University of Strathclyde

A thesis submitted for the degree of Doctor of Philosophy

# Interval order enumeration

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# Declaration

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

# Abstract

This thesis continues the study of interval orders and related structures, containing results on both the labeled and unlabeled variants.

Following a result of Eriksen and Sjöstrand (2014) we identify a link between structures following the Fishburn distribution and Mahonian structures. This is used to detail a technique for the construction of Fishburn structures (structures in bijection with unlabeled interval orders) from appropriate Mahonian structures.

This technique is introduced on a bivincular pattern of Bousquet-Mélou et al. (2010) and then used to introduce a previously unconsidered class of matchings; explicitly, zero alignment matchings according to the number of arcs which are both right-crossed and left-nesting.

The technique is then used to identify a statistic on the factorial posets of Claesson and Linusson (2011) following the Fishburn distribution. Factorial posets mapped to zero by this statistic are canonically labeled factorial posets which may alternatively be viewed as unlabeled interval orders.

As a consequence of our approach we find an identity for the Fishburn numbers in terms of the Mahonian numbers and discuss linear combinations of Fishburn patterns in a manner similar to that of the Mahonian combinations of Babson and Steingrímsson (2001).

To study labeled interval orders we introduce ballot matrices, a signed combinatorial structure whose definition naturally follows from the generating function for labeled interval orders.

A sign reversing involution on ballot matrices is defined. Adapting a bijection of Dukes, Jelínek and Kibitzke (2011), we show that matrices fixed under this involution are in bijection with labeled interval orders and that they decompose to a pair consisting of a permutation and an inversion table.

To fully classify such pairs results pertaining to the enumeration of permutations having a given set of ascent bottoms are given. This allows for a new formula for the number of labeled interval orders.

# Contents

CHAPTER 1

# Introduction

This thesis studies enumerative and bijective results related to *interval orders* and other equinumerous structures (often referred to as *Fishburn structures*).

The key enumerative results of this thesis are new identities for both unlabeled and labeled interval orders.

In chapter 2 we show that the number $i_n$ of unlabeled interval orders of size $n$ can be written in terms of the Mahonian numbers $m_{n,k}$ (A008302) as

$$i_n = \sum_{i=0}^{n-2} (-1)^i \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j},$$

corresponding to OEIS sequence A022493.

This also allows an identity in terms of $m_{n,k}$ for the number $r_n$ of flat interval orders (the subset of interval orders with trivial automorphism group),

$$r_n = \sum_{k=0}^{n} \sum_{i=0}^{k-2} \sum_{j=i}^{\binom{k-i}{2}} (-1)^{n-k+i} \binom{n-1}{k-1} \binom{j}{i} m_{k-i,j}.$$

In chapter 3 the number $\ell_n$ of labeled interval orders of size $n$ are shown to be given by

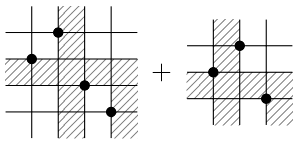
$$\ell_n = \sum_{\{s_1,\ldots,s_k\} \subseteq [n-1]} \left( \det \left[ \binom{n-s_i}{s_{j+1}-s_i} \right] \cdot \prod_{r=1}^{k+1} r^{s_r - s_{r-1}} \right),$$

where $s_0 = 0$ and $s_{k+1} = n$. This corresponds to OEIS sequence A079144.

For unlabeled interval orders the bijective results in this thesis arise from a new technique for the construction of Fishburn structures based on a simple argument by inclusion-exclusion. Such structures are studied with respect to a statistic which follows what we refer to as the *Fishburn distribution*. Structures mapped to zero by this statistic are the Fishburn structures.

The technique is demonstrated on two previously known Fishburn structures and on a third, previously unidentified, Fishburn structure; namely, zero alignment matchings according to the number of arcs which are both left-nesting and right-crossed.

Consequences of the technique prompts consideration of linear combinations of permutation patterns following the Fishburn distribution. We give an example of such a combination, namely



and provide brief commentary for consideration for future work.

For labeled interval orders the bijective results of this thesis include the introduction of *ballot matrices*, a notable subset of which are equinumerous with labeled interval orders. Using ballot matrices as an intermediate object labeled interval orders are shown to be in bijection with pairs of permutations in the following sets. Firstly,

$$\{(\pi, \tau) \in \mathfrak{S}_n \times \mathfrak{S}_n : A(\tau) \subseteq D(\pi)\},$$

where $A(\tau)$ is the set of ascent bottoms of $\tau$, and $D(\pi)$ is the set of descent positions of $\pi$. Secondly,

$$\{(\pi, \tau) \in \mathfrak{S}_n \times \mathfrak{S}_n : D(\pi) \subseteq A(\tau)\}.$$

The work on labeled interval orders was done in collaboration with Anders Claesson.

The thesis is structured as follows:

- The remainder of this chapter provides a literature review and summarizes current techniques to enumerate Fishburn structures.
- Chapter 2 presents results related to unlabeled interval orders and other Fishburn structures.
- Chapter 3 handles the labeled case, presenting a characterization of labeled intervals as pairs of permutations.

The contributions of Chapter 2 can be summarised as the following.

(1) Producing a new technique for enumerating Fishburn structures.
(2) Introducing matchings with no alignments as a Fishburn structure according to the number of arcs which are both left-nesting and right-crossed (definitions are given in the chapter).
(3) Identifying mislabelings, the number of which define a statistic on the factorial posets following the Fishburn distribution.
(4) Demonstrating a linear combination of mesh patterns following the Fishburn distribution.
(5) Finding an identity for the Fishburn distribution in terms of the Mahonian numbers.

The work in Chapter 3 may be summarised thusly.

(1) The introduction of ballot matrices.
(2) Extending work of Dukes et al. [10] to give a surjection between ballot matrices and labeled interval orders.
(3) A proof that labeled interval orders are in bijection with pairs of permutations where the set of ascent bottoms of one permutation is a subset of the set of descent positions of the other.
(4) A proof that labeled interval orders are in bijection with pairs of permutations where the set of descent positions of one permutation is a subset of the set of ascent bottoms of the other.
(5) Finding a new identity for the number of labeled interval orders of size $n$.

## Background

A poset $P$ is said to be an interval order if each $z \in P$ can be assigned a closed interval $[\ell_z, r_z] \subset \mathbb{R}$ such that $x <_P y$ if and only if $r_x < \ell_y$. Such posets are known to be equivalently characterized as those with a linear ordering by inclusion on the predecessor sets for each element (see, for example, Bogart [3]), where the predecessor set of an element $x \in P$ is defined as follows:

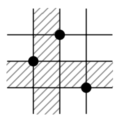
$$\mathrm{pred}\,(x) = \{y \in P : y <_P x\}.$$

An additional, equivalent definition is given by Fishburn [15] as posets with no induced subposet isomorphic to the pair of disjoint chains of length two, the so called $(2+2)$-free posets.

Enumerative work by Haxell, McDonald, and Thomason [16] provided a recursive algorithm to determine $i_n$, the number of unlabeled interval orders of size $n$, suitable for small values of $n$ (the figures for $n \leq 60$ are given in the paper). El-Zahar [13] and Kahmis [18] provided functional equations satisfied by the ordinary generating function for interval orders, however they are not solved at the time they are presented.

**Resurgent interest.** Following a recent resurgence of interest a wide variety of enumerative results pertaining to interval orders as well as bijective relations to other Fishburn structures have been discovered. These often preserve an impressive number of statistics.

This resurgence was born from the introduction of a new class of permutation patterns, *bivincular patterns*, by Bousquet-Mélou et al. [4]. They demonstrated that unlabeled interval orders are in bijection with permutations avoiding the pattern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(for definition and example see page 17) and a subset of fixed point free involutions referred to in the literature as *non-neighbor-nesting matchings* (occasionally *Stoimenow matchings*).

Stoimenow [24] considered non-neighbor-nesting matchings as an upper bound for the dimension of the space of Vassiliev invariants for knots. Zagier [26] determined their ordinary generating function to be

$$\sum_{m \geq 0} \prod_{i=1}^{m} (1 - (1-x)^i).$$

and gave an asymptotic formula for $m_n$, the number of non-neighbor-nesting matchings of size $n$.

$$m_n \sim n! \frac{12\sqrt{3}}{\pi^{\frac{5}{2}}} e^{\frac{\pi^2}{12}} \left(\frac{6}{\pi^2}\right)^n \sqrt{n}.$$

The work of Bousquet-Mélou et al. [4] that interval orders and non neighbor-nesting matchings are equinumerous thus gave that the generating function and asymptotic results of Zagier are equally applicable to interval orders. This, for the first time, explicitly identified the ordinary generating function for interval orders and allowed that $i_n = m_n$.

The bijections used by Bousquet-Mélou et al. [4] are via constructions encoded with an intermediate structure they introduce, namely *ascent sequences*, a subset of inversion tables which are recursively defined. We detail their approach later in this chapter on page 12.

The same approach, using ascent sequences to encode construction, was adopted by Dukes and Parviainen [12] in giving a bijection between *integer matrices* (upper triangular matrices with non-negative entries such that every row and column contains at least one non-zero entry) and ascent sequences.

Taking advantage of the equivalent definition of interval orders, that the strict predecessor sets can be given a total order under inclusion, Dukes, Jelínek and Kubitzke [10] show an intuitive relation between the integer matrices and interval orders where two elements are related in the poset if they share a hook under the diagonal of the matrix. This approach also allows explanation for the labeled counterpart of integer matrices, composition matrices, which are in bijection with labeled interval orders. Their approach provides a more direct relation between both labeled and unlabeled interval orders and their generating function than had appeared in the literature to that point. Details of the hook construction are given on page 13.

Solving a conjecture of Claesson and Linusson [9], Levande [20, 21] identifies an additional, non-trivial, subclass of matchings enumerated by the Fishburn numbers (the non 2-neighbor-nesting matchings). To this end he adopts the use of signed, filled partition shapes, which follow naturally from the generating function, as an intermediary object upon which an involution is used to identify fixed points.

The *dual* of a poset is the mapping taking a poset $P$ to a poset $P'$ defined by taking all relations $x <_P y$ in $P$ to the relations $y <_{P'} x$ in $P'$. A poset $P$ is called a *self-dual* poset if taking the dual returns an isomorphic poset. Jelínek [17], via generating function manipulation, derives the ordinary generating function for *self-dual* interval orders as

$$\sum_{m\geq 0} \frac{1}{(1-x)^{m+1}} \prod_{i=0}^{m-1} \left( \frac{1}{(1-x)^{i+1}} - 1 \right).$$

**The labeled case.** To study labeled interval orders, Claesson, Dukes and Kubitzke [7] introduce composition matrices, a labeled counterpart to integer matrices. They show that composition matrices have exponential generating function

$$\sum_{m\geq 0} \prod_{i=1}^{m} (1 - e^{-xi}),$$

again a function originally considered by Zagier [26]. They present a one-to-one correspondence between labeled interval orders and composition matrices via the Cartesian product of ascent sequences and set partitions.

Prompted by the bijections of Bousquet-Mélou et al. [4], Brightwell and Keller [6] consider enumeration of labeled interval orders of size $n$, denoted $\ell_n$. They give an asymptotic formula for $\ell_n$, that

$$\ell_n \sim (n!)^2 \frac{12\sqrt{3}}{\pi^{\frac{5}{2}}} \left( \frac{6}{\pi^2} \right)^n \sqrt{n}.$$

**Factorial supersets.** Permutations and inversion tables are obvious supersets with cardinality $n!$ of the permutations and ascent sequences studied by Bousquet-Mélou et al. [4] Claesson and Linusson [9] introduce factorial matchings, a subset of naturally labeled $(2+2)$-free posets satisfying an additional labeling property. Furthermore they demonstrate that matchings with no left-nestings are a natural $n!$ superclass of non-neighbor-nesting matchings.

In addition to identifying the generating function for labeled interval orders, Claesson, Dukes and Kubitzke identify the *partition matrices*, a superset of integer matrices which are counted by $n!$.

Eriksen and Sjöstrand [14] provide bijections between various Fishburn structures enumerated by $n!$—including non left-nesting matchings and permutations—and a class of filled partition shapes. In doing so they find the full distribution for these structures according to statistics where previous work had focused on solely in terms of avoidance. This distribution is the aforementioned Fishburn distribution where the row sums are equal to the factorial.

**Flat interval orders.** Khamis [19] and, independently, Dukes, Kitaev, Remmel and Steingrímsson [11] consider the enumeration of *flat* (alternatively *primitive* or *rigid*) interval orders, those with trivial automorphism group.

The generating function for such posets is the following:

$$\sum_{m\geq 0}\prod_{i=0}^{m}\left(1-\frac{1}{(1+x)^i}\right).$$

Brightwell and Keller [6] provide an asymptotic approximation for the number $r_n$ of flat interval orders of size $n$,

$$r_n \sim n!\frac{12\sqrt{3}}{\pi^{\frac{5}{2}}}e^{\frac{-\pi^2}{12}}\left(\frac{6}{\pi^2}\right)^n\sqrt{n}.$$

The set of interval orders may be formed from the set of flat interval orders by substituting entries in flat interval orders with non-empty sets.

Therefore an immediate consequence from the formula for flat interval orders, although seemingly hitherto never explicitly stated, is that the cycle index series $Z_F(x_1, x_2, x_3, \dots)$ for interval orders may be deduced (for information on the cycle index series see, for example, Bergeron et al. [2, Chapter 1, Section 2]). Substituting the cycle index series for non-empty sets into the previous formula yields:

$$Z_F(x_1,x_2,x_3,\dots)=\sum_{m\geq 0}\prod_{i=1}^{m}\left(1-\frac{1}{\left(\exp\left(\frac{x_1}{1}+\frac{x_2}{2}+\frac{x_3}{3}+\cdots\right)\right)^i}\right).$$

The cycle index series encapsulates information on both labeled and unlabeled structures and it is easily checked that:

$$Z_F(x,x^2,x^3,\dots)=\sum_{m\geq 0}\prod_{i=1}^{m}(1-(1-x)^i),$$

and that:

$$Z_F(x, 0, 0, \dots) = \sum_{m \geq 0} \prod_{i=1}^{m} (1 - e^{-xi}),$$

as would be expected.

### Ascent sequence encoding of interval orders

In this section we demonstrate the bijection, omitting proofs, of Bousquet-Mélou et al. [4] between ascent sequences and interval orders. This is to allow the reader to make comparison with the technique presented in Chapter 2. The bijection is dependent on several properties of interval orders, we lead with their introduction.

As noted earlier, an equivalent definition for an interval order is a poset where the predecessor sets can be linearly ordered by inclusion.

Let $D_0 \subset D_1 \subset \cdots \subset D_k$ be the predecessor sets of some poset $P$. Define a function $h$ on posets returning the number of unique predecessor sets, $h(P) = k + 1$. For $x \in P$ let $d(x)$ be the index of the predecessor set, called the *level* of $x$, i.e. defined as follows.

$$\forall x \in D_i \qquad d(x) = i.$$

Analogous to the predecessor set for an element in a poset is the definition of the successor set.

$$\operatorname{Succ}(x) = \{y \in P : x <_P y\}.$$

An element $x \in P$ is a *maximal* element in the poset if $\operatorname{Succ}(x) = \emptyset$. Let $Y$ be the subset of elements in the underlying set of $P$ which are maximal in $P$.

Key to the bijection is a function giving the minimal predecessor index for maximal elements in the poset.

$$m(P) = \min d(y) \qquad y \in Y.$$

We now detail the bijection.

Let $\upsilon = b_1 b_2 \dots b_n$ be an inversion table (i.e. each $b_i \in [0, i-1]$). An ascent in $\upsilon$ is an $i$ such that $b_i > b_{i-1}$. Let $\operatorname{Asc}(\upsilon)$ be the set of ascents for $\upsilon$ and $\operatorname{asc}(\upsilon) = |\operatorname{Asc}(\upsilon)|$.

We define an ascent sequence as an inversion table such that $b_1 = 0$ and each $b_i \in [0, 1 + \operatorname{asc}(b_1 b_2 \dots b_{i-1})]$. Note that this is a recursive specification.

The empty poset and the empty inversion sequence are defined to be in bijection. For $i \in \mathbb{N}$ let $Q$ be the poset constructed on $b_1 b_2 \dots b_{i-1}$ and let $Y$ be the set of maximal elements in $Q$. The following case analysis on $b_i$ defines the insertion procedure of a new element into the poset $Q$.

**Add 1:** If $b_i \leq m(Q)$ then insert a new entry $x$ with $\operatorname{Pred}(x) = D_i$ and empty successor set.

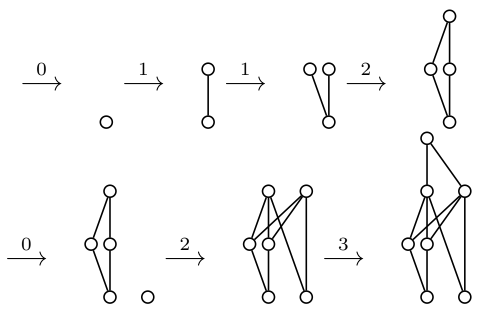
**Add 2:** If $b_i = h(Q)$, add a new entry $x$ covering all entries in $Q$, $\operatorname{Pred}(x) = \{y : y \in Q\}$, and empty successor set.

**Add 3:** If $m(Q) < b_i < h(Q)$ insert a new entry $x$ with $\operatorname{Pred}(x) = D_i$ and empty successor set. Let $M$ be the set of g maximal elements of $Q$ with level less than $i$,

$$M = \{y \in Y : d(y) < i\}.$$

Add additional relations into the poset. For all $x \leq z$ with $z \in M$ set $x <_Q y$ for all $\{y \in Q : i \leq d(y) < h(q)\}$.

Example 1. Consider the ascent sequence 0112023. The construction is as follows.



**Matrix hook bijection**

An *integer matrix* is an upper triangular matrix with non-negative integer entries such that all rows and columns contain at least one non-zero entry. A *composition matrix* is the labeled counterpart: an upper triangular matrix on some underlying set $U$ whose entries are sets which partition $U$ satisfying that there are no rows or columns which contain only the empty set.

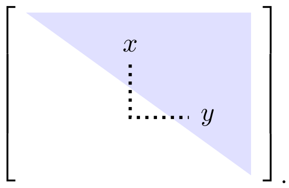



Figure 1. Diagrammatic hook representation

$$\begin{bmatrix} \{1\} & \{6\} & \emptyset & \emptyset \\ & \{4,7\} & \emptyset & \emptyset \\ & & \{2,5\} & \emptyset \\ & & & \{3\} \end{bmatrix} \quad \longleftrightarrow$$

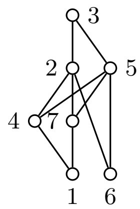



Figure 2. Poset and matrix hook example

Dukes et al. [10] provide a direct bijection between composition matrices and labeled interval orders, equally valid for integer matrices and unlabeled interval orders. We present, proofs omitted, the labeled case. This is for the reader to make comparison to the technique used to enumerate interval orders in Chapter 2 and as background for the adapted surjection we introduce in Chapter 3.

Let $A$ be a composition matrix on $U$. For all $x, y \in U$ if $x$ is an entry in the set in the matrix at position $(i, j)$ and $y$ an element in the set at position $(i', j')$ then define the corresponding poset $P$ by declaring that $x <_P y$ in $P$ if $j < i'$.

This is diagrammatically seen by setting $x <_P y$ in $P$ if the "hook" from $x$ to $y$ passing through $(i', j)$ goes below the diagonal of the matrix as seen in Figure 1. It becomes quickly evident that an equivalent characterization is that the predecessor set of $y$ is the union of columns 1 through $i' - 1$.

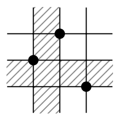
Figure 2 demonstrates this bijection on a labeled variant of our earlier poset.

CHAPTER 2

# Sieved enumeration of Fishburn structures

This chapter details a new technique for the enumeration of Fishburn structures and some consequences thereof. The technique follows from identifying that Eriksen and Sjöstrand's [**14**] refinement of Zagier's formula giving the Fishburn distribution may be written in terms of the $q$-factorial $\boldsymbol{(n)_q!}$.

Eriksen and Sjöstrand consider the distribution of the bivincular pattern

(1)  .

originally introduced by Bousquet-Mélou et al. [**4**] in studying its avoidance. We shall refer to this pattern as $\sigma$ throughout this chapter.

Eriksen and Sjöstrand show that the Fishburn distribution is given by the coefficients $f_{n,k}$ of the following ordinary generating function,

$$\begin{aligned}\sum_{n\geq 0}\sum_{\pi\in\mathfrak{S}_n} x^n y^{\sigma(\pi)} &= \sum_{n\geq 0}\sum_{k\geq 0} f_{n,k}x^n y^k \\ &= \sum_{m\geq 0}(-1)^m \prod_{i=1}^{m}\frac{(1+(y-1)x)^i-1}{1-y}.\end{aligned}$$

By considering the $q$-factorial,

$$\boldsymbol{(n)_q!} = \prod_{i=1}^{n}\frac{1-q^i}{1-q},$$

we note that Eriksen and Sjöstrand's refinement may be written as as the substitution of $q$ by $x(y-1)+1$,

$$\sum_{n\geq 0}\boldsymbol{(n)_{x(y-1)+1}!}x^n.$$

A simple combinatorial explanation corresponding to this generating function substitution exists and forms the basis for the technique in this chapter.

We shall introduce the technique by replicating the result of Eriksen and Sjöstrand [14] counting occurrences of $\sigma$ in permutations. The technique is then formally stated and its application demonstrated by introducing a new Fishburn subset of matchings, those with zero alignments according to the number of arcs that are both left-nesting and right-crossed.

We then construct the factorial posets of Claesson and Linusson [9] marked according to a new feature which we name *mislabelings*. We show that factorial posets with respect to the number of mislabelings follow the Fishburn distribution. A factorial poset with zero mislabelings satisfies a condition of Claesson and Linusson giving that the poset is a canonically labeled interval order.

Our technique gives a close correspondence between certain Mahonian structures (structures enumerated by the $q$-factorial) and the Fishburn structures. Due to the manner in which contemporary interest in interval orders has been prompted, by the introduction of bivincular patterns, it is natural to consider linear combinations of patterns which follow the Fishburn distribution. We present an example motivated from work of Claesson and Brändén [5] and briefly discuss the limitations of our technique towards this purpose.

As a further consequence of the relationship to the $q$-factorial we provide a new identity for coefficients $f_{n,k}$ of the Fishburn distribution with respect to the Mahonian numbers $m_{n,k}$ (A008302), that

$$f_{n,k} = \sum_{i=k}^{n-2} (-1)^{i+k} \binom{i}{k} \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j}.$$

Of particular interest is when $k = 0$, which gives an identity for the $n$th Fishburn number (A022493)

$$\sum_{i=0}^{n-2} (-1)^{i} \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j}.$$

### Terminology and background

For $a, b \in \mathbb{Z}$ with $a < b$ let $[b]$ denote the set $\{1, \ldots, b\}$ and $[a, b]$ the set $\{a, \ldots, b\}$.

$$\begin{array}{c} 1 \\ 1,\ 1 \\ 1,\ 2,\ 2,\ 1 \\ 1,\ 3,\ 5,\ 6,\ 5,\ 3,\ 1 \\ 1,\ 4,\ 9,\ 15,\ 20,\ 22,\ 20,\ 15,\ 9,\ 4,\ 1 \\ 1,\ 5,\ 14,\ 29,\ 49,\ 71,\ 90,\ 101,\ 101,\ 90,\ 71,\ 49,\ 29,\ 14,\ 5,\ 1 \\ \dots \end{array}$$

FIGURE 1. Mahonian Triangle (A008302), $m_{n,k}$

For $U$, a linearly ordered set, and $x \in U$ *not* the maximal element of $U$ then, where there is no ambiguity, we shall abuse notation and use $x + 1$ to refer to the immediate successor of $x$ in $U$.

**Mahonian numbers.** For $n \in \mathbb{N}$ let $\boldsymbol{(n)_q!}$ denote the $q$-factorial, defined as

$$\boldsymbol{(n)_q!} = \prod_{i=1}^{n} \sum_{j=0}^{i-1} q^j = \prod_{i=1}^{n} \frac{1-q^i}{1-q}.$$

The coefficients of the $q$-factorial are known as the Mahonian numbers (A008302). The first few terms are shown in Figure 1. We shall use $m_{n,k}$ to denote the $k$th entry of row $n$.

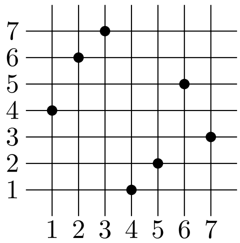
Mahonian numbers derive their name from seminal work identifying permutation statistics by Major MacMahon [**22**]. As a result, and particularly in the case of permutations, structures counted by the $q$-factorial are often referred to as *Mahonian structures*.

**Permutation patterns.** A permutation is a bijection on a finite set $U$. The results in this chapter assume that there is a total order on $U$. We shall therefore assume throughout that, for $n \in \mathbb{N}$, permutations as elements of $\mathfrak{S}_n$ are bijections on the set $[n]$.

For $n, k \in \mathbb{N}$ with $n > k$ take permutations $\pi \in \mathfrak{S}_n$ and $\tau \in \mathfrak{S}_k$. An occurrence of $\tau$ as a *classical permutation pattern* in $\pi$ is a subsequence of $\pi$ whose entries are in the same relative order as in $\tau$. For example taking $\tau = 132$ and $\pi = 4671253$ then the following subsequences of $\pi$ correspond to occurrences of $\tau$,

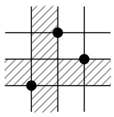
$$465 \quad 475 \quad 153 \quad 253.$$

Permutations may be represented on a grid by dots placed at line intersects such that each line is intersected by exactly one dot. The permutation maps the value of the vertical line to the value of the corresponding horizontal line indicated by the dot placement. For example, the grid below represents the permutation 4671253.




A *mesh pattern*, introduced by Brändén and Claesson [5], consists of a classical permutation pattern and a (potentially empty) set of shaded boxes on the grid representation of that pattern. An occurrence of a mesh pattern consists of an occurrence of the underlying classical permutation such that there are no entries of $\pi$ contained within the shaded boxes.

For example, there are two occurrences of the following mesh pattern in $\pi$,

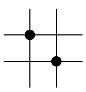
 .

Namely 465 and 253. Whereas, although an occurrence of the underlying classical pattern, 475 is not an occurrence of the above mesh pattern as 6 occurs between the 4 and 7.

A *vincular pattern* is a mesh pattern where only entire columns may be shaded out. A *bivincular pattern* is a mesh pattern where any shaded boxes must contribute to an entire row or column of shaded boxes. The above mesh pattern is also a bivincular pattern.

A permutation with no occurrences of a pattern is said to *avoid* that pattern.

Occurrences of the pattern

 ,

are known as *inversions* and are counted by the $q$-factorial (see MacMahon [22]).

**Posets.** A poset $P$ is defined as a set and an associated binary relation $<_P$ satisfying reflexivity, antisymmetry, and transitivity. A poset constructed on some linearly ordered set $U$ is said to be *naturally labeled* if $i <_P j \implies i <_U j$.

An *interval order* is a poset $P$ where each $z \in P$ can be assigned a closed interval $[\ell_z, r_z] \subset \mathbb{R}$ such that $x <_P y$ if and only if $r_x < \ell_y$. Equivalent conditions are that an interval order is a poset whose predecessor sets can be assigned a total order by inclusion [3] or that a poset is an interval order if it has no induced subposet isomorphic to the pair of disjoint two element chains, i.e. the poset is (2+2)-free [15].

For $i \in P$, let the following notation be used for the predecessor and successor sets of $i$:

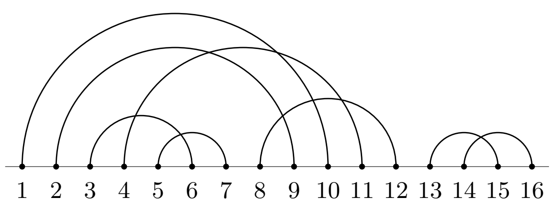
$$\operatorname{Pred} i = \{j \in P : j <_P i\}, \qquad \operatorname{pred} i = |\operatorname{Pred} i|,$$
$$\operatorname{Succ} i = \{\ell \in P : i <_P \ell\}, \qquad \operatorname{succ} i = |\operatorname{Succ} i|.$$

**Matchings.** A matching of size $n$ is a fixed point free involution of semi-length $n$. Matchings are typically represented as a set of ordered pairs $(i, j)$ such that $i < j$. The first entry in the pair is referred to as the arc *opener* and the second entry the *closer*.

Diagrammatically matchings are seen as arcs on the numberline $[2n]$. For example, the matching of size 10

$$\{(1,10),(2,9),(3,6),(4,11),(5,7),(8,12),(13,15),(14,16)\},$$

is represented as




A *nesting* arc in a matching is an arc which entirely encloses another arc when seen diagrammatically, i.e. an $(i, j)$ such that there exists $(k, \ell)$ with $i < k < \ell < j$. The arc which is enclosed is known as a *nested* arc. If $k = i + 1$ then the arcs are called *left-nesting* and *left-nested*, respectively. If $\ell + 1 = j$ then the arcs are *right-nesting* and *right-nested*, respectively.

For example, in the above matching $(2, 9)$ is a nesting arc with respect to the nested arc $(3, 6)$. Furthermore $(2, 9)$ is left-nesting with $(3, 6)$ left-nested.

A *crossing* arc in a matching is the leftmost of two intersecting arcs when seen diagrammatically, i.e. an $(i, j)$ such that there exists $(k, \ell)$ with $i < k < j < \ell$. The same approach as for nestings is taken to define *crossed, left-crossing, left-crossed, right-crossing* and *right-crossed* arcs.

In the above matching $(4, 10)$ is a crossing arc with $(8, 12)$ a crossed arc. Furthermore $(4, 10)$ is a right-crossing with $(8, 12)$ right-crossed,

An alignment in a matching is two arcs $(i, j)$ and $(k, \ell)$ such that $i < j < k < \ell$. For example, $(8, 12)$ and $(13, 15)$ in the above matching are alignments.

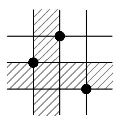
For two arcs $(i, j)$ and $(k, \ell)$, we say that $k$ is an *embraced nested opener* if $k$ is the opener for an arc nested by $(i, j)$.

**Statistics and features.** Given some set of structures $X$ a *statistic* $\psi$ is defined as a function taking a structure to a natural number, i.e. $\psi : X \to \mathbb{N}$.

A *feature* of a structure is a property, aspect or substructure of a combinatorial structure. For example, an inversion in a permutation, or a nesting in a matching are features.

### Original Fishburn permutation

We lead with a previously studied example. Recall the mesh pattern

$\sigma =$  ,

with avoidance originally given by Bousquet-Mélou et al. [4] and the full distribution given by Eriksen and Sjöstrand [14].

In their paper Eriksen and Sjöstrand show a bijection between permutations and filled partition shapes by using the filled entries in the partition shapes to encode the insertion of elements into an ordered list of blocks. Upon completion the block structure is dropped and the elements read left-to-right return the permutation. Their bijection allows that multiple statistics are equidistributed between the two structures and through this they provide the non-commutative generating function with respect to those statistics.

In this section we shall focus on a small part of their work by considering the distribution of occurrences of $\sigma$ in isolation from other statistics. This differs from the work of Eriksen and Sjöstrand in that the proof is based on insertion of entries into a permutation rather than encoding the construction. Our application of the sieve principle is the same.

We begin with the fact that the number of inversions in permutations follow the Mahonian distribution. To construct a permutation of size $n$ with $i$ marked occurrences of $\sigma$ take a permutation of size $n-i$ with $i$ marked inversions. Each marked inversion will be used to insert a new entry which is the first entry of an occurrence of $\sigma$. The sieve principle will then be applied to return those permutations satisfying that *all* occurrences of $\sigma$ are marked.

Define an order on inversions based on the position in the permutation of the first entry in the tuple and value of the second entry in the tuple. For a permutation $a_1a_2\dots a_n$ let $(a_i, a_j)$ and $(a_{i'}, a_{j'})$ be two inversions. If $i = i'$ it follows $a_i = a_{i'}$ and, without loss of generality, we can assume $a_j < a_{j'}$. We then define

$$(a_i, a_j) < (a_{i'}, a_{j'}).$$

Otherwise $i \neq i'$ then, without loss of generality, assume $i < i'$. Then we define

$$(a_i, a_j) < (a_{i'}, a_{j'}).$$

As an example, in the permutation 246531 the following inversions are sorted

$$(4, 1) < (6, 1) < (6, 5).$$

In the above order, each inversion $(a_j, a_k)$ is used to insert a new entry into the permutation. Taking the position before the leftmost entry to be position 0, increment all $a_i > a_k$ by one and insert $a_k + 1$ at position $j - 1$. Thus an occurrence of $\sigma$ is created.

EXAMPLE 2. Take the permutation 246531 where we consider the following inversions to be marked

$$(4,1) < (6,1) < (6,5).$$

As the values in the inversions change, at each step the next inversion to be used will be colored in red. Inserted entries will be marked blue.

The inversion $(4,1)$ is the first inversion under our defined order. Increase all entries greater than 1 by 1

$$357641,$$

and insert 2 at position 1

$$3257641.$$

The next inversion is now labeled $(7,1)$ with the 7 at position 4. Increase all entries greater than 1 and insert 2 at position 3

$$43628751.$$

Applying the process to the final inversion, now labeled $(8,7)$, leads to the permutation

$$436289751.$$

Note that the inserted entries (marked blue) are all the first entries in an occurrence of $\sigma$.

PROPOSITION 3. *The above procedure describes a bijection between permutations of length $n$ with $k$ marked inversions and permutations of length $n+k$ with $k$ first entries in an occurrence of $\sigma$ marked.*

PROOF. To show that this mapping is well defined we need to demonstrate that at each step the insertion of an entry does not remove an occurrence of $\sigma$ previously inserted by this process.

This is enforced by the ordering defined on inversions. Let $(a_i, a_j)$ and $(a_{i'}, a_{j'})$ be inversions.

(1) If $i = i'$ then $a_j < a_{j'}$ and therefore $(a_i, a_j) < (a_{i'}, a_{j'})$. Our insertion process gives that $a_{j'} + 1$ is inserted in the position immediately following that of $a_j$ and thus forming an ascent with $a_j$. Furthermore as $a_{j'} > a_j$, the minimal entry in the occurrence of $\sigma$ that $a_j$ is contained in is not incremented. Therefore the occurrence is preserved with the newly inserted entry $a_{j'}$ taking the role of the largest entry in the occurrence.
(2) If $i < i'$ then $(a_i, a_j) < (a_{i'}, a_{j'})$. As $a_{j'} + 1$ is inserted further to the right in the permutation the ascent that $a_i$ involved cannot be broken. If $a_j < a_{j'}$ the minimal entry in the occurrence of $\sigma$ remains unchanged. If $a_j > a_{j'}$ then all entries in the occurrence of $\sigma$ containing $a_j$ are incremented. If $a_j = a_{j'}$ then $a_{j'}$ replaces the minimal entry of the occurrence of $\sigma$ containing $a_j$.

Thus the mapping is well defined. To show that the mapping is a bijection we demonstrate that it is both injective and surjective.

Injectivity is enforced by the total order on inversions and that an inversion pair uniquely determines the entry which is inserted.

For surjectivity note that the process we have defined inserts the first entries of marked occurrences of $\sigma$ in a left-to-right order within the permutation. We can consider the reverse of the insertion operation taking a permutation with marked occurrences of $\sigma$ to a permutation with marked inversions.

Given a permutation of size $n$ with marked occurrences of $\sigma$, take the rightmost marked occurrence. Removing the first entry contained in the occurrence and standardizing the permutation leaves a permutation of size $n - 1$ with a marked inversion. Surjectivity follows from repeated application. $\square$

Corollary 4. *Permutations with marked occurrences of $\sigma$ are given by the ordinary generating function*

$$u(x,z) = \sum_{n\geq 0} (\boldsymbol{n})_{\boldsymbol{xz+1}}\boldsymbol{!}x^n,$$

*where the coefficient of $x^n z^k$ gives the number of permutations of length $n$ with $k$ marked occurrences of $\sigma$.*

Proof. Permutations with respect to inversions are enumerated by the $q$-factorial. Under the above process an inversion is either marked, in which case a new entry uniquely specifying a marked occurrence of $\sigma$ is inserted, or it is unmarked. This is equivalent to the substitution $(xz+1)$ in place of $q$ in the $q$-factorial with the marking of the occurrence of $\sigma$ denoted by $z$. □

Recreating Eriksen and Sjöstrand's result we now apply the sieve principle to permutations with *subsets* of occurrences of $\sigma$ marked returning those with *all* occurrences of $\sigma$ marked. For more details on this varient of the sieve principle see Wilf [**25**, Chapter 4, Section 2].

Corollary 5. *Permutations with respect to occurrences of $\sigma$ are given by the ordinary generating function*

$$\sum_{n\geq 0} (\boldsymbol{n})_{\boldsymbol{x(y-1)+1}}\boldsymbol{!}x^n,$$

*where the coefficient of $x^n y^k$ gives the number of permutations of length $n$ with* exactly *$k$ occurrences of $\sigma$.*

Proof. The previous corollary gives that permutations with respect to marked occurrences of $\sigma$ are given by the ordinary generating function

$$u(x,z) = \sum_{n\geq 0} (\boldsymbol{n})_{\boldsymbol{xz+1}}\boldsymbol{!}x^n.$$

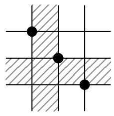
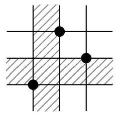
In this set a permutation with $k$ marked occurrences of $\sigma$ occurs a total of $\binom{k}{i}$ times with $i$ occurrences of $\sigma$ marked.

Let $f(x, y)$ be the ordinary generating function for permutations with *all* occurrences of $\sigma$ marked. Consider the substitution of $y$ by $z + 1$. This corresponds to remarking occurrences of $\sigma$ with a $z$, or unmarking them them with the 1. As such each permutation will occur $\binom{k}{i}$ times with $i$ occurrences of $\sigma$ now marked by $z$. Thus we have that

$$u(x, z) = f(x, z + 1).$$

The result then follows through the reverse substitution of $z$ by $y-1$ into $u(x, z)$. $\square$

Remark 6. The distributions of

 and 

given by Eriksen and Sjöstrand [14] can be shown in a near identical manner. Again the key is to note that in occurrences of these patterns each has a point whose value and position are uniquely determined by the other points and that together these other two points form an inversion.

## Technique

We can generalize the previous two corollaries to explicitly state a new technique for constructing Fishburn structures. We present it as the following theorem.

Theorem 7. *Let $\mathcal{F}$ be a Mahonian stucture according to the distribution of some $q$-feature.*

*$\mathcal{F}$ follows the Fishburn distribution with respect to some feature $p$ if we can show that there is a bijection between $\mathcal{F}$ structures of size $n$ with $k$ marked $q$-features and $\mathcal{F}$ structures of size $n + k$ with $k$ marked $p$-features.*

Proof. By definition, the distribution of $q$-features in $\mathcal{F}$ follows the ordinary generating function

$$\sum_{n \geq 0} \boldsymbol{(n)_q!} x^n.$$

Take $\mathcal{F}$ with subsets of $q$-features marked by some variable $w$. As a $q$-feature is either marked or it is not then the generating function for such structures is given by the substitution of $q$ by $w+1$ into the previous equation. We therefore have

$$\sum_{n\geq 0} \boldsymbol{(n)_{w+1}!} x^n.$$

We now use that there exists a bijection between $\mathcal{F}$ structures of size $n$ with $k$ marked $q$-features and $\mathcal{F}$ structures of size $n+k$ with $k$ marked $p$-features. This allows that subsets of $q$-features marked with $w$ can be taken to subsets of $p$-features marked by $z$ with the inclusion of an additional element. In terms of generating function this corresponds to the substitution of $w$ by $xz$.

Therefore the ordinary generating function of $\mathcal{F}$ structures with subsets of marked $p$-features is

$$\sum_{n\geq 0} \boldsymbol{(n)_{xz+1}!} x^n.$$

If subsets of $p$-features are marked, then each $\mathcal{F}$ structure occurs $\binom{k}{j}$ times with $j$ marked $p$-features. By the sieve principle (see Wilf [**25**, Chapter 4, Section 2]), as in the previous corollary, through the substitution of z by $y-1$ it then follows that $\mathcal{F}$ structures with *all* $p$-features marked are given by the ordinary generating function for the Fishburn distribution

$$\sum_{n\geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n.$$

□

**Zero alignment matchings**

In this section we apply Theorem 7 to identify a new Fishburn statistic on a subset of matchings. Explicitly, matchings with zero alignments follow the Fishburn distribution according to the number of arcs which are both left-nesting and right-crossed.

Recall that two arcs $(i,j)$ and $(k,\ell)$ are an alignment if $i<j<k<\ell$.

A matching with no alignments (a zero alignment matching) is equivalently characterized as one where *all* the openers in the diagrammatic representation occur before *all* the closers.

The following proposition is well known.

Proposition 8 (Folklore). *There are $n!$ zero alignment matchings of semi-length $n$.*

*Furthermore they are enumerated by $(\boldsymbol{n})!$ when refined according to the number of nestings.*

Proof. This is easiest seen via recursion with a bijection between matchings with no alignments and inversion tables. Take the empty matching and the empty inversion table to be in bijection.

Let $b_1 b_2 \dots b_n$ be an inversion table with each $b_i \in [0, i-1]$ and $M$ the matching constructed from $b_1 b_2 \dots b_{n-1}$. Label the position to the left of the first closer as 0 and label the positions to the left of an opener right-to-left from 1 to $n-1$. Insert a new arc into $M$ with opener at position $b_n$ and closer at the rightmost position in the matching.

By construction inserted openers occur to the left of all the closers and it is easy to see that entries in the inversion table correspond to the number of nested arcs. $\square$

Recall that a left-nesting arc is an arc $(i, j)$ such that there exists an arc $(i+1, \ell)$ with $\ell < j$. Recall also that $(i, j)$ is right-crossed if there exists an arc $(k, j-1)$ with $k < i$. We shall call an arc which is both left-nesting and right-crossed a *confused* arc.

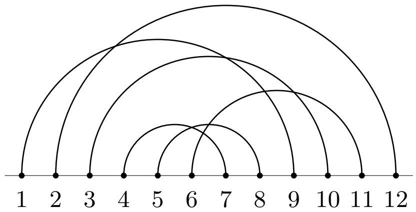
Define an order on embraced nested openers. We shall write embraced nested openers as ordered pairs. Take $((i, j), k)$ and $((i', j'), k')$ where $k$ and $k'$ are openers with $(i, j)$ an arc embracing $k$ and $(i', j')$ an arc embracing $k'$. If $k = k'$ then, without loss of generality, assume $j < j'$ and define

$$((i', j'), k') < ((i, j), k).$$

Otherwise, without loss of generality, assume $k < k'$ and define

$$((i', j'), k') < ((i, j), k).$$

For example, take the matching $\{(1, 9), (2, 12), (3, 10), (4, 7), (5, 8), (6, 11)\}$.

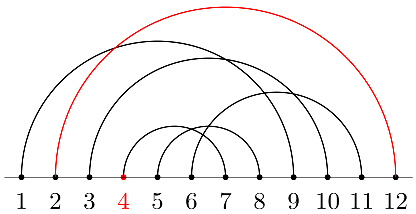



The following subset of embraced nested openers are sorted:

$$((2,12),4) \quad < \quad ((1,9),4) \quad < \quad ((2,12),3).$$

Given a matching with a subset of embraced openers marked, using the above order, for each embraced nested opener $((i,j),k)$ insert a new arc opening immediately to the left of the embraced nested opener $k$ and closing immediately to the right of arc closer $j$. As $i < k$, the new arc is therefore right-crossed, furthermore as the arc with opener $k$ is nested by $(i,j)$ it follows that the newly inserted arc left nests the arc with opener $k$. As both right-crossed and left-nesting the inserted arc is confused.

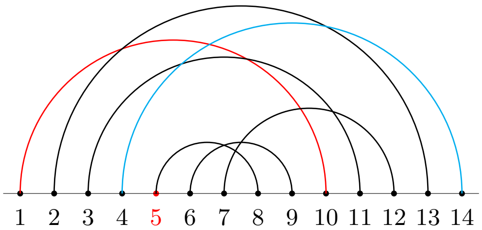
EXAMPLE 9. We demonstrate on our example matching.



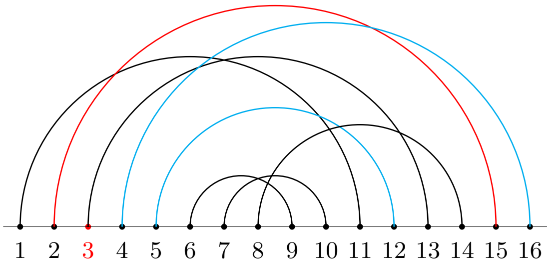

Consider the following nested openers marked.

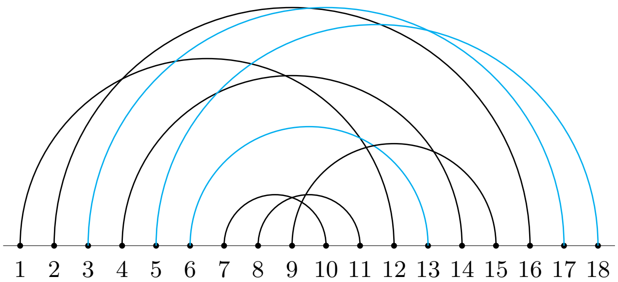
$$((2,12),4) \quad ((1,9),4) \quad ((2,12),3)$$

As in the example for permutations the next nested opener to be considered will be colored red and inserted arcs blue. Inserting a confused arc from the first embraced nested opener results in the following matching.




The next two steps are as follows.




It is easily checked that the inserted arcs are confused and their removal returns the original matching.

PROPOSITION 10. *The above is a bijection between zero alignment matchings of semi-length $n$ with $k$ marked embraced openers and zero alignment matchings of semi-length $n+k$ with $k$ marked confused arcs.*

PROOF. We are required to show that at each stage the process is well defined: that no alignments are introduced and that no previously inserted confused arc has its left nesting or right crossed attributes removed.

That no alignment is introduced can be seen by contradiction. As the inserted arc has its opener to the left of an existing opener and its closer to the right of an existing closer no new alignment can be introduced if the original matching was a zero alignment matching.

That each step of the process does not break the right-crossed or left-nesting property of a previously inserted arc is given by the order on nested openers. If two inserted arcs share the same opener as part of their nested opener, then that both arcs are still left nesting is given by the order on nesting arc closers. If two inserted arcs share the same nesting arc closer as part of their nested opener, then that both arcs are right nesting is given by the order of the opener.

Each inserted arc has its opener and closer uniquely determined by the nested opener. Furthermore it is clear that removing inserted arcs returns the original matching. Injectivity and surjectivy are thus simple. □

The following corollary then results from Theorem 7 and the above proposition.

COROLLARY 11. *Zero alignment matchings with respect to confused arcs follow the Fishburn distribution.*

### Factorial posets

Identifying appropriate statistics and applying the technique given by Theorem 7 to the factorial posets of Claesson and Linusson [**9**] allows for a new method for the enumeration of interval orders which differs from both the recursive construction of Bousquet-Mélou et al. [**4**] and the matrix hook bijection of Dukes, Jelínek and Kubitzke [**10**].

Claesson and Linusson [**9**] define the *factorial posets*, a set of labeled interval orders counted by $n!$, as follows. A factorial poset $P$ on some linearly ordered underlying set $U$ is a naturally labeled poset with the additional condition that, for $i, j, k \in U$,

$$i <_U j <_P k \implies i <_P k.$$

This is referred to as the *factorial condition*.

Easily seen to be equivalent, a poset is factorial if and only if for each $k \in P$ there exists $j \in [0, k-1]$ such that $\operatorname{Pred} k = [1, j]$. As the predecessor sets can be linearly ordered by inclusion it follows that factorial posets are a subset of naturally labeled interval orders.

Claesson and Linusson take advantage of this by using entries of an inversion table to encode the construction of a factorial poset, thus giving that the two structures are in bijection. We include their result for completeness.

THEOREM 12 (Claesson and Linusson [9]). *Factorial posets on* $[n]$ *are in bijection with inversion tables of length* $n$.

PROOF. As a poset is factorial if and only if for all $k \in P$ there exists $j \in [0, k-1]$ such that $\operatorname{Pred} k = [1, j]$. An inversion table $b_1 b_2 \ldots b_n$ is given by setting $b_k$ to the value $j \in [0, k-1]$. $\square$

Claesson and Linusson identify numerous statistics preserved by their bijection. In particular that the number of *incomparable pairs* in factorial posets, defined as

$$|\{(i,j) \in P \times P : i \not<_P j, i <_U j\}|,$$

are counted by the $q$-factorial.

Taking two factorial posets to be equivalent if they are structurally isomorphic, Claesson and Linusson demonstrate that posets satisfying that for all $i \in [n-1]$

$$\operatorname{pred} i \leq \operatorname{pred}(i+1) \quad \text{or} \quad \operatorname{succ} i > \operatorname{succ}(i+1)$$

are unique representatives of their equivalence class.

Again we include their result.

PROPOSITION 13 (Claesson and Linusson [9]). *There is exactly one way to label a* $(2+2)$*-free poset such that it satisfies*

$$\operatorname{pred} i \leq \operatorname{pred}(i+1) \quad \textit{or} \quad \operatorname{succ} i > \operatorname{succ}(i+1).$$

PROOF. A poset satisfying the above condition has that for all $i \in [n]$ the pairs

$$(\operatorname{succ} i, \operatorname{pred} i)$$

are weakly decreasing on the first coordinate and weakly increasing on the second. The factorial condition gives that for $i, j \in [n]$ the pairs $(\operatorname{succ} i, \operatorname{pred} i)$ and

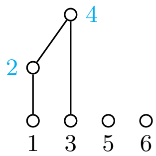
$(\operatorname{succ} j, \operatorname{pred} j)$ are equal if an only if the pairs are indistinguishable within the poset, thus giving a canonical labeling. □

We extend this notion to consider a new feature on factorial posets, explicitly elements which *fail* to satisfy this property.

DEFINITION 14 (Mislabeling). Define a mislabeling in a factorial poset on $[n]$ to be an $i \in [n-1]$ such that

$$\operatorname{pred} i > \operatorname{pred}(i+1) \quad \text{and} \quad \operatorname{succ} i \leq \operatorname{succ}(i+1).$$

EXAMPLE 15. The poset

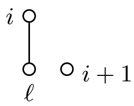



has the set of mislabelings $\{2, 4\}$.

By definition a factorial poset with zero mislabelings satisfies the condition from Proposition 13 and is thus a unique representative of its isomorphism class.

A consequence of the factorial condition is that if $\operatorname{pred} i > \operatorname{pred}(i+1)$ then $i$ and $i+1$ are incomparable as if $i <_P i+1$ then the factorial condition requires that for all $\ell <_U i \implies \ell <_P i+1$. Furthermore, there must exist $\ell$ such that $\ell <_P i$ but that $\ell \not<_P i+1$.

Therefore an equivalent condition to $\operatorname{pred} i > \operatorname{pred}(i+1)$ is that there exists an induced subposet isomorphic to $(2+1)$ with the following labeling




**Sieved enumeration of interval orders.** Recall that we write incomparable pairs as $(i, j)$ with $i <_U j$.

Let $U$ be some linearly ordered set with $|U| = n$. For some $k \in [0, n-1]$ take a poset $P$ built on the first $n-k$ elements of $U$ with $k$ marked incomparable pairs.

Define an order on incomparable pairs. Let $(i, j)$ and $(i', j')$ be two pairs. If $j = j'$ without loss of generality assume $i <_U i'$. Then we define

$$(i', j') < (i, j).$$

Otherwise $j \neq j'$ then, without loss of generality, assume $j <_U j'$. Then we define

$$(i, j) < (i', j').$$

To illustrate, the following pairs are sorted according to the above order.

$$(2, 3) < (1, 3) < (4, 6) < (3, 6).$$

In this order, each pair $(i, j)$ is then used to insert a new element into the poset. This new element has predecessor set

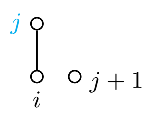
$$\{h \in P : h \leq_U i\},$$

and successor set

$$\operatorname{Succ} j.$$

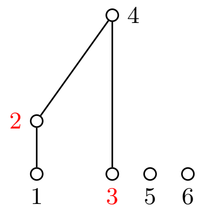
Increment all $k \in P$ with $k \geq_U j$ to its immediate successor in $U$, giving the newly inserted element the value $j$.

By definition this introduces an occurrence of




into the new poset with the inserted element marked in blue. Furthermore as the successor set of the inserted element $j$ is equal to that of the successor set of the element now labeled $j + 1$ it follows that the newly inserted element with label $j$ is a mislabeling.

Example 16. Consider our earlier factorial poset built on [6].



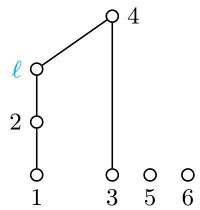

We consider the following incomparable pairs to be marked

$$(2,3) < (1,3) < (4,6) < (3,6)$$

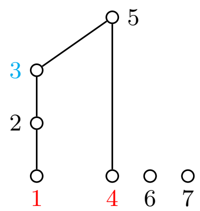
As the values within the pairs change at each stage we will denote the next incomparable pair to be used in red. Inserted elements will be colored blue.

The pair $(2,3)$ specifies the new element $\ell$ to be inserted defined by

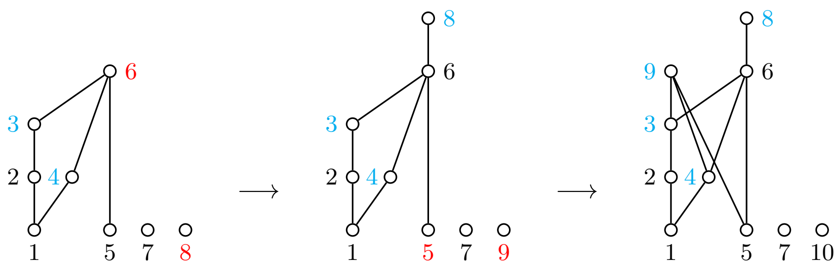
$$\operatorname{Pred} \ell = \{h \in U : h \leq_U 2\} = \{1,2\} \quad \text{and} \quad \operatorname{Succ} \ell = \operatorname{Succ} 3 = \{4\}.$$




All elements with label greater or equal to 3 are incremented by one and the newly inserted element $\ell$ is given the label 3.




The remaining steps are as follows.



Thus we have returned a poset of size 10 with set of marked mislabelings $\{3,4,8,9\}$.

PROPOSITION 17. *The procedure described above gives a bijection between factorial posets on $n$ entries with $k$ marked incomparable pairs and factorial posets on $n+k$ entries with $k$ marked mislabelings.*

PROOF. We shall first show that the process described above is well defined. This is equivalent to showing that at each insertion the following properties are preserved for the resulting poset: it is naturally labeled, it satisfies the factorial condition and it does not remove any mislabelings previously inserted by the process.

For the incomparable pair $(i, j)$ and newly inserted element $\ell$ we have that $\ell$ covers all $\{h \in P : h <_U i\}$ and thus by construction it is both naturally labeled and satisfies the factorial condition. Elements smaller than $j$ under $U$ remain unchanged. The newly inserted element is given the same successor set as the element which previously had that label and thus the insertion does not break the factorial condition for any element larger under $U$ than $j$. This also ensures that the naturally labeled property is preserved.

That no previously inserted mislabelings are removed by the process is given by the order on incomparable pairs and the factorial property, thus the process is a mapping between factorial posets with marked incomparable pairs to factorial posets with marked mislabelings.

Next we show that the mapping is bijective. That it is injective follows from the total order defined on incomparable pairs and that the insertion of a new element is uniquely determined by an incomparable pair.

It remains to show surjectivity. The process we have defined inserts mislabelings in order according to $U$. We can consider the reverse of the insertion operation taking a factorial poset with marked mislabelings to a factorial poset with marked incomparable pairs.

Given a factorial poset of size $n$ with $k$ marked mislabelings take the mislabeling with the largest value $j$ and remove it from the poset. As $j$ is a mislabeling there exists $\ell <_P j$ such that $\ell \not< j+1$. Take the largest such $\ell$ and mark the incomparable pair consisting of $(\ell, j+1)$. Thus we have returned a poset of size $n-1$ with $k-1$

marked mislabelings and 1 marked incomparable pair. Surjectivity follows from repeated application.

As it is both surjective and injective the mapping is a bijection. □

The following corollary then results from Theorem 7 and Proposition 17.

COROLLARY 18. *Factorial posets follow the Fishburn distribution according to the number of mislabelings.*

Substitution of $y = 0$ into the ordinary generating function for the Fishburn distribution,

$$\sum_{n\geq 0} (\boldsymbol{n})_{\boldsymbol{x(y-1)+1}}\boldsymbol{!} x^n|_{y=0},$$

returns the ordinary generating function for factorial posets with no mislabelings. Proposition 13 gives that such posets are unique representatives of their isomorphism class thus yielding, as expected, the result of Bousquet-Mélou et al. [4] that the generating function for unlabeled interval orders is given by

$$\sum_{n\geq 0} (\boldsymbol{n})_{\boldsymbol{-x+1}}\boldsymbol{!} x^n.$$

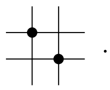
## Linear Combination of Mesh Patterns

In order to motivate this section we briefly review the connections between permutation patterns, Fishburn structures and the Mahonian distribution.

Recall that contemporary study of Fishburn structures has been motivated by the introduction of *bivincular patterns* [4], which were in turn given as a generalization of the *vincular patterns* (né *generalized patterns*) of Babson and Steingrímsson [1]. Furthermore the mesh patterns of Claesson and Brändén [5] evolve as an abstraction of bivincular patterns. We therefore have the following hierarchy of patterns.

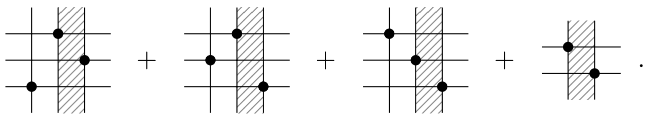
$$\text{classical} \subset \text{vincular} \subset \text{bivincular} \subset \text{mesh}$$

The introduction of vincular patterns by Babson and Steingrímsson was in order to provide a unifying framework via permutation patterns for certain Mahonian permutation statistics, arguably the most simple of which is the number of inversions corresponding to the classical pattern



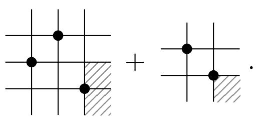
They identify several examples where the sum of occurrences contained in a linear combination of vincular patterns is used to encode both previously studied and new statistics which follow the Mahonian distribution on permutations.

For example, the *major index* on permutations is defined as being the total sum of descent positions. Babson and Steingrímsson identify this as corresponding to the following linear combination of vincular patterns,



As this chapter has identified an explicit link between Mahonian and Fishburn structures and that the study of these structures has been strongly influenced by the aforementioned hierarchy of pattern types it is therefore natural to consider linear combinations of patterns which follow the Fishburn distribution. This section presents one such non-trival example which is based upon a linear combination of mesh patterns introduced by Claesson and Brändén [5].

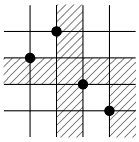
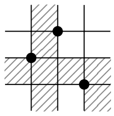
Claesson and Brändén demonstrate that occurrences of the following patterns in permutations are Mahonian.



(we take a trival symmetry of the patterns appearing in the original paper)

Checking via computer where a new entry may be inserted into the above patterns such that the inserted entry has its value and position fixed by pre-existing entries within the pattern leads us to conjecture the following proposition.

PROPOSITION 19. *The linear combination of mesh patterns*

$q_1 + p_2 =$  $+$  ,

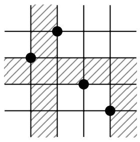
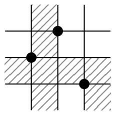
*follows the Fishburn distribution.*

Unfortunately it is not obvious that Theorem 7 can be applied here so we adopt a more direct approach.

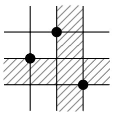
Recall $\sigma$ defined on page 15. An occurrence $abc$ of $\sigma$ in a permutation satisfies one of two conditions:

(1) There exists $d$ with $d < c$ occurring to the right of $c$ in the permutation.
(2) There is no such $d$.

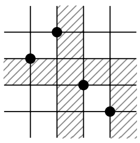
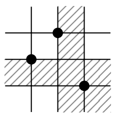
Thus $\sigma$ can be viewed as the following linear combination of mesh patterns.

$p_1 + p_2 =$  $+$  .

In addition to showing that $\sigma$ follows the Fishburn distribution, Eriksen and Sjöstrand show that occurrences of the following are equidistributed with $\sigma$.

$\upsilon =$  .

In a similar manner as before consider the following decomposition of $\upsilon$.

$q_1 + q_2 =$  $+$  .

The patterns in Proposition 19 are the combination of patterns $q_1$ and $p_2$. We prove Proposition 19 via an involution on permutations which takes occurrences of $p_1$ to occurrences of $q_1$ and vice versa whilst preserving the number of occurrences of $p_2$. The proof is provided later in this section. We begin with an observation on occurrences of $p_1$ and $q_1$.

LEMMA 20. *For some fixed permutation let abcd and $a'b'c'd'$ be occurrences of $p_1$. If $b = b'$ then the occurrences are equal i.e. $a = a'$, $b = b'$, $c = c'$ and $d = d'$.*

*Similarly let $efgh$ and $e'f'g'h'$ to be occurrences of $q_1$. If $f = f'$ then the occurrences are equal.*

PROOF. In an occurrence of $p_1$ the entry corresponding to $b$ must immediately follow the entry corresponding to $a$ in the permutation, as $b = b'$ it then follows that $a = a'$. The value of $a$ fixes the value of $c$ and thus $c = c'$. The entry for $d$ is fixed as the first right-to-left minima occurring after $c$ in the permutation, thus $d = d'$.

The proof to the second statement is similar to the above, mutatis mutandis. □

As the previous lemma shows that the second entry in occurrences of both $p_1$ and $q_1$ fix the remaining entries we define an involution on basis of the following cases.

(1) Occurrences of $p_1$ in which the second entry is also the second entry of an occurrence of $q_1$ are left unmodified.
(2) For all occurrences *abcd* of $p_1$ in which $b$ is not the second entry in an occurrence of $q_1$ move the entry in the permutation corresponding to $b$ to the position immediately before the entry corresponding to $c$.
(3) For all occurrences $efgh$ of $q_1$ in which $f$ is not the second entry in an occurrence of $p_1$ do the opposite of the previous case; move the entry corresponding to $f$ to the position immediately to the right of the entry corresponding to $e$ in the permutation.

To demonstrate that this is an involution we are required to show that it is well defined in preserving occurrences of patterns.

LEMMA 21. *The above involution preserves the number of occurrences of $p_2$.*

PROOF. For a fixed permutation let *abcd* be an occurrence of $p_1$ and $efgh$ an occurrence of $q_1$. Take $ijk$ to be some occurrence of $p_2$.

It follows immediately by definition of the decomposition of $\sigma$ that a $b$ in the occurrence of $p_1$ cannot be equal to $j$. As two elements occur below and to the left it cannot be equal to $k$, which is a right-to-left minima, nor $i$ which is one greater to than $k$. Thus the element to be moved in an occurrence of $p_1$ in the involution is not involved as part of an occurrence of $p_2$.

It remains to check if its new location after applying the involution breaks an occurrence of $p_2$. Assume $c = j$, then after applying the involution $b$ is moved to immediately before the $c$ then the occurrence of $p_2$ is broken but as $b > c$ a new occurrence is created and thus the total number is preserved.

The entry corresponding to $f$ in the occurrence of $q_1$ may be equal to $j$, however after applying the involution the $g$ in the occurrence of $q_1$ plays the same role as it is larger than the next right-to-left minima. As $f$ is immediately positioned before an element smaller than it in the permutation it cannot be equal to $i$. Furthermore $f$ cannot be equal to $k$ as there are entries smaller and to the right of it. Thus the element to be moved in an occurrence of $p_1$ in the involution is either not involved as part of an occurrence of $p_2$ or a new occurrence is created after the involution is applied.

Again we ask if the new placement of $f$ after the involution is applied breaks any pre-existing occurrence of $p_2$. Assume $g = j$, then moving $f$ immediately to the left of $g$ in the permutation breaks the occurrence of $p_2$ but as $f > g$ a new occurrence is created and thus the total number is preserved

Therefore the involution preserves the number of occurrences of $p_2$. □

LEMMA 22. *For a fixed permutation the involution allows us to map occurrences of* $p_1$ *to occurrences of* $q_1$ *and vice versa.*

PROOF. Consider the different cases under the involution.

An occurrence of $p_1$ in which the second entry is also the second entry of an occurrence of $q_1$ are mapped to one another.

For $abcd$ a fixed occurrence of $p_1$ and $efgh$ some occurrence of $q_1$ such that $b \neq f$ we have that as the position immediately before $b$ in the permutation is occupied by $a$ with $a < b$ therefore $b \neq g$. As $b$ has two smaller entries to its right it is not $h$, which is a right-to-left minima. Assume $b = e$, applying the involution breaks this occurrence of $q_1$, however a new occurrence is formed.

For $efgh$ a fixed occurrence of $q_1$ and $abcd$ some occurrence of $p_1$ such that $f \neq b$ we have that as the position immediately after $f$ in the permutation is occupied by $g$ with $g < f$ therefore $f \neq a$. As $f$ has two smaller entries to its right it is not a right-to-left minima and thus cannot be $d$ in an occurrence of $q_1$. Assume $f = c$ applying the involution breaks this occurrence of $p_1$, however a new occurrence is formed. □

The above Lemmas allow us to prove Proposition 19.

Proposition 19. Occurrences of $\sigma = p_1 + p_2$ in permutations are known to be enumerated by the Fishburn distribution. Lemma 22 shows that for a given permutation the involution maps occurrences of $p_1$ to occurrences of $q_1$ and vice versa whilst Lemma 21 gives that this preserves occurrences of $p_2$.

Thus the number of occurrences of $q_1 + p_2$ follow the Fishburn distribution. □

Remark 23. It is undeniable that the proof presented above is an unelegant case analysis. Ideally we would want a general approach to construct a set of linear combinations of patterns following the Fishburn distribution from any given Mahonian linear combination of patterns. However, as noted earlier, it is not evident that the technique detailed earlier in this chapter can be used here.

We highlight this as an area worthy of additional study.

## Fishburn distribution

We obtain the following corollaries concerning the Fishburn distribution from Theorem 7 and its proof.

Corollary 24. *For some appropriate structure let $p$ be a feature which follows the Fishburn distribution and $q$ a feature which follows the Mahonian distribution.*

*Letting $u_{n,i}$ denote the number of structures of size $n$ with $i$ marked $p$-features,*

$$\sum_{n\geq 0}\sum_{i\geq 0} u_{n,i} x^n z^i = \sum_{n\geq 0} \boldsymbol{(n)_{xz+1}!} x^n,$$

$$\begin{gathered}1\\2\\6,\ 1\\24,\ 9\\120,\ 72,\ 5\\720,\ 600,\ 98,\ 1\\5040,\ 5400,\ 1450,\ 76\\40320,\ 52920,\ 20100,\ 2200,\ 35\\362880,\ 564480,\ 279300,\ 48750,\ 2299,\ 9\\\cdots\end{gathered}$$

FIGURE 2. Unsieved Fishburn distribution: number of structures of size $n$ with $i$ marked $p$-features, $u_{n,i}$

*we have that*

$$u_{n,i} = \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j}.$$

The first few terms of $u_{n,i}$ are shown in Figure 2

PROOF. Theorem 7 gives that a $q$-factorial structure of size $n-i$ with $i$ marked $q$-features can be extended to a structure of size $n$ with $i$ marked $p$-features.

For a Mahonian structure of size $n-i$ with $j$ $q$-features then $i$ are selected. The number of $q$-factorial structures of size $n-i$ with $j$ $q$-features is given by Mahonian number $m_{n-i,j}$.

The maximum number of $q$-features a $q$-factorial structure of size $n-i$ can have is $\binom{n-i}{2}$. Thus $j$ is bounded as

$$i \leq j \leq \binom{n-i}{2}.$$

□

REMARK 25. The row sums of Figure 2 (A179525), i.e.

$$\sum_{i=0} u_{n,i},$$

have previously been studied by Jelínek [**17**] as counting *primitive row Fishburn matrices*, upper-triangular, binary non-row empty matrices, according to the sum of the entries. Jelínek considers such matrices as part of his work on counting self-dual interval orders; he demonstrates a relation between the generating functions of self-dual interval orders enumerated by a reduced size function and primitive row Fishburn matrices.

We note that the coefficient of $x^n z^k$ in the refined formula

$$\sum_{n\geq 0} \boldsymbol{(n)_{xz+1}!} x^n$$

can be interpreted as counting the number of primitive row Fishburn matrices such that:

(1) There are a total of $k$ entries in the matrix that are not the first to occur in their row.
(2) The entries in the matrix sum to $n$.

Corollary 26. *Recalling that $f_{n,k}$ denotes the coefficient in the Fishburn distribution*

$$\sum_{n\geq 0}\sum_{k\geq 0} f_{n,k} x^n y^k = \sum_{n\geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n,$$

*we have that*

$$\begin{aligned} f_{n,k} &= \sum_{i=k}^{n-2} (-1)^{i+k} \binom{i}{k} u_{n,i} \\ &= \sum_{i=k}^{n-2} (-1)^{i+k} \binom{i}{k} \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j}. \end{aligned}$$

The first few terms are shown in Figure 3

Proof. Again for some appropriate structure let $p$ be a feature which follows the Fishburn distribution.

Recall from the proof of Theorem 7 that to take structures with subsets of $p$-features marked to structures with *all* $p$-features marked corresponds to the substitution of $y-1$ by $z$.

The result then follows from the previous corollary and binomial expansion. □

We get the following corollary from setting $i = 0$ in the above.

$$
\begin{array}{c}
1\\
2\\
5,\ 1\\
15,\ 9\\
53,\ 62,\ 5\\
217,\ 407,\ 95,\ 1\\
1014,\ 2728,\ 1222,\ 76\\
5335,\ 19180,\ 13710,\ 2060,\ 35\\
31240,\ 142979,\ 146754,\ 39644,\ 2254,\ 9\\
\cdots
\end{array}
$$

FIGURE 3. Fishburn distribution, $f_{n,k}$

COROLLARY 27. *The number $i_n$ of Fishburn structures of size $n$ can be written in terms of the Mahonian numbers $m_{n,k}$,*

$$i_n = \sum_{i=0}^{n-2} (-1)^i \sum_{j=i}^{\binom{n-i}{2}} \binom{j}{i} m_{n-i,j}.$$

REMARK 28. In the above corollary the upper bound $n-2$ for the initial summation is justified by noting that an occurrence of $\sigma$ can be uniquely determined by the first element in the occurrence and that two more entries must follow in the permutation. Therefore there can be no more than $n-2$ occurrences of a Fishburn statistic in a structure of size $n$.

This is sufficient for our purposes however we note that this is *not* the least upper bound (easily checked empirically). We leave this as an open question.

QUESTION 29. *Is there an aesthetically pleasing expression for the least upper bound for the value of a Fishburn statistic?*

Recall that we use $r_n$ to denote the number of *flat interval orders*. Brightwell and Keller [6] provide an asymptotic approximation for $r_n$,

$$r_n \sim n! \frac{12\sqrt{3}}{\pi^{\frac{5}{2}}} e^{\frac{-\pi^2}{12}} \left(\frac{6}{\pi^2}\right)^n \sqrt{n}.$$

It is well known that $i_n$ can be derived from $r_n$ by replacing entries of flat interval orders with non-empty sets of entries. Let $I(x)$ and $R(x)$ be the exponential generating functions for interval orders and flat interval orders respectively. Then,

$$I(x) = R\left(\frac{x}{1-x}\right).$$

Simple algebraic manipulation of the coefficients of the above formula returns an identity for $r_n$ in terms of $i_n$,

$$r_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n-1}{k-1} i_k.$$

Via substitution of the result in Corollary 27 into the above equation we get the following.

Corollary 30. *The number $r_n$ of flat interval orders of size $n$ can be written in terms of the Mahonian numbers $m_{n,k}$,*

$$r_n = \sum_{k=0}^{n} \sum_{i=0}^{k-2} \sum_{j=i}^{\binom{k-i}{2}} (-1)^{n-k+i} \binom{n-1}{k-1} \binom{j}{i} m_{k-i,j}.$$

Figures 2 and 3 are expanded in Appendix A.

CHAPTER 3

# Decomposing labeled interval orders

This chapter details work done with Anders Claesson and published in the Electronic Journal of Combinatorics [8].

This chapter interprets the exponential generating function for labeled interval orders,

$$\sum_{m\geq 0}\prod_{i=1}^{m}(1-e^{-xi}),$$

as a combinatorial specification for upper triangular, non-row empty matrices whose entries are ballots.

A bijection of Dukes et al. [10] is adapted to a surjection mapping ballot matrices to labeled interval orders and used to define an equivalence relation on ballot matrices. A sign reversing involution is then used to identify fixed points for which there is exactly one per equivalence class. The decomposition of any single fixed point into a pair consisting of a permutation and an inversion table is then provided. This allows for the main result of this chapter, that the set of labeled interval orders on $[n]$ is in bijection with two separate sets. Firstly,

$$\{(\pi,\tau)\in\mathfrak{S}_n\times\mathfrak{S}_n : A(\tau)\subseteq D(\pi)\},$$

where $A(\tau)$ is the set of ascent bottoms of $\tau$, and $D(\pi)$ is the set of descent positions of $\pi$. Secondly,

$$\{(\pi,\tau)\in\mathfrak{S}_n\times\mathfrak{S}_n : D(\pi)\subseteq A(\tau)\}.$$

As a consequence we derive a new formula for the number of labeled interval orders on $[n]$:

$$\sum_{\{s_1,\ldots,s_k\}\subseteq[n-1]}\left(\det\left[\binom{n-s_i}{s_{j+1}-s_i}\right]\cdot\prod_{r=1}^{k+1} r^{s_r-s_{r-1}}\right)$$

where $s_0=0$ and $s_{k+1}=n$.

### Terminology and preliminaries

For non-negative integers $a$ and $b$ with $a < b$, let $[b]$ denote the set $\{1, \ldots, b\}$ and $[a, b]$ the set $\{a, \ldots, b\}$. This chapter will feature three main combinatorial structures: permutations, inversion tables and ballots. In this section a summary is provided to remind the reader of relevant results pertaining to these structures and to set the notational convention that shall be followed.

**Permutations.** A permutation is a bijection on a finite set. As in the previous chapter, the permutations that we study are assumed to be bijections on a totally ordered sets and we shall take $[n]$ as the default example.

A descent in a permutation $\pi = a_1 a_2 \ldots a_n \in \mathfrak{S}_n$ is a pair $(a_i, a_{i+1})$ where $a_i > a_{i+1}$. Following Stanley [**23**, Section 2.2] let $D(\pi) = \{i : a_i < a_{i+1}\} \subseteq [n-1]$ denote the set of descent positions and define

$$\boldsymbol{\alpha}_n(S) = \{\pi \in \mathfrak{S}_n : D(\pi) \subseteq S\}, \quad \alpha_n(S) = |\boldsymbol{\alpha}_n(S)|,$$

$$\boldsymbol{\beta}_n(S) = \{\pi \in \mathfrak{S}_n : D(\pi) = S\}, \quad \beta_n(S) = |\boldsymbol{\beta}_n(S)|.$$

Let $S = \{s_1, s_2, \ldots, s_k\}$ and $1 \leq s_1 < s_2 < \cdots < s_k < n$. Also, let $s_0 = 0$ and $s_{k+1} = n$. Partitioning $[n]$ into blocks of cardinalities

$$s_1 - s_0,\ s_2 - s_1,\ \ldots,\ s_{k+1} - s_k$$

a permutation is formed by listing elements within the blocks in increasing order and concatenating the blocks. The only position in which a descent *can* occur is at the join between two blocks. Thus,

$$\alpha_n(S) = \binom{n}{s_1 - s_0, s_2 - s_1, \ldots, s_{k+1} - s_k}. \tag{2}$$

By the sieve principle we have that $\beta_n(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \alpha_n(T)$. One can show [**23**, Example 2.2.4] that this leads to the formula

$$\beta_n(S) = \det \left[ \binom{n - s_i}{s_{j+1} - s_i} \right],$$

where $(i, j) \in [0, k] \times [0, k]$.

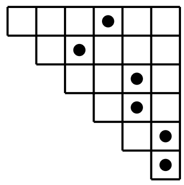


FIGURE 1. Inversion table 231100

**Inversion tables.** Given a permutation $\pi = a_1 a_2 \dots a_n$, an inversion in $\pi$ is a pair $(a_i, a_j)$ where $a_i > a_j$ and $i < j$. An inversion table is an encoding of a permutation where the $i$th value is the number of inversions in which $i$ is involved as the smaller element. The set of inversion tables of length $n$ will be denoted $\mathrm{InvTab}_n$:

$$\mathrm{InvTab}_n = \{\, b_1 b_2 \dots b_n : b_i \in [0, n-i] \,\}.$$

An inversion table may be viewed diagrammatically. To make clear the relationship between inversion tables and $n$ by $n$ upper triangular matrices containing exactly one entry per row we shall break convention and view an inversion table as right aligned, decreasing rows where an entry in row $i$ at column $j$ corresponds to the inversion table with $i$th entry $n - j$. An example is shown in Figure 1.

Define Dent to be the function taking an inversion table to the set of distinct entries it contains. For example, $\mathrm{Dent}(430200) = \{0, 2, 3, 4\}$. We further say that $a \in [n-1]$ is *missing* from a length $n$ inversion table if $a$ is not in its set of distinct entries. For instance, 1 and 5 are both missing from 430200.

**Ballots.** A *ballot*, alternatively known as an ordered set partition, is a collection of pairwise disjoint non-empty sets (referred to as blocks) where the blocks are assigned some total ordering. Adopting a symbolic approach, let $L$ be the construction taking a set $U$ to the set of linear orders built upon $U$. Also, let $E_+$ be the non-empty set construction. That is, $E_+[U] = \{U\}$ if $U$ is non-empty, and $E_+[\emptyset] = \emptyset$. Then define Bal, the construction of ballots, to be the composition $L(E_+)$:

$$\mathrm{Bal} = L(E_+) = \sum_{k \geq 0} (E_+)^k.$$

Consider *signed ballots*, as above but where each ballot is assigned to be either positive or negative. A positive ballot contains an even number of blocks and a negative ballot contains an odd number of blocks. For any species $F$, let $-1 \cdot F = -F$ be as $F$ but with the sign of each object negated. Using $E^{-1}$ to refer to signed ballots—the notation stemming from its role as the symbolic multiplicative inverse of set—we have

$$E^{-1} = L(-E_+) = \sum_{k \geq 0} (-1)^k (E_+)^k.$$

It follows that signed ballots have exponential generating function

$$\frac{1}{1 + (e^x - 1)} = e^{-x} = \sum_{n \geq 0} (-1)^n \frac{x^n}{n!}. \tag{3}$$

See, for example, Bergeron et al. [2, Section 2.5].

We use the notation $(E^{-1})^+$ to refer to the subset of signed ballots which are positive and $(E^{-1})^-$ to refer to the subset which are negative.

### Ballot matrices and interval orders

Equation (3) implies that the number of ballots constructed on some set $U$ with an even number of blocks differ from the number of ballots of $U$ with an odd number of blocks by 1. To be precise

$$|(E^{-1})^+[U]| - |(E^{-1})^-[U]| = (-1)^{|U|}.$$

An involution on ballots witnesses this fact. In the above equation the sign of a ballot with $k$ blocks is $(-1)^k$. Note that we can change the sign of a ballot with $|U| \geq 2$ by splitting a non-singleton block into two blocks or by merging two blocks. Let $\omega = B_1 \dots B_k$ be a ballot in $\mathrm{Bal}[U]$. That is, each $B_i$ is non-empty and $U$ is the disjoint union of the sets $B_1$ through $B_k$.

Take any linear order on $U$. Let $x = \min U$ be smallest element of $U$. If $x \in B_i$ and $B_i$ contains at least two elements, then delete $x$ from $B_i$ and create a new block $\{x\}$ to the immediate right of $B_i$. For example,

$$\omega = \{2,5\}\{1,4,6\}\{3\} \;\mapsto\; \{2,5\}\{4,6\}\{1\}\{3\} = \xi.$$

If $B_i = \{x\}$ and $i > 1$ then delete this block from $\omega$ and add $x$ to $B_{i-1}$. With $\omega$ and $\xi$ as in the example above, we have $\xi \mapsto \omega$. If $B_1 = \{x\}$ then proceed with the next smallest element of $U$ and the ballot $B_2B_3 \dots B_k$. For example,

$$\{1\}\{2\}\{5\}\{4,6\}\{3\} \mapsto \{1\}\{2\}\{5\}\{3,4,6\}.$$

For $U = \{u_1, u_2, \dots, u_n\}$ and $u_1 < u_2 < \cdots < u_n$ the single fixed point under this sign reversing involution is $\{u_1\}\{u_2\}\dots\{u_n\}$.

**Ballot Matrices.** The exponential generating function for the number of labeled interval orders was shown by Claesson et al. [10] to be a function originally studied by Zagier [26],

$$\sum_{m\geq 0}\prod_{i=1}^{m}(1-e^{-xi}) = \sum_{m\geq 0}(-1)^m\prod_{i=1}^{m}(e^{-xi}-1).$$

It it thus natural to consider the signed combinatorial structure

$$\sum_{m\geq 0}(-1)^m\prod_{i=1}^{m}\big((E^{-1})^i-1\big).$$

An $((E^{-1})^i-1)$-structure is a non-empty sequence of $i$ pairwise disjoint ballots. As such, a $(-1)^m\prod_{i=1}^{m}\big((E^{-1})^i-1\big)$-structure is an upper triangular $m\times m$ matrix of pairwise disjoint ballots such that each row is non-empty.

The sign of the matrix is the product of the signs of the ballot entries and the signs of the rows. If $A$ is such a matrix and the total number of blocks of all ballots in $A$ is $\ell$, then the sign of $A$ is $(-1)^{\ell+m}$. We shall call such matrices *Ballot matrices* and use the notation BalMat for the construction with $\mathrm{BalMat}^+$ and $\mathrm{BalMat}^-$ the positive and negative parts respectively. As an example, for $U=\{1,2\}$ we have

$$\mathrm{BalMat}^+[U] = \left\{ \begin{bmatrix}\{1,2\}\end{bmatrix}, \begin{bmatrix}\emptyset & \{1\}\\ & \{2\}\end{bmatrix}, \begin{bmatrix}\emptyset & \{2\}\\ & \{1\}\end{bmatrix}, \begin{bmatrix}\{2\} & \emptyset\\ & \{1\}\end{bmatrix}, \begin{bmatrix}\{1\} & \emptyset\\ & \{2\}\end{bmatrix} \right\}$$

and

$$\mathrm{BalMat}^-[U] = \left\{ \begin{bmatrix}\{1\}\{2\}\end{bmatrix}, \begin{bmatrix}\{2\}\{1\}\end{bmatrix} \right\}.$$

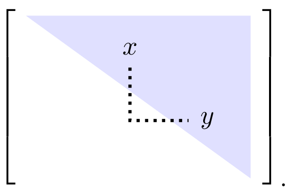
We note the similarity between ballot matrices and the composition matrices of Claesson et al. [**7**]. The entries of composition matrices are sets, which may be viewed as either as ballots with a single block or as ballots where each element is contained within its own singleton block and the blocks are ordered according to the order on $U$. Therefore composition matrices are a subset of ballot matrices. For our purposes we wish to define an involution whose fixed points are either all positive or all negative for any given $U$. However for both interpretations of composition matrices as ballot matrices the sign is not consistent, there exist both positive and negative composition matrices when $|U| \geq 2$, and hence they are not suitable candidates for the fixed points of our involution.

Recall that Dukes et al. [**10**] provide a direct bijection between composition matrices and labeled interval orders, this was detailed on page 13.

We adapt their bijection to define a *surjection* taking ballot matrices to labeled interval orders as follows.

Definition 31. Let $A \in \mathrm{BalMat}[U]$, and let $x$ and $y$ be elements of $U$. Further, let $\omega$ and $\xi$ be the ballot entries $(i, j)$ and $(i', j')$ of $A$ such that $x$ is contained in the underlying set of $\omega$ and $y$ is contained in the underlying set of $\xi$. Define the poset $P(A)$ by declaring that $x < y$ in $P$ if $j < i'$.

Again we have the alternative formulation that $x < y$ in $P$ if the "hook" from $x$ to $y$ passing through $(i', j)$ goes below the diagonal:




Equivalently, the strict downset of $y$ is the union of columns 1 through $i' - 1$. Figure 2 shows an example of a ballot matrix and its corresponding poset.

$$\begin{bmatrix} \emptyset & \{6\} & \emptyset & \{4,5\} \\ & \emptyset & \{3\} & \emptyset \\ & & \{1\} & \emptyset \\ & & & \{2\} \end{bmatrix}$$

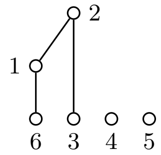



FIGURE 2. A ballot matrix and its corresponding poset

Given a poset $P$, the *predecessor set* (also known as the *downset*) of $x \in P$ is the set of elements smaller than $x$:

$$\text{pred}\,(x) = \{y \in P : y <_P x\}.$$

It is a well known that a poset is an interval order if and only if there is a linear ordering by inclusion on the predecessor set of each element $\{\text{pred}\,(x) : x \in P\}$ (see, for example, Bogart [3]). As the mapping states that the strict downset of $y$ is the union of columns 1 through $i' - 1$ there is a linear ordering on downsets and hence every poset which is mapped to must be an interval order.

In addition, composition matrices are a subset of ballot matrices and as Dukes et al. [10] show that for composition matrices the mapping is a bijection it follows that the adapted mapping is a surjection.

If we declare that two ballot matrices in BalMat$[U]$ are equivalent if they determine the same interval order, then, by definition, there are as many equivalence classes as there are interval orders on $U$. In the next section we define a sign reversing involution that respects this equivalence relation.

## The involution

We now define the involution on ballot matrices. We begin by applying the ballot involution componentwise to entries of BalMat.

Choose some linear order for the entries of the matrix; for instance, order the entries (ballots) with respect to their minimum element, or order them lexicographically with respect to their position $(i, j)$ in the matrix. Then apply the ballot involution to the first entry that is not fixed, if such an element exists, and denote this operation $\psi$. A matrix is a fixed point under this sign reversing involution if and only if each entry of the matrix is fixed, and thus of the form

$$\{a_1\}\{a_2\}\ldots\{a_j\} \quad \text{with} \quad a_1 < a_2 < \cdots < a_j.$$

Note that if $A$ is a $k \times k$ matrix fixed under $\psi$, then the sign of $A$ is $(-1)^{n+k}$, where $n = |U|$. We shall define a sign reversing involution $\varphi$ on the fixed points of $\psi$.

Let $A \in \mathrm{BalMat}[U]$ be a matrix fixed under $\psi$. Let $x \in U$ and assume that $x$ is on row $i$ and column $j$ of $A$. We say that $x$ is a *pivot* element of $A$ if row $i$ contains at least two elements of $U$ and $x$ is the smallest element on row $i$, or the following three conditions are met:

(1) column $i$ is empty;
(2) $\{x\}$ is the only non-empty ballot on its row;
(3) $x$ is smaller than the minimum element of row $i+1$ of $A$.

As an illustration, the pivot elements of the matrix

$$\begin{bmatrix} \emptyset & \{4\} & \emptyset & \emptyset & \emptyset \\ & \{6\}\{8\} & \emptyset & \{3\}\{7\} & \emptyset \\ & & \emptyset & \{2\} & \emptyset \\ & & & \{9\} & \{5\} \\ & & & & \{1\} \end{bmatrix}$$

are 2, 3 and 5.

If the set of pivot elements of $A$ is empty, then let $\varphi(A) = A$. Otherwise, let $x$ be the smallest pivot element of $A$, and assume that $x$ belongs to the $(i, j)$ entry of $A$.

(1) If there is more than one element on row $i$, then remove $x$ from row $i$ and make a new row immediately above row $i$ with the block $\{x\}$ in column $j$ and the rest of the entries empty. Also insert a new empty column $i$, pushing the existing columns one step to the right.

(2) If column $i$ is empty, $\{x\}$ is the only non-empty ballot on its row, and $x$ is smaller than the minimum element of row $i+1$, then remove column $i$ and merge row $i$ with row $i+1$ by inserting the singleton block $x$ at the front of the ballot in position $(i+1, j)$.

Applying $\varphi$ to the example matrix above we get

$$\begin{bmatrix} \emptyset & \{4\} & \emptyset & \emptyset \\ & \{6\}\{8\} & \{3\}\{7\} & \emptyset \\ & & \{2\}\{9\} & \{5\} \\ & & & \{1\} \end{bmatrix}.$$

Note that the smallest pivot element of this matrix is still 2, and applying $\varphi$ to it would bring back the original matrix.

Our main involution $\eta : \mathrm{BalMat}[U] \to \mathrm{BalMat}[U]$ is then defined as the composition of $\psi$ and $\varphi$ in the following sense:

$$\eta(A) = \begin{cases} \varphi(A) & \text{if } \psi(A) = A, \\ \psi(A) & \text{if } \psi(A) \neq A. \end{cases}$$

It is clear that $\eta$ is sign reversing. That any fixed point of $\eta$ has positive sign will be seen in Section 3.

PROPOSITION 32. *The involution $\eta$ preserves the interval order in the following sense. Let $A \in \mathrm{BalMat}[U]$. Let $P$ and $Q$ be the interval orders corresponding to $A$ and $\eta(A)$, respectively. Then $P = Q$.*

PROOF. If $A$ is a fixed point of $\eta$, equality is immediate. Further, the block structure of the elements of $A$ is immaterial to the definition of the poset. Thus, if $\psi(A) \neq A$ and $\eta(A) = \psi(A)$, then equality is immediate. For the remainder of the proof assume that $\eta(A) = \varphi(A) \neq A$.

The proof that the involution preserves the interval order is equivalent to saying that the strict downset of each element is preserved. This follows from a case analysis. Recall that the strict downset of $x$ at position $(i, j)$ in the matrix is the union of columns 1 through $i - 1$.

Let $B = \eta(A)$. The involution has two possibilities. If the minimal pivot element $x$ at position $(i, j)$ in $A$ is not the only element on its row, then $B$ is formed by initially inserting a new empty row above row $i$ and a new empty column before column $i$. The pivot element $x$ is moved to the new row maintaining its column and hence its strict downset is unchanged.

We now demonstrate that the insertion of the new empty row at position $i$ and new empty column at position $i$ preserves hooks below the diagonal. For $y \neq x$ at position $(i', j')$ in $A$ there are three possibilities.

(1) The element $y$ is above the newly inserted row and to the left of the new column, i.e. $y$ remains at position $(i', j')$ in $B$ with $i' < i$ and $j' < i$. Then the new column is inserted to the right of the columns which form the strict downset of $y$ and hence the downset is unchanged.

(2) The element $y$ is to the right of the newly inserted column and above the inserted row, i.e. $y$ is at position $(i', j'+1)$ in $B$ with $i' < i < j'$. Again as $i' < i$ the new column is inserted to the right of the columns which form the strict downset of $y$ and the downset is unchanged.

(3) The element $y$ is below the newly inserted row and to the right but of the new column, i.e. $y$ is at position $(i'+1, j'+1)$ in $B$ with $i < i'$ and $i < j'$. As $i < i'$, the number of columns which form the downset of $y$ is increased by 1. The newly inserted column $i$ is empty and therefore contributes no new entries. As $i < i'$ the previous rightmost column $i' - 1$ is shifted one place to the right to column $i'+1$ in the new matrix. The downset of $y$ in $B$ is therefore the union of elements 1 through $i'$ and hence the downset is unchanged.

Note that $x$ remains the pivot element in the newly constructed matrix $B$, the only non-empty ballot on its row, and with column $i$ empty. Therefore showing that the second possibility of the involution preserves posets follows from taking the reverse of the above cases.

As the strict downsets are equal the posets are equal. □

## Fixed points

A fixed point under the sign reversing involution $\eta$ on BalMat is an $n \times n$ matrix with no pivot elements, equivalently a matrix such that

(1) there is exactly one element per row;

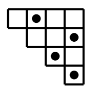
(2) if $a < b$, with $a$ on row $i$, and $b$ on row $i+1$, then column $i$ is non-empty.

Note that the total number of blocks in such a matrix is $n$—each element is in its own block—and thus it has sign $(-1)^{2n} = 1$, positive.

Further, matrices which satisfy these conditions can be decomposed to a pair consisting of a permutation and an inversion table: As there is exactly one element per row, a permutation $\pi = a_1 \dots a_n$ can be read setting each $a_i$ the value held in row $i$. As the matrix is also upper triangular, the position of the element in a row specifies an inversion table $b_1 b_2 \dots b_n$ where each $b_i$ is $n$ minus the column in which the entry in row $i$ occurs.

As an example, consider the matrix below. It decomposes into the permutation 4132 together with the inversion table 2010:

$$\begin{bmatrix} \emptyset & \{4\} & \emptyset & \emptyset \\ & \emptyset & \emptyset & \{1\} \\ & & \{3\} & \emptyset \\ & & & \{2\} \end{bmatrix} \simeq \left( 4132, \quad \right) \simeq (\, 4132,\ 2010\,)\,.$$



Take the equivalence class on ballot matrices where two matrices are equivalent if they correspond to the same interval order. We wish to show that there is exactly one fixed point under $\eta$ per equivalence class. For this purpose and to make explicit the link to previous work we provide a bijection between composition matrices and ballot matrices.

For the following, take the structure of the entries of a composition matrix to be ballots where each element is contained within a singleton block and the blocks are ordered according to the order on the underlying set.

Given an $m \times m$ ballot matrix $A \in \mathrm{BalMat}[U]$, let $u_i$ be the smallest element on the $i$th row of $A$, and define $G(A) = U \setminus \{u_1, u_2, \ldots, u_m\}$.

Assuming that $G(A)$ is non-empty, define $\rho$ to be the following operation. Take $x = \min G(A)$ at position $(i, j)$ in $A$. Insert a new row containing only empty ballots above row $i$ and a new column containing only empty ballots to the left of column $i$. Move $x$ to create a singleton ballot in the new row preserving its column. Note that $|G(\rho(A))| = |G(A)| - 1$. An example with $G(A) = \{4, 6\}$ is given below:

$$\begin{bmatrix} \{2,6\} & \emptyset & \emptyset & \emptyset \\ & \{3\} & \emptyset & \mathbf{\{4\}} \\ & & \emptyset & \{1\} \\ & & & \{5\} \end{bmatrix} \overset{\rho}{\longmapsto} \begin{bmatrix} \{2,6\} & \boldsymbol{\emptyset} & \emptyset & \emptyset & \emptyset \\ & \boldsymbol{\emptyset} & \boldsymbol{\emptyset} & \boldsymbol{\emptyset} & \mathbf{\{4\}} \\ & & \{3\} & \emptyset & \emptyset \\ & & & \emptyset & \{1\} \\ & & & & \{5\} \end{bmatrix}.$$

The inverse operation will be denoted $\rho^{-1}$. To state it explicitly, let $A \in \mathrm{BalMat}[U]$ be a $m \times m$ ballot matrix, and let $u_i$ be the smallest element on the $i$th row of $A$, as before. Then take $H(A)$ to be the subset of $\{u_1, u_2, \ldots, u_{m-1}\}$ consisting of those $u_i$ such that the following three conditions hold: column $i$ is empty; $u_i$ is the sole element on row $i$; and $u_i > u_{i+1}$.

Assuming that $H(A)$ is non-empty, define $\rho^{-1}$ to be the following operation. Take $x = \max H(A)$ at position $(i, j)$ in $A$. Append $x$ in a singleton block at the end of the ballot in position $(i + 1, j)$, then remove row and column $i$.

Proposition 33. *There is a bijection between composition matrices and ballot matrices fixed under $\eta$. As a result there is a unique ballot matrix fixed under $\eta$ per equivalence class.*

Proof. We first show that successive application of the mapping $\rho$ gives an injection from composition matrices into ballot matrices fixed under $\eta$.

The same argument as in Proposition 32 shows that $\rho$ preserves the interval order.

Take a composition matrix. Let $A$ be the matrix returned after repeated application of $\rho$ until the set of elements $G(A)$ is empty. We claim $A$ is a ballot matrix fixed under $\eta$.

From definition we know that $G(A)$ is empty. Therefore there is exactly one element per row. The other requirement to be a fixed point under $\eta$ is that if $a < b$ with $a$ on row $i$ and $b$ on row $i+1$ then column $i$ must be non-empty. As composition matrices have the property that all columns are non-empty and $\rho$ only introduces an empty column $i$ when $a > b$ with $a$ on row $i$, this requirement is met.

Repeated application of $\rho$ is therefore a mapping between composition matrices and ballot matrices fixed under $\eta$ with injectivity following from the preservation of interval order.

As $\rho$ preserves the interval order, the reverse operation $\rho^{-1}$ also preserves the interval order.

Take a fixed point matrix. Let $A$ be the matrix returned after repeated application of $\rho^{-1}$ until the set of elements $H(A)$ is empty. We claim $A$ is a composition matrix.

Composition matrices are neither row nor column empty. Non-row empty is a property of fixed point ballot matrices and $\rho^{-1}$ does not introduce any empty columns. If a fixed point matrix contains an empty column $i$ then from definition there is an $a > b$ with $a$ and $b$ on rows $i$ and $i+1$ respectively. However as $G(A)$ is empty it follows that all empty columns are removed.

Hence all fixed point matrices can be mapped to a composition matrices with the interval order preserved by repeated application of $\rho^{-1}$, giving surjectivity. $\square$

Let $\mathrm{BalMat}^{\eta}[U]$ denote the set of fixed points under $\eta$. Writing simply $x$ for the ballot $\{x\}$, the complete list of matrices in $\mathrm{BalMat}^{\eta}[3]$ is given in Figure 3.

### Permutations from ascent bottoms

In order to examine the fixed points under $\eta$ we shall consider how to characterize the pairs resulting from their decomposition to a permutation and an inversion table. For this purpose, this section is concerned with counting the number of permutations whose set of ascent bottoms is equal to some given set. Bijections between such permutations and two different sets of inversion tables are provided. We make repeated use of the sieve principle and our presentation follows that of Stanley [**23**, Section 2.2].

$$\begin{bmatrix} 3 & \emptyset & \emptyset \\ & 2 & \emptyset \\ & & 1 \end{bmatrix} \begin{bmatrix} 3 & \emptyset & \emptyset \\ & \emptyset & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} \emptyset & 3 & \emptyset \\ & 2 & \emptyset \\ & & 1 \end{bmatrix} \begin{bmatrix} \emptyset & 3 & \emptyset \\ & \emptyset & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} \emptyset & \emptyset & 3 \\ & 2 & \emptyset \\ & & 1 \end{bmatrix}$$

$$\begin{bmatrix} \emptyset & \emptyset & 3 \\ & \emptyset & 2 \\ & & 1 \end{bmatrix} \begin{bmatrix} 3 & \emptyset & \emptyset \\ & 1 & \emptyset \\ & & 2 \end{bmatrix} \begin{bmatrix} \emptyset & 3 & \emptyset \\ & 1 & \emptyset \\ & & 2 \end{bmatrix} \begin{bmatrix} \emptyset & 3 & \emptyset \\ & \emptyset & 1 \\ & & 2 \end{bmatrix} \begin{bmatrix} \emptyset & \emptyset & 3 \\ & 1 & \emptyset \\ & & 2 \end{bmatrix}$$

$$\begin{bmatrix} 2 & \emptyset & \emptyset \\ & 3 & \emptyset \\ & & 1 \end{bmatrix} \begin{bmatrix} 2 & \emptyset & \emptyset \\ & \emptyset & 3 \\ & & 1 \end{bmatrix} \begin{bmatrix} 2 & \emptyset & \emptyset \\ & 1 & \emptyset \\ & & 3 \end{bmatrix} \begin{bmatrix} \emptyset & 2 & \emptyset \\ & 1 & \emptyset \\ & & 3 \end{bmatrix} \begin{bmatrix} \emptyset & 2 & \emptyset \\ & \emptyset & 1 \\ & & 3 \end{bmatrix}$$

$$\begin{bmatrix} \emptyset & \emptyset & 2 \\ & 1 & \emptyset \\ & & 3 \end{bmatrix} \begin{bmatrix} 1 & \emptyset & \emptyset \\ & 3 & \emptyset \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & \emptyset & \emptyset \\ & \emptyset & 3 \\ & & 2 \end{bmatrix} \begin{bmatrix} 1 & \emptyset & \emptyset \\ & 2 & \emptyset \\ & & 3 \end{bmatrix}$$

FIGURE 3. Complete list of matrices in $\mathrm{BalMat}^{\eta}[3]$

Recall the definitions of $\boldsymbol{\alpha}_n(S)$ and $\boldsymbol{\beta}_n(S)$:

$$\boldsymbol{\alpha}_n(S) = \{\tau \in \mathfrak{S}_n : D(\tau) \subseteq S\}, \quad \alpha_n(S) = |\boldsymbol{\alpha}_n(S)|,$$

$$\boldsymbol{\beta}_n(S) = \{\tau \in \mathfrak{S}_n : D(\tau) = S\}, \quad \beta_n(S) = |\boldsymbol{\beta}_n(S)|.$$

In an analogous fashion, for $\pi = a_1 a_2 \dots a_n \in \mathfrak{S}_n$, let

$$A(\pi) = \{a_i : i \in [n-1], a_i < a_{i+1}\}$$

be the set of ascent bottoms of $\pi$. Let

$$\boldsymbol{\kappa}_n(S) = \{\pi \in \mathfrak{S}_n : A(\pi) \subseteq S\}, \quad \kappa_n(S) = |\boldsymbol{\kappa}_n(S)|,$$

$$\boldsymbol{\lambda}_n(S) = \{\pi \in \mathfrak{S}_n : A(\pi) = S\}, \quad \lambda_n(S) = |\boldsymbol{\lambda}_n(S)|.$$

Note that by definition $\kappa_n(S) = \sum_{T \subseteq S} \lambda_n(T)$, and by the sieve principle, $\lambda_n(S) = \sum_{T \subseteq S} (-1)^{|S \setminus T|} \kappa_n(T)$.

The following set of sequences will be convenient as an intermediate structure for later proofs.

DEFINITION 34. For fixed $n$, let $S = \{s_1, \dots, s_k\}$ with $1 \leq s_1 < \cdots < s_k < n$ be given. Also, set $s_0 = 0$ and $s_{k+1} = n$. Define the Cartesian product

$$\mathcal{C}_n(S) = [0,k]^{s_{k+1}-s_k} \times \cdots \times [0,1]^{s_2 - s_1} \times [0,0]^{s_1 - s_0}.$$

We shall call an element of $\mathcal{C}_n(S)$ a *construction choice.*

As example, for $n = 8$ and $S = \{3, 5, 6, 7\}$ we have $s_1 - s_0 = 3$, $s_2 - s_1 = 2$, and $s_3 - s_2 = s_4 - s_3 = s_5 - s_4 = 1$. Thus

$$\mathcal{C}_n(S) = [0, 4] \times [0, 3] \times [0, 2] \times [0, 1] \times [0, 1] \times [0, 0] \times [0, 0] \times [0, 0].$$

An example of a construction choice in $\mathcal{C}_n(S)$ is 42001000, we shall use this as a running example throughout the remainder of this section.

PROPOSITION 35. *For fixed $n$, let $S = \{s_1, \ldots, s_k\}$ and $1 \leq s_1 < \cdots < s_k < n$ be given. Then $\boldsymbol{\kappa}_n(S)$ is in bijection with $\mathcal{C}_n(S)$.*

PROOF. Take a construction choice $c_1c_2 \ldots c_n \in \mathcal{C}_n(S)$. We will use this to construct a permutation by insertion of entries at active sites. Start with the empty permutation. This has a single active site, labeled zero. Reading the construction choice in reverse order, insert elements into the permutation beginning with the minimal element. That is, $c_i$ is the choice of active site for the insertion of $n+1-i$ into the permutation.

A new active site is created when an element of $S$ is introduced into the permutation. The active sites are labeled according to the order in which they are inserted. That is, assuming entries of $S$ are numerically ordered then the active site to the right of $s_i$ in the permutation is labeled $i$. Note that a consequence of this is that $s_i$ is an ascent bottom if and only if $i$ is contained within the construction choice. As a larger element is inserted at each step this ensures that the only place where an ascent can take place is after an entry of in the permutation which is contained within $S$. Therefore only elements of $S$ can be ascent bottoms.

It is easy to see how to reverse this procedure and thus it provides the claimed bijection. $\square$

EXAMPLE 36. For $n = 8$ and $S = \{3, 5, 6, 7\}$ the construction process for the permutation with construction choice 42001000 is as follows. Note the new active site created when an element of $S$ is inserted.

| | |
|---|---|
| $_0$ | |
| $_01$ | Insert 1 at site 0 |
| $_021$ | Insert 2 at site 0 |
| $_03_121$ | Insert 3 at site 0, contained in $S$ |
| $_03_1421$ | Insert 4 at site 1 |
| $_05_23_1421$ | Insert 5 at site 0, contained in $S$ |
| $_06_35_23_1421$ | Insert 6 at site 0, contained in $S$ |
| $_06_35_27_43_1421$ | Insert 7 at site 2, contained in $S$ |
| $_06_35_27_483_1421$ | Insert 8 at site 4 |

So the resulting permutation is $\pi = 65783421$, with $A(\pi) = \{3, 5, 7\}$.

Corollary 37. *For fixed $n$, let $S = \{s_1, \dots, s_k\}$ with $1 \leq s_1 < \cdots < s_k < n$ be given. Then*

$$\kappa_n(S) = \prod_{r=1}^{k+1} r^{s_r - s_{r-1}},$$

*where $s_0 = 0$ and $s_{k+1} = n$.*

Proof. By Proposition 35 we have that $\kappa_n(S)$ is the cardinality of $\mathcal{C}_n(S)$, from which the formula immediately follows. $\square$

We shall now show that construction choices in $\mathcal{C}_n(S)$, and thus permutations in $\boldsymbol{\kappa}_n(S)$, are in bijection with two different sets of inversion tables. Namely

$$\big\{\upsilon \in \mathrm{InvTab}_n : \mathrm{Dent}(\upsilon) \subseteq \{0, s_1, s_2, \dots, s_k\}\big\}$$

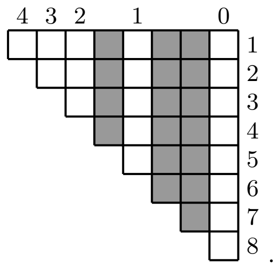
and

$$\big\{\upsilon \in \mathrm{InvTab}_n : [n-1] \setminus \mathrm{Dent}(\upsilon) \subseteq \{n - s_1, \dots, n - s_k\}\big\}.$$

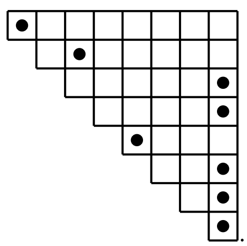
Proposition 38. *For fixed $n$, let $S = \{s_1, \dots, s_k\}$ with $1 \leq s_1 < \cdots < s_k < n$ be given. Then there is a bijection between $\boldsymbol{\kappa}_n(S)$ and inversion tables whose entries* *are a subset of* $\{0\} \cup S$,

$$\big\{\upsilon \in \mathrm{InvTab}_n : \mathrm{Dent}(\upsilon) \subseteq \{0, s_1, s_2, \dots, s_k\}\big\}.$$

PROOF. Again we shall use the construction choice. Entries contained within the inversion table are a subset of $S$. Therefore elements which are in $[n-1]$ but *not* in $S$, that is, elements of $[n-1] \setminus S$, cannot be contained in the inversion table. These entries are therefore forbidden. Label the remaining possible entries right to left from $[0, k]$. In this context it is convenient to use our diagrammatic representation of an inversion table. As an example, let $n = 8$ and $S = \{3, 5, 6, 7\}$. As $[n-1] \setminus S = \{1, 2, 4\}$, the columns $8-1$, $8-2$, and $8-4$ are forbidden (dark, below). Labeling those which remain right-to-left with $[0, 4]$ yields




Given a construction choice $c_1c_2 \dots c_n \in \mathcal{C}_n(S)$, assign the entry on row $i$ to be in the column labeled $c_i$. Note that as a consequence $s_i$ is contained in the inversion table if and only if $i$ is contained within the construction choice. To consider the range of construction choices which are valid, we also note that there are $k+1$ allowed columns for the first $s_k - s_{k-1}$ rows, $k$ choices for the next $s_{k-1} - s_{k-2}$ rows, and so on. This agrees with the definition of $\mathcal{C}_n(n)$. Taking our example construction choice of 42001000 yields the inversion table $\upsilon = 75003000$ where $\mathrm{Dent}(\upsilon) = \{0, 3, 5, 7\}$:



$\square$

Applying the sieve principle to the set of inversion tables from Proposition 38 we arrive at the following result.

COROLLARY 39. *There is a bijection between $\boldsymbol{\lambda}_n(S)$ and inversion tables whose entries are exactly those in $\{0\} \cup S$,*

$$\big\{\upsilon \in \mathrm{InvTab}_n : \mathrm{Dent}(\upsilon) = \{0, s_1, \dots, s_k\}\big\}.$$

To prove the bijection between $\boldsymbol{\kappa}_n(S)$ and the second set of inversion tables, consideration of a set of ballots is useful. The proof of Proposition 40 below shows one way to make a ballot in $\mathrm{Bal}[n]$ (short for $\mathrm{Bal}[[n]]$) from a given construction choice.

PROPOSITION 40. *For fixed $n$, let $S = \{s_1, \dots, s_k\}$ and $1 \le s_1 < \cdots < s_k < n$ be given. Then $\mathcal{C}_n(S)$ is in bijection with the set of ballots*

$$\big\{\, B_1 \dots B_{k+1} \in \mathrm{Bal}[n] : \{\min B_1, \dots, \min B_{k+1}\} = \{1, s_1 + 1, \dots, s_k + 1\}\,\big\}.$$

PROOF. We will show how to construct a ballot from a given construction choice $c_1 c_2 \dots c_n$. Take $k+1$ empty blocks. At any point in the following construction each block will be considered either open or closed, and the open blocks will be numbered 0, 1, …, $k$, from left to right. Initially all blocks are open. For $i$ equal to 1, 2, …, $n$, in that order, let $a = n + 1 - i$ and insert $a$ into the $c_i$th open block. If $a \in \{1, s_1 + 1, \dots, s_k + 1\}$ then also close the block $a$ is inserted into. This way $a$ is guaranteed end up as the minimal element of its block. It is easy to see how to reverse this procedure and thus it provides the claimed bijection. $\square$

EXAMPLE 41. For $n = 8$ and $S = \{3, 5, 6, 7\}$ consider the construction of a ballot whose minimal block elements are $\{1, 4, 6, 7, 8\}$ with construction choice 42001000. Initially we have 5 empty blocks labeled from $[0, 4]$. Note that when a minimal block element is inserted, that block is no longer open and the remaining blocks are relabeled.

| | |
|---|---|
| $\{\}_0\{\}_1\{\}_2\{\}_3\{\}_4$ | |
| $\{\}_0\{\}_1\{\}_2\{\}_3\{8\}_4$ | 8 inserted in block 4, is minimal entry |
| $\{\}_0\{\}_1\{7\}_2\{\}_3\{8\}$ | 7 inserted in block 2, is minimal entry |
| $\{6\}_0\{\}_1\{7\}\{\}_2\{8\}$ | 6 inserted in block 0, is minimal entry |
| $\{6\}\{5\}_0\{7\}\{\}_1\{8\}$ | 5 inserted in block 0 |
| $\{6\}\{5\}_0\{7\}\{4\}\{8\}$ | 4 inserted in block 1, is minimal entry |
| $\{6\}\{3,5\}_0\{7\}\{4\}\{8\}$ | 3 inserted in block 0 |
| $\{6\}\{2,3,5\}_0\{7\}\{4\}\{8\}$ | 2 inserted in block 0 |
| $\{6\}\{1,2,3,5\}_0\{7\}\{4\}\{8\}$ | 1 inserted in block 0, is minimal entry |

Therefore the final ballot is $\{6\}\{1,2,3,5\}\{7\}\{4\}\{8\}$.

PROPOSITION 42. *There is a bijection between $\boldsymbol{\kappa}_n(S)$ and inversion tables whose missing elements are a subset of $n-s_1$, $n-s_2$, $\dots$, $n-s_k$,*

$$\big\{\upsilon \in \mathrm{InvTab}_n : [n-1] \setminus \mathrm{Dent}(\upsilon) \subseteq \{n-s_1,\dots,n-s_k\}\big\}.$$

*Or, equivalently,*

$$\big\{\upsilon \in \mathrm{InvTab}_n : [0,n-1] \setminus \{n-s_1,\dots,n-s_k\} \subseteq \mathrm{Dent}(\upsilon)\big\}.$$

PROOF. As seen in the proof of Equation (2) from Section 3, a ballot can be taken to a permutation by writing the entries within a block in decreasing order and concatenating the blocks. By this method only the minimal element in a block may be an ascent bottom in the permutation, with the exception of the final block whose minimal element is the last element in the permutation.

Hence, for a fixed $n$ and $S$, the ballot construction gives a bijection between permutations whose set of ascent bottoms is a subset of $S$ and permutations whose set of ascent bottoms plus the last element is a subset of $\{1\}\cup\{s_1+1,\dots,s_k+1\}$. Let $\pi = a_1 \dots a_n$ be any such permutation. We shall denote the set of ascent bottoms plus the final element of $\pi$ as $T = \{t_1, t_2, \ldots, t_j\}$:

$$A(\pi) \cup \{a_n\} = T \subseteq \{1\} \cup \{s_1 + 1, \ldots, s_k + 1\}.$$

An element in a permutation can either be an ascent bottom, a descent top, or the final element. Taking the complement of a permutation takes an ascent bottom $t_i$ to a descent top $n+1-t_i$. Letting $\pi^c$ denote the complement of $\pi$, it follows that for $\pi^c$ the set of descent tops and final element is

$$\{n+1-t_1, n+1-t_2, \ldots, n+1-t_j\} \subseteq \{n\} \cup \{n-s_1, \ldots, n-s_k\},$$

which contains at least the element $n$. The set of ascent bottoms in $\pi^c$ contains everything which is not a descent top or the final element.

$$A(\pi^c) = [n] \setminus \{n+1-t_1, \ldots, n+1-t_j\}.$$

As $T \subseteq \{1\} \cup \{s_1 + 1, \ldots, s_k + 1\}$, it follows that

$$[n-1] \setminus \{n-s_1, \ldots, n-s_k\} \subseteq A(\pi^c).$$

From Corollary 39 we have that $\pi^c$ corresponds to an inversion table whose entries are exactly those in $\{0\} \cup A(\pi^c)$, thus giving a unique inversion table satisfying

$$[0, n-1] \setminus \{n-s_1, \ldots, n-s_k\} \subseteq \mathrm{Dent}(\upsilon).$$

This concludes the proof. ☐

Example 43. As in previous examples, let $n = 8$, $S = \{3, 5, 6, 7\}$ and consider the construction choice 42001000. From Example 36 the permutation in $\boldsymbol{\kappa}_n(S)$ that is given by the construction choice is $\pi = 65783421$. We wish to find the inversion table $\upsilon$ corresponding to $\pi$ satisfying

$$[0, 7] \setminus \{8-3, 8-5, 8-6, 8-7\} = \{0, 4, 6, 7\} \subseteq \mathrm{Dent}(\upsilon).$$

From Example 41 the ballot given by the construction choice is $\{6\}\{1, 2, 3, 5\}\{7\}\{4\}\{8\}$. Writing the elements within a block in decreasing order and concatenating the blocks gives the permutation $\tau = 65321748$ with set of ascent bottoms $\{1, 4\}$ and final element $\{8\}$ where

$$\{1,4,8\} \subset \{1, s_1+1, \ldots, s_k+1\} = \{1, 3+1, 5+1, 6+1, 7+1\}.$$

The complement of $\tau$ is $\tau^c = 34678251$ and has set of descent tops $\{9-4, 9-1\} = \{5,8\}$ and final element $9-8=1$. Every other entry in $\tau^c$ is an ascent bottom:

$$A(\tau^c) = \{2,3,4,6,7\}.$$

Taking $S' = A(\tau^c)$, it follows from Proposition 35 that the construction choice uniquely specifying $\tau^c \in \boldsymbol{\kappa}_n(S')$ is 54312000. Applying Proposition 38 and Corollary 39, we can show that $\tau^c$ corresponds to the inversion table 76423000, which, by construction, has set of distinct entries

$$\mathrm{Dent}(76423000) = \{0,2,3,4,6,7\} = \{0\} \cup A(\tau^c).$$

Thus we have constructed $\upsilon$ satisfying $\{0,4,6,7\} \subseteq \{0,2,3,4,6,7\} = \mathrm{Dent}(\upsilon)$.

## Decomposition of fixed points

Recall that matrices fixed under the involution $\eta$ satisfy the properties

(1) there is exactly one element per row;
(2) if $a < b$, with $a$ on row $i$, and $b$ on row $i+1$, then column $i$ is non-empty.

Also recall that a fixed point matrix can be viewed as a pair consisting of a permutation and an inversion table.

For $A \in \mathrm{BalMat}^{\eta}[U]$ where $n = |U|$, let $\pi(A) = a_1 \ldots a_n$ be the permutation defined by setting $a_i$ the value held in the unique nonzero element of row $i$ of $A$. Let an equivalence relation $\sim$ on $\mathrm{BalMat}^{\eta}[U]$ be defined by $A \sim B$ if $\pi(A) = \pi(B)$.

Proposition 44. *For $\pi \in \mathfrak{S}_n$, the equivalence class $[\pi]_\sim$ is determined by the descent set $S = \{s_1, s_2, \ldots, s_n\} = D(\pi)$ of $\pi$ alone. In fact, fixed point matrices in $[\pi]_\sim$ can be viewed as pairs consisting of the permutation $\pi$ and an inversion table whose set of missing entries is a subset of $\{n-s_1, n-s_2, \ldots, n-s_k\}$.*

PROOF. It is a defining property of a fixed point matrix that if $a < b$, with $a$ on row $i$, and $b$ on row $i+1$, then column $i$ is required to be non-empty. This is equivalent to saying that when the matrix is decomposed into a permutation and inversion table, that $n-i$ is an entry contained within the inversion table.

So, if $a > b$ then we have a descent in the associated permutation and therefore column $i$ may or may not be empty. It follows that $n-i$ may or may not be contained in the inversion table.

Therefore, for $\pi \in \mathfrak{S}_n$, if the set of descent positions is $D(\pi) = S = \{s_1, s_2, \ldots, s_k\}$, then the set of inversion tables with which $\pi$ can be paired are exactly those where the set of missing entries is a subset of $\{n-s_1, n-s_2, \ldots, n-s_k\}$. □

THEOREM 45. *Labeled interval orders on $[n]$ are in bijection with the set*

$$\sum_{S\subseteq[n-1]} \boldsymbol{\beta}_n(S) \times \boldsymbol{\kappa}_n(S).$$

*This set may be alternatively written as*

$$\{(\pi,\tau) \in \mathfrak{S}_n \times \mathfrak{S}_n : A(\tau) \subseteq D(\pi)\}.$$

PROOF. The adapted surjection of Dukes et al. is a bijection between labeled interval orders and fixed point ballot matrices. This is given by the equivalence class on ballot matrices according to interval order and Proposition 33 which shows that there is a unique fixed point per equivalence class.

A fixed point matrix can be decomposed into a permutation $\pi$ and an inversion table. If $D(\pi) = \{s_1, s_2, \ldots s_k\}$ Proposition 44 gives that the set of inversion tables with which $\pi$ can be paired are those whose set of missing elements is a subset of $\{n-s_1, n-s_2, \ldots, n-s_k\}$. We know from Proposition 42 that such inversion tables are in bijection with permutations in $\boldsymbol{\kappa}_n(D(\pi))$. □

COROLLARY 46. *The number of labeled interval orders on $[n]$ is given by the formula*

$$\sum_{\{s_1,\ldots,s_k\}\subseteq[n-1]} \left( \det\left[\binom{n-s_i}{s_{j+1}-s_i}\right] \cdot \prod_{r=1}^{k+1} r^{s_r - s_{r-1}} \right),$$

*in which $s_0 = 0$ and $s_{k+1} = n$.*

PROOF. This follows from the formula for $\beta_n$, see Stanley [**23**, Example 2.2.4], and the formula for $\kappa_n$ given by Corollary 37. $\square$

In the above we have taken the permutation to be fixed and considered the set of inversion tables in the equivalence class under $\sim$. It is equally natural to instead take the inversion table as fixed.

As before, for $A \in \mathrm{BalMat}^{\eta}[U]$, let $\upsilon(A) = b_1 b_2 \dots b_n$ be the inversion table from the decomposition of a ballot matrix fixed under $\eta$ defined by setting $b_i$ to $n - j$ where $j$ is the column of the only non-empty ballot entry on row $i$ of $A$.

Let the equivalence relation $\approx$ on $\mathrm{BalMat}^{\eta}[U]$ be defined by $A \approx B$ if $\upsilon(A) = \upsilon(B)$.

PROPOSITION 47. *For $\upsilon \in \mathrm{InvTab}_n$, the equivalence class $[\upsilon]_{\approx}$ is determined by* $\mathrm{Dent}(\upsilon)$ *alone. In fact, fixed point matrices in $[\upsilon]_{\approx}$ can be viewed as pairs consisting of the inversion table $\upsilon$ and a permutation whose descent set is a subset of* $\mathrm{Dent}(\upsilon) \setminus \{0\}$.

PROOF. This proof is similar to that of Proposition 44. Define $S = \{s_1, s_2, \dots, s_k\}$ to be the set of distinct entries in $\upsilon$ with the exception of 0.

$$S = \mathrm{Dent}(\upsilon) \setminus \{0\}.$$

From the definition of the decomposition, matrices in $[\upsilon]_{\approx}$ satisfy that columns $n - s_1$, $n - s_2$, $\dots$, $n - s_k$ are non-empty.

Recall that, for a ballot matrix fixed under $\eta$, if there is an ascent at position $i$, $a_i < a_{i+1}$, then column $i$ must be non-empty. If there is a descent, then it may or may not be non-empty. Therefore the set of ascent positions in the associated permutation must be a subset of $n - s_1, n - s_2, \dots, n - s_k$. Trivially, reversing such a permutation yields a permutation whose descent set is a subset of $s_1, s_2, \dots, s_k$.

Therefore, for any given inversion table where the distinct entries is $\{0\} \cup S$, the set of permutations which can be associated are trivially in bijection with those where the descent set is a subset of $S$. $\square$

THEOREM 48. *Labeled interval orders on* $[n]$ *are in bijection with the set*

$$\sum_{S\subseteq[n-1]} \boldsymbol{\alpha}_n(S) \times \boldsymbol{\lambda}_n(S).$$

*This set may be alternatively written as*

$$\{(\pi,\tau) \in \mathfrak{S}_n \times \mathfrak{S}_n : D(\pi) \subseteq A(\tau)\}.$$

PROOF. Corollary 39 gives that permutations in $\boldsymbol{\lambda}_n(S)$ are in bijection with inversion tables with set of distinct elements $\{0\}\cup S$. Proposition 47 states that the permutations with which an inversion table $\upsilon$ can be paired are those with their descent set a subset of $\mathrm{Dent}(\upsilon)\setminus\{0\}$. From definition, such permutations are those contained within $\boldsymbol{\alpha}_n(S)$. $\square$

APPENDIX A

# Distributions

**Unsieved Fishburn.** Rows $x^n$ coefficients and columns $y^k$ coefficients of ordinary generating function,

$$\sum_{n\geq 0} (\boldsymbol{n})_{\boldsymbol{xy+1}}\boldsymbol{!}x^n.$$

| $n$ $k$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | |
| 2 | 2 | | | | | | | | | | |
| 3 | 6 | 1 | | | | | | | | | |
| 4 | 24 | 9 | | | | | | | | | |
| 5 | 120 | 72 | 5 | | | | | | | | |
| 6 | 720 | 600 | 98 | 1 | | | | | | | |
| 7 | 5040 | 5400 | 1450 | 76 | | | | | | | |
| 8 | 40320 | 52920 | 20100 | 2200 | 35 | | | | | | |
| 9 | 362880 | 564480 | 279300 | 48750 | 2299 | 9 | | | | | |
| 10 | 3628800 | 6531840 | 3998400 | 977550 | 85514 | 1717 | 1 | | | | |
| 11 | 39916800 | 81648000 | 59693760 | 18957120 | 2529968 | 114257 | 923 | | | | |
| 12 | 479001600 | 1097712000 | 934416000 | 367053120 | 67238584 | 5123783 | 119573 | 351 | | | |
| 13 | 6227020800 | 15807052800 | 15367968000 | 7216776000 | 1699673976 | 189227808 | 8396268 | 99426 | 90 | | |
| 14 | 87178291200 | 242853811200 | 265646304000 | 145446840000 | 42140513520 | 6294347136 | 437905656 | 11368850 | 66014 | 14 | |
| 15 | 1307674368000 | 3966612249600 | 4823346528000 | 3020903424000 | 1043267314320 | 197733092760 | 19342656408 | 853067512 | 12891238 | 34900 | 1 |

**Sieved Fishburn.** Rows $x$ coefficients and columns $y$ coefficients in the ordinary generating function

$$\sum_{n\geq 0} \boldsymbol{(n)_{x(y-1)+1}!} x^n.$$

| k | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| 0 | 1 | | | | | | | | | | |
| 1 | 1 | | | | | | | | | | |
| 2 | 2 | | | | | | | | | | |
| 3 | 5 | 1 | | | | | | | | | |
| 4 | 15 | 9 | | | | | | | | | |
| 5 | 53 | 62 | 5 | | | | | | | | |
| 6 | 217 | 407 | 95 | 1 | | | | | | | |
| 7 | 1014 | 2728 | 1222 | 76 | | | | | | | |
| 8 | 5335 | 19180 | 13710 | 2060 | 35 | | | | | | |
| 9 | 31240 | 142979 | 146754 | 39644 | 2254 | 9 | | | | | |
| 10 | 201608 | 1134213 | 1561679 | 652644 | 76944 | 1711 | 1 | | | | |
| 11 | 1422074 | 9577715 | 16873483 | 9961358 | 1972528 | 108719 | 923 | | | | |
| 12 | 10886503 | 85988958 | 187236538 | 146957439 | 43400979 | 4413716 | 117116 | 351 | | | |
| 13 | 89903100 | 819205590 | 2147264370 | 2145907686 | 876005346 | 140933106 | 7702806 | 98706 | 90 | | |
| 14 | 796713190 | 8263024828 | 25537080638 | 31464358302 | 16844072146 | 3901964030 | 360170922 | 10841242 | 65888 | 14 | |
| 15 | 7541889195 | 88038390442 | 315494340997 | 467450351232 | 315782323190 | 98874059884 | 13729207098 | 751193888 | 12577183 | 34890 | 1 |

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