COMPRESSIBLE FLOWS WITH CIRCULAR SECTOR HODOGRAPHS

by

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for the degree of Doctor of Philosophy.

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DECLARATION

I hereby declare that the following thesis is a record of original work, that it has been composed by me, and that it has not been accepted for any other degrees.

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ABSTRACT

By means of finite Fourier transforms the Green's function and hence the general solution is found for the Chaplygin equation of motion for an important class of flows, i.e. 2-dimensional non-viscous compressible 'simple wedge flows' with circular sector hodographs (these flows have been defined and classified according to their hodograph diagrams by Birkhoff and Zarantonello). From the solution expression for velocity potential and physical space coordinates are derived in terms of the hodograph variables.

For the particular case of Réthy flows the solution is used to find the drag coefficient, firstly in an exact analytical form and then, for <u>sonic</u> jet flows past <u>thin</u> wedges, as a series in ascending powers of the wedge angle; comparisons are made with the results obtained from the approximate equations of Tricomi and of Tomotika and Tamada. The study of sonic Réthy flows of small wedge angle is taken further and series which are uniformly valid for all possible source velocities are found for the wedge length and stand-off (i.e. distance of the wedge from the channel) in terms of the (small) wedge angle. From these series certain limitations on the lengths and pressure differences can be determined. Some examples of the general solution (including the solution for Réthy flows) are discussed in relation to earlier published papers and a discrepancy in some Russian papers is explained.

The thesis ends with a theorem on sonic jets. This states that for simple wedge flows involving sonic jets, the physical changes due to the presence of solid boundaries in the flow are completed within a finite distance in those directions in which sonic jet flow prevails.

CONTENTS

CHAPTER I

INTRODUCTION

The application of the hodograph method to the equations of gas dynamics dates back to the turn of the century; it was devised by Molenbrock in 1898 and used fairly extensively in work on gas jets by Chaplygin in 1902. It was not for another thirty years, however, that the full significance of Chaplygin's work was appreciated, when it was used in the researches of Demtchenko and Busemann. More recently (1947) a general theory for the use of the hodograph plane for problems of compressible flows past a body was presented independently by Cherry (6) and Lighthill (15); their work has paved the way for the extensive field of research covered since and still being developed.

Briefly, the method consists in changing the independent variables in the equations of motion from the physical co-ordinates to the components of velocity. For incompressible (steady) flow, the resulting equations for the stream function ψ and the velocity potential ϕ are still Laplacian. The effect of the transformation in compressible flow is to linearise the equations of motion, thus enabling analytic techniques to be applied in attempts at solution. Against this remarkable advantage of linearity, however, has to be set the one drawback of the hodograph method, viz. the difficulty in assessing boundary conditions. In the Lighthill and Cherry method, for example, solutions are obtained by generalisation and analytic continuation from the corresponding solution in incompressible flow; then the boundary has to be determined a posteriori from the generalised solution, the boundaries for incompressible flow being only a limiting case of the compressible flow boundary. But for problems involving jet flows, where the dividing streamline is either a straight line or free (and therefore of constant velocity), no such difficulty is encountered and the boundary value problem can be set up immediately in the hodograph plane.

The equation of motion in the hodograph plane of a two-dimensional, steady, non-viscous irrotational, compressible flow of an ideal polytropic fluid as given by Chaplygin (5) is

$$4\tau^2(1-\tau)\frac{\partial^2\psi}{\partial\tau^2} + 4\tau\left\{1+(\beta-1)\tau\right\}\frac{\partial\psi}{\partial\tau} + \left\{1-(2\beta+1)\tau\right\}\frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad (1.1)$$

where τ is related to the resultant speed q at any point by the relation $\tau = q^2/q_m^{\ 2}$, q_m being the maximum possible speed in the fluid under adiabatic expansion; and θ is the angle made by the flow with some fixed direction, usually taken as the direction of flow in some region where the streamlines are parallel straight lines. β is related to γ, the ratio of specific heats, by $\beta = 1/(\gamma-1)$. A general solution of (1.1) in which the variables are separated was given by Chaplygin himself; a second such solution was defined by Lighthill in 1947. Both solutions involve hypergeometric functions and are quoted in Chapter III.

If the variable τ is transformed to σ where

$$\sigma = \frac{1}{2}\int_{\tau}^{\tau_s}\left(\frac{1-\tau}{1-\tau_s}\right)^{\beta}\frac{d\tau}{\tau} \qquad (1.2)$$

τ_s being the value of τ at sonic speed, Chaplygin's equation assumes the simple form

$$\frac{\partial^2 \psi}{\partial \sigma^2} + k(\sigma) \frac{\partial^2 \psi}{\partial \theta^2} = 0 , \tag{1.3}$$

where $k(\sigma) = (1-\tau_s)^{2\beta}(1-\tau/\tau_s)/(1-\tau)^{2\beta+1}$.

The complexity of the solution of the Chaplygin equation has encouraged many authors to seek approximations to the equation, particularly in the form (1.3). The simplest of these, a replacement of $k(\sigma)$ by the first term in its Maclaurin expansion, leads, with the change of variable $\eta = (\gamma+1)^{2/3}\sigma$, to the equation

$$\frac{\partial^2 \psi}{\partial \eta^2} + \eta \frac{\partial^2 \psi}{\partial \theta^2} = 0.$$

The solution of this equation involves Bessel Functions and has been widely discussed by Tricomi. However, this solution is strictly valid only for small perturbations about $\sigma = 0$ (i.e. $\tau = \tau_s$), $\theta = 0$, and in particular gives a bad fit near the stagnation point (where $\sigma \to \infty$). But it has been widely used (and with good results) for a variety of problems involving near-parallel flows at near-sonic speeds. Another approximation, which tends to the correct limit at both the sonic and the stagnation point is due to Tomotika and Tamada (22). Here

$$k(\sigma) = a\left(1 - e^{-2K\sigma}\right) ; \quad a = \left(\frac{2}{\gamma+1}\right)^{2\beta} , \quad K = \left(\frac{\gamma+1}{2}\right)^{2\beta+1} .$$

The solution again involves only Bessel Functions, but is more complex than the Tricomi solution.

In recent years a great deal of research has gone into the solution of the Chaplygin equation in its exact and approximate forms for particular classes of flow. For example, Germain (11) has shown, with particular reference to the Tricomi and Tomotika and Tamada equations, how transform techniques can be applied to generalised equations of type (1.3). His ideas have been developed by Mackie (16) and used to obtain an exact solution for the "Roshko model" problem. Bergman (3) has used another method, defining operators which transform analytic functions of a complex variable into stream functions defined in a 'pseudo-logarithmic' plane to solve the Chaplygin equation for a wide class of flows.

Yet another method of solving the Chaplygin equation, this time for the class of flows defined by Birkhoff and Zarantonello as "simple flows past wedges" with circular sector hodographs (c.f. Chapter II), forms the first part of this thesis. The method, although not so generally applicable as those just mentioned (of Germain, Mackie and Bergman), nevertheless gives a simple and direct solution to the exact Chaplygin equation itself (rather than the generalised form (1.3)) for an important class of flows. Once the general solution for the stream function ψ (in terms of the velocity components τ and θ) has been found, expressions for the velocity potential ϕ and the complex space co-ordinate z, also in terms of τ and θ , follow directly.

In Chapters V and VI the general results of the previous chapters are applied to the special sub-class of Réthy flows. The wedge length and the drag coefficient are found, first in an exact analytical form, and then, for a sonic jet and small wedge angle α, in the form of a series in ascending powers of α. Similar series for the drag coefficient are subsequently found, starting from the Tomotika and Tamada equation and from the Tricomi equation (in place of Chaplygin); interesting results come out in the comparison of the three series. The study of Rethy flows of sonic jets past thin wedges is then taken further in order to find out exactly how the length of the wedge and its distance from the channel vary with the wedge angle (α), the channel width and the source velocity, and hence to find the variation possible in predetermining initial conditions for setting up such a flow.

Before returning to more general results, one chapter (VII) will be devoted to a study of some recently published papers which are connected in one way or another with the work of this thesis. In particular, several Russian papers contain solutions of the Chaplygin equation (obtained by various methods) for flows which are included in the general class defined here.

Finally a theorem is proved for the general flows defined in Chapter II, when the free stream velocity is sonic. In 1947, Guderley (14) proved that a jet of inviscid gas, issuing with sonic speed on the free boundary from a hole in an infinite straight wall, becomes a uniform parallel sonic jet at a finite distance from the hole. The same result, including a calculation of the 'length' of the jet was given later by Roumieu (20), Oviannikov (19) and Germain and Bader (12). In the final chapter of this thesis, Guderley's result is generalised to all simple flows past wedges with circular sector hodographs; for all such flows it is proved that when the pressure on the jet surface is such that the free stream velocity is sonic, all the disturbances due to the presence of solid boundaries in the flow take place within a finite distance, after which the flow continues as a uniform parallel sonic stream.

CHAPTER II

THE BOUNDARY VALUE PROBLEM

The boundary value problem for the solution of the Chaplygin equation is set up for subsonic and sonic jet flows, whose boundary in the physical plane is in turn in a fixed straight line, at a constant angle to this line, and free, and whose boundary in the hodograph plane is the perimeter of a circular sector of angle not greater than π. These flows include a jet flow against a wall, flow through an aperture, flow from a funnel etc. The corresponding incompressible flows, "simple flows past wedges" with circular sector hodographs, have been classified by Birkhoff and Zarantonello (4) as part of a general classification of simple flows (simple in the Euler sense *) according to their representation in the hodograph plane. The hodograph diagrams are unchanged for compressible flow and can be used to define the flows to be considered here.

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* A complex velocity field $\zeta(z)$, defined on an open domain R with closure $\bar{R}$ is called an Euler simple flow if and only if

(i) R is locally simply covered,

(ii) R is simply connected,

(iii) $\zeta(z)$ is bounded and continuous in $\bar{R}$,

(iv) the boundary $\bar{R}$-R consists of a finite number of rectifiable streamlines turning through a finite total angle.

The Birkhoff and Zarantonello classification is based on the formula

$$n_0 + 2n_1 + 3n_2 = 2 + n_d + 2n_s , \qquad (2.1)$$

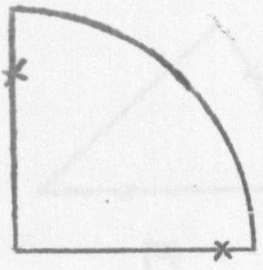
where n_o, n_1 and n_2 are respectively the numbers of jets or tubes, oceans (i.e. semi-infinite streams) and infinite streams, n_d is the number of dividing points on the boundary and n_s the number of internal stagnation points in the flow. The method of classification used is to give n_d and n_s successively larger integral values and interpret corresponding arrangements of n_o, n_1 and n_2 on fixed and free boundaries.

Flows with circular sector hodographs have no internal stagnation points and at most one dividing point on the boundary (at stagnation),

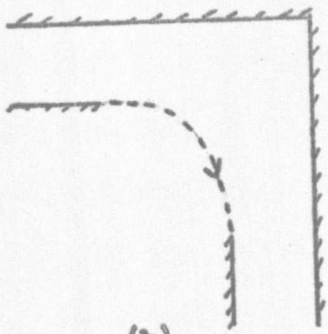
i.e. $n_s = o$, $n_d = o$ or 1 ;

hence, from (2.1) there cannot be more than three singularities on the hodograph boundary; in fact, the left-hand side of (2.1) must assume the value 2 or 3. In addition, in order that a flow be stable (in the sense that it will not "splash"), it is necessary that at every separation point on the boundary, the flow is from the fixed to the free boundary and not the other way round. (A geometrical configuration as shown in Figure II.1 (a) for example is unstable: the flow represented there is the flow round a pipe elbow, sketched in (b), which will splash).

FIGURE II.1



(a)



(b)

The complete classification, according to their hodograph diagrams, of stable simple flows past wedges with circular sector hodographs, is shown in Figure II.2. Limiting forms can be obtained by letting two singularities (denoted by a cross on the boundary) coalesce or by displacing a singularity into a corner of the hodograph diagram. Typical physical plane sketches of the flows in II.2 are given in II.3. Thus II.3 (a), a typical Réthy flow, has a hodograph diagram as shown in II.2 (a) etc. (In figures such as II.3 (a) where only the upper half plane is shown, the dividing streamline is indicated by –·–·–·) .

Examples of flow (a) are the flow past a wedge in a channel, jet flow past a wedge, flow against a wall, jet flow from a funnel etc. An interesting limiting case of (b) is the flow at a cul-de-sac.

FIGURE II.2

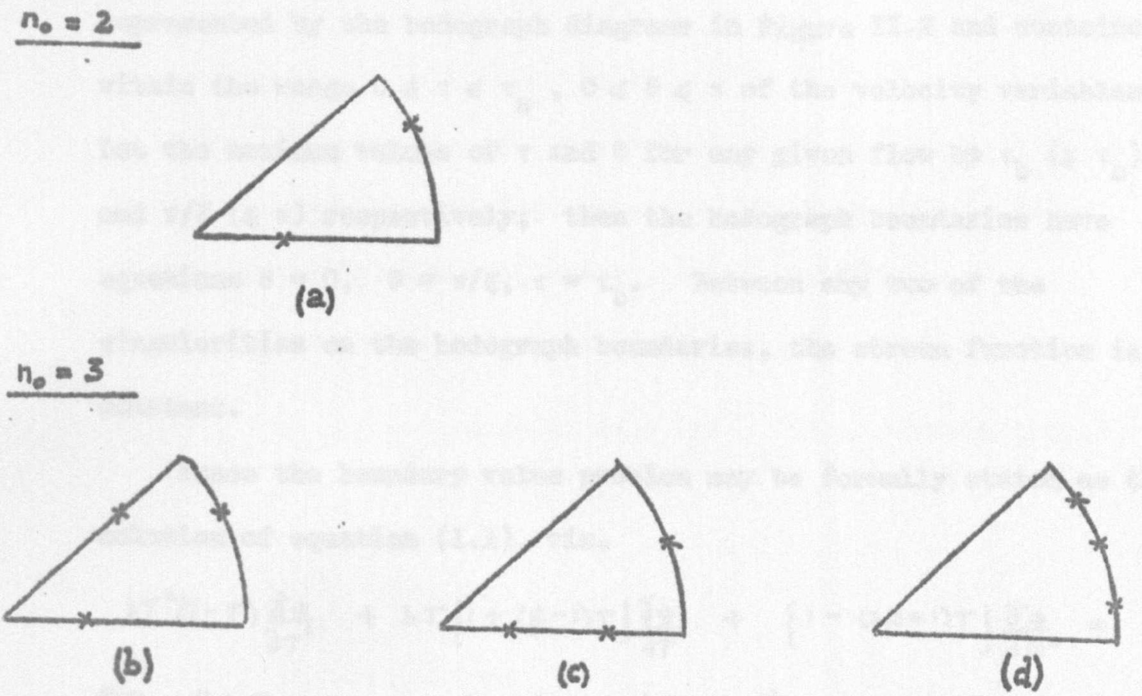
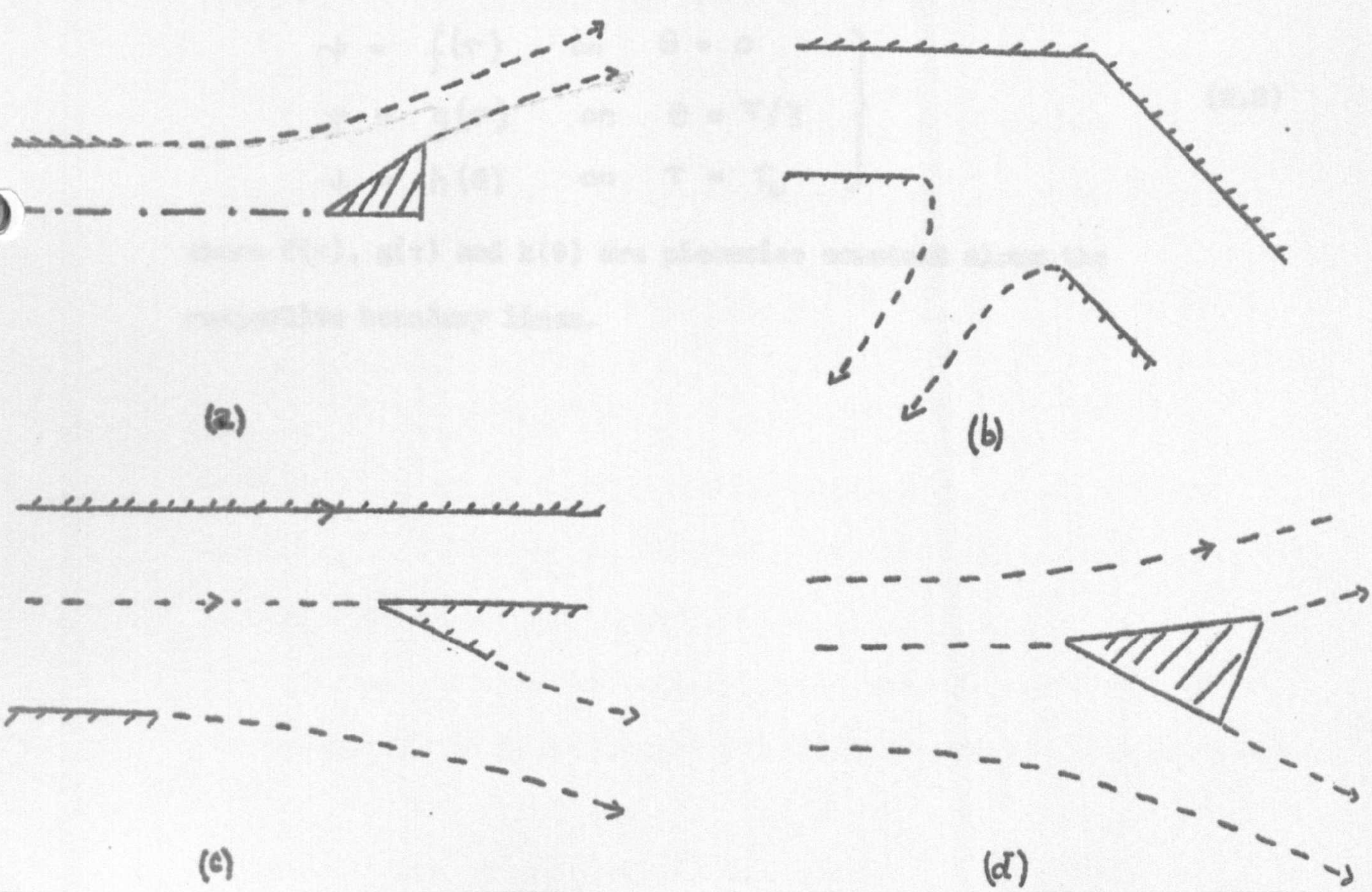


FIGURE II.3



The flows for which this problem is stated are thus all flows represented by the hodograph diagrams in Figure II.2 and contained within the range $0 \leq \tau \leq \tau_s$, $0 \leq \theta \leq \pi$ of the velocity variables. Let the maximum values of τ and θ for any given flow be τ_b ($\leq \tau_s$) and π/ξ ($\leq \pi$) respectively; then the hodograph boundaries have equations $\theta = 0$, $\theta = \pi/\xi$, $\tau = \tau_b$. Between any two of the singularities on the hodograph boundaries, the stream function is constant.

Hence the boundary value problem may be formally stated as the solution of equation (1.1), viz.

$$4\tau^2(1-\tau)\frac{\partial^2\psi}{\partial\tau^2} + 4\tau\{1+(\beta-1)\tau\}\frac{\partial\psi}{\partial\tau} + \{1-(2\beta+1)\tau\}\frac{\partial^2\psi}{\partial\theta^2} = 0$$

for $0 \leq \tau \leq \tau_b$, $0 \leq \theta \leq \pi/\xi$, where $\tau_b \leq \tau_s$, $\pi/\xi \leq \pi$

subject to the conditions

$$\left.\begin{aligned} \psi &= f(\tau) \quad \text{on} \quad \theta = 0 \\ \psi &= g(\tau) \quad \text{on} \quad \theta = \pi/\xi \\ \psi &= h(\theta) \quad \text{on} \quad \tau = \tau_b \end{aligned}\right\} \qquad (2.2)$$

where $f(\tau)$, $g(\tau)$ and $h(\theta)$ are piecewise constant along the respective boundary lines.

CHAPTER III

THE SOLUTION OF THE BOUNDARY VALUE PROBLEM

General solutions of the Chaplygin equation (1.1) in which the variables are separated were found by Chaplygin himself and by Lighthill. Chaplygin's solution, given in 1904 (5) is

$$A_0\theta + \sum_{n=1}^{\infty} A_n \omega_n(\tau) \sin(n\theta + \alpha_n)$$

where A_o, A_n and α_n are constants, and $\omega_n(\tau)$ is the solution, analytic at $\tau = 0$, of the equation

$$4\tau^2(1-\tau)\frac{d^2\omega}{d\tau^2} + 4\tau\left\{1+(\beta-1)\tau\right\}\frac{d\omega}{d\tau} - n^2\left\{1-(2\beta+1)\tau\right\}\omega = 0 \qquad (3.1)$$

It therefore has the form

$$\omega_n(\tau) = \psi_n(\tau) = \tau^{n/2}\,{}_2F_1(a_n, b_n; n+1; \tau)$$

where ${}_2F_1$ is the hypergeometric function in the usual notation, and a_n and b_n are related by the equations

$$a_n + b_n = n - \beta \quad , \quad a_n b_n = -\tfrac{1}{2}\beta n(n+1).$$

Lighthill (15) giving a second solution of the same form, defines

$$\omega_n(\tau) = \psi_n^*(\tau) = \begin{cases} \lim\limits_{m\to -n}\left\{\psi_m(\tau) - \dfrac{mC_n\,\psi_{-m}(\tau)}{m+n}\right\} & \text{when } n \text{ is a positive integer} > 1 \\ \psi_{-n}(\tau) & \text{for all other values of } n \end{cases}$$

Here $C_n = \Gamma(a_n)\Gamma(n+1-b_n)/\Gamma(a_n-n)\Gamma(1-b_n)\{\Gamma(n+1)\}^2$, and has been chosen so as to remove the difficulty of the simple pole in $\psi_n(\tau)$ when n is a negative integer other than -1.

To find the solution satisfying the boundary conditions stated in Chapter II, it is convenient to divide the problem into two parts.

In the first part, the solution ψ_1 is found for which $f(\tau) \equiv g(\tau) \equiv 0$ in (2.2): this is easily obtained as a Fourier Series. In the second part $h(\theta) \equiv 0$, and more advanced techniques are required to find the solution. The two solutions are superimposed to give the final result.

The first part then consists of finding the solution of (1.1) that satisfies the boundary conditions

$$\begin{cases} \psi = 0 & \text{on} \quad \theta = 0 \quad \text{and} \quad \theta = \pi/\xi \\ \psi = h(\theta) & \text{on} \quad \tau = \tau_b. \end{cases} \tag{3.2}$$

Let the singularities on the boundary line $\tau = \tau_b$ of the hodograph plane occur at the points $P_p(\tau_b, \theta_p)$, $p = 0,1,\ldots\bar{p}$, where $0 = \theta_0 \leq \theta_1 \leq \ldots \leq \theta_{\bar{p}} = \pi/\xi$. [The end points $(\tau_b, 0)$ and $(\tau_b, \pi/\xi)$ are assumed to be singular points both here and in the second part of the solution. If in fact no singularity exists in the final solution ψ, the singularities in ψ_1 and ψ_2 will be found to cancel each other out.] As $h(\theta)$ is piecewise constant, it may be defined as

$$h(\theta) = h_p, \quad \theta_p < \theta < \theta_{p+1}, \quad p = 0, 1, \ldots \bar{p}-1. \quad (h_{\bar{p}} = 0)$$

It follows from Figure II.2 and the above remark concerning the end points that $\bar{p} \leq 5$. These boundary conditions are represented in Figure III.1

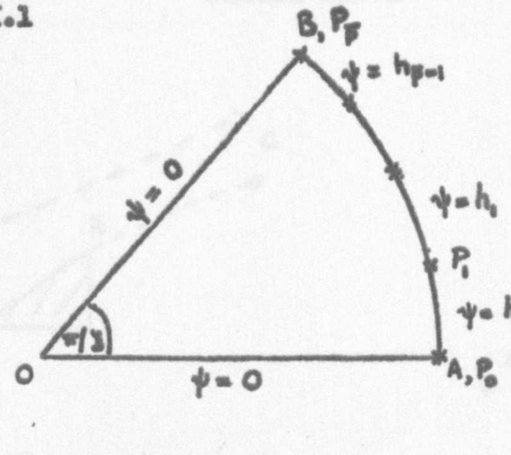


FIGURE III.1

The most general Chaplygin solution satisfying the first of the boundary conditions (3.2) is

$$\psi = \sum_{n=1}^{\infty} A_{n\xi}\, \psi_{n\xi}(\tau) \sin n\xi\theta$$

where the $A_{n\xi}$'s are arbitrary. The second condition of (3.2) requires that these be chosen so that

$$\sum_{n=1}^{\infty} A_{n\xi}\, \psi_{n\xi}(\tau_b) \sin n\xi\theta = h(\theta)$$

Hence, by the theory of Fourier sine series,

$$A_{n\xi}\, \psi_{n\xi}(\tau_b) = \frac{2}{n\pi} \sum_{p=0}^{\bar{p}-1} h_p \left(\cos n\xi\theta_p - \cos n\xi\theta_{p+1}\right).$$

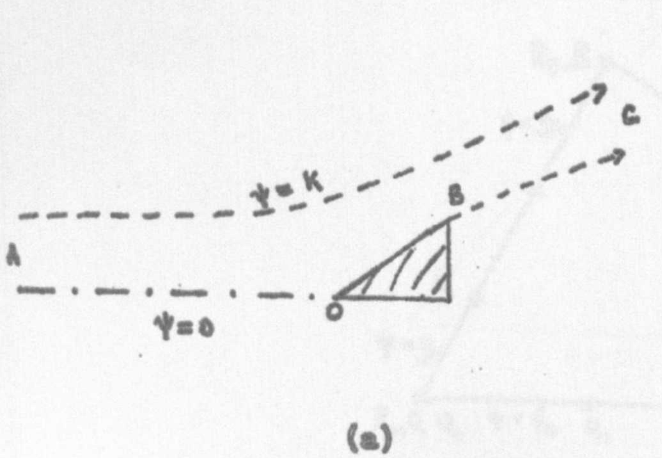
The solution satisfying both boundary conditions is therefore

$$\psi_1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{p=0}^{\bar{p}-1} \frac{1}{n} h_p \left(\cos n\xi\theta_p - \cos n\xi\theta_{p+1}\right) \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} \sin n\xi\theta \qquad (3.3)$$

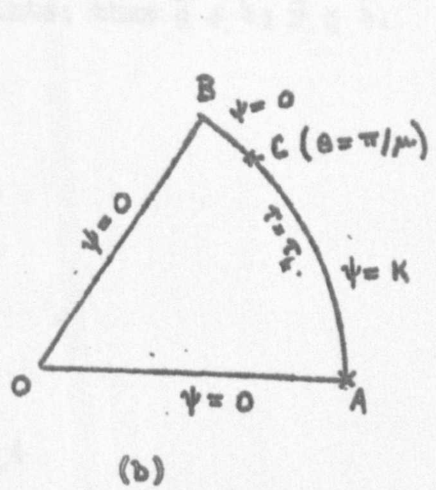
It will be shown in the next chapter that this series is uniformly convergent everywhere in the hodograph sector OAB except at the points P_p, $p = 0, 1, .. \bar{p}$. Thus ψ_1 is continuous in OAB except at P_p, where it has a finite discontinuity of range (h_{p-1}, h_p).

This result alone gives a solution to the equation of motion for such simple problems as the jet flow past a symmetric wedge, as sketched in Figure III.2(a). The hodograph diagram for the upper half of the flow is shown in III.2(b).

FIGURE III.2



(a)



(b)

If $\theta = \pi/\mu$ is the final direction of the flow, the solution by substitution in (3.3) is

$$\psi = \frac{2K}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left(1 - \cos \frac{n\xi\pi}{\mu}\right) \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} \sin n\xi\theta. \qquad (3.4)$$

When $\pi/\xi = \pi/2$, the stream function for the flow past a plate is obtained. When π/μ also takes the value $\pi/2$, the solution for a jet flow against an infinite flat plate or wall is

$$\psi = \frac{4K}{\pi} \sum_{n=0}^{\infty} \frac{1}{2n+1} \frac{\psi_{4n+2}(\tau)}{\psi_{4n+2}(\tau_b)} \sin(4n+2)\theta.$$

In the second part of the problem, the boundary conditions are

$$\begin{cases} \psi = f(\tau) & \text{on } \theta = 0 \\ \psi = g(\tau) & \text{on } \theta = \pi/\xi \\ \psi = 0 & \text{on } \tau = \tau_b . \end{cases}$$

Proceeding as in the first case, $f(\tau)$ and $g(\tau)$ can now be defined as

$$f(\tau) = f_q, \quad \tau_q < \tau < \tau_{q+1}, \quad q = 0, 1, \ldots \bar{q}-1, \text{ where } 0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_{\bar{q}} = \tau_b$$

$$g(\tau) = g_r, \quad \tau_r < \tau < \tau_{r+1}, \quad r = 0, 1, \ldots \bar{r}-1, \text{ where } 0 = \tau_0 \leq \tau_1 \leq \ldots \leq \tau_{\bar{r}} = \tau_b$$

Thus the singularities occur at the points Q_q $(\tau_q, 0)$, $q = 0, \ldots \bar{q}$ and R_r $(\tau_r, \pi/\xi)$, $r = 0, \ldots \bar{r}$, as shown in Figure III.3. Once again the end points are included as singular points; thus $\bar{q} \leq 4$; $\bar{r} \leq 4$.

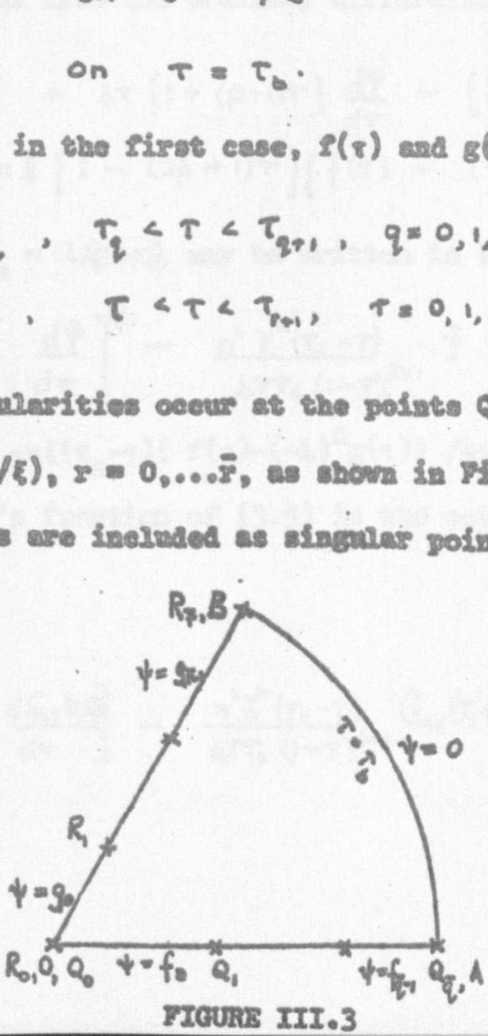


FIGURE III.3

To find the solution, finite Fourier transforms are used to transform the partial differential equation (1.1) with respect to θ. The resulting ordinary differential equation in τ is then solved by finding its Green's function to give the transform of the unknown function $\psi(\tau,\theta)$, the required solution being then obtained as the inverse transform.

Let $\Psi(n\xi,\tau) = \int_0^{\pi/\xi} \psi(\tau,\theta)\sin n\xi\theta d\theta$ be the (finite) Fourier sine transform of $\psi(\tau,\theta)$ with respect to θ. Multiply (1.1) by $\sin n\xi\theta$ and integrate with respect to θ from $\theta = 0$ to $\theta = \pi/\xi$. By using the boundary conditions it is easy to show that

$\int_0^{\pi/\xi} \frac{\partial^2\psi}{\partial\theta^2} \sin n\xi\theta d\theta = n\xi\left[f(\tau) - (-1)^n g(\tau) - n\xi\Psi(n\xi,\tau)\right]$. The equation (1.1) transforms into the ordinary differential equation

$$4\tau^2(1-\tau)\frac{d^2\Psi}{d\tau^2} + 4\tau\{1+(\beta-1)\tau\}\frac{d\Psi}{d\tau} - \{1-(2\beta+1)\tau\}n^2\xi^2\Psi$$
$$= -n\xi\{1-(2\beta+1)\tau\}\{f(\tau) - (-1)^n g(\tau)\},$$

which, since $\tau_s = 1/(2\beta+1)$, may be written in the form

$$\frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{d\Psi}{d\tau}\right] - \frac{n^2\xi^2(\tau_s-\tau)}{4\tau\tau_s(1-\tau)^{\beta+1}}\Psi = F_{n\xi}(\tau), \tag{3.5}$$

where $F_{n\xi}(\tau) = -n\xi(\tau_s-\tau)\{f(\tau)-(-1)^n g(\tau)\}/4\tau\tau_s(1-\tau)^{\beta+1}$.

The Green's function of (3.5) is the solution $G_{n\xi}(\tau,t)$ of the equation

$$\frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{dG_{n\xi}(\tau,t)}{d\tau}\right] - \frac{n^2\xi^2(\tau_s-\tau)}{4\tau\tau_s(1-\tau)^{\beta+1}}G_{n\xi}(\tau,t) = \delta(\tau-t) \tag{3.6}$$

which (i) is finite at $\tau = 0$,

(ii) assumes the value zero on $\tau = \tau_b$

and (iii) is continuous on $\tau = t$.

[In this equation $\delta(\tau - t)$ is the Dirac delta function]. The most general $G_{n\xi}(\tau,t)$ satisfying (i) and (ii) is

$$G_{n\xi}(\tau,t) = \begin{cases} A\psi_{n\xi}(\tau) & , \quad \tau < t \\ B\left[\psi_{n\xi}(\tau)\psi^{*}_{n\xi}(\tau_b) - \psi_{n\xi}(\tau_b)\psi^{*}_{n\xi}(\tau)\right] & , \quad \tau > t \end{cases} \tag{3.7}$$

where A and B are arbitrary functions of t and $n\xi$ (independent of τ), and $\psi_{n\xi}(\tau)$ and $\psi_{n\xi}^{*}(\tau)$ are the Chaplygin and Lighthill functions defined at the beginning of the chapter. To satisfy condition (iii) it is necessary from (3.7) that

$$A\psi_{n\xi}(t) = B\left[\psi_{n\xi}(t)\psi^{*}_{n\xi}(\tau_b) - \psi_{n\xi}(\tau_b)\psi^{*}_{n\xi}(t)\right] \tag{3.8}$$

and from (3.6), on integrating from $\tau = t-0$, to $t+0$, that

$$\left[\tau(1-\tau)^{-\beta}\,\frac{d\,G_{n\xi}(\tau,t)}{d\tau}\right]_{\tau=t-0}^{\tau=t+0} = 1$$

On substituting the value of $\frac{dG}{d\tau}$ obtained from (3.7), this last condition is equivalent to

$$t(1-t)^{-\beta}\left[B\left\{\frac{d\psi_{n\xi}(t)}{dt}\psi^{*}_{n\xi}(\tau_b) - \psi_{n\xi}(\tau_b)\frac{d\psi^{*}_{n\xi}(t)}{dt}\right\} - A\frac{d\psi_{n\xi}(t)}{dt}\right] = 1 \tag{3.9}$$

Equations (3.8) and (3.9) now give

$$B = \frac{(1-t)^{\beta}}{t}\,\frac{\psi_{n\xi}(t)}{\psi_{n\xi}(\tau_b)} \Big/ \left[\frac{d\psi_{n\xi}(t)}{dt}\psi^{*}_{n\xi}(t) - \psi_{n\xi}(t)\frac{d\psi^{*}_{n\xi}(t)}{dt}\right].$$

The expression inside brackets on the denominator is the Wronskian of the functions $\psi_{n\xi}(t)$, and $\psi_{n\xi}^{*}(t)$; its value, derived simply from (3.5), is $K(1-t)^{\beta}/t$; from the limiting values of the expansions of the functions $\psi_{n\xi}(t)$ etc. as t tends to zero, K is determined to be equal to $n\xi$. Hence

$$B = \psi_{n\xi}(t) \,/\, n\xi\, \psi_{n\xi}(\tau_b).$$

The value of A follows at once from equation (3.8). Thus the required Green's function, obtained by substituting for A and B in (3.7), is

$$G_{n\xi}(\tau, t) = \begin{cases} \dfrac{1}{n\xi}\,\dfrac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)}\left[\psi_{n\xi}(t)\psi_{n\xi}^{*}(\tau_b) - \psi_{n\xi}(\tau_b)\psi_{n\xi}^{*}(t)\right], & \tau < t \\[2ex] \dfrac{1}{n\xi}\,\dfrac{\psi_{n\xi}(t)}{\psi_{n\xi}(\tau_b)}\left[\psi_{n\xi}(\tau)\psi_{n\xi}^{*}(\tau_b) - \psi_{n\xi}(\tau_b)\psi_{n\xi}^{*}(\tau)\right], & \tau > t \end{cases} \qquad (3.10)$$

To find the solution of equation (3.5), it is now only necessary to multiply it by its Green's function $G_{n\xi}(\tau,t)$, to multiply (3.6) by $\Psi(n\xi,\tau)$, and subtract, and to integrate with respect to τ from $\tau = 0$ to $\tau = \tau_b$. After simplifying and using the boundary values of G and Ψ, the result is

$$\Psi(n\xi, t) = \int_0^{\tau_b} F_{n\xi}(\tau)\, G_{n\xi}(\tau, t)\, d\tau .$$

The solution of (3.5), obtained by interchanging τ and t, is

$$\Psi(n\xi, \tau) = \int_0^{\tau_b} F_{n\xi}(t)\, G_{n\xi}(t, \tau)\, dt \qquad (3.11)$$

To evaluate this integral, it is necessary to specify $F_{n\xi}(t)$ more fully. This function was originally defined in (3.5) and depends on f(t) and g(t) which in turn were specified exactly in terms of f_q and g_p. It is convenient for the sake of the notation to subdivide the problem at this stage and consider first the case for which $g(\tau) \equiv 0$.

The integral $\Psi(n\xi,\tau)$ for this problem will be denoted by $\Psi_2(n\xi,\tau)$ and the corresponding solution to the original equation by $\psi_2(\tau,\theta)$. When this has been found, the solution $\psi_3(\tau,\theta)$ corresponding to $f(\tau) \equiv 0$ can be written down immediately by analogy. The two solutions superimposed give the required result.

By substitution of the Green's function (3.10) in (3.11),

$$\Psi_2(n\xi,\tau) = \frac{1}{n\xi} \frac{\psi_{n\xi}(\tau)\psi^*_{n\xi}(\tau_b)}{\psi_{n\xi}(\tau_b)} \int_0^{\tau_b} \psi_{n\xi}(t)\, F_{n\xi}(t)\, dt - \frac{1}{n\xi}\psi^*_{n\xi}(\tau)\int_0^{\tau}\psi_{n\xi}(t)F_{n\xi}(t)\,dt - \frac{1}{n\xi}\psi_{n\xi}(\tau)\int_{\tau}^{\tau_b}\psi^*_{n\xi}(t)F_{n\xi}(t)\,dt.$$

Here

$$\psi_{n\xi}(t)\,F_{n\xi}(t) = -\frac{(\tau_b-\tau)\,f(t)}{4t\tau_b(1-t)^{\beta+1}}\; n\xi\,\psi_{n\xi}(t)$$

$$= -\frac{1}{n\xi}\, f(t)\,\frac{d\phi_{n\xi}(t)}{dt} \quad \text{on using the original equation (1.1)}$$

where $\phi_{n\xi}(t) = t(1-t)^{-\beta}\, d\psi_{n\xi}(t)/dt$.

After some algebra, including the use of the Wronskian in the form

$$\phi_{n\xi}(\tau)\,\psi^*_{n\xi}(\tau) - \psi_{n\xi}(\tau)\,\phi^*_{n\xi}(\tau) = n\xi$$

the integral, evaluated in the range $\tau_q < \tau < \tau_{q+1}$, is

$$\Psi_{2,q}(n\xi,\tau) = \frac{1}{n\xi}\left[f_q + a_{n,q}\frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} + b_{n,q}\frac{\psi^*_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)}\right], \quad \tau_q<\tau<\tau_{q+1},\; q=0,1,\ldots\bar{q}-1$$

where

$$\left.\begin{aligned} a_{n,q} &= \frac{1}{n\xi}\left[\sum_1^{\bar{q}} F_q\,\phi_{n\xi}(\tau_q)\psi^*_{n\xi}(\tau_b) - \sum_{q=1}^{\bar{q}} F_q\,\phi^*_{n\xi}(\tau_q)\psi_{n\xi}(\tau_b)\right] \\ b_{n,q} &= -\frac{1}{n\xi}\sum_1^{q} F_q\,\phi_{n\xi}(\tau_q)\,\psi_{n\xi}(\tau_b) \\ \text{and } F_q &= f_q - f_{q-1} \end{aligned}\right\} \quad (3.12)$$

The solution to the boundary value problem is now obtained by means of the inverse transform

$$\psi(\tau, \theta) = \frac{2\xi}{\pi} \sum_{n=1}^{\infty} \bar{\psi}(n\xi, \tau) \sin n\xi\theta$$

This gives

$$\psi_{2,q} = \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left[f_q + a_{n,q} \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} + b_{n,q} \frac{\psi^*_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)}\right] \sin n\xi\theta \qquad (3.13)$$

which, provided $\theta \neq 0$, is equivalent to

$$f_q\left(1 - \frac{\xi\theta}{\pi}\right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left[a_{n,q} \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} + b_{n,q} \frac{\psi^*_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)}\right] \sin n\xi\theta \qquad (3.14)$$

The series in (3.13) is uniformly convergent (cf. next chapter) and equivalent to (3.14) in $\tau_q \leq \tau \leq \tau_{q+1}$, $0 < \theta \leq \pi/\xi$; hence both expressions give the true value of ψ in this range. However, on $\theta = 0$, the series in (3.13) converges to 0 while (3.14) converges to f_q ; hence only the second expression satisfies the required boundary condition.

Also, as $\tau \to \tau_q$, $\psi_{2,q} - \psi_{2,q-1} \to \begin{cases} 0 , & \theta \neq 0 \\ f_q - f_{q-1} , & \theta = 0 . \end{cases}$

The true representation of ψ_2 in $0 \leq \theta \leq \pi/\xi$ is therefore

$$\psi_{2,q} = f_q\left(1 - \frac{\xi\theta}{\pi}\right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n}\left[a_{n,q} \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} + b_{n,q} \frac{\psi^*_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)}\right] \sin n\xi\theta, \qquad (3.15)$$

$$\tau_q \leq \tau \leq \tau_{q+1}, \quad q = 0, 1, \dots \bar{q}-1$$

where $a_{n,q}$ and $b_{n,q}$ are defined in (3.12).

The corresponding solution $\psi_{3,r}(\tau, \theta)$ is

$$\psi_{3,r} = g_r \frac{\xi\theta}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}\left[c_{n,r} \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} + d_{n,r} \frac{\psi^*_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)}\right] \sin n\xi\theta, \qquad (3.16)$$

$$\tau_r \leq \tau \leq \tau_{r+1}, \quad r = 0, 1, \dots \bar{r}-1.$$

where the functions concerned are defined exactly as in (3.12), replacing a by c, b by d, f by g and q by r.

The complete solution ψ is obtained by adding ψ_1 and the appropriate values of ψ_2 and ψ_3 given respectively by (3.3), (3.15) and (3.16). Although each of ψ_1, ψ_2 and ψ_3 has been defined with singularities at two of the corners O, A, B, the sum ψ may be shown to be continuous at any of these points at which there is no singularity in the prescribed value of ψ on the hodograph boundary; the discontinuities in individual parts merely cancel each other out. (The total number of singularities in ψ is in fact two or three, occurring on the hodograph boundary as indicated in Figure II.2.) To sum up, the solution of the Chaplygin equation of motion for flows with boundaries represented in Figure III.4 is

$$\psi = \psi_1 + \psi_2 + \psi_3 ; \qquad (3.17)$$

where
$$\psi_1 = \frac{2}{\pi} \sum_{n=1}^{\infty} \sum_{p=0}^{\bar{p}-1} \frac{1}{n} h_p (\cos n\zeta\theta_p - \cos n\zeta\theta_{p+1}) \frac{\psi_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)} \sin n\zeta\theta ,$$

$$\psi_{2,q} = f_q\left(1 - \frac{\zeta\theta}{\pi}\right) + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \left[a_{n,q} \frac{\psi_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)} + b_{n,q} \frac{\psi^*_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)} \right] \sin n\zeta\theta,$$

$$\tau_q \leq \tau \leq \tau_{q+1}, \quad q = 0, 1, \ldots \bar{q}-1,$$

$$\psi_{3,r} = g_r \frac{\zeta\theta}{\pi} + \frac{2}{\pi} \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \left[c_{n,r} \frac{\psi_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)} + d_{n,r} \frac{\psi^*_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)} \right] \sin n\zeta\theta,$$

$$\tau_r \leq \tau \leq \tau_{r+1}, \quad r = 0, 1, \ldots \bar{r}-1;$$

with
$$a_{n,q} = \frac{1}{n\zeta} \left[\sum_{1}^{\bar{q}} (f_q - f_{q-1}) \phi_{n\zeta}(\tau_q) \psi^*_{n\zeta}(\tau_b) - \sum_{q+1}^{\bar{q}} (f_q - f_{q-1}) \phi^*_{n\zeta}(\tau_q) \psi_{n\zeta}(\tau_b) \right],$$

$$b_{n,q} = -\frac{1}{n\zeta} \sum_{1}^{q} (f_q - f_{q-1}) \phi_{n\zeta}(\tau_q) \psi_{n\zeta}(\tau_b),$$

$$c_{n,r} = \frac{1}{n\zeta} \left[\sum_{1}^{\bar{r}} (g_r - g_{r-1}) \phi_{n\zeta}(\tau_r) \psi^*_{n\zeta}(\tau_b) - \sum_{r+1}^{\bar{r}} (g_r - g_{r-1}) \phi^*_{n\zeta}(\tau_r) \psi_{n\zeta}(\tau_b) \right],$$

$$d_{n,r} = -\frac{1}{n\zeta} \sum_{1}^{r} (g_r - g_{r-1}) \phi_{n\zeta}(\tau_r) \psi_{n\zeta}(\tau_b).$$

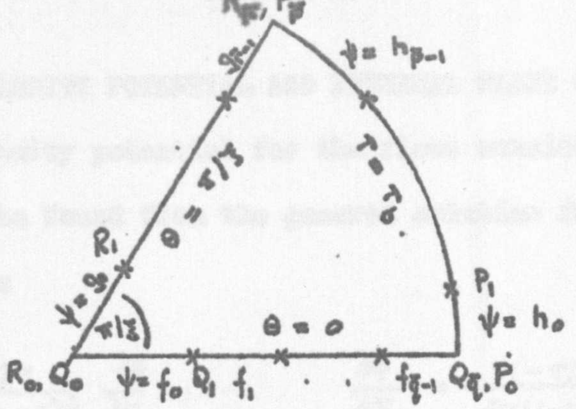


FIGURE III.4

As a check, the above solution will be used to write down the stream function for the flow of a <u>sonic</u> jet through an orifice. The diagrams for this flow are given in Figure III.5

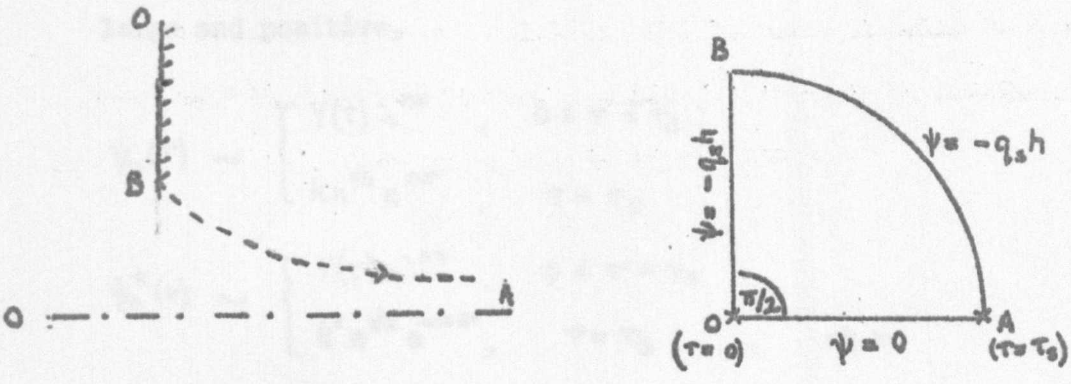


FIGURE III.5

$\psi_2 = 0$, $h_0 = g_0 = -q_s h$, $\pi/\xi = \pi/2$. Hence the solution is

$$\psi = -\frac{2q_s h}{\pi}\left[\theta + \sum_{n=1}^{\infty}\frac{1}{n}\,\frac{\psi_{2n}(\tau)}{\psi_{2n}(\tau_s)}\,\sin n\xi\theta\right],$$

which is the solution found by Germain and Bader (12) by generalisation of the corresponding solution in incompressible flow. In this particularly simple case no analytical continuation is required. More complicated examples will be discussed in Chapter VII.

CHAPTER IV

THE VELOCITY POTENTIAL AND PHYSICAL PLANE COORDINATES

The velocity potential for the flows considered in the last chapter can be found from the general solution for ψ by means of the relations

$$\frac{\partial\phi}{\partial\theta} = \frac{2\tau}{(1-\tau)^{\beta}}\frac{\partial\psi}{\partial\tau}, \qquad \frac{\partial\phi}{\partial\tau} = -\frac{1-\tau/\tau_s}{2\tau(1-\tau)^{\beta+1}}\frac{\partial\psi}{\partial\theta}. \tag{4.1}$$

First, however, it is necessary to examine the infinite series occurring in (3.17) for uniform convergence.

Certain asymptotic expansions will be found useful, and are quoted here for reference. Lighthill (15) has proved that for n large and positive,

$$\psi_n(\tau) \sim \begin{cases} V(\tau)\,e^{ns}, & 0 \le \tau < \tau_s \\ k n^{1/6} e^{n\sigma}, & \tau = \tau_s \end{cases}$$

$$\psi_n^*(\tau) \sim \begin{cases} V(\tau)\,e^{-ns}, & 0 < \tau < \tau_s \\ k' n^{1/6} e^{-n\sigma}, & \tau = \tau_s \end{cases} \tag{4.2}$$

, where

(i) s is a negative monotonic increasing function of τ, increasing from $-\infty$ at $\tau = 0$ to σ at $\tau = \tau_s$,

(ii) k and k' are constants,

(iii) $V(\tau) = \left[\dfrac{(1-\tau)^{2\beta+1}}{1-\tau/\tau_s}\right]^{1/4}$ and is a bounded function of τ when $\tau < \tau_s$.

Hence there exists a constant V such that $0 < V(\tau) \leq V$, for $0 \leq \tau < \tau_s$.

When the above expansions are differentiated, asymptotic expansions for $\phi_n(\tau)$, and $\phi_n^*(\tau)$ when $\tau < \tau_s$ are derived. When $\tau = \tau_s$, an expansion for $\phi_n(\tau_s)$ was found from a result given by Frankl (10)*. The leading terms in the expansions for large positive n are

$$\phi_n(\tau) \sim \begin{cases} W(\tau)\, n\, e^{nS}, & 0 \le \tau < \tau_s \\ \ell\, n^{4/6}\, e^{n\sigma}, & \tau = \tau_s \end{cases} \tag{4.3}$$

$$\phi_n^*(\tau) \sim -\, W(\tau)\, n\, e^{-nS}, \quad 0 < \tau < \tau_s,$$

where $W(\tau) = \dfrac{\tau}{(1-\tau)^{\beta}}\, V(\tau)\, \dfrac{ds}{d\tau}$, which is positive and $< W$ (a constant) for $\tau < \tau_s$ and ℓ is a constant.

The general solution (3.17) can be written in the form

$$\psi = \psi_1 + \psi_2 + \psi_3,$$

$$\psi_i = w_i(\theta) + \frac{2\gamma}{\pi} \sum_{n=1}^{\infty} \frac{1}{n^{\gamma}}\, u_i(n,\tau)\, k_i(n,\theta)$$

$$i = 1, 2, 3.$$

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*The asymptotic expansion given by Frankl is

$$x_n(\tau_s) = \frac{\tau_s}{n} \frac{\psi_{2n}'(\tau_s)}{\psi_{2n}(\tau_s)} \sim A_0 n^{-1/3} + A_1 n^{-1} + A_2 n^{-5/3} + \cdots$$

Together with (4,2) for $\psi_{2n}(\tau_s)$, this yields

$$\psi_n'(\tau_s) \sim \left(B_0 n^{5/6} + B_1 n^{1/6} + B_2 n^{-1/2} + \cdots \right) e^{n\sigma}.$$

Hence $\phi_n(\tau_s)$.

where

$$W_1(\theta) = 0$$

$$W_2(\theta) = W_{2,q}(\theta) = f_q(1 - \zeta\theta/\pi), \quad \tau_q \le \tau \le \tau_{q+1}, \; q = 0,1,\ldots \bar{q}-1$$

$$W_3(\theta) = W_{3,r}(\theta) = g_r \zeta\theta/\pi, \quad \tau_r \le \tau \le \tau_{r+1}, \; r = 0,1,\ldots \bar{r}-1$$

$$u_1(n,\tau) = \frac{\psi_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)}$$

$$u_2(n,\tau) = u_{2,q}(n,\tau) = a_{n,q}\frac{\psi_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)} + b_{n,q}\frac{\psi^*_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)}, \quad \tau_q \le \tau \le \tau_{q+1}$$

$$u_3(n,\tau) = u_{3,r}(n,\tau) = c_{n,r}\frac{\psi_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)} + d_{n,r}\frac{\psi^*_{n\zeta}(\tau)}{\psi_{n\zeta}(\tau_b)}, \quad \tau_r \le \tau \le \tau_{r+1}$$

$$k_1(n,\theta) = \sum_{p=0}^{\bar{p}-1} h_p(\cos n\zeta\theta_p - \cos n\zeta\theta_{p+1})\sin n\zeta\theta$$

$$k_2(n,\theta) = \sin n\zeta\theta$$

$$k_3(n,\theta) = \sin n\zeta(\pi/\zeta - \theta) = e^{(n+1)\pi i} k_2(n,\theta)$$

Consider first the infinite series part of ϕ_1, i.e.

$$\sum_{n=1}^{\infty} \frac{1}{n\zeta} u_1(n,\tau) k_1(n,\theta).$$

Now $\left|\sum_{n+1}^{n+t} h_p \cos n\zeta\theta_p \sin n\zeta\theta\right| \le \dfrac{|h_p|}{\sin\alpha\zeta/2}$ when $0 < \alpha \le \begin{cases}|\theta - \theta_p| \\ |\theta + \theta_p|\end{cases} \le \dfrac{2\pi}{\zeta}$

and t is any positive integer ; hence

$$\text{(i)} \qquad \left|\sum_{n+1}^{n+t} k_1(n,\theta)\right| \le \frac{H}{\sin\alpha\zeta/2}, \qquad 0 \le \theta \le \pi/\zeta, \; \theta \ne \theta_p; \quad t = 1,2,3,\ldots$$

where $H = 2\sum_{0}^{\bar{p}-1} |h_p|$

$u_1(n,\tau) = \psi_{n\zeta}(\tau)/\psi_{n\zeta}(\tau_b)$; from (4.2) this has asymptotic expensions

$$\frac{1}{n\zeta} u_1(n,\tau) \sim \begin{cases} \dfrac{V(\tau)}{V(\tau_b)} \dfrac{1}{n\zeta} e^{n\zeta(s - s_b)}, & 0 \le \tau < \tau_b < \tau_s \\[2ex] \dfrac{V(\tau)}{K} \dfrac{1}{(n\zeta)^{7/6}} e^{n\zeta(s-\sigma)}, & 0 \le \tau < \tau_b = \tau_s \\[2ex] \dfrac{1}{n\zeta}, & \tau = \tau_b = \tau_s . \end{cases} \qquad (4.4)$$

It follows that

(ii) there exists an integer n_1, such that for $n \geqslant n_1$, $\frac{1}{n\xi}\, u_1(n,\tau)$ is a positive monotonic decreasing function of n for $0 \leqslant \tau \leqslant \tau_b \leqslant \tau_s$, and

(iii) $\frac{1}{(n_1+1)\xi}\, u_1(n_1+1,\tau) \leqslant \frac{K}{n_1+1}$, $0 \leqslant \tau \leqslant \tau_b \leqslant \tau_s$ where K is the largest of the upper bounds of $\frac{1}{\xi}\frac{V(\tau)}{V(\tau_b)}$ and $\frac{V(\tau)}{\xi^{7/6}K}$, and $\frac{1}{\xi}$.

From (i), (ii), (iii) and Abel's Inequality, when $n \geqslant n_1$, $t = 1,2,3,\ldots$

$$\sum_{n+1}^{n+t} \frac{1}{n\xi}\, u_1(n,\tau)\, K_1(n,\theta) \leqslant \frac{HK}{(n_1+1)\sin\frac{\alpha\xi}{2}}\, , \quad 0 \leqslant \tau \leqslant \tau_b \leqslant \tau_s\, , \quad 0 \leqslant \theta \leqslant \frac{\pi}{\xi}\, ,\ \theta \neq \theta_p .$$

This proves the infinite series for ψ_1 uniformly convergent in the hodograph sector except along the lines $\theta = \theta_p$.

For all θ,

$$\left|\frac{1}{n\xi}\, u_1(n,\tau)\, k_1(n,\theta)\right| \leqslant H\left|\frac{1}{n\xi}\, u_1(n,\tau)\right|$$

$$\leqslant \begin{cases} HK\dfrac{e^{-n\xi\delta}}{n}\, , & \tau < \tau_b < \tau_s \\ HK\dfrac{e^{-n\xi\delta}}{n^{7/6}}\, , & \tau < \tau_b = \tau_s \\ HK\cdot\dfrac{1}{n}\, , & \tau = \tau_b \leqslant \tau_s \end{cases}$$

when $n \geqslant n_1$, where $|s - \sigma| \geqslant |s - s_b| \geqslant \delta > 0$. The series $\sum_{n=1}^{\infty}\frac{e^{-n\xi\delta}}{n}$ and $\sum_{n=1}^{\infty}\frac{e^{-n\xi\delta}}{n^{7/6}}$ converge when $\delta > 0$, while $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges.

Hence the series for ψ_1 is uniformly convergent for all θ and

$0 \leq \tau < \tau_b$. Combining this with the previous result, the series for ψ_1 is uniformly convergent everywhere in the hodograph sector $0 \leq \tau \leq \tau_b$, $0 \leq \theta \leq \pi/\xi$ except at the points (τ_b, θ_p) (the singular points P_p of Figure III.4).

A similar argument can be used to prove the uniform convergence of the series in ψ_2 and ψ_3. Thus, firstly

$$\left| \sum_{n+1}^{n+t} k_i(n,\theta) \right| \leq \frac{1}{\sin \alpha\xi/2}, \quad t = 1, 2, 3, \ldots$$

for $0 < \alpha \leq \theta < 2\pi/\xi$ and hence for $0 < \theta \leq \pi/\xi$, $i = 2$

and $0 < \alpha \leq \frac{\pi}{\xi} - \theta < \frac{2\pi}{\xi}$ and hence for $0 \leq \theta < \pi/\xi$, $i = 3$.

To prove $\frac{1}{n\xi} u_i(n,\tau)$ monotonic decreasing, asymptotic expansions for the coefficients $a_{n,q}$ etc. have to be found first ; there are

$$a_{n,q} \sim \begin{cases} A_{q+1}\, e^{n\xi(s_b - s_{q+1})}, & \tau_b < \tau_s \\ \bar{A}_{q+1}\, (n\xi)^{1/6}\, e^{n\xi(\sigma - s_{q+1})}, & \tau_b = \tau_s \end{cases} \quad q \neq \bar{q}-1$$

$$a_{n,q} \sim -f_{\bar{q}-1}, \qquad \tau_b \leq \tau_s, \quad q = \bar{q}-1$$

$$b_{n,q} \sim \begin{cases} -A_q\, e^{n\xi(s_q + s_b)}, & \tau_b < \tau_s \\ -\bar{A}_q\, (n\xi)^{1/6}\, e^{n\xi(s_q + \sigma)}, & \tau_b = \tau_s . \end{cases}$$

where $A_q = (f_q - f_{q-1})W(\tau_q)V(\tau)$; $\bar{A}_q = (f_q - f_{q-1})W(\tau_q)k$.

(There are similar expansions for $c_{n,r}$ and $d_{n,r}$). As a result

$$\frac{1}{n\xi} u_{a,q}(n,\tau) \begin{cases} \sim \dfrac{K}{n\xi}\left\{A_{q+1}\, e^{n\xi(s - s_{q+1})} - A_q\, e^{n\xi(s_q - s)}\right\}, & \tau < \tau_b \leq \tau_s \\ = -\dfrac{1}{n\xi} f_{\bar{q}-1}. & \tau = \tau_b \leq \tau_s . \end{cases} \tag{4.5}$$

These are bounded monotonic decreasing functions of n; hence the conditions of Abel's Inequality hold here exactly as before.

The result is the uniform convergence of the series in ψ_2 in the range $0 \leq \tau \leq \tau_b$, $0 < \theta \leq \pi/\xi$ and of ψ_3 in $0 \leq \tau \leq \tau_b$, $0 \leq \theta < \pi/\xi$.

Again, for all θ,

$$\left| \frac{1}{n\xi} u_2(n,\tau)\, k_2(n,\theta) \right| \leq \left| \frac{1}{n\xi} u_{2,q}(n,\tau) \right|, \quad \tau_q \leq \tau \leq \tau_{q+1}, \; q = 0,1,\ldots \bar{q}-1$$

$$\leq \begin{cases} \dfrac{K}{n}\left[A_{q+1} e^{-n\xi\beta_1} + A_q e^{-n\xi\beta_1} \right], & \tau < \tau_b \\ \dfrac{f_{\bar{q}-1}}{n\xi}, & \tau = \tau_b. \end{cases}$$

where $|s - s_{q+1}| \geq \beta_1 > 0$ for $\tau < \tau_{q+1}$

and $|s - s_q| \geq \beta_2 > 0$ for $\tau \geq \tau_q$

Thus the series in $\psi_{2,q}$ is uniformly convergent in $0 \leq \theta \leq \pi/\xi$, $\tau_q < \tau < \tau_{q+1}$. It follows from the first result that the series in ψ_2 is uniformly convergent everywhere in the hodograph sector except at the points $(\tau_q, 0)$. (i.e. the singular points Q_q in Fig. III.4). Similarly the series in ψ_3 converges uniformly everywhere except at points $(\tau_r, \pi/\xi)$ (the points R_r of III.4).

It follows immediately from these results that in the hodograph sector $0 \leq \tau \leq \tau_b$, $0 \leq \theta \leq \pi/\xi$, ψ_1 is a continuous function of τ and θ except at the singular points on the boundary $\tau = \tau_b$, where it has a finite discontinuity (of $h_p - h_{p-1}$ at θ_p) ; $\psi_2(\tau,\theta)$ is continuous except at the singular points on $\theta = 0$, where again the discontinuity is finite ($f_q - f_{q-1}$ at τ_q) ; and $\psi_3(\tau,\theta)$ is continuous except at singular points on $\theta = \pi/\xi$. Adding the three functions, the stream function ψ is a continuous function of τ and θ in the hodograph sector except at the singular points on the boundary.

The velocity potential ϕ_i corresponding to each ψ_i will now be obtained from (4.1). These equations, and (3.17) give

$$\frac{\partial \phi_i}{\partial \theta} = \frac{4\gamma}{\pi} \sum_{n=1}^{\infty} \frac{1}{n\gamma} \frac{\tau}{(1-\tau)^{\beta}} \frac{\partial u_i(n,\tau)}{\partial \tau} k_i(n,\theta) \tag{4.6}$$

$$\frac{\partial \phi_i}{\partial \tau} = -\frac{1-(2\beta+1)\tau}{2\tau(1-\tau)^{\beta+1}} \frac{d w_i}{d\theta} - \frac{\gamma}{\pi} \sum_{n=1}^{\infty} \frac{1}{n\gamma} \frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}} u_i(n,\tau) \frac{\partial k_i(n,\theta)}{\partial \theta}$$

provided the infinite series occurring in these equations are uniformly convergent.

Let $\dfrac{\tau}{(1-\tau)^{\beta}} \dfrac{\partial u_i(n,\tau)}{\partial \tau} = V_i(n,\tau)$

and $\dfrac{1}{n\gamma} \dfrac{\partial k_i(n,\theta)}{\partial \theta} = l_i(n,\theta)$.

[Then $V_1(n,\tau) = \dfrac{\phi_{n\gamma}(\tau)}{\psi_{n\gamma}(\tau_b)}$, $V_{2,q}(n,\tau) = a_{n,q} \dfrac{\phi_{n\gamma}(\tau)}{\psi_{n\gamma}(\tau_b)} + b_{n,q} \dfrac{\phi^{*}_{n\gamma}(\tau)}{\psi_{n\gamma}(\tau_b)}$ etc.

$l_2(n,\theta) = \cos n\gamma\theta$ etc.

From the asymptotic expansions (4.2) and (4.3), it follows that for large n,

$$V_i(n,\tau) \sim \begin{cases} C n\gamma\, u_i(n,\tau), & 0 \le \tau \le \tau_b < \tau_s \\ \bar{C}(n\gamma)^{2/3} u_i(n,\tau), & \tau = \tau_s. \end{cases} \tag{4.7}$$

Thus, using expansions (4.4) and (4.5) it can be easily shown that the above series are uniformly convergent everywhere except on certain boundaries, viz. $\tau = \tau_b < \tau_s$ when $i = 1$; $\tau = \tau_q$ and $\tau = \tau_{q+1}$ for each q-interval when $i = 2$; $\tau = \tau_r$ and $\tau = \tau_{r+1}$ for each r-interval when $i = 3$.

Now, equation (3.1), which is satisfied by both $\psi_n(\tau)$ and $\psi^{*}_n(\tau)$ can be written in the form

$$\frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}}\,\omega_n(\tau) = \frac{4}{n^2}\frac{d}{d\tau}\left[\frac{\tau}{(1-\tau)^\beta}\frac{d\omega_n(\tau)}{d\tau}\right]$$

Hence $$\frac{1-(2\beta+1)\tau}{\tau(1-\tau)^{\beta+1}}\,u_i(n,\tau) = \frac{4}{(n\xi)^2}\frac{d}{d\tau}v_i(n,\tau)$$

The equations (4.6) can now be integrated with respect to θ and τ respectively to give

$$\phi = \phi_1 + \phi_2 + \phi_3,$$

$$\phi_i = s_i(\tau)\frac{\xi}{\pi} - \frac{4\xi}{\pi}\sum_{n=1}^{\infty}\frac{1}{(n\xi)^2}v_i(n,\tau)\,l_i(n,\theta) \tag{4.8}$$

where

$$s_1(\tau) = 0$$

$$s_2(\tau) = s_{2,q}(\tau) = -f_q\left\{\frac{1}{(1-\tau)^\beta} + \int^{\tau}\frac{d\tau}{\tau(1-\tau)^\beta}\right\} + \sum_{q=1}^{q}(f_q - f_{q-1})\left\{\frac{1}{(1-\tau_q)^\beta} + \int^{\tau_q}\frac{d\tau}{\tau(1-\tau)^\beta}\right\},$$
$$\tau_q \le \tau \le \tau_{q+1},\quad q = 0, 1, \ldots \bar{q}-1$$

$$s_3(\tau) = s_{3,r}(\tau) = g_r\left\{\frac{1}{(1-\tau)^\beta} + \int^{\tau}\frac{d\tau}{\tau(1-\tau)^\beta}\right\} - \sum_{r=1}^{r}(g_r - g_{r-1})\left\{\frac{1}{(1-\tau_r)^\beta} + \int^{\tau_r}\frac{d\tau}{\tau(1-\tau)^\beta}\right\}$$
$$\tau_r \le \tau \le \tau_{r+1},\quad r = 0, 1, \ldots \bar{r}-1.$$

$$v_i(n,\tau) = \frac{\tau}{(1-\tau)^\beta}\frac{\partial u_i(n,\tau)}{\partial\tau}$$

$$l_i(n,\theta) = \frac{1}{n\xi}\frac{\partial k_i(n,\theta)}{\partial\theta}$$

(The constants in $s_2(\tau)$ and $s_3(\tau)$ have been chosen so as to make these functions continuous at $\tau = \tau_q$, $q = 1, 2, \ldots \bar{q}-1$, and $\tau = \tau_r$, $r = 1, \ldots \bar{r}-1$ respectively).

Uniform convergence tests can now be applied to the series in (4.8) in exactly the same way as in the series for ϕ_1 making use of the relation (4.7). As a result, the series for ϕ is uniformly convergent for the same range of τ and θ as the series for ϕ except when $\tau_b = \tau$, when the series for ϕ_1 is uniformly convergent at the points (τ_s, θ_p). Thus, when $\tau_b < \tau_s$, ϕ is a continuous function of

τ and θ except at the singular points on the boundary (where it becomes infinite); when $\tau_b = \tau_s$, ϕ is discontinuous only at the singular points on the solid boundaries, and is continuous and finite everywhere on the free boundary.

It now remains to find the coordinates in the physical plane as a function of τ and θ. Let z be the complex position variable; then z is related to τ and θ by the equation

$$dz = \frac{e^{i\theta}}{q}\left[d\phi + \frac{i}{(1-\tau)^{\beta}} d\psi \right].$$

Making use of (4.1) and (4.6), $\frac{\partial z}{\partial \tau}$ and $\frac{\partial z}{\partial \theta}$ can be found in terms of infinite series which can be integrated term by term. The final result is

$$Z = Z_1 + Z_2 + Z_3, \tag{4.9}$$

$$Z_i = \frac{\xi}{\pi q} \cdot \frac{1}{(1-\tau)^{\beta}} \left[d_i e^{i\theta} + {}_{\psi}Z_i + {}_{\phi}Z_i \right] + C_i,$$

where

$${}_{\psi}Z_i = \sum_{n=1}^{\infty} \mu_i(n,\tau)\, K_i(n,\theta),$$

$${}_{\phi}Z_i = -2(1-\tau)^{\beta} \sum_{n=1}^{\infty} \frac{1}{n\xi} \nu_i(n,\tau)\, \lambda_i(n,\theta),$$

$$K_1(n,\theta) = \sum_{p=0}^{\bar{p}-1} h_p \left(\cos n\xi\theta_p - \cos n\xi\theta_{p+1}\right) \left\{ \frac{e^{(n\xi+1)\theta i}}{n\xi+1} - \frac{e^{-(n\xi-1)\theta i}}{n\xi-1} \right\},$$

$$K_2(n,\theta) = \frac{e^{(n\xi+1)\theta i}}{n\xi+1} - \frac{e^{-(n\xi-1)\theta i}}{n\xi-1},$$

$$K_3(n,\theta) = e^{(n+1)\pi i} K_2(n,\theta)$$

and $\lambda_i(n,\theta)$ is the function $K_i(n,\theta)$ with a positive instead of a negative sign between the exponentials; also

$$d_1 = 0; \qquad C_1 = 0$$

$$d_2 = d_{2,q} = -f_q, \qquad C_2 = C_{2,q} = \operatorname{cosec}\frac{\pi}{\xi}\, e^{i\pi/\xi} \sum_{1}^{q} (f_q - f_{q-1}) \frac{1}{q_q(1-\tau_q)^{\beta}}, \quad q = 0 \ldots \bar{q}-1$$

$$d_3 = d_{3,r} = g_r, \qquad C_3 = C_{3,r} = -\operatorname{cosec}\frac{\pi}{\xi} \sum_{1}^{r} (g_r - g_{r-1}) \frac{1}{q_r(1-\tau_r)^{\beta}}, \quad r = 0 \ldots \bar{r}-1.$$

[$C_{2,q}$ was chosen in such a way as to make z_2 continuous at $\tau = \tau_q$ when $\theta \neq 0$; similarly with the above value of $C_{3,r}$, z_3 is continuous at $\tau = \tau_r$, $\theta \neq \pi/\xi$.]

When $K_i(n,\theta)$ and $\lambda_i(n,\theta)$ are split into their real and imaginary parts, it is easily seen that the series ${}_\psi z_i$ and ${}_\phi z_i$ converge uniformly in the same range as ψ_i and ϕ_i respectively. Thus z is a continuous function of τ and θ throughout the hodograph sector except at the singular points on the boundary, (z_1 having discontinuities on $\tau = \tau_b$, z_2 on $\theta = 0$ and z_3 on $\theta = \pi/\xi$).

Further examination of (4.9) shows that at the singular points on $\theta = 0$, where $\tau < \tau_s$, the real part of z becomes infinite (${}_\phi z_2$ diverges), while the imaginary part has a finite discontinuity (of $i(f_1 - f_0) \dfrac{1}{q_1(1-\tau_1)^\beta}$ at τ_1). Similarly at the singular points on $\theta = \pi/\xi$, $\tau < \tau_s$, the real part of $e^{-i\pi/\xi} z$ diverges, while there is a finite discontinuity again at right angles to the flow. On the free boundary, however, the real part of $e^{-i\theta_p} z$ becomes infinite at θ_p, only when $\tau_b < \tau_s$ (when ${}_\phi z_1$ diverges); when $\tau = \tau_s$, ${}_\phi z_1$ is continuous and the only discontinuity is that which occurs in ${}_\psi z_1$, which, like that in ψ_1, is finite. This interesting result for sonic jets forms the basis of a theorem which will be studied in Chapter VIII.

CHAPTER V

RÉTHY FLOWS : DRAG COEFFICIENT

An interesting special case of the flows considered in the previous chapters is a Réthy flow. It is defined by a circular sector hodograph with two singularities, one of which is on a solid boundary and the other on the free boundary ; and comprises such flows as the flow through an aperture, the flow from a funnel, jet flow against a wall or past a symmetrically placed wedge (finite or infinite), flow from a channel past a flat plate etc.

A typical Réthy Flow, the flow from an infinite channel past a symmetrically placed wedge is sketched in Figure V.1

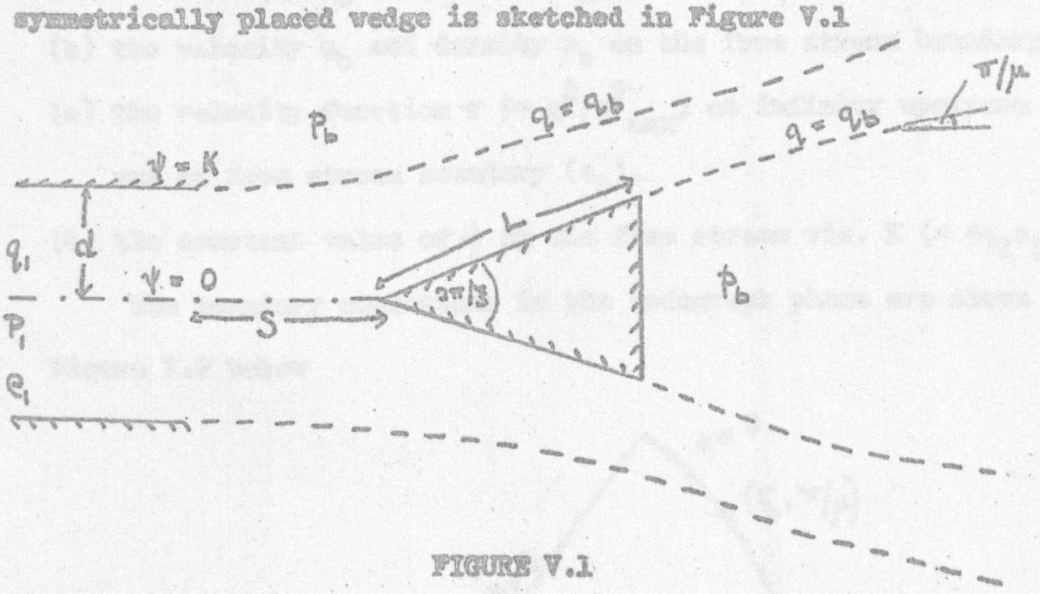


FIGURE V.1

To set up this flow the following initial conditions are required:-

(i) An undisturbed stream with specific heat ratio γ and at pressure $\underline{P_b}$ (this determines the pressure on the boundary streamline of the free jet);

(ii) a wedge of length $\underline{L}$ and angle $\underline{2\pi/\xi}$ symmetrically placed at a distance $\underline{S}$ from an infinite channel of width $\underline{2d}$;

(iii) a flow of density ρ_1, pressure p_1 and velocity q_1 parallel to the channel at infinity upstream, and making an angle of π/μ with the channel direction at infinity downstream.

These 10 constants, however, are not all independent. From the formula for z (4.9), L and S can be found in terms of the other constants. Thus, with a gas of given specific heat ratio (γ), there are at most 7 degrees of freedom in setting up a Réthy Flow. (P_b, π/ξ, d, ρ_1, q_1, p_1, π/μ or P_b, L, π/ξ, S, d and any two of ρ_1, q_1, p_1, π/μ).

From the above constants, Bernoulli's equation and the adiabatic gas equation determine the following boundary conditions immediately:-

(a) the pressure p_0 and density ρ_0 at stagnation (q=0),

(b) the velocity q_b and density ρ_b on the free stream boundary,

(c) the velocity function τ ($= q^2/q^2_{max}$) at infinity upstream (τ_1), and on free stream boundary (τ_b),

(d) the constant value of ψ on the free stream viz. K ($= dq_1\rho_1/\rho_0$).

The boundary conditions in the hodograph plane are shown in Figure V.2 below

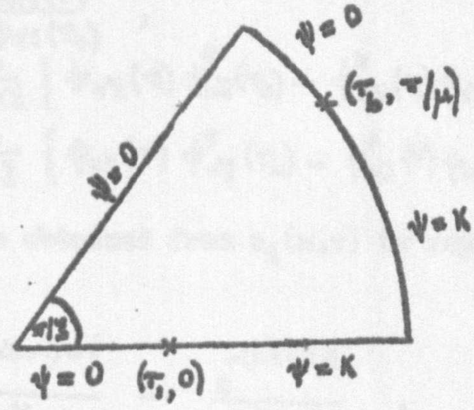


FIGURE V.2

Relations for ψ, ϕ, and z can now be written down immediately from (3.17), (4.8) and (4.9). The results which satisfy the boundary condition of V.2 are:-

$$\psi = \psi_1 + \psi_2 , \qquad (5.1)$$

$$\psi_1 = \frac{2K\xi}{\pi} \sum_{n=1}^{\infty} \frac{1}{n\xi} u_1(n,\tau)\left(1 - \cos\frac{n\xi\pi}{\mu}\right) \sin n\xi\theta$$

$$\psi_{2,q} = f_q\left(1 - \frac{\xi\theta}{\pi}\right) + \frac{2K\xi}{\pi} \sum_{n=1}^{\infty} \frac{1}{n\xi} u_{2,q}(n,\tau) \sin n\xi\theta \quad , \tau_q \le \tau \le \tau_{q+1}, \; q = 0,1.$$

$$\phi = \phi_1 + \phi_2 , \qquad (5.2)$$

$$\phi_1 = \frac{-4K\xi}{\pi} \sum_{n=1}^{\infty} \frac{1}{(n\xi)^2} v_1(n,\tau)\left(1 - \cos\frac{n\xi\pi}{\mu}\right) \cos n\xi\theta$$

$$\phi_{2,q} = -f_q \frac{\xi}{\pi}\left[\frac{1}{(1-\tau)^\beta} - \frac{1}{(1-\tau_q)^\beta} + \int_{\tau_q}^{\tau} \frac{d\tau}{\tau(1-\tau)^\beta}\right] - \frac{4K\xi}{\pi} \sum_{n=1}^{\infty} \frac{1}{(n\xi)^2} v_{2,q}(n,\tau) \cos n\xi\theta$$

$$z = z_1 + z_2 , \qquad (5.3)$$

$$z_1 = \frac{K\xi}{\pi} \frac{1}{q(1-\tau)^\beta} \sum_{n=1}^{\infty} \left(1 - \cos\frac{n\xi\pi}{\mu}\right)\left\{u_1(n,\tau) K(n,\theta) - \frac{2(1-\tau)^\beta}{n\xi} v_1(n,\tau) \lambda(n,\theta)\right\}$$

$$z_{2,q} = f_q \frac{\xi}{\pi}\left[\frac{\pi}{\xi} \operatorname{cosec}\frac{\pi}{\xi} \cdot \frac{1}{q_q(1-\tau_q)^\beta} e^{i\pi/\xi} - \frac{1}{q(1-\tau)^\beta} e^{i\theta}\right] + \frac{K\xi}{\pi} \cdot \frac{1}{q(1-\tau)^\beta} \sum \left\{u_{2,q}(n,\tau) K(n,\theta) - \frac{2(1-\tau)^\beta}{n\xi} v_{2,q}(n,\tau) \lambda(n,\theta)\right\}$$

where

where
$$u_1(n,\tau) = \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} ,$$

$$u_{2,0}(n,\tau) = \frac{1}{n\xi}\left\{\phi_{n\xi}(\tau_1)\psi^*_{n\xi}(\tau_b) - \phi^*_{n\xi}(\tau_1)\psi_{n\xi}(\tau_b)\right\}\frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} - 1 ,$$

$$u_{2,1}(n,\tau) = \frac{1}{n\xi}\left\{\psi_{n\xi}(\tau)\psi^*_{n\xi}(\tau_b) - \psi^*_{n\xi}(\tau)\psi_{n\xi}(\tau_b)\right\}\frac{\phi_{n\xi}(\tau_1)}{\psi_{n\xi}(\tau_b)} - \frac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_b)} ;$$

the functions $v_i(n,\tau)$ are obtained from $u_i(n,\tau)$ by replacing $\psi_{n\xi}(\tau)$ with $\phi_{n\xi}(\tau)$;

$$K(n,\theta) = \frac{e^{(n\xi+1)\theta i}}{n\xi+1} - \frac{e^{-(n\xi-1)\theta i}}{n\xi-1} ,$$

$$\lambda(n,\theta) = \frac{e^{(n\xi+1)\theta i}}{n\xi+1} + \frac{e^{-(n\xi-1)\theta i}}{n\xi-1} ;$$

and $f_q = 0$ when $q = 0$ and K when $q = 1$.

The z-coordinate, as given above, is measured from the stagnation point ($\frac{1}{q}\psi_{n\xi}(\tau)$ and $\frac{1}{q}\phi_{n\xi}(\tau)$ are both zero when $\tau = 0$) ; hence various functions in the physical plane (e.g. wedge length L and standoff S) can be found from (5.3) by substituting the appropriate values of τ and θ. To find the drag coefficient, it is convenient to have a space coordinate measured along the edge of the wedge. This coordinate (ℓ) is given from (5.3) as

$$\ell = |z_1(\tau, \pi/\xi) + z_{2,q}(\tau, \pi/\xi)| \quad , \quad \tau_q \leq \tau \leq \tau_{q+1} \; , \quad q = 0, 1.$$

The remainder of this chapter will be devoted to a study of the drag coefficient for Réthy Flows, an exact expression for which can be found from the above solutions of the Chaplygin equation. For the special case of small wedge angle and sonic free-stream velocity, this drag coefficient will be compared with the corresponding results obtained by using the (approximate) equations of Tricomi and Tomotika and Tamada; with this in mind it is found more convenient at this stage to change the velocity variable from τ to σ as defined in (1.2) (so that the equation of motion takes the general form (1.3)), and to define

$$\begin{aligned} L(\sigma, n) &= -\psi_n(\tau) \big/ \psi_n(\tau_b) \\ R(\sigma, n) &= -\left\{\psi_n(\tau)\psi_n^*(\tau_b) - \psi_n(\tau_b)\psi_n^*(\tau)\right\} \big/ 2n(1-\tau_s)^\beta \end{aligned} \qquad (5.5)$$

as the independent solutions of Chaplygin's equation. (These functions will be shown in Chapter VII to be identical with those defined by Mackie (16)). The solutions of the other two equations can then be written down immediately from the Chaplygin solution by giving

$L(\sigma,n)$ and $R(\sigma,n)$ the appropriate interpretation.

The expression for the ℓ-coordinate in terms of $L(\sigma,n)$ and $R(\sigma,n)$ is given from (5.3) and (5.5) as:

$$\ell = \begin{cases} \dfrac{2K\xi}{\pi}\dfrac{\rho_0}{\rho_s}\dfrac{1}{q}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos n\pi}{n^2\xi^2-1}\left\{\cos\dfrac{n\xi\pi}{\mu} - R'(\sigma_1, n\xi)\right\}\left\{L'(\sigma_1, n\xi) - \dfrac{\rho_s}{\rho}L(\sigma, n\xi)\right\}, & \infty \geq \sigma \geq \sigma_1 \\ \dfrac{K\xi}{\pi}\left\{\dfrac{\pi}{\xi}\operatorname{cosec}\dfrac{\pi}{\xi}\,\dfrac{1}{q_1}\dfrac{\rho_0}{\rho_1} - \dfrac{1}{q_1}\dfrac{\rho_0}{\rho}\right\} & \\ \quad + \dfrac{2K\xi}{\pi}\dfrac{\rho_0}{\rho_s}\dfrac{1}{q}\displaystyle\sum_{n=1}^{\infty}\dfrac{\cos n\pi}{n^2\xi^2-1}\left[\cos\dfrac{n\xi\pi}{\mu}\left\{L'(\sigma, n\xi) - \dfrac{\rho_s}{\rho}L(\sigma, n\xi)\right\} - L'(\sigma_1, n\xi)\left\{R'(\sigma, n\xi) - \dfrac{\rho_s}{\rho}R(\sigma, n\xi)\right\}\right], & \sigma_1 \geq \sigma \geq \sigma_b. \end{cases} \qquad (5.6)$$

(The 'dash' denotes differentiation with respect to σ). In the simplification of the above series, the Wronskian was used in the form

$$R(\sigma, n)\,L'(\sigma, n) - R'(\sigma, n)\,L(\sigma, n) = 1$$

The above series can be expressed as an integral, taken over a contour C parallel to the imaginary axis and cutting the real axis between 1 and 2, (since the integrand is an analytic function of z except for simple poles at $z = n\xi$, and is exponentially small for large z). Thus

$$\ell = \begin{cases} \dfrac{iK}{\pi}\dfrac{\rho_0}{\rho_s}\dfrac{1}{q}\displaystyle\int_C\left\{\cos\dfrac{z\pi}{\mu} - R'(\sigma_1, z)\right\}\left\{L'(\sigma, z) - \dfrac{\rho_s}{\rho}L(\sigma, z)\right\}\dfrac{1}{\sin\frac{z\pi}{\xi}}\,\dfrac{dz}{z^2-1}, & \infty \geq \sigma \geq \sigma_1 \\ \dfrac{K\xi}{\pi}\left\{\dfrac{\pi}{\xi}\operatorname{cosec}\dfrac{\pi}{\xi}\cdot\dfrac{1}{q_1}\dfrac{\rho_0}{\rho_1} - \dfrac{1}{q}\dfrac{\rho_0}{\rho}\right\} & \\ \quad + \dfrac{iK}{\pi}\dfrac{\rho_0}{\rho_s}\dfrac{1}{q}\displaystyle\int_C\left[\cos\dfrac{z\pi}{\mu}\left\{L'(\sigma, z) - \dfrac{\rho_s}{\rho}L(\sigma, z)\right\} - L'(\sigma_1, z)\left\{R'(\sigma, z) - \dfrac{\rho_s}{\rho}R(\sigma, z)\right\}\right]\dfrac{1}{\sin\frac{z\pi}{\xi}}\,\dfrac{dz}{z^2-1}, & \sigma_1 \geq \sigma \geq 0. \end{cases} \qquad (5.7)$$

The drag D is defined by

$$D = \sin\frac{\pi}{\xi}\int_0^L p\,d\ell$$

$$= \sin\frac{\pi}{\xi}\left[p_b L + \int_0^{q_b} \ell\rho q\,dq\right],$$ by Bernoulli's Theorem.

By using (5.7)

$$\int_0^{q_1} l\rho q\,dq = \frac{iK}{\pi}\frac{\rho_0}{\rho_s}\int_c \left\{\cos\frac{z\pi}{\mu} - R'(\sigma_1,z)\right\}\left[\int_0^{q_1}\left\{L'(\sigma,z) - \frac{\rho_s}{\rho}L(\sigma,z)\right\}\rho\,dq\right]\frac{1}{\sin\frac{z\pi}{\gamma}}\frac{dz}{z^2-1}.$$

The inner integral

$$= \int_{\sigma_1}^{\infty}\left\{L'(\sigma,z) - \frac{\rho_s}{\rho}L(\sigma,z)\right\}\rho_s q\,d\sigma$$

$$= -\rho_s q_1 L(\sigma_1,z) \qquad \text{(from (1.2))}$$

Hence

$$\int_0^{q_1} l\rho q\,dq = -\frac{iK}{\pi}\rho_0 q_1\int_c\left\{\cos\frac{z\pi}{\mu} - R'(\sigma_1,z)\right\}\frac{L(\sigma_1,z)}{\sin z\pi/\gamma}\frac{dz}{z^2-1}.$$

Similarly it can be shown that

$$\int_{q_1}^{q_b} l\rho q\,dq = -\frac{K\gamma}{\pi}\rho_0 q_1\left\{\frac{\pi}{\gamma}\operatorname{cosec}\frac{\pi}{\gamma}\times\frac{p_b-p_1}{\rho_1 q_1^2} + \frac{q_b}{q_1} - 1\right\}$$
$$+ \frac{iK}{\pi}\rho_0 q_1\int_c\left[\left\{L(\sigma_1,z) + \frac{q_b}{q_1}\right\}\cos\frac{z\pi}{\mu} - R(\sigma_1,z)L'(\sigma_1,z)\right]\frac{1}{\sin\frac{z\pi}{\gamma}}\frac{dz}{z^2-1}.$$

On adding these two expressions and simplifying,

$$D = \sin\frac{\pi}{\gamma}\left[p_b L - \frac{K\gamma}{\pi}\rho_0 q_b\left\{1 - \frac{q_1}{q_b} + \frac{p_b-p_1}{\rho_1 q_1 q_b}\frac{\pi}{\gamma}\operatorname{cosec}\frac{\pi}{\gamma} - \frac{1}{\gamma_i}\int_c\left(\frac{q_1}{q_b} - \cos\frac{z\pi}{\mu}\right)\frac{1}{\sin\frac{z\pi}{\gamma}}\frac{dz}{z^2-1}\right\}\right]$$

If the pressure is referred to that at boundary velocity, the drag coefficient is defined as

$$C_D = \frac{\sin\pi/\gamma}{\frac{1}{2}\rho_b q_b^2}\;\frac{\int_0^L (p-p_b)\,dl}{L} = \frac{D - \sin\pi/\gamma\, p_b L}{\frac{1}{2}\rho_b q_b^2\cdot L}$$

From (5.7)

$$L = [l]_{\sigma=\sigma_b}$$
$$= -\frac{K\gamma}{\pi}\frac{\rho_0}{\rho_b q_b}\left[1 - \frac{\rho_b q_b}{\rho_1 q_1}\frac{\pi}{\gamma}\operatorname{cosec}\frac{\pi}{\gamma} + \frac{1}{\gamma_i}\frac{\rho_b}{\rho_s}\int_c\cos\frac{z\pi}{\mu}\left\{L'(\sigma_b,z) + \frac{\rho_s}{\rho_b}\right\} - L'(\sigma_1,z)\right\}\frac{1}{\sin\frac{z\pi}{\gamma}}\frac{dz}{z^2-1} \qquad (5.8)$$

Hence

$$C_D = 2\sin\frac{\pi}{\gamma}\,\frac{1 + a_1\frac{\pi}{\gamma}\operatorname{cosec}\frac{\pi}{\gamma} + \frac{1}{\gamma_i}\int_c\cos\frac{z\pi}{\mu}\Big/\sin z\pi/\gamma\cdot\frac{dz}{z^2-1}}{1 + b_1\frac{\pi}{\gamma}\operatorname{cosec}\frac{\pi}{\gamma} + \frac{1}{\gamma_i}\frac{\rho_b}{\rho_s}\int_c\left[\left\{L'(\sigma_b,z) + \frac{\rho_s}{\rho_b}\right\}\cos\frac{z\pi}{\mu} - L'(\sigma_1,z)\right]\frac{1}{\sin\frac{z\pi}{\gamma}}\frac{dz}{z^2-1}} \qquad (5.9)$$

where

$$a_1 = \frac{p_b - p_1 - \rho_1 q_1^2}{\rho_1 q_1 q_b}, \qquad b = -\frac{\rho_b q_b}{\rho_1 q_1}.$$

An interesting special case is obtained by displacing the singularity on $\theta = 0$ to the corner of the hodograph diagram, i.e. by putting $\tau_1 = \tau_b$. This gives the flow of a jet past a symmetric wedge. (The stream function for this flow was found in Chapter III.) Here $a_1 = b_1 = -1$ and hence

$$L = \frac{K}{\pi}\frac{\rho_b}{\rho_s}\frac{1}{q_b^2}\int_C \left\{L^1(\sigma_b, z) + \frac{\rho_s}{\rho_b}\right\}\left\{1 - \cos\frac{z\pi}{\mu}\right\}\frac{1}{\sin z\pi/\xi}\,\frac{dz}{z^2-1}$$

and

$$C_D = 2\sin\frac{\pi}{\xi}\,\frac{\int_C\left\{1-\cos\frac{z\pi}{\mu}\right\}\frac{1}{\sin z\pi/\xi}\,\frac{dz}{z^2-1}}{\int_C\left\{1-\cos\frac{z\pi}{\mu}\right\}\left\{1+\frac{\rho_b}{\rho_s}L^1(\sigma_b, z)\right\}\frac{1}{\sin z\pi/\xi}\,\frac{dz}{z^2-1}}. \tag{5.10}$$

For a sonic jet $\rho_b = \rho_s$ and $\sigma_b = 0$ in these formulae. For this simple case the drag has been derived from first principles, by starting with the series for ψ generalised from the corresponding series in incompressible flow; the method used for finding the drag followed closely that employed by Mackie and Pack (17) for the problem of a wedge in an infinite stream. The results obtained agree with (5.10) (with $\rho_b = \rho_s$ and $\sigma_b = 0$) and provide a check on the more complicated analysis of this chapter.

For small π/ξ (which in future for convenience will be called α), the contour integrals in (5.8) and (5.9) can be evaluated approximately and L and C_D expressed as series in ascending powers of α. The method consists of deforming the contour C (without traversing any singularities) to a new parallel contour C^1 on which

$R\{z\} > \frac{\pi}{2}$ and is therefore large; the expression $1/(z^2-1)$ can then be expanded in series and the functions $L(\sigma,z)$ etc. occurring in the denominator be replaced by their asymptotic expansions for large z. The result will be compared with the corresponding one found from the Tricomi approximation (in which the velocity everywhere is near sonic); hence only the special case for which $\sigma_b = 0$ i.e. $\tau_b = \tau_s$ will be considered.

By using asymptotic formulae for $\psi_z(\tau)$ and $\psi_z'(\tau)$ given by Cherry (7), it can be deduced that, for large $|z|$, and $\sigma_b = 0$,

$$L'(\sigma, z) \sim \left(\frac{1-\tau_s}{1-\tau}\right)^{\beta} \frac{J_z(zt)\left[\{q_1^*(\tau)-1\}z + O(z^{-1})\right] + J_z'(zt)\left[q_0^*(\tau)z^2 + O(z^0)\right]}{J_z(z)\left[z + O(z^{-1})\right] + J_z'(z)\left[\{q_1(\tau_s) - q_0^*(\tau_s)\} + O(z^{-2})\right]} \quad (5.11)$$

where t is a function of τ (and hence of σ) defined by the equation

$$\tanh^{-1}\sqrt{(1-t^2)} - \sqrt{(1-t^2)} = \tanh^{-1}\sqrt{\left(\frac{1-\tau/\tau_s}{1-\tau}\right)} - \tau_s^{-1/2}\tanh^{-1}\sqrt{\left(\frac{\tau_s-\tau}{1-\tau}\right)}$$

and is such that $0 < t \leq 1$ when $0 < \tau \leq \tau_s$, while the q's are functions of τ which have been tabulated in Cherry's paper. Typical values, for $\gamma = 1.4$ ($\tau_s = \frac{1}{6}$) are:

τ	q_0^*	q_1	q_1^*
0	1.000	0.361	0.361
0.08	1.028	0.356	0.384
0.16	1.060	0.350	0.428

In particular $q_0^*(\tau)$ can be expressed analytically as

$$q_0^*(\tau) = \left(\frac{1-6\tau}{1-\tau}\right)^{1/2} \Big/ (1-t^2)^{1/2}$$

When $\tau_s = \frac{1}{6}$, this gives $\quad q_0^*(\tau_s) = \left(\frac{\gamma+1}{2}\right)^{1/3}$.

The Bessel functions in (5.11) can in turn be expanded asymptotically by the series

$$
\begin{aligned}
J_z(zt) &\sim \frac{e^{z(\tanh\theta-\theta)}}{\sqrt{(2\pi\tanh\theta)}}\left[z^{-1/2} + O(z^{-3/2})\right] \\
J_z'(zt) &\sim \frac{e^{z(\tanh\theta-\theta)}}{\sqrt{(2\pi\tanh\theta)}}\left[z^{-1/2}\sinh\theta + O(z^{-3/2})\right] \qquad (5.12)\\
J_z(z) &\sim \frac{\Gamma(\frac{1}{3})}{2^{2/3}\,3^{1/6}\,\pi}\, z^{-1/3} - \frac{2^{2/3}\,3^{1/6}\,\Gamma(2/3)}{140\,\pi}\, z^{-5/3} + o\left(z^{-5/3}\right) \\
J_z'(z) &\sim \frac{3^{1/6}\,\Gamma(2/3)}{2^{1/3}\,\pi}\, z^{-2/3} - \frac{3^{5/6}\,\Gamma(\frac{1}{3})}{2^{2/3}.\,15\pi}\, z^{-4/3} + o\left(z^{-4/3}\right).
\end{aligned}
$$

where $R\{z\}$ is large and positive, $\theta = \operatorname{sech}^{-1} t$, $0 < \theta < \pi/2$ and θ is not too near zero. The exact condition on θ is $\tanh\theta > O(z^{-1/3})$, i.e.$(1 - t) > O(z^{-2/3})$ (cf. Appendix). The first and third of these formulae are given by Watson (27) and (26) respectively; also the first term in the expansion of $J_z'(z)$ (the second term was derived using Watson's method). The expansion for $J_z'(zt)$ is found in the Appendix.

When the results of (5.12) are substituted in (5.11) the required expansions, for use in (5.9) are

$$
\begin{aligned}
L'(\sigma_1, z) &\sim \frac{2^{2/3}\,3^{1/6}\,\pi}{\Gamma(\frac{1}{3})}\left(\frac{1-\tau_s}{1-\tau_1}\right)^{\beta} \frac{e^{z(\tanh\theta_1-\theta_1)}}{\sqrt{(2\pi\tanh\theta_1)}}\left[\underset{\sim}{q}_0^{*}(\tau_1)\sinh\theta_1\, z^{5/6} + O(z^{-1/6})\right] \qquad (5.13)\\
L'(0, z) + 1 &\sim 6^{1/3}\frac{\Gamma(2/3)}{\Gamma(1/3)}\,\underset{\sim}{q}_0^{*}(\tau_s)\, z^{2/3} + \left\{q_1^{*}(\tau_s) - \tfrac{1}{5}\underset{\sim}{q}_0^{*}(\tau_s)\right\} + o(z^0)
\end{aligned}
$$

These expansions, and the analysis which follows, are valid in $0 < t < 1$ provided $1 - t > O(z^{-2/3})$.

The integrals in (5.9) can now be written as a sum of integrals of the form

$$
\frac{1}{\pi i}\int_{C'} \frac{\cos\frac{z\pi}{\mu}}{\sin\frac{z\pi}{3}}\, z^{-m}\,dz \qquad \text{and} \qquad \frac{1}{\pi i}\int_{C'} \frac{e^{-az}}{\sin\frac{z\pi}{3}}\, z^{-m}\,dz\ , \quad m > 1,
$$

where C^1 is the contour parallel to C as mentioned earlier. In the second integral, $a = \theta_1 - \tanh\theta_1 = \frac{1}{3}(1 - t_1^2)^{3/2} > O(z^{-1})$. Therefore $az > O(z^0)$ and $e^{az} z^{-m}$ is exponentially small for all values of m.

The integrals can be evaluated by expressing them as infinite series; the first is equivalent to

$$-\frac{2}{\xi^m}\sum_{n=1}^{\infty} \cos n\pi \cos n\pi\nu / n^m = -\alpha^m f(m, \nu)$$

where $f(m,\nu) = \left\{\zeta\left(1-m, \frac{1+\nu}{2}\right) + \zeta\left(1-m, \frac{1-\nu}{2}\right)\right\} / 2^{1-m}\,\Gamma(m)\cos\frac{m\pi}{2}$,

$\nu = \xi/\mu$ and $\zeta(s, a)$ is the generalised Riemann Zeta Function, while the second integral is equal to

$$-\frac{2}{\xi^m}\sum_{n=1}^{\infty} \cos n\pi\, e^{-an\xi} n^{-m} = -2\left(\frac{\alpha}{\pi}\right)^m F\left(e^{-a\pi/\alpha}, m\right),$$

where $F(z,s) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ for $|z| < 1$ is the function defined by Truesdell (25). It is clear from the remarks made above that the terms of the second series are infinitesimally small compared with those of the first series; so the second integral can be ignored in the series for L and C_D.

After some simplification, the above analysis gives the following results for α small:-

$$L = P\alpha^{-1} + Qf(4/3, \nu)\,\alpha^{1/3} + \{R + Sf(2,\nu)\}\alpha + o(\alpha) \tag{5.14}$$

$$C_D = A\alpha + Bf(4/3, \nu)\,\alpha^{7/3} + \{C + D_1 f(2,\nu)\}\alpha^3 + o(\alpha^3) \tag{5.15}$$

where $P = -\frac{K\rho_0}{\rho_s q_s}(1 + b_1)$ $\qquad Q = \frac{K\rho_0}{\rho_s q_s} 6^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} \underset{\sim}{q}_0^*(\tau_s)$

$R = -\frac{K\rho_0}{\rho_s q_s}\frac{b_1}{6}$ $\qquad S = \frac{K\rho_0}{\rho_s q_s}\left\{\underset{\sim}{q}_1^*(\tau_s) - \frac{1}{5}\underset{\sim}{q}_0^*(\tau_s)\right\}$

$A = \frac{2(1 + a_1)}{1 + b_1}$ $\qquad B = -\frac{AQ}{P}$

$C = -\frac{1 + 2b_1 + a_1 b_1}{3(1 + b_1)^2}$ $\qquad D_1 = \frac{-2\left[(1+b_1) - (1+a_1)\{\underset{\sim}{q}_1^*(\tau_s) - \frac{1}{5}\underset{\sim}{q}_0^*(\tau_s)\}\right]}{(1+b_1)^2}$

[It follows from the continuity equation that $b_1 < -1$; hence the coefficient P is always positive].

It should be noted that in these series the coefficients P,Q etc. depend on the relative values of the source velocity and sonic velocity; hence, care must be taken to ensure that the ratio τ_1/τ_s is not so near the value 1 (i.e. P,A etc. so near 0) as to change the order of significance of the terms of the series. It will be shown in the next chapter that P and Q depend on α in such a way that the order of terms in α is changed when $1 - \tau_1/\tau_s < O(\alpha^{\frac{1}{2}})$; hence in the analysis which follows only values of τ_1 which satisfy the condition $1-\tau_1/\tau_s > O(\alpha^{\frac{1}{2}})$ will be considered.

The series (5.15) will now be compared with the corresponding series obtained using the approximate equations of Tomotika and Tamada and of Tricomi. The solution for ℓ of these equations for Réthy flows of sonic free-stream velocity can be written in the form (5.6) where $L(\sigma,n)$ and $R(\sigma,n)$ are defined not as in (5.5), but as independent solutions of the appropriate (Tomotika and Tamada or Tricomi) equation. The analysis for finding the drag coefficient follows as in the Chaplygin case and C_D is given by (5.9) with $\sigma_b = 0$.

To find the series for C_D it is necessary, as above to find an asymptotic expansion for $L'(\sigma,z)$. For the Tomotika and Tamada equation, (cf. P3) $L(\sigma,n)$ is defined as

$$L(\sigma,n) = -J_m(mt)/J_m(m) \quad ; \quad m = \sqrt{a}\,(n/\kappa) \quad , \quad t = e^{-\kappa\sigma}$$

Thus $$L'(\sigma,n) = n\kappa t\, J_m'(mt)\,/\,J_m(m)$$

which, when combined with formulae (5.12), yields the asymptotic expansions

$$L'(\sigma_1, z) \sim \frac{2^{2/3} 3^{1/6} \pi K}{\Gamma(1/3)} \frac{e^{(\sqrt{a}/\kappa) z (\tanh\phi_1 - \phi_1)}}{\sqrt{(2\pi \tanh\phi_1)}} \left[\left(\frac{\sqrt{a}}{\kappa}\right)^{5/6} \tanh\phi_1 \, z^{5/6} + O(z^{-1/6}) \right]$$

$$L'(0, z) + 1 \sim 6^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} \kappa \left(\frac{\sqrt{a}}{\kappa}\right)^{2/3} z^{2/3} + (1 - \tfrac{1}{5}\kappa) + o(z^0)$$

where $\operatorname{sech} \phi_1 = t_1$ and the same condition on t_1 is imposed as before. These are of the same form as the expressions in (5.13); hence the analysis can proceed exactly as above to give finally (for small α),

$$C_D = A\alpha + Bf(4/3, \nu)\alpha^{7/3} + \{C + D_2 f(2, \nu)\}\alpha^3 + o(\alpha^3) \tag{5.16}$$

where A, B and C are as defined in (5.15) and

$$D_2 = -2\left[(1 + b_1) - (1 + a_1)\{1 - \tfrac{1}{5}\left(\tfrac{\gamma+1}{2}\right)^{2/3}\}\right] / (1 + b_1)^2$$

For the Tricomi equation, (cf. p.3)

$$L(\sigma, n) = \frac{3^{1/6}}{\pi} n^{1/3} \Gamma(2/3) \eta^{1/2} K_{1/3}\left(\tfrac{2}{3} n \eta^{3/2}\right)$$

Hence

$$L'(\sigma, n) = \frac{3^{1/6}}{\pi} \Gamma(2/3)(\gamma+1)^{1/3} n^{1/3} \left\{ \eta n K_{4/3}\left(\tfrac{2}{3} n \eta^{3/2}\right) - \eta^{-1/2} K_{1/3}\left(\tfrac{2}{3} n \eta^{3/2}\right) \right\}$$

giving immediately

$$L'(0, z) + 1 = 3^{1/3} \frac{\Gamma(2/3)}{\Gamma(1/3)} (\gamma+1)^{1/3} z^{2/3} + 1$$

The asymptotic expansion for $L'(\sigma, z)$ is derived by substitution of the expansion for $K_\nu(z)$ given by Watson (27), viz.

$$K_\nu(z) \sim \left(\frac{\pi}{2z}\right)^{1/2} e^{-z} \left[1 + \frac{4\nu^2 - 1}{8z} + \frac{(4\nu^2-1)(4\nu^2-9)}{2!\,(8z)^2} + \cdots \right], \quad |\arg z| < \frac{3\pi}{2}$$

This gives

$$L'(\sigma_1, z) \sim \frac{3^{2/3}}{2\pi^{1/2}} \Gamma(2/3)(\gamma+1)^{1/3} \eta_1^{1/4} e^{-\frac{2}{3}\eta_1^{3/2} z} \left\{ z^{5/6} + O(z^{-1/6}) \right\}.$$

Again, the form of the expansions is the same, and in this case also, the series for C_D can be found immediately as

$$C_D = A\alpha + Bf(4/3, \nu)\alpha^{7/3} + \{C + D_3 f(2, \nu)\}\alpha^3 + o(\alpha^3) \tag{5.17}$$

where

$$D_3 = -2[(1 + b_1) - (1 + a_1)] / (1 + b_1)^2.$$

It is interesting to note that in (5.15), (5.16) and (5.17) the first two terms are in complete agreement; also, the error involved in retaining the third term is of the same order for the Tricomi case as for the (better) approximation of Tomotika and Tamada. A similar result was obtained by Mackie (16) for Helmholtz flow. This suggests that, at least for drag computation, the error introduced by the poor fit of the Tricomi approximation at the stagnation point is not as serious as might have been expected.

To complete the chapter a similar analysis was carried out for a wedge in a sonic free jet. As (5.14) and (5.15) are not uniformly valid for all τ_1, the series for L and C_D had to be worked out from (5.10). The results obtained were

$$L = Q f_1(4/3, \nu)\,\alpha^{1/3} + S f_1(2,\nu)\,\alpha + o(\alpha)$$

where

$$f_1(m,\nu) = f(m,\nu) + 2\pi^{-m}(1-2^{1-m})\,\zeta(m)$$

and Q and S are the coefficients defined in (5.15); and

$$C_D = E\,\frac{f_1(2,\nu)}{f_1(4/3,\nu)}\,\alpha^{5/3} + F_i\left[\frac{f_1(2,\nu)}{f_1(4/3,\nu)}\right]^2 \alpha^{7/3} + o(\alpha^{7/3}) \qquad (5.18)$$

where $E = 2/6^{1/3}\,\dfrac{\Gamma(2/3)}{\Gamma(1/3)}\,\underset{\sim}{q}_0^*(\tau_s)$

$F_1 = -\frac{1}{2}\left\{\underset{\sim}{q}_1^*(\tau_s) - \frac{1}{5}\underset{\sim}{q}_0^*(\tau_s)\right\}E^2$ (from Chaplygin equation)

$F_2 = -\frac{1}{2}\left\{1 - \frac{1}{5}\left(\frac{\gamma+1}{2}\right)^{2\gamma/(\gamma+1)}\right]E^2$ (Tomotika and Tamada)

$F_3 = -\dfrac{E^2}{2}$ (Tricomi).

In the three series obtained for C_D, once again the first terms are identical. The order of the second term is the same in all three cases, and for $\gamma = 1.4$, the coefficients are in the ratio 22 : 40 : 100 for the series derived from the Chaplygin, Tomotika and Tamada, and Tricomi equations of motion respectively.

As $d \to \infty$ and $\nu \to 0$ the sonic jet flow past a wedge tends to Helmholtz flow, for which the drag coefficient has been found by Mackie (16). The above results will be checked against those of Mackie in Chapter VII.

CHAPTER VI

RÉTHY FLOW OF A SONIC JET PAST A THIN WEDGE

It has already been stated (in the last chapter) that a Réthy Flow can be set up if 7 of the 9 constants P_b, L, w/ξ, S, d, p_1, ρ_1, q_1, w/μ are prescribed. For the special case of a sonic jet and a thin wedge (and in a restricted range of q_1), the wedge length L was found in terms of the other constants; it is given by the series

$$L = P\alpha^{-1} + Q f(4/3, \nu)\alpha^{1/3} + O(\alpha) \qquad (5\cdot15)$$

where the coefficients P and Q depend on the ratio q_1/q_s.

Several interesting questions emerge from this relationship. Unless P is small (when q_1 is near q_s), L is of order α^{-1} and must therefore be large (no matter what the value of the other constants). This means that L cannot be prescribed in an arbitrary manner. Is the same true when q_1 is near q_s ? Just how much freedom is there in the choice of L ? And if the length of the wedge is so determined by α, is its distance from the channel (i.e. S) also restricted?

To consider these questions fully, it is clearly desirable to have an expansion for L that is uniformly valid for all possible source velocities up to and including sonic velocity. (5.15) is valid for most values of τ_1, but it breaks down when τ_1 is near τ_s, owing to the restrictions on the asymptotic expansions of $J_s(st)$ and $J_s'(st)$ used in its derivation. To obtain a uniformly valid series it now becomes necessary to look for other expansions of the Bessel Functions which hold in a wider range of t; such expansions were in fact derived by Olver (18) (just before the completion of Chapter V).

Olver's expansions, valid for $|\arg z| < \pi/2$, are

$$J_z(zt) \sim \left(\frac{4\zeta}{1-t^2}\right)^{1/4}\left[\frac{Ai(z^{2/3}\zeta)}{z^{1/3}}\sum_{s=0}^{\infty}\frac{A_s(\zeta)}{z^{2s}} + \frac{Ai'(z^{2/3}\zeta)}{z^{5/3}}\sum_{s=0}^{\infty}\frac{B_s(\zeta)}{z^{2s}}\right] \tag{6.1}$$

$$J_z'(zt) \sim -\frac{2}{t}\left(\frac{1-t^2}{4\zeta}\right)^{1/4}\left[\frac{Ai(z^{2/3}\zeta)}{z^{4/3}}\sum_{s=0}^{\infty}\frac{C_s(\zeta)}{z^{2s}} + \frac{Ai'(z^{2/3}\zeta)}{z^{2/3}}\sum_{s=0}^{\infty}\frac{D_s(\zeta)}{z^{2s}}\right]$$

$Ai(z^{2/3}\zeta)$ and $Ai'(z^{2/3}\zeta)$ are the Airy Function and its derivative defined is the usual way as

$$Ai(z) = \frac{z^{1/2}}{\pi\sqrt{3}}K_{1/3}\left(\tfrac{2}{3}z^{3/2}\right) \quad ; \quad Ai'(z) = -\frac{z}{\pi\sqrt{3}}K_{2/3}\left(\tfrac{2}{3}z^{3/2}\right);$$

ζ is a function of t defined by

$$\frac{2}{3}\zeta^{3/2} = \tanh^{-1}\sqrt{(1-t^2)} - \sqrt{(1-t^2)}$$

and the coefficients $A_s(\zeta)$ etc., are defined by the recurrence relations

$$A_0(\zeta) = 1$$

$$B_s(\zeta) = \frac{1}{2}\zeta^{-1/2}\int_0^{\zeta}\zeta^{-1/2}\left\{f(\zeta)A_s(\zeta) - A_s''(\zeta)\right\}d\zeta$$

$$A_{s+1}(\zeta) = \frac{1}{2}B_s'(\zeta) + \frac{1}{2}\int f(\zeta)B_s(\zeta)\,d\zeta \quad ; \quad A_{s+1}(-\infty) = 0,$$

where
$$f(\zeta) = \frac{5}{16\zeta^2} + \frac{\zeta t^2(t^2+4)}{4(t^2-1)^3}$$

and by

$$C_s(\zeta) = \chi(\zeta)A_s(\zeta) + A_s'(\zeta) + \zeta B_s(\zeta)$$

$$D_s(\zeta) = A_s(\zeta) + \chi(\zeta)B_{s-1}(\zeta) + B_{s-1}'(\zeta) \tag{6.2}$$

where
$$\chi(\zeta) = \frac{1}{4\zeta} - \frac{t^2}{2}\left(\frac{\zeta}{(1-t^2)^3}\right)^{1/2}$$

Before these expansions are substituted in the integral for L, their behaviour in the required range of t, viz. $0 < t \leq 1$ will be examined.

The only possible point of singularity of the coefficients $A_s(\zeta)$ etc. as defined in (5.2) is at $\zeta = 0$ (which corresponds to $t = 1$). However, substitution in (6.2) of the expressions (quoted from Olver's paper)

$$
\begin{aligned}
t(\zeta) &= 1 - 2^{-1/3}\zeta + \frac{3}{10}\,2^{-2/3}\zeta^2 + O(\zeta^3), \\
f(\zeta) &= \frac{2^{1/3}}{70} + O(\zeta),
\end{aligned}
\tag{6.3}
$$

when ζ is near 0, reveals $\chi(\zeta)$, $A_s(\zeta)$, $B_s(\zeta)$, $C_s(\zeta)$ and $D_s(\zeta)$ all to be of order one near $\zeta = 0$. Thus the coefficients are all continuous (and finite) functions of ζ in the range $-\infty < \zeta \leq 0$ (i.e. $0 < t \leq 1$). It follows that

$$\sum_{s=0}^{\infty} \frac{A_s(\zeta)}{z^{2s}} = A_0(\zeta) + o(z^0) \quad \text{etc.,}$$

and the infinite sums in (6.1) are, to the first order, equivalent to $A_0(\zeta)$, $B_0(\zeta)$, $C_0(\zeta)$ and $D_0(\zeta)$ respectively. Near $\zeta = 0$, the above analysis shows these to have the value

$$A_0(\zeta) = 1, \quad B_0(\zeta) = \frac{2^{1/3}}{70} + O(\zeta), \quad C_0(\zeta) = \frac{1}{5.2^{1/3}} + O(\zeta), \quad D_0(\zeta) = 1. \tag{6.4}$$

The behaviour of the Airy Functions in (6.1) depends on the order of magnitude of $z^{2/3}\zeta$. Here (where the Bessel Functions are to be substituted in the formula for $L'(\sigma, z)$ and then integrated along a contour on which $|s| > \pi/2a$), $|z| = O(\alpha^{-1})$ and the value of ζ required is that at source velocity (i.e. ζ_1), which depends on the ratio τ_1/τ_s. So, for any given flow, the Airy functions concerned depend on the relative sizes of α and τ_1/τ_s. It has already been seen (in 5.15) that the order of magnitude of the first term of L is affected also by the relative values of the same functions. A relation between α and τ_1/τ_s is clearly going to be fundamental to the analysis; it can most conveniently be defined by introducing a real number m such that, for small α,

$$\alpha^m = 1 - \tau/\tau_s \tag{6.5}$$

and considering all m in the range $0 < m \leq \infty$ (which corresponds to $0 < \tau_1 \leq \tau_s$).

Various functions of τ_1, can now be expressed in terms of α. From the definition of ζ and t,

$$\tfrac{2}{3}\zeta_1^{3/2} = \tanh^{-1}\sqrt{(1-t_1^2)} - \sqrt{(1-t_1^2)}$$

$$= \tanh^{-1}\sqrt{\left(\frac{1-\tau_1/\tau_s}{1-\tau_1}\right)} - \tau_s^{-1/2}\tanh^{-1}\sqrt{\left(\frac{\tau_s-\tau_1}{1-\tau_1}\right)}$$

When the inverse tanh functions are expanded in series, this gives, after some analysis,

$$\zeta_1 = 2^{-2/3}(1-\tau_s)^{-1/3}\left[\alpha^m + \frac{4\beta-1}{10\beta}\alpha^{2m} + O(\alpha^{3m})\right] \tag{6.6}$$

where $\beta = 1/(\gamma-1)$.

Substitution of (6.5) in (6.3) gives

$$t_1 = 1 - \frac{1}{2(1-\tau_s)^{1/2}}\alpha^m + O(\alpha^{2m}) \tag{6.7}$$

hence

$$\left(\frac{4\zeta_1}{1-t_1^2}\right)^{1/4} = 2^{1/3}\left[1 + O(\alpha^m)\right] \tag{6.8}$$

To return now to the Airy Functions, it is clear from (6.6) that the argument (i.e. $a^{2/3}\zeta_1$) is of order $\alpha^{m-2/3}$, and is thus large or small according as m is less than or greater than 2/3.

(i) <u>m < 2/3</u>

From the inequalities given by Olver,

$$Ai(z) < A(1+|z|^{1/4})^{-1}\left|e^{-\frac{2}{3}z^{3/2}}\right|,$$
$$Ai'(z) < A(1+|z|^{1/4})\left|e^{-\frac{2}{3}z^{3/2}}\right|$$

it can be deduced that at the most,

$$Ai(z^{2/3}\zeta) = O\left[1/e^{\frac{2}{3}\alpha^{-(1-\frac{3}{2}m)}}\right]$$

and $Ai'(z^{2/3}\zeta) = O\left[\alpha^{-1/4(2/3-m)}/e^{2/3\alpha^{-(1-3/2m)}}\right]$.

It follows from (6.1) that the functions $J_z(zt_1)$ and $J_z'(zt_1)$ are of order $1/e^{2/3\alpha^{-(1-3/2\,m)}}$, and are thus exponentially small when α is small.

(ii) <u>m > 2/3</u>

$$\begin{aligned} Ai(z) &= \frac{z^{1/2}}{\pi\sqrt{3}}K_{1/3}\left(\frac{2}{3}z^{3/2}\right) \\ &= \frac{1}{3}\left[e^{i\pi/6}z^{1/2}J_{-1/3}\left(e^{i\pi/2}\cdot\frac{2}{3}z^{3/2}\right) - e^{-i\pi/6}z^{1/2}J_{1/3}\left(e^{i\pi/2}\cdot\frac{2}{3}z^{3/2}\right)\right] \\ &= \frac{3^{-2/3}}{\Gamma(2/3)} - \frac{3^{-4/3}}{\Gamma(4/3)}z + \frac{3^{1/3}}{18\Gamma(2/3)}z^{5/2} + O(z^{7/2}), \end{aligned}$$

when the Bessel Functions are expanded in series, provided $|z| < 1$.

Hence, when $m > 2/3$,

$$Ai(z^{2/3}\zeta) = \frac{3^{-2/3}}{\Gamma(2/3)} - \frac{3^{-4/3}}{\Gamma(4/3)}z^{2/3}\zeta + O\{(z^{2/3}\zeta)^{5/2}\} \qquad (6.9)$$

An expansion for $Ai'(z^{2/3}\zeta)$ can be found in a similar manner; it is

$$Ai'(z^{2/3}\zeta) = -\frac{3^{-1/3}}{\Gamma(1/3)} + \frac{3^{-5/3}}{\Gamma(5/3)}(z^{2/3}\zeta)^2 + O\{(z^{2/3}\zeta)^3\} \qquad (6.10)$$

<u>When $m = 2/3$,</u> $|z^{2/3}\zeta| = O(1)$; hence in this case also the Airy and its derivative are O(1).

For the particular case of <u>m infinite</u> (i.e. $\zeta_1 = 0$),

$Ai(0) = 3^{-2/3}/\Gamma(2/3)$ and $Ai'(0) = -3^{-1/3}/\Gamma(\frac{1}{3})$. Using these, also

(6.4) and (6.8) the expansions (6.1) are reduced to

$$J_z(z) \sim \left(\frac{2}{9}\right)^{1/3} \frac{1}{\Gamma(2/3)} z^{-1/3} - \frac{2^{2/3}}{3^{1/3}} \cdot \frac{1}{70\,\Gamma(1/3)} z^{-5/3} + o(z^{-5/3})$$
$$J_z'(z) \sim \left(\frac{4}{3}\right)^{1/3} \frac{1}{\Gamma(1/3)} z^{-2/3} - \left(\frac{2}{9}\right)^{1/3} \frac{1}{5\,\Gamma(2/3)} z^{-4/3} + o(z^{-4/3}).$$

These expansions are identical to those already quoted (from Watson) in Chapter V.

It follows from the above analysis that for the purposes required here, and in the range $0 < t \leqslant 1$, the expansions (6.1) can be written in the somewhat simpler form

$$\begin{aligned} J_z(zt) &\sim 2^{1/3}\left[Ai(z^{2/3}\zeta)\{z^{-1/3} + o(z^{-1/3})\} + Ai'(z^{2/3}\zeta)\{B_0(\zeta) z^{-5/3} + o(z^{-5/3})\}\right] \\ J_z'(zt) &\sim \frac{-2^{2/3}}{t}\left[Ai(z^{2/3}\zeta)\{C_0(\zeta) z^{-4/3} + o(z^{-4/3})\} + Ai'(z^{2/3}\zeta)\{z^{-2/3} + o(z^{-2/3})\}\right] \end{aligned} \qquad (6.11)$$

(Note that the Airy Functions have to be left in the expansions to maintain their uniform validity).

The required series for L, uniformly valid in $0 < \tau_1 \leqslant \tau_s$, (for small α) can now be obtained, starting from the integral form (5.7) with $\tau_b = \tau_s$, (i.e. $\pi/\xi = \alpha$),

$$L = -B\alpha^{-1}\{1 + b_1 \alpha \operatorname{cosec} \alpha\} + \frac{B}{\pi i}\int_C \left[L'(\sigma_1, z) - \{L'(0, z) + 1\}\cos\frac{z\pi}{\mu}\right] \frac{1}{\sin\frac{z\pi}{\xi}} \frac{dz}{z^2 - 1} \qquad (6.12)$$

where $b_1 = -\dfrac{\rho_s q_s}{\rho_1 q_1}$, and $B = \dfrac{K\rho_0}{\rho_s q_s} = -\dfrac{d}{b_1}$,

(where d is the half-width of the jet).

Firstly, the coefficients B and b_1, which depend on q_1 can be expressed as functions of α, by using the relation (6.5). Expressing b_1 first as a function of τ_1,

$$-b_1 = \left(\frac{1-\tau_1}{1-\tau_s}\right)^{-\beta}\left(\frac{\tau_1}{\tau_s}\right)^{-1/2} = \left(1 + \frac{\alpha^m \tau_s}{1-\tau_s}\right)^{-\beta}(1-\alpha^m)^{-1/2};$$

hence $\quad -b_1 = 1 + \dfrac{\gamma+1}{8}\alpha^{2m} + O(\alpha^{3m})$

and $\quad B = d\left[1 - \dfrac{\gamma+1}{8}\alpha^{2m} + O(\alpha^{3m})\right].$

The contour C is now changed to a parallel contour C^1 on which $z = O(\alpha^{-1})$ as in Chapter V ; the integrand can then be replaced by its asymptotic form. The part $\int_{C^1} \{L'(0,z)+1\} \cos \frac{z\pi}{\mu} \frac{1}{\sin^{z\pi}/\xi} \frac{dz}{z^2-1}$ can be evaluated in terms of the Riemann Zeta Function exactly as in Chapter V; in the remainder of the integral

$$L'(\sigma_1, z) \sim \left(\frac{1-\tau_s}{1-\tau_1}\right)^{\beta} \frac{J_z(zt_1)[\{q_1^*(\tau_1)-1\}z + O(z^{-1})] + J_z'(zt_1)[q_0^*(\tau_1)z^2 + O(z^0)]}{J_z(z)[z + O(z^{-1})] + J_z'(z)[q_1(\tau_s) - q_0^*(\tau_s) + O(z^{-2})]}$$

(as in (5.11))

$$\sim f(\tau_1)\, Ai(z^{2/3}\zeta_1)\{1 + o(z^0)\} - g(\tau_1)\, Ai'(z^{2/3}\zeta_1)\{z^{2/3} + o(z^{2/3})\},$$

on substituting expansions (6.11),

where $f(\tau_1) = 3^{2/3}\,\Gamma(2/3)\left(\frac{1-\tau_s}{1-\tau_1}\right)^{\beta}\left[\, q_1^*(\tau_1) - 1 - \frac{2^{1/3}}{t_1} C_0(\zeta_1) q_0^*(\tau_1)\right]$

and $g(\tau_1) = 2^{1/3}\, 3^{2/3}\,\Gamma(2/3)\left(\frac{1-\tau_s}{1-\tau_1}\right)^{\beta} \frac{1}{t_1}\, q_0^*(\tau_1)$

The functions $q_0(\tau_1)$ etc. occurring in this expansion can be expanded, for τ_1 near τ_s, as power series in $(1-\tau_1/\tau_s)/(1-\tau_1)$ (cf. Cherry (7)); it follows that $q_0(\tau_1) = q_0(\tau_s) + O(\alpha^m)$, with similar expressions for the other q-functions. From this, together with (6.4), (6.5) and (6.6), $f(\tau_1)$ and $g(\tau_1)$ can be expressed as functions of α, viz.

$$f(\tau_1) = 3^{2/3}\,\Gamma(2/3)\left[\, q_1^*(\tau_s) - \tfrac{1}{\xi} q_0^*(\tau_s) - 1\right] + O(\alpha^m)$$

$$g(\tau_1) = 3^{2/3}\, 2^{1/3}\,\Gamma(2/3)\, q_0^*(\tau_s) + O(\alpha^{m'}).$$

$\frac{1}{\pi i}\int_{C^1} L'(\sigma_1, z) \frac{1}{\sin^{z\pi}/\xi} \frac{dz}{z^2-1}$ can now be expressed as a sum of integrals of the form

$$\frac{1}{\pi i}\int_{C'} Ai(z^{2/3}\zeta_1)\, z^{-p} \frac{dz}{\sin z\pi/\xi} = -\frac{2\alpha^{p-1}}{\pi^p} \sum_{n=1}^{\infty} Ai\left\{(n\pi/\alpha)^{2/3}\zeta_1\right\} n^{-p} \cos n\pi ,$$

$$\frac{1}{\pi i}\int_{C'} Ai'(z^{2/3}\zeta_1)\, z^{-p} \frac{dz}{\sin z\pi/\xi} = -\frac{2\alpha^{p-1}}{\pi^p} \sum_{n=1}^{\infty} Ai'\left\{(n\pi/\alpha)^{2/3}\zeta_1\right\} n^{-p} \cos n\pi .$$

(The series are valid since the Airy Function and its derivative are integral functions of their argument, and are bounded as $|z| \to \infty$).

The final result, the series for L in ascending powers of α, uniformly valid for $0 < \tau_1 \leq \tau_s$, is

$$\frac{L}{d} = \frac{\gamma+1}{8}\alpha^{2m-1} + O(\alpha^{3m-1}) + \underline{A}\left\{f(4/3,\nu) - \sum_1(\zeta_1,\alpha)\right\}\alpha^{1/3}$$
$$+ O(\alpha^{1/2+2m}) + \left\{\frac{1}{6} + \underline{B}f(2,\nu) + (\underline{B}-1)\sum_2(\zeta_1,\alpha)\right\}\alpha + O(\alpha^{1+2m}) \qquad (6.13)$$

where $\underline{A} = 6^{1/3}\dfrac{\Gamma(2/3)}{\Gamma(1/3)}\,\underset{\sim}{q}_0^*(\tau_s)$; $\underline{B} = \underset{\sim}{q}_1^*(\tau_s) - \frac{1}{5}\underset{\sim}{q}_0^*(\tau_s)$;

$$\sum_1(\zeta_1,\alpha) = \frac{2.3^{1/3}.\Gamma(\frac{1}{3})}{\pi^{4/3}}\sum_{n=1}^{\infty}\cos n\pi\, Ai'\left\{(n\pi/\alpha)^{2/3}\zeta_1\right\} n^{-4/3},$$

$$\sum_2(\zeta_1,\alpha) = \frac{2.3^{2/3}.\Gamma(2/3)}{\pi^2}\sum_{n=1}^{\infty}\cos n\pi\, Ai\left\{(n\pi/\alpha)^{2/3}\zeta_1\right\} n^{-2},$$

and the Airy Functions occurring in (6.13) are of order less than or equal to one.

The order of importance of the terms of this series depends on the value of m.

A series for S in ascending powers of α can be found in the same way. Starting with

$$S = R_e\left\{z_1(\tau_s, 0) + z_{2,1}(\tau_s, 0)\right\}$$

in (5.3), the series

$$S = -B\alpha^{-1}\left[1 + b_1\alpha\cot\alpha\right] + 2B\alpha^{-1}\sum_{n=1}^{\infty}\frac{1}{n^2\gamma^2-1}\left\{\cos\frac{n\pi\gamma}{\mu}\left(1+L'(0,n\gamma)\right) - L'(\sigma_1,n\gamma)\right\}$$

can be derived, giving finally, for small α,

$$\frac{S}{d} = \frac{\gamma+1}{8}\alpha^{2m-1} + O(\alpha^{3m-1}) + \underline{A}\left\{\bar{f}(4/3,\nu) - \bar{\sum}_1(\zeta_1,\alpha)\right\}\alpha^{1/3}$$
$$+ O(\alpha^{1/3+2m}) + \left\{-\frac{1}{3} + \underline{B}\bar{f}(2,\nu) + (\underline{B}-1)\bar{\sum}_2(\zeta_1,\alpha)\right\}\alpha + O(\alpha^{1+2m}) \qquad (6.14)$$

where $\bar{f}(p,\nu) = 2^{m-1}\left\{\zeta(1-m,\nu/2) + \zeta(1-m,1-\nu/2)\right\}/\Gamma(m)\cos m\pi/2.$

$$\bar{\sum}_1(\zeta_1,\alpha) = \frac{2.3^{1/3}\,\Gamma(1/3)}{\pi^{4/3}} \sum_{n=1}^{\infty} Ai'\left\{(n\pi/\alpha)^{2/3}\zeta_1\right\} n^{-4/3},$$

and $$\bar{\sum}_2(\zeta_1,\alpha) = \frac{2.3^{2/3}\,\Gamma(2/3)}{\pi^2} \sum_{n=1}^{\infty} Ai\left\{(n\pi/\alpha)^{2/3}\zeta_1\right\} n^{-2}.$$

The first observation to be made from (6.13) and (6.14) is that for all m, L and S are of the same order of magnitude. The actual magnitude required depends upon m, i.e. upon the source velocity τ_1, in the following way.

(i) When $0 < m < \frac{1}{2}$, i.e. $0 < \tau_1 < (1-\alpha^{1/2})\tau_s$,

$$\frac{L}{d} = \frac{S}{d} = \frac{\gamma+1}{8}\alpha^{-(1-2m)} + O(\alpha^{-(1-3m)}) :$$

this means that, for small α, L and S have to be equal and large compared with d, i.e. a long wedge placed a long distance from the channel is required to enable the flow to reach sonic velocity at the shoulder.

(ii) When $m = \frac{1}{2}$, i.e. $\tau_1 = (1-\alpha^{1/2})\tau_s$,

$$\frac{L}{d} = \frac{S}{d} = \frac{\gamma+1}{8} + O(\alpha^{1/3}) :$$

here L and S, again equal, must be of the same order of magnitude as the channel width d.

(iii) When $\frac{1}{2} < m < \frac{2}{3}$, i.e. $(1-\alpha^{1/2})\tau_s < \tau_1 < (1-\alpha^{2/3})\tau_s$,

$$\frac{L}{d} = \frac{S}{d} = \frac{\gamma+1}{8}\alpha^{2m-1} + O(\alpha^{1/3})$$

again L and S are equal to a first approximation, but this time they are small compared with d.

(iv) When $m > \frac{2}{3}$, i.e. $\tau_1 > (1-\alpha^{2/3})\tau_s$,

$$\frac{L}{d} = A\left[f(4/3,\nu) - \sum_1(\zeta_1,\alpha)\right]\alpha^{1/3} + \begin{cases} O(\alpha^{2m-1}) & \text{if } m<1 \\ O(\alpha) & \text{if } m>1 \end{cases}$$

$$\frac{S}{d} = A\left[\bar{f}(4/3,\nu) - \bar{\sum}_1(\zeta_1,\alpha)\right]\alpha^{1/3} + \text{''}$$

Σ_1 and $\bar{\Sigma}_1$ can be simplified by substituting for $Ai'(z^{2/3}\zeta_1)$ from

(6.10); as a result

$$\sum_1(\zeta_1,\alpha) = -\frac{2}{\pi^{4/3}}\left(1-2^{-1/3}\right)\zeta(4/3) \tag{6.15}$$

and $\bar{\sum}(\zeta_1,\alpha) = -\frac{2}{\pi^{4/3}}\zeta(4/3)$

Again a short wedge and small standoff are required, but in this case a greater range of values of L and S is possible; this is due to the fact that for the first time L and S depend to the first order on the additional function v (the ratio of the final direction of the flow to the wedge angle). The variation of L with v can most easily be seen by computing a few values of the coefficient of $\alpha^{1/3}$ for different values of v, the results for $\gamma = 1.4$ are shown in the table below

v	0.2	0.4	0.6	0.8
$L/d\alpha^{1/3}$	0.029	0.118	0.284	0.577

From these four cases, certain general observations concerning Réthy flows of sonic jets past thin wedges can be made. Firstly, if a flow is set up with source velocity such that $\tau_1 < (1-\alpha^{2/3})\tau_s$, there is a certain minimum wedge length, depending on α and d, necessary to maintain the flow, viz. $L^* = \left(\frac{\gamma+1}{8}\right)\alpha^{2m-1}d$ .. If the wedge is shorter than this critical length L^*, S will have to be made equal to L and the initial velocity will have to be increased in order to accommodate the pressure differences, i.e. if α, d, and $L < L^*$ are given, S and q_1 cannot be chosen arbitrarily. If $L > L^*$, sonic velocity will be reached at a point distant L^* from the nose of the wedge; provided $S = L^*$, flow will proceed along the wedge and will not change in direction at the shoulder. Thus, if α, d, and $L > L^*$ (which is determined by either S or $q_1(m)$) are given, W/μ is predetermined ($= \alpha$). For flows of high sonic velocity ($\tau_1 > (1-\alpha^{2/3})\tau_s$) on the other hand, L can be chosen in a completely arbitrary manner, since W/μ can adjust itself to make L of the correct magnitude. (It can be shown, using relations for $\zeta(m, \frac{1}{2})$ and its second derivative in terms of $\zeta(m)$ that the coefficient of $\alpha^{1/3}$ in the series for L/d is of order ν^2 for small ν. Hence $L/d = O(\alpha^{1/3}\nu^2)$. If the given wedge length is short, say $O(\alpha^{1/3+\lambda})$, then ν assumes a value $= O(\alpha^{\lambda/2})$ and the flow proceeds). If S is chosen independently of L (and here they need not be equal), then q_1 is determined.

The chapter will be concluded with the series for L for the special case of a sonic free jet flow past a wedge. This case is represented by $m = \infty$ ($\tau_1 = \tau_s$) in the above analysis; hence from (6.13)

$$\frac{L}{d} = A\left\{f\left(\tfrac{4}{3}, \nu\right) - \sum\nolimits_{1}(\zeta_1, \alpha)\right\}\alpha^{1/3} + \left\{\frac{1}{6} + Bf(2, \nu) + (B-1)\sum\nolimits_{2}(\zeta_1, \alpha)\right\}\alpha + o(\alpha)$$

$\Sigma_1(\zeta_1, \alpha)$ is given by (6.15) ; similarly it can be shown that

$$\sum\nolimits_{2}(\zeta_1, \alpha) = \frac{1}{\pi^2}\zeta(2) = \frac{1}{6}.$$

Thus

$$\frac{L}{d} = Af_1(4/3, \nu)\alpha^{1/3} + Bf_1(2, \nu)\alpha + o(\alpha),$$

where $f_1(m, \nu)$ is the function defined in (5.18). This is in agreement with the series for L derived from first principles in Chapter V and quoted in (5.18).

CHAPTER VII

SPECIAL CASES FROM THE LITERATURE

In the last decade several Russian papers have been published on flows which are particular cases of the general class considered in the previous chapters. In addition interesting links can be found with the work of A.G. Mackie (16) on the Roshko Model Cavity Problem and on Helmholtz Flow.

One example of a Réthy Flow, viz. the flow of a gas jet out of a channel and past a flat plate, was studied by Troshin (23). The diagrams for this flow are sketched below.

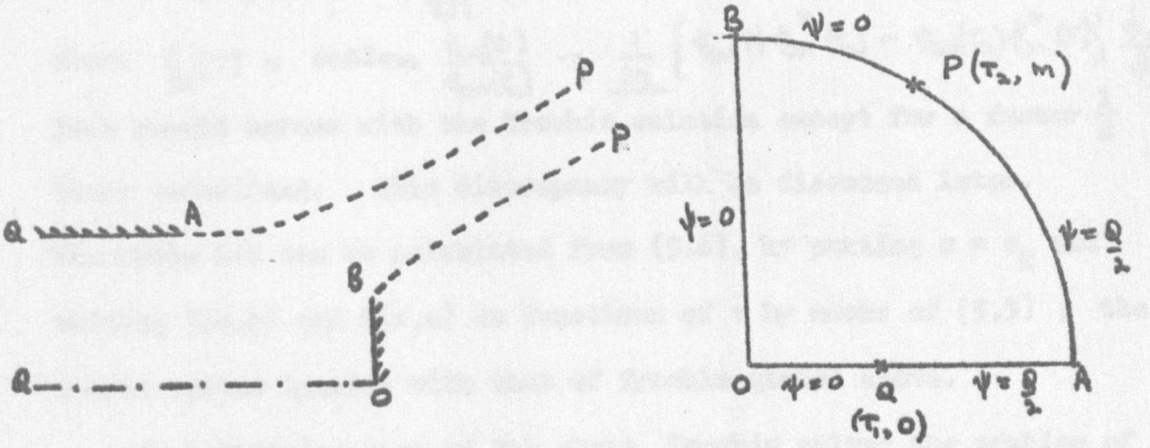


FIGURE VII.1

The solution of the Chaplygin equation for this flow, as given by Troshin, is

$$\psi = \begin{cases} \dfrac{Q}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\left[-\cos 2mn + \frac{1}{n}\left\{\phi_{2n}(\tau_1)\,\psi^*_{2n}(\tau_2) - \phi^*_{2n}(\tau_1)\,\psi_{2n}(\tau_2)\right\}\right]\dfrac{\psi_{2n}(\tau)}{\psi_{2n}(\tau_2)}\sin 2n\theta, & 0 \le \tau \le \tau_1 \\ \dfrac{Q}{\pi}\left(\dfrac{\pi - 2\theta}{2}\right) - \dfrac{Q}{\pi}\displaystyle\sum_{n=1}^{\infty}\frac{1}{n}\, f_n(\tau)\sin 2n\theta, & \tau_1 \le \tau \le \tau_2 \end{cases}$$

where

$$f_n(\tau) = \cos 2mn\,\frac{\psi_{2n}(\tau)}{\psi_{2n}(\tau_2)} - \frac{1}{n}\left\{\psi_{2n}(\tau)\,\psi^*_{2n}(\tau_2) - \psi_{2n}(\tau_2)\,\psi^*_{2n}(\tau)\right\}\frac{\phi_{2n}(\tau_1)}{\psi_{2n}(\tau_2)};$$

and the ratio of the half-length of the plate to the half-width of the jet is

$$\frac{L}{d} = 1-\left(\frac{\tau_1}{\tau_2}\right)^{1/2}\left(\frac{1-\tau_1}{1-\tau_2}\right)^{\beta}\cos m + \frac{2}{\pi}\left(\frac{\tau_1}{\tau_2}\right)^{1/2}\sum_{n=1}^{\infty}\frac{(-1)^{n-1}4n}{4n^2-1}\left[\frac{\psi_{2n}(\tau_1)}{\psi_{2n}(\tau_2)}\varkappa_{2n}(\tau_1) - \left(\frac{1-\tau_1}{1-\tau_2}\right)^{\beta}\varkappa_{2n}(\tau_2)\cos 2mn\right]$$

where $\varkappa_n(\tau) = \frac{\tau}{n}\psi_n'(\tau)/\psi_n(\tau)$.

The solution for ψ can be found from the general solution (5.1) for Réthy flows, by making the substitutions $K = \frac{Q}{2}$, $\xi = 2$, $\pi/\mu = m$, $\tau_b = \tau_2$. The result is

$$\psi = \begin{cases} \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left[-\cos 2mn + \underline{\frac{1}{2n}}\left\{\phi_{2n}(\tau_1)\psi_{2n}^*(\tau_2) - \phi_{2n}^*(\tau_1)\psi_{2n}(\tau_2)\right\}\right]\frac{\psi_{2n}(\tau)}{\psi_{2n}(\tau_2)}\sin 2n\theta, & 0 \le \tau \le \tau_1 \\ \frac{Q}{\pi}\left(\frac{\pi-2\theta}{2}\right) - \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}f_n(\tau)\sin 2n\theta, & \tau_1 \le \tau \le \tau_2 \end{cases} \tag{7.1}$$

where $f_n(\tau) = \cos 2mn\,\frac{\psi_{2n}(\tau)}{\psi_{2n}(\tau_2)} - \underline{\frac{1}{2n}}\left\{\psi_{2n}(\tau)\psi_{2n}^*(\tau_2) - \psi_{2n}(\tau_2)\psi_{2n}^*(\tau)\right\}\frac{\phi_{2n}(\tau_1)}{\psi_{2n}(\tau_2)}$

This result agrees with the Troshin solution except for a factor $\frac{1}{2}$ where underlined. This discrepancy will be discussed later. The ratio L/d can be calculated from (5.6), by putting $\sigma = \sigma_2$ and writing $L(\sigma,n)$ and $R(\sigma,n)$ as functions of τ by means of (5.5) ; the result agrees exactly with that of Troshin quoted above.

As a limiting case of the above, Troshin solved the problem of a free gas jet flow past a flat plate ; the result ($\tau_1 = \tau_2$ in (7.1)) is

$$\psi = \frac{Q}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\left(1-\cos 2mn\right)\frac{\psi_{2n}(\tau)}{\psi_{2n}(\tau_2)}\sin 2n\theta .$$

This agrees with the series for ψ found in (3.4) (when $\xi = 2$).

The substitution $m = 0$ in (7.1) solves two problems - the flow past a plate in a channel of infinite length and the flow out of a symmetric rectangular vessel of finite width and infinite length.

These flows are sketched below in Figure VII.2 (a) and (b) ; (c) represents their (common) hodograph diagram.

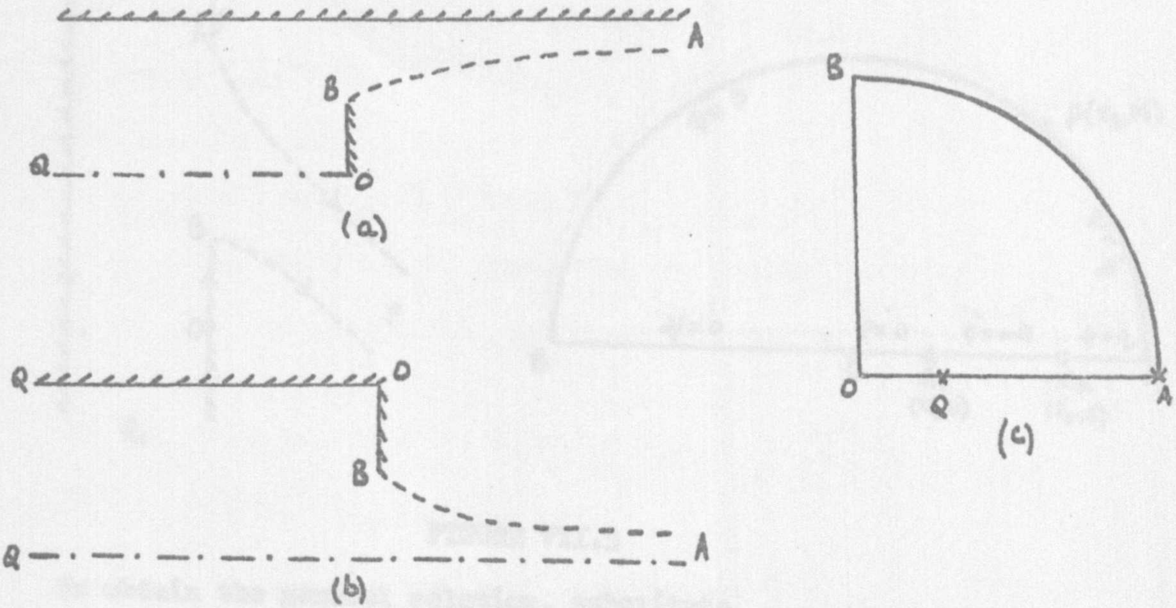


FIGURE VII.2

The solution of this problem was actually found before the publication of Troshin's papers, by Fal'kovich (9) in 1957 and the result has been quoted by Aslanov and Legkova (2). Again a discrepant factor of $\frac{1}{2}$ occurs in both Russian papers. Aslanov and Legkova went on to prove that for a sonic free stream emitting from the vessel in VII.2 (b), a uniform parallel flow is attained at a finite distance (which they calculate) from the vessel mouth ; this is a particular case of the general theorem on sonic jets which will be proved in the next chapter.

An example of a flow with three singularites on the hodograph boundary is the flow of a gas jet through an opening in a channel wall (Troshin (24)).

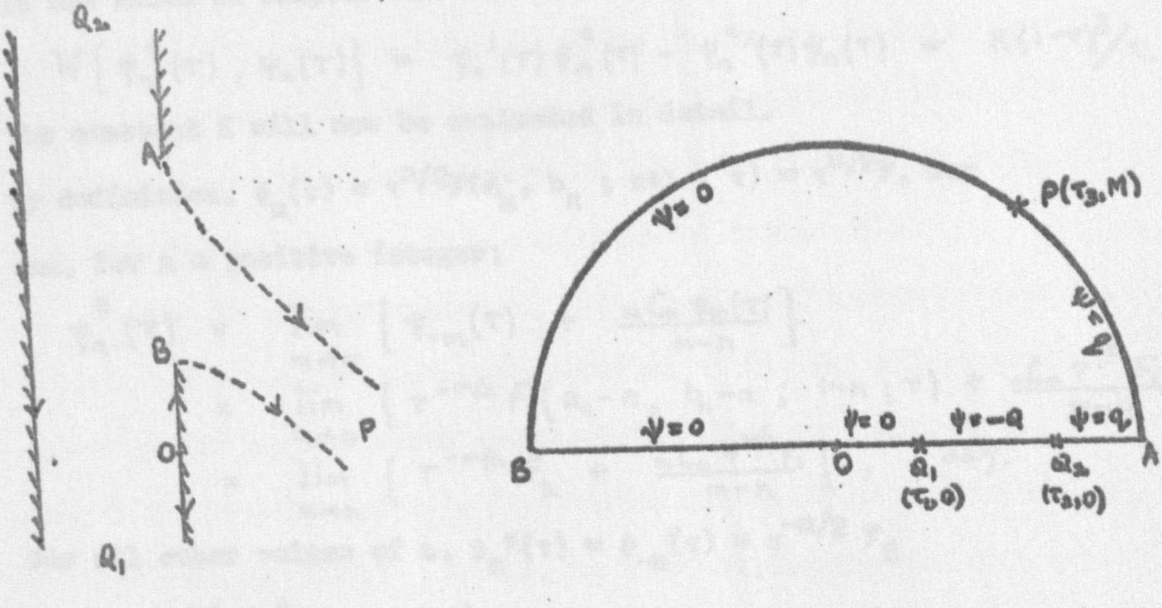


FIGURE VII.3

To obtain the general solution, substitute

$$\xi = 1, \quad \pi/\mu = M, \quad f_0 = 0, \quad f_1 = -Q, \quad f_2 = q, \quad h_0 = q_1, \quad h_1 = 0, \quad g(\tau) \equiv 0$$

in (3.17). The result, for the range $\tau_2 \leq \tau \leq \tau_3$ for example is

$$\psi = q\left(1 - \frac{\theta}{\pi}\right) - \frac{2q}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} f_n(\tau) \sin n\theta$$

where

$$f_n(\tau) = \cos Mn \frac{\phi_n(\tau)}{\psi_n(\tau_3)} + \frac{1}{\pi}\left(\frac{Q}{q}\frac{\phi_n(\tau_1)}{\psi_n(\tau_3)} - \frac{Q+q}{q}\frac{\phi_n(\tau_2)}{\psi_n(\tau_3)}\right)\left(\psi_n^*(\tau_3)\psi_n(\tau) - \psi_n(\tau_3)\psi_n^*(\tau)\right)$$

Yet again there is a discrepancy with Troshin's work; in this case he has a factor 2 in the second term of $f_n(\tau)$.

An investigation of this rather disturbing difference with the Russian authors reveals the root of the trouble to be the evaluation of the Wronskian $W\{\psi_n(\tau), \psi_n^*(\tau)\}$. (In cases where the Wronskian does not occur, e.g. in Troshin's series for L_1 the result obtained by the theory of this thesis agrees completely with that of the Russian).

It was shown in Chapter III that

$$W\{\psi_n^*(\tau),\ \psi_n(\tau)\} = \psi_n'(\tau)\psi_n^*(\tau) - \psi_n^{*\prime}(\tau)\psi_n(\tau) = K(1-\tau)^\beta/\tau.$$

The constant K will now be evaluated in detail.

By definition, $\psi_n(\tau) = \tau^{n/2}F(a_n,\ b_n\ ;\ n+1\ ;\ \tau) = \tau^{n/2}F$, say

and, for n a positive integer;

$$\begin{aligned}\psi_n^*(\tau) &= \lim_{m\to n}\left\{\psi_{-m}(\tau) + \frac{{}_mC_n\psi_m(\tau)}{m-n}\right\}\\ &= \lim_{m\to n}\left\{\tau^{-m/2}F(a_n-n,\ b_n-n\ ;\ 1-n\ ;\ \tau) + \frac{{}_mC_n\tau^{m/2}F_1}{m-n}\right\}\\ &= \lim_{m\to n}\left\{\tau^{-m/2}F_2 + \frac{{}_mC_n\tau^{m/2}F_1}{m-n}\right\},\quad \text{say.}\end{aligned}$$

For all other values of n, $\psi_n^*(\tau) = \psi_{-n}(\tau) = \tau^{-n/2}F_2$

Thus $\quad W\{\psi_n^*(\tau),\ \psi_n(\tau)\}$

$$\begin{aligned}&= \lim_{m\to n}\Bigg[\left(\frac{m}{2}\tau^{m/2-1}F_1 + \tau^{m/2}F_1'\right)\left(\tau^{-m/2}F_2 + \frac{{}_mC_n\tau^{m/2}F_1}{m-n}\right)\\ &\qquad - \tau^{m/2}F_1\left\{-\frac{m}{2}\tau^{-m/2-1}F_2 + \tau^{-m/2}F_2' + \frac{{}_mC_n}{m-n}\left(\frac{m}{2}\tau^{m/2-1}F_1 + \tau^{m/2}F_1'\right)\right\}\Bigg]\\ &= \frac{n}{\tau}F_1F_2 + O(\tau^0)\end{aligned}$$

Hence $\quad K(1-\tau)^\beta = nF_1F_2 + O(\tau)$

Taking the limit as τ tends to zero, K = n, and

$$W\{\psi_n^*(\tau),\ \psi_n(\tau)\} = n(1-\tau)^\beta/\tau.$$

The Russian authors use as their independent Chaplygin solutions the functions

$$z_n(\tau) = \psi_{2n}(\tau) \quad \text{and} \quad \zeta_n(\tau),$$

which is the solution defined by Cherry (6), and which behaves near $\tau = 0$ like $\psi_{2n}^*(\tau)$.

Thus $\quad W\{\zeta_n(\tau),\ z_n(\tau)\} = W\{\psi_{2n}^*(\tau),\ \psi_{2n}(\tau)\} = 2n(1-\tau)^\beta/\tau.$

Fal'kovich has stated explicitly (and the others have used his result) that

$$W\{\zeta_n(\tau),\ z_n(\tau)\} = n(1-\tau)^\beta/\tau.$$

This error explains the discrepancies noted above. (It is interesting to note that Aslanov, in a paper in which he uses the ϕ functions instead of z and ζ, (1) derives the value of the Wronskian as above, thus disagreeing with his own work in his joint paper with Zegkova.) The above result has also been confirmed by Mackie (16).

Mackie, in a paper entitled 'The solution of boundary value problems for a general hodograph equation' (16), has illustrated his theory by solving the Roshko model cavity problem, first for a generalised equation of type (1.3) (i.e. general $k(\sigma)$) and then for the Chaplygin equation. The Roshko model represents the flow of an infinite stream past a symmetrically placed wedge as in Figure VII.4; the stream has velocity $\tau_1 < \tau_s$ at infinity upstream (Q), breaks away from the wedge shoulder with sonic velocity at B, retains sonic speed along the curved portion AB and slows down again to speed τ_1, at infinity downstream (Q'). The hodograph diagram for this flow (Figure VII.4(b)) contains only one singularity, a doublet at $\tau = \tau_1$ on the boundary $\theta = 0$. (This corresponds to the fact that all the streamlines pass through the point $\tau = \tau_1$ at infinity upstream <u>and</u> downstream in the physical plane).

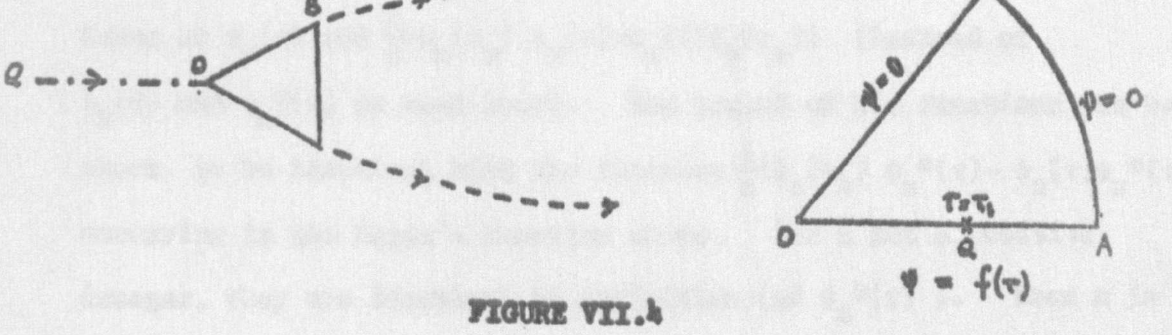


FIGURE VII.4

The Roshko model, since it does not represent a jet flow does not belong to the category of flows considered in this thesis. Nevertheless it is interesting to show that the methods of Chapter III can be applied here to reconstruct Mackie's solution of the Chaplygin equation for this flow.

The boundary conditions for the problem are of the form $f(\tau) = C\delta(\tau-\tau_1)$, $g(\tau) \equiv h(\theta) \equiv 0$. Hence, in equation (3.5), the function $F(\tau)$ is given by

$$F(\tau) = -\frac{C(\tau_s-\tau)}{4\tau_1\tau_s(1-\tau_1)^{\beta+1}}\,\delta(\tau-t)\,n\xi = -b\,\delta(\tau-t)\,n\xi, \quad \text{say.}$$

Substituting this value into (3.11), (with $\tau_b = \tau_s$), the solution is given immediately by inverse transform as

$$\psi = \frac{2b\xi}{\pi}\sum_{n=1}^{\infty} n\xi\, G(\tau,\tau_1)\sin n\xi\theta$$

where, from (3.10)

$$G(\tau,\tau_1) = \begin{cases} \dfrac{1}{n\xi}\,\dfrac{\psi_{n\xi}(\tau)}{\psi_{n\xi}(\tau_s)}\left\{\psi_{n\xi}(\tau_1)\psi^{*}_{n\xi}(\tau_s) - \psi_{n\xi}(\tau_s)\psi^{*}_{n\xi}(\tau_1)\right\} \\[2ex] \dfrac{1}{n\xi}\,\dfrac{\psi_{n\xi}(\tau_1)}{\psi_{n\xi}(\tau_s)}\left\{\psi_{n\xi}(\tau)\psi^{*}_{n\xi}(\tau_s) - \psi_{n\xi}(\tau_s)\psi^{*}_{n\xi}(\tau)\right\}. \end{cases}$$

(The constant C depends on the wedge length and can be determined as shown by Mackie).

In Mackie's work, the fundamental Chaplygin functions have been taken as $\psi_n(\tau)$ and $\frac{1}{n}(\psi_n(\tau_s)\,\psi_n(\tau)-\psi_n(\tau)\psi_n(\tau_s))$ (instead of $\psi_n(\tau)$ and $\psi_n^{*}(\tau)$ as used here). The second of his functions can be shown to be identical with the function $\frac{1}{n}(\psi_n(\tau_s)\,\psi_n^{*}(\tau)-\psi_n(\tau)\psi_n^{*}(\tau_s))$ occurring in the Green's function above. For n not a positive integer, they are identical by definition (of $\psi_n^{*}(\tau)$). When n is a positive integer, (= m say), since $\psi_n(\tau)$ and $\psi_n^*(\tau)$ are regular functions of n,

$$\lim_{n\to m} \left[\psi_n(\tau)\psi_n^*(\tau_s) - \psi_n(\tau_s)\psi_n^*(\tau) \right]$$

$$= \lim_{n\to m} \left[\psi_n(\tau)\left\{ \psi_{-n}(\tau_s) + \frac{nC_m \psi_n(\tau_s)}{m-n} \right\} - \psi_n(\tau_s)\left\{ \psi_{-n}(\tau) + \frac{nC_m \psi_n(\tau)}{m-n} \right\} \right]$$

by definition

$$= \lim_{n\to m} \left[\psi_n(\tau)\,\psi_{-n}(\tau_s) - \psi_n(\tau_s)\,\psi_{-n}(\tau) \right]$$

This limit is finite as the function within the brackets is an integral function of n. Hence the two functions are identical for all n. This establishes agreement between (7.2) and the Mackie solution of the Roshko model cavity problem.

In the same paper, series expansions for small α for the drag coefficient for Helmholtz flow were found. Mackie used in turn the Chaplygin, Tomotika and Tamada and Tricomi equations of motion and compared his results. It will now be shown that these series can be derived as limiting cases of the similar series for more general flows derived (by Mackie's method) in Chapter V.

Helmholtz flow is the flow of a sonic infinite stream past a wedge, breaking away with sonic velocity from the shoulder; it is therefore the limit as $d \to \infty$ and $\nu/\mu \to 0$ of the sonic jet flow past a wedge.

At the end of Chapter V the drag coefficient of a sonic jet past a wedge, obtained by the Chaplygin equation of motion, was given, for small α , as

$$C_D = E\,\frac{f_1(2,\nu)}{f_1(4/3,\nu)}\,\alpha^{5/3} + F_1\left[\frac{f_1(2,\nu)}{f_1(4/3,\nu)}\right]^2 \alpha^{7/3} + o(\alpha^{7/3}) \qquad (5.18)$$

where

$$E = \frac{2}{6^{1/3}} \frac{\Gamma(1/3)}{\Gamma(2/3)} \frac{1}{q_0^{*}(\tau_s)} \quad ; \quad F_1 = \frac{1}{2}\left\{ q_1^{*}(\tau_s) - \frac{1}{5} q_0^{*}(\tau_s)\right\} E^2,$$

(provided $L \leq O(a^{2/3})$).

Let $\varpi/\mu \to a$, i.e. $\nu \to 0$.

Then

$$\lim_{\nu\to 0} \frac{f_1(2,\nu)}{f_1(4/3,\nu)} = \lim_{\nu \to 0} \frac{\partial^2 f_1(2,\nu)/\partial\nu^2}{\partial^2 f_1(4/3,\nu)/\partial\nu^2}$$

$$= \frac{\pi^2\,\Gamma(4/3)}{2^{1/3}\,\pi^{4/3}} \lim_{\nu\to 0} \frac{\frac{\partial^2}{\partial\nu^2}\left[\zeta(-1,\frac{1+\nu}{2}) + \zeta(-1,\frac{1-\nu}{2})\right]}{\frac{\partial^2}{\partial\nu^2}\left[\zeta(-\frac{1}{3},\frac{1+\nu}{2}) + \zeta(-\frac{1}{3},\frac{1-\nu}{2})\right]}.$$

$\zeta(-1,a)$ can be expressed as a Bennoulli Polynomial (cf.(8)) viz.

$$\zeta(-1,a) = -\tfrac{1}{2} B_2(a)$$

$$\therefore \frac{\partial}{\partial a}\zeta(-1,a) = -\tfrac{1}{2}B_2'(a) = -B_1(a) = -(a-\tfrac{1}{2}) \quad ; \quad \frac{\partial^2}{\partial a^2}\zeta(-1,a) = -1.$$

Hence the numerator of the limit is equal to $-\frac{1}{2}$.

To evaluate the denominator, a contour integral representation of $\zeta(s,a)$ (for s not an integer) is used. This, taken again from Erdelyi (8), is

$$2\pi i\,\zeta(s,a) = -\Gamma(1-s)\int_{\infty}^{(0+)} \frac{(-t)^{s-1}\,e^{-wt}}{1-e^{-t}}\,dt \quad , \quad Re(w) > 0,\ |\arg(-t)| \leq \pi$$

and yields

$$2\pi i\,\frac{\partial^2}{\partial a^2}\zeta(s,a) = -\Gamma(1-s)\int_{\infty}^{(0+)} \frac{(-t)^{s-1}\,e^{-wt}}{1-e^{-t}}\,dt$$

$$= \frac{\Gamma(1-s)}{\Gamma(-1-s)}\,.\,2\pi i\,\zeta(s+2,a)$$

$$\therefore \left[\frac{\partial^2}{\partial a^2}\zeta(s,a)\right]_{a=1/2} = s(s+1)(2^{s+2}-1)\,\zeta(s+2)$$

since $\quad \zeta(s,1/2) = (2^s-1)\,\zeta(s).$

$$\therefore \lim_{\nu\to 0}\frac{\partial^2}{\partial\nu^2}\left[\zeta(-\tfrac{1}{3},\tfrac{1+\nu}{2}) + \zeta(-\tfrac{1}{3},\tfrac{1-\nu}{2})\right] = -\tfrac{1}{9}(2^{5/3}-1)\,\zeta(5/3)$$

and

$$\lim_{\nu\to 0}\frac{f_1(2,\nu)}{f_1(\frac{4}{3},\nu)} = \frac{3\pi^{2/3}}{2^{4/3}}\,\Gamma(1/3) \Big/ (2^{5/3}-1)\,\zeta(5/3)$$

Substituting in (5.18) and inverting, the series gives by Mackie, viz.

$$\frac{1}{C_D} = \frac{(2^{5/3}-1)\,\zeta(5/3)\,\Gamma(5/6)}{3^{2/3}\,\Gamma(1/3)\,\pi^{1/2}}\,\frac{(\gamma+1)^{1/3}}{\alpha^{5/3}} + \frac{1}{2\alpha} + O(\alpha^{1/3}).$$

is obtained.

Similar results can be found by taking limits in the C_D series for the Tomotika and Tamada and the Tricomi cases. The first term is the same as for the Chaplygin case; the coefficients of the second term (in the series for C_D) when evaluated for $\gamma = 1.4$ are found to be in the ratio 22 : 40 : 100 (as found in Chapter V). This again agrees with numerical results calcuated by Mackie.

CHAPTER VIII

A THEOREM ON SONIC JET FLOWS

The thesis is concluded with a theorem which is true (when the free stream velocity is sonic) for all the different types of flow that have been discussed in earlier chapters, and which was suggested by certain results which arose in Chapter IV (loc.sit). The theorem states that for all sonic jet flows with hodograph planes as defined in Figure II.2, the physical changes due to the presence of solid boundaries in the flow are completed within a finite distance in those directions in which sonic jet flow prevails. It is characteristic of subsonic flows that they tend monotonically to their final form at an infinite distance, while supersonic jets develop quasi-periodic properties. This property of sonic jets is thus an intermediate phenomenon.

For the special case of a sonic jet flow through a hole in an infinite wall (for which the theorem has been proved by Guderley (14) and others), the theorem states that the free streamline issuing from the wall at A (in Figure VIII.1 (a)) becomes parallel to the streamline O'B' after a finite distance AB, after which the flow is a uniform parallel sonic jet. Again, for the Réthy Flow in VIII.1 (b), the flow at C, where CB is finite becomes a parallel stream inclined at angle π/μ to the horizontal.

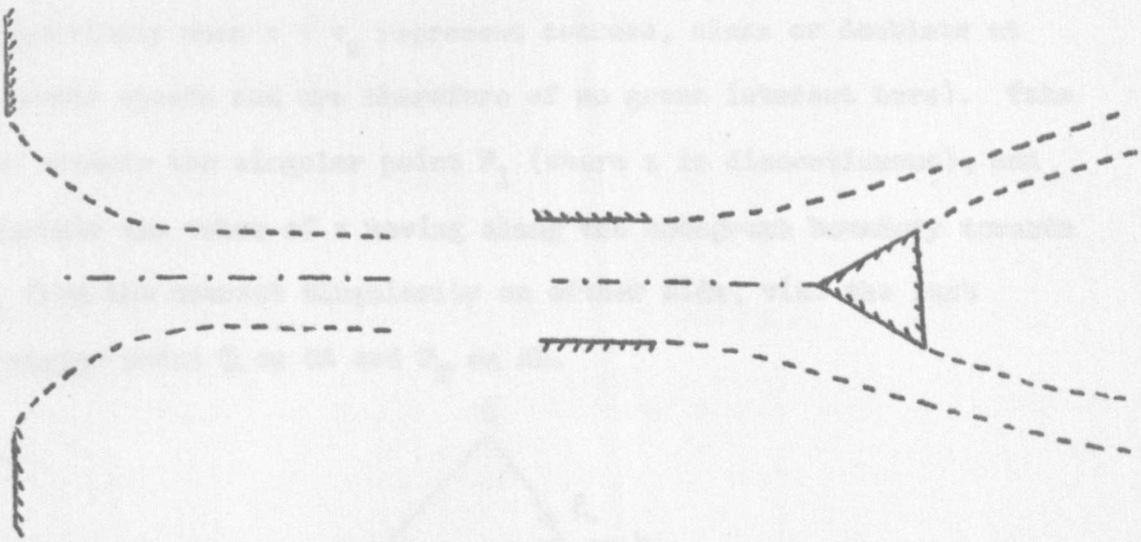


FIGURE VIII.1

To prove the theorem it is necessary to study the properties of the physical coordinate $z(\tau,\theta)$ as τ and θ vary on the hodograph boundary OAB. It was shown in Chapter IV that along each of AB, OA and OB, one of z_1, z_2 and z_3 has singularities while the other two are finite and continuous. Further, between any two singular points, the discontinuous function is either monotonically increasing or decreasing ; this can be proved by considering a particular case, e.g. the boundary line $\tau = \tau_s$ where z_1 is discontinuous. On this line, between any two singularities,

$$\frac{\partial z}{\partial \theta} = \frac{e^{i\theta}}{q_1} \frac{\partial \phi_1}{\partial \theta} , \qquad \text{since } \psi \text{ is constant,}$$

$$= \frac{e^{i\theta}}{q_1} \frac{2\tau}{(1-\tau)^{\beta}} \frac{\partial \psi_1}{\partial \tau}$$

which is monotonic, since ψ_1 always takes locally greatest or least values on the boundary. Thus z_1 increases monotonically with θ on $\tau = \tau_s$. Similar results can be proved for z_2 and z_3 on the other boundary lines.

Of particular interest in the study of z are the singular points

P_p on $\tau = \tau_s$. (The other singular points Q_q and R_r on AO and OB respectively when $\tau < \tau_s$ represent sources, sinks or doublets at subsonic speeds and are therefore of no great interest here). Take for example the singular point P_1 (where z is discontinuous), and consider the value of z moving along the hodograph boundary towards P_1 from the nearest singularity on either side, viz. the last singular point Q on OA and P_2 on AB.

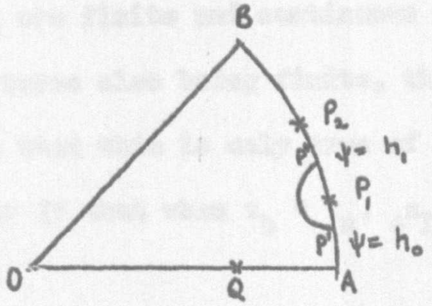


FIGURE XIX

(i) <u>$Q \rightarrow A \rightarrow P_1$</u>

Along QA ($\theta = 0$), z_1 and z_3 are continuous; z_2 is purely real and increases steadily from $-\infty$.

At A, where there is a free-stream breakaway, z is finite. (If there is a singularity at A, z has a <u>finite</u> discontinuity there).

Along AP ($\tau = \tau_s$), z_2 and z_3 are continuous; z_1 increases with θ from A to P_1, where ${}_\psi z_1$ has a discontinuity, and ${}_\phi z_1$ is continuous.

Just below P_1, i.e. $\theta = \theta_1 -$,

$$
\begin{aligned}
{}_\psi z_1 &= \sum_{n=1}^{\infty} \sum_{p=0}^{\bar{p}-1} h_p \left(\cos n\xi\theta_p - \cos n\xi\theta_{p+1}\right) \left\{ \frac{e^{(n\xi+1)\theta i}}{n\xi+1} - \frac{e^{-(n\xi-1)\theta i}}{n\xi-1} \right\} \\
&= \frac{\pi}{\xi} \operatorname{cosec} \frac{\pi}{\xi} \left[h_0 \left\{ e^{i\pi/\xi} - \cos\left(\frac{\pi}{\xi} - \theta_1\right) \right\} + \sum_{p=1}^{\bar{p}-1} h_p \left\{ \cos\left(\frac{\pi}{\xi} - \theta_p\right) - \cos\left(\frac{\pi}{\xi} - \theta_{p+1}\right) \right\} \right]
\end{aligned}
$$

making use of the formulae

$$
\sum_{n=1}^{\infty} \left\{ \frac{e^{(n\xi+1)\theta i}}{n\xi+1} - \frac{e^{-(n\xi-1)\theta i}}{n\xi-1} \right\} = \frac{\pi}{\xi} \frac{e^{i\pi/\xi}}{\sin \pi/\xi} - e^{i\theta},
$$

$$
\sum_{n=1}^{\infty} \left\{ \frac{e^{(n\xi-1)\theta i}}{n\xi-1} - \frac{e^{-(n\xi+1)\theta i}}{n\xi+1} \right\} = -\frac{\pi}{\xi} \frac{e^{-i\pi/\xi}}{\sin \pi/\xi} + e^{i\theta},
$$

both of which are valid for $0 < \theta < 2\pi/\xi$.

Hence, as $\theta \to \theta_1$ from below

$$z \to \frac{\operatorname{cosec} \pi/\xi}{q_s(1-\tau_s)^\beta} \left[h_0\left\{ e^{i\pi/\xi} - \cos\left(\frac{\pi}{\xi} - \theta_1\right)\right\} + \sum_{p=1}^{\bar{p}-1} h_p \left\{\cos\left(\frac{\pi}{\xi} - \theta_p\right) - \cos\left(\frac{\pi}{\xi} - \theta_{p+1}\right)\right\} + \zeta_1 \right] \tag{8.1}$$

where $\zeta_1 = \frac{\pi}{\xi} \sin\frac{\pi}{\xi} \left[{}_\phi z_1 + z_2 + z_3 \right]_{\theta=\theta_1}$.

${}_\phi z_1$, z_2 and z_3 are finite and continuous at θ_1; thus ζ_1 is finite.

The first two terms also being finite, the above value of z is finite. (Note that this is only true of a sonic jet; it was seen in Chapter IV that when $\tau_b < \tau_s$, ${}_\phi z_1$ diverges at P_1).

(ii) <u>$P_2 \to P_1$</u>

Again z_2 and z_3 are continuous; z_1 decreases steadily with θ from P_2 to P_1 where ${}_\psi z_1$ has a discontinuity. Just above P, i.e. $\theta = \theta_1 +$, it can be deduced in the same way as before that

$${}_\psi z_1 = \frac{\pi}{\xi} \operatorname{cosec}\frac{\pi}{\xi} \left[\left\{h_0 + (h_1 - h_0)\cos\theta_1\right\} e^{i\pi/\xi} - h_1 \cos\left(\frac{\pi}{\xi} - \theta_2\right) + \sum_{p=2}^{\bar{p}-1} h_p \left\{\cos\left(\frac{\pi}{\xi} - \theta_p\right) - \cos\left(\frac{\pi}{\xi} - \theta_{p+1}\right)\right\}\right]$$

So as $\theta \to \theta_1$ from above

$$z \to \frac{\operatorname{cosec} \pi/\xi}{q_s(1-\tau_s)^\beta} \left[\left\{h_0 + (h_1 - h_0)\cos\theta_1\right\} e^{i\pi/\xi} - h_1 \cos\left(\frac{\pi}{\xi} - \theta_2\right) + \sum_{p=2}^{\bar{p}-1} h_p \left\{\cos\left(\frac{\pi}{\xi} - \theta_p\right) - \cos\left(\frac{\pi}{\xi} - \theta_{p+1}\right)\right\} + \zeta_1 \right] \tag{8.2}$$

which again is infinite.

(iii) <u>At P</u>

The values of z given in (8.1) $[z(\theta_1-)]$ and (8.2) $[z(\theta_1+)]$ can be compared by writing them in the form

$$z(\theta_1 -) = \frac{h_1 - h_0}{q_s(1-\tau_s)^\beta} \sin\theta_1 + Z_1$$

$$z(\theta_1 +) = \frac{i(h_1 - h_0)}{q_s(1-\tau_s)^\beta} \cos\theta_1 + Z_1$$

where

$$Z_1 = \frac{\operatorname{cosec}\frac{\pi}{3}}{q_s(1-\tau_s)^\beta}\left[h_0 e^{i\pi/3} + (h_1 - h_0)\cos\tfrac{\pi}{3}\cos\theta_1 - h_1\cos\left(\tfrac{\pi}{3}-\theta_1\right) + \sum_{p=2}^{\tilde{p}-1} h_p\left\{\cos\left(\tfrac{\pi}{3}-\theta_p\right) - \cos\left(\tfrac{\pi}{3}-\theta_{p+1}\right)\right\} + \zeta_1\right]$$

Consider a small arc P'P" about P_1.

On P'P", $dz \doteq \frac{i e^{i\theta_1}}{q_s} \frac{1}{(1-\tau_s)^\beta} d\psi_1$,

since ϕ, ψ_2 and ψ_3 are continuous at P.

∴ At any point of the arc,

$$z \doteq \frac{i e^{i\theta_1}}{q_s} \frac{1}{(1-\tau_s)^\beta} \psi_1 + \text{constant},$$

where $h_0 < \psi_1 < h_1$ (assuming $h_1 > h_0$)

Now let the length of the arc tend to zero.

Then the point P_1 in the hodograph plane is the image of the line

$$\frac{h_1 - h_0}{q_s(1-\tau_s)^\beta} \sin\theta_1 + Z_1 \leq z \leq \frac{i(h_1 - h_0)}{q_s(1-\tau_s)^\beta} \cos\theta_1 + Z_1$$

in the physical plane, a line perpendicular to the free streamline at a finite distance from the solid boundary lines. It follows that after this line, the flow becomes a uniform parallel sonic jet at angle θ_1 to the real axis. A similar result can be proved for the other singular points P_p on $\tau = \tau_s$ (of which, as shown in Chapter II, there are at most three in all).

Having proved this theorem for 'simple flows past wedges', the question naturally arises as to whether or not the result can be made even more general and be extended to jet flows with <u>curved</u> boundaries. Recent work by Sedov (21) indicates that in fact this will be the case. He shows that, for the particular case of a jet emerging from a slit in a wall (studied by Ovsiannikov (19)), the effect of curving the surface of the wall is only to add to the solution $\psi(\tau,\theta)$ a function $\psi^*(\tau,\theta)$ which is analytic and continuous near the point of free-stream breakaway (i.e. the singular point on the maximum velocity hodograph boundary). The velocity potential and space coordinate can be found in the usual way; the results again have the added functions $\phi^*(\tau,\theta)$ and $z^*(\tau,\theta)$ respectively, which are both continuous and finite at the singular point. Thus the theorem can easily be proved for this flow also. It seems probable that Sedov's analysis could be applied to other similar flows and perhaps be extended to make the theorem generally true for all sonic jet flows.

APPENDIX

AN ASYMPTOTIC EXPANSION OF $\frac{d}{dt}\left\{J_z(zt)\right\}$

FOR $0 < t < 1$ AND $R(z)$ LARGE AND POSITIVE

The asymptotic expansion of $J_z'(zt)$, (where 'dash' denotes differentiation with respect to t), will be found by means of the recurrence relation

$$J_z'(zt) = \frac{1}{t} J_z(zt) - J_{z+1}(zt) \qquad (1)$$

Asymptotic expansions for $J_z(zt)$ are well known and have been given in several treatises on Bessel Functions; the main part of the analysis therefore consists in finding an expansion for $J_{z+1}(zt)$.

The method used is that given by Watson (27) in §8.6 of his book on Bessel Functions, where Debye's results are extended to the case of Bessel Functions of (large) complex order and argument. This consists in expanding the relevant function, $J_\nu(z)$ say, in terms of two general functions $S_\nu^{(i)}(z)$ and $S_\nu^{(ii)}(z)$ (defined for all ν and z) for which asymptotic expansions have been given. Briefly, these functions are defined by considering the integral

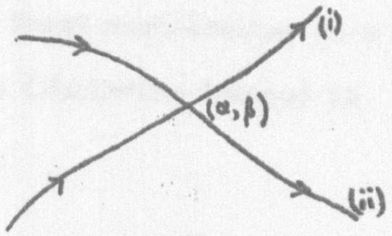
$$\int_C e^{z \sinh w - \nu w}\, dw = \int_C e^{-zf(w)}\, dw,$$

where $f(w) = w \cosh \gamma - \sinh w$, $\cosh \gamma = \nu/z$:

the contour C is chosen as the 'path of steepest descent,' i.e. $\mathcal{I}[f(w)] = \mathcal{I}[f(\gamma)]$, γ being a stationary point of $f(w)$, so that an asymptotic expansion can be derived by expanding the integrand in the neighbourhood of $z = \gamma$. Writing $w = u + iv$ and $\gamma = \alpha + i\beta$, the contour has equation

$$(v-\beta)\cosh\alpha\cos\beta + (u-\alpha)\sinh\alpha\sin\beta - \cosh u\sin v + \cosh\alpha\sin\beta = 0 \qquad (2)$$

Sketches of this curve have been made by Watson to cover all possible values of α and β ; in all cases it has two branches (i) and (ii) say, meeting at (α,β) (i.e. $w = \gamma$), where they have gradients (i) $\pi/4 + \delta$ and (ii) $-\pi/4 + \delta$ where $|\delta| < \pi/2$, and moving off to infinity in both directions.



Tracing the contours in the direction shown in the diagram, the functions $S_\nu(z)$ are defined as

$$S_\nu^{(i)}(z) = \frac{1}{\pi i}\int_{(i)} e^{-zf(w)}\,dw \quad , \qquad S_\nu^{(ii)}(z) = -\frac{1}{\pi i}\int_{(ii)} e^{zf(w)}\,dw$$

and have asymptotic expansions

$$S_\nu^{(i)}(z) \sim \frac{e^{\nu(\tanh\gamma-\gamma) - i\pi/4}}{\sqrt{(-\frac{1}{2}\nu\pi i\tanh\gamma)}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{A_m}{(\frac{1}{2}\nu\tanh\gamma)^m} \qquad (3)$$

$$S_\nu^{(ii)}(z) \sim \frac{e^{-\nu(\tanh\gamma-\gamma) + i\pi/4}}{\sqrt{(-\frac{1}{2}\nu\pi i\tanh\gamma)}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{A_m}{(-\frac{1}{2}\nu\tanh\gamma)^m} \qquad (4)$$

where $\arg(-\frac{1}{2}\nu\pi i\tanh\gamma) = \arg z + \arg(-i\sinh\gamma)$ (5)

and $-\pi/2 < \arg(-i\sinh\gamma) < \pi/2$ (6)

The first two of the A_n's have values

$$A_0 = 1\ , \quad A_1 = \frac{1}{8} - \frac{5}{24}\coth^2\gamma\ .$$

The Schläfli contour integral representation for a Bessel Function (valid here since $|\arg z| < \pi/2$) is

$$J_{z+1}(zt) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} e^{zt\sinh w - (z+1)w}\,dw,$$

which, in accordance with the above can be written in the form

$$J_{z+1}(zt) = \frac{1}{2\pi i}\int_{\infty-\pi i}^{\infty+\pi i} e^{-ztf(w)}\,dw \tag{7}$$

where $f(w) = w\cosh\gamma - \sinh w$, and $\cosh\gamma = (z+1)/zt$. For $0 < t < 1$, $R(z)$ large requires that β be small; if β is restricted to positive values, there is a 1-1 correspondence between $\frac{z+1}{zt}$ and $\cosh\gamma$, and $\alpha \gtrless 0$ according as $I(z) \lesseqgtr 0$. With these restrictions on α and β, the shape of the contour (2) is given (following Watson) in Fig.1 for $\alpha > 0$ and in Fig. 2 for $\alpha < 0$.

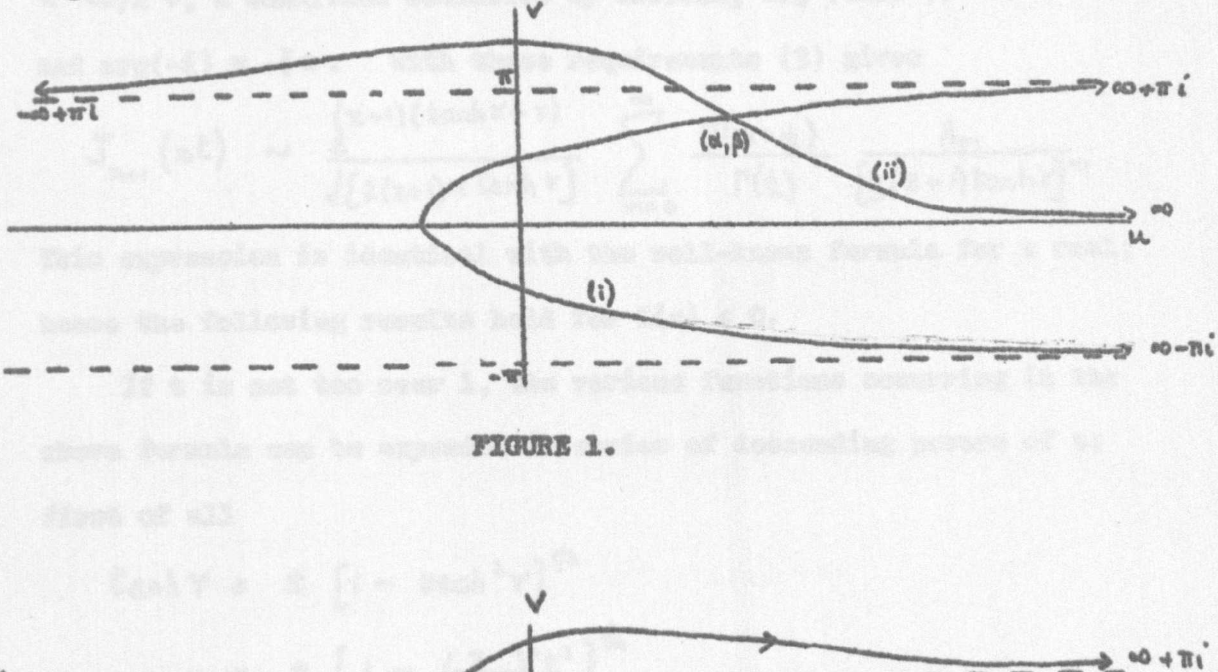


FIGURE 1.

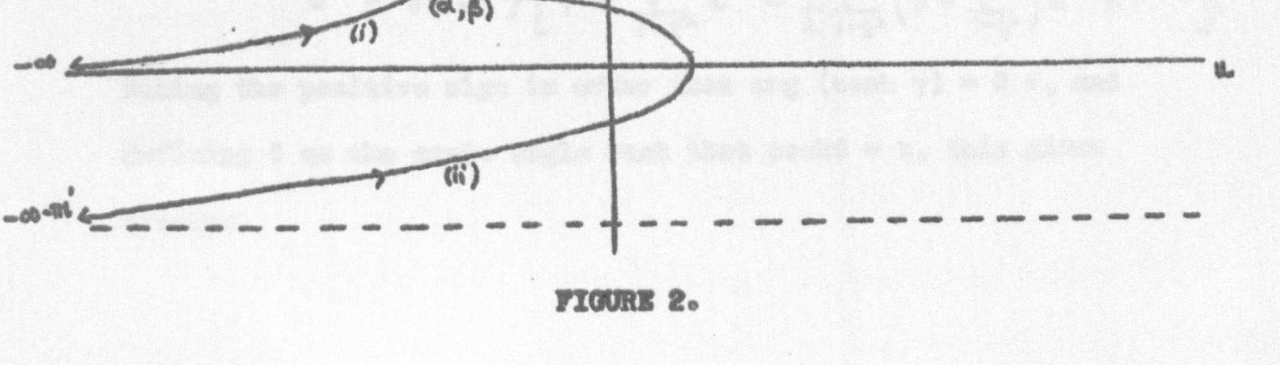


FIGURE 2.

Thus the contour integral (7) can be identified with the function $\frac{1}{2}S^{(i)}_{z+1}(zt)$ when $I(z) < 0$ and with $\frac{1}{2}S^{(ii)}_{z+1}(zt)$ when $I(z) > 0$.

Consider first the case when $I(z) < 0$. Condition (5) requires that in the asymptotic expansion of $J_{z+1}(zt)$,

$$\arg\left[-(z+1)i\tanh\gamma\right] = \arg zt + \arg(-i\sinh\gamma)$$

For $R(z)$ large this requires $\arg(-i\tanh\gamma) \doteq \arg(-i\sinh\gamma)$. Now $\alpha > 0$ and $\beta \sim 0+$; hence $\arg(\sinh\gamma) = 0+$, and from (6), $\arg(i\sinh\gamma) = -\pi/2+$. Hence it is necessary that $\arg(-i\tanh\gamma) = -\pi/2+$, a condition satisfied by choosing $\arg(\tanh\gamma) = 0+$ and $\arg(-i) = -\frac{1}{2}\pi$. With these requirements (3) gives

$$J_{z+1}(zt) \sim \frac{e^{(z+1)(\tanh\gamma-\gamma)}}{\sqrt{[2(z+1)\pi\tanh\gamma]}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{A_m}{\left[\frac{1}{2}(z+1)\tanh\gamma\right]^m}$$

This expression is identical with the well-known formula for z real; hence the following results hold for $I(z) \leqslant 0$.

If t is not too near 1, the various functions occurring in the above formula can be expanded in series of descending powers of z; first of all

$$\begin{aligned}\tanh\gamma &= \pm\left[1-\operatorname{sech}^2\gamma\right]^{1/2} \\ &= \pm\left[1-\left(\frac{z}{z+1}\right)^2 t^2\right]^{1/2} \\ &= \pm\sqrt{(1-t^2)}\left[1+\frac{t}{1-t^2}z^{-1}-\frac{1}{2}\frac{t^2}{1-t^2}\left(3+\frac{t^2}{1-t^2}\right)z^{-2}+\cdots\right].\end{aligned}$$

Taking the positive sign in order that $\arg(\tanh\gamma) = 0+$, and defining θ as the acute angle such that $\operatorname{sech}\theta = t$, this gives finally

$$(z+1)\tanh\gamma \sim z\tanh\theta + \coth\theta - \tfrac{1}{2}z^{-1}\operatorname{cosech}^2\theta\coth\theta + O(z^{-2}) \qquad (9)$$

Also, since $\cosh\gamma = \dfrac{z+1}{zt}$,

$$e^{\gamma} = \frac{z+1}{zt} \pm \left[\left(\frac{z+1}{zt}\right)^2 - 1\right]^{1/2}$$

$$= \frac{1}{t}\left[1 \pm \sqrt{(1-t^2)} + z^{-1}\left\{1 \pm \frac{1}{\sqrt{(1-t^2)}}\right\} \mp z^{-2}\frac{t^2}{2(1-t^2)^{3/2}} + \cdots\right]$$

As $|e^{\gamma}| = e^{\alpha} > 1$, the positive sign has to be taken, giving

$$e^{\gamma} = \frac{1+\sqrt{(1-t^2)}}{t}\left[1 + \frac{1}{\sqrt{(1-t^2)}}z^{-1} - \frac{t^2}{2(1-t^2)^{3/2}}\frac{1}{\{1+\sqrt{(1-t^2)}\}}z^{-2} + \cdots\right]$$

and eventually, since $\log\ \{1+\sqrt{(1-t^2)}\}/t$ is positive and equal to θ,

$$(z+1)\gamma \sim z\theta + \{\theta + \coth\theta\} + z^{-1}\{\coth\theta - \tfrac{1}{2}\coth^3\theta\} + O(z^{-2}) \qquad (10)$$

Substituting expansions (9) and (10) into (8) and using the fact that

$$A_1 = \frac{1}{8} - \frac{5}{24}\coth^2\gamma = \frac{1}{8} - \frac{5}{24}\coth^2\theta + O(z^{-1}),$$

it is finally found after some algebra, that for $R(z)$ large and $I(z) \leqslant 0$,

$$J_{z+1}(zt) \sim \frac{e^{z(\tanh\theta-\theta)-\theta}}{\sqrt{(2\pi\tanh\theta)}}\left[z^{-1/2} - \left\{\tfrac{3}{8}\coth^2\theta + \tfrac{1}{2}\coth^2\theta + \tfrac{5}{24}\coth^3\theta\right\}z^{-3/2} + o(z^{-3/2})\right] \qquad (11)$$

A similar treatment of $J_{z+1}(zt)$ for $I(z) > 0$ shows that it is necessary to take $\arg(+i) = +\frac{1}{2}\pi$ and $\arg(-\tanh\gamma) = 0-$, and hence, from (4)

$$J_{z+1}(zt) \sim \frac{e^{-(z+1)(\tanh\gamma-\gamma)}}{\sqrt{\{-2(z+1)\pi\tanh\gamma\}}}\sum_{m=0}^{\infty}\frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})}\frac{A_m}{\{-\frac{1}{2}(z+1)\tanh\gamma\}^m}$$

The functions occurring in this formula are expanded as above. This time the negative sign is required in the series for $\tanh\gamma$ and e^{γ}; the latter yields in the subsequent analysis the function $\log\{1-\sqrt{(1-t^2)}\}/t$ which has the value $-\theta$. As a result, the series given in (9) and (10) in this case represent $-(z+1)\tanh\gamma$ and $-(z+1)\gamma$ respectively, and the resulting series for $J_{z+1}(zt)$ is as in (11). Thus (11) is the required asymptotic expansion of $J_{z+1}(zt)$ for $R(z)$ large and all values of $I(z)$.

Using this result and the similar result for $J_z(zt)$, (given by Watson among others), viz.

$$J_z(zt) \sim \frac{e^{z(\tanh\theta-\theta)}}{\sqrt{(2\pi z\tanh\theta)}} \sum_{m=0}^{\infty} \frac{\Gamma(m+\frac{1}{2})}{\Gamma(\frac{1}{2})} \frac{A_m}{(\frac{1}{2}z\tanh\theta)^m},$$

in relation (1), it is finally obtained that for $0 < t < 1$ and $R(z)$ large and positive,

$$J_z'(zt) \sim \frac{e^{z(\tanh\theta-\theta)}}{\sqrt{(2\pi\tanh\theta)}}\left[z^{-1/2}\sinh\theta + z^{-3/2}\cosh\theta\left(\frac{7}{24}\coth^2\theta-\frac{3}{8}\right)+o(z^{-5/2})\right] \tag{12}$$

where θ is the acute angle such that $\operatorname{sech}\theta = t$.

This formula, and the Watson formula for $J_z(zt)$, obviously fail to give an adequate representation of the functions when t is too near 1. In fact the expansion for $J_z(zt)$ is valid only if $\frac{A_m}{(z\tanh\theta)^m} < O(1)$. From the definitions given by Watson, it is seen that $A_m = O(\coth^{2m}\theta)$ for small θ .

Thus the required condition on θ is $\tanh\theta > O(z^{-1/3})$; this is equivalent to $1 - t > O(z^{-2/3})$. With this limitation on θ , all the expansions used in the derivation of $J_z'(zt)$ are justified and hence (12) is valid.

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