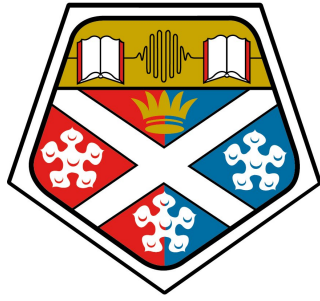




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Reduced-order modelling of thin-film flow over a
spinning disk

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This thesis is submitted to the University of Strathclyde for the
degree of Doctor of Philosophy in the Faculty of Science

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Signed: Laura Milne

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Abstract

This thesis examines the mathematical modelling and analysis of a thin-film flow of Newtonian liquid over a topographically-patterned disk rotating with constant angular velocity. A novel reduced-order model is derived using the weighted residual integral boundary-layer method, which includes the effects of inertia, viscosity, centrifugation, Coriolis force and capillarity. The predictions of this model are tested extensively against direct numerical simulations, with excellent agreement being found throughout. For the case of a planar disk, the fully-developed state exhibits four characteristic spatial regions: the spin-up, inner, middle and outer regions. A small random noisy forcing at the inlet is found to have a strong effect on the dynamics of the system. Analytical insight is obtained into the behaviour of the system in the far field. A linear stability analysis is performed in the inner, middle and outer regions, which are found to be neutrally stable, convectively unstable and stable, respectively. For the case of a topographically-patterned disk, a family of topographies is introduced with parameters controlling the asymmetry, smoothness, amplitude and spatial frequency of the topography. The amplitude of the topography has a significant effect on the flow, even for small amplitudes, while the other three parameters have only a minor effect. Finally, control of the free surface towards a prescribed target free-surface profile in the outer region using the topography of the

disk as the control mechanism is studied. At leading order, an exact solution for the topography is obtained and recovers the target free-surface profile exactly when substituted into the transient problem. At first order, a perturbation solution is obtained for the topography. Hence, this solution produces small errors in the free surface computed from the transient problem compared to the target free-surface profile.

Nomenclature

$\bar{z}$	Rescaled vertical coordinate
δ	Far-field ordering parameter
ϵ	Aspect ratio
η	Film thickness
κ	Interfacial curvature
λ	Inverse Weber number
$\mathcal{T}$	Smoothing parameter
$\mathcal{W}$	Weight
ω	Spatial frequency of topography
Σ	Far-field growth rate
σ	Growth rate
θ	Azimuthal coordinate
$\tilde{\gamma}$	Surface tension

NOMENCLATURE

$\tilde{\nu}$	Kinematic viscosity
$\tilde{\Omega}$	Angular velocity
$\tilde{\rho}$	Density
$\tilde{Q}_{\text{jet}}$	Volume flux
ζ	Far-field outer region ordering parameter
A	Maximum amplitude of inlet forcing
a	Amplitude of topography
B	Far-field topography of disk
b	Topography of disk
c	Asymmetry parameter
E	Far-field film thickness
F	Far-field radial flux
f	Radial flux
G	Far-field azimuthal flux
g	Azimuthal flux
H	Far-field free surface
h	Free surface
H_{targ}	Target free-surface profile
H_N	Initial condition for the far-field free surface

NOMENCLATURE

h_N	Initial condition for the free surface
K	Far-field wave number
k	Wave number
p	Pressure
r	Radial coordinate
R_{disk}	Disk radius
R_{in}	Radial position of transition between inner and middle regions
R_{jet}	Jet radius
R_{out}	Radial position of transition between middle and outer regions
R_{SU}	Radial position of transition between spin-up and inner regions
r_s	Radial position of the self-sustained inception of waves
$S(t)$	Integral measure of shear stress at the surface of the disk
T	Far-field time
t	Time
u	Radial velocity
v	Azimuthal velocity
w	Vertical velocity
$W(t)$	Interfacial waviness
z	Vertical coordinate

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Chapter 1

Introduction

This thesis examines the mathematical modelling and analysis of a thin-film flow of Newtonian liquid over a topographically-patterned disk rotating with constant angular velocity. In Sections 1.1–1.6 we introduce relevant background theory that underpins the research described within this thesis and review the existing literature. In Section 1.7 we provide an outline of the contents of this thesis.

1.1 Free-surface flows

The term “free-surface flows” refers to a specific category of flows in which at least one of the bounding surfaces is free to move (i.e. at least one surface is not bounded by a solid wall).

Free-surface flows are a widely studied topic of scientific interest, as reflected by the review articles that summarise the numerous papers in the field (see, for example, Craster and Matar [28], Oron *et al.* [83], Ruyer-Quil *et al.* [99], and Sellier [107]).



(a)



(b)

Figure 1.1: (a) Kayakers on the river Tummel near Pitlochry. Image courtesy of Veronica Milne. (b) An avalanche of snow and ice in the Himalayas. Reproduced from C. Ancey, “Snow avalanches”, in *Geomorphological Fluid Mechanics*, eds. N. J. Balmforth, and A. Provenzale, Springer Nature, 2001, with permission from Springer Nature.

Free-surface flows occur in many natural settings and have numerous industrial applications. Natural settings in which they occur include avalanches [3, 51], the flow of lava [42], and the flow of water in rivers [9], oceans [78], and tsunamis [39]. Figure 1.1 shows kayakers on the river Tummel near Pitlochry and an avalanche of snow and ice in the Himalayas. Understanding such free-surface flows in nature can help us to understand the dangers of recirculating hydraulic features in rivers and to prepare for natural disasters. Industrial applications include chemical process engineering [63] and microfluidics [120, 145] along with a wealth of applications in marine [22],

automotive [90], and aerospace [61, 143] industries. Free-surface flows are also a topic of interest in mathematics as these flows can exhibit highly complex spatio-temporal dynamics, characterised by a vast range of different dynamical regimes including periodic motion and interacting waves, as well as chaotic motion [87].

Free-surface flows are complicated to analyse because the position of the free surface or surfaces is unknown and must be determined as part of the solution of the governing equations, and boundary and initial conditions of the system. In this thesis, we investigate free-surface flows of incompressible Newtonian liquids, which are governed by the incompressible Navier–Stokes equations [7]:

$$\nabla \cdot \mathbf{u} = 0, \tag{1.1}$$

$$\frac{D\mathbf{u}}{Dt} = \mathbf{g} - \frac{1}{\rho} \nabla p + \nu \nabla^2 \mathbf{u}. \tag{1.2}$$

Here ρ and ν are the constant density and kinematic viscosity of the liquid, respectively. The variables p and $\mathbf{u}$ are functions of space $\mathbf{x}$ and time t and represent the pressure of the liquid and the liquid velocity, respectively. The vector $\mathbf{g}$ denotes the acceleration due to gravity and $D/Dt = \partial/\partial t + \mathbf{u} \cdot \nabla$ is the material derivative. The continuity equation (1.1) enforces the conservation of mass in the liquid and the momentum equation (1.2) represents the conservation of momentum in the liquid. The Navier–Stokes equations (1.1)–(1.2) are a system of nonlinear partial differential equations (PDEs). They are renowned for being analytically intractable and analytical solutions are only attainable for a small number of special cases. Direct numerical simulations (DNS) of the Navier–Stokes equations generally have a prohibitively high computational cost. However, there are simplifying assumptions that can be applied to make the Navier–Stokes equations more tractable to analytical

or numerical analysis; we will introduce some of these in the context of thin-film flows in Section 1.3.

1.2 Thin-film flows

Thin-film flows are a subcategory of the free-surface flows discussed in Section 1.1. A film is said to be thin if the ratio between its characteristic thickness and characteristic length is small, as will be discussed in Section 1.3. This ratio will subsequently be referred to as the aspect ratio of the film. Thin-film flows are a topic of scientific interest and have been the subject of many scientific review articles (see, for example, Craster and Matar [28], Davis [29], Oron *et al.* [83], and Ruyer-Quil *et al.* [99]). Thin-film flows occur in many natural settings and have numerous industrial applications. Natural settings in which they occur include the flow of ice sheets [6, 17], the flow of lava [42], and the flow of rainwater down surfaces [54]. Industrial applications include coating surfaces [67, 90, 134], heat and mass transfer [73], microfluidics [120, 145], and inkjet printers [68, 125]. Figure 1.2 shows a schematic of spray coating on a tubular membrane.

Thin-film flows of incompressible Newtonian liquids are of course also governed by the Navier–Stokes equations (1.1)–(1.2), however, due to the thin nature of the film, simplifications can be made to improve the tractability of the equations. These techniques will be discussed in Section 1.3.

1.3 Reduced-order modelling of thin-film flows

Reduced-order models are powerful tools in fluid mechanics. Given the general analytical intractability of the Navier–Stokes equations and the often prohibitively

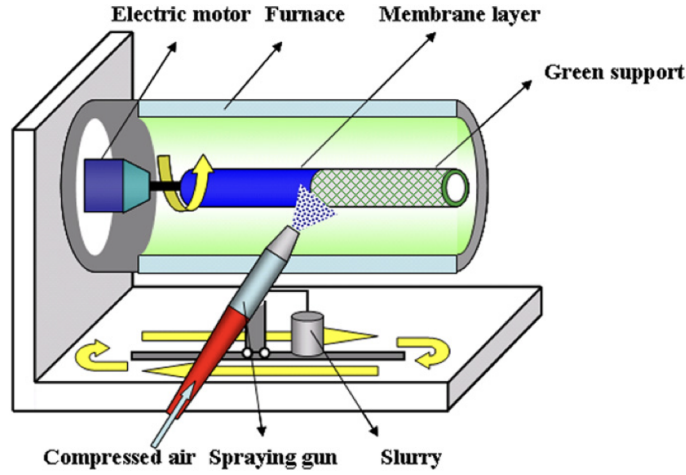


Figure 1.2: Schematic of spray coating on a tubular membrane. Reprinted from Journal of Membrane Science, Vol. 415, Z. Liu, G. Zhang, X. Dong, W. Jiang, W. Jin, and N. Xu, “Fabrication of asymmetric tubular mixed-conducting dense membranes by a combined spin-spraying and co-sintering process”, p. 314, Copyright 2012, with permission from Elsevier.

high computational cost of DNS, reduced-order models that provide an accurate description of the dynamics of the system at a significantly reduced computational cost are highly desirable. Reduced-order models also often enable the determination of analytical and asymptotic expressions for the behaviour of the system, which would be impossible to directly obtain from the Navier–Stokes equations. A situation in which reduced-order models are commonly derived is for thin-film flows, in which the small aspect ratio of the film is exploited, allowing standard asymptotic methods to be applied. This is usually obtained following a nondimensionalisation, which involves condensing the dimensional dependence of a system into dimensionless numbers.

A liquid film is said to be thin when the characteristic cross-stream length scale H is much smaller than the characteristic streamwise length scale L . This disparity in

length scales can be quantified by the aspect ratio ϵ ,

$$\epsilon = \frac{H}{L} \ll 1, \quad (1.3)$$

which is small. If a film is thin then significant mathematical simplification can be applied to the Navier–Stokes equations (1.1)–(1.2), making them more tractable for analysis. In particular, the smallness of the aspect ratio ϵ allows for the application of standard asymptotic methods, including the quantified neglect of suitably small terms. This approximation is usually referred to as either the lubrication or thin-film approximation.

In this thesis, we will develop a reduced-order model for thin-film flow over a spinning disk. However, due to the additional geometric complexity of that situation, we begin by introducing the techniques we will use in the context of a two-dimensional gravity-driven flow down an inclined plane. In particular, we consider the flow of a thin film of Newtonian liquid of constant density ρ and constant kinematic viscosity ν down a plane inclined at angle β to the horizontal. Gravitational acceleration $\mathbf{g}$ acts vertically downwards. We work in a Cartesian coordinate system (x, y) , in which the x -axis coincides with the surface of the plane, and x and y are the dimensionless streamwise and cross-stream coordinates, respectively. The liquid-gas interface is located at $y = h(x, t)$, where t denotes dimensionless time. The liquid is bounded above by an inviscid, hydrodynamically-passive gas at constant pressure p_a and the liquid-gas interface has constant surface tension γ . Figure 1.3 shows a schematic representation of the problem. Note that the reduced-order modelling methods discussed in this section can be extended to many other situations, for example, three-dimensional flow, and to account for a topographically-patterned substrate.

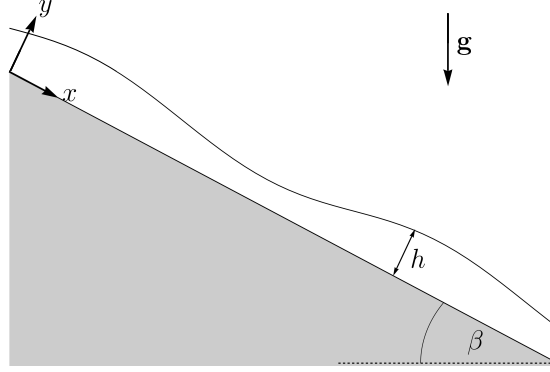


Figure 1.3: Schematic representation of two-dimensional gravity-driven flow down an inclined plane.

The 2D flow shown in Figure 1.3 is governed by the Navier–Stokes equations (1.1)–(1.2) up to second order in ϵ , which take the dimensionless form

$$u_x + v_y = 0, \quad (1.4)$$

$$\epsilon(u_t + uu_x + vv_y) = -\epsilon p_x + \epsilon^2 u_{xx} + u_{yy} + 1, \quad (1.5)$$

$$\epsilon^2(v_t + uv_x + vv_y) = -p_y + \epsilon v_{yy} - \cot \beta, \quad (1.6)$$

where u and v are the streamwise and cross-stream velocities, respectively.

At the surface of the plane $y = 0$, we have the no-slip and impermeability conditions

$$u = v = 0. \quad (1.7)$$

At the free surface $y = h$, we have the normal stress condition, up to second order in ϵ ,

$$-p + 2\epsilon[v_y - h_x(u_y + \epsilon v_x)] = \Gamma h_{xx}, \quad (1.8)$$

where $\Gamma = \gamma/[\rho\nu^{4/3}(g \sin \beta)^{1/3}]$ is the Kapitza number (which measures the relative

importance of surface tension compared to viscous effects), the tangential stress condition up to second order in ϵ ,

$$u_y = \epsilon^2 (4h_x u_x - v_x), \quad (1.9)$$

and kinematic condition up to second order in ϵ ,

$$h_t + u h_x - v = 0. \quad (1.10)$$

As the pressure term appears as ϵp_x in (1.5), the pressure is only required to be known to order $O(\epsilon)$ for the equations to be valid to second order in ϵ . Hence, the cross-stream component of the Navier–Stokes equation (1.6) gives

$$p_y = \epsilon v_{yy} + \cot \beta. \quad (1.11)$$

Integrating (1.11) with respect to the cross-stream coordinate y from y to h subject to the normal and tangential stress conditions (1.8) and (1.9) yields an expression for the pressure to order $O(\epsilon)$, namely

$$p = -\Gamma h_{xx} + (y - h) \cot \beta - \epsilon (u_x + u_x|_{y=h}). \quad (1.12)$$

Substituting (1.12) into the streamwise component of the Navier–Stokes equation (1.5), and using the continuity equation (1.4) to rewrite v_y in terms of u , yields the “boundary layer” equation

$$\epsilon (u_t + u u_x + v u_y) = u_{yy} + 1 - \epsilon (\cot \beta h_x - \Gamma h_{xxx}) + \epsilon^2 [2u_{xx} + (u_x|_{y=h})_x]. \quad (1.13)$$

This terminology is commonly used in literature relating to weighted residual modelling [100, 101, 102, 105, 141] (which we will discuss in detail in Section 1.3.4) even though the flow is not a classical boundary-layer flow. The boundary-layer equation (1.13) is subject to the no-slip condition (1.7), the tangential stress condition (1.9), and the kinematic condition (1.10). We define the dimensionless volume flux $q(x, t)$ by

$$q(x, t) = \int_0^h u \, dy. \quad (1.14)$$

As will prove to be convenient later, the kinematic condition (1.10) can be rewritten in integral form,

$$h_t + q_x = 0. \quad (1.15)$$

There are several techniques we can use to make progress on the system (1.13)–(1.15), which we will discuss in the following subsections.

1.3.1 Gradient expansion method

This method first involves performing a classic perturbation expansion on the boundary-layer equation (1.13), subject to the boundary conditions (1.9) and (1.7), to obtain an asymptotic solution for the streamwise velocity u . The flux q given by (1.14) is then obtained using the expression for the streamwise velocity u , and substituted into the kinematic condition (1.15), allowing for the derivation of a single closed equation for the evolution of the film thickness h . This method was introduced in the seminal work of Benney [8] and is sometimes referred to as the gradient expansion method; this terminology is especially common in the context of weighted residual methods [56, 99, 100, 101, 105]. This leads to a single evolution equation for the dimensionless film thickness h , which has come to be known as the Benney

equation,

$$h_t + \left[\frac{1}{3}h^3 + \epsilon \left(\frac{2}{15}h^6h_x - \frac{1}{3}\cot\beta h^3h_x + \frac{1}{3}\Gamma h h_{xxx} \right) \right]_x = 0. \quad (1.16)$$

In the square brackets in (1.16), the first and third terms correspond to the streamwise and cross-stream components of gravity, respectively, the second term corresponds to inertia, and the fourth term corresponds to surface tension. The streamwise and cross-stream components of gravity have a convective and stabilising effect, respectively, the inertial term has a destabilising effect, and the surface tension term has a stabilising effect. Note that in the original work of Benney [8] the term corresponding to surface tension was omitted as it appears at third order in ϵ (if the Kapitza number $\Gamma = O(1)$). However, subsequent work has shown that this term plays a critical role in the stability and dynamics of thin-film flows and should be retained at first order in ϵ [28, 83]. Hence, it is common to assume that the Kapitza number Γ is of order $O(\epsilon^{-2})$ so that the term corresponding to surface tension is strong enough to enter equation (1.16) at first order in ϵ .

The Benney equation (1.16) correctly predicts the same instability threshold as the linearised Navier–Stokes equations [56]. However, nonlinear simulations of equation (1.16) show “blowup” (i.e. the film thickness becomes infinite) in a finite time [94] for moderate to large inertial effects. In this regime, the inertial term grows rapidly with the film thickness h and surface tension is no longer sufficient to counteract this growth. This blowup is an unphysical result that is not predicted by either DNS of the Navier–Stokes equations or experiments [70, 96, 103, 105], therefore, the Benney equation (1.16) is only valid close to the instability threshold.

1.3.2 Weakly nonlinear analysis

The Benney equation (1.16) can be further simplified by a weakly nonlinear analysis, in which perturbations around the steady solution of a flat film are assumed to be small. Here, we consider the case of a vertical wall (i.e. $\beta = \pi/2$) for simplicity. In particular, we consider small perturbations about the uniform steady state $h = 1$ and set $h = 1 + \epsilon \hat{h}(x, t)$ in the Benney equation (1.16), where $\hat{h}$ is the perturbation of the free surface. Retaining terms up to second order in ϵ yields

$$\epsilon \left(\hat{h}_t + \hat{h}_x \right) + \epsilon^2 \left(2\hat{h}\hat{h}_x + \frac{2}{15}\hat{h}_{xx} + \frac{1}{3}\Gamma\hat{h}_{xxxx} \right) = 0. \quad (1.17)$$

Equation (1.16) is then moved from the laboratory frame to a moving frame of reference, which removes the linear streaming flow. Specifically, the streamwise coordinate x and time t are rescaled according to $\hat{x} = x - t$ and $\hat{t} = \epsilon t$, respectively, yielding

$$\epsilon^2 \left(\hat{h}_{\hat{t}} + 2\hat{h}\hat{h}_{\hat{x}} + \frac{2}{15}\hat{h}_{\hat{x}\hat{x}} + \frac{1}{3}\Gamma\hat{h}_{\hat{x}\hat{x}\hat{x}\hat{x}} \right) = 0. \quad (1.18)$$

The streamwise coordinate $\hat{x}$, time $\hat{t}$ and film thickness $\hat{h}$ are then rescaled according to

$$\hat{x} = \sqrt{\frac{5\Gamma}{2}}X, \quad \hat{t} = \frac{75\Gamma}{4}T, \quad \hat{h} = \frac{1}{15}\sqrt{\frac{2}{5\Gamma}}H, \quad (1.19)$$

to scale out the coefficients in equation (1.18) and obtain

$$H_T + HH_X + H_{XX} + H_{XXXX} = 0, \quad (1.20)$$

which is known as the Kuramoto–Sivashinsky (K–S) equation. The K–S equation has been derived in the context of (and applied to) a variety of physical systems. Notable examples include flame propagation and combustion [80, 115], reaction-

diffusion systems [62], liquid films [116, 124], and control theory [40]. In the present fluid dynamical context, the second, third, and fourth terms in (1.20) correspond to the effects of gravity, inertia, and capillarity, respectively. Notably, only the second term (corresponding to gravitational effects) is nonlinear. However, it is important to emphasise that this equation is only valid for small perturbations about a uniform steady state. The K–S equation offers a relatively tractable model for both analytical and numerical analysis [40]. A feature of the K–S equation is that it can exhibit chaotic solutions [57, 87, 116].

1.3.3 Integral boundary-layer (IBL) method

When the effect of inertia is more significant, an improved reduced-order model is required to overcome the limitations of the Benney equation (1.16) (which exhibits blowup for moderate to large inertial effects). The integral boundary-layer (IBL) method is an alternative to the gradient expansion method described in Section 1.3.1. This method requires a closure condition for the velocity profiles.

The IBL method combines the thin-film approximation with the Kármán–Pohlhausen method, which, in this case, involves assuming a semi-parabolic profile for the streamwise velocity $u(x, y, t)$ that is given by the solution of the boundary-layer equation (1.13) at leading order in ϵ . The boundary-layer equation (1.13) is integrated with respect to the cross-stream coordinate y over the film thickness from 0 to h . In particular, this method yields a coupled system of equations for the dimensionless film thickness $h(x, t)$ and dimensionless volume flux $q(x, t)$ (1.14),

namely

$$h_t + q_x = 0, \quad (1.21)$$

$$\frac{5}{2} \frac{q}{h^2} - \frac{5}{6} h + \epsilon \left[\frac{5}{6} q_t + 2 \frac{q q_x}{h} + \left(\frac{5}{6} \cot \beta h - \left(\frac{q}{h} \right)^2 \right) h_x - \frac{5}{6} \Gamma h h_{xxx} \right] = 0. \quad (1.22)$$

Equations (1.21)–(1.22), derived using the IBL method, are sometimes referred to as the Shkadov model [109]. The IBL method offers an improvement over the gradient expansion method, as the solutions to equations (1.21)–(1.22) do not exhibit the unphysical blowup in finite time of the Benney equation (1.16). Note that equations (1.21)–(1.22) can be used as a primitive system of PDEs on which a gradient expansion can be performed, however, when a gradient expansion is performed on the flux q (i.e. setting $q = q_0(x, t) + \epsilon q_1(x, t) + \dots$) the Benney equation (1.16) is not recovered. As a result, this model incorrectly predicts a different instability threshold from that of the linearised Navier–Stokes equations [56]. This discrepancy arises as the assumption regarding the velocity profile is too restrictive, and the higher-order corrections to the profile have been neglected.

While the IBL method improves upon the Benney equation by eliminating the finite-time blowup and providing a more robust description of the flow, comparisons with DNS reveal that it still lacks sufficient accuracy in regimes where inertial and higher-order effects become important [56, 100, 105].

1.3.4 Weighted residual integral boundary-layer (WRIBL) method

A further improved reduced-order model is required to overcome the limitations of the IBL method, particularly its failure to predict the correct instability threshold. A

suitable method is the weighted residual integral boundary-layer (WRIBL) method, developed by Ruyer-Quil and Manneville [100], for which the resulting model does not experience unphysical blowup in finite time while also predicting the same instability threshold as the Benney equation (1.16) (and hence being in agreement with the linearised Navier–Stokes equations). The WRIBL method offers an improvement over the IBL method as it accounts for higher-order corrections to the velocity profile, as will be discussed in this subsection. The key components of this method involves a separation of variables, as well as the selection of suitable weight functions in the computation of the weighted integrals of the boundary-layer equation (1.13).

Ruyer-Quil and Manneville [100] formulated three reduced-order models: a first-order model, valid to order $O(\epsilon)$; a second-order model, valid to order $O(\epsilon^2)$; and a simplified second-order model, which serves as a hybrid between the previous two that retains viscous (but not inertial) effects to order $O(\epsilon^2)$. We will discuss each of these in turn.

1.3.4.1 First-order model

The leading-order solution for the streamwise velocity u , denoted by $u_0(x, t)$, from (1.13) is

$$u_0 = h^2 \left(\bar{y} - \frac{1}{2} \bar{y}^2 \right), \quad (1.23)$$

where $\bar{y} = y/h$ is the rescaled cross-stream coordinate that maps the film onto $0 \leq \bar{y} \leq 1$, from the balance of viscosity and the convective effect of gravity in (1.13). Following Ruyer-Quil and Manneville [100], we expand the streamwise velocity u as

$$u = a_0(x, t) f_0(\bar{y}) + \epsilon \sum_{n=1}^N a_n(x, t) f_n(\bar{y}), \quad (1.24)$$

corresponding to projecting u onto an appropriate set of $N + 1$ basis functions $f_n(\bar{y})$ for $n = 0, \dots, N$; the basis functions have coefficients $a_n(x, t)$. This is a separation of variables with the basis functions $f_n(\bar{y})$ accounting for the y dependence of the streamwise velocity u (and hence the cross-stream velocity v from the continuity equation (1.1)). Therefore, from the leading-order solution for u (1.23), a suitable choice of the leading-order basis function $f_0(\bar{y})$ is

$$f_0(\bar{y}) = \bar{y} - \frac{1}{2}\bar{y}^2. \quad (1.25)$$

Hence, a convenient choice for the $f_n(\bar{y})$ for $n = 0, \dots, N$ is

$$f_n(\bar{y}) = \bar{y}^{n+1} - \frac{n+1}{n+2}\bar{y}^{n+2}, \quad (1.26)$$

which satisfy the tangential stress condition (1.9) up to first order in ϵ (i.e. $(f_n)_{\bar{y}}(1) = 0$). This form of $f_n(\bar{y})$ automatically satisfies the no-slip (1.7) and tangential stress (1.9) conditions, while also reducing the number of unknown coefficients from seven to five.

In order to determine N , we consider the boundary-layer equation (1.13) truncated at first order in ϵ , which shows that the first-order terms from the streamwise velocity u only enter via the viscous term u_{yy} . Therefore, as the leading-order streamwise velocity u_0 is quadratic in $\bar{y}$, the inertial terms (such as uu_x) are at most quartic in $\bar{y}$. Hence, the first-order terms of the streamwise velocity u are a polynomial of degree at most 6, i.e. we require $N = 4$. In principle, the coefficients a_n can be determined by substituting the expression (1.24) into the boundary-layer equation (1.13), then solving the resulting equation order by order in powers of $\bar{y}$. However, while this is feasible at first order (and indeed this has been followed by Ruyer-Quil

and Manneville [100], Oron and Heining [85], and Wray *et al.* [140]), it is algebraically prohibitive at higher orders. Ruyer-Quil and Manneville [100] demonstrated that this process can be significantly simplified by utilising a weighted integral procedure. In particular, the evolution equation (in terms of the film thickness h and flux q) is determined by suitable weighted integrals, with weights $w_m(x, y, t)$ for $m = 0, \dots, M$, of the boundary-layer equation (1.13) with respect to the cross-stream coordinate y over the film thickness from 0 to h , specifically

$$\int_0^h w_m [\epsilon (u_t + uu_x + vu_y)] \, dy = \int_0^h w_m [u_{yy} + 1 - \epsilon (\cot \beta h_x - \Gamma h_{xxx})] \, dy. \quad (1.27)$$

This process yields $M + 1$ equations for $N + 1$ unknown coefficients a_n for $n = 0, \dots, N$. Therefore, in general, $N + 1$ weights are required (i.e. $M = N$) [100]. However, equation (1.27) shows that the unknown first-order coefficients a_n for $n = 1, \dots, N$ only enter via the leading-order viscous term u_{yy} . Hence, if this term can be written in a form that is independent of a_n for $n = 1, \dots, N$ then (1.27) becomes independent of the a_n and, consequently, of the f_n for $n = 1, \dots, N$. This would allow the weighted integral (1.27) to be computed directly without the need to compute the first-order coefficients a_n or determine the basis functions f_n for $n = 1, \dots, N$. Performing a double integration by parts on the leading-order viscous term u_{yy} yields

$$\begin{aligned} \int_0^h w_m u_{yy} \, dy &= [w_m u_y]_{y=h} - [w_m u_y]_{y=0} \\ &\quad - [(w_m)_y u]_{y=h} + [(w_m)_y u]_{y=0} + \int_0^h (w_m)_{yy} u \, dy, \end{aligned} \quad (1.28)$$

which can be further simplified by applying the no-slip (1.7) and tangential stress (1.9) conditions, eliminating the first and fourth terms on the right-hand side of

(1.28). The resulting expression for the leading-order viscous term is

$$\int_0^h w_m u_{yy} \, dy = -[w_m u_y]_{y=0} - [(w_m)_y u]_{y=h} + \int_0^h (w_m)_{yy} u \, dy. \quad (1.29)$$

Therefore, if a single weight w_0 satisfying $w_0(x, 0, t) = 0$, $(w_0)_y(x, h, t) = 0$ and $(w_0)_{yy}(x, y, t) = A$, where A is a (non-zero) constant, is chosen then the first two terms on the right-hand side of (1.29) are eliminated and the remaining integral can be rewritten in terms of the flux q ,

$$\int_0^h w_0 u_{yy} \, dy = Aq. \quad (1.30)$$

Hence, only the single weight w_0 is required to derive the first-order WRIBL model. Note that due to this choice of weight w_0 , the expression corresponding to the viscous term u_{yy} (1.30) is valid up to first order in ϵ as the flux q is exact. In the case $A = -1$, the weight function becomes $w_0 = f_0(\bar{y})$ and this is suitable for accounting for first-order terms without computation of the coefficients a_n for $n = 1, \dots, N$. This weighted integral formulation is essentially the Galerkin method, as the weights coincide with the basis functions. Concisely, the weight w_0 is given by the adjoint to the leading-order problem and in this case $w_0 = f_0(\bar{y})$ due to the self-adjoint nature of the operator $\partial^2/\partial y^2$ [56].

In practice, it is convenient to rewrite a_0 in terms of the flux q (a quantity that is physically measurable). Integrating (1.24) with respect to the cross-stream coordinate y over the film thickness from 0 to h and solving for a_0 yields

$$a_0 = \frac{q - \epsilon \sum_{n=1}^N a_n \int_0^h f_n(\bar{y}) \, dy}{\int_0^h f_0(\bar{y}) \, dy}. \quad (1.31)$$

Therefore, the streamwise velocity u can be rewritten as

$$u = \frac{3q}{h} \left(\bar{y} - \frac{1}{2} \bar{y}^2 \right) + \epsilon \left[\sum_{n=1}^N a_n(x, t) f_n(\bar{y}) - \frac{3}{h} \left(\bar{y} - \frac{1}{2} \bar{y}^2 \right) \sum_{n=1}^N a_n \int_0^h f_n(\bar{y}) \, d\bar{y} \right]. \quad (1.32)$$

The weighted integral (1.27) can be evaluated using the streamwise velocity u (1.32) to obtain

$$\frac{5}{2} \frac{q}{h^2} + \frac{5}{6} h + \epsilon \left[q_t + \frac{17}{7} \frac{qq_x}{h} + \left(\frac{5}{6} \cot \beta h - \frac{9}{7} \left(\frac{q}{h} \right)^2 \right) h_x - \frac{5}{6} \Gamma h h_{xxx} \right] = 0, \quad (1.33)$$

which, along with the kinematic condition (1.15), forms the first-order WRIBL model.

Note that equation (1.33) only differs from the corresponding equation obtained using the IBL method (1.22) by the coefficients of three terms (namely, the third, fourth, and sixth terms). The first-order WRIBL model does not experience blowup in finite time and recovers the Benney equation (1.16) when a gradient expansion is performed on the flux q . Therefore, the first-order WRIBL model predicts the same instability threshold as the Benney equation (1.16) and consequently the linearised Navier–Stokes equations [56, 100]. The first-order WRIBL model has been shown to have very good agreement with DNS and experiments [27, 84], however, effects such as viscous dispersion are not accounted for by this model.

1.3.4.2 Second-order model

The process of obtaining the second-order model is more algebraically involved than that of obtaining the first-order model. The method for obtaining it essentially

follows the same logic as the first-order model. For full details refer to Ruyer-Quil and Manneville [100], however, the key steps are as follows.

The second-order model requires the velocity u to be known to first order, unlike the first-order model, in which case the leading-order version is sufficient. Analogously to the first-order model, the second-order contributions can only enter via the viscous term u_{yy} , but this can be treated by using the same logic for the choice of weights w_m for $m = 0, \dots, M$. The first-order coefficients a_n for $n = 1, \dots, N$ (which are required for the second-order model) are obtained by substituting the first-order expansion of u (1.24) into the boundary-layer equation (1.13) and solving order by order in $\bar{y}$. The $f_n(\bar{y})$ in the solution for u can be reduced to three orthogonal basis functions $g_n(\bar{y})$ for $n = 0, 1, 2$ via a Gram-Schmidt orthogonalisation (note that $g_0(\bar{y}) = f_0(\bar{y})$), with coefficients b_n for $n = 0, 1, 2$. These basis functions are used as the weights w_m for $m = 0, 1, 2$ as this produces the model with the least algebra required (in a similar manner to the first-order model in Section 1.3.4.1). Therefore, the streamwise velocity u up to first order in ϵ (note that the ϵ dependence is contained within the coefficients b_n for $n = 0, 1, 2$) takes the form

$$u = b_0 g_0(\bar{y}) + b_1 g_1(\bar{y}) + b_2 g_2(\bar{y}). \quad (1.34)$$

Ruyer-Quil and Manneville [100] showed that it is convenient to define the coefficients b_n for $n = 0, 1, 2$ in terms of the flux q and two variables that describe the deviation from the leading-order semi-parabolic velocity profile, $s_1(x, t)$ and $s_2(x, t)$,

$$b_0 = 3 \frac{q - \epsilon s_1 - \epsilon s_2}{h}, \quad b_1 = 45 \frac{\epsilon s_1}{h}, \quad b_2 = 210 \frac{\epsilon s_2}{h}. \quad (1.35)$$

Therefore, the second-order model results in a system of four equations for four

unknowns, namely, the flux q , film thickness h , and two first-order velocity variables s_1 and s_2 . These four equations are lengthy and so are not reproduced here. However, they are given by equations (11) and (38)–(40) in Ruyer-Quil and Manneville [100]. The second-order model includes the effect of viscous dispersion, due to the inclusion of the second-order viscous terms in the boundary-layer equation (1.13), which was not accounted for by the first-order model (1.15) and (1.33).

The second-order model does not experience blowup in finite time and recovers the second-order Benney equation (see equations (5.13)–(5.14c) of Kalliadasis *et al.* [56]) when a gradient expansion is performed on the flux q and the first-order velocity variables s_1 and s_2 . However, the increased complexity (with four unknowns) of the equations limits its amenability to both analytical and numerical analysis. Hence, some further simplifications are often applied.

1.3.4.3 Simplified second-order model

The simplified second-order model is a hybrid between the first-order and second-order models described previously. In particular, viscous terms are included to second order in ϵ (e.g. effects of viscous dispersion are included) while inertial effects are only retained up to first order in ϵ . This significantly simplifies both the derivation of the reduced-order model and the equations themselves. Formally, this means the simplified second-order model is only consistent up to second order in ϵ when the Reynolds number $Re = O(\epsilon)$, but in practice it has been shown to be a good compromise between the first-order and second-order models, even for $Re = O(1)$, while retaining the effects of viscous dispersion.

As discussed above, the simplified second-order model only retains inertial effects up to first order in ϵ (and viscous terms up to second order in ϵ). Therefore, only the

leading-order form of the streamwise velocity u (1.32) with a single basis function f_0 is required. The leading-order viscous term u_{yy} , performing a double integration by parts, can be expressed by (1.28), which can be further simplified by applying the no-slip condition (1.7), to yield

$$\int_0^h w_m u_{yy} \, dy = [w_m u_y]_{y=h} - [w_m u_y]_{y=0} - [(w_m)_y u]_{y=h} + \int_0^h (w_m)_{yy} u \, dy. \quad (1.36)$$

In order to express the leading-order viscous term u_{yy} up to second order in ϵ with only the knowledge of the streamwise velocity profile to leading-order in ϵ , the weight must be chosen such that $w_0(x, 0, t) = 0$, $(w_0)_y(x, h, t) = 0$, and $(w_0)_{yy}(x, y, t) = A$, where A is a (non-zero) constant. With this choice of weight, the second and third terms on the right-hand side of (1.36) are eliminated. The remaining terms are the first and fourth terms on the right-hand side of (1.36), however, it is possible to evaluate these terms explicitly in terms of the position of the free surface h , flux q , and leading-order streamwise velocity profile. The first term on the right-hand side of (1.36) can be evaluated up to second order in ϵ using the tangential stress condition (1.9), which gives an expression for u_y at the free surface $y = h$. The fourth term on the right-hand side of (1.36) can be rewritten in terms of the flux q that is exact. As in Section 1.3.4.1, in the case $A = -1$, the weight is $w_0 = f_0(\bar{y})$.

The weighted integral of the boundary-layer equation (1.13) with respect to y over the film thickness from 0 to h , with weight $w_0 = f_0(\bar{y})$, can be evaluated to obtain

$$\begin{aligned} \frac{5}{2} \frac{q}{h^2} - \frac{5}{6} h + \epsilon \left[q_t + \frac{17}{7} \frac{qq_x}{h} + \left(\frac{5}{6} \cot \beta h - \frac{9}{7} \left(\frac{q}{h} \right)^2 \right) h_x - \frac{5}{6} \Gamma h h_{xxx} \right] \\ + \epsilon^2 \left[-4 \frac{qh_x^2}{h^2} + \frac{9}{2} \frac{q_x h_x}{h} + 6 \frac{qh_{xx}}{h} - \frac{9}{2} q_{xx} \right] = 0, \end{aligned} \quad (1.37)$$

which along with the kinematic condition (1.15) forms the simplified second-order

model.

The simplified second-order model does not recover the Benney equation (1.16) up to second order in ϵ , as it is no longer second-order accurate in the inertial terms [100]. However, this model is significantly more amenable to analytical and numerical analysis than the second-order model discussed in Section 1.3.4.2. Extensive testing of simplified second-order models has shown that they generally exhibit excellent agreement with both DNS and experiments; this includes for falling films on inclined planes (for both planar and topographically-patterned substrates) [23, 30, 85, 105, 139] and films on fibres [31, 102]. They generally provide a good balance between derivation complexity, model complexity and accuracy, and so are commonly used in practice (see, for example, [23, 30, 31, 85, 102, 105, 139, 140, 141]). In this thesis, we will derive a simplified second-order model for thin-film flow over a topographically-patterned spinning disk.

1.4 Analysis of thin-film flows

In this thesis, we utilise asymptotic and numerical methods, and linear stability analysis to analyse the resulting system of PDEs arising from the reduced-order model we derive for our problem. We will briefly outline these numerical and linear stability analysis methods in this section.

1.4.1 Numerical methods

In general, it is challenging to obtain explicit analytical solutions for each field variable for reduced-order models. Numerical methods are often used to simulate reduced-order models and also allow the behaviour in non-linear regimes to be

analysed. In this thesis, the method of lines is utilised to convert the system of PDEs into a system of coupled nonlinear ordinary differential equations (ODEs), using second-order centred finite differences for spatial discretisation, and then integration in time is performed via the backwards Euler method.

Throughout this subsection, we consider a general PDE of the form

$$u_t = F(u, u_x, u_{xx}, \dots), \quad (1.38)$$

where $u = u(x, t)$ and F is a nonlinear differential operator involving u and its spatial derivatives. Equation (1.38) is considered on the spatial domain $0 \leq x \leq L$, where L is the domain length, and is subject to the initial condition

$$u(x, 0) = u_{\text{IC}}(x), \quad (1.39)$$

and periodic boundary conditions

$$\left. \frac{\partial^k u}{\partial x^k} \right|_{x=0} = \left. \frac{\partial^k u}{\partial x^k} \right|_{x=L} \quad (1.40)$$

for $k = 0, 1, 2, \dots$

To solve the system (1.38)–(1.40), the spatial domain is discretised using a uniform grid with $\mathcal{N} + 1$ grid points, denoted by $x = x_i$ for $i = 0, \dots, \mathcal{N}$. The grid points are equally spaced with step size $\Delta x = L/\mathcal{N}$ so that $x_i = i\Delta x$ for $i = 0, \dots, \mathcal{N}$. The time variable t is discretised using $\mathcal{M} + 1$ time steps, denoted by $t = t_j$ for $j = 0, \dots, \mathcal{M}$. The time steps are equally spaced with step size Δt so that $t_j = j\Delta t$ for $j = 0, \dots, \mathcal{M}$. The discretised solution for $u(x, t)$ at $x = x_i$ and $t = t_j$ is denoted by $u_i^j = u(x_i, t_j)$ for $i = 0, \dots, \mathcal{N}$ and $j = 0, \dots, \mathcal{M}$.

Throughout this thesis, we use centred second-order finite differences to calculate the derivatives with respect to the spatial coordinate x (i.e. they have a truncation error of order $O(\Delta x^2)$). Accordingly, the first and second spatial derivatives of u at $x = x_i$ and $t = t_j$ are approximated as

$$\frac{\partial u_i^j}{\partial x} = \frac{1}{2\Delta x} (u_{i+1}^j - u_{i-1}^j), \quad (1.41)$$

$$\frac{\partial^2 u_i^j}{\partial x^2} = \frac{1}{\Delta x^2} (u_{i+1}^j - 2u_i^j + u_{i-1}^j), \quad (1.42)$$

respectively, for $i = 0, \dots, \mathcal{N}$ and $j = 0, \dots, \mathcal{M}$. Higher-order spatial derivatives are easily obtained using this method (see, for example, LeVeque [65] for further details).

The backwards Euler method is an implicit time-stepping method. Specifically, the approximation of the time derivative at the current time $t = t_j$ requires the value of the solution at the next time step $t = t_{j+1}$. Therefore, applying centred finite differences and the backwards difference approximation to the PDE (1.38), then rearranging the terms yields the standard implicit formula for the backwards Euler method, namely

$$u_i^{j+1} = u_i^j + \Delta t F \left(u_i^{j+1}, \frac{\partial u_i^{j+1}}{\partial x}, \dots \right). \quad (1.43)$$

This method is first-order accurate in time and is often employed for stiff problems (i.e. problems with disparate time scales that result in stability, rather than accuracy, considerations dictating very small time steps for explicit methods) as the backwards Euler method offers improved stability compared to explicit methods such as the forwards Euler method, thus allowing larger time steps to be taken while maintaining stability.

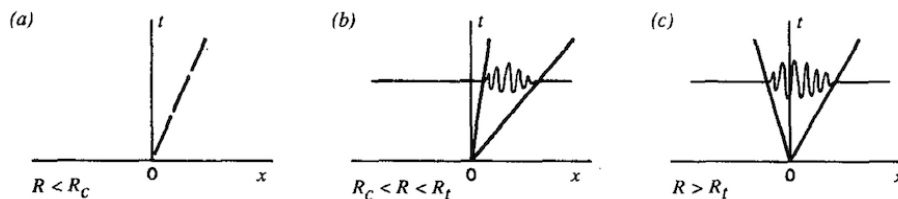


Figure 1.4: Sketches of impulse response for (a) stable, (b) convectively unstable, and (c) absolutely unstable single travelling waves. Used with permission of Annual Reviews, from “Local and global instabilities in spatially developing flows”, P. Huerre and P. A. Monkewitz, Annual Review of Fluid Mechanics, Vol. 22, 1990; permission conveyed through Copyright Clearance Center, Inc.

1.4.2 Linear stability analysis

In Chapter 6, we analyse the temporal and spatial stability of steady base state solutions of the flow. In particular, we analyse small perturbations to the steady solutions to determine the temporal stability, which can exhibit linear instability, stability or neutral stability. Spatial stability is considered for situations that exhibit temporal linear instability. Spatial instability can be either convective, in which case disturbances grow but are advected away by the flow, or absolute, in which case disturbances grow in place. Figure 1.4 shows sketches of impulse response (the system’s reaction to an instantaneous localised disturbance) for stable, convectively unstable, and absolutely unstable single travelling waves.

Throughout this subsection, as an example, we consider the linear stability of the K–S equation (1.20). The K–S equation (1.20) admits the constant (steady) base state solution $H = 0$. We investigate the linear stability of the K–S equation (1.20) by substituting $H = \bar{H}(X, T)$, where $\bar{H}(X, T) \ll 1$ is the small perturbation to the base state, and linearise to obtain

$$\bar{H}_T + \bar{H}_{XX} + \bar{H}_{XXXX} = 0. \quad (1.44)$$

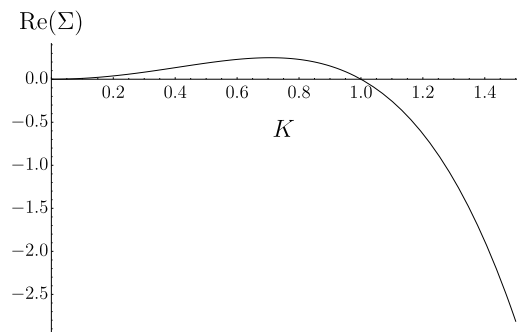


Figure 1.5: Plot of the real part of the growth rate $\text{Re}(\Sigma)$ obtained from (1.46) against the wave number K for the K–S equation.

We consider harmonic disturbances by seeking solutions of the form $\bar{H} = \hat{H}e^{\Sigma T + iKX}$, where $\Sigma(K)$ is the growth rate and K is the wave number, to yield the dispersion relation

$$\Sigma - K^2 + K^4 = 0. \quad (1.45)$$

1.4.2.1 Temporal stability

In order to determine the temporal stability, we consider real wave numbers K (i.e. ones that do not grow or decay in space) and complex growth rate Σ . The temporal growth and decay of the perturbations is determined by the real part of the growth rate $\text{Re}(\Sigma)$. The dispersion relation (1.45) has solution

$$\Sigma = K^2 - K^4. \quad (1.46)$$

Clearly, $\text{Re}(\Sigma) \geq 0$ for $0 \leq K \leq 1$ and $\text{Re}(\Sigma) < 0$ for $K > 1$. Figure 1.5 shows a plot of the real part of the growth rate $\text{Re}(\Sigma)$ obtained from (1.46) against wave number K . Therefore, the K–S equation (1.20) is temporally unstable for a finite range of wave numbers ranging from $K = 0$ to $K = 1$ and temporally stable for wave numbers $K > 1$. The maximum growth rate of $\text{Re}(\Sigma) = 1/4$ occurs at wave number

$$K = 2^{-1/2}.$$

1.4.2.2 Spatial stability

We next investigate the nature of the spatial instability (see, for example, [24, 52]). In order to determine spatial stability, we allow both the wave number K and growth rate Σ to be complex. Spatial instability is determined by the growth rate of waves with (complex) wave number corresponding to zero group velocity (i.e. ones that are fixed in space), which we denote by $K = K_0$, i.e. those waves satisfying

$$\frac{\partial \Sigma}{\partial K}(K_0) = 0. \quad (1.47)$$

The nature of the spatial instability is determined by the behaviour of $\text{Re}(\Sigma(K_0))$, which describes the temporal evolution of waves with wave number K_0 . Specifically, the sign of $\text{Re}(\Sigma(K_0))$ determines whether the instability is convective or absolute. For $\text{Re}(\Sigma(K_0)) > 0$ the flow is absolutely unstable, and for $\text{Re}(\Sigma(K_0)) < 0$ any temporal instability is convectively unstable. The imaginary part of the growth rate $\text{Im}(\Sigma(K_0))$ may be non-zero, however, this has no effect on the absolute or convective criterion described above.

Differentiating the dispersion relation (1.45) with respect to K and applying (1.47) yields

$$2K_0 - 4K_0^3 = 0. \quad (1.48)$$

Equation (1.48) can easily be solved for K_0 ; these solutions are then substituted into the dispersion relation (1.45) to obtain solutions for the growth rate of waves with zero group velocity $\Sigma(K_0)$. The unique non-trivial solution satisfying $\text{Re}(K_0) > 0$

is

$$K_0 = 2^{-1/2}, \quad \Sigma(K_0) = \frac{1}{4}, \quad (1.49)$$

which shows that the flow is absolutely unstable as $\text{Re}(\Sigma(K_0)) > 0$. This result means that localised disturbances will grow in time without being convected downstream.

1.5 Flow over spinning disks

This thesis examines reduced-order modelling of thin-film flow over a spinning disk. This section reviews the existing literature on this topic. Firstly, Section 1.5.1 explores the wide range of applications of flow over a spinning disk. Sections 1.5.2 and 1.5.3 consider the literature in experimental and modelling studies of flow over a spinning disk, respectively. Note that the modelling and analysis techniques introduced in Sections 1.3 and 1.4 can all be applied to the problem of thin-film flow over a spinning disk.

First, we briefly introduce the characteristic behaviour of flow over a spinning disk, which we will discuss in further detail within Sections 1.5.2 and 1.5.3. The flow of a thin film of liquid over a spinning disk has been the subject of many scientific review articles (see, for example, Boodhoo [11] and Sisoiev *et al.* [114]). In this thesis, we consider a disk which is rotating at a constant angular velocity with liquid entering onto the disk via a jet located at the centre of the disk with constant volume flux. Note that we do not consider the related, and widely studied, problem of spin coating, where there is a fixed volume of liquid on the disk (see, for example, [69, 77, 82, 135]). The continuous inlet flow creates a thin film of liquid over the rotating disk and the dynamics of this film are the focus of this thesis. Although



Figure 1.6: Experimental image of the interfacial waves generated by thin-film flow over a spinning disk. Reproduced from Woods [137], licensed under the Creative Commons Attribution 3.0 Licence.

liquid is continuously entering the disk via the jet, the depth-averaged properties of the film are stationary in time [114], meaning that the mean inflow from the jet is balanced by the mean radial outflow and the film behaves as a mass-conserving system. The two natural continuously variable control parameters that affect the dynamics of the flow are the inlet flux and the angular velocity of the disk. Varying these two parameters can create a variety of wave patterns over a disk [114]. The free surface is typically characterised by a waveless region near the inlet and a wavy region further out [73]. Figure 1.6 shows an experimental image of flow over a spinning disk. The characteristics of this wavy region will be discussed further in Sections 1.5.2 and 1.5.3, however, we note that the wavy region is divided into subregions that exhibit different characteristics (as shown in Figure 1.6). The two main competing effects in flow over a spinning disk are centrifugation, which increases with the radial coordinate, and the effect of film thinning, due to conservation of mass.

1.5.1 Applications of flow over a spinning disk

The flow of liquid over spinning disks is a problem of industrial interest. There are many applications that exploit the complex dynamics of the flow over spinning disks, which we will discuss below.

A common use of spinning disks is within spinning disk reactors (SDRs) [79]. Figure 1.7 shows a schematic representation of a SDR (reproduced from [86]). There is the possibility to heat or cool the surface of the disk in a SDR, as shown in Figure 1.7. The dynamics of the flow of liquid over a spinning disk induce intense mixing, resulting in increased heat and mass transfer rates when compared to conventional stirred tank reactors [12, 14, 88]. This is an example of what is commonly referred to as “process intensification” in the literature, which broadly refers to the process of making the production of a desired substance more efficient. SDRs have been used in many chemical processes including manufacture of pharmaceuticals [86], polymerisation [14, 127], biodiesel production [5, 26, 95], and photodegradation processes [33, 66]. One of the benefits of SDRs is that the devices are compact and can cope with full-scale production, avoiding the need to develop methods to scale-up production. For example, a 15 cm SDR could produce up to approximately 8 tons of pharmaceutical products per year [86].

Recently, spinning disks have also been shown to be of use in the production of graphene [117], in which graphite flakes are suspended in a suitable liquid solvent and passed over a spinning disk. Graphene nanosheets are produced by the shear exfoliation of the graphite flakes, which can form by either peel or slip exfoliation routes. The shear exfoliation can cause these graphene nanosheets to fold, partially roll or completely roll-up into scrolls. Figure 1.8 shows transmission electron microscopy of the peel initiation site of graphene from graphite and a case of complete

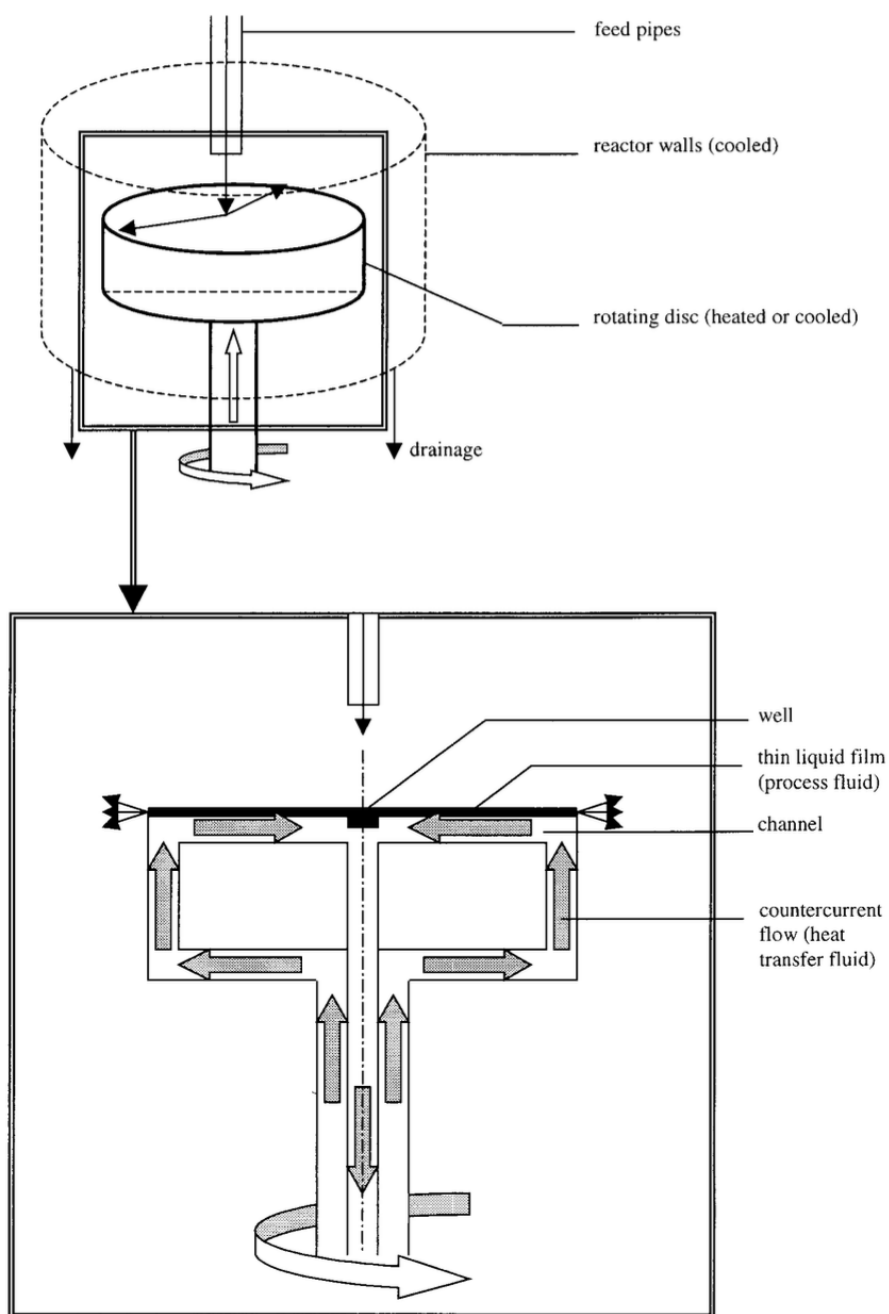


Figure 1.7: Schematic representation of a SDR. Reprinted with permission from Industrial & Engineering Chemistry Research, Vol. 39, Issue 7, P. Oxley, C. Brechtelsbauer, F. Ricard, N. Lewis, and C. Ramshaw, p. 2176, Copyright 2000 American Chemical Society.

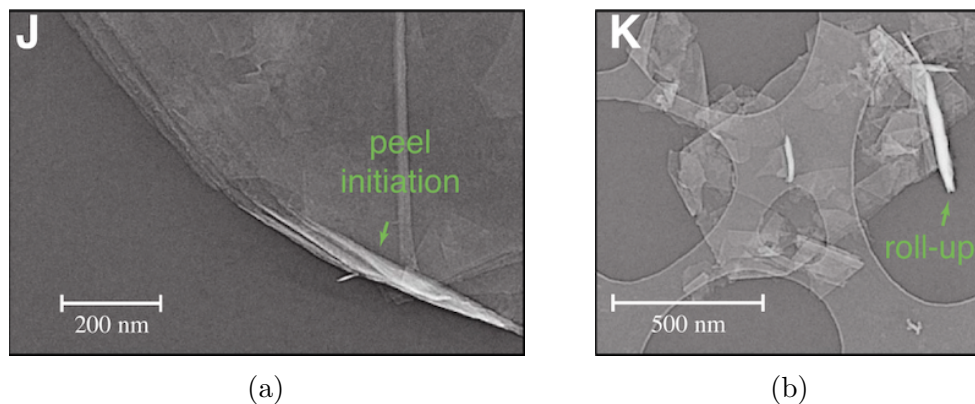


Figure 1.8: Transmission electron microscopy of (a) a peel initiation site of graphene from graphite and (b) a case of complete roll-up of a graphene nanosheet. Reproduced from Stafford *et al.* [117], licensed under the Creative Commons Attribution 4.0 International Licence.

roll up of a graphene nanosheet. The shear stress and residence time (i.e. the time a particle spends in the film on the disk) are the dominant factors which govern the production of graphene. However, these two factors are closely interconnected. For example, increasing the rotational speed increases the shear rate, which enhances shear exfoliation, but an effect of this is that the residence time decreases, meaning that the particles have less time to fully exfoliate. This interplay means that it is challenging to find the conditions for maximising graphene production. Fortunately, due to the nature of SDRs, the flow can be continually controlled and/or adjusted to change the shear stress and residence time in real-time, which allows graphene to be produced consistently and at larger scales.

Spinning disks can also be utilised in the recovery of heat from molten slag (a by-product of the steel industry) [144]. Historically, cold water has been used to cool the molten slag, however, this process uses vast amounts of water and the heat is not reused. Another issue with this method is the air pollution caused by the reaction

between the water and molten slag, which produces sulphur compounds. Therefore, it is of environmental interest to find new ways to reduce this air pollution and water consumption, and to recover the heat. A spinning disk atomiser (SDA) process can be used instead of the traditional water cooling method in which the molten slag flows onto the disk, forming a film that travels radially to the edge of the disk where it forms ligaments that then break into particles. Hot air is obtained from this process can then be used, for example, to produce steam or preheat fuel.

1.5.2 Experimental studies

Experimental studies on spinning disk flows have largely had two focuses: the hydrodynamic behaviour of the film and the associated processes of heat and mass transfer, and chemical reactions. In Section 1.5.2.1 we will discuss the hydrodynamics and in Section 1.5.2.2 we will discuss heat and mass transfer, and chemical reactions.

1.5.2.1 Hydrodynamics

The seminal work of Espig and Hoyle [32] measured the film thickness at a fixed point and visually observed the formation of rivulets for cases of low inlet flow rates compared to the rotational velocity. Above this critical flow rate, they observed a laminar film, concentric waves, and then helical waves superimposed on the concentric waves upon increasing the flow rate. Charwat *et al.* [25] used cameras to study the structure of the flow and the different flow regimes. They demonstrated that the structure of the flow depended on the inlet flow rate, rotational velocity, and the properties of the liquid (i.e. viscosity and surface tension). These could be varied to find a variety of flow structures including a smooth film, concentric waves, spiral

waves, and irregular spiral waves. They observed spiral waves which unwound in the direction of rotation of the disk and found that these spiral waves were stationary in the frame of reference which rotates with the disk. Miyasaka [81] measured the average film thickness and found that the average film thickness either decreases monotonically in the radial direction or that the average film thickness has a local maximum. Butuzov and Pukhovoi [21] observed four concentric regions which they named the “inlet” region, the “first laminar-wave” region, the “turbulent” region and the “second laminar-wave” region. In particular, the inlet region was waveless; the first laminar-wave region consisted of axisymmetric waves; the turbulent region consisted of disordered ripples that have decreasing amplitude as they travel radially outwards; and the second laminar-wave region is then reached. Thomas *et al.* [121] observed two wave regimes: disordered ripples (which occurred at lower rotational speeds) and spiral waves (which occurred at higher rotational speeds). They measured the average film thickness and showed that from the inlet the average film thickness increases to a local maximum before decreasing monotonically in the radial direction (which answered an outstanding question from the work of Miyasaka [81] about the behaviour of the average film thickness). This local maximum will subsequently be referred to as a “shoulder”.

Woods [137] used cameras to study the structure of the flow, and also measured the average film thickness at four radial locations. They observed three concentric regions for which the waves exhibit different behaviours: a laminar region near the centre of the disk, a middle region with concentric waves, and a outer region with spiral waves. The location of these regions depended on the rotational speed of the disk and inlet flow rate. As the rotational speed of the disk increased the waves became more disordered towards the edge of the disk. Leneweit *et al.* [64]

measured the free-surface velocity and used this data to estimate the film thickness. They added dye to the liquid to visualise streaklines and observed the formation of spiral waves. However, they found them to be stationary relative to the laboratory frame of reference (unlike Charwat *et al.* [25]). Burns *et al.* [20] measured the film thickness and identified three concentric regions that exhibit different behaviours: an “injection” region near the inlet, an “acceleration” region where centrifugal effects dominate, and a “synchronisation” region where viscous effects dominate.

Korzhova [60] presented a novel video-based algorithm to detect moving waves, this allowed them to find the wave velocity, wave inclination, and distance between consecutive waves. This analysis was based in the spatial region that we will term as the inner region, in which the behaviour is dictated by the noise at the inlet, hence it is difficult to obtain meaningful comparison with these results using a reduced-order model. Wang *et al.* [130] experimentally measured the average radial velocity of the liquid and used this data to estimate the film thickness. Wang *et al.* [129] extended the work of Woods [137] to further investigate the transition between flow patterns using high-speed cameras. In particular, they identified five flow patterns (shown in Figure 1.9): incomplete wetting flow (Figure 1.9(a)), spiral waves (Figure 1.9(b)–(d)), transitional flow (Figure 1.9(e)), irregular waves (Figure 1.9(f)), and reverse spiral waves (Figure 1.9(g)). They focused on analysing the spiral wave patterns and observed the same three concentric regions as noted by Woods [137].

1.5.2.2 Heat and mass transfer, and chemical reactions

In this thesis we focus on the hydrodynamics of flow over a spinning disk, however, there have been many studies on heat and mass transfer, and chemical reactions due to their industrial applications. In this subsection, we will give a brief outline of the

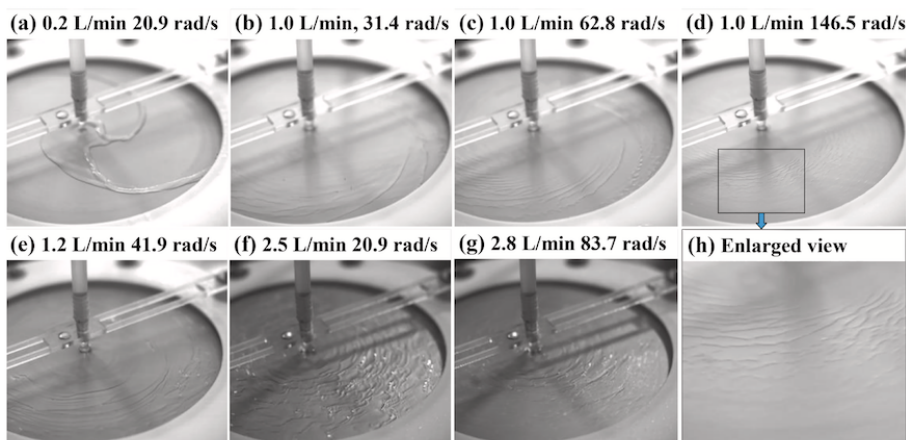


Figure 1.9: Transition of typical flow patterns for 95% glycerol, (a) incomplete wetting flow, (b) long spiral waves, (c) short spiral waves, (d) spiral wavelets, (e) transitional flow, (f) irregular waves, (g) reverse spiral waves, and (h) enlarged view of spiral wavelets. Reprinted from Chemical Engineering Journal, Vol. 451, D. Wang, C. Gu, C. Sun, X. Ling, H. Peng, X. Yang, F. Yuan, J. Du, and W. Yu, “Flow pattern transition and wave characteristics of impinging jet flow on a spinning disk reactor”, Article 138790, Copyright 2023, with permission from Elsevier.

most relevant work by previous authors on this topic to motivate the use of SDRs over other reactors and introduce the potential benefits of topographic patterning on the disk.

Jachuck and Ramshaw [53] considered the heat transfer characteristics of four different disks in a SDR: a normal grooved disk, a grooved disk with undercut grooves, a smooth disk, and a textured disk. They found that, in general, the normal grooved disk yielded a larger heat transfer coefficient compared to the other disk surfaces tested. They also found that for all disk surfaces tested increasing the rotational speed of the disk increased the heat transfer coefficient. Aoune and Ramshaw [4] demonstrated that the characteristics of the flow over a spinning disk created intense mixing, which increased mass transfer. They found that the interfacial

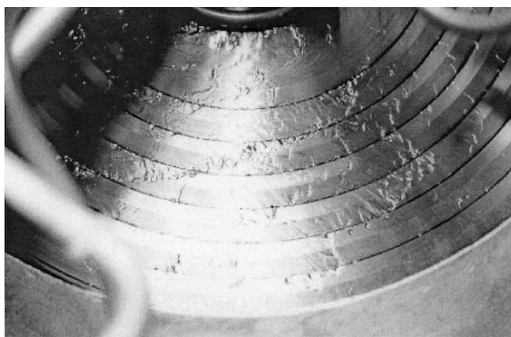


Figure 1.10: Experimental image of interfacial waves generated by thin-film flow over a grooved disk. Reprinted from Applied Thermal Engineering, Vol. 20, Issue 12, K. V. K. Boodhoo and R. J. Jachuck, “Process intensification: spinning disk reactor for styrene polymerisation”, p. 1129, Copyright 2000, with permission from Elsevier.

waves contributed to the increased mass transfer. For water-like liquids, high heat transfer coefficients were achieved for a SDR, however, for more viscous liquids, the heat transfer was significantly reduced. Boodhoo and Jachuck [15] used SDRs for styrene polymerisation with a heated grooved disk and compared the results to a stirred vessel reactor. A grooved disk was chosen due to the enhanced heat transfer characteristics [53]. Figure 1.10 shows an experimental image of interfacial waves generated by thin-film flow over a grooved disk (reproduced from Boodhoo and Jachuck [15]). Their results indicated that the SDR had greater reaction rates and less spread in the molecular weight distribution compared to the stirred vessel reactor. The continuous production of the SDR rather than batch production of the stirred vessel reactor is also likely to improve the safety of production [15]. Increased reaction rates for a SDR compared to traditional production methods were also found for continuous condensation polymerization of esters, styrene free radical polymerisation, and photo polymerisation of n-butyl acrylate by Boodhoo and Jachuck [14], Vicevic *et al.* [128], and Boodhoo *et al.* [13], respectively.

Vivec *et al.* [126] performed isomerisation of α -pinene oxide to campholenic aldehyde

using zinc oxide as a catalyst, which they immobilised by coating the surface of the disk. They found almost 100% conversion rates for angular velocities in the range 100–800 rpm, while higher angular velocities were found to decrease the conversion rate due to the decrease in the residence time. Using this method, the product was catalyst free, so there was no requirement to have an additional process to remove this (which would be required if traditional batch reactors were used).

Burns and Jachuck [19] determined the mass transfer coefficient at different radial positions. Based on their results, they found that Nusselt theory (which corresponds to a waveless, steady base flow solution) provided a method for predicting the minimum mass transfer coefficient over a SDR although this model always under predicted compared to their experimental results.

Boodhoo and Al-Hengari [12] considered the micromixing characteristics of a SDR using an iodide-iodate reaction. In general, they obtained the best micromixing conditions with both a high angular velocity and a high inlet flow rate, with the performance of the SDR exceeding that of a stirred tank reactor. Yao *et al.* [142] extended this work with the addition of a topographically-patterned disk. In particular, they considered a smooth disk, a concentric ridged disk, a spiral ridged disk, and a radial ridged disk, and found that the radial ridged disk produced the greatest micromixing of the four disks tested. Wang *et al.* [131] considered concentric grooves and radial grooves on a disk and found that the radial grooves were preferable for maximising micromixing. The results of Yao *et al.* [142] and Wang *et al.* [131] imply that radial ridges or grooves are the preferable topographies to enhance micromixing. However, in this thesis we will restrict our attention to radially symmetric topographies.

1.5.3 Modelling studies

The reduced-order modelling of flow over a spinning disk (i.e. flow driven by centrifugation) utilises the same techniques as discussed in the context of gravity-driven flow in Section 1.3. The method that has been the most commonly used for reduced-order modelling of thin film flow over a spinning disk is the IBL method, as discussed in Section 1.3.3. A number of simplifications are often applied to obtain reduced-order models, the most common being assuming that the flow is axisymmetric (i.e. there is no variation in the azimuthal direction).

There have been many studies on steady, axisymmetric flow in the limit of large Ekman number E (a dimensionless number that measures the relative importance of viscosity compared to Coriolis force), i.e. the case of high viscosity and/or low Coriolis force, such as Rauscher *et al.* [97] and Woods [137]. These studies determined asymptotic expressions in the limit of large Ekman number for the radial, azimuthal and vertical velocities, pressure and film thickness. Charwat *et al.* [25] performed a linear stability analysis and found that decreasing the Ekman number has a stabilising effect on the flow. Woods [137] applied the lubrication approximation to Navier–Stokes equations and performed a linear stability analysis. They found that there are three spatial regions, each with a different set of equations corresponding to a different asymptotic balance of terms in each region. Aoune and Ramshaw [4] used Nusselt theory to model the flow to validate their experiential findings, however, this model underestimated the mass transfer coefficients compared to the experiments. This discrepancy arose due to the modelling technique neglecting the interfacial waves, highlighting the need for more accurate and sophisticated models.

Scheichl and Kulwick [104] studied steady, axisymmetric flow over a spinning disk. They found that there are four distinct spatial regions in the asymptotic structure of

the flow: the jet regime, jet-bending regime, rotatory boundary layer, and thin-film regime. As their work particularly focused on modelling the effects of the jet itself, only the thin-film region is considered in this thesis, as our objective is to study the behaviour of the wavy region further out. In the thin-film regime, Scheichl and Kulwick [104] obtained boundary-layer equations including the effects of inertia, centrifugation, and viscosity that they studied numerically and analytically. They found that increasing the rotational velocity moves the shoulder in the free surface towards the centre of the disk. For higher rotational velocities, the film thickness decreases monotonically in space.

Next we will review the literature using the reduced-order modelling methodologies described in Section 1.3 and DNS.

1.5.3.1 Gradient expansion method

Gambaryan-Roisman *et al.* [35] studied axisymmetric flow over a heated, topographically-patterned disk. They used the gradient expansion method to derive a Benney-like equation. The reduced-order model includes the effects of centrifugation, Coriolis force, inertia, and Marangoni stresses (i.e. thermocapillarity). The leading-order solution for the film thickness showed qualitative agreement with their own experimental results for the average film thickness over a planar disk, although the film thickness was over predicted for low rotational speeds and under predicted for high rotational speeds. However, in line with the discussion in Section 1.3.1, this model is unlikely to be accurate for moderate to high levels of inertia. Therefore, other techniques such as the IBL method or DNS are often employed.

1.5.3.2 IBL method

Sisoev *et al.* [112] applied the IBL method to flow over a spinning disk for large Ekman number, before later extending the work to include finite Ekman number [111]. They introduced a similarity parameter δ which measures the relative importance of inertia to capillary and viscosity. In this method a closure condition is needed for the velocity profiles, and they chose the radial and azimuthal velocities to be quadratic and quartic polynomials in the vertical coordinate, respectively. For the case of unsteady flow, they derived a set of nonlinear PDEs for localised flow (i.e. only valid close to a given radial position) and performed a linear stability analysis. They found that decreasing the Ekman number had a stabilising effect on the flow, in agreement with Charwat *et al.* [25]. The predictions from this model were also validated against the experimental findings of Woods [137], with which there was found to be good agreement. The same authors later extended this work to include the effect of gas absorption [113]. Matar *et al.* [74] then validated these results numerically using the non-local form of the evolution equations of Sisoev *et al.* [111]. They studied the evolution of the waves in free surface and found a waveless region near the centre of the disk and a wavy region downstream. For fully-developed flow, they found that the system exhibits a radial position at which there is a sustained initiation of waves (as shown in Figure 1.11), the position of which depends on the values of the parameters in the problem.

Matar *et al.* [73] extended the work of Sisoev *et al.* [111] using non-local equations. Their nondimensionalisation introduced an inverse Weber number λ , a dimensionless number that measures the relative importance of surface tension to inertial effects, instead of the Ekman number. They solved the equations numerically over a disk of radius r_{disk} and considered the effect of varying the inverse Weber number. They

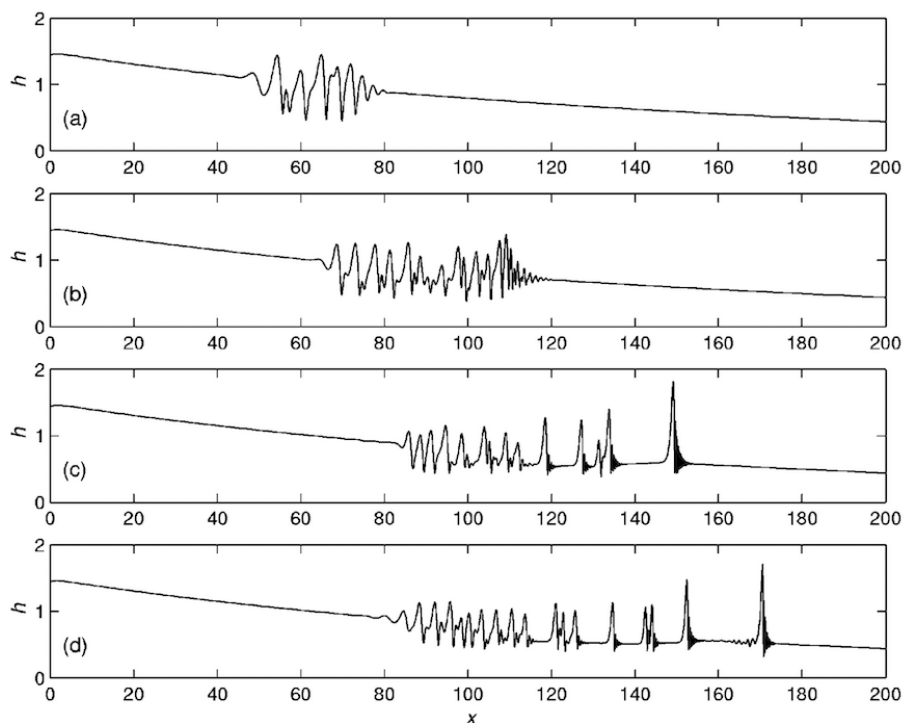


Figure 1.11: Evolution of the free surface h with Ekman number $E = 5$ and similarity parameter $\delta = 0.1$ (in the notation of Matar *et al.* [74]) at dimensionless time (a) $t = 61$, (b) $t = 100$, (c) $t = 150$, and (d) $t = 179$. Reprinted from Physics of Fluids, Vol. 16, Issue 5, O. K. Matar, G. M. Sisoiev, and C. J. Lawrence, “Evolution scales for wave regimes in liquid film flow over a spinning disk”, p. 1536, 2004, with the permission of AIP Publishing.

found that at early times small amplitude waves develop, which travel radially outwards; these waves then steepen and accelerate (as shown in Figures 1.12(a)–(c)). The waves separate and/or coalesce to form large amplitude pulse-like waves, which are preceded by capillary ripples (as shown in Figures 1.12(d)–(h)). Near the centre of the disk, the shoulder forms within the radial region $0.5 \leq r \lesssim 2$ (shown in Figures 1.12(a)–(h)) and undergoes very little change after it forms. In particular, they found that increasing the inverse Weber number causes the waves to appear more regular, increasing the wavelengths and decreasing the amplitudes of the waves. They also

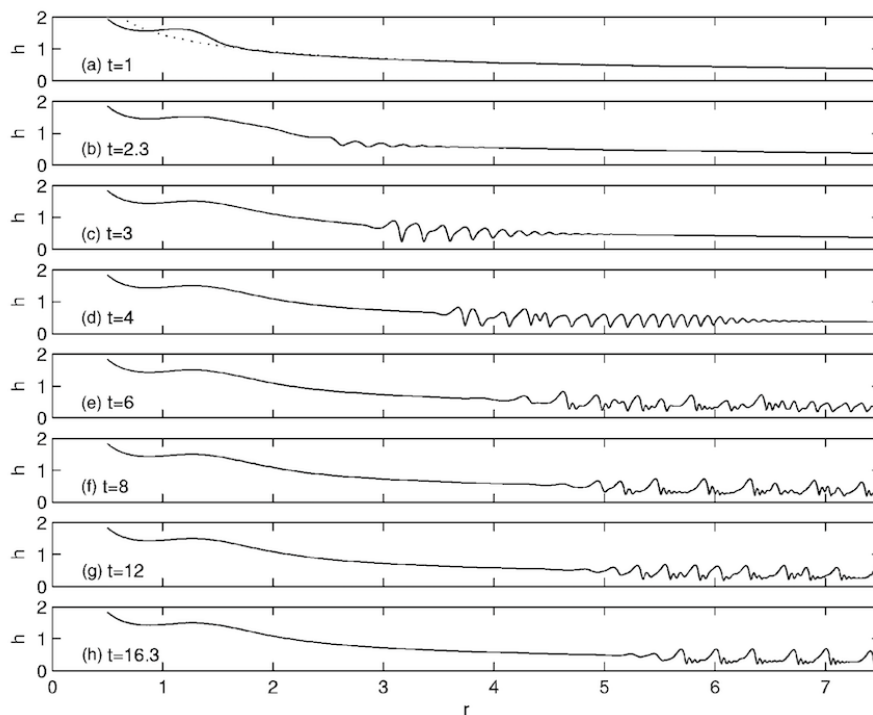


Figure 1.12: Evolution of the free surface h with inverse Weber number $\lambda = 0.01$ for a disk of radius $r_{\text{disk}} = 7.5$ at different dimensionless times. Reprinted from Physics of Fluids, Vol. 17, Issue 5, O. K. Matar, C. J. Lawrence, and G. M. Sisoiev, “The flow of thin liquid films over spinning disks: Hydrodynamics and mass transfer”, Article 052102, 2005, with the permission of AIP Publishing.

considered the evolution of the concentration of a solute within the film. Matar and Lawrence then extended this work to include the effects of an electric field [72] and then surfactant [71].

Matar and Lawrence [75] extended the work of Matar *et al.* [73] to include the effects of a topographically-patterned disk. They considered two types of topography: sawtooth and square-shaped trench, and added a small sinusoidal forcing at the inlet in numerical simulations. In this thesis, we will use the WRIBL method to model the flow and consider the effect of a family of sawtooth topographies (with parameters

for asymmetry, smoothness, amplitude, and spatial frequency) with a small random noisy forcing at the inlet. Matar and Lawrence [75] introduced an integral measure of interfacial waviness to measure the effects of varying topography and inlet forcing, and found that the square-shaped topography caused a greater interfacial waviness than the saw-tooth topography, both exceeding that of the planar disk. The addition of inlet forcing also increased the interfacial waviness. In this thesis, we will extend this work by using the WRIBL method to model the flow and simulate the flow over a larger disk, considering a wider variety of sawtooth topography profiles and obtaining analytical results in the far field.

Sisoev *et al.* [110] later extended the work of Sisoev *et al.* [111] to the non-axisymmetric case to model spiral waves, although they only considered steady flow. They found that the waves follow Archimedean spirals (for which the paths of these spirals depended on the Ekman number) and showed good agreement with their own experimental data. Kim and Kim [59] modelled unsteady non-axisymmetric flow over a planar disk using the IBL method. In this method, they chose to use quartic polynomials for both the radial and azimuthal velocities. They applied simplifications to the inertial terms, inspired by the simplified second-order model of Ruyer-Quil and Manneville [100] (i.e. the inertial terms are computed using a quadratic radial velocity profile). They found asymptotic solutions for small and large radius at steady state and performed a linear stability analysis. The linearised steady equations showed good agreement with DNS of the steady Navier–Stokes equations and the time-averaged experimental data of [25, 64, 81, 121]. However, note that comparisons based on time-averaged quantities are not always representative of the full behaviour of the flow. Prieling and Steiner [93] modelled axisymmetric flow following the same method as Kim and Kim [59], simplifying the inertial terms, and

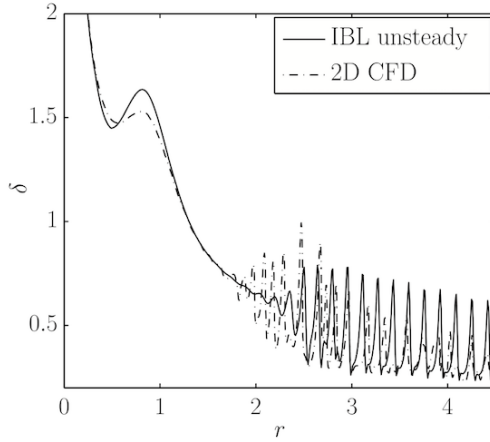


Figure 1.13: Plot of the free surface δ (in the notation of Prieling and Steiner [93]) against radial position r according to the IBL model of Prieling and Steiner [93] and the Navier–Stokes equations. Reprinted from International Journal of Heat and Mass Transfer, Vol. 65, D. Prieling and H. Steiner, “Unsteady thin film flow on spinning disks at large Ekman numbers using an integral boundary layer method”, p. 10–22, Copyright 2013, with permission from Elsevier.

additionally considered heat transfer. They solved the resulting set of equations numerically and compared the results they obtained against the solutions of the axisymmetric Navier–Stokes equations, with which they found good agreement with the time-averaged solutions. They found that there is a radial position at which there is a sustained inception of waves, in agreement with Matar *et al.* [74]. This radial position is well predicted by the reduced-order model when compared to the Navier–Stokes equations (as shown in Figure 1.13). The authors then extended this work to consider mass transfer [92], however, in this case they only considered steady flow. In this work, they used piecewise functions for the velocities and species mass concentration to account for the boundary layers. Due to the balances between these piecewise functions, they found that there exist four spatial regions, with the first region corresponding to the jet flow. The second region is governed by equations

identical to the steady axisymmetric version of the IBL model of Kim and Kim [59]. The equations governing the free surface and the radial and azimuthal depth-averaged velocities in this region are the same in the third and fourth regions, where only the species mass transport equation differs. They again solved the resulting set of equations numerically and compared against the time-averaged solution of the axisymmetric Navier–Stokes equations, with which they found good agreement. Zhao [146] extended the work of Matar *et al.* [73] to the non-axisymmetric case. Their IBL model captured concentric and spiral waves and was validated against DNS. They also considered mass transfer in the non-axisymmetric case, for which they found that the mass transfer increased as the inverse Weber number λ decreased (i.e. as inertial effects increased and/or surface tension decreased).

Kim and Kim [58] also used the IBL method, however, they included the effects of inlet forcing and turbulence for non-axisymmetric flow over a planar disk. They used the same quartic radial and azimuthal velocity profiles as Kim and Kim [59], however, they did not simplify the inertial terms. In particular, their model successfully captured the behaviour of waves compared to their own experimental data (see Figures 1.14(a) and (b)). The model predicted concentric waves near the centre of the disk, which transition to three-dimensional waves, in this case at the radial position $r \approx 1.5$ (as shown in Figure 1.14(c)). They also compared the results of their model against the pre-existing experimental data of [64, 81, 130] using the time-averaged free surface data and also found good agreement.

1.5.3.3 Direct numerical simulations (DNS)

Due to the complexity of the problem, there have been a limited number of studies on DNS for flow over a spinning disk.

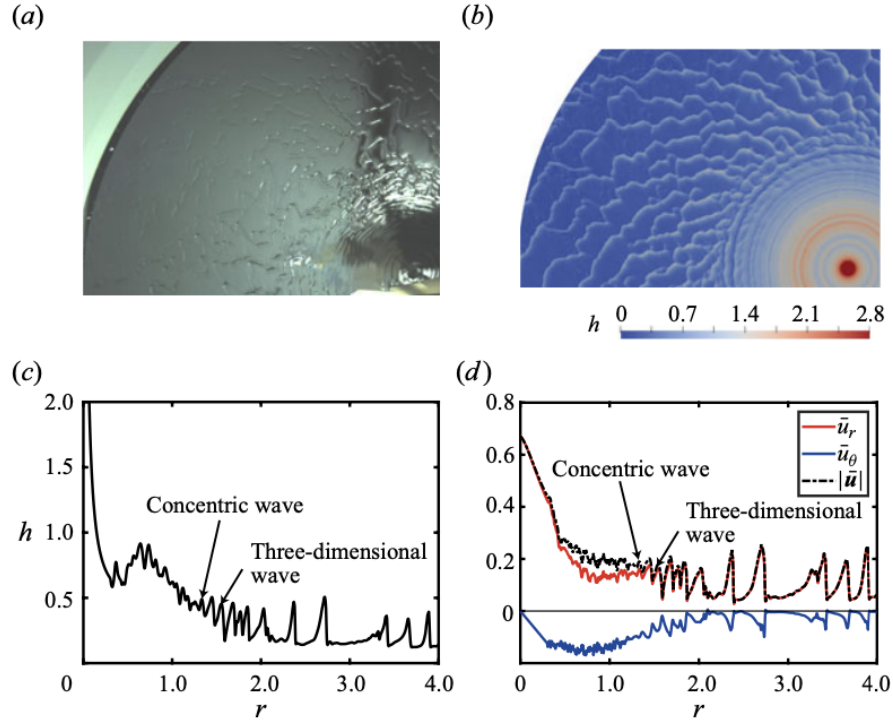


Figure 1.14: (a) Experimental image of free surface waves, (b) contour plot of the free surface h , (c) 2D plot of free surface h , and (d) plot of the depth-averaged velocities. The angular velocity is $\Omega = 52.4 \text{ rad}^{-1}$ and the inlet flux is $Q = 12.5 \text{ mls}^{-1}$. Reproduced from Kim and Kim [58], licensed under the Creative Commons Attribution 4.0 International Licence.

Rice *et al.* [98] used the Volume of Fluid (VOF) technique to model flow over a heated planar disk, and also considered the evaporation of the liquid. They found that the film thickness decreased with increasing rotational velocity and temperature. Wang *et al.* [130] also used the VOF technique to determine the radial positions of the three radial regions identified: the “injection” region (where the liquid velocity decreases to a minimum), the “acceleration” region (where the liquid velocity accelerates up to the rotational speed of the disk), and the “synchronisation” region (where the liquid velocity is approximately constant at the rotational speed of the disk).

They validated this model against their own experimental results and found good agreement for the film thickness. Weerasiri *et al.* [133] used the VOF technique to model axisymmetric flow over a heated rotating cone and disk, and validated the results with the pre-existing experimental data of [98, 121] and found good agreement. They found that there are three distinct spatial regions zones: an inertial region where the shoulder forms, a Coriolis region where film begins to thin non-linearly, and a centrifugal region at outer edge with a uniform reduction in film thickness. Stafford *et al.* [118] used OpenFOAM to solve the Navier–Stokes equations for non-axisymmetric flow over a planar spinning disk. They used a wedge-shaped computational domain and investigated the transition between different wave regimes. They presented a phase diagram showing the wave regimes across a spinning disk for the inverse Weber number λ and a modified Ekman number (as shown in Figure 1.15). The results showed excellent agreement with the pre-existing experiments of Woods [137] and their own experimental results, and that there exist four wave regimes: waveless, 2D waves, transition to 3D waves, and 3D waves.

1.6 Free-surface control

Control of a liquid is a problem of scientific and industrial interest and has been the topic of many scientific review articles (see, for example, Gunzburger [44] and Sellier [107]), with applications including spray coating [34], dip coating [89], and inkjet printing [68]. The control of complex systems (such as liquids) often poses a significant challenge and is usually computationally expensive. Therefore, reduced-order models are often employed to reduce the computational expense.

In this thesis, we consider control of the free surface of a thin film on a spinning disk

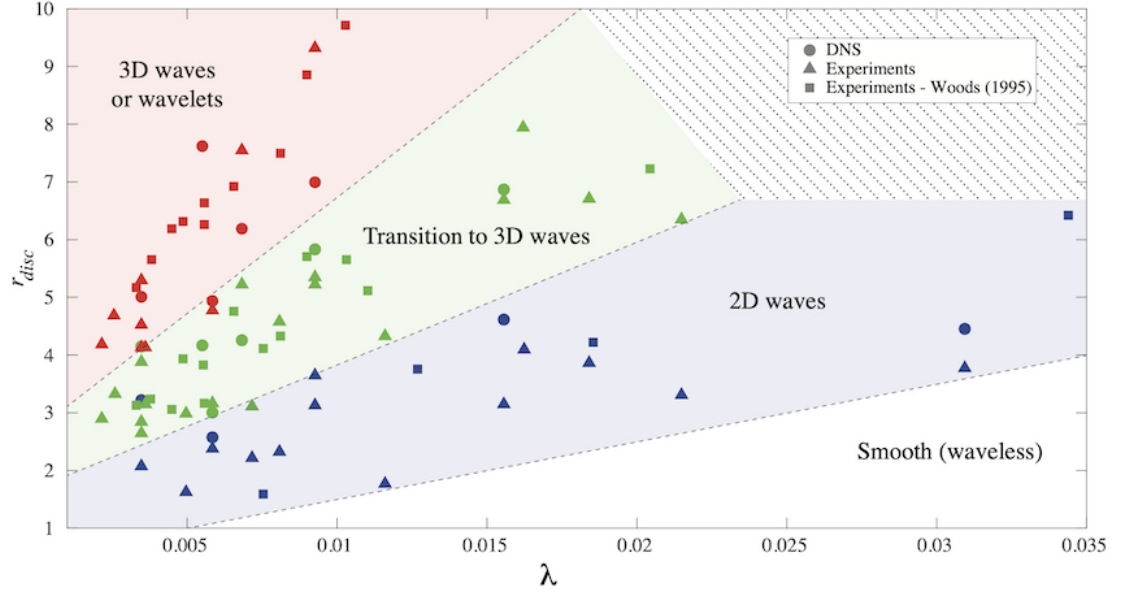


Figure 1.15: Phase diagram showing the wave regimes across a spinning disk for the inverse Weber number λ and a modified Ekman number r_{disc} (in the notation of Stafford *et al.* [118]), according to DNS, experiments of Stafford *et al.* [118], and the experiments of Woods [137]. Reproduced from Stafford *et al.* [118], licensed under the Creative Commons Attribution 4.0 International Licence.

towards a target free-surface profile, which we will refer to as free-surface control. A comprehensive review of control of liquid films in a wider context is given by Sellier [107]. In this thesis, we will use topography as the control mechanism, however, other mechanisms such as electric fields [123, 124, 139] and blowing/suction [27, 122] can also be used. The reconstruction of the topography from a known free-surface profile is sometimes referred to as bathymetry reconstruction in the literature. This problem has been considered in a range of situations, such as glaciers [37, 48], rivers [38], and dam-break flows [1], where in each case free-surface measurements are typically more readily available than the film thickness or topographic measurements. These problems are sometimes referred to as inverse problems in the literature. There are

many methods used to solve control problems. In this thesis, we will use analytic inversion techniques that will be discussed below. However, there are a variety of other techniques used in the literature including machine learning [18] and optimal control [2, 16, 40]. These techniques often have a high computational cost, although they can be applied to a wider variety of problems as they are not reliant on the equations being invertible. In the following, we will discuss inversion techniques for free-surface control and topography reconstruction.

Sellier [106] considered the flow of a liquid film down an inclined plane with topography in 2D. The lubrication approximation is used; however, the normal component of gravity is neglected in their equations. The resulting set of equations at steady state can be inverted to find an explicit expression for the topography in terms of the position of the free surface. The resulting analytical solution is an exact solution to the steady problem and was shown to perform well even in the presence of noise in the free-surface data. Figure 1.16 shows the reproduction of the topography from artificial experimental data with 5% noise for the free surface. However, not every free-surface profile can be achieved, and there is a critical steepness of the free surface beyond which the solution to the inverse problem does not exist. Sellier and Panda [108] then extended this approach to 3D. They found that for the 3D case an explicit expression no longer exists and a first-order linear PDE is required to be solved. Analogously to the 2D case, not every free-surface profile can be achieved, and as in the 2D case there is a critical steepness of the free surface beyond which the solution to the inverse problem does not exist. Heining *et al.* [50] extended the work of Sellier [106] in 2D to include the normal component of gravity. They found an explicit equation for the topography in terms of the position of the free surface and its derivatives. In addition, they were able to determine the velocity field, wall

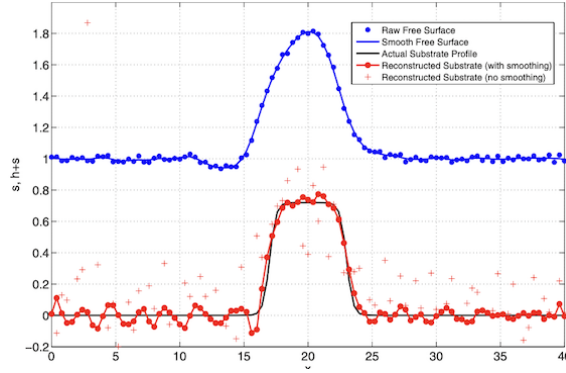


Figure 1.16: Reproduction of the topography s (in the notation of Sellier [106]) from artificial experimental data with 5% noise for the free surface $h + s$ (where h is the film thickness in the notation of Sellier [106]). Reprinted from Physics of Fluids, Vol. 20, Issue 6, M. Sellier, “Substrate design or reconstruction from free surface data for thin film flows”, Article 062106, 2008, with the permission of AIP Publishing.

shear stress, and pressure distribution with only knowledge of the position of the free surface.

Heining and Aksel [45] included inertial effects in the control problem for the flow of a liquid film down an inclined plane with topography in 2D. They modelled the flow using the WRIBL method, deriving a set of two coupled evolution equations for the film thickness and flow rate (as discussed in Section 1.3.4). They derived an equation which allows the topography to be reconstructed from the position of the free surface. For a weakly undulated free surface, Heining and Aksel [45] found that the problem could be solved analytically using a perturbation solution, whereas for a strongly undulated free surface, it is no longer possible to determine an analytical solution and the equation needs to be solved numerically. They then considered the stability of the corresponding direct problem for an undulated free surface, i.e. using the topography computed by the inverse problem then performing a stability analysis on the resulting flow. It was found that the topography, in general, enhances

the stability of the flow. However, when surface tension is sufficiently large, the topography can have a destabilising effect on the flow.

Motivated by the fact that experimentally the free-surface velocity can be measured non-intrusively via methods such as particle image velocimetry and laser-induced fluorescence [41], it is of interest to reconstruct the topography and/or film thickness from this data. Heining *et al.* [50] considered the flow of a liquid film down an inclined plane with topography in 2D, modelled using the lubrication approximation. They were able to compute the position of the free surface, topography, and wall shear stress based only on the free-surface velocity. Determining the position of the free surface requires an ODE to be solved numerically. The authors [47] then extended this work, using the IBL method to include inertial effects. Explicit solutions for the film thickness and position of the free surface were obtained using Fourier series. Hence, a solution for the topography can also be obtained. This method was shown to perform well in the presence of noise in the free-surface velocity, however, this model cannot capture recirculation, so accuracy is limited in these cases. Heining and Sellier [49] extended the work of Heining *et al.* [50] using the lubrication approximation to 3D. The film thickness, topography and pressure field were reconstructed using the free-surface velocity, solving a set of PDEs numerically. Figure 1.17 shows plots of the free-surface velocity input data, the topography, the free surface, and the comparison of the original data and inverse solution for different cross sections. This method was shown to be robust in the presence of noise in the free-surface velocity and the results obtained showed good agreement with the results obtained from the full Navier–Stokes equations and the experimental data of [46].

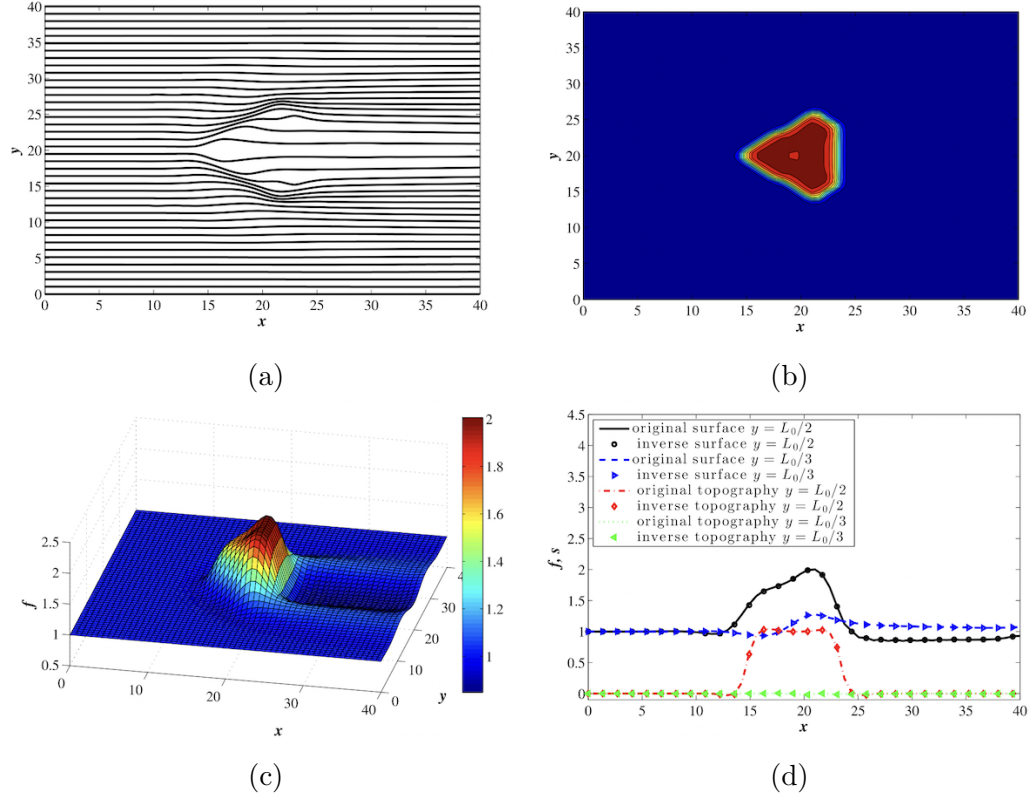


Figure 1.17: Plots of (a) the free-surface velocity input data, (b) the topography, (c) the free surface f (in the notation of Heining and Sellier [49]), and (d) the comparison of the original data and inverse solution for different cross sections. Reprinted from Physics of Fluids, Vol. 29, Issue 1, C. Heining and M. Sellier, “Flow domain identification in three-dimensional creeping flows”, Article 012107, 2017, with the permission of AIP Publishing.

1.7 Objectives and outline of thesis

This thesis examines the mathematical modelling, analysis and control of thin-film flow over a spinning disk. As mentioned in Section 1.3.3, the IBL model (which has been the tool most commonly used to model thin-film flow over a spinning disk) has limitations, therefore, it is important to develop more accurate and sophisticated models for the problem of flow over spinning disks. One of the goals of this thesis is to develop a WRIBL model (as discussed in Section 1.3.4 for the case of a falling film), and then use this model to analyse the dynamics of the flow and investigate the effect of topography. We also use this model to investigate the mathematical reasoning behind the sustained inception of waves (noted by Matar *et al.* [74] and Prieling and Steiner [93]) that has received little analysis. In addition, something that has not been considered before is to study mechanisms to control the dynamics of the free surface. Hence, another important goal of this thesis is to theoretically investigate how to exploit topography to control the free surface of the film on a spinning disk. The objectives of this thesis are as follows.

- To derive a novel reduced-order model for thin-film flow over a spinning disk using the WRIBL method
- To validate the reduced-order model against DNS
- To perform a parametric study to consider the effect of the inverse Weber number λ , a small random noisy forcing at the inlet, and a topographically-patterned disk on the flow
- To perform a stability analysis on the flow
- To control the free surface towards a target free-surface profile

In Chapter 2, we introduce the system we will investigate. In particular, we obtain and nondimensionalise the governing equations and boundary conditions. We then derive a reduced-order model that describes the flow using the WRIBL method.

In Chapter 3, we validate the reduced-order model obtained in Chapter 2 against DNS. In particular, we consider the locations of the peaks in the free surface, the radial and azimuthal fluxes, and the radial and azimuthal velocity profiles of the reduced-order model and DNS.

In Chapter 4, we perform a parametric study for the case of a planar disk. In particular, we consider the effect of the inverse Weber number and a small random noisy forcing at the inlet on the flow.

In Chapter 5, we consider the effect of a topographically-patterned disk on the flow. In particular, we introduce a family of topographies with parameters governing the asymmetry, smoothness, amplitude and spatial frequency of the topography. We then consider the effect of each of these parameters on the flow.

In Chapter 6, we consider the behaviour in the far field for a planar disk. In the far field, we find three spatial regions with corresponding equations and boundary conditions. We investigate the nature of the spatial and temporal stability in each region.

In Chapter 7, we consider control of the free surface towards a target free-surface profile in the outer region (the outermost spatial region in the far field) using the topography of the disk as a control mechanism. In particular, we consider the problem at leading order and at first order in ζ (which will be defined in Chapter 7). In both cases, we obtain a solution at steady state for the topography in terms of the free surface.

In Chapter 8, we give our concluding remarks, provide a summary of our analysis, and highlight possible directions for future research.

1.8 Publications arising from this thesis

The results presented in Chapters 2–6 have been condensed into a single manuscript, which has been submitted for publication and is currently under review.

Chapter 2

Modelling of thin-film flow over a spinning disk

In this chapter, we begin in Section 2.1 by introducing the system that we will consider, in particular, obtaining and nondimensionalising the governing equations and boundary conditions. In Section 2.2, we derive a reduced-order model valid for thin films using the thin-film approximation described at the beginning of Section 1.3 combined with the WRIBL method as described in Section 1.3.4.

2.1 Problem formulation

We consider the flow of a thin film of Newtonian liquid of constant density $\tilde{\rho}$ and constant kinematic viscosity $\tilde{\nu}$ over a rigid impermeable topographically-patterned disk of radius $\tilde{R}_{\text{disk}}$ rotating about its axis with constant angular velocity $\tilde{\Omega}$. The liquid enters on to the disk via a jet of radius $\tilde{R}_{\text{jet}}$ ($< \tilde{R}_{\text{disk}}$) located at the centre of the disk with volume flux $\tilde{Q}_{\text{jet}}$. The liquid is bounded above by an inviscid,

hydrodynamically-passive gas at constant pressure $\tilde{p}_a$ and the liquid-gas interface has constant surface tension $\tilde{\gamma}$. We note that throughout tildes denote dimensional quantities.

We work in the non-inertial frame of reference which rotates with the disk and use a cylindrical coordinate system $(\tilde{r}, \theta, \tilde{z})$ with origin located at the centre of the disk and whose $\tilde{z}$ -axis coincides with that of the disk (i.e. in the case of a planar disk, the surface of disk is located in the plane $\tilde{z} = 0$). The liquid-gas interface and surface of the disk are located at $\tilde{z} = \tilde{h}(\tilde{r}, \theta, \tilde{t})$ and $\tilde{z} = \tilde{b}(\tilde{r}, \theta)$, respectively, where $\tilde{t}$ denotes time. Therefore, the film thickness $\tilde{\eta}(\tilde{r}, \theta, \tilde{t})$ is defined by $\tilde{\eta} = \tilde{h} - \tilde{b}$. The liquid has pressure $\tilde{p}(\tilde{r}, \theta, \tilde{z}, \tilde{t})$ and velocity field $\tilde{\mathbf{u}} = \tilde{u}(\tilde{r}, \theta, \tilde{z}, \tilde{t})\mathbf{e}_{\tilde{r}} + \tilde{v}(\tilde{r}, \theta, \tilde{z}, \tilde{t})\mathbf{e}_{\theta} + \tilde{w}(\tilde{r}, \theta, \tilde{z}, \tilde{t})\mathbf{e}_{\tilde{z}}$, where $\mathbf{e}_{\tilde{r}}$, $\mathbf{e}_{\theta}$ and $\mathbf{e}_{\tilde{z}}$ are the unit vectors in the radial, azimuthal and vertical directions, respectively. The radial and azimuthal volume fluxes $f(\tilde{r}, \theta, \tilde{t})$ and $g(\tilde{r}, \theta, \tilde{t})$, respectively, are defined as

$$\tilde{f} = \tilde{r} \int_{\tilde{b}}^{\tilde{h}} \tilde{u} \, d\tilde{z}, \quad \tilde{g} = \tilde{r} \int_{\tilde{b}}^{\tilde{h}} \tilde{v} \, d\tilde{z}. \quad (2.1)$$

At the inlet $\tilde{r} = \tilde{R}_{\text{jet}}$, we assume that the flow from the inlet jet is directed purely in the radial direction, i.e. $\tilde{f}(\tilde{R}_{\text{jet}}, \theta, \tilde{t}) = \tilde{Q}_{\text{jet}}/(2\pi)$ and $\tilde{g}(\tilde{R}_{\text{jet}}, \theta, \tilde{t}) = 0$. Figure 2.1 shows a schematic representation of the system.

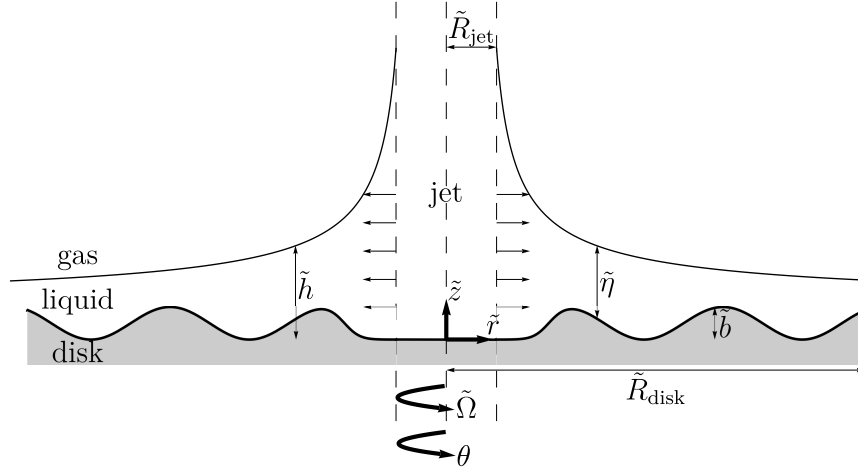


Figure 2.1: Schematic representation of thin-film flow over a topographically-patterned spinning disk.

2.1.1 Governing equations and boundary conditions

The liquid satisfies the continuity and Navier–Stokes equations, which in this frame of reference are

$$\nabla \cdot \tilde{\mathbf{u}} = 0, \quad (2.2)$$

$$\frac{D\tilde{\mathbf{u}}}{D\tilde{t}} + 2\tilde{\boldsymbol{\Omega}} \times \tilde{\mathbf{u}} + \tilde{\boldsymbol{\Omega}} \times (\tilde{\boldsymbol{\Omega}} \times \tilde{\mathbf{X}}) = -\frac{1}{\tilde{\rho}} \nabla \tilde{p} + \tilde{\nu} \nabla^2 \tilde{\mathbf{u}}, \quad (2.3)$$

where $D/D\tilde{t} = \partial/\partial\tilde{t} + \tilde{\mathbf{u}} \cdot \nabla$ is the material derivative, $\tilde{\boldsymbol{\Omega}} = \tilde{\Omega} \mathbf{e}_z$ is the (constant) rotation vector, and $\tilde{\mathbf{X}} = \tilde{r} \mathbf{e}_r + \tilde{z} \mathbf{e}_z$ is the position relative to the origin. In (2.3) the term $2\tilde{\boldsymbol{\Omega}} \times \tilde{\mathbf{u}}$ is the contribution due to the Coriolis force, and the term $\tilde{\boldsymbol{\Omega}} \times (\tilde{\boldsymbol{\Omega}} \times \tilde{\mathbf{X}})$ is the contribution due to the centrifugal force [7]. These two forces are fictitious forces introduced into the governing equations as a result of being in a non-inertial frame of reference. The boundary conditions are no-slip and impermeability at the surface of the disk $\tilde{z} = \tilde{b}$,

$$\tilde{\mathbf{u}} = \mathbf{0}, \quad (2.4)$$

and the normal and tangential stress conditions at the free surface $\tilde{z} = \tilde{h}$,

$$\left[\tilde{\mathbf{n}} \cdot \tilde{\mathbf{T}} \cdot \tilde{\mathbf{n}} \right]_L^G = \tilde{\gamma} \tilde{\kappa}, \quad \left[\tilde{\mathbf{n}} \cdot \tilde{\mathbf{T}} \cdot \tilde{\mathbf{t}}^{(\tilde{r}, \theta)} \right]_L^G = 0, \quad (2.5)$$

where $[\cdot]_L^G$ denotes the change in the given quantity at the free surface between the liquid and gas regions. The stress tensor is given by

$$\tilde{\mathbf{T}} = -\tilde{p}\mathbf{I} + \tilde{\nu}\tilde{\rho} \left[\nabla \tilde{\mathbf{u}} + (\nabla \tilde{\mathbf{u}})^T \right], \quad (2.6)$$

where $\mathbf{I}$ is the identity tensor, the outward unit normal and the tangential vectors in the radial and azimuthal directions to the free surface are

$$\tilde{\mathbf{n}} = \frac{-\tilde{h}_{\tilde{r}}\mathbf{e}_{\tilde{r}} - \left(\tilde{h}_{\theta}/r\right)\mathbf{e}_{\theta} + \mathbf{e}_{\tilde{z}}}{\left(1 + \tilde{h}_{\tilde{r}}^2 + \left(\tilde{h}_{\theta}/r\right)^2\right)^{1/2}}, \quad \tilde{\mathbf{t}}^{(\tilde{r})} = \frac{\mathbf{e}_{\tilde{r}} + \tilde{h}_{\tilde{r}}\mathbf{e}_{\tilde{z}}}{\left(1 + \tilde{h}_{\tilde{r}}^2\right)^{1/2}}, \quad \tilde{\mathbf{t}}^{(\theta)} = \frac{\mathbf{e}_{\theta} + \left(\tilde{h}_{\theta}/r\right)\mathbf{e}_{\tilde{z}}}{\left(1 + \left(\tilde{h}_{\theta}/r\right)^2\right)^{1/2}}, \quad (2.7)$$

respectively, and the interfacial curvature is given by

$$\tilde{\kappa}(\tilde{r}, \theta, \tilde{t}) = \nabla \cdot \tilde{\mathbf{n}}. \quad (2.8)$$

The system (2.2)–(2.8) is closed by the kinematic condition at the free surface $\tilde{z} = \tilde{h}$,

$$\frac{D}{D\tilde{t}}(\tilde{z} - \tilde{h}) = 0. \quad (2.9)$$

2.1.2 Nondimensionalisation

We nondimensionalise to condense the dimensional dependence of the problem into dimensionless numbers. A typical example of a nondimensional number often used in fluid dynamics is the Reynolds number, which is defined as the ratio between

inertial and viscous effects. However, in the present problem the Reynolds number is scaled to be unity. The system (2.2)–(2.9) is nondimensionalised in the same way as Matar *et al.* [73],

$$\begin{aligned} \tilde{t} &= \frac{1}{\tilde{\Omega}} t, \quad \left(\tilde{r}, \tilde{R}_{\text{jet}}, \tilde{R}_{\text{disk}} \right) = \tilde{R} (r, R_{\text{jet}}, R_{\text{disk}}), \quad \left(\tilde{z}, \tilde{h}, \tilde{b}, \tilde{\eta} \right) = \tilde{H} (z, h, b, \eta), \\ \tilde{p} &= \tilde{\rho} \tilde{\Omega}^2 \tilde{R}^2 p, \quad (\tilde{u}, \tilde{v}, \tilde{w}) = \tilde{\Omega} \left(\tilde{R} u, \tilde{R} v, \tilde{H} w \right), \quad \left(\tilde{f}, \tilde{g} \right) = \frac{\tilde{Q}_{\text{jet}}}{2\pi} (f, g), \end{aligned} \quad (2.10)$$

where $\tilde{R}$ and $\tilde{H}$ are the radial and axial length scales, defined by

$$\tilde{R} = \left(\frac{\tilde{Q}_{\text{jet}}}{2\pi} \right)^{1/2} \left(\tilde{\nu} \tilde{\Omega} \right)^{-1/4}, \quad \tilde{H} = \left(\frac{\tilde{\nu}}{\tilde{\Omega}} \right)^{1/2}, \quad (2.11)$$

respectively. The radial length scale is set by equating the scales of the radial and azimuthal fluxes (2.1) and the axial length scale is set by a balance of viscosity and centrifugation in (2.3). The velocity scale of $\tilde{\Omega} \tilde{R}$ is a characteristic velocity of the problem. We utilise the disparity in length scales (i.e. $\tilde{R} \gg \tilde{H}$ as the film is thin) and define the aspect ratio ϵ ,

$$\epsilon = \tilde{H} / \tilde{R} \ll 1. \quad (2.12)$$

We also define the dimensionless parameter

$$\lambda = \frac{1}{\tilde{\Omega} \tilde{R}^2} \sqrt{\frac{\tilde{\gamma} \tilde{H}}{\tilde{\rho}}}, \quad (2.13)$$

which is an inverse Weber number that measures the relative importance of surface tension compared to inertial effects.

We assume axisymmetry of the flow so that there is no variation in the azimuthal direction, i.e. that $h, b, \eta, p, u, v, w, f$ and g are all independent of θ . However,

note that this does not imply that the azimuthal velocity v is zero. In particular, in general we find a non-zero azimuthal velocity $v \neq 0$ across the disk.

The system is governed by the continuity equation (2.2) and the radial, azimuthal and vertical components of the Navier–Stokes equations (2.3), which take the dimensionless form

$$\frac{1}{r}(ru)_r + w_z = 0, \quad (2.14)$$

$$u_t + uu_r + ww_z - \frac{(v+r)^2}{r} = -p_r + u_{zz} + \epsilon^2 \left[\frac{1}{r}(ru)_r \right]_r, \quad (2.15)$$

$$v_t + u(r+v)_r + u \frac{v+r}{r} + wv_z = v_{zz} + \epsilon^2 \left[\frac{1}{r}(rv)_r \right]_r, \quad (2.16)$$

$$\epsilon^2(w_t + uw_r + ww_z) = -p_z + \epsilon^2 w_{zz} + \epsilon^4 \frac{1}{r} [rw_r]_r. \quad (2.17)$$

At the surface of the disk $z = b$, we have the no-slip and impermeability conditions (2.4), which take the dimensionless form

$$u = v = w = 0. \quad (2.18)$$

The stress tensor (2.6) takes the dimensionless form

$$\mathbf{T} = \begin{bmatrix} -p + 2\epsilon u_r & \epsilon^2 \left(v_r - \frac{v}{r} \right) & \epsilon u_z + \epsilon^3 w_r \\ \epsilon^2 \left(v_r - \frac{v}{r} \right) & -p + 2\epsilon \frac{u}{r} & \epsilon v_z \\ \epsilon u_z + \epsilon^3 w_r & \epsilon v_z & -p + 2\epsilon^2 w_z \end{bmatrix}. \quad (2.19)$$

At the free surface $z = h(r, t)$, we have the normal stress condition (2.5), which takes

the dimensionless form

$$-p + \frac{2\epsilon^2}{1 + \epsilon^2 h_r^2} [w_z - h_r (u_z + \epsilon^2 w_r) + \epsilon^2 h_r^2 u_r] = -\lambda^2 \kappa, \quad (2.20)$$

where the dimensionless interfacial curvature (2.8) is given by

$$\kappa = -\frac{1}{r} \left[\frac{r h_r}{(1 + \epsilon^2 h_r^2)^{1/2}} \right]_r, \quad (2.21)$$

and the radial and azimuthal tangential stress conditions (2.5), which take the dimensionless form

$$u_z + \epsilon^2 w_r + \frac{2\epsilon^2 h_r}{1 - \epsilon^2 h_r^2} (w_z - u_r) = 0, \quad (2.22)$$

$$v_z - \epsilon^2 h_r \left(v_r - \frac{v}{r} \right) = 0, \quad (2.23)$$

along with the kinematic condition (2.9), which takes the dimensionless form

$$h_t + u h_r - w = 0. \quad (2.24)$$

We note that the radial and azimuthal fluxes f and g (2.1) take the dimensionless form

$$f = r \int_b^h u \, dz, \quad g = r \int_b^h v \, dz. \quad (2.25)$$

The kinematic condition (2.24) (which is applied at the free surface $z = h$) can be rewritten in terms of the radial flux f (2.25). We first use the continuity equation (2.14) to rewrite w at $z = h$ in terms of u , i.e.

$$h_t + u(r, h, t) h_r + \int_b^h \frac{1}{r} (ru)_r \, dz = 0, \quad (2.26)$$

then expand the integral and use the no-slip condition (2.18) to write

$$h_t + u(r, h, t)h_r - u(r, b, t)b' + \frac{1}{r} \int_b^h u \, dz + \int_b^h u_r \, dz = 0, \quad (2.27)$$

which can be rewritten in the form

$$h_t + \frac{1}{r} \int_b^h u \, dz + \frac{\partial}{\partial r} \int_b^h u \, dz = 0. \quad (2.28)$$

We then use the definition of the radial flux (2.1) to yield

$$h_t + \frac{1}{r} f_r = 0. \quad (2.29)$$

This form will prove to be more convenient to use instead of (2.24) when deriving the reduced-order model in Section 2.2.

As mentioned in Section 2.1, at the inlet $r = R_{\text{jet}}$, we assume that the flow from the inlet jet is directed purely in the radial direction, i.e. $f(R_{\text{jet}}, t) = 1$ and $g(R_{\text{jet}}, t) = 0$, where the value of f at the inlet is imposed by the nondimensionalisation (2.10). In the reduced-order model which we derive in Section 2.2 the field variables become the film thickness η , and the radial and azimuthal fluxes f and g . Therefore, we will require initial conditions for each of these variables.

2.2 Reduced-order modelling

In this section, we derive a reduced-order model using the WRIBL method as described by Ruyer-Quil and Manneville [100]. Before proceeding to the calculation itself, we discuss the procedure we will be using. The WRIBL method, as described in Section 1.3.4, is a two-step procedure. The first step is to exploit a suitable small

aspect ratio in the problem in order to reduce the governing equations to a system of boundary-layer-like equations, as will be discussed in detail in Section 2.2.1, these equations are referred to as boundary-layer equations for thin-film flows. The second step is to project the resulting equations onto a suitable set of basis functions in order to determine an asymptotically consistent model, as will be discussed in detail in Section 2.2.2.

In general, weighted residual formulations formally assume that inertial effects enter into the problem at first order (i.e. that inertial effects are comparatively small) [100, 102, 139, 140]. However, in many practical spinning disk problems, the effects of inertia can enter into the problem at leading order [58, 59, 73, 75, 111]. Therefore, before proceeding further, we discuss the modelling procedure and its anticipated validity.

In the present formulation described in Section 2.1.2, the balance between viscous and inertial effects in equations (2.15) and (2.16) means that inertial effects enter the problem at leading order. As a result, the asymptotic structure used in other weighted residual formulations [100, 102, 139, 140], in which viscosity typically balances the driving force (normally gravity, here centrifugation and Coriolis force) at leading order, breaks down. However, extensive testing of models derived using the WRIBL method has shown that they generally give excellent agreement with both DNS and experiments even when the Reynolds number is sufficiently large that inertial effects should enter the problem at leading order; this includes for falling films on inclined planes [23, 30, 105, 139] and films on fibres [31, 102]. As a consequence, in Section 2.2.2 we will follow the same procedure for deriving a WRIBL model as used in previous publications [100, 102, 139, 140]. Although the model derived will be formally approximate rather than asymptotic due to the size of the inertial terms,

this is consistent with the size of the inertial terms as used in [140]. Note that, in particular, we will derive a model adopting the same approach as the “simplified second-order model”, described in Section 1.3.4.3, in which only the leading order contribution of inertia to the model is retained.

Finally, it is important to emphasise that the validity of the model will be tested extensively a posteriori through validation against DNS in Chapter 3.

2.2.1 Boundary-layer equations

In this subsection, we carry out the first step of the WRIBL method, as described in Section 1.3.4, and derive a system of boundary-layer equations for the radial, azimuthal and vertical velocities u , v and w , respectively. We exploit the small aspect ratio ϵ and retain terms up to second order in ϵ in (2.14)–(2.24), however, in line with the simplified second-order model, inertial terms are only retained up to leading order in ϵ . Hence, the continuity equation (2.14), radial and vertical components of the Navier–Stokes equation (2.15) and (2.16), no-slip and impermeability conditions (2.18), azimuthal tangential stress condition (2.23) and kinematic condition (2.29) remain unchanged. The vertical component of the Navier–Stokes equation (2.17) becomes

$$p_z = \epsilon^2 w_{zz}, \quad (2.30)$$

note that, in line with the simplified second-order model, the terms of order $O(\epsilon^2)$ on the left-hand side of (2.17) are inertial and hence are not retained. The radial tangential stress condition (2.22) becomes

$$u_z = -\epsilon^2 [w_r + 2h_r(w_z - u_r)], \quad (2.31)$$

and the normal stress condition (2.20), applying the radial tangential stress condition (2.31), becomes

$$p = \lambda^2 \kappa + 2\epsilon^2 w_z, \quad (2.32)$$

where we have made use of the fact that the radial tangential stress condition (2.31) indicates $u_z = O(\epsilon^2)$ at the free surface $z = h$. The interfacial curvature κ (2.21) becomes

$$\kappa = -\frac{1}{r} (r h_r)_r. \quad (2.33)$$

Integrating (2.30) with respect to the vertical coordinate z from z to h subject to the normal stress condition (2.32) yields an expression for the pressure p , namely

$$p = \lambda^2 \kappa + \epsilon^2 w_z|_{z=h} + \epsilon^2 w_z. \quad (2.34)$$

Substituting (2.34) into the radial and azimuthal components of the Navier–Stokes equation (2.15) and (2.16), and using the continuity equation (2.14) to rewrite w_z in terms of u , yields the so-called boundary-layer equations,

$$u_t + uu_r + wu_z - \frac{(v+r)^2}{r} = u_{zz} - \lambda^2 \kappa_r + 2\epsilon^2 \left[\frac{1}{r} (ru)_r \right]_r - \epsilon^2 [w_z|_{z=h}]_r, \quad (2.35)$$

$$v_t + u(r+v)_r + u \frac{v+r}{r} + wv_z = v_{zz} + \epsilon^2 \left[\frac{1}{r} (rv)_r \right]_r. \quad (2.36)$$

Equations (2.35) and (2.36) are subject to the no-slip and impermeability conditions (2.18), the tangential stress conditions (2.31) and (2.23), and the kinematic condition (2.24). We are now in a position to begin weighted residual modelling.

2.2.2 Weighted residual modelling

In this subsection, we use the boundary-layer equations (2.35) and (2.36) to develop a reduced-order model using the WRIBL method as outlined by Ruyer-Quil and Manneville [100], as described in Section 1.3.4. We begin by deriving leading-order solutions for the velocities in Section 2.2.2.1 and then extend these solutions to first order in Section 2.2.2.2. We then perform the weighted integral procedure in Section 2.2.2.3 and derive the evolution equations, which are subject to the initial and boundary conditions outlined in Section 2.2.2.4.

2.2.2.1 Leading-order solution for the velocities

We begin by deriving leading-order solutions for the radial and azimuthal velocities u and v from assuming the balance of the viscous and centrifugal terms in (2.35), and from assuming the balance of the viscous and Coriolis terms in (2.36), respectively (in line with the discussion at the beginning of Section 2.2, where typically viscosity balances the driving force at leading-order in weighted residual formulations), subject to the leading-order versions of the boundary conditions (2.18), (2.22) and (2.23), namely

$$u_{zz} = -r \quad \text{subject to} \quad u|_{z=b} = 0, \quad u_z|_{z=h} = 0, \quad (2.37)$$

$$v_{zz} = 2u \quad \text{subject to} \quad v|_{z=b} = 0, \quad v_z|_{z=h} = 0. \quad (2.38)$$

Therefore, the leading-order radial and azimuthal velocities take the form

$$u = -\frac{1}{2}r\eta^2 (\bar{z}^2 - 2\bar{z}), \quad (2.39)$$

$$v = -\frac{1}{12}r\eta^4 (8\bar{z} - 4\bar{z}^3 + \bar{z}^4), \quad (2.40)$$

where $\bar{z}(r, t) = (z - b)/\eta$ is a suitably rescaled vertical coordinate which maps the film onto $0 \leq \bar{z} \leq 1$. In line with the discussion at the beginning of Section 2.2, this assumption of the velocity profiles (due to the size of the inertial terms) means that the model will be formally approximate rather than asymptotic, however, the model and velocity profiles will be validated against DNS in Chapter 3. The vertical velocity w can be written in terms of u by integrating the continuity equation (2.14) with respect to the vertical coordinate z from b to z , i.e.

$$w = -\frac{1}{r} \int_b^z (ru)_r \, dz. \quad (2.41)$$

We note that equation (2.41) is an exact expression and can therefore be applied to higher-order solutions. We compute the expression for the leading-order vertical velocity w by substituting the radial velocity u (2.39) into (2.41)

$$w = \bar{z}\eta^2 \left(\frac{1}{3}\bar{z}^2\eta - \frac{1}{2}\bar{z}(2\eta + r\eta_r + rb') + rb' \right). \quad (2.42)$$

2.2.2.2 First-order solution for the velocities

To proceed to higher order, we project u and v onto appropriate sets of $N + 1$ basis functions $\mathcal{F}_n(\bar{z})$ and $\mathcal{G}_n(\bar{z})$ for $n = 0, 1, 2, \dots, N$, respectively. This is separation of variables and these basis functions account for the z dependence of the velocities. These basis functions have coefficients $a_n(r, t)$ and $b_n(r, t)$ for $n = 0, 1, 2, \dots, N$, respectively. As the vertical velocity w can be written in terms of u , as shown in (2.41), w does not require projection onto another set of basis functions in $\bar{z}$. Hence,

we consider expansions of u and v of the form

$$u = a_0(r, t)\mathcal{F}_0(\bar{z}) + \epsilon \sum_{n=1}^N a_n(r, t)\mathcal{F}_n(\bar{z}), \quad (2.43)$$

$$v = b_0(r, t)\mathcal{G}_0(\bar{z}) + \epsilon \sum_{n=1}^N b_n(r, t)\mathcal{G}_n(\bar{z}), \quad (2.44)$$

where ϵ is the aspect ratio and the leading-order velocities (2.39) and (2.40) require the functions $\mathcal{F}_0$ and $\mathcal{G}_0$ to be of the form

$$\mathcal{F}_0(\bar{z}) = \bar{z}^2 - 2\bar{z}, \quad (2.45)$$

$$\mathcal{G}_0(\bar{z}) = 8\bar{z} - 4\bar{z}^3 + \bar{z}^4. \quad (2.46)$$

In principle, the coefficients a_n and b_n for $n = 0, 1, 2, \dots, N$ can be determined by substituting the expressions (2.43) and (2.44) into the boundary-layer equations (2.35) and (2.36), then solving the resulting system of equations order by order in powers of $\bar{z}$ (this process has been followed by, for example, Ruyer-Quil and Manneville [100], Oron and Heining [85] and Wray *et al.* [140]). However, in general, this process is heavily algebraically involved. As discussed in Section 1.3.4, Ruyer-Quil and Manneville [100] demonstrated that this process can be simplified by utilising a weighted integral procedure. In particular, the evolution equations for the film thickness, and the radial and azimuthal fluxes are therefore determined by suitable weighted integrals of the boundary-layer equations (2.35) and (2.36) with respect to the vertical coordinate z across the film thickness from b to h , with the weights for the integrals being given by the solution to the adjoint version of the problem. This particular choice of weights minimises the algebra required, as shall be demonstrated in what follows. Utilising this weighted integral procedure removes

the need to explicitly calculate the coefficients a_n and b_n and basis functions $\mathcal{F}_n(\bar{z})$ and $\mathcal{G}_n(\bar{z})$, such that only the leading-order solutions for the radial and azimuthal velocities are required to formulate the evolution equations (whilst accounting for higher-order terms). In practice, it is convenient to rewrite a_0 and b_0 in terms of the radial and azimuthal fluxes f and g , as this allows the kinematic equation of the form (2.29) to be applied.

Integrating (2.43) and (2.44) with respect to the vertical coordinate z across the film thickness from b to h , we find

$$f = r \left(a_0 \int_b^h \mathcal{F}_0(\bar{z}) \, dz + \epsilon \sum_{n=1}^N a_n \int_b^h \mathcal{F}_n(\bar{z}) \, dz \right), \quad (2.47)$$

$$g = r \left(b_0 \int_b^h \mathcal{G}_0(\bar{z}) \, dz + \epsilon \sum_{n=1}^N b_n \int_b^h \mathcal{G}_n(\bar{z}) \, dz \right), \quad (2.48)$$

which can be rewritten to give the coefficients a_0 and b_0 in terms of the radial and azimuthal fluxes f and g , namely

$$a_0 = \frac{f - \epsilon r \sum_{n=1}^N a_n \int_b^h \mathcal{F}_n(\bar{z}) \, dz}{r \int_b^h \mathcal{F}_0(\bar{z}) \, dz}, \quad (2.49)$$

$$b_0 = \frac{g - \epsilon r \sum_{n=1}^N b_n \int_b^h \mathcal{G}_n(\bar{z}) \, dz}{r \int_b^h \mathcal{G}_0(\bar{z}) \, dz}. \quad (2.50)$$

Therefore, the radial and azimuthal velocities (2.43) and (2.44) with the substitution of the leading-order coefficients (2.49) and (2.50), and the basis functions (2.45) and

(2.46) take the form

$$\begin{aligned}
 u &= \frac{3f}{\eta r} \left(\bar{z} - \frac{\bar{z}^2}{2} \right) + \epsilon \left[\sum_{n=1}^N a_n \mathcal{F}_n(\bar{z}) - \frac{3}{\eta} \left(\bar{z} - \frac{\bar{z}^2}{2} \right) \sum_{n=1}^N a_n \int_b^h \mathcal{F}_n(\bar{z}) \, dz \right], \quad (2.51) \\
 v &= \frac{5g}{2\eta r} \left(\bar{z} - \frac{\bar{z}^3}{2} + \frac{\bar{z}^4}{8} \right) + \epsilon \left[\sum_{n=1}^N b_n \mathcal{G}_n(\bar{z}) - \frac{5}{2\eta} \left(\bar{z} - \frac{\bar{z}^3}{2} + \frac{\bar{z}^4}{8} \right) \sum_{n=1}^N b_n \int_b^h \mathcal{G}_n(\bar{z}) \, dz \right]. \quad (2.52)
 \end{aligned}$$

2.2.2.3 Weighted integral procedure

We now multiply the boundary-layer equations (2.35) and (2.36) by the appropriate weight $\mathcal{W}(\bar{z})$ (which, as mentioned previously, is the solution to the corresponding adjoint problem) and integrate with respect to the vertical coordinate z across the film thickness from b to h . We will then consider the forms of the weighted integrals with the weight $\mathcal{W}$ that minimises the algebra required. The weighted integrals of the boundary layer equations (2.35) and (2.36) are

$$\begin{aligned}
 \int_b^h \mathcal{W} \left(u_t + uu_r + wu_z - \frac{(v+r)^2}{r} \right) dz &= \int_b^h \mathcal{W} u_{zz} dz - \int_b^h \mathcal{W} \lambda^2 \kappa_r dz \\
 &+ \epsilon^2 \left(\int_b^h 2\mathcal{W} \left[\frac{1}{r} (ru)_r \right]_r dz - \int_b^h \mathcal{W} [w_z|_{z=h}]_r dz \right), \quad (2.53)
 \end{aligned}$$

$$\begin{aligned}
 &\int_b^h \mathcal{W} \left(v_t + u(r+v)_r + u \frac{v+r}{r} + wv_z \right) dz \\
 &= \int_b^h \mathcal{W} v_{zz} dz + \epsilon^2 \int_b^h \mathcal{W} \left[\frac{1}{r} (rv)_r \right]_r dz, \quad (2.54)
 \end{aligned}$$

where u is given by (2.51), v is given by (2.52), and w is given by (2.41). Terms involving the coefficients a_n and b_n for $n = 1, \dots, N$ would enter (2.53) and (2.54) at first order in ϵ , however, in line with the simplified second-order model (described in

Section 1.3.4.3) inertial terms are only retained up to leading order in ϵ . Therefore, the coefficients a_n and b_n for $n = 1, \dots, N$ only enter the problem via the leading-order viscous terms in (2.53) and (2.54) (i.e. the first terms on the right-hand sides of (2.53) and (2.54)). Hence, if the viscous terms can be computed explicitly without having to determine the a_n and b_n for $n = 1, \dots, N$, we can completely avoid the need to determine the a_n and b_n , and hence the $\mathcal{F}_n$ and $\mathcal{G}_n$ for $n = 1, \dots, N$. This can be achieved via a suitable selection of the weight $\mathcal{W}$.

To find the suitable weight $\mathcal{W}(\bar{z})$ for the radial boundary-layer equation (2.53), we evaluate the weighted integral of the leading-order viscous term in (2.53), yielding

$$\begin{aligned} \int_b^h \mathcal{W} u_{zz} \, dz &= [\mathcal{W} u_z]_{z=h} - [\mathcal{W} u_z]_{z=b} \\ &\quad - \frac{1}{\eta} [\mathcal{W}_{\bar{z}} u]_{z=h} + \frac{1}{\eta} [\mathcal{W}_{\bar{z}} u]_{z=b} + \frac{1}{\eta^2} \int_b^h \mathcal{W}_{\bar{z}\bar{z}} u \, dz, \end{aligned} \quad (2.55)$$

where we have performed a double integration by parts. We apply the no-slip condition (2.18) to (2.55) to find

$$\int_b^h \mathcal{W} u_{zz} \, dz = [\mathcal{W} u_z]_{z=h} - [\mathcal{W} u_z]_{z=b} - \frac{1}{\eta} [\mathcal{W}_{\bar{z}} u]_{z=h} + \frac{1}{\eta^2} \int_b^h \mathcal{W}_{\bar{z}\bar{z}} u \, dz. \quad (2.56)$$

In order to eliminate the dependence of (2.56) on the a_n and $\mathcal{F}_n$ for $n = 1, \dots, N$, we choose the weight $\mathcal{W}$ such that

$$\mathcal{W}_{\bar{z}\bar{z}}(\bar{z}) = 1, \quad \mathcal{W}(0) = 0, \quad \mathcal{W}_{\bar{z}}(1) = 0, \quad (2.57)$$

where the first condition in (2.57) is such that the last term on the right-hand side of (2.56) can be written as $f/(r\eta^2)$, the second condition in (2.57) eliminates the second

term on the right-hand side of (2.56), and the third condition in (2.57) eliminates the third term on the right-hand side of (2.56). This leaves the first and fourth terms on the right-hand side of (2.56), however, these terms can be evaluated explicitly. The first term on the right-hand side of (2.56) can be evaluated up to second order in ϵ by using the radial tangential stress condition (2.31) which gives the expression for u_z at the free surface $z = h$. The fourth term can be rewritten in terms of the radial flux f (2.25), and this is valid up to second-order as the radial flux f is exact. Note that the problem (2.57) for $\mathcal{W}$ is identical to the adjoint to the leading-order problem (2.37) for u . Solving (2.57) yields the weight

$$\mathcal{W} = \frac{1}{2} (\bar{z}^2 - 2\bar{z}). \quad (2.58)$$

Therefore, (2.56) can be evaluated explicitly and is given by

$$\int_b^h \mathcal{W} u_{zz} \, dz = \frac{f}{\eta^2 r} - \frac{1}{2} u_z|_{z=h}, \quad (2.59)$$

where an expression for $u_z|_{z=h}$ is given by the radial tangential stress condition (2.31). This gives the viscous term in (2.53) valid up to second order in ϵ (as f is exact and the expression for $u_z|_{z=h}$ is valid up to second order in ϵ from the radial tangential stress condition (2.31)) whilst eliminating the dependence on a_n and $\mathcal{F}_n$ for $n = 1, \dots, N$.

For the azimuthal boundary-layer-equation (2.54), we again consider the leading-order viscous term in (2.54) by performing a double integration by parts and applying the impermeability condition (2.18) to obtain

$$\int_b^h \mathcal{W} v_{zz} \, dz = [\mathcal{W} v_z]_{z=h} - [\mathcal{W} v_z]_{z=b} - \frac{1}{\eta} [\mathcal{W}_{\bar{z}} v]_{z=h} + \frac{1}{\eta^2} \int_b^h \mathcal{W}_{\bar{z}\bar{z}} v \, dz. \quad (2.60)$$

As (2.60) is identical to (2.56) with u replaced by v , the same weight (2.58) is suitable for the explicit calculation of this term. We also note that the adjoint to the leading-order problem (2.38) for v is identical to the problem (2.57) for $\mathcal{W}$. Hence, in this problem, the same weight is appropriate for the weighted integral of both boundary-layer equations, however, we note that in general this does not always hold true. Therefore, we use the same weight (2.58) as previously and can evaluate (2.60) explicitly to obtain

$$\int_b^h \mathcal{W} v_{zz} \, dz = \frac{g}{\eta^2 r} - \frac{1}{2} v_z|_{z=h}, \quad (2.61)$$

where an expression for $v_z|_{z=h}$ is given by the azimuthal tangential stress condition (2.23). This gives the viscous term in (2.54) valid up to second order in ϵ (as g is exact and the expression for $v_z|_{z=h}$ is valid up to second order in ϵ from the azimuthal tangential stress condition (2.23)) whilst eliminating the dependence on the b_n and $\mathcal{G}_n$ for $n = 1, \dots, N$.

The carefully chosen weight $\mathcal{W}$ (2.58) therefore allows higher-order terms to be computed using the radial, azimuthal and vertical velocities, u , v and w , up to leading order in ϵ . The radial and azimuthal velocities, (2.51) and (2.52), up to leading order in ϵ are

$$u = \frac{3f}{\eta r} \left(\bar{z} - \frac{\bar{z}^2}{2} \right), \quad (2.62)$$

$$v = \frac{5g}{2\eta r} \left(\bar{z} - \frac{\bar{z}^3}{2} + \frac{\bar{z}^4}{8} \right), \quad (2.63)$$

respectively. We can determine the expression for the leading-order vertical velocity w in terms of the radial flux f by substituting the leading-order radial velocity u

(2.62) into (2.41) to obtain

$$w = \frac{3f}{\eta r} \left(\bar{z} - \frac{\bar{z}^2}{2} \right) (b' + \bar{z}\eta_r) - \frac{\bar{z}^2 f_r}{2r} (3 - \bar{z}). \quad (2.64)$$

We now follow [100, 138] by computing the weighted integrals of the boundary-layer equations (2.53) and (2.54). The viscous terms are given by (2.59) and (2.61), respectively, and the leading-order radial, azimuthal and vertical velocities, u , v and w , given by (2.62), (2.63) and (2.64), respectively, which are sufficient for computing the higher-order terms due to the use of the weight $\mathcal{W}$ (2.58). This yields the evolution equations, namely

$$h_t + \frac{f_r}{r} = 0, \quad (2.65)$$

$$\begin{aligned} f_t - \frac{9f^2\eta_r}{7\eta^2r} - \frac{9f^2}{7\eta r^2} + \frac{5f}{2\eta^2} + \frac{17ff_r}{7\eta r} - \frac{925g^2}{693\eta r^2} - \frac{85g}{42} - \frac{5}{6}\eta r^2 + \frac{5}{6}\lambda^2\eta r\kappa_r \\ + \epsilon^2 \left[-\frac{4f\eta_r^2}{\eta^2} - \frac{9f\eta_r}{4\eta r} + \frac{9f_r\eta_r}{2\eta} + \frac{6f\eta_{rr}}{\eta} + \frac{9f_r}{2r} - \frac{9f_{rr}}{2} + \frac{5fb_r^2}{\eta^2} + \frac{5fb_r\eta_r}{2\eta^2} + \frac{15fb_{rr}}{4\eta} \right] = 0, \end{aligned} \quad (2.66)$$

$$\begin{aligned} g_t - \frac{22fg\eta_r}{17\eta^2r} + \frac{59gf_r}{51\eta r} + \frac{22fg_r}{17\eta r} + \frac{168f}{85} + \frac{42g}{17\eta^2} + \epsilon^2 \left[-\frac{215g\eta_r^2}{272\eta^2} + \frac{291g_r\eta_r}{272\eta} + \frac{321g\eta_r}{136\eta r} \right. \\ \left. + \frac{3g\eta_{rr}}{2\eta} + \frac{g_r}{r} - g_{rr} + \frac{42gb_r^2}{17\eta^2} + \frac{329gb_r\eta_r}{272\eta^2} + \frac{385gb_r}{136\eta r} + \frac{35g_rb_r}{272\eta} + \frac{35gb_{rr}}{34\eta} \right] = 0. \end{aligned} \quad (2.67)$$

Equations (2.65)–(2.67) constitute a simplified second-order model, i.e. a model which only retains leading-order inertial effects, for the film thickness η and the radial and azimuthal fluxes f and g . This model includes the effects of inertia, viscosity, centrifugation, Coriolis force and capillarity. At leading order in ϵ , (2.65)–(2.67) only differ from the corresponding equations given by Matar *et al.* [75] by

small variations in the values of the coefficients. These differences arise because Matar *et al.* [75] employed the Kármán–Pohlhausen method, in which the velocity profiles are represented using a single prescribed polynomial shape (i.e. a single basis function) for each velocity component. By contrast, the WRIBL method used here refines these profiles by incorporating higher-order corrections through additional basis functions (given by (2.51) and (2.52)), which leads to the variations in the coefficients appearing in (2.65)–(2.67).

2.2.2.4 Initial and boundary conditions

We require appropriate initial and boundary conditions in order to obtain solutions to equations (2.65)–(2.67). We therefore seek steady-state solutions of equations (2.65)–(2.67). The steady-state version of the kinematic condition (2.65) implies that f is a constant and the nondimensionalisation (2.10) imposes $f = 1$ at the radial position $r = R_{\text{jet}}$, and hence $f \equiv 1$. While we cannot solve for the general steady state analytically for the film thickness η and the azimuthal flux g , following Matar *et al.* [73, 75], we can derive an approximation that is valid in the periphery of a large disk (i.e. a disk satisfying $R_{\text{disk}} \gg 1$) by examining the steady solution in the far field (i.e. when $r \gg 1$) at leading order in the aspect ratio ϵ and in r^{-1} , in which we find that the dominant terms in (2.66) and (2.67) are

$$\frac{5}{2\eta^2} - \frac{5}{6}\eta r^2 = 0, \quad \frac{168}{85} + \frac{42g}{17\eta^2} = 0, \quad (2.68)$$

respectively, which have the real solution

$$\eta = \left(\frac{3}{r^2}\right)^{1/3}, \quad g = -\frac{4}{5}\left(\frac{3^2}{r^4}\right)^{1/3}, \quad (2.69)$$

which relates physically to a balance of viscosity and centrifugation in (2.66) and a balance of viscosity and Coriolis force in (2.67). We note that the solution (2.69) is valid for any topography b as the effects of topography only enter at leading order in ϵ through the curvature of the free surface κ in the capillarity term in (2.66). Following Matar *et al.* [73, 75], we impose the initial condition based on the balance of terms in the far field of

$$f(r, 0) = f_N(r) = 1, \quad \eta(r, 0) = \eta_N(r) = \left(\frac{3}{r^2}\right)^{1/3}, \quad g(r, 0) = g_N(r) = 0, \quad (2.70)$$

and use the inlet boundary conditions

$$f(R_{\text{jet}}, t) = 1, \quad \eta(R_{\text{jet}}, t) = \left(\frac{3}{R_{\text{jet}}^2}\right)^{1/3}, \quad g(R_{\text{jet}}, t) = 0. \quad (2.71)$$

We note that the inlet boundary condition $g = 0$ (2.71) arises from the assumption that the flow at the inlet is directed purely in the radial direction. For consistency, we also set $g = 0$ in the initial conditions (2.70), as the solution based on a balance of terms in the far field (2.69) does not satisfy the inlet condition and therefore cannot be used as the initial condition. In contrast, the initial conditions for f and η (2.70) are based on a balance of terms in the far field. Hence, the initial condition (2.70) is not an exact steady state of the system. Therefore, when solving the evolution equations (2.65)–(2.67) numerically, there is a (unknown) finite perturbation away from an exact steady-state solution built into the initial conditions at $t = 0$. After this time, the system will evolve into a fully developed state.

2.3 Concluding remarks

In this chapter, we have derived a novel reduced-order model for thin-film flow over a spinning disk, namely equations (2.65)–(2.67).

In Section 2.1, we introduced the problem and described the relevant governing equations and boundary conditions in Section 2.1.1. The governing equations and boundary conditions were then nondimensionalised in Section 2.1.2, which introduced the aspect ratio ϵ and the inverse Weber number λ .

In Section 2.2, we derived a reduced-order model valid for thin-films using the WRIBL method [100], namely, equations (2.65)–(2.67). In particular, the governing equations were reduced to a system of boundary-layer-like equations in Section 2.2.1, and the application of the WRIBL method was described in Section 2.2.2.

Chapter 3

Validation of the reduced-order model

In this chapter, we validate the reduced-order model (2.65)–(2.67) against DNS for the case of a planar disk (i.e. $b \equiv 0$). In particular, we consider the locations of the peaks in the free surface, the radial and azimuthal fluxes, and the velocity profiles of the reduced-order model and DNS in Sections 3.1, 3.2, 3.3, and 3.4, respectively. We consider the behaviour far from the inlet in a region which we shall show is stable, as will be discussed further numerically in Chapter 4 and then analytically in Chapter 6. This region is chosen for comparison as its stability makes it more tractable for comparisons. In unstable regions, even minute differences in the steady state and the inlet conditions between the reduced-order model and the DNS are amplified by the instability, making meaningful quantitative comparisons difficult.

We implemented a code in C++ to simulate the reduced-order model (2.65)–(2.67), based on a pre-existing code developed by one of the author’s supervisors (Dr

Alexander W. Wray) that has previously been used successfully to solve other problems (see, for example, [76, 139, 141]). The code uses the method of lines to convert the system of PDEs into a system of coupled nonlinear ODEs. It imposes the inlet boundary conditions (2.71) at the radial position $r = R_{\text{jet}}$ and a Neumann boundary condition at the edge of the disk (i.e. $r = R_{\text{disk}}$) and uses a uniform grid. The code uses centred second-order finite differences in space and the backwards Euler method in time, with a uniform grid of $N = 4000$ points, and a time step of 10^{-5} . Newton iterations were stopped at a residual tolerance of 10^{-10} , and simulations were run to $t = 50$. Convergence was verified via grid/time-step refinement in both space and time. Specifically, the backwards Euler method was chosen due to its stability properties. Solving the ODEs requires solving a coupled nonlinear system at each time step. This is done using the Newton–Raphson method. Discretisation using finite differences results in each linear step of the Newton–Raphson involving sparse diagonal matrices that are solved using the Eigen C++ template library for linear algebra [43]. To demonstrate spatial convergence, we performed a refinement test by comparing solutions for h at $t = 10$ across four spatial resolutions $N = 1000, 2000, 4000$ and 8000 , which we denote by U_N . The solution with $N = 8000$ was taken as a reference and the error was quantified using the Euclidean norm, $\|U_N - U_{8000}\|$. Figure 3.1 shows a log-log plot of error at $t = 10$ for the reduced-order model using the spatial resolutions $N = 1000, 2000$ and 4000 . In particular, Figure 3.1 shows that the two coarsest grids, $N = 1000$ and 2000 , are strongly under-resolved, with errors significantly larger than those at finer resolutions. In contrast, the solutions for $N = 4000$ and 8000 are effectively indistinguishable, and the corresponding error is extremely small, indicating that the solution has converged by $N = 4000$.

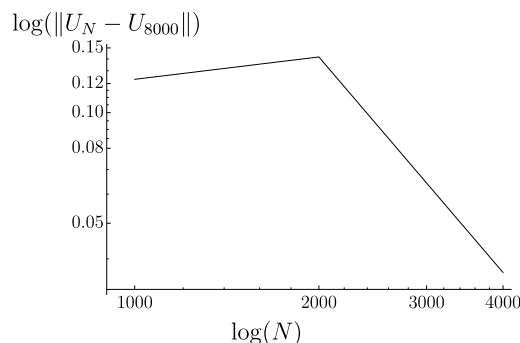


Figure 3.1: Log-log plot of error at $t = 10$ for the reduced-order model of flow over a planar disk with $\lambda = 0.025$, inlet jet radius $R_{\text{jet}} = 25$ and disk radius $R_{\text{disk}} = 40$ without inlet forcing, using the spatial resolutions $N = 1000, 2000$ and 4000 .

We also implemented a second code also in C++ to perform DNS of equations (2.14)–(2.17), i.e. the full Navier–Stokes equations. This is based on the same pre-existing code. It remaps t , r and z onto $\bar{t}$, $\bar{r}$ and $\bar{z}$, respectively, where $\bar{t} = t$, $\bar{r} = r$ and $\bar{z} = z/h(r, t)$. This maps the vertical coordinate from $0 \leq z \leq h$ onto $0 \leq \bar{z} \leq 1$, so that the domain becomes rectilinear and constant in time. Then a uniform grid is used in both the $\bar{r}$ and $\bar{z}$ directions. The DNS requires initial conditions for the radial, azimuthal and vertical velocities u , v and w , and the free surface h . The code imposes a Neumann boundary condition at the edge of the disk (i.e. $r = R_{\text{disk}}$). The inlet boundary conditions for the radial velocity u , azimuthal velocity v , and free surface h are

$$u(R_{\text{jet}}, z, t) = \frac{3}{h_N R_{\text{jet}}} \left(\bar{z} - \frac{\bar{z}^2}{2} \right), \quad v(R_{\text{jet}}, z, t) = 0, \quad h(R_{\text{jet}}, t) = \left(\frac{3}{R_{\text{jet}}^2} \right)^{1/3}, \quad (3.1)$$

which correspond to $f = 1$ and $g = 0$ in the reduced-order model (i.e. zero swirl at the inlet), and (2.71) for the reduced-order model, respectively. The vertical velocity w does not require an inlet boundary condition to be specified as it can be obtained from the continuity equation (2.14), and for the pressure p the Neumann boundary

condition at the edge of the disk is sufficient. At the inlet one-sided differences are used for the vertical velocity w and pressure p . The code uses centred second-order finite differences in space and the backwards Euler method in time, with a uniform grid of 1000 points in the radial direction and 20 points in the vertical direction, and a time step of 10^{-3} . Newton iterations were stopped at a residual tolerance of 10^{-10} , and simulations were run to $t = 50$. Convergence was verified via grid/time-step refinement in both space and time. Again, the Eigen C++ template library for linear algebra [43] was used to perform the linear algebra required.

We use a peak location metric, which allows us to plot the location of each peak at a given time step. We find the locations of the peaks during post processing of h from the code. However, a threshold value for what constitutes a peak needs to be chosen. We set the peak threshold to be the mean of the positive values of $h - \langle h \rangle$, where $\langle h \rangle$ is the mean of h . This choice provides a consistent, data-driven threshold for identifying peaks and is not arbitrary. The peak threshold is then used in the **FindPeaks** function in Mathematica [136] at each time step and this data is then collated into space-time plots of the peaks of the free surface.

The DNS solves equations (2.14)–(2.17) subject to the no-slip and impermeability conditions (2.18), the normal and tangential stress conditions (2.20), (2.22) and (2.23), and the kinematic condition (2.24), which have the field variables h , u , v , w and p . Therefore, the radial and azimuthal fluxes f and g need to be computed during post-processing to allow for comparison against the reduced-order model (2.65)–(2.67), which have the field variables h , f and g . This post-processing was carried out in Mathematica, using the definitions of f and g (2.25). Specifically, the Mathematica function **NIntegrate** was used to perform the numerical integration of u and v with respect to z at each radial grid point, using the trapezoidal method.

We compare the reduced-order model and DNS in the radial interval $25 \leq r \leq 40$, by setting $R_{\text{jet}} = 25$ and $R_{\text{disk}} = 40$. This interval is considered as we shall show that the flow is stable in this region, as will be discussed in further detail in Chapters 4 and 6. The value of R_{jet} lies outside the representative range for real water-based systems listed in Table 4.1 because, for validation, the same inlet radius is imposed for both the reduced-order model and DNS to ensure that comparison is performed in the stable region rather than to represent a physical jet radius. Both the reduced-order model and DNS are allowed to settle to a steady state, then at $t = 35$ a perturbation is introduced to the free surface h of the form $0.01 \exp[-100(r - R_{\text{jet}} - 3)^2]$ (i.e. a Gaussian centred at the radial position $r = R_{\text{jet}} + 3$). The evolution of this disturbance is then considered in both the reduced-order model and DNS. The validation is performed on a single domain length chosen to resolve the long-wave structures of interest; a systematic variation of the domain length was not undertaken. Three values of the inverse Weber number λ are considered: $\lambda = 0.025$, $\lambda = 0.0275$ and $\lambda = 0.03$. These values of λ lie within the representative range for real water-based systems (see Chapter 4 and Table 4.1), lower values of λ are not considered in this Chapter as they make the DNS calculations prohibitively expensive. Throughout this chapter, we take $\epsilon^2 = 10^{-5}$, in line with the representative range for real water-based systems listed in Table 4.1. Results will be presented for peak location, radial flux and azimuthal flux in the following subsections.

3.1 Peak location

Figure 3.2 shows a space-time plot of the peaks of the free surface computed by the reduced-order model and DNS for flow over a planar disk with $\lambda = 0.025$, $\lambda = 0.0275$,

and $\lambda = 0.03$, inlet jet radius $R_{\text{jet}} = 25$ and disk radius $R_{\text{disk}} = 40$ over the time interval from $t = 35$ to $t = 50$. In particular, Figure 3.2 shows that the peaks from the reduced-order model follow those from the DNS with generally good qualitative agreement, with agreement improving for larger values of λ . For all three values of λ presented, the peaks from the reduced-order model travel slightly faster than those from the DNS. The additional peaks form in front and behind of the disturbance due to the weak surface tension effects, due to the relatively low values of λ (which measures the relative importance of surface tension to inertia) considered. This effect becomes more pronounced as λ decreases. These peaks form at earlier times for the reduced-order model than for the DNS, however the agreement between the paths is excellent once the peaks form in the DNS. For all three values of λ presented, there are additional small peaks in front and behind of the disturbance in the reduced-order model compared to the DNS, however, the peaks that are present in both the DNS and reduced-order model closely follow the same trajectory. We note that the reduced-order model gives a good prediction of the radial interval which is affected by the disturbance spread of these peaks. Physically, reducing λ corresponds to decreasing surface tension and/or increasing inertia, both of which have a destabilising effect. Hence, for larger values of λ , the disturbance is expected to develop into fewer peaks (which have smaller amplitude). This effect is seen in Figure 3.2, where there is 1 less peak behind the disturbance at $t = 50$ in the case $\lambda = 0.0275$ compared to the case $\lambda = 0.025$, and this behaviour also occurs in the case $\lambda = 0.03$ compared to the case $\lambda = 0.0275$. These values of λ considered are all relatively small, although they are within the range of plausible/realistic values for real-world systems as will be discussed in Chapter 4. Overall, the peak locations of the free surface as predicted by the DNS are predicted well by the reduced-order model.

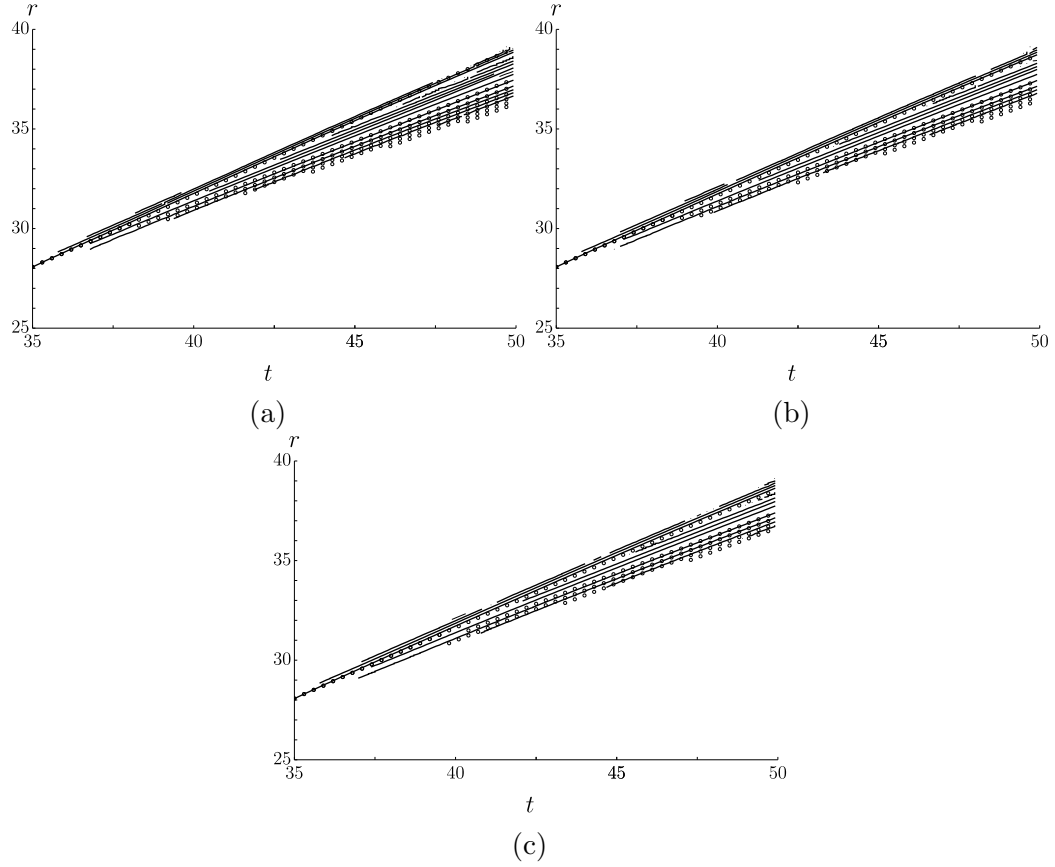


Figure 3.2: Space-time plots of the peaks of the free surface computed by the reduced-order model (2.65)–(2.67) (solid curves) and DNS (2.14)–(2.17) (open circles) for flow over a planar disk with (a) $\lambda = 0.025$, (b) $\lambda = 0.0275$, and (c) $\lambda = 0.03$, inlet jet radius $R_{\text{jet}} = 25$ and disk radius $R_{\text{disk}} = 40$ without inlet forcing over the time interval from $t = 35$ to $t = 50$.

3.2 Radial flux

Figure 3.3 shows plots of the radial flux f computed by the reduced-order model and DNS for flow over a planar disk with $\lambda = 0.025$, $\lambda = 0.0275$, and $\lambda = 0.03$, inlet jet radius $R_{\text{jet}} = 25$ and disk radius $R_{\text{disk}} = 40$ at time $t = 46$. The time $t = 46$ was chosen because the disturbance is well developed while still remaining fully contained within the computational domain for all three values of λ . At earlier times the disturbance is still developing, and at later times it begins to approach R_{disk} , making $t = 46$ the most representative snapshot for comparison. In particular, Figure 3.3 shows that the radial flux f computed by the reduced-order model follows the DNS with generally good qualitative agreement. While there is some disagreement in peak size, the wave speed of the largest peak generated by the disturbance is predicted well by the reduced-order model. The agreement in peak size between the reduced-order model and DNS improves as λ increases. This result is expected as increasing λ corresponds to increasing surface tension and/or reducing inertia, which has a stabilising effect. In all three cases presented, the reduced-order model predicts additional peaks in front and behind of the disturbance. However, the reduced-order model correctly predicts the radial interval that is affected by the disturbance spread of these peaks. Overall, the reduced-order model predicts the radial flux f well in both wave speed and peak size.

3.3 Azimuthal flux

Figure 3.4 shows plots of the azimuthal flux g computed by the reduced-order model and DNS for flow over a planar disk with $\lambda = 0.025$, $\lambda = 0.0275$, and $\lambda = 0.03$, inlet jet radius $R_{\text{jet}} = 25$ and disk radius $R_{\text{disk}} = 40$ at time $t = 46$. In particular,

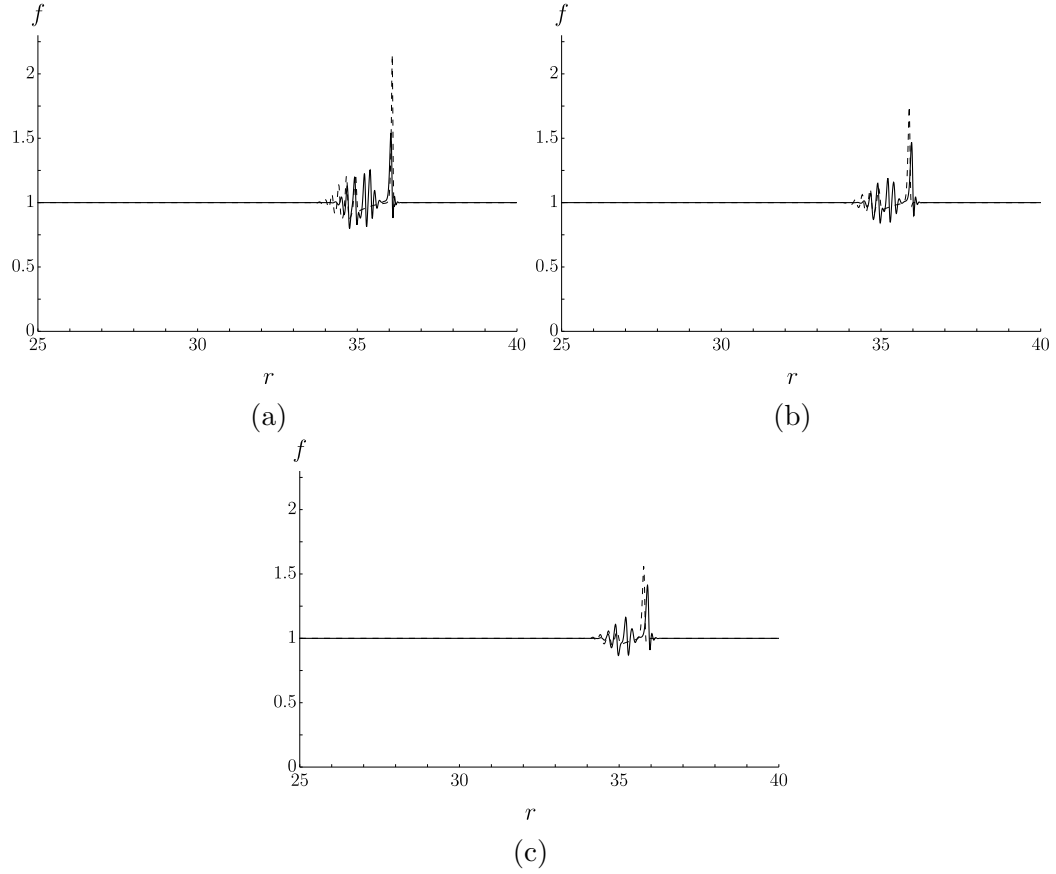


Figure 3.3: Plots of the radial flux computed by the reduced-order model (2.65)–(2.67) (solid curve) and DNS (2.14)–(2.17) (dashed curve) for flow over a planar disk with (a) $\lambda = 0.025$, (b) $\lambda = 0.0275$, and (c) $\lambda = 0.03$, inlet jet radius $R_{\text{jet}} = 25$ and disk radius $R_{\text{disk}} = 40$ without inlet forcing at time $t = 46$.

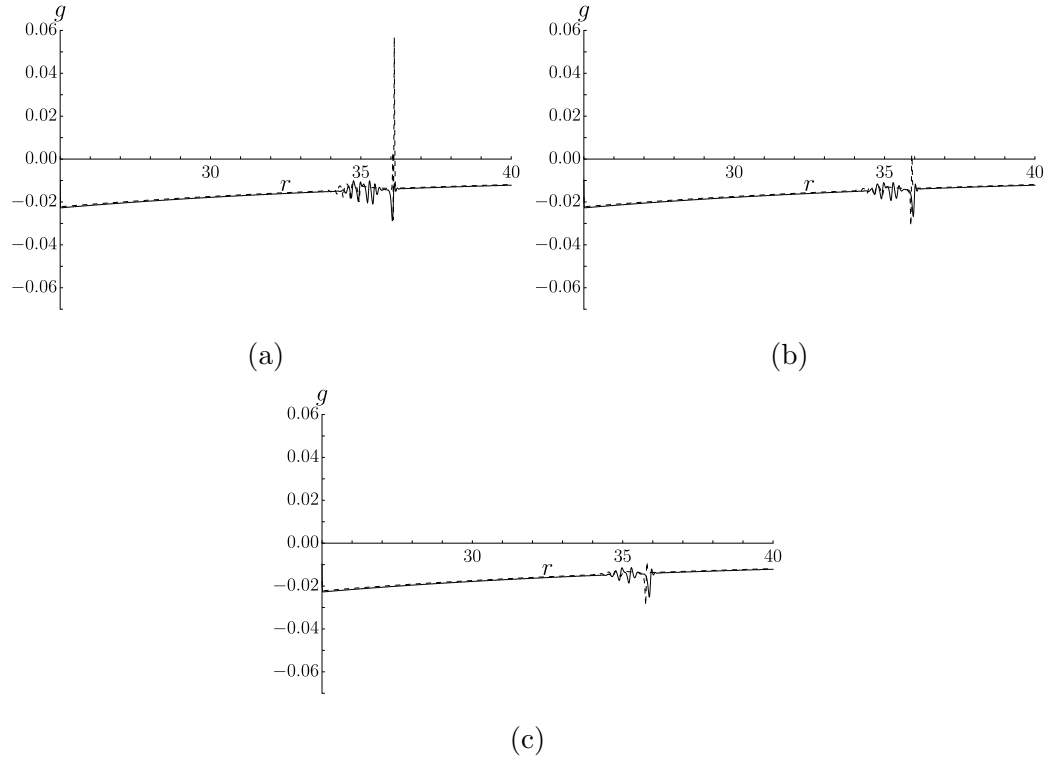


Figure 3.4: Plots of the azimuthal flux computed by the reduced-order model (2.65)–(2.67) (solid curve) and DNS (2.14)–(2.17) (dashed curve) for flow over a planar disk with (a) $\lambda = 0.025$, (b) $\lambda = 0.0275$, and (c) $\lambda = 0.03$, inlet jet radius $R_{\text{jet}} = 25$ and disk radius $R_{\text{disk}} = 40$ without inlet forcing at time $t = 46$.

Figure 3.4 shows that the azimuthal flux g computed by the reduced-order model follows the DNS with generally good qualitative agreement. While there is some disagreement, the reduced-order model correctly predicts the radial interval which is affected by the disturbance. Figure 3.4 shows that the largest trough is predicted with good qualitative agreement in both size and radial position. The cases with $\lambda = 0.025$ and $\lambda = 0.0275$ both have a large peak in front of the largest trough in the DNS that only predicted as a small peak by the reduced-order model with good qualitative agreement in radial position. Due to the sharpness of the large peak, the reduced-order model is expected to have poorer agreement with the DNS as the reduced-order model is derived using a long-wave assumption. The DNS is well resolved in both these cases, hence this is a real phenomenon, which is caused by the azimuthal velocity briefly becoming positive (this will be shown in Section 3.4.2). The exact cause of this is not clear to us, however, it is possibly due to the lower λ value, which physically corresponds to decreasing surface tension and/or increasing inertia. Overall, the reduced-order model qualitatively predicts the azimuthal flux g .

3.4 Validation of the WRIBL radial and azimuthal velocity profiles

In Sections 3.4.1 and 3.4.2, we validate the leading-order version of the radial and azimuthal velocity profiles (2.51) and (2.52) used in the WRIBL method to derive the reduced-order model against the DNS solutions for the radial and azimuthal velocities, respectively. We compare the velocity profiles for u and v in four cases: at a steady state, under a peak, under a trough, and at the midpoint between the

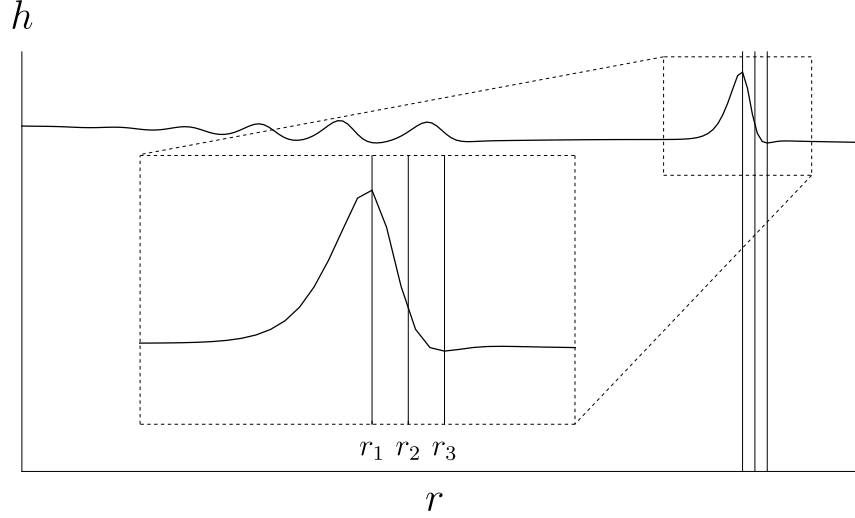


Figure 3.5: Schematic representation of the radial positions under a peak, under a trough, and at the midpoint between these two points in the free surface.

peak and trough. These three radial positions are chosen as the interfacial curvature is high (but of different signs) at the peak and the trough, while the magnitude of the interfacial gradient is high at the midpoint. Figure 3.5 shows a schematic representation of the radial positions at which the velocity profiles are considered, namely under a peak, under a trough, and at the midpoint between these two points. We note that f and g are computed from the radial and azimuthal velocities u and v from the DNS post-processing (as discussed at the beginning of this chapter). Then, for the purpose of validation of the profile assumptions, these values for f and g are used in the leading-order versions of the velocity profiles (2.51) and (2.52). We note that the leading-order version of the radial velocity profile (2.51) is quadratic in z and the leading-order version of the azimuthal velocity profile (2.52) is quartic in z . Kim and Kim [58] used quartic velocity profiles for both u and v in the Kármán–Pohlhausen method based on the solution of Watson [132] for the velocity profile for a steady thin film formed by an impinging liquid jet. The azimuthal velocity profile

used by Kim and Kim [58] is identical to the leading-order version of (2.52). We include the radial velocity profile of Kim and Kim [58], denoted by $u_K(r, z, t)$,

$$u_K = \frac{5f}{2\eta r} \left(\bar{z} - \frac{\bar{z}^3}{2} + \frac{\bar{z}^4}{8} \right), \quad (3.2)$$

in our plots for comparison.

3.4.1 Radial velocity profile

At steady state, the radial velocity profiles computed by DNS and the WRIBL velocity profile (2.51) for flow over a planar disk are essentially identical at the midpoint of the domain (to the nearest grid point). Therefore, only comparisons under a peak, under a trough, and at the midpoint between the peak and trough are presented in the following.

Figure 3.6 shows plots of the radial velocity profiles computed by DNS and the WRIBL velocity profile (2.51) for flow over a planar disk with $\lambda = 0.025$, $\lambda = 0.0275$, and $\lambda = 0.03$, under a peak at the radial position $r = r_1$, at the midpoint between a peak and trough at the radial position $r = r_2$, and under a trough at the radial position $r = r_3$ at time $t = 46$. In particular, Figure 3.6 shows that for all three values of λ presented that the WRIBL radial velocity profile is almost identical to the DNS velocity profile under a peak, at the midpoint between a peak and trough, and under a trough. The WRIBL velocity profile slightly over predicts the velocity near the free surface at the three radial positions for all three values of λ presented. Figure 3.6 shows that the quadratic velocity profile used in the WRIBL method for u performs better against DNS than the quartic velocity profile (3.2) suggested by Kim and Kim [58].

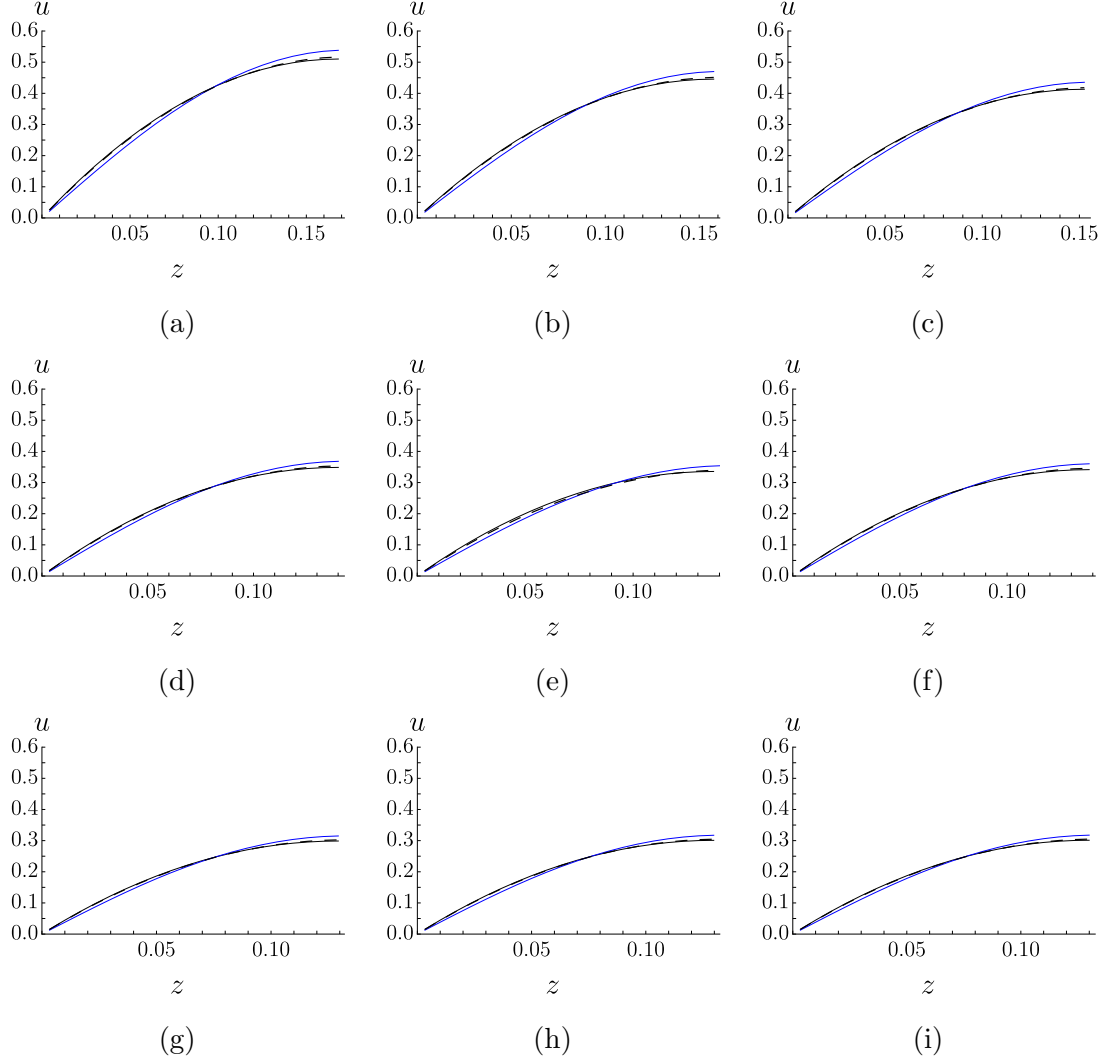


Figure 3.6: Plots of the radial velocity profiles computed by DNS (2.14)–(2.17) (black solid curve) and the WRIBL velocity profile (2.51) (black dashed curve) for flow over a planar disk under a peak in the free surface for (a) $\lambda = 0.025$ at the radial position $r = r_1 = 36.0925$, (b) $\lambda = 0.0275$ at the radial position $r = r_1 = 35.8825$, and (c) $\lambda = 0.03$ at the radial position $r = r_1 = 35.7625$; at the midpoint between a peak and trough in the free surface for (d) $\lambda = 0.025$ at the radial position $r = r_2 = 36.1225$, (e) $\lambda = 0.0275$ at the radial position $r = r_2 = 35.9200$, and (f) $\lambda = 0.03$ at the radial position $r = r_2 = 35.8075$; and under a trough in the free surface for (g) $\lambda = 0.025$ at the radial position $r = r_3 = 36.1525$, (h) $\lambda = 0.0275$ at the radial position $r = r_3 = 35.9575$, and (i) $\lambda = 0.03$ at the radial position $r = r_3 = 35.8525$, without inlet forcing at time $t = 46$. The quartic velocity profile (3.2) of Kim and Kim [58] (blue curve) has also been included for comparison.

3.4.2 Azimuthal velocity profile

At steady state, the azimuthal velocity profiles computed by DNS and the WRIBL velocity profile (2.52) for flow over a planar disk are essentially identical at the midpoint of the domain (to the nearest grid point). Therefore, only comparisons under a peak, under a trough, and at the midpoint between the peak and trough are presented in the following.

Figure 3.7 shows plots of the azimuthal velocity profiles computed by DNS and the WRIBL velocity profile (2.52) for flow over a planar disk with $\lambda = 0.025$, $\lambda = 0.0275$, and $\lambda = 0.03$, under a peak at the radial position $r = r_1$, at the midpoint between a peak and trough at the radial position $r = r_2$, and under a trough at the radial position $r = r_3$ at time $t = 46$. In particular, Figure 3.7 shows that for $\lambda = 0.0275$ and $\lambda = 0.03$ that the WRIBL azimuthal velocity profile is almost identical to the DNS velocity profile under a peak, at the midpoint between a peak and trough, and under a trough. However, for $\lambda = 0.025$, while the agreement is almost identical under a trough, there is some deviation under a peak and at the midpoint between a peak and trough. Under a peak, this deviation occurs due to the azimuthal velocity becoming positive, as expected due to the large peak in the azimuthal flux g in Figure 3.4(a). At the midpoint between a peak and trough, this deviation is likely caused by inertial effects. The leading-order WRIBL velocity profile (which is based on a balance between viscosity and Coriolis force at leading order) does not account for inertial effects. Note that the WRIBL azimuthal velocity profile is still close to the DNS velocity profiles for both of these cases. The WRIBL velocity profile slightly under predicts the velocity near the free surface for all three values of λ presented, while at the midpoint between a peak and trough, the WRIBL velocity profile also slightly over predicts the velocity near the surface of the disk for all three values of

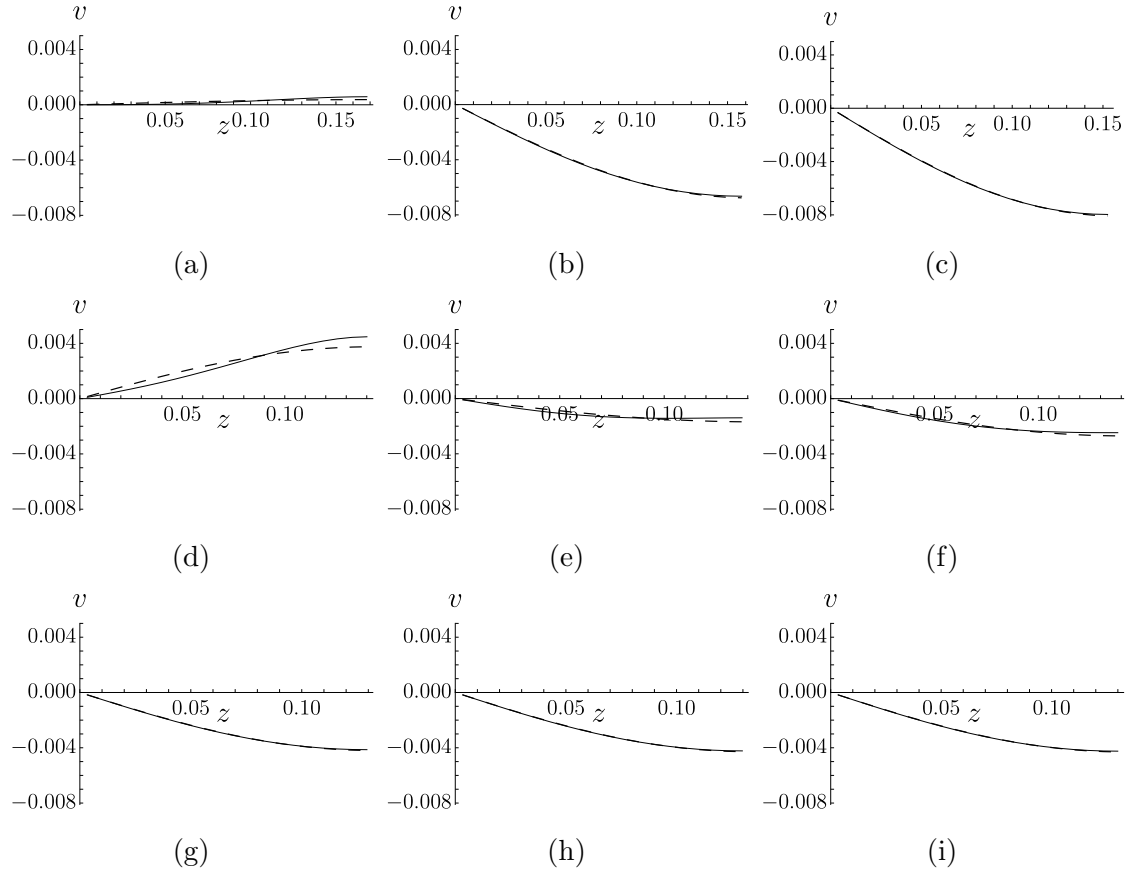


Figure 3.7: Plots of the azimuthal velocity profiles computed by DNS (2.14)–(2.17) (black solid curve) and the WRIBL velocity profile (2.52) (black dashed curve) for flow over a planar disk under a peak in the free surface for (a) $\lambda = 0.025$ at the radial position $r = r_1 = 36.0925$, (b) $\lambda = 0.0275$ at the radial position $r = r_1 = 35.8825$, and (c) $\lambda = 0.03$ at the radial position $r = r_1 = 35.7625$; at the midpoint between a peak and trough in the free surface for (d) $\lambda = 0.025$ at the radial position $r = r_2 = 36.1225$, (e) $\lambda = 0.0275$ at the radial position $r = r_2 = 35.9200$, and (f) $\lambda = 0.03$ at the radial position $r = r_2 = 35.8075$; and under a trough in the free surface for (g) $\lambda = 0.025$ at the radial position $r = r_3 = 36.1525$, (h) $\lambda = 0.0275$ at the radial position $r = r_3 = 35.9575$, and (i) $\lambda = 0.03$ at the radial position $r = r_3 = 35.8525$, without inlet forcing at time $t = 46$.

λ presented.

3.5 Concluding remarks

The reduced-order model qualitatively performs well against the DNS for the free surface, and the radial and azimuthal fluxes. In Section 3.1, we have shown that the reduced-order model predicts additional peaks in the free surface compared to the DNS, however, the radial interval that is affected by the disturbance is correctly predicted. For the peaks that are present in both the reduced-order model and DNS the location and wave speed is well predicted. In Sections 3.2 and 3.3, we have shown that the reduced-order model predicts the radial interval affected by the disturbance, and the location and wave speed of the largest peak is well predicted for the radial and azimuthal fluxes. While there is some disagreement on peak size, this is to be expected for peaks with a short wavelength as the long wave assumption used in the reduced-order model may not hold true for these particular waves. However, we note that nevertheless there is very good qualitative agreement.

In Section 3.4, the leading-order WRIBL radial and azimuthal velocity profiles were validated against DNS. The WRIBL radial velocity profile is almost identical to the DNS for all cases presented. We note that the WRIBL radial velocity profile does slightly over predict the value towards the free surface. The quadratic WRIBL velocity profile used to derive the reduced-order model performs better against the DNS than the quartic velocity profile (3.2) suggested by Kim and Kim [58]. The WRIBL azimuthal velocity profile has good qualitative agreement with the DNS. In particular, the WRIBL azimuthal velocity profile is almost identical to the DNS at steady state and under a trough. Under a peak the WRIBL azimuthal velocity profile

is almost identical to the DNS for $\lambda = 0.0275$ and $\lambda = 0.03$, however, for $\lambda = 0.025$ there is some deviation due to the azimuthal velocity becoming positive. At the midpoint between the peak and trough, while there is good qualitative agreement for all three values of λ , the WRIBL azimuthal velocity profile initially over predicts near the surface of the disk before then under predicting near the free surface compared to the DNS, with agreement improving as λ increases.

Chapter 4

Parametric study of thin-film flow over a planar spinning disk

In this chapter, we consider the flow over a planar disk (i.e. $b \equiv 0$). Therefore, the second-order terms in the evolution equations (2.65)–(2.67) simplify in this case. The parameters governing the present problem are λ , R_{jet} and R_{disk} , which represent the relative importance of surface tension effects compared with inertial effects, the dimensionless radius of the jet, and the dimensionless radius of the disk, respectively. Table 4.1 shows representative values of dimensional and dimensionless parameters for a water-based system [4, 10, 25, 60, 137]. We usually choose to fix typical values of $\lambda = 0.01$, $R_{\text{jet}} = 0.5$ and $R_{\text{disk}} = 15$ and $\epsilon^2 = 10^{-5}$ in our numerical calculations. During numerical testing, we found that changing the values of R_{jet} and R_{disk} has only a minor impact on the behaviour of the flow. Similarly, as ϵ is so small, the terms of order $O(\epsilon^2)$ have only a small impact. Therefore, in Section 4.1 our parametric study for the planar case will focus on quantifying the effect of varying λ .

CHAPTER 4. PARAMETRIC STUDY OF THIN-FILM FLOW OVER A PLANAR SPINNING DISK

Parameter	Symbol	Units	Representative value
Surface tension	$\tilde{\gamma}$	N/m	0.072
Density	$\tilde{\rho}$	kg/m ³	1000
Kinematic viscosity	$\tilde{\nu}$	m ² /s	10 ⁻⁶
Volume flux	$\tilde{Q}_{\text{jet}}$	m ³ /s	$(2 - 8) \times 10^{-5}$
Disk radius	$\tilde{R}_{\text{disk}}$	m	0.1–0.38
Jet radius	$\tilde{R}_{\text{jet}}$	m	0.005–0.02
Aspect ratio squared	ϵ^2	–	$(1-5) \times 10^{-5}$
Inverse Weber number	λ	–	0.0007–0.03
Disk radius	R_{disk}	–	2–20
Jet radius	R_{jet}	–	0.4–1.2

Table 4.1: Representative values of dimensional and dimensionless parameters for a water-based system [4, 10, 25, 60, 137].

In what follows, we consider the effect of inlet forcing on the behaviour of the system. This is important because even the presence of a small amount of noise, as typically present in real-world systems [58], can result in significant differences to the case without inlet forcing. We will investigate the effect of a random inlet forcing in Section 4.2.

As we will show in Section 4.1, the system is characterised by a train of pulse-like waves. We therefore introduce four metrics to help quantify this behaviour: interfacial waviness and shear stress, inter-peak separation, and peak location. The integral measure of the interfacial waviness [75], denoted by $W(t)$, is defined as

$$W(t) = \sqrt{\int_{R_{\text{jet}}}^{R_{\text{disk}}} r(h - h_N)^2 dr}, \quad (4.1)$$

where $h_N = b + \eta_N$ is the initial condition for the free-surface profile (2.70). The

shear stress at the surface of the disk ($z = b$), denoted by $\boldsymbol{\tau}_{\text{disk}}(r, t)$, is given by

$$\begin{aligned}\boldsymbol{\tau}_{\text{disk}} &= \mathbf{T} \cdot \mathbf{n}_{\text{disk}} - (\mathbf{n}_{\text{disk}} \cdot \mathbf{T} \cdot \mathbf{n}_{\text{disk}}) \mathbf{n}_{\text{disk}} \\ &= \frac{\epsilon}{\eta^2 r} \left[-\frac{3f(b_r^2 - 1)}{(b_r^2 + 1)^{3/2}} \mathbf{e}_r + \frac{5g}{2(b_r^2 + 1)^{1/2}} \mathbf{e}_\theta - \frac{3fb_r(b_r^2 - 1)}{(b_r^2 + 1)^{3/2}} \mathbf{e}_z \right],\end{aligned}\tag{4.2}$$

where $\mathbf{n}_{\text{disk}}$ is the outward unit normal of the surface of the disk, up to first order in ϵ . We then define an integral measure of shear stress at the surface of the disk, denoted by $S(t)$, given by the integral of the Euclidean norm of the shear stress there, i.e.

$$\begin{aligned}S(t) &= 2\pi \int_{R_{\text{jet}}}^{R_{\text{disk}}} \|\boldsymbol{\tau}_{\text{disk}}\| r \sqrt{1 + b_r^2} \, dr \\ &= 2\pi\epsilon \int_{R_{\text{jet}}}^{R_{\text{disk}}} \frac{1}{\eta^2} \sqrt{\frac{9f^2(b_r^2 - 1)^2}{b_r^2 + 1} + \frac{25g^2}{4}} \, dr.\end{aligned}\tag{4.3}$$

These two metrics are averaged over space, facilitating comparison across a range of parameter values. The interfacial waviness metric gives us insight into the behaviour of the free surface, whereas the shear stress metric gives us insight into the behaviour of the flow at the surface of the disk. The shear stress metric is also considered as the shear stress has been shown to be one of the dominant factors in graphene production [117] (as discussed in Section 1.5.1). The system exhibits a strong radial dependence due to a combination of factors including radial thinning due to mass conservation (reflected in the far-field solution (2.69)) and the fact that the magnitude of the force due to centrifugation scales like r . These effects have substantial consequences for peak size, separation and interaction. Therefore, we consider the inter-peak separation between the peaks of the pulse-like waves and the locations of these peaks (computed as discussed at the beginning of Chapter 3) to help understand their behaviour. The inter-peak separation is obtained from the distance

between successive peaks at each time step, then the mean and standard deviation is computed to facilitate comparison across a range of parameter values.

We use the code described at the beginning of Chapter 3 to simulate the reduced-order model (2.65)–(2.67), from the inlet ($r = R_{\text{jet}}$) to the edge of the disk ($r = R_{\text{disk}}$). The code uses a uniform grid of 18000 points and a time step of 10^{-5} . Newton iterations were stopped at a residual tolerance of 10^{-10} , and simulations were run to $t = 100$. Convergence was verified via grid/time-step refinement in both space and time.

4.1 Planar disk without inlet forcing

In this section, we consider the case of a planar disk without inlet forcing, i.e. $b \equiv 0$ and $f(R_{\text{jet}}, t) = 1$. We will consider the effect of inlet forcing subsequently in Section 4.2. All results presented within this section were performed for a planar disk with inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$.

Figure 4.1 shows a space-time plot of $h - h_N$ for a planar disk with $\lambda = 0.01$ without inlet forcing from $t = 0$ to $t = 30$. As the initial condition (2.70) is not an exact steady state solution of the system, this results in an initial transient before the system evolves into a fully-developed state (here, the fully-developed state occurs for $t \gtrsim 20$). In particular, the initial transient results in disturbances being generated everywhere throughout the domain at $t = 0$. As shown in Figure 4.1, these waves propagate radially outwards under the effect of centrifugation. The fully-developed state exhibits four characteristic spatial regions. In particular, a spin-up region, an inner region, a middle region, and an outer region. Figure 4.2 shows a schematic representation of the spin-up, inner, middle and outer regions. The first region,

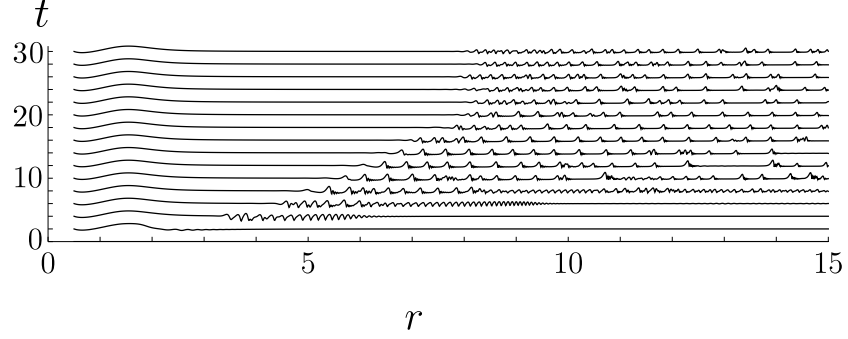


Figure 4.1: Space-time plot of $h - h_N$ for a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ without inlet forcing. In this plot, the time varies from $t = 0$ to $t = 30$ in steps of $t = 2$.

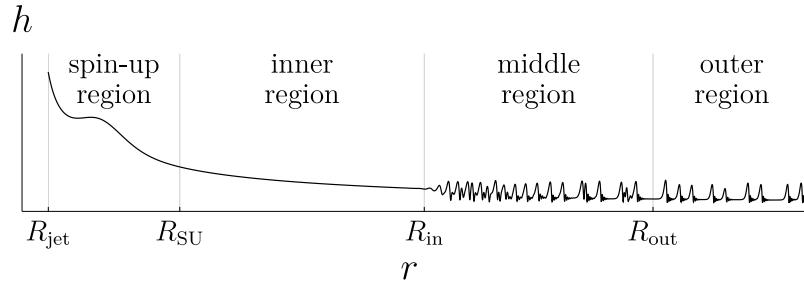


Figure 4.2: Schematic representation of the spin-up, inner, middle and outer regions for fully-developed flow over a planar disk without inlet forcing.

nearest the inlet, is the spin-up region $R_{\text{jet}} \leq r \leq R_{\text{SU}}(\lambda)$, where $R_{\text{SU}}(0.01) \approx 3$ is the radial position of the transition between the spin-up and the next region (the inner region) [73, 75]. Within the spin-up region, the liquid initially resists the rotation of the disk and a shoulder is formed, the liquid then accelerates to match the angular velocity of the disk [20]. The free-surface profile undergoes very little change in this region after the shoulder forms. There are then three further regions: firstly, an inner region $R_{\text{SU}}(\lambda) \leq r \leq R_{\text{in}}(\lambda)$, where $R_{\text{in}}(0.01) \approx 8$ is the radial position of the transition between the inner and the next region (the middle region). For a system without inlet forcing, this region corresponds to the waveless region shown in Figure 4.1 lying between the spin-up region and the radial position at which there is persistent, self-sustained inception of waves, which we denote as $r = r_s$. In the absence of inlet forcing, the two coincide and so in this section $r_s = R_{\text{in}}(\lambda)$; the distinction is made because, in the presence of forcing, the two no longer coincide (see Section 4.2). There are then two further regions which cannot be readily discerned in Figure 4.1, namely a middle region $R_{\text{in}}(\lambda) \leq r \leq R_{\text{out}}(\lambda)$, and an outer region $r \geq R_{\text{out}}$, where R_{out} is the radial position of the transition between the middle and the next region (the outer region). However, we note that the transition between the middle and outer regions is based on the asymptotic balance of terms. The distinction between the middle and outer regions will be explained in Chapter 6, but for present purposes, they are both characterised by fully-developed trains of pulse-like waves. The far-field solutions used for the initial condition (2.70) are a good approximation for $r \gtrsim 3$, which is shown by the horizontal nature of $h - h_N$ for $r \gtrsim 3$ in Figure 4.1. Note that the values of R_{SU} , R_{in} and R_{out} will in general be functions of only λ , and that the time taken for the different regions to emerge depends on both the initial conditions and λ . Finally, we note that in most previous studies, only the spin-up and inner regions have been examined (see, for example, [59, 73, 75, 104, 111]). However,

Matar *et al.* [74] and Prieling and Steiner [93] included the middle and outer regions. They noted the self-sustained inception of waves, but there was limited mathematical analysis of its emergence since they did not consider the problem analytically. Kim and Kim [58] also included the middle and outer regions, however, the behaviours of these regions were hidden due to the inclusion of inlet forcing and turbulence. We will present our findings on the mathematical explanation for the phenomenon of the self-sustained inception of waves in Chapter 6.

Next, we investigate the effect of λ , which represents the relative importance of surface tension compared with inertial effects, on the dynamics of the flow. Figure 4.3 shows the free-surface profile for $\lambda = 0.0125$, $\lambda = 0.01$ and $\lambda = 0.0075$ for flow over a planar disk without inlet forcing at $t = 20$. At this time, the system has fully developed and the self-sustained inception of waves at $r = r_s$ has occurred. For $\lambda = 0.0125$ (see Figure 4.3(a)) the amplitude of the waves is decreased while the separation distance between peaks and radial position of the self-sustained inception of waves ($r_s \approx 8.5$) are both increased relative to their values when $\lambda = 0.01$ (see Figure 4.3(b)). On the other hand, for $\lambda = 0.0075$ (see Figure 4.3(c)) the amplitude of the waves is increased while the separation distance between peaks and radial position of the self-sustained inception of waves ($r_s \approx 7$) are both decreased relative to their values when $\lambda = 0.01$ (see Figure 4.3(b)). Therefore, we conclude that decreasing λ has a destabilising effect on the flow, as this results in waves of higher amplitude travelling over a larger portion of the disk with a decreased inter-peak separation. This is as expected since decreasing λ corresponds to decreasing surface tension and/or increasing inertia, both of which have a destabilising effect. Figure 4.3 also shows that decreasing λ regularises the dynamics of the waves: the waves form a regularly separated train of travelling waves.

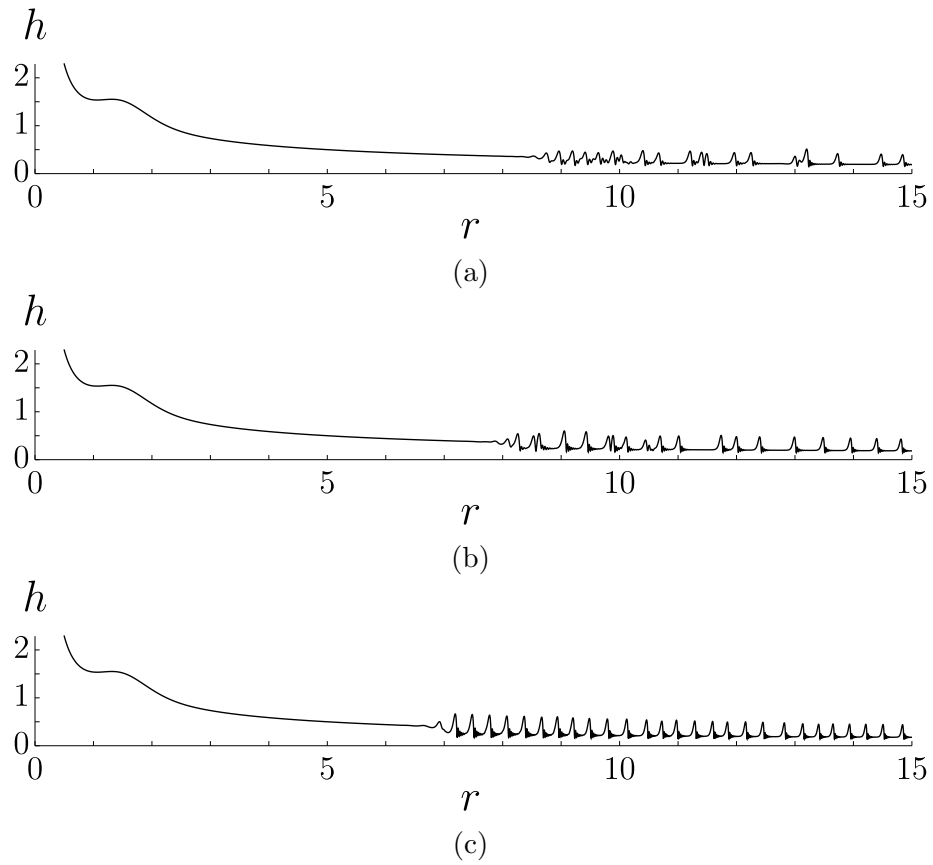


Figure 4.3: The free-surface profile h for flow over a planar disk with (a) $\lambda = 0.0125$, (b) $\lambda = 0.01$ and (c) $\lambda = 0.0075$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ without inlet forcing at $t = 20$.

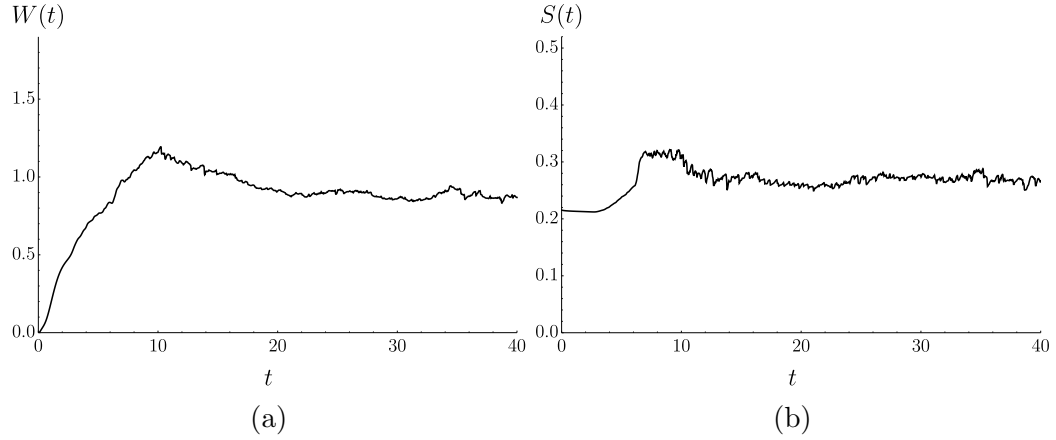


Figure 4.4: Plots of (a) interfacial waviness $W(t)$ and (b) shear stress $S(t)$ as functions of t for flow over a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ without inlet forcing.

Given the complexity of the behaviour shown in Figures 4.1 and 4.3, in the following subsections we focus on the following key metrics: interfacial waviness and shear stress, inter-peak separation, and peak location. We now discuss each of these metrics in turn for the case without inlet forcing with the typical values $\lambda = 0.01$, $R_{\text{jet}} = 0.5$ and $R_{\text{disk}} = 15$.

4.1.1 Interfacial waviness and shear stress

Figure 4.4 shows the interfacial waviness $W(t)$ and shear stress $S(t)$ as functions of t for flow over a planar disk with $\lambda = 0.01$ without inlet forcing.

Figure 4.4(a) shows that the initial transient induces a sharp increase in the interfacial waviness $W(t)$. After this initial increase, the interfacial waviness reaches a maximum of $W(t) \approx 1.2$ at $t \approx 10$, which is the time at which the waves become fully developed in the middle and outer regions. However, the radial position of the inception of waves takes time to recede to $r = r_s$, at which the waves become

self-sustained in the fully-developed state at $t \approx 20$. It is this recession that is responsible for the nearly linear decrease in $W(t)$ between $t \approx 10$ and $t \approx 20$. After this decrease, $W(t)$ fluctuates around a constant value of approximately 0.85 for $t \gtrsim 20$. These small fluctuations are caused by two factors: the formation of waves due to the self-sustained inception of waves and waves leaving the edge disk.

Figure 4.4(b) shows that the shear stress increases until time $t \approx 7$, then fluctuates around a value of $S(t) \approx 0.31$ for $7 \lesssim t \lesssim 10$ before decreasing to fluctuate around a value of approximately 0.26 for $t \gtrsim 11$. Waves are present in the middle and outer regions for $t \gtrsim 7$ and these waves become fully developed at time $t \approx 10$, and are responsible for the shear stress reaching its maximum value in this time interval. Once the fully developed pulse-like waves emerge across the outer region, the shear stress decreases to fluctuate around a constant value, indicating that the recession of the radial position of self-sustained waves has negligible effect on the value of the shear stress.

4.1.2 Inter-peak separation

Figure 4.5 shows a bar chart of the mean and standard deviation of the inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a planar disk with $\lambda = 0.01$ without inlet forcing, over the time interval from $t = 20$ to $t = 100$. These intervals in r and t have been chosen as the middle and outer regions shown in Figure 4.1 are present and fully developed, and the intervals are long enough in time for averages to be suitably representative. In particular, Figure 4.5 shows that both the mean and standard deviation of the inter-peak separation increase with r , reaching maximum values of approximately 0.22 and 0.15, respectively, in the region $12 \leq r < 13$. This implies that on average inter-peak separation is increasing as

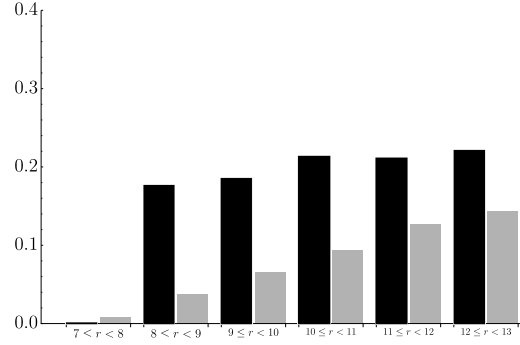


Figure 4.5: Bar chart of the mean (black) and standard deviation (grey) of inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ without inlet forcing, over the time interval from $t = 20$ to $t = 100$.

the waves travel across the disk, however, the inter-peak separation becomes more dispersed.

4.1.3 Peak location

Figure 4.6 shows a space-time plot of the peaks of the free surface for flow over a planar disk with $\lambda = 0.01$ without inlet forcing. In particular, Figure 4.6 shows that the waves due to the initial transient travel at a nonuniform speed and accelerate towards the edge of the disk. Note that in the areas in Figure 4.6 which appear solid black, the peaks are so closely spaced that the individual plotted points overlap and merge into a continuous block. The peaks which form after the initial transient in the inner region then travel at a more uniform speed, which is illustrated by the nearly linear paths in Figure 4.6. The radial position of the self-sustained inception of waves is at $r_s \approx 8$ is also evident in Figure 4.6 from the presence of peaks at $r \approx 8$ from $t \approx 20$ onwards. The peaks formed in the middle region due to the self-sustained inception of waves also travel at nonuniform speed. This behaviour is

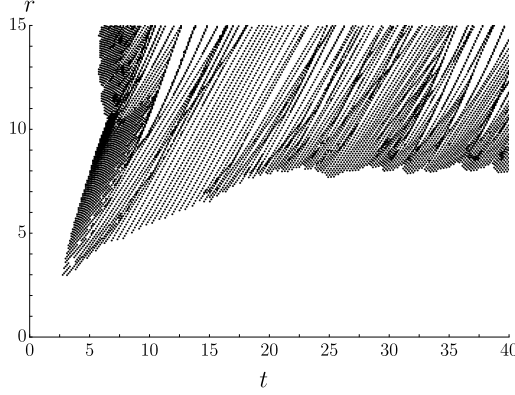


Figure 4.6: Space-time plot of the peaks of the free surface for flow over a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ without inlet forcing.

due to a competition between the destabilising force of centrifugation (which scales like r) and the stabilising effect of film thinning (as h decreases like $r^{-2/3}$, reflected in (2.69)). This nonuniform speed is caused by successive coalescence of peaks, creating larger peaks which travel faster due to the effect of centrifugation.

4.2 Planar disk with inlet forcing

We next consider the effect of inlet forcing. Small variations in the flux at the inlet are typically present in real-world systems as in practice inlet flow rates are often non-constant [58]. We therefore add a small random noisy forcing to the radial flux at the inlet (2.71). Specifically, we replace f in the inlet boundary condition (2.71) to implement inlet forcing using the expression,

$$f(R_{\text{jet}}, t) = 1 + \frac{A}{\sum_{i=1}^{50} \mathcal{R}_{\text{amp},i}} \sum_{i=1}^{50} \mathcal{R}_{\text{amp},i} \cos [2\pi (\mathcal{R}_{\text{freq},i} t + \mathcal{R}_{\text{off},i})], \quad (4.4)$$

where $\mathcal{R}_{\text{amp},i}, \mathcal{R}_{\text{off},i} \in [0, 1]$ and $\mathcal{R}_{\text{freq},i} \in [0, 10]$ are uniformly distributed random numbers and A is the maximum amplitude of the inlet forcing. While we selected a random set of values of $\mathcal{R}_{\text{amp},i}, \mathcal{R}_{\text{off},i}, \mathcal{R}_{\text{freq},i}$ in (4.4), these were held fixed for all of the computations. All results presented in this section were performed for a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$, and disk radius $R_{\text{disk}} = 15$.

Figure 4.7 shows space-time plots of $h - h_N$ for flow over a planar disk with inlet forcing of maximum amplitude $A = 0.1$, $A = 0.01$ and $A = 0.001$ from $t = 0$ to $t = 30$. In particular, Figure 4.7 shows that the behaviour of the flow is rather different in the presence of inlet forcing compared to the case without inlet forcing (previously discussed in Section 4.1). In the case $A = 0.1$ the radial position of the self-sustained inception of waves is at $r_s \approx 3$ (shown in Figure 4.7(a)), whereas when $A = 0.001$ it is at $r_s \approx 4.5$ (shown in Figure 4.7(c)). In contrast, this radial position is at $r_s = R_{\text{in}}(0.01) \approx 8$ in the case without inlet forcing $A = 0$. This shows that in the presence of inlet forcing, the behaviour of the flow, particularly in the inner region, is very different to that in the absence of inlet forcing. In particular, the inlet forcing induces waves in this region that are absent for $A = 0$. As a consequence, the radial position of the self-sustained inception of waves lies in the interval $R_{\text{SU}} < r_s < R_{\text{in}}$, with r_s moving closer to R_{SU} as A increases (due to the stronger excitation). Again, the far-field solutions used for the initial condition (2.70) are a good approximation for $r \gtrsim 3$, which is shown by the horizontal nature of $h - h_N$ for $r \gtrsim 3$ in Figure 4.7. Furthermore, comparing Figures 4.1 and 4.7 shows that the peaks of the free surface near $r = r_s$ have a greater variation in amplitudes and frequencies as A increases. In Figure 4.7(a), the disturbances in the spin-up region are now large enough to be discernible by eye as the shoulder is visibly affected by the inlet forcing. However, in Figures 4.7(b) and 4.7(c), the shoulder remains largely unaffected by the presence

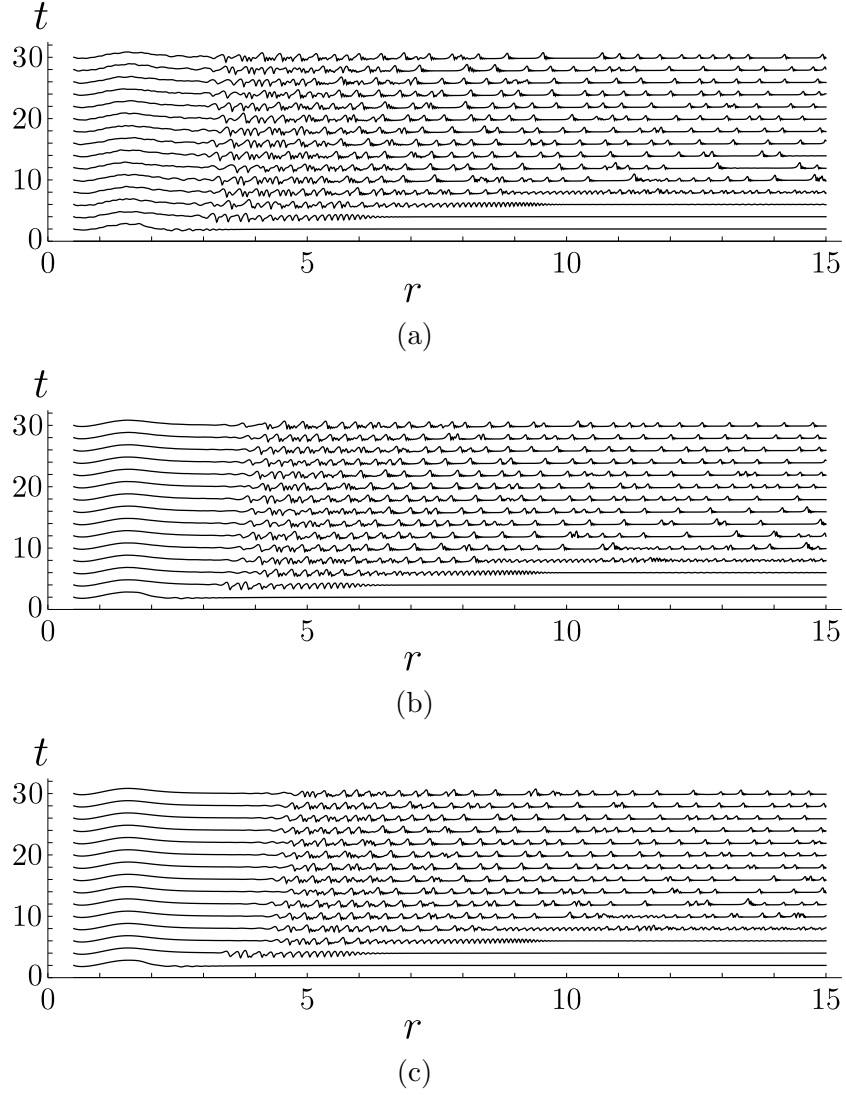


Figure 4.7: Space-time plots of $h - h_N$ for flow over a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$, with inlet forcing of maximum amplitude (a) $A = 0.1$, (b) $A = 0.01$ and (c) $A = 0.001$. In these plots, the time varies from $t = 0$ to $t = 30$ in steps of $t = 2$.

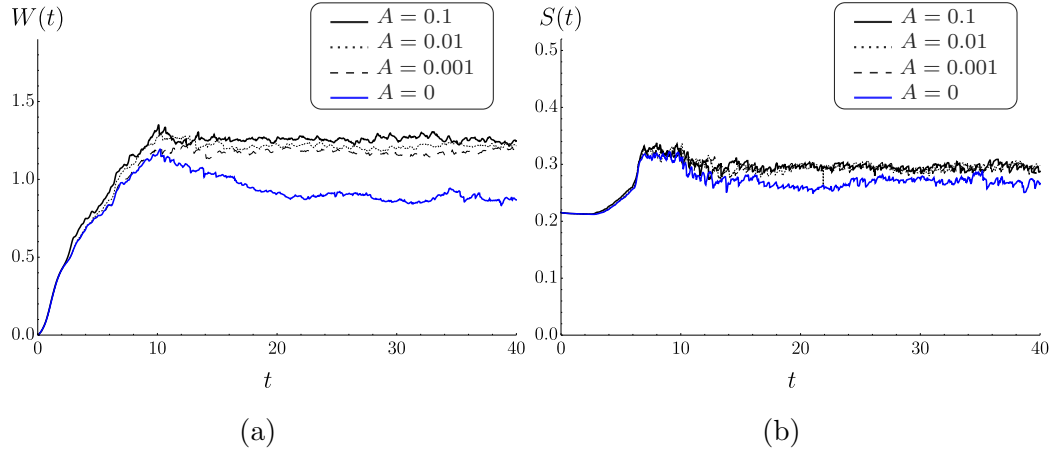


Figure 4.8: Plots of (a) interfacial waviness $W(t)$ and (b) shear stress $S(t)$ as functions of t for flow over a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$, and disk radius $R_{\text{disk}} = 15$, with inlet forcing of maximum amplitude $A = 0.1$ (solid curve), $A = 0.01$ (dotted curve) and $A = 0.001$ (dashed curve). The case of no inlet forcing $A = 0$ (blue curve) has also been included for comparison.

of inlet forcing.

We next consider the effect of the maximum amplitude of the inlet forcing A on the following key metrics: interfacial waviness and shear stress, inter-peak separation, and peak location.

4.2.1 Interfacial waviness and shear stress

Figure 4.8 shows the interfacial waviness $W(t)$ and shear stress $S(t)$ as functions of t for flow over a planar disk with inlet forcing of maximum amplitude $A = 0.1$, $A = 0.01$ and $A = 0.001$ together with the case of no inlet forcing $A = 0$.

Figure 4.8(a) shows that the interfacial waviness $W(t)$ is rather different in presence of inlet forcing compared to the case of no inlet forcing, previously discussed in Section 4.1. Figure 4.8(a) shows that when inlet forcing is present the interfacial

waviness fluctuates around a constant value of $W(t) \approx 1.2$ for $t \gtrsim 10$ that is almost independent of A . However, note that increasing the maximum amplitude of inlet forcing A does slightly increase the interfacial waviness. The behaviour of $W(t)$ and its dependence on the presence and magnitude of the inlet forcing A is due to the time taken for the system to reach a fully-developed state. In particular, the middle and outer regions do not become fully developed until $t \approx 10$. This is what is responsible for $W(t)$ increasing up until this time for all values of A . However, the subsequent behaviour of $W(t)$ is dictated by the behaviour in the inner region: for $A \neq 0$ the waves in the inner region are fully developed, and so have roughly constant values of $W(t)$. However, in the absence of forcing (discussed in Section 4.1.1), after the waves in the middle and outer regions have fully developed the inception of waves takes additional time to recede to the radial position $r = r_s$ (at which the waves become self-sustained) at $t \approx 20$, resulting in the nearly linear decrease in $W(t)$ between $t \approx 10$ and $t \approx 20$, as discussed in Section 4.1.1.

Figure 4.8(b) shows that when inlet forcing is present the shear stress fluctuates around a constant value of $S(t) \approx 0.29$ for $t \gtrsim 11$ that is essentially independent of the maximum amplitude of inlet forcing A . Figure 4.8(b) also shows that the shear stress is relatively unaffected for $t \lesssim 10$ by the presence of inlet forcing, however for $t \gtrsim 11$ the presence of inlet forcing increases the shear stress when compared to the case without inlet forcing. Therefore, we conclude that the presence of inlet forcing increases the shear stress, but increasing A has a negligible effect on the shear stress.

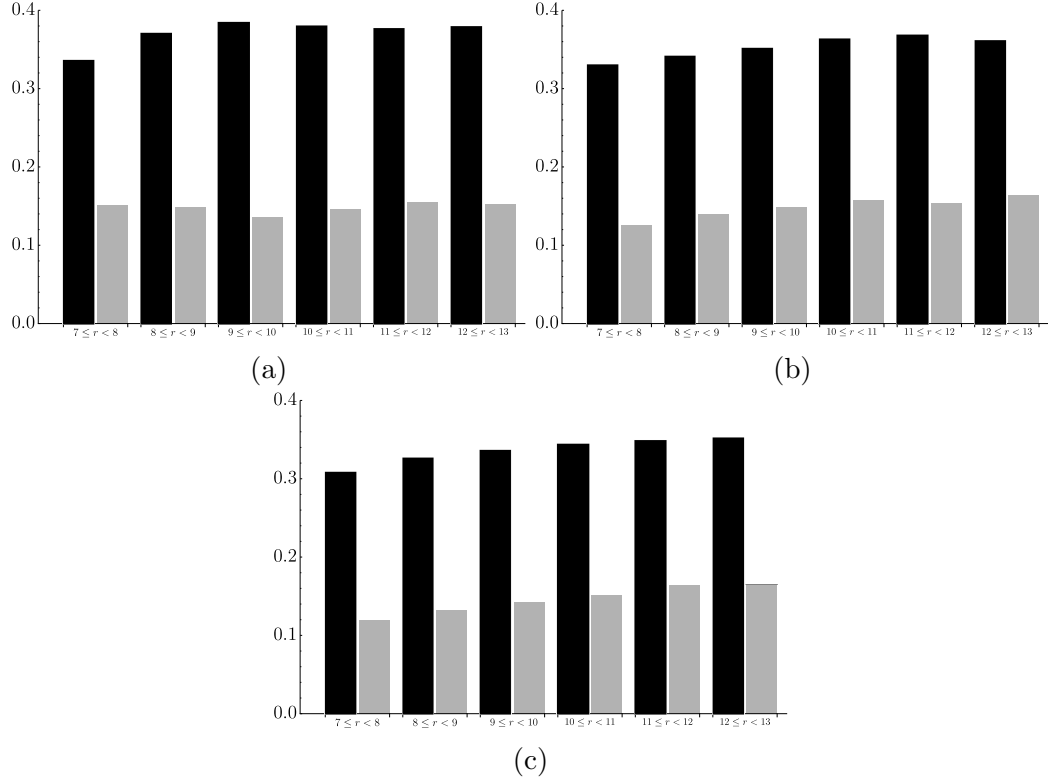


Figure 4.9: Bar charts of the mean (black) and standard deviation (grey) of inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a planar disk with $\lambda = 0.01$, with inlet jet radius $R_{\text{jet}} = 0.5$, and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude (a) $A = 0.1$, (b) $A = 0.01$ and (c) $A = 0.001$, over the time interval from $t = 20$ to $t = 100$.

4.2.2 Inter-peak separation

Figure 4.9 shows bar charts of the mean and standard deviation of inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a planar disk, with inlet forcing of maximum amplitude $A = 0.1$, $A = 0.01$ and $A = 0.001$, over the time interval from $t = 20$ to $t = 100$. The initial time of $t = 20$ was chosen as this is the time by which the system without inlet forcing has fully developed and the interfacial waviness has reached a nearly constant value (see Figure 4.4(a)). In the

presence of inlet forcing the fully developed behaviour occurs for $t \gtrsim 10$ (see Figure 4.8(a)). However, we consider the inter-peak separation from $t = 20$ to $t = 100$ for consistency with the results presented in Section 4.1.2.

For all three values of A presented in this subsection, the radial dependence of inter-peak separation is relatively weak, but some general trends can nevertheless be discerned. This behaviour is in contrast to the case without inlet forcing (see Figure 4.5) and highlights the significant effect even small variations at the inlet have on the system. In the case $A = 0.1$ the mean inter-peak separation is nearly constant for $r \geq 8$ (as shown in Figure 4.9(a)), in the case $A = 0.01$ the mean inter-peak separation is nearly constant for $r \geq 10$ (as shown in Figure 4.9(b)), whereas in the case $A = 0.001$ the mean separation increases gradually throughout $7 \leq r < 13$ (as shown in Figure 4.9(c)). This latter trend is akin to that in the case without inlet forcing for $8 \leq r < 13$ shown in Figure 4.5. Furthermore, comparing Figures 4.5 and 4.9 shows that the presence of inlet forcing significantly increases the mean inter-peak separation for $7 \leq r < 13$, however, further increasing A only marginally increases the mean inter-peak separation. In the case $A = 0.1$ the standard deviation of the inter-peak separation remains relatively constant between $r = 7$ and $r = 13$ at approximately 0.15 (shown in Figure 4.9(a)). This value of the standard deviation occurs for $A = 0.01$ and $A = 0.001$ for $r \geq 10$ (shown in Figure 4.9(b)) and $r \geq 11$ (shown in Figure 4.9(c)), respectively. Prior to these radial positions, the inter-peak separation is still increasing, which causes the standard deviation of the inter-peak separation to be lower than 0.15 (the approximately constant value for the case $A = 0.1$). The case of no inlet forcing ($A = 0$) shown in Figure 4.5, also has a standard deviation of approximately 0.15 in the region $12 \leq r < 13$. Increasing A helps the system to reach a regime in which the mean inter-peak separation is

independent of the radial distance. On the other hand, for smaller values of A , the mean inter-peak separation depends on r . In all cases tested (i.e. $A = 0.1$, $A = 0.01$, $A = 0.001$ and $A = 0$) the standard deviation tends to approximately 0.15 in the region $12 \leq r < 13$. Therefore, we conclude that the inlet forcing has little effect on the standard deviation of the inter-peak separation. Loosely speaking, this means that the level of disorder, as characterised by the standard deviation, is essentially unaffected by the value of A .

4.2.3 Peak location

Figure 4.10 shows space-time plots of the peaks of the free surface for flow over a planar disk with inlet forcing of maximum amplitude $A = 0.1$, $A = 0.01$ and $A = 0.001$. Note that in the areas in Figure 4.10 which appear solid black, the peaks are so closely spaced that the individual plotted points overlap and merge into a continuous block. In particular, Figure 4.10 shows that increasing the maximum amplitude of the inlet forcing A causes the radial position of the self-sustained inception of waves r_s to occur nearer to R_{SU} , consistent with the behaviour shown in Figure 4.7. Increasing A also causes the self-sustained inception of waves to occur at slightly earlier times, in particular at $t \approx 3$ when $A = 0.1$, $t \approx 5$ when $A = 0.01$ and $t \approx 6$ when $A = 0.001$. In all three cases, the self-sustained inception of waves occurs significantly earlier than in the case of no inlet forcing where it occurs from $t \approx 20$ onwards, as shown in Figure 4.6. These results confirm that the system is strongly sensitive to variations at the inlet, as discussed in Sections 4.2.1 and 4.2.2. Due to the small variations at the inlet, waves are present in the inner region $R_{SU} \leq r \leq R_{in}$, with r_s moving closer to R_{SU} as A increases. Therefore, the larger the maximum amplitude of the inlet forcing A is, the earlier the self-sustained inception of waves

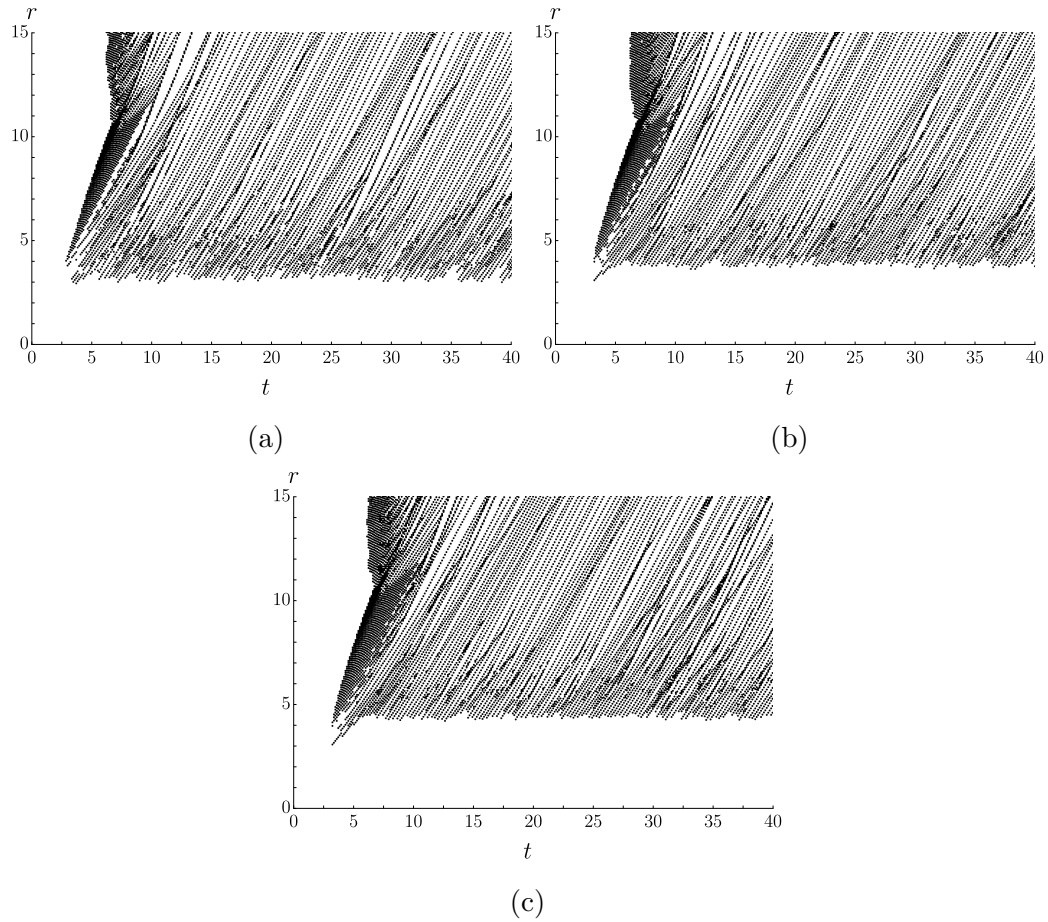


Figure 4.10: Space-time plots of the peaks of the free surface for flow over a planar disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude (a) $A = 0.1$, (b) $A = 0.01$ and (c) $A = 0.001$.

occurs. Figure 4.10 also shows that after the initial disturbance waves due to the initial condition propagate radially outwards, the peaks travel at a nearly constant speed across the disk, as shown by the nearly linear paths in Figures 4.10(a)–(c). On the other hand, the initial disturbance waves travel along a curved path as discussed in Section 4.1.3. We note that, while after the initial transient most waves have a nearly constant velocity, there are some interesting exceptions. For example, in the case $A = 0.1$ (see Figure 4.10(a)) there is a deviation from the nearly linear paths that begins at $t \approx 23$. This is due to the large maximum amplitude of the inlet forcing causing the formation of a peak that is slightly larger than others. Therefore, this peak travels slightly faster due to the effect of centrifugation, hence catching up to and coalescing with some of the peaks ahead of it. In wake of this peak, there are larger inter-peak separations for a short time until the system returns to its preferred inter-peak separation.

4.3 Concluding remarks

In Section 4.1, we have shown that the fully-developed state exhibits four characteristic spatial regions: the spin-up, inner, middle and outer regions. The flow has different characteristics in each of these regions. The radial positions of these regions depend only on the inverse Weber number λ . In the spin-up region, the shoulder forms; the inner region is characterised by a waveless region; and the middle and outer regions are characterised by fully-developed trains of pulse-like waves. In Section 4.2, we have shown that small amounts of inlet forcing can have a strong effect on the dynamics of the system. In the subsequent numerical simulations, we include inlet forcing of maximum amplitude $A = 0.01$ (equivalent to the inlet flux varying by $\pm 1\%$) that is of the same order of magnitude as Kim and

Kim [58] found when comparing the amplitude of a random noisy inlet forcing in numerical simulations with experimental data. This value of A does not overwhelm the underlying dynamics of the system, as demonstrated by the shoulder remaining largely unaffected in Figure 4.7(b).

Chapter 5

Parametric study of thin-film flow over a topographically-patterned spinning disk

In this chapter, we consider the effect of topographic patterning of the disk on the dynamics of the flow. We introduce a multi-parameter family of topographies with parameters controlling the asymmetry, smoothness, amplitude and spatial frequency of the topography, and investigate the effect of each parameter on the flow. In particular, we create a piecewise linear function $\mathcal{B}(r, \bar{t})$, where $\bar{t}$ is a pseudo-time, with initial condition

$$\mathcal{B}(r, 0) = \begin{cases} \frac{r}{(1+c)\pi} & \text{for } 0 \leq r \leq (1+c)\pi, \\ \frac{2\pi-r}{(1-c)\pi} & \text{for } (1+c)\pi \leq r \leq 2\pi, \end{cases} \quad (5.1)$$

where $c \in (-1, 1)$ is the asymmetry parameter. We note that a positive value of c creates a right-skewed topography, $c = 0$ creates a topography with no skew, and a negative value of c creates a left-skewed topography. We then evolve $\mathcal{B}$ according to the heat equation $\mathcal{B}_{\bar{t}} = \mathcal{B}_{rr}$ from $\bar{t} = 0$ to $\bar{t} = \mathcal{T}$ to smooth the topography, so that the value of $\mathcal{T}$ controls the smoothness and is therefore referred to as the smoothing parameter. This ensures that $\mathcal{B}$ is differentiable at all points, as derivatives up to third order of the topography b are present in the reduced-order model (2.65)–(2.67). We then rescale $\mathcal{B}$ horizontally and vertically to have maximum amplitude a and spatial frequency ω , then extend $\mathcal{B}$ periodically to $r = R_{\text{disk}}$. Finally, we apply a mollifier $m(r)$ that sets the topography near the inlet and the edge of the disk to be flat, and to decay with radius as $r^{-2/3}$ (consistent with the film thickness (2.69)), which is chosen to be

$$m = \frac{r^{-2/3}}{4} \left(1 + \tanh [5 (r - R_{\text{jet}} - 1)] \right) \left(1 - \tanh [5 (r - R_{\text{disk}} + 2)] \right). \quad (5.2)$$

The topography of the disk is then $b(c, \mathcal{T}, a, \omega, r) = m\mathcal{B}$. Figure 5.1 shows the effects of the asymmetry parameter c , smoothing parameter $\mathcal{T}$, amplitude a and spatial frequency ω on the topography of the disk. We will discuss the effect of each of these four parameters in turn. We note that the results in this chapter will show that the amplitude of the topography has the most significant effect on the behaviour of the flow and that the asymmetry, smoothness and spatial frequency of the topography have a relatively weak effect on the behaviour of the flow.

All results presented in this chapter were performed with a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. Throughout this section, unless otherwise stated, typical topography parameters of $c = 0$, $\mathcal{T} = 0.001$, $a = 0.2$ and

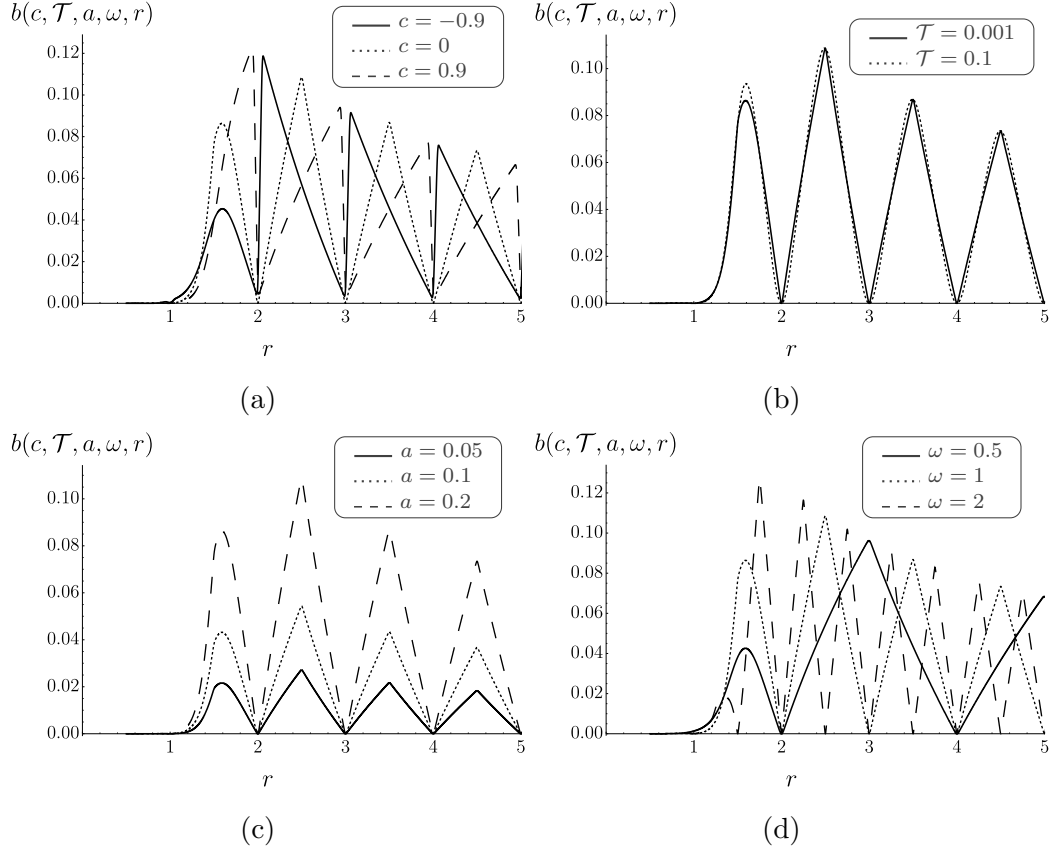


Figure 5.1: Effect of the topography parameters on the topography of the disk $b(c, \mathcal{T}, a, \omega, r)$, (a) effect of the asymmetry parameter $c = -0.9$ (solid curve), $c = 0$ (dotted curve) and $c = 0.9$ (dashed curve), where the other parameters are $\mathcal{T} = 0.001$, $a = 0.2$ and $\omega = 1$; (b) effect of the smoothing parameter $\mathcal{T} = 0.001$ (solid curve) and $\mathcal{T} = 0.1$ (dotted curve), where the other parameters are $c = 0$, $a = 0.2$ and $\omega = 1$; (c) the effect of amplitude $a = 0.05$ (solid curve), $a = 0.1$ (dotted curve) and $a = 0.2$ (dashed curve), where the other parameters are $c = 0$, $\mathcal{T} = 0.001$ and $\omega = 1$; and (d) effect of spatial frequency $\omega = 0.5$ (solid curve), $\omega = 1$ (dotted curve) and $\omega = 2$ (dashed curve), where the other parameters are $c = 0$, $\mathcal{T} = 0.001$ and $a = 0.2$.

$\omega = 1$ will be used. We use the code described at the beginning of Chapter 3 to simulate the reduced-order model (2.65)–(2.67) from the inlet ($r = R_{\text{jet}}$) to the edge of the disk ($r = R_{\text{disk}}$). A soft boundary condition is implemented at $r = R_{\text{disk}}$ by introducing a small smoothly increasing diffusion term near the edge of the disk to stabilise outgoing waves, however, we have verified that this condition has no effect on the dynamics upstream. The code uses a uniform grid of 23000 points and a time step of 10^{-5} . Newton iterations were stopped at a residual tolerance of 10^{-10} , and simulations were run to $t = 100$. Convergence was verified via grid/time-step refinement in both space and time. Therefore, in this chapter, results are presented over the radial interval $R_{\text{jet}} \leq r \leq R_{\text{disk}} - 1$, to exclude the effect of the soft boundary condition.

5.1 Effect of asymmetry

We first consider the effect of the asymmetry parameter c , the effect of which on the topography is shown in Figure 5.1(a).

Figure 5.2 shows space-time plots of $h - h_N$ for flow over a topographically-patterned disk with asymmetry parameter $c = -0.9$, $c = -0.45$, $c = 0$, $c = 0.45$ and $c = 0.9$. In particular, Figure 5.2 shows that the radial position of the self-sustained inception of waves remains at $r_s \approx 3.75$ irrespective of the value of c . This is the same radial position of the self-sustained inception of waves as in the planar case with inlet forcing of maximum amplitude $A = 0.01$ shown in Figure 4.7(b), which indicates that r_s is unaffected by both the addition of topography and the value of the asymmetry parameter c . Figure 5.2 also shows that the asymmetry parameter c has little effect on the general behaviour of the free surface. However, close inspection of Figure 5.2

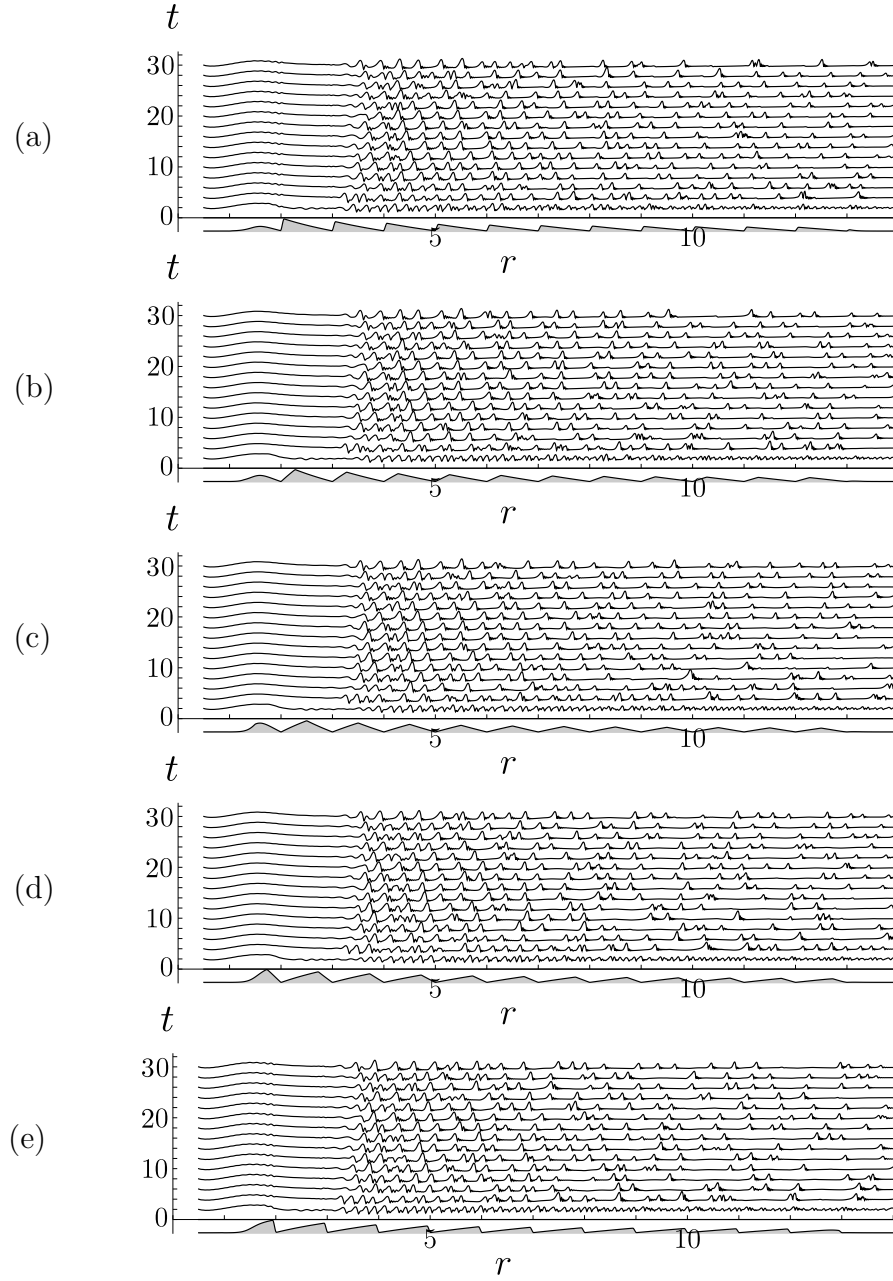


Figure 5.2: Space-time plots of $h - h_N$ for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter (a) $c = -0.9$, (b) $c = -0.45$, (c) $c = 0$, (d) $c = 0.45$ and (e) $c = 0.9$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$; and spatial frequency $\omega = 1$. In these plots, the time varies from $t = 0$ to $t = 30$ in steps of $t = 2$ and the topography is magnified by a factor of 10 (shaded).

reveals that the value of the asymmetry parameter c does have an effect on peak size and inter-peak separation; this will be analysed further in the following subsections. This effect is caused by additional small waves being generated by the topography, which then coalesce with the larger waves. This creates larger peaks that travel faster than the case of a planar disk due to the effect of centrifugation.

We now consider the effect of the asymmetry parameter c on the following key metrics: interfacial waviness and shear stress, inter-peak separation, and peak location.

5.1.1 Interfacial waviness and shear stress

Figure 5.3 shows the interfacial waviness $W(t)$ and shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with asymmetry parameter $c = -0.9$, $c = -0.45$, $c = 0$, $c = 0.45$ and $c = 0.9$.

Figure 5.3(a) shows that, although there are small variations, the results for the interfacial waviness for all five values of the asymmetry parameter c considered yield results that follow the same general trend. In particular, Figure 5.3(a) shows that the interfacial waviness increases steeply until $t \approx 3$, at which time it reaches its maximum value. The interfacial waviness then decreases gradually before fluctuating around a constant value of $W(t) \approx 1.65$ for $t \gtrsim 10$, which is almost independent of the value of the asymmetry parameter c . This behaviour is rather different to that in the planar case with inlet forcing described in Section 4.2.1 in which the interfacial waviness sharply increases before fluctuating around a constant value of $W(t) \approx 1.2$ for $t \gtrsim 10$ (shown in Figure 4.8(a) and reproduced in Figure 5.11(a) for comparison). The increase in the constant value around which the interfacial waviness fluctuates is due to the topography generating additional small waves that coalesce with the

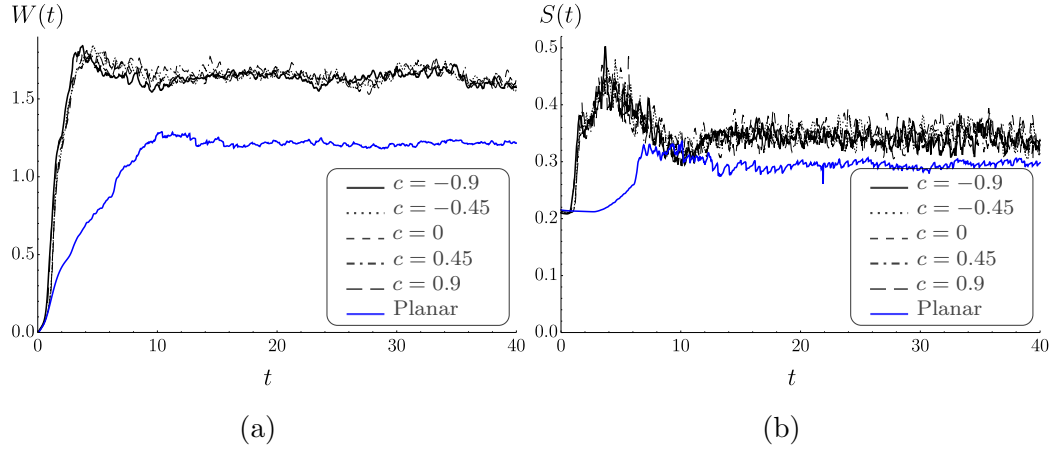


Figure 5.3: Plots of (a) interfacial waviness $W(t)$ and (b) shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = -0.9$ (solid curve), $c = -0.45$ (dotted curve), $c = 0$ (dashed curve), $c = 0.45$ (dot-dashed curve) and $c = 0.9$ (long dashed curve); smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$; and spatial frequency $\omega = 1$. The planar case with inlet forcing of maximum amplitude $A = 0.01$ (blue curve) has also been included for comparison.

larger waves. However, the approach to this is somewhat different to that observed in the planar case with inlet forcing. This is because the initial condition (2.70), in which the film thickness is set to $\eta = 3^{1/3}r^{-2/3}$, corresponds to a strongly-varying interface due to the presence of topography. This initial disturbance to the interface, in combination with the presence of the topography, generates additional waves (compared to the planar case with and without inlet forcing) in early stages of evolution, corresponding to the interfacial waviness increasing more sharply than in the planar case with inlet forcing. In fact, reaching its maximum value at $t \approx 3$ in contrast to at $t \approx 10$ in the planar case both with and without inlet forcing. The decrease in the interfacial waviness (a trend akin to the planar case without inlet forcing) over the time interval $3 \lesssim t \lesssim 10$ is caused by the larger waves on the disk coalescing with the disturbances created by the topography. This results in a smaller number of waves that have larger amplitudes, which decreases the interfacial waviness. At $t \approx 10$ the system reaches a fully-developed state, after which the interfacial waviness fluctuates around a constant value. This fluctuation is due to three factors: the topography generating small waves that coalesce with the larger waves in addition to the two factors previously discussed in Section 4.1, namely the formation of waves due to the self-sustained inception of waves, and waves leaving the edge of the disk.

Figure 5.3(b) shows that the value of the shear stress initially increases sharply before reaching its maximum value at $t \approx 4$. The value of shear stress then decreases until $t \approx 9$ before increasing and then fluctuating around a constant value of $S(t) \approx 0.35$ for $t \gtrsim 11$ that is essentially independent of the asymmetry parameter c . Again, the time at which the shear stress fluctuates around a constant value occurs is the time at which the waves become fully developed in the middle and outer regions.

This behaviour is rather different to that in the planar case with inlet forcing. In particular, Figure 5.3(b) shows that the shear stress increases more sharply in the presence of topography than in the planar case with inlet forcing, reaching its maximum value at $t \approx 3$ compared to $t \approx 7$ in the planar case with inlet forcing. However, for both a topographically-patterned disk and a planar disk, the shear stress fluctuates around a constant (but different) value for $t \gtrapprox 11$. In particular, Figure 5.3(b) shows that the topography causes the shear stress to fluctuate around a higher value than in the planar case with inlet forcing for $t \gtrapprox 11$. Therefore, we conclude that the shear stress is greater for a topographically-patterned disk than for a planar disk; however, the shear stress is relatively unaffected by the value of the asymmetry parameter c .

5.1.2 Inter-peak separation

Figure 5.4 shows bar charts of the mean and standard deviation of inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a topographically-patterned disk with asymmetry parameter $c = -0.9$, $c = -0.45$, $c = 0$, $c = 0.45$ and $c = 0.9$, over the time interval from $t = 20$ to $t = 100$. We consider the time interval from $t = 20$ to $t = 100$ for consistency with the results presented in Sections 4.1 and 4.2. In particular, Figure 5.4 shows that over the subintervals between $7 \leq r < 13$ the mean inter-peak separation decreases with r for all five values of the asymmetry parameter c presented, with the value of the mean inter-peak separation in the interval $7 \leq r < 8$ approximately 0.34 in all five cases. This is in contrast to the behaviour in the planar case with inlet forcing (shown in Figure 4.9(b)) in which the mean inter-peak separation exhibits relatively weak radial dependence, with a general trend of increasing across the subintervals within $7 \leq r < 13$. In the

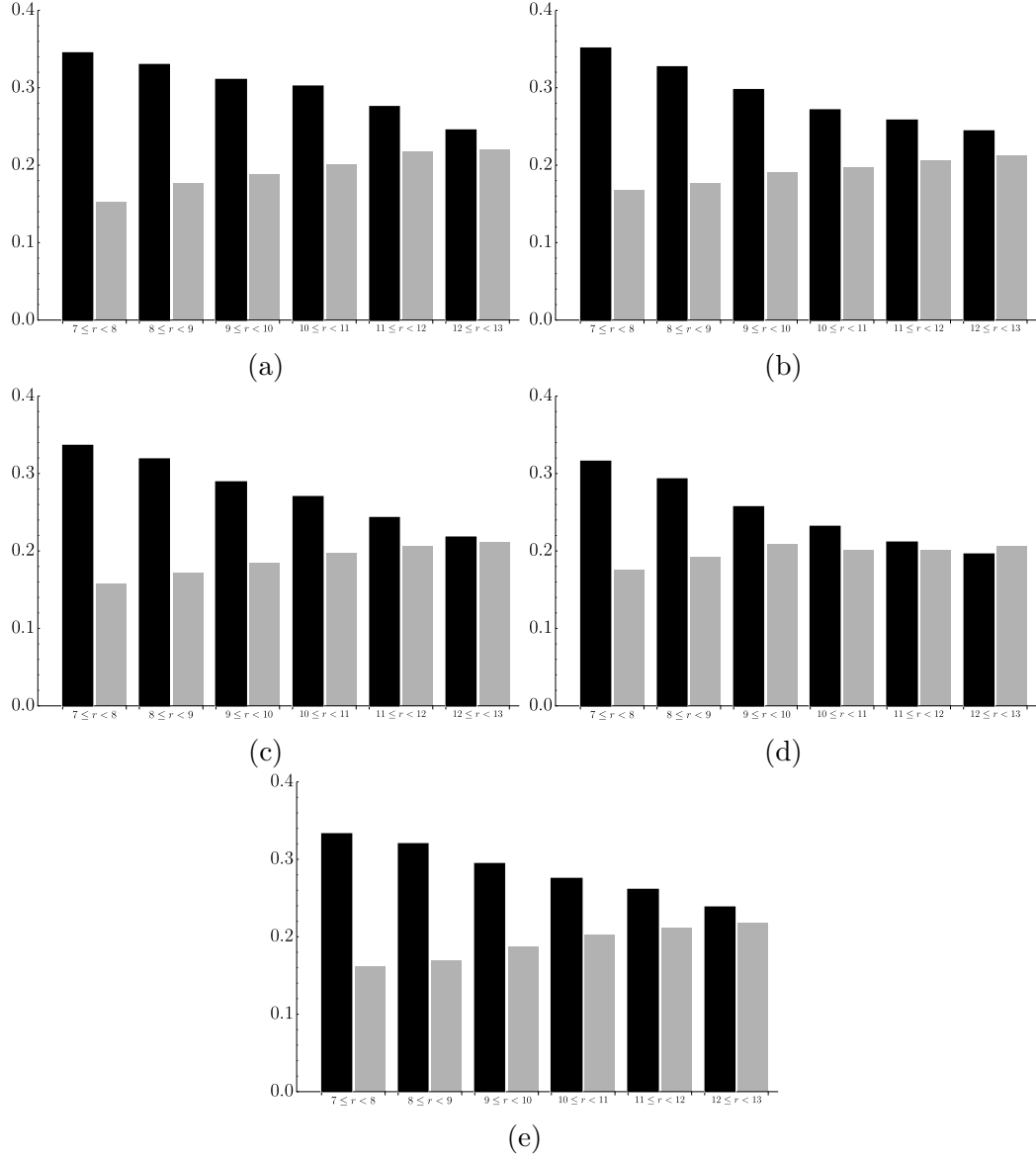


Figure 5.4: Bar charts of the mean (black) and standard deviation (grey) of inter-peak separation in unit intervals from $r = 7$ to $r = 13$, for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$, over the time interval from $t = 20$ to $t = 100$. The topography parameters are given by asymmetry parameter (a) $c = -0.9$, (b) $c = -0.45$, (c) $c = 0$, (d) $c = 0.45$ and (e) $c = 0.9$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$; and spatial frequency $\omega = 1$.

planar case with inlet forcing, the mean inter-peak separation is also approximately 0.34 in the interval $7 \leq r < 8$, indicating that this interval is relatively unaffected by the presence of topography. Figure 5.4 shows that the value of the asymmetry parameter c has a minor effect on the mean inter-peak separation. In particular, the asymmetry parameter value of $c = -0.9$ produces the greatest mean inter-peak separation on average across the subintervals within $7 \leq r < 13$, whereas the asymmetry parameter value of $c = 0.45$ produces the lowest mean inter-peak separation on average across the same interval. Figure 5.4 shows that in all five cases the standard deviation of the mean inter-peak separation increases with r across the subintervals within $7 \leq r < 13$, reaching a maximum value of approximately 0.21 in the interval $12 \leq r < 13$. This is in contrast to the behaviour of the planar case with inlet forcing (shown in Figure 4.9(b)) in which the maximum value reached is approximately 0.15. Therefore, the addition of topography significantly increases the level of disorder, as characterised by the standard deviation. The extreme asymmetry parameter values of $c = -0.9$ and $c = 0.9$ produce a marginally greater standard deviation of inter-peak separation in the interval $12 \leq r < 13$ than the other three values of c considered.

5.1.3 Peak location

Figure 5.5 shows space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with asymmetry parameter $c = -0.9$, $c = -0.45$, $c = 0$, $c = 0.45$ and $c = 0.9$. In particular, Figure 5.5 shows that for all five values of the asymmetry parameter c presented, the self-sustained inception of the waves occurs at $t \approx 5$. This occurs at a slightly earlier time than in the planar case with inlet forcing in which it occurs at $t \approx 6$ (see Figure 4.10(b)). Figure 5.5 also shows

CHAPTER 5. PARAMETRIC STUDY OF THIN-FILM FLOW OVER A TOPOGRAPHICALLY-PATTERNED SPINNING DISK

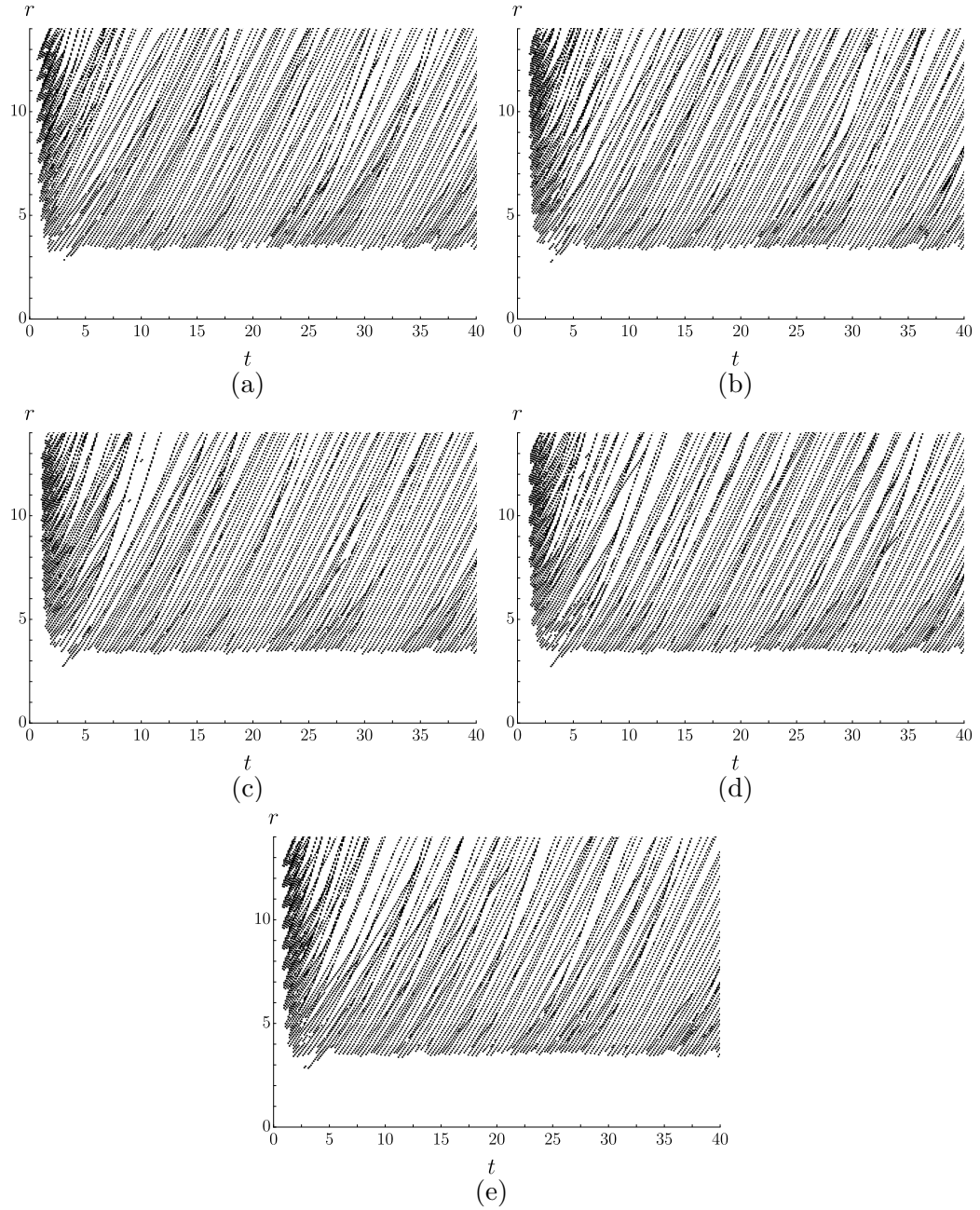


Figure 5.5: Space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter (a) $c = -0.9$, (b) $c = -0.45$, (c) $c = 0$, (d) $c = 0.45$ and (e) $c = 0.9$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$; and spatial frequency $\omega = 1$.

that the waves due to the initial transient exhibit dependence on the asymmetry parameter c . This is due to the initial condition setting the film thickness to be $\eta = 3^{1/3}r^{-2/3}$, which causes peaks to be present for $4 \lesssim r \leq 15$ at $t \approx 1.5$ due to the topography in contrast to the planar case with inlet forcing in which peaks emerge at $r \approx 3$ at $t \approx 3$. After the initial transient, there are more coalescence and separation events for all five values of c presented than in the planar case with inlet forcing, which causes more variation in the inter-peak separation, explaining the increase in the standard deviation of the inter-peak separation with the addition of topography. We consider the case of $c = -0.9$ (shown in Figure 5.5(a)) that produces the greatest inter-peak separation on average over the subintervals within $7 \leq r < 13$. In this case, successive coalescence of peaks occur more frequently compared to the other four cases presented. For example, in Figure 5.5(a) at the radial position $r \approx 6$ at $t \approx 25$ a peak begins to have multiple successive coalescence events. This behaviour is the cause of the greater mean inter-peak separation produced for $c = -0.9$. The coalescence of peaks creates larger peaks that travel faster (due to the effect of centrifugation), which increase the inter-peak separation.

5.2 Effect of smoothing

We next consider the effect of the smoothing parameter $\mathcal{T}$, the effect of which on the topography is shown in Figure 5.1(b).

Figure 5.6 shows space-time plots of $h - h_N$ for flow over a topographically-patterned disk with smoothing parameter $\mathcal{T} = 0.001$ and $\mathcal{T} = 0.1$. In particular, Figure 5.6 shows that the radial position of the self-sustained inception of waves remains at $r_s \approx 3.75$ irrespective of the value of $\mathcal{T}$. This is the same radial position of the

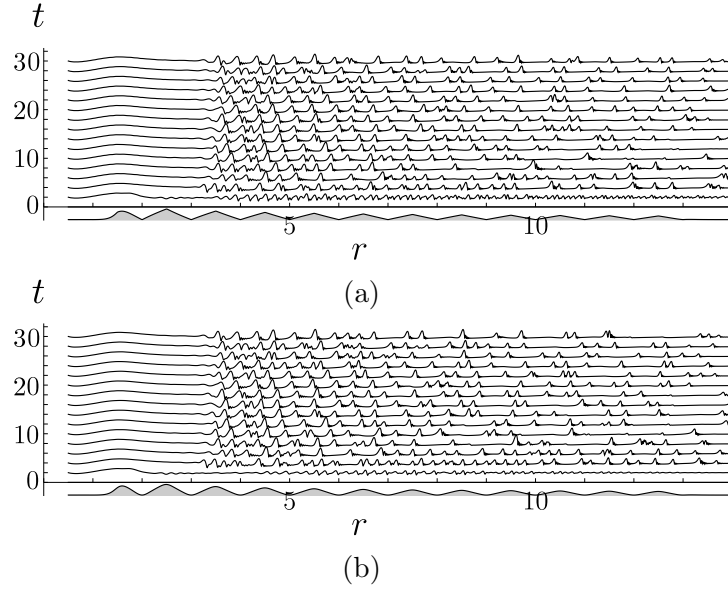


Figure 5.6: Space-time plots of $h - h_N$ for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter (a) $\mathcal{T} = 0.001$ and (b) $\mathcal{T} = 0.1$; amplitude $a = 0.2$; and spatial frequency $\omega = 1$. In these plots, the time varies from $t = 0$ to $t = 30$ in steps of $t = 2$ and the topography is magnified by a factor of 10 (shaded).

self-sustained inception of waves as in the planar case with inlet forcing of maximum amplitude $A = 0.01$ (shown in Figure 4.7(b)), which indicates that r_s is unaffected by both the addition of topography and the value of the smoothing parameter $\mathcal{T}$. Figure 5.6 shows that the smoothing parameter $\mathcal{T}$ has little effect on the general behaviour of the free surface. However, close inspection of Figure 5.6 reveals that the value of the smoothing parameter $\mathcal{T}$ does have an effect on peak size and inter-peak separation; this will be analysed further in the following subsections.

We now investigate the effect of the smoothing parameter $\mathcal{T}$ on the interfacial waviness and shear stress, inter-peak separation, and peak location.

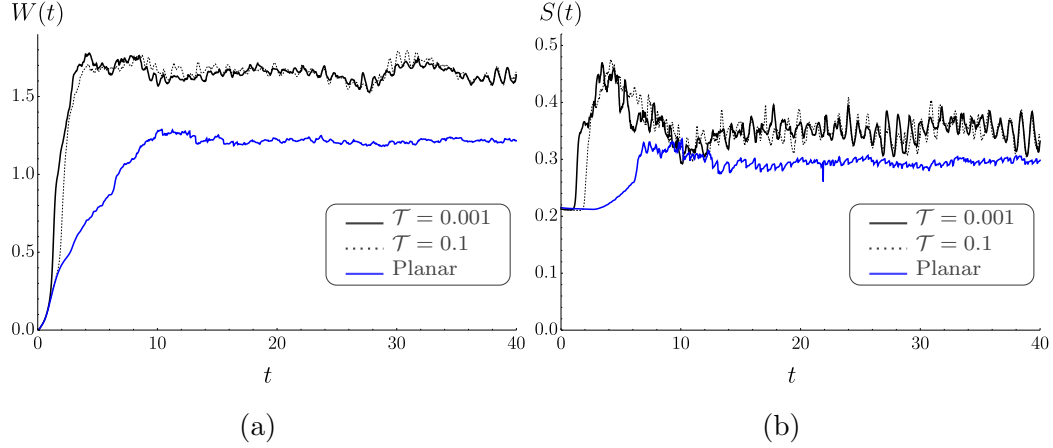


Figure 5.7: Plots of (a) interfacial waviness $W(t)$ and (b) shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$ (solid curve) and $\mathcal{T} = 0.1$ (dotted curve); amplitude $a = 0.2$; and spatial frequency $\omega = 1$. The planar case with inlet forcing of maximum amplitude $A = 0.01$ (blue curve) has also been included for comparison.

5.2.1 Interfacial waviness and shear stress

Figure 5.7 shows the interfacial waviness $W(t)$ and shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with smoothing parameter $\mathcal{T} = 0.001$ and $\mathcal{T} = 0.1$.

Figure 5.7(a) shows that the interfacial waviness for the two values of the smoothing parameter presented follow the same general trend, however, there are small variations between the two. In particular, Figure 5.7(a) shows that the interfacial waviness increases steeply until $t \approx 3$, at which time it reaches its maximum value. The interfacial waviness then decreases gradually before fluctuating around a constant value of $W(t) \approx 1.65$ for $t \gtrsim 10$, which is almost independent of the value of the smoothing parameter $\mathcal{T}$. Figure 5.7(a) shows that initially the case

$\mathcal{T} = 0.001$ increases more steeply than the case $\mathcal{T} = 0.1$ for $0 \leq t \lesssim 3$. This causes the interfacial waviness in the case $\mathcal{T} = 0.1$ to not reach its maximum at $t \approx 3$. The maximum value in this case is not reached until $t \approx 30$, behaviour that is caused by a larger fluctuation around the constant value compared to the rest of the time interval. Therefore, the smoothing parameter $\mathcal{T}$ has a minimal effect on the interfacial waviness, although a higher value of the smoothing parameter causes the interfacial waviness to increase less sharply initially and hence not to reach its maximum value until a later time.

Figure 5.7(b) shows that the value of the shear stress initially increases sharply before reaching its maximum value at $t \approx 4$. The value of shear stress then decreases until $t \approx 9$ before increasing and then fluctuating around a constant value of $S(t) \approx 0.35$ for $t \gtrsim 11$ that is essentially independent of the smoothing parameter $\mathcal{T}$. During the decrease for times $4 \lesssim t \lesssim 9$, the value of the shear stress approaches that of the planar case as during this time the waves in the middle and outer regions have not fully developed. The case $\mathcal{T} = 0.1$ takes a slightly longer time to begin the sharp increase, however, after $t \approx 3$ the same trend is followed and there are only minor differences in the fluctuations. Therefore, the value of the smoothing parameter $\mathcal{T}$ only meaningfully impacts the shear stress for $0 \leq t \lesssim 3$, and after this interval it has a negligible effect.

5.2.2 Inter-peak separation

Figure 5.8 shows bar charts of the mean and standard deviation of inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a topographically-patterned disk with smoothing parameter $\mathcal{T} = 0.001$ and $\mathcal{T} = 0.1$, over the time interval from $t = 20$ to $t = 100$. We consider the time interval from $t = 20$ to $t = 100$

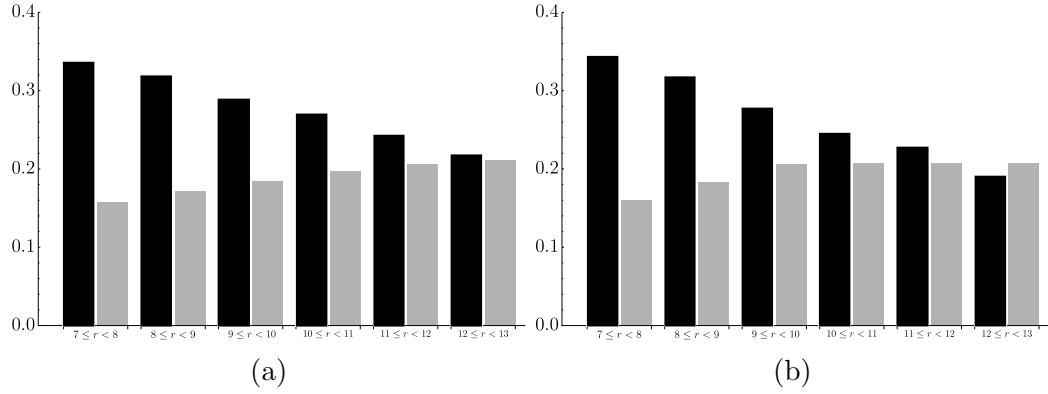


Figure 5.8: Bar charts of the mean (black) and standard deviation (grey) of inter-peak separation in unit intervals from $r = 7$ to $r = 13$, for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$, over the time interval from $t = 20$ to $t = 100$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter (a) $\mathcal{T} = 0.001$ and (b) $\mathcal{T} = 0.1$; amplitude $a = 0.2$; and spatial frequency $\omega = 1$.

for consistency with the results presented in Sections 4.1 and 4.2. In particular, Figure 5.8 shows that the two values of the smoothing parameter $\mathcal{T}$ presented follow the same general trend in the mean and standard deviation of inter-peak separation. However, there are some small differences, for example, for the case $\mathcal{T} = 0.1$ the mean inter-peak separation is slightly higher in the interval $7 \leq r < 8$ and slightly lower in the interval $12 \leq r < 13$ than the case $\mathcal{T} = 0.001$. In the case $\mathcal{T} = 0.1$, the standard deviation of inter-peak separation increases throughout the interval $7 \leq r < 9$ then remains nearly constant at a value of approximately 0.2 for the interval $9 \leq r < 13$, whereas, in the case $\mathcal{T} = 0.001$, the standard deviation increases throughout the interval $7 \leq r < 13$ reaching a maximum value of approximately 0.2 in the interval $12 \leq r < 13$. Therefore, the smoothing parameter $\mathcal{T}$ has only a minor impact on the mean and standard deviation of the inter-peak separation. However, increasing the smoothing parameter does slightly decrease the mean inter-

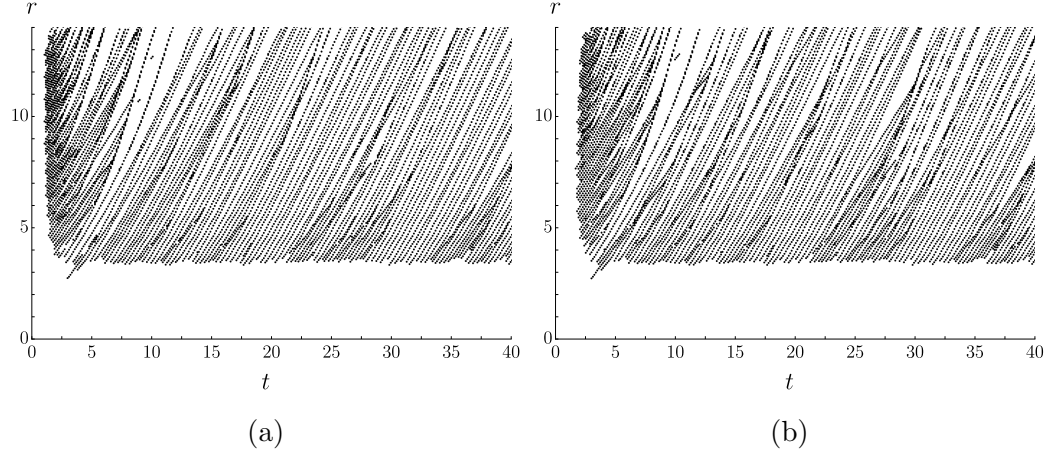


Figure 5.9: Space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter (a) $\mathcal{T} = 0.001$ and (b) $\mathcal{T} = 0.1$; amplitude $a = 0.2$; and spatial frequency $\omega = 1$.

peak separation and increase the standard deviation of the inter-peak separation in the region $12 \leq r < 13$.

5.2.3 Peak location

Figure 5.9 shows space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with smoothing parameter $\mathcal{T} = 0.001$ and $\mathcal{T} = 0.1$. In particular, Figure 5.9 shows that for both values of the smoothing parameter $\mathcal{T}$ presented, the self-sustained inception of waves r_s occurs at $t \approx 5$. The case $\mathcal{T} = 0.1$ has more variation in inter-peak separation (due to more coalescence events) than the case $\mathcal{T} = 0.001$. This explains the slight increase in the standard deviation of inter-peak separation for $\mathcal{T} = 0.1$ compared to $\mathcal{T} = 0.001$ previously discussed in Section 5.2.2. Therefore, increased smoothing of the topography causes more

variation in inter-peak separation, however, this effect is minor when averaged over time, as shown in Section 5.2.2.

5.3 Effect of amplitude

We next consider the effect of varying the amplitude of the topography a , the effect of which on the topography is shown in Figure 5.1(c).

Figure 5.10 shows space-time plots of $h - h_N$ for flow over a topographically-patterned disk with amplitude $a = 0.05$, $a = 0.1$ and $a = 0.2$. Figure 5.10 shows that the radial position of the self-sustained inception of waves remains at $r_s \approx 3.75$ irrespective of the value of a . This is the same radial position of the self-sustained inception of waves as in the planar case with inlet forcing (shown in Figure 4.8(a)), which indicates that r_s is unaffected by both the addition of topography and the amplitude of the topography a . Figure 5.10 shows that the amplitude of the topography a has little effect on the general behaviour of the free surface. However, close inspection of Figure 5.10 reveals that the amplitude of the topography a does have an effect on peak size and inter-peak separation; this effect will be analysed further in the following subsections.

We now consider the effect of the amplitude a on the interfacial waviness and shear stress, inter-peak separation, and peak location.

5.3.1 Interfacial waviness and shear stress

Figure 5.11 shows the interfacial waviness $W(t)$ and shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with amplitude $a = 0.05$, $a = 0.1$ and $a = 0.2$.

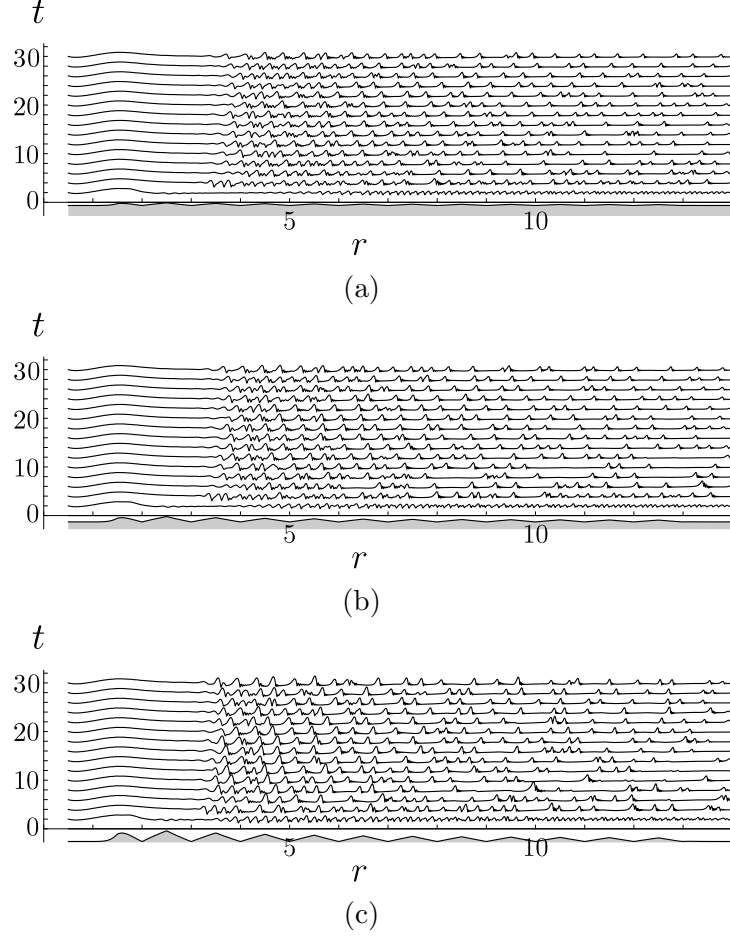


Figure 5.10: Space-time plots of $h - h_N$ for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude (a) $a = 0.05$, (b) $a = 0.1$ and (c) $a = 0.2$; and spatial frequency $\omega = 1$. In these plots, the time varies from $t = 0$ to $t = 30$ in steps of $t = 2$ and the topography is magnified by a factor of 10 (shaded).

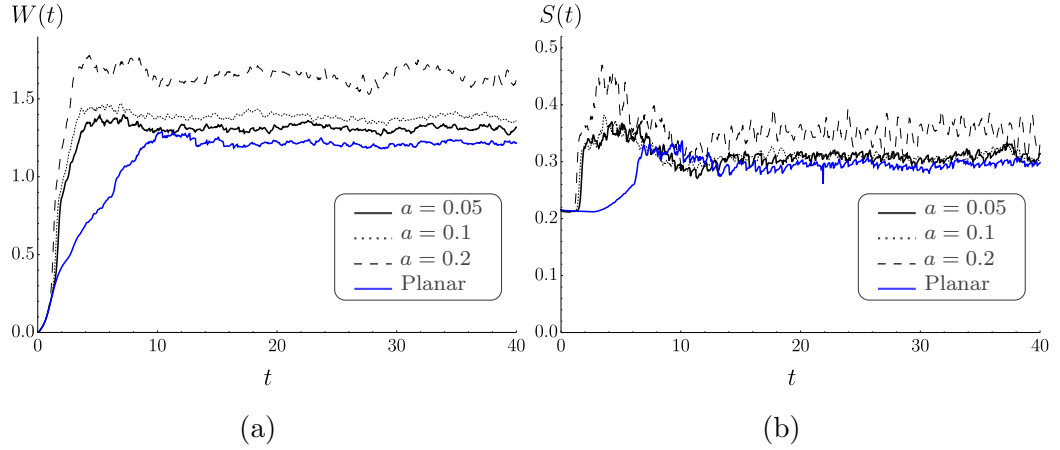


Figure 5.11: Plots of (a) interfacial waviness $W(t)$ and (b) shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.05$ (solid curve), $a = 0.1$ (dotted curve) and $a = 0.2$ (dashed curve); and spatial frequency $\omega = 1$. The planar case with inlet forcing of maximum amplitude $A = 0.01$ (blue curve) has also been included for comparison.

Figure 5.11(a) shows that the amplitude a has a significant effect on the interfacial waviness. In particular, Figure 5.11(a) shows that for all three values of a presented, the interfacial waviness increases steeply until $t \approx 3$, at which time it reaches its maximum value. The interfacial waviness then decreases gradually before fluctuating around a constant value for $t \gtrsim 10$. Figure 5.11(a) shows that both the maximum value and the constant value depend on the amplitude a . In particular, the maximum value reached is $W(t) \approx 1.4$, $W(t) \approx 1.45$ and $W(t) \approx 1.75$ for the amplitudes $a = 0.05$, $a = 0.1$ and $a = 0.2$, respectively. The constant value that the interfacial waviness $W(t)$ fluctuates around for $t \gtrsim 10$ is $W(t) \approx 1.3$, $W(t) \approx 1.375$ and $W(t) \approx 1.65$ for the amplitudes $a = 0.05$, $a = 0.1$ and $a = 0.2$, respectively. Therefore, increasing the amplitude a increases the interfacial waviness. However, the case $a = 0.1$ only shows a small increase compared to the case $a = 0.05$. In comparison to the planar case with inlet forcing, the effect of even topography with small amplitude a on the interfacial waviness is pronounced. In particular, the value of the interfacial waviness for the planar case with inlet forcing increases then fluctuates around the value of $W(t) \approx 1.2$ for $t \gtrsim 10$. Hence, even the inclusion of topography with small amplitude a causes interfacial waviness to increase more sharply and also increase the constant value that the interfacial waviness fluctuates around for $t \gtrsim 10$.

Figure 5.11(b) shows that the value of the shear stress initially increases sharply before reaching its maximum value at $t \approx 4$, which depends on the amplitude a . The value of shear stress then decreases until $t \approx 9$ before increasing and then fluctuating around a constant value for $t \gtrsim 11$, which also depends on the amplitude a . In particular, the maximum value reached is $S(t) \approx 0.36$, $S(t) \approx 0.38$ and $S(t) \approx 0.47$ for the amplitudes $a = 0.05$, $a = 0.1$ and $a = 0.2$, respectively. The constant value

that the shear stress $S(t)$ fluctuates around for $t \gtrsim 11$ is $S(t) \approx 0.31$, $S(t) \approx 0.31$ and $S(t) \approx 0.35$ for the amplitudes $a = 0.05$, $a = 0.1$ and $a = 0.2$, respectively. Therefore, increasing the amplitude a increases the shear stress, with this effect being less pronounced for smaller amplitudes a . The cases $a = 0.05$ and $a = 0.1$ have almost the same value of shear stress, with the case $a = 0.1$ only reaching a slightly greater maximum value. The case $a = 0.2$ shows a significant increase in shear stress compared to the other two amplitudes a presented. In comparison to the planar case with inlet forcing, even small amplitudes have a significant effect on the shear stress. In particular, the planar case with inlet forcing reaches its maximum value at $t \approx 7$ then decreases before fluctuating around a constant value of $S(t) \approx 0.29$ for $t \gtrsim 11$. Note that the amplitudes $a = 0.05$ and $a = 0.1$ only produce a minor increase in the constant value that the shear stress fluctuates around compared to the planar case due to the smallness of the amplitudes. Hence, even the inclusion of topography with small amplitude a causes the shear stress to increase more sharply and fluctuate around an increased value for $t \gtrsim 11$.

5.3.2 Inter-peak separation

Figure 5.12 shows bar charts of the mean and standard deviation of inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a topographically-patterned disk with amplitude $a = 0.05$, $a = 0.1$ and $a = 0.2$, over the time interval from $t = 20$ to $t = 100$. We consider the time interval from $t = 20$ to $t = 100$ for consistency with the results presented in Sections 4.1 and 4.2. Figure 5.12 shows that as the amplitude a increases, on average the mean inter-peak separation decreases and the standard deviation of the inter-peak separation increases. This result is consistent with the interfacial waviness increasing with the amplitude a ,

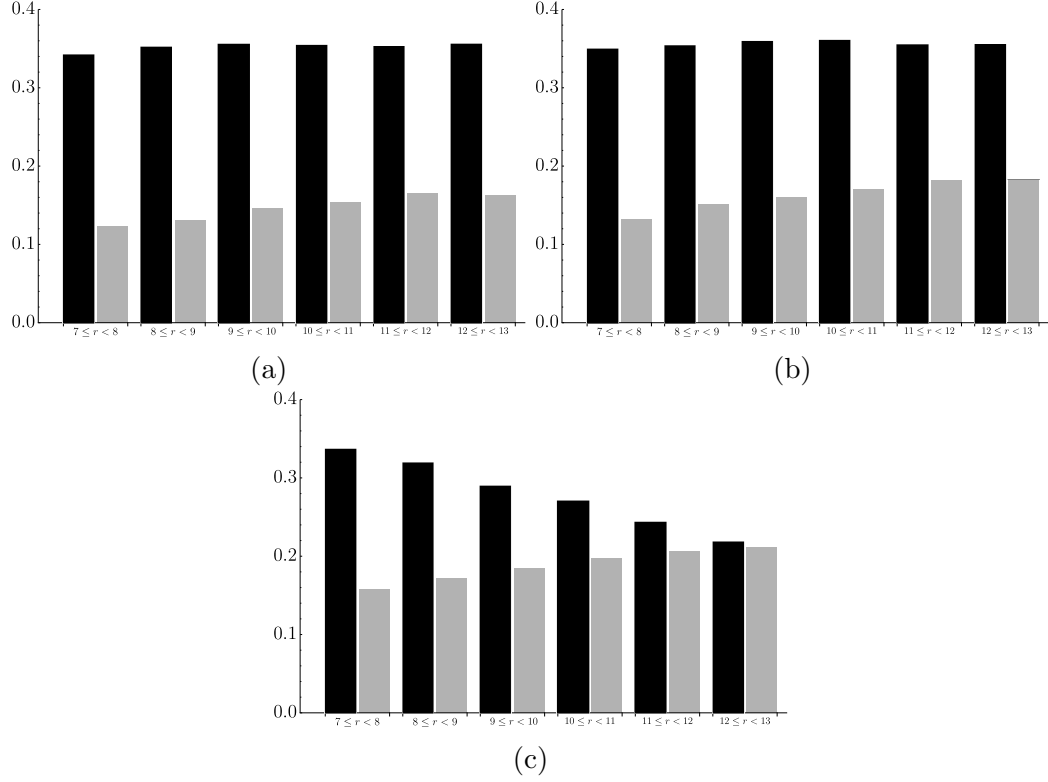


Figure 5.12: Bar charts of the mean (black) and standard deviation (grey) of inter-peak separation in unit intervals from $r = 7$ to $r = 13$, for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$, over the time interval from $t = 20$ to $t = 100$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude (a) $a = 0.05$, (b) $a = 0.1$ and (c) $a = 0.2$; and spatial frequency $\omega = 1$.

as discussed in Section 5.3.1. The cases $a = 0.05$ and $a = 0.1$ only show minor differences in the mean and standard deviation of inter-peak separation (which is again consistent with the results presented in Section 5.3.1), the mean inter-peak separation remains approximately constant at a value of 0.35 across the interval $7 \leq r < 13$ and the standard deviation increases with r across the interval $7 \leq r < 13$ up to approximately 0.16. In the case $a = 0.2$, the behaviour of the mean inter-peak separation is rather different to that for the other two amplitudes a presented. In the case $a = 0.2$, the mean inter-peak separation decreases with r , in contrast to the behaviour for the other two amplitudes a presented and the planar case with inlet forcing (shown in Figure 4.9(b)) in which the mean inter-peak separation has a relatively weak radial dependence. In the case $a = 0.2$, the standard deviation increases with r across the interval $7 \leq r < 13$ up to approximately 0.21. Therefore, increasing a causes the mean inter-peak separation to decrease, however, for larger values of a the inter-peak separation exhibits a radial dependence and decreases with r . Increasing the amplitude a causes the standard deviation of inter-peak separation to increase, indicating that the level of disorder (as characterised by the standard deviation) increases with a .

5.3.3 Peak location

Figure 5.13 shows space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with amplitude $a = 0.05$, $a = 0.1$ and $a = 0.2$. In particular, Figure 5.13 shows that while the peaks follow the same general trend for all three amplitudes a presented, there are some visible differences on close inspection. The cases $a = 0.05$ and $a = 0.1$ show minimal differences, in line with the results presented in Sections 5.3.1 and 5.3.2. However, the case $a = 0.2$ has more coalescence

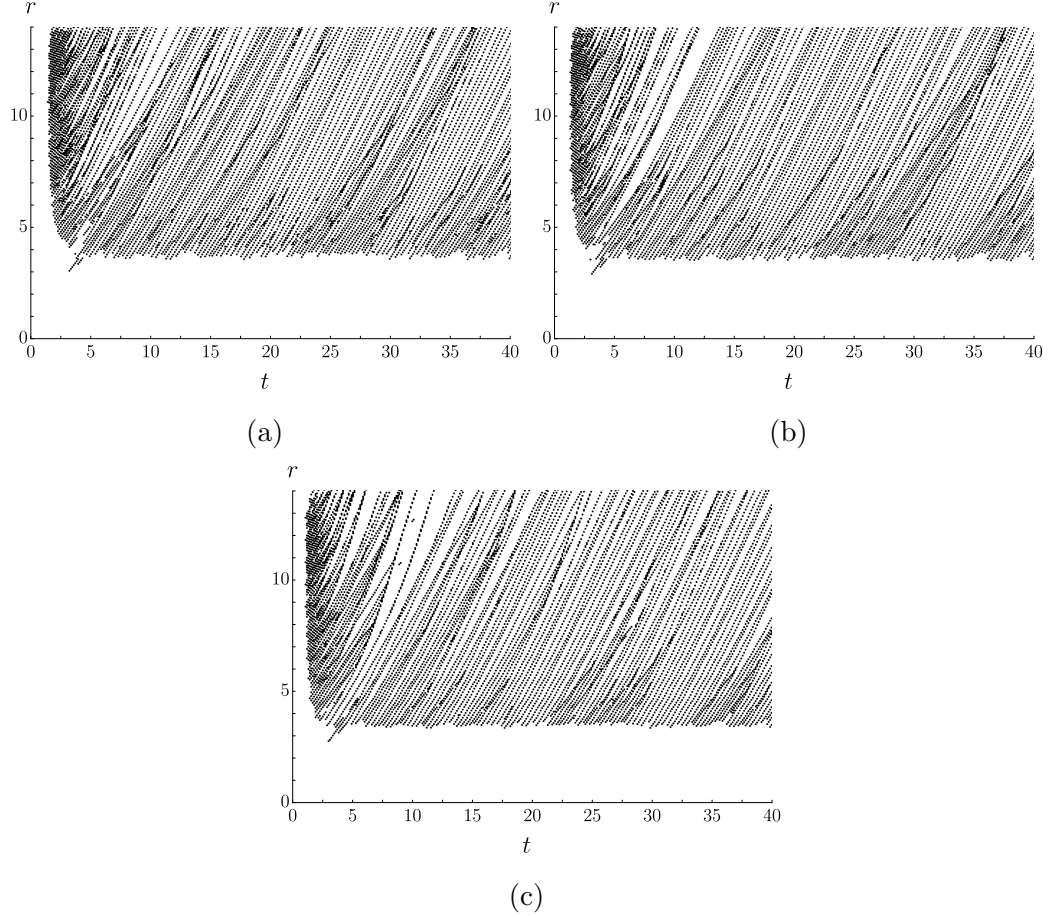


Figure 5.13: Space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude (a) $a = 0.05$, (b) $a = 0.1$ and (c) $a = 0.2$; and spatial frequency $\omega = 1$.

and separation events occurring compared to the other two amplitudes a presented. This is the reason for the differences in interfacial waviness and inter-peak separation for $a = 0.2$ compared to the other two amplitudes a presented. These additional coalescence and separation events result in the inter-peak separation having greater variation, which explains the increase in standard deviation compared to the other two amplitudes a presented.

5.4 Effect of spatial frequency

Finally, we consider the effect of the spatial frequency of the topography ω , the effect of which on the topography is shown in Figure 5.1(d).

Figure 5.14 shows space-time plots of $h - h_N$ for flow over a topographically-patterned disk with spatial frequency $\omega = 0.5$, $\omega = 1$, $\omega = 2$ and $\omega = 4$. Figure 5.14 shows that the radial position of the self-sustained inception of waves remains at $r_s \approx 3.75$ irrespective of the value of ω . This is the same radial position of the self-sustained inception of waves as in the planar case with inlet forcing (shown in Figure 4.8(a)), which indicates that r_s is unaffected by both the addition of topography and varying the spatial frequency ω . Figure 5.14 shows that the spatial frequency ω has little effect on the general behaviour of the free surface. However, close inspection of Figure 5.14 reveals that the spatial frequency ω does have an effect on peak size and inter-peak separation; this effect will be analysed further in the following subsections.

We now investigate the effect of the spatial frequency ω on the interfacial waviness and shear stress, inter-peak separation, and peak location.

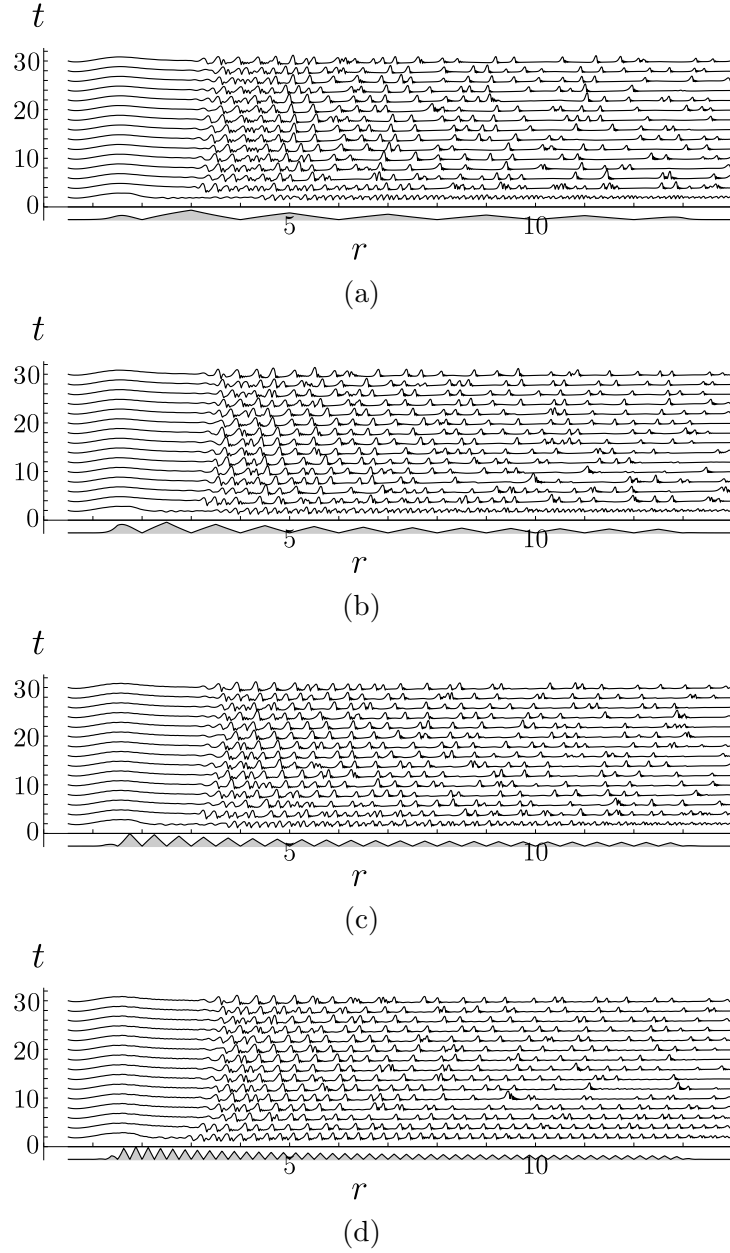


Figure 5.14: Space-time plots of $h - h_N$ for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$; and spatial frequency (a) $\omega = 0.5$, (b) $\omega = 1$, (c) $\omega = 2$ and (d) $\omega = 4$. In these plots, the time varies from $t = 0$ to $t = 30$ in steps of $t = 2$ and the topography is magnified by a factor of 10 (shaded).

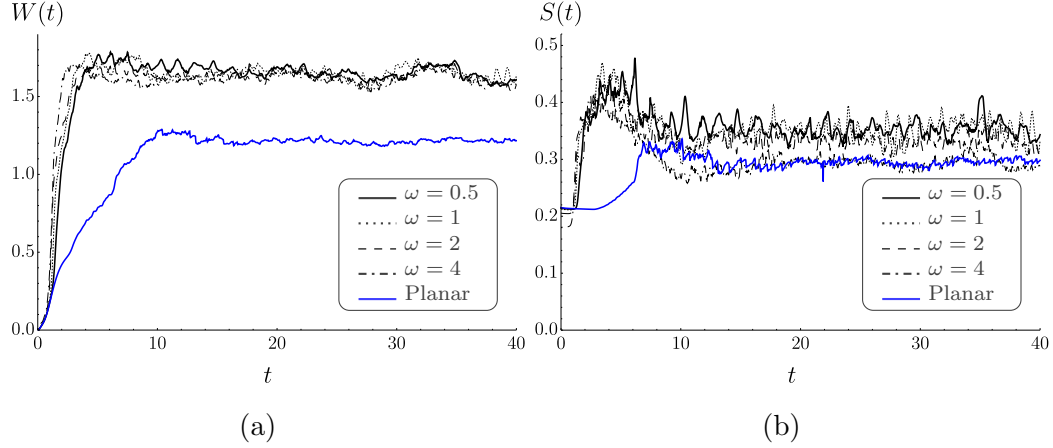


Figure 5.15: Plots of (a) interfacial waviness $W(t)$ and (b) shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$; and spatial frequency $\omega = 0.5$ (solid curve), $\omega = 1$ (dotted curve), $\omega = 2$ (dashed curve) and $\omega = 4$ (dot-dashed curve). The planar case with inlet forcing of maximum amplitude $A = 0.01$ (blue curve) has also been included for comparison.

5.4.1 Interfacial waviness and shear stress

Figure 5.15 shows the interfacial waviness $W(t)$ and shear stress $S(t)$ as functions of t for flow over a topographically-patterned disk with spatial frequency $\omega = 0.5$, $\omega = 1$, $\omega = 2$ and $\omega = 4$.

Figure 5.15(a) shows that the interfacial waviness follows the same general trend for all four values of the spatial frequency ω presented, however, there are small variations between the four cases. In particular, Figure 5.15(a) shows that the interfacial waviness increases steeply until $t \approx 3$, at which time it reaches its maximum value. The interfacial waviness then decreases gradually before fluctuating around a constant value of $W(t) \approx 1.65$ for $t \gtrsim 10$, which is almost independent of

the value of the spatial frequency ω . Figure 5.15(a) shows that in the initial steep increase of interfacial waviness between $0 \leq t \lesssim 3$, increasing the value of the spatial frequency ω causes the interfacial waviness to increase more steeply. Figure 5.15(a) shows that increasing the spatial frequency ω marginally decreases the maximum value of interfacial waviness. Therefore, the spatial frequency ω has a minimal effect on the interfacial waviness.

Figure 5.15(b) shows that for all four values of spatial frequency ω presented, the shear stress initially increases sharply before reaching its maximum value at $t \approx 4$. The value of the shear stress then decreases until $t \approx 9$ before increasing and then fluctuating around a constant value for $t \gtrsim 11$, which depends on the spatial frequency ω . The case $\omega = 0.5$ produces the greatest maximum value of shear stress. The cases $\omega = 0.5$, $\omega = 1$ and $\omega = 2$ all fluctuate around a constant value of $S(t) \approx 0.35$ for $t \gtrsim 11$. However, the case $\omega = 4$ fluctuates around a constant value of $S(t) \approx 0.29$ for $t \gtrsim 11$, which is the same value as for the planar case with inlet forcing discussed in Section 4.2.1. The exact reason for this is currently unknown; however, it is possibly caused by the spatial frequency of the topography being such that it oscillates too rapidly for the film to respond. Once the flow reaches a fully developed state, the topography is effectively averaged out, and the flow behaves as though the substrate were planar. Therefore, the shear stress is significantly lower in the case $\omega = 4$ for $t \gtrsim 11$ compared to the three other values of spatial frequency ω presented.

5.4.2 Inter-peak separation

Figure 5.16 shows bar charts of the mean and standard deviation of inter-peak separation in unit intervals from $r = 7$ to $r = 13$ for flow over a topographically-

patterned disk with spatial frequency $\omega = 0.5$, $\omega = 1$, $\omega = 2$ and $\omega = 4$, over the time interval from $t = 20$ to $t = 100$. We consider the time interval from $t = 20$ to $t = 100$ for consistency with the results presented in Sections 4.1 and 4.2. Figure 5.16 shows that all four values of spatial frequency ω presented follow the same general trend, namely that the mean inter-peak separation decreases and the standard deviation of the inter-peak separation increases across the interval $7 \leq r < 13$. In the interval $7 \leq r < 8$, all four values of spatial frequency ω presented have a mean inter-peak separation of approximately 0.34. However, after this interval, there begins to be some variation. In particular, in the case $\omega = 0.5$ the mean inter-peak separation is lower over the interval $8 \leq r < 13$ than for the other three values of spatial frequency ω presented. The variations are most pronounced in the interval $12 \leq r < 13$. In this interval, the mean inter-peak separation is approximately 0.17, 0.21, 0.22 and 0.23 for the frequencies $\omega = 0.5$, $\omega = 1$, $\omega = 2$ and $\omega = 4$, respectively. This implies that the mean-inter-peak separation increases with the spatial frequency ω . For the cases $\omega = 0.5$ and $\omega = 2$, the standard deviation increases with r across the interval $7 \leq r < 10$ and then is approximately constant over the interval $10 \leq r < 13$. For the cases $\omega = 1$ and $\omega = 4$, the standard deviation increases with r across the interval $7 \leq r < 13$. The case $\omega = 0.5$ has the highest standard deviation across the interval $7 \leq r < 13$ in comparison to the other three values of spatial frequency ω presented. In conclusion, the spatial frequency ω has only minor effects on the inter-peak separation, however, from the values of spatial frequency ω presented, $\omega = 0.5$ produces a lower mean and higher standard deviation of inter-peak separation.

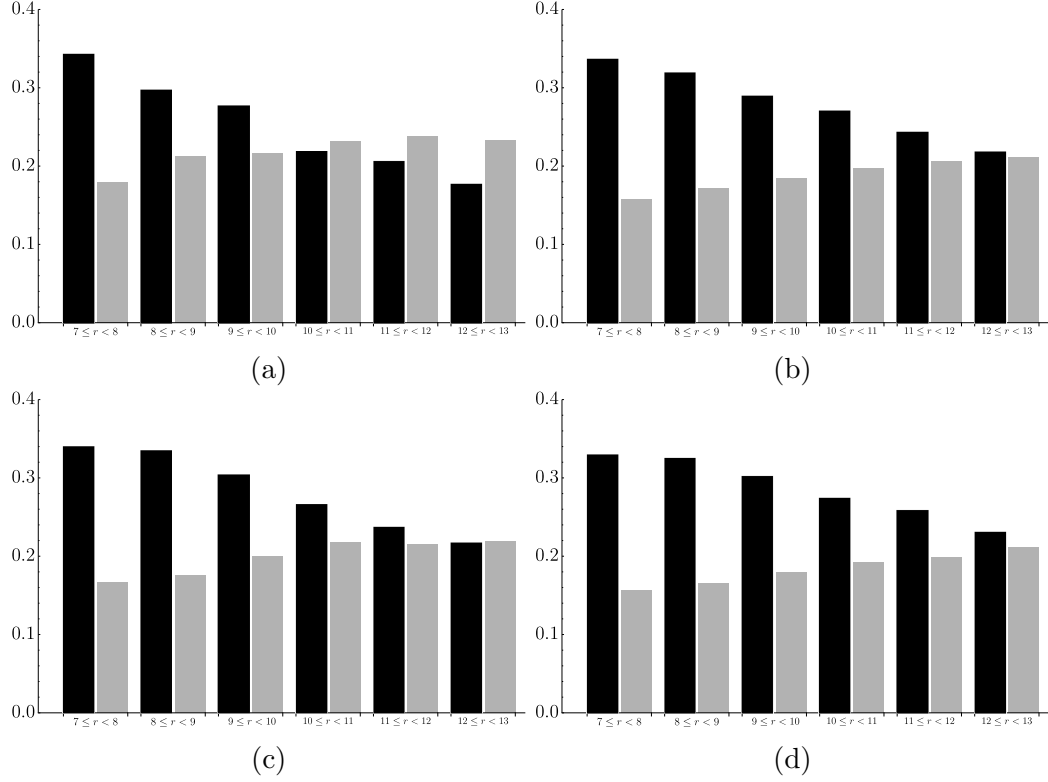


Figure 5.16: Bar charts of the mean (black) and standard deviation (grey) of inter-peak separation in unit intervals from $r = 7$ to $r = 13$, for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$, over the time interval from $t = 20$ to $t = 100$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$; and spatial frequency (a) $\omega = 0.5$, (b) $\omega = 1$, (c) $\omega = 2$ and (d) $\omega = 4$.

5.4.3 Peak location

Figure 5.17 shows space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with spatial frequency $\omega = 0.5$, $\omega = 1$, $\omega = 2$ and $\omega = 4$. In particular, Figure 5.17 shows that while the peaks follow the same general trend for all four spatial frequencies ω presented, there are some visible differences on close inspection. The case $\omega = 0.5$ has more separation and coalescence events occurring compared to the other three spatial frequencies ω presented, which implies that $\omega = 0.5$ interacts non-resonantly with the preferred inter-peak separation of the pulse-like train of waves. As the wavelength of the topography is much larger than the preferred pulse spacing, the topography disrupts the pulses, leading to more variation in the inter-peak separation in this case. This explains the slight decrease in mean inter-peak separation and increase in standard deviation of inter-peak separation for $\omega = 0.5$ compared to the other three values of spatial frequency ω presented, which was previously discussed in Section 5.4.2.

5.5 Concluding remarks

In this chapter, we have introduced a multi-parameter family of topographies with four parameters controlling the asymmetry, smoothness, amplitude and spatial frequency. We then performed a parametric study to understand the effect of each of these parameters on the flow. The results presented within this chapter have shown that the amplitude of the topography has the most significant impact on the flow. Variations in the asymmetry, smoothing and spatial frequency of the topography have only minor effects on the flow. The addition of topography amplitude $a = 0.05$ and $a = 0.1$ causes the mean inter-peak separation to remain approximately constant

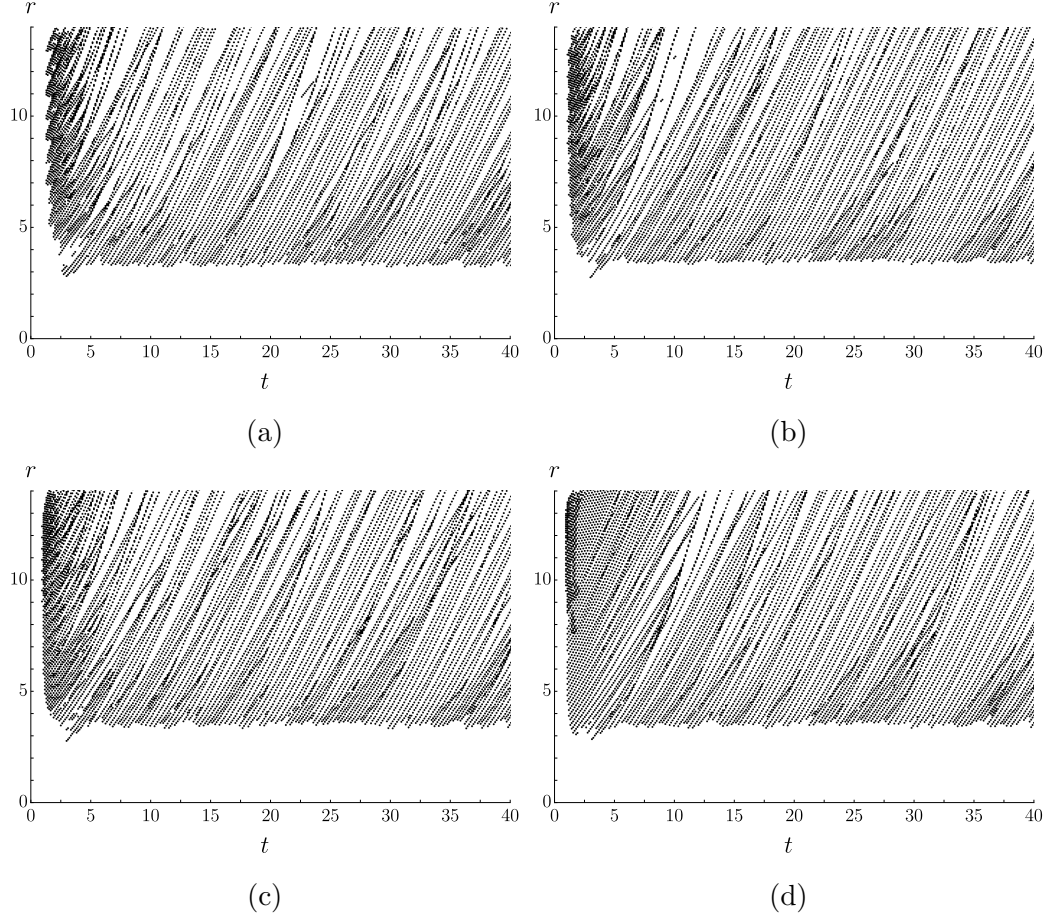


Figure 5.17: Space-time plots of the peaks of the free surface for flow over a topographically-patterned disk with $\lambda = 0.01$, inlet jet radius $R_{\text{jet}} = 0.5$ and disk radius $R_{\text{disk}} = 15$ with inlet forcing of maximum amplitude $A = 0.01$. The topography parameters are given by asymmetry parameter $c = 0$; smoothing parameter $\mathcal{T} = 0.001$; amplitude $a = 0.2$ and spatial frequency (a) $\omega = 0.5$, (b) $\omega = 1$, (c) $\omega = 2$ and (d) $\omega = 4$.

across the interval $7 \leq r < 13$, reducing the weak radial dependence exhibited in the planar case with inlet forcing in Section 4.2.2. Increasing the amplitude of the topography to $a = 0.2$ causes the mean inter-peak separation to decrease with r across the interval $7 \leq r < 13$. The interfacial waviness and shear stress both increase with the amplitude a . While the asymmetry, smoothness and spatial frequency of the topography have only minor effects on the flow, these parameters do have some effect. In particular, the asymmetry parameter value $c = -0.9$ produces a marginally greater mean inter-peak separation on average. Increasing the smoothing parameter $\mathcal{T}$ marginally decreases the mean inter-peak separation and marginally increases the standard deviation of the inter-peak separation. The spatial frequency $\omega = 0.5$ produces a lower mean inter-peak separation and higher standard deviation of inter-peak separation than the other values of spatial frequency tested. The spatial frequency $\omega = 4$ reduces the shear stress for $t \gtrsim 11$ compared to the other values of spatial frequency tested, in fact, reducing the shear stress to the value of the planar case with inlet forcing.

Chapter 6

Stability analysis of the behaviour in the far field

In this chapter, we obtain some analytical insight into the behaviour of the system. To make analytical progress, we consider the behaviour in the far field and discuss the appropriate scalings in the far field in Section 6.1. We then obtain three systems of equations and boundary conditions corresponding to the inner, middle and outer regions, and perform a stability analysis for each region in Sections 6.2–6.4.

6.1 Far-field scalings

In order to obtain additional insight into the behaviour of the system we consider the spatial and temporal stability of the evolution equations (2.65)–(2.67). The self-sustained inception of waves discussed in Chapters 4 and 5 is of particular interest. This occurs at the radial position $r = r_s$, where $r_s \gtrapprox 3$ for the parameter values considered in the present work. We can therefore make analytical progress by

considering the behaviour in the far field. The inner, middle and outer regions, first discussed numerically in Chapter 4, are sufficiently far out that they lie in the far field, due to the steady far-field solution (2.69) being a good approximation for the free surface h for $r \gtrsim 3$ (discussed in Sections 4.1 and 4.2). In Sections 6.2, 6.3 and 6.4, we obtain three different systems of equations and boundary conditions describing the leading-order behaviour in the inner, middle and outer regions depending on the appropriate balance of terms in each region. In each region, we describe the leading-order asymptotic behaviour and then perform a stability analysis, providing analytic insight into the behaviour of the flow.

We retain only terms of leading order in the aspect ratio $\epsilon \ll 1$ and, for simplicity, consider the case of a planar disk (i.e. the case $b \equiv 0$ and $\eta \equiv h$). We consider the behaviour in the far field, defined as $r \sim 1/\delta \gg 1$, where $\delta \ll 1$, in which a suitable rescaling is given by

$$r = \frac{1}{\delta} + \lambda^\alpha \delta^\beta R, \quad (6.1)$$

where λ is the inverse Weber number previously defined in (2.13), α and β are positive exponents, and $R = O(1)$ is the appropriately rescaled local radial coordinate. In line with the typical values of λ (see Table 4.1) we assume $\lambda \ll 1$. In the far field, as previously discussed in Section 2.2.2, the kinematic condition (2.65) and initial condition (2.70) set $f \sim 1$ and as previously shown in (2.69), balancing the largest terms for $\delta \ll 1$ in equations (2.66)–(2.67) we find

$$h \sim \delta^{2/3}, \quad g \sim \delta^{4/3}, \quad (6.2)$$

where the behaviour of h arises from a balance of the effects of viscosity and centrifugation, while the behaviour of g arises from a balance of the effects of viscosity

and Coriolis force. By balancing the largest terms in the kinematic equation (2.65), we find that the corresponding time scale is

$$t \sim \lambda^\alpha \delta^{\beta-1/3}. \quad (6.3)$$

We therefore rescale the variables in the far field as

$$t = \lambda^\alpha \delta^{\beta-1/3} T, \quad h = \delta^{2/3} H(R, T), \quad (f, g) = (F(R, T), \delta^{4/3} G(R, T)). \quad (6.4)$$

Under these far-field scalings, the evolution equations (2.65)–(2.67) up to leading order in the aspect ratio ϵ become

$$\delta^{1-\beta} \lambda^{-\alpha} (H_T + F_R) = 0, \quad (6.5)$$

$$\begin{aligned} & \delta^{1/3-\beta} \lambda^{-\alpha} \left(F_T - \frac{9F^2 H_R}{7H^2} + \frac{17FF_R}{7H} \right) - \delta^{1/3} \frac{9F^2}{7H} + \delta^{-4/3} \frac{5F}{2H^2} \\ & - \delta^3 \frac{925G^2}{693H} - \delta^{4/3} \frac{85G}{42} - \delta^{2/3} \frac{5}{6} H \left(\frac{1}{\delta} + \lambda^\alpha \delta^\beta R \right)^2 \\ & - \delta^{1/3-3\beta} \lambda^{2-3\alpha} \frac{5}{6} H [H_{RRR} + \delta^{1+\beta} \lambda^\alpha (RH_{RRR} + H_{RR} - \delta^{1+\beta} \lambda^\alpha H_R)] = 0, \end{aligned} \quad (6.6)$$

$$\delta^{5/3-\beta} \lambda^{-\alpha} \left(G_T - \frac{22FGH_R}{17H^2} + \frac{59GF_R}{51H} + \frac{22FG_R}{17H} \right) + \frac{168F}{85} + \frac{42G}{17H^2} = 0, \quad (6.7)$$

subject to the initial condition (2.70), namely

$$F(R, 0) = F_N(R) = 1, \quad H(R, 0) = H_N(R) = 3^{1/3}, \quad G(R, 0) = G_N(R) = 0. \quad (6.8)$$

Note that this notation, introduced in equations (6.8), is not required in this chapter but will be used in Chapter 7. In equation (6.6) note that the first four terms from

the left-hand side correspond to inertia, the fifth term corresponds to viscosity, the sixth and seventh terms correspond to Coriolis force, the eighth term corresponds to centrifugation, and the terms in the square brackets correspond to capillarity.

We find that, depending on the size of δ , there are three different possible balances of terms in equations (6.5)–(6.7), corresponding to the inner, middle and outer regions. In particular, the far field consists of an inner region where $\delta^{-1} \ll \lambda^{-3/5}$, a middle region where $\delta^{-1} \sim \lambda^{-3/5}$, and an outer region where $\delta^{-1} \gg \lambda^{-3/5}$. These regions are indicated schematically in Figure 4.2. We note that in (6.6) inertia scales like $\delta^{1/3-\beta}\lambda^{-\alpha}$, viscosity and centrifugation scale like $\delta^{-4/3}$, and capillarity scales like $\delta^{1/3-3\beta}\lambda^{2-3\alpha}$, which represent the three possible balances of terms depending on the size of δ . In the inner region, the effects of inertia and capillarity balance; in the middle region, all four effects balance; and in the outer region, viscosity, centrifugation, and capillarity balance.

We wish to investigate the nature of the spatial and temporal stability of the solution in the inner, middle and outer regions. Therefore, we will consider the analytically tractable case of the spatial and temporal stability of the evolution equations (2.65)–(2.67) in the far field for the case of a planar disk at leading order in the aspect ratio ϵ . We now discuss each of the three regions in turn. In particular, we will examine the temporal stability of each region and in any region exhibiting instability, we will determine whether that instability is convective or absolute.

6.2 Inner region ($\delta^{-1} \ll \lambda^{-3/5}$)

In this region, the dominant effects are inertia and capillarity corresponding to the values of the exponents $\alpha = 1$ and $\beta = 0$. At leading order in δ , equations (6.5)–(6.6)

become

$$H_T + F_R = 0, \quad (6.9)$$

$$F_T - \frac{9F^2 H_R}{7H^2} + \frac{17FF_R}{7H} - \frac{5}{6}HH_{RRR} = 0, \quad (6.10)$$

where the first three terms in (6.10) correspond to inertia and the last term corresponds to capillarity. Note that equations (6.9)–(6.10) are parameter free. In this region, the leading-order terms in (6.6) appearing in (6.10) are of order $O(\delta^{1/3}\lambda^{-1})$ while the largest term involving g is of order $O(\delta^{4/3})$, which is smaller as $\lambda \ll \delta^{5/3}$. Hence g is decoupled from the behaviour of f . Equations (6.9)–(6.10) admit the constant (steady) base state solutions

$$F = F_0 = 1, \quad H = H_0 = 3^{1/3}, \quad (6.11)$$

satisfying the initial conditions (6.8). We investigate the linear stability of this solution by substituting $F = F_0 + \bar{F}(R, T)$ and $H = H_0 + \bar{H}(R, T)$ into (6.9)–(6.10), where $\bar{F}(R, T)$, $\bar{H}(R, T) \ll 1$ are small perturbations, and linearising to obtain

$$\bar{H}_T + \bar{F}_R = 0, \quad (6.12)$$

$$\bar{F}_T - \frac{9F_0^2 \bar{H}_R}{7H_0^2} + \frac{17F_0 \bar{F}_R}{7H_0} - \frac{5}{6}H_0 \bar{H}_{RRR} = 0. \quad (6.13)$$

We consider harmonic disturbances by seeking solutions of the form $(\bar{F}, \bar{H}) = (\hat{F}, \hat{H})e^{\Sigma T + iKR}$, where $\Sigma(K)$ is the (complex) growth rate, and K is the (real) wave number of the disturbances. This yields the dispersion relation

$$\Sigma^2 + \frac{17K\Sigma i}{7(3^{1/3})} + \frac{5(3^{1/3})K^4}{6} - \frac{9K^2}{7(3^{2/3})} = 0. \quad (6.14)$$

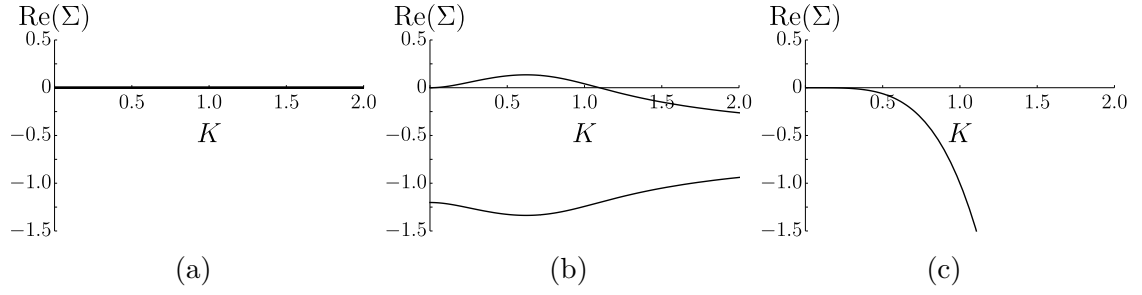


Figure 6.1: Plots of the real part of the growth rate $\text{Re}(\Sigma)$ against the wave number K in the (a) inner region (6.14), (b) middle region (6.18), and (c) outer region (6.25).

The temporal growth and decay of the perturbations are determined by the real part of the growth rate $\text{Re}(\Sigma)$. The dispersion relation (6.14) has solutions

$$\Sigma = \frac{iK}{42} \left[-17 (3^{2/3}) \pm 3^{1/6} (111 + 1470K^2)^{1/2} \right], \quad (6.15)$$

which shows that solutions for F and H have two modes. Clearly, $\text{Re}(\Sigma) \equiv 0$ for all K in both modes. In particular, these modes are both neutrally stable meaning that a linear disturbance neither grows nor decays over time. Figure 6.1(a) shows a plot of the growth rate $\text{Re}(\Sigma) \equiv 0$ against the wave number K in the inner region (6.14) for completeness.

This neutral linear stability is evident in the corresponding numerical results given previously in Chapter 4. In the case without inlet forcing ($A = 0$), the flow at the inlet is waveless and hence continues to be waveless throughout the inner region (seen in Figure 4.1) due to the neutral linear stability. However, in the case with inlet forcing ($A \neq 0$), disturbances are induced at the inlet and so waves remain present through the inner region (seen in Figure 4.7) again due to the neutral linear stability. In fact, the waves can grow and coalesce in this inner region, indicating that, while linear disturbances are neutrally stable, there is a nonlinear instability.

6.3 Middle region ($\delta^{-1} \sim \lambda^{-3/5}$)

In this region, all four effects balance, namely inertia, viscosity, centrifugation and capillarity corresponding again to the values of the exponents $\alpha = 1$ and $\beta = 0$. At leading order in δ , equations (6.5)–(6.6) become

$$H_T + F_R = 0, \quad (6.16)$$

$$F_T - \frac{9F^2 H_R}{7H^2} + \frac{17FF_R}{7H} + \frac{5F}{2H^2} - \frac{5}{6}H - \frac{5}{6}HH_{RRR} = 0, \quad (6.17)$$

which up to differences in scaling recovers the first-order falling film equation of Ruyer-Quil and Manneville [100] (other than the term corresponding to the normal component of gravity in their equations, which does not have an analogous effect here). The first three terms in (6.17) correspond to inertia, the fourth term corresponds to viscosity, the fifth term corresponds to centrifugation, and the last term corresponds to capillarity. The centrifugation term in (6.17) has an identical form to the term corresponding to the streamwise component of gravity in the falling film equation of Ruyer-Quil and Manneville [100]. Physically, these are both body forces and although the force due to centrifugation varies as a function of r , in the far field the radial dependence becomes order $O(\delta^{-1})$. In this region, the leading-order terms in (6.17) are of order $O(\delta^{-4/3})$ and the largest term involving g is of order $O(\delta^{4/3})$, which is smaller as $\delta \ll 1$. Hence, g is decoupled from the behaviour of f . Equations (6.16)–(6.17) are parameter free and determine the behaviour of the system in the middle region, which is characterised by the self-sustained inception of waves in the case without inlet forcing discussed in Section 4.1.

We investigate the linear stability following the same procedure as in the inner region,

in Section 6.2, and find the dispersion relation

$$\Sigma^2 + \left(\frac{5}{2(3^{2/3})} + \frac{17Ki}{7(3^{1/3})} \right) \Sigma + \frac{5(3^{1/3})K^4}{6} - \frac{9K^2}{7(3^{2/3})} + \frac{15Ki}{6} = 0. \quad (6.18)$$

The dispersion relation (6.18) has solutions

$$\Sigma = \frac{1}{84} \left[-35(3^{1/3}) - 34(3^{2/3})Ki \pm \frac{(3675 - 10500(3^{2/3})Ki - 444(3^{2/3})K^2 - 5880(3^{2/3})K^4)^{1/2}}{3^{1/6}} \right], \quad (6.19)$$

which shows that the solutions for F and H have two modes. Figure 6.1(b) shows a plot of the real part of the growth rate $\text{Re}(\Sigma)$ against wave number K in the middle region (6.19). In particular, for the unstable mode, $\text{Re}(\Sigma) \geq 0$ for $0 \leq K \leq K_c \approx 1.09545$, where K_c is the critical wave number, and $\text{Re}(\Sigma) < 0$ for $K > K_c$. Therefore, the system is temporally unstable for a finite range of wave numbers extending from $K = 0$ to $K = K_c$ and temporally stable for $K > K_c$ (in line with the long wave theory). The maximum growth rate occurs at wave number $K = K_m \approx 0.62249$.

We next investigate the spatial nature of the instability (see, for example, [24, 52]), as described in Section 1.4.2.2. The spatial instability is determined by the growth rate of waves with (complex) wave number corresponding to zero group velocity, which we denote by $K = K_0$, i.e. those waves satisfying

$$\frac{\partial \Sigma}{\partial K}(K_0) = 0. \quad (6.20)$$

The nature of the spatial instability is determined by the behaviour of $\text{Re}(\Sigma(K_0))$,

which describes the temporal evolution of waves with wave number K_0 . Specifically, the sign of $\text{Re}(\Sigma(K_0))$ determines whether the instability is convective or absolute. For $\text{Re}(\Sigma(K_0)) > 0$ the flow is absolutely unstable, and for $\text{Re}(\Sigma(K_0)) < 0$ any temporal instability is convectively unstable.

Differentiating the dispersion relation (6.18) with respect to K and applying (6.20) yields

$$\frac{17i\Sigma(K_0)}{7H_0} + \frac{10H_0K_0^3}{3} - \frac{18K_0}{7H_0^2} + \frac{15i}{6} = 0. \quad (6.21)$$

Equation (6.21) along with the dispersion relation (6.18) evaluated at $K = K_0$, form a pair of equations which can be solved to obtain the unique non-trivial solution satisfying $\text{Re}(K_0) > 0$ for the unstable mode,

$$K_0 = 0.645812 - 0.498727i, \quad \Sigma(K_0) = -0.423388 - 1.08094i, \quad (6.22)$$

which shows that the flow is convectively unstable as $\text{Re}(\Sigma(K_0)) < 0$. This conclusion is consistent with the behaviour of the falling film equation of Ruyer-Quil and Manneville, which has also been shown to exhibit convective instability [101]. This result means that in the middle region linear disturbances will grow, but will be convected outwards across the disk.

The consequence of this convective instability can be seen in the numerical results in Sections 4.1 and 4.2. In the case without inlet forcing ($A = 0$), the inner region is waveless and the temporal instability explains the self-sustained inception point of the waves at the radial position $r = r_s$ in the middle region. This instability initiates the formation of the pulse-like structures at the radial position $r = r_s$, as shown in Figure 4.1, once the system has reached a fully developed state. The pulses are then convected radially outwards across the disk. In the case with inlet forcing ($A \neq 0$),

the flow is unstable in the inner region and therefore the inner and middle regions are not easily distinguishable in Figure 4.7. We also note this convective instability always occurs irrespective of the value of λ ; only the radial position at which it occurs changes.

6.4 Outer region ($\delta^{-1} \gg \lambda^{-3/5}$)

In this region, the dominant effects are viscosity, centrifugation and capillarity corresponding to the values of the exponents $\alpha = 2/3$ and $\beta = 5/9$. At leading order in δ , equations (6.5)–(6.6) become

$$H_T + F_R = 0, \quad (6.23)$$

$$\frac{5F}{2H^2} - \frac{5}{6}H - \frac{5}{6}HH_{RRR} = 0, \quad (6.24)$$

where the first term in (6.24) corresponds to viscosity, the second term corresponds to centrifugation and the third term corresponds to capillarity. Note that equations (6.23)–(6.24) are parameter free. In this region, the leading-order terms in (6.24) are of order $O(\delta^{-4/3})$ and the largest term involving g is of order $O(\delta^{4/3})$, which is smaller as $\delta \ll 1$. Hence, g is decoupled from the behaviour of f .

We investigate the linear stability following the same procedure as in the inner region, in Section 6.2, and find the dispersion relation

$$\frac{\Sigma}{2(3^{2/3})} + \frac{(3^{1/3})K^4}{6} + \frac{Ki}{2} = 0. \quad (6.25)$$

The dispersion relation (6.25) has the unique solution

$$\Sigma = -K^4 - (3^{2/3}) Ki, \quad (6.26)$$

and since $\text{Re}(\Sigma) \leq 0$ for all K , this region is linearly stable and linear disturbances decay over time. Figure 6.1(c) shows a plot of the growth rate $\text{Re}(\Sigma)$ against wave number K in the outer region (6.25). This result is in agreement with Woods [137], who found that the flow became linearly stable once the film thickness reduced below a critical value.

However, as the linearly stable nature of the outer region is not seen in the numerical results in Chapter 4, we further analyse the temporal decay of disturbances in this region. We undo the far-field scalings (6.4) and return to the original non-dimensional variables, in order to determine the decay rate of disturbances. Equations (6.23)–(6.24) become

$$\delta^{-1}h_t + f_r = 0, \quad (6.27)$$

$$\frac{5f}{2h^2} - \delta^{-2}\frac{5}{6}h - \lambda^2\delta^{-1}\frac{5}{6}hh_{rrr} = 0. \quad (6.28)$$

Rewriting the growth rate (6.26) in terms of the original variables, we find that the real part of the growth rate is $\text{Re}(\sigma) = -\lambda^2\delta^2k^4$. Therefore, even though the outer region is linearly stable, the decay rates are very slow as $\lambda^2\delta^2 \ll 1$. This slow decay rate is reflected by the fact that in our numerical results in Sections 4.1 and 4.2 the outer region cannot be easily distinguished from the middle region.

6.5 Concluding remarks

In this chapter, we performed a stability analysis in the inner, middle and outer regions for a planar disk. We derived three systems of equations and boundary conditions describing the leading-order asymptotic behaviour in the inner, middle and outer regions by applying the far-field scalings described in Section 6.1. In Section 6.2, we found the inner region has linear neutral stability; in Section 6.3, we found that the middle region is convectively unstable; and in Section 6.4, we found that the outer region is linearly stable. These analytical results are consistent with the numerical results presented within Chapter 4. These analytical findings provide a mathematical explanation for the phenomenon of the self-sustained inception of waves (which occurs at the radial position $r = r_s$) for flow over a spinning disk without and with inlet forcing discussed numerically in Sections 4.1 and 4.2, respectively. The addition of inlet forcing causes the radial position of the self-sustained inception of waves r_s to move from the middle region (which is convectively unstable) to the inner region (which has neutral linear stability). This behaviour is caused by the neutral linear stability of the inner region. In the case without inlet forcing, the flow from the inlet is waveless and hence continues to be waveless throughout the inner region, whereas in the case with inlet forcing the disturbances induced at the inlet cause waves to be present within this region. This provides a mathematical explanation for small amounts of inlet forcing having a strong effect on the dynamics of the system.

Chapter 7

Control of the free surface in the outer region

In this chapter, we consider control of the free surface towards a prescribed target free-surface profile using the topography of the disk as the control mechanism. The approach taken here is an inverse one: we determine the substrate shape that would be required to produce a prescribed target free-surface profile. Since the solution reflects the steepness and variation of the chosen target profile, some of the resulting topographies appear unusual, as we will demonstrate. When gentler target profiles are used, the corresponding topographies become smoother and more physically plausible. As in Chapter 6, we consider the behaviour of the flow in the far field in order to make analytical progress. In particular, we consider only the outer region as in this region the flow is stable (as shown in Section 6.4); therefore, control is analytically tractable. In unstable regions, even minute disturbances are amplified by such instability, rendering these regions unsuitable for control. It is convenient

to define a small parameter $\zeta \ll 1$,

$$\zeta = \frac{\delta^{10/9}}{\lambda^{2/3}}, \quad (7.1)$$

which arises in the far-field equations in the outer region (equations (6.5)–(6.7), where the exponents take the values $\alpha = 2/3$ and $\beta = 5/9$) and in Sections 7.1 and 7.2, the problem is considered at leading order and first order in ζ , respectively.

We extend the far-field scalings (6.4), originally derived for the case of a planar disk in Chapter 6, to the case of a topographically-patterned disk by rescaling the topography b and film thickness η as

$$(b, \eta) = \delta^{2/3} (B(R, T), E(R, T)). \quad (7.2)$$

Under these far-field scalings, the evolution equations (2.65)–(2.67) in the outer region ($\alpha = 2/3$ and $\beta = 5/9$) become

$$\delta^{4/9} \lambda^{-2/3} (H_T + F_R) = 0, \quad (7.3)$$

$$\begin{aligned} & \zeta \left(F_T - \frac{9F^2 E_R}{7E^2} + \frac{17FF_R}{7E} \right) - \delta^{5/3} \frac{9F^2}{7E} + \frac{5F}{2E^2} \\ & - \delta^{13/3} \frac{925G^2}{693E} - \delta^{8/3} \frac{85G}{42} - \delta^2 \frac{5}{6} E \left(\frac{1}{\delta} + \delta^{5/9} \lambda^{2/3} R \right)^2 \end{aligned} \quad (7.4)$$

$$- \frac{5}{6} E [H_{RRR} + \delta^{14/9} \lambda^{2/3} (RH_{RRR} + H_{RR} - \delta^{14/9} \lambda^{2/3} H_R)] = 0,$$

$$\zeta \left(G_T - \frac{22FGE_R}{17E^2} + \frac{59GF_R}{51E} + \frac{22FG_R}{17E} \right) + \frac{168F}{85} + \frac{42G}{17E^2} = 0. \quad (7.5)$$

For both the leading-order and first-order problems, we prescribe a target free-surface profile and seek an analytical solution for the topography. This is an inversion

technique similar to that used by Sellier [106] for gravity-driven thin-film flow, as described in Section 1.6. We then test the solution for the topography numerically on a domain of length $L = 200$ using a transient code including the effect of introducing a nonlinear disturbance.

7.1 Leading-order problem

At leading order in ζ , the evolution equations (7.3)–(7.5) in the outer region (i.e. the region in which $\delta^{-1} \gg \lambda^{-3/5}$) become

$$H_T + F_R = 0, \tag{7.6}$$

$$\frac{5F}{2E^2} - \frac{5}{6}E - \frac{5}{6}EH_{RRR} = 0, \tag{7.7}$$

where the first term in (7.7) corresponds to viscosity, the second term corresponds to centrifugation and the third term corresponds to capillarity. While the topography B does not enter equations (7.6)–(7.7) explicitly, its effects are included within the capillarity term in (7.7), which depends on the free surface H rather than the film thickness E (as $H = E + B$). We note that $H_T \equiv \eta_T$ as the topography B is a function of only R . We can rearrange (7.7) for the radial flux F to find

$$F = \frac{E^3}{3} (1 + H_{RRR}). \tag{7.8}$$

This expression for the radial flux F can be substituted into the kinematic condition (7.6) to obtain the single equation which governs the leading-order behaviour of the

flow in the outer region, namely,

$$H_T + \left[\frac{E^3}{3} (1 + H_{RRR}) \right]_R = 0. \quad (7.9)$$

We consider the inverse problem, i.e. the situation in which the free surface position H is prescribed and the topography B is unknown. We rewrite the film thickness in terms of the free surface position as $E = H - B$ in equation (7.9) to obtain

$$H_T + \left[\frac{(H - B)^3}{3} (1 + H_{RRR}) \right]_R = 0, \quad (7.10)$$

which recovers equations (1)–(2) of Sellier [106] up to differences in the nondimensionalisation.

Following Sellier [106], we consider the steady version of (7.10) to find an analytical solution for the topography. The steady version of the kinematic condition (7.6) together with the initial condition (6.8), give $F = 1$. We substitute $F = 1$ into (7.7) and solve for the film thickness E to obtain

$$E = \left(\frac{3}{1 + H_{RRR}} \right)^{1/3}. \quad (7.11)$$

An expression for topography B is then obtained from (7.11) using $B = H - E$, namely,

$$B = H - \left(\frac{3}{1 + H_{RRR}} \right)^{1/3}. \quad (7.12)$$

This equation is subject to a constraint, namely that the film thickness E must be positive by definition (i.e. $H - B > 0$). Evidently, this constraint can only be satisfied

if

$$H_{RRR} > -1. \quad (7.13)$$

We denote the steady target free-surface profile by $H_{\text{targ}}(R)$ and consider four specific steady target free-surface profiles: a Gaussian target free-surface profile,

$$H_{\text{targ}}^{\text{G}} = 3^{1/3} + 0.5e^{-0.6(R-L/2)^2}, \quad (7.14)$$

a trench target free-surface profile,

$$H_{\text{targ}}^{\text{T}} = 3^{1/3} - 0.3 \left(\tanh [1.1(R - L + 10)] - \tanh [1.1(R - L - 10)] \right), \quad (7.15)$$

and both symmetric and asymmetric triple-step target free-surface profiles of the form

$$H_{\text{targ}}^{\text{TS}} = 3^{1/3} + \sum_{i=1}^6 S_i \tanh[\mathcal{C}_i(R - R_i)], \quad (7.16)$$

where S_i is the amplitude of each step up/down, R_i is the position of each step up/down, and $\mathcal{C}_i$ controls the gradient of the slope of each step up/down for $i = 1, 2, \dots, 6$. For the symmetric triple-step target free-surface profile, we choose $S_1 = S_3 = S_5 = 0.45$, and $S_2 = S_4 = S_6 = -0.45$; $R_1 = L - 40$, $R_2 = L - 20$, $R_3 = L - 10$, $R_4 = L + 10$, $R_5 = L + 20$ and $R_6 = L + 40$; $\mathcal{C}_1 = \mathcal{C}_2 = \mathcal{C}_3 = \mathcal{C}_4 = \mathcal{C}_5 = \mathcal{C}_6 = 1$. For the asymmetric triple-step target free-surface profile, the S_i and R_i are identical to the symmetric case but we choose $\mathcal{C}_1 = \mathcal{C}_3 = \mathcal{C}_5 = 1$ and $\mathcal{C}_2 = \mathcal{C}_4 = \mathcal{C}_6 = 0.2$. These four specific target free-surface profiles include relatively short and sharp profiles of the Gaussian (7.14) and trench (7.15) target free-surface profiles, as well as the two triple-step target free-surface profiles (7.16), which cover a comparatively large portion of the computational

domain. The asymmetric triple-step target free-surface profile (7.16) is chosen to consider the effect of asymmetry. We note that all four of these target free-surface profiles are chosen to be close to the limit of the constraint $H_{RRR} > -1$ and all are centred at $R = L/2$. In particular, the minimum values of H_{RRR} for the Gaussian, trench, symmetric triple-step and asymmetric triple-step are approximately -0.907 , -0.799 , -0.900 and -0.901 , respectively. For target free-surface profiles with H_{RRR} values far from the limit of the constraint $H_{RRR} > -1$, the topography required is almost identical to the shape of the target free-surface profile. We note that for all of the target free-surface profiles H_{targ} considered, the deviation from the uniform film solution ($H = 3^{1/3}$) is contained within the radial interval $L/2 - 50 \leq R \leq L/2 + 50$. Figure 7.1 shows plots of the target free-surface profile H_{targ} and topography B (7.12) for a Gaussian, trench target free-surface profile, symmetric triple-step target free-surface profile, and asymmetric triple-step target free-surface profile. In particular, Figure 7.1 shows that the topography B is asymmetric even in the cases of a symmetric target free-surface profile. We note that for the case of flow over a simple step-shaped topography, the free surface forms a depression ahead of the step up and the depression is replaced by a bulge ahead of a step down [36, 55, 119]; these effects are induced by capillarity. However, in this problem the design of the topography counteracts these effects. Hence, the topography B involves a trench before a step up and a ridge before a step down (as shown in Figures 7.1(a)–(d)). These results are in agreement with the results of Sellier [106]. In the asymmetric triple-step case (shown in Figure 7.1(d)), we note that the three trenches in the topography B are of different depths. The first trench has an approximate depth of -1.217 , while the second and third trenches have approximate depths of -1.210 . This effect is caused by the steps overlapping slightly due to the reduced gradient on the right-hand side of the steps, hence, decreasing the depth of the last two trenches in the topography

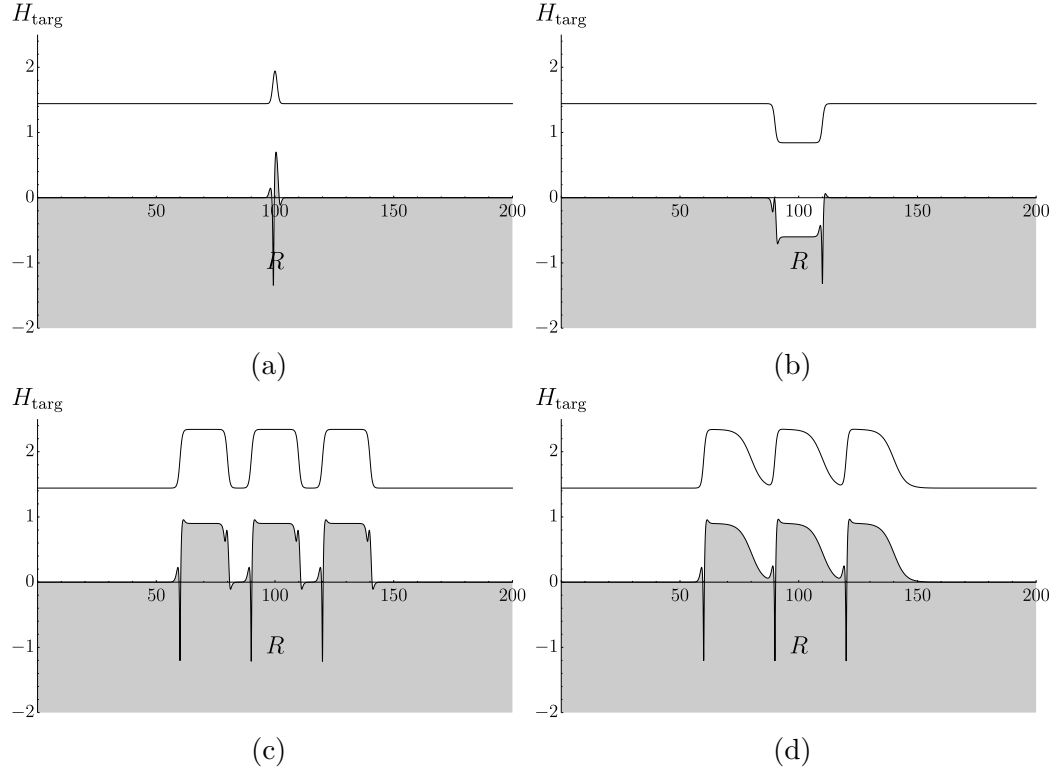


Figure 7.1: Plots of the target free-surface profile H_{targ} and topography B (shaded) computed from equation (7.12) for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile.

B .

We calculate the steady solution for the topography B (7.12) using the target free-surface profile H_{targ} . This topography B is substituted into the transient equation (7.10), which we solve numerically on a domain of length $L = 200$. In particular, we adapt the code in C++ used in Chapters 3, 4 and 5 to simulate equation (7.10). The code uses the initial condition for the free surface $H = H_{\text{targ}}$, imposes periodic boundary conditions at the ends of the domain, and uses a uniform grid in space. The code uses centred second-order finite differences in space and the backwards Euler method in time. As the target free-surface profile H_{targ} is a steady solution, we initiate a Gaussian disturbance to the free surface at $T = 0$ given by $0.3 \exp[-(R - L/4)^2]$ with maximum located at $R = L/4$ that travels radially outwards across the domain. This was chosen as a representative disturbance and used for all simulations within this chapter, in all cases tested the flow tends to a steady state after the disturbance travels radially outwards.

7.1.1 Effect of a disturbance

Figure 7.2 shows space-time plots for a Gaussian, trench, symmetric triple-step, and asymmetric triple-step target free-surface profile from $T = 0$ to $T = 45$. In particular, Figure 7.2 shows that for all four cases presented the free surface returns to the target free-surface profile after the disturbance has travelled radially outwards across the domain. Therefore, the flow returns to its steady state after introducing a nonlinear disturbance. The disturbance travels radially outward and eventually leaves the domain. However, if simulated over a sufficiently long domain the disturbance eventually dies out.

We are interested in the deviation of the free surface H computed by (7.10) from the

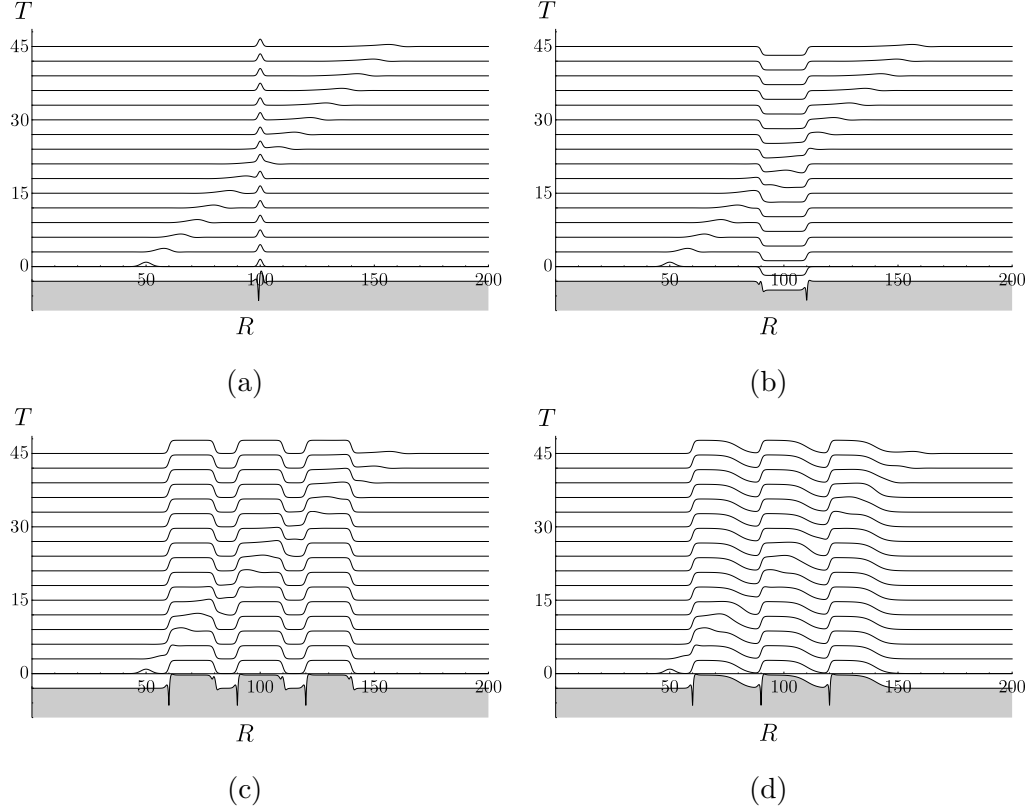


Figure 7.2: Space-time plots of $H - H_N$ computed from equation (7.10) for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile. In these plots the time varies from $T = 0$ to $T = 45$ in steps of $T = 3$.

target free-surface profile H_{targ} and the effect of the topography on the disturbance, which will be discussed in turn.

7.1.2 Deviation of the free surface from the target free-surface profile

We consider the deviation of the free surface H computed by equation (7.10) from the target free-surface profile H_{targ} . In particular, we consider the radial interval $50 \leq R \leq 150$, within which the target free-surface profiles are located. We consider the time $T = 68$ as this is after the disturbance has passed through the radial interval and has no lasting transient effect. Figure 7.3 shows plots of the free surface H compared to the target free-surface profile H_{targ} for a Gaussian, trench, symmetric triple-step, and asymmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$. In particular, Figure 7.3 shows that for all cases tested the free surface H and the target free-surface profile H_{targ} are in excellent agreement. In fact, the agreement between the free surface H and the target free-surface profile is numerically exact (up to the six decimal places output by the code). Therefore, the topography B determined by (7.12) accurately produces the target free-surface profile H_{targ} when substituted into the leading-order transient problem (7.10).

7.1.3 Effect of the topography on the disturbance

We next consider the effect of the topography on the disturbance. Therefore, we compare the disturbance after it has travelled radially outwards over the Gaussian, trench and triple-step target free-surface profiles with the planar case (i.e. $B = 0$) at time $T = 56$, a time after the disturbance has travelled radially outwards across the topography in all four cases. Figure 7.4 shows plots of the free surface H for

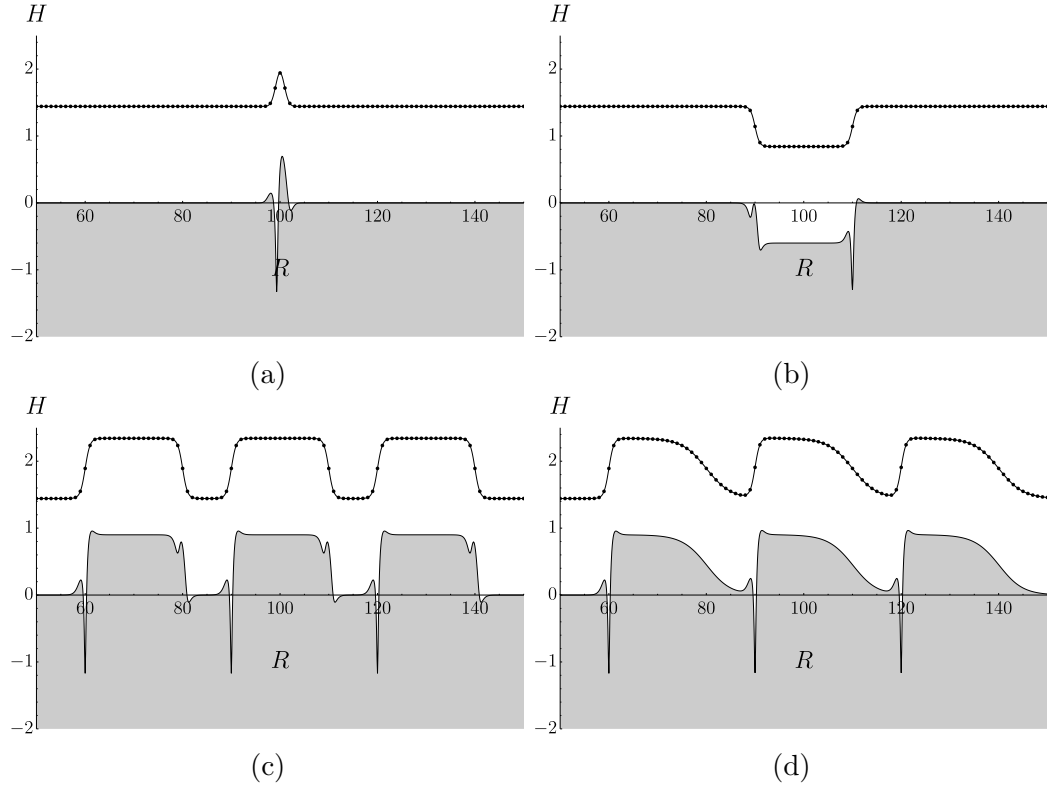


Figure 7.3: Plots of the free surface H computed from equation (7.10) (solid curve) and target free-surface profile H_{targ} (closed circles) for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$.

a Gaussian, trench, symmetric triple-step, and asymmetric triple-step target free-surface profile from $R = 155$ to $R = 200$ at $T = 56$ compared to the planar case. In particular, Figure 7.4 shows that the topography has little effect on the disturbance in all four cases presented. However, we note that in all four cases presented there is a small but non-zero difference compared to the planar case. The difference is more pronounced in the symmetric and asymmetric triple-step target free-surface profile cases. We hypothesise that this is due to the topography covering a larger portion of the computational domain in the two triple-step cases, hence, the disturbance is subjected to its effects for a longer period of time. In the Gaussian and trench target free-surface profile cases, the topography is relatively short and the disturbance passes over it in a shorter period of time, therefore, having a weaker effect.

7.2 First-order problem

To first order in ζ , the evolution equations (7.3)–(7.5) in the outer region (i.e. the region in which $\delta^{-1} \gg \lambda^{-3/5}$) become

$$H_T + F_R = 0, \quad (7.17)$$

$$\frac{5F}{2E^2} - \frac{5}{6}E - \frac{5}{6}EH_{RRR} + \zeta \left(F_T - \frac{9F^2 E_R}{7E^2} + \frac{17FF_R}{7E} \right) = 0, \quad (7.18)$$

where the first term in (7.18) corresponds to viscosity, the second term corresponds to centrifugation, the third term corresponds to capillarity, and the last three terms on the left-hand side correspond to inertia. Hence, the inclusion of terms to first order in ζ adds inertial effects to the evolution equations, which are not present in the leading-order case.

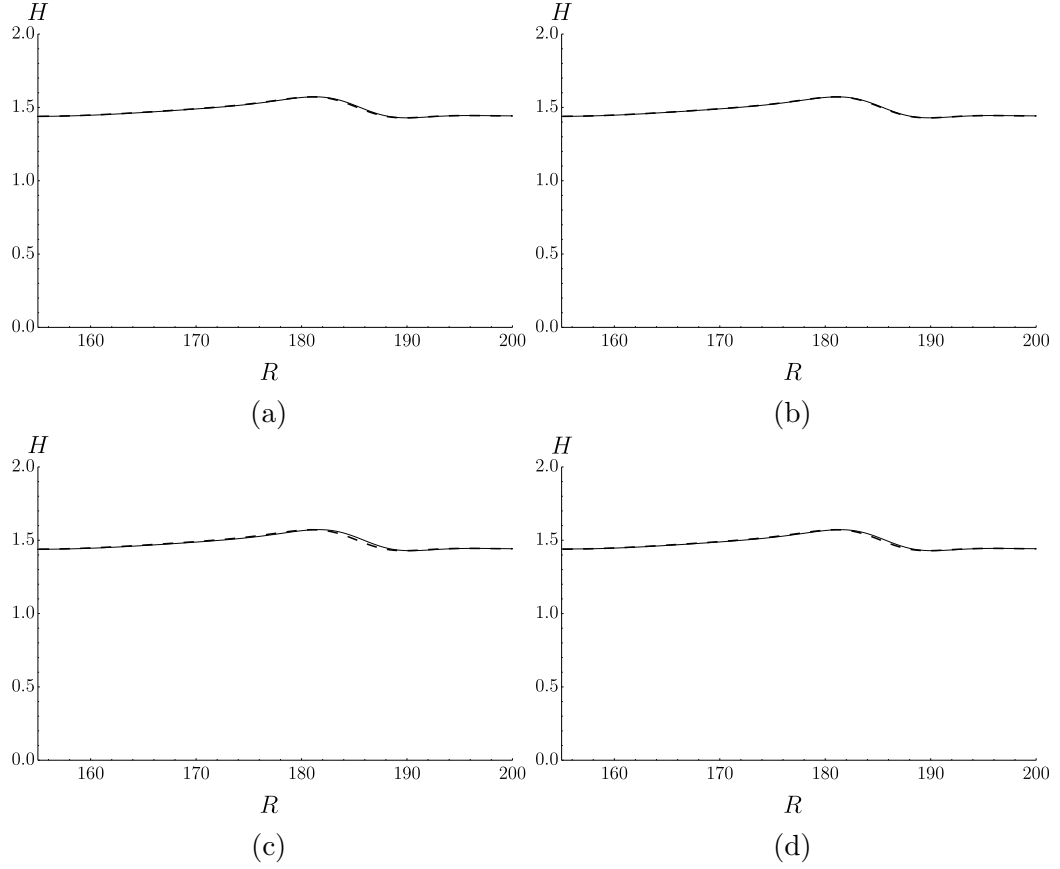


Figure 7.4: Plots of the free surface H computed from equation (7.10) for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile from $R = 155$ to $R = 200$ at $T = 56$ compared to the planar case (dashed curve).

We solve the steady version of (7.17)–(7.18) to find an analytical solution for the topography. The steady version of the kinematic condition (7.17) together with the initial condition (6.8), give $F = 1$. Therefore, the steady version of (7.18) is

$$\frac{5}{2E^2} - \frac{5}{6}E - \frac{5}{6}EH_{RRR} + \zeta \left(-\frac{9E_R}{7E^2} \right) = 0. \quad (7.19)$$

We seek an asymptotic solution for the film thickness of the form $E = E_0 + \zeta E_1 + O(\zeta^2)$ and then solve equation (7.19) at each order in ζ . At $O(1)$, equation (7.19) recovers the leading-order problem, hence,

$$E_0 = \left(\frac{3}{1 + H_{RRR}} \right)^{1/3}. \quad (7.20)$$

At $O(\zeta)$, equation (7.19) along with the expression for leading-order film thickness E_0 (7.20) gives

$$-\frac{5(3^{2/3})(1 + H_{RRR})^{1/3}E_1}{2} + \frac{3(3^{1/3}H_{RRRR})}{7(1 + H_{RRR})^{4/3}} = 0, \quad (7.21)$$

which can be solved for E_1 to obtain

$$E_1 = \frac{2(3^{2/3})H_{RRRR}}{35(1 + H_{RRR})^{5/3}}. \quad (7.22)$$

Hence, the film thickness E is given by

$$E = \left(\frac{3}{1 + H_{RRR}} \right)^{1/3} + \zeta \frac{2(3^{2/3})H_{RRRR}}{35(1 + H_{RRR})^{5/3}} + O(\zeta^2). \quad (7.23)$$

The expression for topography B to first order in ζ is then obtained using the film

thickness (7.23) as $B = H - E$, namely

$$B = H - \left(\frac{3}{1 + H_{RRR}} \right)^{1/3} - \zeta \left(\frac{2(3^{2/3})H_{RRRR}}{35(1 + H_{RRR})^{5/3}} \right) + O(\zeta^2). \quad (7.24)$$

As in the leading-order problem, equation (7.24) is subject to the constraint that the film thickness E must be positive, which can only be satisfied if

$$H_{RRR} > -1. \quad (7.25)$$

We consider the same target free-surface profiles H_{targ} as in the leading-order problem: a Gaussian target free-surface profile (7.14), a trench target free-surface profile (7.15), and a symmetric and asymmetric triple-step target free-surface profile (7.16). Figure 7.5 shows plots of the target free-surface profile H_{targ} and topography B (7.12) with $\zeta = 0.13$ for a Gaussian, trench target free-surface profile, symmetric triple-step target free-surface profile, and asymmetric triple-step target free-surface profile. In particular, Figure 7.5 shows that as expected, the topography B is asymmetric even in the cases of a symmetric target free-surface profile, in agreement with the results for the leading-order problem in Section 7.1. The topography B requires slightly deeper trenches to produce the two triple-step target free-surface profiles H_{targ} in the first-order problem compared to the leading-order problem (shown in Figure 7.1), for example, the symmetric triple-step requires trenches with approximate depths of -1.216 and -1.328 for the leading-order and first-order problems, respectively. To show the difference in the topography B between the leading-order and first-order problems, we plot $B_F - B_L$, where B_F and B_L denote the topography for the leading-order (7.12) and first-order (7.24) problems, respectively. Figure 7.6 shows plots of $B_F - B_L$ with $\zeta = 0.13$ for a Gaussian,

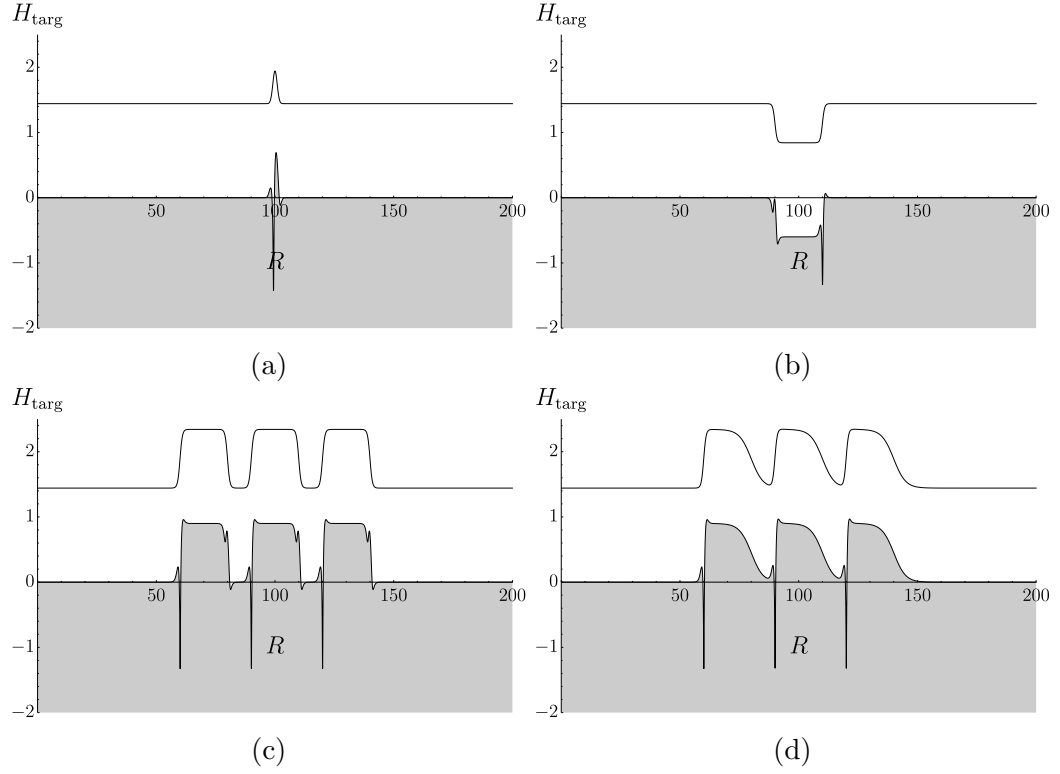


Figure 7.5: Plots of the target free-surface profile H_{targ} and topography B (shaded) computed from equation (7.24) with $\zeta = 0.13$ for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile.

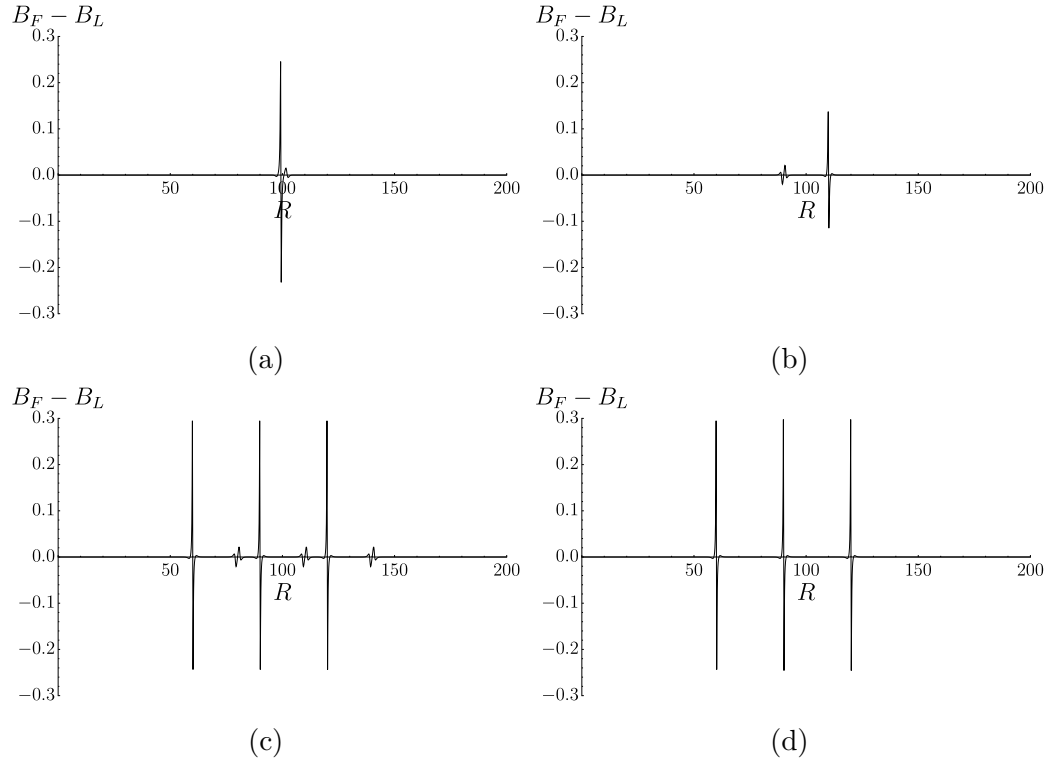


Figure 7.6: Plots of $B_F - B_L$ with $\zeta = 0.13$ for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile.

trench target free-surface profile, symmetric triple-step target free-surface profile, and asymmetric triple-step target free-surface profile. In particular, Figure 7.6 shows that the largest difference between the leading-order and first-order topographies is in the trenches. The difference is larger than expected here as the trench is shifted slightly downstream by the first-order terms. Figure 7.6 also shows that the difference is larger where the target free-surface profile forms a step up than when it forms a step down.

We use the code described in Section 7.1 to simulate equations (7.17)–(7.18). Again, we consider a domain of length $L = 200$, the free surface has initial condition

$H = H_{\text{targ}}$ and we initiate a Gaussian disturbance to the free surface at $T = 0$ given by $0.3 \exp[-(R - L/4)^2]$ with maximum located at $R = L/4$ that travels radially outwards across the domain. As a coupled set of equations is required to be solved in the first-order problem, an additional initial condition for the radial flux given by $F = 1$ is required.

7.2.1 Effect of a disturbance

Figure 7.7 shows space-time plots of $H - H_N$ with $\zeta = 0.13$ for a Gaussian, trench, symmetric triple-step and asymmetric triple-step target free-surface profile from $T = 0$ to $T = 45$. In particular, Figure 7.7 shows that for all four cases presented the free surface returns to the target free-surface profile after the disturbance has travelled radially outwards across the domain. Therefore, the flow returns to its steady state after introducing a nonlinear disturbance.

7.2.2 Deviation of the free surface from the target free-surface profile

We next consider the deviation of the free surface H computed from equations (7.17)–(7.18) from the target free-surface profile H_{targ} . We consider the same radial interval $50 \leq R \leq 150$ at time $T = 68$ as for the leading-order problem in Section 7.1. Figure 7.8 shows plots of the free surface H compared to the target free-surface profile H_{targ} with $\zeta = 0.13$ for a Gaussian, trench, symmetric triple-step and asymmetric triple-step from $R = 50$ to $R = 150$ at $T = 68$. In particular, Figure 7.8 shows that the free surface H is in excellent agreement with the target free-surface profile H_{targ} and the two are visually indistinguishable, however, there are some minor differences between them (i.e. the first-order problem is not numerically exact as in

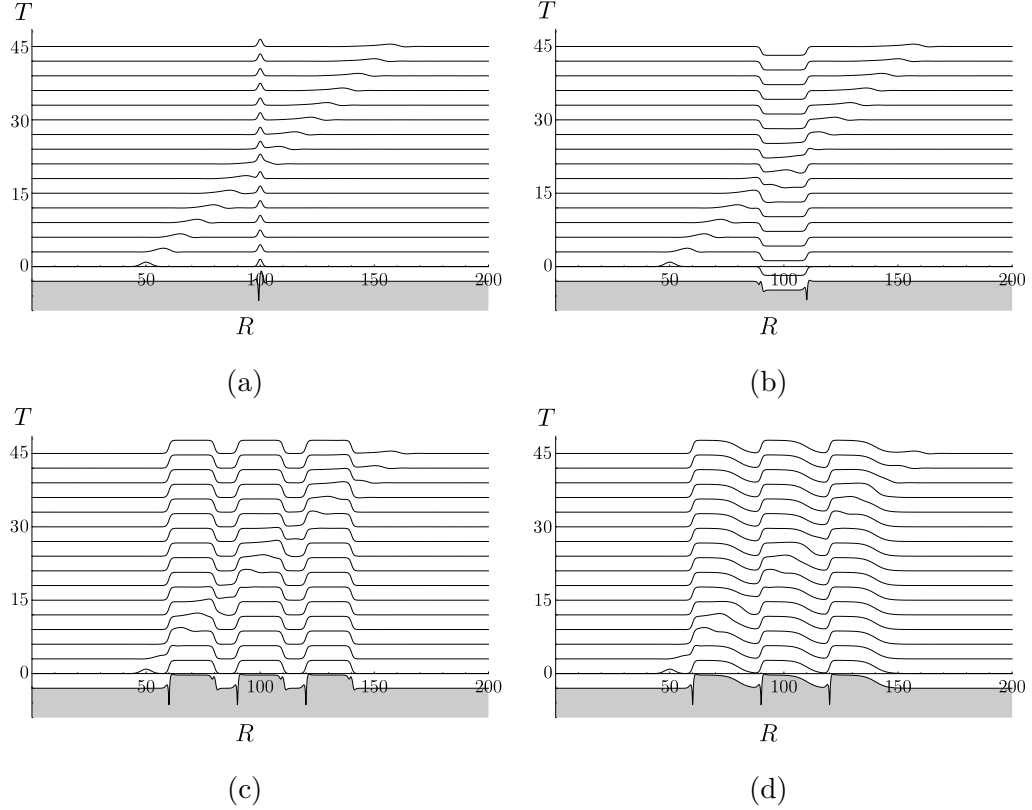


Figure 7.7: Space-time plots of $H - H_N$ computed from equations (7.17)–(7.18) with $\zeta = 0.13$ for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile. In these plots the time varies from $T = 0$ to $T = 45$ in steps of $T = 3$.

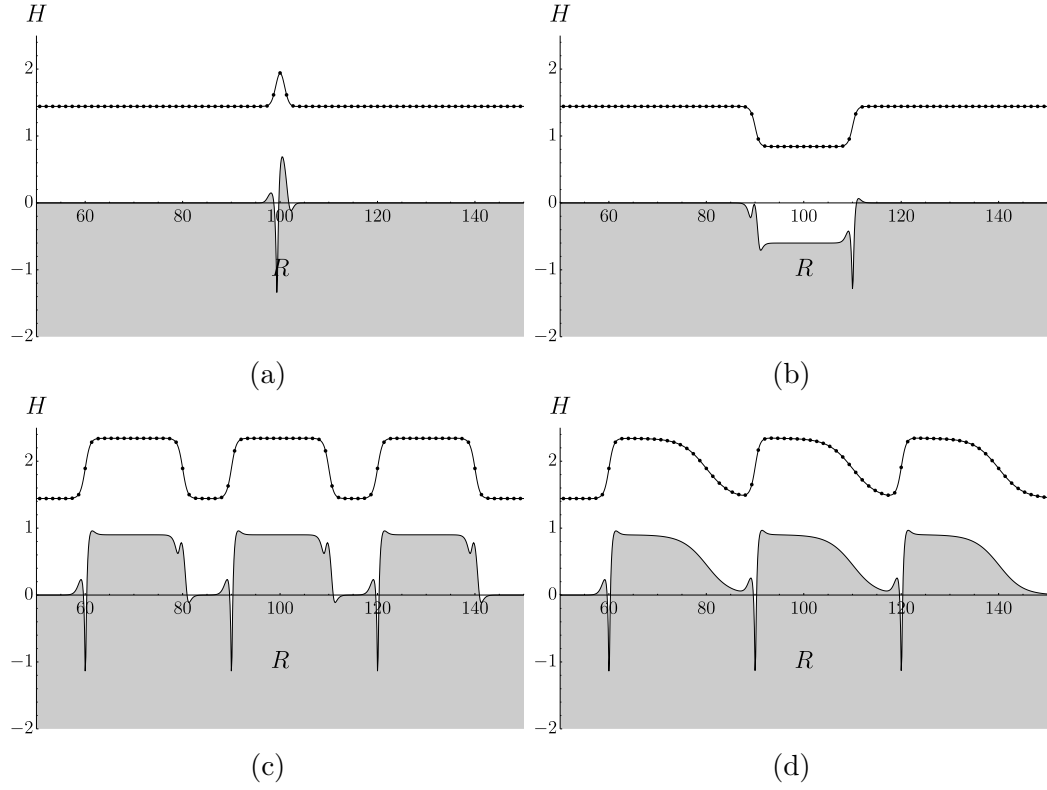


Figure 7.8: Plots of the free surface H computed from equations (7.17)–(7.18) (solid curve) and target free-surface profile H_{targ} (closed circles) with $\zeta = 0.13$ for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$.

the leading-order problem). To quantify these differences, we consider the percentage error of the free surface H from the target free-surface profile H_{targ} over the radial interval $50 \leq R \leq 150$. Specifically, we compute the magnitude of the deviation of the free surface H from the target free-surface profile H_{targ} , normalised by the target free-surface profile H_{targ} , at each grid point and express this as a percentage difference. Figure 7.9 shows the percentage error of the free surface H from the target free-surface profile H_{targ} with $\zeta = 0.13$ for a Gaussian, trench, symmetric triple-step and asymmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$. In particular, Figure 7.9 shows that for all four cases presented the error between the free surface H and the target free-surface profile H_{targ} is relatively small, with the largest error in the case of the Gaussian target free-surface profile that reaches a maximum of approximately 0.65%. The percentage error reaches a maximum of approximately 0.4% for the trench target free-surface profile and 0.34% for both the symmetric and asymmetric triple-step target free-surface profiles. The Gaussian target free-surface profile exhibits the largest error due to its short, sharp nature. The first-order problem has these small errors because the perturbation solution obtained for the topography B (7.24) is not an exact solution to the steady equation (7.19). Therefore, errors of order $O(\zeta^2)$ are expected, which is in agreement with the results shown in Figure 7.9. Figures 7.9(b)–(c) also show that the error is greater when the free surface forms a step up than when it forms a step down.

7.2.3 Effect of the topography on the disturbance

We also consider the effect of the topography on the disturbance. Figure 7.10 shows plots of the free surface H with $\zeta = 0.13$ for a Gaussian, trench, symmetric triple-step, and asymmetric triple-step target free-surface profile from $R = 155$ to

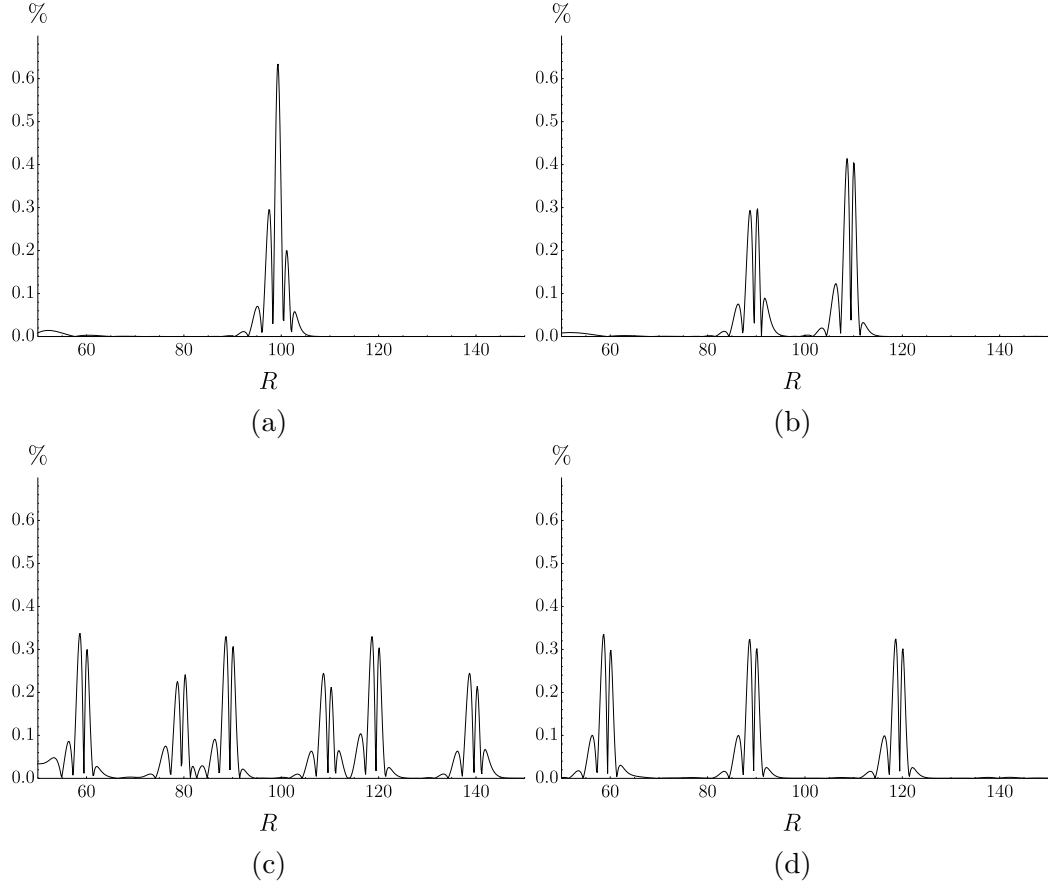


Figure 7.9: Plots of percentage error of the free surface H computed from equations (7.17)–(7.18) compared to the target free-surface profile H_{targ} with $\zeta = 0.13$ for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$.

$R = 200$ at $T = 56$ compared to the planar case. In particular, Figure 7.10 shows that the topography has a negligible effect on the disturbance in all four cases presented. However, we note that in all four cases presented there is a small but non-zero difference compared to the planar case. The effect is more pronounced in the symmetric and asymmetric triple-step target free-surface profile cases. We hypothesise that this is due to the topography covering a larger portion of the domain in the two triple-step cases, as discussed in Section 7.1.

7.2.4 Effect of ζ

In this subsection, we consider the effect of the small parameter ζ that determines the strength of the inertial effects in the first-order equations (7.17)–(7.18). We consider the symmetric triple-step case and vary the value of ζ . In particular, we consider four values of the small parameter ζ : $\zeta = 0.13$, $\zeta = 0.0975$, $\zeta = 0.065$ and $\zeta = 0.0325$. The value of ζ has negligible effect in the space-time plots of $H - H_N$ and plots of the free surface H compared to the target free-surface profile H_{targ} , which is visually indistinguishable. For brevity these results are not given here but (for completeness) they are presented in Appendix A.

7.2.4.1 Deviation of the free surface from the target free-surface profile

Figure 7.11 shows the percentage error of the free surface H from the target free-surface profile H_{targ} with $\zeta = 0.13$, $\zeta = 0.0975$, $\zeta = 0.065$ and $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$. In particular, Figure 7.11 shows that the maximum percentage error is approximately 0.34%, 0.26%, 0.17% and 0.09% for $\zeta = 0.13$, $\zeta = 0.0975$, $\zeta = 0.065$ and $\zeta = 0.0325$,

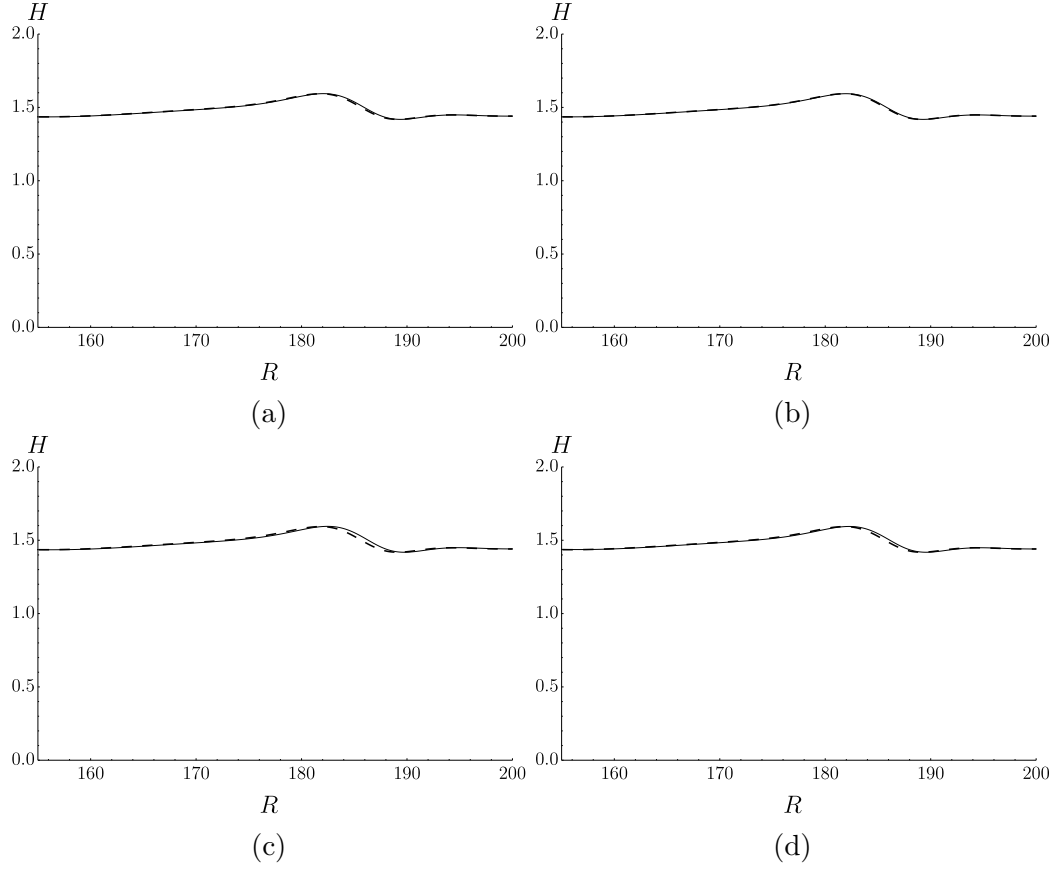


Figure 7.10: Plots of the free surface H computed from equations (7.17)–(7.18) with $\zeta = 0.13$ for (a) a Gaussian target free-surface profile, (b) a trench target free-surface profile, (c) a symmetric triple-step target free-surface profile, and (d) an asymmetric triple-step target free-surface profile from $R = 155$ to $R = 200$ at $T = 56$ compared to the planar case (dashed curve).

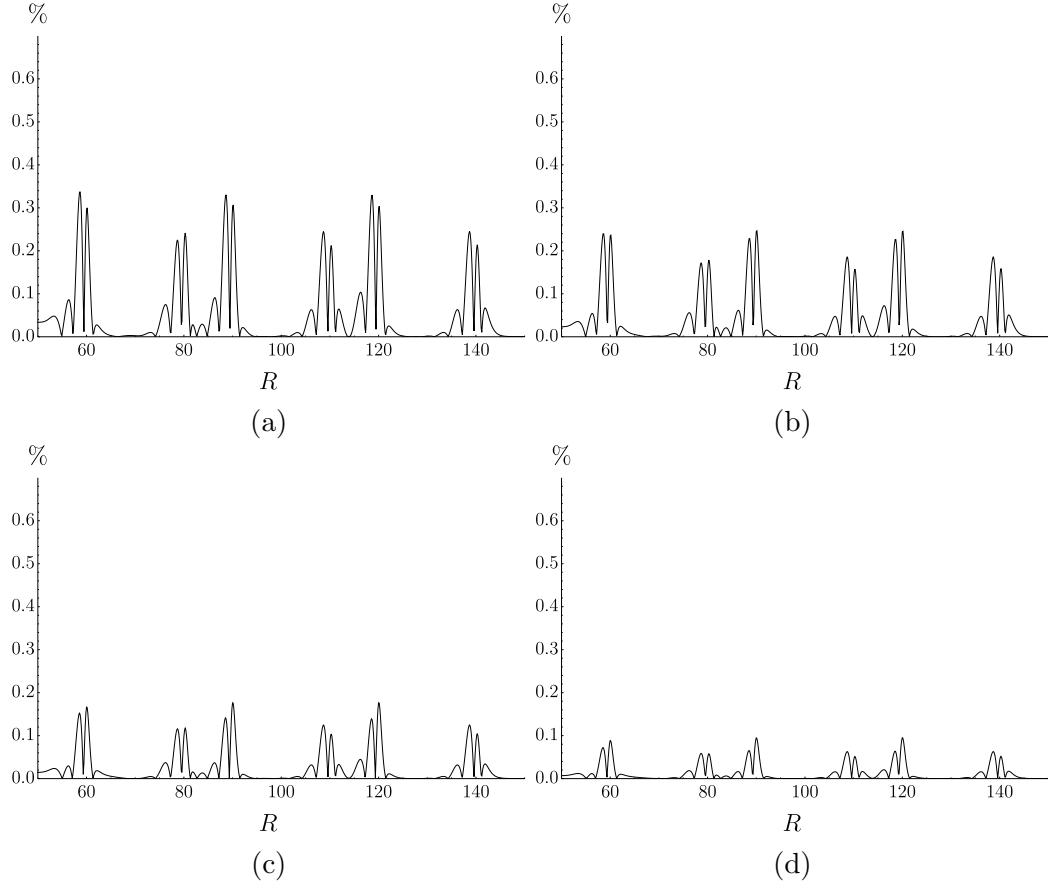


Figure 7.11: Plots of percentage error of the free surface H computed from equations (7.17)–(7.18) compared to the target free-surface profile H_{targ} with (a) $\zeta = 0.13$, (b) $\zeta = 0.0975$, (c) $\zeta = 0.065$, and (d) $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$.

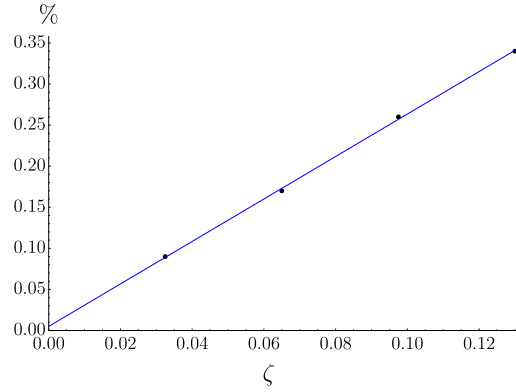


Figure 7.12: Plot of ζ against the maximum percentage error for a symmetric triple-step target free-surface profile (closed circles), together with the least-squares linear fit (blue curve).

respectively. Therefore, the percentage error decreases as ζ decreases (i.e. as inertial effects decrease). This is expected as the topography B (7.24) was obtained using a perturbation solution and is not an exact solution to the steady equation (7.19). Therefore, errors of order $O(\zeta^2)$ are expected, which is in agreement with the results of Figure 7.11. For all four values of ζ presented, the error is larger when creating a step up in the free surface than for a step down in the free surface. This effect was previously noted in Section 7.2.2.

We next investigate the relationship between ζ and the maximum percentage error using the results presented in Figure 7.11 for a symmetric triple-step target free-surface profile. Figure 7.12 shows a plot of ζ against the maximum percentage error for a symmetric triple-step target free-surface profile, together with the least-squares linear fit. In particular, Figure 7.12 shows that the maximum percentage error increases linearly with ζ . The close agreement between the measured points and the least-squares fit confirms that a linear model captures the behaviour accurately, with negligible deviation across the sampled values. The fitted slope of 2.58462 provides

a measure of the rate at which the error increases with ζ .

7.2.4.2 Effect of the topography on the disturbance

Figure 7.13 shows plots of the free surface H with $\zeta = 0.13$, $\zeta = 0.0975$, $\zeta = 0.065$ and $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile from $R = 155$ to $R = 200$ at $T = 56$ compared to the planar case. In particular, Figure 7.13 shows that the topography has very little effect on the disturbance for all four values of ζ presented. However, we note that for all four values of ζ presented there is a small but non-zero difference compared to the planar case. Figure 7.13 also shows that there is slightly more spread in the disturbance as ζ decreases (i.e. as inertial effects decrease).

7.3 Concluding remarks

In this chapter, we have determined suitable topography profiles to control the free surface H towards a prescribed target free-surface profile H_{targ} in the outer region. In Section 7.1, we considered the leading-order problem, for which we found an exact analytical solution for the topography B in the steady problem. The leading-order problem recovered the results of Sellier [106], which considered a gravity-driven thin-film, up to differences in the nondimensionalisation. The topography B was found to be asymmetric even in the case of a symmetric target free-surface profile H_{targ} . We tested the topography B numerically in the transient problem (7.10) including the effect of introducing a nonlinear disturbance. The free surface returned to the target free-surface profile H_{targ} after the disturbance has travelled radially outwards across the domain. In the leading-order problem, the agreement between the free surface H computed from equation (7.10) and the target free-surface profile was numerically

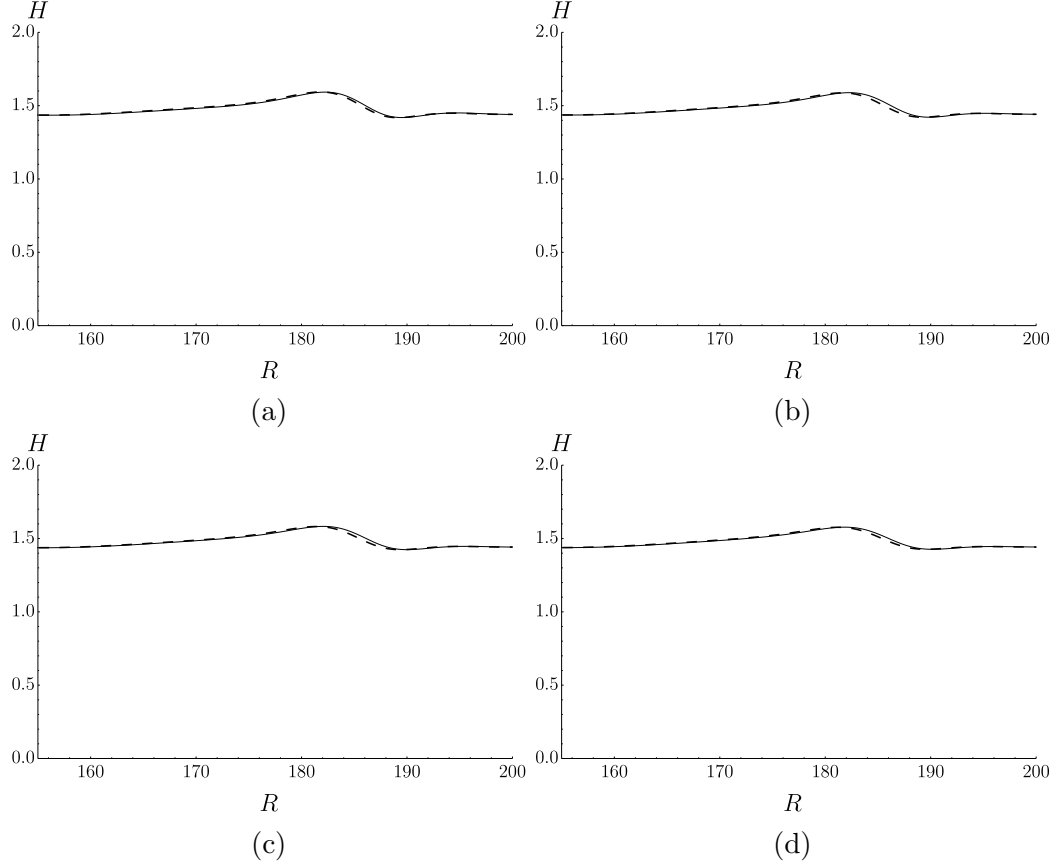


Figure 7.13: Plots of the free surface H computed from equations (7.17)–(7.18) with (a) $\zeta = 0.13$, (b) $\zeta = 0.0975$, (c) $\zeta = 0.065$, and (d) $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile from $R = 155$ to $R = 200$ at $T = 56$ compared to the planar case (dashed curve).

exact (up to the six decimal places output by the code). The topography B had very little effect on the profile of the disturbance compared to the planar case, however there was a non-zero difference in each case. The deviation was slightly larger in the two triple-step cases, which we hypothesised was due to the topography B covering a larger portion of the domain compared to the other cases presented. Hence, the disturbance was subjected to the effects of the topography B for a longer period of time.

In Section 7.2, we considered the first-order problem, for which an exact analytical solution could not be obtained for the topography B in the steady problem. We found a perturbation solution for the topography B up to first order in ζ . We tested the topography B numerically in the transient problem (7.17)–(7.18) including the effect of introducing a nonlinear disturbance. The free surface returned to the target free-surface profile H_{targ} after the disturbance has travelled radially outwards across the domain. In the first-order problem, the free surface H computed from equations (7.17)–(7.18) had small errors compared to the target free-surface profile H_{targ} . Errors of order $O(\zeta^2)$ were expected as the perturbation solution for the topography B (7.24) was not an exact solution to the steady problem (7.19). In particular, for the cases presented, the error reaches a maximum of approximately 0.65% for the Gaussian target free-surface profile with $\zeta = 0.13$. In all cases presented, the error was greater when the free surface forms a step up than for a step down. The topography B had very little effect on the profile of the disturbance. However, we note that for each case presented there was a small but non-zero difference compared to the planar case. The deviation was slightly greater in the two triple-step cases as found in the leading-order problem. In Section 7.2.4, we considered the effect of ζ . The error between the free surface H and the target free-surface profile

H_{targ} decreased with ζ . The error being of order $O(\zeta^2)$ was in agreement with the results presented within this subsection. The topography had very little effect on the disturbance for all four values of ζ presented, however, there was slightly more spread in the disturbance as ζ decreased.

Chapter 8

Conclusions and further work

In this chapter, we summarise and conclude the work reported within this thesis. In particular, in Section 8.1 we summarise the methods, key findings and conclusions, and in Section 8.2 we suggest some possible directions for further study.

8.1 Summary and conclusions

In this thesis, we considered the reduced-order modelling and analysis of thin-film flow over a spinning disk. Specifically, we have developed a novel reduced-order model using the WRIBL method and validated this model against DNS. We then performed parametric studies on varying the inverse Weber number λ and the maximum amplitude of inlet forcing A for the case of a planar disk and varying the asymmetry, smoothness, amplitude and spatial frequency of the topography on the disk. Analytical results were obtained for the linear stability in the far field and to control the free surface towards a prescribed target free-surface profile using the topography of the disk in the outer region. Next we discuss our findings.

In Chapter 2 we derived a novel reduced-order model for thin-film flow over a topographically-patterned spinning disk (2.65)–(2.67), which was derived using the simplified second-order WRIBL method. This reduced-order model retains inertial effects to leading order, but retains viscous effects up to second order. The reduced-order model derived is formally approximate rather than asymptotic due to inertial effects entering the problem at leading order. However, we validated the model against DNS of flow over a planar disk in Chapter 3 and, in particular, found that the reduced-order model qualitatively gave good predictions when compared against the DNS for the free surface, and the radial and azimuthal fluxes. The leading-order radial and azimuthal velocity profiles used in the WRIBL method to derive the reduced-order model were also validated against DNS, for which there was, in general, excellent agreement.

Due to the complex dynamics of the flow, it was first essential to understand flow over a planar disk before studying flow over a topographically-patterned disk. Therefore we performed a parametric study for varying the inverse Weber number λ and the maximum amplitude of inlet forcing A , and a linear stability analysis for a planar disk. This parametric study was undertaken numerically in Chapter 4 and then the linear stability analysis analytically in Chapter 6. We found that the fully-developed flow exhibits four characteristic spatial regions: the spin-up region, the inner region, the middle region and the outer region. The radial positions of these regions depend only on value of the inverse Weber number λ . In the case without inlet forcing, the shoulder in the free surface forms within the spin-up region; the inner region is characterised by a waveless region; and the middle and outer regions are characterised by fully-developed trains of pulse-like waves. These pulse-like waves emerge at the radial position $r = r_s$, we referred to this phenomenon as the self-

sustained inception of waves. The linear stability of the inner, middle and outer regions was investigated, and it was found that the inner region is neutrally stable, the middle region is convectively unstable, and the outer region is stable. However, in the outer region the decay rates are very slow, and so the middle and outer regions were not easily distinguished in our numerical results. For the case with inlet forcing, the radial location of the self-sustained inception of waves also depends on the maximum amplitude of the inlet forcing A , specifically, waves emerge nearer the centre of the disk as the maximum amplitude of the inlet forcing A increases. For fully-developed flow, the interfacial waviness increases compared to the case without inlet forcing, fluctuating around a constant value of approximately 1.2 (the interfacial waviness does slightly increase with the maximum amplitude of inlet forcing A), and the shear stress increases compared to the case without inlet forcing before fluctuating around a constant value, which is essentially independent of the maximum amplitude of inlet forcing A . The presence of inlet forcing significantly increases the inter-peak separation compared to the case without inlet forcing, however, increasing the maximum amplitude of inlet forcing only marginally increases the inter-peak separation. The inter-peak separation has a relatively weak radial dependence in the presence of inlet forcing, and increasing the maximum amplitude of inlet forcing A helps the system reach a regime in which the inter-peak separation is independent of the radial distance.

The effect of a topographically-patterned disk on the flow was considered in Chapter 5 (note that we included inlet forcing of maximum amplitude $A = 0.01$ in this chapter). In particular, for the family of topographies studied with parameters controlling the asymmetry, smoothness, amplitude and spatial frequency of the topography, the amplitude of the topography has the most significant impact on

the flow. The addition of topographies with amplitudes $a = 0.05$ and $a = 0.1$ cause the mean inter-peak separation to remain approximately constant across the radial interval $7 \leq r \leq 13$, reducing the weak radial dependence exhibited for a planar disk. Further increasing the amplitude to $a = 0.2$ causes the mean inter-peak separation to decrease with r across the radial interval $7 \leq r \leq 13$. The interfacial waviness increases with the amplitude of the topography and the shear stress also increases with the amplitude of the topography for fully-developed flow. The asymmetry, smoothness and spatial frequency of the topography have only minor effects on the flow. However, for all topographies considered, the addition of topography causes the interfacial waviness to increase compared to a planar disk, and the shear stress to increase for fully-developed flow compared to a planar disk. The radial position of the self-sustained inception of waves is not affected by the presence of topography.

Finally, in Chapter 7 we considered controlling the free surface towards a prescribed target free-surface profile using the topography as the control mechanism. In particular, we considered the behaviour in the (stable) outer region. We used an inversion technique to seek an analytical solution for the topography given a prescribed target free-surface profile. In all of the cases studied, the resulting topography was found to be asymmetric even for the case of a symmetric target free-surface profile. For the leading-order problem, an exact analytical solution for the topography was obtained for the steady problem. Numerical testing of the topography in the transient problem (7.10) found that when a nonlinear disturbance was introduced, the free surface returned to the target free-surface profile after the disturbance had travelled radially outwards across the domain, and that the free surface computed was numerically exact (up to the six decimal places output by the code) to the target free-surface profile. The topography had a small but non-zero

effect on the disturbance in each case tested. This effect was slightly greater in the two triple-step cases, which we hypothesised was due to the topography covering a larger portion of the domain. For the first-order problem, an exact analytical solution for the topography could not be obtained in the steady problem. However, we found a perturbation solution up to first order in ζ . As for the leading-order problem, when a nonlinear disturbance was introduced, the free surface returned to the target free-surface profile after it had travelled radially outwards across the domain. However, the computed free surface had small errors compared to the target free-surface profile, with the maximum error for the cases studied reaching approximately 0.65% for the Gaussian target free-surface profile with $\zeta = 0.13$. These small errors of order $O(\zeta^2)$ were due to the perturbation solution not being an exact solution to the steady problem.

Overall, the results presented in this thesis have shown that the dynamics of the flow are very sensitive to small variations in the inlet flow, with the behaviour of the inner region being dictated by these small variations. This makes comparison to experiments challenging as in practice inlet flows are often non-constant, however, the exact variations are, in general, unknown. The linear stability analysis has provided mathematical explanation for the phenomenon of the self-sustained inception of waves seen in our numerical simulations, and the numerical simulations of Matar *et al.* [74] and Prieling and Steiner [93]. The self-sustained inception of waves is caused by the neutrally stable inner region and the convectively unstable middle region for the cases with and without inlet forcing, respectively. The amplitude is the topography parameter which has the most significant effect, implying that the amplitude matters more than the shape of the topography. In relation to the applications described in Section 1.5.1, the reduced-order model presented in this

thesis offers a useful initial tool for evaluating flow conditions in a SDR, helping to minimise the number of experiments undertaken to find the correct angular velocity and inlet flow rate. Stafford *et al.* [117] showed that the shear stress and residence time (i.e. the time a particle spends in the film on the disk) are the dominant factors that govern the production of graphene. Our results in Chapter 5 showed that for all the topographies studied the shear stress increased compared to a planar disk and the presence of topography should not significantly alter the residence time. Therefore, based on these results, a topographically-patterned disk would likely increase the production of graphene compared to that of a planar disk.

8.2 Further work

In this section, we discuss some possible directions for future work that would build upon the results presented in this thesis.

Firstly, the reduced-order model (2.65)–(2.67) derived in Chapter 2 is based on the assumption that the flow is axisymmetric (i.e. there is no variation in the azimuthal direction). This is a common modelling assumption that is widely used in the literature, however, there have been some studies on modelling non-axisymmetric flow over a spinning disk using the IBL method (see, for example, [58, 59, 110, 146]). Relaxing this assumption would make the results obtained more general. The evolution equations and boundary conditions for the reduced-order model in this case would be relatively easy to obtain. However, challenges would arise in solving the resulting model numerically. Relaxing the axisymmetric assumption introduces the additional independent variable θ , which would need to be discretised. Care would need to be taken as due to the geometry of the problem, a uniform grid in

(r, θ) space does not correspond to a uniform grid in (x, y) space. In particular, in (x, y) space, the grid points would spread more the further they travel from the centre of the disk. The grid point spacing would be dictated by the middle and outer regions where the dynamics of the flow become more complex, in turn leading to more closely spaced grid points in the inner region where this is not required. This spread in points may cause issues with convergence in space and time. It is unlikely that the non-axisymmetric problem could be simulated over as large a disk as presented in this thesis (disk radius $R_{\text{disk}} = 15$) due to the number of grid points that would be required in each direction.

Another direction would be to extend the family of topographies studied. As the amplitude of the topography was found to have the most significant effect on the flow compared to the other topography parameters studied, it would be of interest to consider larger amplitude topographies than studied in this thesis. However, doing this could eventually enter a regime in which the reduced-order model is not valid, as it would ultimately violate the small amplitude topography assumption. This problem could be considered either using DNS or experimentally. In this thesis, we have only considered a planar disk and the family of topographies based on a sawtooth profile in Chapter 5. However, axisymmetric square trench topographies were studied numerically by Matar *et al.* [75], and non-axisymmetric topographies were studied experimentally by Yao *et al.* [142] and Wang *et al.* [131]. It would be of interest to study the effect of other topographies on the flow using either the reduced-order model derived in this thesis or an extended non-axisymmetric model.

The production of graphene is a problem of scientific interest. For example, Stafford *et al.* [117] used a SDR to produce graphene from graphite flakes suspended in a

solvent via shear exfoliation (as discussed in Section 1.5.1). They showed that the shear stress and residence time (i.e. the time a particle spends in the film on the disk) are the dominant factors that govern the production of graphene. This problem poses an interesting control problem as the two governing factors of shear stress and residence time are generally in competition, e.g. increasing the angular velocity of the disk increases the shear stress, however, this also decreases the residence time. A problem could be posed using the reduced-order model to optimise the shear stress and residence time using the volume flux at the inlet, angular velocity of the disk and topography as the variables. A method such as optimal control could be applied to solve this challenging problem numerically.

The free-surface control results presented in Chapter 7 offer many possible directions for further study. Firstly, we have only considered topography as a control mechanism, therefore, other mechanisms such as electric fields could be considered. This would involve the introduction of an additional equation to incorporate the electrostatic effects. Also, we only considered the problem up to first order in ζ , so another avenue to explore would be to consider the problem to higher order. Extending the method used in Chapter 7 to higher order would involve higher derivatives of the free surface H with respect to R . Finding the direct solution to the first-order steady problem would also be of interest, as we have only found the perturbative solution up to first order in ζ . This would involve solving equation (7.19) numerically as a boundary value problem for the film thickness E , from which the topography is easily obtainable.

Another possible avenue would be to apply weakly-interacting pulse theory to flow over a spinning disk. For instance, Pradas *et al.* [91] applied this theory to the case of gravity-driven flow (using the simplified second-order WRIBL model) and considered

the interaction of two pulses. For flow over a spinning disk, the dynamics of the flow are generally governed by pulse-like waves, which cover a large portion of the disk. These pulse-like waves separate and/or coalesce. Applying weakly-interacting pulse theory could further enhance the understanding of the dynamics of the flow.

In summary, while we believe that this thesis describes a worthwhile contribution to the understanding of thin-film flow over a spinning disk, there are still many possible open questions on this topic worthy of further investigation.

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Appendix A

Control of the free surface: effect of ζ

In this appendix, we present space-time plots of $H - H_N$ and plots of the free surface H compared to the target free-surface profile H_{targ} for each value of ζ .

A.1 Effect of a disturbance

Figure A.1 shows space time plots of $H - H_N$ with $\zeta = 0.13$, $\zeta = 0.0975$, $\zeta = 0.065$ and $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile from $T = 0$ to $T = 45$. In particular, Figure A.1 shows that for all four values of ζ presented the free surface returns to the target free-surface profile after the disturbance has convected across the domain. Therefore, the flow returns to its steady state after introducing a nonlinear disturbance. Figure A.1 shows that the value of ζ has little effect on both the disturbance travelling radially outwards across the domain and the topography B .

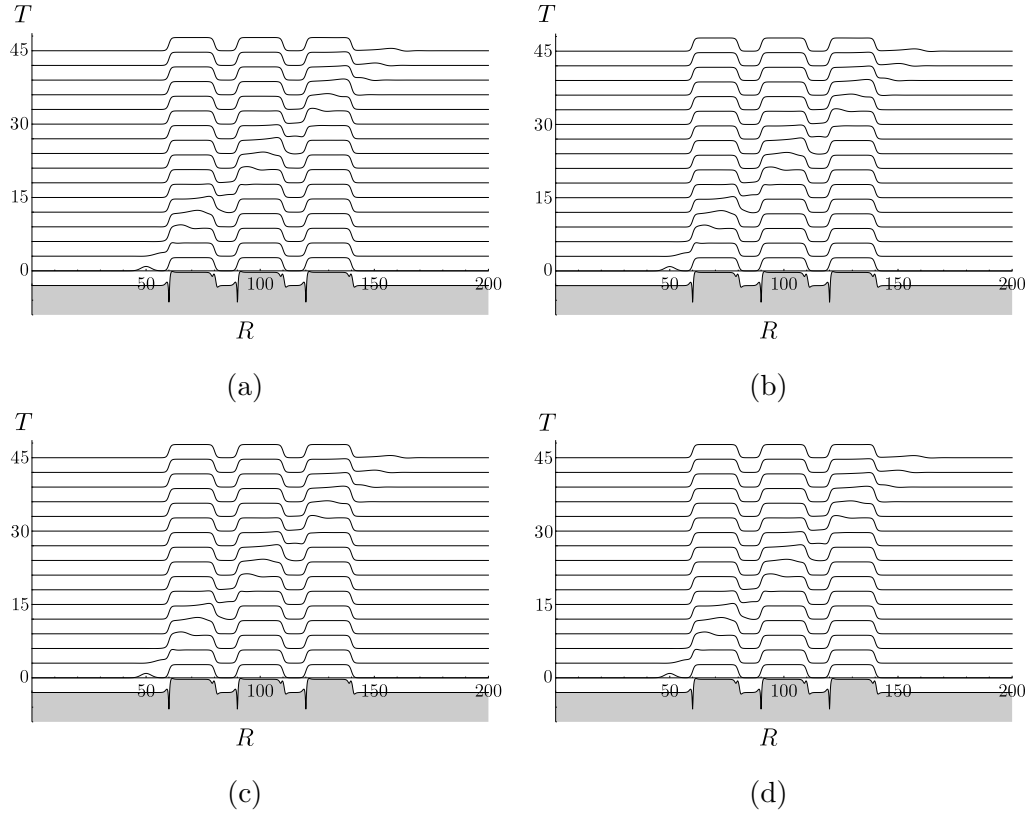


Figure A.1: Space-time plots of $H - H_N$ computed from equations (7.17)–(7.18) with (a) $\zeta = 0.13$, (b) $\zeta = 0.0975$, (c) $\zeta = 0.065$, and (d) $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile. In these plots the time varies from $T = 0$ to $T = 45$ in steps of $T = 3$.

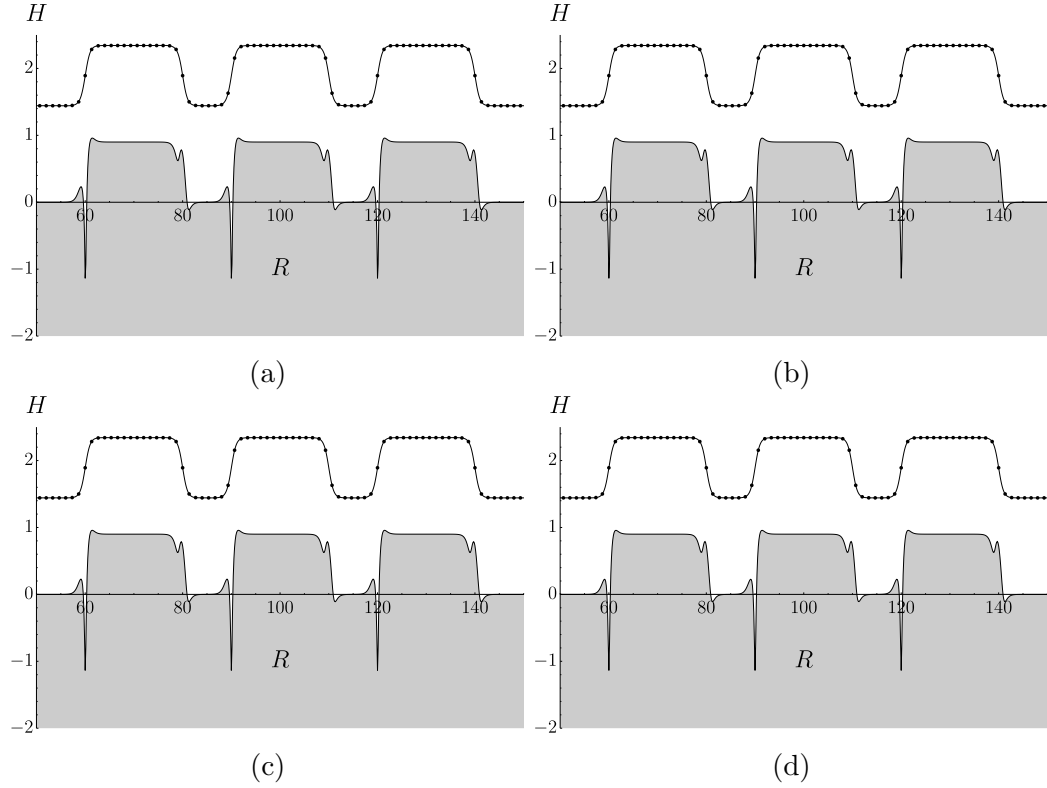


Figure A.2: Plots of the free surface H computed from equations (7.17)–(7.18) (solid curve) and target free-surface profile H_{targ} (closed circles) with (a) $\zeta = 0.13$, (b) $\zeta = 0.0975$, (c) $\zeta = 0.065$, and (d) $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$.

A.2 Deviation of the free surface profile from the target free-surface profile

Figure A.2 shows plots of the free surface H compared to the target free-surface profile H_{targ} with $\zeta = 0.13$, $\zeta = 0.0975$, $\zeta = 0.065$ and $\zeta = 0.0325$ for a symmetric triple-step target free-surface profile from $R = 50$ to $R = 150$ at $T = 68$. In particular, Figure A.2 shows that for all four values of ζ presented the free surface H has excellent agreement with the target free-surface profile H_{targ} . However, there are

some minor differences between them. We investigate these differences by considering the percentage error of the free surface H from the target free-surface profile H_{targ} , presented in Section 7.2.4.