Truncated Euler Maruyama Numerical Method for Stochastic Differential (Delay) Equations Models

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Abstract

In this thesis, our focus has been on enhancing the applicability and reliability of the truncated Euler-Maruyama (EM) numerical method for stochastic differential equations (SDEs) and stochastic delay differential equations (SDDEs), initially introduced by Mao [21]. Building upon this method, our contributions span several chapters. In Chapter 3, we pointed out its limitations in determining the convergence rate over a finite time interval and established a new result for SDEs whose diffusion coefficients may not satisfy the global Lipschitz condition. We extended our exploration to include time delays in Chapter 4, allowing for varying delays over time. The chapter also introduces additional lemmas to ensure the convergence rates of the method to the solution at specific time points and over finite intervals. However, the global Lipschitz condition on the diffusion coefficient is currently required. In Chapter 5, we focused on the Lotka-Volterra model, introducing modifications such as the Positive Preserving Truncated EM (PPTEM) and Nonnegative Preserving Truncated EM (NPTEM) methods to handle instances where the truncated EM method generated nonsensical negative solutions. The proposed adjustments, guided by Assumption 5.1.1, ensure that the numerical solutions remain meaningful and interpretable. Chapter 6 extends these concepts to the stochastic delay Lotka-Volterra model with a variable time delay, demonstrating the adaptability and applicability of our methods. Despite we also assume the stronger condition 6.1.1 to prove the convergence of numerical solutions, future research aims to explore relaxed conditions, broadening the applicability of these numerical methods. Overall, this thesis contributes to establishing convergence rates for SDEs under local Lipschitz diffusion coefficients, extending the methodology to address time delays and modifying the truncated EM method to ensure positive and nonnegative numerical solutions. These advancements are demonstrated through applications to the stochastic variable time delay Lotka-Volterra model, emphasizing the meaningfulness and interpretability of the solutions.

Contents

Notations

positive	:	> 0.
nonpositive	:	≤ 0.
negative	:	< 0.
nonnegative	:	≥ 0.
a.s.	:	almost surely, or with probaility 1.
$A := B$	:	A is defined by B or A is denoted by B.
$A(x) \equiv B(x)$	:	$A(x)$ and $B(x)$ are indentically equal, i.e. $A(x) = B(x)$ for all x.
$\emptyset$	:	the empty set.
$\mathbb{1}_A$	:	the indicator function of a set A i.e. $\mathbb{1}_A(x) = 1$ if $x \in A$ or otherwise 0.
A^c	:	the complement of A in Ω, i.e. $A^c = \Omega - A$.
$a \wedge b$	:	$\min\{a, b\}$.
$a \vee b$	:	$\max\{a, b\}$.
$f : A \to B$	:	the mapping f from A to B.
$\mathbb{R} = \mathbb{R}^1$	:	the real line.
$\mathbb{R}_+$	:	the set of all nonnegative real numbers, i.e. $\mathbb{R}_+ = [0, \infty)$.
$\mathbb{R}^d$	:	the d-dimensional Euclidean space.
$\mathbb{R}^d_+$	:	$= \left\{x \in \mathbb{R}^d : x_i > 0, 1 \leq i \leq d\right\}$, i.e. the positive cone.
$\bar{\mathbb{R}}^d_+$	:	$= \left\{x \in \mathbb{R}^d : x_i \geq 0, 1 \leq i \leq d\right\}$.
$\|x\|$	:	the Euclidean norm of a vector x.
A^T	:	the transpose of a vector or matrix A.
trace$A$	:	the trace of a square matrix $A = (a_{i,j})_{d\times d}$, i.e. $\text{trace}A = \sum_{1\leq i\leq d} a_{ii}$.
$\|A\|$	:	$\sqrt{\text{trace}\,(A^T A)}$, i.e. the trace norm of a matrix A.
$C(D; \mathbb{R}^d)$	:	the family of continuous $\mathbb{R}^d$-valued functions defined on D.
$C^m(D; \mathbb{R}^d)$	:	the family of continuous $m-$times differentiable $\mathbb{R}^d$-valued functions defined on D.

$C_0^m(D;\mathbb{R}^d)$	:	the family of functions in $C^m(D;\mathbb{R}^d)$ with compact support in D.
$C^{2,1}(D\times\mathbb{R}_+;\mathbb{R})$	:	the family of all real-valued functions $V(x,t)$ defined on $D\times\mathbb{R}_+$ which are continuously twice differentiable in $x\in D$ and once differentiable in $t\in\mathbb{R}_+$.
V_x	:	$=(V_{x_1},\cdots,V_{x_d})=\left(\frac{\partial V}{\partial x_1},\cdots,\frac{\partial V}{\partial x_d}\right)$.
V_{xx}	:	$=\left(V_{x_ix_j}\right)_{d\times d}=\left(\frac{\partial^2 V}{\partial x_i\partial x_j}\right)_{d\times d}$.
$\lVert\xi\rVert_{L^p}$	:	$=(\mathbb{E}\,\lvert\xi\rvert^p)^{1/p}$.
$L^p(\Omega;\mathbb{R}^d)$	:	the family of $\mathbb{R}^d$-valued random variables ξ with $\mathbb{E}\,\lvert\xi\rvert^p<\infty$.
$L^p_{\mathcal{F}_t}(\Omega;\mathbb{R}^d)$	:	the family of $\mathbb{R}^d$-valued $\mathcal{F}_t$-measurable random variables ξ with $\mathbb{E}\,\lvert\xi\rvert^p<\infty$.
$C([-\tau,0];\mathbb{R}^d)$	:	the space of all continuous $\mathbb{R}^d$-valued functions φ defined on $[-\tau,0]$ with a norm $\lVert\varphi\rVert=\sup_{-\tau\le\theta\le0}\lvert\varphi(\theta)\rvert$.
$L^p_{\mathcal{F}}([-\tau,0];\mathbb{R}^d)$	:	the family of all $C([-\tau,0];\mathbb{R}^d)$-valued random variables ϕ such that $\mathbb{E}\lVert\phi\rVert^p<\infty$.
$L^p_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$	:	the family of all $\mathcal{F}_t$-measurable $C([-\tau,0];\mathbb{R}^d)$-valued random variables ϕ such that $\mathbb{E}\lVert\phi\rVert^p<\infty$.
$C^b_{\mathcal{F}_t}([-\tau,0];\mathbb{R}^d)$	:	the family of all $\mathcal{F}_t$-measurable bounded $C([-\tau,0];\mathbb{R}^d)$ -valued random variables.
$L^p([a,b];\mathbb{R}^d)$	:	the family of Borel measurable functions $h:[a,b]\to\mathbb{R}^d$ such that $\int_a^b\lvert h(t)\rvert^p dt<\infty$.
$\mathcal{L}^p([a,b];\mathbb{R}^d)$	:	the family of $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b\lvert f(t)\rvert^p dt<\infty$.
$\mathcal{M}^p([a,b];\mathbb{R}^d)$	:	the family of processes $\{f(t)\}_{a\le t\le b}\in\mathcal{L}^p([a,b];\mathbb{R}^d)$ such that $\mathbb{E}\int_a^b\lvert f(t)\rvert^p dt<\infty$.
$\mathcal{L}^p(\mathbb{R}_+;\mathbb{R}^d)$	:	the family of processes $\{f(t)\}_{t\ge0}$ such that for every $T>0$, $\{f(t)\}_{0\le t\le T}\in\mathcal{L}^p([0,T];\mathbb{R}^d)$.
$\mathcal{M}^p(\mathbb{R}_+;\mathbb{R}^d)$	:	the family of processes $\{f(t)\}_{t\ge0}$ such that for every $T>0$, $\{f(t)\}_{0\le t\le T}\in\mathcal{M}^p([0,T];\mathbb{R}^d)$.

Additional notations will be clarified when they first appear.

Chapter 1

Introduction

A stochastic differential equation (SDE) is a mathematical framework used to model systems affected by both deterministic and random factors. It extends the principles of ordinary differential equations (ODEs) to encompass elements of chance and is employed in fields like physics, finance, biology, and engineering where randomness and unpredictable variations are significant.

A standard SDE is usually expressed as follows:

$$dX(t) = f(X(t), t)dt + g(X(t), t)dB(t)$$

In this equation, $X(t)$ is the stochastic process and represents the system's changing state at time $t > 0$. $f(X(t), t)$ characterizes the deterministic drift, signifying the system's expected behaviour, while $g(X(t), t)$ describes stochastic diffusion, modelling random fluctuations. And $dB(t)$ denotes the incremental change of a Wiener process or Brownian motion, capturing the inherent uncertainty, [15, 19, 30]

If the drift and diffusion are complex, determining the explicit solutions of SDEs is difficult. Numerical methods for SDEs have become a focal point of research in this area. Until 2002, most of the existing strong convergence theory in this area necessitated global Lipschitz continuous coefficients for SDEs (see, e.g., [15, 19, 29]). In 2002, Higham, Mao, and Stuart's publication [10] initiated a new phase, focusing on the strong convergence issue for numerical approximations under the local Lipschitz condition.

Given that the classical Euler-Maruyama (EM) method may struggle with SDEs under the local Lipschitz condition but without the linear growth condition (i.e., highly nonlinear SDEs) (see, e.g., [11, 13]), implicit methods have naturally been employed to study numerical solutions for highly nonlinear SDEs (see, e.g., [26, 33, 34]). Despite this, the explicit EM method possesses a straightforward algebraic structure, cost-effectiveness, and an acceptable convergence rate under the global Lipschitz condition. Several modified EM methods have recently been developed for the highly nonlinear SDEs. These include the tamed EM method [14, 31, 32], the tamed Milstein method [35] and the stopped EM method [18].

In 2015, Mao introduced the modified EM approximate method for SDEs which is called the truncated EM method, [21]. Using this method, he demonstrated that the truncated EM solution converges in the L^q norm to the exact solution under both local Lipschitz and Khasminskii-type conditions. The following year, Mao further presented the convergence rate at a finite time $T > 0$, see [22]. To calculate the numerical solution with the rate of [22], unfortunately, the step size is sometimes required very small, or we can say it is inapplicable.

In contrast, Hu, Li, and Mao (2018) asserted in [12] that the step size can be flexible within the interval $(0, 1]$, and the convergence rate at time T exhibits a degree of similarity. In practical applications, a convergence rate at time T suffices for scenarios, for example, requiring the approximation of European put or call option values. Yet, for accurate estimations of path-dependent quantities, encompassing the entire lifespan of options like barrier options and bonds is essential, for more information see [9].

Mao (2016) [22] previously established the convergence rate of the truncated EM method over a finite time interval. However, the challenge persists as the small step size requirement remains. Additionally, the proof of convergence rate over a finite time interval in [22] relied on assuming global Lipschitz continuity in the diffusion coefficient. Despite the enhancement seen in the truncated EM method, as demonstrated in [12], it effectively addresses the challenge associated with the step size. Generally, numerous financial models exhibit diffusion coefficients, as exemplified by the Ait-Sahalia model [5], that do not adhere to the global Lipschitz condition. In Chapter 3, we aim to relax the requirement of global Lipschitz continuity for the diffusion coefficient to achieve the convergence rate of the truncated EM method over a finite time interval.

A stochastic differential delay equation (SDDE) expands the SDE framework by introducing both stochastic components and time delays in the system's dynamics. SDDEs are used to model systems where the current state depends not only on past states and random influences but also on states at previous time points.

A general form of a stochastic differential delay equation is as follows:

$$dX(t) = f(X(t), X(t-\tau), t)dt + g(X(t), X(t-\tau), t)dB(t)$$

In this equation, $X(t)$ represents the state of the system at time t. The term $f(X(t), X(t-\tau), t)dt$ captures the deterministic part of the evolution, representing how the system changes over time based on its current state and a delayed state. The delayed term $X(t-\tau)$ introduces a time lag, reflecting the impact of past states on the current dynamics. The term $g(X(t), X(t-\tau), t)dB(t)$ introduces stochasticity to the system. The function g quantifies how random fluctuations affect the system, and $dB(t)$ represents the differential increment of a Brownian motion. The stochastic component accounts for inherent uncertainties and external influences in the system. The inclusion of time delays in SDDEs can lead to intricate and diverse dynamics, making them especially suitable for modelling systems exhibiting phenomena like feedback loops, memory effects, or history-dependent behaviour, see more [19].

Similarly to SDEs, early research of SDDEs focuses on the numerical solutions under conditions that their coefficients satisfy the linear growth condition and the global Lipschitz condition. Consequently, to reduce the global Lipschitz condition, many researchers developed the numerical solution under the linear growth condition and the local Lipschitz condition, [2, 3, 4, 16]. To apply more SDDE models, the generalized Khasminskii-type condition was applied to SDDEs instead of the linear growth condition, [20]. In 2018, Guo, Mao and Yue modified the truncated EM method for SDDEs with a constant time delay under the generalized Khasminskii-type condition and the local Lipschtiz condition. Fei et. al., in 2020, fixed the problem that the step size required [8] too very small and also provided the rate of convergence both at a time T and over a finite time interval.

According to the result in [25], Mao and Sabanis applied the EM numerical method for SDDEs with variable time delay. In Chapter 4, we aim to find the convergence rate of the SDDEs, with a variable time delay under the generalized Khasminskii-type condition and the local Lipschtiz condition, at a specific time T and over a finite time interval.

Although the truncated EM method is the new method that can be applied to SDEs and SDDEs with the nonlinear coefficients as described above, on some

SDE models, the truncated EM method can generate negative numerical solutions which are uninterpretable. For example, the stochastic Lotka–Volterra model for interacting multi-species in ecology should have positive solutions (see, e.g., [1, 23, 19]). The SDE SIS model in epidemiology also has positive solutions (see, e.g., [7]). These SDE models are all highly nonlinear. Therefore a positive solution is important to make the meaningfulness and interpretability of the solution.

Chapter 5 mainly focuses on the modification of the truncated EM method to create a new positivity preserving truncated EM (PPTEM) for the well-known stochastic Lotka–Volterra model for interacting multi-species in ecology. The reason why we will concentrate on this model is because it has typical features: highly nonlinear, positive solution and multi-dimensional. It is not worthless to note that our approach is to establish a new nonnegative preserving truncated EM (NPTEM) and then the more desired PPTEM since some other SDE models, in applications, have their solutions taking nonnegative values. Furthermore, it would be natural based on mathematics to determine the nonnegative solutions and follow the positive ones.

As a consequence, in SDDE models, the truncated EM numerical solutions also take a negative value. The next aim is to modify the truncated EM method to have the positive preserving or the nonnegative preserving properties by applying the stochastic delay Lokta-Volterra model. Additionally, we combine the idea of a variable time delay to this model and describe the methodology in Chapter 6.

To be more clear about this thesis, we organise this thesis as follows: We provide mathematical background such as Probability theory, Itô formula and other useful inequalities in Chapter 2. Chapter 3 is our first aim which is the convergence rate over a finite time interval of SDEs with a nonlinear diffusion coefficient. In Chapter 4, we investigate the rate of convergence of SDDEs with a variable time delay. Chapter 5 is extracted from the paper *Positivity Preserving Truncated Euler-Maruyama Method for Stochastic Lotka-Volterra Competition Model*, [27], which I co-authored with Prof. Mao Xuerong and Prof. Wei Fengying. In this chapter, we define the new methodologies PPTEN and NPTEM for the stochastic Lokta-Volterra model. The last aim in Chapter 6 is an extension of PPTEM and NPTEM to the stochastic delay Lokta-Volterra model with a variable time delay. The thesis concludes by engaging in discussions that encompass the drawbacks, limitations, potential further applications, and future extensions of the theoretical discoveries outlined in Chapter 7.

Chapter 2

Preliminaries

In order to make this thesis self-sufficient, we will cover the fundamental mathematical tools. In this chapter, we will explore the basics of SDEs by starting with some concepts from the probability theory. Afterwards, we will introduce the stochastic processes and dive into the essential ideas of Brownian motion, stochastic integrals and Itô calculus. We will also introduce stochastic differential equations and SDDEs, and end up with the well-known mathematical inequalities. Just so you know, we've drawn inspiration and content from references [19], [28], and [30].

2.1 Basic probability concepts

Let us begin with the fundamental mathematical principles of probability theory. Let Ω be a given set. A family of subsets of Ω called as *σ-algebra*, $\mathcal{F}$, on Ω if it satisfies the following properties:

(i). $\emptyset \in \mathcal{F}$;

(ii). if $A \in \mathcal{F}$ then $A^c \in \mathcal{F}$;

(iii). if $\{A_i\}_{i\geq 1} \in \mathcal{F}$ then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The pair of $(\Omega, \mathcal{F})$ is called a *measurable space*, and the individual elements within $\mathcal{F}$ are referred to as *$\mathcal{F}$-measurable sets* or simply *events*. If C is a family of subsets of Ω, then there exists a smallest σ-algebra $\sigma(C)$ on Ω which contains C. The $\sigma(C)$ is also called the *σ-algebra generated by C*. For a specific case $\Omega = \mathbb{R}^d$ and the family of all open sets C in $\mathbb{R}^d$, $\mathcal{B}^d = \sigma(C)$ is called the *Borel σ-algebra* and the elements of $\mathcal{B}^d$ are called the *Borel sets*.

A real-valued function $X : \Omega \to \mathbb{R}$ is said to be *$\mathcal{F}$-measurable* if

$$\{\omega : X(\omega) \leq a\} \in \mathcal{F} \text{ for all } a \in \mathbb{R}.$$

The function X can also be referred to as a real-valued ($\mathcal{F}$-measurable) *random variable*. An $\mathbb{R}^d$-valued function $X(\omega) = (X_1(\omega), X_2(\omega), \cdots, X_d(\omega))^T$ is said to be *$\mathcal{F}$-measurable* if all the elements X_i are $\mathcal{F}$-measurable. Similarly, a $d \times m$-matrix-valued function $X(\omega) = (X_{ij}(\omega))_{d\times m}$ is said to be *$\mathcal{F}$-measurable* if all the elements X_{ij} are $\mathcal{F}$-measurable.

The indicator function $\mathbb{1}_A$ of a set $A \subseteq \Omega$ is defined by

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \text{for } \omega \in A, \\ 0 & \text{for } \omega \notin A. \end{cases}$$

The indicator function $\mathbb{1}_A$ is $\mathcal{F}$-measurable if and only if A is an $\mathcal{F}$-measurable set, i.e. $A \in \mathcal{F}$. If the measurable space is $(\mathbb{R}^d, \mathcal{B}^d)$, a $\mathcal{B}^d$-measurable function is then called a *Borel measurable function*.

More generally, let $(\Omega', \mathcal{F}')$ be another measurable space. A mapping $X : \Omega \to \Omega'$ is said to be $(\mathcal{F}, \mathcal{F}')$-measurable if

$$\{\omega : X(\omega) \in A'\} \in \mathcal{F} \text{ for all } A' \in \mathcal{F}.$$

The mapping X is then called an Ω'-valued $(\mathcal{F}, \mathcal{F}')$-measurable (or simply, $\mathcal{F}$-measurable) random variable.

For a given function $X : \Omega \to \mathbb{R}^d$, the *$\sigma$-algebra $\sigma(X)$ generated by X* is the smallest σ-algebra on Ω containing all the sets $\{\omega : X(\omega) \in U\}, U \subseteq \mathbb{R}^d$ open. That is

$$\sigma(X) = \sigma(\{\omega : X(\omega) \in U\} : U \subseteq \mathbb{R}^d open).$$

Clearly, in this case, X becomes $\sigma(X)$-measurable and $\sigma(X)$ is the smallest σ-algebra with the property. If X is $\mathcal{F}$-measurable, then $\sigma(X) \subseteq \mathcal{F}$, in other words, X generates a sub-σ-algebra of $\mathcal{F}$. If $\{X_i : i \in I\}$ is a collection of $\mathbb{R}^d$-valued functions, define

$$\sigma(X_i : i \in I) = \sigma\left(\bigcup_{i \in I} \sigma(X_i)\right)$$

which is called the *σ-algebra generated by* $\{X_i : i \in I\}$. It is the smallest σ-algebra with respect to which every X_i is measurable.

A *probability measure* $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that

(i). $\mathbb{P}(\Omega) = 1$, and

(ii). if $A_1, A_2, A_3, \dots$ is a sequence in $\mathcal{F}$ such that $A_i \cap A_j = \emptyset$ for all $i \neq j$, then

$$\mathbb{P}\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \mathbb{P}(A_n).$$

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*.

If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we set

$$\bar{\mathcal{F}} = \{A \in \Omega : \text{ there exist } B, C \in \mathcal{F} \text{ such that } B \subseteq A \subseteq C, \mathbb{P}(B) = \mathbb{P}(C)\}.$$

Then $\bar{\mathcal{F}}$ is a σ-algebra and is referred to as the *completion* of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be *complete*. In consequence, we refer to a given complete probability space as $(\Omega, \mathcal{F}, \mathbb{P})$.

A *random variable* X is an $\mathcal{F}$-measurable function $X : \Omega \to \mathbb{R}^d$. Every random variable induces a probability measure μ_X on the Borel measurable space $(\mathbb{R}^d, \mathcal{B}^d)$, defined by

$$\mu_X(B) = \mathbb{P}\{\omega : X(\omega) \in B\} \text{ for } B \in \mathcal{B}^d,$$

and μ_X is called the *distribution* of X.

If X is a real-valued random variable and is integrable with respect to the probability measure $\mathbb{P}$, then the number

$$\mathbb{E}X = \int_{\Omega} X(\omega) d\mathbb{P}(\omega) = \int_{\mathbb{R}^d} x d\mu_X(x)$$

is called the *expectation* of X with respect to $\mathbb{P}$. The number

$$\mathbb{V}(X) = \mathbb{E}(X - \mathbb{E}(X))^2$$

is called the *variance* of X.

More generally, if $f : \mathbb{R}^d \to \mathbb{R}^m$ is Borel measurable and $\int_\Omega |f(X(\omega))| d\mathbb{P}(\omega) < \infty$, then we have

$$\mathbb{E}f(X) = \int_\Omega f(X(\omega)) d\mathbb{P}(\omega) = \int_{\mathbb{R}_d} f(x) d\mu_X(x).$$

The number $\mathbb{E}|X|^p$ for $p > 0$ is called the pth moment of X i.e. $\mathbb{E}|X|^p = \int_\Omega |X(\omega)|^p d\mathbb{P}(\omega)$. For $p \in (0, \infty)$, let $L^p = L^p(\Omega; \mathbb{R}^d)$ be the family of $\mathbb{R}^d$-valued random variables X with $\mathbb{E}|X|^p < \infty$. In L^1, we have $|\mathbb{E}X| \leq \mathbb{E}|X|$. Moreover, the following three inequalities hold true.

(i). **Hölder's inequality :** if $p > 1$, $1/p + 1/q = 1$, $X \in L^p$ and $Y \in L^q$, then

$$|\mathbb{E}(X^T Y)| \leq (\mathbb{E}|X|^p)^{1/p} (\mathbb{E}|Y|^q)^{1/q};$$

(ii). **Minkovski's inequality :** if $p > 1$ and $X, Y \in L^p$, then

$$(\mathbb{E}|X + Y|^p)^{1/p} \leq (\mathbb{E}|X|^p)^{1/p} + (\mathbb{E}|Y|^p)^{1/p};$$

(iii). **Chebyshev's inequality :** if $c > 0$, $p > 0$ and $X \in L^p$, then

$$\mathbb{P}\{\omega : |X(\omega)| \geq c\} \leq \frac{1}{c^p} \mathbb{E}|X|^p.$$

A simple application of Hölder's inequality implies

$$(\mathbb{E}|X|^r)^{1/r} \leq (\mathbb{E}|X|^p)^{1/p}$$

if $0 < r < p < \infty, X \in L^p$.

2.2 Stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A *filtration* is a family $\{\mathcal{F}\}_{t\geq 0}$ of increasing sub-σ-algebras of $\mathcal{F}$ (i.e. $F_t \subseteq \mathcal{F}_s \subseteq \mathcal{F}$ for all $0 \leq t < s < \infty$). The filtration is said to be *right continuous* if $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geq 0$. When the probability space is complete, the filtration is considered to satisfy the usual conditions if it is both right continuous and F_0 contains all $\mathbb{P}$-null sets.

From now on, unless otherwise specified, we assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space with a filtration $\{F_t\}_{t\geq 0}$ satisfying the usual conditions. We also define $F_\infty = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$, i.e. the σ-algebra generated by $\bigcup_{t\geq 0} \mathcal{F}_t$.

A family $\{X_t\}_{t\in I}$ of $\mathbb{R}^d$-valued random variables is called a *stochastic process* with *parameter set* or *index set* I and *state space* $\mathbb{R}^d$. The parameter set I is usually the half-line $\mathbb{R}_+ = [0, \infty)$, but it may also be an interval $[a, b]$, the non-negative integers or even subsets of $\mathbb{R}^d$. For each fixed $t \in I$, we have a random variable

$$\Omega \ni \omega \to X_t(\omega) \in \mathbb{R}^d$$

On the other hand, for each fixed $\omega \in \Omega$, we have a function

$$I \ni t \to X_t(\omega) \in \mathbb{R}^d$$

which is called a *sample path* of the process, and we shall write $X_\bullet(\omega)$ for the path. For convenience, we often write $X(t, \omega)$ instead of $X_t(\omega)$. The stochastic process can be seen as a function of two variables (t, ω), mapping from $I \times \Omega$ to $\mathbb{R}^d$. We commonly represent the stochastic process $\{X_t\}_{t\geq 0}$ as simply $\{X_t\}$, X_t or $X(t)$. In this work, we use the variable $x(t)$ to refer to a *stochastic process*.

Let $\{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued stochastic process. The stochastic process is said to be *continuous (resp. right continuous, left continuous)* if for almost all $\omega \in \Omega$, the function $X_t(\omega)$ is continuous (resp. right continuous, left continuous) on $t \geq 0$. It is said to be *càdlàg* (right continuous and left limit) if it is right continuous and for almost all $\omega \in \Omega$, the left limit $\lim_{s\uparrow t} X_s(\omega)$ exists and is finite for all $t > 0$. It is said to be *integrable* if for every $t \geq 0$, X_t is an integrable random variable. It is said to be $\{\mathcal{F}_t\}$-*adapted* if for every $t \geq 0$, X_t is F_t-measurable. It is said to be *measurable* if the stochastic process regarded as a function of two random variables (t, ω) from $\mathbb{R}_+ \times \Omega$ to $\mathbb{R}^d$ is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{F}$-measurable, where $\mathcal{B}(\mathbb{R}_+)$ is the family of all Borel sub-sets of $\mathbb{R}_+$. The stochastic process is said to be *progressively* *measurable* or *progressive* if for every $T \geq 0$, $\{X_t\}_{0\leq t\leq T}$ regarded as a function of (t,w) from $[0,T]\times\Omega$ to $\mathbb{R}^d$ is $\mathcal{B}([0,T])\times\mathcal{F}_T$-measurable, where $\mathcal{B}([0,T])$ is the family of all Borel sub-sets of $[0,T]$.

A random variable $\tau:\Omega\to[0,\infty]$ (it may take the value ∞) is called $\{\mathcal{F}_t\}$ *stopping time* if $\{\omega:\tau(\omega)\leq t\}\in\mathcal{F}_t$ for any $t\geq 0$.

Theorem 2.2.1. *If $\{X_t\}_{t\geq 0}$ is a progressively measurable process and τ is a stopping time, then $X_\tau \mathbb{1}_{\{\tau<\infty\}}$ is $\mathcal{F}_\tau$-measurable. In particular, if τ is finite, then X_τ is $\mathcal{F}_\tau$ measurable.*

Theorem 2.2.2. *Let $\{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued càdlàg $\{\mathcal{F}_t\}$-adapted process, and D an open subset of $\mathbb{R}^d$. Define*

$$\tau = \inf\{t\geq 0 : X_t \notin D\},$$

where we use the convention $\inf\emptyset=\infty$. Then τ is an $\{\mathcal{F}_t\}$-stopping time, and is called the first exit time from D. Moreover, if ρ is a stopping time, then

$$\theta = \inf\{t\geq \rho : X_t \notin D\}$$

is also called $\{\mathcal{F}_t\}$-stopping time, and is called the first exit time from D after ρ.

An $\mathbb{R}^d$-valued $\{\mathcal{F}_t\}$-adapted integrable process $\{M_t\}_{t\geq 0}$ is called a martingale with respect to $\{\mathcal{F}_t\}$ (or simply, martingale) if

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \text{ a.s. for all } 0\leq s<t<\infty.$$

Keep in mind that every martingale has a càdlàg modification because we consistently assume that the filtration $\mathcal{F}_t$ is right continuous.

If $X=\{X_t\}_{t\geq 0}$ is a progressively measurable process and τ is a *stopping time*, then $X_\tau=\{X_{\tau\wedge t}\}_{t\geq 0}$ is called a *stopped process* of X. The following is the well-known Doob martingale stopping theorem.

Theorem 2.2.3. *Let $\{M_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued martingale with respect to $\{\mathcal{F}_t\}$, and let θ,ρ be two finite stopping times. Then*

$$\mathbb{E}(M_\theta|\mathcal{F}_\rho) = M_{\theta\wedge\rho} \ a.s.$$

In particular, if τ is a stopping time, then

$$\mathbb{E}(M_{\tau\wedge t}|\mathcal{F}_s) = M_{\tau\wedge s} \ a.s.$$

holds for $0 \le s < t < \infty$. That is, the stopped process $M_\tau = \{M_{\tau\wedge t}\}$ is still martingale with respect to the same filtration $\{\mathcal{F}_t\}$.

A stochastic process $X = \{X_t\}_{t\geq 0}$ is called *square-integrable* if it satisfies the condition $\mathbb{E}|X_t|^2 < \infty$ for every $t \geq 0$. If $M = \{M_t\}_{t\geq 0}$ is a real-valued square-integrable continuous martingale, then there exists a unique continuous integrable adapted increasing process denoted by $\langle M, M\rangle_t$ such that $\{M_t^2 - \langle M, M\rangle_t\}$ is a continuous martingale vanishing at $t = 0$. The process $\{\langle M, M\rangle_t\}$ is called the *quadratic variation* of M. In particular, for any finite stopping time τ,

$$\mathbb{E}M_\tau^2 = \mathbb{E}\left\langle M, M\right\rangle_\tau .$$

If $N = \{N_t\}_{t\geq 0}$ is another real-valued square-integrable continuous martingale, we define

$$\langle M, N\rangle_t = \frac{1}{2}\left(\langle M+N, M+N\rangle_t - \langle M, M\rangle_t - \langle N, N\rangle_t\right),$$

and call $\{\langle M, N\rangle_t\}$ the *joint quadratic variation* of M and N. It is useful to know that $\{\langle M, N\rangle_t\}$ is the unique continuous integrable adapted process of finite variation such that $\{M_tN_t - \langle M, N\rangle_t\}$ is a continuous martingale vanishing at $t = 0$. In particular, for any finite stopping time τ,

$$\mathbb{E}M_\tau N_\tau = \mathbb{E}\left\langle M, N\right\rangle_\tau .$$

A right continuous adapted process $M = \{M_t\}_{t\geq 0}$ is called a *local martingale* if there exists a nondecreasing sequence $\tau_{k k\geq 1}$ of stopping times such that $\tau_k \uparrow \infty$ a.s. Furthermore, for each $k > 1$ the process $\{M_{\tau_k\wedge t} - M_0\}_{t\geq 0}$ is a martingale. It's worth noting that every martingale is also a local martingale, as indicated by Theorem 2.2.3, but the converse is not necessary true. If $M = \{M_t\}_{t\geq 0}$ and $N = \{N_t\}_{t\geq 0}$ are two real-valued continuous local martingales, their *joint quadratic variation* $\{\langle M, N\rangle\}_{t\geq 0}$ is the unique continuous adapted process of finite variation. This process has the property that $\{M_tN_t - \langle M, N\rangle_t\}_{t\geq 0}$ is a continuous local martingale vanishing at $t = 0$. When M and N are equal, $\{\langle M, M\rangle\}_{t\geq 0}$ is known as the *quadratic variation* of M.

The next outcome is the valuable strong law of large numbers.

Theorem 2.2.4 (Strong law of large numbers). *Let $M = \{M_t\}_{t\geq 0}$ be a real-valued continuous local martingale varnishing at $t = 0$. Then*

$$\textit{if } \lim_{t\to\infty} \langle M, M\rangle_t = \infty \textit{ a.s.} \quad \textit{then } \lim_{t\to\infty} \frac{M_t}{\langle M, M\rangle_t} = 0 \textit{ a.s.}$$

and also

$$\textit{if } \limsup_{t\to\infty} \frac{\langle M, M\rangle_t}{t} < \infty \textit{ a.s.} \quad \textit{then } \lim_{t\to\infty} \frac{M_t}{t} = 0 \textit{ a.s.}$$

More generally, if $A = \{A_t\}_{t\geq 0}$ is a continuous adapted increasing process such that

$$\lim_{t\to\infty} A_t = \infty \textit{ and } \int_0^\infty \frac{d\langle M, M\rangle_t}{(1 + A_t)^2} < \infty \textit{ a.s.}$$

then

$$\lim_{t\to\infty} \frac{M_t}{A_t} = 0 \textit{ a.s.}$$

A real-valued $\{\mathcal{F}_t\}$-adapted integrable process $\{M_t\}_{t\geq 0}$ is called a *supermartingale (with respect to $\{\mathcal{F}_t\}$)* if

$$\mathbb{E}(M_t|\mathcal{F}_s) \leq M_s,$$

and a *submartingale (with respect to $\{\mathcal{F}_t\}$)* if

$$\mathbb{E}(M_t|\mathcal{F}_s) \geq M_s \text{ a.s. for all } 0 \leq s < t < \infty.$$

Obviously, $\{M_t\}$ is submartingale if and only if $\{-M_t\}$ is a supermartingale. For a real-valued martingale $\{M_t\}$, both $\left\{M_t^+ := \max(M_t, 0)\right\}$ and $\left\{M_t^- := \max(0, -M_t)\right\}$ are submartingales. In the case of a supermartingale (or submartingale), the expected value $\mathbb{E}M_t$ is monotonically decreasing (or increasing). Furthermore, if $p \geq 1$ and $\{M_t\}$ is an $\mathbb{R}^d$-valued martingale with $M_t \in L^p(\Omega; \mathbb{R}^d)$, then $\{|M_t|^p\}$ is a non-negative submartingale. It's worth noting that Doob's stopping Theorem 2.2.3 can applies to supermartingales and submartingales as well.

2.3 Brownian motions

In 1828, the Scottish botanist Robert Brown made an observation that pollens suspended in liquid exhibited irregular motion. This motion was later explained as a result of random collisions with the molecules of the liquid. To mathematically describe this motion, it is natural to employ the concept of a stochastic process $B_t(\omega)$, which can be interpreted as the position of the pollen grain ω at a given time t. This stochastic process is known as Brownian motion and is one of the fundamental continuous-time stochastic processes. It finds valuable applications in several stochastic systems and lays the groundwork for stochastic analysis. Let us now provide the mathematical definition of Brownian motion.

Definition 2.3.1. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A (standard) one-dimensional Brownian motion is a real-valued continuous $\mathcal{F}_t$-adapted process $B_{tt\geq 0}$ with the following properties:*

(i). $B_0 = 0$ a.s.;

(ii). for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is normally distributed with mean zero and variance $t - s$;

(iii). for $0 \leq s < t < \infty$, the increment $B_t - B_s$ is independent of $\mathcal{F}_s$.

(iv). Almost surely, the sample path $t \to B_t(\omega)$ is continuous.

Let $\{B_t\}_{0\leq t\leq T}$ on $[0, T]$ for some $T > 0$. If $\{B_t\}_{t\geq 0}$ is Brownian motion and $0 \leq t_0 < t_1 < \cdots < t_k < \infty$, then the increments $B_{t_i} - B_{t_{i-1}}$, $1 \leq i \leq k$ are independent, and we say that the Brownian motion has independent increments. Moreover, the distribution of $B_{t_i} - B_{t_{i-1}}$ depends only on the difference $t_i - t_{i-1}$, and we say that the Brownian motion has stationary increments. The filtration $\{\mathcal{F}_t\}$ is a part of the definition of Brownian motion.

The following are important properties of Brownian motion.

(i). $\{-B_t\}$ is a Brownian motion with respect to the same filtration $\{\mathcal{F}_t\}$.

(ii). Let $c > 0$. Define $X_t = \frac{B_{ct}}{\sqrt{c}}$ for $t \geq 0$. The $\{X_t\}$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_{ct}\}$.

(iii). $\{B_t\}$ is a continuous square-integrable martingale and its quadratic variation $\langle B, B\rangle_t = t$ for all $t \geq 0$.

(iv). The strong law of large numbers states that

$$\lim_{t\to\infty} \frac{B_t}{t} = 0 \text{ a.s.}$$

(v). For almost every $\omega \in \Omega$, the Brownian sample path B.( ) is nowhere differentiable.

(vi). For almost every $\omega \in \Omega$, the Brownian sample path $B_\bullet(\omega)$ is locally Hölder continuous with exponent δ if $\delta \in (0, 1/2)$. However, for almost every $\omega \in \Omega$, the Brownian sample path $B_\bullet(\omega)$ is nowhere Hölder continuous with exponent $\delta > 1/2$.

2.4 Stochastic integrals

In this section, we introduce the mathematical framework for stochastic integral. Now, let us establish the definition of the stochastic integral

$$\int_0^t f(s)dB_s$$

with respect to an m-dimensional Brownian motion $\{B_t\}$ for a class of $d \times m$-matrix-valued stochastic processes $\{f(t)\}$. Due to the fact that, for almost all $\omega \in \Omega$, the sample path of Brownian motion $B_\bullet(\omega)$ exhibits infinite variation and is nowhere differentiable, the integral cannot be defined using the usual methods. The concept of this integral was first defined by K. Itˆo in 1949 and is now known as *Itô stochastic integral*.

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $B = \{B_t\}_{t\geq 0}$ be a one-dimensional Brownian motion defined on the probability space adapted to the filtration.

Definition 2.4.1. *Let $0 \leq a < b < \infty$. Denote by $\mathcal{M}^2([a, b]; \mathbb{R})$ the space of all real-valued measurable $\{\mathcal{F}_t\}-$adapted processes $f = \{f(t)\}_{a\leq t\leq b}$ such that*

$$\|f\|^2_{a,b} = \mathbb{E}\int_a^b |f(t)|^2 dt < \infty.$$

We identify f and $\bar{f}$ in $\mathcal{M}^2([a,b];\mathbb{R})$ if $\|f-\bar{f}\|_{a,b}^2 = 0$. In this case, we say that f and $\bar{f}$ are equivalent and write $f = \bar{f}$.

The stochastic processes $f \in \mathcal{M}^2([a,b];\mathbb{R})$ would help define the Itô stochastic integral. The approach is quite intuitive: first we define the integral $\int_a^b g(t)dB_t$ for a class of simple processes g. Then, we demonstrate that each $f \in \mathcal{M}^2([a,b];\mathbb{R})$ can be approximated by such simple processes g s and we define the limit of $\int_a^b g(t)dB_t$ as the integral of $\int_a^b f(t)dB_t$. Let us introduce the concept of simple processes.

Definition 2.4.2. *A real-valued stochastic process $g = \{g(t)\}_{a\le t\le b}$ is called a simple (or step) process if there exists a partition $a = t_0 < t_1 < \cdots < t_k = b$ of $[a,b]$, and bounded random variables ξ_i, $0 \le i \le k-1$ such that ξ_i is $\mathcal{F}_{t_i}$-measurable and*

$$g(t) = \xi_0 \mathbb{1}_{[t_0,t_1]}(t) + \sum_{i=1}^{k-1} \xi_i \mathbb{1}_{(t_i,t_{i+1}]}(t). \tag{2.1}$$

Denote by $\mathcal{M}_0([a,b];\mathbb{R})$ the family of all such processes.

Evidently, $\mathcal{M}_0([a,b];\mathbb{R}) \subseteq \mathcal{M}^2([a,b];\mathbb{R})$. Now, we will proceed to present the definition of Itô stochastic integral for these simple processes.

Definition 2.4.3. *For a simple process g with the form of (2.1) in $\mathcal{M}_0([a,b];\mathbb{R})$, define*

$$\int_a^b g(t)dB_t = \sum_{i=0}^{k-1} \xi_i \left(B_{t_{i+1}} - B_{t_i}\right) \tag{2.2}$$

and name it the stochastic integral of g with respect to the Brownian motion $\{B_t\}$ or the Itô integral.

It's evident that the stochastic integral $\int_a^b g(t)dB_t$ is $\mathcal{F}_b$-measurable. By extending the idea from Equation (2.2) into $\mathcal{M}^2([a,b];\mathbb{R})$, we arrive at the following definition.

Definition 2.4.4. *Let $f \in \mathcal{M}^2([a,b];\mathbb{R})$. The Itô integral of f with respect to $\{B_t\}$ is defined by*

$$\int_a^b f(t)dB_t = \lim_{n\to\infty} \int_a^b g_n(t)dB_t \text{ in } L^2(\Omega;\mathbb{R}),$$

where $\{g_n\}$ *is a sequence of simple processes such that*

$$\lim_{n\to\infty} \mathbb{E}\int_a^b |f(t)-g_n(t)|^2 dt = 0.$$

Let present the following useful properties of Itô integral.

Theorem 2.4.5. *Let* $f, g \in \mathcal{M}^2([a,b];\mathbb{R})$*, and let* α, β *be two real numbers. Then*

1. $\int_a^b f(t)dB_t$ *is* $\mathcal{F}_b$*-measurable;*
2. $\mathbb{E}\int_a^b f(t)dB_t = 0$*;*
3. $\mathbb{E}\left|\int_a^b f(t)dB_t\right|^2 = \mathbb{E}\int_a^b |f(t)|^2 dt$*;*
4. $\int_a^b [\alpha f(t) + \beta g(t)]dB_t = \alpha\int_a^b f(t)dB_t + \beta\int_a^b g(t)dB_t$*.*

The indefinite Itô integral is defined below.

Definition 2.4.6. *Let* $f \in \mathcal{M}^2([a,b];\mathbb{R})$*. Define*

$$I(t) = \int_0^t f(s)dB_s \text{ for } 0 \le t \le T,$$

where, by definition, $I(0) = \int_0^0 f(s)dB_s = 0$*. We call* $I(t)$ *the indefinite Itô integral of* f*.*

Obviously, $\{I(t)\}$ is $\{\mathcal{F}_t\}$-adapted. We now present the crucial martingale property of the indefinite Itô integral.

Theorem 2.4.7. *Let* $f \in \mathcal{M}^2([a,b];\mathbb{R})$*, then the indefinite Itô integral* $\{I(t)\}_{0\le t\le T}$ *is a square-integrable martingale with respect to the filtration* $\{\mathcal{F}_t\}$*. In particular,*

$$\mathbb{E}\left[\sup_{0\le t\le T}\left|\int_0^t f(s)dB_s\right|^2\right] \le 4\mathbb{E}\int_0^T |f(s)|^2 ds.$$

Theorem 2.4.8. *If* $f \in \mathcal{M}^2([a,b];\mathbb{R})$*, then the indefinite Itô integral* $\{I(t)\}_{0\le t\le T}$ *has a continuous version.*

Theorem 2.4.9. *Let* $f \in \mathcal{M}^2([a,b];\mathbb{R})$*. Then the indefinite Itô integral* $I = \{I(t)\}_{0\le t\le T}$ *is a square-integrable continuous martingale and its quadatic variation* *is given by*

$$\langle I, I \rangle_t = \int_0^t |f(s)|^2 \, ds, \quad 0 \le t \le T.$$

2.5 The Itô formula

We employ the Itô formula to evaluate the Itô integral, which allows us to simplify stochastic integrals into Lebesgue integrals for easy evaluation. In this section, we will begin by establishing the one-dimensional Itˆo formula and then generalise it to the multi-dimensional case.

Let $B = \{B_t\}_{t\ge 0}$ be a one-dimensional Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to the filtration $\{\mathcal{F}_t\}_{t\ge 0}$. Let $\mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ denote the family of all $\mathbb{R}^d$-valued measurable $\{\mathcal{F}_t\}$-adapted processes $f = \{f(t)\}_{t\ge 0}$ such that

$$\int_0^T |f(t)| dt < \infty \text{ a.s. for every } T > 0.$$

To define the Itô formula, we first need the Itô process. Let's proceed to define the Itô process.

Definition 2.5.1. *A d-dimensional Itô process is an $\mathbb{R}^d$-valued continuous adapted process $x(t) = (x_1(t), \cdots, x_d(t))^T$ on $t \ge 0$ of the form*

$$x(t) = x(0) + \int_0^t f(s) ds + \int_0^t g(s) dB(s),$$

where $f = (f_1, \cdots, f_d)^T \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g = (g_{ij})_{d\times m} \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d\times m})$. We shall say that $x(t)$ has stochastic differential $dx(t)$ on $t \ge 0$ given by

$$dx(t) = f(t)dt + g(t)dB(t).$$

Let $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$ denote the family of all real-valued functions $V(x, t)$ defined on $\mathbb{R}^d \times \mathbb{R}_+$ such that these functions exhibit continuous second-order differentiability in x and first-order differentiability in t. If $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$, we set

$$V_t = \frac{\partial V}{\partial t}, V_x = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d} \right)$$

and

$$V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{d\times d} = \begin{pmatrix} \frac{\partial^2 V}{\partial x_1 \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_1 \partial x_d} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 V}{\partial x_d \partial x_1} & \cdots & \frac{\partial^2 V}{\partial x_d \partial x_d} \end{pmatrix}.$$

Theorem 2.5.2 (The multi-dimensional Itô formula)**.** *Let $x(t)$ be a d-dimensional Itô process on $t \geq 0$ with the stochastic differential*

$$dx(t) = f(t)dt + g(t)dB(t),$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d\times m})$. Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t), t)$ is again an Itô process with the stochastic differential given by

$$\begin{aligned} dV(x(t), t) = & \left[V_t(x(t), t) + V_x(x(t), t)f(t) + \frac{1}{2}\text{trace}(g^T(t)V_{xx}(x(t), t)g(t)) \right] dt \\ & + V_x(x(t), t)g(t)dB(t) \ \ a.s. \end{aligned}$$

Let us now present formally a multiplication table:

$$\begin{aligned} & dtdt = 0, \quad dB_i dt = 0, \\ & dB_i dB_i = dt, \quad dB_i dB_j = 0 \quad \text{ if } i \neq j. \end{aligned}$$

Then, for example,

$$dx_i(t)dx_j(t) = \sum_{k=1}^{m} g_{ik}(t)g_{jk}(t)dt.$$

Moreover, the Itô formula can be written as

$$dV(x(t), t) = V_t(x(t), t)dt + V_x(x(t), t)dx(t) + \frac{1}{2}dx^T(t)V_{xx}(x(t), t)dx(t).$$

2.6 Stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Throughout this section, we let $B(t) = (B_1(t), \cdots, B_m(t))^T$, $t \geq 0$ be an m-dimensional Brownian motion defined on the space. Let $0 \leq t_0 < T < \infty$. Let x_0 be an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^d$-valued random variable such that $\mathbb{E}|x_0|^2 < \infty$. Let $f : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d\times m}$ be both Borel measurable. Consider the d-dimensional stochastic differential equation of Itô type

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t) \text{ on } t_0 \leq t \leq T, \tag{2.3}$$

with initial value $x(t_0) = x_0$. By the definition of stochastic differential, this equation is equivalent to the following integral equation:

$$x(t) = x_0 + \int_{t_0}^{t} f(x(s), s)ds + \int_{t_0}^{t} g(x(s), s)dB(s) \text{ on } t_0 \leq t \leq T. \tag{2.4}$$

Let us now provide the definition of the solution.

Definition 2.6.1. *An $\mathbb{R}^d$-valued stochastic process $\{x(t)\}_{t_0\leq t\leq T}$ is called a solution of equation (2.3) if it has the following properties:*

(i). $\{x(t)\}$ is continuous and $\mathcal{F}_t$-adapted;

(ii). $\{f(x(t), t)\} \in \mathcal{L}^1([t_0, T]; \mathbb{R}^d)$ and $\{g(x(t), t)\} \in \mathcal{L}^2([t_0, T]; \mathbb{R}^{d\times m})$;

(iii). equation (2.4) holds for every $t \in [t_0, T]$ with probability 1.

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$\mathbb{P}\{x(t) = \bar{x}(t) \text{ for all } t_0 \leq t \leq T\} = 1.$$

The following theorem provides conditions to guarantee existence and uniqueness of the solution to SDE (2.3)

Theorem 2.6.2. *Assume that there exist two positive constants $\bar{K}$ and K such that* *(i). (Lipschitz condition) for all $x, \bar{x} \in \mathbb{R}^d$ and $t \in [t_0, T]$*

$$|f(x,t) - f(\bar{x},t)|^2 \bigvee |g(x,t) - g(\bar{x},t)|^2 \leq \bar{K}|x - \bar{x}|^2; \tag{2.5}$$

(ii). (Linear growth condition) for all $(x,t) \in \mathbb{R}^d \times [t_0, T]$

$$|f(x,t)|^2 \bigvee |g(x,t)|^2 \leq K(1 + |x|^2). \tag{2.6}$$

Then there exists a unique solution $x(t)$ to equation (2.3) *and the solution belongs $\mathcal{M}^2([t_0,T];\mathbb{R}^d)$.*

The Lipschitz condition described in equation (2.5) implies that the coefficients $f(x,t)$ and $g(x,t)$ do not change rapidly than a linear function of x when x changes. This implies the continuity of $f(x,t)$ and $g(x,t)$ in terms of x for all $t \in [t_0, T]$. As a result, functions that are discontinuous with respect to x are excluded as the coefficients. This shows that the Lipschitz condition is too restrictive. The following theorem is the generalisation of Theorem 2.6.2 by replacing the (uniform) Lipschitz condition with a local Lipschitz condition.

Theorem 2.6.3. *Assume that the linear growth condition* (2.6) *holds. However, instead of the Lipschitz condition* (2.5)*, we apply the following local Lipschitz condition: For every integer $n \geq 1$, there exists a positive constant K_n such that, for all $t \in [t_0, T]$ and all $x, \bar{x} \in \mathbb{R}^d$ with $|x| \vee |\bar{x}| \leq n$*

$$|f(x,t) - f(\bar{x},t)|^2 \bigvee |g(x,t) - g(\bar{x},t)|^2 \leq K_n|x - \bar{x}|^2. \tag{2.7}$$

Then there exists a unique solution $x(t)$ to equation (2.3) *and the solution belongs $\mathcal{M}^2([t_0,T];\mathbb{R}^d)$.*

The local Lipschitz condition broadens the range of allowable functions significantly. Nonetheless, the linear growth condition still excludes some important functions. The following result serves to enhance the situation.

Theorem 2.6.4. *Assume that the local Lipschitz condition* (2.7) *holds. However, insteasd of the linear growth condition* (2.6)*, we replace the following monotone condition: There exists a positive constant K such that for all $(x,t) \in \mathbb{R}^d \times [t_0, T]$*

$$x^T f(x,t) + \frac{1}{2}|g(x,t)|^2 \leq K(1 + |x|^2). \tag{2.8}$$

Then there exists a unique solution $x(t)$ to equation (2.3) *in* $\mathcal{M}^2([t_0,T];\mathbb{R}^d)$.

2.7 Stochastic differential delay equations

In this section, we will start with the stochastic functional differential equations and consider the stochastic differential delay equations as a special case. We set the notation same as the previous section (SDEs) as follow: $(\Omega,\mathcal{F},\mathbb{P})$ is a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. $\{B(t)\}_{t>0}$ is an m-dimensional Brownian motion defined on the space. Now, let $\tau > 0$ and denote $0 \leq t_0 < T < \infty$. Let $f : \mathbb{R}^d \times \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \times [t_0,T] \to \mathbb{R}^{d\times m}$ be both Borel measurable. Consider the d-dimensional stochastic functional differential equation

$$dx(t) = f(x_t,t)dt + g(x_t,t)dB(t) \text{ on } t_0 \leq t \leq T, \tag{2.9}$$

where $x_t = \{x(t+\theta) : -\tau \leq \theta \leq 0\} \in C([-\tau,0];\mathbb{R}^d)$ with the initial data

$$x_{t_0} = \xi = \{\xi(\theta) : -\tau \leq \theta \leq 0\} \in C([-\tau,0];\mathbb{R}^d). \tag{2.10}$$

Now, we provide the definition of the solution.

Definition 2.7.1. *An $\mathbb{R}^d$-valued stochastic process $x(t)_{t_0\leq t\leq T}$ is called a solution to equation* (2.9) *with initial data* (2.10) *if it has the following properties:*

(i). it is continuous and $\{x_t\}_{t_0\leq t\leq T}$ is $\mathcal{F}$-measurable;

(ii). $\{f(x_t,t)\} \in \mathcal{L}^1([t_0,T];\mathbb{R}^d)$ *and* $\{g(x_t,t)\} \in \mathcal{L}^2([t_0,T];\mathbb{R}^{d\times m})$*;*

(iii). $x_{t_0} = \xi$ *and for every* $t_0 \leq t \leq T$,

$$x(t) = \xi(0) + \int_{t_0}^t f(x_s,s)ds + \int_{t_0}^t g(x_s,s)dB(s) \qquad a.s.$$

A solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$\mathbb{P}\{x(t) = \bar{x}(t) \textit{ for all } t_0 - \tau \leq t \leq T\} = 1.$$

Similarly with the SDEs, the existance and uniqueness of (2.9) can be obtained if its coefficients satisfy the local Lipschitz condition and the linear growth condition as the following theorem.

Theorem 2.7.2. *Assume that for every integer $n \geq 1$, there exists a positive constant K_n such that, for all $t \in [t_0, T]$ and all $\phi, \varphi \in C([-\tau, 0]; \mathbb{R}^d)$ with $\|\phi\| \vee \|\varphi\| \leq n$*

$$|f(\phi,t) - f(\varphi,t)|^2 \bigvee |g(\phi,t) - g(\varphi,t)|^2 \leq K_n \|\phi - \varphi\|^2;$$

and there exists additionally a $K > 0$ such that for all $(\phi, t) \in C([-\tau, 0]; \mathbb{R}^d) \times [t_0, T]$,

$$|f(\phi,t)|^2 \bigvee |g(\phi,t)|^2 \leq K \left(1 + \|\phi\|^2\right).$$

Then there exists a unique solution to equation (2.9) *and the solution belongs to $\mathcal{M}^2([t_0 - \tau, \infty]; \mathbb{R}^d)$.*

Now, we define the an important special case of a stochastic functional differential equations is the SDDEs. And equations are defined as follow:

$$dx(t) = f(x(t), x(t-\tau), t)dt + g(x(t), x(t-\tau), t)dB(t), \tag{2.11}$$

on $t \in [t_0, T]$ with initial data (2.10), where $f : \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T] \to \mathbb{R}^{d \times m}$. As you can see, if we define $f_1(\phi, t) = f(x(t), x(t-\tau), t)$ and $g_1(\phi, t) = g(x(t), x(t-\tau), t)$, the equation (2.11) can be transfered to equation (2.9), so equation (2.11) can be applied the existance and uniqueness with this equation. For example, let f and g satisfy the local Lipschitz condition and the linear growth condition. That means, for every integer $n \geq 1$, there exists a positive constant K_n such that, for all $t \in [t_0, T]$ and all $x, \bar{x}, y, \bar{y} \in \mathbb{R}^d$ with $|x| \vee |\bar{x}| \vee y \vee \bar{y} \leq n$

$$|f(x,y,t) - f(\bar{x},\bar{y},t)|^2 \bigvee |g(x,y,t) - g(\bar{x},\bar{y},t)|^2 \leq K_n \left(|x - \bar{x}|^2 + |y - \bar{y}|^2\right);$$

and there exists additionally a $K > 0$ such that for all $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times [t_0, T]$,

$$|f(x,y,t)|^2 \bigvee |g(x,y,t)|^2 \leq K \left(1 + |x|^2 + |y|^2\right).$$

Then there exists a unique solution to the delay equation (2.11).

In realisticity, the time delay can be depended up on a time t. We, hence, let $\delta : [t_0, T] \to [-\tau, t_0]$ be a Borel measurable function . Consider the stochastic differential delay equation.

$$dx(t) = f(x(t), x(\delta(t)), t)dt + g(x(t), x(\delta(t)), t)dB(t) \tag{2.12}$$

on $t \in [t_0, T]$ with intital data (2.10). By setting the $f_1(\phi, t) = f(x(t), x(\delta(t)), t)$ and $g_1(\phi, t) = g(x(t), x(\delta(t)), t)$, we obtain the existance and uniqueness property of equation (2.12).

2.8 Mathematical inequalities

Let us also present some useful inequalities which are used frequently in this thesis. Let us start with the simplest inequality

$$2ab \leq a^2 + b^2, \text{ for all } a, b \in \mathbb{R}.$$

From this follows

$$2ab \leq \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \text{ for all } a, b \in \mathbb{R} \text{ and all } \varepsilon > 0.$$

Let us also proceed to the Young inequality

$$|a|^{\beta} |b|^{(1-\beta)} \leq \beta |a| + (1-\beta)|b|, \text{ for all } a, b \in \mathbb{R} \text{ and all } \beta \in [0, 1].$$

Theorem 2.8.1 (Jensen's inequality)**.** *If $\varphi : \Omega \to \mathbb{R}$ is a convex function while $\xi : \mathbb{R} \to \mathbb{R}$ is a random varaible on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that $\mathbb{E}|\xi| < \infty$, then*

$$\varphi(\mathbb{E}\xi) \leq \mathbb{E}(\varphi(\xi)).$$

Theorem 2.8.2 (Doob's martingale inequalities)**.** *Let $\{M_t\}_{t \geq 0}$ be an $\mathbb{R}^d$-valued martingale. Let $[a, b]$ be a bounded interval in $\mathbb{R}_+$.*

(i). If $p \geq 1$ and $M_t \in L^p(\Omega; \mathbb{R}^d)$, then

$$\mathbb{P}\left\{\omega : \sup_{a \leq t \leq b} |M_t(\omega)| \geq c\right\} \leq \frac{\mathbb{E}|M_b|^p}{c^p}$$

holds for all $c > 0$.

(ii). If $p > 1$ and $M_t \in L^p(\Omega; \mathbb{R}^d)$, then

$$\mathbb{E}\left(\sup_{a \leq t \leq b} |M_t|^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_b|^p.$$

Theorem 2.8.3. *Let $p \geq 2$. Let $g \in \mathcal{M}^2([0,T]; \mathbb{R}^{d \times m})$ such that*

$$\mathbb{E}\int_0^T |g(s)|^p ds < \infty.$$

Then

$$\mathbb{E}\left|\int_0^T g(s)dB(s)\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} T^{\frac{p-2}{2}} \mathbb{E}\int_0^T |g(s)|^p ds.$$

In particular, for $p = 2$, there is equality.

Theorem 2.8.4. *Let $p \geq 2$. Let $g \in \mathcal{M}^2([0,T]; \mathbb{R}^{d \times m})$ such that*

$$\mathbb{E}\int_0^T |g(s)|^p ds < \infty.$$

Then

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \int_0^t |g(s)|^p ds\right).$$

Theorem 2.8.5 (Burkholder-Davis-Gundy inequality)**.** *Let $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d \times m})$. Define, for $t \geq 0$,*

$$x(t) = \int_0^t g(s)dB(s) \text{ and } A(t) = \int_0^t |g(s)|^2 ds.$$

Then for every $p > 0$, there exist universal positive constants c_p, C_p (depending on *only p), such that*

$$c_p\mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}\left(\sup_{0\leq s\leq t}|x(s)|^p\right) \leq C_p\mathbb{E}|A(t)|^{\frac{p}{2}}$$

for all $t \geq 0$. In particular, one may take

$$\begin{array}{lll} c_p = \left(\frac{p}{2}\right)^p, & C_p = \left(\frac{32}{p}\right)^{\frac{p}{2}} & \text{if } \ 0 < p < 2; \\ c_p = 1, & C_p = 4 & \text{if } \ p = 2; \\ c_p = (2p)^{-\frac{p}{2}}, & C_p = \left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} & \text{if } \ p > 2. \end{array}$$

Theorem 2.8.6 (Gronwall's inequality)**.** *Let $T > 0$ and $c \geq 0$. Let $u(\bullet)$ be a Borel measurable bounded non-negative function on $[0, T]$, and let $v(\bullet)$ be a non-negative integrable function on $[0, T]$. If*

$$u(t) \leq c + \int_0^t v(s)u(s)ds \quad \text{for all } \ 0 \leq t \leq T,$$

then

$$u(t) \leq c\exp\left(\int_0^t v(s)ds\right) \quad \text{for all } \ 0 \leq t \leq T.$$

Chapter 3

Truncated Euler-Maruyama for Stochastic Differential Equations with non-linear coefficients

3.1 Introduction

In 2015, Mao introduced a modified method for approximating SDEs called the truncated Euler-Maruyama approximate method, see [21]. With this method, he proved that the truncated EM solution is L^q convergent to the exact solution for $q \geq 2$ under the local Lipschitz and Khasminskii-type conditions. The year after, he also determined the rate of convergence at a finite time $T > 0$, see [22]. To calculate the numerical solution with the rate of Mao (2016), unfortunately, the step size is sometime required very small, or we can say it is inapplicable. Hu, Li and Mao (2018), [12], showed the step size can be flexible in $(0, 1]$ and the convergence rate at the time T is slightly similar.

In application, the convergence rate at a time T is sufficient for applications that need to approximate the European put or call option value. We, nevertheless, need to approximate the path-dependent quantities that have to take all parts of life of the option such as a barrier option, and a bond, see [9]. Mao (2016), [22], established the convergence rate of the truncated EM method over a finite time interval, but the result also got the problem that the step size is required very small. Moreover, to prove the rate of convergence over a finite time interval, he assumed the global Lipschitz on the diffusion coefficient. Even though there is an improvement for the truncated EM method, like [12], they just solved the step size problem. There are, in general, many models in finance that the diffusion coefficient does not satisfy the global Lipschitz such as the Ait-Sahalia model, see [5]. In this research, we will relax the diffusion coefficient to not satisfy the global Lipschitz for the strong convergence of the truncated EM method over a finite time interval.

In this chapter, we consider a d-dimensional non-linear SDE,

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \quad t \geq 0, \tag{3.1}$$

with the initial value $x(0) = x_0 \in \mathbb{R}^d$, where $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are Borel measurable. We assume two common assumptions for equation (3.1) which are the local Lipschitz condition and the Khasminskii-type condition as follow.

Assumption 3.1.1 (Local Lipschitz condition)**.** *For every integer $n \geq 1$, there exists a positive constant K_n such that, for all $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \leq n$,*

$$|f(x) - f(y)|^2 \vee |g(x) - g(y)|^2 \leq K_n |x - y|^2 . \tag{3.2}$$

Assumption 3.1.2 (Khasminskii-type condition)**.** *For any $p > 2$ there is $K_p > 0$ such that for all $x \in \mathbb{R}^d$*

$$x^T f(x) + \frac{p-1}{2} |g(x)|^2 \leq K_p(1 + |x|^2). \tag{3.3}$$

Lemma 3.1.3. *Under Assumptions 3.1.1 and 3.1.2, the SDE* (3.1) *has a unique global solution $x(t)$. Moreover,*

$$\sup_{0\leq t\leq T} \mathbb{E}\, |x(t)|^p < \infty, \quad \forall T > 0. \tag{3.4}$$

To introduce the truncated EM method which is defined in [21], let $\mathbb{R}^+$ be a set of positive real numbers and $\mu : [1, \infty) \to \mathbb{R}^+$ be a strictly increasing continuous function such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\leq u} (|f(x)| \vee |g(x)|) \leq \mu(u), \quad \forall u \geq 1. \tag{3.5}$$

Denote the μ^{-1} is an inverse function of μ which is a strictly increasing continuous function from $[\mu(1), \infty)$. We also choose a constant $\hat{h} \geq 1 \vee \mu(1)$ and a strictly increasing function $h : (0, 1] \to [\mu(1), \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0, 1]. \tag{3.6}$$

For a given step size $\Delta \in (0, 1]$, define the truncated mapping $\pi_\Delta : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\pi_\Delta(x) = \left(|x| \wedge \mu^{-1}(h(\Delta))\right) \frac{x}{|x|}, \tag{3.7}$$

where we set $x/|x| = 0$ if $x = 0$. That means, π_Δ truncates x to $\mu^{-1}(h(\Delta))(x/|x|)$ if $|x| > \mu^{-1}(h(\Delta))$. Define the truncated functions

$$f_\Delta(x) = f(\pi_\Delta(x)) \quad \text{and} \quad g_\Delta(x) = g(\pi_\Delta(x)), \tag{3.8}$$

for all $x \in \mathbb{R}^d$. Hence, $|f_\Delta(x)| \vee |g_\Delta(x)| \leq \mu(\mu^{-1}(h(\Delta))) = h(\Delta)$ for all $x \in \mathbb{R}^d$. The discrete time truncated EM solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ are formed by setting $X_\Delta(0) = x_0$ and computing

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta(X_\Delta(t_k))\Delta + g_\Delta(X_\Delta(t_k))\Delta B_k$$

for $k = 0, 1, \ldots,$ where $\Delta B_k = B_{t_{k+1}} - B_{t_k}$. There are two versions of the continuous-time truncated EM solution. The first one is defined by

$$\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \quad \text{for } t \geq 0.$$

This is a simple step process so its sample paths are not continuous. We will refer to it as the continuous-time step-process truncated EM solution. The other one is defined by

$$x_\Delta(t) = \int_0^t f_\Delta(\bar{x}_\Delta(s))ds + \int_0^t g_\Delta(\bar{x}_\Delta(s))dB(s) \tag{3.9}$$

for $t \geq 0$. We will refer to it as the continuous-time continuous-sample truncated EM solution. Notice that $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$ for all $k \geq 0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = f_\Delta(\bar{x}_\Delta(t))dt + g_\Delta(\bar{x}_\Delta(t))dB(t)$$

The following lemma confirms that the truncated functions satisfy Assumption 3.1.2, see the proof in [12, 21]:

Lemma 3.1.4. *Assume Assumption 3.1.2 hold. Then, for* $\Delta \in (0,1]$*, for any* $p > 2$ *there is* $\widehat{K_p} > 0$ *such that for all* $x \in \mathbb{R}^d$

$$x^T f_\Delta(x) + \frac{p-1}{2} |g_\Delta(x)|^2 \leq \widehat{K_p} \left(1 + |x|^2\right).$$

Then, we can recall the useful lemmas in [21] as the following:

Lemma 3.1.5. *For any* $\Delta \in (0,1]$ *and* $p > 0$*. Then, for all* $t > 0$*,*

$$\mathbb{E} |x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq C_p \Delta^{p/2} h(\Delta)^p,$$

where $C_p = \begin{cases} 2^{p-1}\left(1 + \left(\dfrac{p(p-1)}{2}\right)^{\frac{p}{2}}\right) & \text{if } p \geq 2, \\ 2^p & \text{if } p \in (0,2). \end{cases}$

Lemma 3.1.6. *Let Assumptions 3.1.1 and 3.1.2 hold. Then,*

$$\sup_{0<\Delta\leq 1} \left(\sup_{0\leq t\leq T} \mathbb{E} |x_\Delta(t)|^p \right) \leq C_p,$$

where $C_p = \left[\mathbb{E} |x_0|^p + 2^p T \widehat{K_p} + 2^{\frac{p}{2}} \left(1 + \left(\dfrac{p(p-2)}{8}\right)^{\frac{p}{4}}\right) T\right] e^{2^{p+1}\widehat{K_p}}.$

Lemma 3.1.7. *Let* $p \geq 2$*,* $\Delta \in (0,1]$ *and* $\varepsilon \in \left(0, \frac{1}{4}\right]$ *be given. For a sufficiently large interger* n *for which*

$$\left(\frac{2n}{2n-1}\right)^p (T+1)^{\frac{p}{2n}} \leq 2 \text{ and } \frac{1}{n} < \varepsilon, \tag{3.10}$$

we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |x_\Delta(t) - \bar{x}_\Delta(t)|^p\right) \leq C_p \Delta^{\frac{p}{2}(1-\varepsilon)} \left(h(\Delta)\right)^p, \tag{3.11}$$

where $C_p = 2^{p+1} n^{\frac{p}{2}}$.

Recall from [12], they assumed the following conditions to their main result, which is Theorem 3.1.10. Moreover, this result plays a significant roll to prove our main theorem.

Assumption 3.1.8. *Assume that there is a pair of constants $q > 2$ and $K_1 > 0$ such that for all $x \in \mathbb{R}^d$*

$$(x-y)^T(f(x)-f(y)) + \frac{q-1}{2}|g(x)-g(y)|^2 \le K_1(1+|x|^2).$$

Assumption 3.1.9. *Assume that there is a pair of constants ρ and $K_2 > 0$ such that for all $x, y \in \mathbb{R}^d$*

$$|f(x)-f(y)|^2 \vee |g(x)-g(y)|^2 \le K_2(1+|x|^\rho+|y|^\rho)|x-y|^2.$$

Theorem 3.1.10. *Let Assumptions 3.1.1, 3.1.2, 3.1.8 and 3.1.9 hold and assume that $2p > (2+\rho)q$. Then, for any $\bar{q} \in [2, q)$ and $\Delta \in (0, 1]$,*

$$\mathbb{E}|x(t)-x_\Delta(t)|^{\bar{q}} \le C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\rho)\bar{q}}{2}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right),$$

and

$$\mathbb{E}|x(t)-\bar{x}_\Delta(t)|^{\bar{q}} \le C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\rho)\bar{q}}{2}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right),$$

for all $0 \le t \le T$ and C_p is a constant independence with Δ.

3.2 Convergence over a finite time interval

Recall, in the introduction part, the convergence over a time interval is very important when it is applied with a barrier option or a bond. To point out our main result, according to [22], Mao proved the convergence over a finite time interval by assuming condition as follow.

Assumption 3.2.1. *Assume there exists a pair of constants $H_0, \gamma_0 > 0$ such that for any $x, y \in \mathbb{R}^d$,*

$$(x-y)^T(f(x)-f(y)) \le H_0|x-y|^2 \tag{3.12}$$

$$|f(x)-f(y)|^2 \le H_0(1+|x|^{\gamma_0}+|y|^{\gamma_0})|x-y|^2 \tag{3.13}$$

$$|g(x)-g(y)|^2 \le H_0|x-y|^2. \tag{3.14}$$

The inequality (3.14) implies that the diffusion coefficient satisfies the global

Lipschitz condition. There are, however, a lot of SDEs with non global Lipschitz diffusion coefficient. Our main result will prove the convergence over a finite time interval for non global Lipschitz SDEs by modifying some conditions from 3.2.1.

Assumption 3.2.2. *Assume there exist constants $H, \gamma, \beta > 0$ and $q > 2$ such that for all $x, y \in \mathbb{R}^d$*

$$(x-y)^T(f(x)-f(y)) + \frac{q-1}{2}|g(x)-g(y)|^2 \le H|x-y|^2 \tag{3.15}$$

$$|f(x)-f(y)|^2 \le H(1+|x|^\gamma+|y|^\gamma)|x-y|^2 \tag{3.16}$$

$$|g(x)-g(y)|^2 \le H(1+|x|^\beta+|y|^\beta)|x-y|^2 . \tag{3.17}$$

On inequality (3.17), we allow the diffusion coefficient can be polynomials such as $x^{3/2}$ which is not global Lipschitz, in detail see in section Examples with Simulations. So, we state the main result in the next theorem. Throughout this thesis, we will further use C and C_p to stand for generic positive real constants *independent of the step size* Δ, this means in each step the constants may be different. Moreover, we also assume C_p is dependent on p whlie C is not.

Theorem 3.2.3. *Let Assumptions 3.1.2 and 3.2.2 hold. For each $\bar{q} \in [2, q)$, choose a positive number p such that $p > \frac{(2+(\gamma\vee\beta))q}{2} \vee \frac{(q+\bar{q})\beta}{q-\bar{q}}$. Then, for any $\Delta \in (0, 1]$ and $\varepsilon \in (0, \frac{1}{4}]$,*

$$\mathbb{E}\left(\sup_{0\le u\le T}|x(u)-x_\Delta(u)|^{\bar{q}}\right) \le C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{4p\bar{q}-(2+(\gamma\vee\beta))(q+\bar{q})\bar{q}}{2(q+\bar{q})}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right), \tag{3.18}$$

$$\mathbb{E}\left(\sup_{0\le u\le T}|x(u)-\bar{x}_\Delta(u)|^{\bar{q}}\right) \le C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{4p\bar{q}-(2+(\gamma\vee\beta))(q+\bar{q})\bar{q}}{2(q+\bar{q})}} + \Delta^{\bar{q}(1-\varepsilon)/2}h(\Delta)^{\bar{q}}\right). \tag{3.19}$$

Proof. Fix $\bar{q} \in [2, q)$, and let $p > \frac{(2+(\gamma\vee\beta))\bar{q}}{2} \vee \frac{(q+\bar{q})(\beta+1)}{q-\bar{q}}$ and $\Delta \in (0, 1]$. For any $n \ge |x_0|$, define $\theta_n = \inf\{t \ge 0 : |x(t)| \vee |x_\Delta(t)| \ge n\}$. Let $e_\Delta(t) = x(t) - x_\Delta(t)$

for $t \geq 0$. By Itô's formula,

$$
\begin{aligned}
& d\left|e_\Delta(s)\right|^{\bar q} \\
& \leq \bar q \left|e_\Delta(s)\right|^{\bar q-2}\left[e_\Delta^T(s)(f(x(s)) - f_\Delta(\bar x_\Delta(s)) + \frac{\bar q-1}{2}\left|g(x(s)) - g_\Delta(\bar x_\Delta(s))\right|^2\right]ds \\
& \quad + \bar q \left|e_\Delta(s)\right|^{\bar q-2} e_\Delta^T(s)(g(x(s)) - g_\Delta(\bar x_\Delta(s)))dB(s)
\end{aligned}
$$

Then, for $0 \leq t \leq T$,

$$
\begin{aligned}
& \mathbb{E}\left(\sup_{0\leq u\leq t\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar q}\right) \\
& \leq \mathbb{E}\Bigg(\sup_{0\leq u\leq t\wedge\theta_n}\int_0^u \bar q\left|e_\Delta(s)\right|^{\bar q-2}\Big[e_\Delta^T(s)(f(x(s)) - f_\Delta(\bar x_\Delta(s)) \\
& \qquad + \frac{\bar q-1}{2}\left|g(x(s)) - g_\Delta(\bar x_\Delta(s))\right|^2\Big]ds\Bigg) \\
& \quad + \mathbb{E}\left(\sup_{0\leq u\leq t\wedge\theta_n}\int_0^u \bar q\left|e_\Delta(s)\right|^{\bar q-2} e_\Delta^T(s)(g(x(s)) - g_\Delta(\bar x_\Delta(s)))dB(s)\right) \\
& \leq \mathbb{E}\Bigg(\sup_{0\leq u\leq t\wedge\theta_n}\int_0^u \bar q\left|e_\Delta(s)\right|^{\bar q-2}\Big[e_\Delta^T(s)(f(x(s)) - f(x_\Delta(s))) \\
& \qquad + \frac{q-1}{2}\left|g(x(s)) - g(x_\Delta(s))\right|^2\Big]ds\Bigg) \\
& \quad + \mathbb{E}\left(\sup_{0\leq u\leq t\wedge\theta_n}\int_0^u \bar q\left|e_\Delta(s)\right|^{\bar q-2}\left[e_\Delta^T(s)(f(x_\Delta(s)) - f_\Delta(\bar x_\Delta(s)))\right]ds\right) \\
& \quad + \mathbb{E}\left(\int_0^{t\wedge\theta_n}\left(\frac{\bar q(\bar q-1)(q-1)}{2(q-\bar q)}\right)\left|e_\Delta(s)\right|^{\bar q-2}\left|g(x_\Delta(s)) - g_\Delta(\bar x_\Delta(s))\right|^2 ds\right) \\
& \quad + \mathbb{E}\left(\left[\sup_{0\leq u\leq t\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar q}\right]^{\frac12}\left[32\bar q^2\int_0^{t\wedge\theta_n}\left|e_\Delta(s)\right|^{\bar q-2}\left|g(x(s)) - g_\Delta(\bar x_\Delta(s))\right|^2 ds\right]^{\frac12}\right)
\end{aligned}
$$

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}|e_\Delta(u)|^{\bar q}\right)\\
&\le \mathbb{E}\Bigg(\sup_{0\le u\le t\wedge\theta_n}\int_0^u \bar q\,|e_\Delta(s)|^{\bar q-2}\Big[e_\Delta^T(s)(f(x(s))-f(x_\Delta(s)))\\
&\qquad\qquad + \frac{q-1}{2}\,|g(x(s))-g(x_\Delta(s))|^2\Big]ds\Bigg)\\
&\quad + \mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}\int_0^u \bar q\,|e_\Delta(s)|^{\bar q-2}\left[e_\Delta^T(s)(f(x_\Delta(s))-f_\Delta(\bar x_\Delta(s)))\right]ds\right)\\
&\quad + \mathbb{E}\left(\int_0^{t\wedge\theta_n}\left(\frac{\bar q(\bar q-1)(q-1)}{2(q-\bar q)}\right)|e_\Delta(s)|^{\bar q-2}\,|g(x_\Delta(s))-g_\Delta(\bar x_\Delta(s))|^2\,ds\right)\\
&\quad + \frac{1}{2}\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}|e_\Delta(u)|^{\bar q}\right)\\
&\quad + \mathbb{E}\left(\left[32\bar q^2\int_0^{t\wedge\theta_n}|e_\Delta(s)|^{\bar q-2}\,|g(x_\Delta(s))-g_\Delta(\bar x_\Delta(s))|^2\,ds\right]\right)\\
&\quad + \mathbb{E}\left(\left[32\bar q^2\int_0^{t\wedge\theta_n}|e_\Delta(s)|^{\bar q-2}\,|g(x(s))-g(x_\Delta(s))|^2\,ds\right]\right)\\
&= \frac{1}{2}\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}|e_\Delta(u)|^{\bar q}\right) + J_1+J_2+J_3+J_4,
\end{aligned}
$$

where

$$
\begin{aligned}
J_1 &= \mathbb{E}\Bigg(\sup_{0\le u\le t\wedge\theta_n}\int_0^u \bar q\,|e_\Delta(s)|^{\bar q-2}\Big[e_\Delta^T(s)(f(x(s))-f(x_\Delta(s)))\\
&\qquad + \frac{q-1}{2}\,|g(x(s))-g(x_\Delta(s))|^2\Big]ds\Bigg),\\
J_2 &= \mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}\int_0^u \bar q\,|e_\Delta(s)|^{\bar q-2}\left[e_\Delta^T(s)(f(x_\Delta(s))-f_\Delta(\bar x_\Delta(s)))\right]ds\right),\\
J_3 &= \mathbb{E}\left(\int_0^{t\wedge\theta_n}\left(\frac{\bar q(\bar q-1)(q-1)}{2(q-\bar q)}+32\bar q^2\right)|e_\Delta(s)|^{\bar q-2}\,|g(x_\Delta(s))-g_\Delta(\bar x_\Delta(s))|^2\,ds\right)\\
J_4 &= \mathbb{E}\left(\left[32\bar q^2\int_0^{t\wedge\theta_n}|e_\Delta(s)|^{\bar q-2}\,|g(x(s))-g(x_\Delta(s))|^2\,ds\right]\right).
\end{aligned}
$$

By Assumption 3.2.2,

$$
\begin{aligned}
J_1 &= \mathbb{E}\Bigg(\sup_{0 \le u \le t \wedge \theta_n} \int_0^u \bar{q}\, |e_\Delta(s)|^{\bar{q}-2} \Big[e_\Delta^T(s)(f(x(s)) - f(x_\Delta(s))) \\
&\quad + \frac{q-1}{2} |g(x(s)) - g(x_\Delta(s))|^2 \Big] ds \Bigg), \\
&\le \bar{q} H \mathbb{E}\left(\int_0^{t\wedge\theta_n} |e_\Delta(s)|^{\bar{q}}\, ds \right) \\
&\le \bar{q} H \int_0^t \mathbb{E}\left(\sup_{0\le u \le s\wedge\theta_n} |e_\Delta(u)|^{\bar{q}}\, ds \right), \\
J_2 &= \mathbb{E}\left(\sup_{0 \le u \le t\wedge\theta_n} \int_0^u \bar{q}\, |e_\Delta(s)|^{\bar{q}-2} \left[e_\Delta^T(s)(f(x_\Delta(s)) - f_\Delta(\bar{x}_\Delta(s))) \right] ds \right) \\
&\le \mathbb{E}\left(\sup_{0 \le u \le t\wedge\theta_n} \int_0^u \bar{q} H\, |e_\Delta(s)|^{\bar{q}-1} |f(x_\Delta(s)) - f_\Delta(\bar{x}_\Delta(s))|\, ds \right) \\
&\le \mathbb{E}\left(\sup_{0 \le u \le t\wedge\theta_n} \int_0^u \bar{q} H \left(|e_\Delta(s)|^{\bar{q}} \right)^{1-1/\bar{q}} \left(|f(x_\Delta(s)) - f_\Delta(\bar{x}_\Delta(s))|^{\bar{q}} \right)^{1/\bar{q}} ds \right) \\
&\le \mathbb{E}\left(\sup_{0 \le u \le t\wedge\theta_n} \int_0^u H \left((\bar{q}-1) |e_\Delta(s)|^{\bar{q}} + |f(x_\Delta(s)) - f_\Delta(\bar{x}_\Delta(s))|^{\bar{q}} \right) ds \right) \\
&\le \mathbb{E}\Bigg(\int_0^{t\wedge\theta_n} H\Big((\bar{q}-1) |e_\Delta(s)|^{\bar{q}} \\
&\quad + \Big[(1 + |x_\Delta(s)|^\gamma + |\pi_\Delta(\bar{x}_\Delta(s))|^\gamma)^{\frac{\bar{q}}{2}} |x_\Delta(s) - \pi_\Delta(\bar{x}_\Delta(s))|^{\bar{q}} \Big] \Big) ds \Bigg) \\
&\le (\bar{q}-1) H \int_0^t \mathbb{E}\, |e_\Delta(s\wedge\theta_n)|^{\bar{q}}\, ds \\
&\quad + 3^{\bar{q}/2-1} H \mathbb{E}\Bigg(\int_0^{t\wedge\theta_n} \Big[(1 + |x_\Delta(s)|^{\frac{\gamma\bar{q}}{2}} + |\pi_\Delta(\bar{x}_\Delta(s))|^{\frac{\gamma\bar{q}}{2}}) \\
&\qquad \times |x_\Delta(s) - \pi_\Delta(\bar{x}_\Delta(s))|^{\bar{q}} \Big] ds \Bigg) \\
&\le (\bar{q}-1) H \int_0^t \mathbb{E}\, |e_\Delta(s\wedge\theta_n)|^{\bar{q}}\, ds \\
&\quad + 3^{\bar{q}/2-1} H \int_0^T \mathbb{E}\left[(1 + |x_\Delta(s)|^{\frac{\gamma\bar{q}}{2}} + |\pi_\Delta(\bar{x}_\Delta(s))|^{\frac{\gamma\bar{q}}{2}}) |x_\Delta(s) - \pi_\Delta(\bar{x}_\Delta(s))|^{\bar{q}} \right] ds
\end{aligned}
$$

By Hölder's inequality, Lemma 3.1.6 and the fact that $|\pi_\Delta(x)|^p \le |x|^p$,

$$\begin{aligned}
J_2 &\le (\bar q - 1)H\int_0^t \mathbb{E}\,|e_\Delta(s\wedge\theta_n)|^{\bar q}\,ds\\
&\quad + 3^{\bar q/2-1}H\int_0^T\Bigg(\left[3^{\frac{2p}{\gamma\bar q}-1}\left(1+\mathbb{E}\,|x_\Delta(s)|^p+\mathbb{E}\,|\bar x_\Delta(s)|^p\right)\right]^{\frac{\gamma\bar q}{2p}}\\
&\qquad \times\left[\mathbb{E}\,|x_\Delta(s)-\pi_\Delta(\bar x_\Delta(s))|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]^{\frac{2p-\gamma\bar q}{2p}}\Bigg)ds\\
&\le (\bar q-1)H\int_0^t \mathbb{E}\left(\sup_{0\le u\le s\wedge\theta_n}|e_\Delta(u)|^{\bar q}\right)ds\\
&\quad + \left[3^{\frac{\bar q}{2}-\frac{\gamma\bar q}{2p}}H\left(1+2C_p\right)^{\frac{\gamma\bar q}{2p}}\right]\int_0^T\left[\mathbb{E}\,|x_\Delta(s)-\pi_\Delta(\bar x_\Delta(s))|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]^{\frac{2p-\gamma\bar q}{2p}}ds.
\end{aligned}$$

Let consider the last integral $\int_0^T\left[\mathbb{E}\,|x_\Delta(s)-\pi_\Delta(\bar x_\Delta(s))|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]^{\frac{2p-\gamma\bar q}{2p}}ds$. Noting that by the definition of π_Δ, we obtain the inequality $|\pi_\Delta(x)-\pi_\Delta(y)|^p\le|x-y|^p$,

$$\begin{aligned}
&\int_0^T\left[\mathbb{E}\,|x_\Delta(s)-\pi_\Delta(\bar x_\Delta(s))|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]^{\frac{2p-\gamma\bar q}{2p}}ds\\
&\le 2^{\frac{\gamma\bar q}{2p}}\int_0^T\left(\mathbb{E}\left[|x_\Delta(s)-\pi_\Delta(x_\Delta(s))|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]\right)^{\frac{2p-\gamma\bar q}{2p}}ds\\
&\quad + 2^{\frac{\gamma\bar q}{2p}}\int_0^T\left(\mathbb{E}\left[|\pi_\Delta(x_\Delta(s))-\pi_\Delta(\bar x_\Delta(s))|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]\right)^{\frac{2p-\gamma\bar q}{2p}}ds\\
&\le 2^{\frac{\gamma\bar q}{2p}}\int_0^T\left(\mathbb{E}\left[I_{\{|x_\Delta(s)|>\mu^{-1}(h(\Delta))\}}\,|x_\Delta(s)|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]\right)^{\frac{2p-\gamma\bar q}{2p}}ds\\
&\quad + 2^{\frac{\gamma\bar q}{2p}}\int_0^T\left(\mathbb{E}\left[|x_\Delta(s)-\bar x_\Delta(s)|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]\right)^{\frac{2p-\gamma\bar q}{2p}}ds\\
&\le 2^{\frac{\gamma\bar q}{2p}}\int_0^T\left(\left[\mathbb{P}\left\{|x_\Delta(s)|>\mu^{-1}(h(\Delta))\right\}\right]^{\frac{2p-(2+\gamma)\bar q}{2p-\gamma\bar q}}\left[\mathbb{E}\,|x_\Delta(s)|^p\right]^{\frac{2\bar q}{2p-\gamma\bar q}}\right)^{\frac{2p-\gamma\bar q}{2p}}ds\\
&\quad + 2^{\frac{\gamma\bar q}{2p}}\int_0^T\left(\mathbb{E}\left[|x_\Delta(s)-\bar x_\Delta(s)|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]\right)^{\frac{2p-\gamma\bar q}{2p}}ds\\
&\le 2^{\frac{\gamma\bar q}{2p}}\Bigg[\int_0^T\left(\left[\frac{\mathbb{E}\,|x_\Delta(s)|^p}{(\mu^{-1}(h(\Delta)))^p}\right]^{\frac{2p-(2+\gamma)\bar q}{2p}}\left[\mathbb{E}\,|x_\Delta(s)|^p\right]^{\frac{2\bar q}{2p}}\right)ds\\
&\quad + \int_0^T\left(\mathbb{E}\left[|x_\Delta(s)-\bar x_\Delta(s)|^{\frac{2p\bar q}{2p-\gamma\bar q}}\right]\right)^{\frac{2p-\gamma\bar q}{2p}}ds\Bigg]
\end{aligned}$$

$$\int_0^T \left[\mathbb{E}\left|x_\Delta(s) - \pi_\Delta(\bar{x}_\Delta(s))\right|^{\frac{2p\bar{q}}{2p-\gamma\bar{q}}}\right]^{\frac{2p-\gamma\bar{q}}{2p}} ds$$
$$\leq 2^{\frac{\gamma\bar{q}}{2p}}\left[\int_0^T \left(\left[\frac{1}{(\mu^{-1}(h(\Delta)))^p}\right]^{\frac{2p-(2+\gamma)\bar{q}}{2p}} \left[\mathbb{E}\left|x_\Delta(s)\right|^p\right]^{\frac{2p-\gamma\bar{q}}{2p}}\right) ds \right.$$
$$\left. + \int_0^T \left(\mathbb{E}\left|x_\Delta(s) - \bar{x}_\Delta(s)\right|^p\right)^{\frac{\bar{q}}{p}} ds\right].$$

By Lemma 3.1.5 and Lemma 3.1.6,

$$\int_0^T \left[\mathbb{E}\left|x_\Delta(s) - \pi_\Delta(\bar{x}_\Delta(s))\right|^{\frac{2p\bar{q}}{2p-\gamma\bar{q}}}\right]^{\frac{2p-\gamma\bar{q}}{2p}} ds$$
$$\leq 2^{\frac{\gamma\bar{q}}{2p}} T\left[(C_p)^{\frac{2p-\gamma\bar{q}}{2p}} \left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\gamma)\bar{q}}{2}} + (C_p)^{\bar{q}/p}\,\Delta^{\bar{q}/2} h(\Delta)^{\bar{q}}\right].$$

Substituting this into J_2,

$$J_2 \leq (\bar{q}-1)H\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s\wedge\theta_n} \left|e_\Delta(u)\right|^{\bar{q}}\right) ds$$
$$+ C_p\left[\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\gamma)\bar{q}}{2}} + \Delta^{\bar{q}/2} h(\Delta)^{\bar{q}}\right].$$

By Assumption 3.2.2, we can derive J_3 similarly J_2, so

$$J_3 = \left(\frac{\bar{q}(\bar{q}-1)(q-1)}{2(q-\bar{q})} + 32\bar{q}^2\right)\mathbb{E}\left(\int_0^{t\wedge\theta_n} \left|e_\Delta(s)\right|^{\bar{q}-2}\left|g(x_\Delta(s)) - g_\Delta(\bar{x}_\Delta(s))\right|^2 ds\right)$$
$$\leq C\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s\wedge\theta_n} \left|e_\Delta(u)\right|^{\bar{q}}\right) ds + C_p\left[\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\beta)\bar{q}}{2}} + \Delta^{\bar{q}/2} h(\Delta)^{\bar{q}}\right].$$

Consider J_4, by Hölder's inequality, let $q_1 = \frac{q+\bar{q}}{2}$,

$$J_4 = \mathbb{E}\left[32\bar{q}^2\int_0^{t\wedge\theta_n} \left|e_\Delta(s)\right|^{\bar{q}-2}\left|g(x(s)) - g(x_\Delta(s))\right|^2 ds\right]$$
$$\leq 32\bar{q}^2 H\left(\int_0^T \mathbb{E}\left(\left|e_\Delta(s)\right|^{\bar{q}}\left(1 + |x(s)|^\beta + |x_\Delta(s)|^\beta\right)\right) ds\right)$$
$$\leq 32\bar{q}^2 H\left(\int_0^T \left(\mathbb{E}\left|e_\Delta(s)\right|^{q_1}\right)^{\bar{q}/q_1}\left(\mathbb{E}(1 + |x(s)|^\beta + |x_\Delta(s)|^\beta)^{\frac{q_1}{q_1-\bar{q}}}\right)^{\frac{q_1-\bar{q}}{q_1}} ds\right)$$
$$\leq 32\bar{q}^2 H\left(\int_0^T \left(\mathbb{E}\left|e_\Delta(s)\right|^{q_1}\right)^{\bar{q}/q_1} 3^{1-\beta/p}(1 + \mathbb{E}\,|x(s)|^p + \mathbb{E}\,|x_\Delta(s)|^p)^{\beta/p} ds\right)$$

$$J_4 \leq 32\bar{q}^2 H 3^{1-\beta/p}(1+C_p)^{\beta/p}\left(\int_0^T \left(\mathbb{E}\left|e_\Delta(s)\right|^{q_1}\right)^{\bar{q}/q_1} ds\right).$$

By Theorem 3.1.10, let $\rho = \gamma \vee \beta$

$$\begin{aligned}\left(\mathbb{E}\left|e_\Delta(s)\right|^{q_1}\right)^{\bar{q}/q_1} &\leq C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\rho)q_1}{2}} + \Delta^{q_1/2}h(\Delta)^{q_1}\right)^{\bar{q}/q_1} \\ &\leq C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p\bar{q}-(2+\rho)q_1\bar{q}}{2q_1}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right).\end{aligned}$$

Then,

$$\begin{aligned}J_4 &\leq 32\bar{q}^2 H 3^{1-\beta/p}(1+C_p)^{\beta/p} T C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p\bar{q}-(2+\rho)q_1\bar{q}}{2q_1}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right) \\ &= C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{4p\bar{q}-(2+\rho)(q+\bar{q})\bar{q}}{2(q+\bar{q})}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right).\end{aligned}$$

Combine J_1, J_2, J_3 and J_4,

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq u\leq t\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar{q}}\right) \\ &\leq \frac{1}{2}\mathbb{E}\left(\sup_{0\leq u\leq t\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar{q}}\right) \\ &\quad + \bar{q}H\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar{q}} ds\right) \\ &\quad + (\bar{q}-1)H\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar{q}}\right) ds \\ &\quad + C_p\left[\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\gamma)\bar{q}}{2}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right] \\ &\quad + C\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar{q}}\right) ds + C_p\left[\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\beta)\bar{q}}{2}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right] \\ &\quad + C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{4p\bar{q}-(2+\rho)(q+\bar{q})\bar{q}}{2(q+\bar{q})}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\right) \\ &\leq C\int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s\wedge\theta_n}\left|e_\Delta(u)\right|^{\bar{q}}\right) ds + C_p\Delta^{\bar{q}/2}h(\Delta)^{\bar{q}} \\ &\quad + C_p\left[\left(\mu^{-1}(h(\Delta))\right)^{-\frac{2p-(2+\rho)\bar{q}}{2}} + \left(\mu^{-1}(h(\Delta))\right)^{-\frac{4p\bar{q}-(2+\rho)(q+\bar{q})\bar{q}}{2(q+\bar{q})}}\right]\end{aligned}$$

By the Gronwall inequality and $\frac{4p\bar{q}-(2+\rho)(q+\bar{q})\bar{q}}{2(q+\bar{q})} \leq \frac{2p-(2+\rho)\bar{q}}{2}$,

$$\mathbb{E}\left(\sup_{0\leq u\leq t\wedge\theta_n}|e_\Delta(u)|^{\bar{q}}\right) \leq C_p\Bigg(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{4p\bar{q}-(2+\rho)(q+\bar{q})\bar{q}}{2(q+\bar{q})}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\Bigg)$$

Using the Fatou lemma, we can let $n \to \infty$ to get equation (3.18). Moreover, equation (3.19) follows from lemma 3.1.7. □

Observing inequalities (3.16) and (3.17), $|f(x)| \vee |g(x)| \leq Hx^{(2+(\gamma\vee\beta))/2}$.

Corollary 3.2.4. *Assume Assumptions 3.1.2 and 3.2.2 hold and $\varepsilon \in (0, 1/4]$ be arbitrary. Let $\rho = \gamma \vee \beta$. If $\mu(u) = Hu^{(2+\rho)/2}$ for $u \geq 0$ and $h(\Delta) = \Delta^{-\varepsilon}$ and $\hat{h} \geq 1$. Then, for any $\bar{q} \in [2, q)$*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|x(t) - x_\Delta(t)|^{\bar{q}}\right) = O\left(\Delta^{\frac{\bar{q}}{2}(1-2\varepsilon)}\right) \tag{3.20}$$

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|x(t) - \bar{x}_\Delta(t)|^{\bar{q}}\right) = O\left(\Delta^{\frac{\bar{q}}{2}(1-3\varepsilon)}\right) \tag{3.21}$$

Proof. By definition of $\mu(u)$, we get the inverse $\mu^{-1}(h(\Delta)) = C\,(\Delta)^{-\frac{2\varepsilon}{2+\rho}}$. It follows from Theorem 3.2.3 that

$$\begin{aligned}\mathbb{E}\left(\sup_{0\leq u\leq t\wedge\theta_n}|e_\Delta(u)|^{\bar{q}}\right) &\leq C_p\Bigg(\left(\mu^{-1}(h(\Delta))\right)^{-\frac{4p\bar{q}-(2+\rho)(q+\bar{q})\bar{q}}{2(q+\bar{q})}} + \Delta^{\bar{q}/2}h(\Delta)^{\bar{q}}\Bigg)\\ &\leq C_p\Bigg(\Delta^{\frac{\varepsilon(4p\bar{q}-(2+\rho)(q+\bar{q})\bar{q})}{(q+\bar{q})(2+\rho)}} + \Delta^{\frac{\bar{q}}{2}(1-2\varepsilon)}\Bigg)\end{aligned}$$

Since p is arbitrary, we can choose

$$p > \frac{(q+\bar{q})(2+\rho)}{8\varepsilon}.$$

We can assert equations (3.20) and get (3.21) as similar. □

3.3 Comparisons with known results

First of all, we will compare our result Theorem 3.2.3 with the result in [12], namely Theorem 3.1.10. Recall that Assumptions 3.1.1, 3.1.8, and 3.1.9 are similar with Assumption 3.2.2, which means that our contribution uses the common conditions same with the main result in [12]. Theorem 3.2.3, therefore, also keep some properties which are the step size Δ can be arbitrary in $(0,1]$ and the rate of convergence is almost similar with Theorem 3.1.10. There are, however, some differences as follow:

- The key improvement of Theorem 3.2.3 is that the convergence over a finite time interval under the slightly stronger conditions with Theorem 3.1.10.

- Theorem 3.2.3 needs to use the Khasminskii-type condition that holds for any parameter p as defined in Assumption 3.1.2, while Theorem 3.1.10 can hold for some $p > (2+\rho q)/2$, see more [12].

Recall the result in [22],

Theorem 3.3.1. *Let Assumption 3.2.1 hold. Let $R > |x_0|$ be a real number and let $\Delta \in (0,1)$ be sufficiently small such that $\mu^{-1}(h(\Delta)) \geq R$. Let $\theta_{\Delta,R} = \inf\{t \geq 0 : |x(t)| \vee |x_\Delta(t)|\} \geq R$. Let $q \geq 2$ be arbitrary. Then,*

$$\mathbb{E}\left(\sup_{0\leq u\leq T\wedge\theta_{\Delta,R}} |x(t)-x_\Delta(t)|^q\right) \leq C\Delta^{q/2}\left(h(\Delta)\right)^q, \quad \forall T > 0. \tag{3.22}$$

Moreover, define $\mu(u) = Hu^{1+\gamma_0}, u \geq 0$ and $h(\Delta) = \Delta^{-\varepsilon/2}, \Delta \in (0,1]$ be sufficiently small. Then, for any $\varepsilon \in (0,1/2)$

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |x(t)-x_\Delta(t)|^q\right) = O\left(\Delta^{\frac{q}{2}(1-\varepsilon)}\right), \tag{3.23}$$

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |x(t)-\bar{x}_\Delta(t)|^q\right) = O\left(\Delta^{\frac{q}{2}(1-\varepsilon)}\right). \tag{3.24}$$

Now, let compare the convergence over a finite time interval, that means we will compare Theorem 3.2.3 with Theorem 3.3.1. So, we let show the differences as follow:

- Theorem 3.2.3 holds for non global Lipschitz diffusion coefficient while Theorem 3.3.1 needs the diffusion coefficient to satisfy the global Lipschitz, as you can see (3.17) and (3.14) respectively.

- The assertions of Theorem 3.2.3 hold for any $\Delta \in (0,1]$ while the assertions of Theorem 3.3.1 hold for a sufficiently small Δ.

- Theorem 3.2.3 needs a stronger condition on Khasminskii-type condition, namely Khasminskii-type condition needs to satisfy for any parameter p.
- Even though the rate of convergence of Theorem 3.2.3 looks worse than Theorem 3.3.1, if p is large enough then the rate of convergence of both theorems is the same.

The key advantage of our Theorem 3.2.3 is that the diffusion coefficient is not required to satisfy the global Lipschitz. That means we are allowed to apply more functions with the SDEs, and we will show the example in the next section.

3.4 Examples with simulations

In this section, we will illustrate the example of the non global Lipschitz diffusion coefficient SDE, which is the special case of the Ait-Sahalia model as the following equation. Let a, b, c be positive constants,

$$dx(t) = ax(t) - bx^3(t)dt + cx^{3/2}(t)dB(t) \tag{3.25}$$

To apply with our Theorem 3.2.3, we write equation (3.25) as the SDE (3.1) in $\mathbb{R}$ by defining

$$\begin{cases} f(x) = ax - bx^3, g(x) = cx^{3/2} & \text{for } \; x \geq 0, \\ f(x) = g(x) = 0 & \text{for } \; x < 0. \end{cases} \tag{3.26}$$

We will show (3.26) satisfies Assumptions 3.1.2 and 3.2.2. For any $p > 2$

$$xf(x) + \frac{p-1}{2} |g(x)|^2 = ax^2 - bx^4 + \frac{p-1}{2} c^2 |x|^3 = \left(a - bx^2 + \frac{p-1}{2} c^2 |x| \right) |x|^2 ,$$

for all $x \in \mathbb{R}$. Since $a - bx^2 + \frac{p-1}{2} c^2 |x|$ is bounded above by some positive constant K_p,

$$xf(x) + \frac{p-1}{2} |g(x)|^2 \leq K_p |x|^2 \leq K_p(1 + |x|^2).$$

Then, equation (3.26) satisfies Assumption 3.1.2. Now, any $x, y \in \mathbb{R}$, By the Mean

Value Theorem,

$$\begin{aligned}|f(x)-f(y)|^2 &\le \left(\left(a+3b|x|^2+3b|y|^2\right)|x-y|\right)^2\\ &\le \left(3a^2+27b^2|x|^4+27b^2|y|^4\right)|x-y|^2.\\ |g(x)-g(y)|^2 &\le \left(\frac{3c}{2}\left||x|^{1/2}+|y|^{1/2}\right||x-y|\right)^2\\ &\le \frac{9c^2}{2}\left(|x|+|y|\right)|x-y|^2.\end{aligned}$$

Then, equation (3.26) satisfies inequalities (3.16) and (3.17). Therefore, we will show equation (3.26) satisfies (3.15) by letting $q=3$, let $x,y\in\mathbb{R}$. If $x,y<0$, it is obvious. Then, assume x or y is negative, without loss of generality, we assume $x\le 0<y$, so

$$\begin{aligned}&(x-y)^T\left(f(x)-f(y)\right)+|g(x)-g(y)|^2\\ &\quad=(x-y)\left(-ay+by^3\right)+c^2y^3\\ &\quad=-axy+bxy^3-by^2\left(\left(y-\frac{c^2}{2b}\right)^2+\frac{c^4-4ab}{4b^2}\right)\\ &\quad\le -axy-y^2\left(\frac{c^4-4ab}{4b}\right)\\ &\quad\le \frac{a}{2}|x-y|^2+|x-y|^2\left|\frac{c^4-4ab}{4b}\right|\\ &\quad=\left(\frac{a}{2}+\left|\frac{c^4-4ab}{4b}\right|\right)|x-y|^2.\end{aligned}$$

Assume $x,y\ge 0$, then

$$\begin{aligned}&(x-y)^T\left(f(x)-f(y)\right)+|g(x)-g(y)|^2\\ &\quad\le (x-y)\left(ax-bx^3-ay+by^3\right)+\frac{9c^2}{2}\left(|x|+|y|\right)|x-y|^2\\ &\quad=a(x-y)^2-b(x-y)^2\left(x^2+xy+y^2\right)+\frac{9c^2}{2}\left(|x|+|y|\right)|x-y|^2\\ &\quad=|x-y|^2\left[a-0.5b\left(x^2+y^2+(x+y)^2\right)+\frac{9c^2}{2}|x|+\frac{9c^2}{2}|y|\right]\\ &\quad\le |x-y|^2\left[a-0.5b\left(|x|^2+|y|^2\right)+\frac{9c^2}{2}|x|+\frac{9c^2}{2}|y|\right].\end{aligned}$$

Since $\left[a - 0.5b\left(|x|^2 + |y|^2\right) + \frac{9c^2}{2}|x| + \frac{9c^2}{2}|y|\right]$ is bounded, we can bound this term by some constant H. Then, equation (3.25) satisfies Assumption 3.2.2. We apply Theorem 3.2.3 to see what we get. Obvious that for each $\bar{q} \in [2, 3)$, we can choose p such that $p > \frac{2+4(3)}{2} \vee \frac{3+\bar{q}}{3-\bar{q}}$. It's follow from (3.26) that $|f(x)| \vee |g(x)| \leq H_1 x^3$ for all $x > 1$ and some positive constant H_1. Now, we can let $\mu(u) = H_1 u^3$ and $h(\Delta) = \Delta^{-1/8}$ and $\bar{q} = 2$, by Corollary 3.2.4, we get

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|x(t) - x_\Delta(t)|^2\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T}|x(t) - \bar{x}_\Delta(t)|^2\right) \leq C\Delta^{\frac{3}{4}}.$$

For numerical simulations, we compare our method with the numerical solution by the backward Euler-Maruyama (BEM) and we let $a = 10$, $b = 1$, $c = 0.5$ and $x_0 = 3$ and choose the step size $\Delta = 10^{-3}$ for the left and $\Delta = 10^{-4}$ for the right in Figure 3.1. These show both sample paths generated by the truncated EM method are very closed to the BEM. More precisely, these simulations are desired to produce the squares of the maximum differences of solutions between the truncated EM $X_\Delta(t_k)$ and the BEM $Y_\Delta(t_k)$:

$$\begin{aligned} &\max_{0\leq k\leq 10^3}|X_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.03853134 \\ \text{and } &\max_{0\leq k\leq 10^4}|X_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.002820902, \end{aligned}$$

where $\Delta = 10^{-3}$ and $\Delta = 10^{-4}$ respectively.

For wide application, we model another example in more than one-dimension as the following:

$$\begin{aligned} dS(t) &= (V(t) - 4S^3(t))dt + V(t)\sin(V(t))dB(t), \\ dV(t) &= (3S(t) - 8V^3(t))dt + S(t)\cos(S(t))d\tilde{B}(t). \end{aligned} \tag{3.27}$$

Therefore, we have the coefficients

$$f\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2 - 4x_1^3 \\ 3x_1 - 8x_2^3 \end{bmatrix} \text{ and } g\left(\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\right) = \begin{bmatrix} x_2\sin x_2 & 0 \\ 0 & x_1\cos x_1 \end{bmatrix}. \tag{3.28}$$

To apply with our Theorem 3.2.3, we will show (3.28) satisfies Assumptions 3.1.2

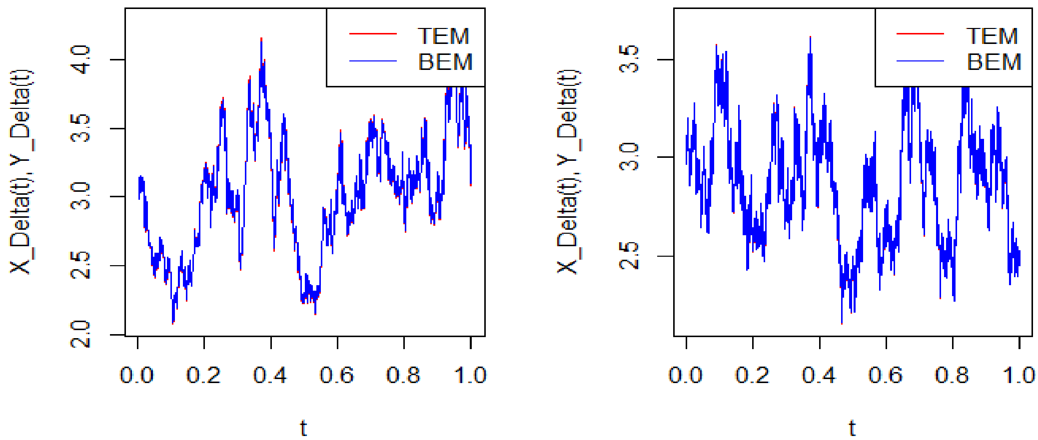

Figure 3.1: The computer simulations of the sample paths of the solution of equation (3.25) by the BEM and the truncated EM. Left: $\Delta = 10^{-3}$. Right: $\Delta = 10^{-4}$.

and 3.2.2. For any $x \in \mathbb{R}^2$,

$$x^T f(x) = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} x_2 - 4x_1^3 \\ 3x_1 - 8x_2^3 \end{bmatrix} = x_1x_2 - 4x_1^4 + 3x_1x_2 - 8x_2^4 = 4x_1x_2 - 4x_1^4 - 8x_2^4.$$

$$\begin{aligned} |g(x)|^2 &= \text{trace}\left(\begin{bmatrix} x_2 \sin x_2 & 0 \\ 0 & x_1 \cos x_1 \end{bmatrix} \begin{bmatrix} x_2 \sin x_2 & 0 \\ 0 & x_1 \cos x_1 \end{bmatrix} \right) \\ &= x_1^2 \cos^2 x_1 + x_2^2 \sin^2 x_2 \\ &\le x_1^2 + x_2^2. \end{aligned}$$

Therefore, for any $p > 2$,

$$\begin{aligned} x^T f(x) + \frac{p-1}{2} |g(x)|^2 &\le 4x_1x_2 - 4x_1^4 - 8x_2^4 + \frac{p-1}{2} \left(x_1^2 + x_2^2 \right) \\ &\le 2x_1^2 + 2x_2^2 - 4x_1^4 - 8x_2^4 + \frac{p-1}{2} \left(x_1^2 + x_2^2 \right). \end{aligned}$$

Since $2x_1^2 - 4x_1^4 + \left(\frac{p-1}{2}\right) x_1^2$ and $2x_2^2 - 8x_2^4 + \left(\frac{p-1}{2}\right) x_2^2$ are bounded above, there is

$K_p > 0$ such that

$$x^T f(x) + \frac{p-1}{2} |g(x)|^2 \le K_p x_1^2 + K_p x_2^2 = K_p |x|^2 \le K_p \left(1 + |x|^2\right).$$

Now before we show conditions (3.16) and (3.17) hold, define $h_1, h_2 : \mathbb{R} \to \mathbb{R}$ by $h_1(x) = x \sin x$ and $h_2(x) = x \cos x$. Then, for any $x \in \mathbb{R}$, $h_1'(x) = x \cos x + \sin x$ and $h_2'(x) = \cos x - x \sin x$. By the Mean Value Theorem, for any $x \le y$ there exists a constant $k \in (x, y)$ such that

$$\begin{aligned}
|h_1(x) - h_1(y)|^2 &\le |h_1'(k)|^2 |x-y|^2 = (k \cos k + \sin k)^2 |x-y|^2 \\
&\le (1 + |x| + |y|)^2 |x-y|^2, \\
|h_2(x) - h_2(y)|^2 &\le |h_2'(k)|^2 |x-y|^2 = (\cos k - k \sin k)^2 |x-y|^2 \\
&\le (1 + |x| + |y|)^2 |x-y|^2.
\end{aligned}$$

Then, let consider

$$\begin{aligned}
|f(x) - f(y)|^2 &= \left| \begin{matrix} (x_2 - y_2) - 4(x_1^3 - y_1^3) \\ 2(x_1 - y_1) - 8(x_2^3 - y_2^3) \end{matrix} \right|^2 \\
&= ((x_2 - y_2) - 4(x_1^3 - y_1^3))^2 + (3(x_1 - y_1) - 8(x_2^3 - y_2^3))^2 \\
&\le \left(2(x_2 - y_2)^2 + 8(x_1 - y_1)^2 (x_1^2 + x_1 y_1 + y_1^2)^2\right) \\
&\quad + \left(6(x_1 - y_1)^2 + 16(x_2 - y_2)^2 (x_2^2 + x_2 y_2 + y_2^2)^2\right) \\
&\le 36(1 + x_1^4 + y_1^4)(x_1 - y_1)^2 + 72(1 + x_2^4 + y_2^4)(x_2 - y_2)^2 \\
&\le 72(1 + |x|^4 + |y|^4) |x-y|^2 \\
|g(x) - g(y)|^2 &= \left| \begin{matrix} x_2 \sin x_2 - y_2 \sin y_2 & 0 \\ 0 & x_1 \cos x_1 - y_1 \cos y_1 \end{matrix} \right|^2 \\
&= (x_2 \sin x_2 - y_2 \sin y_2)^2 + (x_1 \cos x_1 - y_1 \cos y_1)^2 \\
&\le (1 + |x_2| + |y_2|)^2 (x_2 - y_2)^2 + (1 + |x_1| + |y_1|)^2 (x_1 - y_1)^2 \\
&\le 3(1 + |x|^2 + |y|^2) |x-y|^2
\end{aligned}$$

Then, equations (3.16) and (3.17) assert. To show equation (3.28) satisfies (3.15), let $x, y \in \mathbb{R}^2$. Therefore,

$$
\begin{aligned}
&(x-y)^T\left(f(x)-f(y)\right)\\
&\qquad = (x-y)^T \begin{vmatrix} (x_2-y_2)-4(x_1^3-y_1^3) \\ 3(x_1-y_1)-8(x_2^3-y_2^3) \end{vmatrix}\\
&\qquad = \left((x_2-y_2)-4(x_1^3-y_1^3)\right)(x_1-y_1)+\left(3(x_1-y_1)-8(x_2^3-y_2^3)\right)(x_2-y_2)\\
&\qquad = 4(x_1-y_1)(x_2-y_2)-4(y_1^2+y_1x_1+x_1^2)(x_1-y_1)^2\\
&\qquad\quad -8(y_2^2+y_2x_2+x_2^2)(x_2-y_2)^2\\
&\qquad \le 2(x_1-y_1)^2+2(x_2-y_2)^2-2(x_1^2+y_1^2)(x_1-y_1)^2-4(x_2^2+y_2^2)(x_2-y_2)^2\\
&\qquad \le (2-2\left|x_1\right|^2-2\left|y_1\right|^2)(x_1-y_1)^2+(2-4\left|x_2\right|^2-4\left|y_2\right|^2)(x_2-y_2)^2.
\end{aligned}
$$

Now, we will show equation (3.27) satisfies condition (3.15), by letting $q=\frac{7}{3}$, we have

$$
\begin{aligned}
&(x-y)^T(f(x)-f(y))+\frac{2}{3}\left|g(x)-g(y)\right|^2\\
&\qquad \le (2-2\left|x_1\right|^2-2\left|y_1\right|^2)(x_1-y_1)^2+(2-4\left|x_2\right|^2-4\left|y_2\right|^2)(x_2-y_2)^2\\
&\qquad\quad +\frac{2}{3}\left[3(1+\left|x_2\right|^2+\left|y_2\right|^2)(x_2-y_2)^2+3(1+\left|x_1\right|^2+\left|y_1\right|^2)(x_1-y_1)^2\right]\\
&\qquad = 4(x_1-y_1)^2+(4-2\left|x_2\right|^2-2\left|y_2\right|^2)(x_2-y_2)^2
\end{aligned}
$$

Since $4-2\left|x_2\right|^2-2\left|y_2\right|^2$ is bounded above, there exists a positive constant H such that

$$
(x-y)^T(f(x)-f(y))+\frac{2}{3}\left|g(x)-g(y)\right|^2 \le H(x_1-y_1)^2+H(x_2-y_2)^2 = H\left|x-y\right|^2.
$$

We compare our method with the numerical solution by the backward Euler-Maruyama (BEM) of the equations

$$
\begin{aligned}
dX(t) &= (Y(t)-4X^3(t))dt+Y(t)\sin(Y(t))dB(t),\\
dY(t) &= (3X(t)-8Y^3(t))dt+X(t)\cos(X(t))d\tilde{B}(t).
\end{aligned}
\tag{3.29}
$$

We let $x_0=-0.5=s_0$ and $y_0=0.5=v_0$ for the step size $\Delta=10^{-3}$ in Figure 3.2 and $\Delta=10^{-4}$ in Figure 3.3, respectively. These show the truncated EM numerical solutions are very closed to the BEM numerical solutions. More precisely, these simulations are desired to produce the squares of the maximum differences of solutions between the truncated EM $S_\Delta(t_k)$ with the BEM $X_\Delta(t_k)$ and the truncated EM $V_\Delta(t_k)$ with the BEM $Y_\Delta(t_k)$:

$$\max_{0\le k\le 10^4} |S_\Delta(t_k) - X_\Delta(t_k)|^2 = 0.00204972$$
$$\text{and} \quad \max_{0\le k\le 10^4} |V_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.002800053,$$

where $\Delta = 10^{-3}$, and

$$\max_{0\le k\le 10^5} |S_\Delta(t_k) - X_\Delta(t_k)|^2 = 0.0002813802$$
$$\text{and} \quad \max_{0\le k\le 10^5} |V_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.0002510902,$$

where $\Delta = 10^{-4}$.

3.5 Summary

In this chapter, we relaxed the global Lipschitz condition on the diffusion coefficient over a finite time interval to be the local Lipschitz condition by using a bit stronger Khasminskii-type condition. Moreover, we also showed the convergence rate of the truncated Euler-Maruyama numerical solutions in L^q closed to $q/2$, a half of the order.

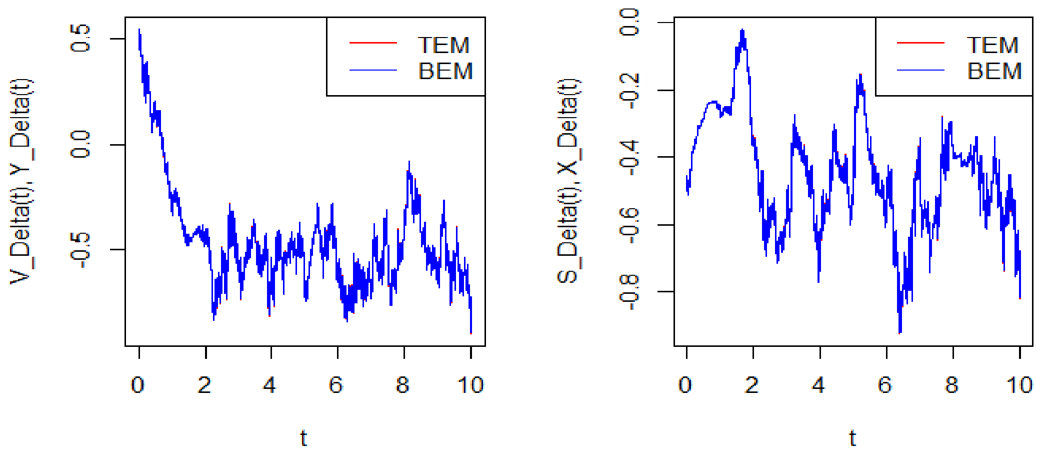

Figure 3.2: The computer simulations of the sample paths of the solution of equation (3.27) by the BEM and the truncated EM with $\Delta = 10^{-3}$.

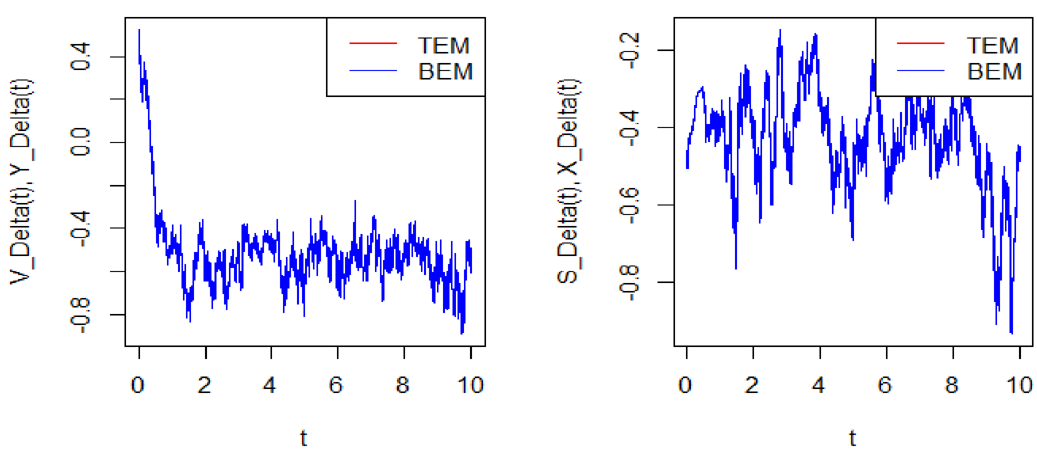

Figure 3.3: The computer simulations of the sample paths of the solution of equation (3.27) by the BEM and the truncated EM with $\Delta = 10^{-4}$.

Chapter 4

Truncated Euler-Maruyama for Stochastic Differential Delay Equations with non-constant delay

4.1 Introduction

In the previous chapter, we showed the result of stochastic differential equations. Nevertheless, there are a lot of phenomena which have some delay of time before their occurrence, i.e. the delay of illustrating an illness of infected patients. Fei, W. et.al. (2020) studied the convergence rate of non-linear stochastic differential delay equations (SDDEs) with a constant delay, as shown in equation (4.1), by utilizing the truncated Euler-Maruyama numerical approach, [6].

$$dx(t) = f(x(t), x(t-\tau))dt + g(x(t), x(t-\tau))dB(t) \tag{4.1}$$

In realistic situations, however, the time-delay is not necessary to be constant all the time. Mao, X. and Sabanis, S. (2003) studied the SDDEs with non-constant time delay under the global Lipschitz coefficients by using the Euler-Maruyama numerical method, see [25]. In this chapter, we get inspiration from [6, 8, 25]. We apply the truncated EM method with the variable time delay SDDEs. We, moreover, find the convergence rate of the truncated numerical solutions at a time

T and over a finite time interval.

Throughout this chapter, let τ and T be positive constants. We consider the d-dimensional stochastic differential delay equation,

$$dx(t) = f(x(t), x(\delta(t)))dt + g(x(t), x(\delta(t)))dB(t), \quad t \geq 0, \tag{4.2}$$

with the initial data $\{x(\theta) : -\tau \leq \theta \leq 0\} = \xi \in C\left([-\tau, 0]; \mathbb{R}^d\right)$ and functions $f : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^{d\times m}$. As the standing hypothesis we always assume that the Lipschitz continuous function $\delta : [0, \infty) \to \mathbb{R}$ stands for the time delay satisfying

$$-\tau \leq \delta(t) \leq t \quad \text{ and } \quad |\delta(t) - \delta(s)| \leq \gamma\, |t - s|\,, \quad \forall t, s \geq 0 \tag{4.3}$$

for some positive constant γ.

We also assume the following assumptions to guarantee the existence and uniqueness of the solution.

Assumption 4.1.1 (Local Lipschitz condition)**.** *For every integer $n \geq 1$, there is a positive constant K_n such that, for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq n$,*

$$|f(x, y) - f(\bar{x}, \bar{y})|^2 \vee |g(x, y) - g(\bar{x}, \bar{y})|^2 \leq K_n \left(|x - \bar{x}|^2 + |y - \bar{y}|^2\right). \tag{4.4}$$

Assumption 4.1.2 (Khasminskii-type condition)**.** *For any $p > 2$ there is $K_p > 0$ such that for all $x, y \in \mathbb{R}^d$*

$$x^T f(x, y) + \frac{p-1}{2}\, |g(x, y)|^2 \leq K_p(1 + |x|^2 + |y|^2). \tag{4.5}$$

By the result of [24], we can state the following lemma.

Lemma 4.1.3. *Under Assumptions 4.1.1 and 4.1.2, equation* (4.2) *has a unique global solution $x(t)$ on $t \in [-\tau, \infty)$. Moreover,*

$$\sup_{-\tau \leq t \leq T} \mathbb{E}\, |x(t)|^p < \infty, \quad \forall T > 0. \tag{4.6}$$

To introduce the truncated EM method for SDDEs which defined in [6, 8], let $\mathbb{R}^+$ be a set of positive real numbers and $\mu : [1, \infty) \to \mathbb{R}^+$ be a strictly increasing continuous function such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{|x|\vee|y|\leq u} (|f(x,y)| \vee |g(x,y)|) \leq \mu(u), \quad \forall u \geq 1. \tag{4.7}$$

Therefore, an inverse function μ^{-1} of μ is a strictly increasing continuous function from $[\mu(1), \infty)$. We, additionally, set a constant $\hat{h} \geq 1 \vee \mu(1)$ and a strictly increasing function $h : (0,1] \to [\mu(1), \infty)$ satisfying

$$\lim_{\Delta\to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \leq \hat{h}, \quad \forall \Delta \in (0,1]. \tag{4.8}$$

For a given step size $\Delta \in (0,1]$, define the truncated mapping $\pi_\Delta : \mathbb{R}^d \to \mathbb{R}^d$ by

$$\pi_\Delta(x) = \left(|x| \wedge \mu^{-1}(h(\Delta))\right) \frac{x}{|x|}, \tag{4.9}$$

where we set $x/|x| = 0$ if $x = 0$. That means, x is restricted at $\mu^{-1}(h(\Delta))\,(x/|x|)$ if $x > \mu^{-1}(h(\Delta))$. Therefore, the truncated functions can be defined by

$$f_\Delta(x,y) = f(\pi_\Delta(x), \pi_\Delta(y)) \quad \text{and} \quad g_\Delta(x,y) = g(\pi_\Delta(x), \pi_\Delta(y)), \tag{4.10}$$

for all $x, y \in \mathbb{R}^d$. Hence, $|f_\Delta(x,y)| \vee |g_\Delta(x,y)| \leq \mu(\mu^{-1}(h(\Delta))) = h(\Delta)$ for all $x, y \in \mathbb{R}^d$.

From now on, we will define the step size Δ be a fraction of τ. That means, $\Delta = \tau/M$ for some positive integer M. Define $t_k = k\Delta$ for all $k = -M, -(M-1), \cdots, 0, 1, 2, \cdots$. Define the discrete time truncated EM solutions $X_\Delta(t_k) = \xi(t_k)$ for $k = -M, -(M-1), \cdots, 0$ and compute, for $k = 0, 1, \ldots,$

$$\begin{aligned} X_\Delta(t_{k+1}) = X_\Delta(t_k) &+ f_\Delta(X_\Delta(t_k), X_\Delta(I_\Delta[\delta(t_k)]\Delta))\Delta \\ &+ g_\Delta(X_\Delta(t_k), X_\Delta(I_\Delta[\delta(t_k)]\Delta))\Delta B_k, \end{aligned}$$

where $\Delta B_k = B_{t_{k+1}} - B_{t_k}$ and $I_\Delta[u]$ is the largest integer less than or equal to u/Δ for u in $\mathbb{R}$. Then,

$$-\tau \leq I_\Delta[\delta(t_k)]\Delta \leq t_k \text{ for } k \geq 0. \tag{4.11}$$

Before defining the continuous-time truncated EM solution, let us define two step processes

$$z_1(t) = \sum_{k=0}^{\infty} X_\Delta(t_k)\mathbb{1}_{[t_k,t_{k+1})}(t),$$
$$z_2(t) = \sum_{k=0}^{\infty} X_\Delta(I_\Delta[\delta(t_k)]\Delta)\mathbb{1}_{[t_k,t_{k+1})}(t),$$

for $t \geq 0$. These are simple step processes so their sample paths are not continuous. Now, we refer to the continuous-time (continuous process) truncated EM solution defining by $x_\Delta(t) = \xi(t)$ for $t \in [-\tau, 0]$, and

$$x_\Delta(t) = \xi(0) + \int_0^t f_\Delta(z_1(s), z_2(s))ds + \int_0^t g_\Delta(z_1(s), z_2(s))dB(s) \text{ for } t \geq 0. \tag{4.12}$$

Notice that $x_\Delta(t_k) = X_\Delta(t_k)$ for all $k \geq -M$. Moreover, $x_\Delta(t)$ is an Itô process on $t \geq 0$ with its Itô differential

$$dx_\Delta(t) = f_\Delta(z_1(t), z_2(t))dt + g_\Delta(z_1(t), z_2(t))dB(t).$$

The following lemma is a result from [24] to confirm that the truncated functions satisfy Assumption 4.1.2. The proof of this Lemma is straightforward similar to Lemma 3.1.4.

Lemma 4.1.4. *Assume Assumption 4.1.2 holds. Then, for each $p > 2$ there exists $\widehat{K_p} > 0$ such that for all $x \in \mathbb{R}^d$*

$$x^T f_\Delta(x, y) + \frac{p-1}{2} |g_\Delta(x, y)|^2 \leq \widehat{K_p} \left(1 + |x|^2 + |y|^2\right).$$

Recall the following notation and assumptions on both the initial data and the coefficients which are required for illustrating the convergence rate of the numerical solution, [6, 8].

Assumption 4.1.5. *There is a pair of constants $K_1 > 0$ and $\beta \in (0, 1]$ such that the initial data ξ satisfies*

$$|\xi(u) - \xi(v)| \leq K_1 |u - v|^\beta, \text{ for } -\tau \leq v < u \leq 0. \tag{4.13}$$

Let $\mathcal{U}$ be the family of continuous functions $U : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^+$ such that for each $b > 0$, there is a positive constant K_b for which

$$U(x, \bar{x}) \leq K_b |x - \bar{x}|^2, \quad \text{for } x, \bar{x} \in \mathbb{R}^n \text{ with } |x| \vee |\bar{x}| \leq b. \tag{4.14}$$

Assumption 4.1.6. *There is a pair of constants α and $K_2 > 0$ and a function $U \in \mathcal{U}$ such that for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$ and positive constants λ_1, λ_2*

$$\begin{aligned}
&(x - \bar{x})^T (f(x, y) - f(\bar{x}, \bar{y})) + \frac{1+\alpha}{2} |g(x, y) - g(\bar{x}, \bar{y})|^2 \\
&\leq K_2(|x - \bar{x}|^2 + |y - \bar{y}|^2) - \lambda_1 U(x, \bar{x}) + \lambda_2 U(y, \bar{y}).
\end{aligned} \tag{4.15}$$

Assumption 4.1.7. *There is a pair of constants ρ and $K_3 > 0$ such that for all $x, y, \bar{x}, \bar{y} \in \mathbb{R}^d$*

$$\begin{aligned}
&|f(x, y) - f(\bar{x}, \bar{y})|^2 \vee |g(x, y) - g(\bar{x}, \bar{y})|^2 \\
&\leq K_3 \left(|x - \bar{x}|^2 + |y - \bar{y}|^2\right) (1 + |x|^\rho + |\bar{x}|^\rho + |y|^\rho + |\bar{y}|^\rho).
\end{aligned} \tag{4.16}$$

By the definition of x_Δ, we obtain three of the following lemmas which have an important role to prove our main Theorems 4.2.1 and 4.3.2.

Lemma 4.1.8. *There is a positive constant C_p independent of Δ such that*

$$\mathbb{E} |x_\Delta(t) - z_1(t)|^p \leq C_p \Delta^{p/2} (h(\Delta))^p \ \text{ for } p > 0 \text{ and } t \geq 0. \tag{4.17}$$

This C_p may be different from the C_p before but we use C_p to stand for generic constants dependent on p and they may change from place to place.

Proof. For each $t \in [0, T]$, choose a k such that $t \in [t_k, t_{k+1}]$. Then, let $p \geq 2$

$$\begin{aligned}
\mathbb{E}\left|x_\Delta(t) - z_1(t)\right|^p &= \mathbb{E}\left|x_\Delta(t) - X_\Delta(t_k)\right|^p \\
&\leq \mathbb{E}\left|\int_{t_k}^{t} f_\Delta(z_1(s), z_2(s))ds + \int_{t_k}^{t} g_\Delta(z_1(s), z_2(s))dB(s)\right|^p \\
&\leq 2^{p-1}\left[\Delta^{p-1}\mathbb{E}\int_{t_k}^{t} \left|f_\Delta(z_1(s), z_2(s))\right|^p ds\right. \\
&\quad \left. + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \Delta^{\frac{p-2}{2}}\mathbb{E}\int_{t_k}^{t} \left|g_\Delta(z_1(s), z_2(s))\right|^p ds\right] \\
&\leq 2^{p-1}\left[\Delta^p \left(h(\Delta)\right)^p + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} \Delta^{\frac{p}{2}} \left(h(\Delta)\right)^p\right] \\
&\leq 2^{p-1}\left(1 + \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}\right) \Delta^{\frac{p}{2}} \left(h(\Delta)\right)^p.
\end{aligned}$$

For $0 < p < 2$, by Hölder's inequality,

$$\mathbb{E}\left|x_\Delta(t) - z_1(t)\right|^p \leq \left(\mathbb{E}\left|x_\Delta(t) - z_1(t)\right|^2\right)^{\frac{p}{2}} \leq \left(4\Delta\left(h(\Delta)\right)^2\right)^{\frac{p}{2}} \leq 2^p\Delta^{\frac{p}{2}}\left(h(\Delta)\right)^p.$$

By setting C_p be the coefficient before $\Delta^{\frac{p}{2}}\left(h(\Delta)\right)^p$, (4.17) is done. □

Lemma 4.1.9. *There is a positive constant C_Δ dependent of Δ such that*

$$\mathbb{E}\left|x_\Delta(\delta(t)) - z_2(t)\right|^p \leq C_p\left(\Delta^{p\beta} + \Delta^{\frac{p}{2}}\left(h(\Delta)\right)^p\right) \text{ for } p > 0 \text{ and } t \geq 0. \tag{4.18}$$

Proof. For each $t \in [0, T]$, choose a k such that $t \in [t_k, t_{k+1}]$. Then,

$$x_\Delta(\delta(t)) - z_2(t) = x_\Delta(\delta(t)) - X_\Delta(I_\Delta[\delta(t_k)]\Delta) = x_\Delta(\delta(t)) - x_\Delta(I_\Delta[\delta(t_k)]\Delta)$$

Note that $\delta(t_k) - \Delta \leq I_\Delta[\delta(t_k)]\Delta \leq \delta(t_k)$. Let $p \geq 2$.
Case 1 : If $\delta(t) \geq I_\Delta[\delta(t_k)]\Delta \geq 0$, then by the Lipschitz property of δ

$$\delta(t) - I_\Delta[\delta(t_k)]\Delta \leq \delta(t) - \delta(t_k) + \Delta \leq \gamma\left|t - t_k\right| + \Delta \leq (\gamma + 1)\Delta.$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left|x_\Delta(\delta(t)) - z_2(t)\right|^p \\
&= \mathbb{E}\left|\int_{I_\Delta[\delta(t_k)]\Delta}^{\delta(t)} f_\Delta(z_1(s), z_2(s))ds + \int_{I_\Delta[\delta(t_k)]\Delta}^{\delta(t)} g_\Delta(z_1(s), z_2(s))dB\right|^p \\
&\leq 2^{p-1}((\gamma+1)\Delta)^{p-1}\mathbb{E}\int_{I_\Delta[\delta(t_k)]\Delta}^{\delta(t)} \left|f_\Delta(z_1(s), z_2(s))\right|^p ds \\
&\quad + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}((\gamma+1)\Delta)^{\frac{p-2}{2}}\mathbb{E}\int_{I_\Delta[\delta(t_k)]\Delta}^{\delta(t)} \left|g_\Delta(z_1(s), z_2(s))\right|^p ds \\
&\leq 2^{p-1}((\gamma+1)\Delta)^p\left(h(\Delta)\right)^p + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}((\gamma+1)\Delta)^{\frac{p}{2}}\left(h(\Delta)\right)^p \\
&\leq C_p\Delta^{\frac{p}{2}}\left(h(\Delta)\right)^p.
\end{aligned}
$$

Case 2 : If $0 \leq \delta(t) \leq I_\Delta[\delta(t_k)]\Delta$, then by the Lipschitz property of δ,

$$I_\Delta[\delta(t_k)]\Delta - \delta(t) \leq \delta(t_k) - \delta(t) + \Delta \leq \gamma\left|t_k - t\right| \leq \gamma\Delta.$$

Then,

$$
\begin{aligned}
&\mathbb{E}\left|x_\Delta(\delta(t)) - z_2(t)\right|^p \\
&= \mathbb{E}\left|\int_{\delta(t)}^{I_\Delta[\delta(t_k)]\Delta} f_\Delta(z_1(s), z_2(s))ds + \int_{\delta(t)}^{I_\Delta[\delta(t_k)]\Delta} g_\Delta(z_1(s), z_2(s))dB\right|^p \\
&\leq 2^{p-1}(\gamma\Delta)^{p-1}\mathbb{E}\int_{\delta(t)}^{I_\Delta[\delta(t_k)]\Delta} \left|f_\Delta(z_1(s), z_2(s))\right|^p ds \\
&\quad + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(\gamma\Delta)^{\frac{p-2}{2}}\mathbb{E}\int_{\delta(t)}^{I_\Delta[\delta(t_k)]\Delta} \left|g_\Delta(z_1(s), z_2(s))\right|^p ds \\
&\leq 2^{p-1}(\gamma\Delta)^p\left(h(\Delta)\right)^p + 2^{p-1}\left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}}(\gamma\Delta)^{\frac{p}{2}}\left(h(\Delta)\right)^p \\
&\leq C_p\Delta^{\frac{p}{2}}\left(h(\Delta)\right)^p.
\end{aligned}
$$

Case 3 : If $0 \geq \delta(t) \geq I_\Delta[\delta(t_k)]\Delta$ or $0 \geq I_\Delta[\delta(t_k)]\Delta \geq \delta(t)$, then

$$
\begin{aligned}
&0 \leq \delta(t) - I_\Delta[\delta(t_k)]\Delta \leq \delta(t) - \delta(t_k) + \Delta \leq (\gamma+1)\Delta, \\
\text{or } &0 \leq I_\Delta[\delta(t_k)]\Delta - \delta(t) \leq \delta(t_k) - \delta(t) \leq \gamma\Delta \leq (\gamma+1)\Delta.
\end{aligned}
$$

Then,

$$\begin{aligned}\mathbb{E}\left|x_\Delta(\delta(t)) - z_2(t)\right|^p &= \mathbb{E}\left|x_\Delta(\delta(t)) - x_\Delta(I_\Delta[\delta(t_k)]\Delta)\right|^p \\ &\le \mathbb{E}\left|\xi(\delta(t)) - \xi(I_\Delta[\delta(t_k)]\Delta)\right|^p \\ &\le K_2\mathbb{E}\left|\delta(t) - I_\Delta[\delta(t_k)]\Delta\right|^{p\beta} \\ &\le K_2\left((\gamma+1)\Delta\right)^{p\beta}.\end{aligned}$$

Case 4 : If $I_\Delta[\delta(t_k)]\Delta \ge 0 \ge \delta(t)$, then

$$\begin{aligned}0 \le I_\Delta[\delta(t_k)]\Delta \le \delta(t_k) \le \delta(t_k) - \delta(t) \le \gamma\Delta,\\ \text{and } 0 \le -\delta(t) \le \delta(t_k) - \delta(t) \le \gamma\Delta.\end{aligned}$$

Then,

$$\begin{aligned}&\mathbb{E}\left|x_\Delta(\delta(t)) - z_2(t)\right|^p \\ &= \mathbb{E}\left|\xi(\delta(t)) - x_\Delta(I_\Delta[\delta(t_k)]\Delta)\right|^p \\ &\le 2^{p-1}\mathbb{E}\left|\xi(\delta(t)) - \xi(0)\right|^p + 2^{p-1}\mathbb{E}\left|\xi(0) - x_\Delta(I_\Delta[\delta(t_k)]\Delta)\right|^p \\ &\le 2^{p-1}K_2\left(\gamma\Delta\right)^{p\beta} \\ &\quad + 2^{p-1}\mathbb{E}\left|\int_0^{I_\Delta[\delta(t_k)]\Delta} f_\Delta(z_1(s), z_2(s))ds + \int_0^{I_\Delta[\delta(t_k)]\Delta} g_\Delta(z_1(s), z_2(s))dB\right|^p \\ &\le 2^{p-1}K_2\left(\gamma\Delta\right)^{p\beta} + C_p\Delta^{\frac{p}{2}}\left(h(\Delta)\right)^p.\end{aligned}$$

Case 5 : If $\delta(t) \ge 0 \ge I_\Delta[\delta(t_k)]\Delta$, then

$$\begin{aligned}0 \le -I_\Delta[\delta(t_k)]\Delta \le -\delta(t_k) + \Delta \le \delta(t) - \delta(t_k) + \Delta \le (\gamma+1)\Delta,\\ \text{and } 0 \le \delta(t) \le \delta(t) - \delta(t_k) \le \gamma\Delta.\end{aligned}$$

Then,

$$\begin{aligned}
&\mathbb{E}\left|x_\Delta(\delta(t)) - z_2(t)\right|^p \\
&= \mathbb{E}\left|x_\Delta(\delta(t)) - \xi(I_\Delta[\delta(t_k)]\Delta)\right|^p \\
&\le 2^{p-1}\mathbb{E}\left|x_\Delta(\delta(t)) - \xi(0)\right|^p + 2^{p-1}\mathbb{E}\left|\xi(0) - \xi(I_\Delta[\delta(t_k)]\Delta)\right|^p \\
&\le 2^{p-1}\mathbb{E}\left|\int_0^{\delta(t)} f_\Delta(z_1(s), z_2(s))ds + \int_0^{\delta(t)} g_\Delta(z_1(s), z_2(s))dB\right|^p + 2^{p-1}K_2\left((\gamma+1)\Delta\right)^{p\beta} \\
&\le C_p\Delta^{\frac{p}{2}}\left(h(\Delta)\right)^p + 2^{p-1}K_2\left((\gamma+1)\Delta\right)^{p\beta}.
\end{aligned}$$

By combining the above cases, (4.18) is claimed. For $0 < p < 2$, (4.18) is held by using the Hölder inequality with the above result. □

Lemma 4.1.10. *Let Assumptions 4.1.1 and 4.1.2 hold. Then,*

$$\sup_{0<\Delta\le 1}\left(\sup_{-\tau\le t\le T}\mathbb{E}\left|x_\Delta(t)\right|^p\right) \le C_p. \tag{4.19}$$

Proof. Again we need only to prove for $p \ge 2$. Let $\Delta \in (0,1]$ and $T > 0$. By Itô's formula, we have, for $0 \le t \le T$,

$$\begin{aligned}
&\mathbb{E}\left|x_\Delta(t)\right|^p \\
&\le \mathbb{E}\left|\xi(0)\right|^p + \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\left(x_\Delta^T(s)f_\Delta(z_1(s), z_2(s)) + \frac{p-1}{2}\left|g_\Delta(z_1(s), z_2(s))\right|^2\right)ds \\
&= \mathbb{E}\left|\xi(0)\right|^p + \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\left(z_1^T(s)f_\Delta(z_1(s), z_2(s)) + \frac{p-1}{2}\left|g_\Delta(z_1(s), z_2(s))\right|^2\right)ds \\
&\quad + \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\left(x_\Delta(s) - z_1\right)^T f_\Delta(z_1(s), z_2(s))ds.
\end{aligned}$$

By Young's inequality and Lemma 4.1.4,

$$\begin{aligned}
\mathbb{E}\left|x_\Delta(t)\right|^p &\le \mathbb{E}\left|\xi(0)\right|^p + \mathbb{E}\int_0^t p\left|x_\Delta(s)\right|^{p-2}\widehat{K_p}\left(1 + \left|z_1(s)\right|^2 + \left|z_2(s)\right|^2\right)ds \\
&\quad + \mathbb{E}\int_0^t\left[(p-2)\left|x_\Delta(s)\right|^p + 2\left|x_\Delta(s) - z_1(s)\right|^{p/2}\left|f_\Delta(z_1(s), z_2(s))\right|^{p/2}\right]ds \\
&\le \mathbb{E}\left|\xi(0)\right|^p + \widehat{K_p}\mathbb{E}\int_0^t\left[2(p-2)\left|x_\Delta(s)\right|^p + 3^p + 3^p\left|z_1(s)\right|^p + 3^p\left|z_2(s)\right|^p\right]ds \\
&\quad + 2(h(\Delta))^{p/2}\int_0^T\mathbb{E}\left|x_\Delta(s) - z_1(s)\right|^{p/2}ds
\end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\,|x_\Delta(t)|^p \\
&\leq \mathbb{E}\,|\xi(0)|^p + \widehat{K_p}\int_0^t \left[2(p-2)\mathbb{E}\,|x_\Delta(s)|^p + 3^p + 3^p\mathbb{E}\,|z_1(s)|^p + 3^p\mathbb{E}\,|z_2(s)|^p\right] ds \\
&\quad + C_p\Delta^{p/4}(h(\Delta))^p T \\
&\leq \mathbb{E}\,|\xi(0)|^p + 3^p T\widehat{K_p} + \widehat{K_p}\left(2(p-2)\vee 3^p\right)\int_0^t \left[\mathbb{E}\,|x_\Delta(s)|^p + \mathbb{E}\,|z_1(s)|^p + \mathbb{E}\,|z_2(s)|^p\right] ds \\
&\quad + C_p\Delta^{p/4}(h(\Delta))^p \\
&\leq C_p\int_0^t \left(\sup_{0\leq u\leq s}\mathbb{E}\,|x_\Delta(u)|^p\right) ds + C_p\Delta^{p/4}(h(\Delta))^p.
\end{aligned}
$$

Hence, by the definition of h which is $\Delta^{1/4}h(\Delta)\leq 1$,

$$
\sup_{0\leq u\leq t}\mathbb{E}\,|x_\Delta(u)|^p \leq C_p + C_p\int_0^t \left(\sup_{0\leq u\leq s}\mathbb{E}\,|x_\Delta(u)|^p\right) ds.
$$

By Gronwall's inequality,

$$
\sup_{0\leq u\leq T}\mathbb{E}\,|x_\Delta(u)|^p \leq C_p.
$$

For $-\tau\leq t<0$, by definition of x_Δ, $\mathbb{E}\,|x_\Delta(t)|^p$ is bounded, so

$$
\sup_{-\tau\leq u\leq T}\mathbb{E}\,|x_\Delta(u)|^p \leq C_p.
$$

Since Δ is arbitrary on $(0,1]$ the right hand side is independence of Δ,

$$
\sup_{0<\Delta\leq 1}\left(\sup_{-\tau\leq t\leq T}\mathbb{E}\,|x_\Delta(t)|^p\right) \leq C_p. \qquad \square
$$

By Lemma 4.1.10, it can be implied that

$$
\sup_{0<\Delta\leq 1}\left(\sup_{0\leq t\leq T}\mathbb{E}\,|z_1(t)|^p\right) \leq C_p \tag{4.20}
$$

$$
\sup_{0<\Delta\leq 1}\left(\sup_{0\leq t\leq T}\mathbb{E}\,|z_2(t)|^p\right) \leq C_p. \tag{4.21}
$$

4.2 Convergence rate at a finite time T

Now, we state the main theorem by assuming some conditions to fit with the variable time delay, as shown.

Theorem 4.2.1. *Let Assumptions 4.1.2, 4.1.5, 4.1.6 and 4.1.7 hold. Assume that* $\frac{d\delta(t)}{dt} \geq \frac{\lambda_2}{\lambda_1}$ *for all* $t \in [-\tau, T]$ *and* $p \geq \rho + 2$. *Then, for* $\Delta \in (0, 1]$,

$$\mathbb{E}\left|x(T) - x_\Delta(T)\right|^2 \leq C_p \Bigg(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)} \vee \Delta h(\Delta)^2 \vee \Delta^{2\rho} \Bigg), \tag{4.22}$$

$$\mathbb{E}\left|x(T) - z_1(T)\right|^2 \leq C_p \Bigg(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)} \vee \Delta h(\Delta)^2 \vee \Delta^{2\rho} \Bigg). \tag{4.23}$$

Proof. Let $n \geq |x_0|$, define $\theta_n = \inf\{t \geq 0 : |x(t)| \vee |x_\Delta(t)| \geq n\}$. Let $e_\Delta(t) = x(t) - x_\Delta(t)$ for $-\tau \leq t \leq T$. By Itô's formula,

$$\begin{aligned} d\left|e_\Delta(t)\right|^2 \leq 2 \Bigg[& e_\Delta^T(t) \left(f(x(t), x(\delta(t))) - f_\Delta(z_1(t), z_2(t)) \right) \\ & + \frac{1}{2} \left| g(x(t), x(\delta(t))) - g_\Delta(z_1(t), z_2(t)) \right|^2 \Bigg] dt \\ & + 2 e_\Delta^T(t) \left(g(x(t), x(\delta(t))) - g_\Delta(z_1(t), z_2(t)) \right) dB(t). \end{aligned}$$

Then, for $0 \leq t \leq T$,

$$\begin{aligned} & \mathbb{E}\left|e_\Delta(t \wedge \theta_n)\right|^2 \\ & = \mathbb{E}\Bigg\{ \int_0^{t\wedge\theta_n} \Big[2e_\Delta^T(s)(f(x(s), x(\delta(s))) - f_\Delta(z_1(s), z_2(s))) \\ & \quad + \left| g(x(s), x(\delta(s))) - g_\Delta(z_1(s), z_2(s)) \right|^2 \Big] ds \Bigg\} \end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\left|e_\Delta(t\wedge\theta_n)\right|^2 \\
&\quad\le \mathbb{E}\Bigg\{\int_0^{t\wedge\theta_n}\Big[2e_\Delta^T(s)(f(x(s),x(\delta(s)))-f(x_\Delta(s),x_\Delta(\delta(s))))\\
&\qquad+(1+\alpha)\left|g(x(s),x(\delta(s)))-g(x_\Delta(s),x_\Delta(\delta(s)))\right|^2\Big]ds\\
&\qquad+\int_0^{t\wedge\theta_n}\left(2e_\Delta^T(s)(f(x_\Delta(s),x_\Delta(\delta(s)))-f_\Delta(z_1(s),z_2(s)))\right)ds\\
&\qquad+\int_0^{t\wedge\theta_n}\left(1+\alpha^{-1}\right)\left|g(x_\Delta(s),x_\Delta(\delta(s)))-g_\Delta(z_1(s),z_2(s))\right|^2ds\Bigg\}\\
&\quad\le \mathbb{E}\Bigg\{\int_0^{t\wedge\theta_n}\Big[2K_2\left(|x(s)-x_\Delta(s)|^2+|x(\delta(s))-x_\Delta(\delta(s))|^2\right)\\
&\qquad-2\lambda_1U(x(s),x_\Delta(s))+2\lambda_2U(x(\delta(s)),x_\Delta(\delta(s)))\Big]ds\\
&\qquad+\int_0^{t\wedge\theta_n}2\left|e_\Delta^T(s)\right|\left|(f(x_\Delta(s),x_\Delta(\delta(s)))-f_\Delta(z_1(s),z_2(s)))\right|ds\\
&\qquad+\int_0^{t\wedge\theta_n}\left(1+\alpha^{-1}\right)\left|g(x_\Delta(s),x_\Delta(\delta(s)))-g_\Delta(z_1(s),z_2(s))\right|^2ds\Bigg\}\\
&\quad\le \mathbb{E}\Bigg\{\int_0^{t\wedge\theta_n}\Big[2K_2\left(|e_\Delta(s)|^2+|e_\Delta(\delta(s))|^2\right)+|e_\Delta(s)|^2\\
&\qquad-2\lambda_1U(x(s),x_\Delta(s))+2\lambda_2U(x(\delta(s)),x_\Delta(\delta(s)))\Big]ds\Bigg\}\\
&\qquad+\mathbb{E}\int_0^{t\wedge\theta_n}\left|(f(x_\Delta(s),x_\Delta(\delta(s)))-f_\Delta(z_1(s),z_2(s)))\right|^2ds\\
&\qquad+\mathbb{E}\int_0^{t\wedge\theta_n}\left(1+\alpha^{-1}\right)\left|g(x_\Delta(s),x_\Delta(\delta(s)))-g_\Delta(z_1(s),z_2(s))\right|^2ds\\
&\quad\le \mathbb{E}\Bigg\{\int_0^{t\wedge\theta_n}\Big[2K_2\left(|e_\Delta(s)|^2+|e_\Delta(\delta(s))|^2\right)+|e_\Delta(s)|^2\\
&\qquad-2\lambda_1U(x(s),x_\Delta(s))+2\lambda_2U(x(\delta(s)),x_\Delta(\delta(s)))\Big]ds\Bigg\}\\
&\qquad+\mathbb{E}\int_0^{t\wedge\theta_n}\left(2+\alpha^{-1}\right)K_3\left(|x_\Delta(s)-\pi_\Delta(z_1(s))|^2+|x_\Delta(\delta(s))-\pi_\Delta(z_2(s))|^2\right)\\
&\qquad\quad(1+|x_\Delta(s)|^\rho+|\pi_\Delta(z_1(s))|^\rho+|x_\Delta(\delta(s))|^\rho+|\pi_\Delta(z_2(s))|^\rho)\,ds\\
&\quad\le H_1+H_2,
\end{aligned}
$$

where

$$H_1 = \mathbb{E}\Bigg\{\int_0^{t\wedge\theta_n} \Big[2K_2 \left(|e_\Delta(s)|^2 + |e_\Delta(\delta(s))|^2\right) + |e_\Delta(s)|^2$$

$$- 2\lambda_1 U(x(s), x_\Delta(s)) + 2\lambda_2 U(x(\delta(s)), x_\Delta(\delta(s)))\Big] ds\Bigg\}$$

$$H_2 = \mathbb{E}\int_0^{t\wedge\theta_n} \left(2+\alpha^{-1}\right) K_3 \left(|x_\Delta(s) - \pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s)) - \pi_\Delta(z_2(s))|^2\right)$$

$$\times \left(1 + |x_\Delta(s)|^\rho + |\pi_\Delta(z_1(s))|^\rho + |x_\Delta(\delta(s))|^\rho + |\pi_\Delta(z_2(s))|^\rho\right) ds.$$

Since $\int_{-\tau}^0 |e_\Delta(s)|^2\, ds = 0$, we can derive by setting $a = \delta(s)$

$$\begin{aligned}
\int_0^{t\wedge\theta_n} |e_\Delta(\delta(s))|^2\, ds &= \int_{\delta(0)}^{\delta(t\wedge\theta_n)} |e_\Delta(a)|^2\, da \cdot \left(\frac{d\delta(s)}{ds}\right)^{-1} \\
&\le \int_{-\tau}^{t\wedge\theta_n} |e_\Delta(s)|^2\, ds \left(\frac{\lambda_1}{\lambda_2}\right) \\
&= \int_0^{t\wedge\theta_n} |e_\Delta(s)|^2\, ds \left(\frac{\lambda_1}{\lambda_2}\right).
\end{aligned}$$

Note that $U(x(s), x_\Delta(s)) = 0$ for all $s \in [-\tau, 0]$, then

$$\begin{aligned}
\int_0^{t\wedge\theta_n} U(x(\delta(s)), x_\Delta(\delta(s)))ds &= \int_{\delta(0)}^{\delta(t\wedge\theta_n)} U(x(a), x_\Delta(a))da \cdot \left(\frac{d\delta(s)}{ds}\right)^{-1} \\
&\le \left(\frac{\lambda_1}{\lambda_2}\right) \int_{-\tau}^{t\wedge\theta_n} U(x(s), x_\Delta(s))ds \\
&= \left(\frac{\lambda_1}{\lambda_2}\right) \int_0^{t\wedge\theta_n} U(x(s), x_\Delta(s))ds.
\end{aligned}$$

Hence,

$$\begin{aligned}
H_1 &\le \mathbb{E}\int_0^{t\wedge\theta_n} \left(2K_2\left(1 + \frac{\lambda_1}{\lambda_2}\right) + 1\right) |e_\Delta(s)|^2\, ds \\
&\le \left(2K_2\left(1 + \frac{\lambda_1}{\lambda_2}\right) + 1\right) \int_0^t \mathbb{E}\, |e_\Delta(s\wedge\theta_n)|^2\, ds.
\end{aligned}$$

By Lemmas 4.1.8, 4.1.9, 4.1.10 and the fact that $|\pi_\Delta(x)|^p \le |x|^p$,

$$
\begin{aligned}
& H_2 \\
&= \mathbb{E}\int_0^{t\wedge\theta_n} \left(2+\alpha^{-1}\right) K_3 \left(|x_\Delta(s)-\pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s))-\pi_\Delta(z_2(s))|^2\right) \\
&\quad \times \left(1+|x_\Delta(s)|^\rho + |\pi_\Delta(z_1(s))|^\rho + |x_\Delta(\delta(s))|^\rho + |\pi_\Delta(z_2(s))|^\rho\right) ds \\
&\le \mathbb{E}\int_0^{t\wedge\theta_n} 2\left(2+\alpha^{-1}\right) K_3 \Bigg[\Bigg(|x_\Delta(s)-\pi_\Delta(x_\Delta(s))|^2 + |\pi_\Delta(x_\Delta(s))-\pi_\Delta(z_1(s))|^2 \\
&\quad + |x_\Delta(\delta(s))-\pi_\Delta(x_\Delta(\delta(s)))|^2 + |\pi_\Delta(x_\Delta(\delta(s)))-\pi_\Delta(z_2(s))|^2 \Bigg) \\
&\quad \times \left(1+|x_\Delta(s)|^\rho + |\pi_\Delta(z_1(s))|^\rho + |x_\Delta(\delta(s))|^\rho + |\pi_\Delta(z_2(s))|^\rho\right)\Bigg] ds \\
&\le 2\left(2+\alpha^{-1}\right) K_3 \int_0^T \mathbb{E}\Bigg[\Bigg(|x_\Delta(s)-\pi_\Delta(x_\Delta(s))|^2 + |\pi_\Delta(x_\Delta(s))-\pi_\Delta(z_1(s))|^2 \\
&\quad + |x_\Delta(\delta(s))-\pi_\Delta(x_\Delta(\delta(s)))|^2 + |\pi_\Delta(x_\Delta(\delta(s)))-\pi_\Delta(z_2(s))|^2 \Bigg) \\
&\quad \times \left(1+|x_\Delta(s)|^\rho + |z_1(s)|^\rho + |x_\Delta(\delta(s))|^\rho + |z_2(s)|^\rho\right)\Bigg] ds \\
&\le C_p \int_0^T \Bigg[4^{\frac{\rho}{p}}\Bigg(\mathbb{E}\,|x_\Delta(s)-\pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|\pi_\Delta(x_\Delta(s))-\pi_\Delta(z_1(s))|^{\frac{2p}{p-\rho}} \\
&\quad + \mathbb{E}\,|x_\Delta(\delta(s))-\pi_\Delta(x_\Delta(\delta(s)))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|\pi_\Delta(x_\Delta(\delta(s)))-\pi_\Delta(z_2(s))|^{\frac{2p}{p-\rho}} \Bigg)^{\frac{p-\rho}{p}} \\
&\quad \times 5^{\frac{p-\rho}{p}} \left(1+\mathbb{E}\,|x_\Delta(s)|^p + \mathbb{E}\,|z_1(s)|^p + \mathbb{E}\,|x_\Delta(\delta(s))|^p + \mathbb{E}\,|z_2(s)|^p\right)^{\frac{\rho}{p}}\Bigg] ds \\
&\le C_p \int_0^T \Bigg(\mathbb{E}\,|x_\Delta(s)-\pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|x_\Delta(s)-z_1(s)|^{\frac{2p}{p-\rho}} \\
&\quad + \mathbb{E}\,|x_\Delta(\delta(s))-\pi_\Delta(x_\Delta(\delta(s)))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|x_\Delta(\delta(s))-z_2(s)|^{\frac{2p}{p-\rho}} \Bigg)^{\frac{p-\rho}{p}} ds \\
&\le C_p \int_0^T \Bigg[\left(\mathbb{E}\,|x_\Delta(s)-\pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} + \left(\mathbb{E}\,|x_\Delta(\delta(s))-\pi_\Delta(x_\Delta(\delta(s)))|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} \\
&\quad + \Delta\,(h(\Delta))^2 + \left(\Delta^{2\beta} + \Delta\,(h(\Delta))^2\right)\Bigg] ds
\end{aligned}
$$

$$
\begin{aligned}
H_2 \leq C_p \int_0^T &\left(\mathbb{E}\left|x_\Delta(s) - \pi_\Delta(x_\Delta(s))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds + C_p\left(\Delta^{2\beta} + \Delta\left(h(\Delta)\right)^2\right) \\
&+ C_p \int_0^T \left(\mathbb{E}\left|x_\Delta(\delta(s)) - \pi_\Delta(x_\Delta(\delta(s)))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds.
\end{aligned}
$$

Consider the integral $\int_0^T \left(\mathbb{E}\left|x_\Delta(s) - \pi_\Delta(x_\Delta(s))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds$,

$$
\begin{aligned}
&\int_0^T \left(\mathbb{E}\left|x_\Delta(s) - \pi_\Delta(x_\Delta(s))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds \\
&\quad \leq \int_0^T \left(\mathbb{E}\left[I_{\{|x_\Delta(s)|>\mu^{-1}(h(\Delta))\}}\left|x_\Delta(s)\right|^{\frac{2p}{p-\rho}}\right]\right)^{\frac{p-\rho}{p}} ds \\
&\quad \leq \int_0^T \left(\left[\mathbb{P}\left\{|x_\Delta(s)| > \mu^{-1}(h(\Delta))\right\}\right]^{\frac{p-\rho-2}{p-\rho}}\left[\mathbb{E}\left|x_\Delta(s)\right|^p\right]^{\frac{2}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds \\
&\quad \leq \int_0^T \left(\left[\frac{\mathbb{E}\left|x_\Delta(s)\right|^p}{(\mu^{-1}(h(\Delta)))^p}\right]^{\frac{p-\rho-2}{p}}\left[\mathbb{E}\left|x_\Delta(s)\right|^p\right]^{\frac{2}{p}}\right) ds \\
&\quad \leq \int_0^T \left(\left[\frac{1}{(\mu^{-1}(h(\Delta)))^p}\right]^{\frac{p-\rho-2}{p}}\left[\mathbb{E}\left|x_\Delta(s)\right|^p\right]^{\frac{p-\rho}{p}}\right) ds.
\end{aligned}
$$

By Lemma 4.1.10,

$$
\int_0^T \left(\mathbb{E}\left|x_\Delta(s) - \pi_\Delta(x_\Delta(s))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds \leq C_p\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)}.
$$

Similarly, we can derive

$$
\int_0^T \left(\mathbb{E}\left|x_\Delta(\delta(s)) - \pi_\Delta(x_\Delta(\delta(s)))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds \leq C_p\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)}.
$$

Hence,

$$
H_2 \leq C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)} + \Delta^{2\beta} + \Delta\left(h(\Delta)\right)^2\right).
$$

Now combine H_1 and H_2,

$$
\begin{aligned}
\mathbb{E}\left|e_\Delta(t\wedge\theta_n)\right|^2 \leq{}& C_p\int_0^t \mathbb{E}\left|e_\Delta(s\wedge\theta_n)\right|^2 ds \\
&+ C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)} \vee \Delta^{2\beta} \vee \Delta\left(h(\Delta)\right)^2\right).
\end{aligned}
$$

By the Gronwall inequality,

$$\mathbb{E}\left|e_\Delta(t\wedge\theta_n)\right|^2 \le C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)} \vee \Delta^{2\beta} \vee \Delta\left(h(\Delta)\right)^2\right).$$

By letting $n\to\infty$, (4.22) is proved. Hence, (4.23) is done by Lemma 4.1.8. □

4.3 Convergence rate over a finite time interval

To find the rate of convergence over a finite time interval, we also need a strong condition on the diffusion coefficient.

Assumption 4.3.1. *There is a pair of constants ρ and $K_4>0$ such that for all $x,y,\bar{x},\bar{y}\in\mathbb{R}^d$*

$$|f(x,y)-f(\bar{x},\bar{y})|^2 \le K_4\left(|x-\bar{x}|^2+|y-\bar{y}|^2\right)\left(1+|x|^\rho+|\bar{x}|^\rho+|y|^\rho+|\bar{y}|^\rho\right)$$

and

$$|g(x,y)-g(\bar{x},\bar{y})|^2 \le K_4\left(|x-\bar{x}|^2+|y-\bar{y}|^2\right).$$

Theorem 4.3.2. *Let Assumptions 4.1.2, 4.1.5, 4.1.6 and 4.3.1 hold. Assume that $\frac{d\delta(t)}{dt}\ge\frac{\lambda_2}{\lambda_1}$ for all $t\in[-\tau,T]$ and $p\ge\rho+2$. Then, for any $\Delta\in(0,1]$,*

$$\mathbb{E}\left(\sup_{0\le u\le T}|x(u)-x_\Delta(u)|^2\right) \le C_p\Bigg(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)} \vee \Delta h(\Delta)^2 \vee \Delta^{2\rho}\Bigg). \tag{4.24}$$

Proof. For any $n\ge|x_0|$, define $\theta_n=\inf\{t\ge0:|x(t)|\vee|x_\Delta(t)|\ge n\}$. Let $e_\Delta(t)=x(t)-x_\Delta(t)$ for $t\ge0$. By Itô's formula,

$$\begin{aligned} d\left|e_\Delta(t)\right|^2 \le 2\Bigg[&e_\Delta^T(t)\left(f(x(t),x(\delta(t)))-f_\Delta(z_1(t),z_2(t))\right)\\ &+\frac{1}{2}\left|g(x(t),x(\delta(t)))-g_\Delta(z_1(t),z_2(t))\right|^2\Bigg]dt\\ &+2e_\Delta^T(t)\left(g(x(t),x(\delta(t)))-g_\Delta(z_1(t),z_2(t))\right)dB(t)\end{aligned}$$

Then, for $0 \le t \le T$,

$$\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n} |e_\Delta(u)|^2\right) \le H_3 + H_4,$$

where,

$$\begin{aligned}
H_3 = \mathbb{E} \sup_{0\le u\le t\wedge\theta_n} \Bigg\{ \int_0^u \Big[& 2e_\Delta^T(s)(f(x(s), x(\delta(t))) - f_\Delta(z_1(s), z_2(s))) \\
& + |g(x(s), x(\delta(s))) - g_\Delta(z_1(s), z_2(s))|^2 \Big] ds \Bigg\} \\
H_4 = \mathbb{E} \Bigg(\sup_{0\le u\le t\wedge\theta_n} & \int_0^u 2e_\Delta^T(s)(g(x(s), x(\delta(t))) - g_\Delta(z_1(s), z_2(s)))dB(s) \Bigg).
\end{aligned}$$

We now estimate H_3 and H_4, respectively.

$$\begin{aligned}
H_3 \le \mathbb{E} \sup_{0\le u\le t\wedge\theta_n} \Bigg\{ & \int_0^u \Big[2e_\Delta^T(s)(f(x(s), x(\delta(s))) - f(x_\Delta(s), x_\Delta(\delta(s)))) \\
& + (1+\alpha) |g(x(s), x(\delta(s))) - g(x_\Delta(s), x_\Delta(\delta(s)))|^2 \Big] ds \\
& + \int_0^u \left(2e_\Delta^T(s)(f(x_\Delta(s), x_\Delta(\delta(s))) - f_\Delta(z_1(s), z_2(s)))\right) ds \\
& + \int_0^u \left(1+\alpha^{-1}\right) |g(x_\Delta(s), x_\Delta(\delta(s))) - g_\Delta(z_1(s), z_2(s))|^2 ds \Bigg\} \\
\le \mathbb{E} \sup_{0\le u\le t\wedge\theta_n} \Bigg\{ & \int_0^u \Big[2K_2 \left(|x(s) - x_\Delta(s)|^2 + |x(\delta(s)) - x_\Delta(\delta(s))|^2\right) \\
& - 2\lambda_1 U(x(s), x_\Delta(s)) + 2\lambda_2 U(x(\delta(s)), x_\Delta(\delta(s))) \Big] ds \\
& + \int_0^u 2\left|e_\Delta^T(s)\right| |(f(x_\Delta(s), x_\Delta(\delta(s))) - f_\Delta(z_1(s), z_2(s)))| ds \\
& + \int_0^u \left(1+\alpha^{-1}\right) K_4 \left(|x_\Delta(s) - \pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s)) - \pi_\Delta(z_2(s))|^2\right) ds \Bigg\}
\end{aligned}$$

$$
\begin{aligned}
H_3 \le\ & \mathbb{E} \sup_{0\le u\le t\wedge\theta_n} \left\{ \int_0^u \Big[2K_2 \left(|e_\Delta(s)|^2 + |e_\Delta(\delta(s))|^2 \right) + |e_\Delta(s)|^2 \right. \\
& \left. - 2\lambda_1 U(x(s), x_\Delta(s)) + 2\lambda_2 U(x(\delta(s)), x_\Delta(\delta(s))) \Big] ds \right\} \\
& + \mathbb{E} \sup_{0\le u\le t\wedge\theta_n} \int_0^u |(f(x_\Delta(s), x_\Delta(\delta(s))) - f_\Delta(z_1(s), z_2(s)))|^2 \, ds \\
& + \mathbb{E} \int_0^{t\wedge\theta_n} \left(1+\alpha^{-1}\right) K_4 \left(|x_\Delta(s) - \pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s)) - \pi_\Delta(z_2(s))|^2 \right) ds \\
\le\ & \mathbb{E} \sup_{0\le u\le t\wedge\theta_n} \left\{ \int_0^u \Big[2K_2 \left(|e_\Delta(s)|^2 + |e_\Delta(\delta(s))|^2 \right) + |e_\Delta(s)|^2 \right. \\
& \left. - 2\lambda_1 U(x(s), x_\Delta(s)) + 2\lambda_2 U(x(\delta(s)), x_\Delta(\delta(s))) \Big] ds \right\} \\
& + \mathbb{E} \int_0^{t\wedge\theta_n} K_4 \left(|x_\Delta(s) - \pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s)) - \pi_\Delta(z_2(s))|^2 \right) \\
& \quad (1 + |x_\Delta(s)|^\rho + |\pi_\Delta(z_1(s))|^\rho + |x_\Delta(\delta(s))|^\rho + |\pi_\Delta(z_2(s))|^\rho) \, ds \\
& + \mathbb{E} \int_0^{t\wedge\theta_n} \left(1+\alpha^{-1}\right) K_4 \left(|x_\Delta(s) - \pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s)) - \pi_\Delta(z_2(s))|^2 \right) ds \\
\le\ & H_{31} + H_{32},
\end{aligned}
$$

where

$$
\begin{aligned}
H_{31} =\ & \mathbb{E} \sup_{0\le u\le t\wedge\theta_n} \left\{ \int_0^u \Big[2K_2 \left(|e_\Delta(s)|^2 + |e_\Delta(\delta(s))|^2 \right) + |e_\Delta(s)|^2 \right. \\
& \left. - 2\lambda_1 U(x(s), x_\Delta(s)) + 2\lambda_2 U(x(\delta(s)), x_\Delta(\delta(s))) \Big] ds \right\} \\
H_{32} =\ & \mathbb{E} \int_0^{t\wedge\theta_n} K_4 \left(|x_\Delta(s) - \pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s)) - \pi_\Delta(z_2(s))|^2 \right) \\
& \times \left(\left(2+\alpha^{-1}\right) + |x_\Delta(s)|^\rho + |\pi_\Delta(z_1(s))|^\rho + |x_\Delta(\delta(s))|^\rho + |\pi_\Delta(z_2(s))|^\rho \right) ds.
\end{aligned}
$$

Since $\int_{-\tau}^{0} |e_\Delta(s)|^2\, ds = 0$, we can derive by setting $a = \delta(s)$

$$\begin{aligned}\int_0^u |e_\Delta(\delta(s))|^2\, ds &= \int_{\delta(0)}^{\delta(u)} |e_\Delta(a)|^2\, da \cdot \left(\frac{d\delta(s)}{ds}\right)^{-1}\\ &\leq \int_{-\tau}^{u} |e_\Delta(s)|^2\, ds \left(\frac{\lambda_1}{\lambda_2}\right)\\ &= \int_0^u |e_\Delta(s)|^2\, ds \left(\frac{\lambda_1}{\lambda_2}\right).\end{aligned}$$

Note that $U(x(s), x_\Delta(s)) = 0$ for all $s \in [-\tau, 0]$, then

$$\begin{aligned}\int_0^u U(x(\delta(s)), x_\Delta(\delta(s)))ds &= \int_{\delta(0)}^{\delta(u)} U(x(a), x_\Delta(a))da \cdot \left(\frac{d\delta(s)}{ds}\right)^{-1}\\ &\leq \left(\frac{\lambda_1}{\lambda_2}\right) \int_{-\tau}^{u} U(x(s), x_\Delta(s))ds\\ &= \left(\frac{\lambda_1}{\lambda_2}\right) \int_0^u U(x(s), x_\Delta(s))ds.\end{aligned}$$

Hence,

$$\begin{aligned}H_{31} &\leq \mathbb{E} \sup_{0\leq u\leq t\wedge\theta_n} \left(\int_0^u \left(2K_2\left(1+\frac{\lambda_1}{\lambda_2}\right)+1\right) |e_\Delta(s)|^2\, ds\right)\\ &\leq \left(2K_2\left(1+\frac{\lambda_1}{\lambda_2}\right)+1\right) \int_0^t \mathbb{E}\left(\sup_{0\leq u\leq s\wedge\theta_n} |e_\Delta(u)|^2\right) ds.\end{aligned}$$

By Lemmas 4.1.8, 4.1.9, 4.1.10 and the fact that $|\pi_\Delta(x)|^p \leq |x|^p$,

$$\begin{aligned}H_{32} = \mathbb{E} &\int_0^{t\wedge\theta_n} K_4 \left(|x_\Delta(s) - \pi_\Delta(z_1(s))|^2 + |x_\Delta(\delta(s)) - \pi_\Delta(z_2(s))|^2\right)\\ &\times \left(\left(2+\alpha^{-1}\right) + |x_\Delta(s)|^\rho + |\pi_\Delta(z_1(s))|^\rho + |x_\Delta(\delta(s))|^\rho + |\pi_\Delta(z_2(s))|^\rho\right) ds\\ \leq \mathbb{E} &\int_0^{t\wedge\theta_n} 2K_4 \Bigg[\Bigg(|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^2 + |\pi_\Delta(x_\Delta(s)) - \pi_\Delta(z_1(s))|^2\\ &+ |x_\Delta(\delta(s)) - \pi_\Delta(x_\Delta(\delta(s)))|^2 + |\pi_\Delta(x_\Delta(\delta(s))) - \pi_\Delta(z_2(s))|^2 \Bigg)\\ &\times \left(\left(2+\alpha^{-1}\right) + |x_\Delta(s)|^\rho + |\pi_\Delta(z_1(s))|^\rho + |x_\Delta(\delta(s))|^\rho + |\pi_\Delta(z_2(s))|^\rho\right)\Bigg] ds\end{aligned}$$

$$
\begin{aligned}
&\leq 2K_4 \int_0^T \mathbb{E}\Bigg[\Bigg(|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^2 + |\pi_\Delta(x_\Delta(s)) - \pi_\Delta(z_1(s))|^2 \\
&\quad + |x_\Delta(\delta(s)) - \pi_\Delta(x_\Delta(\delta(s)))|^2 + |\pi_\Delta(x_\Delta(\delta(s))) - \pi_\Delta(z_2(s))|^2 \Bigg) \\
&\quad \times \left(\left(2 + \alpha^{-1}\right) + |x_\Delta(s)|^\rho + |z_1(s)|^\rho + |x_\Delta(\delta(s))|^\rho + |z_2(s)|^\rho \right) \Bigg] ds \\
&\leq 2K_4 \int_0^T \Bigg[4^{\frac{\rho}{p}} \Bigg(\mathbb{E}\,|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|\pi_\Delta(x_\Delta(s)) - \pi_\Delta(z_1(s))|^{\frac{2p}{p-\rho}} \\
&\quad + \mathbb{E}\,|x_\Delta(\delta(s)) - \pi_\Delta(x_\Delta(\delta(s)))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|\pi_\Delta(x_\Delta(\delta(s))) - \pi_\Delta(z_2(s))|^{\frac{2p}{p-\rho}} \Bigg)^{\frac{p-\rho}{p}} \\
&\quad \times 5^{\frac{p-\rho}{p}} \left(\left(2 + \alpha^{-1}\right) + \mathbb{E}\,|x_\Delta(s)|^p + \mathbb{E}\,|z_1(s)|^p + \mathbb{E}\,|x_\Delta(\delta(s))|^p + \mathbb{E}\,|z_2(s)|^p \right)^{\frac{\rho}{p}} \Bigg] ds \\
&\leq C_p \int_0^T \Bigg(\mathbb{E}\,|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|x_\Delta(s) - z_1(s)|^{\frac{2p}{p-\rho}} \\
&\quad + \mathbb{E}\,|x_\Delta(\delta(s)) - \pi_\Delta(x_\Delta(\delta(s)))|^{\frac{2p}{p-\rho}} + \mathbb{E}\,|x_\Delta(\delta(s)) - z_2(s)|^{\frac{2p}{p-\rho}} \Bigg)^{\frac{p-\rho}{p}} ds \\
&\leq C_p \int_0^T \left(\mathbb{E}\,|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}} \right)^{\frac{p-\rho}{p}} ds \\
&\quad + C_p \int_0^T \left(\mathbb{E}\,|x_\Delta(\delta(s)) - \pi_\Delta(x_\Delta(\delta(s)))|^{\frac{2p}{p-\rho}} \right)^{\frac{p-\rho}{p}} ds + C_p \left(\Delta^{2\beta} + \Delta\,(h(\Delta))^2 \right).
\end{aligned}
$$

Consider the integral $\int_0^T \left(\mathbb{E}\,|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}} \right)^{\frac{p-\rho}{p}} ds$,

$$
\begin{aligned}
&\int_0^T \left(\mathbb{E}\,|x_\Delta(s) - \pi_\Delta(x_\Delta(s))|^{\frac{2p}{p-\rho}} \right)^{\frac{p-\rho}{p}} ds \\
&\qquad \leq \int_0^T \left(\mathbb{E}\left[I_{\{|x_\Delta(s)| > \mu^{-1}(h(\Delta))\}} \,|x_\Delta(s)|^{\frac{2p}{p-\rho}} \right] \right)^{\frac{p-\rho}{p}} ds \\
&\qquad \leq \int_0^T \left(\left[\mathbb{P}\left\{ |x_\Delta(s)| > \mu^{-1}(h(\Delta)) \right\} \right]^{\frac{p-\rho-2}{p-\rho}} \left[\mathbb{E}\,|x_\Delta(s)|^p \right]^{\frac{2}{p-\rho}} \right)^{\frac{p-\rho}{p}} ds \\
&\qquad \leq \int_0^T \left(\left[\frac{\mathbb{E}\,|x_\Delta(s)|^p}{(\mu^{-1}(h(\Delta)))^p} \right]^{\frac{p-\rho-2}{p}} \left[\mathbb{E}\,|x_\Delta(s)|^p \right]^{\frac{2}{p}} \right) ds \\
&\qquad \leq \int_0^T \left(\left[\frac{1}{(\mu^{-1}(h(\Delta)))^p} \right]^{\frac{p-\rho-2}{p}} \left[\mathbb{E}\,|x_\Delta(s)|^p \right]^{\frac{p-\rho}{p}} \right) ds.
\end{aligned}
$$

By Lemma 4.1.10,

$$\int_0^T \left(\mathbb{E}\left|x_\Delta(s)-\pi_\Delta(x_\Delta(s))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds \le C_p\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)}.$$

Similarly, we can derive

$$\int_0^T \left(\mathbb{E}\left|x_\Delta(\delta(s))-\pi_\Delta(x_\Delta(\delta(s)))\right|^{\frac{2p}{p-\rho}}\right)^{\frac{p-\rho}{p}} ds \le C_p\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)}.$$

Hence,

$$H_{32} \le C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)} + \Delta^{2\beta} + \Delta\left(h(\Delta)\right)^2\right).$$

Now we consider H_4,

$$\begin{aligned}
H_4 &= \mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}\int_0^u 2e_\Delta^T(s)(g(x(s),x(\delta(s))) - g_\Delta(z_1(s),z_2(s)))dB(s)\right)\\
&\le 4\sqrt{2}\mathbb{E}\left(\int_0^{t\wedge\theta_n}\left|2e_\Delta^T(s)(g(x(s),x(\delta(s))) - g_\Delta(z_1(s),z_2(s)))\right|^2 ds\right)^{\frac{1}{2}}\\
&\le 8\sqrt{2}\mathbb{E}\left(\int_0^{t\wedge\theta_n}\left|e_\Delta(s)\right|^2\left|g(x(s),x(\delta(s))) - g_\Delta(z_1(s),z_2(s))\right|^2 ds\right)^{\frac{1}{2}}\\
&\le \mathbb{E}\left(\left[\sup_{0\le u\le t\wedge\theta_n}\left|e_\Delta(u)\right|^2\right]\right)^{\frac{1}{2}}\\
&\quad\times\left(128\int_0^{t\wedge\theta_n}\left|g(x(s),x(\delta(s))) - g(\pi_\Delta(z_1(s)),\pi_\Delta(z_2(s)))\right|^2 ds\right)^{\frac{1}{2}}\\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}\left|e_\Delta(u)\right|^2\right)\\
&\quad+64\mathbb{E}\int_0^{t\wedge\theta_n}\left|g(x(s),x(\delta(s))) - g(\pi_\Delta(z_1(s)),\pi_\Delta(z_2(s)))\right|^2 ds\\
&\le \frac{1}{2}\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}\left|e_\Delta(u)\right|^2\right)\\
&\quad+128\mathbb{E}\int_0^{t\wedge\theta_n}\left|g(x(s),x(\delta(s))) - g(x_\Delta(s),x_\Delta(\delta(s)))\right|^2 ds\\
&\quad+128\mathbb{E}\int_0^{t\wedge\theta_n}\left|g(x_\Delta(s),x_\Delta(\delta(t))) - g(\pi_\Delta(z_1(s)),\pi_\Delta(z_2(s)))\right|^2 ds
\end{aligned}$$

$$
\begin{aligned}
H_4 \le& \frac{1}{2}\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}|e_\Delta(u)|^2\right)\\
&+128K_4\mathbb{E}\int_0^{t\wedge\theta_n}\left(|x(s)-x_\Delta(s)|^2+|x(\delta(s))-x_\Delta(\delta(s))|^2\right)ds\\
&+128K_4\mathbb{E}\int_0^{t\wedge\theta_n}\left(|x_\Delta(s)-\pi_\Delta(z_1(s))|^2+|x_\Delta(\delta(s))-\pi_\Delta(z_2(s))|^2\right)ds\\
\le& \frac{1}{2}\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}|e_\Delta(u)|^2\right)+128K_4\left(1+\frac{\lambda_1}{\lambda_2}\right)\mathbb{E}\int_0^{t\wedge\theta_n}|e_\Delta(s)|^2\,ds\\
&+C_p\left(\Delta^{2\beta}+\Delta\left(h(\Delta)\right)^2\right)\\
\le& \frac{1}{2}\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}|e_\Delta(u)|^2\right)+128K_4\left(1+\frac{\lambda_1}{\lambda_2}\right)\int_0^{t}\mathbb{E}\left(\sup_{0\le u\le s}|e_\Delta(u)|^2\right)ds\\
&+C_p\left(\Delta^{2\beta}+\Delta\left(h(\Delta)\right)^2\right).
\end{aligned}
$$

Combining the estimates of H_3 and H_4, one has

$$
\begin{aligned}
\mathbb{E}\left(\sup_{0\le u\le t\wedge\theta_n}|e_\Delta(u)|^2\right)\le& C\int_0^t\mathbb{E}\left(\sup_{0\le u\le s}|e_\Delta(u)|^2\right)ds\\
&+C_p\left(\left(\mu^{-1}(h(\Delta))\right)^{-(p-\rho-2)}\vee\Delta^{2\beta}\vee\Delta\left(h(\Delta)\right)^2\right).
\end{aligned}
$$

By the Gronwall inequality and letting $n\to\infty$, this theorem is proved. □

In practice, calculating the solution using continuous time continuous process can sometimes be difficult. Therefore, the next lemma becomes crucial in determining the solution using the continuous time step process. Additionally, the proof of the next lemma is straightforward following from [6].

Lemma 4.3.3. *Let $\Delta\in(0,1]$ and $\varepsilon\in(0,1/4]$. Let ν be a sufficiently large integer for which*

$$
\left(\frac{2\nu}{2\nu-1}\right)^p(T+1)^{p/2\nu}\le 2 \text{ and } \frac{1}{\nu}<\varepsilon. \tag{4.25}
$$

Then, for $p\ge 2$

$$
\mathbb{E}\left(\sup_{0\le u\le T}|x_\Delta(u)-z_1(u)|^p\right)\le C_p\left(\Delta^{(p/2)(1-\varepsilon)}(h(\Delta))^p\right). \tag{4.26}
$$

We therefore obtain the following corollary illustrating that the solution can be computed by the step process.

Corollary 4.3.4. *Let Assumptions 4.1.2, 4.1.5, 4.1.6 and 4.3.1 hold. Assume that* $\frac{d\delta(t)}{dt} \geq \frac{\lambda_2}{\lambda_1}$ *for all* $t \in [-\tau, T]$ *and* $p \geq \rho + 2$. *If* $\mu(u) = Cu^{(2+\rho)/2}$ *and* $h(\Delta) = \Delta^{-\varepsilon}$, *then*

$$\mathbb{E}\left(\sup_{0\leq u\leq T} |x(u) - z_1(u)|^2\right) \leq C_p \left(\Delta^{2\rho\wedge(1-3\varepsilon)}\right). \tag{4.27}$$

4.4 Comparision and Summary

In this chapter, we have enhanced the truncated EM numerical approach for stochastic differential delay equations with a variable time delay denoted as $\delta(t)$. This improvement combines the truncated EM method, as defined in [6] and [8], with the model involving varying time delays presented in [25]. Furthermore, we have calculated the convergence rate at specific time points and over finite time intervals using some techniques in the previous chapter. When comparing our results to those of [6], we observed that the convergence rate in both cases of the solution with variable delay are similar to the results obtained in the case of constant delay as demonstrated in [6]. However, it is essential to emphasize that our method imposes specific conditions that must be satisfied in the case of variable delay.

Chapter 5

Positive Preserving Truncated Euler-Maruyama Numerical Method

5.1 Introduction

SDE models in applications have their special properties. For example, the square root process and mean-reverting square root process in finance have nonnegative solutions (see, e.g., [17, 19]). The stochastic Lotka–Volterra model for interacting multi-species in ecology has positive solutions (see, e.g., [1, 23, 19]). The SDE SIS model in epidemiology has positive solutions (see, e.g., [7]). These SDE models are all highly nonlinear. If we apply the modified EM methods (including the tamed EM method [14, 31, 32], the tamed Milstein method [35], the stopped EM method [18], the truncated EM method [21, 22]) to these SDEs, they do not maintain the non-negativity or positivity of the solutions.

Therefore the aim of this chapter is to modify the truncated EM method to establish a positivity preserving truncated EM (PPTEM). We, moreover, focus on applying this technique to the well-known stochastic Lotka-Volterra model, which describes the dynamics of interacting multiple species in ecology. The rationale for selecting this model is because its characteristic features: highly nonlinear, positive solution and multi-dimensional. Consequently, the methods developed in this chapter can be applied for broader applications, extending to other SDE models, such as the SDE SIS model.

Our approach is to establish a new nonnegative preserving truncated EM (NPTEM) and then the more desired PPTEM. Although the solution of the stochastic Lotka–Volterra model is positive, there are some SDE models whose solutions remain the nonnegative values. For instance, in finance, well-known models like the square root process and mean-reverting square root process necessitate nonnegative solutions, (see, e.g., [17, 19]). Moreover, from a mathematical perspective, the way to prove the convergence of the PPTEM solution to the true solution would be more natural if we start to establish the convergence of the NPTEM solutions as an intial step.

As explained in the previous, we consider the d-dimensional stochastic Lotka–Volterra model (see, e.g., [1, 19])

$$dx(t) = \text{diag}(x_1(t), x_2(t), ..., x_d(t))[(b - Ax(t))dt + \sigma dB(t)], \tag{5.1}$$

where $x(t) = (x_1(t), \cdots, x_d(t))^T$ is the state of the d interacting species and the system parameters $b = (b_1, \cdots, b_d)^T \in \mathbb{R}^d$, $\sigma = (\sigma_1, \cdots, \sigma_d)^T \in \mathbb{R}^d$, $A = (a_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$. It is worth noting that the scalar Brownian motion $B(t)$ in this chapter can be generalised into a multi-dimensional one without any difficulty. We impose the following assumption as a standing hypothesis, which is the only one for this chapter.

Assumption 5.1.1. *All elements of A are nonnegative, namely $a_{ij} \geq 0$ for all $1 \leq i, j \leq d$.*

From the ecological point of view, this assumption means that the d interacting species are competitive. The SDE (5.1) has been studied intensively by many authors. For example, it is known (see, e.g., [19]) that under Assumption 5.1.1, for any initial value $x(0) \in \mathbb{R}^d_+$, the SDE (5.1) has a unique global solution $x(t)$ on $t \geq 0$ and the solution will remain to be in $\mathbb{R}^d_+$ with probability one (namely, $x(t) \in \mathbb{R}^d_+$ a.s. for all $t \geq 0$).

Throughout this chapter, we set

$$\bar{b} = \max_{1\leq i\leq d} |b_i|, \quad \bar{\sigma} = \max_{1\leq i\leq d} |\sigma_i|, \quad \bar{a} = \max_{1\leq i,j\leq d} a_{ij}. \tag{5.2}$$

From now on, we will fix the initial value $x(0) \in \mathbb{R}^d_+$ arbitrarily and, of course, $x(t)$ is the corresponding solution. We will also fix two real numbers $T > 0$ and $p \geq 2$ arbitrarily. Let us present two lemmas which will play their useful role in this chapter. And recalling that we use C and C_p to stand for generic positive real constants which C_p is dependent on p whlie C is not.

Lemma 5.1.2. *Under Assumption 5.1.1,*

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x(t)|^p\Big)\le C_p. \tag{5.3}$$

Proof. Recalling that $x(t)\in\mathbb{R}^d_+$ and applying the Itô formula and Assumption 5.1.1, we can easily show from (5.1) that

$$d(x_i(t))^p\le p[\bar b+0.5(p-1)\bar\sigma^2](x_i(t))^p dt+p\sigma_i(x_i(t))^p dB(t),$$

for $t\ge 0$ and every $i=1,\cdots,d$. By the Burkholder–Davis–Gundy inequality, it is straightforward to show that

$$\mathbb{E}\Big(\sup_{0\le u\le t}(x_i(u))^p\Big)\le C_p+C_p\int_0^t\mathbb{E}\Big(\sup_{0\le u\le s}(x_i(u))^p\Big)ds,\quad \text{for all } t\in[0,T].$$

An application of the well-known Gronwall inequality gives

$$\mathbb{E}\Big(\sup_{0\le u\le T}(x_i(u))^p\Big)\le C_p.$$

This implies the required assertion (5.3). □

Lemma 5.1.3. *Under Assumption 5.1.1,*

$$\mathbb{E}\Big(\sup_{0\le t\le T}[x_i(t)-1-\log(x_i(t))]\Big)\le C,\quad 1\le i\le d. \tag{5.4}$$

Proof. For each i, by the Itô formula, we have

$$\begin{aligned}&d[x_i(t)-1-\log(x_i(t))]\\ \le&\Big(-b_i+0.5\sigma_i^2+b_ix_i(t)+\sum_{j=1}^d a_{ij}x_j(t)\Big)dt+\sigma_i(x_i(t)-1)dB(t).\end{aligned}$$

By Lemma 5.1.2, the first and second moments of the solution is bounded (by C) for $t\in[0,T]$. Applying the Burkholder–Davis–Gundy inequality again, we can then derive that

$$
\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}[x_i(t)-1-\log(x_i(t))]\Big)\\
\le& C+\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t\sigma_i(x_i(s)-1)dB(s)\Big)\\
\le& C+3\mathbb{E}\Big(\int_0^T|\sigma_i(x_i(s)-1)|^2ds\Big)^{1/2}\\
\le& C+3\bar{\sigma}\Big(\int_0^T 2(\mathbb{E}|x_i(s)|^2+1)ds\Big)^{1/2}\\
\le& C+3\bar{\sigma}\sqrt{2T(C+1)},
\end{aligned}
$$

which is the desired assertion (5.4). □

5.2 Definitions of New Numerical Schemes

In this section, we will develop two numerical schemes. The first one will be called the NPTEM scheme, while the second one the PPTEM scheme. We have explained in the previous why we do not only study the PPTEM but also the NPTEM in this chapter, although the solution of the underlying SDE (5.1) is positive with probability one.

5.2.1 Nonnegativity preserving truncated EM method

To define the NPTEM scheme, it would be convenient to treat the SDE (5.1) in $\mathbb{R}^d$ instead of $\mathbb{R}^d_+$. For this purpose, we need to extend the definition of the coefficients of the SDE from $\mathbb{R}^d_+$ to $\mathbb{R}^d$. We denote the coefficients by

$$F_1(x)=(b_1x_1,\cdots,b_dx_d)^T,\;\; F_2(x)=-\text{diag}(x_1,\cdots,x_d)Ax,\;\; G(x)=(\sigma_1x_1,\cdots,\sigma_dx_d)^T$$

for $x\in\bar{\mathbb{R}}^d_+$. Define a mapping $\pi_0:\mathbb{R}^d\to\bar{\mathbb{R}}^d_+$ by

$$\pi_0(x)=(x_1\vee 0,\cdots,x_d\vee 0)^T\quad\text{for } x\in\mathbb{R}^d.$$

Define $f_1,f_2,g:\mathbb{R}^d\to\mathbb{R}^d$ by

$$f_1(x)=F_1(\pi_0(x)),\quad f_2(x)=F_2(\pi_0(x)),\quad g(x)=G(\pi_0(x))\quad\text{for } x\in\mathbb{R}^d.$$

Obviously, $f_1(x) = F_1(x)$ etc. if $x \in \bar{\mathbb{R}}^d_+$. In other words, f_1, f_2, g are the extended functions of F_1, F_2, G, respectively. Recalling that the solution of the SDE (5.1) has the property that $x(t) \in \mathbb{R}^d_+$ a.s. for all $t \geq 0$, we can therefore write the SDE (5.1) as the following equation

$$dx(t) = [f_1(x(t)) + f_2(x(t))]dt + g(x(t))dB(t) \tag{5.5}$$

in $\mathbb{R}^d$. We observe that f_1 and g are linearly bounded, namely

$$|f_1(x)| \leq \bar{b}|x|, \quad |g(x)| \leq \bar{\sigma}|x|, \quad \text{for all } x \in \mathbb{R}^d, \tag{5.6}$$

but f_2 is not. The classical EM method is therefore not applicable to the SDE (see, e.g., [11, 14]). The truncated EM method established by [21, 22] may be applied but it cannot preserve nonnegativity, not mentioning positivity.

The aim of this subsection is to modify the truncated EM method in order to create a new NPTEM method. For this purpose, we first choose a strictly increasing continuous function $\mu : [1, \infty) \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{x \in \mathbb{R}^d,\, |x| \leq u} |f_2(x)| = \sup_{x \in \bar{\mathbb{R}}^d_+,\, |x| \leq u} |F_2(x)| \leq \mu(u), \qquad \text{for all } u \geq 1. \tag{5.7}$$

Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(1), \infty)$ to $\mathbb{R}_+$. We also choose a constant $\hat{h} \geq 1 \vee \mu(1) \vee |x(0)|$ and a strictly decreasing function $h : (0, 1] \to [\mu(1), \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \leq \hat{h}, \qquad \text{for all } \Delta \in (0, 1]. \tag{5.8}$$

Note that for $x \in \bar{\mathbb{R}}^d_+$,

$$|F_2(x)|^2 = \sum_{i=1}^d x_i^2 \Big(\sum_{j=1}^d a_{ij} x_j \Big)^2 \leq \sum_{i=1}^d x_i^2 \Big(\sum_{j=1}^d a_{ij}^2 \Big) |x|^2 \leq |A|^2 |x|^4,$$

where $|A| = \sqrt{\text{trace}(A^T A)}$ is the trace norm of a matrix A. We can hence let $\mu(u) = |A| u^2$, while let $h(\Delta) = \hat{h} \Delta^{-\theta}$ for some $\theta \in (0, 1/4]$. In other words, there are lots of choices for $\mu(\cdot)$ and $h(\cdot)$.

For a given step size $\Delta \in (0, 1]$, let us define the truncation mapping $\pi_\Delta : \mathbb{R}^d \to$

$\{x \in \mathbb{R}^d : |x| \le \mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x) = \big(|x| \wedge \mu^{-1}(h(\Delta))\big)\frac{x}{|x|},$$

where we set $x/|x| = 0$ when $x = 0$. That is, π_Δ maps x to itself or $\mu^{-1}(h(\Delta))x/|x|$ depending on $|x| \le \mu^{-1}(h(\Delta))$ or not. It is useful to see that for all $x \in \mathbb{R}^d$,

$$f_2(\pi_0(\pi_\Delta(x))) = F_2(\pi_0(\pi_0(\pi_\Delta(x)))) = F_2(\pi_0(\pi_\Delta(x))) = f_2(\pi_\Delta(x)). \tag{5.9}$$

Hence

$$|f_2(\pi_0(\pi_\Delta(x)))| = |f_2(\pi_\Delta(x))| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta). \tag{5.10}$$

Moreover, noting $\pi_0(\pi_\Delta(x)) = (|x| \wedge \mu^{-1}(h(\Delta)))\pi_0(x)/|x|$, we also have

$$x^T f_2(\pi_0(\pi_\Delta(x))) = x^T f_2(\pi_\Delta(x)) = (\pi_0(x))^T F_2(\pi_0(\pi_\Delta(x))) \le 0. \tag{5.11}$$

We can now form the discrete-time NPTEM solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ by setting $\bar{X}_\Delta(0) = X_\Delta(0) = x(0)$ and computing

$$\bar{X}_\Delta(t_{k+1}) = \bar{X}_\Delta(t_k) + [f_1(\bar{X}_\Delta(t_k)) + f_2(X_\Delta(t_k))]\Delta + g(\bar{X}_\Delta(t_k))\Delta B_k, \tag{5.12}$$

$$X_\Delta(t_{k+1}) = \pi_0(\pi_\Delta(\bar{X}_\Delta(t_{k+1}))), \tag{5.13}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Please note that $\bar{X}_\Delta(t_{k+1})$ is an intermediate step in order to get the NPTEM solution $X_\Delta(t_{k+1})$. We extend the definitions of both $\bar{X}_\Delta(\cdot)$ and $X_\Delta(\cdot)$ from the grid points t_k to the whole $t \ge 0$ by defining

$$\bar{X}_\Delta(t) = \sum_{k=0}^{\infty} \bar{X}_\Delta(t_k)\mathbb{1}_{[t_k, t_{k+1})}(t) \tag{5.14}$$

and

$$X_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k)\mathbb{1}_{[t_k, t_{k+1})}(t) \tag{5.15}$$

for $t \ge 0$. Clearly, $X_\Delta(t) = \pi_0(\pi_\Delta(\bar{X}_\Delta(t)))$ so it preserves the nonnegativity although $\bar{X}_\Delta(t)$ does not.

5.2.2 Positivity preserving truncated EM method

For each step size $\Delta \in (0, 1]$, define one more truncation mapping $\pi_+ : \mathbb{R}^d \to \mathbb{R}_+^d$ by

$$\pi_+(x) = (\Delta \vee x_1, \cdots, \Delta \vee x_d)^T, \quad x \in \mathbb{R}^d.$$

Please note that π_+ maps $\mathbb{R}^d$ to $\mathbb{R}_+^d$ while π_0 to $\bar{\mathbb{R}}_+^d$ so they are different. The PPTEM solution is defined by

$$X_\Delta^+(t) = \pi_+(\pi_\Delta(\bar{X}_\Delta(t))), \quad t \geq 0, \tag{5.16}$$

where $\bar{X}_\Delta(t)$ has already been defined by (5.14).

At this step, the upcoming question is that can we define the PPTEM is a similar fashion at the NPTEM, namely by replacing π_0 in (5.13) with π_+ while keeping everything else unchanged. This is certainly possible but the mathematics will become slightly more complicated because π_+ does not preserve the nice property that π_Δ has while π_0 does. More precisely, π_Δ maps $\mathbb{R}^d$ into the ball in $\mathbb{R}^d$ with center 0 and radius $\mu^{-1}(h(\Delta))$ but π_+ may map some x in the ball outside the ball. For a mathematical reason, we have

$$|\pi_\Delta(x)| \leq \mu^{-1}(h(\Delta)), \quad \forall x \in \mathbb{R}^d$$

but we may have

$$|\pi_+(\pi_\Delta(x))| > \mu^{-1}(h(\Delta))$$

for some $x \in \mathbb{R}^d$ with $|x| \leq \mu^{-1}(h(\Delta))$. For example, if $x = (\mu^{-1}(h(\Delta)), 0, \cdots, 0)^T$, then $\pi_+(\pi_\Delta(x)) = (\mu^{-1}(h(\Delta)), \Delta, \cdots, \Delta)^T$ and

$$|\pi_+(\pi_\Delta(x))| = \sqrt{(\mu^{-1}(h(\Delta)))^2 + (d-1)\Delta^2} > \mu^{-1}(h(\Delta)).$$

5.3 Main Theorems

Our aim of this chapter is to show that both NPTEM solution $X_\Delta(t)$ and PPTEM solution $X_\Delta^+(t)$ converge to the true solution $x(t)$ in L^p for any $p \geq 2$ as described in the following theorems.

Theorem 5.3.1. *Under Assumption 5.1.1, one has*

$$\lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \le t \le T} |X_\Delta(t) - x(t)|^p \Big) = 0. \tag{5.17}$$

Theorem 5.3.2. *Under Assumption 5.1.1, one has*

$$\lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \le t \le T} |X_\Delta^+(t) - x(t)|^p \Big) = 0. \tag{5.18}$$

The proof of these theorems are highly technical. To make it more understandable, we break it into a number of lemmas in the next subsection and prove the theorems afterward.

5.3.1 Lemmas

For the mathematical analysis, we need to define a new process

$$x_\Delta(t) = x(0) + \int_0^t [f_1(\bar{X}_\Delta(s)) + f_2(X_\Delta(s))]ds + \int_0^t g(\bar{X}_\Delta(s))dB(s) \tag{5.19}$$

for $t \ge 0$. We observe that $x_\Delta(t_k) = \bar{X}_\Delta(t_k)$ for all $k \ge 0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = [f_1(\bar{X}_\Delta(t)) + f_2(X_\Delta(t))]dt + g(\bar{X}_\Delta(t))dB(t). \tag{5.20}$$

We also denote the ith component of $x_\Delta(t)$, $X_\Delta(t)$ or $\bar{X}_\Delta(t)$ by $x_{\Delta,i}(t)$, $X_{\Delta,i}(t)$ or $\bar{X}_{\Delta,i}(t)$, respectively.

By (5.6) and (5.10), it is easy to see from (5.12) that for any $q \ge 2$, $\mathbb{E}|\bar{X}_\Delta(t_1)|^q < \infty$ and then, by induction, $\mathbb{E}|\bar{X}_\Delta(t_k)|^q < \infty$ for all $k \ge 1$. By (5.19) we can then further see that $\mathbb{E}|x_\Delta(t)|^q < \infty$ for all $t \ge 0$. But we will show a better result (see Lemma 5.3.4).

We start with the following lemma which shows that $x_\Delta(t)$ and $\bar{X}_\Delta(t)$ are close to each other in the sense of L^p.

Lemma 5.3.3. *For any $\Delta \in (0, 1]$, we have*

$$\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p \le C_p \Delta^{p/2}(h(\Delta))^p, \quad \forall t \in [0, T]. \tag{5.21}$$

Consequently

$$\lim_{\Delta\to 0} \mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p = 0. \tag{5.22}$$

Proof. By (5.10),

$$|f_2(X_\Delta(t))| = |f_2(\pi_0(\pi_\Delta(\bar{X}_\Delta(t))))| \le h(\Delta). \tag{5.23}$$

Using this and (5.6), we can easily show from (5.19) that

$$\mathbb{E}|x_\Delta(t)|^p \le C_p(h(\Delta))^p + C_p \int_0^t \mathbb{E}|\bar{X}_\Delta(s)|^p ds$$

for $t \in [0, T]$. This implies

$$\begin{aligned}\sup_{0\le u\le t} \mathbb{E}|x_\Delta(u)|^p &\le C_p(h(\Delta))^p + C_p \int_0^t \mathbb{E}|\bar{X}_\Delta(s)|^p ds \\ &\le C_p(h(\Delta))^p + C_p \int_0^t \Big(\sup_{0\le u\le s} \mathbb{E}|x_\Delta(u)|^p\Big) ds.\end{aligned}$$

The well-known Gronwall inequality shows

$$\sup_{0\le u\le T} \mathbb{E}|x_\Delta(u)|^p \le C_p(h(\Delta))^p. \tag{5.24}$$

Now, for any $t \in [0, T]$, there is a unique $k \ge 0$ such that $t \in [t_k, t_{k+1})$ and hence $\bar{X}_\Delta(t) = \bar{X}_\Delta(t_k) = x_\Delta(t_k)$. It then follows from (5.19) that

$$\begin{aligned}&\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p = \mathbb{E}|x_\Delta(t) - x_\Delta(t_k)|^p \\ &\le C_p\Delta^{p-1}\mathbb{E}\int_{t_k}^t [|f_1(\bar{X}_\Delta(s)|^p + |f_2(X_\Delta(s))|^p] ds + C_p\Delta^{(p-2)/2}\int_{t_k}^t |g(\bar{X}_\Delta(s)|^p ds.\end{aligned}$$

This, along with (5.6), (5.10) and (5.24), implies

$$\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p \le C_p\Delta^p(h(\Delta))^p + C_p\Delta^{p/2}(h(\Delta))^p \le C_p\Delta^{p/2}(h(\Delta))^p$$

which is the first assertion. Noting from (5.8) that $\Delta^{p/2}(h(\Delta))^p \le \Delta^{p/4}$, we obtain the second assertion from the first one immediately. $\square$

The following lemma shows a much better result than (5.24).

Lemma 5.3.4. *Let Assumption 5.1.1 hold. Then*

$$\sup_{0<\Delta\le 1} \mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t)|^p\Big) \le C_p. \tag{5.25}$$

Proof. Fix any $\Delta \in (0,1]$. By the Itô formula and the Burkholder-Davis-Gundy inequality etc., it is almost routine (see, e.g., [19, pp.59-63]) to show that

$$\mathbb{E}\Big(\sup_{0\le u\le t} |x_\Delta(u)|^p\Big) \le C_p + C_p \int_0^t \mathbb{E}\Big(\sup_{0\le u\le s} |x_\Delta(u)|^p\Big) ds + J_1(t) \tag{5.26}$$

for $t \in [0,T]$, where

$$J_1(t) = \mathbb{E}\Big(\sup_{0\le u\le t} \int_0^u p|x_\Delta(s)|^{p-2} x_\Delta^T(s) f_2(X_\Delta(s)) ds\Big).$$

By (5.10) and (5.11), we have

$$\begin{aligned} x_\Delta^T(s) f_2(X_\Delta(s)) &= \big([x_\Delta(s) - \bar{X}_\Delta(s)]^T + \bar{X}_\Delta^T(s)\big) f_2(\pi_0(\pi_\Delta(\bar{X}_\Delta(s)))) \\ &\le h(\Delta)|x_\Delta(s) - \bar{X}_\Delta(s)|. \end{aligned}$$

Hence

$$J_1(t) \le \mathbb{E}\int_0^t p|x_\Delta(s)|^{p-2} h(\Delta)|x_\Delta(s) - \bar{X}_\Delta(s)| ds.$$

Using the Young inequality

$$pa^{p-2}b \le (p-2)a^p + 2b^{p/2}, \quad \forall a, b \ge 0,$$

as well as Lemma 5.3.3, we can then derive that

$$\begin{aligned} J_1(t) &\le \mathbb{E}\int_0^t \big[(p-2)|x_\Delta(s)|^p + 2(h(\Delta))^{p/2}|x_\Delta(s) - \bar{X}_\Delta(s)|^{p/2}\big] ds \\ &\le (p-2)\int_0^t \mathbb{E}|x_\Delta(s)|^p ds + 2(h(\Delta))^{p/2}\int_0^T (\mathbb{E}|x_\Delta(s) - \bar{X}_\Delta(s)|^p)^{1/2} ds \\ &\le (p-2)\int_0^t \mathbb{E}|x_\Delta(s)|^p ds + C_p \Delta^{p/4}(h(\Delta))^p \\ &\le (p-2)\int_0^t \mathbb{E}\Big(\sup_{0\le u\le s} |x_\Delta(u)|^p\Big) ds + C_p, \end{aligned}$$

where we have used (5.8) in the last step. Substituting this into (5.26) yields

$$\mathbb{E}\Big(\sup_{0\le u\le t}|x_\Delta(u)|^p\Big)\le C_p+C_p\int_0^t\mathbb{E}\Big(\sup_{0\le u\le s}|x_\Delta(u)|^p\Big)ds.$$

An application of the Gronwall inequality gives

$$\mathbb{E}\Big(\sup_{0\le u\le T}|x_\Delta(u)|^p\Big)\le C_p.$$

As this holds for any $\Delta\in(0,1]$ while C_p is independent of Δ, we see the required assertion (5.25). □

The following lemma improves the second assertion in Lemma 5.3.3.

Lemma 5.3.5. *Let Assumption 5.1.1 hold. Then*

$$\lim_{\Delta\to0}\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar X_\Delta(t)|^p\Big)=0. \tag{5.27}$$

Proof. Let m be the integer part of T/Δ. Then, by (5.6) and (5.23) as well as Lemma 5.3.4, we derive that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar X_\Delta(t)|^p\Big)\\
\le&\mathbb{E}\Big(\max_{0\le k\le m}\sup_{t_k\le t\le t_{k+1}}\Big|[f_1(\bar X_\Delta(t_k))+f_2(X_\Delta(t_k))](t-t_k)+g(\bar X_\Delta(t_k))(B(t)-B(t_k))\Big|^p\Big)\\
\le&C_p\mathbb{E}\Big(\max_{0\le k\le m}[|\bar X_\Delta(t_k)|^p+(h(\Delta))^p]\Delta^p\Big)+J_2\\
\le&C_p\Delta^p\mathbb{E}\Big(\max_{0\le k\le m}|x_\Delta(t_k)|^p+(h(\Delta))^p\Big)+J_2\\
\le&C_p\Delta^p[C+(h(\Delta))^p]+J_2\le C_p\Delta^p(h(\Delta))^p+J_2,
\end{aligned}\tag{5.28}$$

where

$$J_2=C_p\mathbb{E}\Big(\max_{0\le k\le m}\Big[|\bar X_\Delta(t_k))|^p\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^p\Big]\Big).$$

Now, choose a sufficiently large integer $n\ge 3\vee p$, dependent on p and T, for which

$$\Big(\frac{2n}{2n-1}\Big)^p(T+1)^{p/2n}\le 2. \tag{5.29}$$

But, by the Hölder inequality,

$$J_2 \le C_p \Big\{ \mathbb{E}\Big(\max_{0\le k\le m} \Big[|\bar{X}_\Delta(t_k))|^{2n} \sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^{2n} \Big] \Big) \Big\}^{p/2n}$$
$$\le C_p \Big(\sum_{k=0}^{m} \mathbb{E}\Big[|\bar{X}_\Delta(t_k))|^{2n} \sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^{2n} \Big] \Big)^{p/2n}.$$

But, by Lemma 5.3.4 (replacing p there by $2n$ though n here depends on p), $\mathbb{E}|\bar{X}_\Delta(t_k))|^{2n}$ is bounded by C_p for every t_k. Note also that for each k, $\bar{X}_\Delta(t_k)$ is independent of $\sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^{2n}$. We hence have

$$J_2 \le C_p \Big(\sum_{k=0}^{m} \mathbb{E}|\bar{X}_\Delta(t_k))|^{2n}\, \mathbb{E}\Big[\sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^{2n} \Big] \Big)^{p/2n}$$
$$\le C_p \Big(\sum_{k=0}^{m} \mathbb{E}\Big[\sup_{t_k\le t\le t_{k+1}} |B(t)-B(t_k)|^{2n} \Big] \Big)^{p/2n}.$$

By the Doob martingale inequality (see, e.g., [19, Theorem 4.8 on p.14], we further derive that

$$J_2 \le C_p \Big(\sum_{k=0}^{m} \Big[\frac{2n}{2n-1} \Big]^{2n} \mathbb{E}|B(t_{k+1})-B(t_k)|^{2n} \Big)^{p/2n}$$
$$\le C_p \Big(\sum_{k=0}^{m} \Big[\frac{2n}{2n-1} \Big]^{2n} (2n-1)!!\Delta^n \Big)^{p/2n}$$
$$\le C_p \Big(\Big[\frac{2n}{2n-1} \Big]^{2n} (T+1)(2n-1)!!\Delta^{n-1} \Big)^{p/2n},$$

where $(2n-1)!! = (2n-1)\times(2n-3)\times\cdots\times 3\times 1$. But

$$[(2n-1)!!]^{1/n} \le \frac{1}{n}\sum_{i=1}^{n}(2i-1) = n.$$

Thus

$$J_2 \le C_p n^{p/2} \Big(\frac{2n}{2n-1} \Big)^p (T+1)^{p/2n} \Delta^{p(n-1)/2n}.$$

Using (5.29) while noting $(n-1)/2n \ge 1/3$ as we choose $n \ge 3$, we obtain

$$J_2 \le C_p \Delta^{p/3}.$$

Substituting this into (5.28) gives

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar X_\Delta(t)|^p\Big)\le C_p(h(\Delta))^p\Delta^p+C_p\Delta^{p/3}\le C_p(h(\Delta))^p\Delta^{p/3}.$$

But, by (5.8),

$$(h(\Delta))^p\Delta^{p/3}=\Delta^{p/12}(\Delta^{1/4}h(\Delta))^p\le\Delta^{p/12}.$$

We hence obtain

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar X_\Delta(t)|^p\Big)\le C_p\Delta^{p/12}.$$

This implies the required assertion (5.27). □

In the remaining of this section, we need a couple of new notations. For each $r>|x_0|$, define the stopping times

$$\tau_r=\inf\{t\ge 0:|x(t)|\ge r\}$$

and

$$\rho_{\Delta,r}=\inf\{t\ge 0:|x_\Delta(t)|\ge r\},$$

where throughout this thesis we set $\inf\emptyset=\infty$. Moreover, we set

$$\theta_{\Delta,r}=\tau_r\wedge\rho_{\Delta,r}$$

and define the closed ball

$$S_r=\{x\in\mathbb{R}^d:|x|\le r\}.$$

The following lemma shows both $x(t\wedge\theta_{\Delta,r})$ and $x_\Delta(t\wedge\theta_{\Delta,r})$ are close to each other.

Lemma 5.3.6. *Let Assumption 5.1.1. Then for each $r>|x_0|$, there is a $\Delta_1=\Delta_1(r)\in(0,1]$ such that*

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x(t\wedge\theta_{\Delta,r})-x_\Delta(t\wedge\theta_{\Delta,r})|^p\Big)\le C_r\Delta^{p/2},\quad \text{for all } \Delta\in(0,\Delta_1],\tag{5.30}$$

where C_r is a positive constant dependent on r,p,T etc. but independent of Δ.

Proof. Define

$$f_{2,r}(x)=f_2\big((|x|\wedge r)x/|x|\big)\quad\text{for } x\in\mathbb{R}^d.$$

Obviously, $f_{2,r}(\cdot)$ is bounded and globally Lipschitz continuous in $\mathbb{R}^d$ but its Lipschitz constant depends on r. Consider the SDE

$$dy(t) = [f_1(y(t)) + f_{2,r}(y(t))]dt + g(y(t))dB(t) \tag{5.31}$$

on $t \geq 0$ with the initial value $y(0) = x(0)$. It has a unique global solution $y(t)$ on $t \geq 0$. For each step size $\Delta \in (0, 1]$, we can apply the EM method to the SDE (5.31). That is, we form the EM solutions $Y_\Delta(t_k) \approx y(t_k)$ for $t_k = k\Delta$ by setting $Y_\Delta(0) = x(0)$ and computing

$$Y_\Delta(t_{k+1}) = Y_\Delta(t_k) + [f_1(Y_\Delta(t_k)) + f_{2,r}(Y_\Delta(t_k))]\Delta + g(Y_\Delta(t_k))\Delta B_k, \tag{5.32}$$

for $k = 0, 1, \cdots$. Extend the definitions of $Y_\Delta(\cdot)$ from the grid points t_k to the whole $t \geq 0$ by setting

$$Y_\Delta(t) = \sum_{k=0}^{\infty} Y_\Delta(t_k) \mathbb{1}_{[t_k, t_{k+1})}(t), \tag{5.33}$$

and then define the Itô process

$$y_\Delta(t) = x(0) + \int_0^t [f_1(Y_\Delta(s)) + f_{2,r}(Y_\Delta(s))]ds + \int_0^t g(Y_\Delta(s))dB(s) \tag{5.34}$$

for $t \geq 0$. It is well known (see, e.g., [15, 19]) that

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |y(t) - y_\Delta(t)|^p\Big) \leq C_r \Delta^{p/2}. \tag{5.35}$$

Let us relate $y(t)$ and $y_\Delta(t)$ to $x(t)$ and $x_\Delta(t)$, respectively. It is straightforward to see that

$$x(t \wedge \tau_r) = y(t \wedge \tau_r) \quad \text{a.s for all } t \in [0, T]. \tag{5.36}$$

We now choose $\Delta_1 \in (0, 1]$ sufficiently small for $\mu^{-1}(h(\Delta_1)) \geq r$. Obviously, for all $\Delta \in (0, \Delta_1]$,

$$f_2(\pi_\Delta(x)) = f_{2,r}(x), \quad \forall x \in S_r.$$

This, together with (5.9), yields

$$f_2(\pi_0(\pi_\Delta(x))) = f_{2,r}(x), \quad \forall x \in S_r.$$

Comparing (5.12), (5.19) with (5.32),(5.34), we then see that

$$x_\Delta(t \wedge \rho_{\Delta,r}) = y_\Delta(t \wedge \rho_{\Delta,r}) \quad \text{a.s for all } t \in [0,T] \tag{5.37}$$

provided $\Delta \in (0, \Delta_1]$. Combining (5.35) - (5.37), we obtain the desired assertion (5.30) immediately. □

5.3.2 Proof of Theorem 5.3.1

We are finally in a position to prove our main theorems. We prove Theorem 5.3.1 first in this subsection and then Theorem 5.3.2 next. Obviously,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta(t) - x(t)|^p\Big) \le 3^{p-1}(J_3(\Delta) + J_4(\Delta) + J_5(\Delta)), \tag{5.38}$$

where

$$\begin{aligned}
J_3(\Delta) &= \mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta(t) - \bar{X}_\Delta(t)|^p\Big),\\
J_4(\Delta) &= \mathbb{E}\Big(\sup_{0\le t\le T}|\bar{X}_\Delta(t) - x_\Delta(t)|^p\Big),\\
J_5(\Delta) &= \mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t) - x(t)|^p\Big).
\end{aligned}$$

By Lemma 5.3.5, we already have that $J_4(\Delta) \to 0$ as $\Delta \to 0$. To complete the proof, we hence only need to show both $J_3(\Delta)$ and $J_5(\Delta)$ tend to 0.

Let us first show $J_5(\Delta) \to 0$. Let $\varepsilon \in (0,1)$ be arbitrary. By Lemmas 5.1.2 and 5.3.4,

$$\begin{aligned}
&\mathbb{P}(\theta_{r,\Delta} \le T) \le \mathbb{P}(\tau_r \le T) + \mathbb{P}(\rho_{r,\Delta} \le T)\\
=&\frac{1}{r^p}\Big[\mathbb{E}\big(|x(\tau_r)|^p \mathbb{1}_{\{\tau_r \le T\}}\big) + \mathbb{E}\big(|x_\Delta(\rho_{r,\Delta})|^p \mathbb{1}_{\{\rho_{r,\Delta} \le T\}}\big)\Big]\\
\le&\frac{1}{r^p}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|x(t)|^p\Big) + \mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)|^p\Big)\Big]\\
\le&\frac{C_p}{r^p}.
\end{aligned}$$

We can hence choose a real number $r = r(\varepsilon)$ so large that

$$\mathbb{P}(\theta_{r,\Delta} \le T) \le \varepsilon^2.$$

For this r, by Lemma 5.3.6, we have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x(t\wedge\theta_{\Delta,r})-x_\Delta(t\wedge\theta_{\Delta,r})|^p\Big)\le C_r\Delta^{p/2},\quad \text{for all } \Delta\in(0,\Delta_1],$$

where Δ_1 now depends on ε (as r dependent on ε). Thus, for $\Delta\in(0,\Delta_1]$, we derive

$$\begin{aligned}
J_5(\Delta)&=\mathbb{E}\Big(\mathbb{1}_{\{\theta_{r,\Delta}\le T\}}\sup_{0\le t\le T}|x_\Delta(t)-x(t)|^p\Big)+\mathbb{E}\Big(\mathbb{1}_{\{\theta_{r,\Delta}>T\}}\sup_{0\le t\le T}|x_\Delta(t)-x(t)|^p\Big)\\
&\le\big[\mathbb{P}(\theta_{r,\Delta}\le T)\big]^{1/2}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-x(t)|^{2p}\Big)\Big]^{1/2}\\
&+\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t\wedge\theta_{r,\Delta})-x(t\wedge\theta_{r,\Delta})|^p\Big)\\
&\le C_p\varepsilon+C_r\Delta^{p/2}.
\end{aligned}$$

But, by Lemma 5.3.4 (recalling p is arbitrary once again),

$$\begin{aligned}
&\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-x(t)|^{2p}\Big)\Big]^{1/2}\\
\le&2^{(p-1)/2}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)|^{2p}\Big)+\mathbb{E}\Big(\sup_{0\le t\le T}|x(t)|^{2p}\Big)\Big]^{1/2}\le C_p.
\end{aligned}$$

We then have

$$J_5(\Delta)\le C_p\varepsilon+C_r\Delta^{p/2},\quad\forall\Delta\in(0,\Delta_1].$$

This implies

$$\limsup_{\Delta\to0}J_5(\Delta)\le C_p\varepsilon.$$

As ε is arbitrary, we must have that $J_5(\Delta)\to0$ as $\Delta\to0$.

Let us finally show $J_3(\Delta)\to0$ to complete our proof of Theorem 5.3.1. By Lemmas 5.1.2 and 5.3.4, we can find a positive number $r=r(\varepsilon)$ so large that

$$\mathbb{P}(\Omega_1)\ge1-\varepsilon/3,\tag{5.39}$$

where

$$\Omega_1=\{|x(t)|\vee|x_\Delta(t)|<r\text{ for all }0\le t\le T\}.$$

For a sufficiently small $\delta \in (0,1)$, define

$$\zeta_{\delta,i} = \inf\{t \geq 0 : x_i(t) \leq \delta\}, \quad 1 \leq i \leq d.$$

By Lemma 5.1.3,

$$\begin{aligned}\mathbb{P}(\zeta_{\delta,i} \leq T) &= \mathbb{E}\Big(\mathbb{1}_{\{\zeta_{\delta,i} \leq T\}} \frac{x_i(\zeta_{\delta,i}) - 1 - \log(x_i(\zeta_{\delta,i}))}{\delta - 1 - \log(\delta)}\Big) \\ &\leq \frac{1}{\delta - 1 - \log(\delta)} \mathbb{E}\Big(\sup_{0 \leq t \leq T} [x_i(t) - 1 - \log(x_i(t))]\Big) \leq \frac{C}{\delta - 1 - \log(\delta)}.\end{aligned}$$

Noting that $\delta - 1 - \log(\delta) \to \infty$ as $\delta \to 0$, we can find a $\delta = \delta(\varepsilon)$ so small that

$$\mathbb{P}(\zeta_{\delta,i} \leq T) \leq \frac{\varepsilon}{3d}, \quad 1 \leq i \leq d.$$

Set $\zeta_\delta = \min_{1 \leq i \leq d} \zeta_{\delta,i}$. Then

$$\mathbb{P}(\zeta_\delta \leq T) \leq \mathbb{P}\big(\cup_{i=1}^d \{\zeta_{\delta,i} \leq T\}\big) \leq \sum_{i=1}^d \mathbb{P}(\zeta_{\delta,i} \leq T) \leq \varepsilon/3.$$

So $\mathbb{P}(\zeta_\delta > T) \geq 1 - \varepsilon/3$. This implies

$$\mathbb{P}(\Omega_2) \geq 1 - \varepsilon/3, \tag{5.40}$$

where

$$\Omega_2 = \Big\{\min_{1 \leq i \leq d} \inf_{0 \leq t \leq T} x_i(t) > \delta\Big\}.$$

On the other hand, for the pair of chosen r and δ, define

$$\Omega_\Delta = \Big\{\sup_{0 \leq t \leq T} |x(t \wedge \theta_{\Delta,r}) - x_\Delta(t \wedge \theta_{\Delta,r})| < \delta/2\Big\}.$$

By Lemma 5.3.6 and the Chebyshev inequality, we see that there is a $\Delta_1 = \Delta_1(\varepsilon)$ (as $r = r(\varepsilon)$) such that $\mu^{-1}(h(\Delta_1)) \geq r$ and

$$\mathbb{P}(\Omega_\Delta^c) = \mathbb{P}\Big(\sup_{0 \leq t \leq T} |x(t \wedge \theta_{\Delta,r}) - x_\Delta(t \wedge \theta_{\Delta,r})| \geq \delta/2\Big) \leq \frac{C_r \Delta^{p/2}}{(\delta/2)^p}, \quad \forall \Delta \in (0, \Delta_1].$$

Consequently, there is a $\Delta_2 = \Delta_2(\varepsilon) \in (0, \Delta_1]$ such that

$$\mathbb{P}(\Omega_\Delta) \geq 1 - \varepsilon/3, \quad \forall \Delta \in (0, \Delta_2]. \tag{5.41}$$

Set $\Omega_{3,\Delta} = \Omega_1 \cap \Omega_2 \cap \Omega_\Delta$. Combining (5.39) - (5.41) gives

$$\mathbb{P}(\Omega_{3,\Delta}) \geq 1 - \varepsilon, \quad \forall \Delta \in (0, \Delta_2]. \tag{5.42}$$

From now on, we consider any step size $\Delta \in (0, \Delta_2]$. Note that for every $\omega \in \Omega_{3,\Delta}$, $\theta_{\Delta,r} > T$,

$$\sup_{0\leq t\leq T} |\bar{X}_\Delta(t)| \leq \sup_{0\leq t\leq T} |x_\Delta(t)| \leq r \leq \mu^{-1}(h(\Delta_1)) \leq \mu^{-1}(h(\Delta)), \tag{5.43}$$

and

$$\begin{aligned}\inf_{0\leq t\leq T} \bar{X}_{\Delta,i}(t) &\geq \inf_{0\leq t\leq T} x_{\Delta,i}(t) \geq \inf_{0\leq t\leq T} x_i(t) - \sup_{0\leq t\leq T} |x_i(t) - x_{\Delta,i}(t)| \\ &> \delta - \sup_{0\leq t\leq T} |x(t) - x_\Delta(t)| > \delta - \delta/2 = \delta/2. \end{aligned} \tag{5.44}$$

In other words, for every $\omega \in \Omega_{3,\Delta}$, $\bar{X}_\Delta(t) \in \mathbb{R}^d_+$ with $|\bar{X}_\Delta(t)| \leq \mu^{-1}(h(\Delta))$, whence $X_\Delta(t) = \pi_0(\pi_\Delta(\bar{X}_\Delta(t))) = \bar{X}_\Delta(t)$ for all $t \in [0, T]$. Consequently,

$$\begin{aligned} J_3(\Delta) &= \mathbb{E}\Big(\mathbb{1}_{\Omega^c_{3,\Delta}} \sup_{0\leq t\leq T} |X_\Delta(t) - \bar{X}_\Delta(t)|^p\Big) \\ &\leq \big[\mathbb{P}(\Omega^c_{3,\Delta})\big]^{1/2}\Big[\mathbb{E}\Big(\sup_{0\leq t\leq T} |X_\Delta(t) - \bar{X}_\Delta(t)|^{2p}\Big)\Big]^{1/2} \\ &\leq 2^p\sqrt{\varepsilon}\Big[\mathbb{E}\Big(\sup_{0\leq t\leq T} |x_\Delta(t)|^{2p}\Big)\Big]^{1/2} \\ &\leq C_p\sqrt{\varepsilon} \end{aligned}$$

provided $\Delta \in (0, \Delta_2]$, where Lemma 5.3.4 has been used once again. As ε is arbitrary, we must have that $J_3(\Delta) \to 0$ as $\Delta \to 0$. This completes our proof of Theorem 5.3.1. □

5.3.3 Proof of Theorem 5.3.2

Once again, it is obvious that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-x(t)|^p\Big)\le 3^{p-1}(J_4(\Delta)+J_5(\Delta)+J_6(\Delta)), \tag{5.45}$$

where $J_4(\Delta), J_5(\Delta)$ have been defined before and

$$J_6(\Delta)=\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-\bar{X}_\Delta(t)|^p\Big).$$

Clearly, all we need to do is to show that $J_6(\Delta)\to 0$ as $\Delta\to 0$. Let $\Delta\in(0,\Delta_2\wedge(\delta/2)]$ be arbitrary. We see from (5.43) and (5.44) that for every $\omega\in\Omega_{3,\Delta}$, $\bar{X}_\Delta(t)\in\mathbb{R}_+^d$ with $|\bar{X}_\Delta(t)|\le\mu^{-1}(h(\Delta))$ and $\inf_{0\le t\le T}\bar{X}_{\Delta,i}(t)>\delta/2$, whence $X_\Delta^+(t)=\pi_+(\pi_\Delta(\bar{X}_\Delta(t)))=\bar{X}_\Delta(t)$ for all $t\in[0,T]$. Consequently,

$$\begin{aligned}J_6(\Delta)&=\mathbb{E}\Big(\mathbb{1}_{\Omega_{3,\Delta}^c}\sup_{0\le t\le T}|X_\Delta^+(t)-\bar{X}_\Delta(t)|^p\Big)\\&\le\big[\mathbb{P}(\Omega_{3,\Delta}^c)\big]^{1/2}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-\bar{X}_\Delta(t)|^{2p}\Big)\Big]^{1/2}\\&\le 2^p\sqrt{\varepsilon}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)|^{2p}\Big)+\mathbb{E}\Big(\sup_{0\le t\le T}|\bar{X}_\Delta(t)|^{2p}\Big)\Big]^{1/2}.\end{aligned}$$

But, by Lemma 5.3.4,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\bar{X}_\Delta(t)|^{2p}\Big)\le C_p.$$

On the other hand, for any $x\in\mathbb{R}^d$,

$$\begin{aligned}|\pi_+(x)|^{2p}&=\Big(\sum_{i=1}^d(\Delta\vee x_i)^2\Big)^p\le\Big(\sum_{i=1}^d(\Delta^2+|x_i|^2)\Big)^p\\&\le(d+|x|^2)^p\le d^{p-1}(d^p+|x|^{2p}).\end{aligned}$$

So

$$\begin{aligned}|X_\Delta^+(t)|^{2p}&=|\pi_+(\pi_\Delta(\bar{X}_\Delta(t)))|^{2p}\le 2^pd^{p-1}(d^p+|\pi_\Delta(\bar{X}_\Delta(t))|^{2p})\\&\le d^{p-1}(d^p+|\bar{X}_\Delta(t)|^{2p}).\end{aligned}$$

Consequently

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)|^{2p}\Big)\le C_p.$$

In other words, we have showed that

$$J_6(\Delta)\le C_p\sqrt{\varepsilon}$$

provided $\Delta\in(0,\Delta_2\wedge(\delta/2)]$. As ε is arbitrary, we must have that $J_6(\Delta)\to 0$ as $\Delta\to 0$. This completes our proof of Theorem 5.3.2. $\Box$

5.4 An Example with Simulations

To illustrate as well as to verify our new PPTEM scheme, we consider the scalar stochastic Lotka–Volterra competitive model

$$dx(t)=x(t)[(b-ax(t))dt+\sigma dB(t)] \tag{5.46}$$

for a single species, where individuals within the species are competitive, $x(t)\in(0,\infty)$, b,a,σ are all positive numbers. The main reason we discuss this model is because it has an explicit solution so that we can compare it with the NPTEM numerical solution in order to verify the NPTEM scheme.

We write the Lotka–Volterra model (5.46) as the SDE (5.5) in $\mathbb{R}$ by defining

$$\begin{cases} f_1(x)=bx,\quad f_2(x)=-ax^2,\quad g(x)=\sigma x \quad \text{for } x\ge 0,\\ f_1(x)=f_2(x)=g(x)=0 \quad \text{for } x<0.\end{cases} \tag{5.47}$$

Define $\mu:\mathbb{R}_+\to\mathbb{R}_+$ by $\mu(u)=au^2$ for $u\ge 1$. Its inverse function of $\mu:[a,\infty)\to\mathbb{R}_+$ has the form $\mu^{-1}(u)=\sqrt{u/a}$. Let $\hat h=1\vee a\wedge x(0)$ and define the strictly decreasing function $h:(0,1]\to[\mu(1),\infty)$ by $h(\Delta)=\hat h\Delta^{-\theta}$ for some $\theta\in(0,1/4]$. Hence $\mu^{-1}(h(\Delta))=\sqrt{\hat h/a\Delta^\theta}$. The mapping $\pi_+(\pi_\Delta(\cdot)):\mathbb{R}\to\Big[\Delta,\sqrt{\hat h/a\Delta^\theta}\Big]$ has the form

$$\pi_+(\pi_\Delta(x))=(\Delta\vee x)\wedge\sqrt{\hat h/a\Delta^\theta},\quad \text{for } x\in\mathbb{R}.$$

We first apply the NPTEM to the Lotka–Volterra model (5.46) (namely the SDE (5.5) with f_1,f_2 and g being defined by (5.47)). That is, set $\bar X_\Delta(0)=X_\Delta(0)=x(0)$

and compute

$$\bar{X}_\Delta(t_{k+1}) = \bar{X}_\Delta(t_k) + [f_1(\bar{X}_\Delta(t_k)) + f_2(X_\Delta(t_k))]\Delta + g(\bar{X}_\Delta(t_k))\Delta B_k, \tag{5.48}$$

$$X_\Delta(t_{k+1}) = (0 \vee \bar{X}_\Delta(t_{k+1})) \wedge \sqrt{\hat{h}/a\Delta^\theta} \tag{5.49}$$

for $k = 0, 1, \cdots$, and then extend the definitions of $X_\Delta(\cdot)$ from the grid points t_k to the whole $t \geq 0$ by (5.15). The PPTEM solution is then defined by

$$X_\Delta^+(t) = (\Delta \vee X_\Delta(t)) \wedge \sqrt{\hat{h}/a\Delta^\theta}, \quad t \geq 0.$$

By Theorem 5.3.2, we can conclude that $X_\Delta^+(T)$ converges to $x(t)$ defined by (5.51) in the sense that

$$\lim_{\Delta\to 0} \mathbb{E}\Big(\sup_{0\leq t\leq T} |X_\Delta^+(t) - x(t)|^p \Big) = 0. \tag{5.50}$$

Given an initial value $x(0) > 0$, the solution $x(t)$ remains to be positive. Let $z(t) = 1/x(t)$. By the Itô formula,

$$dz(t) = [a + (\sigma^2 - b)z(t)]dt - \sigma z(t)dB(t).$$

By the variation-of-constants formula (see, e.g., [19, Theorem 4.1 on p.96]),

$$z(t) = \exp\big(- [b - 0.5\sigma^2]t - \sigma B(t)\big)\Big(z(0) + a \int_0^t \exp\big([b - 0.5\sigma^2]s + \sigma B(s)\big)ds\Big).$$

This gives the explicit solution of (5.46):

$$x(t) = \exp\big([b - 0.5\sigma^2]t + \sigma B(t)\big)\Big(\frac{1}{x(0)} + a \int_0^t \exp\big([b - 0.5\sigma^2]s + \sigma B(s)\big)ds\Big)^{-1}. \tag{5.51}$$

Although the integration in this formula cannot be calculated analytically, it can be approximated numerically by the Riemann sum. More precisely, define

$$\phi(t) = \exp\big([b - 0.5\sigma^2]t + \sigma B(t)\big), \quad 0 \leq t \leq T. \tag{5.52}$$

In the remaining of this example, we set $\Delta = T/N$ for an integer $N > T$ and

$t_k = k\Delta$ for $0 \le k \le N$. We approximate $\int_0^{t_k} \phi(s)ds$ by

$$\Psi_\Delta(t_k) = \sum_{i=0}^{k-1} 0.5\Delta[\phi(t_i) + \phi(t_{i+1})], \quad 0 \le k \le N \tag{5.53}$$

and of course set $\Psi(t_0) = 0$. We then form the discrete-time Riemann approximate solutions $Y_\Delta(t_k) \approx x(t_k)$ by

$$Y_\Delta(t_k) = \phi(t_k)/(1/x(0) + a\Psi_\Delta(t_k)), \quad 0 \le k \le N. \tag{5.54}$$

We will show in appendix A that

$$\lim_{N\to\infty} \mathbb{E}\Big(\sup_{0\le k\le N} |Y_\Delta(t_k) - x(t_k)|^2 \Big) = 0. \tag{5.55}$$

Although it is sufficient to compare our new PPTEM solutions $X_\Delta^+(t_k)$ with $Y_\Delta(t_k)$, we will do better by comparing it with the well-known backward Euler-Maruyama (BEM) scheme (see, e.g., [10]) as well. To be more precise, the BEM applied to the Lotka–Volterra model is to form the discrete-time BEM solutions $Z_\Delta(t_k) \approx x(t_k)$ by setting $Z_\Delta(0) = x(0)$ and computing

$$Z_\Delta(t_{k+1}) = Z_\Delta(t_k) + [f_1(Z_\Delta(t_k)) + f_2(Z_\Delta(t_{k+1}))]\Delta + g(Z_\Delta(t_k))\Delta B_k$$

for $k \ge 0$. It is known that

$$\lim_{N\to\infty} \mathbb{E}\Big(\sup_{0\le k\le N} |Z_\Delta(t_k) - x(t_k)|^2 \Big) = 0.$$

For numerical simulations, we let $b = 10$, $a = 1$, $\sigma = 0.5$, $x(0) = 6$ and choose $\theta = 1/4$, $\hat{h} = 1000$, whence $\mu^{-1}(h(\Delta)) = \sqrt{\hat{h}/\Delta^\theta}$. The simulations in Figure 4.1 show the sample paths of the solution for $t \in [0, 10]$ by three schemes of the PPTEM, Riemann and BEM. The simulations in the left graph use $\Delta = 10^{-3}$ while in the right $\Delta = 10^{-4}$.

The simulations show that three sample paths generated by the three schemes are very close to each other. More precisely, the simulations are designed to produce the squares of the max differences between PPTEM and Riemann as well as

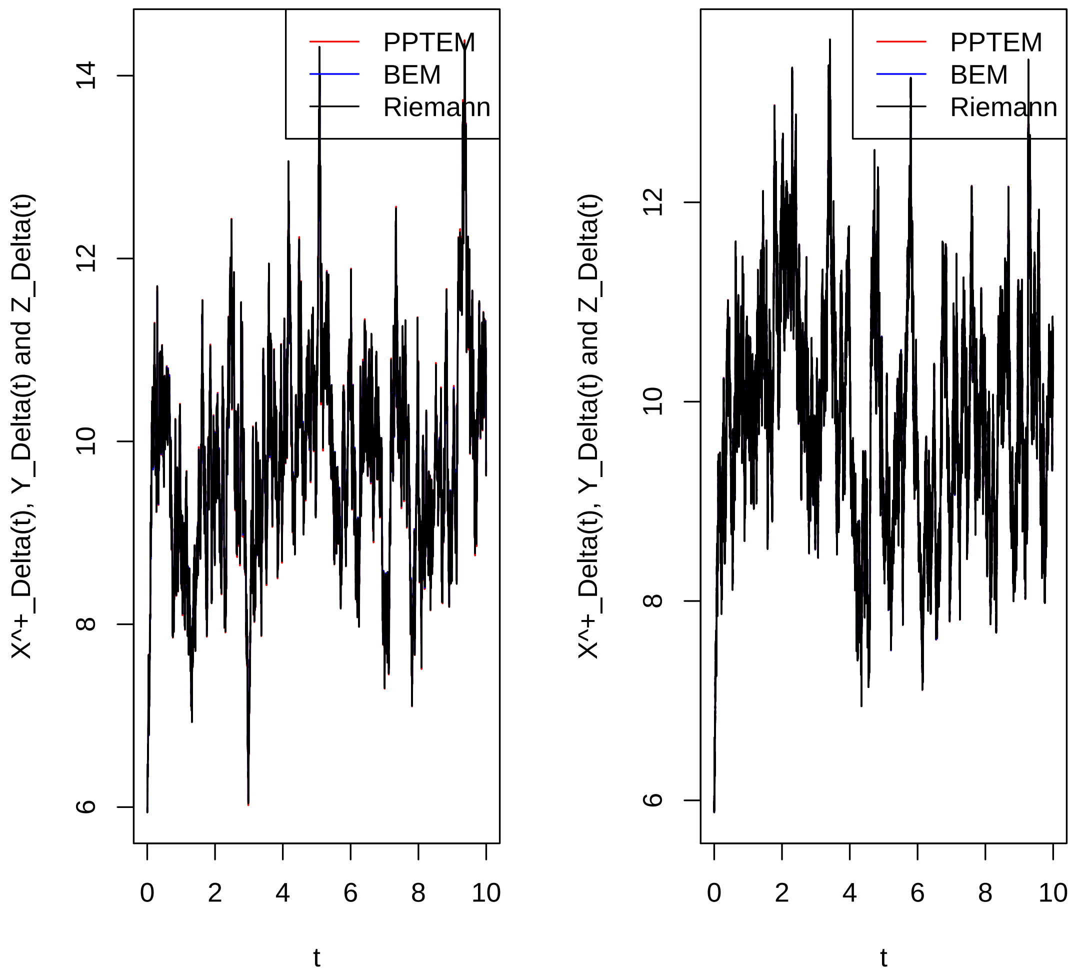

Figure 4.1: The computer simulations of the sample paths of the solution to equation (5.46) by PPTEM, Riemann and BEM. Left: $\Delta = 10^{-3}$. Right: $\Delta = 10^{-4}$.

BEM and Riemann:

$$\max_{0 \le k \le 10^4} |X^+_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.002809 \quad \text{and} \quad \sup_{0 \le k \le 10^4} |Z_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.005086$$

when $\Delta = 10^{-3}$; while

$$\max_{0 \le k \le 10^5} |X^+_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.0002235 \quad \text{and} \quad \sup_{0 \le k \le 10^5} |Z_\Delta(t_k) - Y_\Delta(t_k)|^2 = 0.0002527$$

when $\Delta = 10^{-4}$. These seem to indicate that PPTEM is closer to Riemann than BEM. To confirm this, we repeat the above simulations 100 times (namely, simulate 100 sample paths for each of the three scheme) and produce the mean squares (MS) of the max differences:

$$\frac{1}{100}\sum_{j=1}^{100}\Big(\sup_{0\le k\le N}|X_{\Delta}^{+,j}(t_k) - Y_{\Delta}^{j}(t_k)|^2\Big) \text{ and } \frac{1}{100}\sum_{j=1}^{100}\Big(\sup_{0\le k\le N}|Z_{\Delta}^{j}(t_k) - Y_{\Delta}^{j}(t_k)|^2\Big),$$

where j stands for the jth sample paths. To reduce the time of simulations without losing any necessary illustration, we only simulate the paths for $t \in [0,1]$ but we make comparisons for $\Delta = 10^{-2}, 10^{-3}, 10^{-4}$ and 10^{-5}. The outcomes of the simulations are shown in Figure 4.2. They show that our new PPTEM solutions are closer to Riemann solutions than BEM slightly. They also indicates that our new PPTEM solutions converge to the true solution with the rate of order 0.5, though we have not proved this in theory yet but we will tackle it in the future.

Chapter 6

Positive Preserving Truncated Euler-Maruyama Numerical Method for Stochastic Delay Differential Equations

6.1 Introduction

In the previous chapter, we introduced modifications to the truncated EM method, namely PPTEM and NPTEM, for the stochastic Lotka-Volterra model, which describes the population growth of d interacting species. In reality, ecological systems often exhibit time lags and delayed responses to changes in population dynamics, resource availability, environmental conditions, and other factors. The delay Lotka-Volterra model is better suited to capture these real-world complexities by accounting for delays in predator-prey interactions and other ecological processes, see [1, 19].

In this chapter, we apply the ideas from Chapter 5 to establish the PPTEM and NPTEM numerical solutions for the stochastic delay Lotka–Volterra model. We also adopt the concepts of the variable time delay from Chapter 4 to the delay equations. Our approach follows a similar procedure as before, starting with NPTEM and then PPTEM. Nonetheless, it's worth noting that we encounter the need for slightly stronger conditions, which may be subject to relaxation in future research, to establish the convergence of these two models.

Now, let us introduce the d-dimensional stochastic delay Lotka–Volterra model (see, e.g., [1, 19])

$$dx(t) = \mathrm{diag}(x_1(t), x_2(t), ..., x_d(t))[(b - Ax(t) - \bar{A}x(\delta(t)))dt + \sigma dB(t)], \qquad (6.1)$$

where $x(t) = (x_1(t), \cdots, x_d(t))^T$ is the state of the d interacting species and the system parameters $b = (b_1, \cdots, b_d)^T \in \mathbb{R}^d$, $\sigma = (\sigma_1, \cdots, \sigma_d)^T \in \mathbb{R}^d$, $A = (a_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$, $\bar{A} = (\bar{a}_{ij})_{d\times d} \in \mathbb{R}^{d\times d}$. We define the initial condition $\{x(\theta) : -\tau \le \theta \le 0\} = \xi \in C\left([-\tau, 0] : \mathbb{R}^d_+\right)$ and $-\tau \le \delta(t) \le t$ (including condition that there are $\gamma > 0$ and $\beta \in (0, 1]$, $|\xi(u) - \xi(v)| \le \gamma\, |u - v|^\beta$ for $u, v \in [-\tau, 0]$).

Assumption 6.1.1. *All elements of A and $\bar{A}$ are nonnegative, namely $a_{ij} \ge 0$ and $\bar{a}_{ij} \ge 0$ for all $1 \le i, j \le d$.*

It is known that, to confirm the existence and uniquness of equation (6.1), it is sufficient to assume only A are nonnegative with the initial value $\{x(\theta) : -\tau \le \theta \le 0\}$. However, in this thesis, we add a condition $\bar{A}$ is nonnegative to Assumption 6.1.1 as well to help us model the PPTEM and NPTEM for equation 6.1, see, e.g., [19].

Throughout this chapter, we set

$$b' = \max_{1\le i\le d} |b_i|, \quad \sigma' = \max_{1\le i\le d} |\sigma_i|, \quad a' = \max_{1\le i,j\le d} a_{ij}, \quad \bar{a}' = \max_{1\le i,j\le d} \bar{a}_{ij}. \qquad (6.2)$$

From this point forward, we choose an arbitrary initial data $x(\theta) \in \mathbb{R}^d_+$ for $\theta \in [-\tau, 0]$ and naturally, $x(t)$, represent the corresponding solution. We will also select two real numbers $T > 0$ and $p \ge 2$ arbitrarily. Recalling that we also continue to use C and C_p to stand for generic positive real constants independent on the step size Δ. Now, we introduce two lemmas which will serve a valuable purpose in this chapter.

Lemma 6.1.2. *Under Assumption 6.1.1, we then have*

$$\mathbb{E}\left(\sup_{-\tau\le t\le T} |x(t)|^p\right) \le C_p. \qquad (6.3)$$

Proof. As we know that $x(t) \in \mathbb{R}^d_+$, by applying the Itô formula and Assumption

6.1.1, we can easily show from (6.1) that

$$\begin{aligned} d(x_i(t))^p \leq & \Bigg[p\,(x_i(t))^{p-1} \left(b_i x_i(t) - \sum_{j=1}^d a_{ij} x_i(t) x_j(t) - \sum_{j=1}^d \bar{a}_{ij} x_i(t) x_j(\delta(t)) \right) \\ & + \frac{1}{2} p(p-1)\sigma_i^2](x_i(t))^p \Bigg] dt + p\sigma_i (x_i(t))^p dB(t) \\ & \leq p[\bar{b} + 0.5(p-1)\bar{\sigma}^2](x_i(t))^p dt + p\sigma_i (x_i(t))^p dB(t), \end{aligned}$$

for $t \geq 0$ and every $i = 1, \cdots, d$. By the Burkholder-Davis-Gundy inequality, it is straightforward to show that

$$\mathbb{E}\Big(\sup_{-\tau \leq u \leq t} (x_i(u))^p \Big) \leq C_p + C_p \int_0^t \mathbb{E}\Big(\sup_{-\tau \leq u \leq s} (x_i(u))^p \Big) ds, \quad \forall t \in [0, T].$$

An application of the Gronwall inequality gives

$$\mathbb{E}\Big(\sup_{-\tau \leq u \leq T} (x_i(u))^p \Big) \leq C_p.$$

This implies the required assertion (6.3). □

Lemma 6.1.3. *Under Assumption 6.1.1,*

$$\mathbb{E}\Big(\sup_{0 \leq t \leq T} [x_i(t) - 1 - \log(x_i(t))] \Big) \leq C, \quad 1 \leq i \leq d. \tag{6.4}$$

Proof. For each i, by the Itô formula, we have

$$\begin{aligned} & d[x_i(t) - 1 - \ln(x_i(t))] \\ & \leq \Big(-b_i + 0.5\sigma_i^2 + b_i x_i(t) + \sum_{j=1}^d a_{ij} x_j(t) + \sum_{j=1}^d \bar{a}_{ij} x_j(\delta(t)) \Big) dt + \sigma_i (x_i(t) - 1) dB(t). \end{aligned}$$

By Lemma 6.1.2, the first and second moments of the solution is bounded (by C) for $t \in [0, T]$. Applying the Burkholder–Davis–Gundy inequality again, we can derive the boundary in a manner similar to that in Theorem 5.1.3, that is

$$\begin{aligned}\mathbb{E}\Big(\sup_{0\le t\le T}[x_i(t)-1-\log(x_i(t))]\Big) \le& C+\mathbb{E}\Big(\sup_{0\le t\le T}\int_0^t \sigma_i(x_i(s)-1)dB(s)\Big)\\ \le& C+3\mathbb{E}\Big(\int_0^T |\sigma_i(x_i(s)-1)|^2ds\Big)^{1/2}\\ \le& C+3\bar{\sigma}\Big(\int_0^T 2(\mathbb{E}|x_i(s)|^2+1)ds\Big)^{1/2}\\ \le& C+3\bar{\sigma}\sqrt{2T(C+1)},\end{aligned}$$

which is the desired assertion (6.4). □

6.2 Definitions of New Numerical Schemes

We are now going to develop two numerical methods: NPTEM for the first and PPTEM for the second.

6.2.1 Nonnegativity preserving truncated EM method

To define the NPTEM scheme, it would be advantageous to work with the SDDE (6.1) in $\mathbb{R}^d$ rather than $\mathbb{R}^d_+$. We, therefore, must expand the definition of the SDDE coefficients from $\mathbb{R}^d_+$ to $\mathbb{R}^d$. These coefficients are denoted as follows:

$$\begin{aligned}F_1(x)=(b_1x_1,\cdots,b_dx_d)^T,\quad F_2(x)=-\mathrm{diag}(x_1,\cdots,x_d)Ax,\\ F_3(x,y)=-\mathrm{diag}(x_1,\cdots,x_d)\bar{A}y,\quad G(x)=(\sigma_1x_1,\cdots,\sigma_dx_d)^T\end{aligned}$$

for $x\in\bar{\mathbb{R}}^d_+$. Define a mapping $\pi_0:\mathbb{R}^d\to\bar{\mathbb{R}}^d_+$ by

$$\pi_0(x)=(x_1\vee 0,\cdots,x_d\vee 0)^T \quad \text{for } x\in\mathbb{R}^d.$$

Define $f_1,f_2,g:\mathbb{R}^d\to\mathbb{R}^d$ by

$$f_1(x)=F_1(\pi_0(x)),\quad f_2(x)=F_2(\pi_0(x)),f_3(x,y)=F_3(\pi_0(x),\pi_0(y))\quad g(x)=G(\pi_0(x))$$

for $x\in\mathbb{R}^d$. Obviously, f_1,f_2,f_3,g represent the extended functions of F_1,F_2,F_3,G, respectively. Considering that the solution of the SDDE (6.1) obtains the property that $x(t)\in\mathbb{R}^d_+$ a.s. for all $t\ge 0$, we can express the SDDE (6.1) as the following equation

$$dx(t) = [f_1(x(t)) + f_2(x(t)) + f_3(x(t), x(\delta(t))]dt + g(x(t))dB(t) \tag{6.5}$$

in $\mathbb{R}^d$. We observe that f_1 and g are linearly bounded, namely

$$|f_1(x)| \le \bar{b}|x|, \quad |g(x)| \le \bar{\sigma}|x|, \quad \forall x \in \mathbb{R}^d, \tag{6.6}$$

while f_2 and f_3 are not.

We, then, choose a strictly increasing continuous function $\mu : [1, \infty) \to \mathbb{R}_+$ such that $\mu(u) \to \infty$ as $u \to \infty$ and

$$\sup_{x \in \mathbb{R}^d,\, |x| \le u} |f_2(x)| = \sup_{x \in \bar{\mathbb{R}}^d_+,\, |x| \le u} |F_2(x)| \le \mu(u), \tag{6.7}$$

$$\text{and} \quad \sup_{x,y \in \mathbb{R}^d,\, |x| \vee |y| \le u} |f_3(x, y)| = \sup_{x,y \in \bar{\mathbb{R}}^d_+,\, |x| \vee |y| \le u} |F_3(x, y)| \le \mu(u). \tag{6.8}$$

for all $u \ge 1$ Denote by μ^{-1} the inverse function of μ and we see that μ^{-1} is a strictly increasing continuous function from $[\mu(1), \infty)$ to $\mathbb{R}_+$. We also choose a constant $\hat{h} \ge 1 \vee \mu(1) \vee |x(0)|$ and a strictly decreasing function $h : (0, 1] \to [\mu(1), \infty)$ such that

$$\lim_{\Delta \to 0} h(\Delta) = \infty \quad \text{and} \quad \Delta^{1/4} h(\Delta) \le \hat{h}, \quad \forall \Delta \in (0, 1]. \tag{6.9}$$

Note that for $x \in \bar{\mathbb{R}}^d_+$,

$$|F_2(x)|^2 = \sum_{i=1}^d x_i^2 \left(\sum_{j=1}^d a_{ij} x_j \right)^2 \le \sum_{i=1}^d x_i^2 \left(\sum_{j=1}^d a_{ij}^2 \right) |x|^2 \le |A|^2 |x|^4,$$

$$|F_3(x, y)|^2 = \sum_{i=1}^d x_i^2 \left(\sum_{j=1}^d a_{ij} y_j \right)^2 \le \sum_{i=1}^d x_i^2 \left(\sum_{j=1}^d a_{ij}^2 \right) |y|^2 \le |A|^2 |x|^2 |y|^2$$

We can hence let $\mu(u) = |A|u^2$, while let $h(\Delta) = \hat{h}\Delta^{-\theta}$ for some $\theta \in (0, 1/4]$. In other words, there are lots of choices for $\mu(\cdot)$ and $h(\cdot)$.

For a given step size $\Delta \in (0, 1]$, let us define the truncation mapping $\pi_\Delta : \mathbb{R}^d \to \{x \in \mathbb{R}^d : |x| \le \mu^{-1}(h(\Delta))\}$ by

$$\pi_\Delta(x) = \big(|x| \wedge \mu^{-1}(h(\Delta))\big) \frac{x}{|x|},$$

where we set $\frac{x}{|x|} = 0$ when $x = 0$. In other word, π_Δ maps x to itself or $\mu^{-1}(h(\Delta))\frac{x}{|x|}$ based on whether $|x| \le \mu^{-1}(h(\Delta))$ or not. It is beneficial to note that for any $x, y \in \mathbb{R}^d$,

$$f_2(\pi_0(\pi_\Delta(x))) = F_2(\pi_0(\pi_0(\pi_\Delta(x)))) = F_2(\pi_0(\pi_\Delta(x))) = f_2(\pi_\Delta(x)), \tag{6.10}$$

$$\begin{aligned} f_3(\pi_0(\pi_\Delta(x)), \pi_0(\pi_\Delta(y))) &= F_3(\pi_0(\pi_0(\pi_\Delta(x))), \pi_0(\pi_0(\pi_\Delta(y)))) \\ &= F_3(\pi_0(\pi_\Delta(x)), \pi_0(\pi_\Delta(y))) \\ &= f_3(\pi_\Delta(x), \pi_\Delta(y)). \end{aligned} \tag{6.11}$$

Hence

$$|f_2(\pi_0(\pi_\Delta(x)))| = |f_2(\pi_\Delta(x))| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta), \tag{6.12}$$

$$|f_3(\pi_0(\pi_\Delta(x)), \pi_0(\pi_\Delta(y)))| = |f_3(\pi_\Delta(x), \pi_\Delta(y))| \le \mu(\mu^{-1}(h(\Delta))) = h(\Delta). \tag{6.13}$$

Moreover, noting $\pi_0(\pi_\Delta(x)) = (|x| \wedge \mu^{-1}(h(\Delta)))\frac{\pi_0(x)}{|x|}$, we also have

$$x^T f_2(\pi_0(\pi_\Delta(x))) = x^T f_2(\pi_\Delta(x)) = (\pi_0(x))^T F_2(\pi_0(\pi_\Delta(x))) \le 0, \tag{6.14}$$

$$\begin{aligned} x^T f_3(\pi_0(\pi_\Delta(x)), \pi_0(\pi_\Delta(y))) &= x^T f_3(\pi_\Delta(x), \pi_\Delta(y)) \\ &= (\pi_0(x))^T F_3(\pi_0(\pi_\Delta(x)), \pi_0(\pi_\Delta(y))) \le 0. \end{aligned} \tag{6.15}$$

From now on, let Δ be a faction of τ, that means $\Delta = \frac{\tau}{M}$ for some positive integer M, and also define $t_k = k\Delta$ for all $k = -M, -M+1, \cdots, 0, 1, 2, \cdots$. We can now form the discrete-time NPTEM solutions $X_\Delta^0(t_k) \approx x(t_k)$ for $t_k = k\Delta$ by setting $X_\Delta^0(t_k) = \bar{X}_\Delta(t_k) = X_\Delta(t_k) = x(t_k) = \xi(t_k)$ for $k = -M, -M+1, \cdots, 0$. Then, we compute

$$\begin{aligned} \bar{X}_\Delta(t_{k+1}) = \bar{X}_\Delta(t_k) &+ [f_1(\bar{X}_\Delta(t_k)) + f_2(X_\Delta(t_k)) + f_3(X_\Delta(t_k), X_\Delta(I_\Delta[\delta(t_k)]\Delta))]\Delta \\ &+ g(\bar{X}_\Delta(t_k))\Delta B_k, \end{aligned} \tag{6.16}$$

$$X_\Delta(t_{k+1}) = \pi_0(\pi_\Delta(\bar{X}_\Delta(t_{k+1}))), \tag{6.17}$$

for $k = 0, 1, \cdots$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Please note that $\bar{X}_\Delta(t_{k+1})$ is an intermediate step in order to get the NPTEM solution $X_\Delta^0(t_{k+1})$. We extend the above definitions from the grid points t_k to the whole $t \geq 0$ by defining

$$\bar{X}_\Delta(t) = \sum_{k=0}^{\infty} \bar{X}_\Delta(t_k) \mathbb{1}_{[t_k, t_{k+1})}(t) \tag{6.18}$$

$$z_1(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) \mathbb{1}_{[t_k, t_{k+1})}(t) \tag{6.19}$$

$$z_2(t) = \sum_{k=0}^{\infty} X_\Delta(I_\Delta[\delta(t_k)]\Delta) \mathbb{1}_{[t_k, t_{k+1})}(t) \tag{6.20}$$

for $t \geq 0$. Clearly, $X^0_\Delta(t) = z_1(t) = \pi_0(\pi_\Delta(\bar{X}_\Delta(t)))$ so it preserves the nonnegativity.

We define one more truncation mapping $\pi_+ : \mathbb{R}^d \to \mathbb{R}^d_+$ by

$$\pi_+(x) = (\Delta \vee x_1, \cdots, \Delta \vee x_d)^T, \quad x \in \mathbb{R}^d.$$

Hence, the Positive preserving truncated EM (PPTEM) solution is defined by

$$X^+_\Delta(t) = \pi_+(\pi_\Delta(\bar{X}_\Delta(t))), \quad t \geq 0, \tag{6.21}$$

where $\bar{X}_\Delta(t)$ has already been defined by (6.18).

6.3 Main Results

6.3.1 Statement of main results

The objective of this section is to show that both NPTEM solution $X^0_\Delta(t)$ and PPTEM solution $X^+_\Delta(t)$, which are defined in the previous section converge to the true solution $x(t)$ in L^p for any $p \geq 2$.

Theorem 6.3.1. *Under Assumption 6.1.1, it holds that*

$$\lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \leq t \leq T} |X^0_\Delta(t) - x(t)|^p \Big) = 0. \tag{6.22}$$

Theorem 6.3.2. *Under Assumption 6.1.1, it holds that*

$$\lim_{\Delta \to 0} \mathbb{E}\Big(\sup_{0 \leq t \leq T} |X^+_\Delta(t) - x(t)|^p \Big) = 0. \tag{6.23}$$

For clarity, we will divide this process into multiple lemmas in the following subsection and then proceed to prove the theorems.

6.3.2 Lemmas

For the mathematical analysis, we define a new process starting with $x_\Delta(t) = \xi(t)$ for $t \in [-\tau, 0]$ and

$$x_\Delta(t) = \xi(0) + \int_0^t [f_1(\bar{X}_\Delta(s)) + f_2(z_1(s)) + f_3(z_1(s), z_2(s))]ds + \int_0^t g(\bar{X}_\Delta(s))dB(s) \tag{6.24}$$

for $t \geq 0$. It's worth to noting that $x_\Delta(t_k) = \bar{X}_\Delta(t_k)$ for all $k \geq 1$. Additionally, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = [f_1(\bar{X}_\Delta(t)) + f_2(z_1(t)) + f_3(z_1(t), z_2(t)]dt + g(\bar{X}_\Delta(t))dB(t). \tag{6.25}$$

We also use the notation of the ith component of $x_\Delta(t)$ or $\bar{X}_\Delta(t)$ by $x_{\Delta,i}(t)$ or $\bar{X}_{\Delta,i}(t)$, respectively.

From (6.6) and (6.12), it is straightforward to deduce from (6.16) that, for any $q \geq 2$, $\mathbb{E}|\bar{X}_\Delta(t_k)|^q < \infty$ for all $k \geq 1$. By (6.24), we can consequently establish that $\mathbb{E}|x_\Delta(t)|^q < \infty$ for all $t \geq 0$. However, the better result will be obtained (see Lemma 6.3.4).

Before showing Lemma 6.3.4, we will start by proving that $x_\Delta(t)$ and $\bar{X}_\Delta(t)$ are close to each other in the sense of L^p.

Lemma 6.3.3. *For any* $\Delta \in (0, 1]$, *we have*

$$\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p \leq C_p \Delta^{p/2}(h(\Delta))^p, \quad \forall t \in [0, T]. \tag{6.26}$$

Consequently

$$\lim_{\Delta \to 0} \mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p = 0. \tag{6.27}$$

Proof. By (6.12) and (6.13), we have

$$|f_2(z_1(t))| = |f_2(\pi_0(\pi_\Delta(\bar{X}_\Delta(t))))| \leq h(\Delta), \tag{6.28}$$

$$|f_3(z_1(t), z_2(t))| = \left|f_3(\pi_0(\pi_\Delta(\bar{X}_\Delta(t))), \pi_0(\pi_\Delta(\bar{X}_\Delta(I_\Delta[\delta(t)]\Delta))))\right| \leq h(\Delta). \tag{6.29}$$

We can easily show that

$$\sup_{-\tau\le u\le T} \mathbb{E}|x_\Delta(u)|^p \le C_p(h(\Delta))^p. \tag{6.30}$$

Now, for any $t \in [0,T]$, there is a unique $k \ge 0$ such that $t \in [t_k, t_{k+1})$ and hence $\bar{X}_\Delta(t) = \bar{X}_\Delta(t_k) = x_\Delta(t_k)$, $z_1(t) = \pi_0(\pi_\Delta(\bar{X}_\Delta(t_k)))$, $z_2(t) = \pi_0(\pi_\Delta(\bar{X}_\Delta(I_\Delta[\delta(t_k)]\Delta)))$. It then follows from (6.24) that

$$\begin{aligned}
&\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p = \mathbb{E}|x_\Delta(t) - x_\Delta(t_k)|^p \\
&\le C_p\Delta^{p-1}\mathbb{E}\bigg[\int_{t_k}^t \Big(\big|f_1(\bar{X}_\Delta(s))\big|^p + |f_2(z_1(s))|^p + |f_3(z_1(s), z_2(s))|^p\Big)\, ds \\
&\quad + C_p\Delta^{\frac{p-2}{2}}\int_{t_k}^t |g(\bar{X}_\Delta(s))|^p ds\bigg].
\end{aligned}$$

This, along with (6.6), (6.12), (6.13) and (6.30), implies

$$\mathbb{E}|x_\Delta(t) - \bar{X}_\Delta(t)|^p \le C_p\Delta^p(h(\Delta))^p + C_p\Delta^{\frac{p}{2}}(h(\Delta))^p \le C_p\Delta^{\frac{p}{2}}(h(\Delta))^p$$

which is the first assertion. Noting from (6.9) that $\Delta^{\frac{p}{2}}(h(\Delta))^p \le \Delta^{\frac{p}{4}}$, we obtain the second assertion from the first one immediately. □

Now, we can state the lemma which is the better result than (6.30).

Lemma 6.3.4. *Let Assumption 6.1.1 hold. Then*

$$\sup_{0<\Delta\le 1} \mathbb{E}\Big(\sup_{0\le t\le T} |x_\Delta(t)|^p\Big) \le C. \tag{6.31}$$

Proof. Fix any $\Delta \in (0,1]$. By the Itô formula and the Burkholder-Davis-Gundy inequality, it is staightforward to show that

$$\mathbb{E}\Big(\sup_{0\le u\le t} |x_\Delta(u)|^p\Big) \le C_p + C_p\int_0^t \mathbb{E}\Big(\sup_{0\le u\le s} |x_\Delta(u)|^p\Big)ds + J_1(t) + J_2(t) \tag{6.32}$$

for $t \in [0, T]$, where

$$J_1(t) = \mathbb{E}\left(\sup_{0 \le u \le t} \int_0^u p|x_\Delta(s)|^{p-2} x_\Delta^T(s) f_2(z_1(s)) ds\right),$$
$$J_2(t) = \mathbb{E}\left(\sup_{0 \le u \le t} \int_0^u p|x_\Delta(s)|^{p-2} x_\Delta^T(s) f_3(z_1(s), z_2(s)) ds\right).$$

By applying (6.12), (6.14) and Lemma 6.3.3, we can derive J_1 in the same way as Lemma 5.3.4 was prove that

$$J_1(t) \le (p-2) \int_0^t \mathbb{E}\Big(\sup_{0 \le u \le s} |x_\Delta(u)|^p\Big) ds + C_p,$$

By (6.13) and (6.15), we have

$$\begin{aligned} &x_\Delta^T(s) f_3(z_1(s), z_2(s)) \\ &= \big([x_\Delta(s) - \bar{X}_\Delta(s)]^T + \bar{X}_\Delta^T(s)\big) f_3(\pi_0(\pi_\Delta(\bar{X}_\Delta(s))), \pi_0(\pi_\Delta(\bar{X}_\Delta(I_\Delta[\delta(s)]\Delta)))) \\ &\le h(\Delta)|x_\Delta(s) - \bar{X}_\Delta(s)|. \end{aligned}$$

Then J_2 can be derived that

$$J_2(t) \le (p-2) \int_0^t \mathbb{E}\Big(\sup_{0 \le u \le s} |x_\Delta(u)|^p\Big) ds + C_p.$$

Substituting these into (6.32) yields

$$\mathbb{E}\Big(\sup_{0 \le u \le t} |x_\Delta(u)|^p\Big) \le C_p + C_p \int_0^t \mathbb{E}\Big(\sup_{0 \le u \le s} |x_\Delta(u)|^p\Big) ds.$$

By the Gronwall inequality, we can conclude

$$\mathbb{E}\Big(\sup_{0 \le u \le T} |x_\Delta(u)|^p\Big) \le C_p.$$

As this holds for any $\Delta \in (0, 1]$ while C_p is independent of Δ, we see the required assertion (6.31). $\square$

The following result mimics Lemma 5.3.5, demonstrating the improvement of the second assertion in Lemma 6.3.3. We, however, present the proof as follows.

Lemma 6.3.5. *Let Assumption 6.1.1 hold. Then*

$$\lim_{\Delta\to 0}\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar{X}_\Delta(t)|^p\Big)=0. \tag{6.33}$$

Proof. Let ℓ be an integer such that $t_\ell \le T < t_{\ell+1}$. Then, by (6.6) and (6.28) as well as Lemma 6.3.4, we derive that

$$\begin{aligned}
&\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-\bar{X}_\Delta(t)|^p\Big)\\
&\le \mathbb{E}\Big(\max_{0\le k\le \ell}\sup_{t_k\le t\le t_{k+1}}\Big|[f_1(\bar{X}_\Delta(t_k))+f_2(X_\Delta(t_k))+f_3(X_\Delta(t_k),X_\Delta(I_\Delta[\delta(t_k)]\Delta))](t-t_k)\\
&\quad + g(\bar{X}_\Delta(t_k))(B(t)-B(t_k))\Big|^p\Big)\\
&\le C_p\mathbb{E}\Big(\max_{0\le k\le \ell}[|\bar{X}_\Delta(t_k)|^p+(h(\Delta))^p]\Delta^p\Big)+J_3\\
&\le C_p\Delta^p\mathbb{E}\Big(\max_{0\le k\le \ell}|x_\Delta(t_k)|^p+(h(\Delta))^p\Big)+J_3\\
&\le C_p\Delta^p[C_p+(h(\Delta))^p]+J_3\le C_p\Delta^p(h(\Delta))^p+J_3,
\end{aligned} \tag{6.34}$$

where

$$J_3=C_p\mathbb{E}\Big(\max_{0\le k\le \ell}\Big[|\bar{X}_\Delta(t_k))|^p\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^p\Big]\Big).$$

Now, choose a sufficiently large integer $n\ge 3\vee p$, dependent on p and T, for which

$$\Big(\frac{2n}{2n-1}\Big)^p(T+1)^{p/2n}\le 2. \tag{6.35}$$

But, by the Hölder inequality,

$$\begin{aligned}
J_3 &\le C\Big\{\mathbb{E}\Big(\max_{0\le k\le \ell}\Big[|\bar{X}_\Delta(t_k))|^{2n}\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big]\Big)\Big\}^{p/2n}\\
&\le C\Big(\sum_{k=0}^{\ell}\mathbb{E}\Big[|\bar{X}_\Delta(t_k))|^{2n}\sup_{t_k\le t\le t_{k+1}}|B(t)-B(t_k)|^{2n}\Big]\Big)^{p/2n}.
\end{aligned}$$

But, by Lemma 6.3.4 (replacing p there by $2n$ though n here depends on p), $\mathbb{E}|\bar{X}_\Delta(t_k))|^{2n}$ is bounded by C for every t_k. Note also that for each k, $\bar{X}_\Delta(t_k)$ is independent of $\sup_{t_k \le t \le t_{k+1}} |B(t) - B(t_k)|^{2n}$. We hence have

$$\begin{aligned} J_3 &\le C\Big(\sum_{k=0}^{\ell} \mathbb{E}|\bar{X}_\Delta(t_k))|^{2n}\, \mathbb{E}\Big[\sup_{t_k \le t \le t_{k+1}} |B(t) - B(t_k)|^{2n}\Big]\Big)^{p/2n} \\ &\le C\Big(\sum_{k=0}^{\ell} \mathbb{E}\Big[\sup_{t_k \le t \le t_{k+1}} |B(t) - B(t_k)|^{2n}\Big]\Big)^{p/2n}. \end{aligned}$$

By the Doob martingale inequality, we can derive that

$$\begin{aligned} J_3 &\le C\Big(\sum_{k=0}^{\ell} \Big[\frac{2n}{2n-1}\Big]^{2n} \mathbb{E}|B(t_{k+1}) - B(t_k)|^{2n}\Big)^{p/2n} \\ &\le C\Big(\sum_{k=0}^{\ell} \Big[\frac{2n}{2n-1}\Big]^{2n} (2n-1)!!\Delta^n\Big)^{p/2n} \\ &\le C\Big(\Big[\frac{2n}{2n-1}\Big]^{2n} (T+1)(2n-1)!!\Delta^{n-1}\Big)^{p/2n}, \end{aligned}$$

where $(2n-1)!! = (2n-1) \times (2n-3) \times \cdots \times 3 \times 1$. But

$$[(2n-1)!!]^{1/n} \le \frac{1}{n}\sum_{i=1}^{n}(2i-1) = n.$$

Thus

$$J_3 \le Cn^{p/2}\Big(\frac{2n}{2n-1}\Big)^p (T+1)^{p/2n}\Delta^{p(n-1)/2n}.$$

Using (6.35) while noting $(n-1)/2n \ge 1/3$ as we choose $n \ge 3$, we obtain

$$J_3 \le C\Delta^{p/3}.$$

Substituting this into (6.34) gives

$$\mathbb{E}\Big(\sup_{0 \le t \le T} |x_\Delta(t) - \bar{X}_\Delta(t)|^p\Big) \le C(h(\Delta))^p\Delta^p + C\Delta^{p/3} \le C(h(\Delta))^p\Delta^{p/3}.$$

But, by (6.9),

$$(h(\Delta))^p\Delta^{p/3} = \Delta^{p/12}(\Delta^{1/4}h(\Delta))^p \le \Delta^{p/12}.$$

We hence obtain

$$\mathbb{E}\Big(\sup_{0 \le t \le T} |x_\Delta(t) - \bar{X}_\Delta(t)|^p\Big) \le C\Delta^{p/12}.$$

This implies the required assertion (6.33). □

In the remaining of this section, we set a few notations. For each $r > |x_0|$, define the stopping times

$$\tau_r = \inf\{t \geq 0 : |x(t)| \geq r\}$$

and

$$\rho_{\Delta,r} = \inf\{t \geq 0 : |x_\Delta(t)| \geq r\},$$

where throughout this paper we set $\inf \emptyset = \infty$. Moreover, we set

$$\theta_{\Delta,r} = \tau_r \wedge \rho_{\Delta,r}$$

and define the closed ball

$$S_r = \{x \in \mathbb{R}^d : |x| \leq r\}.$$

The following lemma shows both $x(t \wedge \theta_{\Delta,r})$ and $x_\Delta(t \wedge \theta_{\Delta,r})$ are close to each other.

Lemma 6.3.6. *Let Assumption 6.1.1. Then for each $r > |x_0|$, there is a $\Delta_1 = \Delta_1(r) \in (0,1]$ such that*

$$\mathbb{E}\Big(\sup_{0\leq t\leq T} |x(t \wedge \theta_{\Delta,r}) - x_\Delta(t \wedge \theta_{\Delta,r})|^p\Big) \leq C_r \left(\Delta^{p\beta} + \Delta^{\frac{p}{2}}(h(\Delta))^p\right), \tag{6.36}$$

for all $\Delta \in (0, \Delta_1]$ and C_r is a positive constant independent of Δ.

Proof. Define for $x \in \mathbb{R}^d$

$$f_{2,r}(x) = f_2\left((|x| \wedge r)\frac{x}{|x|}\right), \quad f_{3,r}(x,y) = f_3\left((|x| \wedge r)\frac{x}{|x|}, (|y| \wedge r)\frac{y}{|y|}\right).$$

Therefore, $f_{2,r}$ and $f_{3,r}$ are bounded and globally Lipschitz continuous in $\mathbb{R}^d$ but their Lipschitz constant depend on r. Consider the SDDE

$$dy(t) = [f_1(y(t)) + f_{2,r}(y(t)) + f_{3,r}(y(t), y(\delta(t)))]dt + g(y(t))dB(t) \tag{6.37}$$

on $t \geq 0$ with the initial value $y(\theta) = x(\theta)$ for $-\tau \leq \theta \leq 0$. There exists the unique global solution $y(t)$ on $t \geq 0$. For each step size $\Delta \in (0,1]$, we can apply the EM method to the SDDE (6.37). This involves generating approximate EM solutions $Y_\Delta(t_k) \approx y(t_k)$ for $t_k = k\Delta$ by initializing $Y_\Delta(\theta) = x(\theta)$ and computing

$$\begin{aligned} Y_\Delta(t_{k+1}) = Y_\Delta(t_k) &+ [f_1(Y_\Delta(t_k)) + f_{2,r}(Y_\Delta(t_k)) + f_{3,r}(Y_\Delta(t_k), Y_\Delta(I_\Delta[\delta(t_k)]\Delta))]\Delta \\ &+ g(Y_\Delta(t_k))\Delta B_k, \end{aligned} \tag{6.38}$$

for $k = 0, 1, \cdots$. Extend the definitions of Y_Δ from the grid points t_k to the whole $t \geq 0$ by setting

$$\begin{aligned} z_1(t) &= \sum_{k=0}^{\infty} Y_\Delta(t_k) I_{[t_k, t_{k+1})}(t), \\ z_2(t) &= \sum_{k=0}^{\infty} Y_\Delta(I_\Delta[\delta(t_k)]\Delta) I_{[t_k, t_{k+1})}(t), \end{aligned}$$

and then define the Itô process

$$y_\Delta(t) = x(0) + \int_0^t [f_1(z_1(s)) + f_{2,r}(z_1(s)) + f_{3,r}(z_1(s), z_2(s))]ds + \int_0^t g(z_1(s))dB(s) \tag{6.39}$$

for $t \geq 0$. By applying Lemma 4.1.8 and 4.1.9, for an arbitrary $T_1 \in [0, T]$,

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\le t\le T_1} |y(t) - y_\Delta(t)|^p\right) \\ &\le \mathbb{E} \sup_{0\le t\le T_1} \bigg| \int_0^t f_1(y(s)) - f_1(z_1(s)) + f_{2,r}(y(s)) - f_{2,r}(z_1(s)) \\ &\quad + f_{3,r}(y(s), y(\delta(s))) - f_{3,r}(z_1(s), z_2(s))ds \bigg|^p \\ &\quad + \mathbb{E} \sup_{0\le t\le T_1} \left| \int_0^t g(y(s)) - g(z_1(s))dB(s) \right|^p \\ &\le C\mathbb{E} \sup_{0\le t\le T_1} \bigg[\int_0^t |f_1(y(s)) - f_1(z_1(s))|^p \, ds + \int_0^t |f_{2,r}(y(s)) - f_{2,r}(z_1(s))|^p \, ds \\ &\quad + \int_0^t |f_{3,r}(y(s), y(\delta(s))) - f_{3,r}(z_1(s), z_2(s))|^p \, ds \bigg] \\ &\quad + C\mathbb{E} \int_0^{T_1} |g(y(s)) - g(z_1(s))|^p \, ds \end{aligned}$$

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T_1}|y(t)-y_\Delta(t)|^p\right)\\
&\le C_r\mathbb{E}\int_0^{T_1}|y(s)-z_1(s)|^p+|y(\delta(s))-z_2(s)|^p\,ds\\
&\le C_r\mathbb{E}\int_0^{T_1}|y(s)-y_\Delta(s)|^p+|y_\Delta(s)-z_1(s)|^p+|y(\delta(s))-y_\Delta(\delta(s))|^p\\
&\quad+|y_\Delta(\delta(s))-z_2(s)|^p\,ds\\
&\le C_r\int_0^{T_1}\mathbb{E}\sup_{0\le u\le s}|y(u)-y_\Delta(u)|^p\,ds+C_r\left(\Delta^{p\beta}+\Delta^{\frac{p}{2}}(h(\Delta))^p\right)
\end{aligned}
$$

By Gronwall's inequality,

$$
\mathbb{E}\Big(\sup_{0\le t\le T_1}|y(t)-y_\Delta(t)|^p\Big)\le C_r\left(\Delta^{p\beta}+\Delta^{\frac{p}{2}}(h(\Delta))^p\right). \tag{6.40}
$$

Let us relate $y(t)$ and $y_\Delta(t)$ to $x(t)$ and $x_\Delta(t)$, respectively. It is straightforward to see that

$$
x(t\wedge\tau_r)=y(t\wedge\tau_r)\ \text{ a.s for all } t\in[0,T]. \tag{6.41}
$$

We now choose $\Delta_1\in(0,1]$ sufficiently small for $\mu^{-1}(h(\Delta_1))\ge r$. Obviously, for all $\Delta\in(0,\Delta_1]$,

$$
f_2(\pi_\Delta(x))=f_{2,r}(x),\quad f_3(\pi_\Delta(x),\pi_\Delta(y))=f_{3,r}(x,y),\ \text{ for all } x,y\in S_r.
$$

This, together with (6.10) and (6.11), yields

$$
f_2(\pi_0(\pi_\Delta(x)))=f_{2,r}(x),\quad f_3(\pi_0(\pi_\Delta(x)),\pi_0(\pi_\Delta(y)))=f_{3,r}(x,y),\ \text{ for all } x,y\in S_r.
$$

Comparing (6.16), (6.24) with (6.38),(6.39), we then see that

$$
x_\Delta(t\wedge\rho_{\Delta,r})=y_\Delta(t\wedge\rho_{\Delta,r})\ \text{ a.s for all } t\in[0,T] \tag{6.42}
$$

provided $\Delta\in(0,\Delta_1]$. Combining (6.40) - (6.42), we obtain the desired assertion (6.36) immediately. $\square$

6.3.3 Proof of Theorem 6.3.1

We are ready to prove our main theorems. Same as Chapter 5, we start to prove Theorem 6.3.1 first in this subsection and then Theorem 6.3.2 next. Obviously,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X^0_\Delta(t)-x(t)|^p\Big)\le 3^{p-1}(J_4(\Delta)+J_5(\Delta)+J_6(\Delta)), \tag{6.43}$$

where

$$\begin{aligned} J_4(\Delta)&=\mathbb{E}\Big(\sup_{0\le t\le T}|X^0_\Delta(t)-\bar{X}_\Delta(t)|^p\Big),\\ J_5(\Delta)&=\mathbb{E}\Big(\sup_{0\le t\le T}|\bar{X}_\Delta(t)-x_\Delta(t)|^p\Big),\\ J_6(\Delta)&=\mathbb{E}\Big(\sup_{0\le t\le T}|x_\Delta(t)-x(t)|^p\Big). \end{aligned}$$

According to Lemma 6.3.5, it has already been proved that $J_5(\Delta)\to 0$ as $\Delta\to 0$. To finalize the proof, we only need to demonstrate that both $J_4(\Delta)$ and $J_6(\Delta)$ converge to 0.

Let us first show $J_6(\Delta)\to 0$. Let $\varepsilon\in(0,1)$ be arbitrary. By Lemmas 6.1.2 and 6.3.4, we have shown in Chapter 5 that

$$\mathbb{P}(\theta_{r,\Delta}\le T)\le\frac{C_p}{r^p}.$$

By choosing a real number $r=r(\varepsilon)$ so large, we obtain

$$\mathbb{P}(\theta_{r,\Delta}\le T)\le\varepsilon^2.$$

For this r, by Lemma 6.3.6, we have

$$\mathbb{E}\Big(\sup_{0\le t\le T}|x(t\wedge\theta_{\Delta,r})-x_\Delta(t\wedge\theta_{\Delta,r})|^p\Big)\le C_r\left(\Delta^{p\beta}+\Delta^{\frac{p}{2}}(h(\Delta))^p\right),$$

for all $\Delta\in(0,\Delta_1]$. Note that Δ_1 depends on ε (as r dependent on ε). Thus, for

$\Delta \in (0, \Delta_1]$, by Lemma 6.3.4 (recalling p is arbitrary once again), we derive

$$\begin{aligned}
J_6(\Delta) &= \mathbb{E}\Big(\mathbb{1}_{\{\theta_{r,\Delta} \le T\}} \sup_{0 \le t \le T} |x_\Delta(t) - x(t)|^p\Big) + \mathbb{E}\Big(\mathbb{1}_{\{\theta_{r,\Delta} > T\}} \sup_{0 \le t \le T} |x_\Delta(t) - x(t)|^p\Big) \\
&\le \big[\mathbb{P}(\theta_{r,\Delta} \le T)\big]^{1/2} \Big[\mathbb{E}\Big(\sup_{0 \le t \le T} |x_\Delta(t) - x(t)|^{2p}\Big)\Big]^{1/2} \\
&\quad + \mathbb{E}\Big(\sup_{0 \le t \le T} |x_\Delta(t \wedge \theta_{r,\Delta}) - x(t \wedge \theta_{r,\Delta})|^p\Big) \\
&\le \big[\mathbb{P}(\theta_{r,\Delta} \le T)\big]^{1/2} 2^{(p-1)/2} \Big[\mathbb{E}\Big(\sup_{0 \le t \le T} |x_\Delta(t)|^{2p}\Big) + \mathbb{E}\Big(\sup_{0 \le t \le T} |x(t)|^{2p}\Big)\Big]^{1/2} \\
&\quad + C_r \left(\Delta^{p\beta} + \Delta^{\frac{p}{2}} (h(\Delta))^p\right) \\
&\le C\varepsilon + C_r \left(\Delta^{p\beta} + \Delta^{\frac{p}{2}} (h(\Delta))^p\right).
\end{aligned}$$

We then have

$$J_6(\Delta) \le C\varepsilon + C_r \left(\Delta^{p\beta} + \Delta^{\frac{p}{2}} (h(\Delta))^p\right), \quad \text{for all } \Delta \in (0, \Delta_1].$$

This implies

$$\limsup_{\Delta \to 0} J_6(\Delta) \le C\varepsilon.$$

Since ε is arbitrary, we must have that $J_6(\Delta) \to 0$ as $\Delta \to 0$.

Let us finally show $J_4(\Delta) \to 0$ to complete our proof of Theorem 6.3.1. In this proof, we apply the similar technique with Theorem 5.3.1. By Lemmas 6.1.2 and 6.3.4, we can find a positive number $r = r(\varepsilon)$ so large that

$$\mathbb{P}(\Omega_1) \ge 1 - \frac{\varepsilon}{3}, \tag{6.44}$$

where

$$\Omega_1 = \{|x(t)| \vee |x_\Delta(t)| < r \text{ for all } 0 \le t \le T\}.$$

For a sufficiently small $\alpha \in (0, 1)$, define

$$\zeta_{\alpha,i} = \inf\{t \ge 0 : x_i(t) \le \alpha\}, \quad 1 \le i \le d.$$

By Lemma 6.1.3,

$$\begin{aligned}
\mathbb{P}(\zeta_{\alpha,i} \le T) &= \mathbb{E}\Big(\mathbb{1}_{\{\zeta_{\alpha,i} \le T\}} \frac{x_i(\zeta_{\alpha,i}) - 1 - \log(x_i(\zeta_{\alpha,i}))}{\delta - 1 - \log(\alpha)}\Big) \\
&\le \frac{1}{\alpha - 1 - \log(\alpha)} \mathbb{E}\Big(\sup_{0 \le t \le T} [x_i(t) - 1 - \log(x_i(t))]\Big) \\
&\le \frac{C}{\alpha - 1 - \log(\alpha)}.
\end{aligned}$$

Noting that $\alpha - 1 - \log(\alpha) \to \infty$ as $\alpha \to 0$, we can find a $\alpha = \alpha(\varepsilon)$ so small that

$$\mathbb{P}(\zeta_{\alpha,i} \le T) \le \frac{\varepsilon}{3d}, \quad 1 \le i \le d.$$

Set $\zeta_\alpha = \min_{1 \le i \le d} \zeta_{\alpha,i}$. Then

$$\mathbb{P}(\zeta_\alpha \le T) \le \mathbb{P}\big(\bigcup_{i=1}^{d} \{\zeta_{\alpha,i} \le T\}\big) \le \sum_{i=1}^{d} \mathbb{P}(\zeta_{\alpha,i} \le T) \le \frac{\varepsilon}{3}.$$

So $\mathbb{P}(\zeta_\alpha > T) \ge 1 - \frac{\varepsilon}{3}$. This implies

$$\mathbb{P}(\Omega_2) \ge 1 - \frac{\varepsilon}{3}, \tag{6.45}$$

where

$$\Omega_2 = \big\{ \min_{1 \le i \le d} \inf_{0 \le t \le T} x_i(t) > \alpha \big\}.$$

On the other hand, for the pair of chosen r and α, define

$$\Omega_\Delta = \big\{ \sup_{0 \le t \le T} |x(t \wedge \theta_{\Delta,r}) - x_\Delta(t \wedge \theta_{\Delta,r})| < \alpha/2 \big\}.$$

By Lemma 6.3.6 and applying the Chebyshev inequality, there exists a $\Delta_1 = \Delta_1(\varepsilon)$ (as $r = r(\varepsilon)$) such that $\mu^{-1}(h(\Delta_1)) \ge r$ and

$$\begin{aligned}
\mathbb{P}(\Omega_\Delta^c) &= \mathbb{P}\Big(\sup_{0 \le t \le T} |x(t \wedge \theta_{\Delta,r}) - x_\Delta(t \wedge \theta_{\Delta,r})| \ge \alpha/2\Big) \\
&\le \frac{C_r \left(\Delta^{p\beta} + \Delta^{\frac{p}{2}} (h(\Delta))^p\right)}{(\alpha/2)^p}, \quad \text{for all } \Delta \in (0, \Delta_1].
\end{aligned}$$

Consequently, there is a $\Delta_2 = \Delta_2(\varepsilon) \in (0, \Delta_1]$ such that

$$\mathbb{P}(\Omega_\Delta) \geq 1 - \frac{\varepsilon}{3}, \quad \text{for all } \Delta \in (0, \Delta_2]. \tag{6.46}$$

Set $\Omega_{3,\Delta} = \Omega_1 \cap \Omega_2 \cap \Omega_\Delta$. Combining (6.44) - (6.46) gives

$$\mathbb{P}(\Omega_{3,\Delta}) \geq 1 - \varepsilon, \quad \text{for all } \Delta \in (0, \Delta_2]. \tag{6.47}$$

From now on, we consider any step size $\Delta \in (0, \Delta_2]$. Note that for every $\omega \in \Omega_{3,\Delta}$, $\theta_{\Delta,r} > T$,

$$\sup_{0\leq t\leq T} |\bar{X}_\Delta(t)| \leq \sup_{0\leq t\leq T} |x_\Delta(t)| \leq r \leq \mu^{-1}(h(\Delta_1)) \leq \mu^{-1}(h(\Delta)), \tag{6.48}$$

and

$$\begin{aligned}\inf_{0\leq t\leq T} \bar{X}_{\Delta,i}(t) \geq \inf_{0\leq t\leq T} x_{\Delta,i}(t) &\geq \inf_{0\leq t\leq T} x_i(t) - \sup_{0\leq t\leq T} |x_i(t) - x_{\Delta,i}(t)| \\ &> \alpha - \sup_{0\leq t\leq T} |x(t) - x_\Delta(t)| > \alpha - \frac{\alpha}{2} = \frac{\alpha}{2}.\end{aligned} \tag{6.49}$$

In other words, for every $\omega \in \Omega_{3,\Delta}$, $\bar{X}_\Delta(t) \in \mathbb{R}^d_+$ with $|\bar{X}_\Delta(t)| \leq \mu^{-1}(h(\Delta))$, whence $X^0_\Delta(t) = \pi_0(\pi_\Delta(\bar{X}_\Delta(t))) = \bar{X}_\Delta(t)$ for all $t \in [0, T]$. Consequently,

$$\begin{aligned}J_4(\Delta) &= \mathbb{E}\Big(\mathbb{1}_{\Omega^c_{3,\Delta}} \sup_{0\leq t\leq T} |X^0_\Delta(t) - \bar{X}_\Delta(t)|^p\Big) \\ &\leq \big[\mathbb{P}(\Omega^c_{3,\Delta})\big]^{1/2} \Big[\mathbb{E}\Big(\sup_{0\leq t\leq T} |X^0_\Delta(t) - \bar{X}_\Delta(t)|^{2p}\Big)\Big]^{1/2} \\ &\leq 2^p\sqrt{\varepsilon}\Big[\mathbb{E}\Big(\sup_{0\leq t\leq T} |x_\Delta(t)|^{2p}\Big)\Big]^{1/2} \\ &\leq C_p\sqrt{\varepsilon}\end{aligned}$$

provided $\Delta \in (0, \Delta_2]$, where Lemma 6.3.4 has been used once again. Since ε is arbitrary, we must have that $J_4(\Delta) \to 0$ as $\Delta \to 0$. This completes our proof of Theorem 6.3.1. □

6.3.4 Proof of Theorem 6.3.2

Once again, it is obvious that

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-x(t)|^p\Big)\le 3^{p-1}(J_5(\Delta)+J_6(\Delta)+J_7(\Delta)), \tag{6.50}$$

where $J_5(\Delta), J_6(\Delta)$ have been defined before and

$$J_7(\Delta)=\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-\bar{X}_\Delta(t)|^p\Big).$$

Clearly, all we need to do is to show that $J_7(\Delta)\to 0$ as $\Delta\to 0$. Let $\Delta\in(0,\Delta_2\wedge(\alpha/2)]$ be arbitrary. By equations (6.48) and (6.49), we obtained that for every $\omega\in\Omega_{3,\Delta}$, $\bar{X}_\Delta(t)\in\mathbb{R}_+^d$ with $|\bar{X}_\Delta(t)|\le\mu^{-1}(h(\Delta))$ and $\inf_{0\le t\le T}\bar{X}_{\Delta,i}(t)>\alpha/2$, whence $X_\Delta^+(t)=\pi_+(\pi_\Delta(\bar{X}_\Delta(t)))=\bar{X}_\Delta(t)$ for all $t\in[0,T]$. Consequently,

$$\begin{aligned}J_6(\Delta)&=\mathbb{E}\Big(\mathbb{1}_{\Omega_{3,\Delta}^c}\sup_{0\le t\le T}|X_\Delta^+(t)-\bar{X}_\Delta(t)|^p\Big)\\&\le\Big[\mathbb{P}(\Omega_{3,\Delta}^c)\Big]^{1/2}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)-\bar{X}_\Delta(t)|^{2p}\Big)\Big]^{1/2}\\&\le 2^p\sqrt{\varepsilon}\Big[\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)|^{2p}\Big)+\mathbb{E}\Big(\sup_{0\le t\le T}|\bar{X}_\Delta(t)|^{2p}\Big)\Big]^{1/2}.\end{aligned}$$

But, by Lemma 6.3.4,

$$\mathbb{E}\Big(\sup_{0\le t\le T}|\bar{X}_\Delta(t)|^{2p}\Big)\le C_p.$$

On the other hand, for any $x\in\mathbb{R}^d$,

$$\begin{aligned}|\pi_+(x)|^{2p}&=\Big(\sum_{i=1}^d(\Delta\vee x_i)^2\Big)^p\le\Big(\sum_{i=1}^d(\Delta^2+|x_i|^2)\Big)^p\\&\le(d+|x|^2)^p\le d^{p-1}(d^p+|x|^{2p}).\end{aligned}$$

So

$$\begin{aligned}|X_\Delta^+(t)|^{2p}&=|\pi_+(\pi_\Delta(\bar{X}_\Delta(t)))|^{2p}\le 2^pd^{p-1}(d^p+|\pi_\Delta(\bar{X}_\Delta(t))|^{2p})\\&\le d^{p-1}(d^p+|\bar{X}_\Delta(t)|^{2p}).\end{aligned}$$

Consequently

$$\mathbb{E}\Big(\sup_{0\le t\le T}|X_\Delta^+(t)|^{2p}\Big) \le C_p.$$

In other words, we have showed that

$$J_7(\Delta) \le C_p\sqrt{\varepsilon}$$

provided $\Delta \in (0, \Delta_2 \wedge (\alpha/2)]$. As ε is arbitrary, we must have that $J_7(\Delta) \to 0$ as $\Delta \to 0$. This completes our proof of Theorem 6.3.2. $\square$

Chapter 7

Conclusion

In this chapter, we summarize the contributions made in this thesis. This thesis we mainly focus on a numerical method for stochastic differential equations (SDEs), namely the truncated Euler-Maruyama (EM) numerical method, and their convergence rates. This method was introduced in [21]. However, there are many points to improve about this method for applying more situations. For example, in [22], to find the rate of convergence over a finite time interval, they required the global Lipschitz property on the diffusion coefficient however we improved this in Chapter 3. Furthermore, we also modified the truncated EM numerical method with the stochastic delay differential equations (SDDEs) by combining concepts of the variable time delay and provides their convergence rate over a finite time interval, as shown in Chapter 4. However, the truncated EM method could generates the negative solutions which do not have meanings to some SDE models. In Chapter 5 and 6, we modified the new methods, called positive preserving truncated EM (PPTEM) method and nonnegative preserving truncated EM (NPTEM) method, to apply for SDEs and SDDEs that their solution cannot be negative.

The truncated Euler-Maruyama (EM) method, introduced in [21], provides a novel approach to address SDEs with nonlinear coefficients. However, limitations in determining the convergence rate over a finite time interval were identified in previous research, as indicated in [22]. We, in Chapter 3, apply the concepts from [12] to establish convergence over a finite time interval. As a result, our main theorem, namely Theorem 3.2.3, provides the rate of convergence over a finite time interval which is similar to the rate of convergence at a time T in [12]. To achieve a stronger result, we also need a stronger condition on the Khasminskii-type condition that is satisfied for any parameter p as shown in Assumption 3.1.2.

Furthermore, we apply Theorem 3.2.3 to SDEs with nonlinear diffusion coefficient to determine the rate of convergence over a finite time interval, a capability not present in the results of [22].

Considering the widespread application of SDEs to real-world systems, we extend our exploration by incorporating time delays into these equations, following the insights from [6, 8]. This extension forms the basis of our contribution in Chapter 4. We not only apply the truncated EM method to SDDEs but also allow the time delay to vary over time, represented by $\delta(t)$ as defined in equation (4.3). On the way to find the rate of convergence, we need some extra Lemmas 4.1.8 and 4.1.9, illustrating that both non-delay and delay parts are close to the numerical solution. Both lemmas also play an important role in providing convergence rates of the truncated EM method to the solution at both a specific time point T, and over a finite time interval. In this work, we also require the global Lipschitz condition on the diffusion coefficient of the SDDE models to collect the convergence rate over a finite time interval. Nevertheless, we hope to reduce the global Lipschitz condition to the local Lipschitz condition in future research.

The subsequent contribution unfolds in Chapter 5, where we delve into the numerical solutions of a population model, which is the Lotka-Volterra model. Recognizing that the truncated EM method may generate nonsensical negative solutions in certain instances, we introduce modifications, resulting in the positivity preserving truncated EM (PPTEM) and nonnegative preserving truncated EM (NPTEM) methods. From a mathematical point of view, it would be natural to define the NPTEM method before the PPTEM method. To define the NPTEM method, we begin with extending the domain of the population model to the entire $\mathbb{R}^d$, mapping negative values to be 0 (represented by π_0). We can, consequently, apply the idea of the normal truncated EM method with the extended model. As we focus on obtaining a nonnegative numerical solution, we ensure, at each step, to map the numerical solution with π_0 again to confirm the nonnegative preserving property. After iterating this process, we obtain the NPTEM numerical solution. For the PPTEM method, we employ a similar idea as NPTEM, mapping the positive delta (denoted as π_+) into each step of the iteration to guarantee the positive preserving property. Again, this process results in the creation of the PPTEM numerical solution. Additionally, to demonstrate the convergence of both numerical solutions, we assume only one assumption which is Assumption 5.1.1. These adjustments ensure that the numerical solutions remain meaningful and interpretable.

In Chapter 6, we applied the concepts from Chapter 5 to derive PPTEM and NPTEM numerical solutions for the stochastic delay Lotka-Volterra model with a variable time delay. Therefore, the idea to approach the PPTEM and NPTEM methods is slightly similar to Chapter 5. We also have to deal with the term of variable time delay. However, we apply the methodology in Chapter 4 to separately approximate the numerical solutions for non-delay and delay terms, say z_1 and z_2 respectively. We, moreover, assume the strong Assumption 6.1.1, which forces that the matrix coefficients have all positive elements. In practice, Assumption 5.1.1 is sufficient for the stochastic delay Lotka-Volterra model to have a unique solution. Although Assumption 6.1.1 provides favourable properties and allows us to assert theorems and lemmas similar to those in Chapter 5, in future research, we also aim to explore the relaxation of these conditions to broaden the scope of applicability.

In summary, we follow the aims to establish the rate of convergence over a finite time interval of SDEs under the local Lipschitz diffusion coefficient. We further extended our exploration to incorporate time delays, addressing SDDEs and allowing time delays to vary over time. Moreover, we also modified the truncated EM method to be PPTEM and NPTEM for positive and nonnegative numerical solutions to maintain the meaningfulness and interpretability of the solutions. These modifications are also extended to the stochastic variable time delay Lotka-Volterra model.

Bibliography

[1] Arifah Bahar and Xuerong Mao. Stochastic delay population dynamics. *International Journal of Pure and Applied Mathematics*, 11:377–400, 2004.

[2] Christopher TH Baker and Evelyn Buckwar. Numerical analysis of explicit one-step methods for stochastic delay differential equations. *LMS Journal of Computation and Mathematics*, 3:315–335, 2000.

[3] Christopher TH Baker and Evelyn Buckwar. Exponential stability in p-th mean of solutions, and of convergent Euler-type solutions, of stochastic delay differential equations. *Journal of Computational and Applied Mathematics*, 184(2):404–427, 2005.

[4] Tomás Caraballo, Peter E Kloeden, and José Real. Discretization of asymptotically stable stationary solutions of delay differential equations with a random stationary delay. *Journal of dynamics and differential equations*, 18:863–880, 2006.

[5] Coffie Emmanuel and Xuerong Mao. Truncated EM numerical method for generalised Ait-Sahalia-type interest rate model with delay. *Journal of Computational and Applied Mathematics*, 383:113–137, 2021.

[6] Weiyin Fei, Liangjian Hu, Xuerong Mao, and Dengfeng Xia. Advances in the truncated Euler-Maruyama method for stochastic differential delay equations. *Communications on Pure and Applied Analysis*, 19(4):2081–2100, 2020.

[7] Alison Gray, David Greenhalgh, Liangjian Hu, Xuerong Mao, and Jiafeng Pan. A stochastic differential equation SIS epidemic model. *SIAM Journal on Applied Mathematics*, 71(3):876–902, 2011.

[8] Qian Guo, Xuerong Mao, and Rongxian Yue. The truncated Euler–Maruyama method for stochastic differential delay equations. *Numerical Algorithms*, 78:599–624, 2018.

[9] Desmond J Higham and Xuerong Mao. Convergence of Monte Carlo simulations involving the mean–reverting square root process. *Journal of Computational Finance*, 8(3):35–61, 2005.

[10] Desmond J Higham, Xuerong Mao, and Andrew M Stuart. Strong convergence of Euler-type methods for nonlinear stochastic differential equations. *SIAM journal on numerical analysis*, 40(3):1041–1063, 2002.

[11] Desmond J Higham, Xuerong Mao, and Andrew M Stuart. Exponential mean-square stability of numerical solutions to stochastic differential equations. *LMS Journal of Computation and Mathematics*, 6:297–313, 2003.

[12] Liangjian Hu, Xiaoyue Li, and Xuerong Mao. Convergence rate and stability of the truncated Euler–Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 337:274–289, 2018.

[13] Martin Hutzenthaler, Arnulf Jentzen, and Peter E Kloeden. Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 467(2130):1563–1576, 2011.

[14] Martin Hutzenthaler, Arnulf Jentzen, and Peter E Kloeden. Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients. 2012.

[15] Peter E Kloeden and Eckhard Platen. *Numerical solution of stochastic differential equations*. Springer-Verlog, Berlin, 1992.

[16] Uwe Küchler and Eckhard Platen. Strong discrete time approximation of stochastic differential equations with time delay. *Mathematics and Computers in Simulation*, 54(1-3):189–205, 2000.

[17] Alan L Lewis. Option valuation under stochastic volatility ii. *Finance Press*, 2009.

[18] Wei Liu and Xuerong Mao. Strong convergence of the stopped Euler–Maruyama method for nonlinear stochastic differential equations. *Applied Mathematics and Computation*, 223:389–400, 2013.

[19] Xuerong Mao. *Stochastic differential equations and applications*. 2007.

[20] Xuerong Mao. Numerical solutions of stochastic differential delay equations under the generalized Khasminskii-type conditions. *Applied Mathematics and Computation*, 217(12):5512–5524, 2011.

[21] Xuerong Mao. The truncated Euler–Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 290:370–384, 2015.

[22] Xuerong Mao. Convergence rates of the truncated Euler–Maruyama method for stochastic differential equations. *Journal of Computational and Applied Mathematics*, 296:362–375, 2016.

[23] Xuerong Mao, Glenn Marion, and Eric Renshaw. Environmental Brownian noise suppresses explosions in population dynamics. *Stochastic Processes and their Applications*, 97(1):95–110, 2002.

[24] Xuerong Mao and Matina John Rassias. Khasminskii-type theorems for stochastic differential delay equations. *Stochastic analysis and applications*, 23(5):1045–1069, 2005.

[25] Xuerong Mao and Sotirios Sabanis. Numerical solutions of stochastic differential delay equations under local Lipschitz condition. *Journal of computational and applied mathematics*, 151(1):215–227, 2003.

[26] Xuerong Mao and Lukasz Szpruch. Strong convergence and stability of implicit numerical methods for stochastic differential equations with non-globally Lipschitz continuous coefficients. *Journal of Computational and Applied Mathematics*, 238:14–28, 2013.

[27] Xuerong Mao, Fengying Wei, and Teerapot Wiriyakraikul. Positivity preserving truncated Euler–Maruyama method for stochastic Lotka–Volterra competition model. *Journal of Computational and Applied Mathematics*, 394:113566, 2021.

[28] Xuerong Mao and Chenggui Yuan. *Stochastic differential equations with Markovian switching*. Imperial college press, 2006.

[29] Grigori N Milstein and Michael V Tretyakov. *Stochastic numerics for mathematical physics*, volume 39. Springer, 2004.

[30] Bernt Oksendal. *Stochastic differential equations: an introduction with applications*. Springer Science & Business Media, 2013.

[31] Sotirios Sabanis. A note on tamed Euler approximations. 2013.

[32] Sotirios Sabanis. Euler approximations with varying coefficients: the case of superlinearly growing diffusion coefficients. 2016.

[33] Lukasz Szpruch, Xuerong Mao, Desmond J Higham, and Jiazhu Pan. Numerical simulation of a strongly nonlinear Ait-Sahalia-type interest rate model. *BIT Numerical Mathematics*, 51:405–425, 2011.

[34] Michael V Tretyakov and Zhongqiang Zhang. A fundamental mean-square convergence theorem for SDEs with locally Lipschitz coefficients and its applications. *SIAM Journal on Numerical Analysis*, 51(6):3135–3162, 2013.

[35] Xiaojie Wang and Siqing Gan. The tamed Milstein method for commutative stochastic differential equations with non-globally Lipschitz continuous coefficients. *Journal of Difference Equations and Applications*, 19(3):466–490, 2013.