|x(t)|^p \le \mathbb{E}\hat{V}(x(t),t) \le \mathbb{E}V(x(0),0). \qquad (A.59)$$

Then

$$\mathbb{E}|x(t)|^p \le Ce^{-\int_0^t \tilde{\beta}(s)ds}. \qquad (A.60)$$

So we have

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) \\
\le &\limsup_{t\to\infty} \frac{-1}{t}\int_0^t \tilde{\beta}(s)ds = -\liminf_{k\to\infty,\Delta t\in[0,T)} \frac{\int_0^{kT+\Delta t}\tilde{\beta}(s)ds}{kT+\Delta t}, \qquad (A.61)
\end{aligned}
$$

where

$$\frac{\int_0^{kT+\Delta t}\tilde{\beta}(s)ds}{kT+\Delta t} = \frac{kv + \int_0^{\Delta t}\tilde{\beta}(s)ds}{kT+\Delta t} \ge \frac{v}{T} - \frac{\max_{z\in[0,T)}\int_z^T \tilde{\beta}(s)ds}{kT+\Delta t}.$$

Substituting into (A.61) gives

$$
\begin{aligned}
&\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}|x(t)|^p) \\
\le &-\frac{v}{T} - \liminf_{k\to\infty,\Delta t\in[0,T)} -\frac{\max_{z\in[0,T)}\int_z^T \tilde{\beta}(s)ds}{kT+\Delta t} = -\frac{v}{T} - \limsup_{k\to\infty,\Delta t\in[0,T)} \frac{\max_{z\in[0,T)}\int_z^T \tilde{\beta}(s)ds}{kT+\Delta t} \\
\le &-\frac{v}{T}. \qquad (A.62)
\end{aligned}
$$

Hence we obtain the assertion (5.37).

Similarly to the proof of (3.49) in Chapter 3, according to Theorem 2.5.5 (or see [9, Theorem 6.2 on page 175]), we obtain the assertion (5.38). The proof is complete. □

Bibliography

[1] Bachelier, L. 2011. *Louis Bachelier's theory of speculation: the origins of modern finance*. Translated and with commentary by Mark Davis and Alison Etheridge. Princeton University Press.

[2] Einstein, A. 1905. On the movement of small particles suspended in stationary liquids required by the molecular-kinetic theory of heat. *Annalen der Physik*, 17, pp.549-560. https://pdfs.semanticscholar.org/9c1d/91a9f0a37e578ee9a6605b224ad554ec6e86.pdf

[3] Piasecki, J. 2007. Centenary of Marian Smoluchowski's theory of Brownian motion. *Acta Physica Polonica B*, 38(5).

[4] Davis, M. *Louis Bacheliers Theory of Speculation*. Imperial College. https://f-origin.hypotheses.org/wp-content/blogs.dir/1596/files/2014/12/Mark-Davis-Talk.pdf

[5] Wiener, N. 1923. Differential-Space. *Journal of Mathematics and Physics*, 2, pp.131-174.

[6] Itô, K. 1944. Stochastic integral. *Proc. Imp. Acad.*, 20(8), pp.519-524. doi:10.3792/pia/1195572786. https://projecteuclid.org/euclid.pja/1195572786

[7] Itô, K. 1951. On a formula concerning stochastic differentials. *Nagoya Math. J.*, 3, pp.55-65. https://projecteuclid.org/euclid.nmj/1118799221

[8] Simund, K. *Kolmogorov and population dynamics.* Translated by Elizabeth Strouse. http://homepage.univie.ac.at/Karl.Sigmund/Kolmogorov.pdf

[9] Mao, X. 2007. *Stochastic differential equations and applications*, second Edition. Horwood Publishing Limited, Chichester.

[10] Gutiérrez-Sanchez, R., Melchor, M.C. and Ramos-Ábalos, E. 2014. A stochastic Gompertz model highlighting internal and external therapy function for tumour growth. *Applied Mathematics and Computation*, 246, pp.1-11.

[11] Boxler, P. 1989. A stochastic version of center manifold theory. *Probability Theory and Related Fields*, 83(4), pp.509-545.

[12] Mθller, C.M., 1993. A stochastic version of Thiele's differential equation. *Scandinavian Actuarial Journal*, 1993, pp.1-16.

[13] Mao, X., 1999. Stochastic versions of the LaSalle theorem. *Journal of Differential Equations*, 153(1), pp.175-195.

[14] Särkkä, S. *Applied Stochastic Differential Equations.* Version as of November 21, 2012. Written material for the course held in Autumn 2012. https://users.aalto.fi/~ssarkka/course_s2012/pdf/sde_course_booklet_2012.pdf

[15] Gawalwad, B.G. and Sharma, S.N. 2016. Coloured Noise Analysis of a Phase-Locked Loop System: Beyond It and Stratonovich Stochastic Calculi. *Differential Equations and Dynamical Systems*, 24(2), pp.231-245.

[16] Phase-locked loop. https://searchnetworking.techtarget.com/definition/phase-locked-loop Posted by Margaret Rouse.

[17] Fournier, N., 1999. Strict positivity of the density for a Poisson driven SDE. *Stochastics: An International Journal of Probability and Stochastic Processes*, 68(1-2), pp.1-43.

[18] Denis, L., 2000. A criterion of density for solutions of Poisson-driven SDEs. *Probability Theory and Related Fields*, 118(3), pp.406-426.

[19] Fischer, E., Lu, S., Samorodnitsky, G., Gong, W. and Towsley, D. April 15, 2016. Cornell University, University of Massachusetts, Amherst. A poisson driven stochastic differential equation model with regularly varying tails. http://www.orie.cornell.edu/orie/research/groups/multheavytail/upload/Samorodnitsky1_041516.pdf

[20] Lasota, A. and Traple, J. 2003. Invariant measures related with Poisson driven stochastic differential equation. *Stochastic processes and their applications*, 106(1), pp.81-93.

[21] Ishikawa, Y. and Kunita, H. 2006. Malliavin calculus on the WienerPoisson space and its application to canonical SDE with jumps. *Stochastic processes and their applications*, 116(12), pp.1743-1769.

[22] Grigoriu, M., 2009. Numerical solution of stochastic differential equations with Poisson and Lévy white noise. *Physical Review E*, 80(2), pp.026704.

[23] Liu, D., Yang, G. and Zhang, W., 2011. The stability of neutral stochastic delay differential equations with Poisson jumps by fixed points. *Journal of Computational and Applied Mathematics*, 235(10), pp.3115-3120.

[24] Basharin, G.P., Langville, A.N. and Naumov, V.A., 2004. The life and work of A. A. Markov. *Linear algebra and its applications*, 386, pp.3-26.

[25] Gagniuc, P.A. 2017. *Markov Chains: From Theory to Implementation and Experimentation*. John Wiley & Sons.

[26] Mao, X. and Yuan, C. 2006. *Stochastic Differential Equations with Markovian Switching*. Imperial College Press.

[27] Van Der Schaft, A.J. and Schumacher, J.M., 2000. *An introduction to hybrid dynamical systems* (Vol. 251). London: Springer. https://www.ecse.rpi.edu/~agung/course/vanderschaft.pdf

[28] Goebel, R., Sanfelice, R.G. and Teel, A.R., 2009. Hybrid dynamical systems. *IEEE Control Systems*, 29(2), pp.28-93.

[29] Lynch, N. and Krogh, B. eds., 2007. *Hybrid Systems: Computation and Control: Third International Workshop, HSCC 2000 Pittsburgh, PA, USA, March 23-25, 2000 Proceedings.* Lecture Notes in Computer Science 1790. Springer Science & Business Media.

[30] Majumdar, R. and Tabuada, P., 2009. *Hybrid systems: computation and control: 12th International Conference, HSCC 2009 San Francisco, CA, USA, April 2009 Proceedings.* Lecture Notes in Computer Science 5469. Springer Science & Business Media.

[31] Garajayewa, G. and Hofreiter, M. 2011. Markov Chains for Fault Diagnosis. *XXVI. ASR '2001 Seminar, Instruments and Control, Ostrava, April 26 - 27, 2001.* `http://akce.fs.vsb.cz/2001/asr2001/Proceedings/papers/21.pdf`

[32] LeGland, F. and Mevel, L. 2000. Fault detection in hidden Markov models: A local asymptotic approach. In *Decision and Control, 2000. Proceedings of the 39th IEEE Conference on*, 5, pp.4686-4690. IEEE.

[33] Dugan, J.B., Bavuso, S.J. and Boyd, M.A. 1993. Fault trees and Markov models for reliability analysis of fault-tolerant digital systems. *Reliability Engineering & System Safety*, 39(3), pp.291-307.

[34] Jovanović, D.P. and Pollett, P.K. 2014. Distributed fault detection using consensus of Markov chains. In *Optimization and Control Methods in Industrial Engineering and Construction*, pp. 85-105. Springer, Dordrecht.

[35] Song, G., Zheng, B. C., Luo, Q. and Mao, X. 2016. Stabilisation of hybrid stochastic differential equations by feedback control based on discrete-time observations of state and mode. *IET Control Theory Appl.* 11(3), pp. 301–307.

[36] Differences Between Feedback Control & Feed Forward Control. By Milton Kazmeyer. `https://www.techwalla.com/articles/differences-between-feedback-control-feed-forward-control`

[37] Lecture notes chapter 15 of CHE 360: Chemical Process Control, Cockrell School of Engineering. http://www.che.utexas.edu/course/che360/lecture_notes.html

[38] Kolmanovskii, V. and Myshkis, A. 1992. *Applied theory of functional differential equations.* Kluwer Academic Publishers, Springer Science & Business Media Dordrecht.

[39] Lyapunov, A. M. 1892. The General Problem of the Stability of Motion (In Russian), Doctoral dissertation, Univ. Kharkov. English translations: The General Problem of the Stability of Motion, (A. T. Fuller trans.) Taylor & Francis, London 1992.

[40] Murray, R.M., Li, Z. and Sastry, S.S. 1994. *A mathematical introduction to robotic manipulation.* CRC press, Taylor & Francis Group, Boca Raton, FL. pp.179-188.

[41] Khas'minskii, R.Z. 2011. *Stochastic stability of differential equations*, (Vol. 66). Originally published in Russian, by Nauka, Moscow 1969. 1st English ed. published in 1980 under R.Z. Khas'minskii in the series Mechanics: Analysis by Sijthoff & Noordhoff. Completely Revised and Enlarged 2nd Edition, Springer-Verlag Berlin Heidelberg, Springer Science & Business Media.

[42] Bucy, R.S. 1965. Stability and positive supermartingales. *Journal of Differential Equations*, 1(2), pp.151-155.

[43] Khas'minskii, R.Z. 1967. Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems. *Theory of Probability & Its Applications*, 12(1), pp.144-147. Translated by B. Seckler.

[44] Khas'minskii, R.Z., 1967. Necessary and sufficient conditions for the asymptotic stability of linear stochastic systems. *Theory of Probability & Its Applications*, 12(1), pp.144-147.

[45] Curtain, R.F., 1981. Stability of stochastic partial differential equation. *Journal of Mathematical Analysis and Applications*, 79(2), pp.352-369.

[46] Arnold, L., Oeljeklaus, E. and Pardoux, E., 1986. Almost sure and moment stability for linear It equations. In *Lyapunov exponents*, pp. 129-159. Springer, Berlin, Heidelberg.

[47] Mao, X., 1990. Eventual asymptotic stability for stochastic differential systems with respect to semimartingales. *The Quarterly Journal of Mathematics*, 41(1), pp.71-77.

[48] Chang, M.H., Ladde, G. and Liu, P.T., 1974. Stability of stochastic functional differential equations. *Journal of Mathematical Physics*, 15(9), pp.1474-1478.

[49] Li, X., Zhu, Q. and O'Regan, D. 2014. pth moment exponential stability of impulsive stochastic functional differential equations and application to control problems of NNs. *Journal of the Franklin Institute*, 351, pp. 4435–4456.

[50] Mao, X., 1996. Razumikhin-type theorems on exponential stability of stochastic functional differential equations. *Stochastic processes and their applications*, 65(2), pp.233-250.

[51] Mao, X., 2000. The LaSalle-type theorems for stochastic functional differential equations. *Nonlinear Studies*, 7(2), pp.307-328.

[52] Shen, Y., Luo, Q. and Mao, X., 2006. The improved LaSalle-type theorems for stochastic functional differential equations. *Journal of Mathematical Analysis and Applications*, 318(1), pp.134-154.

[53] Li, X. and Mao, X. 2012. The improved LaSalle-type theorems for stochastic differential delay equations. *Stochastic Analysis and Applications* 30(4) pp.568-589.

[54] Li, X. and Mao, X. 2012. A note on almost sure asymptotic stability of neutral stochastic delay differential equations with Markovian switching. *Automatica*, 48(9), pp.2329-2334.

[55] Mao, X. 2007. Stability and stabilization of stochastic differential delay equations. *IET Control Theory & Applications*, 1(6), pp. 1551–1566.

[56] Huang, H., Ho, D.W.C. and Lam, J. 2005. Stochastic stability analysis of fuzzy Hopfield neural networks with time-varying delays. *IEEE Trans. Circuits and Systems II: Express Briefs*, 52, pp.251–255.

[57] Huang, C., He, Y., Huang, L. and Zhu, W. 2008. pth moment stability analysis of stochastic recurrent neural networks with time-varying delays. *Information Sciences*, 178, pp. 2194–2203.

[58] Gao, H., Wang, C. and Wang, J. 2005. On H_∞ performance analysis for continuous-time stochastic systems with polytopic uncertainties. *Circuits Systems Signal Processing*, 24, pp. 415–429.

[59] Protter, P., 1978. $\mathcal{H}^p$ tability of solutions of stochastic differential equations. *Zeitschrift für Wahrscheinlichkeitstheorie und verwandte Gebiete*, 44(4), pp.337-352.

[60] Kozin, F., 1972. Stability of the linear stochastic system. In *Stability of Stochastic Dynamical Systems*, pp. 186-229. Springer, Berlin, Heidelberg.

[61] Ryashko, L.B. and Schurz, H., 1997. Mean square stability analysis of some linear stochastic systems. *Dynamic Systems and Applications*, 6, pp.165-190.

[62] Hu, L., Ren, Y. and Xu, T. 2014. p-Moment stability of solutions to stochastic differential equations driven by G-Brownian motion. *Applied Mathematics and Computation*, 230, pp. 231–237.

[63] Xu, Y., He, Z. and Wang, P., 2015. pth moment asymptotic stability for neutral stochastic functional differential equations with Lévy processes. *Applied Mathematics and Computation*, 269, pp.594-605.

[64] Basak, G.K., Bisi, A. and Ghosh, M.K. 1996. Stability of a random diffusion with linear drift. *J. Math. Anal. Appl.* 202, pp. 604–622.

[65] Mao, X. 1999. Stability of stochastic differential equations with Markovian switching. *Sto. Proc. Their Appl.* ,79, pp. 45–67.

[66] Mao, X., Matasov, A. and Piunovskiy, A.B. 2000. Stochastic differential delay equations with Markovian switching. *Bernoulli*, 6(1), pp. 73–90.

[67] Yuan, C. and Mao, X., 2003. Asymptotic stability in distribution of stochastic differential equations with Markovian switching. *Stochastic processes and their applications*, 103(2), pp.277-291.

[68] Mao, X., Yin, G. and Yuan, C. 2007. Stabilization and destabilization of hybrid systems of stochastic differential equations. *Automatica*, 43, pp. 264–273.

[69] Hu, L., Mao, X. and Shen, Y. 2013. Stability and boundedness of nonlinear hybrid stochastic differential delay equations. *Systems & Control Letters*, 62, pp. 178–187.

[70] Wu, X., Zhang, W. and Tang, Y. 2013. pth Moment stability of impulsive stochastic delay differential systems with Markovian switching. *Commun Nonlinear Sci Numer Simulat*, 18, pp. 1870–1879.

[71] Zhu, E., Tian, X. and Wang, Y. 2015. On pth moment exponential stability of stochastic differential equations with Markovian switching and time-varying delay. *Journal of Inequalities and Applications*, 2015:137. DOI 10.1186/s13660-015-0657-9.

[72] Ji, Y. and Chizeck, H.J. 1990. Controllability, stabilizability and continuous-time Markovian jump linear quadratic control. *IEEE Trans. Automat. Control*, 35, pp. 777–788.

[73] Mariton, M., 1990. *Jump linear systems in automatic control.* New York: Marcel Dekker.

[74] Chen,T. and Francis, B. 1995. *Optimal sampled-data control systems.* Springer-Verlag, London.

[75] Tian, Y.P. and Li, S., 2002. Exponential stabilization of nonholonomic dynamic systems by smooth time-varying control. *Automatica*, 38(7), pp.1139-1146.

[76] Ge, S.S., Wang, Z. and Lee, T.H., 2003. Adaptive stabilization of uncertain nonholonomic systems by state and output feedback. *Automatica*, 39(8), pp.1451-1460.

[77] Allwright, J.C., Astolfi, A. and Wong, H.P. 2005. A note on asymptotic stabilization of linear systems by periodic, piecewise constant, output feedback. *Automatica* , 41(2), pp.339–344.

[78] Sun, M., Lam, J., Xu, S. and Zou, Y. 2007. Robust exponential stabilization for Markovian jump systems with mode-dependent input delay. *Automatica*, 43, pp. 1799–1807.

[79] Niu, Y., Ho, D.W.C. and Lam, J. 2005. Robust integral sliding mode control for uncertain stochastic systems with time-varying delay. *Automatica*, 41, pp. 873–880.

[80] Xu, S., Lam, J., Yang, G.H. and Wang J.L. 2006. Stabilization and H_∞ control for uncertain stochastic time-delay systems via non-fragile controllers. *Asian Journal of Control* ,8, pp. 197–200.

[81] Tsakalis, K.S. and Ioannou, P.A. 1989. Adaptive control of linear time-varying plants: A new model reference controller structure. *IEEE Transactions on Automatic Control*, 34(10), pp. 1038–1046.

[82] Jiang, Z.P., 1999. A combined backstepping and small-gain approach to adaptive output feedback control. *Automatica*, 35(6), pp.1131-1139.

[83] Zhou, Q., Shi, P., Lu, J. and Xu, S., 2011. Adaptive output-feedback fuzzy tracking control for a class of nonlinear systems. *IEEE Transactions on Fuzzy Systems*, 19(5), pp.972-982.

[84] Zhou, Q., Shi, P., Xu, S. and Li, H., 2013. Observer-based adaptive neural network control for nonlinear stochastic systems with time delay. *IEEE Transactions on Neural Networks and Learning Systems*, 24(1), pp.71-80.

[85] Li, F. and Liu, Y., 2017. General stochastic convergence theorem and stochastic adaptive output-feedback controller. *IEEE Transactions on Automatic Control*, 62(5), pp.2334-2349.

[86] Scheutzow, M., 1985. Noise can create periodic behavior and stabilize nonlinear diffusions. *Stochastic processes and their applications*, 20(2), pp.323-331.

[87] Scheutzow, M. 1993. Stabilization and destabilization by noise in the plane. *Stochastic Analysis and Applications*, 11, pp. 97–113.

[88] Mao, X. 1994. Stochastic stabilization and destabilization. *Systems & control letters*, 23, pp. 279–290.

[89] Li, F. and Liu, Y., 2015. Global stabilization via time-varying output-feedback for stochastic nonlinear systems with unknown growth rate. *Systems & Control Letters*, 77, pp.69-79.

[90] Li, F. and Liu, Y., 2016. Stabilization and destabilization via time-varying noise for uncertain nonlinear systems. *ESAIM: Control, Optimisation and Calculus of Variations*, 22(3), pp.610-624.

[91] Mao, X. 2016. Almost sure exponential stabilization by discrete-time stochastic feedback control. *IEEE Transactions on Automatic Control*, 61(6), pp. 1619–1624.

[92] Fernholz, R. and Karatzas, I. 2005. Relative arbitrage in volatility-stabilized markets. *Annals of Finance*, 1, pp. 149-177.

[93] Shkolnikov, M. 2013. Large volatility-stabilized markets. *Stochastic Processes and their Applications*, 123, pp. 212–228.

[94] Mao,X. 2013. Stabilization of continuous-time hybrid stochastic differential equations by discrete-time feedback control. *Automatica*, 49(12), pp. 3677-3681.

[95] Hagiwara,T. and Araki, M. 1988. Design of stable state feedback controller based on the multirate sampling of the plant output. *IEEE Trans. Automat. Control*, 33(9), pp. 812–819.

[96] Mao, X., Liu, W., Hu, L., Luo, Q., and Lu, J. 2014. Stabilisation of hybrid stochastic differential equations by feedback control based on discrete-time state observations. *Systems & Control Letters*, 73, pp. 88–95.

[97] You, S., Liu, W., Lu, J., Mao, X. and Qiu, Q. 2015. Stabilization of hybrid systems by feedback control based on discrete-time state observations. *SIAM J. Control Optim.* 53(2), pp. 905–925.

[98] You, S., Hu, L., Mao, W. and Mao, X. 2015. Robustly exponential stabilization of hybrid uncertain systems by feedback controls based on discrete-time observations. *Statistics* & *Probability Letters*, 102, pp.8-16.

[99] Qiu, Q., Liu, W., Hu, L., Mao, X. and You, S. 2016 . Stabilization of stochastic differential equations with Markovian switching by feedback control based on discrete-time state observation with a time delay. *Statistics* & *Probability Letters* ,115, pp. 16–26.

[100] Ruan, X., Zhu, W., Huang, J. and Zhang, J.E. 2016. Equilibrium asset pricing under the Lévy process with stochastic volatility and moment risk premiums. *Economic Modelling* , 54, pp. 326–338.

[101] Do, H.X., Brooks, R., Treepongkaruna, S. and Wu, E. 2016. Stock and currency market likages: New evidence from realized spillovers in higher moments. *Internatinal Review of Economics & Finance* , 42, pp. 167–185.

[102] Da Fonseca, J. 2016. On moment non-explosions for Wishart-based stochastic volatility models. *European Journal of Operational Research* ,254, pp. 889–894.

[103] Singh, C. and Upneja, R. 2014. Accurate calculation of high order pseudo-Zernike moments and their numerical stability. *Digital Signal Processing*, 27, pp. 95–106.

[104] Dong, R and Mao, X. 2017. On Pth Moment Stabilization of Hybrid Systems by Discrete-time Feedback Control. *Stochastic Analysis and Applications*, 35(5), pp. 803–822

[105] Lu, J. and Chen, G. 2005. A time-varying complex dynamical network model and its controlled synchronization criteria. *IEEE Transactions on Automatic Control* ,50(6), pp. 841–846.

[106] Wang, J.L., Wu, H.N. and Huang, T. 2015. Passivity-based synchronization of a class of complex dynamical networks with time-varying delay. *Automatica*, 56, pp.105–112.

[107] Morin, P., Eudes, A. and Scandaroli, G. 2017. Uniform Observability of Linear Time-Varying Systems and Application to Robotics Problems. *In International Conference on Geometric Science of Information* (pp. 336-344). Springer, Cham.

[108] Zhang, Q. 2002. Adaptive observer for multiple-input-multiple-output (MIMO) linear time-varying systems. *IEEE transactions on automatic control*, 47, pp. 525–529.

[109] Dong, R., 2018. Almost sure exponential stabilization by stochastic feedback control based on discrete-time observations. *Stochastic Analysis and Applications*,36(4), pp.561-583.

[110] Lu, J., Li, Y., Mao, X. and Qiu, Q., 2017. Stabilization of Hybrid Systems by Feedback Control Based on DiscreteTime State and Mode Observations. *Asian Journal of Control*, 19(6), pp.1943–1953.

[111] Li, Y., Lu, J., Kou, C., Mao, X. and Pan, J., 2017. Robust stabilization of hybrid uncertain stochastic systems by discretetime feedback control. *Optimal Control Applications and Methods*, 38(5), pp.847–859.

[112] Zhu, Q. and Zhang, Q. 2017. pth moment exponential stabilisation of hybrid stochastic differential equations by feedback controls based on discrete-time state observations with a time delay. *IET Control Theory & Applications*, 11(12), pp.1992-2003.

[113] Arnold, L. and Tudor, C. 1998. Stationary and almost periodic solutions of almost periodic affine stochastic differential equations. *Stochastics and Stochastic Reports* 64, pp. 177–193.

[114] Bezandry, P.H. and Diagana, T. 2011. *Almost periodic stochastic processes.* Springer Science and Business Media.

[115] Sabo, J.L. and Post, D.M. 2008. Quantifying periodic, stochastic, and catastrophic environmental variation. *Ecological Monographs*, 78(1), pp. 19–40.

[116] Tsiakas, I. 2005. Periodic stochastic volatility and fat tails. *Journal of Financial Econometrics*, 4(1) , pp. 90-135.

[117] Wang, T. and Liu, Z. 2012. Almost periodic solutions for stochastic differential equations with Lévy noise. *Nonlinearity*, 25(10), 2803.

[118] Zhang, X., Hu, G. and Wang, K., 2014. Stochastic periodic solutions of stochastic periodic differential equations. *Filomat*, 28(7), pp.1353-1362.

[119] Wang, C. and Agarwal, R.P. 2017. Almost periodic solution for a new type of neutral impulsive stochastic LasotaWazewska timescale model. *Applied Mathematics Letters*, 70, pp. 58–65.

[120] Wang, C., Agarwal, R.P. and Rathinasamy, S. 2017. Almost periodic oscillations for delay impulsive stochastic Nicholsons blowflies timescale model. *Computational and Applied Mathematics*, pp.1-22.

[121] Rifhat, R., Wang, L. and Teng, Z. 2017. Dynamics for a class of stochastic SIS epidemic models with nonlinear incidence and periodic coefficients. *Physica A: Statistical Mechanics and its Applications*, 481: 176–190.

[122] Dong, R. and Mao, X. 2018. Stabilization of Continuous-time Periodic Stochastic Systems by Feedback Control Based on Discrete-time Periodic Observations. Manuscript submitted for publication.

[123] Skorokhod, A.V., 1989. *Asymptotic methods in the theory of stochastic differential equations.* American Mathematical Society.

[124] Slotine, J.J.E. and Li, W. 1991. *Applied Nonlinear Control.* Prentice Hall, New Jersey.

[125] Θksendal, B.K., 2003. *Stochastic Differential Equations: An Introduction with Applications.* 6th ed. Springer, Berlin.

[126] Professor Karl Sigman's Lecture Notes on Monte Carlo Simulation, Simulation of Markov chains, 2007. Columbia University. `http://www.columbia.edu/~ks20/4703-Sigman/4703-07-Notes-MC.pdf`

[127] Ross, K., 2008. Stochastic control in continuous time. *Lecture Notes on Continuous Time Stochastic Control.* Spring. `http://statweb.stanford.edu/~kjross/Stat220notes.pdf`