Convex Hulls of Planar Random Walks

20th April 2017

Chang Xu

Department of Mathematics and Statistics

University of Strathclyde

Glasgow, UK

This thesis is submitted to the University of Strathclyde for the degree of Doctor of Philosophy in the Faculty of Science.

Acknowledgements

My deepest gratitude goes to my supervisor Doctor Andrew Wade and Professor Xuerong Mao. This thesis would not exist without their constant support.

I also need to thank Prof. Mikhail Menshikov at Durham University, my current and former colleagues Andreas Wachtel, Dr. Hongrui Wang, Dr. Wei Liu, Dr. Jiafeng Pan et al. for plenty of valuable advice and help during my PhD study.

Last, I would like to express my thanks to my family and the University of Strathclyde for the financial support.

Abstract

For the perimeter length L_n and the area A_n of the convex hull of the first n steps of a planar random walk, this thesis study $n \to \infty$ mean and variance asymptotics and establish distributional limits. The results apply to random walks both with drift (the mean of random walk increments) and with no drift under mild moments assumptions on the increments.

Assuming increments of the random walk have finite second moment and non-zero mean, Snyder and Steele showed that $n^{-1}L_n$ converges almost surely to a deterministic limit, and proved an upper bound on the variance $\mathbb{V}\text{ar}[L_n] = O(n)$. We show that $n^{-1}\mathbb{V}\text{ar}[L_n]$ converges and give a simple expression for the limit, which is non-zero for walks outside a certain degenerate class. This answers a question of Snyder and Steele. Furthermore, we prove a central limit theorem for L_n in the non-degenerate case.

Then we focus on the perimeter length with no drift and area with both drift and zero-drift cases. These results complement and contrast with previous work and establish non-Gaussian distributional limits. We deduce these results from weak convergence statements for the convex hulls of random walks to scaling limits defined in terms of convex hulls of certain Brownian motions. We give bounds that confirm that the limiting variances in our results are non-zero.

Contents

Notations

			Page
S_n, Z_i	:	random walk with location S_n and increments Z_i	1
$\mathrm{hull}(S_0, \ldots, S_n)$	:	the convex hull of random walk S_n	6
$\mathcal{S}_n$	:	the random walk $\{S_0, S_1, \ldots, S_n\}$	26
L_n	:	the perimeter length of $\mathrm{hull}(S_0, \ldots, S_n)$	6
A_n	:	the area of $\mathrm{hull}(S_0, \ldots, S_n)$	6
$\\|\cdot\\|$	:	the Euclidean norm	6
μ	:	the mean drift vector	6
Σ	:	the covariance matrix associated with Z_i	6
$\Sigma^{1/2}$	:	the matrix square-root of Σ	31
σ^2	:	$= \operatorname{tr} \Sigma$	6
$\hat{\mu}$	:	$= \\|\mu\\|^{-1}\mu$ for $\mu \neq 0$	7
σ_μ^2	:	$= \mathbb{E}\left[((Z_1 - \mu) \cdot \hat{\mu})^2\right]$	7
$\sigma_{\mu_\perp}^2$	:	$= \sigma^2 - \sigma_\mu^2$	6
$\mathcal{C}([0,T]; \mathbb{R}^d)$	:	the class of continuous functions from $[0,T]$ to $\mathbb{R}^d$	20
$\mathcal{C}^0([0,T]; \mathbb{R}^d)$	:	$= \{f \in \mathcal{C}([0,T]; \mathbb{R}^d) : f(0) = \mathbf{0}\}$	20
$\rho_\infty(\cdot\,,\cdot)$	:	the supremum metric	20
$\rho(\mathbf{x}, A)$	:	$= \inf_{\mathbf{y} \in A} \rho(\mathbf{x}, \mathbf{y})$ for $A \subseteq \mathbb{R}^d$ and a point $\mathbf{x} \in \mathbb{R}^d$	20
$\mathcal{C}_d$	:	$= \mathcal{C}([0,1]; \mathbb{R}^d)$	20
$\mathcal{C}_d^0$	:	$= \{f \in \mathcal{C}_d : f(0) = \mathbf{0}\}$	20
$\mathbb{S}_{d-1}$	:	$= \{\mathbf{u} \in \mathbb{R}^d : \\|\mathbf{u}\\| = 1\}$, the unit sphere in $\mathbb{R}^d$	20
$\mathcal{K}_d$	:	the collection of convex compact sets in $\mathbb{R}^d$	21
$\mathcal{K}_d^0$	:	$= \{A \in \mathcal{K}_d : \mathbf{0} \in A\}$	21
$\rho_H(\cdot\,,\cdot)$	:	the Hausdorff metric	21
$\pi_r(\cdot)$	:	the parallel body at distance r	21
b	:	$= (b(s))_{s \in [0,1]}$, standard Brownian motion in $\mathbb{R}^d$	22
$\mathcal{A}(\cdot)$	:	the area of convex compact sets in the plane	22
$\mathcal{L}(\cdot)$	:	the perimeter length of convex compact sets	22
$h_A(\cdot)$	:	the support function of $A \in \mathcal{K}_d^0$	21
$H(f)$	:	$= \mathrm{hull}\,(f[0,1])$ for $f \in \mathcal{C}_d$	27
h_t	:	the convex hull of the Brownian path up to time t	31

ℓ_t	:	$= \mathcal{L}(h_t)$	31
a_t	:	$= \mathcal{A}(h_t)$	31
$\Rightarrow$	:	weak convergence	22
w	:	$= (w(s))_{s\in[0,1]}$, standard Brownian motion in $\mathbb{R}$	32
$\tilde{b}(s)$	:	$= (s, w(s))$, for $s \in [0,1]$	32
$\tilde{h}_t$	:	$= \operatorname{hull} \tilde{b}[0,t] \in \mathcal{K}_2^0$	32
$\tilde{a}_t$	:	$= \mathcal{A}(\tilde{h}_t)$	32
$\mathbf{1}\{\cdot\}$	:	the indicator function	37
x^+	:	$= x\,\mathbf{1}\{x > 0\}$	38
x^+	:	$= -x\,\mathbf{1}\{x < 0\}$	38
$u_0(\Sigma)$	:	$= \mathbb{V}\mathrm{ar}\mathcal{L}(\Sigma^{1/2}h_1)$	66
v_+, v_0	:	defined in equation (6.1)	69

Chapter 1

Introduction

1.1 Background on Random Walk

Let $Z_1, Z_2, \ldots$ be independent identically distributed (i.i.d.) random variables taking values in $\mathbb{R}^d$ and let $S_n = \sum_{i=1}^{n} Z_i$. S_n is a *random walk* [30, p. 88].

Random walk theory is a classical and well-studied topic in probability theory. In 1905, Albert Einstein studied the Brownian motion in his paper *"On the Movement of Small Particles Suspended in a Stationary Liquid Demanded by the Molecular-Kinetic Theory of Heat"*. Brownian motion is the random motion of particles in a fluid which is found by the botanist Robert Brown in 1827 [32, Sec. 2.1]. He noted that the pollen grains in water kept moved through randomly. Einstein explained in details how the motion that Brown had observed was a result of the pollen being moved by individual water molecules.

Scientists then gave the mathematical formalisation for the Brownian motion and its generalisation: random walk. The term *random walk* was first used by Karl Pearson in 1905. In a letter to Nature, he gave a simple model to describe a mosquito infestation in a forest. At each time step, a single mosquito moves a fixed length in a randomly chosen direction. Pearson wanted to know the distribution of the mosquitoes after many steps had been taken. The letter was answered by Lord Rayleigh, who had already solved a more general form of this problem in 1880, in the context of sound waves in heterogeneous materials. Modelling a sound wave travelling through the material can be thought of as summing up a sequence of random wave-vectors of constant amplitude but random phase since sound waves in the material have roughly constant wavelength, but their directions are altered at scattering sites within the material.

There are some classical results we need to bear in mind when we study random walks. First we need to introduce the concepts of recurrence and transience. A random walk S_n taking values in $\mathbb{R}^d$ is called *point-recurrent* if

$$\mathbb{P}(S_n = 0 \text{ infinitely often}) = 1$$

and *point-transient* if

$$\mathbb{P}(S_n = 0 \text{ infinitely often}) = 0.$$

If the random walk is not discrete then these definitions are not very useful. Instead we say that the random walk is *neighbourhood-recurrent* if for some $\varepsilon > 0$,

$$\mathbb{P}(|S_n| < \varepsilon \text{ infinitely often}) = 1$$

and *neighbourhood-transient* if

$$\mathbb{P}(|S_n| < \varepsilon \text{ infinitely often}) = 0.$$

In the discrete case, for a simple random walk we have the Pólya's theorem [48]. A random walk $S_n = \sum_{i=1}^n Z_i$ on $\mathbb{Z}^d$ is *simple* if for any $i \in \mathbb{N}$,

$$\mathbb{P}(Z_i = e) = \begin{cases} (2d)^{-1} & \text{if } e \in \mathbb{Z}^d \text{ and } \|e\| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 1.1 (Pólya)**.** *A simple random walk $S_n = \sum_{i=1}^n Z_i$ in $\mathbb{Z}^d$ is recurrent for $d = 1$ or $d = 2$ and transient for $d \geq 3$.*

This theorem was generalised by Chung and Fuchs [15] in 1951.

Theorem 1.2 (Chung–Fuchs)**.** *Let S_n be a random walk in $\mathbb{R}^d$. Then,*

(i) If $d = 1$ and $n^{-1}S_n \to 0$ in probability, then S_n is neighbourhood-recurrent.

(ii) If $d = 2$ and $n^{-1/2}S_n$ converges in distribution to a centred normal distribution, then S_n is neighbourhood-recurrent.

(iii) If $d \geq 3$ and the random walk is not contained in a lower-dimensional subspace, then it is neighbourhood-transient.

1.2 Background on geometric probability

A central theme of classical geometric probability or stochastic geometry concerns the study of the properties of random point sets in Euclidean space and associated structures. For example, a large literature is devoted to study of the lengths of graphs on random vertex sets in Euclidean space $\mathbb{R}^d$, $d \geq 2$. The interests are primarily in the lengths of those graphs representing the solutions to problems in Euclidean combinatorial optimization (see [60] or [67]). In the classical setting, the random point sets are generated by i.i.d. random variables. Some typical problems involve the construction of the shortest possible network of some kind:

Let $X_0, X_1, \ldots, X_n$ be i.i.d. random points with common distribution on $\mathbb{R}^d$ and $V = \{X_i\}_{i=0}^n$.

(i) Travelling salesman problem. Find the length of shortest closed path traversing each vertex in V exactly once.

(ii) Minimal spanning tree. Find the minimal total edge length of a spanning tree through V.

(iii) Minimal Euclidean matching. Find the minimal total edge length of a Euclidean matching of points in V.

Many of the questions of geometric probability or stochastic geometry are equally valid for point sets generated by random walk trajectories.

1.3 Random convex hulls

We first define the convex hull here. A set C in $\mathbb{R}^d$ is *convex* if it has the following property [29, p. 42]:

$$(1-\lambda)x + \lambda y \in C \text{ for any } x, y \in C, 0 \leq \lambda \leq 1.$$

Given a set A in $\mathbb{R}^d$, its *convex hull* is the intersection of all convex sets in $\mathbb{R}^d$ which contain A. Since the intersection of convex sets is always convex, the convex hull of A is convex and it is the smallest convex set in $\mathbb{R}^d$ with respect to set inclusion, which contains A.

One of the motivations to study the convex hulls is to find the extreme values in the random points. For the 1-dimensional case, the extreme values are just the maximum and minimum values. For higher dimensional cases, the extreme values could be determined by the convex hulls.

However, the extreme values have different meanings in these two different main settings of classical stochastic geometry. For the setting of i.i.d. random points, one important concern is the outlier detection in random sample. For the setting of trajectories of stochastic processes, extremes are important for study of record values. It gives two related but different streams of research, with different underlying probabilistic models and different motivating questions, though generally the motivations are all comes from multidimensional theory of extremes. See for example [4], [5], [6] and [45].

1.3.1 i.i.d. random points

Convex hulls of iid. random points, also known as *random polytopes*, were first studied by Geffroy [24] (1961), Rényi and Sulanke [50] (1963), and Efron [18] (1965). In the case where the points are normally distributed, the resulting convex hulls are known as *Gaussian polytopes*. See Reitzner [49, Random polytopes, pp. 45-76] (2010) and Hug [31] (2013) for recent surveys.

Motivation arises in statistics (multivariate extremes) and convex geometry (approximation of convex sets), and there are connection to the isotropic constant in functional analysis: see Reitzner [49]. He also listed some other applications including to the analysis of algorithms and optimization.

For the multivariate extremes, let $X_0, X_1, \ldots, X_n$ be the iid. random points with common distribution on $\mathbb{R}^d$ and $V = \{X_i\}_{i=0}^n$. In the case of $d = 1$, iid. points extremes are used in outlier detection in statistics. In the case of $d \geq 2$, Green [28] describes the peeling algorithm for detection of multivariate outliers via the iterated removal of points on the boundary of the convex hulls.

For the approximation of convex sets, Reitzner [49] insulates the algorithms to efficiently compute convex hull for large point set in $\mathbb{R}^d$.

1.3.2 Trajectories of stochastic process

Before the study of random polytopes, Lévy [40] had considered the convex hull of planar Brownian motion. The study of convex hull of random walk goes back to Spitzer and Widom [58]. Generally, the convex hull of a stochastic process is an interesting geometrical object, related to extremes of the stochastic processes, giving a multivariate analogue of *record values*.

In one dimension, a value of a process is a record value if it is either less than all previous values (a lower record) or greater than all previous values (an upper record). In higher dimensions, a natural definition of "record" is then a point that lies outside the convex hull of all previous values.

More recent work on convex hull of Brownian motion includes Burdzy [11] (1985), Cranston, Hsu and March [13] (1989), Eldan [20] (2014), Evans [21] (1985), Pitman and Ross [47] (2012).

For general stochastic processes, convex hulls and related convex *minorants* or *majorants*, are studied by Bass [8] (1982) and Sinai [55] (1998).

1.4 Applications for convex hulls of random walks

In recent studies of random walks, attention has focussed on various geometrical aspects of random walk trajectories. Many of the questions of stochastic geometry, traditionally concerned with functionals of independent random points, are also of interest for point sets generated by random walks.

Study of the convex hull of planar random walk goes back to Spitzer and Widom [58] and the continuum analogue, convex hull of planar Brownian motion, to Lévy [40, §52.6, pp. 254–256]; both have received renewed interest recently, in part motivated by applications arising for example in modelling the 'home range' of animals. Random walks have been extensively used to model the movement of animals; Karl Pearson's original motivation for the random walk problem originated with modelling the migration of animal species such as mosquitoes, and subsequently random walks have been used to model the locomotion of microbes: see [16, 56] for surveys. If the trajectory of the random walker represents the locations visited by a roaming animal, then the convex hull is a natural estimate of the 'home range' of the animal [65,66]. Natural properties of interest are the perimeter length and area of the convex hull. See [42] for a recent survey of motivation and previous work. The method of Chapter 3 in part relies on an analysis of *scaling limits*, and thus links the discrete and continuum settings.

1.5 Introduction of the model

On each unsteady step, a drunken gardener deposits one of n seeds. Once the flowers have bloomed, what is the minimum length of fencing required to enclose the garden?

Let $Z_1, Z_2, \ldots$ be a sequence of independent, identically distributed (i.i.d.) random vectors on $\mathbb{R}^2$. Write $\mathbf{0}$ for the origin in $\mathbb{R}^2$. Define the random walk $(S_n; n \in \mathbb{Z}_+)$ by $S_0 := \mathbf{0}$ and for $n \geq 1$, $S_n := \sum_{i=1}^n Z_i$. Let $\operatorname{hull}(S_0, \ldots, S_n)$ be the convex hull of positions of the walk up to and including the nth step, which is the smallest convex set that contains $S_0, S_1 \ldots, S_n$. Let L_n denote the length of the perimeter of $\operatorname{hull}(S_0, \ldots, S_n)$ and A_n be the area of the convex hull. (See Figure 1.1.)

We will impose a moments condition of the following form:

(M$_p$) Suppose that $\mathbb{E}\,[\|Z_1\|^p] < \infty$.

For almost everything that follows, we will assume that at least the $p = 1$ case of (M$_p$) holds, and frequently we will assume the $p = 2$ case. For several of our results we assume that (M$_p$) holds for some $p > 2$. In any case, we will be explicit about which case we assume at any particular point.

Given that (M$_p$) holds for some $p \geq 1$, then $\mu := \mathbb{E}\, Z_1 \in \mathbb{R}^2$, the mean drift vector of the walk, is well defined. If (M$_p$) holds for some $p \geq 2$, then $\Sigma := \mathbb{E}\,[(Z_1 - \mu)(Z_1 - \mu)^\top]$, the covariance matrix associated with Z, is well defined; Σ is positive semidefinite and symmetric. We write $\sigma^2 := \operatorname{tr} \Sigma = \mathbb{E}\,[\|Z_1 - \mu\|^2]$. Here and elsewhere Z_1 and μ are viewed as column vectors, and $\|\cdot\|$ is the Euclidean norm. We also introduce the decomposition $\sigma^2 = \sigma^2_\mu + \sigma^2_{\mu_\perp}$ with

$$\sigma^2_\mu := \mathbb{E}\,\left[((Z_1 - \mu) \cdot \hat{\mu})^2\right] = \mathbb{E}\,[(Z_1 \cdot \hat{\mu})^2] - \|\mu\|^2 \in \mathbb{R}_+.$$

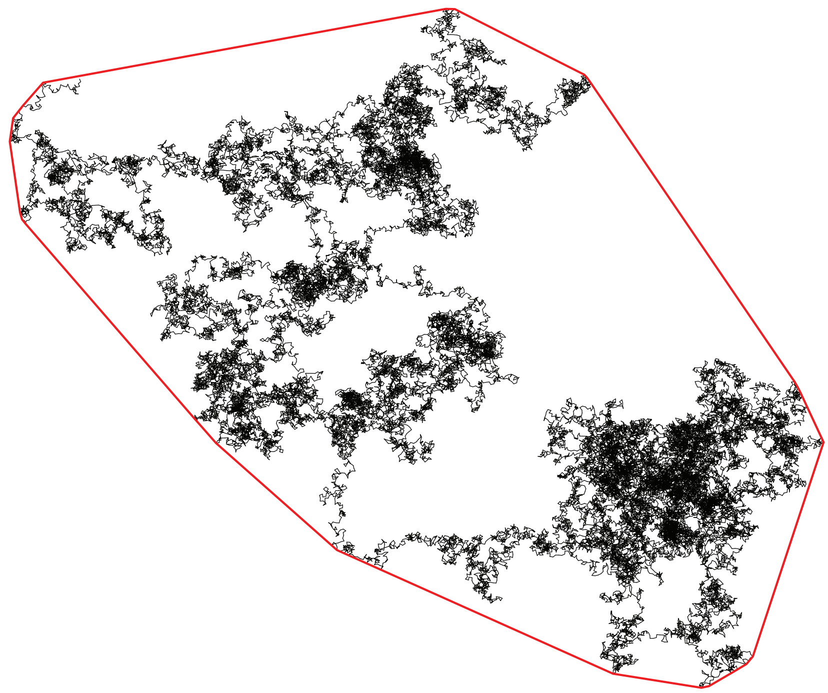

Figure 1.1: Simulated path of a zero-drift random walk and its convex hull.

Here and elsewhere, '$\cdot$' denotes the scalar product, $\hat{\mu} := \|\mu\|^{-1}\mu$ for $\mu \neq 0$, and $\mathbb{R}_+ := [0, \infty)$.

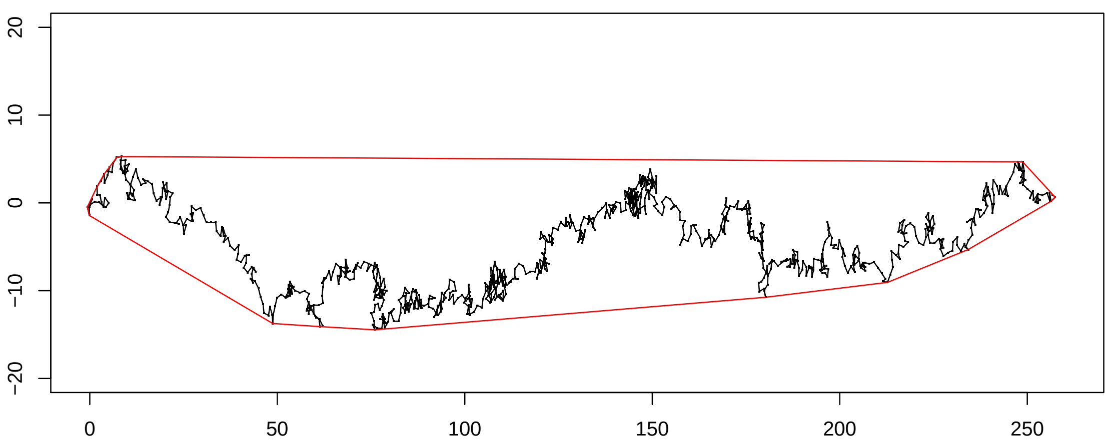

Figure 1.2: Example with mean drift $\mathbb{E}[Z_1]$ of magnitude $\|\mu\| = 1/4$ and $n = 10^3$ steps.

Convex hulls of random points have received much attention over the last several decades: see [42] for an extensive survey, including more than 150 bibliographic references, and sources of motivation more serious than our drunken gardener, such as modelling the 'home-range' of animal populations. An important tool in the study of random convex hulls is provided by a result of Cauchy in classical convex geometry. Spitzer and Widom [58], using Cauchy's formula, and later Baxter [9], using a combinatorial argument, showed that

$$\mathbb{E}[L_n] = 2\sum_{i=1}^{n} \frac{1}{i}\mathbb{E}\|S_i\|. \tag{1.1}$$

Note that $\mathbb{E}[L_n]$ thus scales like n in the case where the one-step mean drift vector $\mathbb{E}[Z_1] \neq \mathbf{0}$ but like $n^{1/2}$ in the case where $\mathbb{E}[Z_1] = \mathbf{0}$ (provided $\mathbb{E}[\|Z_1\|^2] < \infty$). The Spitzer–Widdom–Baxter result, in common with much of the literature, is concerned with first-order properties of L_n: see [42] for a summary of results in this direction for various random convex hulls, with a specific focus on (driftless) planar Brownian motion.

Much less is known about higher-order properties of L_n. There is a clear distinction between the zero drift case ($\mathbb{E}[Z_1] = \mathbf{0}$) and the non-zero drift case ($\|\mathbb{E}[Z_1]\| > 0$). For example, denote $r_n := \inf_{\mathbf{x}\in\partial\mathrm{hull}(S_0,\ldots,S_n)} \|\mathbf{x}\|$. Note that r_n is non decreasing in n, because $S_0 = \mathbf{0} \in \mathrm{hull}(S_0, \ldots, S_n) \subseteq \mathrm{hull}(S_0, \ldots, S_{n+1})$. We investigated the asymptotic behaviour of r_n in the following two different cases.

Proposition 1.3. *(i) Suppose* $\mathbb{E}[\|Z_1\|^2] < \infty$ *and* $\mathbb{E}[Z_1] = \mathbf{0}$. *Then* $\lim_{n\to\infty} r_n = \infty$ *a.s.*

(ii) Suppose $\mathbb{E}\|Z_1\| < \infty$ *and* $\mathbb{E}[Z_1] \neq \mathbf{0}$. *Then* $\lim_{n\to\infty} r_n < \infty$ *a.s.*

Proof. (i) In the first case, the random walk $(S_n; n \in \mathbb{Z}_+)$ is recurrent (see e.g. [17]). There exists $h \in \mathbb{R}_+$, depending on the distributioon of Z_1, such that S_n will visit any ball of radius at least h infinitely often (e.g., in the case of simple symmetric random walk on $\mathbb{Z}^2$, it suffices to take $h = 1$). Let $r > 0$. Then, S_n will visit $B((r + h)\mathbf{y}; h)$ infinitely often for each $\mathbf{y} \in \{(1, 1), (-1, 1), (1, -1), (-1, -1)\}$. Here the notation $B(\mathbf{x}; r)$ is a Euclidean ball (a disk) with centre $\mathbf{x} \in \mathbb{R}^2$ and radius $r \in \mathbb{R}_+$.

So there exists some random time N with $N < \infty$ a.s. such that $\{S_0, \ldots, S_N\}$ contains a point in each of these four balls, and so $\mathrm{hull}(S_0, \ldots, S_N)$ contains the square with these points as its corners, which in turn contains $B(\mathbf{0}; r)$. So $\liminf_{n\to\infty} r_n \geq r$ for any $r \in \mathbb{R}_+$. So $\lim_{n\to\infty} r_n = \infty$.

(ii) In the second case, the random walk is transient (see [17]). Let W_i be a wedge with apex S_i with a angle $\theta < \pi$ (say $\theta = \pi/4$) so that θ is bisected by $\mathbb{E}\, Z_1$. By the Strong Law of Large Numbers, $\|S_n/n - \mathbb{E}\, Z_1\| \to 0$ a.s. and so $S_n/n \cdot \mathbb{E}\, Z_1^{\perp} \to 0$ a.s., where $\mathbb{E}\, Z_1^{\perp}$ is the normal vector of $\mathbb{E}\, Z_1$. This implies the number of points outside the wedge W_i is finite for any $i \in \mathbb{Z}^+$. We take some S_k inside the wedge W_0 and denote the set of finitely many points outside W_k by $\{S_{\sigma_j} : j = 1, 2, \ldots, m\}$. Note that S_0 is outside W_k so the set $\{S_{\sigma_j}\}$ is non-empty. Hence, there must be some $S_{\sigma_t} \in \{S_{\sigma_j}\}$ standing on the boundary of the convex hull, $S_{\sigma_t} \in \partial\text{hull}(S_0, \ldots, S_n)$ for all $n \geq \sigma_t$. Then, $\limsup_{n\to\infty} r_n \leq \|S_{\sigma_t}\| < \infty$, which implies $\lim_{n\to\infty} r_n < \infty$ a.s. since r_n is non decreasing.

□

Remark 1.1. The key property for (i) is not (compact set) recurrence, but *angular recurrence* in the sense that S_n visits any cone with apex at $\mathbf{0}$ and non-zero angle infinitely often. Thus the same distinction between (i) and (ii) persists for random walks in $\mathbb{R}^d$, $d \geq 3$, with the notation extended in the natural way.

Because of this distinction, we always separate the arguments of L_n and A_n into the cases of non-zero and zero drift.

To illustrate our model, here we give some pictures of simulation examples (see Figure 1.3).

1.6 Outline of the thesis

Chapter 2 is some necessary mathematical prerequisites for our results. It includes the concepts of the study objects and the essential tools used in the rest chapters.

In Chapter 3 we describe our scaling limit approach, and carry it through after presenting the necessary preliminaries; the main new results of this chapter, Theorems 3.6 and 3.8, give weak convergence statements for convex hulls of random walks in the case of zero and non-zero drift, respectively. Armed with these weak convergence results, we present asymptotics for expectations and variances of the quantities L_n and A_n in Section 5.4, 6.4 and 6.5; the arguments in this section rely in part on the scaling limit apparatus, and in part on direct random walk

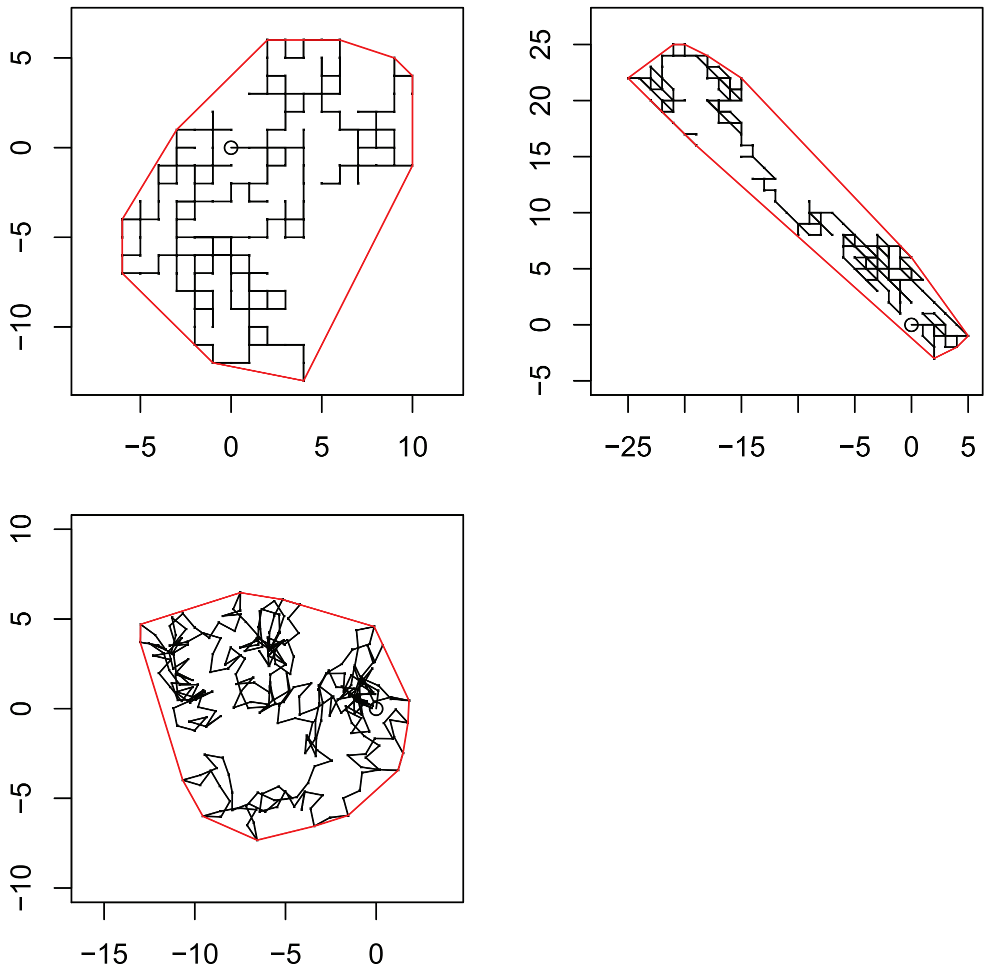

Figure 1.3: The number of steps $n = 300$ for all three examples. The top left: Simple random walk on $\mathbb{Z}^2$. Z_i takes $(\pm 1, 0)$, $(0, \pm 1)$ each with probability 1/4. The top right: Z_i takes $(\pm 1, 0)$, $(0, \pm 1)$, $(-1, 1)$, $(1, -1)$ each with probability 1/6. The bottom left: Pearson–Rayleigh random walk. Z_i takes value uniformly on the unit circle.

computations. This section concludes with upper and lower bounds we found for the limiting variances.

Snyder and Steele [57] showed that $n^{-1} L_n$ converges almost surely to a deterministic limit, and proved an upper bound on the variance $\mathbb{V}\mathrm{ar}[L_n] = O(n)$ [57]. In Chapter 4, we give a different approach to prove their major results, which includes the fact that $n^{-1} \mathbb{E}[L_n]$ converges (Proposition 4.7) and a simple expression for the limit in Proposition 4.5. For the zero drift case, we give a new improved limit expression in Proposition 4.9.

Chapter 5 gives the convergence of $n^{-1}\mathbb{V}\mathrm{ar}[L_n]$ in Proposition 5.4, which is first proved by Snyder and Steele [57]. They also gave the law of large numbers for L_n in the non-zero drift case. But we found it also valid for the zero drift case (Proposition 5.5). Apart from that, the following of major results in this chapter are new. For the non-zero drift case, we give a simple expression for the limit of $n^{-1}\mathbb{V}\mathrm{ar}[L_n]$ in Theorem 5.13 [63, Theorem 1.1], which is non-zero for walks outside a certain degenerate class. This answers a question of Snyder and Steele. It is also the only case where the perimeter length L_n is Gaussian. So we give a central limit theorem for L_n in this case in Theorem 5.14 [63, Theorem 1.2]. For the non-zero drift case, the limit expression of $n^{-1}\mathbb{V}\mathrm{ar}[L_n]$ is given in Proposition 5.15 [64, Proposition 3.5] and its upper and lower bounds are given by Proposition 5.16 [64, Proposition 3.7].

Chapter 6 is an analogue of Chapter 5 for the area A_n. In Theorem 6.8 we give the asymptotic for the expected area $\mathbb{E}\, A_n$ with zero drift, which is a bit more general than the form given by Barndorff–Nielsen and Baxter [3]. Apart from that, the following of major results in this chapter are new. We give the asymptotic for the expected area $\mathbb{E}\, A_n$ with drift in Proposition 6.9 [64, Proposition 3.4] and also the asymptotics for their variance $\mathbb{V}\mathrm{ar} A_n$ in both zero drift (Proposition 6.12 [64, Proposition 3.5]) and non-zero drift cases (Proposition 6.13 [64, Proposition 3.6]). Meanwhile, some upper and lower variance bounds are provided by the last section of this chapter.

Chapter 2

Mathematical prerequisites

2.1 Convergence of random variables

First of all, we define the different modes of convergence we will need in this thesis.

Let X and $X_1, X_2, \dots$ be random variables in $\mathbb{R}$.

X_n converges *almost surely* to X ($X_n \xrightarrow{a.s.} X$) as $n \to \infty$ iff

$$\mathbb{P}\left(\{\omega : X_n(\omega) \to X(\omega) \text{ as } n \to \infty\}\right) = 1.$$

X_n converges *in probability* to X ($X_n \xrightarrow{p} X$) as $n \to \infty$ iff, for every $\varepsilon > 0$,

$$\mathbb{P}\left(|X_n - X| > \varepsilon\right) \to 0 \text{ as } n \to \infty.$$

The L^p *norm* of X is defined by

$$\|X\|_p := \left(\mathbb{E}\,|X|^p\right)^{1/p}.$$

X_n converges *in* L^p to X ($X_n \xrightarrow{L^p} X$) for $p \geq 1$, as $n \to \infty$ iff

$$\mathbb{E}\left(|X_n - X|^p\right) \to 0, \text{ i.e. } \|X_n - X\|_p \to 0, \text{ as } n \to \infty.$$

Let $F_X(x) = \mathbb{P}(X \leq x), x \in \mathbb{R}$, be the distribution function of X and let $C(F_X) = \{x : F_X(x) \text{ is continuous at } x\}$ be the continuity set of F_X. X_n converges *in distribution* to X ($X_n \xrightarrow{d} X$) as $n \to \infty$ iff

$$F_{X_n}(x) \to F_X(x) \text{ as } n \to \infty, \text{ for all } x \in C(F_X).$$

The concept of convergence in distribution extends to random variables in $\mathbb{R}^d$ in terms of the joint distribution functions $\mathbb{P}[X_n^{(1)} \le x^{(1)}, \dots, X_n^{(d)} \le x^{(d)}]$.

These modes of convergence have the following logical relationships.

$$
\begin{array}{ccc}
X_n \xrightarrow{L^p} X \ \Searrow & & \\
 & X_n \xrightarrow{p} X \Longrightarrow X_n \xrightarrow{d} X \\
X_n \xrightarrow{a.s.} X \ \Nearrow & &
\end{array}
$$

Now we collect some basic results on deducing convergence lemmas and theorems.

Lemma 2.1 (Dominated convergence [30] p.57)**.** *Let X, Y and $X_1, X_2, \dots$ be random variables. Suppose that $|X_n| \le Y$ for all n, where $\mathbb{E}\,Y < \infty$, and that $X_n \to X$ a.s. as $n \to \infty$. Then*

$$\mathbb{E}\,|X_n - X| \to 0 \text{ as } n \to \infty,$$

In particular,

$$\mathbb{E}\,X_n \to \mathbb{E}\,X \text{ as } n \to \infty.$$

Lemma 2.2 (Pratt's lemma [30] p.221)**.** *Let X and $X_1, X_2, \dots$ be random variables. Suppose that $X_n \to X$ almost surely as $n \to \infty$, and that*

$$|X_n| \le Y_n \textit{ for all } n, \quad Y_n \to Y \text{ a.s.}, \quad \mathbb{E}\,Y_n \to \mathbb{E}\,Y \text{ as } n \to \infty.$$

Then

$$X_n \to X \textit{ in } L^1 \quad \textit{and} \quad \mathbb{E}\,X_n \to \mathbb{E}\,X \text{ as } n \to \infty.$$

Lemma 2.3 (The Borel–Cantelli lemma [30] p.96, 98)**.** *Let $\{A_n, n \ge 1\}$ be arbitrary events. Then*

$$\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty \Longrightarrow \mathbb{P}(A_n \ i.o.) = 0.$$

Moreover, suppose that $X_1, X_2, \dots$ are random variables. Then,

$$\sum_{n=1}^{\infty} \mathbb{P}(|X_n| > \varepsilon) < \infty \textit{ for any } \varepsilon > 0 \Longrightarrow X_n \to 0 \text{ a.s. as } n \to \infty.$$

Lemma 2.4 (Slutsky's theorem [30] p.249). *Let $X_1, X_2, \ldots$ and $Y_1, Y_2, \ldots$ be sequences of random variables, Suppose that*

$$X_n \xrightarrow{d} X \text{ and } Y_n \xrightarrow{p} a \text{ as } n \to \infty,$$

where a is some constant. Then,

$$X_n + Y_n \xrightarrow{d} X + a \text{ and } X_n \cdot Y_n \xrightarrow{d} X \cdot a.$$

Here we also introduce some useful concepts of uniform integrability.

A collection of random variables X_i, $i \in I$, is said to be *uniformly integrable* if

$$\lim_{M \to \infty} \left(\sup_{i \in I} \mathbb{E}\left(|X_i| \, \mathbf{1}(|X_i| > M)\right) \right) = 0.$$

Lemma 2.5. *Let X and $X_1, X_2, \ldots$ be random variables. If $X_n \to X$ in probability then the following are equivalent:*

(i) $\{X_n\}_{i=1}^{\infty}$ *is uniformly integrable.*

(ii) $X_n \to X$ *in* L^1.

(iii) $\mathbb{E}\,|X_n| \to \mathbb{E}\,|X| < \infty$.

Lemma 2.6 (convergence of means [35] p.45). *Let $X, X_1, X_2, \ldots$ be $\mathbb{R}_+$-valued random variables with $X_n \xrightarrow{d} X$. If $\{X_i\}_{i=1}^{\infty}$ is uniformly integrable, then $\mathbb{E}\,X_n \to \mathbb{E}\,X$ as $n \to \infty$.*

2.2 Martingales

A sequence $\{X_n\}_{i=1}^{\infty}$ of random variables is $\{\mathcal{F}_n\}$-*adapted* if X_n is $\mathcal{F}_n$-measurable for all n, which means for any $k \in \mathbb{R}$, $\{\omega : X_n(\omega) \le k\} \in \mathcal{F}_n$.

An integrable $\{\mathcal{F}_n\}$-adapted sequence X_n is called a *martingale* if

$$\mathbb{E}\left(X_{n+1} \mid \mathcal{F}_n\right) = X_n \text{ a.s. for all } n \ge 0.$$

It is called a *submartingale* if

$$\mathbb{E}\left(X_{n+1} \mid \mathcal{F}_n\right) \ge X_n \text{ a.s. for all } n \ge 0,$$

and a *supermartingale* if

$$\mathbb{E}\left(X_{n+1} \mid \mathcal{F}_n\right) \leq X_n \text{ a.s. for all } n \geq 0.$$

An integrable, $\{\mathcal{F}_n\}$-adapted sequence $\{D_n\}$ is called a *martingale difference sequence* if

$$\mathbb{E}\left(D_{n+1} \mid \mathcal{F}_n\right) = 0 \text{ for all } n \geq 0.$$

Then, the sequence of $M_n := \sum_{k=1}^{n} D_k$ is $\{\mathcal{F}_n\}$-martingale since

$$\mathbb{E}\left[M_{n+1} - M_n \mid \mathcal{F}_n\right] = \mathbb{E}\left[D_{n+1} \mid \mathcal{F}_n\right] = 0,$$

which indicate

$$\mathbb{E}\left[M_{n+1} \mid \mathcal{F}_n\right] = M_n.$$

Lemma 2.7 (Orthogonality of martingale differences [30] p.488)**.** *Let $\{D_n\}_{n=0}^{\infty}$ be a martingale difference sequence. Then $\mathbb{E}\left[D_m D_n\right] = 0$ for $m \neq n$. Hence,*

$$\mathbb{V}\text{ar}\left(\sum_{i=0}^{n} D_i\right) = \sum_{i=0}^{n} \mathbb{V}\text{ar}(D_i).$$

We use a standard martingale difference construction based on resampling. Consider the functional on $\mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}$. Let $Y_1, Y_2, \ldots, Y_n$ be iid. random variables and $W_n = f(Y_1, \ldots, Y_n)$. Let $Y_1', Y_2', \ldots, Y_n'$ be independent copies of $Y_1, Y_2, \ldots, Y_n$ and

$$W_n^{(i)} = f(Y_1, \ldots, Y_{i-1}, Y_i', Y_{i+1}, \ldots, Y_n).$$

Let $D_{n,i} = \mathbb{E}\left[W_n - W_n^{(i)} \mid \mathcal{F}_i\right]$ where $\mathcal{F}_i = \sigma(Y_1, \ldots, Y_i)$.

Lemma 2.8. *Let $n \in \mathbb{N}$. Then*

(i) $W_n - \mathbb{E}\, W_n = \sum_{i=1}^{n} D_{n,i}$;

(ii) $\mathbb{V}\text{ar}(W_n) = \sum_{i=1}^{n} \mathbb{E}\left[D_{n,i}^2\right]$ *whenever the latter sum is finite.*

Proof. The idea is well known. Since $W_n^{(i)}$ is independent of Y_i,

$$\mathbb{E}\left[W_n^{(i)} \mid \mathcal{F}_i\right] = \mathbb{E}\left[W_n^{(i)} \mid \mathcal{F}_{i-1}\right] = \mathbb{E}\left[W_n \mid \mathcal{F}_{i-1}\right].$$

So,

$$D_{n,i} = \mathbb{E}\left[W_n \mid \mathcal{F}_i\right] - \mathbb{E}\left[W_n \mid \mathcal{F}_{i-1}\right].$$

Hence $D_{n,i}$ is martingale differences, since

$$\mathbb{E}\left[D_{n,i} \mid \mathcal{F}_{i-1}\right] = \mathbb{E}\left[W_n \mid \mathcal{F}_{i-1}\right] - \mathbb{E}\left[W_n \mid \mathcal{F}_{i-1}\right] = 0$$

and

$$\sum_{i=1}^{n} D_{n,i} = \mathbb{E}\left[W_n \mid \mathcal{F}_n\right] - \mathbb{E}\left[W_n \mid \mathcal{F}_0\right] = W_n - \mathbb{E}\, W_n.$$

So,

$$\mathbb{E}\left[\left(\sum_{i=1}^{n} D_{n,i}\right)^2\right] = \mathbb{V}\mathrm{ar}(W_n).$$

But by orthogonality of martingale differences, (Lemma 2.7),

$$\mathbb{V}\mathrm{ar}(W_n) = \sum_{i=1}^{n} \mathbb{E}\left[D_{n,i}^2\right].$$

□

Note that by the conditional Jensen's inequality $\left(\mathbb{E}\left(\left[\xi \mid \mathcal{F}\right]\right)\right)^2 \leq \mathbb{E}\left[\xi^2 \mid \mathcal{F}\right]$, we have

$$D_{n,i}^2 \leq \mathbb{E}\left[\left(W_n - W_n^{(i)}\right)^2 \mid \mathcal{F}_i\right].$$

So from part (ii) of Lemma 2.8,

$$\mathbb{V}\mathrm{ar}(W_n) \leq \sum_{i=1}^{n} \mathbb{E}\left[\left(W_n^{(i)} - W_n\right)^2\right].$$

This gives a upper bound for the variance of W_n, which is a factor of 2 larger than the upper bound obtained from the Efron–Stein inequality (equation (2.3) in [57]):

Lemma 2.9.

$$\mathbb{V}\mathrm{ar}(W_n) \leq \frac{1}{2} \sum_{i=1}^{n} \mathbb{E}\left[\left(W_n^{(i)} - W_n\right)^2\right].$$

2.3 Reflection principle for Brownian motion

Lemma 2.10 (Reflection principle [44] p.44)**.** *If T is a stopping time and $\{w(t) : t \geq 0\}$ is a standard 1-dimensional Brownian motion, then the process $\{w^*(t) : t \geq 0\}$ called Brownian motion reflected at T and defined by*

$$w^*(t) = w(t)\, \mathbf{1}\{t \leq T\} + (2w(T) - w(t))\, \mathbf{1}\{t > T\}$$

is also a standard Brownian motion.

Corollary 2.11. *Suppose $r > 0$ and $\{w(t) : t \geq 0\}$ is a standard 1-dimensional Brownian motion. Then,*

$$\mathbb{P}\left(\sup_{0\leq s\leq t} w(s) > r\right) = 2\mathbb{P}\left(w(t) > r\right).$$

2.4 Useful inequalities

We collect some useful inequalities which is useful in the next chapters.

Lemma 2.12 (Markov's inequality [30] p.120)**.** *Let X be a random variable. Suppose that $\mathbb{E}\,|X|^r < \infty$ for some $r > 0$, and let $x > 0$. Then,*

$$\mathbb{P}(|X| > x) \leq \frac{\mathbb{E}\,|X|^r}{x^r}.$$

Lemma 2.13 (Chebyshev's inequality [30] p.121)**.** *Let X be a random variable. Suppose that $\mathbb{V}\mathrm{ar}X < \infty$. Then for $x > 0$,*

$$\mathbb{P}(|X - \mathbb{E}\,X| > x) \leq \frac{\mathbb{V}\mathrm{ar}X}{x^2}.$$

Lemma 2.14 (The Cauchy–Schwarz inequality [30] p.130)**.** *Suppose that random variables X and Y have finite variances. Then,*

$$|\mathbb{E}\,XY| \leq \mathbb{E}\,|XY| \leq \|X\|_2\|Y\|_2 = \sqrt{\mathbb{E}\,(X^2)\mathbb{E}\,(Y^2)}.$$

The next result generalises the Cauchy–Schwarz inequality.

Lemma 2.15 (The Hölder inequality [30] p.129)**.** *Let X and Y be random variables. Suppose that $p^{-1} + q^{-1} = 1$, $\mathbb{E}\,|X|^p < \infty$ and $\mathbb{E}\,|Y|^q < \infty$, then*

$$|\mathbb{E}\,XY| \leq \mathbb{E}\,|XY| \leq \|X\|_p\|Y\|_q = (\mathbb{E}\,X^p)^{1/p}(\mathbb{E}\,Y^q)^{1/q}.$$

Lemma 2.16 (The Minkowski inequality [30] p.129)**.** *Let $p \geq 1$. Suppose that X and Y are random variables, such that $\mathbb{E}\,|X|^p < \infty$ and $\mathbb{E}\,|Y|^p < \infty$. Then,*

$$\|X + Y\|_p \leq \|X\|_p + \|Y\|_p.$$

This is the triangle inequality for the L^p norm.

Now we introduce some inequalities on martingales.

Lemma 2.17 (Doob's inequality [17] p.214)**.** *If X_n is a martingale, then for $1 < p < \infty$,*

$$\mathbb{E}\left[\left(\max_{0\le m\le n}|X_m|\right)^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\,(|X_n|^p).$$

Lemma 2.18 (Azuma–Hoeffding inequality [46] p.33)**.** *Let $D_{n,i}$ $(i = 1,\dots,n)$ be a martingale difference sequence adapted to a filtration $\mathcal{F}_i$, which means $D_{n,i}$ is $\mathcal{F}_i$-measurable and $\mathbb{E}\,[D_{n,i}|\mathcal{F}_{i-1}] = 0$. Then, for any $t > 0$,*

$$\mathbb{P}\left(\Big|\sum_{i=1}^{n} D_{n,i}\Big| > t\right) \le 2\exp\left(-\frac{t^2}{2nd_\infty^2}\right),$$

where d_∞ is such that $|D_{n,i}| \le d_\infty$ a.s. for all n, i.

We also introduce some inequalities for sums of independent random variables.

Lemma 2.19 (Marcinkiewicz–Zygmund inequality [30] p.151)**.** *Let $p \ge 1$. Suppose that $X, X_1, X_2, \dots, X_n$ are independent, identically distributed random variables with mean 0 and $\mathbb{E}\,|X|^p < \infty$. Set $S_n = \sum_{k=1}^n X_k$. Then there exists a constant B_p depending only on p, such that*

$$\mathbb{E}\,|S_n|^p \le \begin{cases} B_p n\mathbb{E}\,|X|, & \text{if } 1 \le p \le 2,\\ B_p n^{p/2}\mathbb{E}\,|X|^{p/2}, & \text{if } p > 2.\end{cases}$$

Lemma 2.20 (Rosenthal's inequality [30] p.151)**.** *Let $p \ge 1$. Suppose that $X_1, X_2, \dots, X_n$ are independent random variables such that $E|X_k|^p < \infty$ for all k. Set $S_n = \sum_{k=1}^n X_k$. Then,*

$$\mathbb{E}\,|S_n|^p \le \max\left\{2^p\sum_{k=1}^{n}\mathbb{E}\,|X_k|^p,\, 2^{p^2}\left(\sum_{k=1}^{n}\mathbb{E}\,|X_k|\right)^p\right\}.$$

2.5 Useful theorems and lemmas

Lemma 2.21 (Fubini's theorem [30] p.65)**.** *Let $(\Omega_1, \mathcal{F}_1, P_1)$ and $(\Omega_2, \mathcal{F}_2, P_2)$ be probability spaces, and consider the product space $(\Omega_1\times\Omega_2, \mathcal{F}_1\times\mathcal{F}_2, P)$, where $P = P_1 \times P_2$ is the product measure. Suppose that $\mathbf{X} = (X_1, X_2)$ is a two-dimensional random variable, and that g is $\mathcal{F}_1 \times \mathcal{F}_2$-measurable, and (i) non-negative or (ii) integrable. Then,*

$$\mathbb{E}\,g(\mathbf{X}) = \int_\Omega g(\mathbf{X})\,\mathrm{d}P = \int_{\Omega_1}\left(\int_{\Omega_2} g(\mathbf{X})\,\mathrm{d}P_2\right)\mathrm{d}P_1 = \int_{\Omega_2}\left(\int_{\Omega_1} g(\mathbf{X})\,\mathrm{d}P_1\right)\mathrm{d}P_2.$$

Lemma 2.22. *Let $\{y_n\}_{n=1}^{\infty}$ be a sequence of real numbers and let $y \in \mathbb{R}$. If $y_n \to y$ as $n \to \infty$, then $n^{-1}\sum_{i=1}^{n} y_i \to y$ as $n \to \infty$.*

Proof. By assumption, for any $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that $|y_n - y| \leq \varepsilon$ for all $n \geq n_0$. Then,

$$\begin{aligned}\left|\frac{1}{n}\sum_{i=1}^{n} y_i - y\right| &= \left|\frac{1}{n}\sum_{i=1}^{n}(y_i - y)\right| \\ &\leq \left|\frac{1}{n}\sum_{i=1}^{n_0}(y_i - y)\right| + \left|\frac{1}{n}\sum_{i=n_0+1}^{n}(y_i - y)\right| \\ &\leq \frac{1}{n}\sum_{i=1}^{n_0}|y_i - y| + \frac{1}{n}\sum_{i=n_0+1}^{n}|y_i - y| \\ &\leq \frac{1}{n}\sum_{i=1}^{n_0}|y_i - y| + \varepsilon \\ &\leq 2\varepsilon,\end{aligned}$$

for all n big enough. Since $\varepsilon > 0$ was arbitrary, the result follows. □

2.6 Multivariate normal distribution

Let Σ be a symmetric positive semi-definite $(d \times d)$ matrix. Then, there exists an unique positive semi-definite symmetric matrix $\Sigma^{1/2}$ such that $\Sigma = \Sigma^{1/2}\Sigma^{1/2}$ [41]. The matrix $\Sigma^{1/2}$ can also be regarded as a linear transform of $\mathbb{R}^d$ given by $\mathbf{x} \mapsto \Sigma^{1/2}\mathbf{x}$.

For a random variable Y, the notation $Y \sim \mathcal{N}(0, \Sigma)$ means Y has d dimensional normal distribution with mean 0 and covariance matrix Σ. In the degenerate case, all entries of the covariance matrix is 0, $\Sigma = 0$, which means that $Y = 0$ almost surely.

Lemma 2.23. *Suppose $X \sim \mathcal{N}(0, I)$ and let $Y = \Sigma^{1/2}X$. Then $Y \sim \mathcal{N}(0, \Sigma)$.*

Lemma 2.24 (Multidimensional Central Limit Theorem [41] p.62)**.** *Suppose $\{Z_i\}_{i=1}^{\infty}$ is a sequence of i.i.d. random variables on $\mathbb{R}^d$. $S_n = \sum_{i=1}^{n} Z_i$ is a random walk on $\mathbb{R}^d$. If $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, $\mathbb{E}\, Z_1 = 0$ and $\mathbb{E}\left(Z_1 Z_1^\top\right) = \Sigma$, then*

$$n^{-1/2} S_n \xrightarrow{d} \mathcal{N}(0, \Sigma).$$

2.7 Analytic and Geometric prerequisites

We recall a few basic facts from real analysis: [53] is an excellent general reference. The *Heine–Borel theorem* states that a set in $\mathbb{R}^d$ is compact if and only if it is closed and bounded [53, p. 40]. Compactness is preserved under continuous mappings: if (X, ρ_X) is a compact metric space and (Y, ρ_Y) is a metric space, and $f : (X, \rho_X) \to (Y, \rho_Y)$ is continuous, then the image $f(X)$ is compact [53, p. 89]; moreover f is uniformly continuous on X [53, p. 91]. For any such uniformly continuous f, there is a monotonic *modulus of continuity* $\mu_f : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\rho_Y(f(x_1), f(x_2)) \leq \mu_f(\rho_X(x_1, x_2))$ for all $x_1, x_2 \in X$, and for which $\mu_f(\rho) \downarrow 0$ as $\rho \downarrow 0$ (see e.g. [35, p. 57]).

Let d be a positive integer. For $T > 0$, let $\mathcal{C}([0, T]; \mathbb{R}^d)$ denote the class of continuous functions from $[0, T]$ to $\mathbb{R}^d$. Endow $\mathcal{C}([0, T]; \mathbb{R}^d)$ with the supremum metric

$$\rho_\infty(f, g) := \sup_{t \in [0,T]} \rho(f(t), g(t)), \text{ for } f, g \in \mathcal{C}([0, T]; \mathbb{R}^d).$$

Let $\mathcal{C}^0([0, T]; \mathbb{R}^d)$ denote those functions in $\mathcal{C}([0, T]; \mathbb{R}^d)$ that map 0 to the origin in $\mathbb{R}^d$.

Usually, we work with $T = 1$, in which case we write simply

$$\mathcal{C}_d := \mathcal{C}([0, 1]; \mathbb{R}^d), \quad \text{and} \quad \mathcal{C}_d^0 := \{f \in \mathcal{C}_d : f(0) = \mathbf{0}\}.$$

For $f \in \mathcal{C}([0, T]; \mathbb{R}^d)$ and $t \in [0, T]$, define $f[0, t] := \{f(s) : s \in [0, t]\}$, the image of $[0, t]$ under f. Note that, since $[0, t]$ is compact and f is continuous, the *interval image* $f[0, t]$ is compact. We view elements $f \in \mathcal{C}([0, T]; \mathbb{R}^d)$ as *paths* indexed by time $[0, T]$, so that $f[0, t]$ is the section of the path up to time t.

We need some notation and concepts from convex geometry: we found [29] to be very useful, supplemented by [58] as a convenient reference for a little integral geometry. Let d be a positive integer. Let $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ denote the Euclidean distance between $\mathbf{x}$ and $\mathbf{y}$ in $\mathbb{R}^d$. For a set $A \subseteq \mathbb{R}^d$, write ∂A for the boundary of A (the intersection of the closure of A with the closure of $\mathbb{R}^d \backslash A$), and $\text{int}(A) := A \backslash \partial A$ for the interior of A. For $A \subseteq \mathbb{R}^d$ and a point $\mathbf{x} \in \mathbb{R}^d$, set $\rho(\mathbf{x}, A) := \inf_{\mathbf{y} \in A} \rho(\mathbf{x}, \mathbf{y})$, with the usual convention that $\inf \emptyset = +\infty$. We write λ_d for Lebesgue measure on $\mathbb{R}^d$. Write $\mathbb{S}_{d-1} := \{\mathbf{u} \in \mathbb{R}^d : \|\mathbf{u}\| = 1\}$ for the unit sphere in $\mathbb{R}^d$.

Let $\mathcal{K}_d$ denote the collection of convex compact sets in $\mathbb{R}^d$, and write

$$\mathcal{K}_d^0 := \{A \in \mathcal{K}_d : \mathbf{0} \in A\}$$

for those sets in $\mathcal{K}_d$ that include the origin. The Hausdorff metric on $\mathcal{K}_d^0$ will be denoted

$$\rho_H(A,B) := \max\left\{\sup_{\mathbf{x}\in B}\rho(\mathbf{x},A), \sup_{\mathbf{y}\in A}\rho(\mathbf{y},B)\right\} \text{ for } A,B \in \mathcal{K}_d.$$

Given $A \in \mathcal{K}_d$, for $r > 0$ set

$$\pi_r(A) := \{\mathbf{x} \in \mathbb{R}^d : \rho(\mathbf{x},A) \leq r\},$$

the *parallel body* of A at distance r. Note that, two equivalent descriptions of ρ_H (see e.g. Proposition 6.3 of [29]) are for $A, B \in \mathcal{K}_d^0$,

$$\rho_H(A,B) = \inf\{r \geq 0 : A \subseteq \pi_r(B) \text{ and } B \subseteq \pi_r(A)\}\,; \text{ and} \tag{2.1}$$

$$\rho_H(A,B) = \sup_{e\in\mathbb{S}_{d-1}} |h_A(e) - h_B(e)|\,, \tag{2.2}$$

where $h_A(\mathbf{x}) := \sup_{\mathbf{y}\in A}(\mathbf{x}\cdot\mathbf{y})$ is the *support function* of A and $\mathbf{x}\cdot\mathbf{y}$ is the inner product of $\mathbf{x}$ and $\mathbf{y}$, i.e. $(x_1,y_1)\cdot(x_2,y_2) = x_1x_2 + y_1y_2$.

2.8 Continuous mapping theorem and Donsker's Theorem

We consider random walks in $\mathbb{R}^d$ in this section. First we need to define the weak convergence in $\mathbb{R}^d$.

Suppose $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space and (M, ρ) is a metric space. For $n \geq 1$, suppose that

$$X_n, X : \Omega \longrightarrow M$$

are random variables taking values in M. If

$$\mathbb{E}\, f(X_n) \to \mathbb{E}\, f(X) \text{ as } n \to \infty,$$

for all bounded, continuous functional $f : M \longrightarrow \mathbb{R}$, then we say that X_n *converges weakly* to X and write $X_n \Rightarrow X$. The weak convergence generalises the concept of convergence in distribution for random variables on $\mathbb{R}^d$.

Lemma 2.25 (continuous mapping theorem [35] p.41). *Fix two metric spaces* (M_1, ρ_1) *and* (M_2, ρ_2). *Let* $X, X_1, X_2, \ldots$ *be random variables taking values in* M_1 *with* $X_n \Rightarrow X$. *Suppose* f *is a mapping on* $(M_1, \rho_1) \to (M_2, \rho_2)$, *which is continuous everywhere in* M_1 *apart from possible on a set* $A \subseteq M_1$ *with* $\mathbb{P}(X \in A) = 0$. *Then,* $f(X_n) \Rightarrow f(X)$.

We generalise the definition of Z_i and S_n a little in this section. Let $\{Z_i\}_{i=1}^{\infty}$ be a i.i.d. random vectors on $\mathbb{R}^d$ and $S_n = \sum_{i=1}^{n} Z_i$. For each $n \in \mathbb{N}$ and all $t \in [0, 1]$, define

$$X_n(t) := S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\left(S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor}\right) = S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}.$$

Let $b := (b(s))_{s \in [0,1]}$ denote standard Brownian motion in $\mathbb{R}^d$, started at $b(0) = 0$.

Lemma 2.26 (Donsker's Theorem). *Let* $d \in \mathbb{N}$. *Suppose that* $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, $\|\mathbb{E}\, Z_1\| = 0$, *and* $\mathbb{E}\left[Z_1 Z_1^{\top}\right] = \Sigma$. *Then, as* $n \to \infty$,

$$n^{-1/2} X_n \Rightarrow \Sigma^{1/2} b,$$

in the sense of weak convergence on $(\mathcal{C}_d^0, \rho_\infty)$.

Remark 2.1. Donsker's theorem generalizes the multidimensional central limit theorem (Lemma 2.24) to a *functional* central limit theorem, because weak convergence of paths implies convergence in distribution of the endpoints. Indeed, taking $t = 1$ in Donsker's Theorem, the marginal convergence gives

$$n^{-1/2} X_n(1) = n^{-1/2} S_n \xrightarrow{d} \Sigma^{1/2} b(1).$$

Here by Lemma 2.23, $\Sigma^{1/2} b(1) \sim \mathcal{N}(0, \Sigma)$ since $b(1) \sim \mathcal{N}(0, I)$. Then we have $n^{-1/2} S_n \xrightarrow{d} \mathcal{N}(0, \Sigma)$, which is Lemma 2.24.

2.9 Cauchy formula

For this section we take $d = 2$. We consider the $\mathcal{A} : \mathcal{K}_2 \to \mathbb{R}_+$ and $\mathcal{L} : \mathcal{K}_2 \to \mathbb{R}_+$ given by the area and the perimeter length of convex compact sets in the plane. Formally, we may define

$$\mathcal{A}(A) := \lambda_2(A), \quad \text{and} \quad \mathcal{L}(A) := \lim_{r \downarrow 0} \left(\frac{\lambda_2(\pi_r(A)) - \lambda_2(A)}{r} \right), \text{ for } A \in \mathcal{K}_2. \qquad (2.3)$$

The limit in (2.3) exists by the *Steiner formula* of integral geometry (see e.g. [54]), which expresses $\lambda_2(\pi_r(A))$ as a quadratic polynomial in r whose coefficients are given in terms of the *intrinsic volumes* of A:

$$\lambda_2(\pi_r(A)) = \lambda_2(A) + r\mathcal{L}(A) + \pi r^2 \,\mathbf{1}\{A \neq \emptyset\}. \tag{2.4}$$

In particular,

$$\mathcal{L}(A) = \begin{cases} \mathcal{H}_1(\partial A) & \text{if } \operatorname{int}(A) \neq \emptyset, \\ 2\mathcal{H}_1(\partial A) & \text{if } \operatorname{int}(A) = \emptyset, \end{cases}$$

where $\mathcal{H}_d$ is d-dimensional Hausdorff measure on Borel sets. We observe the translation-invariance and scaling properties

$$\mathcal{L}(x + \alpha A) = \alpha \mathcal{L}(A), \quad \text{and} \quad \mathcal{A}(x + \alpha A) = \alpha^2 \mathcal{A}(A),$$

where for $A \in \mathcal{K}_2$, $x + \alpha A = \{x + \alpha y : y \in A\} \in \mathcal{K}_2$.

For $A \in \mathcal{K}_2$, Cauchy obtained the following formula:

$$\mathcal{L}(A) = \int_0^\pi \left(\sup_{\mathbf{y} \in A} (\mathbf{y} \cdot \mathbf{e}_\theta) - \inf_{\mathbf{y} \in A} (\mathbf{y} \cdot \mathbf{e}_\theta) \right) \mathrm{d}\theta. \tag{2.5}$$

We will need the following consequence of (2.5).

Proposition 2.27. *Let* $K = \{\mathbf{z}_0, \ldots, \mathbf{z}_n\}$ *be a finite point set in* $\mathbb{R}^2$*, and let* $\mathcal{C} = \operatorname{hull}(K)$*. Then*

$$\mathcal{L}(\mathcal{C}) = \int_0^\pi \left(\max_{0 \le i \le n} (\mathbf{z}_i \cdot \mathbf{e}_\theta) - \min_{0 \le i \le n} (\mathbf{z}_i \cdot \mathbf{e}_\theta) \right) \mathrm{d}\theta. \tag{2.6}$$

In particular, for the case of our random walk, (2.6) says

$$L_n = \mathcal{L}(\operatorname{hull}(S_0, \ldots, S_n)) = \int_0^\pi \left(\max_{0 \le i \le n} (S_i \cdot \mathbf{e}_\theta) - \min_{0 \le i \le n} (S_i \cdot \mathbf{e}_\theta) \right) \mathrm{d}\theta. \tag{2.7}$$

An immediate but useful consequence of (2.7) is that

$$L_{n+1} \ge L_n, \ \text{a.s.} \tag{2.8}$$

In the case where K is a finite point set, $\operatorname{hull}(K)$ is a convex polygon, the boundary of which contains vertices $\mathcal{V} \subseteq K$ (extreme points of the convex hull) and the line-segment edges connecting them; note that $\operatorname{hull}(K) = \operatorname{hull}(\mathcal{V})$.

Now, by convexity,

$$\sup_{\mathbf{y}\in\mathcal{C}}(\mathbf{y}\cdot\mathbf{e}_\theta) = \max_{0\le i\le n}(\mathbf{z}_i\cdot\mathbf{e}_\theta) = \sup_{\mathbf{y}\in\mathcal{V}}(\mathbf{y}\cdot\mathbf{e}_\theta),$$

and similarly for the infimum. So (2.5) does indeed imply (2.6). However, to keep this presentation as self-contained as possible, we give a direct proof of (2.6) without appealing to the more general result (2.5).

Proof of Proposition 2.27. The above discussion shows that it suffices to consider the case where $\mathcal{V} = K$ in which all of the $\mathbf{z}_i$ are on the boundary of the convex hull. Without loss of generality, suppose that $\mathbf{0}\in\mathcal{C}$. Then we may rewrite (2.6) as

$$\mathcal{L}(\mathcal{C}) = \int_0^{2\pi} \max_{0\le i\le n}(\mathbf{z}_i\cdot\mathbf{e}_\theta)\,\mathrm{d}\theta.$$

Suppose also that $\mathbf{z}_i = \|\mathbf{z}_i\|\mathbf{e}_{\theta_i}$ in polar coordinates, labelled so that $0\le\theta_0<\theta_1<\cdots<\theta_n<2\pi$. Thus starting from the rightmost point of $\partial\mathcal{C}$ on the horizontal axis and traversing the boundary anticlockwise, one visits the vertices $\mathbf{z}_0,\mathbf{z}_1,\ldots,\mathbf{z}_n$ in order.

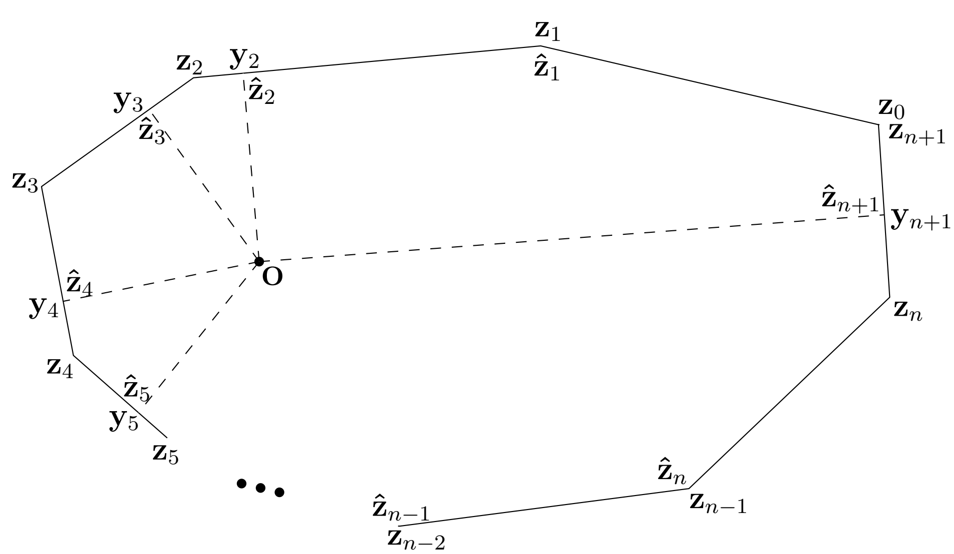

Figure 2.1: Proof of Proposition 2.27

Let $\mathbf{z}_{n+1} := \mathbf{z}_0$. Draw the perpendicular line of $\mathbf{z}_k - \mathbf{z}_{k-1}$ passing through point

$\mathbf{0}$ and denote the foot as $\mathbf{y}_k$. For $1 \le k \le n+1$, let

$$\hat{\mathbf{z}}_k := \begin{cases} \mathbf{y}_k, & \text{if } \mathbf{y}_k \in \text{ line segment } \overline{\mathbf{z}_{k-1}\mathbf{z}_k} \\ \mathbf{z}_k, & \text{if } \mathbf{y}_k \in \text{ extended line of } \overrightarrow{\mathbf{z}_{k-1}\mathbf{z}_k} \\ \mathbf{z}_{k-1}, & \text{if } \mathbf{y}_k \in \text{ extended line of } \overrightarrow{\mathbf{z}_k\mathbf{z}_{k-1}} \end{cases}$$

and let $\hat{\mathbf{z}}_0 := \hat{\mathbf{z}}_{n+1}$. Notice that $\hat{\mathbf{z}}_1, \dots, \hat{\mathbf{z}}_{n+1}$ are ordered in the same way as $\mathbf{z}_0, \dots, \mathbf{z}_n$ (see Figure 2.1). Therefore,

$$\partial\mathcal{C} = \bigcup_{k=0}^{n} \left[(\hat{\mathbf{z}}_{k+1} - \mathbf{z}_k) \cup (\mathbf{z}_k - \hat{\mathbf{z}}_k)\right].$$

Write $\hat{\mathbf{z}}_i = \|\hat{\mathbf{z}}_i\| \mathbf{e}_{\hat{\theta}_i}$ for $0 \le i \le n+1$ in the polar coordinates, we have

$$\int_0^{2\pi} \max_{0\le i\le n} (\mathbf{z}_i \cdot \mathbf{e}_\theta)\, \mathrm{d}\theta = \sum_{k=0}^{n} \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \mathbf{z}_k \cdot \mathbf{e}_\theta \, \mathrm{d}\theta.$$

Consider $\int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \mathbf{z}_k \cdot \mathbf{e}_\theta \, \mathrm{d}\theta$. Let $\mathbf{z}_k := (\alpha_1, \beta_1)$, $\mathbf{z}_{k+1} := (\alpha_2, \beta_2)$ and $\mathbf{z}_{k-1} := (\alpha_0, \beta_0)$. Without loss of generality, we can set $\beta_1 = 0$ and $\alpha_1 > 0$. Then we have $\beta_2 \ge 0$, $\beta_0 \le 0$, $0 \le \hat{\theta}_{k+1} \le \pi/2$ and $-\pi/2 \le \hat{\theta}_k \le 0$. So,

$$\begin{aligned} \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \mathbf{z}_k \cdot \mathbf{e}_\theta \, \mathrm{d}\theta = & \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} (\alpha_1, 0) \cdot (\cos\theta, \sin\theta)\, \mathrm{d}\theta \\ =& \alpha_1 (\sin\hat{\theta}_{k+1} - \sin\hat{\theta}_k) \\ =& \alpha_1 \left(\frac{\|\hat{\mathbf{z}}_{k+1} - \mathbf{z}_k\|}{\alpha_1} - \frac{-\|\mathbf{z}_k - \hat{\mathbf{z}}_k\|}{\alpha_1} \right) \\ =& \|\hat{\mathbf{z}}_{k+1} - \mathbf{z}_k\| + \|\mathbf{z}_k - \hat{\mathbf{z}}_k\|. \end{aligned}$$

Hence,

$$\int_0^{2\pi} \max_{0\le i\le n} (\mathbf{z}_i \cdot \mathbf{e}_\theta)\, \mathrm{d}\theta = \sum_{k=0}^{n} \int_{\hat{\theta}_k}^{\hat{\theta}_{k+1}} \mathbf{z}_k \cdot \mathbf{e}_\theta \, \mathrm{d}\theta = \sum_{k=0}^{n} \left(\|\hat{\mathbf{z}}_{k+1} - \mathbf{z}_k\| + \|\mathbf{z}_k - \hat{\mathbf{z}}_k\|\right) = L(\mathcal{C}). \quad \square$$

Chapter 3

Scaling limits for convex hulls

3.1 Overview

For some of the results that follow, scaling limit ideas are useful. Recall that $S_n = \sum_{k=1}^{n} Z_k$ is the location of our random walk in $\mathbb{R}^2$ after n steps. Write $\mathcal{S}_n := \{S_0, S_1, \dots, S_n\}$. Our strategy to study properties of the random convex set $\operatorname{hull} \mathcal{S}_n$ (such as L_n or A_n) is to seek a weak limit for a suitable scaling of $\operatorname{hull} \mathcal{S}_n$, which we must hope to be the convex hull of some scaling limit representing the walk $\mathcal{S}_n$.

In the case of zero drift ($\mu = 0$) a candidate scaling limit for the walk is readily identified in terms of planar Brownian motion. For the case $\mu \neq 0$, the 'usual' approach of centering and then scaling the walk (to again obtain planar Brownian motion) is not useful in our context, as this transformation does not act on the convex hull in any sensible way. A better idea is to scale space differently in the direction of μ and in the orthogonal direction.

In other words, in either case we consider $\phi_n(\mathcal{S}_n)$ for some *affine* continuous scaling function $\phi_n : \mathbb{R}^2 \to \mathbb{R}^2$. The convex hull is preserved under affine transformations, so

$$\phi_n(\operatorname{hull} \mathcal{S}_n) = \operatorname{hull} \phi_n(\mathcal{S}_n),$$

the convex hull of a random set which will have a weak limit. We will then be able to deduce scaling limits for quantities L_n and A_n provided, first, that we work in suitable spaces on which our functionals of interest enjoy continuity, so that we can appeal to the continuous mapping theorem for weak limits, and, second, that

ϕ_n acts on length and area by simple scaling. The usual $n^{-1/2}$ scaling when $\mu = 0$ is fine; for $\mu \neq 0$ we scale space in one coordinate by n^{-1} and in the other by $n^{-1/2}$, which acts nicely on area, but *not* length. Thus these methods work exactly in the three cases corresponding to (6.8).

In view of the scaling limits that we expect, it is natural to work not with point sets like $\mathcal{S}_n$, but with continuous *paths*; instead of $\mathcal{S}_n$ we consider the interpolating path constructed as follows. For each $n \in \mathbb{N}$ and all $t \in [0, 1]$, define

$$X_n(t) := S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor)\left(S_{\lfloor nt \rfloor + 1} - S_{\lfloor nt \rfloor}\right) = S_{\lfloor nt \rfloor} + (nt - \lfloor nt \rfloor) Z_{\lfloor nt \rfloor + 1}.$$

Note that $X_n(0) = S_0$ and $X_n(1) = S_n$. Given n, we are interested in the convex hull of the image in $\mathbb{R}^2$ of the interval $[0, 1]$ under the continuous function X_n. Our scaling limits will be of the same form.

3.2 Convex hulls of paths

In this section we study some basic properties of the map from a continuous path to its convex hull. Let $f \in \mathcal{C}([0, T], \mathbb{R}^d)$. For any $t \in [0, T]$, $f[0, t]$ is compact, and so Carathéodory's theorem for convex hulls (see Corollary 3.1 of [29, p. 44]) shows that $\operatorname{hull}(f[0, t])$ is compact. So $\operatorname{hull}(f[0, t]) \in \mathcal{K}_d$ is convex, bounded, and closed; in particular, it is a Borel set.

For reasons that we shall see, it mostly suffices to work with paths parametrized over the interval $[0, 1]$. For $f \in \mathcal{C}_d$, define

$$H(f) := \operatorname{hull}\left(f[0, 1]\right).$$

First we prove continuity of the map $f \mapsto H(f)$.

Lemma 3.1. *For any $f, g \in \mathcal{C}_d^0$, we have*

$$\rho_H(H(f), H(g)) \leq \rho_\infty(f, g). \tag{3.1}$$

Hence the function $H : (\mathcal{C}_d^0, \rho_\infty) \to (\mathcal{K}_d^0, \rho_H)$ is continuous.

Proof. Let $f, g \in \mathcal{C}_d^0$. Then $H(f)$ and $H(g)$ are non-empty, as they both contain $f(0) = g(0) = \mathbf{0}$. Consider $\mathbf{x} \in H(f)$. Since the convex hull of a set is the set of all convex combinations of points of the set (see Lemma 3.1 of [29, p. 42]), there exist a finite positive integer n, weights $\lambda_1, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^n \lambda_i = 1$, and $t_1, \ldots, t_n \in [0,1]$ for which $\mathbf{x} = \sum_{i=1}^n \lambda_i f(t_i)$. Then, taking $\mathbf{y} = \sum_{i=1}^n \lambda_i g(t_i)$, we have that $\mathbf{y} \in H(g)$ and, by the triangle inequality,

$$\rho(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^n \lambda_i \rho(f(t_i), g(t_i)) \leq \rho_\infty(f, g).$$

Thus, writing $r = \rho_\infty(f, g)$, every $\mathbf{x} \in H(f)$ has $\mathbf{x} \in \pi_r(H(g))$, so $H(f) \subseteq \pi_r(H(g))$. The symmetric argument gives $H(g) \subseteq \pi_r(H(f))$. Thus, by (2.1), we obtain (3.1). □

Given $f \in \mathcal{C}_d$, let $E(f) := \text{ext}(H(f))$, the extreme points of the convex hull (see [29, p. 75]). The set $E(f)$ is the smallest set (by inclusion) that generates $H(f)$ as its convex hull, i.e., for any A for which $\text{hull}(A) = H(f)$, we have $E(f) \subseteq A$; see Theorem 5.5 of [29, p. 75]. In particular, $E(f) \subseteq f[0,1]$.

Lemma 3.2. *Let $f \in \mathcal{C}_d$. Let $q : \mathbb{R}^d \to \mathbb{R}$ be continuous and convex. Then q attains its supremum over $H(f)$ at a point of f, i.e.,*

$$\sup_{\mathbf{x} \in H(f)} q(\mathbf{x}) = \max_{t \in [0,1]} q(f(t)).$$

Proof. Theorem 5.6 of [29, p. 76] shows that any continuous convex function on $H(f)$ attains its maximum at a point of $E(f)$. Hence, since $E(f) \subseteq f[0,1]$,

$$\sup_{\mathbf{x} \in H(f)} q(\mathbf{x}) = \sup_{\mathbf{x} \in E(f)} q(\mathbf{x}) \leq \sup_{\mathbf{x} \in f[0,1]} q(\mathbf{x}).$$

On the other hand, $f[0,1] \subseteq H(f)$, so $\sup_{\mathbf{x} \in f[0,1]} q(\mathbf{x}) \leq \sup_{\mathbf{x} \in H(f)} q(\mathbf{x})$. Hence

$$\sup_{\mathbf{x} \in H(f)} q(\mathbf{x}) = \sup_{\mathbf{x} \in f[0,1]} q(\mathbf{x}) = \sup_{t \in [0,1]} q(f(t)).$$

Since $q \circ f$ is the composition of two continuous functions, it is itself continuous, and so the supremum is attained in the compact set $[0,1]$. □

For $A \in \mathcal{K}_d^0$, the *support function* of A is $h_A : \mathbb{R}^d \to \mathbb{R}_+$ defined by

$$h_A(\mathbf{x}) := \sup_{\mathbf{y} \in A} (\mathbf{x} \cdot \mathbf{y}).$$

For $A \in \mathcal{K}_2^0$, *Cauchy's formula* (2.5) states

$$\mathcal{L}(A) = \int_{\mathbb{S}_1} h_A(\mathbf{u})\mathrm{d}\mathbf{u} = \int_0^{2\pi} h_A(\mathbf{e}_\theta)\mathrm{d}\theta.$$

We end this section by showing that the map $t \mapsto \mathrm{hull}(f[0,t])$ on $[0,T]$ is continuous if f is continuous on $[0,T]$, so that the continuous trajectory $t \mapsto f(t)$ is accompanied by a continuous 'trajectory' of its convex hulls. This observation was made by El Bachir [19, pp. 16–17]; we take a different route based on the path space result Lemma 3.1. First we need a lemma.

Lemma 3.3. *Let $T > 0$ and $f \in \mathcal{C}([0,T];\mathbb{R}^d)$. Then the map defined for $t \in [0,T]$ by $t \mapsto g_t$, where $g_t : [0,1] \to \mathbb{R}^d$ is given by $g_t(s) = f(ts)$, $s \in [0,1]$, is a continuous function from $([0,T],\rho)$ to $(\mathcal{C}_d,\rho_\infty)$.*

Proof. First we fix $t \in [0,T]$ and show that $s \mapsto g_t(s)$ is continuous, so that $g_t \in \mathcal{C}_d$ as claimed. Since f is continuous on the compact interval $[0,T]$, it is uniformly continuous, and admits a monotone modulus of continuity μ_f. Hence

$$\rho(g_t(s_1), g_t(s_2)) = \rho(f(ts_1), f(ts_2)) \le \mu_f(\rho(ts_1, ts_2)) = \mu_f(t\rho(s_1,s_2)),$$

which tends to 0 as $\rho(s_1,s_2) \to 0$. Hence $g_t \in \mathcal{C}_d$.

It remains to show that $t \mapsto g_t$ is continuous. But on $\mathcal{C}_d$,

$$\begin{aligned} \rho_\infty(g_{t_1}, g_{t_2}) &= \sup_{s\in[0,1]} \rho(f(t_1 s), f(t_2 s)) \\ &\le \sup_{s\in[0,1]} \mu_f(\rho(t_1 s, t_2 s)) \\ &\le \mu_f(\rho(t_1,t_2)), \end{aligned}$$

which tends to 0 as $\rho(t_1,t_2) \to 0$, again using the uniform continuity of f. □

Here is the path continuity result for convex hulls of continuous paths; cf [19, p. 16–17].

Corollary 3.4. *Let $T > 0$ and $f \in \mathcal{C}^0([0,T];\mathbb{R}^d)$ with $f(0) = \mathbf{0}$. Then the map defined for $t \in [0,T]$ by $t \mapsto \mathrm{hull}(f[0,t])$ is a continuous function from $([0,T],\rho)$ to $(\mathcal{K}_d^0,\rho_H)$.*

Proof. By Lemma 3.3, $t \mapsto g_t$ is continuous, where $g_t(s) = f(ts)$, $s \in [0,1]$. Note that, since $f(0) = \mathbf{0}$, $g_t \in \mathcal{C}_d^0$. But the sets $f[0,t]$ and $g_t[0,1]$ coincide, so $\text{hull}(f[0,t]) = H(g_t)$, and, by Lemma 3.1, $g_t \mapsto H(g_t)$ is continuous. Thus $t \mapsto H(g_t)$ is the composition of two continuous functions, hence itself a continuous function:

$$\begin{array}{ccccc} [0,T] & \longrightarrow & \mathcal{C}_d^0 & \longrightarrow & \mathcal{K}_d^0 \\ t & \mapsto & g_t & \mapsto & H(g_t) \end{array}$$ □

Recall definitions of the functionals for perimeter length $\mathcal{L}$ and area $\mathcal{A}$ in (2.3). We give the following inequalities in the metric spaces.

Lemma 3.5. *Suppose that $A, B \in \mathcal{K}_2^0$. Then*

$$\rho(\mathcal{L}(A), \mathcal{L}(B)) \leq 2\pi\rho_H(A,B); \tag{3.2}$$

$$\rho(\mathcal{A}(A), \mathcal{A}(B)) \leq \pi\rho_H(A,B)^2 + (\mathcal{L}(A) \vee \mathcal{L}(B))\rho_H(A,B). \tag{3.3}$$

Hence, the functions $\mathcal{L}$ and $\mathcal{A}$ are both continuous from $(\mathcal{K}_2^0, \rho_H)$ to $(\mathbb{R}_+, \rho)$.

Proof. First consider $\mathcal{L}$. By Cauchy's formula,

$$\begin{aligned} |\mathcal{L}(A) - \mathcal{L}(B)| &= \left| \int_{\mathbb{S}_1} (h_A(\mathbf{u}) - h_B(\mathbf{u}))\, \mathrm{d}\mathbf{u} \right| \\ &\leq \int_{\mathbb{S}_1} \sup_{\mathbf{u} \in \mathbb{S}_1} |h_A(\mathbf{u}) - h_B(\mathbf{u})|\, \mathrm{d}\mathbf{u} = 2\pi\rho_H(A,B), \end{aligned}$$

by the triangle inequality and then (2.2). This gives (3.2).

Now consider $\mathcal{A}$. Set $r = \rho_H(A,B)$. Then, by (2.1), $A \subseteq \pi_r(B)$. Hence

$$\mathcal{A}(A) \leq \mathcal{A}(\pi_r(B)) \leq \mathcal{A}(B) + r\mathcal{L}(B) + \pi r^2,$$

by (2.4). With the analogous argument starting from $B \subseteq \pi_r(A)$, we get (3.3). □

3.3 Brownian convex hulls as scaling limits

Now we return to considering the random walk $S_n = \sum_{k=1}^n Z_k$ in $\mathbb{R}^2$. The two different scalings outlined in Section 3.1, for the cases $\mu = 0$ and $\mu \neq 0$, lead to different scaling limits for the random walk. Both are associated with Brownian motion.

In the case $\mu = 0$, the scaling limit is the usual planar Brownian motion, at least when $\Sigma = I$, the identity matrix. Let $b := (b(s))_{s\in[0,1]}$ denote standard Brownian motion in $\mathbb{R}^2$, started at $b(0) = 0$. For convenience we may assume $b \in \mathcal{C}_2^0$ (we can work on a probability space for which continuity holds for all sample points, rather than merely almost all). For $t \in [0, 1]$, let

$$h_t := \operatorname{hull} b[0,t] \in \mathcal{K}_2^0 \tag{3.4}$$

denote the convex hull of the Brownian path up to time t. By Corollary 3.4, $t \mapsto h_t$ is continuous. Much is known about the properties of h_t: see e.g. [13, 19, 21, 36]. We also set

$$\ell_t := \mathcal{L}(h_t), \quad \text{and} \quad a_t := \mathcal{A}(h_t), \tag{3.5}$$

the perimeter length and area of the standard Brownian convex hull. By Lemma 3.5, the processes $t \mapsto \ell_t$ and $t \mapsto a_t$ also have continuous sample paths.

We also need to work with the case of general covariances Σ; to do so we introduce more notation and recall some facts about multivariate Gaussian random vectors. For definiteness, we view vectors as Cartesian column vectors when required. Since Σ is positive semidefinite and symmetric, there is a (unique) positive semidefinite symmetric matrix square-root $\Sigma^{1/2}$ for which $\Sigma = (\Sigma^{1/2})^2$. The map $x \mapsto \Sigma^{1/2}x$ associated with $\Sigma^{1/2}$ is a linear transformation on $\mathbb{R}^2$ with Jacobian $\det \Sigma^{1/2} = \sqrt{\det \Sigma}$; hence $\mathcal{A}(\Sigma^{1/2}A) = \mathcal{A}(A)\sqrt{\det \Sigma}$ for any measurable $A \subseteq \mathbb{R}^2$.

If $W \sim \mathcal{N}(0, I)$, then by Lemma 2.23, $\Sigma^{1/2}W \sim \mathcal{N}(0, \Sigma)$, a bivariate normal distribution with mean 0 and covariance Σ; the notation permits $\Sigma = 0$, in which case $\mathcal{N}(0, 0)$ stands for the degenerate normal distribution with point mass at 0. Similarly, given b a standard Brownian motion on $\mathbb{R}^2$, the diffusion $\Sigma^{1/2}b$ is *correlated* planar Brownian motion with covariance matrix Σ. Recall that '$\Rightarrow$' (see Section 2.8) indicates weak convergence.

Theorem 3.6. *Suppose that* $\mathbb{E}\,(\|Z_1\|^2) < \infty$ *and* $\mu = 0$. *Then, as* $n \to \infty$,

$$n^{-1/2}\operatorname{hull}\{S_0, S_1, \ldots, S_n\} \Rightarrow \Sigma^{1/2}h_1,$$

in the sense of weak convergence on $(\mathcal{K}_2^0, \rho_H)$.

Proof. Donsker's theorem (see Lemma 2.26) implies that $n^{-1/2}X_n \Rightarrow \Sigma^{1/2}b$ on $(\mathcal{C}_2^0, \rho_\infty)$. Now, the point set $X_n[0, 1]$ is the union of the line segments $\{S_k +$

$\theta(S_{k+1} - S_k) : \theta \in [0,1]\}$ over $k = 0, 1, \ldots, n-1$. Since the convex hull is preserved under affine transformations,

$$H(n^{-1/2}X_n) = n^{-1/2}H(X_n) = n^{-1/2}\,\mathrm{hull}\{S_0, S_1, \ldots, S_n\}.$$

By Lemma 3.1, H is continuous, and so the continuous mapping theorem (see Lemma 2.25) implies that

$$n^{-1/2}\,\mathrm{hull}\{S_0, S_1, \ldots, S_n\} \Rightarrow H(\Sigma^{1/2}b) \text{ on } (\mathcal{K}_2^0, \rho_H).$$

Finally, invariance of the convex hull under affine transformations shows $H(\Sigma^{1/2}b) = \Sigma^{1/2}H(b) = \Sigma^{1/2}h_1$. □

Theorem 3.6 together with the continuous mapping theorem and Lemma 3.5 implies the following distributional limit results in the case $\mu = 0$. Recall that '$\xrightarrow{d}$' (see Section 2.1) denotes convergence in distribution for $\mathbb{R}$-valued random variables.

Corollary 3.7. *Suppose that $\mathbb{E}(\|Z_1\|^2) < \infty$ and $\mu = 0$. Then, as $n \to \infty$,*

$$n^{-1/2}L_n \xrightarrow{d} \mathcal{L}(\Sigma^{1/2}h_1), \quad \textit{and} \quad n^{-1}A_n \xrightarrow{d} \mathcal{A}(\Sigma^{1/2}h_1) = a_1\sqrt{\det \Sigma}.$$

Remark 3.1. Recall that $a_1 = \mathcal{A}(h_1)$ is the area of the standard 2-dimensional Brownian convex hull run for unit time. The distributional limits for $n^{-1/2}L_n$ and $n^{-1}A_n$ in Corollary 3.7 are supported on $\mathbb{R}_+$ and, as we will show in Proposition 5.16 and Proposition 6.14 below, are non-degenerate if Σ is positive definite; hence they are *non-Gaussian* excluding trivial cases.

In the case $\mu \neq 0$, the scaling limit can be viewed as a space-time trajectory of one-dimensional Brownian motion. Let $w := (w(s))_{s\in[0,1]}$ denote standard Brownian motion in $\mathbb{R}$, started at $w(0) = 0$; similarly to above, we may take $w \in \mathcal{C}_1^0$. Define $\tilde{b} \in \mathcal{C}_2^0$ in Cartesian coordinates via

$$\tilde{b}(s) = (s, w(s)), \text{ for } s \in [0,1];$$

thus $\tilde{b}[0,1]$ is the space-time diagram of one-dimensional Brownian motion run for unit time. For $t \in [0,1]$, let $\tilde{h}_t := \mathrm{hull}\,\tilde{b}[0,t] \in \mathcal{K}_2^0$, and define $\tilde{a}_t := \mathcal{A}(\tilde{h}_t)$. (Closely related to $\tilde{h}_t$ is the greatest *convex minorant* of w over $[0,t]$, which is of interest in its own right, see e.g. [47] and references therein.)

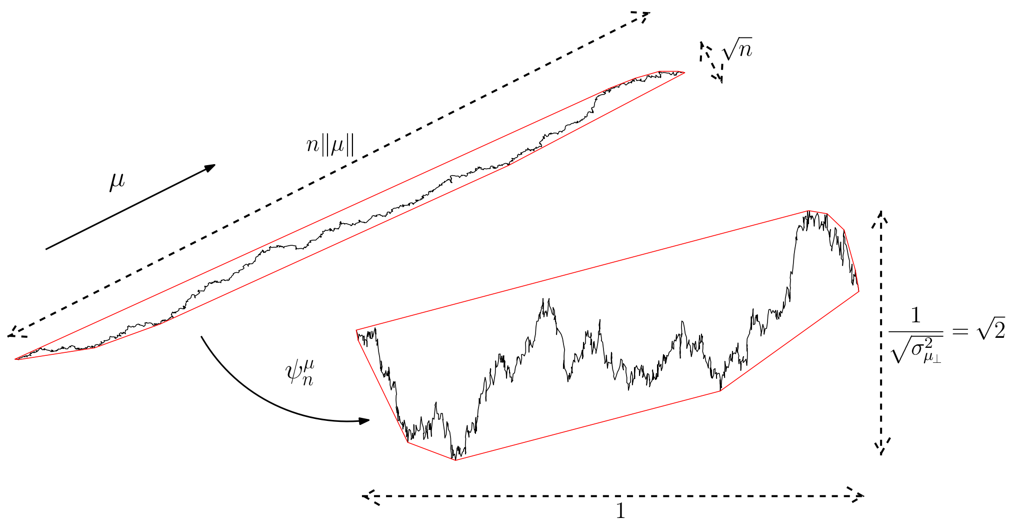

Figure 3.1: Simulated path of $n = 1000$ steps a random walk with drift $\mu = (\frac{1}{2}, \frac{1}{4})$ and its convex hull (*top left*) and (not to the same scale) the image under ψ_n^μ (*bottom right*).

Suppose $\mu \neq 0$ and $\sigma^2_{\mu_\perp} \in (0, \infty)$. Given $\mu \in \mathbb{R}^2 \setminus \{0\}$, let $\hat{\mu}_\perp$ be the unit vector perpendicular to μ obtained by rotating $\hat{\mu}$ by $\pi/2$ anticlockwise. For $n \in \mathbb{N}$, define $\psi_n^\mu : \mathbb{R}^2 \to \mathbb{R}^2$ by the image of $x \in \mathbb{R}^2$ in Cartesian components:

$$\psi_n^\mu(x) = \left(\frac{x \cdot \hat{\mu}}{n\|\mu\|}, \frac{x \cdot \hat{\mu}_\perp}{\sqrt{n\sigma^2_{\mu_\perp}}} \right).$$

In words, ψ_n^μ rotates $\mathbb{R}^2$, mapping $\hat{\mu}$ to the unit vector in the horizontal direction, and then scales space with a horizontal shrinking factor $\|\mu\|n$ and a vertical factor $\sqrt{n\sigma^2_{\mu_\perp}}$; see Figure 3.1 for an illustration.

Theorem 3.8. *Suppose that* $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, $\mu \neq 0$, *and* $\sigma^2_{\mu_\perp} > 0$. *Then, as* $n \to \infty$,

$$\psi_n^\mu(\text{hull}\{S_0, S_1, \ldots, S_n\}) \Rightarrow \tilde{h}_1,$$

in the sense of weak convergence on $(\mathcal{K}_2^0, \rho_H)$.

Proof. Observe that $\hat{\mu} \cdot S_n$ is a random walk on $\mathbb{R}$ with one-step mean drift $\hat{\mu} \cdot \mu = \|\mu\| \in (0, \infty)$, while $\hat{\mu}_\perp \cdot S_n$ is a walk with mean drift $\hat{\mu}_\perp \cdot \mu = 0$ and increment variance

$$\mathbb{E}\left[(\hat{\mu}_\perp \cdot Z)^2\right] = \mathbb{E}\left[(\hat{\mu}_\perp \cdot (Z - \mu))^2\right]$$

$$
\begin{aligned}
&= \mathbb{E}\left[\|Z-\mu\|^2\right] - \mathbb{E}\left[(\hat{\mu}\cdot(Z-\mu))^2\right] = \sigma^2 - \sigma_\mu^2 \\
&= \sigma_{\mu_\perp}^2 .
\end{aligned}
$$

According to the strong law of large numbers, for any $\varepsilon > 0$ there exists $N_\varepsilon \in \mathbb{N}$ a.s. such that $|m^{-1}\hat{\mu}\cdot S_m - \|\mu\|| < \varepsilon$ for $m \geq N_\varepsilon$. Now we have that

$$
\begin{aligned}
\sup_{N_\varepsilon/n \leq t \leq 1} \left| \frac{\hat{\mu}\cdot S_{\lfloor nt \rfloor}}{n} - t\|\mu\| \right| &\leq \sup_{N_\varepsilon/n \leq t \leq 1} \left(\frac{\lfloor nt \rfloor}{n}\right) \left| \frac{\hat{\mu}\cdot S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \|\mu\| \right| \\
&\quad + \|\mu\| \sup_{0\leq t\leq 1} \left| \frac{\lfloor nt \rfloor}{n} - t \right| \\
&\leq \sup_{N_\varepsilon/n \leq t \leq 1} \left| \frac{\hat{\mu}\cdot S_{\lfloor nt \rfloor}}{\lfloor nt \rfloor} - \|\mu\| \right| + \frac{\|\mu\|}{n} \\
&\leq \varepsilon + \frac{\|\mu\|}{n}.
\end{aligned}
$$

On the other hand,

$$
\sup_{0\leq t \leq N_\varepsilon/n} \left| \frac{\hat{\mu}\cdot S_{\lfloor nt \rfloor}}{n} - t\|\mu\| \right| \leq \frac{1}{n}\max\{\hat{\mu}\cdot S_0, \ldots, \hat{\mu}\cdot S_{N_\varepsilon}\} + \frac{N_\varepsilon\|\mu\|}{n} \to 0, \text{ a.s.},
$$

since $N_\varepsilon < \infty$ a.s. Combining these last two displays and using the fact that $\varepsilon > 0$ was arbitrary, we see that

$$
\sup_{0\leq t\leq 1} \left| n^{-1}\hat{\mu}\cdot S_{\lfloor nt \rfloor} - t\|\mu\| \right| \to 0, \text{ a.s. (the functional version of the strong law).}
$$

Similarly,

$$
\sup_{0\leq t\leq 1} \left| n^{-1}\hat{\mu}\cdot S_{\lfloor nt \rfloor+1} - t\|\mu\| \right| \to 0, \text{ a.s. as well.}
$$

Since $X_n(t)$ interpolates $S_{\lfloor nt \rfloor}$ and $S_{\lfloor nt \rfloor+1}$, it follows that

$$
\sup_{0\leq t\leq 1} \left| n^{-1}\hat{\mu}\cdot X_n(t) - t\|\mu\| \right| \to 0, \text{ a.s.}
$$

In other words, $(n\|\mu\|)^{-1}X_n\cdot\hat{\mu}$ converges a.s. to the identity function $t \mapsto t$ on $[0, 1]$.

For the other component, Donsker's theorem (Lemma 2.26) gives $(n\sigma_{\mu_\perp}^2)^{-1/2}X_n\cdot \hat{\mu}_\perp \Rightarrow w$ on $(\mathcal{C}_1^0, \rho_\infty)$. It follows that, as $n\to\infty$, $\psi_n^\mu(X_n) \Rightarrow \tilde{b}$, on $(\mathcal{C}_2^0, \rho_\infty)$. Hence by Lemma 3.1 and since ψ_n^μ acts as an affine transformation on $\mathbb{R}^2$,

$$
\psi_n^\mu(H(X_n)) = H(\psi_n^\mu(X_n)) \Rightarrow H(\tilde{b}),
$$

on $(\mathcal{K}_2^0, \rho_H)$, and the result follows. $\square$

Theorem 3.8 with the continuous mapping theorem (Lemma 2.25), Lemma 3.5, and the fact that $\mathcal{A}(\psi_n^\mu(A)) = n^{-3/2}\|\mu\|^{-1}(\sigma^2_{\mu_\perp})^{-1/2}\mathcal{A}(A)$ for measurable $A \subseteq \mathbb{R}^2$, implies the following distributional limit for A_n in the case $\mu \neq 0$.

Corollary 3.9. *Suppose that* $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$*,* $\mu \neq 0$*, and* $\sigma^2_{\mu_\perp} > 0$*. Then*

$$n^{-3/2}A_n \xrightarrow{d} \|\mu\|(\sigma^2_{\mu_\perp})^{1/2}\tilde{a}_1, \ \ \textit{as } n \to \infty.$$

Remarks 3.2. (i) Only the $\sigma^2_{\mu_\perp} > 0$ case is non-trivial, since $\sigma^2_{\mu_\perp} = 0$ if and only if Z is parallel to $\pm\mu$ a.s., in which case all the points $S_0, \ldots, S_n$ are collinear and $A_n = 0$ a.s. for all n.
(ii) The limit in Corollary 3.9 is non-negative and non-degenerate (see Proposition 6.14 below) and hence non-Gaussian.

The framework of this chapter shows that whenever a discrete-time process in $\mathbb{R}^d$ converges weakly to a limit on the space of continuous paths, the corresponding convex hulls converge. It would be of interest to extend the framework to admit discontinuous limit processes, such as Lévy processes with jumps [36] that arise as scaling limits of random walks whose increments have infinite variance.

Chapter 4

Spitzer–Widom formula for the expected perimeter length and its consequences

4.1 Overview

Our contribution in this Chapter is giving a new proof of the Spitzer–Widom formula in Section 4.2 and giving the asymptotics for the expected perimeter length in Section 4.3 by using that formula. Firstly, we show how to deduce the Spitzer–Widom formula from the Cauchy formula.

The following theorem is Theorem 2 in [58].

Theorem 4.1 (Spitzer–Widom formula)**.** *Suppose that* $\mathbb{E}\,\|Z_1\| < \infty$*. Then*

$$\mathbb{E}\,L_n = 2\sum_{k=1}^{n} \frac{1}{k}\mathbb{E}\,\|S_k\|.$$

The basis for our derivation of the Spitzer–Widdom formula is an analogous result for *one-dimensional* random walk, stated in Lemma 4.3 below, which is itself a consequence of the combinatorial result given in Lemma 4.2. Lemma 4.2 was stated by Kac [34, pp. 502–503 and Theorem 4.2 on p. 508] and attributed to Hunt; the proof given is due to Dyson. Lemma 4.3 is variously attributed to Chung, Hunt, Dyson and Kac; it is also related to results of Sparre Andersen [1] and is a special case of what has become known as the Spitzer or Spitzer–Baxter identity [35, Ch. 9] for random walks, which is a more sophisticated result usually deduced from Wiener–Hopf Theory.

4.2 Derivation of Spitzer–Widom formula

Let $X_1, X_2, \ldots$ be i.i.d. random variables. Let $T_n = \sum_{i=1}^n X_i$ and $M_n = \max\{0, T_1, \ldots, T_n\}$. Let $\sigma : (1, 2, \ldots, n) \mapsto (\sigma_1, \sigma_2, \ldots, \sigma_n) \in \mathbb{Z}_+^n$ be a permutation on $\{1, \ldots, n\}$. Then $(\pi_n; \circ)$ is a group consisting of σ under the composition operation. For $\sigma \in \pi_n$, let $T_n^\sigma = \sum_{i=1}^n X_{\sigma_i}$ and $M_n^\sigma = \max\{0, T_1^\sigma, \ldots, T_n^\sigma\}$.

Lemma 4.2.

$$\sum_{\sigma \in \pi_n} M_n^\sigma = \sum_{\sigma \in \pi_n} X_{\sigma_1} \sum_{k=1}^n \mathbf{1}\{T_k^\sigma > 0\}.$$

Proof. Note that if $T_k^\sigma \leq 0$, then $M_k^\sigma - M_{k-1}^\sigma = 0$. If $T_k^\sigma > 0$, then

$$M_k^\sigma = \max(T_1^\sigma, T_2^\sigma, \ldots, T_k^\sigma) = X_{\sigma_1} + \max(0, X_{\sigma_2}, X_{\sigma_2} + X_{\sigma_3}, \ldots, \sum_{l=2}^k X_{\sigma_l}).$$

Combining these two cases, we get

$$\begin{aligned} M_k^\sigma - M_{k-1}^\sigma = \mathbf{1}\{T_k^\sigma > 0\} \Bigg[& X_{\sigma_1} + \max\Big(0, X_{\sigma_2}, X_{\sigma_2} + X_{\sigma_3}, \ldots, \sum_{l=2}^k X_{\sigma_l}\Big) \\ & - \max\Big(0, X_{\sigma_1}, X_{\sigma_1} + X_{\sigma_2}, \ldots, \sum_{j=1}^{k-1} X_{\sigma_j}\Big)\Bigg]. \end{aligned}$$

Fix $k \in \{1, \ldots, n\}$. Let $G(\omega_{k+1}, \ldots, \omega_n)$ be the subset of π_n consisting of permutations whose last $(n-k)$ indices are $\omega_{k+1}, \ldots, \omega_n$, where $1 \leq \omega_i \leq n$. Then π_n is decomposed into $\frac{n!}{k!}$ disjoint subsets $G(\omega_{k+1}, \ldots, \omega_n)$ of size $k!$.

Denote

$$f(\sigma_1, \ldots, \sigma_{k-1}, \sigma_k) := \max\Big(0, X_{\sigma_1}, X_{\sigma_1} + X_{\sigma_2}, \ldots, \sum_{j=1}^{k-1} X_{\sigma_j}\Big).$$

Then,

$$M_k^\sigma - M_{k-1}^\sigma = \mathbf{1}\{T_k^\sigma > 0\} \left[X_{\sigma_1} + f(\sigma_2, \ldots, \sigma_k, \sigma_1) - f(\sigma_1, \ldots, \sigma_{k-1}, \sigma_k)\right].$$

Summing both sides of the equation over $\{\sigma \in \pi_n\}$, since

$$\sum_{\sigma\in\pi_n} = \sum_{1\le\sigma_{k+1},\ldots,\sigma_n\le n}\ \sum_{\sigma\in G(\sigma_{k+1},\ldots,\sigma_n)},$$

and

$$\sum_{\sigma\in G(\sigma_{k+1},\ldots,\sigma_n)} f(\sigma_2,\ldots,\sigma_k,\sigma_1) = \sum_{\sigma\in G(\sigma_{k+1},\ldots,\sigma_n)} f(\sigma_1,\ldots,\sigma_{k-1},\sigma_k),$$

we get

$$\sum_{\sigma\in\pi_n}\left(M_k^\sigma - M_{k-1}^\sigma\right) = \sum_{\sigma\in\pi_n} X_{\sigma_1}\,\mathbf{1}\{T_k^\sigma > 0\}. \tag{4.1}$$

The result is implied by summing both sides of the equation (4.1) from $k=1$ to n. Note that $M_0^\sigma = \max(0) = 0$. $\square$

Here we use the notation $x^+ := x\,\mathbf{1}\{x>0\}$ and $x^- := -x\,\mathbf{1}\{x<0\}$ for $x\in\mathbb{R}$. So $x = x^+ - x^-$ and $|x| = x^+ + x^-$.

The following result on the expected maximum of 1-dimensional random walk is variously attributed to Chung, Hunt, Dyson and Kac. A combinatorial proof similar to the one given here can be found on page 301-302 of [14].

Lemma 4.3. *Suppose that $\mathbb{E}\,|X_k| < \infty$. Then,*

$$\mathbb{E}\,M_n = \sum_{k=1}^n \frac{\mathbb{E}\,(T_k^+)}{k}.$$

Proof. By Lemma 4.2, we have

$$\begin{aligned}\mathbb{E}\,M_n = \mathbb{E}\,M_n^\sigma &= \frac{1}{n!}\sum_{\sigma\in\pi_n}\mathbb{E}\,M_n^\sigma\\ &= \frac{1}{n!}\sum_{\sigma\in\pi_n}\mathbb{E}\left[X_{\sigma_1}\sum_{k=1}^n \mathbf{1}\{T_k^\sigma>0\}\right]\\ &= \mathbb{E}\left[X_1\sum_{k=1}^n\mathbf{1}\{T_k>0\}\right],\end{aligned}$$

since the X_i are i.i.d., $\mathbb{E}\,(X_1\,\mathbf{1}\{T_k>0\}) = \mathbb{E}\,(X_i\,\mathbf{1}\{T_k>0\})$ for any $1\le i\le k$. Also, $\mathbb{E}\,(X_1\,\mathbf{1}\{T_k>0\}) = k^{-1}\mathbb{E}\,(T_k\,\mathbf{1}\{T_k>0\})$. Then,

$$\mathbb{E}\left[X_1\sum_{k=1}^n\mathbf{1}\{T_k>0\}\right] = \sum_{k=1}^n\mathbb{E}\left[X_1\,\mathbf{1}\{T_k>0\}\right]$$

$$= \sum_{k=1}^{n} \mathbb{E}\left[\frac{T_k}{k}\mathbf{1}\{T_k > 0\}\right]$$
$$= \sum_{k=1}^{n} \frac{\mathbb{E}\,(T_k^+)}{k}. \qquad \square$$

Remark 4.1. *Fluctuation theory* for one-dimensional random walks concerns a series of important identities involving the distributions of M_n, T_n, and other quantities associated with the random walk path. A cornerstone of the theory is the celebrated double generating-function identity of Spitzer which states that

$$\sum_{n=0}^{\infty} t^n \mathbb{E}\,[\mathrm{e}^{iuM_n}] = \exp\left\{\sum_{k=1}^{\infty} \frac{t^k}{k}\mathbb{E}\,[\mathrm{e}^{iuT_k^+}]\right\}$$

for $|t| < 1$. Lemma 3.3 is a corollary to Spitzer's identity, obtained on differentiating with respect to u and setting $u = 0$. The proof of Spitzer's identity may be approached from an analytic perspective, using the Wiener–Hopf factorization (see e.g. Resnick [51, Ch. 7]), or from a combinatorial one (see e.g. Karlin and Taylor [37, Ch. 17]). These references discuss many other aspects of fluctuation theory, as do Chung [14, §§8.4 & 8.5], Feller [23], Asmussen [2, Ch. VIII], and Takács [62]. In particular, Chung [14, pp. 301–302] gives a direct proof of Lemma 4.3 closely related to the one presented here; essentially the same proof is in [2, p. 232].

Proof of the Spitzer–Widom formula.

Denote $M_n(\theta) := \max_{0\leq i\leq n}(S_i \cdot \mathbf{e}_\theta)$ and $m_n(\theta) := \min_{0\leq i\leq n}(S_i \cdot \mathbf{e}_\theta)$. Note that $M_n(\theta) \geq 0$ and $m_n(\theta) \leq 0$ since $\mathbf{0} \in \mathcal{H}_n$.

Applying Fubini's theorem (see Lemma 2.21) in Cauchy formula (2.7), we get

$$\mathbb{E}\,L_n = \int_0^\pi (\mathbb{E}\,M_n(\theta) - \mathbb{E}\,m_n(\theta))\,\mathrm{d}\theta.$$

Observe that $S_n \cdot \mathbf{e}_\theta$ is a one-dimensional random walk on $\mathbb{R}$. Take $T_k = S_k \cdot \mathbf{e}_\theta$ in Lemma 4.3. Then,

$$\mathbb{E}\,M_n(\theta) = \sum_{k=1}^{n} \frac{\mathbb{E}\,[(S_k \cdot \mathbf{e}_\theta)^+]}{k} \quad \text{and} \quad \mathbb{E}\,m_n(\theta) = -\sum_{k=1}^{n} \frac{\mathbb{E}\,[(-S_k \cdot \mathbf{e}_\theta)^+]}{k},$$

since $m_n(\theta) = -\max_{0\leq i\leq n}(-S_i \cdot \mathbf{e}_\theta)$. So, since $x^- = (-x)^+$,

$$\mathbb{E}\,L_n = \int_0^\pi \sum_{k=1}^{n} \frac{1}{k}\mathbb{E}\,\left[(S_k \cdot \mathbf{e}_\theta)^+ + (S_k \cdot \mathbf{e}_\theta)^-\right]\mathrm{d}\theta$$

$$= \int_0^\pi \sum_{k=1}^n \frac{\mathbb{E}\,|S_k \cdot \mathbf{e}_\theta|}{k} \mathrm{d}\theta.$$

Then, by Fubini's theorem,

$$\begin{aligned}\mathbb{E}\, L_n =& \sum_{k=1}^n \frac{1}{k} \int_0^\pi \mathbb{E}\,|S_k \cdot \mathbf{e}_\theta|\,\mathrm{d}\theta \\ =& \sum_{k=1}^n \frac{1}{k} \mathbb{E} \int_0^\pi |S_k \cdot \mathbf{e}_\theta|\,\mathrm{d}\theta \\ =& 2\sum_{k=1}^n \frac{\mathbb{E}\,\|S_k\|}{k}.\end{aligned}$$

□

4.3 Asymptotics for the expected perimeter length

To investigate the first-order properties of $\mathbb{E}\, L_n$, we suggested by the Spitzer-Widom formula (1.1) that the first-order properties of $\mathbb{E}\,\|S_n\|$ need to be studied first.

Lemma 4.4. *If $\mathbb{E}\,\|Z_1\| < \infty$, then $n^{-1}\mathbb{E}\,\|S_n\| \to \|\mu\|$ as $n \to \infty$.*

Proof. The strong law of large numbers for S_n says $\|S_n/n - \mathbb{E}\, Z_1\| \to 0$ a.s. as $n \to \infty$. Then by the triangle inequality,

$$\|S_n/n\| = \|S_n/n - \mathbb{E}\, Z_1 + \mathbb{E}\, Z_1\| \le \|S_n/n - \mathbb{E}\, Z_1\| + \|\mathbb{E}\, Z_1\|$$

and

$$\|\mathbb{E}\, Z_1\| \le \|\mathbb{E}\, Z_1 - S_n/n\| + \|S_n/n\|.$$

So, $\|S_n\|/n \to \|\mathbb{E}\, Z_1\|$ a.s. as $n \to \infty$.

Similarly, let $Y_n = \sum_{i=1}^n \|Z_i\|$, then $Y_n/n \to \mathbb{E}\,\|Z_1\|$ a.s. as $n \to \infty$. Also we simply have $\mathbb{E}\,[Y_n/n] = \mathbb{E}\,\|Z_1\|$ and $0 \le \|S_n\|/n \le Y_n/n$. Hence, the result is proved by Pratt's Lemma (see Lemma 2.2). □

The following asymptotic result for $\mathbb{E}\, L_n$ was obtained as equation (2.16) by Snyder & Steele [57] under the stronger condition $\mathbb{E}\,(\|Z_1\|^2) < \infty$; as Lemma 4.4 shows, a finite first moment is sufficient.

Proposition 4.5. *Suppose* $\mathbb{E}\,\|Z_1\| < \infty$, *then* $n^{-1}\mathbb{E}\,L_n \to 2\|\mu\|$, *as* $n \to \infty$.

Proof. The result is implied by the Spitzer–Widom formula (1.1) and Lemma 2.22 with $y_n = n^{-1}\mathbb{E}\,\|S_n\|$, since $y_n \to \|\mu\|$ by Lemma 4.4. $\square$

Remarks 4.2. (i) Proposition 4.5 says that if $\mu \neq 0$ then $\mathbb{E}\,L_n$ is of order n. If $\mu = 0$, it says $\mathbb{E}\,L_n = o(n)$. We will show later in Proposition 4.9 that under mild extra conditions in the $\mu = 0$ case, $n^{-1/2}\mathbb{E}\,L_n$ has a limit.

(ii) Snyder and Steele [57, p. 1168] showed that if $\mathbb{E}\,(\|Z_1\|^2) < \infty$ and $\mu \neq 0$, then in fact $n^{-1}L_n \to 2\|\mu\|$ a.s. as $n \to \infty$. We give a proof of this in Proposition 5.5 below.

For the zero drift case $\mu = 0$, we have the following.

Lemma 4.6. *Suppose* $\mathbb{E}\,(\|Z_1\|^2) < \infty$ *and* $\mu = 0$, *then* $\mathbb{E}\,(\|S_n\|^2) = O(n)$ *and* $\mathbb{E}\,\|S_n\| = O(n^{1/2})$.

Proof. Consider $\|S_n\|^2$,

$$\|S_{n+1}\|^2 = \|S_n + Z_{n+1}\|^2 = \|S_n\|^2 + 2S_n \cdot Z_{n+1} + \|Z_{n+1}\|^2. \tag{4.2}$$

So,

$$\mathbb{E}\,(\|S_{n+1}\|^2) - \mathbb{E}\,(\|S_n\|^2) = \mathbb{E}\,(\|Z_1\|^2),$$

since S_n and Z_{n+1} are independent and Z_{n+1} has mean 0, so $\mathbb{E}\,(S_n \cdot Z_{n+1}) = \mathbb{E}\,S_n \cdot \mathbb{E}\,Z_{n+1} = 0$. Then sum from $n = 0$ to $m - 1$ to get

$$\mathbb{E}\,(\|S_m\|^2) - \mathbb{E}\,(\|S_0\|^2) = m\mathbb{E}\,(\|Z_1\|^2).$$

Hence, $\mathbb{E}\,(\|S_n\|^2) = O(n)$. The last result is given by Jensen's inequality, $\mathbb{E}\,\|S_n\| \leq (\mathbb{E}\,[\|S_n\|^2])^{1/2}$. $\square$

Remark 4.3. Lemma 4.6 only gives the upper bound for the order of $\mathbb{E}\,\|S_n\|$. Under the mild assumption $\mathbb{P}(\|Z_1\| = 0) < 1$, $n^{-1/2}\mathbb{E}\,\|S_n\|$ in fact has a positive limit, as we will see in the proof of Proposition 4.9 below. This extra condition is of course necessary for the positive limit, since if $Z_1 \equiv 0$ then $\mathbb{E}\,\|S_n\| \equiv 0$.

Proposition 4.7. *Suppose* $\mathbb{E}\,(\|Z_1\|^2) < \infty$ *and* $\mu = 0$, *then* $\mathbb{E}\,L_n = O(n^{1/2})$.

Proof. By Lemma 4.6 and Spitzer–Widom formula (1.1), for some constant C,

$$\mathbb{E}\, L_n \le 2\sum_{i=1}^{n} \frac{C\sqrt{i}}{i} = 2C\sum_{i=1}^{n} i^{-1/2} = O(n^{1/2}). \quad \square$$

Lemma 4.8. *Let $p > 1$. Suppose that $\mathbb{E}\,[\|Z_1\|^p] < \infty$.*

(i) For any $e \in \mathbb{S}_1$ such that $e \cdot \mu = 0$, $\mathbb{E}\,[\max_{0\le m\le n} |S_m \cdot e|^p] = O(n^{1\vee(p/2)})$.

(ii) Moreover, if $\mu = 0$, then $\mathbb{E}\,[\max_{0\le m\le n} \|S_m\|^p] = O(n^{1\vee(p/2)})$.

(iii) On the other hand, if $\mu \neq 0$, then $\mathbb{E}\,[\max_{0\le m\le n} |S_m \cdot \hat{\mu}|^p] = O(n^p)$.

Proof. Given that $\mu \cdot e = 0$, $S_n \cdot e$ is a martingale, and hence, by convexity, $|S_n \cdot e|$ is a non-negative submartingale. Then, for $p > 1$,

$$\mathbb{E}\left[\max_{0\le m\le n} |S_m \cdot e|^p\right] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}\,\left[|S_n \cdot e|^p\right] = O(n^{1\vee(p/2)}),$$

where the first inequality is Doob's L^p inequality (see Lemma 2.17) and the second is the Marcinkiewicz–Zygmund inequality (see Lemma 2.19). This gives part (i).

Part (ii) follows from part (i): take $\{e_1, e_2\}$ an orthonormal basis of $\mathbb{R}^2$ and apply (i) with each basis vector. Then by the triangle inequality

$$\max_{0\le m\le n} \|S_m\| \le \max_{0\le m\le n} |S_m \cdot e_1| + \max_{0\le m\le n} |S_m \cdot e_2|$$

together with Minkowski's inequality (see Lemma 2.16), we have

$$\begin{aligned}
\mathbb{E}\left[\max_{0\le m\le n} \|S_m\|^p\right] &\le \mathbb{E}\left[\left(\max_{0\le m\le n} |S_m \cdot e_1| + \max_{0\le m\le n} |S_m \cdot e_2|\right)^p\right] \\
&= \left\|\max_{0\le m\le n} |S_m \cdot e_1| + \max_{0\le m\le n} |S_m \cdot e_2|\right\|_p^p \\
&\le \left(\left\|\max_{0\le m\le n} |S_m \cdot e_1|\right\|_p + \left\|\max_{0\le m\le n} |S_m \cdot e_2|\right\|_p\right)^p \\
&= O(n^{1\vee(p/2)}).
\end{aligned}$$

Part (iii) follows from the fact that

$$\max_{0\le m\le n} |S_m \cdot \hat{\mu}| \le \sum_{k=1}^{n} |Z_k \cdot \hat{\mu}| \le \sum_{k=1}^{n} \|Z_k\|$$

and an application of Rosenthal's inequality (see Lemma 2.20) to the latter sum gives

$$\begin{aligned}
\mathbb{E}\left[\max_{0\le m\le n}\|S_m\cdot\hat{\mu}\|^p\right] &\le \mathbb{E}\left[\left(\sum_{k=1}^{n}\|Z_k\|\right)^p\right]\\
&\le \max\left\{2^p\sum_{k=1}^{n}\mathbb{E}\,\|Z_k\|^p,\ 2^{p^2}\left(\sum_{k=1}^{n}\mathbb{E}\,\|Z_k\|\right)^p\right\}\\
&\le \max\{O(n), O(n^p)\}\\
&\le O(n^p). \qquad\square
\end{aligned}$$

Proposition 4.7 gives the order of $\mathbb{E}\,L_n$. Now we can have the exact limit by the following result, the statement of which is similar to an example on p. 508 of [58].

Proposition 4.9. *Suppose* $\mathbb{E}\,(\|Z_1\|^2) < \infty$ *and* $\mu = 0$. *Then, for* $Y \sim \mathcal{N}(\mathbf{0}, \Sigma)$,

$$\lim_{n\to\infty} n^{-1/2}\mathbb{E}\,L_n = \mathbb{E}\,\mathcal{L}(\Sigma^{1/2}h_1) = 4\mathbb{E}\,\|Y\|.$$

Proof. The finite point-set case of Cauchy's formula gives

$$L_n = \int_{\mathbb{S}_1}\max_{0\le k\le n}(S_k\cdot e)\mathrm{d}e \le 2\pi\max_{0\le k\le n}\|S_k\|. \tag{4.3}$$

Then by Lemma 4.8(ii) we have $\sup_n \mathbb{E}\,[(n^{-1/2}L_n)^2] < \infty$. Hence $n^{-1/2}L_n$ is uniformly integrable, so that Theorem 3.6 yields $\lim_{n\to\infty} n^{-1/2}\mathbb{E}\,L_n = \mathbb{E}\,\mathcal{L}(\Sigma^{1/2}h_1)$.

It remains to show that $\lim_{n\to\infty} n^{-1/2}\mathbb{E}\,L_n = 4\mathbb{E}\,\|Y\|$. One can use Cauchy's formula to compute $\mathbb{E}\,\mathcal{L}(\Sigma^{1/2}h_1)$; instead we give a direct random walk argument, following [58]. The central limit theorem for S_n implies that $n^{-1/2}\|S_n\| \to \|Y\|$ in distribution. Under the given conditions, $\mathbb{E}\,[\|S_{n+1}\|^2] = \mathbb{E}\,[\|S_n\|^2] + \mathbb{E}\,[\|Z_{n+1}\|^2]$, so that $\mathbb{E}\,[\|S_n\|^2] = O(n)$. It follows that $n^{-1/2}\|S_n\|$ is uniformly integrable, and hence

$$\lim_{n\to\infty} n^{-1/2}\mathbb{E}\,\|S_n\| = \mathbb{E}\,\|Y\|.$$

So for any $\varepsilon > 0$, there is some $n_0 \in \mathbb{N}$ such that $\left|k^{-1/2}\mathbb{E}\,\|S_k\| - \mathbb{E}\,\|Y\|\right| < \varepsilon$ for all $k \ge n_0$. Then by the S–W formula (1.1), we have

$$\left|\frac{\mathbb{E}\,L_n}{\sqrt{n}} - 2\mathbb{E}\,\|Y\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}k^{-1/2}\right|$$

$$
\begin{aligned}
&= \frac{2}{\sqrt{n}} \left| \sum_{k=1}^{n} \left(\frac{\mathbb{E}\,\|S_k\|}{k} - \mathbb{E}\,\|Y\| k^{-1/2} \right) \right| \\
&\le \frac{2}{\sqrt{n}} \sum_{k=1}^{n} \left| \frac{\mathbb{E}\,\|S_k\|}{\sqrt{k}} - \mathbb{E}\,\|Y\| \right| k^{-1/2} \\
&= \frac{2}{\sqrt{n}} \left(\sum_{k=1}^{n_0} + \sum_{i=n_0+1}^{n} \right) \left| \frac{\mathbb{E}\,\|S_k\|}{\sqrt{k}} - \mathbb{E}\,\|Y\| \right| k^{-1/2} \\
&\le \frac{D}{\sqrt{n}} + \frac{2}{\sqrt{n}} \sum_{k=n_0+1}^{n} \left| \frac{\mathbb{E}\,\|S_k\|}{\sqrt{k}} - \mathbb{E}\,\|Y\| \right| k^{-1/2} \\
&\le \frac{D}{\sqrt{n}} + \frac{2\varepsilon}{\sqrt{n}} \sum_{k=n_0+1}^{n} k^{-1/2},
\end{aligned}
$$

for some constant D and the n_0 mentioned above.

Also notice the fact that $\lim_{n\to\infty} n^{-1/2} \sum_{k=1}^{n} k^{-1/2} = 2$. This can be proved by the monotonicity,

$$
2\left[(n+1)^{1/2} - 1\right] = \int_1^{n+1} x^{-1/2}\,\mathrm{d}x \le \sum_{k=1}^{n} k^{-1/2} \le \int_0^n x^{-1/2}\,\mathrm{d}x = 2n^{1/2}.
$$

Taking $n \to \infty$ in the displayed inequality gives

$$
\limsup_{n\to\infty} \left| \frac{\mathbb{E}\,L_n}{\sqrt{n}} - 2\mathbb{E}\,\|Y\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} k^{-1/2} \right| \le 4\varepsilon.
$$

Since $\varepsilon > 0$ was arbitrary, it follows that

$$
\lim_{n\to\infty} \left| \frac{\mathbb{E}\,L_n}{\sqrt{n}} - 2\mathbb{E}\,\|Y\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} k^{-1/2} \right| = 0.
$$

Therefore,

$$
\lim_{n\to\infty} \frac{\mathbb{E}\,L_n}{\sqrt{n}} = \lim_{n\to\infty} 2\mathbb{E}\,\|Y\| \frac{1}{\sqrt{n}} \sum_{k=1}^{n} k^{-1/2} = 4\mathbb{E}\,\|Y\|. \quad \square
$$

Cauchy's formula applied to the line segment from 0 to Y with Fubini's theorem implies $2\mathbb{E}\,\|Y\| = \int_{\mathbb{S}_1} \mathbb{E}\,[(Y \cdot e)^+]\mathrm{d}e$. Here $Y \cdot e = e^\top Y$ is univariate normal with mean 0 and variance $e^\top \Sigma e = \|\Sigma^{1/2} e\|^2$, so that $\mathbb{E}\,[(Y \cdot e)^+]$ is $\|\Sigma^{1/2} e\|$ times one half of the mean of the square-root of a χ_1^2 random variable. Hence

$$
\mathbb{E}\,\|Y\| = (8\pi)^{-1/2} \int_{\mathbb{S}_1} \|\Sigma^{1/2} e\|\,\mathrm{d}e,
$$

which in general may be expressed via a complete elliptic integral of the second kind in terms of the ratio of the eigenvalues of Σ. In the particular case $\Sigma = I$, $\mathbb{E}\,\|Y\| = \sqrt{\pi/2}$ so then Proposition 4.9 implies that

$$\lim_{n\to\infty} n^{-1/2}\mathbb{E}\,L_n = \sqrt{8\pi},$$

matching the formula $\mathbb{E}\,\ell_1 = \sqrt{8\pi}$ of Letac and Takács [39, 61] (see Lemma 4.10 below). We also note the bounds

$$\pi^{-1/2}\sqrt{\operatorname{tr}\Sigma} \le \mathbb{E}\,\|Y\| \le \sqrt{\operatorname{tr}\Sigma}; \tag{4.4}$$

the upper bound here is from Jensen's inequality and the fact that $\mathbb{E}\,[\|Y\|^2] = \operatorname{tr}\Sigma$. The lower bound in (4.4) follows from the inequality

$$\mathbb{E}\,\|Y\| \ge \sup_{e\in\mathbb{S}_1}\mathbb{E}\,|Y\cdot e| = \sqrt{2/\pi}\sup_{e\in\mathbb{S}_1}(\mathbb{V}\mathrm{ar}[Y\cdot e])^{1/2}$$

together with the fact that

$$\sup_{e\in\mathbb{S}_1}\mathbb{V}\mathrm{ar}[Y\cdot e] = \sup_{e\in\mathbb{S}_1}\|\Sigma^{1/2}e\|^2 = \|\Sigma^{1/2}\|_{\mathrm{op}}^2 = \|\Sigma\|_{\mathrm{op}} = \lambda_\Sigma \ge \frac{1}{2}\operatorname{tr}\Sigma,$$

where $\|\cdot\|_{\mathrm{op}}$ is the matrix operator norm and λ_Σ is the largest eigenvalue of Σ; in statistical terminology, λ_Σ is the variance of the first principal component associated with Y.

We give a proof of the formula of Letac and Takács [39, 61].

Lemma 4.10. *Let $\ell_1 = \mathcal{L}(h_1)$ (see equation (3.5)) be the perimeter length of convex hull of a standard Brownian motion on $[0,1]$ in $\mathbb{R}^2$. Then, $\mathbb{E}\,\ell_1 = \sqrt{8\pi}$.*

Proof. Applying Fubinis theorem (Lemma 2.21) in Cauchy formula (2.5) for ℓ_1,

$$\ell_1 = \int_0^{2\pi}\sup_{t\in[0,1]}(b(t)\cdot\mathbf{e}_\theta)\,d\theta,$$

we have

$$\begin{aligned}
\mathbb{E}\,\ell_1 &= \int_0^{2\pi}\mathbb{E}\sup_{t\in[0,1]}(b(t)\cdot\mathbf{e}_\theta)\,d\theta \\
&= 2\pi\mathbb{E}\sup_{t\in[0,1]}(b(t)\cdot\mathbf{e}_\theta),\ \text{where } b(t)\cdot\mathbf{e}_\theta \text{ is a 1 dimensional Brownian motion,} \\
&= 2\pi\mathbb{E}\sup_{t\in[0,1]} w(t).
\end{aligned}$$

Here $w(t)$ is defined as a standard 1-dimensional Brownian motion, which is the same as in Corollary 2.11. Then we have

$$\begin{aligned}
\mathbb{E} \sup_{t\in[0,1]} w(t) &= \int_0^\infty \mathbb{P}\left(\sup_{t\in[0,1]} w(t) > r\right) \mathrm{d}r \\
&= 2\int_0^\infty \mathbb{P}\left(w(1) > r\right) \mathrm{d}r, \text{ by Reflection principle (Corollary 2.11),} \\
&= 2\int_0^\infty \frac{\mathrm{d}r}{\sqrt{2\pi}} \int_r^\infty e^{-y^2/2}\,\mathrm{d}y \\
&= \sqrt{\frac{2}{\pi}} \int_0^\infty \mathrm{d}y \int_0^y e^{-y^2/2}\,\mathrm{d}r, \text{ by changing orders of integrals,} \\
&= \sqrt{\frac{2}{\pi}}
\end{aligned}$$

Hence, the result follows. $\square$

Chapter 5

Asymptotics for perimeter length of the convex hull

5.1 Overview

To start this chapter we discuss some simulations. We considered a specific form of random walk with increments $Z_i - \mathbb{E}[Z_i] = (\cos \Theta_i, \sin \Theta_i)$, where Θ_i was uniformly distributed on $[0, 2\pi)$, corresponding to a uniform distribution on a unit circle centred at $\mathbb{E}[Z_i] = \mu$. We took one example with $\mu = \mathbf{0}$, and two examples with $\mu \neq \mathbf{0}$ of different magnitudes.

For the expected perimeter length, the simulations (see Figure 5.1) are consistent with the Spitzer–Widdom–Baxter result (see the argument below (1.1)), Proposition 4.9 and Proposition 5.5. In the case of $\mu = \mathbf{0}$, the result in Proposition 4.9 take the form: $\lim_{n\to\infty} n^{-1/2}\mathbb{E}\, L_n = 4\mathbb{E}\,\|Y\| = 4$. In the case of $\mu \neq \mathbf{0}$, the result in Proposition 5.5 take the form: $n^{-1}L_n \xrightarrow{a.s.} 2\|\mu\| = 0.4$ or 0.72.

For the variance of perimeter length with drift, the result in Theorem 5.13 take the form: $\lim_{n\to\infty} \mathbb{V}\mathrm{ar}[L_n] = 4\mathbb{E}[\cos^2\Theta_1] = 2$ and in Theorem 5.14, $(2n)^{-1/2}(L_n - \mathbb{E}[L_n])$ converges in distribution to a standard normal distribution. The corresponding pictures in Figures 5.2 and 5.3 show an agreement between the simulations and the theory. In the zero drift case, the simulations (the leftmost plot in Figure 5.2) suggest that $\lim_{n\to\infty} n^{-1}\mathbb{V}\mathrm{ar}[L_n]$ exists but Figure 5.3 does not appear to be consistent with a normal distribution as a limiting distribution.

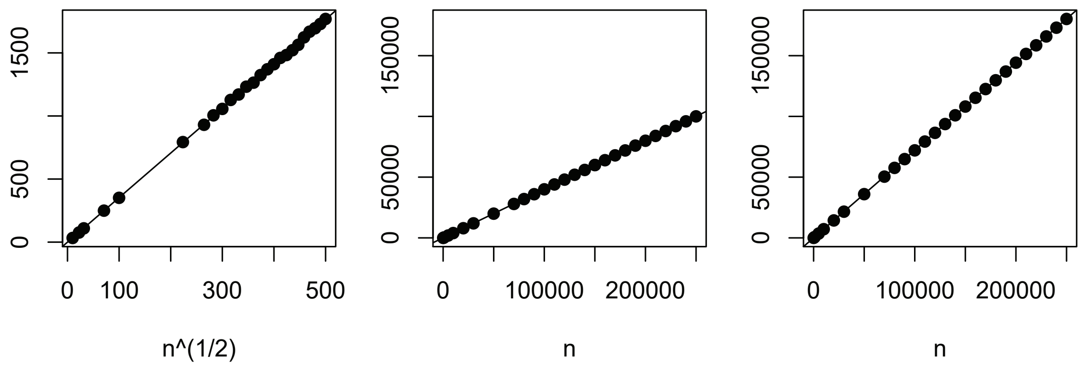

Figure 5.1: Plots of $y = \mathbb{E}[L_n]$ estimates against $x =$ (left to right) $n^{1/2}$, n, n for about 25 values of n in the range 10^2 to 2.5×10^5 for 3 examples with $\|\mu\| =$ (left to right) 0, 0.2, 0.36. Each point is estimated from 10^3 repeated simulations. Also plotted are straight lines $y = 3.532x$ (leftmost plot), $y = 0.40x$ (middle plot) and $y = 0.721x$ (rightmost plot).

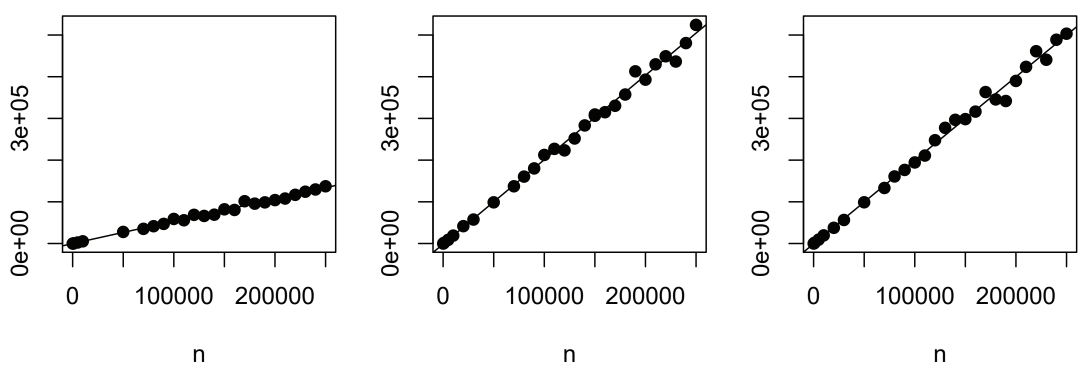

Figure 5.2: Plots of $y = \mathbb{V}\mathrm{ar}[L_n]$ estimates against $x = n$ for the three examples described in Figure 5.1. Also plotted are straight lines $y = 0.536x$ (leftmost plot) and $y = 2x$ (other two plots).

We will show in Proposition 5.15 that

$$\text{if } \mu = 0: \quad \lim_{n\to\infty} n^{-1}\mathbb{V}\mathrm{ar} L_n = u_0(\Sigma),$$

where $u_0(\cdot)$ is finite and positive provided $\sigma^2 < \infty$. For the constant $u_0(I)$ (I being the identity matrix), Table 5.1 gives numerical evaluation of rigorous bound that we prove in Proposition 5.16 below, plus estimate from simulations. See also Section 7.2 for an explicit integral expression for $u_0(I)$.

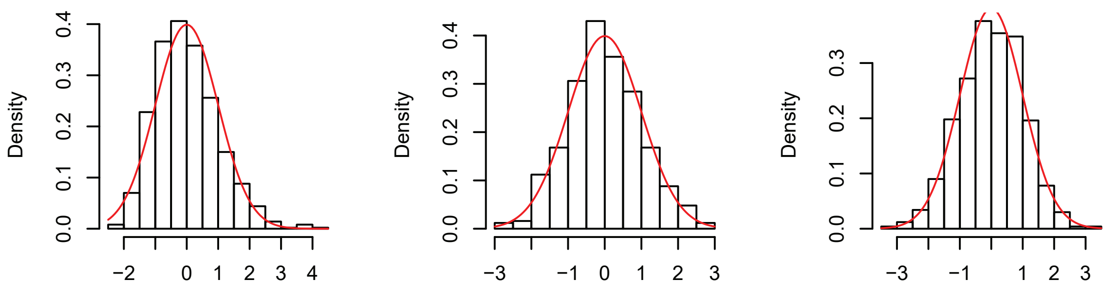

Figure 5.3: Simulated histogram estimates for the distribution of $\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\mathbb{V}\mathrm{ar}[L_n]}}$ with $n = 5 \times 10^3$ in the three examples described in Figure 5.1. Each histogram is compiled from 10^3 samples.

	lower bound	simulation estimate	upper bound
$u_0(I)$	2.65×10^{-3}	1.08	9.87

Table 5.1: The simulation estimate is based on 10^5 instances of a walk of length $n = 10^5$. The final decimal digit in the numerical upper (lower) bounds has been rounded up (down).

5.2 Upper bound for the variance

Assuming that $\mathbb{E}[\|Z_1\|^2] < \infty$, Snyder and Steele [57] obtained an upper bound for $\mathbb{V}\mathrm{ar}[L_n]$ using Cauchy's formula together with a version of the Efron–Stein inequality. Snyder and Steele's result (Theorem 2.3 of [57]) can be expressed as

$$n^{-1}\mathbb{V}\mathrm{ar}[L_n] \leq \frac{\pi^2}{2}\left(\mathbb{E}[\|Z_1\|^2] - \|\mathbb{E}[Z_1]\|^2\right), \quad (n \in \mathbb{N} := \{1, 2, \ldots\}). \tag{5.1}$$

As far as we are aware, there are no lower bounds for $\mathbb{V}\mathrm{ar}[L_n]$ in the literature. According to the discussion in [57, §5], Snyder and Steele had "no compelling reason to expect that $O(n)$ is the correct order of magnitude" in their upper bound for $\mathbb{V}\mathrm{ar}[L_n]$, and they speculated that perhaps $\mathbb{V}\mathrm{ar}[L_n] = o(n)$ (maybe with a distinction between the cases of zero and non-zero drift). Our first main result settles this question under minimal conditions, confirming that (5.1) is indeed of the correct order, apart from in certain degenerate cases, while demonstrating that the constant on the right-hand side of (5.1) is not, in general, sharp.

The first step in looking for the variance upper bound is a martingale difference argument, based on resampling members of the sequence $Z_1, \ldots, Z_n$, to get an expression for $\mathbb{V}\mathrm{ar}[L_n]$ amenable to analysis: see Section 2.2. Let $\mathcal{F}_0$ denote the trivial σ-algebra, and for $n \in \mathbb{N}$ set $\mathcal{F}_n := \sigma(Z_1, \ldots, Z_n)$, the σ-algebra generated by the first n steps of the random walk. Then S_n is $\mathcal{F}_n$-measurable, and for $n \in \mathbb{N}$ we can write $L_n = \Lambda_n(Z_1, \ldots, Z_n)$ for $\Lambda_n : \mathbb{R}^{2n} \to [0, \infty)$ a measurable function.

Let $Z'_1, Z'_2, \ldots$ be an independent copy of the sequence $Z_1, Z_2, \ldots$. Fix $n \in \mathbb{N}$. For $i \in \{1, \ldots, n\}$, we 'resample' the ith increment, replacing Z_i with Z'_i, as follows. Set

$$S_j^{(i)} := \begin{cases} S_j & \text{if } j < i \\ S_j - Z_i + Z'_i & \text{if } j \geq i; \end{cases} \tag{5.2}$$

then $(S_j^{(i)}; 0 \leq j \leq n)$ is a modification of the random walk $(S_j; 0 \leq j \leq n)$ that keeps all the components apart from the ith step which is independently resampled. We let $L_n^{(i)}$ denote the perimeter length of the corresponding convex hull for this modified walk, namely $\mathrm{hull}(S_0^{(i)}, \ldots, S_n^{(i)})$, i.e.,

$$L_n^{(i)} := \Lambda_n(Z_1, \ldots, Z_{i-1}, Z'_i, Z_{i+1}, \ldots, Z_n).$$

For $i \in \{1, \ldots, n\}$, define

$$D_{n,i} := \mathbb{E}\,[L_n - L_n^{(i)} \mid \mathcal{F}_i]; \tag{5.3}$$

in other words, $-D_{n,i}$ is the expected change in the perimeter length of the convex hull, given $\mathcal{F}_i$, on replacing Z_i by Z'_i. The point of this construction is the following result.

Lemma 5.1. *Let $n \in \mathbb{N}$. Then (i) $L_n - \mathbb{E}\,[L_n] = \sum_{i=1}^n D_{n,i}$; and (ii) $\mathbb{V}\mathrm{ar}[L_n] = \sum_{i=1}^n \mathbb{E}\,[D_{n,i}^2]$, whenever the latter sum is finite.*

Proof. Take $W_n = L_n$ in Lemma 2.8. Then the results follow. □

Remark 5.1. Lemma 5.1 with the conditional Jensen's inequality gives the bound

$$\mathbb{V}\mathrm{ar}[L_n] \leq \sum_{i=1}^n \mathbb{E}\left[\left(L_n^{(i)} - L_n\right)^2\right],$$

which is a factor of 2 larger than the upper bound obtained from the Efron–Stein inequality: $\mathbb{V}\mathrm{ar}[L_n] \leq 2^{-1} \sum_{i=1}^n \mathbb{E}\left[(L_n^{(i)} - L_n)^2\right]$ (see equation (2.3) in [57]).

Let $\mathbf{e}_\theta = (\cos\theta, \sin\theta)$ be the unit vector in direction $\theta \in (-\pi, \pi]$. For $\theta \in [0, \pi]$, define

$$M_n(\theta) := \max_{0 \le j \le n} (S_j \cdot \mathbf{e}_\theta), \text{ and } m_n(\theta) := \min_{0 \le j \le n} (S_j \cdot \mathbf{e}_\theta).$$

Note that since $S_0 = \mathbf{0}$, we have $M_n(\theta) \ge 0$ and $m_n(\theta) \le 0$, a.s. In the present setting (see equation (2.7)), Cauchy's formula for convex sets yields

$$L_n = \int_0^\pi (M_n(\theta) - m_n(\theta))\, \mathrm{d}\theta = \int_0^\pi R_n(\theta) \mathrm{d}\theta,$$

where $R_n(\theta) := M_n(\theta) - m_n(\theta) \ge 0$ is the *parametrized range function*. Similarly, when the ith increment is resampled,

$$L_n^{(i)} = \int_0^\pi \left(M_n^{(i)}(\theta) - m_n^{(i)}(\theta) \right) \mathrm{d}\theta = \int_0^\pi R_n^{(i)}(\theta) \mathrm{d}\theta,$$

where $R_n^{(i)}(\theta) = M_n^{(i)}(\theta) - m_n^{(i)}(\theta)$, defining

$$M_n^{(i)}(\theta) := \max_{0 \le j \le n} (S_j^{(i)} \cdot \mathbf{e}_\theta), \text{ and } m_n^{(i)}(\theta) := \min_{0 \le j \le n} (S_j^{(i)} \cdot \mathbf{e}_\theta).$$

Thus to study $D_{n,i} = \mathbb{E}\left[L_n - L_n^{(i)} \mid \mathcal{F}_i\right]$ we will consider

$$L_n - L_n^{(i)} = \int_0^\pi \left(R_n(\theta) - R_n^{(i)}(\theta) \right) \mathrm{d}\theta = \int_0^\pi \Delta_n^{(i)}(\theta) \mathrm{d}\theta, \tag{5.4}$$

where $\Delta_n^{(i)}(\theta) := R_n(\theta) - R_n^{(i)}(\theta)$. For $\theta \in [0, \pi]$, let

$$\underline{J}_n(\theta) := \arg\min_{0 \le j \le n} (S_j \cdot \mathbf{e}_\theta), \text{ and } \bar{J}_n(\theta) := \arg\max_{0 \le j \le n} (S_j \cdot \mathbf{e}_\theta),$$

so $m_n(\theta) = S_{\underline{J}_n(\theta)} \cdot \mathbf{e}_\theta$ and $M_n(\theta) = S_{\bar{J}_n(\theta)} \cdot \mathbf{e}_\theta$. Similarly, recalling (5.2), define

$$\underline{J}_n^{(i)}(\theta) := \arg\min_{0 \le j \le n} (S_j^{(i)} \cdot \mathbf{e}_\theta), \text{ and } \bar{J}_n^{(i)}(\theta) := \arg\max_{0 \le j \le n} (S_j^{(i)} \cdot \mathbf{e}_\theta).$$

(Apply the following conventions in the event of ties: arg min takes the maximum argument among tied values, and arg max the minimum.)

We will use the following simple bound repeatedly in the arguments that follow. This upper bound for $|\Delta_n^{(i)}(\theta)|$ is also given in Lemma 2.1 of [57]. But we have a different way to prove here.

Lemma 5.2. *Almost surely, for any $\theta \in [0, \pi]$ and any $i \in \{1, 2, \ldots, n\}$,*

$$|\Delta_n^{(i)}(\theta)| \le |(Z_i - Z_i') \cdot \mathbf{e}_\theta| \le \|Z_i\| + \|Z_i'\|. \tag{5.5}$$

Proof. Consider the effect on $S_k \cdot \mathbf{e}_\theta$ when Z_i is replaced by Z_i'. If $i > k$, then $S_k \cdot \mathbf{e}_\theta = S_k^{(i)} \cdot \mathbf{e}_\theta$. If $i \leq k$, then $S_k \cdot \mathbf{e}_\theta = S_k^{(i)} \cdot \mathbf{e}_\theta + (Z_i - Z_i') \cdot \mathbf{e}_\theta$. Hence, for all i,

$$S_k \cdot \mathbf{e}_\theta \leq S_k^{(i)} \cdot \mathbf{e}_\theta + ((Z_i - Z_i') \cdot \mathbf{e}_\theta \vee 0).$$

Therefore,

$$\max_{1\leq k\leq n} S_k \cdot \mathbf{e}_\theta \leq \max_{1\leq k\leq n} S_k^{(i)} \cdot \mathbf{e}_\theta + ((Z_i - Z_i') \cdot \mathbf{e}_\theta \vee 0).$$

Similarly, we have

$$\min_{1\leq k\leq n} S_k \cdot \mathbf{e}_\theta \geq \min_{1\leq k\leq n} S_k^{(i)} \cdot \mathbf{e}_\theta + ((Z_i - Z_i') \cdot \mathbf{e}_\theta \wedge 0).$$

Combining these two inequalities with maximum and minimum, we get

$$\begin{aligned} R_n(\theta) - R_n^{(i)}(\theta) &\leq ((Z_i - Z_i') \cdot \mathbf{e}_\theta \vee 0) - ((Z_i - Z_i') \cdot \mathbf{e}_\theta \wedge 0) \\ &= |(Z_i - Z_i') \cdot \mathbf{e}_\theta|. \end{aligned}$$

Also similarly, we can get $R_n^{(i)}(\theta) - R_n(\theta) \leq |(Z_i' - Z_i) \cdot \mathbf{e}_\theta|$. Thus, the result follows from the triangle inequality. □

The following is Lemma 2.2 in [57].

Lemma 5.3. *For all* $1 \leq i \leq n$,

$$\mathbb{E}\left[\left(\int_0^\pi |(Z_i - Z_i') \cdot \mathbf{e}_\theta| \,\mathrm{d}\theta\right)^2\right] \leq \pi^2 \left(\mathbb{E}\,\|Z_1\|^2 - \|\mu\|^2\right) = \pi^2\sigma^2.$$

Proof. By Cauchy-Schwarz Inequality, we have

$$\mathbb{E}\left[\left(\int_0^\pi |(Z_i - Z_i') \cdot \mathbf{e}_\theta| \,\mathrm{d}\theta\right)^2\right] \leq \pi\mathbb{E}\left(\int_0^\pi |(Z_i - Z_i') \cdot \mathbf{e}_\theta|^2 \,\mathrm{d}\theta\right).$$

Then, since Z_i, Z_i' are identically and independently distributed,

$$\begin{aligned} \mathbb{E}\left[|Z_i \cdot \mathbf{e}_\theta - Z_i' \cdot \mathbf{e}_\theta|^2\right] &= \mathbb{E}\left[(Z_i \cdot \mathbf{e}_\theta)^2\right] + \mathbb{E}\left[(Z_i' \cdot \mathbf{e}_\theta)^2\right] - 2\mathbb{E}\left[(Z_i \cdot \mathbf{e}_\theta)(Z_i' \cdot \mathbf{e}_\theta)\right] \\ &= 2\mathbb{V}\mathrm{ar}[Z_1 \cdot \mathbf{e}_\theta] \\ &= 2\left(\sigma_\mu^2 \cos^2\theta + \sigma_{\mu_\perp}^2 \cos^2\theta + 2\cos\theta\sin\theta\rho_{\mu\mu_\perp}\sigma_\mu\sigma_{\mu_\perp}\right), \end{aligned}$$

where $\rho_{\mu\mu_\perp}$ is the covariance of $(Z_1 - \mu) \cdot \hat{\mu}$ and $(Z_1 - \mu) \cdot \hat{\mu}_\perp$. So,

$$\mathbb{E}\int_0^\pi |(Z_i - Z_i') \cdot \mathbf{e}_\theta|^2 \,\mathrm{d}\theta = 2\left(\sigma_\mu^2 \int_0^\pi \cos^2\theta \,\mathrm{d}\theta + \sigma_{\mu_\perp}^2 \int_0^\pi \sin^2\theta \,\mathrm{d}\theta\right)$$

$$
\begin{aligned}
&+ 4\rho_{\mu\mu_\perp}\sigma_\mu\sigma_{\mu_\perp}\int_0^\pi \cos\theta\sin\theta\,\mathrm{d}\theta \\
&= \pi(\sigma_\mu^2 + \sigma_{\mu_\perp}^2).
\end{aligned}
$$

This proves the lemma. □

The next result is a version of Theorem 2.3 in [57]. But they get better right-hand side by using Efron–Stein inequality

Proposition 5.4. *Suppose* $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$. *Then*

$$
\mathbb{V}\mathrm{ar}(L_n) \leq \frac{\pi^2\sigma^2}{2} n. \tag{5.6}
$$

Proof. By Lemma 2.9, equation (5.4) and (5.5),

$$
\begin{aligned}
\mathbb{V}\mathrm{ar}[L_n] &\leq \frac{1}{2}\sum_{i=1}^n \mathbb{E}\left[\left(\int_0^\pi \Delta_n^{(i)}(\theta)\mathrm{d}\theta\right)^2\right] \\
&\leq \frac{1}{2}\sum_{i=1}^n \mathbb{E}\left[\left(\int_0^\pi |(Z_i - Z_i')\cdot \mathbf{e}_\theta|\,\mathrm{d}\theta\right)^2\right] \\
&\leq \frac{1}{2}\sum_{i=1}^n \pi^2\sigma^2 \\
&= \frac{n\pi^2\sigma^2}{2},
\end{aligned}
$$

since Z_i are independent identically distributed. □

5.3 Law of large numbers

As we mentioned earlier in Remarks 4.2, Snyder and Steele [57] has shown the asymptotic behaviour of L_n/n. They state their law of large numbers only for $\mu \neq 0$ but the case with $\mu = 0$ works equally well. Here we give a different proof of the law of large numbers by using the variance bound.

Proposition 5.5. *If* $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, *then* $n^{-1}L_n \to 2\|\mu\|$ a.s. *as* $n \to \infty$.

Proof. We have $n^{-1}\mathbb{E}\,L_n \to 2\|\mu\|$ by Proposition 4.5 and the variance bound $\mathbb{V}\mathrm{ar}L_n \leq Cn$ by Proposition 5.4. Chebyshev's inequality says, for any $\varepsilon > 0$,

$$
\mathbb{P}\left(\left|\frac{L_n}{n} - \frac{\mathbb{E}\,L_n}{n}\right| > \varepsilon\right) \leq \frac{\mathbb{V}\mathrm{ar}(n^{-1}L_n)}{\varepsilon^2} \leq \frac{C}{\varepsilon^2 n}.
$$

Take $n = n_k = k^2$, then

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\left| \frac{L_{n_k}}{n_k} - \frac{\mathbb{E}\, L_{n_k}}{n_k} \right| > \varepsilon \right) \le \frac{C}{\varepsilon^2} \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty.$$

So the Borel–Cantelli lemma (see Lemma 2.3) implies that $|n_k^{-1} L_{n_k} - n_k^{-1} \mathbb{E}\, L_{n_k}| \to 0$ a.s. as $k \to \infty$. Hence

$$\left| \frac{L_{n_k}}{n_k} - 2\|\mu\| \right| \le \left| \frac{L_{n_k}}{n_k} - \frac{\mathbb{E}\, L_{n_k}}{n_k} \right| + \left| \frac{\mathbb{E}\, L_{n_k}}{n_k} - 2\|\mu\| \right| \to 0 \text{ a.s. as } k \to 0.$$

For any n, let $k = \lfloor \sqrt{n} \rfloor$. Then $n_k \le n < n_{k+1}$. Since L_n is non-decreasing in n by (2.8), we have

$$\frac{L_n}{n} \le \frac{L_{n_{k+1}}}{n} \le \frac{L_{n_{k+1}}}{n_{k+1}} \cdot \frac{n_{k+1}}{n} \le \frac{L_{n_{k+1}}}{n_{k+1}} \cdot \frac{n_{k+1}}{n_k},$$

and also

$$\frac{L_n}{n} \ge \frac{L_{n_k}}{n} \ge \frac{L_{n_k}}{n_k} \cdot \frac{n_k}{n} \ge \frac{L_{n_k}}{n_k} \cdot \frac{n_k}{n_{k+1}}.$$

Then as $n \to \infty$, $k \to \infty$ so

$$\frac{L_{n_k}}{n_k} \overset{a.s.}{\to} 2\|\mu\| \quad \text{and} \quad \frac{n_k}{n_{k+1}} = \frac{(\lfloor \sqrt{n} \rfloor)^2}{(\lfloor \sqrt{n} \rfloor + 1)^2} \to 1.$$

Therefore $n^{-1} L_n \to 2\|\mu\|$ a.s. □

Proposition 5.5 says that if $\mathbb{E}\,[\|Z_1\|^2] < \infty$ and $\mu = 0$, then $n^{-1} L_n \to 0$ a.s. But Proposition 4.7 says that $\mathbb{E}\, L_n = O(n^{1/2})$, so we might expect to be able to improve on this 'law of large numbers'. Indeed, we have the following.

Proposition 5.6. *Suppose* $\mathbb{E}\,[\|Z_1\|^2] < \infty$.

(i) For any $\alpha > 1/2$, *as* $n \to \infty$,

$$\frac{L_n - \mathbb{E}\, L_n}{n^\alpha} \to 0, \text{ in probability.}$$

(ii) If, in addition, $\mu = 0$, *then for any* $\alpha > 1/2$, $n^{-\alpha} L_n \to 0$ *a.s. as* $n \to \infty$.

Proof. Similarly to the proof of Proposition 5.5, Chebyshev's inequality gives, for $\varepsilon > 0$,

$$\mathbb{P}\left(\frac{|L_n - \mathbb{E}\, L_n|}{n^\alpha} > \varepsilon \right) \le \frac{C}{\varepsilon^2} n^{1-2\alpha}. \tag{5.7}$$

The right-hand side here tends to 0 as $n \to \infty$ provided $\alpha > 1/2$, giving (i).

For part (ii), take $n = n_k = 2^k$ in (5.7). Then

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{|L_{n_k} - \mathbb{E}\, L_{n_k}|}{n_k^{\alpha}} > \varepsilon\right) < \infty,$$

provided $\alpha > 1/2$. So

$$\lim_{k\to\infty} \frac{|L_{n_k} - \mathbb{E}\, L_{n_k}|}{n_k^{\alpha}} = 0, \text{ a.s.}$$

But

$$\lim_{k\to\infty} \frac{\mathbb{E}\, L_{n_k}}{n_k^{\alpha}} = \lim_{n\to\infty} \frac{\mathbb{E}\, L_n}{n^{\alpha}} = 0,$$

by Proposition 4.7, and hence

$$\lim_{k\to\infty} \frac{L_{n_k}}{n_k^{\alpha}} = 0, \text{ a.s.}$$

For every positive integer n, there exists $k(n) \in \mathbb{Z}_+$ for which $2^{k(n)} \le n < 2^{k(n)+1}$ and $k(n) \to \infty$ as $n \to \infty$. Hence, by (2.8),

$$\frac{L_n}{n^{\alpha}} \le \frac{L_{2^{k(n)+1}}}{(2^{k(n)})^{\alpha}} = 2^{\alpha} \frac{L_{2^{k(n)+1}}}{(2^{k(n)+1})^{\alpha}},$$

which tends to 0 a.s. as $n \to \infty$. □

Moreover, $(L_n - \mathbb{E}\, L_n)n^{-\alpha}$ in Proposition 5.6(i) is also convergent to 0 almost surely, if we assume $\|Z_1\|$ is upper bounded by some constant. To show this, we need to use Azuma–Hoeffding inequality (see Lemma 2.18).

Lemma 5.7. *Assume $\|Z_1\| \le B$ a.s. for some constant B. Then, for any $t > 0$,*

$$\mathbb{P}\left(|L_n - \mathbb{E}\, L_n| > t\right) \le 2 \exp\left(-\frac{t^2}{8\pi^2 B^2 n}\right).$$

Proof. Let $D_{n,i} = \mathbb{E}\,[L_n - L_n^{(i)} | \mathcal{F}_i]$, where $\mathcal{F}_0$ denote the trivial σ-algebra, and for $i \in \mathbb{N}$, $\mathcal{F}_i = \sigma(Z_1, \dots, Z_i)$ is the σ-algebra generated by the first n steps of the random walk. So $D_{n,i}$ is $\mathcal{F}_i$-measurable. Since $L_n^{(i)}$ is independent of Z_i,

$$\mathbb{E}\,[L_n^{(i)} | \mathcal{F}_i] = \mathbb{E}\,[L_n^{(i)} | \mathcal{F}_{i-1}] = \mathbb{E}\,[L_n | \mathcal{F}_{i-1}],$$

so that $D_{n,i} = \mathbb{E}\,[L_n | \mathcal{F}_i] - \mathbb{E}\,[L_n | \mathcal{F}_{i-1}]$. Hence, $\mathbb{E}\,[D_{n,i} | \mathcal{F}_{i-1}] = 0$.

By using equation (5.5) and our assumption that $\|Z_1\| \leq B$ a.s., we can deduce an upper bound for $|D_{n,i}|$ as follows.

$$|D_{n,i}| \leq \mathbb{E}\left[\int_0^\pi |\Delta_n^{(i)}(\theta)|\mathrm{d}\theta \Big| \mathcal{F}_i\right] \leq \pi(\|Z_i\| + \|Z_i'\|) \leq 2\pi B.$$

Hence, the result follows Lemma 2.18 with $d_\infty = 2\pi B$. $\square$

Proposition 5.8. *Suppose $\|Z_1\| \leq B$ for some constant B. Then for any $\alpha > 1/2$,*

$$\frac{L_n - \mathbb{E}\, L_n}{n^\alpha} \to 0 \text{ a.s.}$$

Proof. The result follows Lemma 5.7 by using Borel–Cantelli Lemma (see Lemma 2.3). $\square$

5.4 Central limit theorem for the non-zero drift case

5.4.1 Control of extrema

For the remainder of this section, without loss of generality, we suppose that $\mathbb{E}[Z_1] = \mu \mathbf{e}_{\pi/2}$ with $\mu \in (0, \infty)$. Observe that $(S_j \cdot \mathbf{e}_\theta; 0 \leq j \leq n)$ is a one-dimensional random walk: indeed, $S_j \cdot \mathbf{e}_\theta = \sum_{k=1}^j Z_k \cdot \mathbf{e}_\theta$. The mean drift of this one-dimensional random walk is

$$\mathbb{E}[Z_1 \cdot \mathbf{e}_\theta] = \mathbb{E}[Z_1] \cdot \mathbf{e}_\theta = \mu \sin\theta. \tag{5.8}$$

Note that the drift $\mu \sin\theta$ is positive if $\theta \in (0, \pi)$. This crucial fact gives us control over the behaviour of the extrema such as $M_n(\theta)$ and $m_n(\theta)$ that contribute to (5.4), and this will allow us to estimate the conditional expectation of the final term in (5.4) (see Lemma 5.10 below).

For $\gamma \in (0, 1/2)$ and $\delta \in (0, \pi/2)$ (two constants that will be chosen to be suitably small later in our arguments), we denote by $E_{n,i}(\delta, \gamma)$ the event that the following occur:

- for all $\theta \in [\delta, \pi - \delta]$, $\underline{J}_n(\theta) < \gamma n$ and $\bar{J}_n(\theta) > (1 - \gamma)n$;
- for all $\theta \in [\delta, \pi - \delta]$, $\underline{J}_n^{(i)}(\theta) < \gamma n$ and $\bar{J}_n^{(i)}(\theta) > (1 - \gamma)n$.

We write $E^{\mathrm{c}}_{n,i}(\delta,\gamma)$ for the complement of $E_{n,i}(\delta,\gamma)$. The idea is that $E_{n,i}(\delta,\gamma)$ will occur with high probability, and on this event we have good control over $\Delta_n^{(i)}(\theta)$. The next result formalizes these assertions. For $\gamma \in (0,1/2)$, define $I_{n,\gamma} := \{1,\dots,n\} \cap [\gamma n, (1-\gamma)n]$.

Lemma 5.9. *For any $\gamma \in (0,1/2)$ and any $\delta \in (0,\pi/2)$, the following hold.*

(i) If $i \in I_{n,\gamma}$, then, a.s., for any $\theta \in [\delta, \pi-\delta]$,

$$\Delta_n^{(i)}(\theta)\,\mathbf{1}(E_{n,i}(\delta,\gamma)) = (Z_i - Z_i') \cdot \mathbf{e}_\theta\,\mathbf{1}(E_{n,i}(\delta,\gamma)). \tag{5.9}$$

(ii) If $\mathbb{E}\,\|Z_1\| < \infty$ and $\|\mathbb{E}\,[Z_1]\| \neq 0$, then $\min_{1 \le i \le n} \mathbb{P}[E_{n,i}(\delta,\gamma)] \to 1$ as $n \to \infty$.

Proof. First we prove part (i). Suppose that $i \in I_{n,\gamma}$, so $\gamma n \le i \le (1-\gamma)n$. Suppose that $\theta \in [\delta, \pi-\delta]$. Then on $E_{n,i}(\delta,\gamma)$, we have $\underline{J}_n(\theta) < i < \bar{J}_n(\theta)$ and $\underline{J}_n^{(i)}(\theta) < i < \bar{J}_n^{(i)}(\theta)$. Then from (5.2) it follows that in fact $\underline{J}_n(\theta) = \underline{J}_n^{(i)}(\theta)$ and $\bar{J}_n(\theta) = \bar{J}_n^{(i)}(\theta)$. Hence $m_n(\theta) = m_n^{(i)}(\theta)$ and

$$M_n^{(i)}(\theta) = S^{(i)}_{\bar{J}_n(\theta)} \cdot \mathbf{e}_\theta = M_n(\theta) + (Z_i' - Z_i) \cdot \mathbf{e}_\theta, \text{ by (5.2)}.$$

Equation (5.9) follows.

Next we prove part (ii). Suppose that $\mu = \|\mathbb{E}\,[Z_1]\| > 0$. Since $\mathbb{E}\,\|Z_1\| < \infty$, the strong law of large numbers implies that $\|n^{-1}S_n - \mathbb{E}\,[Z_1]\| \to 0$, a.s., as $n \to \infty$. In other words, for any $\varepsilon_1 > 0$, there exists $N := N(\varepsilon_1)$ such that $\mathbb{P}[N < \infty] = 1$ and $\|n^{-1}S_n - \mathbb{E}\,[Z_1]\| < \varepsilon_1$ for all $n \ge N$. In particular, for $n \ge N$, by (5.8),

$$\left|n^{-1}S_n \cdot \mathbf{e}_\theta - \mu \sin\theta\right| = \left|n^{-1}S_n \cdot \mathbf{e}_\theta - \mathbb{E}\,[Z_1] \cdot \mathbf{e}_\theta\right| \le \left\|n^{-1}S_n - \mathbb{E}\,[Z_1]\right\| < \varepsilon_1, \tag{5.10}$$

for all $\theta \in [0, 2\pi)$.

Take $\varepsilon_1 < \mu \sin\delta$. If $n \ge N$, then, by (5.10),

$$S_n \cdot \mathbf{e}_\theta > (\mu \sin\theta - \varepsilon_1)n \ge (\mu\sin\delta - \varepsilon_1)n,$$

provided $\theta \in [\delta, \pi-\delta]$. By choice of ε_1, the last term in the previous display is strictly positive. Hence, for $n \ge N$, for any $\theta \in [\delta, \pi-\delta]$, $S_n \cdot \mathbf{e}_\theta > 0$. But, $S_0 \cdot \mathbf{e}_\theta = 0$. So $\underline{J}_n(\theta) < N$ for all $\theta \in [\delta, \pi-\delta]$, and

$$\mathbb{P}\left[\cap_{\theta \in [\delta, \pi-\delta]} \{\underline{J}_n(\theta) < \gamma n\}\right] \ge \mathbb{P}[N < \gamma n] \to 1,$$

as $n \to \infty$, since $N < \infty$ a.s.

Now,

$$\max_{0 \le j \le (1-\gamma)n} S_j \cdot \mathbf{e}_\theta \le \max \left\{ \max_{0 \le j \le N} S_j \cdot \mathbf{e}_\theta, \max_{N \le j \le (1-\gamma)n} S_j \cdot \mathbf{e}_\theta \right\}. \tag{5.11}$$

For the final term on the right-hand side of (5.11), (5.10) implies that

$$\max_{N \le j \le (1-\gamma)n} S_j \cdot \mathbf{e}_\theta \le \max_{0 \le j \le (1-\gamma)n} (\mu \sin\theta + \varepsilon_1) j \le (\mu \sin\theta + \varepsilon_1)(1-\gamma)n.$$

On the other hand, if $n \ge N$, then (5.10) implies that $S_n \cdot \mathbf{e}_\theta \ge (\mu \sin\theta - \varepsilon_1)n$. Here

$$\mu \sin\theta - \varepsilon_1 \ge (\mu \sin\theta + \varepsilon_1)(1-\gamma) \text{ if } \varepsilon_1 < \frac{\gamma\mu\sin\theta}{2-\gamma}.$$

Now we choose $\varepsilon_1 < \frac{\gamma\mu\sin\delta}{2}$. Then, for any $\theta \in [\delta, \pi-\delta]$, we have that, for $n \ge N$,

$$S_n \cdot \mathbf{e}_\theta > \max_{N \le j \le (1-\gamma)n} S_j \cdot \mathbf{e}_\theta.$$

Hence, by (5.11),

$$\begin{aligned} \mathbb{P}\left[\cap_{\theta\in[\delta,\pi-\delta]} \{\bar{J}_n(\theta) > (1-\gamma)n\}\right] &\ge \mathbb{P}\left[\cap_{\theta\in[\delta,\pi-\delta]} \left\{ S_n \cdot \mathbf{e}_\theta > \max_{0\le j\le(1-\gamma)n} S_j \cdot \mathbf{e}_\theta \right\}\right] \\ &\ge \mathbb{P}\left[N \le n, \cap_{\theta\in[\delta,\pi-\delta]} \left\{ S_n \cdot \mathbf{e}_\theta > \max_{0\le j\le N} S_j \cdot \mathbf{e}_\theta \right\}\right]. \end{aligned}$$

Also, for $n \ge N$, $S_n \cdot \mathbf{e}_\theta > (1 - \frac{\gamma}{2})\mu n \sin\delta$, so we obtain

$$\mathbb{P}\left[\cap_{\theta\in[\delta,\pi-\delta]} \{\bar{J}_n(\theta) > (1-\gamma)n\}\right] \ge \mathbb{P}\left[N \le n, \max_{0\le j\le N} \|S_j\| \le \left(1 - \frac{\gamma}{2}\right) \mu n \sin\delta \right],$$

using the fact that $\max_{0\le j\le N} S_j \cdot \mathbf{e}_\theta \le \max_{0\le j\le N} \|S_j\|$ for all θ.

Now, as $n \to \infty$, $\mathbb{P}[N > n] \to 0$, and

$$\mathbb{P}\left[\max_{0\le j\le N} \|S_j\| > \left(1 - \frac{\gamma}{2}\right) \mu n \sin\delta \right] \to 0,$$

since $N < \infty$ a.s. So we conclude that

$$\mathbb{P}\left[\cap_{\theta\in[\delta,\pi-\delta]} \{\underline{J}_n(\theta) < \gamma n, \bar{J}_n(\theta) > (1-\gamma)n\}\right] \to 1,$$

as $n \to \infty$, and the same result holds for $\underline{J}_n^{(i)}(\theta)$ and $\bar{J}_n^{(i)}(\theta)$, uniformly in $i \in \{1, \dots, n\}$, since resampling Z_i does not change the distribution of the trajectory.

$\square$

5.4.2 Approximation for the martingale differences

The following result is a key component to our proof. Recall that $D_{n,i} = \mathbb{E}\,[L_n - L_n^{(i)} \mid \mathcal{F}_i]$.

Lemma 5.10. *Suppose that* $\mathbb{E}\,\|Z_1\| < \infty$, $\gamma \in (0, 1/2)$, *and* $\delta \in (0, \pi/2)$. *For any* $i \in I_{n,\gamma}$,

$$\left| D_{n,i} - \frac{2(Z_i - \mathbb{E}\,[Z_1]) \cdot \mathbb{E}\,[Z_1]}{\|\mathbb{E}\,[Z_1]\|} \right| \leq 4\delta \|Z_i\| + 4\delta \mathbb{E}\,\|Z_1\| + 3\pi \|Z_i\| \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] + 3\pi \mathbb{E}\,[\|Z_i'\|\, \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i], \text{ a.s.} \tag{5.12}$$

Proof. Taking (conditional) expectations in (5.4), we obtain

$$D_{n,i} = \int_0^\pi \mathbb{E}\,[\Delta_n^{(i)}(\theta)\, \mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta + \int_0^\pi \mathbb{E}\,[\Delta_n^{(i)}(\theta)\, \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta. \tag{5.13}$$

For the second term on the right-hand side of (5.13), we have

$$\left| \int_0^\pi \mathbb{E}\,[\Delta_n^{(i)}(\theta)\, \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta \right| \leq \int_0^\pi \mathbb{E}\,[|\Delta_n^{(i)}(\theta)|\, \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta. \tag{5.14}$$

Applying the bound (5.5), we obtain

$$\begin{aligned} \int_0^\pi \mathbb{E}\,[|\Delta_n^{(i)}(\theta)|\, \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta &\leq \pi \mathbb{E}\,[(\|Z_i\| + \|Z_i'\|)\, \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \\ &= \pi \|Z_i\| \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] + \pi \mathbb{E}\,[\|Z_i'\|\, \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i], \end{aligned} \tag{5.15}$$

since Z_i is $\mathcal{F}_i$-measurable with $\mathbb{E}\,\|Z_i\| < \infty$.

We decompose the first integral on the right-hand side of (5.13) as $I_1 + I_2 + I_3$, where

$$\begin{aligned} I_1 &:= \int_0^\delta \mathbb{E}\,[\Delta_n^{(i)}(\theta)\, \mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta, \\ I_2 &:= \int_\delta^{\pi-\delta} \mathbb{E}\,[\Delta_n^{(i)}(\theta)\, \mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta, \\ I_3 &:= \int_{\pi-\delta}^{\pi} \mathbb{E}\,[\Delta_n^{(i)}(\theta)\, \mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \mathrm{d}\theta. \end{aligned}$$

First we deal with I_1 and I_3. We have

$$|I_1| \leq \int_0^\delta \mathbb{E}\,[|\Delta_n^{(i)}(\theta)| \mid \mathcal{F}_i] \mathrm{d}\theta \leq \delta \mathbb{E}\,[\|Z_i\| + \|Z_i'\| \mid \mathcal{F}_i], \text{ a.s.},$$

by another application of (5.5). Here $\mathbb{E}[\|Z_i\| \mid \mathcal{F}_i] = \|Z_i\|$, since Z_i is $\mathcal{F}_i$-measurable, and, since Z_i' is independent of $\mathcal{F}_i$, $\mathbb{E}[\|Z_i'\| \mid \mathcal{F}_i] = \mathbb{E}\|Z_i'\| = \mathbb{E}\|Z_1\|$. A similar argument applies to I_3, so that

$$|I_1 + I_3| \leq 2\delta\|Z_i\| + 2\delta\mathbb{E}\|Z_1\|, \text{ a.s.} \tag{5.16}$$

We now consider I_2. From (5.9), since $i \in I_{n,\gamma}$, we have

$$\begin{aligned} I_2 &= \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mathbf{1}(E_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta \\ &= \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta - \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta. \end{aligned}$$

Here, by the triangle inequality,

$$\begin{aligned} &\left| \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta \right| \\ &\leq \int_0^\pi \mathbb{E}[(\|Z_i\| + \|Z_i'\|)\mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i]\mathrm{d}\theta \\ &= \pi\|Z_i\|\mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] + \pi\mathbb{E}[\|Z_i'\|\mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i], \end{aligned} \tag{5.17}$$

similarly to (5.15). Finally, similarly to (5.16),

$$\begin{aligned} &\left| \int_\delta^{\pi-\delta} \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta - \int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta \right| \\ &\qquad \leq 2\delta\mathbb{E}[\|Z_i\| + \|Z_i'\| \mid \mathcal{F}_i] = 2\delta(\|Z_i\| + \mathbb{E}\|Z_1\|). \end{aligned} \tag{5.18}$$

We combine (5.13) with (5.14) and the bounds in (5.15)–(5.18) to give

$$\begin{aligned} \left| D_{n,i} - \int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta \right| &\leq 4\delta\|Z_i\| + 4\delta\mathbb{E}\|Z_1\| + 3\pi\|Z_i\|\mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] \\ &\quad + 3\pi\mathbb{E}[\|Z_i'\|\mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i], \text{ a.s.} \end{aligned} \tag{5.19}$$

To complete the proof of the lemma, we compute the integral on the left-hand side of (5.19). First note that $\mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i] = (Z_i - \mathbb{E}[Z_i']) \cdot \mathbf{e}_\theta$, since Z_i is $\mathcal{F}_i$-measurable and Z_i' is independent of $\mathcal{F}_i$, so that

$$\int_0^\pi \mathbb{E}[(Z_i - Z_i') \cdot \mathbf{e}_\theta \mid \mathcal{F}_i]\mathrm{d}\theta = \int_0^\pi (Z_i - \mathbb{E}[Z_i]) \cdot \mathbf{e}_\theta \mathrm{d}\theta.$$

To evaluate the last integral, it is convenient to introduce the notation $Z_i - \mathbb{E}[Z_i] = R_i\mathbf{e}_{\Theta_i}$ where $R_i = \|Z_i - \mathbb{E}[Z_i]\| \geq 0$ and $\Theta_i \in [0, 2\pi)$. Then

$$\int_0^\pi (Z_i - \mathbb{E}[Z_i]) \cdot \mathbf{e}_\theta \mathrm{d}\theta = \int_0^\pi R_i\mathbf{e}_{\Theta_i} \cdot \mathbf{e}_\theta \mathrm{d}\theta = R_i \int_0^\pi \cos(\theta - \Theta_i)\mathrm{d}\theta$$

$$= 2R_i \sin\Theta_i = 2R_i \mathbf{e}_{\Theta_i} \cdot \mathbf{e}_{\pi/2}.$$

Now (5.12) follows from (5.19), and the proof is complete. □

5.4.3 Proofs for the central limit theorem

For ease of notation, we write $Y_i := 2\|\mathbb{E}[Z_1]\|^{-1}(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]$, and define

$$W_{n,i} := D_{n,i} - Y_i.$$

The upper bound for $|W_{n,i}|$ in Lemma 5.10 together with Lemma 5.9(ii) will enable us to prove the following result, which will be the basis of our proof of Theorem 5.12.

Lemma 5.11. *Suppose that* $\mathbb{E}[\|Z_1\|^2] < \infty$ *and* $\|\mathbb{E}[Z_1]\| \neq 0$. *Then*

$$\lim_{n\to\infty} n^{-1} \sum_{i=1}^{n} \mathbb{E}[W_{n,i}^2] = 0.$$

Proof. Fix $\varepsilon > 0$. We take $\gamma \in (0, 1/2)$ and $\delta \in (0, \pi/2)$, to be specified later. We divide the sum of interest into two parts, namely $i \in I_{n,\gamma}$ and $i \notin I_{n,\gamma}$. Now from (5.4) with (5.5) we have $|L_n^{(i)} - L_n| \leq \pi(\|Z_i\| + \|Z_i'\|)$, a.s., so that

$$|D_{n,i}| \leq \pi \mathbb{E}[\|Z_i\| + \|Z_i'\| \mid \mathcal{F}_i] = \pi(\|Z_i\| + \mathbb{E}\|Z_i\|).$$

It then follows from the triangle inequality that

$$|W_{n,i}| \leq |D_{n,i}| + 2\|Z_i - \mathbb{E}[Z_i]\| \leq (\pi + 2)(\|Z_i\| + \mathbb{E}\|Z_i\|).$$

So provided $\mathbb{E}[\|Z_1\|^2] < \infty$, we have $\mathbb{E}[W_{n,i}^2] \leq C_0$ for all n and all i, for some constant $C_0 < \infty$, depending only on the distribution of Z_1. Hence

$$\frac{1}{n} \sum_{i \notin I_{n,\gamma}} \mathbb{E}[W_{n,i}^2] \leq \frac{1}{n} 2\gamma n C_0 = 2\gamma C_0,$$

using the fact that there are at most $2\gamma n$ terms in the sum. From now on, choose $\gamma > 0$ small enough so that $2\gamma C_0 < \varepsilon$.

Now consider $i \in I_{n,\gamma}$. For such i, (5.12) shows that, for some constant $C_1 < \infty$,

$$|W_{n,i}| \leq C_1(1 + \|Z_i\|)\delta + C_1\|Z_i\| \mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta, \gamma) \mid \mathcal{F}_i]$$

$$+ C_1 \mathbb{E}\,[\|Z_i'\|\,\mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i], \text{ a.s.} \tag{5.20}$$

Here, for any $B_1 \in (0,\infty)$, a.s.,

$$\begin{aligned}\mathbb{E}\,[\|Z_i'\|\,\mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] &\leq \mathbb{E}\,[\|Z_i'\|\,\mathbf{1}\{\|Z_i'\| > B_1\} \mid \mathcal{F}_i] + B_1 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] \\ &= \mathbb{E}\,[\|Z_i'\|\,\mathbf{1}\{\|Z_i'\| > B_1\}] + B_1 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i],\end{aligned}$$

since Z_i' is independent of $\mathcal{F}_i$. Here, since $\mathbb{E}\,\|Z_i'\| = \mathbb{E}\,\|Z_1\| < \infty$, the dominated convergence theorem (see Lemma 2.1) implies that $\mathbb{E}\,[\|Z_i'\|\,\mathbf{1}\{\|Z_i'\| > B_1\}] \to 0$ as $B_1 \to \infty$. So we can choose $B_1 = B_1(\delta)$ large enough so that

$$\mathbb{E}\,[\|Z_i'\|\,\mathbf{1}(E^{\mathrm{c}}_{n,i}(\delta,\gamma)) \mid \mathcal{F}_i] \leq \delta + B_1 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i], \text{ a.s.}$$

Combining this with (5.20) we see that there is a constant $C_2 < \infty$ for which

$$|W_{n,i}| \leq C_2(1 + \|Z_i\|)\left(\delta + B_1 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i]\right), \text{ a.s.}$$

Hence

$$\begin{aligned}W_{n,i}^2 &\leq C_2^2(1+\|Z_i\|)^2\left(\delta^2 + 2B_1\delta\mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] + B_1^2 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i]^2\right) \\ &\leq C_3^2(1+\|Z_i\|)^2\left(\delta + B_1^2 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i]\right),\end{aligned}$$

for some constant $C_3 < \infty$, using the facts that $\delta < \pi/2 < 2$ and $\mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] \leq 1$. Taking expectations we get

$$\mathbb{E}\,[W_{n,i}^2] \leq C_3^2 \delta \mathbb{E}\,[(1+\|Z_i\|)^2] + C_3^2 B_1^2 \mathbb{E}\,\left[(1+\|Z_i\|)^2 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i]\right].$$

Provided $\mathbb{E}\,[\|Z_1\|^2] < \infty$, there is a constant $C_4 < \infty$ such that the first term on the right-hand side of the last display is bounded by $C_4\delta$. Now fix $\delta > 0$ small enough so that $C_4\delta < \varepsilon$; this choice also fixes B_1. Then

$$\mathbb{E}\,[W_{n,i}^2] \leq \varepsilon + C_3^2 B_1^2 \mathbb{E}\,\left[(1+\|Z_i\|)^2 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i]\right]. \tag{5.21}$$

For the final term in (5.21), observe that, for any $B_2 \in (0,\infty)$, a.s.,

$$\begin{aligned}(1+\|Z_i\|)^2 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] &\leq (1+B_2)^2 \mathbb{P}[E^{\mathrm{c}}_{n,i}(\delta,\gamma) \mid \mathcal{F}_i] \\ &\quad + (1+\|Z_i\|)^2\,\mathbf{1}\{\|Z_i\| > B_2\}.\end{aligned} \tag{5.22}$$

Here $\mathbb{E}[(1+\|Z_i\|)^2 \mathbf{1}\{\|Z_i\| > B_2\}] \to 0$ as $B_2 \to \infty$, provided $\mathbb{E}[\|Z_1\|^2] < \infty$, by the dominated convergence theorem. Hence, since δ and B_1 are fixed, we can choose $B_2 = B_2(\varepsilon) \in (0,\infty)$ such that

$$C_3^2 B_1^2 \mathbb{E}\left[(1+\|Z_i\|)^2 \mathbf{1}\{\|Z_i\| > B_2\}\right] < \varepsilon.$$

Then taking expectations in (5.22) we obtain from (5.21) that

$$\mathbb{E}[W_{n,i}^2] \le 2\varepsilon + C_3^2 B_1^2 (1+B_2)^2 \mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)].$$

Now choose n_0 such that $C_3^2 B_1^2 (1+B_2)^2 \mathbb{P}[E_{n,i}^{\mathrm{c}}(\delta,\gamma)] < \varepsilon$ for all $n \ge n_0$, which we may do by Lemma 5.9(ii). So for the given $\varepsilon > 0$ and $\gamma \in (0,1/2)$, we can choose n_0 such that for all $i \in I_{n,\gamma}$ and all $n \ge n_0$, $\mathbb{E}[W_{n,i}^2] \le 3\varepsilon$. Hence

$$\frac{1}{n}\sum_{i \in I_{n,\gamma}} \mathbb{E}[W_{n,i}^2] \le 3\varepsilon,$$

for all $n \ge n_0$.

Combining the estimates for $i \in I_{n,\gamma}$ and $i \notin I_{n,\gamma}$, we see that

$$\frac{1}{n}\sum_{i=1}^{n} \mathbb{E}[W_{n,i}^2] \le 2\gamma C_0 + 3\varepsilon \le 4\varepsilon,$$

for all $n \ge n_0$. Since $\varepsilon > 0$ was arbitrary, the result follows. □

Now we can claim and prove our main theorems.

Theorem 5.12. *Suppose that $\mathbb{E}[\|Z_1\|^2] < \infty$ and $\|\mathbb{E}[Z_1]\| \neq 0$. Then, as $n \to \infty$,*

$$n^{-1/2}\left| L_n - \mathbb{E}[L_n] - \sum_{i=1}^{n} \frac{2(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]}{\|\mathbb{E}[Z_1]\|} \right| \to 0, \ \text{in } L^2.$$

Proof. First note that

$$\mathbb{E}[W_{n,i} \mid \mathcal{F}_{i-1}] = \mathbb{E}[D_{n,i} \mid \mathcal{F}_{i-1}] - \mathbb{E}[Y_i \mid \mathcal{F}_{i-1}] = 0 - \mathbb{E}[Y_i],$$

since $D_{n,i}$ is a martingale difference sequence and Y_i is independent of $\mathcal{F}_{i-1}$. Here, by definition, $\mathbb{E}[Y_i] = 0$, and so $W_{n,i}$ is also a martingale difference sequence. Therefore, by orthogonality,

$$n^{-1}\mathbb{E}\left[\left(\sum_{i=1}^{n} W_{n,i}\right)^2\right] = n^{-1}\sum_{i=1}^{n} \mathbb{E}\left[W_{n,i}^2\right] \to 0 \text{ as } n \to \infty, \text{ by Lemma 5.11.}$$

In other words, $n^{-1/2}\sum_{i=1}^{n} W_{n,i} \to 0$ in L^2, which, with Lemma 5.1(i), implies the statement in the theorem. □

Theorem 5.13. *Suppose that* $\mathbb{E}[\|Z_1\|^2] < \infty$ *and* $\|\mathbb{E}[Z_1]\| \neq 0$. *Then*

$$\lim_{n\to\infty} n^{-1}\mathbb{V}\mathrm{ar}[L_n] = \frac{4\mathbb{E}[((Z_1 - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1])^2]}{\|\mathbb{E}[Z_1]\|^2} = 4\sigma_\mu^2. \tag{5.23}$$

Remarks 5.2. (i) The assumptions $\mathbb{E}[\|Z_1\|^2] < \infty$ and $\|\mathbb{E}[Z_1]\| \neq 0$ ensure $4\sigma_\mu^2 < \infty$.

(ii) To compare the limit result (5.23) with Snyder and Steele's upper bound (5.1), observe that

$$4\sigma_\mu^2 = 4\left(\frac{\mathbb{E}[(Z_1 \cdot \mathbb{E}[Z_1])^2] - \|\mathbb{E}[Z_1]\|^4}{\|\mathbb{E}[Z_1]\|^2}\right) \leq 4\left(\mathbb{E}[\|Z_1\|^2] - \|\mathbb{E}[Z_1]\|^2\right).$$

(iii) The limit $4\sigma_\mu^2$ is zero if and only if $(Z_1 - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1] = 0$ with probability 1, i.e., if $Z_1 - \mathbb{E}[Z_1]$ is always orthogonal to $\mathbb{E}[Z_1]$. In such a degenerate case, (5.23) says that $\mathbb{V}\mathrm{ar}[L_n] = o(n)$. This is the case, for example, if Z_1 takes values $(1,1)$ and $(1,-1)$ each with probability $1/2$. Note that the Snyder–Steele bound (5.1) applied in this example says only that $\mathbb{V}\mathrm{ar}[L_n] \leq (\pi^2/2)n$, which is not the correct order. Here, the two-dimensional trajectory can be viewed as a space-time trajectory of a *one-dimensional* simple symmetric random walk. We conjecture that in fact $\mathbb{V}\mathrm{ar}[L_n] = O(\log n)$. Steele [59] obtains variance results for the *number of faces* of the convex hull of one-dimensional simple random walk, and comments that such results for L_n seem "far out of reach" [59, p. 242].

Proof. Write

$$\xi_n = \frac{L_n - \mathbb{E}[L_n]}{\sqrt{n}}; \quad \text{and} \quad \zeta_n = \frac{1}{\sqrt{n}}\sum_{i=1}^n Y_i, \text{ where } Y_i = \frac{2(Z_i - \mathbb{E}[Z_1]) \cdot \mathbb{E}[Z_1]}{\|\mathbb{E}[Z_1]\|}. \tag{5.24}$$

Then Theorem 5.12 shows that $|\xi_n - \zeta_n| \to 0$ in L^2 as $n \to \infty$. Also, with $4\sigma_\mu^2$ as given by (5.23), $\mathbb{E}[\zeta_n^2] = 4\sigma_\mu^2$. Then a computation shows that

$$n^{-1}\mathbb{V}\mathrm{ar}[L_n] = \mathbb{E}[\xi_n^2] = \mathbb{E}[(\xi_n - \zeta_n)^2] + \mathbb{E}[\zeta_n^2] + 2\mathbb{E}[(\xi_n - \zeta_n)\zeta_n].$$

Here, by the L^2 convergence, $\mathbb{E}[(\xi_n - \zeta_n)^2] \to 0$ and, by the Cauchy–Schwarz inequality (see Lemma 2.14),

$$|\mathbb{E}[(\xi_n - \zeta_n)\zeta_n]| \leq \left(\mathbb{E}[(\xi_n - \zeta_n)^2]\mathbb{E}[\zeta_n^2]\right)^{1/2} \to 0 \text{ as well.}$$

So $\mathbb{E}[\xi_n^2] \to 4\sigma_\mu^2$ as $n \to \infty$. $\square$

In the case where $\mathbb{E}[\|Z_1\|^2] < \infty$ and $\|\mathbb{E}[Z_1]\| = \mu > 0$, Snyder and Steele deduce from their bound (5.1) a strong law of large numbers for L_n, namely $\lim_{n\to\infty} n^{-1}L_n = 2\mu$, a.s. (see [57, p. 1168]). Given this and the variance asymptotics of Theorem 5.13, it is natural to ask whether there is an accompanying central limit theorem. Our next result gives a positive answer in the non-degenerate case, again with essentially minimal assumptions.

In the proof of Theorem 5.14 we will use two facts about convergence in distribution that we now recall (see Lemma 2.4). First, if sequences of random variables ξ_n and ζ_n are such that $\zeta_n \to \zeta$ in distribution for some random variable ζ and $|\xi_n - \zeta_n| \to 0$ in probability, then $\xi_n \to \zeta$ in distribution (this is *Slutsky's theorem*). Second, if $\zeta_n \to \zeta$ in distribution and $\alpha_n \to \alpha$ in probability, then $\alpha_n\zeta_n \to \alpha\zeta$ in distribution.

Theorem 5.14. *Suppose that $\mathbb{E}[\|Z_1\|^2] < \infty$, $\|\mathbb{E}[Z_1]\| \neq 0$ and $\sigma_\mu^2 > 0$. Then for any $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\mathbb{V}\mathrm{ar}[L_n]}} \leq x\right] = \lim_{n\to\infty} \mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{4\sigma_\mu^2 n}} \leq x\right] = \Phi(x), \tag{5.25}$$

where Φ is the standard normal distribution function.

Proof. Use the notation for ξ_n and ζ_n as given by (5.24). Then, by Theorem 5.12, $|\xi_n - \zeta_n| \to 0$ in L^2, and hence in probability.

In the sum ζ_n, the Y_i are i.i.d. random variables with mean 0 and variance $\mathbb{E}[Y_i^2] = 4\sigma_\mu^2$. Hence the classical central limit theorem (see e.g. [17, p. 93]) shows that ζ_n converges in distribution to a normal random variable with mean 0 and variance $4\sigma_\mu^2$. Slutsky's theorem then implies that ξ_n has the same distributional limit. Hence, for any $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{\xi_n}{\sqrt{4\sigma_\mu^2}} \leq x\right] = \lim_{n\to\infty} \mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{4\sigma_\mu^2 n}} \leq x\right] = \Phi(x),$$

where Φ is the standard normal distribution function. Moreover,

$$\mathbb{P}\left[\frac{L_n - \mathbb{E}[L_n]}{\sqrt{\mathbb{V}\mathrm{ar}[L_n]}} \leq x\right] = \mathbb{P}\left[\frac{\xi_n\alpha_n}{\sqrt{4\sigma_\mu^2}} \leq x\right],$$

where $\alpha_n = \sqrt{\frac{4\sigma_\mu^2 n}{\mathbb{V}\mathrm{ar}[L_n]}} \to 1$ by Theorem 5.13. Thus we verify the limit statements in (5.25). □

5.5 Asymptotics for the zero drift case

Recall that h_1 is defined in (3.4) and Σ is a covariance matrix (see Section 3.3), which is positive semidefinite and symmetric. Let

$$u_0(\Sigma) := \mathbb{V}\mathrm{ar}\mathcal{L}(\Sigma^{1/2}h_1), \tag{5.26}$$

we have the following results.

Proposition 5.15. *Suppose that* (M_p) *holds for some* $p > 2$*, and* $\mu = 0$*. Then*

$$\lim_{n\to\infty} n^{-1}\mathbb{V}\mathrm{ar}L_n = u_0(\Sigma).$$

Proof. From (4.3) and Lemma 4.8(ii), for $p > 2$ we have $\sup_n \mathbb{E}\,[(n^{-1}L_n^2)^{p/2}] < \infty$. Hence $n^{-1}L_n^2$ is uniformly integrable, and we deduce convergence of $n^{-1}\mathbb{V}\mathrm{ar}L_n$ in Corollary 3.7. □

The next result gives bounds on $u_0(\Sigma)$ defined in (5.26).

Proposition 5.16.

$$\frac{263}{1080}\pi^{-3/2}\mathrm{e}^{-144/25}\,\mathrm{tr}\,\Sigma \leq u_0(\Sigma) \leq \frac{\pi^2}{2}\,\mathrm{tr}\,\Sigma. \tag{5.27}$$

In addition, if $\Sigma = I$ *we have the following sharper form of the lower bound:*

$$\mathbb{V}\mathrm{ar}\ell_1 = u_0(I) \geq \frac{2}{5}\left(1 - \frac{8}{25\pi}\right)\mathrm{e}^{-25\pi/16} > 0.$$

For the proof of this result, we rely on a few facts about one-dimensional Brownian motion, including the bound (see e.g. equation (2.1) of [33]), valid for all $r > 0$,

$$\mathbb{P}\left[\sup_{0\leq s\leq 1}|w(s)| \leq r\right] \geq \frac{4}{\pi}\left(\mathrm{e}^{-\pi^2/(8r^2)} - \frac{1}{3}\mathrm{e}^{-9\pi^2/(8r^2)}\right). \tag{5.28}$$

We let Φ denote the distribution function of a standard normal random variable; we will also need the standard Gaussian tail bound (see e.g. [17, p. 12])

$$1 - \Phi(x) = \frac{1}{\sqrt{2\pi}}\int_x^\infty \mathrm{e}^{-y^2/2}\mathrm{d}y \geq \frac{1}{x\sqrt{2\pi}}\left(1 - \frac{1}{x^2}\right)\mathrm{e}^{-x^2/2}, \quad \text{for } x > 0. \tag{5.29}$$

We also note that for $e \in \mathbb{S}_1$ the diffusion $e \cdot (\Sigma^{1/2}b)$ is one-dimensional Brownian motion with variance parameter $e^\top\Sigma e$.

The idea behind the variance lower bounds is elementary. For a random variable X with mean $\mathbb{E}\,X$, we have, for any $\theta \geq 0$,

$$\mathbb{V}\mathrm{ar} X = \mathbb{E}\,\left[(X - \mathbb{E}\,X)^2\right] \geq \theta^2 \mathbb{P}\left[|X - \mathbb{E}\,X| \geq \theta\right].$$

If $\mathbb{E}\,X \geq 0$, taking $\theta = \alpha \mathbb{E}\,X$ for $\alpha > 0$, we obtain

$$\mathbb{V}\mathrm{ar} X \geq \alpha^2 (\mathbb{E}\,X)^2 \big(\mathbb{P}[X \leq (1-\alpha)\mathbb{E}\,X] + \mathbb{P}[X \geq (1+\alpha)\mathbb{E}\,X]\big), \tag{5.30}$$

and our lower bounds use whichever of the latter two probabilities is most convenient.

Proof of Proposition 5.16. We start with the upper bounds. Snyder and Steele's bound (5.6) with the statement for $\mathbb{V}\mathrm{ar} L_n$ in Proposition 5.15 gives the upper bound in (5.27).

We now move on to the lower bounds. Let $e_\Sigma \in \mathbb{S}_1$ denote an eigenvector of Σ corresponding to the principal eigenvalue λ_Σ. Then since $\Sigma^{1/2} h_1$ contains the line segment from 0 to any (other) point in $\Sigma^{1/2} h_1$, we have from monotonicity of $\mathcal{L}$ that

$$\mathcal{L}(\Sigma^{1/2} h_1) \geq 2 \sup_{0 \leq s \leq 1} \|\Sigma^{1/2} b(s)\| \geq 2 \sup_{0 \leq s \leq 1} \big(e_\Sigma \cdot (\Sigma^{1/2} b(s))\big).$$

Here $e_\Sigma \cdot (\Sigma^{1/2} b)$ has the same distribution as $\lambda_\Sigma^{1/2} w$. Hence, for $\alpha > 0$,

$$\begin{aligned}\mathbb{P}\left[\mathcal{L}(\Sigma^{1/2} h_1) \geq (1+\alpha)\mathbb{E}\,\mathcal{L}(\Sigma^{1/2} h_1)\right] &\geq \mathbb{P}\left[\sup_{0 \leq s \leq 1} w(s) \geq \frac{1+\alpha}{2} \lambda_\Sigma^{-1/2} \mathbb{E}\,\mathcal{L}(\Sigma^{1/2} h_1)\right] \\ &\geq \mathbb{P}\left[\sup_{0 \leq s \leq 1} w(s) \geq 2(1+\alpha)\sqrt{2}\right],\end{aligned}$$

using the fact that $\lambda_\Sigma \geq \frac{1}{2} \operatorname{tr} \Sigma$ and the upper bound in (4.4). Applying (5.30) to $X = \mathcal{L}(\Sigma^{1/2} h_1) \geq 0$ gives, for $\alpha > 0$,

$$\begin{aligned}\mathbb{V}\mathrm{ar} \mathcal{L}(\Sigma^{1/2} h_1) &\geq \alpha^2 (\mathbb{E}\,\mathcal{L}(\Sigma^{1/2} h_1))^2 \mathbb{P}\left[\sup_{0 \leq s \leq 1} w(s) \geq 2(1+\alpha)\sqrt{2}\right] \\ &\geq \frac{32}{\pi} \alpha^2 (\operatorname{tr} \Sigma) \left(1 - \Phi(2(1+\alpha)\sqrt{2})\right),\end{aligned}$$

using the lower bound in (4.4) and the fact that $\mathbb{P}[\sup_{0 \leq s \leq 1} w(s) \geq r] = 2\mathbb{P}[w(1) \geq r] = 2(1 - \Phi(r))$ for $r > 0$, which is a consequence of the reflection principle.

Numerical curve sketching suggests that $\alpha = 1/5$ is close to optimal; this choice of α gives, using (5.29),

$$\mathbb{V}\mathrm{ar}\mathcal{L}(\Sigma^{1/2}h_1) \geq \frac{32}{25\pi}\,(\mathrm{tr}\,\Sigma)\left(1-\Phi(12\sqrt{2}/5)\right) \geq \frac{263}{1080}\pi^{-3/2}\,(\mathrm{tr}\,\Sigma)\exp\left\{-\frac{144}{25}\right\},$$

which is the lower bound in (5.27). We get a sharper result when $\Sigma = I$ and $\mathcal{L}(h_1) = \ell_1$, since we know $\mathbb{E}\,\ell_1 = \sqrt{8\pi}$ explicitly. Then, similarly to above, we get

$$\mathbb{V}\mathrm{ar}\ell_1 \geq 8\pi\alpha^2\mathbb{P}\left[\sup_{0\leq s\leq 1} w(s) \geq (1+\alpha)\sqrt{2\pi}\right], \text{ for } \alpha > 0,$$

which at $\alpha = 1/4$ yields the stated lower bound. $\square$

Chapter 6

Results on area of the convex hull

6.1 Overview

The aims of the present chapter are to provide first and second-order information for A_n in both the cases $\mu = 0$ and $\mu \neq 0$. We start by some simulations. We considered the same form of random walk as in Section 5.1.

For the expected area, the simulations (see Figure 6.1) are consistent with Theorem 6.8 and Theorem 6.9. In the case of $\mu = \mathbf{0}$, Theorem 6.8 implies: $\lim_{n\to\infty} n^{-1}\mathbb{E}\, A_n = \frac{\pi}{2}\sqrt{\det \Sigma} = 0.785$. In the case of $\mu \neq 0$, Theorem 6.9 takes the form: $\lim_{n\to\infty} n^{-3/2}\mathbb{E}\, A_n = \frac{1}{3}\|\mu\|\sqrt{2\pi\sigma^2_{\mu_\perp}} = 0.236$ or 0.425.

For the variance of area, Proposition 6.12 and 6.13 show that the limits for variance exist in both zero and non-zero drift cases. For example, we will show that

$$\begin{aligned} \text{if } \mu \neq 0: \quad & \lim_{n\to\infty} n^{-3}\mathbb{V}\mathrm{ar} A_n = v_+\|\mu\|^2\sigma^2_{\mu_\perp}; \\ \text{if } \mu = 0: \quad & \lim_{n\to\infty} n^{-2}\mathbb{V}\mathrm{ar} A_n = v_0 \det \Sigma, \end{aligned} \tag{6.1}$$

where v_0 and v_+ are finite and positive, and these quantities are in fact variances associated with convex hulls of Brownian scaling limits for the walk. These scaling limits provide the basis of the analysis in this chapter; the methods are necessarily quite different from those in [63]. For the constants v_0 and v_+, Table 6.1 gives numerical evaluations of rigorous bounds that we prove in Proposition 6.14 below, plus estimates from simulations. The variance limits we deduced in the simulations (see Figure 6.2) are indeed lie in the variance bounds given by Proposition 6.14.

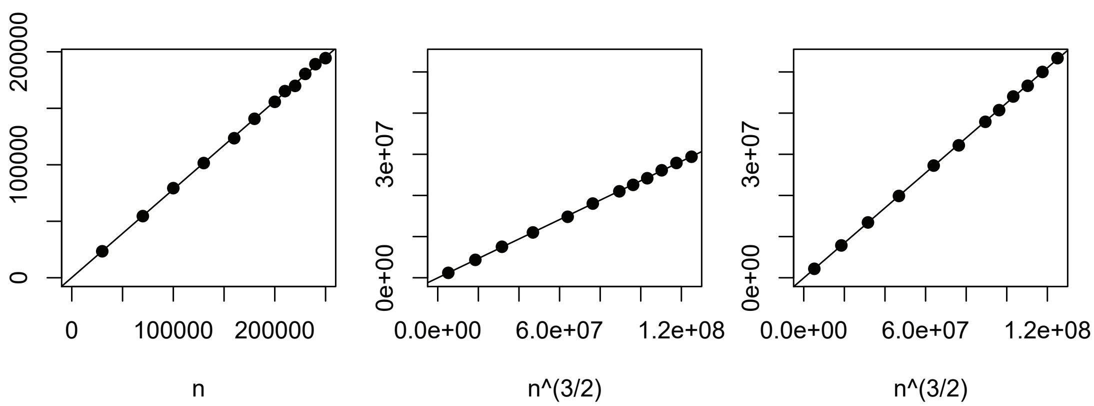

Figure 6.1: Plots of $y = \mathbb{E}[A_n]$ estimates against $x =$ (left to right) n, $n^{3/2}$, $n^{3/2}$ for about 25 values of n in the range 10^2 to 2.5×10^5 for 3 examples with $\|\mu\| =$ (left to right) 0, 0.4, 0.72. Each point is estimated from 10^3 repeated simulations. Also plotted are straight lines $y = 0.781x$ (leftmost plot), $y = 0.236x$ (middle plot) and $y = 0.425x$ (rightmost plot).

	lower bound	simulation estimate	upper bound
v_0	8.15×10^{-7}	0.30	5.22
v_+	1.44×10^{-6}	0.019	2.08

Table 6.1: Each of the simulation estimates is based on 10^5 instances of a walk of length $n = 10^5$. The final decimal digit in each of the numerical upper (lower) bounds has been rounded up (down).

6.2 Upper bound for the expected value and variance for the area

Proposition 6.1. *Let $p \geq 1$. Suppose that $\mathbb{E}[\|Z_1\|^{2p}] < \infty$.*

(i) We have $\mathbb{E}[A_n^p] = O(n^{3p/2})$. Suppose in addition $\mathbb{E}(\|Z_1\|^{4p}) < \infty$, then $\mathbb{V}\mathrm{ar}(A_n^p) = O(n^{3p})$.

(ii) Moreover, if $\mu = 0$ we have $\mathbb{E}[A_n^p] = O(n^p)$. Suppose in addition $\mathbb{E}(\|Z_1\|^{4p}) < \infty$, then $\mathbb{V}\mathrm{ar}(A_n^p) = O(n^{2p})$.

Proof. For part (i), it suffices to suppose $\mu \neq 0$. Then, bounding the convex hull

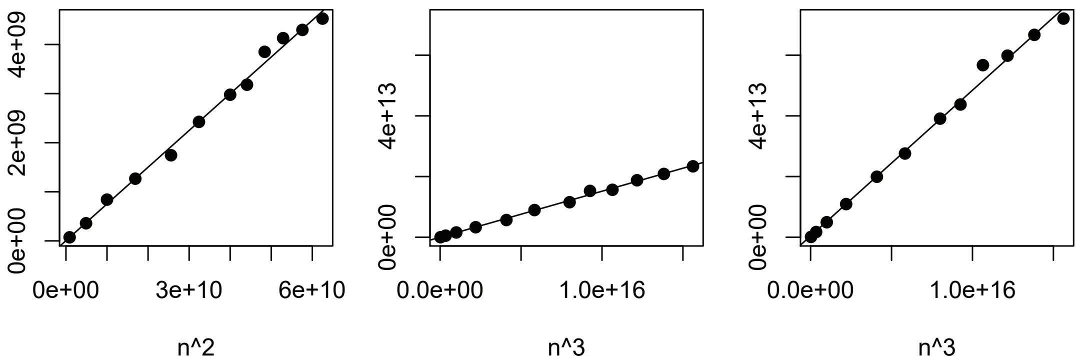

Figure 6.2: Plots of $y = \mathbb{V}\mathrm{ar}[A_n]$ estimates against $x =$ (left to right) n^2, n^3, n^3 for the three examples described in Figure 6.1. Also plotted are straight lines $y = 0.0748x$ (leftmost plot), $y = 0.00152x$ (middle plot) and $y = 0.00480x$ (rightmost plot).

by a rectangle,

$$\begin{aligned} A_n &\leq \left(\max_{0\leq m\leq n} S_m \cdot \hat{\mu} - \min_{0\leq m\leq n} S_m \cdot \hat{\mu} \right) \left(\max_{0\leq m\leq n} S_m \cdot \hat{\mu}_\perp - \min_{0\leq m\leq n} S_m \cdot \hat{\mu}_\perp \right) \\ &\leq 4 \left(\max_{0\leq m\leq n} |S_m \cdot \hat{\mu}| \right) \left(\max_{0\leq m\leq n} |S_m \cdot \hat{\mu}_\perp| \right). \end{aligned}$$

Hence, by the Cauchy–Schwarz inequality, we have

$$\mathbb{E}\,[A_n^p] \leq 4^p \left(\mathbb{E} \left[\max_{0\leq m\leq n} |S_m \cdot \hat{\mu}|^{2p} \right] \right)^{1/2} \left(\mathbb{E} \left[\max_{0\leq m\leq n} |S_m \cdot \hat{\mu}_\perp|^{2p} \right] \right)^{1/2}.$$

Now an application of Proposition 4.8(i) and (iii) gives $\mathbb{E}\,[A_n^p] = O(n^{3p/2})$.

Suppose in addition $\mathbb{E}\,(\|Z_1\|^{4p}) < \infty$. By the same process as above, we have

$$A_n^{2p} \leq 4^{2p} \left(\max_{0\leq m\leq n} |S_m \cdot \hat{\mu}|^{2p} \right) \left(\max_{0\leq m\leq n} |S_m \cdot \hat{\mu}_\perp|^{2p} \right),$$

and $\mathbb{E}\,(A_n^{2p}) = O(n^{3p})$. Hence, $\mathbb{V}\mathrm{ar}(A_n^p) = \mathbb{E}\,(A_n^{2p}) - (\mathbb{E}\,A_n^p)^2 = O(n^{3p})$.

For part (ii), $\mu = 0$. Since the convex hull$(S_0, \ldots, S_n)$ is contained in the disk of radius $\max_{0\leq m\leq n} \|S_m\|$ and centre 0, $A_n^p \leq \pi^p(\max_{0\leq m\leq n} \|S_m\|^{2p})$ a.s. Proposition 4.8(ii) then yields $\mathbb{E}\,[A_n^p] = O(n^p)$.

Suppose in addition $\mathbb{E}\,(\|Z_1\|^{4p}) < \infty$. By the same process as above, we have $\mathbb{E}\,[A_n^{2p}] = O(n^{2p})$. Therefore, $\mathbb{V}\mathrm{ar}(A_n^p) = O(n^{2p})$. □

Remark 6.1. We will show below in Theorem 6.9 $n^{-3/2}\mathbb{E}\,A_n$ has a limit in the non-zero drift case and, in Proposition 6.8, $n^{-1}\mathbb{E}\,A_n$ has a limit in the zero drift case.

6.3 Asymptotics for the expected area

Let $T(\mathbf{u}, \mathbf{v})$ ($\mathbf{u}, \mathbf{v} \in \mathbb{R}^2$) be the area of a triangle with sides of $\mathbf{u}, \mathbf{v}$ and $\mathbf{u} + \mathbf{v}$. Then,

$$T(\mathbf{u}, \mathbf{v}) = \frac{1}{2}\sqrt{\|\mathbf{u}\|^2\|\mathbf{v}\|^2 - (\mathbf{u} \cdot \mathbf{v})^2}.$$

For $\alpha, \beta > 0$, $T(\alpha\mathbf{u}, \beta\mathbf{v}) = \alpha\beta T(\mathbf{u}, \mathbf{v})$.

Lemma 6.2. *Suppose* $\mathbb{E}\left(\|Z_1\|^2\right) < \infty$, $\mathbb{E}\, Z_1 = \mathbf{0}$ *and* $\mathbb{E}\left(Z_1^T Z_1\right) = \Sigma$. *Then as* $m \to \infty$ *and* $(k - m) \to \infty$,

$$\frac{\mathbb{E}\, T(S_m, S_k - S_m)}{\sqrt{m(k-m)}} \to \mathbb{E}\, T(Y_1, Y_2),$$

where Y_1, Y_2 *are iid. rvs.* $Y_1, Y_2 \sim \mathcal{N}(\mathbf{0}, \Sigma)$.

Proof. By Central Limit Theorem in $\mathbb{R}^2$ (see [17]), $n^{-1/2}S_n \xrightarrow{d.} \mathcal{N}(\mathbf{0}, \Sigma)$. Since S_m and $S_k - S_m$ are independent, as m and $k - m \to \infty$,

$$\left(\frac{S_m}{\sqrt{m}}, \frac{S_k - S_m}{\sqrt{k-m}}\right) \xrightarrow{d.} T(Y_1, Y_2).$$

Using the fact T is continuous,

$$\frac{T(S_m, S_k - S_m)}{\sqrt{m(k-m)}} = T\left(\frac{S_m}{\sqrt{m}}, \frac{S_k - S_m}{\sqrt{k-m}}\right) \xrightarrow{d.} T(Y_1, Y_2).$$

Also, by Lemma 4.6,

$$\begin{aligned}\mathbb{E}\left(\left[\frac{\mathbb{E}\, T(S_m, S_k - S_m)}{\sqrt{m(k-m)}}\right]^2\right) &\leq \frac{\mathbb{E}\left(\|S_m\|^2\|S_k - S_m\|^2\right)}{m(k-m)} \\ &\leq \frac{\mathbb{E}\,\|S_m\|^2}{m} \cdot \frac{\mathbb{E}\,\|S_k - S_m\|^2}{k-m} < \infty.\end{aligned}$$

That means $m^{-1/2}(k-m)^{-1/2}T(S_m, S_k - S_m)$ is uniformly integrable over (m, k) with $m \geq 1$, $k \geq m + 1$. So the result follows. □

We state the following result without proof. It is a higher dimensional analogue of S–W formula (1.1). See Barndorff–Nielson and Baxter [9] for the proof.

Lemma 6.3 (Barndorff Nielsen & Baxter)**.**

$$\mathbb{E}\left(A_n\right) = \sum_{k=2}^{n}\sum_{m=1}^{k-1} \frac{\mathbb{E}\left[T(S_m, S_k - S_m)\right]}{m(k-m)}. \tag{6.2}$$

Lemma 6.4.

$$\lim_{k\to\infty}\sum_{m=1}^{k-1}\frac{1}{m^{1/2}(k-m)^{1/2}}=\pi.$$

Proof. Let $f(m,k)=m^{-1/2}(k-m)^{-1/2}$. For any $\delta\in(0,1)$, we have $f(m,k)\leq f(m-\delta,k)$ if $m\leq k/2$ and $f(m,k)\geq f(m-\delta,k)$ if $m\geq k/2$. Consider the sum as two parts,

$$\sum_{m=1}^{k-1}f(m,k)=\left(\sum_{m=1}^{\lfloor k/2\rfloor}+\sum_{m=\lfloor k/2\rfloor+1}^{k-1}\right)f(m,k).$$

Then,

$$\begin{aligned}\sum_{m=1}^{k-1}f(m,k)&\geq\int_1^{\lfloor k/2\rfloor}f(m,k)\,\mathrm{d}m+\int_{\lfloor k/2\rfloor+1}^{k-1}f(m-1,k)\,\mathrm{d}m,\\ &\qquad\text{by letting } u=\frac{m}{k}\text{ and } v=\frac{m-1}{k},\\ &=\int_{1/k}^{\lfloor\frac{k}{2}\rfloor/k}\frac{1}{\sqrt{u(1-u)}}\,\mathrm{d}u+\int_{\lfloor\frac{k}{2}\rfloor/k}^{1-\frac{2}{k}}\frac{1}{\sqrt{v(1-v)}}\,\mathrm{d}v\\ &=\int_{1/k}^{1-2/k}\frac{1}{\sqrt{u(1-u)}}\,\mathrm{d}u.\end{aligned}$$

Also,

$$\begin{aligned}\sum_{m=1}^{k-1}f(m,k)&\leq\int_1^{\lfloor k/2\rfloor}f(m-1,k)\,\mathrm{d}m+\int_{\lfloor k/2\rfloor+1}^{k-1}f(m,k)\,\mathrm{d}m\\ &=\int_0^{\lfloor\frac{k}{2}\rfloor/k-1/k}\frac{1}{\sqrt{u(1-u)}}\,\mathrm{d}u+\int_{\lfloor\frac{k}{2}\rfloor/k+1/k}^{1-\frac{1}{k}}\frac{1}{\sqrt{v(1-v)}}\,\mathrm{d}v\\ &\leq\int_0^{1-1/k}\frac{1}{\sqrt{u(1-u)}}\,\mathrm{d}u.\end{aligned}$$

Therefore,

$$\lim_{k\to\infty}\sum_{m=1}^{k-1}f(m,k)=\int_0^1[u(1-u)]^{-1/2}\,\mathrm{d}u=B\left(\frac{1}{2},\frac{1}{2}\right)=\Gamma\left(\frac{1}{2}\right)^2=\pi,\tag{6.3}$$

where $B(\cdot,\cdot)$ is the Beta function and $\Gamma(\cdot)$ is the Gamma function. □

Lemma 6.5.

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=2}^{n}\sum_{m=1}^{k-1}\frac{1}{m^{1/2}(k-m)^{1/2}}=\pi.$$

Proof. The result follows from Lemma 2.22 and Lemma 6.4. □

Proposition 6.6. *Suppose* $\mathbb{E}(\|Z_1\|^2) < \infty$ *and* $\mu = 0$. *Then,*

$$\lim_{n\to\infty} \frac{\mathbb{E}\, A_n}{n} = \pi \mathbb{E}\, T(Y_1, Y_2),$$

where Y_1, Y_2 *are iid. rvs.* $Y_1, Y_2 \sim \mathcal{N}(\mathbf{0}, \Sigma)$ *and* $\Sigma = \mathbb{E}(Z_1^T Z_1)$.

Proof. In (6.2), denote $g(k,m) := m^{-1/2}(k-m)^{-1/2}\mathbb{E}\left[T(S_m, S_k - S_m)\right]$. Then,

$$\mathbb{E}\, A_n = \sum_{k=2}^{n}\sum_{m=1}^{k-1} \frac{g(k,m)}{m^{1/2}(k-m)^{1/2}}. \tag{6.4}$$

and by Lemma 6.2,

$$\lim_{m\to\infty,\ k-m\to\infty} g(k,m) = \mathbb{E}\, T(Y_1, Y_2) := \lambda. \tag{6.5}$$

So, for every $\varepsilon > 0$, there exists $m_0 \in \mathbb{Z}_+$ such that for any $m \geq m_0$ and $k - m \geq m_0$ we have $|g(k,m) - \lambda| \leq \varepsilon$.

For the upper bound of $\mathbb{E}\, A_n$, Separate the inner sum as

$$\begin{aligned}
\mathbb{E}\, A_n &= \left(\sum_{k=2}^{m_0} + \sum_{k=m_0+1}^{n}\right)\sum_{m=1}^{k-1} \frac{g(k,m)}{m^{1/2}(k-m)^{1/2}} \\
&= \sum_{k=m_0+1}^{n}\sum_{m=1}^{k-1} \frac{g(k,m)}{m^{1/2}(k-m)^{1/2}} + O(1) \\
&= \sum_{k=m_0+1}^{n}\left(\sum_{m=1}^{m_0} + \sum_{m=k-m_0}^{k-1} + \sum_{m=m_0+1}^{k-m_0-1}\right) \frac{g(k,m)}{m^{1/2}(k-m)^{1/2}} + O(1),
\end{aligned}$$

where

$$\begin{aligned}
&\sum_{k=m_0+1}^{n}\left(\sum_{m=1}^{m_0} + \sum_{m=k-m_0}^{k-1}\right) \frac{g(k,m)}{m^{1/2}(k-m)^{1/2}} \\
&\leq m_0 \sum_{k=m_0+1}^{n} \frac{\max_{1\leq m\leq m_0} g(k,m)}{(k-m_0)^{1/2}} + m_0 \sum_{k=m_0+1}^{n} \frac{\max_{k-m_0\leq m\leq k} g(k,m)}{(k-m_0)^{1/2}} \\
&\leq \lambda' \sum_{k=m_0+1}^{n} \frac{2m_0}{(k-m_0)^{1/2}}, \quad \text{since} \max_{1\leq k,m\leq n} g(k,m) < \infty, \\
&\leq O(n^{1/2}),
\end{aligned} \tag{6.6}$$

where λ' is some constant, and

$$\sum_{k=m_0+1}^{n}\sum_{m=m_0+1}^{k-m_0-1}\frac{g(k,m)}{m^{1/2}(k-m)^{1/2}} \le (\lambda+\varepsilon)\sum_{k=2}^{n}\sum_{m=1}^{k-1}\frac{1}{m^{1/2}(k-m)^{1/2}}.$$

By Lemma 6.5,

$$\limsup_{n\to\infty}\frac{1}{n}\sum_{k=m_0+1}^{n}\sum_{m=m_0+1}^{k-m_0-1}\frac{g(k,m)}{m^{1/2}(k-m)^{1/2}} \le (\lambda+\varepsilon)\pi.$$

Hence, $\limsup_{n\to\infty} n^{-1}\mathbb{E}\,A_n \le (\lambda+\varepsilon)\pi$ by (6.6). So $\limsup_{n\to\infty} n^{-1}\mathbb{E}\,A_n \le \lambda\pi$, since $\varepsilon > 0$ was arbitrary.

For the lower bound

$$\begin{aligned}
\mathbb{E}\,A_n &\ge \sum_{k=2}^{n}\sum_{m=m_0}^{k-m_0}\frac{g(k,m)}{m^{1/2}(k-m)^{1/2}}\\
&\ge (\lambda-\varepsilon)\sum_{k=2}^{n}\sum_{m=m_0}^{k-m_0}\frac{1}{m^{1/2}(k-m)^{1/2}}\\
&\ge (\lambda-\varepsilon)\sum_{k=2}^{n}\left(\sum_{m=1}^{k-1}-\sum_{m=1}^{m_0-1}-\sum_{m=k-m_0+1}^{k-1}\right)\frac{1}{m^{1/2}(k-m)^{1/2}}\\
&\ge (\lambda-\varepsilon)\sum_{k=2}^{n}\sum_{m=1}^{k-1}\frac{1}{m^{1/2}(k-m)^{1/2}}-(\lambda-\varepsilon)\sum_{k=2}^{n}\frac{2(m_0-1)}{(k-1)^{1/2}}.
\end{aligned}$$

By Lemma 6.5, $\liminf_{n\to\infty} n^{-1}\mathbb{E}\,A_n \ge (\lambda-\varepsilon)\pi$. Therefore $\liminf_{n\to\infty} n^{-1}\mathbb{E}\,A_n \ge \lambda\pi$, since $\varepsilon > 0$ was arbitrary. Then the result follows. □

Lemma 6.7. *If Y_1, Y_2 are iid. rvs. $Y_1, Y_2 \sim \mathcal{N}(\mathbf{0},\Sigma)$ and $\Sigma = \mathbb{E}\,(Z_1^T Z_1)$ Then,*

$$\mathbb{E}\,T(Y_1,Y_2) = \frac{1}{2}\sqrt{\det\Sigma}.$$

Proof. With $\Sigma = (\Sigma^{1/2})^2$, we have that (Y_1, Y_2) is equal in distribution to $(\Sigma^{1/2}W_1, \Sigma^{1/2}W_2)$ where W_1 and W_2 are independent $\mathcal{N}(0, I)$ random vectors. Since $\Sigma^{1/2}$ acts as a linear transformation on $\mathbb{R}^2$ with Jacobian $\sqrt{\det\Sigma}$,

$$\mathbb{E}\,T(Y_1,Y_2) = \mathbb{E}\,T(\Sigma^{1/2}W_1, \Sigma^{1/2}W_2) = \sqrt{\det\Sigma}\,\mathbb{E}\,T(W_1,W_2).$$

Here

$$\mathbb{E}\,T(W_1,W_2) = \frac{1}{2}\mathbb{E}\,[\|W_1\|\|W_2\|\sin\Theta],$$

where the minimum angle Θ between W_1 and W_2 is uniform on $[0,\pi]$, and $(\|W_1\|,\|W_2\|,\Theta)$ are independent. Hence

$$\mathbb{E}\,T(W_1,W_2) = \frac{1}{2}(\mathbb{E}\,\|W_1\|)^2(\mathbb{E}\,\sin\Theta) = \frac{1}{2},$$

using the fact that $\mathbb{E}\,\sin\Theta = 2/\pi$ and $\|W_1\|$ is the square-root of a χ_2^2 random variable, so $\mathbb{E}\,\|W_1\| = \sqrt{\pi/2}$ and the result follows. $\square$

Theorem 6.8. *Suppose that $\mathbb{E}\,\|Z_1\|^2 < \infty$ and $\mu = 0$. Then,*

$$\lim_{n\to\infty} n^{-1}\mathbb{E}\,A_n = \frac{\pi}{2}\sqrt{\det\Sigma}.$$

Proof. The result follows from Proposition 6.6 combining with Lemma 6.7. $\square$

Theorem 6.9. *Suppose that* (M_p) *holds for some $p > 2$, $\mu \neq 0$, and $\sigma^2_{\mu_\perp} > 0$. Then*

$$\lim_{n\to\infty} n^{-3/2}\mathbb{E}\,A_n = \|\mu\|(\sigma^2_{\mu_\perp})^{1/2}\mathbb{E}\,\tilde{a}_1 = \frac{1}{3}\|\mu\|\sqrt{2\pi\sigma^2_{\mu_\perp}}.$$

In particular, $\mathbb{E}\,\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$.

Proof. Recall that $\tilde{a}_1 = \mathcal{A}(\tilde{h}_1)$ is the convex hull area of the space-time diagram of one-dimensional Brownian motion run for unit time.

Given $\mathbb{E}\,[\|Z_1\|^p] < \infty$ for some $p > 2$, Proposition 6.1(i) shows that $\mathbb{E}\,[A_n^{p/2}] = O(n^{3p/4})$, so that $\mathbb{E}\,[(n^{-3/2}A_n)^{p/2}]$ is uniformly bounded. Hence $n^{-3/2}A_n$ is uniformly integrable, so Corollary 3.9 implies that

$$\lim_{n\to\infty} n^{-3/2}\mathbb{E}\,A_n = \|\mu\|(\sigma^2_{\mu_\perp})^{1/2}\mathbb{E}\,\tilde{a}_1. \tag{6.7}$$

In light of (6.7), it remains to identify $\mathbb{E}\,\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$. It does not seem straightforward to work directly with the Brownian limit; it turns out again to be simpler to work with a suitable random walk. We choose a walk that is particularly convenient for computations.

Let $\xi \sim \mathcal{N}(0,1)$ be a standard normal random variable, and take Z to be distributed as $Z = (1,\xi)$ in Cartesian coordinates. Then $S_n = (n, \sum_{k=1}^n \xi_k)$ is the space-time diagram of the symmetric random walk on $\mathbb{R}$ generated by i.i.d. copies $\xi_1, \xi_2, \ldots$ of ξ.

For $Z = (1,\xi)$, $\mu = (1,0)$ and $\sigma^2 = \sigma^2_{\mu_\perp} = \mathbb{E}\,[\xi^2] = 1$. Thus by (6.7), to complete the proof of Theorem 6.9 it suffices to show that for this walk $\lim_{n\to\infty} n^{-3/2}\mathbb{E}\,A_n =$

$\frac{1}{3}\sqrt{2\pi}$. If $u, v \in \mathbb{R}^2$ have Cartesian components $u = (u_1, u_2)$ and $v = (v_1, v_2)$, then we may write $T(u, v) = \frac{1}{2}|u_1 v_2 - v_1 u_2|$. Hence

$$T(S_m, S_k - S_m) = \frac{1}{2}\left|(k-m)\sum_{j=1}^{m}\xi_j - m\sum_{j=m+1}^{k}\xi_j\right|.$$

By properties of the normal distribution, the right-hand side of the last display has the same distribution as $\frac{1}{2}|\xi\sqrt{km(k-m)}|$. Hence

$$\frac{\mathbb{E}\, T(S_m, S_k - S_m)}{\sqrt{m(k-m)}} = \frac{1}{2}\mathbb{E}\,|\xi\sqrt{k}| = \frac{1}{2}\sqrt{2k/\pi},$$

using the fact that $|\xi|$ is distributed as the square-root of a χ_1^2 random variable, so $\mathbb{E}\,|\xi| = \sqrt{2/\pi}$. Hence, by (6.4), this random walk enjoys the exact formula

$$\mathbb{E}\, A_n = \frac{1}{\sqrt{2\pi}}\sum_{k=2}^{n}\sum_{m=1}^{k-1}\frac{\sqrt{k}}{\sqrt{m(k-m)}}.$$

Then from (6.3) we obtain $\mathbb{E}\, A_n \sim \sqrt{\pi/2}\sum_{k=2}^{n} k^{1/2}$, which gives the result. □

Remark 6.2. The idea used in the proof of Theorem 6.9, first establishing the existence of a limit for a class of models and then choosing a particular model for which the limit can be conveniently evaluated, goes back at least to Kac; see [34, p. 293].

6.4 Law of large numbers for the area

Proposition 6.10. *Suppose* $\mathbb{E}\,(\|Z_1\|^4) < \infty$ *and* $\|\mathbb{E}\, Z_1\| = 0$. *Then for any* $\alpha > 1$, $n^{-\alpha} A_n \to 0$ *a.s. as* $n \to \infty$.

Proof. By Chebyshev's inequality for A_n,

$$\mathbb{P}\left(\frac{|A_n - \mathbb{E}\, A_n|}{n^\alpha} \geq \varepsilon\right) = \mathbb{P}(|A_n - \mathbb{E}\, A_n| \geq \varepsilon n^\alpha) \leq \frac{\mathbb{V}\mathrm{ar}(A_n)}{\varepsilon^2 n^{2\alpha}}.$$

Since $\mathbb{V}\mathrm{ar}(A_n) = O(n^2)$ by Proposition 6.1(ii), for any $\alpha > 1$, as $n \to \infty$ we have

$$\mathbb{P}\left(\frac{|A_n - \mathbb{E}\, A_n|}{n^\alpha} \geq \varepsilon\right) = O(n^{2-2\alpha}).$$

So $n^{-\alpha}(A_n - \mathbb{E}\, A_n) \to 0$ in probability.

Take $n = n_k = 2^k$ for $k \in \mathbb{N}$, we have

$$\mathbb{P}\left(\frac{|A_{n_k} - \mathbb{E}\, A_{n_k}|}{n_k^\alpha} \geq \varepsilon\right) = O(n_k^{2-2\alpha}) = O(4^{k(1-\alpha)}).$$

So for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{|A_{n_k} - \mathbb{E}\, A_{n_k}|}{n_k^\alpha} \geq \varepsilon\right) < \infty.$$

By Borel–Cantelli Lemma (Lemma 2.3), as $k \to \infty$

$$\frac{A_{n_k} - \mathbb{E}\, A_{n_k}}{n_k^\alpha} \to 0 \ \text{ a.s.}$$

By Proposition 6.1(ii), $n_k^{-\alpha}\mathbb{E}\, A_{n_k} \to 0$ as $n \to \infty$, we get

$$\frac{A_{n_k}}{n_k^\alpha} \to 0 \ \text{ a.s. as } k \to \infty.$$

For any $n \in \mathbb{N}$, there exists $k(n) \in N$ such that $2^{k(n)} \leq n < 2^{k(n)+1}$. By monotonicity of A_n,

$$2^{-\alpha}\frac{A_{n_{k(n)}}}{n_{k(n)}^\alpha} = \frac{A_{2^{k(n)}}}{(2^{k(n)+1})^\alpha} \leq \frac{A_n}{n^\alpha} \leq \frac{A_{2^{k(n)+1}}}{(2^{k(n)})^\alpha} = 2^\alpha \frac{A_{n_{k(n)+1}}}{n_{k(n)+1}^\alpha}.$$

The result follows by the Squeezing Theorem. □

Proposition 6.11. *Suppose* $\mathbb{E}\,(\|Z_1\|^4) < \infty$*. Then, for any* $\alpha > 3/2$*,* $n^{-\alpha}A_n \to 0$ *a.s. as* $n \to \infty$*.*

Proof. By Chebyshev's inequality for A_n,

$$\mathbb{P}\left(\frac{|A_n - \mathbb{E}\, A_n|}{n^\alpha} \geq \varepsilon\right) = \mathbb{P}(|A_n - \mathbb{E}\, A_n| \geq \varepsilon n^\alpha) \leq \frac{\mathbb{V}\text{ar}(A_n)}{\varepsilon^2 n^{2\alpha}}.$$

Since $\mathbb{V}\text{ar}(A_n) = O(n^3)$ by Proposition 6.1(i), for any $\alpha > 3/2$, as $n \to \infty$ we have

$$\mathbb{P}\left(\frac{|A_n - \mathbb{E}\, A_n|}{n^\alpha} \geq \varepsilon\right) = O(n^{3-2\alpha}).$$

So $n^{-\alpha}(A_n - \mathbb{E}\, A_n) \to 0$ in probability.

Take $n = n_k = 2^k$ for $k \in \mathbb{N}$, we have

$$\mathbb{P}\left(\frac{|A_{n_k} - \mathbb{E}\, A_{n_k}|}{n_k^\alpha} \geq \varepsilon\right) = O(n_k^{3-2\alpha}) = O(4^{k(3/2-\alpha)}).$$

So for any $\varepsilon > 0$,

$$\sum_{k=1}^{\infty} \mathbb{P}\left(\frac{|A_{n_k} - \mathbb{E}\, A_{n_k}|}{n_k^{\alpha}} \geq \varepsilon \right) < \infty.$$

By Borel–Cantelli Lemma (Lemma 2.3), as $k \to \infty$

$$\frac{A_{n_k} - \mathbb{E}\, A_{n_k}}{n_k^{\alpha}} \to 0 \quad \text{a.s.}$$

By Proposition 6.1(i), $n_k^{-\alpha} \mathbb{E}\, A_{n_k} \to 0$ as $n \to \infty$, we get

$$\frac{A_{n_k}}{n_k^{\alpha}} \to 0 \quad \text{a.s. as } k \to \infty.$$

For any $n \in \mathbb{N}$, there exists $k(n) \in N$ such that $2^{k(n)} \leq n < 2^{k(n)+1}$. By monotonicity of A_n,

$$2^{-\alpha} \frac{A_{n_{k(n)}}}{n_{k(n)}^{\alpha}} = \frac{A_{2^{k(n)}}}{(2^{k(n)+1})^{\alpha}} \leq \frac{A_n}{n^{\alpha}} \leq \frac{A_{2^{k(n)+1}}}{(2^{k(n)})^{\alpha}} = 2^{\alpha} \frac{A_{n_{k(n)+1}}}{n_{k(n)+1}^{\alpha}}.$$

The result follows by the Squeezing Theorem. □

6.5 Asymptotics for the variance

Recall that Proposition 5.15 shows $\lim_{n\to\infty} n^{-1} \mathbb{V}\mathrm{ar} L_n = u_0(\Sigma)$. In this section, we will show that

$$\begin{aligned} &\text{if } \mu \neq 0: \quad \lim_{n\to\infty} n^{-3} \mathbb{V}\mathrm{ar} A_n = v_+ \|\mu\|^2 \sigma^2_{\mu_\perp}; \\ &\text{if } \mu = 0: \quad \lim_{n\to\infty} n^{-2} \mathbb{V}\mathrm{ar} A_n = v_0 \det \Sigma. \end{aligned} \tag{6.8}$$

The quantities v_0 and v_+ in (6.8) are finite and positive, as is $u_0(\cdot)$ provided $\sigma^2 \in (0, \infty)$, and these quantities are in fact variances associated with convex hulls of Brownian scaling limits for the walk.

Proposition 6.12. *Suppose that* (M_p) *holds for some* $p > 4$*, and* $\mu = 0$*. Then*

$$\lim_{n\to\infty} n^{-2} \mathbb{V}\mathrm{ar} A_n = v_0 \det \Sigma.$$

Proof. Lemma 6.1(ii) shows that $\mathbb{E}\,[A_n^{2(p/4)}] = O(n^{p/2})$, so that $\mathbb{E}\,[(n^{-2}A_n^2)^{p/4}]$ is uniformly bounded. Hence $n^{-2}A_n^2$ is uniformly integrable, and we deduce convergence of $n^{-2}\mathbb{V}\mathrm{ar} A_n$ in Corollary 3.7. □

For the case with drift, we have the following variance result.

Proposition 6.13. *Suppose that* (M_p) *holds for some* $p > 4$ *and* $\mu \neq 0$*. Then*

$$\lim_{n\to\infty} n^{-3}\mathbb{V}\mathrm{ar} A_n = v_+ \|\mu\|^2 \sigma^2_{\mu_\perp}.$$

Proof. Given $\mathbb{E}[\|Z_1\|^p] < \infty$ for some $p > 4$, Lemma 6.1(i) shows that $\mathbb{E}[A_n^{2(p/4)}] = O(n^{3p/4})$, so that $\mathbb{E}[(n^{-3}A_n^2)^{p/4}]$ is uniformly bounded. Hence $n^{-3}A_n^2$ is uniformly integrable, so Corollary 3.9 yields the result. □

6.6 Variance bounds

Proposition 6.14. *We have* $u_0(\Sigma) = 0$ *if and only if* $\operatorname{tr}\Sigma = 0$*. The following inequalities for the quantities defined at* (5.26) *hold.*

$$0 < \frac{4}{49}\left(\mathrm{e}^{-7\pi^2/12} - \frac{1}{3}\mathrm{e}^{-21\pi^2/4}\right)^2 \leq v_0 \leq 16(\log 2)^2 - \frac{\pi^2}{4}; \tag{6.9}$$

$$0 < \frac{2}{225}\left(\mathrm{e}^{-25\pi/9} - \frac{1}{3}\mathrm{e}^{-25\pi}\right) \leq v_+ \leq 4\log 2 - \frac{2\pi}{9}. \tag{6.10}$$

Proof. Bounding $\tilde{a}_1$ by the area of a rectangle, we have

$$\tilde{a}_1 \leq r_1 \leq 2 \sup_{0\leq s\leq 1} |w(s)|, \text{ a.s.}, \tag{6.11}$$

where $r_1 := \sup_{0\leq s\leq 1} w(s) - \inf_{0\leq s\leq 1} w(s)$. A result of Feller [22] states that $\mathbb{E}[r_1^2] = 4\log 2$. So by the first inequality in (6.11), we have $\mathbb{E}[\tilde{a}_1^2] \leq 4\log 2$, and by Theorem 6.9 we have $\mathbb{E}\,\tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$; the upper bound in (6.10) follows.

Similarly, for any orthonormal basis $\{e_1, e_2\}$ of $\mathbb{R}^2$, we bound a_1 by a rectangle

$$a_1 \leq \left(\sup_{0\leq s\leq 1} e_1 \cdot b(s) - \inf_{0\leq s\leq 1} e_1 \cdot b(s)\right)\left(\sup_{0\leq s\leq 1} e_2 \cdot b(s) - \inf_{0\leq s\leq 1} e_2 \cdot b(s)\right),$$

and the two (orthogonal) components are independent, so $\mathbb{E}[a_1^2] \leq (\mathbb{E}[r_1^2])^2 = 16(\log 2)^2$, which with the fact that $\mathbb{E}\,a_1 = \frac{\pi}{2}$ gives the upper bound in (6.9).

We now move on to the lower bounds. Tractable upper bounds for a_1 and $\tilde{a}_1$ are easier to come by than lower bounds, and thus we obtain a lower bound on the variance by showing the appropriate area has positive probability of being smaller than the corresponding mean.

Consider a_1; note $\mathbb{E}\, a_1 = \pi/2$ [19]. Since, for any orthonormal basis $\{e_1, e_2\}$ of $\mathbb{R}^2$,

$$a_1 \le \pi \sup_{0\le s\le 1} \|b(s)\|^2 \le \pi \sup_{0\le s\le 1} |e_1 \cdot b(s)|^2 + \pi \sup_{0\le s\le 1} |e_2 \cdot b(s)|^2,$$

using the fact that $e_1 \cdot b$ and $e_2 \cdot b$ are independent one-dimensional Brownian motions,

$$\mathbb{P}[a_1 \le r] \ge \mathbb{P}\left[\sup_{0\le s\le 1} |w(s)|^2 \le \frac{r}{2\pi} \right]^2, \text{ for } r > 0.$$

We apply (5.30) with $X = a_1$ and $\alpha \in (0,1)$, and set $r = (1-\alpha)\frac{\pi}{2}$ to obtain

$$\begin{aligned} \mathbb{V}\mathrm{ar}\, a_1 &\ge \alpha^2 \frac{\pi^2}{4} \mathbb{P}\left[\sup_{0\le s\le 1} |w(s)| \le \frac{\sqrt{1-\alpha}}{2} \right]^2 \\ &\ge 4\alpha^2 \left(\exp\left\{ -\frac{\pi^2}{2(1-\alpha)} \right\} - \frac{1}{3} \exp\left\{ -\frac{9\pi^2}{2(1-\alpha)} \right\} \right)^2, \end{aligned}$$

by (5.28). Taking $\alpha = 1/7$ is close to optimal, and gives the lower bound in (6.9).

For $\tilde{a}_1$, we apply (5.30) with $X = \tilde{a}_1$ and $\alpha \in (0,1)$. Using the fact that $\mathbb{E}\, \tilde{a}_1 = \frac{1}{3}\sqrt{2\pi}$ (from Theorem 6.9) and the weaker of the two bounds in (6.11), we obtain

$$\begin{aligned} \mathbb{V}\mathrm{ar}\, \tilde{a}_1 &\ge \alpha^2 \frac{2\pi}{9} \mathbb{P}\left[\sup_{0\le s\le 1} |w(s)| \le \frac{(1-\alpha)\sqrt{2\pi}}{6} \right] \\ &\ge \frac{8}{9}\alpha^2 \left(\exp\left\{ -\frac{9\pi}{4(1-\alpha)^2} \right\} - \frac{1}{3} \exp\left\{ -\frac{81\pi}{4(1-\alpha)^2} \right\} \right), \end{aligned}$$

by (5.28). Taking $\alpha = 1/10$ is close to optimal, and gives the lower bound in (6.10). □

Remark 6.3. The main interest of the lower bounds in Proposition 6.14 is that they are *positive*; they are certainly not sharp. The bounds can surely be improved. We note just the following idea. A lower bound for $\tilde{a}_1$ can be obtained by conditioning on $\theta := \sup\{s \in [0,1] : w(s) = 0\}$ and using the fact that the maximum of w up to time θ is distributed as the maximum of a scaled Brownian bridge; combining this with the previous argument improves the lower bound on v_+ to 2.09×10^{-6}.

Chapter 7

Conclusions and open problems

7.1 Summary of the limit theorems

We summarize in general the asymptotic behaviour of the expectation and variance of L_n and A_n as the following table.

		limit exists for $\mathbb{E}$	limit exists for $\mathbb{V}\mathrm{ar}$	limit law
$\mu = 0$	L_n	$n^{-1/2}\mathbb{E}\, L_n{}^{\S}$	$n^{-1}\mathbb{V}\mathrm{ar} L_n$	non-Gaussian
	A_n	$n^{-1}\mathbb{E}\, A_n{}^{\P}$	$n^{-2}\mathbb{V}\mathrm{ar} A_n$	non-Gaussian
$\mu \neq 0$	L_n	$n^{-1}\mathbb{E}\, L_n{}^{\S\dagger}$	$n^{-1}\mathbb{V}\mathrm{ar} L_n{}^{\ddagger}$	Gaussian‡
	A_n	$n^{-3/2}\mathbb{E}\, A_n$	$n^{-3}\mathbb{V}\mathrm{ar} A_n$	non-Gaussian

Table 7.1: Results originate from: § [58]; † [57]; ‡ [63]; ¶ [3] (in part); the rest are new. The limit laws exclude degenerate cases when associated variances vanish.

Table 7.2 collets the lower and upper bounds and simulation estimates for the constants defined at equation (5.26) and equation (6.8).

Claussen et al. [12] give some numerical estimations that $\mathbb{V}\mathrm{ar}\, l_1 \approx 1.075$ and $\mathbb{V}\mathrm{ar}\, a_1 \approx 0.31$, which is a good agreement with our limit estimations 1.08 and 0.30.

	lower bound	simulation estimate	upper bound
$u_0(I)$	2.65×10^{-3}	1.08	9.87
v_0	8.15×10^{-7}	0.30	5.22
v_+	1.44×10^{-6}	0.019	2.08

Table 7.2: Each of the simulation estimates is based on 10^5 instances of a walk of length $n = 10^5$. The final decimal digit in each of the numerical upper (lower) bounds has been rounded up (down).

7.2 Exact evaluation of limiting variances

It would, of course, be of interest to evaluate any of u_0, v_0, or v_+ exactly. In general this looks hard. The paper [52] provides a key component to a possible approach to evaluating u_0. By Cauchy's formula and Fubini's theorem,

$$\mathbb{E}\left[\ell_1^2\right] = \int_{\mathbb{S}_1}\int_{\mathbb{S}_1} \mathbb{E}\left[\left(\sup_{0\le s\le 1}(e_1\cdot b(s))\right)\left(\sup_{0\le t\le 1}(e_2\cdot b(t))\right)\right] \mathrm{d}e_1\mathrm{d}e_2.$$

Here, the two standard one-dimensional Brownian motions $e_1 \cdot b$ and $e_2 \cdot b$ have correlation determined by the cosine of the angle ϕ between them, i.e.,

$$\mathbb{E}\left[(e_1\cdot b(s))(e_2\cdot b(t))\right] = (s\wedge t)\, e_1\cdot e_2 = (s\wedge t)\cos\phi.$$

The result of Rogers and Shepp [52] then shows that

$$\mathbb{E}\left[\left(\sup_{0\le s\le 1}(e_1\cdot b(s))\right)\left(\sup_{0\le t\le 1}(e_2\cdot b(t))\right)\right] = c(\cos\phi),$$

where the function c is given explicitly in [52]. Using this result, we obtain

$$\mathbb{E}\left[\ell_1^2\right] = 4\pi\int_{-\pi/2}^{\pi/2} c(\sin\theta)\mathrm{d}\theta = 4\pi\int_{-\pi/2}^{\pi/2}\mathrm{d}\theta\int_0^\infty \mathrm{d}u\cos\theta\frac{\cosh(u\theta)}{\sinh(u\pi/2)}\tanh\left(\frac{(2\theta+\pi)u}{4}\right).$$

We have not been able to deal with this integral analytically, but numerical integration gives $\mathbb{E}\left[\ell_1^2\right] \approx 26.1677$, which with the fact that $\mathbb{E}\,\ell_1 = \sqrt{8\pi}$ gives $u_0(I) = \mathbb{V}\mathrm{ar}\ell_1 \approx 1.0350$, in reasonable agreement with the simulation estimate in Table 6.1.

Another possible approach to evaluating u_0 is suggested by a remarkable computation of Goldman [27] for the analogue of $u_0(I) = \mathbb{V}\mathrm{ar}\ell_1$ for the planar *Brownian* *bridge*. Specifically, if b'_t is the standard Brownian bridge in $\mathbb{R}^2$ with $b'_0 = b'_1 = 0$, and $\ell'_1 = \mathcal{L}(\operatorname{hull} b'[0,1])$ the perimeter length of its convex hull, [27, Théorème 7] states that

$$\mathbb{V}\mathrm{ar}\ell'_1 = \frac{\pi^2}{6}\left(2\pi \int_0^\pi \frac{\sin\theta}{\theta}\mathrm{d}\theta - 2 - 3\pi\right) \approx 0.34755.$$

7.3 Open problems

7.3.1 Degenerate case for L_n when $\mu \neq 0$ and $\sigma_\mu^2 = 0$

Recall Remark 5.2(iii) for Theorem 5.13. For example, consider

$$Z_1 = \begin{cases} (1,1), & \text{with probability } 1/2; \\ (1,-1), & \text{with probability } 1/2. \end{cases}$$

Then the σ_μ^2 in Theorem 5.13 is zero and our results on the second-order properties of L_n in Chapter 5 can not be applied in this degenerate case. See Figure 7.1 for an example of random walk in this case.

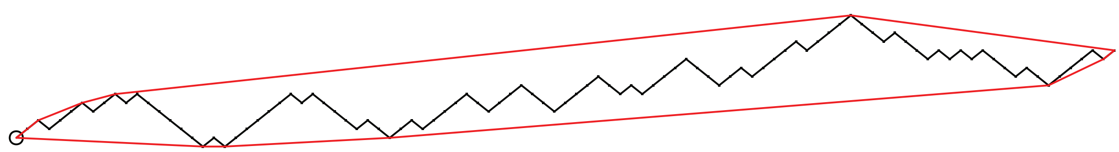

Figure 7.1: Example of the degenerate case with $n = 100$.

For this example, we conjecture $\frac{\mathbb{V}\mathrm{ar} L_n}{\log n} \to$ constant, based on some simulations. See Figure 7.2 below.

A second open question is whether in this case $\frac{L_n - \mathbb{E}\, L_n}{\sqrt{\mathbb{V}\mathrm{ar} L_n}}$ has a distributional limit. If so, is that limit normal? We conjecture that there is a limit, but it is not normal (see Figure 7.3).

7.3.2 Heavy-tailed increments

All main results from previous chapters are based on the assumption M_p for $p = 2$, that the second moments of increments are finite. But what happens in the heavy-tail problems, in which $\mathbb{E}\left(\|Z_1\|^2\right) = \infty$? We give two simulation examples.

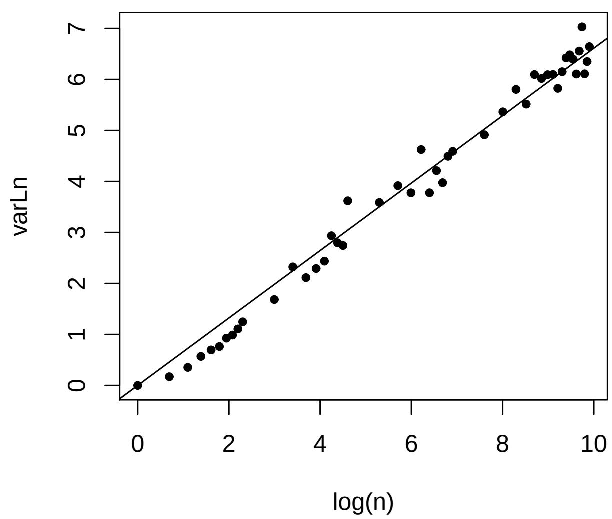

Figure 7.2: Simulation for the degenerate case $\mathbb{V}\mathrm{ar} L_n = 0.6612 \log(n)$.

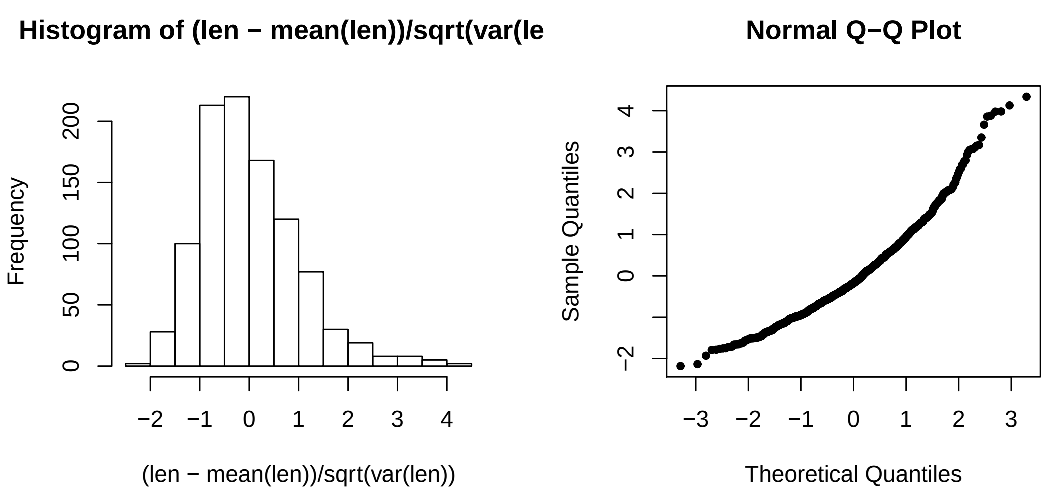

Figure 7.3: Simulations for the degenerate case.

7.3.3 Centre-of-mass process

We can associate to a random walk trajectory $S_0, S_1, S_2, \ldots$ its *centre-of-mass* process $G_0, G_1, G_2, \ldots$ defined by $G_0 := S_0 = 0$ and for $n \geq 1$ by $G_n = \frac{1}{n} \sum_{k=1}^{n} S_k$. By convexity, the convex hull of $\{G_0, G_1, \ldots, G_n\}$ is contained in the convex hull of $\{S_0, S_1, \ldots, S_n\}$. What can one say about its perimeter length or area? Note that one may express G_n as a weighted sum of the increments of the walk as

$$G_n = \sum_{k=1}^{n} \left(\frac{n-k+1}{n} \right) Z_k.$$

Then, for example, we expect that the method of Section 5.4 carries through to this case; this is one direction for future work.

7.3.4 Higher dimensions

Most of the analysis of L_n in this thesis is restricted to $d = 2$ because we rely on the Cauchy formula for planar convex sets. In higher dimensions, the analogues of L_n and A_n are the *intrinsic volumes* of the convex body. Analogues of Cauchy's formula are available, but these seem more difficult to use as the basis for analysis.

However, the scaling limit theories in Chapter 3 may have some relatively straightforward corollaries in higher dimensions. So, some analogous results for A_n in Chapter 6 may not be so difficult to figure out.

Bibliography

[1] E. S. Andersen, On the fluctuations of sums of random variables II, *Math. Scand.* **2** (1954) 195–223.

[2] S. Asmussen, *Applied Probability and Queues*, 2nd ed., Springer-Verlag, New York, 2003.

[3] O. Barndorff–Nielsen and G. Baxter, Combinatorial lemmas in higher dimensions, *Trans. Amer. Math. Soc.* **108** (1963) 313–325.

[4] V. Barnett, The ordering of multivariate data, *J. Roy. Statist. Soc. Ser. A* **139** (1976) no.3, 318–355.

[5] V. Barnett, Outliers and order statistics, *Comm. Statist. Theory Methods* **17** (1988) no.7, 2109–2118.

[6] V. Barnett and T. Lewis, *Outliers in statistical data*, 3rd ed., Wiley, Chichester-New York-Brisbane, 1994.

[7] F. Bartumeus, M. G. E. Da Luz, G. M. Viswanathan and J. Catalan, Animal search strategies: a quantitative random-walk analysis, *Ecology* **86** no.11, (2005) 3078–3087.

[8] R. F. Bass, Markov processes and convex minorants, *Seminaire de Probabilities*, LNM (1982) 1059.

[9] G. Baxter, A combinatorial lemma for complex numbers, *Ann. Math. Statist.* **32** (1961) 901–904.

[10] P. Billingsley, *Convergence of Probability Measures*, 2nd ed., Wiley, New York, 1999.

[11] K. Burdzy, Brownian paths and cones, *Ann. Prob.* **13** no.3, (1985) 1006–1010.

[12] G. Claussen, A. K. Hartmann, and S. N. Majumdar, Convex hulls of random walks: Large-deviation properties, *Phys. Rev.* **91** (2015) 052104.

[13] M. Cranston, P. Hsu, and P. March, Smoothness of the convex hull of planar Brownian motion, *Ann. Probab.* **17** (1989) 144–150.

[14] K. L. Chung, *A Course in Probability Theory*, 3rd ed., Academic Press, San Diego, 2001.

[15] K. L. Chung and W. H. J. Fuchs, On the distribution of values of sums of random variables, *Mem. Amer. Math. Soc.* **6** (1951).

[16] E. A. Codling, M. J. Plank and S. Benhamou, Random walk models in biology, *J. R. Soc. Interface* **5** (2008) 813–834.

[17] R. Durrett, *Probability: Theory and Examples*, Wadsworth & Brooks/Cole, Pacific Grove, CA, 1991.

[18] B. Efron, The convex hull of a random set of points, *Biometrika* **52** no. 3/4, (1965) 331–343.

[19] M. El Bachir, *L'enveloppe convex du mouvement Brownien*, Ph.D. thesis, Université Toulouse III—Paul Sabatier, 1983.

[20] R. Eldan, Volumetric properties of the convex hull of an n-dimensional Brownian motion, *Electron. J. Prob.* **19** no.45, (2014) 1–34.

[21] S. N. Evans, On the Hausdorff dimension of Brownian cone points, *Math. Proc. Camb. Philos. Soc.* **98** (1985) 343–353.

[22] W. Feller, The asymptotic distribution of the range of sums of independent random variables, *Ann. Math. Statist.* **22** (1951) 427–432.

[23] W. Feller, *An Introduction to Probability Theory and its Applications. Vol. II.*, 2nd ed., Wiley, New York, 1971.

[24] J. Geffroy, Localisation asymptotique du polyèdre d'appui d'un échantillon laplacien à k dimensions, *Publ. Inst. Stat. Univ. Paris* **10** (1961) 213–228.

[25] L. Giuggioli, J. R. Potts and S. Harris, Animal interactions and the emergence of territoriality, *PLoS. Comput. Biol.* **7**(3) (2011) e1002008. doi:10.1371/journal.pcbi.1002008

[26] R. H. Glendinning, The convex hull of a dependent vector-valued process, *J. Statist. Comput. Simul.* **38** (1991) 219–237.

[27] A. Goldman, Le spectre de certaines mosaïques poissoniennes du plan et l'enveloppe convex du pont brownien, *Probab. Theory Relat. Fields* **105** (1996) 57–83.

[28] P. J. Green, Peeling bivariate data, pp. 3–19 in *Interpreting Multivariate Data*, V. Barnett (ed.), Wiley, 1981.

[29] P. M. Gruber, *Convex and Discrete Geometry*, Springer, Berlin, 2007.

[30] A. Gut, *Probability: A Graduate Course*, Springer, Uppsala, 2005.

[31] D. Hug, Random polytopes, Chapter 7 in *Stochastic Geometry, Spatial Statistics and Random Fields*, Springer, 2013.

[32] B. Hughes, *Random Walks and Random Environments, Vol. I.*, Oxford, 1995.

[33] N. C. Jain and W. E. Pruitt, The other law of the iterated logarithm, *Ann. Probab.* **3** (1975) 1046–1049.

[34] M. Kac, Toeplitz matrices, translation kernels and a related problem in probability theory, *Duke Math. J.* **21** (1954) 501–509.

[35] O. Kallenberg, *Foundations of Modern Probability*, 2nd ed., Springer, New York, 2002.

[36] J. Kampf, G. Last, and I. Molchanov, On the convex hull of symmetric stable processes, *Proc. Amer. Math. Soc.* **140** (2012) 2527–2535.

[37] S. Karlin, and H. M. Taylor, *A Second Course in Stochastic Processes*, Academic Press, New York, 1981.

[38] G. Letac, An explicit calculation of the mean of the perimeter of the convex hull of a plane random walk, *J. Theor. Prob.* **6** (1993) 385–387.

[39] G. Letac, Advanced problem 6230, *Amer. Math. Monthly* **85** (1978) 686.

[40] P. Lévy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.

[41] K. V. Mardia, J. T. Kent and J. M. Bibby, *Multivariate Analysis*, Academic Press, London, 1979.

[42] S. N. Majumdar, A. Comtet, and J. Randon-Furling, Random convex hulls and extreme value statistics, *J. Stat. Phys.* **138** (2010) 955–1009.

[43] C. O. Mohr, Table of equivalent populations of north American small mammals, *Amer. Midland Naturalist* **37** no.1, (1947) 223–249.

[44] P. Mörters and Y. Peres, *Brownian Motion*, Cambridge, 2010.

[45] V. B. Nevzorov, *Records: Mathematical Theory*, Amer. Math. Soc., 2001.

[46] M. Penrose, *Random Geometric Graphs*, Oxford, 2003.

[47] J. Pitman and N. Ross, The greatest convex minorant of Brownian motion, meander, and bridge, *Probab. Theory Relat. Fields* **153** (2012) 771–807.

[48] G. Pólya, Über eine Aufgabe der Wahrscheinlichkeitsrechnung betreffend die Irrfahrt im Strassennetz, *Math. Ann.* **84** (1921) 149-160.

[49] . Reitzner, pp. 45–76 in *New Perspectives in Stochastic Geometry*, W.S. Kendall & I. Molchanov (eds.), OUP, 2010.

[50] A. Rényi and R. Sulanke, Über die konvexe hülle von n zufällig gewählten punkten, *Z. Wahrscheinlichkeitstheorie* **2** (1963) 75–84.

[51] S. Resnick, *Adventures in Stochastic Processes*, Birkhäuser, Boston, 1992.

[52] L.C.G. Rogers and L. Shepp, The correlation of the maxima of correlated Brownian motions, *J. Appl. Probab.* **43** (2006) 880–883.

[53] W. Rudin, *Principles of Mathematical Analysis*, 3rd ed., McGraw-Hill, 1976.

[54] R. Schneider and W. Weil, Classical stochastic geometry, pp. 1–42 in *New Perspectives in Stochastic Geometry*, W.S. Kendall & I. Molchanov (eds.), OUP, 2010.

[55] Ya. G. Sinai, Convex hulls of random processes, *Amer. Math. Soc. Transl.* **186** (1998).

[56] P.E. Smouse, S. Focardi, P.R. Moorcroft, J.G. Kie, J.D. Forester and J.M. Morales, Stochastic modelling of animal movement, *Phil. Trans. R. Soc. B* **365** (2010) 2201–2211.

[57] T. L. Snyder and J. M. Steele, Convex hulls of random walks, *Proc. Amer. Math. Soc.* **117** (1993) 1165–1173.

[58] F. Spitzer and H. Widom, The circumference of a convex polygon, *Proc. Amer. Math. Soc.* **12** (1961) 506–509.

[59] J. M. Steele, The Bohnenblust–Spitzer algorithm and its applications, *J. Comput. Appl. Math.* **142** (2002) 235–249.

[60] J. M. Steele, *Probability Theory and Combinatorial Optimization*, Soc. for Industrial and Applied Math., 1997.

[61] L. Takács, Expected perimeter length, *Amer. Math. Monthly* **87** (1980) 142.

[62] L. Takács, *Combinatorial Methods in the Theory of Stochastic Processes*, Wiley, New York, 1967.

[63] A.R. Wade and C. Xu, Convex hulls of planar random walks with drift, *Proc. Amer. Math. Soc.* **143** (2015) 433–445.

[64] A.R. Wade and C. Xu, Convex hulls of planar random walks and their scaling limits, *Stoc. Proc. and their Appl.* **125** (2015) 4300–4320.

[65] B. J. Worton, A review of models of home range for animal movement, *Ecol. Modelling* **38** (1987) 277–298.

[66] B. J. Worton, A convex hull-based estimator of home-range size, *Biometrics* **51** no.4, (1995) 1206–1215.

[67] J. E. Yukich, *Probability Theory of Classical Euclidean Optimization Problems*, Springer, 1998.