Stabilization of Hybrid Systems by Discrete-time Feedback Controls

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Abstract

Hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching) have been used to model many practical systems where they may experience abrupt changes in their structure and parameters. One of the important issues in the study of hybrid systems is the automatic control, with consequent emphasis being placed on the asymptotic analysis of stability. One classical topic in this field is the problem of stabilization. The stability of hybrid systems by feedback control based on continuous-time observations has been studied extensively in the past decades.

Recently, Mao [52] initiates the study on the mean-square exponential stabilization of continuous-time hybrid stochastic differential equations by feedback controls based on discrete-time state observations. Mao [52] also obtains an upper bound on the duration τ between two consecutive state observations. However, it is due to the general technique used there that the bound on τ is not very sharp.

In this thesis, we will consider a couple of important classes of hybrid SDEs. Making full use of their special features, we will be able to establish a better bound on τ. Our new theory enables us to observe the system state less frequently (so cost less) but still to be able to design the feedback control based on the discrete-time state observations to stabilize the given hybrid SDEs in the sense of mean-square exponential stability.

Moreover, we will be able to establish a better bound on τ making use of Lyapunov functionals. By this method, we will not only discuss the stabilization in the sense of exponential stability but also in other sense of H_∞ stability or asymptotic stability as well. We will not only consider the mean square stability but also the almost sure stability.

It is easy to observe that the feedback control there still depends on the

continuous-time observations of the mode. However, it usually costs to identify the current mode of the system in practice. So we can further improve the control to reduce the control cost by identifying the mode at discrete times when we make observations for the state. Therefore, we will also design such a type of feedback control based on the discrete-time observations of both state and mode to stabilize the given hybrid stochastic differential equations (SDEs) in the sense of mean-square exponential stability in this thesis. Similarly, we can extend our discussion to the stabilization of continuous-time hybrid stochastic differential equations by feedback controls based on not only discrete-time state observations but also discrete-time mode observations by Lyapunov method.

At last, we will investigate stability of Stochastic differential delay equations with Markovian switching by feedback control based on discrete-time State and mode observations by using Lyapunov functional. Hence, we will get the upper bound on the duration τ between two consecutive state and mode observations.

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Notations

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$a.e.$	:	Almost everywhere.
$a.s.$	:	Almost surely, or with probability 1.
$\emptyset$	:	The empty set.
$A := B$	:	A is defined by B or B is denoted by A.
$\mathbb{I}_A$	:	The indicator function of set A, i.e. $\mathbb{I}_A(x) = 1$ if x A or otherwise 0.
A^C	:	The complement of A in Ω, i.e. $A^C = \Omega - A$.
$A \subset B$	:	$A \cap B^C = \emptyset$.
$A \subset B \quad a.s.$	:	$\mathbb{P}(A \cap B^C = \emptyset) = 1$.
$\sigma(C)$	:	The σ-algebra generated by C.
$a \vee b$	:	The maximum of a and b.
$a \wedge b$	:	The minimum of a and b.
$f : A \to B$	:	The mapping f from A to B.
$R = R^1$	:	The real line.
R_+	:	The set of all nonnegative real numbers, i.e. $R_+ = [0, \infty)$.
R^d	:	The d-dimensional Euclidean space.
R^d_+	:	$= \{x \in R^d, x_i > 0, 1 \le i \le d\}$, i.e. the positive cone.
$\mathcal{B} = \mathcal{B}^1$	:	The Borel-σ-algebra on R.
$\mathcal{B}^n$	:	The Borel-σ-algebra on R^n.
$\lvert\cdot\rvert$	:	The Euclidean norm of a vector.
$C(D; R^n)$	:	The family of continuous R^n-valued functions defined on D.
$C^m(D; R^n)$	:	The family of continuously m-times differentiable R^n-valued functions defined on D.
$\mathcal{C}^{2,1}(D \times R_+; R)$	:	The family of all real-valued functions $V(x,t)$ defined on $D \times R_+$ which are continuously twice differentiable in $x \in D$ and once differentiable in $t \in R_+$.
S_h	:	$= \{x \in R^d : \lvert x\rvert \le h\}$.
A^T	:	The transpose of a vector of a matrix A.
trace $\quad A$	:	$= \Sigma_{1 \le i \le d} a_{ii}$ for a square matrix $A = (a_{ij})_{d \times d}$.
$\lambda_{min}(A)$	:	The smallest eigenvalue of a matrix A.
$\lambda_{max}(A)$	:	The largest eigenvalue of a matrix A.

$\lvert A\rvert$	:	$\sqrt{\text{trace}(A^T A)}$.
$\lVert A\rVert$	:	$= sup\{\lvert Ax\rvert : \lvert x\rvert = 1\} = \sqrt{\lambda_{max}(A^T A)}$.
∇	:	$= \left(\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n}\right)$.
V_x	:	$= \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n}\right)$.
V_{xx}	:	$= \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{n\times n}$.
$\lVert\xi\rVert_{L^p}$	:	$= (\mathbb{E}\,\lvert\xi\rvert^p)^{1/p}$.
$(\Omega, \mathcal{F}, \mathbb{P})$	:	a complete probability space.
$\mathcal{L}^p(\Omega; R^d)$	:	The family of R^d-valued random variables ξ with $\mathbb{E}\,\lvert\xi\rvert^p < \infty$.
$\mathcal{L}^p_{\mathcal{F}_t}(\Omega; R^d)$	:	The family of R^d-valued $\mathcal{F}_t$-measurable random variables ξ with $\mathbb{E}\,\lvert\xi\rvert^p < \infty$.
$C([-\tau, 0]; R^d)$	:	The space of all continuous R^d-valued functions φ defined on $[-\tau, 0]$ with a norm $\lVert\varphi\rVert = sup_{-\tau\le\theta\le 0}\lvert\varphi(\theta)\rvert$.
$\mathcal{L}^p_{\mathcal{F}_t}([-\tau, 0]; R^d)$	:	The family of $\mathcal{F}_t$-measurable bounded $C([-\tau, 0]; R^d)$-valued random variables ϕ such that $\mathbb{E}\,\lVert\phi\rVert^p < \infty$.
$C^b_{\mathcal{F}_t}([-\tau, 0]; R^d)$	:	The family of $\mathcal{F}_t$-measurable bounded $C([-\tau, 0]; R^d)$-valued random variables.
$\mathcal{L}^p([a, b]; R^d)$	:	The family of Borel measuable functions $h : [a, b] \to R^d$ such that $\int_a^b \lvert h(t)\rvert^p dt < \infty$.
$\mathcal{L}^p_{\mathcal{F}_t}([a, b]; R^d)$	:	The family of R^d-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b \lvert f(t)\rvert^p dt < \infty$ $a.s.$.
$\mathcal{L}^p(R_+; R^d)$	:	The family of processes $\{f(t)\}_{t\ge 0}$ such that for every $T > 0$, $\{f(t)\}_{0\le t\le T} \in \mathcal{L}^p([0, T]; R^d)$.

Chapter 1

Introduction

1.1 Literature Review

A hybrid system is a dynamical system where the behavior of interest is determined by interacting continuous and discrete dynamics. The concept of hybrid systems was established in the 1990s by combining the classical time-driven systems with event-driven systems (also called Discrete Event Systems [11]), which evolving as the merging of these two complementary points of dynamic systems. Much of the studies on hybrid systems has concentrated on deterministic models which completely predicts the future states of the system without allowing any uncertainty. But in practice, it is often required to consider some uncertainty in the models. Therefore, non-deterministic hybrid systems are developed to allow uncertainty to take place in some places: choice of continuous evolution, choice of discrete transition destination, or choice between continuous evolution and a discrete transition. Deterministic and non-deterministic hybrid system have been widely used and played an important role in some application areas, such as automotive control, communication networks, traffic management, manufacturing, chemical processes and so on (see review on [3]). However, non-deterministic hybrid systems can not distinguish solutions that means only worst case analysis can be done with these systems. This implies that only qualitative, yes-no type questions can be put forward. Therefore, stochastic models are employed to provide the quantitative probabilistic analysis of uncertain hybrid systems, which are so-called stochastic hybrid systems (see [10, 67]). Stochastic hybrid systems are of course more realistic in practice, which have been widely applied in biology, finance and some other areas (see [2, 34–36, 73]).

In stochastic systems, uncertainty is often described by Brownian motion. Brownian motion is the irregular random motion of tiny pollen particles in water under a microscope first observed by Robert Brown in 1828. In the next more than 70 years, numerous explanations of such motion of the small pollen grains were developed until A. Einstein put forward a clear theoretical explanation that Brownian motion was resulted of the incessant bombardment of the pollen grains by the water molecules in 1905 (see [15, 54]). For example, On the Movement of Small Particles Suspended in Stationary Liquids Required by the Molecular-Kinetic Theory of Heat [15] concerns the Brownian motion of such particles. As time going by, in 1923, Norbert Wiener published a book that gave the formal mathematical definition of Brownian motion [80]. Based on Brownian motion process, Itô firstly proposed the definition of stochastic integral in 1944 (see [29]). Since then, the study on the Itô stochastic differential equations (also known as SDEs), differential equations driven by Brownian motion process, started to bloom [55]. In this thesis, all our studies are focused on Itô stochastic differential equations (SDEs).

A Markov chain, which is named after Russian mathematician Andrei Andreevich Markov in 1906 [53], is a stochastic process with Markov property, which means given the current state, the future state is independent of the past states (see [20, 22, 71]). In hybrid stochastic systems, Marov chains with discrete time states are often employed to model the process of deciding the mode of system.

One of the important issues in the study of hybrid stochastic differential equations (SDEs with Markovian switching) is the automatic control, with consequent emphasis being placed on the asymptotic analysis of stability. The concept of stability means insensitivity of the state of the system to small changes in the initial state or the parameters of the system [47]. Stabilization by continuous-time feedback controls for hybrid SDEs has been studied by some authors (see [5, 14, 30, 59, 61, 63, 72, 74, 75, 78, 86]). In particular, [48, 50] are two of most cited papers while [62] is the first book in this area. In this thesis, our main aims is to investigate the exponential stability, almost sure stability, asymptotic stability and H_∞ stability (detailed definition will be stated in Chapter 2).

Time delay is the property of a dynamical system by which the response to an applied force (action) is delayed in its effect, which means the future evolution of a system depends not only on its current state but also on its past. Time delays abound in the world because whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. The distance and transmission speed decides the value of delay. Time delays appear in various systems such as biological, social, engineering systems etc. For example, the central bank in a country often adjusts interest rates to influence the economy; the effect of a interest rates change takes months to be translated into an impact on the economy. There are some more examples for real-life systems with time delays (see [19, 65, 89]). Stochastic delay systems are time delay systems affected by Wiener process, which also has been studied by some authors (see [6, 8, 13, 32, 44, 46, 47, 49, 50, 56, 60, 90]). In recent years, Huang and Mao studied a lot about stability and stabilization of stochastic delay systems (see [38–43, 57]). Particularly, a delayed-state-feedback controller that exponentially stabilizes hybrid stochastic systems in mean square was proposed in [57].

In 2013, Mao studied on the mean-square exponential stabilization of the hybrid SDEs by discrete-time-state feedback controls and managed to get a upper bound for duration between two consecutive observations [52]. This is the first paper to study on the stabilization of hybrid systems by discrete-time-state feedback controls. Since then, we started to consider about the stabilization problems of SDEs by discrete-time feedback controls. We study on the mean-square exponential stabilization of hybrid SDEs by discrete-time-state feedback controls by a new method and manage to improve the existing result of the bound on the duration between two consecutive observations. Meanwhile, we investigate some more types of stabilizations for hybrid SDEs. The results are stated in Chapter 3 and 4. In addition, Geromel and Gabriel [31] state that it is important and reasonable to consider the feedback controls based on discrete-time mode (markov chain) observations, which also motivates us to study on the stabilization problems of SDEs by feedback controls based on discrete-time state and mode (markov chain) observations as well. Similarly, we study on the mean-square exponential stabilization at first and extend our discussion to some other types of stabilizations for hybrid

SDEs. The results are stated in Chapter 5, 6 and 7.

1.2 Outline of the study

This thesis focuses on developing the theory for the stabilization of hybrid systems by feedback control based on discrete-time observations. As far as we know, hybrid stochastic differential equations (SDEs) (also known as SDEs with Markovian switching) have been used to model many practical systems where they may experience abrupt changes in their structure and parameters. This thesis studies on stability of stochastic delay systems with Markovian switching as well.

Chapter 2 introduces the basic theory of stochastic analysis. It begins with elementary probability definitions and discusses with the basic theory of stochastic integral, Markov chains, stochastic differential equations and stability of SDEs. It should be mentioned that concepts and theorems in this chapter may be found in many mathematical books or papers on stochastic analysis (see [16,37,67–69,76]). In addition, Mao's books [47,62] are the main sources of reference for this chapter.

In Chapter 3, we stabilize the given hybrid stochastic differential equations by feedback control based on discrete-time state observations by using the mathematical features of the SDEs. The type of stability studied in this chapter is exponential stability in mean square sense. Meanwhile, we estabilishes an upper bound for duration τ between two consecutive observations.

By employing a Lyapunov functional, we manage to prove some more types of stability of hybrid SDEs controlled by discrete-time feedback control studied in Chapter 3 (H_∞ stability in the sense, mean-square asymptotical stability and almost sure asymptotically). In Chapter 4, we also find an upper bound for duration τ, which improve the existing result found in Chapter 3.

However, it is usually costs to identify the current mode of the system in practice. So we further improve the control to reduce the control cost by discrete the mode observations as well the state observations in Chapter 5 and 6. [31] also illustrates that it is reasonable and important to study on the discrete-time markov chain observations. We also establish an corresponding upper bound for duration τ.

In Chapter 7, we extend our discussion to hybrid stochastic differential delay equations. We stabilize the given delay system by feedback control based on discrete-time state and mode observations. The technique is employing a Lyapunov functional.

Some publications of our main results in Chapter 3-5 are listed as follows:

- Xuerong Mao, Wei Liu, Liangjian Hu, Qi Luo and Jianqiu Lu, Stabilization of Hybrid Stochastic Differential Equations by Feedback Control based on Discrete-time State Observations, Systems and Control Letters 73 (2014), 88–95.

- Surong You, Wei Liu, Jianqiu Lu, Xuerong Mao and Qinwei Qiu, Stabilization of Hybrid Systems by Feedback Control based on Discrete-time State Observations, SIAM Journal on Control and Optimization 53(2) (2015), 905–925.

- Yuyuan Li, Jianqiu Lu, Xuerong Mao, Qinwei Qiu, Stabilization of Hybrid Systems by Feedback Control Based on Discrete-Time State and Mode Observations, Asian Journal of Control 20(1) (2018), 1–11.

Chapter 2

Stochastic Analysis

2.1 Random Variables

Probability theory copes with mathematical models of trials whose outcomes depend on chance. All the possible outcomes (the elementary events) are grouped together to form a set, Ω, with typical element, $\omega \in \Omega$. The subsets of Ω which are of interest, are grouped together as a family $\mathcal{F}$. A family $\mathcal{F}$ with the following three properties is called a σ-algebra:

(i) $\emptyset \in \mathcal{F}$,

(ii) $A \in \mathcal{F} \Rightarrow A^C \in \mathcal{F}$,

(iii) $\{A_i\}_{i\geq 1} \subset \mathcal{F} \Rightarrow \cup_{i=1}^{\infty} A_i \in \mathcal{F}$,

where $\emptyset$ denotes the empty set and A^C denotes the complement of A in Ω. The pair $(\Omega, \mathcal{F})$ is called a measurable space, and the elements of $\mathcal{F}$ is henceforth called $\mathcal{F}$-measurable sets.

A real-valued function X: $\Omega \to R$ is said to be $\mathcal{F}$-measurable if

$$\{\omega : X(\omega) \leq a\} \in \mathcal{F}, \text{ for all } a \in R.$$

The function X is also called a real-valued $\{\mathcal{F}$-measurable$\}$ random variable. An R^d-valued function $X(\omega) = (X_1(\omega), X_2(\omega), ..., X_d(\omega))^T$ is said to be $\mathcal{F}$-measurable if all the elements X_i are $\mathcal{F}$-measurable. The indicator function $\mathbf{1_A}$ of a set $A \subset \Omega$ is defined by

$$\mathbf{1_A}(\omega) = \begin{cases} 1 & \text{for } \omega \in A, \\ 0 & \text{for } \omega \notin A. \end{cases}$$

The indicator function $\mathbf{1_A}$ is $\mathcal{F}$-measurable if and only if A is an $\mathcal{F}$-measurable set, i.e. $A \in \mathcal{F}$.

A probability measure $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P}$: $\mathcal{F} \to [0,1]$ such that

(i) $\mathbb{P}(\Omega) = 1$;

(ii) for any disjoint sequence $\{A_i\}_{i\geq 1} \subset \mathcal{F}$ (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$), $\mathbb{P}(\cup_{i=1}^{\infty} A_i) = \Sigma_{i=1}^{\infty} \mathbb{P}(A_i)$.

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. We set

$$\bar{\mathcal{F}} = \{A \subset \Omega : \exists B, C \in \mathcal{F} \text{ such that } B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}.$$

Then $\bar{\mathcal{F}}$ is a σ-algebra and is called the completion of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete.

If X is a real-valued random variable and is integrable with respect to the probability measure $\mathbb{P}$, then the number $\mathbb{E}\, X = \int_{\Omega} X(\omega) d\mathbb{P}(w)$ is called the expectation of X (with respect to $\mathbb{P}$). The number $V(X) = \mathbb{E}\,(X - \mathbb{E}\, X)^2$ is called the variance of X. And the number $\mathbb{E}\,|X|^p (p > 0)$ is called the pth moment of X.

For $p \in (0, \infty)$, let $L^p = L^p(\Omega; R^d)$ be the family of R^d-valued random variables X with $\mathbb{E}\,|X|^p < \infty$. In L^1, we have $|\mathbb{E}\, X| \leq \mathbb{E}\,|X|$. Moreover, the following three inequalities are useful in this thesis:

(i) **Hölder's inequality**

$$|\mathbb{E}\,(X^T Y)| \leq (\mathbb{E}\,|X|^p)^{1/p} (\mathbb{E}\,|Y|^q)^{1/q}$$

if $p > 1$, $1/p + 1/q = 1$, $X \in L^p$, $Y \in L^q$. This is also known as the Cauchy-Schwarz inequality when $p = 2$;

(ii) **Minkovski's inequality**

$$(\mathbb{E}\,|X + Y|^p)^{1/p} \leq (\mathbb{E}\,|X|^p)^{1/p} + (\mathbb{E}\,|Y|^p)^{1/p}$$

if $p > 1$, $X, Y \in L^p$;

(iii) **Chebyshev's inequality**

$$\mathbb{P}\{\omega : |X(\omega)| \geq c\} \leq c^{-p} \mathbb{E}\,|X|^p$$

if $c > 0$, $p > 0$, $X \in L^p$.

Let X and $X_k, k \geq 1$ be R^d-valued random variables. The following four concepts discuss about convergence:

(i) If there exists a P-null set $\Omega_0 \in \mathcal{F}$ such that for every $\omega \notin \Omega_0$, the sequence $\{X_k(\omega)\}$ converges to $X(\omega)$ in the usual sense in R^d, then $\{X_k\}$ is said to converge to X almost surely or with probability 1, and written as $\lim_{k\to\infty} X_k = X$ a.s.

(ii) If for every $\varepsilon > 0$, $\mathbb{P}\{\omega : |X_k(\omega) - X(\omega)| > \varepsilon\} \to 0$ as $k \to \infty$, then $\{X_k\}$ is said to converge to X stochastically or in probability.

(iii) If X_k and X belong to L^p and $E|X_k - X|)^p \to 0$, then $\{X_k\}$ is said to converge to X in pth moment or in L^p.

(iv) If for every real-valued continuous bounded function g defined on R^d, $\lim_{k\to\infty} Eg(X_k) = Eg(X)$, then $\{X_k\}$ is said to converge to X in distribution.

Now we state two very important integration convergence theorems.

Theorem 2.1.1. ***(Monotonic convergence theorem)*** *If $\{X_k\}$ is an increasing sequence of nonnegative random variables, then*

$$\lim_{k\to\infty} \mathbb{E}\, X_k = \mathbb{E}\,(\lim_{k\to\infty} X_k).$$

Theorem 2.1.2. ***(Dominated convergence theorem)*** *Let $p \geq 1$, $\{X_k\} \subset L^p(\Omega; R^d)$ and $Y \in L^p(\Omega; R)$. Assume that $|X_k| \leq Y a.s.$ and $\{X_k\}$ converges to X in probability. Then $X \in L^p(\Omega; R)$, $\{X_k\}$ converges to X in L^p, and*

$$\lim_{k\to\infty} \mathbb{E}\, X_k = \mathbb{E}\,(X).$$

When Y is bounded, this theorem is also referred as the bounded convergence theorem.

Let $\{A_k\}$ be a sequence of sets in $\mathcal{F}$. Define the upper limt of the sets by

$$\limsup_{k\to\infty} A_k = \{\omega : \omega \in A_k \quad \text{for } infinitely \quad many \quad k\} = \cap_{i=1}^{\infty} \cup_{k=i}^{\infty} A_k.$$

Then we have the following well-known Borel-Cantelli lemma.

Lemma 2.1.3. ***(Borel-Cantelli's lemma)***
(1) If $\{A_k\} \subset \mathcal{F}$ and $\Sigma_{k=1}^{\infty}\mathbb{P}(A_k) < \infty$, then

$$\mathbb{P}(\limsup_{k\to\infty} A_k) = 0.$$

(2) If the sequence $\{A_k\} \subset \mathcal{F}$ is independent and $\Sigma_{k=1}^{\infty}\mathbb{P}(A_k) = \infty$, then

$$\mathbb{P}(\limsup_{k\to\infty} A_k) = 1.$$

Conditional expectation plays an important role in this thesis. Therefore we illustrate the general concept of conditional expection in the following. Let $X \in$

$L^1(\Omega; R)$, and $\mathcal{G} \subset \mathcal{F}$ be a sub-σ-algebra of $\mathcal{F}$ so $(\Omega, \mathcal{G})$ is a measurable space. In general X is not $\mathcal{G}$-measurable. We now seek an integrable $\mathcal{G}$-measurable random variable Y such that it has the same values as X on the average in the sense that

$$\mathbb{E}\,(I_G Y) = \mathbb{E}\,(I_G X) \quad i.e. \quad \int_G Y(\omega) d\mathbb{P}(\omega) = \int_G X(\omega) d\mathbb{P}(\omega) \quad \text{for } all \quad G \in \mathcal{G}.$$

By the Radon-Nikodym theorem, there exists a unique Y a.s. It is called the conditional expectation of X under the condition $\mathcal{G}$, denoted by

$$Y = \mathbb{E}\,(X|\mathcal{G}).$$

If $\mathcal{G}$ is the σ-algebra generated by a random variable Y, we write

$$\mathbb{E}\,(X|\mathcal{G}) = \mathbb{E}\,(X|Y).$$

For example, consider a collection of sets $\{A_k\} \subset \mathcal{F}$ with

$$\cup_k A_k = \Omega, \quad \mathbb{P}(A_k) > 0, \quad A_k \cap A_i = \emptyset, \quad if \quad k \neq i.$$

Let $\mathcal{G} = \sigma(\{A_k\})$, i.e. $\mathcal{G}$ is generated by $\{A_k\}$. Then $\mathbb{E}\,(X|\mathcal{G})$ is a step function on Ω given by

$$\mathbb{E}\,(X|\mathcal{G}) = \Sigma_k \frac{\mathbf{1_{A_k}} \mathbb{E}\,(\mathbf{I_{A_k} X})}{\mathbb{P}(A_k)}.$$

In other words, if $\omega \in A_k$,

$$\mathbb{E}\,(X|\mathcal{G})(\omega) = \frac{\mathbb{E}\,(\mathbf{1_{A_k} X})}{\mathbb{P}(A_k)}.$$

It follows from the definition that

$$\mathbb{E}\,(\mathbb{E}\,(X|\mathcal{G})) = \mathbb{E}\,(X)$$

and

$$|\mathbb{E}\,(X|\mathcal{G})| \leq \mathbb{E}\,(|X||\mathcal{G}) \quad a.s.$$

Some other important properties of the conditional expectation are listed as following:

(a) $X \geq 0 \Rightarrow \mathbb{E}\,(X|\mathcal{G}) \geq 0$;

(b) X is $\mathcal{G}$-measurable $\Rightarrow \mathbb{E}\,(X|\mathcal{G}) = X$;

(c) $X = c = const. \Rightarrow \mathbb{E}\,(X|\mathcal{G}) = c$;

(d) $a, b \in R \Rightarrow \mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G});$ a.s.
(e) $X \leq Y \Rightarrow \mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G});$
(f) X is $\mathcal{G}$-measurable $\Rightarrow \mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G});$
(g) X, Y are independent $\Rightarrow \mathbb{E}(X|Y) = \mathbb{E}(X);$
(h) $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F} \Rightarrow \mathbb{E}(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1) = \mathbb{E}(X|\mathcal{G}_1).$

Finally, if $X = (X_1, X_2, ..., X_d)^T \in L^1(\Omega; R^d)$, its conditional expectation under $\mathcal{G}$ is defined as

$$\mathbb{E}(X|\mathcal{G}) = (\mathbb{E}(X_1|\mathcal{G}), ..., \mathbb{E}(X_d|\mathcal{G}))^T.$$

2.2 Stochastic Processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration is a family $\{\mathcal{F}_t\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$, that is $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F} for \quad all \quad 0 \leq t < s < \infty$. The filtration is said to be right continuous if $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geq 0$. When the probability space is complete, if the filtration is right continuous and $\mathcal{F}_0$ contains all P-null sets, the filtration satisfies the usual conditions. In this thesis, unless otherwise specified, we always work on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions.

A family of R^d-valued random variables $\{X_t\}_{t\in I}$ is called a stochastic process with index set I and state space R^d. In this thesis, the index set I is set to $R_+ = [0, \infty)$ and we consider the index t as time on $[0, \infty)$. For each fixed $t \in [0, \infty)$, we have a random variable $X_t(\omega) \in R^d$. On the other hand, for each fixed $\omega \in \Omega$, we have a function of t, $X_t(\omega) \in R^d$, which is called a sample path of the process.

Let $\{X_t\}_{t\geq 0}$ be an R^d-valued stochastic process. It is said to be continuous if for almost all $\omega \in \Omega$ function $X_t(\omega)$ is continuous on $t \geq 0$. It is said to be cadlag if it is right continuous and for almost all $\omega \in \Omega$ the left limit $\lim_{s\to t} X_s(\omega)$ exists and is finite for all $t > 0$. It is said to be integrable if for every $t \geq 0$, X_t is an integrable random variable. It is said to be adapted if for every t, X_t is $\mathcal{F}_t$-measurable. It is said to be progressively measurable if for every $T \geq 0$, $\{X_t\}_{0\leq t\leq T}$ regarded as a function of (t, ω) from $[0, T] \times \Omega$ to R^d is $\mathcal{B}([0, T]) \times \mathcal{F}_T$-measurable, where $\mathcal{B}([0, T])$ is the family of all Borel sub-sets of $[0, T]$. A real-valued stochastic process $\{A_t\}_{t\geq 0}$ is called an increasing process if for almost all $\omega \in \Omega$, $A_t(\omega)$ is nonnegative nondecreasing right continuous on $t \geq 0$.

A random variable α: $\Omega \to [0,\infty]$ is called an $\{\mathcal{F}_t\}$-stopping time if $\{\omega : \alpha(\omega) \le t\} \in \mathcal{F}_t$ for any $t \ge 0$. Stopping time is used in the proof process in this thesis so we quote the following two useful theorems.

Theorem 2.2.1. *If $\{x_t\}_{t\ge0}$ is a progressively measurable process and α is a stopping time, then $x_\alpha \mathbf{1}_{\alpha<\infty}$ is $\mathcal{F}_\alpha$-measurable. In particular, if α is finite, then x_α is $\mathcal{F}_\alpha$-measurable.*

Theorem 2.2.2. *Let $\{x_t\}_{t\ge0}$ be an R^d-valued cadlag $\{\mathcal{F}_t\}$-adapted process, and D an open subset of R^d. Define $\alpha = \inf\{t \ge 0 : X_t \notin D\}$, where $\inf\emptyset = \infty$. Then α is an $\{\mathcal{F}_t\}$-stopping time, and is called the first exit time from D.*

An R^d-valued $\{\mathcal{F}_t\}$-adapted integrable process $\{M_t\}_{t\ge0}$ is called a martingale with respect to $\{\mathcal{F}_t\}$ if

$$\mathbb{E}\,(M_t|\mathcal{F}_s) = M_s \quad a.s. \qquad \text{for } all \quad 0 \le s < t < \infty.$$

If $X = \{X_t\}_{t\ge0}$ is a progressively measurable process and α is a stopping time, the $X^\alpha = \{X_{\alpha\wedge t}\}_{t\ge0}$ is called a stopped process of X. The following is the well-known Doob martingale stopping theorem.

Theorem 2.2.3. *Let $\{M_t\}_{t\ge0}$ be an R^d-valued martingale with respect to $\{\mathcal{F}_t\}$, and let θ, ρ be two finite stopping times. Then*

$$\mathbb{E}\,(M_\theta|\mathcal{F}_\rho) = M_{\theta\wedge\rho} \quad a.s.$$

Particularly, if α is a stopping time, then

$$\mathbb{E}\,(M_{\alpha\wedge t}|\mathcal{F}_s) = M_{\alpha\wedge s} \quad a.s.$$

holds for all $0 \le s < t < \infty$. That is, the stopped process $M^\alpha = \{M_{\alpha\wedge t}\}$ is still a martingale with respect to the same filtration $\{\mathcal{F}_t\}$.

A real-valued $\{\mathcal{F}_t\}$-adapted integrable process $\{M_t\}_{t\ge0}$ is called a supermartingale (with respect to $\{\mathcal{F}_t\}$) if

$$\mathbb{E}\,(M_t|\mathcal{F}_s) \le M_s \quad a.s. \qquad \text{for } all \quad 0 \le s < t < \infty.$$

Moreover , it is called a submaringale, if

$$\mathbb{E}\,(M_t|\mathcal{F}_s) \ge M_s \quad a.s. \qquad \text{for } all \quad 0 \le s < t < \infty.$$

The well-known Doob's maringale inequalities are stated as follows.

Theorem 2.2.4. ***(Doob's martingale inequalities)*** *Let $\{M_t\}_{t\geq 0}$ be an R^d-valued martingale. Let $[a,b]$ be a bounded interval in R_+.*
(i) If $p \geq 1$ and $M_t \in L^p(\Omega; R^d)$, then

$$\mathbb{P}\{\omega : \sup_{a\leq t\leq b} |M_t(\omega)| \geq c\} \leq \frac{\mathbb{E}\,|M_b|^p}{c^p}$$

holds for all $c > 0$.
(ii) If $p > 1$ and $M_t \in L^p(\Omega; R^d)$, then

$$\mathbb{E}\,(\sup_{a\leq t\leq b} |M_t|^p) \leq (\frac{p}{p-1})^p\,\mathbb{E}\,|M_b|^p.$$

A stochastic process $X = \{X_t\}_{t\geq 0}$ is called square-integrable if $\mathbb{E}\,|X_t|^2 < \infty$ for every $t \geq 0$. A right continuous adapted process $M = \{M_t\}_{t\geq 0}$ is called a local martingale if there exists a nondecreasing sequence $\{\alpha_k\}_{k\geq 1}$ of stopping times with $\alpha_k \uparrow \infty$ *a.s.* such that every $\{M_{\alpha_k\wedge t} - M_0\}_{t\geq 0}$ is a martingale. By Theorem 2.2.6, we can see that every martingale is a local martingale, but the converse is not always true. If $M = \{M_t\}_{t\geq 0}$ and $N = \{N_t\}_{t\geq 0}$ are two real-valued continuous local martingales, their joint quadratic variation $\{\langle M,N\rangle\}_{t\geq 0}$ is the unique continuous adapted process of finite variation such that $\{M_tN_t - \langle M,N\rangle_t\}_{t\geq 0}$ is a continuous local martingale vanishing at $t = 0$. In particular, for any finite stopping time τ^*, $\mathbb{E}\,M^2_{\tau^*} = \mathbb{E}\,\langle M,M\rangle_{\tau^*}$ and $\langle M,N\rangle_t = \frac{1}{2}(\langle M+N,M+N\rangle_t - \langle M,M\rangle_t - \langle N,N\rangle_t)$. The following result is the useful strong law of large numbers.

Theorem 2.2.5. ***(Strong law of large numbers)*** *Let $M = \{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\lim_{t\to\infty} \langle M,M\rangle_t = \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{\langle M,M\rangle_t} = 0 \quad a.s.$$

and also

$$\limsup_{t\to\infty} \frac{\langle M,M\rangle_t}{t} < \infty \quad a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.$$

More generally, if $A = \{A_t\}_{t\geq 0}$ is a continuous adapted increasing process such that

$$\lim_{t\to\infty} A_t = \infty \quad and \quad \int_0^\infty \frac{d\langle M,M\rangle_t}{(1+A_t)^2} < \infty \quad a.s.$$

then

$$\lim_{t\to\infty} \frac{M_t}{A_t} = 0 \quad a.s.$$

To close this section, we state a useful convergence theorem as following. This theorem plays an important role in the stability analysis.

Theorem 2.2.6. *Let $\{A_t\}_{t\geq 0}$ and $\{U_t\}_{t\geq 0}$ be two continuous adapted increasing processes with $A_0 = U_0 = 0$ a.s. Let $\{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale with $M_0 = 0$ a.s. Let ξ be a nonnegative $\mathcal{F}_0$-measurable random variable. Define*

$$X_t = \xi + A_t - U_t + M_t \qquad \text{for } t \geq 0.$$

If X_t is nonnegative, then

$$\{\lim_{t\to\infty} A_t < \infty\} \subset \{\lim_{t\to\infty} X_t \quad exists \quad and \quad is \quad finite\} \cap \{\lim_{t\to\infty} U_t < \infty\} \quad a.s.$$

where $B \subset D$ a.s. means $\mathbb{P}(B \cap D^c) = 0$. In particular, if $\lim_{t\to\infty} A_t < \infty$ a.s., then for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} X_t(\omega) \quad exists \quad and \quad is \quad finite, \quad \lim_{t\to\infty} U_t(\omega) < \infty.$$

2.3 Brownian Motion

Brownian motion describes the irregular motion of pollen grains suspended in water which was initially observed by the Scottish botanist Robert Brown in 1828. Later, Norbert Wiener obtained the mathematical foundation for Brownian motion as a stochastic process in 1931. To describe the motion mathematically it is natural to use the concept of a stochastic process $W_t(\omega)$, considered as the position of the pollen grain ω at time t. Let us now give the mathematical definition of Brownian motion.

Definition 2.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A (standard) one-dimensional Brownian motion is a real-valued continuous $\mathcal{F}_t$-adapted process $\{W_t\}_{t\geq 0}$ with the following properties:*
(i) $W_0 = 0 \quad$ a.s.;
(ii) for $0 \leq s < t < \infty$, the increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$;
(iii) for $0 \leq s < t < \infty$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$.

Throughout this thesis, unless otherwise specified, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions, and the one-dimensional Brownian motion $\{W_t\}$ is defined on it. The Brownian motion has many important properties, and some are summarized as follows:

(a) $\{-W_t\}$ is a Brownian motion with respect to the same filtration $\{\mathcal{F}_t\}$.

(b) Let $c > 0$. Define

$$X_t = \frac{W_{ct}}{\sqrt{c}}, \quad \text{for } t \geq 0.$$

Then $\{X_t\}$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_{ct}\}$.

(c) $\{W_t\}$ is a continuous square-integrable martingale and its quadratic variation $\langle W, W \rangle_t = t$ for all $t \geq 0$.

(d) The strong law of large numbers states that

$$\lim_{t \to \infty} \frac{W_t}{t} = 0 \quad a.s.$$

(e) For almost all $\omega \in \Omega$, the Brownian motion sample path $W(t, \omega)$ is nowhere differentiable.

Now we generalize the d-dimensional definition as following.

Definition 2.3.2. *A d-dimensional process $\{W_t = (W_t^1, \ldots, W_t^d)\}_{t \geq 0}$ is called a d-dimensional Brownian motion if every $\{W_t^i\}$ is a one-dimensional Brownian motion, and $\{W_t^1\}, \ldots, \{W_t^d\}$ are independent.*

From Definition 2.3.2, the similiar properties of one-dimensional Brownian motion hold for d-dimensional Brownian motion as well. In addition, it is easy to see that a d-dimensional Brownian motion is a d-dimensional continuous martingale with the joint quadratic variations

$$\langle W^i, W^j \rangle_t = \delta_{ij} t \qquad \text{for } 1 \leq i, j \leq d,$$

where

$$\delta_{ij} = \begin{cases} 1 & \text{for } i = j, \\ 0 & \text{for } i \neq j. \end{cases}$$

It turns out that this property characterizes Brownian motion among continuous local martingales, which is described by the following well-known Lévy theorem.

Theorem 2.3.3. ***(Lévy theorem)*** *Let* $\{M_t\} = \{(M_t^1, \ldots, M_t^d)\}_{t\geq 0}$ *be a d-dimensional continuous local martingale with respect to the filtration* $\{\mathcal{F}_t\}$ *and* $M_0 = 0$ *a.s. If*

$$\langle M_t^i, M_t^j \rangle_t = \delta_{ij} t \qquad \text{for } 1 \leq i, j \leq d,$$

then $\{M_t\} = \{(M_t^1, \ldots, M_t^d)\}_{t\geq 0}$ *is a d-dimensional Brownian motion with respect to* $\{\mathcal{F}_t\}$*.*

As an application of Theorem 2.3.3, we have the following useful theorem.

Theorem 2.3.4. *Let* $M = \{M_t\}_{t\geq 0}$ *be a real-valued continuous local martingale such that* $M_0 = 0$ *and* $\lim_{t\to\infty} \langle M, M\rangle_t = \infty$ *a.s. For each* $t \geq 0$*, define the stopping time*

$$\alpha_t = \inf\{s : \langle M, M\rangle_s > t\}.$$

Then $\{M_{\alpha_t}\}_{t\geq 0}$ *is a Brownian motion with respect to the filtration* $\{\mathcal{F}_{\alpha_t}\}_{t\geq 0}$*.*

2.4 Stochastic integrals and Itô's formula

In this section, we define the stochastic integral at first

$$\int_0^t f(s) dW_s$$

with respect to an m-dimensional Brownian motion $\{W_t\}$ for a class of $d \times m$-matrix-valued stochastic process $\{f(t)\}$. This integral was first defined by K. Itô in 1949 and is now known as Itô stochastic integral. Now we start to define the stochastic integral step by step.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $W = \{W_t\}_{t\geq 0}$ be a one-dimensional Brownian motion defined on the probability space adapted to the filtration.

Definition 2.4.1. *Let* $0 \leq a < b < \infty$*. Denote by* $\mathcal{M}^2([a,b];\mathbb{R})$ *the space of all real-valued measurable* $\{\mathcal{F}_t\}$*-adapted processes* $f = \{f(t)\}_{a\leq t\leq b}$ *such that*

$$\|f\|_{a,b}^2 = \mathbb{E}\int_a^b |f(t)|^2 dt < \infty. \tag{2.1}$$

We identify f *and* $\bar{f}$ *in* $\mathcal{M}^2([a,b];\mathbb{R})$ *if* $\|f - \bar{f}\|_{a,b}^2 = 0$*. in this case we say that* f *and* $\bar{f}$ *are equivalent and write* $f = \bar{f}$*.*

A real-valued stochastic process $g = \{g(t)\}_{a\le t\le b}$ is called a simple (or step) process if there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$, and bounded random variables ξ_i, $0 \le i \le k-1$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable and

$$g(t) = \xi_0 \mathbf{1}_{[\mathbf{t_0},\mathbf{t_1}]}(\mathbf{t}) + \mathbf{\Sigma}_{\mathbf{i=1}}^{\mathbf{k-1}} \xi_\mathbf{i} \mathbf{1}_{(\mathbf{t_i},\mathbf{t_{i+1}}]}(\mathbf{t}). \tag{2.2}$$

And we denote by $\mathcal{M}_0([a,b];\mathbb{R})$ the family of all such processes. It is easy to know that $\mathcal{M}_0([a,b];\mathbb{R}) \subset \mathcal{M}^2([a,b];\mathbb{R})$. We now define the Itô integral for such simple processes as following.

Definition 2.4.2. ***(Part 1 of the definition of Itô's integral)*** *For a simple process g with the form of (2.2) in $\mathcal{M}_0([a,b];\mathbb{R})$, define*

$$\int_a^b g(t)dW_t = \Sigma_{i=0}^{k-1} \xi_i (W_{t_{i+1}} - W_{t_i}) \tag{2.3}$$

and call it the stochastic integral of g with respect to the Brownian motion $\{W_t\}$ or the Itô integral.

Clearly, the stochastic integral is $\mathcal{F}_b$-measurable. And we have the following useful lemmas.

Lemma 2.4.3. *If $g \in \mathcal{M}_0([a,b];\mathbb{R})$, then*

$$\mathbb{E} \int_a^b g(t)dW_t = 0, \tag{2.4}$$

$$\mathbb{E} \left| \int_a^b g(t)dW_t \right|^2 = \mathbb{E} \int_a^b |g(t)|^2 dt. \tag{2.5}$$

Lemma 2.4.4. *Let $g_1, g_2 \in \mathcal{M}_0([a,b];\mathbb{R})$ and let c_1, c_2 be two real numbers. Then $c_1 g_1 + c_2 g_2 \in \mathcal{M}_0([a,b];\mathbb{R})$ and*

$$\int_a^b [c_1 g_1(t) + c_2 g_2(t)] dW_t = c_1 \int_a^b g_1(t)dW_t + c_2 \int_a^b g_2(t)dW_t.$$

Lemma 2.4.5. *For any $f \in \mathcal{M}^2([a,b];\mathbb{R})$, there exists a sequence $\{g_n\}$ of simple processes such that*

$$\lim_{k\to\infty} \mathbb{E} \int_a^b |f(t) - g_n(t)|^2 dt = 0.$$

Hence by Lemma 2.4.5, there is a sequence $\{g_k\}_{k\geq 1}$ of simple processes such that $\lim_{k\to\infty} \mathbb{E} \int_a^b |f(t) - g_k(t)|^2 dt = 0$. Thus by (2.5) and Lemma 2.4.5,

$$\mathbb{E}\,|\int_a^b g_k(t) - g_j(t) dW_t|^2 = \mathbb{E}\,|\int_a^b g_k(t) - g_j(t) dW_t|^2 \to 0 \quad as \quad k, j \to \infty.$$

In other words, $\{\int_a^b g_k(t)dW_t\}$ is a Cauchy sequence in $L^2(\Omega; \mathbb{R})$. So the limit exists and is independent of the choice of sequences of simple processes approximating f. This limit is defined as the stochastic integral. This leads to the following definition.

Definition 2.4.6. ***(Part 2 of the definition of Itô's integral)*** *Let $f \in \mathcal{M}^2([a, b]; \mathbb{R})$. The Itô integral of f with respect to $\{W_t\}$ is defined by*

$$\int_a^b f(t)dW_t = \lim_{k\to\infty} \int_a^b g_k(t)dW_t \quad in \quad L^2(\Omega; \mathbb{R}),$$

where $\{g_k\}$ is a sequence of simple process such that

$$\lim_{k\to\infty} \mathbb{E} \int_a^b |f(t) - g_k(t)|^2 dt = 0.$$

The Itô stochastic integral has many properties, some of them used in this thesis are listed as follows. Let $f, g \in \mathcal{M}^2([a, b]; \mathbb{R})$. Then

(i) $\int_a^b f(t)dW_t$ is $\mathcal{F}_b$-measurable;
(ii) $\mathbb{E} \int_a^b f(t)dW_t = 0$;
(iii) $\mathbb{E}\,| \int_a^b f(t)dW_t|^2 = \mathbb{E} \int_a^b |f(t)|^2 dt$;
(iv) $\mathbb{E}\,(\int_a^b f(t)dW(t)|\mathcal{F}_\alpha) = 0$;
(v) $\mathbb{E}\,(| \int_a^b f(t)dW(t)|^2|\mathcal{F}_\alpha) = \mathbb{E}\,(\int_a^b |f(t)|^2 dt|\mathcal{F}_\alpha) = \int_a^b \mathbb{E}\,(|f(t)|^2|\mathcal{F}_\alpha)dt$.

Definition 2.4.7. *Let $f \in \mathcal{M}^2([0, T]; \mathbb{R})$. Define*

$$I(t) = \int_0^t f(s)dW_s \qquad for\ 0 \leq t \leq T,$$

where $I(0) = \int_0^0 f(s)dW_s = 0$. We call $I(t)$ the indefinite Itô integral of f.

Then we know that $I = \{I(t)\}_{0\leq t\leq T}$ is a square-integrable continuous martingale and its quadratic variation is given by

$$\langle I, I\rangle_t = \int_0^t |f(s)|^2 ds, \quad 0 \leq t \leq T. \tag{2.6}$$

However the definition of the integrals is not very convenient in evaluating a given integral. This is similar to the situation for classical Lebesgue integrals, where we do not use the basic definition but rather the fundamental theorem of calculus plus the chain rule in the explicit calculations. Therefore, we establish the stochastic version of the chain rule for the Itô integrals, which is called Itô's formula. This plays a key role in stochastic analysis.

An n-dimensional Itô process is an $\mathbb{R}^n$-valued continuous adapted process $x(t) = (x_1(t), \ldots, x_n(t))^T$ on $t \geq 0$ of the form

$$x(t) = x(0) + \int_0^t f(s)ds + \int_0^t g(s)dW(s), \tag{2.7}$$

where $f = (f_1, \ldots, f_n)^T \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^n)$ and $g = (g_{ij})_{n\times m} \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$. We shall say that $x(t)$ has a stochastic differential $dx(t)$ on $t \geq 0$ given by

$$dx(t) = f(t)dt + g(t)dW(t). \tag{2.8}$$

Before stating the well-known Itô's formula, we introduce some basic notations. Let $C^{2,1}(\mathbb{R} \times \mathbb{R}_+; \mathbb{R})$ denote the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}^n \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t. If $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, we set

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = (\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_n}), \quad V_{xx} = (\frac{\partial^2 V}{\partial x_i \partial x_j})_{n\times n}.$$

Following is the well-known Itô's formula.

Theorem 2.4.8. ***(Itô's formula)*** *Let $x(t)$ be an n-dimensional Itô process on $t \geq 0$ with the stochastic differential*

$$dx(t) = f(t)dt + g(t)dW(t),$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^n)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{n\times m})$. Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t),t)$ is a real-valued Itô process with its stochastic differential given by

$$\begin{aligned} dV(x(t),t) = {} & [V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}trace(g^T(t)V_{xx}(x(t),t)g(t)]dt \\ & +V_x(x(t),t)g(t)dW(t) \quad a.s. \end{aligned} \tag{2.9}$$

In addition, we can generate a multiplication table:

$$\begin{aligned} &dtdt = 0, \quad dW_i dt = 0, \\ &dW_i dW_i = dt, \quad dW_i dW_j = 0 \quad if \quad i \neq j. \end{aligned}$$

2.5 Markov chains

In this section, we recall some basic facts about Markov chain at first (see [4,62,84]). A stochastic process $X = \{X_t\}_{t\geq 0}$ defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, with values in a countable set $\mathbb{S}$ (state space), is called a continuous-time Markov chain if for any finite set $0 \leq t_1 < t_2 < t_3 < \cdots < t_n < t_{n+1}$ of "times", and corresponding set $i_1, i_2, \ldots, i_{n-1}, i, j$ of states in $\mathbb{S}$ such that $\mathbb{P}\{X(t_n) = i, X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1\} > 0$, we have

$$\mathbb{P}\{X(t_{n+1}) = j | X(t_n) = i, X(t_{n-1}) = i_{n-1}, \ldots, X(t_1) = i_1\}$$
$$= \mathbb{P}\{X(t_{n+1}) = j | X(t_n) = i\}.$$

If for all s, t such that $0 \leq s \leq t < \infty$ and all $i, j \in \mathbb{S}$ the conditional probability $\mathbb{P}\{X(t) = j | X(s) = i\}$ depends only on $t - s$, we say that the process X is homogeneous. In this case, $\mathbb{P}\{X(t) = j | X(s) = i\} = \mathbb{P}\{X(t - s) = j | X(0) = i\}$, and the function

$$p_{ij}(s, t) := \mathbb{P}\{X(t) = j | X(s) = i\}, \quad i, j \in \mathbb{S}, t \geq 0,$$

is called the transition function or the transition probability of the process X. In addition, the matrix $P(s, t) = (p_{ij}(s, t))_{i,j\in\mathbb{S}}$ is called the transition matrix of X if the following properties are satisfied:

(i) $p_{ij}(s, t) = \mathbb{P}(X(t) = j | X(s) = i)$ for all $0 \leq s \leq t$ and $i, j \in \mathbb{S}$;

(ii) $p_{ij}(s, s) = \delta_{ij}$ for all $s \geq 0$ and $i, j \in \mathbb{S}$;

(iii) $\Sigma_{j\in\mathbb{S}} p_{ij}(s, t) = 1$ for all $0 \leq s \leq t$ and $i \in \mathbb{S}$;

(iv) the Kolmogorov-Chapman equation

$$p_{ij}(s, t) = \Sigma_{k\in\mathbb{S}} p_{ik}(s, u) p_{kj}(u, t),$$

or

$$P(s, t) = P(s, u) P(u, t)$$

holds for all $0 \leq s \leq u \leq t < \infty$.

The Markov chain X is said to be stationary if its transition probabilities $p_{ij}(s, t)$, $i, j \in \mathbb{S}$, are stationary i.e. $p_{ij}(s, t)$ depends only on the difference $t - s$ for all $0 \leq s \leq t < \infty$ and $i, j \in \mathbb{S}$. This implies $P(s, s + u) = P(u)$ for all $s \geq 0$ and $u \geq 0$. The transition matrix $P(t) = (p_{ij}(t))_{i,j\in\mathbb{S}}$ is said to be standard if $\lim_{t\to 0} p_{ii}(t) = 1$ for all $i \in \mathbb{S}$.

Theorem 2.5.1. *Let $P_{ij}(t)$ be a standard transition function, then $\gamma_i = \lim_{t\to 0}[1-P_{ii}(t)]/t$ exists (but may be ∞) for all $i \in \mathbb{S}$. A state $i \in \mathbb{S}$ is said to be stable if $\gamma_i < \infty$.*

Theorem 2.5.2. *Let $P_{ij}(t)$ be a standard transition function, and let j be a stable state. Then $\gamma_{ij} = P'_{ij}(0)$ exists and is finite for all $i \in \mathbb{S}$.*

Let $\gamma_{ii} = -\gamma_i$ and $\Gamma = (\gamma_{ij})_{i,j\in\mathbb{S}}$. Γ is called the generator of the Markov chain. If the state space is finite which we can take to be $\mathbb{S} = \{1, 2, \ldots, N\}$, then the process is called a continuous-time finite Markov chain. Throughout this thesis, we assume all Markov chains are finite and all states are stable. For such a Markov chain, almost every sample path is a right continuous step function.

Theorem 2.5.3. *Let $P(t) = (P_{ij}(t))_{N\times N}$ be the transition probability matrix and $\Gamma = (\gamma_{ij})_{N\times N}$ be the generator of a finite Markov chain. Then*

$$P(t) = e^{t\Gamma}.$$

It is useful to recall that a continuous-time Markov chain X with generator $\Gamma = (\gamma_{ij})_{N\times N}$ can be represented as a stochastic integral with respect to a Poisson random measure (see [5, 17, 18, 70, 87]). Indeed, let Δ_{ij} be consecutive, left-closed, right-open intervals of the real line having length γ_{ij} such that

$$\begin{aligned}
\Delta_{12} &= [0, \gamma_{12}),\\
\Delta_{13} &= [\gamma_{12}, \gamma_{12} + \gamma_{13}),\\
&\vdots\\
\Delta_{1N} &= [\Sigma_{j=2}^{N-1}\gamma_{1j}, \Sigma_{j=2}^{N}\gamma_{1j}),\\
\Delta_{21} &= [\Sigma_{j=2}^{N}\gamma_{1j}, \Sigma_{j=2}^{N}\gamma_{1j} + \gamma_{21}),\\
\Delta_{23} &= [\Sigma_{j=2}^{N}\gamma_{1j} + \gamma_{21}, \Sigma_{j=2}^{N}\gamma_{1j} + \gamma_{21} + \gamma_{23}),\\
&\vdots\\
\Delta_{2N} &= [\Sigma_{j=2}^{N}\gamma_{1j} + \Sigma_{j=1,j\neq 2}^{N-1}\gamma_{2j}, \Sigma_{j=2}^{N}\gamma_{1j} + \Sigma_{j=1,j\neq 2}^{N}\gamma_{2j})
\end{aligned}$$

and so on. Define a function $h : \mathbb{S} \times \mathbb{R} \to \mathbb{R}$ by

$$h(i, y) = \begin{cases} j - i & \text{if } y \in \Delta_{ij},\\ 0 & \text{otherwise.}\end{cases} \tag{2.10}$$

Then

$$dX(t) = \int_{\mathbb{R}} h(X(t-, y)v(dt, dy), \tag{2.11}$$

with initial condition $X(0) = i_0$, where $v(dt, dy)$ is a Poisson random measure with intensity $dt \times \mu(dy)$, in which μ is the Lebesgue measure on $\mathbb{R}$.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost all sample paths of $r(t)$ are constant except for a finite number of simple jumps in any finite subinterval of R_+. We stress that almost all sample paths of $r(t)$ are right continuous.

In this thesis, we consider the paired process$(x(t), r(t))$ and we state how a function $V : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ map $(x(t), r(t))$ into another process $V(x(t), r(t), t)$ as follows. For $\forall i \in \mathbb{S}$, let $V(x,t) \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+; \mathbb{R})$, we denote an operator LV from $\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned} LV(x,i,t) = & V_t(x,i,t) + V_x(x,i,t)f(t) \\ & + \frac{1}{2}[trace(g^T(t)V_{xx}(x,i,t)g(t)] + \Sigma_{j=1}^N \gamma_{ij} V(x,j,t). \end{aligned} \tag{2.12}$$

Theorem 2.5.4. ***(Generalized Itô Formula)*** *If* $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$, *then for any* $t \geq 0$

$$\begin{aligned} & V(x(t), r(t), t) \\ = & V(x(0), r(0), 0) + \int_0^t LV(x(s), r(s), s)ds \\ + & \int_0^t V_x(x(s), r(s), s)g(x(s), r(s), s)dW(s) \end{aligned}$$

$$+\int_0^t \int_{\mathbb{R}} (V(x(s), i_0, s + h(r(s), l)) - V(x(s), r(s), s))\mu(ds, dl), \tag{2.13}$$

where the function h is defined as (2.10) and $\mu(ds, dl) = v(ds, dl) - \mu(dl)ds$ is a martingale measure (see also [5, 17, 18, 87]).

Taking expectation on both sides of (2.13) yields

Lemma 2.5.5. *Let $V \in C^{2,1}(\mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$ and ρ_1, ρ_2 be bounded stopping times such that $0 \le \rho_1 \le \rho_2$ a.s. If $V(x(t), r(t), t)$ and $LV(x(t), r(t), t)$ are bounded on $t \in [\rho_1, \rho_2]$ with probability 1, then*

$$\mathbb{E}\, V(x(\rho_2), r(\rho_2), \rho_2) = \mathbb{E}\, V(x(\rho_2), r(\rho_2), \rho_2) + \mathbb{E} \int_{\rho_1}^{\rho_2} LV(x(s), r(s), s)ds. \tag{2.14}$$

2.6 Stochastic differential equations with Markovian switching

In this thesis, we focus on the stochastic differential equations with Markovian switching (also known as hybrid SDEs), which of the form as follows

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dW(t), \quad t_0 \le t \le T \tag{2.15}$$

with initial data $x(t_0) = x_0 \in \mathcal{L}^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R}^n)$ and $r(t_0) = r_0$, where r_0 is an $\mathbb{S}$-valued $\mathcal{F}_{t_0}$-measurable random variable and

$$f : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^n \quad and \quad g : \mathbb{R}^n \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{n\times m}.$$

By the definition of stochastic differential, (2.15) is equivalent to the following stochastic integral equation

$$x(t) = x_0 + \int_{t_0}^t f(x(s), r(s), s)ds + \int_{t_0}^t g(x(s), r(s), s)dW(s), \quad \forall t \in [t_0, T]. \tag{2.16}$$

An $\mathbb{R}^n$-valued stochastic process $\{x(t)\}_{t_0 \le t \le T}$ is called a solution of equation (2.15) if it has the following properties:

(i) $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;

(ii) $\{f(x(t), t, r(t))\}_{t_0 \le t \le T} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^n)$ while $\{g(x(t), t, r(t))\}_{t_0 \le t \le T} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{n\times m})$;

(iii) equation (2.16) holds with probability 1.

Moreover, a solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$ i.e. $\mathbb{P}\{x(t) = \bar{x}(t) \quad \text{for } all \quad t_0 \le t \le T\} = 1$. The following theorem is the well-known existence and uniqueness theorem of solutions of SDEs.

Theorem 2.6.1. *Assume that there exist two positive constants $\bar{K}$ and K such that*

(i) ***(Lipschitz continuous condition)*** *for all $x, y \in \mathbb{R}^n$ and $t \in [t_0, T]$, any $i \in \mathbb{S}$*

$$|f(x,t,i) - f(y,t,i)|^2 \vee |g(x,t,i) - g(y,t,i)|^2 \le \bar{K}|x-y|^2; \tag{2.17}$$

(ii) ***(Linear growth condition)*** *for all $(x,t,i) \in \mathbb{R} \times [t_0, T] \times \mathbb{S}$*

$$|f(x,t,i)|^2 \vee |g(x,t,i)|^2 \le \bar{K}(1 + |x|^2). \tag{2.18}$$

Then there exists a unique solution $x(t)$ to equation (2.5.1) and, moreover,

$$\mathbb{E}\,(\sup_{t_0 \le t \le T} |x(t)|^2) \le (1 + 3\mathbb{E}\,|x_0|^2)e^{3K(T-t_0)(T-t_0+4)} \tag{2.19}$$

so the solution belongs to $\mathcal{M}^2([t_0, T]; \mathbb{R}^n)$.

In addition the following theorem relax the Lipschitz continuous condition to local Lipschitz condition, and the existence and uniqueness of solution to (2.5.1) still hold. This can be stated as the following theorem.

Theorem 2.6.2. ***(Local Lipschitz condition)*** *Assume that for every integer $k \ge 1$, there exists a positive constant h_k such that, for all $t \in [t_0, T]$, $i \in \mathbb{S}$ and those $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \le k$,*

$$|f(x,t,i) - f(y,t,i)|^2 \vee |g(x,t,i) - g(y,t,i)|^2 \le h_k|x-y|^2. \tag{2.20}$$

Assume the linear growth condition (2.18) holds. Then there exists a unique solution to equation (2.15).

The proof of the existence and uniqueness theorem can be referred to [47, 62]. To close this section, we introduce the well-known Gronwall's inequality, which has been widely applied in the theory of SDEs to prove the results on existence, uniqueness, boundedness, comparison and stability etc. Therefore, Gronwall's inequality is important in this thesis.

Theorem 2.6.3. ***(Gronwall's inequality)*** *Let $T > 0$ and $c \geq 0$. Let $u(.)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let $v(.)$ be a nonnegative integrable function on $[0, T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)ds \qquad \text{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq c\exp(\int_0^t v(s)ds) \qquad \text{for all} \quad 0 \leq t \leq T.$$

Theorem 2.6.4. *Let $T > 0$, $a \in [0, 1)$ and $c \geq 0$. Let $u(\cdot)$ be a Borel measurable bounded nonnegative function on $[0, T]$, and let $v(\cdot)$ be a nonnegative integrable function on $[0, T]$. If*

$$u(t) \leq c + \int_0^t v(s)[u(s)]^a ds \qquad \text{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq \left(c^{1-a} + (1-a)\int_0^t v(s)ds\right)^{\frac{1}{1-a}} \qquad \text{for all} \quad 0 \leq t \leq T.$$

2.7 Stability of SDEs

Stability of a process refers to the consistency of the process with respect to important process characteristics such as the average value of a key dimension or the variation in that key dimension. If the process behaves consistently over time, it is said to be stable. This property turns out to be of utmost importance. It should be emphasized that an individual predictable process can be physically realized only if it is stable in the corresponding natural sense. The main technique in this area is the method of Lyapunov functions, known as Lyapunov's second method. This method has gained increasing significance and has given decisive impetus for modern development of stability theory of dynamic systems during the past decades. A manifest advantage of this method is that it does not require the knowledge of solutions of equations and therefore has a great power in applications. Lyapunov function transforms a given complicated stochastic differential system into relatively simpler differential equations and so it is sufficient to study the properties of solutions of this simpler differential equation. Now let us propose the concept of trivial solution before introducing the definitions of stability.

Assumption 2.7.1. *Assume that for each $k = 1, 2, \dots,$ there is an $h_k > 0$ such that*

$$|f(x, i, t)| \vee |g(x, i, t)| \le h_k |x|$$

for all $0 \le t \le k, i \in \mathbb{S}$ and those $x \in \mathbb{R}^n$ with $|x| \le k$.

It is easy to see that Assumption 2.7.1 implies $f(0, t, i) \equiv 0$ and $g(0, t, i) \equiv 0$. Therefore, we observe that the solution $x(t)$ of equation (2.15) will remain to be zero if it starts from zero, namely $x(t; t_0, 0, r_0) \equiv 0$. This solution is often called a trivial solution. In addition, any solution of equation (2.15) starting from a non-zero state will remain to be non-zero.

In this section we shall investigate various types of stability for stochastic differential equation (2.15) defined as follows.

(1) Stability in distribution: the solution $x(t)$ of (2.15) is said to be asymptotically stable in distribution if there exists a probability measure $\pi(\cdot \times \cdot)$ on $\mathbb{R}^n \times \mathbb{S}$ such that the transition probability $p(t, y, i, dx \times \{j\})$ of $x(t)$ converges weakly to $\pi(dx \times \{j\})$ as $t \to \infty$ for every $(y, i) \in \mathbb{R}^n \times \mathbb{S}$. Equation (2.15) is said to be asymptotically stable in distribution if $x(t)$ is asymptotically stable in distribution (see [62, 88]).

(2) Stability in probability: system (2.15) is said to be

(a) stochastically stable or stable in probability if for every pair of $\varepsilon \in (0, 1)$ and $r > 0$, there exists a $\delta = \delta(\varepsilon, r, t_0) > 0$ such that

$$\mathbb{P}\{|x(t; t_0, x_0)| < r \qquad \text{for } \textit{all} \quad t \ge t_0\} \ge 1 - \varepsilon$$

whenever $|x_0| < \delta$.

(b) stochastically aysmptotically stable if it is stochastically stable, and, moreover, for every $\varepsilon \in (0, 1)$, there exists a $\delta_0 = \delta_0(\varepsilon, t_0) > 0$ such that

$$\mathbb{P}\{\lim_{t \to \infty} x(t; t_0, x_0) = 0\} \ge 1 - \varepsilon$$

whenever $|x_0| < \delta$.

Theorem 2.7.2. *If there exists a positive-definite function $V(x, t) \in C^{2,1}(\mathbb{S}_h \times [t_0, \infty); \mathbb{R}_+)$ such that $LV(x, t) \le 0$ for all $(x, t) \in \mathbb{S}_h \times [t_0, \infty)$, then the solution of equation (2.15) is stochastically stable.*

If $LV(x, t)$ is negative-definite, then the solution of equation (2.15) is stochastically asymptotically stable.

(3) Moment stability: the solution of equation (2.15) is said to be asymptotically stable in pth ($p > 0$) moment if

$$\lim_{t\to\infty} \mathbb{E}\left(|x(t;t_0,x_0,r_0)|^p\right) = 0$$

for all $(t_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$. When $p = 2$, it is said to be asymptotically stable in mean square.

(4) Almost sure stability: the solution of equation (2.15) is said to be almost surely asymptotically stable if

$$\mathbb{P}\{\lim_{t\to\infty} x(t;t_0,x_0,r_0) = 0\} = 1 \quad a.s.$$

for all $(t_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$.

Moreover, in this thesis, we mainly discuss on the exponential stability and also refer to H_∞-stability as follows.

(5) Exponential stability in pth ($p > 0$) moment sense: system (2.15) is said to be pth ($p > 0$) moment exponentially stable if there is a pair of positive constants C and ε such that

$$\mathbb{E}\,|x(t;t_0,x_0,r_0)|^p \le C|x_0|^p e^{-\varepsilon t}, \quad \forall t \ge t_0, \tag{2.21}$$

for all $(t_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$, which also implies that

$$\lim_{t\to\infty} \sup \frac{1}{t} \log(\mathbb{E}\,|x(t;x_0)|^p) \le -\varepsilon \tag{2.22}$$

for all $(t_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$. When $p = 2$, it is said to be exponentially stable in mean square.

(6) Exponential stability in almost sure sense: system (2.15) is said to be exponentially stable in almost sure sense if there is a positive constant ε such that

$$\lim_{t\to\infty} \sup \frac{1}{t} \log|x(t;t_0,x_0,r_0)| \le -\varepsilon \quad a.s. \tag{2.23}$$

for all $(t_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$.

(7) H_∞ stability: system (2.15) is said to be H_∞-stable in the sense if

$$\int_0^\infty \mathbb{E}\,|x(t;t_0,x_0,r_0)|^2 < \infty, \tag{2.24}$$

for all $(t_0, x_0, r_0) \in \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{S}$.

Chapter 3

Stabilization of hybrid stochastic differential equations by feedback control based on discrete-time state observations

3.1 Introduction

Mao [52] investigates the following stabilization problem by a feedback control based on the discrete-time state observations: Consider an unstable hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t), \tag{3.1}$$

where $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a Markov chain (please see Section 2 for the formal definitions) which represents the system mode, and the SDE is in the Itô sense. The aim is to design a feedback control $u(x([t/\tau]\tau), r(t), t)$ in the drift part so that the controlled system

$$\begin{aligned} dx(t) = &\big(f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t)\big)dt \\ &+ g(x(t), r(t), t)dw(t) \end{aligned} \tag{3.2}$$

becomes stable, where $\tau > 0$ is a constant and $[t/\tau]$ is the integer part of t/τ. The key feature here is that the feedback control $u(x([t/\tau]\tau), r(t), t)$ is designed based on the discrete-time observations of the state $x(t)$ at times $0,\ \tau, 2\tau,\ \cdots$. This is significantly different from the stabilization by a continuous-time (regular) feedback control $u(x(t), r(t), t)$, based on the current state, where the aim is to design $u(x(t), r(t), t)$ in order for the controlled system

$$\begin{aligned} dx(t) = &\big(f(x(t), r(t), t) + u(x(t), r(t), t)\big)dt \\ &+ g(x(t), r(t), t)dw(t) \end{aligned} \tag{3.3}$$

to be stable. In fact, the regular feedback control requires the continuous observations of the state $x(t)$ for all $t \geq 0$, while the feedback control $u(x([t/\tau]\tau), r(t), t)$ needs only the discrete-time observations of the state $x(t)$ at times $0,\ \tau, 2\tau,\ \cdots$. The latter is clearly more realistic and costs less in practice. To the best knowledge of the authors, Mao [52] is the first paper that studies this stabilization problem by feedback controls based on the discrete-time state observations in the area of SDEs, although the corresponding problem for the deterministic differential equations has been studied by many authors (see e.g. [1, 7, 9, 24, 25]).

Mao [52] shows that if the continuous-time controlled SDE (3.3) is mean-square exponentially stable, then so is the discrete-time-state feedback controlled system (3.2) provided τ is sufficiently small. This is of course a very general result. However, it is due to the general technique used there that the bound on τ is not very sharp. In this chapter, we will consider a couple of important classes of hybrid SDEs. Making full use of their special features, we will be able to establish a better bound on τ.

Mathematically speaking, the key technique in Mao [52] is to compare the discrete-time-state feedback controlled system (3.2) with the the continuous-time controlled SDE (3.3) and then prove the stability of system (3.2) by making use of the stability of the SDE (3.3). However, in this chapter, we will work directly on the discrete-time-state feedback controlled system (3.2) itself. To cope with the mixture of the continuous-time state $x(t)$ and the discrete-time state $x([t/\tau]\tau)$ in the system, we have developed some new techniques.

In addition, the key condition imposed in Mao [52] is the global Lipschitz condition on the coefficients of the underlying SDEs, while in this chapter we only require a local Lipschitz condition and hence our new theory is applicable in much more general fashion.

Let us begin to develop these new techniques and to establish our new theory.

3.2 Problem Statement

Throughout this thesis, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $w(t) = (w_1(t), \cdots, w_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. If A is a vector or matrix, its transpose is denoted by A^T. If $x \in R^n$, then $|x|$ is its Euclidean norm. If A is a matrix, we let $|A| = \sqrt{\text{trace}(A^T A)}$ be its trace norm and $\|A\| = \max\{|Ax| : |x| = 1\}$ be the operator norm. If A is a symmetric matrix ($A = A^T$), denote by $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$ its smallest and largest eigenvalue, respectively. By $A \leq 0$ and $A < 0$, we mean A is non-positive and negative definite, respectively. If A is a subset of Ω, denote by I_A its indicator function; that is $I_A(\omega) = 1$ when $\omega \in A$ and 0 otherwise.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & \text{if } i = j, \end{cases}$$

where $\Delta > 0$. Here $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while

$$\gamma_{ii} = -\sum_{j\neq i} \gamma_{ij}.$$

We assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $w(\cdot)$. It is known that almost all sample paths of $r(t)$ are constant except for a finite number of simple jumps in any finite subinterval of R_+ $(:= [0, \infty))$. We stress that almost all sample paths of $r(t)$ are right continuous.

Consider an n-dimensional linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t) \tag{3.4}$$

on $t \geq 0$, with initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n)$. Here $A,\ B_k : S \to R^{n\times n}$ and we will often write $A(i) = A_i$ and $B_k(i) = B_{ki}$. Suppose that this given equation is unstable and we are required to design a feedback control $u(x(\delta(t)), r(t))$ based on the discrete-time state observations in the drift part so that the controlled SDE

$$dx(t) = [A(r(t))x(t) + u(x(\delta(t)), r(t))]dt + \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t) \quad (3.5)$$

will be mean-square exponentially stable, where u is a mapping from $R^n \times S$ to R^n, $\tau > 0$ and

$$\delta(t) = [t/\tau]\tau \quad \text{for } t \geq 0, \quad (3.6)$$

in which $[t/\tau]$ is the integer part of t/τ. We repeat that the feedback control $u(x(\delta(t)), r(t))$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$, though the given hybrid SDE (3.4) is of continuous-time.

As the given SDE (3.4) is linear, it is natural to use a linear feedback control. One of the most common linear feedback controls is the structure control of the form $u(x, i) = F(i)G(i)x$, where F and G are mappings from S to $R^{n\times l}$ and $R^{l\times n}$, respectively, and one of them is given while the other needs to be designed. These two cases are known as:

- State feedback: design $F(\cdot)$ when $G(\cdot)$ is given;
- Output injection: design $G(\cdot)$ when $F(\cdot)$ is given.

Again, we will often write $F(i) = F_i$ and $G(i) = G_i$. As a result, the controlled system (3.5) becomes

$$dx(t) = [A(r(t))x(t) + F(r(t))G(r(t))x(\delta(t))]dt + \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t). \quad (3.7)$$

We observe that equation (3.7) is in fact a stochastic differential delay equation (SDDE) with a bounded variable delay. Indeed, if we define the bounded variable delay $\zeta : [0, \infty) \to [0, \tau]$ by

$$\zeta(t) = t - v\tau \quad \text{for } v\tau \leq t < t(v + 1)\tau, \quad (3.8)$$

and $v = 0, 1, 2, \cdots$, then equation (3.7) can be written as

$$dx(t) = [A(r(t))x(t) + F(r(t))G(r(t))x(t - \zeta(t))]dt$$

$$+\sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t) \tag{3.9}$$

It is therefore known (see e.g. [62]) that equation (3.7) has a unique solution $x(t)$ such that $\mathbb{E}\,|x(t)|^2 < \infty$ for all $t \geq 0$.

3.3 Main Results

In this section, we will first write $F(r(t))G(r(t)) = D(r(t))$ and establish the stability theory for the following hybrid SDE

$$\begin{aligned} dx(t) =&[A(r(t))x(t) + D(r(t))x(\delta(t))]dt \\ &+\sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t). \end{aligned} \tag{3.10}$$

We will then design either $G(\cdot)$ given $F(\cdot)$ or $F(\cdot)$ given $G(\cdot)$ in order for the controlled SDE (3.7) to be stable.

3.3.1 Stability of SDE (3.10)

Let us begin with a useful lemma.

Lemma 3.3.1. *Set*

$$M_A = \max_{i\in S} \|A_i\|^2, \quad M_D = \max_{i\in S} \|D_i\|^2,$$

$$M_B = \max_{i\in S} \sum_{k=1}^{m} \|B_{ki}\|^2,$$

and define

$$K(\tau) = [6\tau(\tau M_A + M_B) + 3\tau^2 M_D]e^{6\tau(\tau M_A + M_B)} \tag{3.11}$$

for $\tau > 0$. If τ is sufficiently small for $2K(\tau) < 1$, then the solution $x(t)$ of the SDE (3.10) satisfies

$$\mathbb{E}\,|x(t) - x(\delta(t))|^2 \leq \frac{2K(\tau)}{1 - 2K(\tau)}\mathbb{E}\,|x(t)|^2 \tag{3.12}$$

for all $t \geq 0$.

Proof. Fix any integer $v \geq 0$. For $t \in [v\tau, (v+1)\tau)$, we have $\delta(t) = v\tau$. It follows from (3.10) that

$$\begin{aligned} &x(t) - x(\delta(t)) = x(t) - x(v\tau) \\ =&\int_{v\tau}^{t} [A(r(s))x(s) + D(r(s))x(v\tau)]ds \\ +&\sum_{k=1}^{m} \int_{v\tau}^{t} B_k(r(s))x(s)dw_k(s). \end{aligned}$$

We can then derive

$$\begin{aligned} &\mathbb{E}\,|x(t) - x(\delta(t))|^2 \\ \leq &3(\tau M_A + M_B) \int_{v\tau}^{t} \mathbb{E}\,|x(s)|^2 ds + 3\tau^2 M_D \mathbb{E}\,|x(k\tau)|^2 \\ \leq &6(\tau M_A + M_B) \int_{v\tau}^{t} \mathbb{E}\,|x(s) - x(\delta(s))|^2 ds \\ +&[6\tau(\tau M_A + M_B) + 3\tau^2 M_D]\mathbb{E}\,|x(v\tau)|^2. \end{aligned}$$

The well-known Gronwall inequality shows

$$\mathbb{E}\,|x(t) - x(\delta(t))|^2 \leq K(\tau)\mathbb{E}\,|x(v\tau)|^2.$$

Consequently

$$\begin{aligned} &\mathbb{E}\,|x(t) - x(\delta(t))|^2 \\ \leq &2K(\tau)\Big(\mathbb{E}\,|x(t) - x(\delta(t))|^2 + \mathbb{E}\,|x(t)|^2\Big). \end{aligned}$$

This implies that (3.12) holds for $t \in [v\tau, (v+1)\tau)$. But $v \geq 0$ is arbitrary so the desired assertion (3.12) must hold for all $t \geq 0$. The proof is complete. □

Theorem 3.3.2. *Assume that there are symmetric positive definite matrices $Q(i) = Q_i$ ($i \in S$) such that*

$$\begin{aligned} \bar{Q}(i) = \bar{Q}_i :=&Q_i(A_i + D_i) + (A_i + D_i)^T Q_i \\ &+ \sum_{k=1}^{m} B_{ki}^T Q_i B_{ki} + \sum_{j=1}^{N} \gamma_{ij} Q_j \end{aligned} \tag{3.13}$$

are all negative-definite matrices. Set

$$-\lambda := \max_{i \in S} \lambda_{\max}(\bar{Q}_i) \quad \textit{and} \quad M_{QD} = \max_{i \in S} \|Q_i D_i\|^2$$

(and of course $\lambda > 0$). If τ is sufficiently small for $\lambda > 2\lambda_\tau$, where

$$\lambda_\tau := \sqrt{\frac{2M_{QD}K(\tau)}{1-2K(\tau)}}, \tag{3.14}$$

then the solution of the SDE (3.10) satisfies

$$\mathbb{E}\,|x(t)|^2 \le \frac{\lambda_M}{\lambda_m}\,\mathbb{E}\,|x_0|^2 e^{-\theta t}, \quad \forall t \ge 0, \tag{3.15}$$

where $\lambda_M = \max_{i\in S}\lambda_{\max}(Q_i)$, $\lambda_m = \min_{i\in S}\lambda_{\min}(Q_i)$, $K(\tau)$ has been defined in Lemma 3.3.1 and

$$\theta = \frac{\lambda - 2\lambda_\tau}{\lambda_M}. \tag{3.16}$$

In other words, the SDE (3.10) is exponentially stable in mean square.

Proof. Applying the generalized Itô formula (see Theorem 2.5.4 or [62, Theorem 1.14 on page 48]) to $x^T(t)Q(r(t))x(t)$ we get

$$\begin{aligned}
&d[x^T(t)Q(r(t))x(t)]\\
=&\Big(2x^T(t)Q(r(t))[A(r(t))x(t) + D(r(t))x(\delta(t))]\\
&+\sum_{k=1}^{m} x^T(t)B_k^T(r(t))Q(r(t))B_k(r(t))x(t)\\
&+\sum_{j=1}^{N}\gamma_{r(t),j}x^T(t)Q_jx(t)\Big)dt + dM_1(t)\\
=&\Big(x^T(t)\bar{Q}(r(t))x(t)\\
&-2x^T(t)Q(r(t))D(r(t))(x(t)-x(\delta(t)))\Big)dt + dM_1(t).
\end{aligned}$$

Here $M_1(t)$, and the following $M_2(t)$ are martingales with $M_1(0) = M_2(0) = 0$. Their forms are not used so are not specified here as we will take expectations later and their means are zero. Applying the generalized Itô formula now to $e^{\theta t}x^T(t)Q(r(t))x(t)$, we then have

$$\begin{aligned}
&d[e^{\theta t}x^T(t)Q(r(t))x(t)]\\
=&e^{\theta t}\Big(\theta x^T(t)Q(r(t))x(t) + x^T(t)\bar{Q}(r(t))x(t)\\
&-2x^T(t)Q(r(t))D(r(t))(x(t)-x(\delta(t)))\Big)dt + dM_2(t).
\end{aligned}$$

This implies

$$
\begin{aligned}
&\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2 \\
\leq&\lambda_M\mathbb{E}\,|x_0|^2 + \int_0^t (\theta\lambda_M - \lambda)e^{\theta s}\mathbb{E}\,|x(s)|^2 ds \\
+&\int_0^t 2e^{\theta s}\sqrt{M_{QD}}\,\mathbb{E}\,(|x(s)||x(s)-x(\delta(s))|)ds. \qquad (3.17)
\end{aligned}
$$

But, by Lemma 3.3.1, we have

$$
\begin{aligned}
&2\sqrt{M_{QD}}\,\mathbb{E}\,(|x(s)||x(s)-x(\delta(s))|) \\
\leq&\lambda_\tau\mathbb{E}\,|x(s)|^2 + \frac{M_{QD}}{\lambda_\tau}\mathbb{E}\,|x(s)-x(\delta(s))|^2 \\
\leq&\lambda_\tau\mathbb{E}\,|x(s)|^2 + \frac{M_{QD}}{\lambda_\tau}\,\frac{2K(\tau)}{1-2K(\tau)}\,\mathbb{E}\,|x(t)|^2 \\
=&2\lambda_\tau\mathbb{E}\,|x(s)|^2. \qquad (3.18)
\end{aligned}
$$

Substituting this into (3.17) yields

$$
\begin{aligned}
&\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2 \\
\leq&\lambda_M\mathbb{E}\,|x_0|^2 + \int_0^t (\theta\lambda_M + 2\lambda_\tau - \lambda)e^{\theta s}\mathbb{E}\,|x(s)|^2 ds.
\end{aligned}
$$

But, by (3.16), $\theta\lambda_M + 2\lambda_\tau - \lambda = 0$. Thus

$$\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2 \leq \lambda_M\mathbb{E}\,|x_0|^2,$$

which implies the desired assertion (3.15). The proof is complete. □

3.3.2 State feedback: design $F(\cdot)$ when $G(\cdot)$ is given

We can now begin to consider the case of state feedback. In this case, $G(\cdot)$ is given so our aim is to design $F(\cdot)$ such that the controlled SDE (3.7) is exponentially stable in mean square. One technique used frequently in the study of stability of linear SDEs is the method of linear matrix inequalities (LMIs) (see e.g. [23, 26, 66, 85, 86]), although there are other methods (see e.g. the survey paper [51]). We will use the technique of LMIs to design $F(\cdot)$ in this section.

According to Theorem 3.3.2, it is sufficient if we can design $G(\cdot)$, namely G_i for $i \in S$, so that we can further find positive-definite symmetric matrices Q_i ($i \in S$)

in order for

$$
\begin{aligned}
&Q_i(A_i + F_iG_i) + (A_i + F_iG_i)^T Q_i \\
&+ \sum_{k=1}^{m} B_{ki}^T Q_i B_{ki} + \sum_{j=1}^{N} \gamma_{ij} Q_j < 0, \quad i \in S.
\end{aligned} \tag{3.19}
$$

We observe that the above matrix inequalities are not linear in Q_i and F_i's. However, if we set $Y_i = Q_i F_i$, then they become the following LMIs

$$
\begin{aligned}
&Q_i A_i + Y_i G_i + A_i^T Q_i + G_i^T Y_i^T \\
&+ \sum_{k=1}^{m} B_{ki}^T Q_i B_{ki} + \sum_{j=1}^{N} \gamma_{ij} Q_j < 0, \quad i \in S.
\end{aligned} \tag{3.20}
$$

If these LMIs have their solutions $Q_i = Q_i^T > 0$ and Y_i ($i \in S$), then, setting $F_i = Q_i^{-1} Y_i$, we have (3.19). Applying Theorem 3.3.2, we hence obtain the following corollary.

Corollary 3.3.3. *Assume that the LMIs in (3.20) have their solutions $Q_i = Q_i^T > 0$ and Y_i. Set $F_i = Q_i^{-1} Y_i$ and $D_i = F_i G_i$. Then the conclusion of Theorem 3.3.2 holds. In other words, the controlled SDE (3.7) will be exponentially stable in mean square if we set $F_i = Q_i^{-1} Y_i$ and make sure $\tau > 0$ be sufficiently small for $\lambda > 2\lambda_\tau$.*

3.3.3 Output injection: design $G(\cdot)$ when $F(\cdot)$ is given

Let us now consider the case of output injection. In this case, $F(\cdot)$ is given and our aim is to design $G(\cdot)$ so that the controlled SDE (3.7) is exponentially stable in mean square. Once again, based on Theorem 3.3.2, it is sufficient if we can design $F(\cdot)$, namely F_i for $i \in S$, so that we can further find positive-definite symmetric matrices Q_i ($i \in S$) in order for the matrix inequalities (3.19) to hold. Multiplying Q_i^{-1} from left and then from right, and writing $Q_i^{-1} = X_i$, we see that the matrix inequalities (3.19) are equivalent to the following matrix inequalities

$$
\begin{aligned}
&A_i X_i + F_i G_i X_i + X_i A_i^T + X_i G_i^T F_i^T \\
&+ \sum_{k=1}^{m} X_i B_{ki}^T X_i^{-1} B_{ki} X_i \\
&+ \sum_{j=1}^{N} \gamma_{ij} X_i X_j^{-1} X_i < 0, \quad i \in S.
\end{aligned} \tag{3.21}
$$

By setting $G_iX_i = Y_i$, these matrix inequalities become

$$\begin{aligned} &A_iX_i + F_iY_i + X_iA_i^T + Y_i^TF_i^T + \gamma_{ii}X_i \\ &+ \sum_{k=1}^{m} X_iB_{ki}^TX_i^{-1}B_{ki}X_i \\ &+ \sum_{j\neq i}^{N} \gamma_{ij}X_iX_j^{-1}X_i < 0, \quad i \in S. \end{aligned} \tag{3.22}$$

By the well-known Schur complements (see [62, Theorem 2.8 on page 64]), we see these matrix inequalities are equivalent to the following LMIs

$$\begin{bmatrix} M_{i1} & M_{i2} & M_{i3} \\ M_{i2}^T & -M_{i4} & 0 \\ M_{i3}^T & 0 & -M_{i5} \end{bmatrix} < 0, \quad i \in S, \tag{3.23}$$

where

$$\begin{aligned} M_{i1} &= A_iX_i + F_iY_i + X_iA_i^T + Y_i^TF_i^T + \gamma_{ii}X_i, \\ M_{i2} &= [X_iB_{1i}^T, \cdots, X_iB_{mi}^T], \\ M_{i3} &= [\sqrt{\gamma_{i1}}X_i, \cdots, \sqrt{\gamma_{i(i-1)}}X_i, \sqrt{\gamma_{i(i+1)}}X_i, \cdots, \sqrt{\gamma_{iN}}X_i], \\ M_{i4} &= \mathrm{diag}[X_i, \cdots, X_i], \\ M_{i5} &= \mathrm{diag}[X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_N]. \end{aligned}$$

In other words, if the LMIs in (3.23) have their solutions $X_i = X_i^T > 0$ and Y_i ($i \in S$), then, setting $Q_i = X_i^{-1}$ and $G_i = Y_iX_i^{-1}$, we have (3.19). Applying Theorem 3.3.2, we hence obtain the following corollary.

Corollary 3.3.4. *Assume that the LMIs in (3.23) have their solutions $X_i = X_i^T > 0$ and Y_i ($i \in S$). Set $Q_i = X_i^{-1}$ and $G_i = Y_iX_i^{-1}$. Then the conclusion of Theorem 3.3.2 holds. In other words, the controlled SDE (3.7) will be exponentially stable in mean square if we set $G_i = Y_iX_i^{-1}$ and make sure $\tau > 0$ be sufficiently small for $\lambda > 2\lambda_\tau$.*

3.4 Stabilization of Nonlinear Hybrid SDEs

Let us now discuss a more general nonlinear problem. Assume that the underlying system is now described by a nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t) \tag{3.24}$$

on $t \geq 0$ with the initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n)$. Here, $f : R^n \times S \times R_+ \to R^n$ and $g : R^n \times S \times R_+ \to R^{n\times m}$. Assume that both f and g are locally Lipschitz continuous and obey the linear growth condition (see e.g. [62]). We also assume that $f(0, i, t) = 0$ and $g(0, i, t) = 0$ for all $i \in S$ and $t \geq 0$ so that $x = 0$ is an equilibrium point for (3.24).

Suppose that the given SDE (3.24) is unstable and we are required to design a linear feedback control $F(r(t))G(r(t))x(\delta(t))$ based on the discrete-time state observations in the drift part so that the controlled system

$$dx(t) = [f(x(t), r(t), t) + F(r(t))G(r(t))x(\delta(t))]dt \\ + g(x(t), r(t), t)dw(t) \tag{3.25}$$

will be mean-square exponentially stable. Recalling the definition of ζ by (3.8), we see that the SDE (3.25) can be written as an SDDE

$$dx(t) = [f(x(t), r(t), t) + F(r(t))G(r(t))x(t - \zeta(t))]dt \\ + g(x(t), r(t), t)dw(t). \tag{3.26}$$

It is therefore known (see e.g. [62]) that equation (3.25) has a unique solution $x(t)$ such that $\mathbb{E}\,|x(t)|^2 < \infty$ for all $t \geq 0$.

Given that we use a linear control to stabilize a nonlinear system, it is natural to impose some conditions on the nonlinear coefficients f and g.

Assumption 3.4.1. *For each $i \in S$, there is a pair of symmetric $n \times n$-matrices Q_i and $\hat{Q}_i$ with Q_i being positive-definite such that*

$$2x^T Q_i f(x, i, t) + g^T(x, i, t)Q_i g(x, i, t) \leq x^T \hat{Q}_i x$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Assumption 3.4.2. *There is a pair of positive constants δ_1 and δ_2 such that*

$$|f(x, i, t)|^2 \leq \delta_1 |x|^2 \quad \text{and} \quad |g(x, i, t)|^2 \leq \delta_2 |x|^2$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Let us first present a useful lemma.

Lemma 3.4.3. *Let Assumption 3.4.2 hold. Set*

$$\delta_3 = \max_{i\in S} \sum_{k=1}^{m} \|F_i G_i\|^2,$$

and define

$$H(\tau) = [6\tau(\tau\delta_1 + \delta_2) + 3\tau^2\delta_3]e^{6\tau(\tau\delta_1+\delta_2)} \tag{3.27}$$

for $\tau > 0$. If τ is sufficiently small for $2H(\tau) < 1$, then the solution $x(t)$ of the SDE (3.25) satisfies

$$\mathbb{E}\,|x(t) - x(\delta(t))|^2 \le \frac{2H(\tau)}{1-2H(\tau)}\mathbb{E}\,|x(t)|^2 \tag{3.28}$$

for all $t \ge 0$.

This lemma can be proved in the same way as Lemma 3.3.1 was proved so we omit the proof.

Theorem 3.4.4. *Let Assumptions 3.4.1 and 3.4.2 hold. Assume that the following LMIs*

$$U_i := \hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + \sum_{j=1}^{N} \gamma_{ij} Q_j < 0, \quad i \in S, \tag{3.29}$$

have their solutions F_i ($i \in S$) in the case of feedback control (i.e. G_i's are given), or their solutions G_i in the case of output injection (i.e. F_i's are given). Set

$$-\gamma := \max_{i\in S} \lambda_{\max}(U_i) \quad \textit{and} \quad \delta_4 = \max_{i\in S} \|Q_i F_i G_i\|^2$$

If τ is sufficiently small for $\gamma > 2\gamma_\tau$, where

$$\gamma_\tau := \sqrt{\frac{2\delta_4 H(\tau)}{1-2H(\tau)}}, \tag{3.30}$$

then the solution of the SDE (3.25) satisfies

$$\mathbb{E}\,|x(t)|^2 \le \frac{\lambda_M}{\lambda_m}\,\mathbb{E}\,|x_0|^2 e^{-\theta t}, \quad \forall t \ge 0, \tag{3.31}$$

where $\lambda_M = \max_{i\in S}\lambda_{\max}(Q_i)$, $\lambda_m = \min_{i\in S}\lambda_{\min}(Q_i)$, $H(\tau)$ has been defined in Lemma 3.4.3 and

$$\theta = \frac{\gamma - 2\gamma_\tau}{\lambda_M}. \tag{3.32}$$

Proof. This theorem can be proved in a similar way as Theorem 3.3.2 was proved so we only give the key steps. Applying the generalized Itô formula to $x^T(t)Q(r(t))x(t)$ we get

$$\begin{aligned}
&d[x^T(t)Q(r(t))x(t)]\\
=&\Big(x^T(t)U(r(t))x(t)\\
&-2x^T(t)Q(r(t))F(r(t))G(r(t))(x(t)-x(\delta(t)))\Big)dt\\
&+dM_3(t),
\end{aligned}$$

where $M_3(t)$ is a martingale with $M_3(0) = 0$. Applying the generalized Itô formula further to $e^{\theta t}x^T(t)Q(r(t))x(t)$, we can then obtain

$$\begin{aligned}
&\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2\\
\le&\lambda_M\mathbb{E}\,|x_0|^2+\int_0^t(\theta\lambda_M-\gamma)e^{\theta s}\mathbb{E}\,|x(s)|^2ds\\
&+\int_0^t 2e^{\theta s}\sqrt{\delta_4}\,\mathbb{E}\left(|x(s)||x(s)-x(\delta(s))|\right)ds.
\end{aligned}\tag{3.33}$$

But, by Lemma 3.4.3, we can show

$$2\sqrt{\delta_4}\,\mathbb{E}\left(|x(s)||x(s)-x(\delta(s))|\right)\le 2\gamma_\tau\mathbb{E}\,|x(s)|^2.\tag{3.34}$$

Substituting this into (3.33) yields

$$\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2\le\lambda_M\mathbb{E}\,|x_0|^2,$$

which implies the desired assertion (3.31). The proof is complete. □

To apply Theorem 3.4.4, we need two steps:

1 we first need to look for the $2N$ matrices Q_i and $\hat{Q}_i$ for Assumption 3.4.1 to hold;

2 we then need to solve the LMIs in (3.29) for their solutions F_i (or G_i).

There are available computer softwares e.g. Matlab for step 2 so in the remaining part of this section we will develop some ideas for step 1. To make our ideas more clear, *we will only consider the case of feedback control, but the same ideas work for the case of output injection.*

In theory, it is flexible to use $2N$ matrices Q_i and $\hat{Q}_i$ in Assumption 3.4.1. But, in practice, it means more work to be done in finding these $2N$ matrices. It is in this spirit that we introduce a stronger assumption.

Assumption 3.4.5. *There are $N+1$ symmetric $n \times n$-matrices Z and Z_i ($i \in S$) with $Z > 0$ such that*

$$2x^T Z f(x,i,t) + g^T(x,i,t) Z g(x,i,t) \le x^T Z_i x$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Under this assumption, if we let $Q_i = q_i Z$ and $\hat{Q}_i = q_i Z_i$ for some positive numbers q_i, then Assumption 3.4.1 holds. Moreover, the LMIs in (3.29) become

$$q_i Z_i + q_i Z F_i G_i + q_i G_i^T F_i^T Z + \sum_{j=1}^{N} \gamma_{ij} q_j Z < 0, \quad i \in S.$$

If we set $Y_i := q_i F_i$, then these become the following LMIs in q_i and Y_i:

$$q_i Z_i + Z Y_i G_i + G_i^T Y_i^T Z + \sum_{j=1}^{N} \gamma_{ij} q_j Z < 0, \quad i \in S. \tag{3.35}$$

We hence have the following corollary.

Corollary 3.4.6. *Let Assumptions 3.4.5 and 3.4.2 hold. Assume that the LMIs (3.35) have their solutions $q_i > 0$ and Y_i ($i \in S$). Then Theorem 3.4.4 holds by setting $Q_i = q_i Z$, $\hat{Q}_i = q_i Z_i$ and $F_i = q_i^{-1} Y_i$. In other words, the controlled SDE (3.25) will be exponentially stable in mean square if we set $F_i = q_i^{-1} Y_i$ and make sure $\tau > 0$ be sufficiently small for $\gamma > 2\gamma_\tau$.*

An even simpler (but in fact stronger) condition is:

Assumption 3.4.7. *There are constants z_i ($i \in S$) such that*

$$2x^T f(x,i,t) + |g(x,i,t)|^2 \le z_i |x|^2$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Under this assumption, if we let $Q_i = q_i I$ and $\hat{Q}_i = q_i z_i I$ for some positive numbers q_i, where I is the $n \times n$ identity matrix, then Assumption 3.4.1 holds. Moreover, the LMIs in (3.29) become

$$q_i z_i I + q_i F_i G_i + q_i G_i^T F_i^T + \sum_{j=1}^{N} \gamma_{ij} q_j I < 0, \quad i \in S.$$

If we set $Y_i := q_i F_i$, then these become the following LMIs in q_i and Y_i:

$$q_i z_i I + Y_i G_i + G_i^T Y_i^T + \sum_{j=1}^{N} \gamma_{ij} q_j I < 0, \quad i \in S. \tag{3.36}$$

We hence have another corollary.

Corollary 3.4.8. *Let Assumptions 3.4.7 and 3.4.2 hold. Assume that the LMIs (3.36) have their solutions $q_i > 0$ and Y_i $(i \in S)$. Then Theorem 3.4.4 holds by setting $Q_i = q_i I$, $\hat{Q}_i = q_i z_i I$ and $F_i = q_i^{-1} Y_i$. In other words, the controlled SDE (3.25) will be exponentially stable in mean square if we set $F_i = q_i^{-1} Y_i$ and make sure $\tau > 0$ be sufficiently small for $\gamma > 2\gamma_\tau$.*

3.5 Examples

Let us now discuss some examples to illustrate our theory.

Example 3.5.1. Let us first consider the same example as discussed in Mao [52], namely the linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + B(r(t))x(t)dw(t) \tag{3.37}$$

on $t \geq t_0$. Here $w(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

and the system matrices are

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The SDE (3.37) may be regarded as a system which switches between two operation modes, say mode 1 and mode 2, and the switching obeys the law of the Markov chain, where in mode 1, the system evolves according to the SDE

$$dx(t) = A_1 x(t)dt + B_1 x(t)dw(t),$$

while in mode 2, according to the other SDE

$$dx(t) = A_2x(t)dt + B_2x(t)dw(t).$$

The computer simulation (Figure 3.1) shows this hybrid SDE is not mean square exponentially stable. (The simulation of the paths is sufficient to illustrate since it is known that the mean square exponential stability implies the almost sure exponential stability [62].)

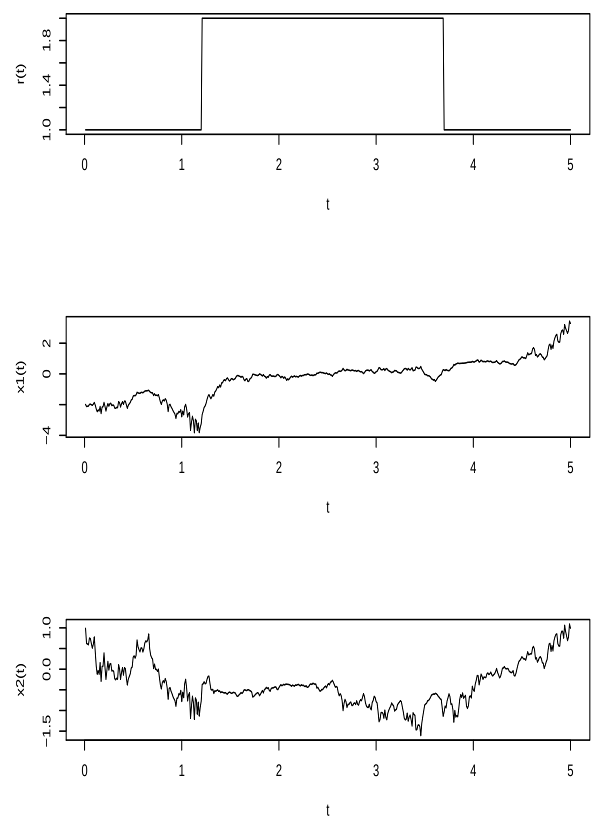

Figure 3.1: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (3.37) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

Let us now design a discrete-time-state feedback control to stabilize the system. Assume that the controlled hybrid SDE has the form

$$dx(t) = [A(r(t))x(t) + F(r(t))G(r(t))x(\delta(t))]dt$$

$$+ B(r(t))x(t)dw(t), \tag{3.38}$$

where

$$G_1 = (1, 0), \quad G_2 = (0, 1).$$

Our aim here is to seek for F_1 and F_2 in $R^{2\times 1}$ and then make sure τ is sufficiently small for this controlled SDE to be exponentially stable in mean square. To apply Corollary 3.3.3, we first find that the following LMIs

$$\begin{aligned}\bar{Q}_i :=& Q_iA_i + Y_iG_i + A_i^TQ_i + G_i^TY_i^T + B_i^TQ_iB_i \\ &+ \sum_{j=1}^{2} \gamma_{ij}Q_j < 0, \quad i = 1, 2,\end{aligned}$$

have the following set of solutions $Q_1 = Q_2 = I$ (the 2×2 identity matrix) and

$$Y_1 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 \\ -10 \end{bmatrix},$$

and for these solutions we have

$$\bar{Q}_1 = \begin{bmatrix} -16 & 0 \\ 0 & -8 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} -8 & 0 \\ 0 & -16 \end{bmatrix}.$$

Hence, we have

$$-\lambda = \max_{i=1,2} \lambda_{\max}(\hat{Q}_i) = -8, \; M_{YG} = \max_{i=1,2} \|Y_iG_i\|^2 = 100.$$

To determine λ_τ, we compute

$$M_A = 27.42, \; M_B = 2, \; M_D = 100, \; M_{QD} = 100.$$

Hence

$$\lambda_\tau = \sqrt{\frac{200K(\tau)}{1 - 2K(\tau)}},$$

where $K(\tau) = [6\tau(27.42\tau+2)+300\tau^2]e^{6\tau(27.42\tau+2)}$. It is easy to show that $\lambda > 2\lambda_\tau$ whenever $\tau < 0.0046$. By Corollary 3.3.3, if we set $F_1 = Y_1$ and $F_2 = Y_2$, and make sure that $\tau < 0.0046$, then the discrete-time-state feedback controlled hybrid SDE (3.38) is mean-square exponentially stable. The computer simulation (Figure 3.2) supports this result clearly. It should be pointed out that it is required for $\tau < 0.0000308$ in Mao [52], while applying our new theory we only need $\tau < 0.0046$. In other words, our new theory has improved the existing result significantly.

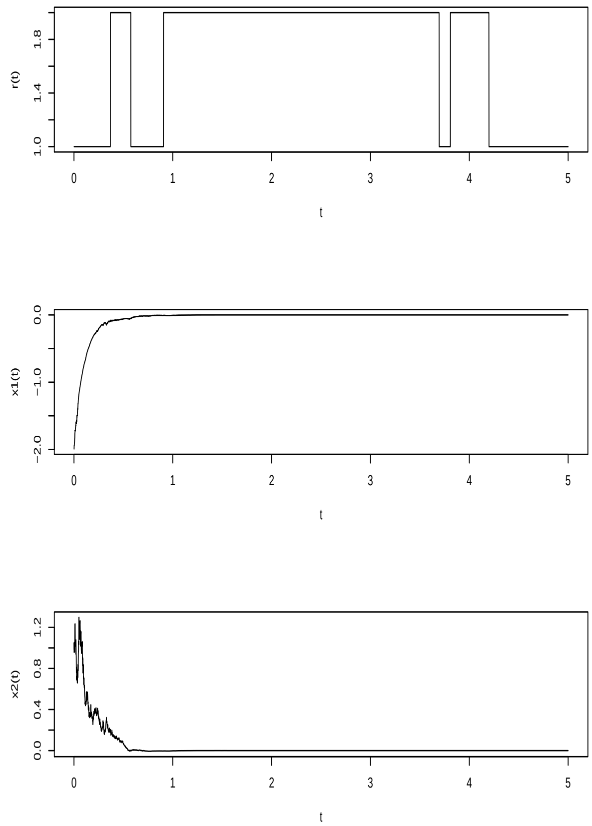

Figure 3.2: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the controlled hybrid SDE (3.38) with $\tau = 10^{-3}$ using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

Example 3.5.2. Let us now discuss one more example, where we will not only illustrate our theory but also explain a new concept which may motivate a further research.

Let $r(t)$, $t \geq 0$, be a right-continuous Markov chain on the probability space taking values in the state space $S = \{1, 2\}$ with generator

$$\Gamma = \begin{bmatrix} -\gamma_{12} & \gamma_{12} \\ \gamma_{21} & -\gamma_{21} \end{bmatrix},$$

where both $\gamma_{12} > 0$ and $\gamma_{21} > 0$. Consider an unstable nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t). \tag{3.39}$$

Here, f and g are both mappings from $R^n \times S \times R_+$ to R^n. This SDE may be regarded as a system which switches between two operation modes, say mode 1 and mode 2, and the switching obeys the law of the Markov chain, where in mode 1, the system evolves according to the SDE

$$dx(t) = f(x(t), 1, t)dt + g(x(t), 1, t)dw(t),$$

while in mode 2, according to the other SDE

$$dx(t) = f(x(t), 2, t)dt + g(x(t), 2, t)dw(t).$$

Assume that in mode 1, the state $x(t)$ can be observed at discrete times (intermittent time instants) but in mode 2, it is not observable. Therefore, we can design a feedback control based on discrete-time observations of the state in mode 1, but we cannot have a feedback control in mode 2. In terms of mathematics, the controlled SDE is

$$\begin{aligned} dx(t) = {} & [f(x(t), r(t), t) + F(r(t))G(r(t))x(\delta(t))]dt \\ & + g(x(t), r(t), t)dw(t), \end{aligned} \tag{3.40}$$

where $G_1 = I$, the $n \times n$ identity matrix but $G_2 = 0$. Given $G_2 = 0$ we can simply set $F_2 = 0$. Hence, the stabilization problem becomes: can we find a matrix $F_1 \in R^{n \times n}$ so that the controlled SDE (3.40) becomes exponentially stable in mean square?

To give a positive answer to the question, we assume that f and g obey Assumptions 3.4.2 and 3.4.5. To apply Corollary 3.4.6, we only need to look for the solutions $q_1, q_2 > 0$ and $Y_1 \in R^{n \times n}$ to the following LMIs

$$q_1 Z_1 + ZY_1 + Y_1^T Z - \gamma_{12} q_1 Z + \gamma_{12} q_2 Z < 0 \tag{3.41}$$

and

$$q_2 Z_2 + \gamma_{21} q_1 Z - \gamma_{21} q_2 Z < 0. \tag{3.42}$$

It is easy to see from (3.42) that we have to assume

$$Z_2 - \gamma_{21} Z < 0. \tag{3.43}$$

This means that the rate at which the system switches from the unobservable mode 2 to the observable mode 1 should be sufficiently large. This is reasonable because the system in mode 2 is not controllable while it is controllable (hence stabilizable) in mode 1. Let us now choose $q_1 = 1$. Under condition (3.43), we can further choose

$$q_2 > \frac{\gamma_{21}\lambda_{\max}(Z)}{\lambda_{\min}(\gamma_{21}Z - Z_2)} \tag{3.44}$$

for (3.42) to hold. Finally, we can choose Y_1 to be symmetric for

$$q_1 Z_1 + 2ZY_1 - \gamma_{12} q_1 Z + \gamma_{12} q_2 Z = -I, \tag{3.45}$$

where I is the $n \times n$ identity matrix. That is, we set

$$Y_1 = 0.5Z^{-1}(-I - q_1 Z_1 + \gamma_{12}(q_1 - q_2)Z), \tag{3.46}$$

which guarantees (3.41). Let us summarize what we have so far: Under condition (3.43), we can choose $q_1 = 1$ and q_2 sufficiently large for (3.44) to hold and then compute Y_1 by (3.46) and set $F_1 = Y_1$.

To determine τ, we note that $\delta_3 = \delta_4 = \|F_1\|^2$. We then compute

$$-\gamma = \max_{i=1,2} \lambda_{\max}(U_i),$$

where

$$U_1 = -I, \ U_2 = q_2 Z_2 + \gamma_{21}(1 - q_2)Z.$$

Finally, make sure $\tau > 0$ is sufficiently small for $2\gamma_\tau < \gamma$, where γ_τ can be computed by (3.30) and (3.27). Then, by Corollary 3.4.6, the controlled system (3.40) is exponentially stable in mean square.

3.6 Summary

In this chapter we first show that unstable linear hybrid SDEs can be stabilized by the linear feedback controls based on the discrete-time state observations. We then generalize the theory to a class of nonlinear hybrid SDEs and release the condition of coefficients from global Lipschitz condition to local Lipschitz and linear growth conditions comparised with the results in [52]. Making full use of their special features, we have established a better bound on τ (because the bound for the duration in this chapter is larger that in [52] in the same example) and this is supported particularly by Example 3.5.1. Of course, the bound on τ obtained in this chapter is certainly not optimal. We will further develop this result in the next chapter.

The theory established works well for linear hybrid SDEs or a class of nonlinear hybrid SDEs which satisfy Assumptions 3.4.1 and 3.4.2. These assumptions are somehow restrictive. It is useful and interesting to replace these by weaker conditions. Moreover, we assume in this thesis that the mode $r(t)$ is available for all time although we only require the state $x(t)$ be available at discrete times. This is the case, for example, when hybrid SDEs are used to model electric power systems [79] and the evasive target tracking problem [63].

Chapter 4

Lyapunov Approach and More types of Stability of Hybrid SDEs by Feedback Control based on Discrete-time state Observations

4.1 Introduction

Let us recall SDE (3.1) and (3.2) before starting this chapter. Previously, we have investigated the following problem: Consider an unstable hybrid SDE (3.1)

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t),$$

where $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a Markov chain with finite state space which represents the system mode, and the SDE is in the Itô sense. Our aim is to design a feedback control $u(x([t/\tau]\tau), r(t), t)$ based on the discrete-time observations of the state $x(t)$ at times $0, \tau, 2\tau, \cdots$ so that the controlled system (3.2)

$$\begin{aligned} dx(t) = & \big(f(x(t), r(t), t) + u(x([t/\tau]\tau), r(t), t)\big)dt \\ & + g(x(t), r(t), t)dw(t) \end{aligned}$$

becomes stable, where $\tau > 0$ is a constant and $[t/\tau]$ is the integer part of t/τ.

We also proved that *under the local Lipschitz and linear growth condition*, the discrete-time-state feedback controlled system (3.2) is exponentially stable in mean square provided duration τ is sufficiently small. This is of course a very general result. However, it is due to the general technique used there that the bound on τ is not very sharp. In this chapter, we will use the method of the Lyapunov functionals to study the stabilization problem. We will be able to improve the bound on τ. The key features which differ from those in Chapter 3 are as follows:

- Chapter 3 has only discussed the stabilization in the sense of mean square exponential stability. In this chapter, in addition to the mean square exponential stability, we will investigate the stabilization in the sense of H_∞ stability as well as asymptotic stability. We will not only consider the mean square stability but also the almost sure stability, and the proof of the later is much more technical than that of former (please see the proof of Theorem 4.3.4 below).
- The key technique in Mao [52] is to work directly on the discrete-time-state feedback controlled system (3.2) and then prove the stability of system (3.2) by making use of its main features. However, in this chapter, we will study on the discrete-time-state feedback controlled system (3.2) itself using the method of the Lyapunov functionals. To cope with the mixture of the continuous-time state $x(t)$ and the discrete-time state $x([t/\tau]\tau)$ in the same system, we have developed some new techniques.

Let us begin to develop these new techniques and to establish our new theory.

4.2 Stabilization Problem

Consider an n-dimensional controlled hybrid SDE

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(t), t)\big)dt + g(x(t), r(t), t)dw(t) \tag{4.1}$$

on $t \geq 0$, with initial data $x(0) = x_0 \in R^n$ and $r(0) = r_0 \in S$ at time zero. Here

$$f, u : R^n \times S \times R_+ \to R^n \quad \text{and} \quad g : R^n \times S \times R_+ \to R^{n\times m},$$

while $\tau > 0$ and

$$\delta_t = [t/\tau]\tau, \tag{4.2}$$

in which $[t/\tau]$ is the integer part of t/τ. Our aim here is to design the feedback control $u(x(\delta_t), r(t), t)$ so that this controlled hybrid SDE becomes mean-square asymptotically stable, though the given uncontrolled system

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t) \tag{4.3}$$

may not be stable. We observe that the feedback control $u(x(\delta_t), r(t), t)$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$, though the given hybrid SDE (4.3) is of continuous-time. In this paper we impose the following standing hypotheses.

Assumption 4.2.1. *Assume that the coefficients f and g are all locally Lipschitz continuous (see e.g. [45–47, 62]). Moreover, they satisfy the following linear growth condition*

$$|f(x, i, t)| \le K_1|x| \quad \text{and} \quad |g(x, i, t)| \le K_2|x| \tag{4.4}$$

for all $(x, i, t) \in R^n \times S \times R_+$, where both K_1 and K_2 are positive numbers.

We observe that (4.4) forces

$$f(0, i, t) = 0, \qquad g(0, i, t) = 0 \tag{4.5}$$

for all $(i, t) \in S \times R_+$. This is of course for the stability purpose of this paper. For a technical reason, we require a global Lipschitz condition on the controller function u. More precisely, we impose the following hypothesis.

Assumption 4.2.2. *Assume that there exists a positive constant K_3 such that*

$$|u(x, i, t) - u(y, i, t)| \le K_3|x - y| \tag{4.6}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$. Moreover,

$$u(0, i, t) = 0 \tag{4.7}$$

for all $(i, t) \in S \times R_+$.

Once again, condition (4.7) is for the stability purpose of this paper. We also see that Assumption 4.2.2 implies the following linear growth condition on the controller function

$$|u(x,i,t)| \le K_3|x| \tag{4.8}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

We observe that equation (4.1) is in fact a stochastic differential delay equation (SDDE) with a bounded variable delay. Indeed, if we define the bounded variable delay $\zeta : [0,\infty) \to [0,\tau]$ by

$$\zeta(t) = t - k\tau \quad \text{for } k\tau \le t < (k+1)\tau,\ k = 0,1,2,\cdots,$$

then equation (4.1) can be written as

$$dx(t) = \big(f(x(t),r(t),t) + u(x(t-\zeta(t)),r(t),t)\big)dt + g(x(t),r(t),t)dw(t). \tag{4.9}$$

It is therefore known (see e.g. [62]) that under Assumptions 4.2.1 and 4.2.2, the SDDE (4.9) (namely the controlled system (4.1)) has a unique solution $x(t)$ such that $\mathbb{E}|x(t)|^2 < \infty$ for all $t \ge 0$. Of course, we should point out that equation (4.9) is a special SDDE in the sense we need to know only the initial data $x(0)$ and $r(0)$ at $t = 0$ in order to determine the unique solution $x(t)$ on $t \ge 0$. However, if we are given data $x(s)$ and $r(s)$ for some $s \in (k\tau,(k+1)\tau)$, we will not be able to determine the solution $x(t)$ on $t \ge s$ unless we also know $x(k\tau)$.

The observation above also shows that the stability and stabilization problem of equation (4.1) can be regarded as the problem of the hybrid SDDE (4.9) with a bounded variable delay. On the other hand, as far as the authors know, the existing results on the stability of the hybrid SDDE require the bounded variable delay be differentiable and the derivative be less than one (see e.g. [28, p.182] or [62, p.285]). However, the bounded variable delay $\zeta(t)$ defined above is not differentiable when $t = k\tau$, $k = 1,2,\cdots$, while its derivative $d\zeta(t)/dt = 1$ for $t \in ((k-1)\tau, k\tau)$. Therefore, the existing results on the stability of the hybrid SDDEs are not applicable here and we need to develop our new theory.

4.3 Asymptotic Stabilization

For our stabilization purpose related to the controlled system (2.1) we will use a Lyapunov functional on the segments $\hat{x}_t := \{x(t+s) : -2\tau \le s \le 0\}$ and

$\hat{r}_t := \{r(t+s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \le t < 2\tau$, we set $x(s) = x_0$ and $r(s) = r_0$ for $-2\tau \le s \le 0$. The Lyapunov functional used in this chapter will be of the form

$$V(\hat{x}_t, \hat{r}_t, t) = U(x(t), r(t), t) + \theta \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau |f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dvds \tag{4.10}$$

for $t \ge 0$, where θ is a positive number to be determined later and we set

$$f(x,i,s) = f(x,i,0), \quad u(x,i,s) = u(x,i,0), \quad g(x,i,s) = f(x,i,0)$$

for $(x,i,s) \in R^n \times S \times [-2\tau, 0)$. Of course, the functional above uses $r(u)$ only on $t - \tau \le u \le t$ so we could have defined $\hat{r}_t := \{r(t+s) : -\tau \le s \le 0\}$. But, to be consistent with the definition of $\hat{x}_t$, we define $\hat{r}_t$ as above and this does not lose any generality. For $\forall i \in \mathbb{S}$, we also require $U \in C^{2,1}(R^n \times R_+; R_+)$, the family of non-negative functions $U(x,i,t)$ defined on $(x,i,t) \in R^n \times S \times R_+$ which are twice continuously differentiable in x and once in t. For $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$, let us define $\mathcal{L}U : R^n \times S \times R_+ \to R$ by

$$\mathcal{L}U(x,i,t) = U_t(x,i,t) + U_x(x,i,t)[f(x,i,t) + u(x,i,t)] + \tfrac{1}{2}\text{trace}[g^T(x,i,t)U_{xx}(x,i,t)g(x,i,t)] + \sum_{j=1}^{N} \gamma_{ij} U(x,j,t), \tag{4.11}$$

where

$$U_t(x,i,t) = \frac{\partial U(x,i,t)}{\partial t}, \quad U_x(x,i,t) = \Big(\frac{\partial U(x,i,t)}{\partial x_1}, \cdots, \frac{\partial U(x,i,t)}{\partial x_n}\Big),$$

and

$$U_{xx}(x,i,t) = \Big(\frac{\partial^2 U(x,i,t)}{\partial x_i \partial x_j}\Big)_{n \times n}.$$

Let us now impose a new assumption on U.

Assumption 4.3.1. *Assume that there is a function $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$ and two positive numbers λ_1, λ_2 such that*

$$\mathcal{L}U(x,i,t) + \lambda_1 |U_x(x,i,t)|^2 \le -\lambda_2 |x|^2 \tag{4.12}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Let us comment on this assumption. Condition (4.12) implies

$$\mathcal{L}U(x,i,t) \le -\lambda_2|x|^2, \tag{4.13}$$

which guarantees the asymptotic stability (in mean square etc.) of the controlled system (3.2). In other words, the continuous-time feedback control $u(x(t), r(t), t)$ will stabilize the system. However, in order for the discrete-time feedback control $u(x(\delta_t), r(t), t)$ to do the job, we need a slightly stronger condition, namely we add a new term $\lambda_1|U_x(x,i,t)|^2$ into the left-hand-side of (4.13) to form (4.12). As demonstrated in Sections 5 and 6 later, we will see this is quite easy to achieve by choosing λ_1 sufficiently small when the derivative vector $U_x(x,i,t)$ is bounded by a linear function of x. We can now state our first result.

Theorem 4.3.2. *Let Assumptions 4.2.1, 4.2.2 and 4.3.1 hold. If $\tau > 0$ is sufficiently small for*

$$\lambda_2 > \frac{\tau K_3^2}{\lambda_1}[2\tau(K_1^2 + 2K_3^2) + K_2^2] \quad \text{and} \quad \tau \le \frac{1}{4K_3}, \tag{4.14}$$

then the controlled system (4.1) is H_∞-stable in the sense that

$$\int_0^\infty \mathbb{E}\,|x(s)|^2 ds < \infty. \tag{4.15}$$

for all initial data $x_0 \in R^n$ and $r_0 \in S$.

Proof. Fix any $x_0 \in R^n$ and $r_0 \in S$. Applying the generalized Itô formula (see Theorem 2.5.4 or [50, 62]) to the Lyapunov functional defined by (4.10) yields

$$dV(\hat{x}_t, \hat{r}_t, t) = LV(\hat{x}_t, \hat{r}_t, t)dt + dM(t) \tag{4.16}$$

for $t \ge 0$, where $M(t)$ is a continuous martingale with $M(0) = 0$ (the explicit form of $M(t)$ is of no use in this chapter so we do not state it here) and

$$\begin{aligned}
&LV(\hat{x}_t, \hat{r}_t, t)\\
&= U_t(x(t), r(t), t) + U_x(x(t), r(t), t)[f(x(t), r(t), t) + u(x(\delta_t), r(t), t)]\\
&+ \tfrac{1}{2}\text{trace}[g^T(x(t), r(t), t)U_{xx}(x(t), r(t), t)g(x(t), r(t), t)]\\
&+ \sum_{j=1}^N \gamma_{r(t),j} U(x(t), j, t)
\end{aligned}$$

$$+\,\theta\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(t),t)|^2+|g(x(t),r(t),t)|^2\Big]$$
$$-\,\theta\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds. \qquad (4.17)$$

To see why (4.16) holds, we regard the solution $x(t)$ of equation (4.1) as an Itô process and apply the generalized Itô formula to $U(x(t),r(t),t)$ to get

$$dU(x(t),r(t),t)=\Big(U_t(x(t),r(t),t)+U_x(x(t),r(t),t)[f(x(t),r(t),t)+u(x(\delta_t),r(t),t)]$$
$$+\tfrac{1}{2}\text{trace}[g^T(x(t),r(t),t)U_{xx}(x(t),r(t),t)g(x(t),r(t),t)]$$
$$+\sum_{j=1}^{N}\gamma_{r(t),j}U(x(t),j,t)\Big)dt+dM(t).$$

On the other hand, the fundamental theory of calculus shows

$$d\Big(\int_{t-\tau}^{t}\int_{s}^{t}\Big[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(v),v)|^2+|g(x(v),r(v),v)|^2\Big]dvds\Big)$$
$$=\Big(\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(t),t)|^2+|g(x(t),r(t),t)|^2\Big]$$
$$-\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds\Big)dt.$$

Combining these two equalities gives (4.16).

Recalling (4.11), we can re-write (4.17) as

$$LV(\hat{x}_t,\hat{r}_t,t)$$
$$=\mathcal{L}U(x(t),r(t),t)-U_x(x(t),r(t),t)[u(x(t),r(t),t)-u(x(\delta_t),r(t),t)]$$
$$+\,\theta\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(t),t)|^2+|g(x(t),r(t),t)|^2\Big]$$
$$-\,\theta\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds. \qquad (4.18)$$

But, by Assumption 4.2.2,

$$-\,U_x(x(t),r(t),t)[u(x(t),r(t),t)-u(x(\delta_t),r(t),t)]$$
$$\le\lambda_1|U_x(x(t),r(t),t)|^2+\frac{1}{4\lambda_1}|u(x(t),r(t),t)-u(x(\delta_t),r(t),t)|^2$$
$$\le\lambda_1|U_x(x(t),r(t),t)|^2+\frac{K_3^2}{4\lambda_1}|x(t)-x(\delta_t)|^2. \qquad (4.19)$$

Moreover, by Assumptions 4.2.1 and 4.2.2, we have

$$\theta\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(t),t)|^2+|g(x(t),r(t),t)|^2\Big]$$

$$\leq \theta\tau\Big[2\tau(K_1^2|x(t)|^2 + K_3^2|x(\delta_t)|^2) + K_2^2|x(t)|^2\Big]$$
$$\leq \theta\tau[2\tau(K_1^2 + 2K_3^2) + K_2^2]|x(t)|^2 + 4\theta\tau^2K_3^2|x(t) - x(\delta_t)|^2. \tag{4.20}$$

Substituting (4.19) and (4.20) yields

$$\begin{aligned} LV(\hat{x}_t,\hat{r}_t,t) \leq{}& \mathcal{L}U(x(t),r(t),t) + \lambda_1|U_x(x(t),r(t),t)|^2 \\ &+ \theta\tau[2\tau(K_1^2+2K_3^2)+K_2^2]|x(t)|^2 + \Big(\frac{K_3^2}{4\lambda_1} + 4\theta\tau^2K_3^2\Big)|x(t)-x(\delta_t)|^2 \\ &- \theta\int_{t-\tau}^t \Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2 + |g(x(s),r(s),s)|^2\Big]ds. \end{aligned} \tag{4.21}$$

It then follows from (4.21) and Assumption 4.3.1 that

$$\begin{aligned} LV(\hat{x}_t,\hat{r}_t,t) \leq{}& -\lambda|x(t)|^2 + \Big(\frac{K_3^2}{4\lambda_1} + 4\theta\tau^2K_3^2\Big)|x(t)-x(\delta_t)|^2 \\ &- \theta\int_{t-\tau}^t \Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2 + |g(x(s),r(s),s)|^2\Big]ds, \end{aligned} \tag{4.22}$$

where

$$\lambda = \lambda(\theta,\tau) := \lambda_2 - \theta\tau[2\tau(K_1^2+2K_3^2)+K_2^2]. \tag{4.23}$$

Noting that $t-\delta_t \leq \tau$ for all $t \geq 0$, we can show easily from (4.1) that

$$\begin{aligned} &\mathbb{E}\,|x(t)-x(\delta_t)|^2 \\ \leq{}&2\mathbb{E}\int_{\delta_t}^t \Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2 + |g(x(s),r(s),s)|^2\Big]ds. \end{aligned} \tag{4.24}$$

Let us now choose

$$\theta = \frac{K_3^2}{\lambda_1} \quad \text{and} \quad \tau \leq \frac{1}{4K_3}. \tag{4.25}$$

It then follows from (4.22) and (4.24) that

$$\mathbb{E}\,(LV(\hat{x}_t,\hat{r}_t,t)) \leq -\lambda\mathbb{E}\,|x(t)|^2, \tag{4.26}$$

and by condition (4.14) we have $\lambda > 0$. By (4.16), we hence have

$$\mathbb{E}\,(V(\hat{x}_t,\hat{r}_t,t)) \leq C_1 - \lambda\int_0^t \mathbb{E}\,|x(s)|^2ds, \tag{4.27}$$

for $t \geq 0$, where

$$\begin{aligned} C_1 &= V(\hat{x}_0, \hat{r}_0, 0) \\ &= U(x_0, r_0, 0) + 0.5\theta\tau^2 \Big[\tau|f(x_0, r_0, 0) + u(x_0, r_0, 0)|^2 + |g(x_0, r_0, 0)|^2\Big], \quad (4.28) \end{aligned}$$

so C_1 is a positive number. It follows from (4.27) immediately that

$$\int_0^\infty \mathbb{E}\,|x(s)|^2 ds \leq C_1/\lambda.$$

This implies the desired assertion (4.15). □

In general, it does not follow from (4.15) that $\lim_{t\to\infty} \mathbb{E}\,(|x(t)|^2) = 0$. But, in our case, this is possible. We state this as our second result.

Theorem 4.3.3. *Under the same assumptions of Theorem 4.3.2, the solution of the controlled system (4.1) satisfies*

$$\lim_{t\to\infty} \mathbb{E}\,|x(t)|^2 = 0$$

for all initial data $x_0 \in R^n$ and $r_0 \in S$. That is, the controlled system (4.1) is asymptotically stable in mean square.

Proof. Again, fix any $x_0 \in R^n$ and $r_0 \in S$. By the Itô formula, we have

$$\mathbb{E}\,(|x(t)|^2) = |x_0|^2 + \mathbb{E}\int_0^t \Big(2x(s)[f(x(s), r(s), s) + u(x(\delta_s), r(s), s)] + |g(x(s), r(s), s)|^2\Big)dt$$

for all $t \geq 0$. By Assumptions 4.2.1 and 4.2.2, it is easy to show that

$$\mathbb{E}\,|x(t)|^2 \leq |x_0|^2 + C\int_0^t \mathbb{E}\,|x(s)|^2 ds + C\int_0^t \mathbb{E}\,|x(s) - x(\delta_s)|^2 ds, \quad (4.29)$$

where, and in the remaining part of this chapter, C denotes a positive constant that may change from line to line but its special form is of no use. For any $s \geq 0$, there is a unique integer $v \geq 0$ for $s \in [v\tau, (v+1)\tau)$. Moreover, $\delta_z = v\tau$ for $z \in [v\tau, s]$. It follows from (4.1) that

$$\begin{aligned} &x(s) - x(\delta_s) = x(s) - x(v\tau) \\ &= \int_{v\tau}^s [f(x(z), r(z), z) + u(x(v\tau), r(z), z)]dz + \int_{v\tau}^s g(x(z), r(z), z)dw(z). \end{aligned}$$

By Assumptions 4.2.1 and 4.2.2, we can then derive

$$\mathbb{E}\,|x(s) - x(\delta_s)|^2$$

$$
\begin{aligned}
&\leq 3(\tau K_1^2 + K_2^2)\int_{v\tau}^{s} \mathbb{E}\,|x(z)|^2 dz + 3\tau^2 K_3^2 \mathbb{E}\,|x(v\tau)|^2 \\
&\leq 3(\tau K_1^2 + K_2^2)\int_{\delta_s}^{s} \mathbb{E}\,|x(z)|^2 dz + 6\tau^2 K_3^2 (\mathbb{E}\,|x(s)|^2 + \mathbb{E}\,|x(s) - x(\delta_s)|^2).
\end{aligned}
$$

Noting that $6\tau^2 K_3^2 < 1$ by condition (4.14), we hence have

$$
\mathbb{E}\,|x(s) - x(\delta_s)|^2 \leq \frac{3(\tau K_1^2 + K_2^2)}{1 - 6\tau^2 K_3^2}\int_{\delta_s}^{s} \mathbb{E}\,|x(z)|^2 dz + \frac{6\tau^2 K_3^2}{1 - 6\tau^2 K_3^2}\mathbb{E}\,|x(s)|^2. \quad (4.30)
$$

Substituting this into (4.29) yields

$$
\mathbb{E}\,|x(t)|^2 \leq |x_0|^2 + C\int_0^t \mathbb{E}\,|x(s)|^2 ds + C\int_0^t \int_{\delta_s}^{s} \mathbb{E}\,|x(z)|^2 dz ds. \quad (4.31)
$$

But, it is easy to derive that

$$
\begin{aligned}
&\int_0^t \int_{\delta_s}^{s} \mathbb{E}\,|x(z)|^2 dz ds \leq \int_0^t \int_{s-\tau}^{s} \mathbb{E}\,|x(z)|^2 dz ds \\
&\leq \int_{-\tau}^{t} \mathbb{E}\,|x(z)|^2 \int_z^{z+\tau} ds dz \leq \tau \int_{-\tau}^{t} \mathbb{E}\,|x(z)|^2 dz.
\end{aligned}
$$

Substituting this into (4.31) and then applying Theorem 4.3.2, we obtain that

$$
\mathbb{E}\,|x(t)|^2 \leq C \quad \forall t \geq 0. \quad (4.32)
$$

By the Itô formula, we have

$$
\begin{aligned}
&\mathbb{E}\,|x(t_2)|^2 - \mathbb{E}\,|x(t_1)|^2 \\
=&\mathbb{E}\int_{t_1}^{t_2} \Big(2x(t)[f(x(t), r(t), t) + u(x(\delta_t), r(t), t)] + |g(x(t), r(t), t)|^2\Big) dt
\end{aligned}
$$

for any $0 \leq t_1 < t_2 < \infty$. Using (4.32) and Assumptions 4.2.1 and 4.2.2, we can then easily show that

$$
|\mathbb{E}\,|x(t_2)|^2 - \mathbb{E}\,|x(t_1)|^2| \leq C(t_2 - t_1).
$$

That is, $\mathbb{E}\,|x(t)|^2$ is uniformly continuous in t on R_+. It then follows from (4.15) that $\lim_{t\to\infty} \mathbb{E}\,|x(t)|^2 = 0$ as required. $\square$

In general, we cannot imply $\lim_{t\to\infty} |x(t)| = 0$ a.s. from $\lim_{t\to\infty} \mathbb{E}\,(|x(t)|^2) = 0$. But, in our case, this is once again possible. We state this as our third result.

Theorem 4.3.4. *Under the same assumptions of Theorem 4.3.2, the solution of the controlled system (4.1) satisfies*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s.$$

for all initial data $x_0 \in R^n$ and $r_0 \in S$. That is, the controlled system (4.1) is almost surely asymptotically stable.

Proof. The proof is very technical so we divide it into three steps.

Step 1. Again we fix any $x_0 \in R^n$ and $r_0 \in S$. It follows from Theorem 4.3.3 and the well known Fubini theorem that

$$\mathbb{E}\int_0^\infty |x(t)|^2 dt < \infty. \tag{4.33}$$

This implies

$$\int_0^\infty |x(t)|^2 dt < \infty \quad a.s.$$

We must therefore have

$$\liminf_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{4.34}$$

We now claim that

$$\lim_{t\to\infty} |x(t)| = 0 \quad a.s. \tag{4.35}$$

If this is false, then

$$\mathbb{P}\Big(\limsup_{t\to\infty} |x(t)| > 0\Big) > 0$$

We hence can find a positive number ε, sufficiently small, for

$$\mathbb{P}(\Omega_1) \geq 3\varepsilon, \tag{4.36}$$

where

$$\Omega_1 = \Big\{\limsup_{t\to\infty} |x(t)| > 2\varepsilon\Big\}.$$

Step 2. Let $h > |x_0|$ be a number. Define the stopping time

$$\beta_h = \inf\{t \geq 0 : |x(t)| \geq h\},$$

where throughout this chapter we set $\inf \emptyset = \infty$ (in which $\emptyset$ denotes the empty set as usual). Then, by the Itô formula, we have

$\mathbb{E}\,|x(t \vee \beta_h)|^2$

$$= |x_0|^2 + \mathbb{E}\int_0^{t\vee\beta_h}\Big(2x(s)[f(x(s),r(s),s)+u(x(\delta_s),r(s),s)]+|g(x(s),r(s),s)|^2\Big)dt$$

for all $t \geq 0$. By Assumptions 4.2.1 and 4.2.2 as well as Theorem 4.3.2, it is easy to show that

$$\mathbb{E}\,|x(t\vee\beta_h)|^2 \leq C.$$

Hence

$$h^2\mathbb{P}(\beta_h \leq t) \leq C.$$

Letting $t \to \infty$ and then choosing h sufficiently large, we get

$$\mathbb{P}(\beta_h < \infty) \leq \frac{C}{h^2} \leq \varepsilon.$$

This implies

$$\mathbb{P}(\Omega_2) \geq 1 - \varepsilon, \tag{4.37}$$

where

$$\Omega_2 = \{|x(t)| < h \text{ for all } 0 \leq t < \infty\}.$$

It then follows easily from (4.36) and (4.37) that

$$\mathbb{P}(\Omega_1 \cap \Omega_2) \geq 2\varepsilon. \tag{4.38}$$

Step 3. Define a sequence of stopping times:

$$\begin{aligned}
\alpha_1 &= \inf\{t \geq 0 : |x(t)|^2 \geq 2\varepsilon\},\\
\alpha_{2i} &= \inf\{t \geq \alpha_{2i-1} : |x(t)|^2 \leq \varepsilon\}, \quad i = 1, 2, \cdots,\\
\alpha_{2i+1} &= \inf\{t \geq \alpha_{2i} : |x(t)|^2 \geq 2\varepsilon\}, \quad i = 1, 2, \cdots.
\end{aligned}$$

We observe from (4.34) and the definitions of Ω_1 and Ω_2 that $\alpha_{2i} < \infty$ whenever $\alpha_{2i-1} < \infty$, and moreover,

$$\beta_h(\omega) = \infty \text{ and } \alpha_i(\omega) < \infty \text{ for all } i \geq 1 \text{ whenever } \omega \in \Omega_1 \cap \Omega_2. \tag{4.39}$$

By (4.33), we derive

$$\begin{aligned}
\infty > \mathbb{E}\int_0^\infty |x(t)|^2 dt &\geq \sum_{i=1}^\infty \mathbb{E}\Big(I_{\{\alpha_{2i-1}<\infty,\beta_h=\infty\}}\int_{\alpha_{2i-1}}^{\alpha_{2i}} |x(t)|^2 dt\Big)\\
&\geq \varepsilon\sum_{i=1}^\infty \mathbb{E}\Big(I_{\{\alpha_{2i-1}<\infty,\beta_h=\infty\}}[\alpha_{2i}-\alpha_{2i-1}]\Big).
\end{aligned} \tag{4.40}$$

Let use now define

$$F(t) = f(x(t), r(t), t) + u(x(\delta_t), r(t), t) \quad \text{and} \quad G(t) = g(x(t), r(t), t)$$

for $t \geq 0$. By Assumptions 4.2.1 and 4.2.2, we see that

$$|F(t)|^2 \vee |G(t)|^2 \leq K_h \quad \forall t \geq 0$$

whenever $|x(t) \vee |x(\delta_t)| \leq h$ (in particular, for $\omega \in \Omega_2$), where K_h is a positive constant. By the Hölder inequality and the Doob martingale inequality, we then derive that, for any $T > 0$,

$$\begin{aligned}
&\mathbb{E}\left(I_{\{\beta_h \vee \alpha_{2i-1} < \infty\}} \sup_{0 \leq t \leq T} |x(\beta_h \vee (\alpha_{2i-1} + t)) - x(\beta_h \vee \alpha_{2i-1})|^2\right) \\
\leq & 2\mathbb{E}\left(I_{\{\beta_h \vee \alpha_{2i-1} < \infty\}} \sup_{0 \leq t \leq T} \left| \int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1} + t)} F(s) ds \right|^2\right) \\
+ & 2\mathbb{E}\left(I_{\{\beta_h \vee \alpha_{2i-1} < \infty\}} \sup_{0 \leq t \leq T} \left| \int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1} + t)} G(s) dw(s) \right|^2\right) \\
\leq & 2T\mathbb{E}\left(I_{\{\beta_h \vee \alpha_{2i-1} < \infty\}} \int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1} + T)} |F(s)|^2 ds\right) \\
+ & 8\mathbb{E}\left(I_{\{\beta_h \vee \alpha_{2i-1} < \infty\}} \int_{\beta_h \vee \alpha_{2i-1}}^{\beta_h \vee (\alpha_{2i-1} + T)} |G(s)|^2 ds\right) \\
\leq & 2K_h T(T + 4).
\end{aligned} \tag{4.41}$$

Let $\theta = \varepsilon/(2h)$. It is easy to see that

$$||x|^2 - |y|^2| < \varepsilon \text{ whenever } |x - y| < \theta,\ |x| \vee |y| \leq h. \tag{4.42}$$

Choose T sufficiently small for

$$\frac{2K_h T(T + 4)}{\theta^2} < \varepsilon. \tag{4.43}$$

It then follows from (4.41) that

$$\begin{aligned}
&\mathbb{P}\Big(\{\beta_h \vee \alpha_{2i-1} < \infty\} \cap \Big\{\sup_{0 \leq t \leq T} |x(\beta_h \vee (\alpha_{2i-1} + t)) - x(\beta_h \vee \alpha_{2i-1})| \geq \theta\Big\}\Big) \\
&\leq \frac{2K_h T(T + 4)}{\theta^2} < \varepsilon.
\end{aligned}$$

Therefore

$$\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0 \leq t \leq T} |x(\alpha_{2i-1} + t) - x(\alpha_{2i-1})| \geq \theta\Big\}\Big)$$

$$=\mathbb{P}\Big(\{\beta_h \vee \alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\beta_h \vee (\alpha_{2i-1}+t)) - x(\beta_h \vee \alpha_{2i-1})| \ge \theta\Big\}\Big)$$
$$\le\mathbb{P}\Big(\{\beta_h \vee \alpha_{2i-1} < \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\beta_h \vee (\alpha_{2i-1}+t)) - x(\beta_h \vee \alpha_{2i-1})| \ge \theta\Big\}\Big)$$
$$\le\varepsilon.$$

Using (4.38) and (4.39), we then have

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\alpha_{2i-1}+t) - x(\alpha_{2i-1})| < \theta\Big\}\Big)\\
=&\mathbb{P}(\{\alpha_{2i-1} < \infty, \beta_h = \infty\})\\
-&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\alpha_{2i-1}+t) - x(\alpha_{2i-1})| \ge \theta\Big\}\Big)\\
\ge&2\varepsilon - \varepsilon = \varepsilon.
\end{aligned}$$

By (4.42), we get

$$\begin{aligned}
&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} \big||x(\alpha_{2i-1}+t)|^2 - |x(\alpha_{2i-1})|^2\big| < \varepsilon\Big\}\Big)\\
\ge&\mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \Big\{\sup_{0\le t\le T} |x(\alpha_{2i-1}+t) - x(\alpha_{2i-1})| < \theta\Big\}\Big)\\
\ge&\varepsilon. \qquad (4.44)
\end{aligned}$$

Set

$$\hat{\Omega}_i = \Big\{\sup_{0\le t\le T} \big||x(\alpha_{2i-1}+t)|^2 - |x(\alpha_{2i-1})|^2\big| < \varepsilon\Big\}.$$

Note that

$$\alpha_{2i}(\omega) - \alpha_{2i-1}(\omega) \ge T \quad \text{if } \omega \in \{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \hat{\Omega}_i.$$

Using (4.40) and (4.44), we finally derive that

$$\begin{aligned}
\infty > \varepsilon &\sum_{i=1}^{\infty} \mathbb{E}\left(I_{\{\alpha_{2i-1}<\infty,\beta_h=\infty\}}[\alpha_{2i} - \alpha_{2i-1}]\right)\\
\ge \varepsilon &\sum_{i=1}^{\infty} \mathbb{E}\left(I_{\{\alpha_{2i-1}<\infty,\beta_h=\infty\}\cap\hat{\Omega}_i}[\alpha_{2i} - \alpha_{2i-1}]\right)\\
\ge \varepsilon T &\sum_{i=1}^{\infty} \mathbb{P}\Big(\{\alpha_{2i-1} < \infty, \beta_h = \infty\} \cap \hat{\Omega}_i\Big)\\
\ge \varepsilon T &\sum_{i=1}^{\infty} \varepsilon = \infty, \qquad (4.45)
\end{aligned}$$

which is a contradiction. Hence, (4.35) must hold. The proof is complete. □

4.4 Exponential Stabilization

In the previous section, we have discussed various asymptotic stabilities by feedback controls based on discrete-time state observations. However, all these stabilities do not reveal the rate at which the solution tends to zero. In this section, we will discuss the exponential stabilization by feedback controls. For this purpose, we need to impose another condition.

Assumption 4.4.1. *Assume that there is a pair of positive numbers* c_1 *and* c_2 *such that*

$$c_1|x|^2 \le U(x,i,t) \le c_2|x|^2 \tag{4.46}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

The following theorem shows that the controlled system (4.1) can be stabilized in the sense of both mean square and almost sure exponential stability.

Theorem 4.4.2. *Let Assumptions 4.2.1, 4.2.2, 4.3.1 and 4.4.1 hold. Let* $\tau > 0$ *be sufficiently small for (4.14) to hold and set*

$$\theta = \frac{K_3^2}{\lambda_1} \quad \text{and} \quad \lambda = \lambda_2 - \theta\tau[2\tau(K_1^2 + 2K_3^2) + K_2^2]$$

(so $\lambda > 0$*). Then the solution of the controlled system (4.1) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}\,|x(t)|^2) \le -\gamma \tag{4.47}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) \le -\frac{\gamma}{2} \quad a.s. \tag{4.48}$$

for all initial data $x_0 \in R^n$ *and* $r_0 \in S$*, where* $\gamma > 0$ *is the unique root to the following equation*

$$2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_2) + \gamma c_2 = \lambda, \tag{4.49}$$

in which

$$H_1 = \theta\tau(2\tau(K_1^2 + 2K_3^2) + K_2^2) + \frac{24\tau^3 K_3^4}{1 - 6\tau^2 K_3^2}, \quad H_2 = \frac{12\theta\tau^2 K_3^2(\tau K_1^2 + K_2^2)}{1 - 6\tau^2 K_3^2}. \tag{4.50}$$

Proof. By the generalized Itô formula, we have

$$\mathbb{E}\left[e^{\gamma t} V(\hat{x}_t, \hat{r}_t, t)\right] = V(\hat{x}_0, \hat{r}_0, t) + \mathbb{E}\int_0^t e^{\gamma z}[\gamma V(\hat{x}_z, \hat{r}_z, z) + LV(\hat{x}_z, \hat{r}_z, z)]dz$$

for $t \geq 0$. Using (4.26), (4.28) and (4.46), we get

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \leq C_1 + \int_0^t e^{\gamma z}[\gamma \mathbb{E}\,(V(\hat{x}_z, \hat{r}_z, z)) - \lambda \mathbb{E}\,|x(z)|^2]dz. \tag{4.51}$$

Define

$$\bar{V}(\hat{x}_t, \hat{r}_t, t) := \theta \int_{t-\tau}^t \int_s^t \Big[\tau |f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dv ds. \tag{4.52}$$

By (4.10) and (4.46), we then have

$$\mathbb{E}\,(V(\hat{x}_z, \hat{r}_z, z)) \leq c_2 \mathbb{E}\,|x(z)|^2 + \mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)). \tag{4.53}$$

Moreover, by Assumptions 4.2.1 and 4.2.2,

$$\begin{aligned} \mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)) &\leq \theta\tau \int_{z-\tau}^z \Big[(2\tau K_1^2 + K_2^2)\mathbb{E}\,|x(v)|^2 + 2\tau K_3^2 \mathbb{E}\,|x(\delta_v)|^2\Big] dv \\ &\leq \theta\tau \int_{z-\tau}^z \Big[(2\tau(K_1^2 + 2K_3^2) + K_2^2)\mathbb{E}\,|x(v)|^2 + 4\tau K_3^2 \mathbb{E}\,|x(v) - x(\delta_v)|^2\Big] dv. \end{aligned} \tag{4.54}$$

By Theorem 4.3.2, we see that $\mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z))$ is bounded on $z \in [0, 2\tau]$. For $z \geq 2\tau$, by (4.30), we have

$$\mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)) \leq H_1 \int_{z-\tau}^z \mathbb{E}\,|x(v)|^2 dv + H_2 \int_{z-\tau}^z \int_{\delta_v}^v \mathbb{E}\,|x(y)|^2 dy dv. \tag{4.55}$$

where both H_1 and H_2 have been defined by (4.50). But

$$\int_{z-\tau}^z \int_{\delta_v}^v \mathbb{E}\,|x(y)|^2 dy dv \leq \int_{z-\tau}^z \int_{v-\tau}^v \mathbb{E}\,|x(y)|^2 dy dv \leq \tau \int_{z-2\tau}^z \mathbb{E}\,|x(y)|^2 dy.$$

We hence have

$$\mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)) \leq (H_1 + \tau H_2) \int_{z-2\tau}^z \mathbb{E}\,|x(y)|^2 dy. \tag{4.56}$$

Substituting this into (4.53) and then putting the resulting inequality further to (4.51), we get that, for $t \geq 2\tau$,

$$\begin{aligned} c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \leq & C + \gamma(H_1 + \tau H_2) \int_{2\tau}^t e^{\gamma z}\Big(\int_{z-2\tau}^z \mathbb{E}\,|x(y)|^2 dy\Big) dz \\ & - (\lambda - \gamma c_2) \int_0^t e^{\gamma z}\mathbb{E}\,|x(z)|^2 dz. \end{aligned} \tag{4.57}$$

But

$$\int_{2\tau}^{t} e^{\gamma z}\Big(\int_{z-2\tau}^{z} \mathbb{E}\,|x(y)|^2 dy\Big)dz \le \int_0^t \mathbb{E}\,|x(y)|^2\Big(\int_y^{y+2\tau} e^{\gamma z}dz\Big)dy \le 2\tau e^{2\tau\gamma}\int_0^t e^{\gamma y}\mathbb{E}\,|x(y)|^2 dy.$$

Substituting this into (4.57) yields

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \le C + \Big(2\tau\gamma e^{2\tau\gamma}(H_1+\tau H_2) + \gamma c_2 - \lambda\Big)\int_0^t e^{\gamma z}\mathbb{E}\,|x(z)|^2 dz. \quad (4.58)$$

Recalling (4.49), we see

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \le C \quad \forall t \ge 2\tau. \quad (4.59)$$

The assertion (4.47) follows immediately. Finally by [62, Theorem 8.8 on page 309], we can obtain the another assertion (4.48) from (4.59). The proof is therefore complete. □.

4.5 Corollaries

The use of our theorems established in the previous two sections depends on Assumptions 4.2.1, 4.2.2, 4.3.1 and 4.4.1. Among these, Assumption 4.3.1 is the critical one as the others can be verified easily. In other words, it is critical if we can design a control function $u(x,i,t)$ which satisfies Assumption 4.2.2 so that we can then further find a Lyapunov function $U(x,i,t)$ that fulfills Assumption 4.3.1.

It is known that the stabilization problem (4.1) by the continuous-time (regular) feedback control has been discussed by several authors e.g. [30, 57, 61]. That is, to a certain degree, we know how to design a control function $u(x,i,t)$ which satisfies Assumption 4.2.2 so that we can then further find a Lyapunov function $U(x,i,t)$ that obeys (4.13). If the derivative vector $U_x(x,i,t)$ of this Lyapunov function is bounded by a linear function of x, we can then verify Assumption 4.3.1. This motivates us to propose the following alternative assumption.

Assumption 4.5.1. *Assume that there is a function* $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ *and two positive numbers* λ_3, λ_4 *such that*

$$\mathcal{L}U(x,i,t) \le -\lambda_3 |x|^2 \quad (4.60)$$

and

$$|U_x(x,i,t)| \le \lambda_4 |x| \quad (4.61)$$

for all $(x,i,t) \in R^n \times S \times R_+$.

In this case, if we choose a positive number $\lambda_1 < \lambda_3/\lambda_4^2$, then

$$\mathcal{L}U(x,i,t) + \lambda_1 |U_x(x,i,t)|^2 \le -(\lambda_3 - \lambda_1\lambda_4^2)|x|^2. \tag{4.62}$$

But this is the desired condition (4.12) if we set $\lambda_2 = \lambda_3 - \lambda_1\lambda_4^2$. In other words, we have shown that Assumption 4.5.1 implies Assumption 4.3.1. The following corollary is therefore clear.

Corollary 4.5.2. *All the theorems in Sections 3 and 4 hold if Assumption 4.3.1 is replaced by Assumption 4.5.1.*

In practice, we often use the quadratic functions as the Lyapunov functions. That is, we use $U(x,i,t) = x^T Q_i x$, where Q_i's are all symmetric positive-definite $n \times n$ matrices. In this case, Assumption 4.4.1 holds automatically with $c_1 = \min_{i\in S}\lambda_{\min}(Q_i)$ and $c_2 = \max_{i\in S}\lambda_{\max}(Q_i)$. Moreover, condition (4.61) holds as well with $\lambda_4 = 2\max_{i\in S}\|Q_i\|$. So all we need is to find Q_i's for (4.60) to hold. This motivate us to propose the following another assumption.

Assumption 4.5.3. *Assume that there are symmetric positive-definite matrices $Q_i \in R^{n\times n}$ $(i \in S)$ and a positive number λ_3 such that*

$$\begin{aligned} 2x^T Q_i[f(x,i,t) + u(x,i,t)] &+ \text{trace}[g^T(x,i,t)Q_i(x,i,t)g(x,i,t)] \\ &+ \sum_{j=1}^N \gamma_{ij} x^T Q_j x \le -\lambda_3 |x|^2, \end{aligned} \tag{4.63}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

The following corollary follows immediately from Theorem 4.4.2.

Corollary 4.5.4. *Let Assumptions 4.2.1, 4.2.2 and 4.5.3. Set*

$$c_1 = \min_{i\in S}\lambda_{\min}(Q_i), \quad c_2 = \max_{i\in S}\lambda_{\max}(Q_i), \lambda_4 = 2\max_{i\in S}\|Q_i\|.$$

Choose $\lambda_1 < \lambda_3/\lambda_4^2$ and then set $\lambda_2 = \lambda_3 - \lambda_1\lambda_4^2$. Let $\tau > 0$ be sufficiently small for (4.14) to hold and set

$$\theta = \frac{K_3^2}{\lambda_1} \quad \textit{and} \quad \lambda = \lambda_2 - \theta\tau[2\tau(K_1^2 + 2K_3^2) + K_2^2]$$

(so $\lambda > 0$). Then the assertions of Theorem 4.4.2 hold.

4.6 Examples

Let us now discuss some examples to illustrate our theory.

Example 4.6.1. We first consider the same example as discussed in Mao [52], namely the linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + B(r(t))x(t)dw(t) \tag{4.64}$$

on $t \geq t_0$. Here $w(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

and the system matrices are

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The computer simulation (Figure 4.1) shows this hybrid SDE is not almost surely exponentially stable.

Let us now design a discrete-time-state feedback control to stabilize the system. Assume that the controlled hybrid SDE has the form

$$\begin{aligned} dx(t) = {} & [A(r(t))x(t) + F(r(t))G(r(t))x(\delta_t)]dt \\ & + B(r(t))x(t)dw(t), \end{aligned} \tag{4.65}$$

namely, our controller function has the form $u(x, i, t) = F_i G_i x$. Here, we assume that

$$G_1 = (1, 0), \quad G_2 = (0, 1),$$

and our aim is to seek for F_1 and F_2 in $R^{2\times 1}$ and then make sure τ is sufficiently small for this controlled SDE to be exponentially stable in mean square and almost surely as well. To apply Corollary 4.5.4, we observe that Assumptions 4.2.1 and

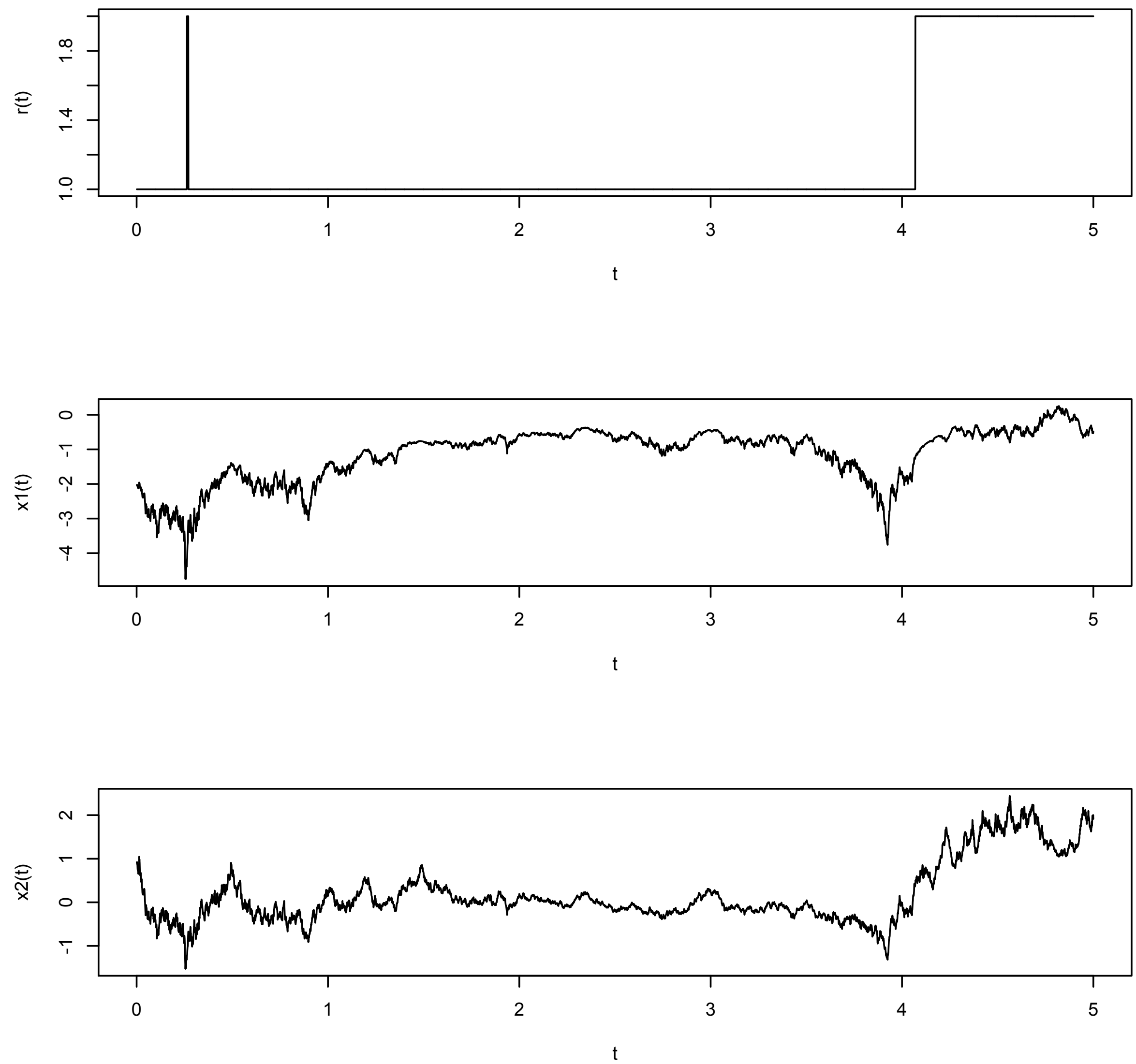

Figure 4.1: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (4.64) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

4.2.2 hold with $K_1 = 5.236$ and $K_2 = \sqrt{2}$. We need to verify Assumption 4.5.3. It is easy to see the left-hand-side term of (4.63) becomes $x^T\bar{Q}_ix$ $(i = 1, 2)$, where

$$\bar{Q}_i := Q_i(A_i + F_iG_i) + (A_i^T + G_i^T F_i^T)Q_i + B_i^T Q_i B_i + \sum_{j=1}^{2} \gamma_{ij} Q_j.$$

Let us now choose $Q_1 = Q_2 = I$ (the 2×2 identity matrix) and

$$F_1 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 0 \\ -10 \end{bmatrix}.$$

We then have

$$\bar{Q}_1 = \begin{bmatrix} -16 & 0 \\ 0 & -8 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} -8 & 0 \\ 0 & -16 \end{bmatrix}.$$

Hence, $x^T\bar{Q}_ix \le -8|x|^2$. In other words, (4.63) holds with $\lambda_3 = 8$. It is also easy to verify that Assumptions 4.2.1 and 4.2.2 hold with $K_1 = 5.236$, $K_3 = 10$ and $K_2 = \sqrt{2}$. We further compute the parameters specified in Corollary 4.5.4: $c_1 = c_2 = 1$ and $\lambda_4 = 2$. Choosing $\lambda_1 = 1$, we then have $\lambda_2 = 4$. Consequently, condition (4.14) becomes

$$4 > 200\tau(227.42\tau + 1), \quad \tau \le 1/40.$$

These hold as long as $\tau < 0.0074$. By Corollary 4.5.4, if we set F_i as above and make sure that $\tau < 0.0074$, then the discrete-time-state feedback controlled hybrid SDE (4.65) is exponentially stable in mean square and almost surely as well. The computer simulation (Figure 4.2) supports this result clearly. It should be pointed out that it is required for $\tau < 0.0000308$ in Mao [52], while applying our new theory we only need $\tau < 0.0074$. In other words, our new theory has improved the existing result significantly.

Example 4.6.2. Let us now return to the nonlinear uncontrolled system (4.3). Given that its coefficients satisfy the linear growth condition (4.4), we consider a linear controller function of the form $u(x, i, t) = A_ix$, where $A_i \in R^{n\times n}$ for all $i \in S$. That is, the controlled hybrid SDE has the form

$$dx(t) = \big(f(x(t), r(t), t) + A_{r(t)}x(\delta_t)\big)dt + g(x(t), r(t), t)dw(t). \qquad (4.66)$$

We observe that Assumption 4.2.2 holds with $K_3 = \max_{i\in S} \|A_i\|$. Let us now establish Assumption 4.5.3 in order to apply Corollary 4.5.4. We choose $Q_i = q_iI$,

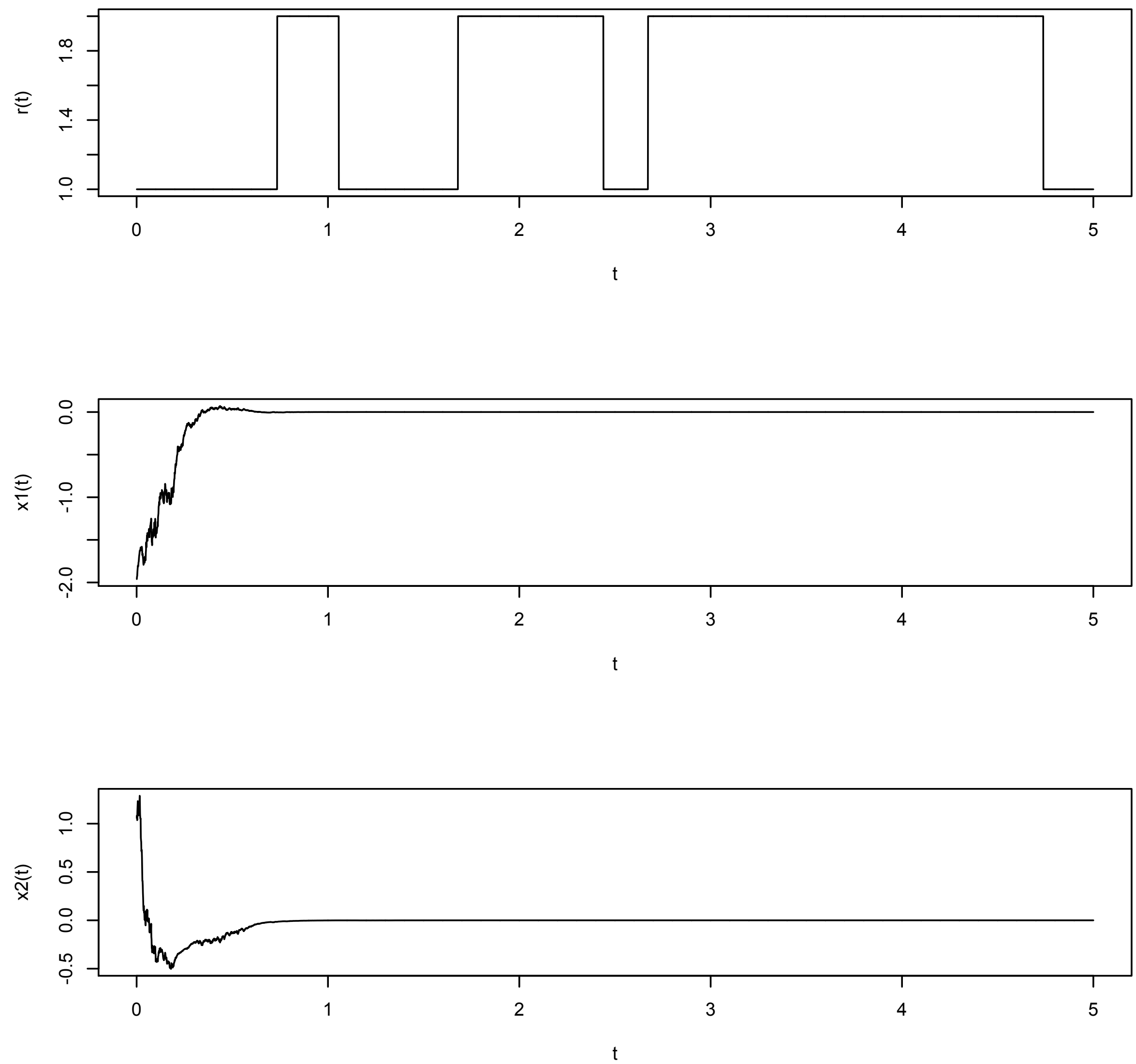

Figure 4.2: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the controlled hybrid SDE (4.65) with $\tau = 10^{-3}$ using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

where $q_i > 0$ and I is the $n \times n$ identity matrix. We estimate the right-hand-side of (4.63):

$$2x^TQ_i[f(x,i,t)+u(x,i,t)]+\text{trace}[g^T(x,i,t)Q_i(x,i,t)g(x,i,t)]+\sum_{j=1}^N\gamma_{ij}x^TQ_jx$$
$$\le q_i(2K_1+K_2^2)|x|^2+2q_ix^TA_ix+\sum_{j=1}^N\gamma_{ij}q_j|x|^2$$
$$=x^T\Big(q_i(2K_1+K_2^2)I+q_i(A_i+A_i^T)+\sum_{j=1}^N\gamma_{ij}q_jI\Big)x. \tag{4.67}$$

We now assume that the following linear matrix inequalities

$$q_i(2K_1+K_2^2)I+Y_i+Y_i^T+\sum_{j=1}^N\gamma_{ij}q_jI<0 \tag{4.68}$$

have their solutions of $q_i>0$ and $Y_i\in R^{n\times n}$ $(i\in S)$. Set $A_i=q_i^{-1}Y_i$ and

$$-\lambda_3=\max_{i\in S}\lambda_{\max}\Big(q_i(2K_1+K_2^2)I+Y_i+Y_i^T+\sum_{j=1}^N\gamma_{ij}q_jI\Big). \tag{4.69}$$

We then see Assumption 4.5.3 is satisfied. The corresponding parameters in Corollary 4.5.4 becomes

$$c_1=\min_{i\in S}q_i,\quad c_2=\max_{i\in S}q_i,\quad \lambda_4=2c_2.$$

Choose $\lambda_1<\lambda_3/\lambda_4^2$ and then set $\lambda_2=\lambda_3-\lambda_1\lambda_4^2$. Let $\tau>0$ be sufficiently small for (4.14) to hold. Then, by Corollary 4.5.4, the controlled system (4.66) is exponentially stable in mean square and almost surely as well.

4.7 Summary

In this chapter we have discussed the stabilization of continuous-time hybrid stochastic differential equations by feedback controls based on discrete-time state observations. The stabilities discussed in this chapter includes exponential stability and asymptotic stability, in both mean square and almost sure sense, as well as the H_∞ stability. One of the significant contributions here is the better bound obtained on the duration τ between two consecutive state observations. This is achieved by the method of Lyapunov functionals.

Chapter 5

Stabilization of hybrid systems by feedback control based on discrete-time state and mode observations

5.1 Introduction

The regular problem of stabilization is stated as following: designing a control function $u(x(t), r(t), t)$ which usually appears in the drift part such that the controlled system (3.3)

$$dx(t) = [f(x(t), r(t), t) + u(x(t), r(t), t)]dt + g(x(t), r(t), t)dw(t)$$

will be stable though the original system (3.1) with $u(x(t), r(t), t) = 0$ is unstable, where $t \geq 0$, $r(t)$ is a Markov chain, $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion and the SDE is in the Itô sense.

Wang et al. in [81] designed a state feedback controller to stabilize bilinear uncertain time-delay stochastic systems with Markovian jumping parameters in mean square sense. In [27], the problem of almost sure exponential stabilization of stochastic systems by state-feedback control had been discussed. A robust delayed-state-feedback controller that exponentially stabilizes uncertain stochastic systems was proposed in [38]. It is observed that the state feedback controllers in these papers require continuous observations of the system state $x(t)$ for all time $t \geq t_0$. Recently, Mao [52] first proposed to design a discrete-time feedback control $u(x(\delta(t, t_0, \tau)), r(t), t)$ in order to make the controlled system

$$\begin{aligned} dx(t) =&[f(x(t), r(t), t) + u(x(\delta(t, t_0, \tau)), r(t), t)]dt \\ &+g(x(t), r(t), t)dw(t) \end{aligned} \tag{5.1}$$

become exponentially stable in mean square. Here $\tau > 0$ is a constant and

$$\delta(t, t_0, \tau) = t_0 + [(t - t_0)/\tau]\tau, \tag{5.2}$$

in which $[(t - t_0)/\tau]$ is the integer part of $(t - t_0)/\tau$. This problem has been studied further more in Chapter 3 and 4. The advantage of such a discrete-time feedback control has been discussed in Chapter 3 as well. Despite all this, we can still try a further step. We find that the feedback control in Mao [52] and the previous two chapters is based on the discrete-time observations of the state $x(t_0 + k\tau)(k = 0, 1, 2, \cdots)$ but still depends on the continuous-time observations of the mode $r(t)$ on $t \geq t_0$. This is perfectly fine if the mode of the system can be fully observed at no cost. However, it usually costs to identify the current mode of the system in practice. So we can further improve the control to reduce the control cost by identifying the mode at discrete times when we make observations for the state. [31] supports our idea. In addition, we want to point out that there is significant difference between the feedback control based on discrete-time state observations and mode observations, i.e. $x([t/\tau]\tau)$ and $r([t/\tau]\tau)$. Because $x([t/\tau]\tau)$ tends to $x(t)$ when τ tends to zero, while $r([t/\tau]\tau)$ may not tends to $r(t)$ as a jumping process when τ tends to zero. Therefore, in this chapter, we will consider an n-dimensional controlled hybrid system

$$\begin{aligned} dx(t) =&[f(x(t), r(t), t) \\ &+ u(x(\delta(t, t_0, \tau)), r(\delta(t, t_0, \tau), t)]dt \\ &+ g(x(t), r(t), t)dw(t) \end{aligned} \tag{5.3}$$

on $t \geq t_0$, where our new feedback control is based on the discrete observations of state $x(t_0 + k\tau)$ and mode $r(t_0 + k\tau)$.

Due to the difficulties arisen from the discrete-time Markov chain $r(t_0 + k\tau)$, the analysis in this chapter will be much more complicated in comparison with the related previous chapters and new techniques will be developed.

5.2 Problem Statement

Consider an n-dimensional uncontrolled unstable linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t) \tag{5.4}$$

on $t \geq 0$, with initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n)$. Here $A,\ B_k : S \to R^{n\times n}$ and we will often write $A(i) = A_i$ and $B_k(i) = B_{ki}$. Now we are required to design a feedback control $u(x(\delta(t)), r(\delta(t)))$ based on the discrete-time state and mode observations in the drift part so that the controlled linear SDE

$$\begin{aligned} dx(t) =& [A(r(t))x(t) + u(x(\delta(t)), r(\delta(t)))]dt \\ &+ \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t) \end{aligned} \tag{5.5}$$

will be mean-square exponentially stable, where u is a mapping from $R^n \times S$ to R^n, $\tau > 0$ and

$$\delta(t) = [t/\tau]\tau \quad \text{for } t \geq 0, \tag{5.6}$$

in which $[t/\tau]$ is the integer part of t/τ. As the given SDE (5.4) is linear, it is natural to use a linear feedback control. One of the most common linear feedback controls is the structure control of the form $u(x, i) = F(i)G(i)x$, where F and G are mappings from S to $R^{n\times l}$ and $R^{l\times n}$, respectively, and one of them is given while the other needs to be designed. These two cases are known as:

- State feedback: design $F(\cdot)$ when $G(\cdot)$ is given;
- Output injection: design $G(\cdot)$ when $F(\cdot)$ is given.

Again, we will often write $F(i) = F_i$ and $G(i) = G_i$. Then the controlled system (5.5) becomes

$$\begin{aligned} dx(t) =& [A(r(t))x(t) + F(r(\delta(t)))G(r(\delta(t)))x(\delta(t))]dt \\ &+ \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t). \end{aligned} \tag{5.7}$$

It is observed that equation (5.7) is in fact a stochastic differential delay equation (SDDE) with a bounded variable delay(see e.g. [52]). So equation (5.7) has a unique solution $x(t)$ such that $\mathbb{E}\,|x(t)|^2 < \infty$ for all $t \geq 0$ (see e.g. [62]).

5.3 Stabilization of linear hybrid SDEs

We will first denote $F(r(\delta(t)))G(r(\delta(t))) = D(r(\delta(t)))$ and discuss the stability of the following hybrid stochastic system

$$\begin{aligned} dx(t) =& [A(r(t))x(t) + D(r(\delta(t)))x(\delta(t))]dt \\ &+ \sum_{k=1}^{m} B_k(r(t))x(t)dw_k(t) \end{aligned} \tag{5.8}$$

in this section. And then design either $G(\cdot)$ given $F(\cdot)$ or $F(\cdot)$ given $G(\cdot)$ in order for the controlled SDE (5.7) to be stable.

Let us first give two lemmas for preparation.

Lemma 5.3.1. *Let $x(t)$ be the solution of system (5.8). Set*

$$M_A = \max_{i \in S} \|A_i\|^2, \quad M_D = \max_{i \in S} \|D_i\|^2,$$

$$M_B = \max_{i \in S} \sum_{k=1}^{m} \|B_{ki}\|^2$$

and define

$$K(\tau) = [6\tau(\tau M_A + M_B) + 3\tau^2 M_D] e^{6\tau(\tau M_A + M_B)} \tag{5.9}$$

for $\tau > 0$. If τ is small enough for $2K(\tau) < 1$, then for any $t \geq 0$,

$$\mathbb{E}\,|x(t) - x(\delta(t))|^2 \leq \frac{2K(\tau)}{1 - 2K(\tau)} \mathbb{E}\,|x(t)|^2. \tag{5.10}$$

Proof. Fix any integer $v \geq 0$. For $t \in [v\tau, (v+1)\tau)$, we have $\delta(t) = v\tau$. It follows from (5.8) that

$$\begin{aligned} x(t) - x(\delta(t)) =& x(t) - x(v\tau) \\ =& \int_{v\tau}^{t} [A(r(s))x(s) + D(r(v\tau))x(v\tau)]ds \end{aligned}$$

$$+\sum_{k=1}^{m}\int_{v\tau}^{t}B_k(r(s))x(s)dw_k(s).$$

Using the fundamental inequality $|a+b+c|^2 \leq 3|a|^2+3|b|^2+3|c|^2$ as well as $H\ddot{o}lder's$ inequality and Doob's martingale inequality, we can then derive

$$\begin{aligned}\mathbb{E}\,|x(t)-x(\delta(t))|^2 \leq & 3(\tau M_A+M_B)\int_{v\tau}^{t}\mathbb{E}\,|x(s)|^2ds \\ & +3\tau^2M_D\mathbb{E}\,|x(v\tau)|^2 \\ \leq & 6(\tau M_A+M_B)\int_{v\tau}^{t}\mathbb{E}\,|x(s)-x(\delta(s))|^2ds \\ & +[6\tau(\tau M_A+M_B)+3\tau^2M_D]\mathbb{E}\,|x(v\tau)|^2.\end{aligned}$$

By the well-known Gronwall inequality, we have

$$\mathbb{E}\,|x(t)-x(\delta(t))|^2 \leq K(\tau)\mathbb{E}\,|x(v\tau)|^2.$$

Consequently

$$\mathbb{E}\,|x(t)-x(\delta(t))|^2 \leq 2K(\tau)\Big(\mathbb{E}\,|x(t)-x(\delta(t))|^2+\mathbb{E}\,|x(t)|^2\Big).$$

This implies that (5.10) holds for $t\in[v\tau,(v+1)\tau)$. But $v\geq 0$ is arbitrary, so the desired assertion (5.10) must hold for all $t\geq 0$. The proof is complete. □

Lemma 5.3.2. *For any $t\geq 0, v>0$ and $i\in S$,*

$$\begin{aligned}&\mathbb{P}(r(s)\neq i \text{ for some } s\in[t,t+v]|r(t)=i) \\ &\leq 1-e^{-\bar{\gamma}v},\end{aligned} \tag{5.11}$$

in which

$$\bar{\gamma}=\max_{i\in S}(-\gamma_{ii}). \tag{5.12}$$

Proof. Given $r(t)=i$, define the stopping time

$$\rho_i=\inf\{s\geq t: r(s)\neq i\},$$

where and throughout this chapter we set $\inf\emptyset=\infty$ (in which $\emptyset$ denotes the empty set as usual). It is well known (see e.g. [62]) that ρ_i-t has the exponential distribution with parameter $-\gamma_{ii}$. Hence

$$\mathbb{P}(r(s)\neq i \text{ for some } s\in[t,t+v]|r(t)=i)$$

$$
\begin{aligned}
&= \mathbb{P}(\rho_i - t \le v | r(t) = i) = \int_0^v -\gamma_{ii} e^{\gamma_{ii} s} ds \\
&= 1 - e^{\gamma_{ii} v} \le 1 - e^{-\bar{\gamma} v}
\end{aligned}
$$

as desired. □

We now state the main result on the exponential stability in mean-square of system (3.1).

Theorem 5.3.3. *If there exist positive definite symmetric matrices* $Q(i) = Q_i$*,* $i \in S$*, such that*

$$
\begin{aligned}
\bar{Q}(i) = \bar{Q}_i := & Q_i(A_i + D_i) + (A_i + D_i)^T Q_i \\
& + \sum_{k=1}^{m} B_{ki}^T Q_i B_{ki} + \sum_{j=1}^{N} \gamma_{ij} Q_j
\end{aligned}
\tag{5.13}
$$

are all negative-definite matrices. Set

$$
M_{QD} = \max_{i \in S} \|Q_i D_i\|^2, \quad N_D = \max_{i,j \in S} \|D_j - D_i\|^2
$$
$$
\text{and} \quad -\lambda := \max_{i \in S} \lambda_{\max}(\bar{Q}_i)
$$

(of course $\lambda > 0$*). If* τ *is sufficiently small for* $\lambda > 2\lambda_\tau + 2\lambda_M \mu_\tau$*, where*

$$
\lambda_\tau := \sqrt{\frac{2M_{QD} K(\tau)}{1 - 2K(\tau)}}, \quad \mu_\tau := \sqrt{\frac{2N_D(1 - e^{-\bar{\gamma}\tau})}{1 - 2K(\tau)}}, \tag{5.14}
$$

then the solution of the SDE (5.8) satisfies

$$
\mathbb{E}\,|x(t)|^2 \le \frac{\lambda_M}{\lambda_m}\,\mathbb{E}\,|x_0|^2 e^{-\theta t}, \quad \forall t \ge 0, \tag{5.15}
$$

where $K(\tau)$ *has been defined in Lemma 5.3.1 and*

$$
\lambda_M = \max_{i \in S} \lambda_{\max}(Q_i), \quad \lambda_m = \min_{i \in S} \lambda_{\min}(Q_i),
$$

$$
\theta = \frac{\lambda - 2\lambda_\tau - 2\lambda_M \mu_\tau}{\lambda_M}. \tag{5.16}
$$

In other words, the SDE (5.8) is exponentially stable in mean square.

Proof. Let $V(x(t), r(t)) = x^T(t)Q(r(t))x(t)$. Applying the generalized Itô formula (see Theorem 2.5.4 or [62]) to V, we get

$$dV(x(t), r(t)) = \mathcal{L}V(x(t), r(t))dt + dM_1(t),$$

where $M_1(t)$ is a martingale with $M_1(0) = 0$ and

$$\begin{aligned}
&\mathcal{L}V(x(t), r(t)) \\
=&2x^T(t)Q(r(t))[A(r(t))x(t) + D(r(\delta(t)))x(\delta(t))] \\
&+\sum_{k=1}^{m} x^T(t)B_k^T(r(t))Q(r(t))B_k(r(t))x(t) \\
&+\sum_{j=1}^{N} \gamma_{r(t),j}x^T(t)Q_jx(t) \\
=&x^T(t)\bar{Q}(r(t))x(t) \\
&-2x^T(t)Q(r(t))D(r(t))(x(t) - x(\delta(t))) \\
&-2x^T(t)Q(r(t))(D(r(t)) - D(r(\delta(t))))x(\delta(t)) \\
\leq& - \lambda|x(t)|^2 + 2\sqrt{M_{QD}}|x(t)||x(t) - x(\delta(t))| \\
&-2x^T(t)Q(r(t))(D(r(t)) - D(r(\delta(t))))x(\delta(t))
\end{aligned} \tag{5.17}$$

Applying the generalized Itô formula now to $e^{\theta t}x^T(t)Q(r(t))x(t)$, we then have

$$\begin{aligned}
&e^{\theta t}x^T(t)Q(r(t))x(t) = x^T(0)Q(r(0))x(0) \\
&+\int_0^t e^{\theta s}[\theta x^T(s)Q(r(s))x(s) + \mathcal{L}V(x(s), r(s))]ds \\
&+M_2(t),
\end{aligned}$$

where $M_2(t)$ is also a martingale with $M_2(0) = 0$. Combining this with (5.17) yields

$$\begin{aligned}
&\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2 \\
\leq& E(e^{\theta t}x^T(t)Q(r(t))x(t)) \\
\leq& \lambda_M\mathbb{E}\,|x_0|^2 + \int_0^t (\theta\lambda_M - \lambda)e^{\theta s}\mathbb{E}\,|x(s)|^2 ds \\
&+\int_0^t 2e^{\theta s}\sqrt{M_{QD}}\mathbb{E}\,(|x(s)||x(s) - x(\delta(s))|)ds
\end{aligned}$$

$$-\int_0^t 2e^{\theta s}\mathbb{E}\left(x^T(s)Q(r(s))(D(r(s))\right.$$
$$\left. -D(r(\delta(s))))x(\delta(s))\right)ds. \tag{5.18}$$

But, by Lemma 5.3.1 and 5.3.2, we have

$$\begin{aligned}
&-2e^{\theta s}\mathbb{E}\left(x^T(s)Q(r(s))(D(r(s)) - D(r(\delta(s))))x(\delta(s))\right) \\
&\le e^{\theta s}\mathbb{E}\left(\lambda_M\mu_\tau|x(s)|^2\right. \\
&\left.+\frac{\lambda_M}{\mu_\tau}\|D(r(s)) - D(r(\delta(s)))\|^2|x(\delta(s))|^2\right) \\
&= e^{\theta s}\lambda_M\{\mu_\tau\mathbb{E}\,|x(s)|^2 \\
&+\frac{1}{\mu_\tau}\mathbb{E}\left(\mathbb{E}\left(\|D(r(s)) - D(r(\delta(s)))\|^2|x(\delta(s))|^2|\mathcal{F}_{\delta(s)}\right)\right)\} \\
&\le e^{\theta s}\lambda_M\{\mu_\tau\mathbb{E}\,|x(s)|^2 \\
&+\frac{1}{\mu_\tau}\mathbb{E}\left(|x(\delta(s))|^2\sum_{r(\delta(s))=i}\mathcal{I}_{\{r(\delta(s))=i\}}\max_{i,j\in S}\|D_j - D_i\|^2\right)\} \\
&\le e^{\theta s}\lambda_M\{\mu_\tau\mathbb{E}\,|x(s)|^2 + \frac{N_D(1-e^{-\bar\gamma\tau})}{\mu_\tau}\mathbb{E}\,|x(\delta(s))|^2\} \\
&\le e^{\theta s}\lambda_M\{\mu_\tau\mathbb{E}\,|x(s)|^2 \\
&+\frac{N_D(1-e^{-\bar\gamma\tau})}{\mu_\tau}\frac{2}{1-2K(\tau)}\mathbb{E}\,|x(s)|^2)\} \\
&= 2e^{\theta s}\lambda_M\ \mu_\tau\mathbb{E}\,|x(s)|^2
\end{aligned} \tag{5.19}$$

and

$$\begin{aligned}
&2\sqrt{M_{QD}}\,\mathbb{E}\left(|x(s)||x(s) - x(\delta(s))|\right) \\
&\le \lambda_\tau\mathbb{E}\,|x(s)|^2 + \frac{M_{QD}}{\lambda_\tau}\mathbb{E}\,|x(s) - x(\delta(s))|^2 \\
&\le \lambda_\tau\mathbb{E}\,|x(s)|^2 + \frac{M_{QD}}{\lambda_\tau}\frac{2K(\tau)}{1-2K(\tau)}\,\mathbb{E}\,|x(s)|^2 \\
&= 2\lambda_\tau\mathbb{E}\,|x(s)|^2.
\end{aligned} \tag{5.20}$$

Substituting (5.19) and (5.20) into (5.18) gives

$$\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2 \le \lambda_M\mathbb{E}\,|x_0|^2$$
$$+\int_0^t(\theta\lambda_M + 2\lambda_\tau + 2\lambda_M\mu_\tau - \lambda)e^{\theta s}\mathbb{E}\,|x(s)|^2ds.$$

But, by (5.16), $\theta\lambda_M + 2\lambda_\tau + 2\lambda_M\mu_\tau - \lambda = 0$. Thus

$$\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2 \le \lambda_M \mathbb{E}\,|x_0|^2,$$

which implies the desired assertion (5.15). The proof is complete. □

The following two corollaries provide us with an LMI method to design the controller based on discrete-time observations of both state and mode to stabilize the unstable system (5.4). Corollary 5.3.4 and 5.3.5 demonstrate the case of state feedback and output injection, respectively.

Corollary 5.3.4. *Assume that there are solutions $Q_i = Q_i^T > 0$ and Y_i $(i \in S)$ to the following LMIs*

$$\begin{aligned} &Q_iA_i + Y_iG_i + A_i^TQ_i + G_i^TY_i^T \\ &+\sum_{k=1}^{m} B_{ki}^TQ_iB_{ki} + \sum_{j=1}^{N}\gamma_{ij}Q_j < 0. \end{aligned} \tag{5.21}$$

Then by setting $F_i = Q_i^{-1}Y_i$ and $D_i = F_iG_i$, the controlled SDE (5.7) will be exponentially stable in mean square if $\tau > 0$ is sufficiently small for $\lambda > 2\lambda_\tau + 2\lambda_M\mu_\tau$.

Proof. Recalling $F_i = Q_i^{-1}Y_i$ and $D_i = F_iG_i$, we find that (5.21) is equivalent to the condition that matrices in (5.13) are all negative-definite. So the required assertion follows directly from Theorem 5.3.3.

Corollary 5.3.5. *Assume that there are solutions $X_i = X_i^T > 0$ and Y_i $(i \in S)$ to the following LMIs*

$$\begin{bmatrix} M_{i1} & M_{i2} & M_{i3} \\ M_{i2}^T & -M_{i4} & 0 \\ M_{i3}^T & 0 & -M_{i5} \end{bmatrix} < 0, \tag{5.22}$$

where

$$\begin{aligned} M_{i1} &= A_iX_i + F_iY_i + X_iA_i^T + Y_i^TF_i^T + \gamma_{ii}X_i, \\ M_{i2} &= [X_iB_{1i}^T, \cdots, X_iB_{mi}^T], \\ M_{i3} &= [\sqrt{\gamma_{i1}}X_i, \cdots, \sqrt{\gamma_{i(i-1)}}X_i, \sqrt{\gamma_{i(i+1)}}X_i, \cdots, \sqrt{\gamma_{iN}}X_i], \\ M_{i4} &= \mathrm{diag}[X_i, \cdots, X_i], \end{aligned}$$

$$M_{i5} = \text{diag}[X_1, \cdots, X_{i-1}, X_{i+1}, \cdots, X_N].$$

Then by setting $Q_i = X_i^{-1}$, $G_i = Y_i X_i^{-1}$ *and* $D_i = F_i G_i$, *the controlled SDE (5.7) will be exponentially stable in mean square if* $\tau > 0$ *is sufficiently small for* $\lambda > 2\lambda_\tau + 2\lambda_M \mu_\tau$.

Proof. We first observe that by the well-known Schur complements (see [62]), the LMIs (5.22) are equivalent to the following matrix inequalities

$$\begin{aligned} &A_iX_i + F_iY_i + X_iA_i^T + Y_i^TF_i^T + \gamma_{ii}X_i \\ &+ \sum_{k=1}^{m} X_iB_{ki}^TX_i^{-1}B_{ki}X_i + \sum_{j\neq i}^{N} \gamma_{ij}X_iX_j^{-1}X_i < 0. \end{aligned} \tag{5.23}$$

Recalling that $G_i = Y_iX_i^{-1}$ and $X_i = X_i^T$, we have

$$\begin{aligned} &A_iX_i + F_iG_iX_i + X_iA_i^T + X_iG_i^TF_i^T \\ &+ \sum_{k=1}^{m} X_iB_{ki}^TX_i^{-1}B_{ki}X_i + \sum_{j=1}^{N} \gamma_{ij}X_iX_j^{-1}X_i < 0. \end{aligned} \tag{5.24}$$

Multiplying X_i^{-1} from left and then from right, and noting $Q_i = X_i^{-1}$, $D_i = F_iG_i$, we see that the matrix inequalities (5.25) are equivalent to the following matrix inequalities

$$\begin{aligned} &Q_iA_i + Q_iD_i + A_i^TQ_i + D_i^TQ_i + \\ &\sum_{k=1}^{m} B_{ki}^TQ_iB_{ki} + \sum_{j=1}^{N} \gamma_{ij}Q_j < 0, \end{aligned} \tag{5.25}$$

which yields matrices in (5.13) are all negative-definite. Again, the required assertion follows directly from Theorem 5.10.

5.4 Stabilization of nonlinear hybrid SDEs

Let us now develop our theory to cope with the more general nonlinear stabilization problem. For an unstable nonlinear hybrid SDE

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t) \tag{5.26}$$

on $t \geq 0$ with the initial data $x(0) = x_0 \in L^2_{\mathcal{F}_0}(R^n)$. Here, $f : R^n \times S \times R_+ \to R^n$ and $g : R^n \times S \times R_+ \to R^{n\times m}$. Assume that both f and g are locally Lipschitz continuous and obey the linear growth condition (see e.g. [62]).

Assumption 5.4.1. *Assume that the coefficients f and g are all locally Lipschitz continuous (see e.g. [45–47,62]). That is, for each integer $k \geq 1$, we have a positive constant L_k, such that*

$$\begin{aligned}|f(x,i,t) - f(y,i,t)| \vee |g(x,i,t) - g(y,i,t)| \\ \leq L_k|x-y|,\end{aligned} \tag{5.27}$$

for $x, y \in R^n$ with $|x| \vee |y| \leq k$ and $(i,t) \in S \times R_+$.

We also assume that f, g satisfy the following linear growth condition

$$|f(x,i,t)| \leq K_1|x| \quad \text{and} \quad |g(x,i,t)| \leq K_2|x| \tag{5.28}$$

It is easy to see that $f(0,i,t) = 0$ and $g(0,i,t) = 0$ for all $i \in S$ and $t \geq 0$ so that $x = 0$ is an equilibrium point for (5.26).

We are required to design a linear feedback control $F(r(\delta(t)))G(r(\delta(t)))x(\delta(t))$ based on the discrete-time state and mode observations in the drift part so that the controlled system

$$\begin{aligned}dx(t) = [f(x(t),r(t),t) + F(r(\delta(t)))G(r(\delta(t)))x(\delta(t))]dt \\ + g(x(t),r(t),t)dw(t)\end{aligned} \tag{5.29}$$

will be mean-square exponentially stable. Defining ζ as $\zeta : [0,\infty) \to [0,\tau]$ by

$$\zeta(t) = t - v\tau \quad \text{for} \quad v\tau \leq t < t(v+1)\tau, \tag{5.30}$$

and $v = 0, 1, 2, \cdots$, then we see that the SDE (5.29) can be written as an SDDE

$$\begin{aligned}dx(t) = [f(x(t),r(t),t)+ \\ F(r(t-\zeta(t)))G(r(t-\zeta(t)))x(t-\zeta(t))]dt \\ + g(x(t),r(t),t)dw(t).\end{aligned} \tag{5.31}$$

It is therefore known (see e.g. [62]) that equation (5.29) has a unique solution $x(t)$ such that $\mathbb{E}\,|x(t)|^2 < \infty$ for all $t \geq 0$.

In order to stabilize a nonlinear system by a linear control, we impose some conditions on the nonlinear coefficients f and g as follows.

Assumption 5.4.2. *For each $i \in S$, there is a pair of symmetric $n \times n$-matrices Q_i and $\hat{Q}_i$ with Q_i being positive-definite such that*

$$2x^T Q_i f(x,i,t) + g^T(x,i,t) Q_i g(x,i,t) \le x^T \hat{Q}_i x$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Assumption 5.4.3. *There is a pair of positive constants δ_1 and δ_2 such that*

$$|f(x,i,t)|^2 \le \delta_1 |x|^2 \quad \text{and} \quad |g(x,i,t)|^2 \le \delta_2 |x|^2$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Let us first present a useful lemma.

Lemma 5.4.4. *Let Assumption 5.4.3 hold. Set*

$$\delta_3 = \max_{i \in S} \sum_{k=1}^{m} \|F_i G_i\|^2,$$

and define

$$H(\tau) = [6\tau(\tau\delta_1 + \delta_2) + 3\tau^2\delta_3] e^{6\tau(\tau\delta_1+\delta_2)} \tag{5.32}$$

for $\tau > 0$. If τ is sufficiently small for $2H(\tau) < 1$, then the solution $x(t)$ of the SDE (5.29) satisfies

$$\mathbb{E}\,|x(t) - x(\delta(t))|^2 \le \frac{2H(\tau)}{1 - 2H(\tau)} \mathbb{E}\,|x(t)|^2 \tag{5.33}$$

for all $t \ge 0$.

This lemma can be proved in the same way as Lemma 5.3.1 was proved so we omit the proof.

Theorem 5.4.5. *Let Assumptions 5.4.2 and 5.4.3 hold. Assume that the following LMIs*

$$U_i := \hat{Q}_i + Q_i F_i G_i + G_i^T F_i^T Q_i + \sum_{j=1}^{N} \gamma_{ij} Q_j < 0, \quad i \in S, \tag{5.34}$$

have their solutions F_i ($i \in S$) in the case of feedback control (i.e. G_i's are given), or their solutions G_i in the case of output injection (i.e. F_i's are given). Set

$$-\gamma := \max_{i\in S} \lambda_{\max}(U_i) \quad \text{and} \quad \delta_4 = \max_{i\in S} \|Q_i F_i G_i\|^2,$$
$$\delta_5 = \max_{i,j\in S} \|F_i G_i - F_j G_j\|^2.$$

If τ is sufficiently small for $\gamma > 2\gamma_\tau + 2\lambda_M \eta_\tau$, where

$$\gamma_\tau := \sqrt{\frac{2\delta_4 H(\tau)}{1 - 2H(\tau)}}, \quad \eta_\tau := \sqrt{\frac{2\delta_5(1 - e^{-\bar{\gamma}\tau})}{1 - 2H(\tau)}} \tag{5.35}$$

then the solution of the SDE (5.29) satisfies

$$\mathbb{E}\,|x(t)|^2 \le \frac{\lambda_M}{\lambda_m}\,\mathbb{E}\,|x_0|^2 e^{-\theta t}, \quad \forall t \ge 0, \tag{5.36}$$

where $H(\tau)$ has been defined in Lemma 5.4.4 and

$$\lambda_M = \max_{i\in S} \lambda_{\max}(Q_i), \quad \lambda_m = \min_{i\in S} \lambda_{\min}(Q_i),$$

$$\theta = \frac{\gamma - 2\gamma_\tau - 2\lambda_M \eta_\tau}{\lambda_M}. \tag{5.37}$$

Proof. This theorem can be proved in a similar way as Theorem 5.3.3 was proved so we only give the key steps. Applying the generalized Itô formula to $x^T(t)Q(r(t))x(t)$ we get

$$\begin{aligned}
&d[x^T(t)Q(r(t))x(t)] \\
=&\Big(x^T(t)U(r(t))x(t) \\
&-2x^T(t)Q(r(t))F(r(t))G(r(t))(x(t) - x(\delta(t))) \\
&-2x^T(t)Q(r(t)) \\
&\quad F(r(t) - r(\delta(t)))G(r(t) - r(\delta(t)))x(\delta(t))\Big)dt \\
&+dM_3(t),
\end{aligned}$$

where $M_3(t)$ is a martingale with $M_3(0) = 0$. Applying the generalized Itô formula further to $e^{\theta t}x^T(t)Q(r(t))x(t)$, we can then obtain

$$\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2$$

$$\begin{aligned}
\le&\lambda_M \mathbb{E}\,|x_0|^2 + \int_0^t (\theta\lambda_M - \gamma)e^{\theta s}\mathbb{E}\,|x(s)|^2 ds \\
&+ \int_0^t 2e^{\theta s}\sqrt{\delta_4}\,\mathbb{E}\,(|x(s)||x(s) - x(\delta(s))|)ds \\
&+ \int_0^t 2\mathbb{E}\left(e^{\theta s}x^T(s)Q(r(s))(F(r(s))G(r(s)) \right. \\
&\left. - F(r(\delta(s)))G(r(\delta(s))))x(\delta(s))\right)ds.
\end{aligned} \tag{5.38}$$

But, by Lemma 5.4.4, we can show

$$2\sqrt{\delta_4}\,\mathbb{E}\,(|x(s)||x(s) - x(\delta(s))|) \le 2\gamma_\tau \mathbb{E}\,|x(s)|^2, \tag{5.39}$$

while by Lemma 5.3.2 and (5.35) we can prove that

$$\begin{aligned}
&2\mathbb{E}\left(e^{\theta s}x^T(s)Q(r(s))(F(r(s))G(r(s)) \right. \\
&\left. - F(r(\delta(s)))G(r(\delta(s))))x(\delta(s))\right) \\
\le&2e^{\theta s}\lambda_M\eta_\tau\mathbb{E}\,|x(s)|^2.
\end{aligned} \tag{5.40}$$

Substituting this into (5.38) yields

$$\lambda_m e^{\theta t}\mathbb{E}\,|x(t)|^2 \le \lambda_M \mathbb{E}\,|x_0|^2,$$

which implies the desired assertion (5.36). The proof is complete. □

To apply Theorem 5.4.5, we need two steps:

1 we first need to look for the $2N$ matrices Q_i and $\hat{Q}_i$ for Assumption 5.4.2 to hold;

2 we then need to solve the LMIs in (5.34) for their solutions F_i (or G_i).

There are available computer softwares e.g. Matlab for step 2 so in the remaining part of this section we will develop some ideas for step 1. To make our ideas more clear, *we will only consider the case of feedback control, but the same ideas work for the case of output injection.*

In theory, it is flexible to use $2N$ matrices Q_i and $\hat{Q}_i$ in Assumption 5.4.2. But, in practice, it means more work to be done in finding these $2N$ matrices. It is in this spirit that we introduce a stronger assumption.

Assumption 5.4.6. *There are $N+1$ symmetric $n \times n$-matrices Z and Z_i ($i \in S$) with $Z > 0$ such that*

$$2x^T Zf(x,i,t) + g^T(x,i,t)Zg(x,i,t) \le x^T Z_i x$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Under this assumption, if we let $Q_i = q_i Z$ and $\hat{Q}_i = q_i Z_i$ for some positive numbers q_i, then Assumption 5.4.2 holds. Moreover, the LMIs in (5.34) become

$$\begin{aligned} & q_i Z_i + q_i Z F_i G_i + q_i G_i^T F_i^T Z \\ & + \sum_{j=1}^N \gamma_{ij} q_j Z < 0, \quad i \in S. \end{aligned}$$

If we set $Y_i := q_i F_i$, then these become the following LMIs in q_i and Y_i:

$$\begin{aligned} & q_i Z_i + ZY_i G_i + G_i^T Y_i^T Z \\ & + \sum_{j=1}^N \gamma_{ij} q_j Z < 0, \quad i \in S. \end{aligned} \tag{5.41}$$

We hence have the following corollary.

Corollary 5.4.7. *Let Assumptions 5.4.6 and 5.4.3 hold. Assume that the LMIs (5.41) have their solutions $q_i > 0$ and Y_i ($i \in S$). Then Theorem 5.4.5 holds by setting $Q_i = q_i Z$, $\hat{Q}_i = q_i Z_i$ and $F_i = q_i^{-1} Y_i$. In other words, the controlled SDE (5.29) will be exponentially stable in mean square if we set $F_i = q_i^{-1} Y_i$ and make sure $\tau > 0$ be sufficiently small for $\gamma > 2\gamma_\tau + 2\lambda_M \eta_\tau$.*

An even simpler (but in fact stronger) condition is:

Assumption 5.4.8. *There are constants z_i ($i \in S$) such that*

$$2x^T f(x,i,t) + |g(x,i,t)|^2 \le z_i |x|^2$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Under this assumption, if we let $Q_i = q_i I$ and $\hat{Q}_i = q_i z_i I$ for some positive numbers q_i, where I is the $n \times n$ identity matrix, then Assumption 5.4.2 holds. Moreover, the LMIs in (5.34) become

$$q_i z_i I + q_i F_i G_i + q_i G_i^T F_i^T$$

$$+\sum_{j=1}^{N}\gamma_{ij}q_jI < 0, \quad i \in S.$$

If we set $Y_i := q_iF_i$, then these become the following LMIs in q_i and Y_i:

$$\begin{aligned} &q_iz_iI + Y_iG_i + G_i^TY_i^T \\ &+\sum_{j=1}^{N}\gamma_{ij}q_jI < 0, \quad i \in S. \end{aligned} \tag{5.42}$$

We hence have another corollary.

Corollary 5.4.9. *Let Assumptions 5.4.8 and 5.4.3 hold. Assume that the LMIs (5.42) have their solutions $q_i > 0$ and Y_i ($i \in S$). Then Theorem 5.4.5 holds by setting $Q_i = q_iI$, $\hat{Q}_i = q_iz_iI$ and $F_i = q_i^{-1}Y_i$. In other words, the controlled SDE (5.29) will be exponentially stable in mean square if we set $F_i = q_i^{-1}Y_i$ and make sure $\tau > 0$ be sufficiently small for $\gamma > 2\gamma_\tau + 2\lambda_M\eta_\tau$.*

5.5 Example

Let us consider an unstable linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + B(r(t))x(t)dw(t) \tag{5.43}$$

on $t \geq t_0$. Here $w(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix};$$

and the system matrices are

$$A_1 = \begin{bmatrix} 1 & -1 \\ 1 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -5 & -1 \\ 1 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix}.$$

The computer simulation (Fig 5.1) shows this hybrid SDE is not mean square exponentially stable.

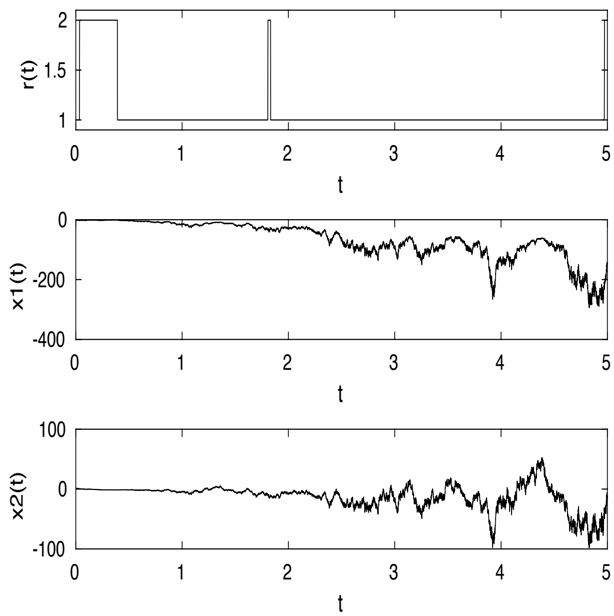

Figure 5.1: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (5.43) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

Let us now design a discrete-time-state feedback control to stabilize the system. Assume that the controlled hybrid SDE has the form

$$\begin{aligned} dx(t) = &[A(r(t))x(t) + F(r(\delta(t)))G(r(\delta(t)))x(\delta(t))]dt \\ &+ B(r(t))x(t)dw(t), \end{aligned} \tag{5.44}$$

where

$$G_1 = [1, 0], \quad G_2 = [0, 1].$$

Our aim is to find F_1 and F_2 in $R^{2\times 1}$ and then make sure τ is sufficiently small for this controlled SDE to be exponentially stable in mean square. To apply Corollary 5.3.4, we first find that the following LMIs

$$\begin{aligned} \bar{Q}_i :=&Q_iA_i + Y_iG_i + A_i^TQ_i + G_i^TY_i^T + B_i^TQ_iB_i \\ &+ \sum_{j=1}^{2} \gamma_{ij}Q_j < 0, \quad i = 1, 2, \end{aligned}$$

have the following set of solutions

$$Q_1 = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix},$$

and

$$Y_1 = \begin{bmatrix} -10 \\ 0 \end{bmatrix}, \quad Y_2 = \begin{bmatrix} 0 \\ -10 \end{bmatrix},$$

and for these solutions we have

$$\bar{Q}_1 = \begin{bmatrix} -7 & 0 \\ 0 & -1 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} -1 & 0 \\ 0 & -7 \end{bmatrix}.$$

Hence, we have

$$-\lambda = \max_{i=1,2} \lambda_{\max}(\hat{Q}_i) = -1, \ M_{YG} = \max_{i=1,2} \|Y_i G_i\|^2 = 100.$$

It is easy to compute that

$$M_A = 27.42, \ M_B = 2, \ M_D = 100,$$

$$M_{QD} = 100, \ N_D = 100.$$

Hence

$$\lambda_\tau = \sqrt{\frac{200K(\tau)}{1-2K(\tau)}}, \quad \mu_\tau = \sqrt{\frac{200(1-e^{-\bar{\gamma}\tau})}{1-2K(\tau)}}$$

where $K(\tau) = [6\tau(27.42\tau + 2) + 300\tau^2]e^{6\tau(27.42\tau+2)}$. By calculating, we get that $\lambda > 2\lambda_\tau + 2\lambda_M \mu_\tau$ whenever $\tau < 0.000015$. By Corollary 5.3.4, if we set $F_1 = Y_1$ and $F_2 = Y_2$, and make sure that $\tau < 1.5 \times 10^{-5}$, then the discrete-time-state feedback controlled hybrid SDE (5.44) is mean-square exponentially stable. The computer simulation (Fig 5.2) supports this result clearly.

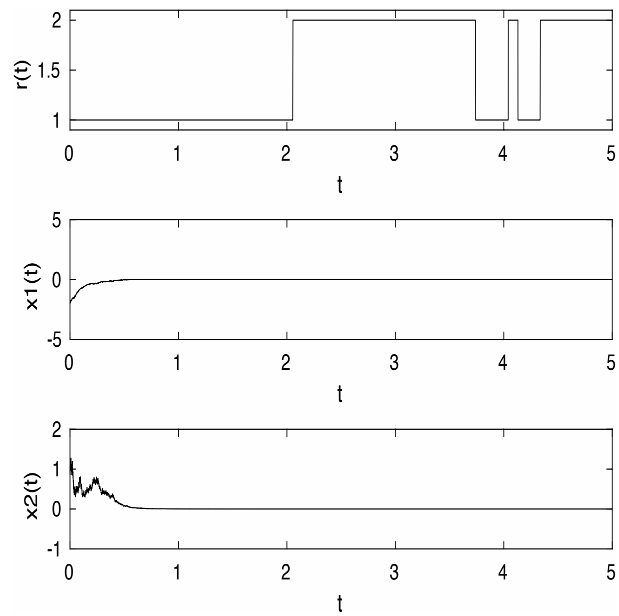

Figure 5.2: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the controlled hybrid SDE (5.44) with $\tau = 10^{-3}$ using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

5.6 Summary

In this chapter, we have proved that unstable linear hybrid SDEs, in the form of (5.4), can be stabilized by a feedback control based on discrete-time state and mode observations. Moreover, we have generalised the theory to a class of nonlinear systems.

Chapter 6

Generalisation of the results in Chapter 5 by Lyapunov Approach

6.1 Introduction

In Chapter 5, we have investigated the following stabilization problem by a feedback control based on the discrete-time state observations: Given an unstable hybrid SDE (3.1)

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t),$$

our aim is to design a feedback control $u(x([t/\tau]\tau), r([t/\tau]\tau), t)$ in order for the controlled system (6.1)

$$\begin{aligned} dx(t) = \big(f(x(t), r(t), t) + u(x([t/\tau]\tau), r([t/\tau]\tau), t)\big)dt \\ + g(x(t), r(t), t)dw(t) \end{aligned} \tag{6.1}$$

becomes exponentially stable in mean square. And we obtain an corresponding upper bound for duration τ.

In Chapter 4, we have shown that *under the local Lipschitz condition*, the discrete-time state observations controlled SDE (3.2) is stable by using Lyapunov function provided τ is sufficiently small. The stabilities discussed in Chapter 4 include exponential stability and asymptotic stability, in both mean square and almost sure sense, as well as the H_∞ stability.

In this chapter, we study on how to use Lyapunov function to stabilize the hybrid SDEs (3.1) by feedback control based on discrete-time state observations and discrete-time mode observations as well. In addition, we will obtain the upper bound on the duration τ.

6.2 Notation and Stabilization Problem

Consider an n-dimensional controlled hybrid SDE

$$dx(t) = \big(f(x(t), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), r(t), t)dw(t) \tag{6.2}$$

on $t \geq 0$, with initial data $x(0) = x_0 \in R^n$ and $r(0) = r_0 \in S$ at time zero. Here

$$f, u : R^n \times S \times R_+ \to R^n \quad \text{and} \quad g : R^n \times S \times R_+ \to R^{n \times m},$$

while $\tau > 0$ and

$$\delta_t = [t/\tau]\tau, \tag{6.3}$$

in which $[t/\tau]$ is the integer part of t/τ. Our aim here is to design the feedback control $u(x(\delta_t), r(t), t)$ so that this controlled hybrid SDE becomes mean-square asymptotically stable, though the given uncontrolled system

$$dx(t) = f(x(t), r(t), t)dt + g(x(t), r(t), t)dw(t) \tag{6.4}$$

may not be stable. We observe that the feedback control $u(x(\delta_t), r(\delta_t), t)$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$, though the given hybrid SDE (6.4) is of continuous-time. In this paper we impose the following *local Lipschitz conditions.*

Assumption 6.2.1. *Assume that the coefficients f and g are all locally Lipschitz continuous (see e.g. [45–47, 62]). Moreover, they satisfy the following linear growth condition*

$$|f(x, i, t)| \leq K_1|x| \quad \textit{and} \quad |g(x, i, t)| \leq K_2|x| \tag{6.5}$$

for all $(x, i, t) \in R^n \times S \times R_+$, where both K_1 and K_2 are positive numbers.

We observe that (6.5) forces

$$f(0, i, t) = 0, \qquad g(0, i, t) = 0 \tag{6.6}$$

for all $(i, t) \in S \times R_+$. This is of course for the stability purpose of this paper. For a technical reason, we require a global Lipschitz condition on the controller function u. More precisely, we impose the following hypothesis.

Assumption 6.2.2. *Assume that there exists a positive constant K_3 such that*

$$|u(x,i,t) - u(y,i,t)| \le K_3|x-y| \tag{6.7}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$. Moreover,

$$u(0,i,t) = 0 \tag{6.8}$$

for all $(i,t) \in S \times R_+$.

Once again, condition (6.8) is for the stability purpose of this paper. We also see that Assumption 6.2.2 implies the following linear growth condition on the controller function

$$|u(x,i,t)| \le K_3|x| \tag{6.9}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

We find that equation (6.2) is in fact a stochastic differential delay equation (SDDE) with a bounded variable delay. Therefore, if we define the bounded variable delay $\zeta : [0,\infty) \to [0,\tau]$ by

$$\zeta(t) = t - k\tau \quad \text{for } k\tau \le t < (k+1)\tau,\ k = 0,1,2,\cdots,$$

then equation (6.2) can be written as

$$dx(t) = \big(f(x(t),r(t),t) + u(x(t-\zeta(t)), r(t-\zeta(t)),t)\big)dt + g(x(t),r(t),t)dw(t). \tag{6.10}$$

It is therefore known (see e.g. [62]) that under Assumptions 6.2.1 and 6.2.2, for the initial data $x(0)$ and $r(0)$, the SDDE (6.10) (namely the controlled system (6.2)) has a unique solution $x(t)$ such that $\mathbb{E}\,|x(t)|^2 < \infty$ for all $t \ge 0$. However, it is almost impossible to determine the solution $x(t)$ on $t \ge s$ if we are given data $x(s)$ and $r(s)$ for some $s \in (k\tau,(k+1)\tau)$, unless we also know $x(k\tau)$ and $r(k\tau)$.

6.3 Asymptotic Stabilization

For our stabilization purpose related to the controlled system (6.2) we will use a Lyapunov functional on the segments $\hat{x}_t := \{x(t+s) : -2\tau \le s \le 0\}$ and $\hat{r}_t := \{r(t+s) : -2\tau \le s \le 0\}$ for $t \ge 0$. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for

$0 \le t < 2\tau$, we set $x(s) = x_0$ and $r(s) = r_0$ for $-2\tau \le s \le 0$. The Lyapunov functional used in this paper will be of the form

$$V(\hat{x}_t, \hat{r}_t, t) = U(x(t), r(t), t) \\ + \theta \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau |f(x(v), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dv ds \tag{6.11}$$

for $t \ge 0$, where θ is a positive number to be determined later and we set

$$f(x, i, s) = f(x, i, 0), \quad u(x, i, s) = u(x, i, 0), \quad g(x, i, s) = f(x, i, 0)$$

for $(x, i, s) \in R^n \times S \times [-2\tau, 0)$. Of course, the functional above uses $r(u)$ only on $t - \tau \le u \le t$ so we could have defined $\hat{r}_t := \{r(t+s) : -\tau \le s \le 0\}$. But, to be consistent with the definition of $\hat{x}_t$, we define $\hat{r}_t$ as above and this does not lose any generality. We also require $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$, the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in R^n \times S \times R_+$ which are continuously twice differentiable in x and once in t. For $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$, let us define $\mathcal{L}U : R^n \times S \times R_+ \to R$ by

$$\mathcal{L}U(x, i, t) = U_t(x, i, t) + U_x(x, i, t)[f(x, i, t) + u(x, i, t)] \\ + \tfrac{1}{2}\text{trace}[g^T(x, i, t) U_{xx}(x, i, t) g(x, i, t)] + \sum_{j=1}^{N} \gamma_{ij} U(x, j, t), \tag{6.12}$$

where

$$U_t(x, i, t) = \frac{\partial U(x, i, t)}{\partial t}, \quad U_x(x, i, t) = \Big(\frac{\partial U(x, i, t)}{\partial x_1}, \cdots, \frac{\partial U(x, i, t)}{\partial x_n}\Big),$$

and

$$U_{xx}(x, i, t) = \Big(\frac{\partial^2 U(x, i, t)}{\partial x_i \partial x_j}\Big)_{n \times n}.$$

Let us put forward a new assumption on U.

Assumption 6.3.1. *Assume that there is a function $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$ and two positive numbers λ_1, λ_2 such that*

$$\mathcal{L}U(x, i, t) + \lambda_1 |U_x(x, i, t)|^2 \le -\lambda_2 |x|^2 \tag{6.13}$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Obviously, condition (6.13) implies

$$\mathcal{L}U(x,i,t) \leq -\lambda_2|x|^2, \tag{6.14}$$

which means the asymptotic stability (in mean square etc.) of the controlled system (6.1). In other words, the system based on the continuous-time feedback control $u(x(t),r(t),t)$ will be stable. However, in order for the discrete-time feedback control $u(x(\delta_t),r(\delta_t),t)$ to stablize the system, we need to add a new term $\lambda_1|U_x(x,i,t)|^2$ into the left-hand-side of (6.14) to form (6.13). Let us recall a useful lemma before stating the main results.

Lemma 6.3.2. *For any $t \geq 0, v > 0$ and $i \in S$,*

$$\begin{aligned}&\mathbb{P}(r(s) \neq i \text{ for some } s \in [t,t+v]|r(t)=i)\\ &\leq 1-e^{-\bar{\gamma}v},\end{aligned}$$

in which

$$\bar{\gamma} = \max_{i\in S}(-\gamma_{ii}).$$

The proof process can be referred to Lemma 5.3.2 in Chapter 5. We can now state our first result in this chapter.

Theorem 6.3.3. *Let Assumptions 6.2.1, 6.2.2 and 6.3.1 hold. And we have Lemma 6.3.2. If $\tau > 0$ is sufficiently small for*

$$\lambda_2 > \frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})+\tau[\frac{2K_3^2}{\lambda_1}+\frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})][2\tau(K_1^2+2K_3^2)+K_2^2] \quad \text{and} \quad \tau \leq \frac{1}{4K_3}, \tag{6.15}$$

then the controlled system (6.2) is H_∞-stable in the sense that

$$\int_0^\infty \mathbb{E}\,|x(s)|^2 ds < \infty. \tag{6.16}$$

for all initial data $x_0 \in R^n$ and $r_0 \in S$.

Proof. Fix any $x_0 \in R^n$ and $r_0 \in S$. Regarding the solution $x(t)$ of equation (6.2) as an Itô process and apply the generalized Itô formula (see Theorem 2.5.4 or [50,62]) to $U(x(t),r(t),t)$, we can get

$$dU(x(t),r(t),t) = \Big(U_t(x(t),r(t),t) + U_x(x(t),r(t),t)[f(x(t),r(t),t) + u(x(\delta_t),r(\delta_t),t)]$$

$$+\tfrac{1}{2}\mathrm{trace}[g^T(x(t),r(t),t)U_{xx}(x(t),r(t),t)g(x(t),r(t),t)]$$
$$+\sum_{j=1}^{N}\gamma_{r(t),j}U(x(t),j,t)\Big)dt+dM(t).$$

On the other hand, the fundamental theory of calculus shows

$$d\Big(\int_{t-\tau}^{t}\int_{s}^{t}\Big[\tau|f(x(v),r(v),v)+u(x(\delta_v),r(\delta_v),v)|^2+|g(x(v),r(v),v)|^2\Big]dvds\Big)$$
$$=\Big(\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)|^2+|g(x(t),r(t),t)|^2\Big]$$
$$-\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds\Big)dt.$$

Then combining these two equalities and using the generalized Itô formula (see Theorem 2.5.4 or [50, 62]) to the Lyapunov functional defined by (6.11) yields

$$dV(\hat{x}_t,\hat{r}_t,t)=LV(\hat{x}_t,\hat{r}_t,t)dt+dM(t) \tag{6.17}$$

for $t\geq 0$, where $M(t)$ is a continuous martingale with $M(0)=0$ and

$$LV(\hat{x}_t,\hat{r}_t,t)$$
$$=U_t(x(t),r(t),t)+U_x(x(t),r(t),t)[f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)]$$
$$+\tfrac{1}{2}\mathrm{trace}[g^T(x(t),r(t),t)U_{xx}(x(t),r(t),t)g(x(t),r(t),t)]$$
$$+\sum_{j=1}^{N}\gamma_{r(t),j}U(x(t),j,t)$$
$$+\theta\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)|^2+|g(x(t),r(t),t)|^2\Big]$$
$$-\theta\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds. \tag{6.18}$$

Recalling (6.12), we can re-write (6.18) as

$$LV(\hat{x}_t,\hat{r}_t,t)$$
$$=\mathcal{L}U(x(t),r(t),t)-U_x(x(t),r(t),t)[u(x(t),r(t),t)-u(x(\delta_t),r(t),t)$$
$$+u(x(\delta_t),r(t),t)-u(x(\delta_t),r(\delta_t),t)]$$
$$+\theta\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)|^2+|g(x(t),r(t),t)|^2\Big]$$
$$-\theta\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(\delta_s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds. \tag{6.19}$$

But, by Assumption 6.2.2 and Lemma 6.3.2,

$$\begin{aligned}
&\mathbb{E}\big(-U_x(x(t),r(t),t)[u(x(t),r(t),t)-u(x(\delta_t),r(t),t)+u(x(\delta_t),r(t),t)-u(x(\delta_t),r(\delta_t),t)]\big)\\
\le&\mathbb{E}\big(\lambda_1|U_x(x(t),r(t),t)|^2\\
&+\frac{1}{4\lambda_1}|u(x(t),r(t),t)-u(x(\delta_t),r(t),t)+u(x(\delta_t),r(t),t)-u(x(\delta_t),r(\delta_t),t)|^2\big)\\
\le&\mathbb{E}\big(\lambda_1|U_x(x(t),r(t),t)|^2+\frac{K_3^2}{2\lambda_1}|x(t)-x(\delta_t)|^2+\frac{1}{2\lambda_1}|u(x(\delta_t),r(t),t)-u(x(\delta_t),r(\delta_t),t)|^2\big)\\
\le&\mathbb{E}\,(\lambda_1|U_x(x(t),r(t),t)|^2)+\frac{K_3^2}{2\lambda_1}\mathbb{E}\,(|x(t)-x(\delta_t)|^2)+\frac{2K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})\mathbb{E}\,|x(\delta_t)|^2)\\
\le&\mathbb{E}\,(\lambda_1|U_x(x(t),r(t),t)|^2)+[\frac{K_3^2}{2\lambda_1}+\frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})]\mathbb{E}\,(|x(t)-x(\delta_t)|^2)+\frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})\mathbb{E}\,(|x(t)|^2).
\end{aligned} \tag{6.20}$$

In addition, by Assumptions 6.2.1 and 6.2.2, we have

$$\begin{aligned}
&\theta\tau\Big[\tau|f(x(t),r(t),t)+u(x(\delta_t),r(\delta_t),t)|^2+|g(x(t),r(t),t)|^2\Big]\\
\le&\theta\tau\Big[2\tau(K_1^2|x(t)|^2+K_3^2|x(\delta_t)|^2)+K_2^2|x(t)|^2\Big]\\
\le&\theta\tau[2\tau(K_1^2+2K_3^2)+K_2^2]|x(t)|^2+4\theta\tau^2K_3^2|x(t)-x(\delta_t)|^2.
\end{aligned} \tag{6.21}$$

Substituting (6.20) and (6.21) yields

$$\begin{aligned}
\mathbb{E}\,(LV(\hat{x}_t,\hat{r}_t,t))\le&\ \mathbb{E}\,(\mathcal{L}U(x(t),r(t),t)+\lambda_1|U_x(x(t),r(t),t)|^2)\\
&+\big(\frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})+\theta\tau[2\tau(K_1^2+2K_3^2)+K_2^2]\big)\mathbb{E}\,(|x(t)|^2)\\
&+\Big(\frac{K_3^2}{2\lambda_1}+\frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})+4\theta\tau^2K_3^2\Big)\mathbb{E}\,(|x(t)-x(\delta_t)|^2)\\
&-\theta\mathbb{E}\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds).
\end{aligned} \tag{6.22}$$

By Assumption 6.3.1, it follows that

$$\begin{aligned}
\mathbb{E}\,(LV(\hat{x}_t,\hat{r}_t,t))\le&-\lambda\mathbb{E}\,(|x(t)|^2)+\Big(\frac{K_3^2}{2\lambda_1}+\frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau})+4\theta\tau^2K_3^2\Big)\mathbb{E}\,(|x(t)-x(\delta_t)|^2)\\
&-\theta\mathbb{E}\,(\int_{t-\tau}^{t}\Big[\tau|f(x(s),r(s),s)+u(x(\delta_s),r(s),s)|^2+|g(x(s),r(s),s)|^2\Big]ds),
\end{aligned} \tag{6.23}$$

where

$$\lambda = \lambda(\theta, \tau) := \lambda_2 - \frac{4K_3^2}{\lambda_1}(1 - e^{-\bar{\gamma}t}) - \theta\tau[2\tau(K_1^2 + 2K_3^2) + K_2^2]. \tag{6.24}$$

Clearly that $t - \delta_t \leq \tau$ for all $t \geq 0$, we can prove from (2.1) that

$$\begin{aligned} &\mathbb{E}\,|x(t) - x(\delta_t)|^2 \\ \leq &2\mathbb{E}\int_{\delta_t}^{t} \Big[\tau|f(x(s), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^2 + |g(x(s), r(s), s)|^2\Big] ds. \end{aligned} \tag{6.25}$$

If we now choose

$$\theta = \frac{2K_3^2}{\lambda_1} + \frac{4K_3^2}{\lambda_1}(1 - e^{-\frac{\bar{\gamma}}{4K_3^2}}) \quad \text{and} \quad \tau \leq \frac{1}{4K_3^2}. \tag{6.26}$$

It then follows from (6.23) and (6.25) that

$$\mathbb{E}\,(LV(\hat{x}_t, \hat{r}_t, t)) \leq -\lambda\mathbb{E}\,|x(t)|^2, \tag{6.27}$$

and by condition (6.15) we have $\lambda > 0$. Hence, we have

$$\mathbb{E}\,(V(\hat{x}_t, \hat{r}_t, t)) \leq C_1 - \lambda\int_0^t \mathbb{E}\,|x(s)|^2 ds, \tag{6.28}$$

for $t \geq 0$, where

$$\begin{aligned} C_1 &= V(\hat{x}_0, \hat{r}_0, 0) \\ &= U(x_0, r_0, 0) + 0.5\theta\tau^2\Big[\tau|f(x_0, r_0, 0) + u(x_0, r_0, 0)|^2 + |g(x_0, r_0, 0)|^2\Big], \end{aligned} \tag{6.29}$$

C_1 is a positive number. It follows from (6.28) immediately that

$$\int_0^\infty \mathbb{E}\,|x(s)|^2 ds \leq C_1/\lambda.$$

This implies the desired assertion (6.16). □

Therefore, since $\mathbb{E}\,(|x(t)|^2)$ is uniformly continuous (see Chapter 4), $\lim_{t\to\infty}\mathbb{E}\,(|x(t)|^2) = 0$ holds. We will omit the proof here. This implies the controlled system (6.2) is asymptotically stable in mean square.

6.4 Exponential Stability

In this section, we will discuss the exponential stability of the controlled system (6.1). Before stating our result, let us impose another condition.

Assumption 6.4.1. *Assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^2 \le U(x,i,t) \le c_2|x|^2 \tag{6.30}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

Theorem 6.4.2. *Let Assumptions 6.2.1, 6.2.2, 6.3.1, 6.4.1 and Lemma 6.3.2 hold. Let $\tau > 0$ be sufficiently small for (6.15) to hold and set*

$$\theta = \frac{2K_3^2}{\lambda_1} + \frac{4K_3^2}{\lambda_1}(1-e^{-\frac{\bar{\gamma}}{4K_3^2}}) \quad and \quad \lambda = \lambda_2 - \frac{4K_3^2}{\lambda_1}(1-e^{-\bar{\gamma}t}) - \theta\tau[2\tau(K_1^2+2K_3^2)+K_2^2]$$

(so $\lambda > 0$). Then the solution of the controlled system (6.2) satisfies

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}\,|x(t)|^2) \le -\gamma \tag{6.31}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\gamma}{2} \quad a.s. \tag{6.32}$$

for all initial data $x_0 \in R^n$ and $r_0 \in S$, where $\gamma > 0$ is the unique root to the following equation

$$2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_2) + \gamma c_2 = \lambda, \tag{6.33}$$

in which

$$H_1 = \theta\tau(2\tau(K_1^2+2K_3^2)+K_2^2) + \frac{24\tau^3K_3^4}{1-6\tau^2K_3^2}, \quad H_2 = \frac{12\theta\tau^2K_3^2(\tau K_1^2+K_2^2)}{1-6\tau^2K_3^2}. \tag{6.34}$$

Proof. It is easy to show from (6.2) that

$$\begin{aligned} &x(s) - x(\delta_s) = x(s) - x(v\tau) \\ =&\int_{v\tau}^{s}[f(x(z),r(z),z) + u(x(v\tau),r(v\tau),z)]dz + \int_{v\tau}^{s} g(x(z),r(z),z)dw(z). \end{aligned}$$

By Assumptions 6.2.1 and 6.2.2, we have

$$\begin{aligned} &\mathbb{E}\,|x(s)-x(\delta_s)|^2 \\ \le&3(\tau K_1^2+K_2^2)\int_{v\tau}^{s}\mathbb{E}\,|x(z)|^2dz + 3\tau^2K_3^2\mathbb{E}\,|x(v\tau)|^2 \\ \le&3(\tau K_1^2+K_2^2)\int_{\delta_s}^{s}\mathbb{E}\,|x(z)|^2dz + 6\tau^2K_3^2(\mathbb{E}\,|x(s)|^2 + \mathbb{E}\,|x(s)-x(\delta_s)|^2). \end{aligned}$$

Noting that $6\tau^2 K_3^2 < 1$ by condition (6.15), we hence get

$$\mathbb{E}\,|x(s) - x(\delta_s)|^2 \leq \frac{3(\tau K_1^2 + K_2^2)}{1 - 6\tau^2 K_3^2} \int_{\delta_s}^{s} \mathbb{E}\,|x(z)|^2 dz + \frac{6\tau^2 K_3^2}{1 - 6\tau^2 K_3^2} \mathbb{E}\,|x(s)|^2. \quad (6.35)$$

Moreover, by the generalized Itô formula, we can prove

$$\mathbb{E}\left[e^{\gamma t} V(\hat{x}_t, \hat{r}_t, t)\right] = V(\hat{x}_0, \hat{r}_0, t) + \mathbb{E} \int_0^t e^{\gamma z} [\gamma V(\hat{x}_z, \hat{r}_z, z) + LV(\hat{x}_z, \hat{r}_z, z)] dz$$

for $t \geq 0$. Using (6.27), (6.29) and (6.30), we get

$$c_1 e^{\gamma t} \mathbb{E}\,|x(t)|^2 \leq C_1 + \int_0^t e^{\gamma z} [\gamma \mathbb{E}\,(V(\hat{x}_z, \hat{r}_z, z)) - \lambda \mathbb{E}\,|x(z)|^2] dz. \quad (6.36)$$

Define

$$\bar{V}(\hat{x}_t, \hat{r}_t, t) := \theta \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau |f(x(v), r(v), v) + u(x(\delta_v), r(v), v)|^2 + |g(x(v), r(v), v)|^2\Big] dv ds. \quad (6.37)$$

By (6.11) and (6.30), we have

$$\mathbb{E}\,(V(\hat{x}_z, \hat{r}_z, z)) \leq c_2 \mathbb{E}\,|x(z)|^2 + \mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)). \quad (6.38)$$

Moreover, by Assumptions 6.2.1 and 6.2.2,

$$\begin{aligned} &\mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)) \\ &\leq \theta\tau \int_{z-\tau}^{z} \Big[(2\tau(K_1^2 + 2K_3^2) + K_2^2) \mathbb{E}\,|x(v)|^2 + 4\tau K_3^2 \mathbb{E}\,|x(v) - x(\delta_v)|^2\Big] dv. \end{aligned} \quad (6.39)$$

By Theorem 6.3.3, we see that $\mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z))$ is bounded on $z \in [0, 2\tau]$. For $z \geq 2\tau$, by (6.35), we have

$$\mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)) \leq H_1 \int_{z-\tau}^{z} \mathbb{E}\,|x(v)|^2 dv + H_2 \int_{z-\tau}^{z} \int_{\delta_v}^{v} \mathbb{E}\,|x(y)|^2 dy dv. \quad (6.40)$$

where both H_1 and H_2 defined by (6.34). But

$$\int_{z-\tau}^{z} \int_{\delta_v}^{v} \mathbb{E}\,|x(y)|^2 dy dv \leq \int_{z-\tau}^{z} \int_{v-\tau}^{v} \mathbb{E}\,|x(y)|^2 dy dv \leq \tau \int_{z-2\tau}^{z} \mathbb{E}\,|x(y)|^2 dy.$$

We hence have

$$\mathbb{E}\,(\bar{V}(\hat{x}_z, \hat{r}_z, z)) \leq (H_1 + \tau H_2) \int_{z-2\tau}^{z} \mathbb{E}\,|x(y)|^2 dy. \quad (6.41)$$

Substituting this into (6.38) and then putting the resulting inequality further to (6.36), we get that, for $t \geq 2\tau$,

$$\begin{aligned} c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \leq & C + \gamma(H_1 + \tau H_2)\int_{2\tau}^{t} e^{\gamma z}\Big(\int_{z-2\tau}^{z}\mathbb{E}\,|x(y)|^2 dy\Big)dz \\ & - (\lambda - \gamma c_2)\int_0^t e^{\gamma z}\mathbb{E}\,|x(z)|^2 dz. \end{aligned} \tag{6.42}$$

But

$$\int_{2\tau}^{t} e^{\gamma z}\Big(\int_{z-2\tau}^{z}\mathbb{E}\,|x(y)|^2 dy\Big)dz \leq \int_0^t \mathbb{E}\,|x(y)|^2\Big(\int_y^{y+2\tau} e^{\gamma z}dz\Big)dy \leq 2\tau e^{2\tau\gamma}\int_0^t e^{\gamma y}\mathbb{E}\,|x(y)|^2 dy.$$

Substituting this into (6.42) yields

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \leq C + \Big(2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_2) + \gamma c_2 - \lambda\Big)\int_0^t e^{\gamma z}\mathbb{E}\,|x(z)|^2 dz. \tag{6.43}$$

Recalling (6.33), we see

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \leq C \quad \forall t \geq 2\tau. \tag{6.44}$$

The assertion (6.31) follows immediately. Finally by [62, Theorem 8.8 on page 309], we can obtain the another assertion (6.32) from (6.44). The proof is therefore complete. □.

6.5 Corollaries

Under Assumptions 6.2.1, 6.2.2, 6.3.1 and 6.4.1, we can apply the theorems established in the previous sections. Among these, Assumption 6.3.1 is the critical one while the others can be verified easily. Therefore, it is critical if we can design a control function $u(x, i, t)$ which satisfies Assumption 6.2.2 so that we can then further find a Lyapunov function $U(x, i, t)$ that fulfills Assumption 6.3.1.

Assumption 6.5.1. *Assume that there is a function $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and two positive numbers λ_3, λ_4 such that*

$$\mathcal{L}U(x, i, t) \leq -\lambda_3 |x|^2 \tag{6.45}$$

and

$$|U_x(x, i, t)| \leq \lambda_4 |x| \tag{6.46}$$

for all $(x, i, t) \in R^n \times S \times R_+$.

In this case, if we choose a positive number $\lambda_1 < \lambda_3/\lambda_4^2$, then

$$\mathcal{L}U(x,i,t) + \lambda_1 |U_x(x,i,t)|^2 \le -(\lambda_3 - \lambda_1\lambda_4^2)|x|^2. \tag{6.47}$$

But this is the desired condition (6.13) if we set $\lambda_2 = \lambda_3 - \lambda_1\lambda_4^2$. In other words, we have shown that Assumption 6.5.1 implies Assumption 6.3.1. The following corollary is therefore clear.

Corollary 6.5.2. *All the theorems in Sections 3 and 4 hold if Assumption 6.3.1 is replaced by Assumption 6.5.1.*

In practice, we often use the quadratic functions as the Lyapunov functions. That is, we use $U(x,i,t) = x^T Q_i x$, where Q_i's are all symmetric positive-definite $n \times n$ matrices. In this case, Assumption 6.4.1 holds automatically with $c_1 = \min_{i\in S} \lambda_{\min}(Q_i)$ and $c_2 = \max_{i\in S} \lambda_{\max}(Q_i)$. Moreover, condition (6.46) holds as well with $\lambda_4 = 2\max_{i\in S} \|Q_i\|$. So all we need is to find Q_i's for (6.45) to hold. This gives us the following another assumption.

Assumption 6.5.3. *Assume that there are symmetric positive-definite matrices $Q_i \in R^{n\times n}$ $(i \in S)$ and a positive number λ_3 such that*

$$\begin{aligned} 2x^T Q_i[f(x,i,t) + u(x,i,t)] + \mathrm{trace}[g^T(x,i,t) Q_i(x,i,t) g(x,i,t)] \\ + \sum_{j=1}^N \gamma_{ij} x^T Q_j x \le -\lambda_3 |x|^2, \end{aligned} \tag{6.48}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

The following corollary follows immediately from Theorem 6.4.2.

Corollary 6.5.4. *Let Assumptions 6.2.1, 6.2.2 and 6.5.3 hold. Set*

$$c_1 = \min_{i\in S} \lambda_{\min}(Q_i), \quad c_2 = \max_{i\in S} \lambda_{\max}(Q_i), \lambda_4 = 2\max_{i\in S} \|Q_i\|.$$

Choose $\lambda_1 < \lambda_3/\lambda_4^2$ and then set $\lambda_2 = \lambda_3 - \lambda_1\lambda_4^2$. Let $\tau > 0$ be sufficiently small for (6.15) to hold and set

$$\theta = \frac{2K_3^2}{\lambda_1} + \frac{4K_3^2}{\lambda_1}(1 - e^{-\frac{\bar\gamma}{4K_3^2}}) \quad \textit{and} \quad \lambda = \lambda_2 - \frac{4K_3^2}{\lambda_1}(1 - e^{-\bar\gamma t}) - \theta\tau[2\tau(K_1^2 + 2K_3^2) + K_2^2$$

(so $\lambda > 0$). Then the assertions of Theorem 6.4.2 hold.

6.6 Example

Example 6.6.1. We first consider the same example as discussed in Mao [52], namely the linear hybrid SDE

$$dx(t) = A(r(t))x(t)dt + B(r(t))x(t)dw(t) \tag{6.49}$$

on $t \geq t_0$. Here $w(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix};$$

and the system matrices are

$$A_1 = \begin{bmatrix} 1 & 3 \\ 4 & -5 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1 & 7 \\ 6 & 2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 2 & 3 \\ 0 & 1 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix}.$$

The computer simulation (Figure 6.1) shows this hybrid SDE is not almost surely exponentially stable.

Let us now design a discrete-time-state feedback control to stabilize the system. Assume that the controlled hybrid SDE has the form

$$\begin{aligned} dx(t) = {} & [A(r(t))x(t) + F(r(\delta_t))G(r(\delta_t))x(\delta_t)]dt \\ & + B(r(t))x(t)dw(t), \end{aligned} \tag{6.50}$$

namely, our controller function has the form $u(x, i, t) = F_i G_i x$. Here, we assume that

$$G_1 = (-1.41, -1.4402), \quad G_2 = (3.1016, 1.9571),$$

and our aim is to seek for F_1 and F_2 in $R^{2\times 1}$ and then make sure τ is sufficiently small for this controlled SDE to be exponentially stable in mean square and almost surely as well. To apply Corollary 6.5.4, we observe that Assumptions 6.2.1 and 6.2.2 hold with $K_1 = 8.1003$ and $K_2 = 3.7025$. We need to verify Assumption 6.5.3. It is easy to see the left-hand-side term of (6.48) becomes $x^T \bar{Q}_i x$ $(i = 1, 2)$, where

$$\bar{Q}_i := Q_i(A_i + F_i G_i) + (A_i^T + G_i^T F_i^T)Q_i + B_i^T Q_i B_i + \sum_{j=1}^{2} \gamma_{ij} Q_j.$$

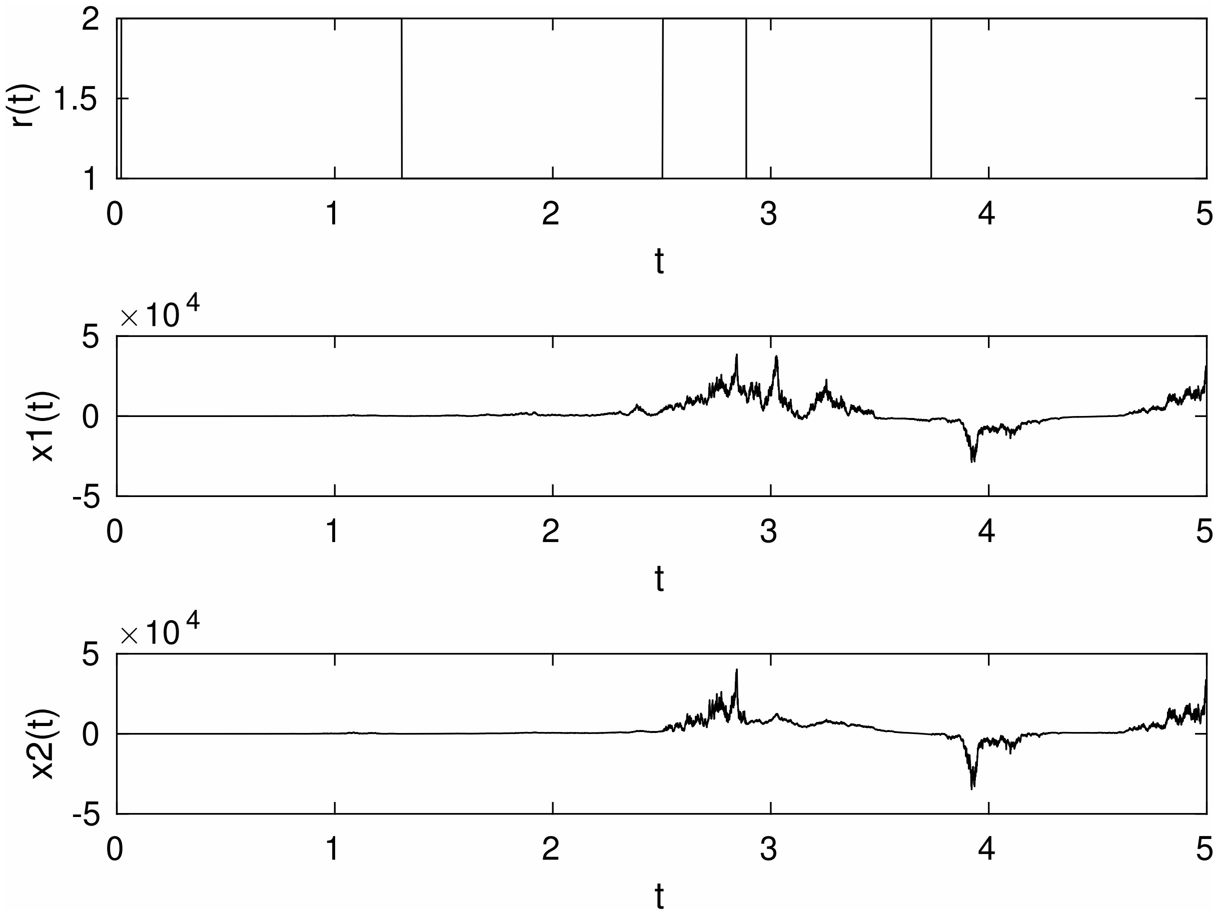

Figure 6.1: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (6.49) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

Let us now choose

$$Q_1 = \begin{bmatrix} 2.5048 & 0.9239 \\ 0.9239 & 3.1738 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5.3836 & 3.0928 \\ 3.0928 & 3.3392 \end{bmatrix}.$$

and

$$F_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 \\ -10 \end{bmatrix}.$$

We then have

$$\bar{Q}_1 = \begin{bmatrix} -14.9558 & -4.2385 \\ -4.2385 & -35.3341 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} -53.8195 & -36.4580 \\ -36.4580 & -30.9954 \end{bmatrix}.$$

Hence, $x^T\bar{Q}_i x \le -80.6096|x|^2$. In other words, (6.48) holds with $\lambda_3 = 80.6096$. It is also easy to verify that Assumptions 6.2.1 and 6.2.2 hold with $K_1 = 8.1003$, $K_3 = 36.8565$ and $K_2 = 3.7025$. We further compute the parameters specified in Corollary 6.5.4: $c_1 = 1.1041$, $c_2 = 7.6187$ and $\lambda_4 = 15.2374$. Choosing $\lambda_1 = 0.3$, we then have $\lambda_2 = 10.9561$. Then, condition (6.15) becomes

$$10.9561 > 18112(1-e^{-2\tau})+\tau[9056+18112(1-e^{-2\tau})][5564.8\tau+13.7082], \quad \tau \le 0.0068.$$

These hold as long as $\tau < 0.000069$. By Corollary 6.5.4, if we set F_i as above and make sure that $\tau < 0.000069$, then the discrete-time-state feedback controlled hybrid SDE (6.50) is exponentially stable in mean square and almost surely as well. The computer simulation (Figure 6.2) supports this result clearly.

Example 6.6.2. Let us now return to the nonlinear uncontrolled system (6.4). Given that its coefficients satisfy the linear growth condition (6.5), we consider a linear controller function of the form $u(x, i, t) = A_i x$, where $A_i \in R^{n\times n}$ for all $i \in S$. That is, the controlled hybrid SDE has the form

$$dx(t) = \big(f(x(t), r(t), t) + A_{r(\delta_t)}x(\delta_t)\big)dt + g(x(t), r(t), t)dw(t). \tag{6.51}$$

We observe that Assumption 6.2.2 holds with $K_3 = \max_{i,j\in S} \|A_i - A_j\|(1 - e^{-\bar{\gamma}\tau})$. Let us now establish Assumption 6.5.3 in order to apply Corollary 6.5.4. We choose $Q_i = q_i I$, where $q_i > 0$ and I is the $n \times n$ identity matrix. We estimate the right-hand-side of (6.48):

$$\begin{aligned} &2x^T Q_i[f(x, i, t) + u(x, i, t)] + \mathrm{trace}[g^T(x, i, t)Q_i(x, i, t)g(x, i, t)] + \sum_{j=1}^N \gamma_{ij} x^T Q_j x \\ &\le q_i(2K_1 + K_2^2)|x|^2 + 2q_i x^T A_i x + \sum_{j=1}^N \gamma_{ij} q_j |x|^2 \\ &= x^T\Big(q_i(2K_1 + K_2^2)I + q_i(A_i + A_i^T) + \sum_{j=1}^N \gamma_{ij} q_j I\Big)x. \end{aligned} \tag{6.52}$$

We now assume that the following linear matrix inequalities

$$q_i(2K_1 + K_2^2)I + Y_i + Y_i^T + \sum_{j=1}^N \gamma_{ij} q_j I < 0 \tag{6.53}$$

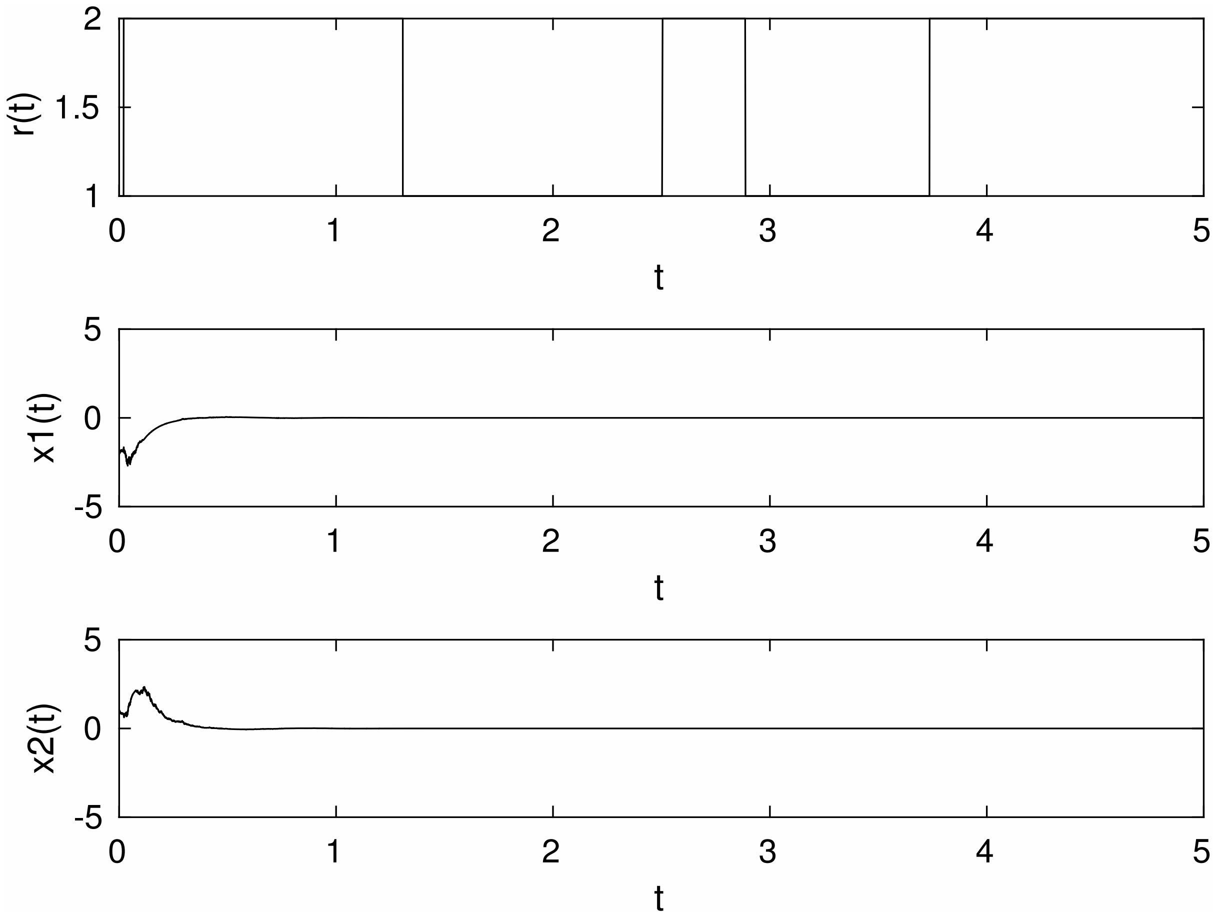

Figure 6.2: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the controlled hybrid SDE (6.50) with $\tau = 10^{-3}$ using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -2$ and $x_2(0) = 1$.

have their solutions of $q_i > 0$ and $Y_i \in R^{n\times n}$ $(i \in S)$. Set $A_i = q_i^{-1}Y_i$ and

$$-\lambda_3 = \max_{i\in S} \lambda_{\max}\Big(q_i(2K_1 + K_2^2)I + Y_i + Y_i^T + \sum_{j=1}^{N} \gamma_{ij} q_j I\Big). \tag{6.54}$$

We then see Assumption 6.5.3 is satisfied. The corresponding parameters in Corollary 6.5.4 becomes

$$c_1 = \min_{i\in S} q_i, \quad c_2 = \max_{i\in S} q_i, \quad \lambda_4 = 2c_2.$$

Choose $\lambda_1 < \lambda_3/\lambda_4^2$ and then set $\lambda_2 = \lambda_3 - \lambda_1\lambda_4^2$. Let $\tau > 0$ be sufficiently small for (6.15) to hold. Then, by Corollary 6.5.4, the controlled system (6.51) is exponentially stable in mean square and almost surely as well.

6.7 Summary

We have proved the stabilization of continuous-time hybrid stochastic differential equations by *feedback controls based on discrete-time state and mode observations* in this chapter. The stabilities here mainly referred to the H_∞ stability, mean squared asymptotic stability and mean squared exponential stability. Moreover, we also managed to build the upper bound on the duration τ between two consecutive state observations. This is achieved by the method of Lyapunov functionals.

Chapter 7

Stabilization of Hybrid Delay Systems by feedback control based on discrete-time state and mode observations

7.1 Introduction

Studying on using hybrid stochastic differential equations (with Markovian switching) to model practical systems has received a lot of attentions in the recent years ([1, 5, 23, 30, 47, 48, 61–63, 72, 79, 82]). However, in many branches of science and industry, systems may not only depend on the current state, but also be decided by the past states. Therefore, stochastic delay systems have also been studied intensively (see e.g. [26, 28, 38–43, 51, 57, 62, 74, 75, 77, 83, 86]). For example, Mao et al. studied stability and stabilization of stochastic delay systems with Markovian swithing (see e.g. [21, 28, 38–43, 47, 51, 57–59, 62]). He et al. studied stability of fuzzy Hopfield neural networks with time-varying delays ([26]). Xu et al. studied uncertain stochastic time-delay systems ([85]).

One classical problem in this field is stabilization ([51, 59, 62]). Huang [57] has proved that hybrid stochastic delay systems can be stabilized exponentially in mean square. In this thesis, we have also proved that it is reasonable to study on the stabilization problems of hybrid systems by discrete-time-state-and-mode-observations feedback controls. However, as far as the authors knowledge, discrete-time feedback control theory has not been applied to study the stabilization problem of hybrid delay system. Therefore, in this chapter, we aim to prove the stability of hybrid SDDEs by discrete-time feedback control.

Consider an unstable hybrid SDDE

$$dx(t) = f(x(t), x(t-h), r(t), t)dt + g(x(t), x(t-h), r(t), t)dw(t), \tag{7.1}$$

where $h > 0$ and $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a continuous-time Markov chain (please see Section 2 for the formal definitions) which represents the system mode. It is reasonable to design a feedback control $u(x([t/\tau]\tau), r([t/\tau]\tau), t)$ in order for the controlled system

$$\begin{aligned} dx(t) = &\big(f(x(t), x(t-h), r(t), t) + u(x([t/\tau]\tau), r([t/\tau]\tau), t)\big)dt \\ &+ g(x(t), x(t-h), r(t), t)dw(t). \end{aligned} \tag{7.2}$$

Due to the technical difficulites arisen in dealing with the relationship between h and τ, it is hard to use the same method in Chapter 3 and 5 to study on the stability of the controlled system (7.2), which is invesitgating the stability of the system from its main features. Therefore, we employ a Lyapunov functional to help us solve our stability problem in this chapter, which makes it possible to study on the stability problems of nonlinear SDDEs.

7.2 Notation and Stabilization Problem

Consider an n-dimensional unstable hybrid SDDE

$$dx(t) = f(x(t), x(t-h), r(t), t)dt + g(x(t), x(t-h), r(t), t)dw(t) \tag{7.3}$$

on $h \geq 0$ and $t \geq 0$, with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ (such that $\mathbb{E}\,||\xi||^2 < \infty$) and $r(0) = r_0 \in S$ at time zero. Here

$$f : R^n \times R^n \times S \times R_+ \to R^n \quad \text{and} \quad g : R^n \times R^n \times S \times R_+ \to R^{n\times m}.$$

Our aim is to design a feedback control $u(x(\delta_t), r(\delta_t), t)$ so that the controlled hybrid SDDE

$$dx(t) = \big(f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), x(t-h), r(t), t)dw(t) \tag{7.4}$$

becomes stable, where $\tau > 0$ and

$$\delta_t = [t/\tau]\tau, \tag{7.5}$$

in which $[t/\tau]$ is the integer part of t/τ, and $u : R^n \times S \times R_+ \to R^n$.

We observe that the feedback control $u(x(\delta_t), r(\delta_t), t)$ is designed based on the discrete-time state observations $x(0), x(\tau), x(2\tau), \cdots$, and discrete-time mode observations
$r(0), r(\tau), r(2\tau), \cdots$ as well, though the given hybrid SDE (7.3) is of continuous-time. In this chapter we impose the following *local Lipschitz conditions*.

Assumption 7.2.1. *Assume that the coefficients f and g are all locally Lipschitz continuous and obey linear growth condition(see e.g. [45–47, 62]).*

1. Local Lipschitz Condition. *For each integer $k \geq 1$ there is a positive constant L_k such that*

$$|f(x, y, i, t) - f(\bar{x}, \bar{y}, i, t)| \bigvee |g(x, y, i, t) - g(\bar{x}, \bar{y}, i, t)| \leq L_k(|x - \bar{x}| + |y - \bar{y}|) \tag{7.6}$$

for those $x, y, \bar{x}, \bar{y} \in R^n$ with $|x| \bigvee |y| \bigvee |\bar{x}| \bigvee |\bar{y}| \leq k$ and $(i, t) \in S \times R_+$.

2. Linear Growth Condition. *There is a constant $L > 0$ such that*

$$|f(x, y, i, t)| \bigvee |g(x, y, i, t)| \leq L(|x| + |y|) \tag{7.7}$$

for all $(x, y, i, t) \in R^n \times R^n \times S \times R_+$.

Therefore, we can show that

$$f(0, 0, i, t) = 0, \qquad g(0, 0, i, t) = 0 \tag{7.8}$$

for all $(i, t) \in S \times R_+$.

This is of course for the stability purpose of this chapter. For a technical reason, we require a global Lipschitz condition on the controller function u. More precisely, we impose the following hypothesis.

Assumption 7.2.2. *Assume that there exists a positive constant K such that*

$$|u(x,i,t) - u(y,i,t)| \le K|x-y| \tag{7.9}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$. Moreover,

$$u(0,i,t) = 0 \tag{7.10}$$

for all $(i,t) \in S \times R_+$.

Once again, condition (7.10) is for the stability purpose of this chapter. We also see that Assumption 7.2.2 implies the following linear growth condition on the controller function

$$|u(x,i,t)| \le K|x| \tag{7.11}$$

for all $(x,i,t) \in R^n \times S \times R_+$.

We observe that equation (7.3) is in fact a stochastic differential equation with several delays (SDDE). If we define the delay $\zeta_1 : [0,\infty) \to [0,\tau]$, $\zeta_2 : [0,\infty) \to [0,\infty]$ by

$$\zeta_1(t) = t - k\tau \quad for \quad k\tau \le t < (k+1)\tau, \quad k = 0,1,2,\dots,$$

$$\zeta_2(t) = h,$$

then the equation (7.4) can be written as

$$\begin{aligned} dx(t) =& \big(f(x(t), x(t-\zeta_2(t)), r(t), t) + u(x(t-\zeta_1(t)), r(t-\zeta_1(t)), t\big)dt \\ &+ g(x(t), x(t-h), r(t), t)dw(t). \end{aligned} \tag{7.12}$$

Therefore, under Assumptions (7.2.1) and (7.2.2), the SDDE (7.12) (namely the controlled system (7.4) has a unique solution $x(t)$ such that $\mathbb{E}\left(\sup_{-h\le t<\infty} |x(t)|^2\right) < \infty$ (see e.g. [62]). When studying on the stability problem of this SDDE (7.12), it is natural to consider employing a Lyapunov function and using Razumikhin-type theorem [51]. This method has been widely used in the proof of stability problems of SDFEs (stochastic differential functional equations). However, this method can only be applied to prove the p-th moment exponential stability of nonlinear delay systems. Therefore, we develop our new theory by employing a Lyapunov functional to study on some more kinds of stability problem of nonlinear delay system in the following.

7.3 Asymptotic Stabilization

For our stabilization purpose related to the controlled system (7.4) we will use a Lyapunov functional on the segments $\{\hat{x}_t : -2\tau^* \le t \le 0\} = \varphi \in C([-2\tau^*, 0]; R^n)$, where $\tau^* = h \vee \tau$, and $\hat{r}_t := \{r(t+s) : -2\tau^* \le s \le 0\}$ for $t \ge 0$ without losing consistency and generality. For $\hat{x}_t$ and $\hat{r}_t$ to be well defined for $0 \le t < 2\tau^*$, we set $x(s) = \varphi \in C([-2\tau^*, 0]; R^n)$, $r(s) = r_0$ for $-2\tau^* \le s \le 0$. The Lyapunov functional used in this chapter will be of the form

$$\begin{aligned} V(\hat{x}_t, \hat{r}_t, t) &= U(x(t), r(t), t) + \int_{t-h}^{t} x(s)^T P(r(t)) x(s) ds \\ &+ \theta \int_{t-\tau}^{t} \int_{s}^{t} \Big[\tau |f(x(v), x(v-h), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^2 \\ &+ |g(x(v), x(v-h), r(v), v)|^2 \Big] dv ds \end{aligned} \tag{7.13}$$

for $t \ge 0$, where $P(r(t))$ are symmetric positve-define matrices and θ is a positive number to be determined later and we set

$$f(x, y, i, s) = f(x, y, i, 0), \quad u(x, i, s) = u(x, i, 0), \quad g(x, y, i, s) = g(x, y, i, 0)$$

for $(x, y, i, s) \in R^n \times R^n \times S \times [-2\tau^*, 0)$ and $(x, i, s) \in R^n \times S \times [-2\tau^*, 0)$. We also require $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$, the family of non-negative functions $U(x, i, t)$ defined on $(x, i, t) \in R^n \times S \times R_+$ which are continuously twice differentiable in x and once in t. For $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$, let us define $\mathcal{L}U : R^n \times R^n \times S \times R_+ \to R$ by

$$\begin{aligned} \mathcal{L}U(x, y, i, t) &= U_t(x, i, t) + U_x(x, i, t)[f(x, y, i, t) + u(x, i, t)] \\ &+ \tfrac{1}{2}\text{trace}[g^T(x, y, i, t) U_{xx}(x, i, t) g(x, y, i, t)] + \sum_{j=1}^{N} \gamma_{ij} U(x, j, t), \end{aligned} \tag{7.14}$$

where

$$U_t(x, i, t) = \frac{\partial U(x, i, t)}{\partial t}, \quad U_x(x, i, t) = \Big(\frac{\partial U(x, i, t)}{\partial x_1}, \cdots, \frac{\partial U(x, i, t)}{\partial x_n} \Big),$$

and

$$U_{xx}(x, i, t) = \Big(\frac{\partial^2 U(x, i, t)}{\partial x_i \partial x_j} \Big)_{n \times n}.$$

Let us put forward a new assumption on U.

Assumption 7.3.1. *Assume that there is a function* $\forall i \in \mathbb{S}$, $U \in C^{2,1}(R^n \times R_+; R_+)$ *and three positive numbers* λ_1, λ_2 *and* λ_3 *such that*

$$\mathcal{L}U(x,y,i,t) + \lambda_1 |U_x(x,i,t)|^2 \le -\lambda_2 |x|^2 + \lambda_3 |y|^2 \tag{7.15}$$

for all $(x,i,t) \in R^n \times S \times R_+$, $(x,y,i,t) \in R^n \times R^n \times S \times R_+$.

Again, let us recall a useful lemma.

Lemma 7.3.2.

$$For \quad any \quad t \ge 0, \quad v > 0 \quad and \quad i \in S,$$

$$\mathbb{P}(r(s) \ne i \quad for \quad some \quad s \in [t, t+v] | r(t) = i) \le 1 - e^{-\bar{\gamma} v},$$

in which

$$\bar{\gamma} = \max_{i \in S}(-\gamma_{ii}).$$

We can now state our first result.

Theorem 7.3.3. *Let Assumptions 7.2.1, 7.2.2 and 7.3.1 hold. Assume that there exist positive-define symmetric matrices* $P_i (i \in S)$ *such that* $\lambda_2 > \lambda_{PM} := \max_{i \in S} \lambda_{\max}(P_i)$,
$\lambda_3 \le \lambda_{Pm} := \min_{i \in S} \lambda_{\min}(P_i)$. *Set*

$$\theta = \frac{2K^2}{\lambda_1}\big(1 + 8(1 - e^{-\frac{\bar{\gamma}}{4K}})\big). \tag{7.16}$$

If $\tau > 0$ *is sufficiently small for*

$$\lambda_2 > \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau}) + \theta\tau(4\tau + 2)L^2 + 4\theta\tau^2 K^2 + \lambda_{PM}, \tag{7.17}$$

$$\lambda_3 \le \lambda_{Pm} - \theta\tau(4\tau + 2)L^2 \quad and \quad \tau \le \frac{1}{4K}, \tag{7.18}$$

then the controlled system (7.4) is H_∞*-stable in the sense that*

$$\int_0^\infty \mathbb{E}\,|x(s)|^2 ds < \infty. \tag{7.19}$$

for every initial data $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ *and* $r_0 \in S$.

Proof. Fix any $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$. Regarding the solution $x(t)$ of equation (7.4) as an Itô process and apply the generalized Itô formula (see Theorem 2.5.4 or [50, 62]) to $U(x(t), r(t), t)$, we can get

$$\begin{aligned} dU(x(t), r(t), t) = \Big(&U_t(x(t), r(t), t) + U_x(x(t), r(t), t)[f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)] \\ &+ \tfrac{1}{2}\text{trace}[g^T(x(t), x(t-h), r(t), t)U_{xx}(x(t), r(t), t)g(x(t), x(t-h), r(t), t)] \\ &+ \sum_{j=1}^{N} \gamma_{r(t),j} U(x(t), j, t)\Big)dt + dM(t). \end{aligned}$$

On the other hand, the fundamental theory of calculus shows

$$\begin{aligned} &d\Big(\int_{t-\tau}^{t}\int_{s}^{t} \Big[\tau|f(x(v), x(v-h), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^2 + |g(x(v), x(v-h), r(v), v)|^2\Big]dvds\Big) \\ &= \Big(\tau\Big[\tau|f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^2 + |g(x(t), x(t-h), r(t), t)|^2\Big] \\ &- \int_{t-\tau}^{t} \Big[\tau|f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^2 + |g(x(s), x(s-h), r(s), s)|^2\Big]ds\Big)dt. \end{aligned}$$

Then combining these two equalities and using the generalized Itô formula to the Lyapunov functional defined by (7.13) yields

$$dV(\hat{x}_t, \hat{r}_t, t) = LV(\hat{x}_t, \hat{r}_t, t)dt + dM(t) \tag{7.20}$$

for $t \geq 0$, where $M(t)$ is a continuous martingale with $M(0) = 0$ and

$$\begin{aligned} &LV(\hat{x}_t, \hat{r}_t, t \\ &= U_t(x(t), r(t), t) + U_x(x(t), r(t), t)[f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)] \\ &+ \tfrac{1}{2}\text{trace}[g^T(x(t), x(t-h), r(t), t)U_{xx}(x(t), r(t), t)g(x(t), x(t-h), r(t), t)] \\ &+ \sum_{j=1}^{N} \gamma_{r(t),j} U(x(t), j, t) + x(t)^T P(r(t))x(t) - x(t-h)^T P(r(t))x(t-h) \\ &+ \theta\tau\Big[\tau|f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^2 + |g(x(t), x(t-h), r(t), t)|^2\Big] \\ &- \theta\int_{t-\tau}^{t} \Big[\tau|f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^2 + |g(x(s), x(s-h), r(s), s)|^2\Big]ds. \end{aligned} \tag{7.21}$$

Recalling (7.14), we can re-write (7.21) as

$$LV(\hat{x}_t, \hat{r}_t, t))$$

$$
\begin{aligned}
&= \mathbb{E}\,(\mathcal{L}U(x(t), r(t), t) - U_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(t), t)\\
&+ u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)]\\
&+ x(t)^T P(r(t))x(t) - x(t-h)^T P(r(t))x(t-h)\\
&+ \theta\tau\Big[\tau|f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^2 + |g(x(t), x(t-h), r(t), t)|^2\Big]\\
&- \theta\int_{t-\tau}^{t}\Big[\tau|f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)|^2 + |g(x(s), x(s-h), r(s), s)|^2\Big]ds.
\end{aligned}
\tag{7.22}
$$

But, by Assumption 7.2.2 and Lemma 7.3.2,

$$
\begin{aligned}
&\mathbb{E}\,(-U_x(x(t), r(t), t)[u(x(t), r(t), t) - u(x(\delta_t), r(t), t) + u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)])\\
\le&\mathbb{E}\,(\lambda_1|U_x(x(t), r(t), t)|^2\\
&+ \frac{1}{4\lambda_1}|u(x(t), r(t), t) - u(x(\delta_t), r(t), t) + u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^2)\\
\le&\mathbb{E}\,(\lambda_1|U_x(x(t), r(t), t)|^2 + \frac{K^2}{2\lambda_1}|x(t) - x(\delta_t)|^2 + \frac{1}{2\lambda_1}|u(x(\delta_t), r(t), t) - u(x(\delta_t), r(\delta_t), t)|^2)\\
\le&\mathbb{E}\,(\lambda_1|U_x(x(t), r(t), t)|^2 + \frac{K^2}{2\lambda_1}|x(t) - x(\delta_t)|^2) + \frac{2K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau})\mathbb{E}\,(|x(\delta_t)|^2)\\
\le&\mathbb{E}\,(\lambda_1|U_x(x(t), r(t), t)|^2 + [\frac{K^2}{2\lambda_1} + \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau})]|x(t) - x(\delta_t)|^2) + \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau})\mathbb{E}\,(|x(t)|^2).
\end{aligned}
\tag{7.23}
$$

In addition, by Assumptions 7.2.1 and 7.2.2, we have

$$
\begin{aligned}
&\theta\tau\Big[\tau|f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)|^2 + |g(x(t), x(t-h), r(t), t)|^2\Big]\\
&+ x(t)^T P(r(t))x(t) - x(t-h)^T P(r(t))x(t-h)\\
\le&\theta\tau\Big[2\tau(L^2(|x(t)| + |x(t-h)|)^2 + K^2|x(\delta_t)|^2) + L^2(|x(t)| + |x(t-h)|)^2\Big]\\
&+ \lambda_{PM}|x(t)|^2 - \lambda_{Pm}|x(t-h)|^2\\
\le&\theta\tau\Big[2\tau(2L^2|x(t)|^2 + 2L^2|x(t-h)|^2 + K^2|x(\delta_t)|^2) + 2L^2|x(t)|^2 + 2L^2|x(t-h)|^2\Big]\\
&+ \lambda_{PM}|x(t)|^2 - \lambda_{Pm}|x(t-h)|^2\\
=&\big[\theta\tau(4\tau + 2)L^2 + \lambda_{PM}\big]|x(t)|^2 + \big[\theta\tau(4\tau + 2)L^2 - \lambda_{Pm}\big]|x(t-h)|^2 + 2\theta\tau^2K^2|x(\delta_t)|^2\\
\le&\big[\theta\tau(4\tau + 2)L^2 + 4\theta\tau^2K^2 + \lambda_{PM}\big]|x(t)|^2 + \big[\theta\tau(4\tau + 2)L^2 - \lambda_{Pm}\big]|x(t-h)|^2\\
&+ 4\theta\tau^2K^2|x(t) - x(\delta_t)|^2.
\end{aligned}
\tag{7.24}
$$

Substituting (7.23) and (7.24) yields

$$
\mathbb{E}\,(LV(\hat{x}_t,\hat{r}_t, t)) \le \mathbb{E}\,(\mathcal{L}U(x(t), r(t), t) + \lambda_1|U_x(x(t), r(t), t)|^2
$$

$$
\begin{aligned}
&+ \Big(\frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau}) + \theta\tau(4\tau+2)L^2 + 4\theta\tau^2K^2 + \lambda_{PM}\Big)|x(t)|^2 \\
&+ \big[\theta\tau(4\tau+2)L^2 - \lambda_{Pm}\big]|x(t-h)|^2 \\
&+ \Big(\frac{K^2}{2\lambda_1} + \frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau}) + 4\theta\tau^2K^2\Big)|x(t)-x(\delta_t)|^2 \\
&- \theta\int_{t-\tau}^{t}\Big[\tau|f(x(s),x(s-h),r(s),s) + u(x(\delta_s),r(s),s)|^2 \\
&+ |g(x(s),x(s-h),r(s),s)|^2\Big]ds). \qquad (7.25)
\end{aligned}
$$

By Assumption 7.3.1, it follows that

$$
\begin{aligned}
\mathbb{E}\,(LV(\hat{x}_t,\hat{r}_t,t)) \le \mathbb{E}\,(-\lambda|x(t)|^2 &+ \Big(\frac{K^2}{2\lambda_1} + \frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau}) + 4\theta\tau^2K^2\Big)|x(t)-x(\delta_t)|^2 \\
&- \theta\int_{t-\tau}^{t}\Big[\tau|f(x(s),x(s-h),r(s),s) + u(x(\delta_s),r(s),s)|^2 \\
&+ |g(x(s),x(s-h),r(s),s)|^2\Big]ds), \qquad (7.26)
\end{aligned}
$$

where

$$
\lambda = \lambda(\theta,\tau) := \lambda_2 - \frac{4K^2}{\lambda_1}(1-e^{-\bar{\gamma}\tau}) - \theta\tau(4\tau+2)L^2 - 4\theta\tau^2K^2 - \lambda_{PM}. \qquad (7.27)
$$

Clearly that $t-\delta_t \le \tau$ for all $t \ge 0$, we can prove from (7.4) that

$$
\begin{aligned}
&\mathbb{E}\,|x(t)-x(\delta_t)|^2 \\
\le& 2\mathbb{E}\int_{\delta_t}^{t}\Big[\tau|f(x(s),x(s-h),r(s),s) + u(x(\delta_s),r(\delta_s),s)|^2 \\
&+ |g(x(s),x(s-h),r(s),s)|^2\Big]ds. \qquad (7.28)
\end{aligned}
$$

If we now choose

$$
\theta = \frac{2K^2}{\lambda_1}\big(1 + 8(1-e^{-\frac{\bar{\gamma}}{4K}})\big) \quad \text{and} \quad \tau \le \frac{1}{4K}. \qquad (7.29)
$$

It then follows from (7.26) and (7.28) that

$$
\mathbb{E}\,(LV(\hat{x}_t,\hat{r}_t,t)) \le -\lambda\mathbb{E}\,|x(t)|^2, \qquad (7.30)
$$

and by condition (7.17) we have $\lambda > 0$. Hence, we have

$$
\mathbb{E}\,(V(\hat{x}_t,\hat{r}_t,t)) \le C_1 - \lambda\int_0^t \mathbb{E}\,|x(s)|^2 ds, \qquad (7.31)
$$

and for $(\varphi, r_0, t_0) \in C([-2\tau^*, 0]; R^n) \times S \times [-2\tau^*, 0]$ we have

$$\begin{aligned} C_1 &= V(\varphi, \hat{r}_0, 0) \\ &\le U(\varphi, r_0, 0) + h\lambda_{PM}||\varphi||^2 + 4\theta\tau^2[(2\tau+1)L^2 + 2\tau K^2]||\varphi||^2, \end{aligned} \tag{7.32}$$

C_1 is a positive number. It follows from (7.31) immediately that

$$\int_0^\infty \mathbb{E}\,|x(s)|^2 ds \le C_1/\lambda.$$

This implies the desired assertion (7.19). □

Theorem 7.3.4. *Under the same assumptions of Theorem 7.3.3 and Theorem 7.3.3, the solution of the controlled system (7.4) satisfies*

$$\lim_{t\to\infty} \mathbb{E}\,|x(t)|^2 = 0$$

for every initial data $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ *and* $r_0 \in S$*. That is the controlled system (7.4) is asymptotically stable in mean square.*

Proof. Fix any $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$. By the Itô formula, we have

$$\begin{aligned} \mathbb{E}\,(|x(t)|^2) = \|\varphi\|^2 + \mathbb{E}\int_0^t \Big(&2x(s)[f(x(s), x(s-h), r(s), s) + u(x(\delta_s), r(\delta_s), s)] \\ &+ |g(x(s), x(s-h), r(s), s)|^2\Big)dt \end{aligned}$$

for all $t \ge 0$. Under Assumptions 7.2.1 and 7.2.2, we show that

$$\mathbb{E}\,|x(t)|^2 \le \|\varphi\|^2 + C\int_0^t \mathbb{E}\,|x(s)|^2 ds + C\int_0^t \mathbb{E}\,|x(s-h)|^2 ds + C\int_0^t \mathbb{E}\,|x(s) - x(\delta_s)|^2 ds, \tag{7.33}$$

where, and in the remaining part of this chapter, C denotes a positive constant that may change from line to line but its special form is of no use. For any $s \ge 0$, there is a unique integer $v \ge 0$ for $s \in [v\tau, (v+1)\tau)$. Moreover, $\delta_z = v\tau$ for $z \in [v\tau, s]$. It follows from (7.4) that

$$\begin{aligned} &x(s) - x(\delta_s) = x(s) - x(v\tau) \\ &= \int_{v\tau}^s [f(x(z), x(z-h), r(z), z) + u(x(v\tau), r(v\tau), z)]dz \end{aligned}$$

$$+\int_{v\tau}^{s} g(x(z), x(z-h), r(z), z)dw(z).$$

By Assumptions 7.2.1 and 7.2.2, we can derive

$$\begin{aligned}
&\mathbb{E}\,|x(s)-x(\delta_s)|^2 \\
\leq &3(\tau+1)L^2\int_{v\tau}^{s}\mathbb{E}\,|x(z)+x(z-h)|^2dz+3\tau^2K^2\mathbb{E}\,|x(v\tau)|^2 \\
\leq &6(\tau+1)L^2[\int_{\delta_s}^{s}\mathbb{E}\,|x(z)|^2dz+\int_{\delta_s}^{s}\mathbb{E}\,|x(z-h)|^2dz] \\
&+6\tau^2K^2(\mathbb{E}\,|x(s)|^2+\mathbb{E}\,|x(s)-x(\delta_s)|^2).
\end{aligned}$$

Noting that $6\tau^2K^2<1$ by condition (7.17), we hence have

$$\begin{aligned}
\mathbb{E}\,|x(s)-x(\delta_s)|^2 \leq &\frac{6(\tau+1)L^2}{1-6\tau^2K^2}[\int_{\delta_s}^{s}\mathbb{E}\,|x(z)|^2dz+\int_{\delta_s}^{s}\mathbb{E}\,|x(z-h)|^2dz] \\
&+\frac{6\tau^2K^2}{1-6\tau^2K^2}\mathbb{E}\,|x(s)|^2. \qquad (7.34)
\end{aligned}$$

Substituting this into (7.33) yields

$$\begin{aligned}
\mathbb{E}\,|x(t)|^2 \leq &\|\varphi\|^2+C\int_0^t\mathbb{E}\,|x(s)|^2ds+C\int_0^t\mathbb{E}\,|x(s-h)|^2ds \\
&+C\int_0^t\int_{\delta_s}^{s}[\mathbb{E}\,|x(z)|^2+\mathbb{E}\,|x(z-h)|^2]dzds. \qquad (7.35)
\end{aligned}$$

But, it is easy to show that

$$\int_0^t\mathbb{E}\,|x(s-h)|^2ds=\int_{-h}^{t-h}\mathbb{E}\,|x(z)|^2dz,$$

$$\begin{aligned}
&\int_0^t\int_{\delta_s}^{s}\mathbb{E}\,|x(z)|^2dzds\leq\int_0^t\int_{s-\tau}^{s}\mathbb{E}\,|x(z)|^2dzds \\
\leq &\int_{-\tau}^{t}\mathbb{E}\,|x(z)|^2\int_z^{z+\tau}dsdz\leq\tau\int_{-\tau}^{t}\mathbb{E}\,|x(z)|^2dz
\end{aligned}$$

and

$$\begin{aligned}
&\int_0^t\int_{\delta_s}^{s}\mathbb{E}\,|x(z-h)|^2dzds\leq\int_0^t\int_{s-\tau}^{s}\mathbb{E}\,|x(z-h)|^2dzds \\
=&\int_0^t\int_{s-\tau-h}^{s-h}\mathbb{E}\,|x(y)|^2dyds\leq\int_{-\tau-h}^{t-h}\mathbb{E}\,|x(y)|^2\int_{y+h}^{y+h+\tau}dsdy
\end{aligned}$$

$$\leq \tau \int_{-\tau-h}^{t-h} \mathbb{E}\,|x(y)|^2 dy.$$

Substituting these into (7.35) and then applying Theorem 7.3.3, it is easy to obtain that

$$\mathbb{E}\,|x(t)|^2 \leq C \quad \forall t \geq 0. \tag{7.36}$$

By the Itô formula, we have

$$\begin{aligned}
&\mathbb{E}\,|x(t_2)|^2 - \mathbb{E}\,|x(t_1)|^2 \\
=&\mathbb{E} \int_{t_1}^{t_2} \Big(2x(t)[f(x(t), x(t-h), r(t), t) + u(x(\delta_t), r(\delta_t), t)] + |g(x(t), x(t-h), r(t), t)|^2\Big) dt
\end{aligned}$$

for any $0 \leq t_1 < t_2 < \infty$. Using (7.36) and Assumptions 7.2.1 and 7.2.2, we can then easily show that

$$|\mathbb{E}\,|x(t_2)|^2 - \mathbb{E}\,|x(t_1)|^2| \leq C(t_2 - t_1).$$

That is, $\mathbb{E}\,|x(t)|^2$ is uniformly continuous in t on R_+. It then follows from (7.19) that $\lim_{t\to\infty} \mathbb{E}\,|x(t)|^2 = 0$ as required. $\Box$

7.4 Exponential Stability

In this section, we will discuss the exponential stability of the system (7.1) by feedback controls. Before stating our result, let us impose another condition.

Assumption 7.4.1. *Assume that there is a pair of positive numbers c_1 and c_2 such that*

$$c_1|x|^2 \leq U(x, i, t) \leq c_2|x|^2 \tag{7.37}$$

for all $(x, i, t) \in R^n \times S \times R_+$.

Theorem 7.4.2. *Let Assumptions 7.2.1, 7.2.2, 7.3.1, 7.4.1 and Lemma 7.3.2 hold. Let $\tau > 0$ be sufficiently small for (7.17) and (7.18) to hold and recall that*

$$\theta = \frac{2K^2}{\lambda_1}\big(1 + 8(1 - e^{-\frac{\bar{\gamma}}{4K}})\big)$$

and

$$\lambda = \lambda_2 - \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau}) - \theta\tau(4\tau + 2)L^2 - 4\theta\tau^2K^2 - \lambda_{PM}$$

(so $\lambda > 0$). Then the solution of the controlled system (7.4) satisfies

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}\,|x(t)|^2) \le -\gamma \tag{7.38}$$

and

$$\limsup_{t\to\infty} \frac{1}{t} \log(|x(t)|) \le -\frac{\gamma}{2} \quad a.s. \tag{7.39}$$

for every initial data $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$, where $\gamma > 0$ is the unique root to the following equation

$$2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_3) + 2\tau\gamma e^{(2\tau+h)\gamma}(H_2 + \tau H_3) + \gamma(c_2 + h\lambda_{PM}) = \lambda, \tag{7.40}$$

$$H_1 = 4\theta\tau^2(L^2 + K^2) + 2\theta\tau L^2 + \frac{24\theta\tau^4 K^4}{1 - 6\tau^2 K^2}, \quad H_2 = 2\theta\tau L^2(2\tau + 1), \tag{7.41}$$

$$H_3 = \frac{24\theta\tau^2(\tau + 1)K^2 L^2}{1 - 6\tau^2 K^2}. \tag{7.42}$$

Proof. For any $s \ge 0$, there is a unique integer $v \ge 0$ for $s \in [v\tau, (v+1)\tau)$. Moreover, $\delta_z = v\tau$ for $z \in [v\tau, s]$. From the previous section, we have

$$\begin{aligned}\mathbb{E}\,|x(s) - x(\delta_s)|^2 \le & \frac{6(\tau+1)L^2}{1 - 6\tau^2K^2}[\int_{\delta_s}^s \mathbb{E}\,|x(z)|^2 dz + \int_{\delta_s}^s \mathbb{E}\,|x(z-h)|^2 dz] \\ & + \frac{6\tau^2K^2}{1 - 6\tau^2K^2}\mathbb{E}\,|x(s)|^2. \end{aligned} \tag{7.43}$$

Moreover, by the generalized Itô formula, we can prove

$$\mathbb{E}\left[e^{\gamma t} V(\hat{x}_t, \hat{r}_t, t)\right] = V(\varphi, \hat{r}_0, 0) + \mathbb{E}\int_0^t e^{\gamma z}[\gamma V(\hat{x}_z, \hat{r}_z, z) + LV(\hat{x}_z, \hat{r}_z, z)]dz$$

for $t \ge 0$. Using (7.30), (7.32) and (7.37), we get

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \le C_1 + \int_0^t e^{\gamma z}[\gamma \mathbb{E}\,(V(\hat{x}_z, \hat{r}_z, z)) - \lambda \mathbb{E}\,|x(z)|^2]dz. \tag{7.44}$$

Define

$$\bar{V}(\hat{x}_t, \hat{r}_t, t) := \int_{t-h}^t x(s)^T P(r(t)) x(s) ds$$

$$+\theta \int_{t-\tau}^t \int_s^t \Big[\tau |f(x(v), x(v-h), r(v), v) + u(x(\delta_v), r(\delta_v), v)|^2$$

$$+|g(x(v),x(v-h),r(v),v)|^2\Big]dvds. \tag{7.45}$$

By (7.13) and (7.37), we have

$$\mathbb{E}\,(V(\hat{x}_z,\hat{r}_z,z)) \le c_2\mathbb{E}\,|x(z)|^2 + \mathbb{E}\,(\bar{V}(\hat{x}_z,\hat{r}_z,z)). \tag{7.46}$$

Moreover, by Assumptions 7.2.1 and 7.2.2,

$$\begin{aligned}
&\mathbb{E}\,(\bar{V}(\hat{x}_z,\hat{r}_z,z))\\
&\le h\lambda_{PM}\mathbb{E}\,|x(z)|^2\\
&+\theta\tau\int_{z-\tau}^{z}\Big[(4\tau(L^2+K^2)+2L^2)\mathbb{E}\,|x(v)|^2+2L^2(2\tau+1)\mathbb{E}\,|x(v-h)|^2\\
&+4\tau K^2\mathbb{E}\,|x(v)-x(\delta_v)|^2\Big]dv.
\end{aligned} \tag{7.47}$$

By Theorem 7.3.3, we see that $\mathbb{E}\,(\bar{V}(\hat{x}_z,\hat{r}_z,z))$ is bounded on $z\in[0,2\tau^*]$. For $z\ge 2\tau^*$, by (7.43), we have

$$\begin{aligned}
\mathbb{E}\,(\bar{V}(\hat{x}_z,\hat{r}_z,z)) \le& h\lambda_{PM}\mathbb{E}\,|x(z)|^2 + H_1\int_{z-\tau}^{z}\mathbb{E}\,|x(v)|^2dv + H_2\int_{z-\tau}^{z}\mathbb{E}\,|x(v-h)|^2dv\\
&+H_3\int_{z-\tau}^{z}\int_{\delta_v}^{v}(\mathbb{E}\,|x(y)|^2+\mathbb{E}\,|x(y-h)|^2)dydv.
\end{aligned} \tag{7.48}$$

where both H_1, H_2 and H_3 defined by (7.70). But

$$\begin{aligned}
&\int_{z-\tau}^{z}\int_{\delta_v}^{v}(\mathbb{E}\,|x(y)|^2+\mathbb{E}\,|x(y-h)|^2)dydv \le \int_{z-\tau}^{z}\int_{v-\tau}^{v}(\mathbb{E}\,|x(y)|^2+\mathbb{E}\,|x(y-h)|^2)dydv\\
&\le \tau\int_{z-2\tau}^{z}(\mathbb{E}\,|x(y)|^2+\mathbb{E}\,|x(y-h)|^2)dy = \tau\int_{z-2\tau}^{z}\mathbb{E}\,|x(y)|^2dy + \tau\int_{z-2\tau}^{z}\mathbb{E}\,|x(y-h)|^2)dy.
\end{aligned} \tag{7.49}$$

We hence have

$$\begin{aligned}
\mathbb{E}\,(\bar{V}(\hat{x}_z,\hat{r}_z,z)) \le& h\lambda_{PM}\mathbb{E}\,|x(z)|^2 + (H_1+\tau H_3)\int_{z-2\tau}^{z}\mathbb{E}\,|x(y)|^2dy\\
&+(H_2+\tau H_3)\int_{z-2\tau}^{z}\mathbb{E}\,|x(y-h)|^2dy.
\end{aligned} \tag{7.50}$$

Substituting this into (7.46) and then putting the resulting inequality further to (7.44), we get that, for $t\ge 2\tau^*$,

$$c_1e^{\gamma t}\mathbb{E}\,|x(t)|^2 \le C + \gamma(H_1+\tau H_3)\int_{2\tau}^{t}e^{\gamma z}\Big(\int_{z-2\tau}^{z}\mathbb{E}\,|x(y)|^2dy\Big)dz$$

$$+ \gamma(H_2 + \tau H_3) \int_{2\tau}^{t} e^{\gamma z}\Big(\int_{z-2\tau}^{z} \mathbb{E}\,|x(y-h)|^2 dy\Big) dz$$
$$- (\lambda - \gamma(c_2 + h\lambda_{PM})) \int_0^t e^{\gamma z}\mathbb{E}\,|x(z)|^2 dz. \quad (7.51)$$

But

$$\int_{2\tau}^{t} e^{\gamma z}\Big(\int_{z-2\tau}^{z} \mathbb{E}\,|x(y)|^2 dy\Big) dz \le \int_0^t \mathbb{E}\,|x(y)|^2 \Big(\int_{y}^{y+2\tau} e^{\gamma z} dz\Big) dy \le 2\tau e^{2\tau\gamma} \int_0^t e^{\gamma y}\mathbb{E}\,|x(y)|^2 dy.$$

and

$$\int_{2\tau}^{t} e^{\gamma z}\Big(\int_{z-2\tau}^{z} \mathbb{E}\,|x(y-h)|^2 dy\Big) dz \le \int_0^t \mathbb{E}\,|x(y-h)|^2 \Big(\int_{y}^{y+2\tau} e^{\gamma z} dz\Big) dy$$
$$\le 2\tau e^{2\tau\gamma} \int_0^t e^{\gamma y}\mathbb{E}\,|x(y-h)|^2 dy \le 2\tau e^{(2\tau+h)\gamma} \int_{-h}^{t-h} e^{\gamma y}\mathbb{E}\,|x(y)|^2 dy$$
$$\le C + 2\tau e^{(2\tau+h)\gamma} \int_0^t e^{\gamma y}\mathbb{E}\,|x(y)|^2 dy$$

Substituting this into (7.51) yields

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \le$$
$$C + \Big(2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_3) + 2\tau\gamma e^{(2\tau+h)\gamma}(H_2 + \tau H_3)$$
$$+ \gamma(c_2 + h\lambda_{PM}) - \lambda\Big) \int_0^t e^{\gamma z}\mathbb{E}\,|x(z)|^2 dz. \quad (7.52)$$

Recalling (7.69), we see

$$c_1 e^{\gamma t}\mathbb{E}\,|x(t)|^2 \le C \quad \forall t \ge 2\tau. \quad (7.53)$$

The assertion (7.67) follows immediately. Finally by [62, Theorem 8.8 on page 309], we can obtain the another assertion (7.68) from (7.53). The proof is therefore complete. □.

7.5 Corollaries

Assumption 7.5.1. *Assume that there is a function $U \in C^{2,1}(R^n \times S \times R_+; R_+)$ and three positive numbers λ_4, λ_5 and λ_6 such that*

$$\mathcal{L}U(x, y, i, t) \le -\lambda_4 |x|^2 + \lambda_5 |y|^2 \quad (7.54)$$

and

$$|U_x(x,i,t)| \le \lambda_6|x| \tag{7.55}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$ *and* $(x,i,t) \in R^n \times S \times R_+$.

Under this condition, if we choose a positive number $\lambda_1 < \lambda_4/\lambda_6^2$, then

$$\mathcal{L}U(x,y,i,t) + \lambda_1|U_x(x,i,t)|^2 \le -(\lambda_4 - \lambda_1\lambda_6^2)|x|^2 + \lambda_5|y|^2. \tag{7.56}$$

But this is a special condition of (7.15) if we set $\lambda_2 = \lambda_4 - \lambda_1\lambda_6^2$. In other words, we have shown that Assumption 7.5.1 implies Assumption 7.3.1. The following corollary states this.

Corollary 7.5.2. *All the theorems in Sections 3 and 4 hold if Assumption 7.3.1 is replaced by Assumption 7.5.1.*

In practice, the quadratic functions are widely used to be the Lyapunov functions. That is, we use $U(x,i,t) = x^T Q_i x$, where Q_i's are all symmetric positive-definite $n \times n$ matrices. In this case, Assumption 7.4.1 holds automatically with $c_1 = \min_{i\in S}\lambda_{\min}(Q_i)$ and $c_2 = \max_{i\in S}\lambda_{\max}(Q_i)$. Moreover, condition (7.55) holds as well with $\lambda_6 = 2\max_{i\in S}\|Q_i\|$. So all we need is to find Q_i's for (7.54) to hold. This gives us the following another assumption.

Assumption 7.5.3. *Assume that there are symmetric positive-definite matrices* $Q_i \in R^{n\times n}$ *($i \in S$) and two positive numbers* λ_4, λ_5 *such that*

$$\begin{aligned} 2x^T Q_i[f(x,y,i,t) + u(x,i,t)] + \mathrm{trace}[g^T(x,y,i,t)Q_i(x,i,t)g(x,y,i,t)] \\ + \sum_{j=1}^N \gamma_{ij}x^T Q_j x \le -\lambda_4|x|^2 + \lambda_5|y|^2, \end{aligned} \tag{7.57}$$

for all $(x,y,i,t) \in R^n \times R^n \times S \times R_+$ *and* $(x,i,t) \in R^n \times S \times R_+$.

The following corollary follows immediately from Theorem 7.4.2.

Corollary 7.5.4. *Let Assumptions 7.2.1, 7.2.2 and 7.5.3 hold. Set*

$$c_1 = \min_{i\in S}\lambda_{\min}(Q_i), \quad c_2 = \max_{i\in S}\lambda_{\max}(Q_i), \lambda_6 = 2\max_{i\in S}\|Q_i\|.$$

Choose $\lambda_1 < \lambda_4/\lambda_6^2$ *and then set* $\lambda_2 = \lambda_4 - \lambda_1\lambda_6^2$. *Let* $\tau > 0$ *be sufficiently small for (7.17) to hold and set*

$$\theta = \frac{2K^2}{\lambda_1}\big(1 + 8(1 - e^{-\frac{\bar{\gamma}}{4K}})\big)$$

and

$$\lambda = \lambda_2 - \frac{4K^2}{\lambda_1}(1 - e^{-\bar{\gamma}\tau}) - \theta\tau(4\tau + 2)L^2 - 4\theta\tau^2 K^2 - \lambda_{PM}$$

(so $\lambda > 0$). Then the assertions of Theorem 7.4.2 hold.

7.6 Example

Example 7.6.1. We first consider an unstable linear hybrid SDE

$$dx(t) = \big(A(r(t))x(t) + A_d(r(t))x(t-h)\big)dt + \big(B(r(t))x(t) + B_d(r(t))x(t-h)\big)dw(t) \tag{7.58}$$

on $t \geq 0$ with initial value $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$. Here $h = 0.1$, $w(t)$ is a scalar Brownian motion; $r(t)$ is a Markov chain on the state space $S = \{1, 2\}$ with the generator

$$\Gamma = \begin{bmatrix} -2 & 2 \\ 1 & -1 \end{bmatrix};$$

and the system matrices are

$$A_1 = \begin{bmatrix} 0.9 & 3.2 \\ 4.05 & -5.02 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.94 & 6.93 \\ 6.02 & 2.01 \end{bmatrix},$$

$$A_{d1} = \begin{bmatrix} 0.1 & -0.2 \\ -0.05 & 0.02 \end{bmatrix}, \quad A_{d2} = \begin{bmatrix} 0.06 & 0.07 \\ -0.02 & -0.01 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1.98 & 3.04 \\ 0 & 1.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 2.01 & 1.04 \\ 2.08 & 2 \end{bmatrix}.$$

$$B_{d1} = \begin{bmatrix} 0.02 & -0.04 \\ 0 & 0.05 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0.01 & -0.04 \\ -0.08 & 0 \end{bmatrix}.$$

The computer simulation (Figure 7.1) shows this hybrid SDE is not mean square exponentially stable.

Let us now design a feedback control based on discrete time state and mode observations to stabilize the system. Assume that the controlled hybrid SDE has the form

$$dx(t) = [A(r(t))x(t) + A_d(r(t))x(t-h) + F(r(\delta_t))G(r(\delta_t))x(\delta_t)]dt$$

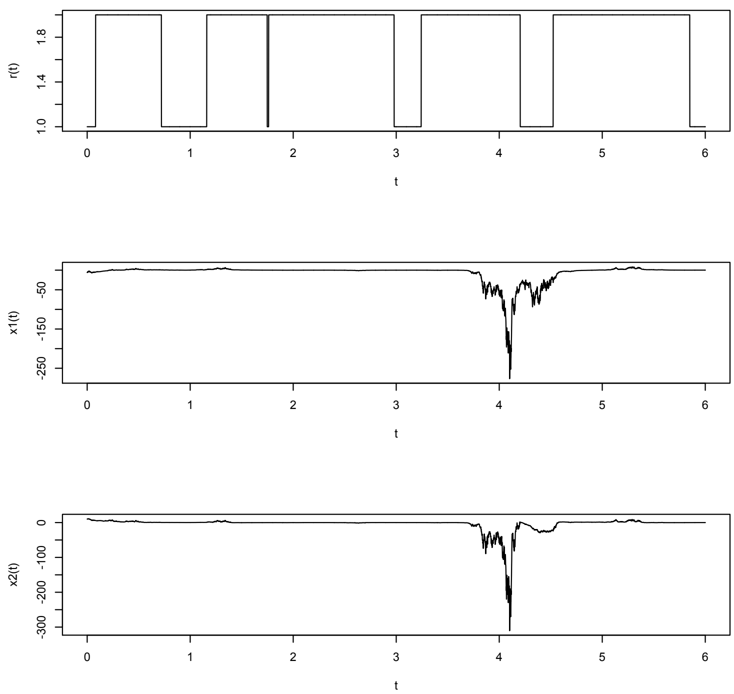

Figure 7.1: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (7.58) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -6$ and $x_2(0) = 10$.

$$+ \big(B(r(t))x(t) + B_d(r(t))x(t-h)\big)dw(t), \tag{7.59}$$

namely, our controller function has the form $u(x, i, t) = F_iG_ix$. Here, we assume that

$$G_1 = (-1.41, -1.4402), \quad G_2 = (3.1016, 1.9571),$$

and our aim is to seek for F_1 and F_2 in $R^{2\times 1}$ and then make sure τ is sufficiently small for this controlled SDE to be exponentially stable in mean square and almost surely as well. To apply Corollary 7.5.4, we observe that by Assumptions 7.2.1 and 7.2.2, it is easy to know $L = 8.0406$ and $K = 11.7523$. Then we need to verify Assumption 7.5.3. It is easy to see the left-hand-side term of (7.57) becomes $x^T\bar{Q}_ix$ $(i = 1, 2)$, where

$$\bar{Q}_i := Q_i(A_i + A_{di} + F_iG_i) + (A_i^T + A_{di}^T + G_i^TF_i^T)Q_i + B_i^TQ_iB_i + \sum_{j=1}^{2}\gamma_{ij}Q_j.$$

Let us now choose

$$Q_1 = \begin{bmatrix} 2.5048 & 0.9239 \\ 0.9239 & 3.1738 \end{bmatrix}, \quad Q_2 = \begin{bmatrix} 5.3836 & 3.0928 \\ 3.0928 & 3.3392 \end{bmatrix}.$$

and

$$F_1 = \begin{bmatrix} 5 \\ 3 \end{bmatrix}, \quad F_2 = \begin{bmatrix} 1 \\ -10 \end{bmatrix}.$$

We then have

$$\bar{Q}_1 = \begin{bmatrix} -14.9558 & -4.2385 \\ -4.2385 & -35.3341 \end{bmatrix}, \quad \bar{Q}_2 = \begin{bmatrix} -53.8195 & -36.4580 \\ -36.4580 & -30.9954 \end{bmatrix}.$$

Hence, $x^T \bar{Q}_i x \leq -80.6096|x|^2$. We also know that $\lambda_3 = 0.09$. In other words, (7.57) holds with $\lambda_4 = 80.6996$. We further compute the parameters specified in Corollary 7.5.4: $c_1 = 1.1041$, $c_2 = 7.6187$ and $\lambda_6 = 15.2374$. Choosing $\lambda_1 = 0.3$, we then have $\lambda_2 = 10.9561$. In addition, we choose $\lambda_P M = 5.6$ and $\lambda_P m = 5.42$. Then, conditions (7.17) and (7.18) becomes

$$10.9561 > 1841.554(1 - e^{-\tau}) + 69553.04\tau(4\tau + 2) + 594353.7\tau^2 + 5.6,$$

$$69553.04\tau(4\tau + 2) \leq 5.33 \quad \tau \leq 0.021.$$

These hold as long as $\tau < 0.0000383$. By Corollary 7.5.4, if we set F_i as above and make sure that $\tau < 0.0000383$, then the discrete-time-state-and-mode feedback controlled hybrid SDE (7.59) is exponentially stable in mean square and almost surely as well. The computer simulation (Figure 7.2) supports this result clearly.

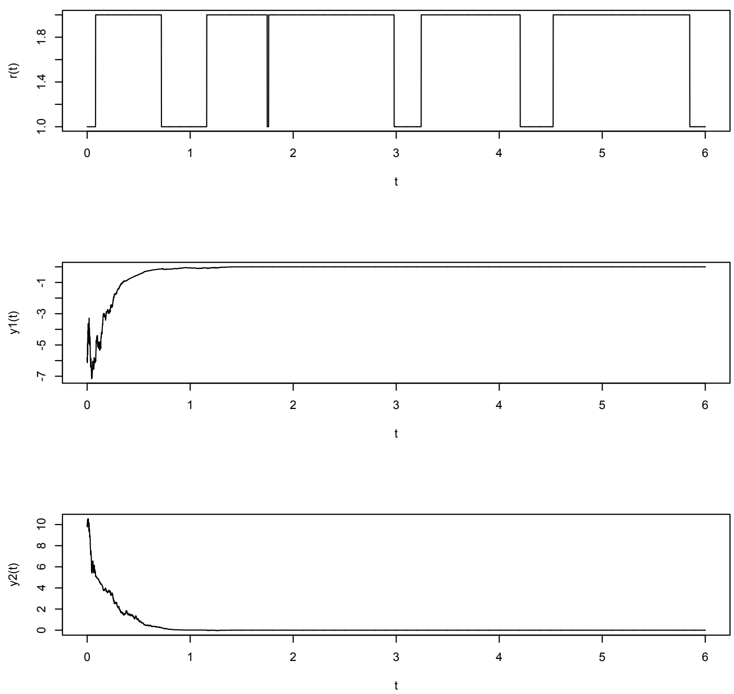

Figure 7.2: Computer simulation of the paths of $r(t)$, $x_1(t)$ and $x_2(t)$ for the hybrid SDE (7.59) using the Euler–Maruyama method with step size 10^{-6} and initial values $r(0) = 1$, $x_1(0) = -6$ and $x_2(0) = 10$.

Example 7.6.2. Let us now consider a nonlinear uncontrolled system (7.3). Given that its coefficients f and g satisfy the linear growth condition (7.6), we consider a linear controller function of the form $u(x, i, t) = D_i x$, where $D_i \in R^{n\times n}$ for all $i \in S$. That is, the controlled hybrid SDE has the form

$$dx(t) = \big(f(x(t), x(t-h), r(t), t) + D_{r(\delta_t)}x(\delta_t)\big)dt + g(x(t), x(t-h), r(t), t)dw(t). \tag{7.60}$$

It is easy to observe that Assumption 7.2.2 holds with $K = \max_{i\in S}\|D_i\|$. Let us now establish Assumption 7.5.3 in order to apply Corollary 7.5.4. We choose $Q_i = q_i I$, where $q_i > 0$ and I is the $n \times n$ identity matrix. We estimate the left-hand-side of (7.57):

$$2x^T Q_i[f(x, y, i, t) + u(x, i, t)] + \text{trace}[g^T(x, y, i, t)Q_i(x, i, t)g(x, y, i, t)] + \sum_{j=1}^{N}\gamma_{ij}x^T Q_j x$$

$$\leq 2q_iL|x|(|x|+|y|) + 2q_ix^TD_ix + q_iL^2(|x|+|y|)^2 + \sum_{j=1}^{N}\gamma_{ij}q_j|x|^2$$

$$\leq 2q_iL|x|^2 + q_iL|x|^2 + q_iL|y|^2 + 2q_ix^TD_ix + 2q_iL^2(|x|^2+|y|^2) + \sum_{j=1}^{N}\gamma_{ij}q_j|x|^2$$

$$= x^T\Big(q_i(3L+2L^2)I + q_i(D_i+D_i^T) + \sum_{j=1}^{N}\gamma_{ij}q_jI\Big)x + y^T(q_i(L+2L^2))y. \tag{7.61}$$

Assume that the following linear matrix inequalities

$$q_i(3L+2L^2)I + Y_i + Y_i^T + \sum_{j=1}^{N}\gamma_{ij}q_jI < 0 \tag{7.62}$$

have their solutions of $q_i > 0$ and $Y_i \in R^{n\times n}$ $(i \in S)$. Set $D_i = q_i^{-1}Y_i$ and

$$-\lambda_4 = \max_{i\in S}\lambda_{\max}\Big(q_i(3L+2L^2)I + Y_i + Y_i^T + \sum_{j=1}^{N}\gamma_{ij}q_jI\Big) \tag{7.63}$$

$$\lambda_5 = \max_{i\in S}\lambda_{\max}\big(q_i(L+2L^2)I\big). \tag{7.64}$$

We then see Assumption 7.5.3 is satisfied. The corresponding parameters in Corollary 7.5.4 becomes

$$c_1 = \min_{i\in S} q_i, \quad c_2 = \max_{i\in S} q_i, \quad \lambda_6 = 2c_2.$$

Choose $\lambda_1 < \lambda_4/\lambda_6^2$ and then set $\lambda_2 = \lambda_4 - \lambda_1\lambda_6^2$. Let $\tau > 0$ be sufficiently small for (7.17) and (7.18) to hold. Then, by Corollary 7.5.4, the controlled system (7.60) is exponentially stable in mean square.

For example, if we have the same Markov chain as that in Example 7.6.1, and set

$$f(t) = \begin{bmatrix} 0.2\sin x_2(t) & 1 \\ 0 & 0.5\cos x_1(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.01\cos x_2(t) & 0 \\ 0.02 & 0.01\sin x_1(t) \end{bmatrix} x(t-h),$$

$$g(t) = \begin{bmatrix} 0.8\sin 2x_2(t) & 0 \\ -1 & 0.8\cos 2x_1(t) \end{bmatrix} x(t) + \begin{bmatrix} 0.01\cos 2x_2(t) & 0.03 \\ 0 & 0.01\sin 2x_1(t) \end{bmatrix} x(t-h).$$

and $h = 0.1$. Hence we observe that $L = 1.4434$. Then subsitute into the linear matrix inequalities (7.62) and get their solutions $q_1 = 1$, $q_2 = 2$,

$$Y_1 = \begin{bmatrix} -6 & 1 \\ 0 & -8 \end{bmatrix} \quad \text{and} \quad Y_2 = \begin{bmatrix} -9 & 4 \\ -2 & -10 \end{bmatrix}.$$

Then we get

$$D_1 = \begin{bmatrix} -6 & 1 \\ 0 & -8 \end{bmatrix} \quad \text{and} \quad D_2 = \begin{bmatrix} -4.5 & 2 \\ -1 & -5 \end{bmatrix}.$$

Hence $K = 8.1359$. We also observe that $\lambda_4 = 0.77$, $\lambda_5 = 11.2204$, $c_1 = 1$, $c_2 = 2$ and $\lambda_6 = 4$. Choose $\lambda_1 = 0.02$ and set $\lambda_2 = 0.45$. Let $\tau < 6.54 \times 10^{-6}$, then by Corollary 7.5.4,the controlled system (7.60) is exponentially stable in mean square.

7.7 Generalization

In this section, we will discuss a more general case. Consider an unstable hybrid SDDE

$$dx(t) = f(x(t), x(t-h(t)), r(t), t)dt + g(x(t), x(t-h(t)), r(t), t)dw(t), \quad (7.65)$$

where $t \geq 0$, $x(t) \in R^n$ is the state, $w(t) = (w_1(t), \cdots, w_m(t))^T$ is an m-dimensional Brownian motion, $r(t)$ is a continuous-time Markov chain. But h is now defined on the entire R_+, namely $h : R_+ \to [0, \bar{\tau}]$, and we assume that h is differentiable and its derivative is bounded by a constant $\bar{h} \in [0, 1)$, that is $\dot{h}(t) \leq \bar{h}$, for any t. In addition, SDDE (7.65) has initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-\bar{\tau}, 0]; R^n)$ (such that $\mathbb{E}\,||\xi||^2 < \infty$) and $r(0) = r_0 \in S$ at time zero.

We aim to design a feedback control $u(x(\delta_t), r(\delta_t), t)$ so that the controlled hybrid SDDE

$$dx(t) = \big(f(x(t), x(t-h(t)), r(t), t) + u(x(\delta_t), r(\delta_t), t)\big)dt + g(x(t), x(t-h(t)), r(t), t)dw(t) \quad (7.66)$$

becomes H_∞-stable, asymptotically stable and exponentially stable in mean square, where $\tau > 0$, δ_t defined as (7.5) and $u : R^n \times S \times R_+ \to R^n$.

By employing the same Lyapunov functional as (7.13), all the results still hold in this chapter. But Theorem (7.4.2) experiences changes in some coefficients. We state this result in the following theorem.

Theorem 7.7.1. *Let Assumptions 7.2.1, 7.2.2, 7.3.1, 7.4.1 and Lemma 7.3.2 hold. Let $\tau > 0$ be sufficiently small for (7.17) and (7.18) to hold. Recall that θ is* *defined as (7.16) and λ is defined as (7.27) (so $\lambda > 0$). Then the solution of the controlled system (7.66) satisfies*

$$\limsup_{t\to\infty} \frac{1}{t}\log(\mathbb{E}\,|x(t)|^2) \le -\gamma \tag{7.67}$$

and

$$\limsup_{t\to\infty} \frac{1}{t}\log(|x(t)|) \le -\frac{\gamma}{2} \quad a.s. \tag{7.68}$$

for every initial data $x_0 = \varphi \in C^b_{\mathcal{F}_0}([-2\tau^*, 0]; R^n)$ and $r_0 \in S$, where $\gamma > 0$ is the unique root to the following equation

$$2\tau\gamma e^{2\tau\gamma}(H_1 + \tau H_3) + \frac{2\tau e^{(2\tau+\tau^*)\gamma}}{1-\bar{h}}(H_2 + \tau H_3) + \gamma(c_2 + h\lambda_{PM}) = \lambda, \tag{7.69}$$

$$H_1 = 4\theta\tau^2(L^2 + K^2) + 2\theta\tau L^2 + \frac{24\theta\tau^4 K^4}{1 - 6\tau^2 K^2}, \quad H_2 = 2\theta\tau L^2(2\tau + 1), \tag{7.70}$$

$$H_3 = \frac{24\theta\tau^2(\tau+1)K^2L^2}{1 - 6\tau^2K^2}. \tag{7.71}$$

7.8 Summary

In this chapter, we have proved the stabilization of continuous-time hybrid stochastic delay differential equations by *feedback controls based on discrete-time state and mode observations*. The stabilities here mainly referred to the H_∞ stability, mean squared asymptotic stability and mean squared exponential stability. Moreover, we also managed to build the upper bound on the duration τbetween two consecutive state observations. We achieved these by employing Lyapunov functional.

Chapter 8

Conclusions and Future work

In this thesis, we have developed our new theory about stabilization of hybrid systems by feedback controls based discrete time observations, as well as of hybrid delay systems.

In Chapter 3, we have shown that unstable linear hybrid SDEs can be stabilized by the linear feedback controls based on the discrete-time state observations. We then generalize the theory to a class of nonlinear hybrid SDEs. Making full use of their special features, we have established a better bound on duration τ. However, by the method used in Chapter 3, we can only proved the mean square exponential stability of controlled system (3.2) and the upper bound on τ is not very sharp.

Therefore, we extend our discussion in Chapter 4 to shown that continuous-time hybrid stochastic differential equations can be stabilized by feedback controls based on discrete-time state observations in some other types by the method of Lyapunov functionals. The stabilities discussed in this chapter includes exponential stability and asymptotic stability, in both mean square and almost sure sense, as well as the H_∞ stability. One of the significant contributions here is the better bound obtained on the duration τ between two consecutive state observations.

In Chapter 5 and 6, we have proved that unstable hybrid SDEs can be stabilized by feedback controls based on discrete-time state and mode observations by simular methods used in Chapter 3 and 4 respectively. We also managed to build the corresponding upper bound on the duration τ between two consecutive state observations.

In chapter 7, we have extended our study to continuous-time hybrid stochastic delay differential equations by *feedback controls based on discrete-time state and mode observations*. The stabilities here mainly referred to the H_∞ stability, mean squared asymptotic stability and mean squared exponential stability. Moreover, we also managed to build the upper bound on the duration τbetween two consecutive state observations. We achieved these by employing Lyapunov functional as well.

However, there are still some problems worthy of consideration after this thesis. For instance, we only study on the hybrid stochastic systems in this thesis, stabilization problems of some other types of stochastic systems by discrete-time feedback controls may also be considered. In addition, we mainly study on the stability in mean square sense in this thesis, we can extend our discussion to the sability in pth-moment sense as well. At the same time, we can also consider the stabilization problem of SDEs with Markov process which has infinite number of states.

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