Instabilities in Smectic A Liquid Crystals

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June 2012

This thesis is submitted to the University of Strathclyde for the degree of Doctor of Philosophy in the Faculty of Science.

Signed:

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Acknowledgements

I would like to thank my supervisor Prof. Iain Stewart for all his help and guidance during my years at Strathclyde, both as an undergraduate and postgraduate student. His expertise and enthusiasm helped me to understand and enjoy the complex world of liquid crystals. I would also like to thank my two examiners Prof. Riccardo Rosso and Dr André Sonnet for all the time and effort they have given my thesis. Thanks must also be given to EPSRC for the financial support they provided.

Friendships are an important part of life and I have some truly amazing people who I call friends. The list is long, and I thank you all, but I would like to pay special mention to Susan, Cat and Alejandro for being there for me and providing me with cinema trips, buckets of fries, and whisky, not to mention laughter and tears.

The biggest thank you of all must go to my family. Mum, Dad, Ross, Eda and Olly: you have been there for me and supported me in everything I have done, especially when things were hard, and I thank you from the bottom of my heart. I wouldn't be where I am today without your love, support and encouragement. And, finally, I would like to thank Gabriel for being there for me in every way imaginable.

"Le coeur a ses raisons que la raison ne connaît point."
- Blaise Pascal

Abstract

In this thesis, the behaviour and stability of smectic A and smectic C liquid crystals are considered under the application of external influences. The behaviour of smectic A liquid crystals under oscillatory shear will be modelled using dynamic continuum theory recently developed by Stewart (2007). The dynamic continuum theory developed by Leslie, Stewart and Nakagawa (1991) is used to model smectic C liquid crystals under the effects of oscillatory shear flow and electric fields.

Equilibrium solutions are presented for smectic A liquid crystals in which the director $\mathbf{n}$ and unit normal $\mathbf{a}$ are not forced to coincide. The stability of these solutions is then investigated and conclusions are drawn on the effect of changing geometries. The two experimental geometries studied consist of planar homeotropically aligned smectic layers and bookshelf aligned layers. The case in which $\mathbf{n}$ and $\mathbf{a}$ are allowed to decouple is then considered for the bookshelf aligned layers, with full nonlinear solutions presented along with a linear stability analysis.

Bookshelf aligned smectic A will be considered subject to an oscillatory shear in both the finite and semi-infinite domains. Planar aligned smectic C will also be considered subject to an oscillatory shear in the cases when the director is aligned parallel and perpendicular to the oscillation.

Smectic C* liquid crystals are analysed under the influence of an electric field. This is based on work already in the literature which is then extended to include elastic effects. Finally, the effect of surface anchoring on the behaviour of lipid bilayers is briefly discussed.

Contents

Chapter 1

Introduction

1.1 Introduction

The study of liquid crystals began in 1888 when Austrian botanist Friedrich Reinitzer observed during experiments that the substance cholesteryl benzoate appeared to have two melting points. When heating the substance to 145.5 °C it transformed from a solid to a cloudy liquid, then when heated further to 178.5 °C the liquid turned clear. This observation encouraged further experiments and research into similar substances which continued up until the second world war. Interest in liquid crystals decreased greatly after this time until the early 1960's when new applications of liquid crystals became evident. For a general history of liquid crystals we direct the reader to [22, 45].

Liquid crystals are extremely sensitive to outside forces, in particular magnetic and electric fields, and this sensitivity has been used to great effect for many purposes. It is possible to alter the orientation of the molecules inside a liquid crystal simply by applying a field, hence changing the optical properties of the sample. This opened up the possibility for nematic liquid crystals to be used in display devices. The first major breakthrough in this application came in 1970 when Leslie [25] theoretically studied the twisted nematic device, which was later applied by Schadt and Helfrich [17, 43]. This opened up a whole new industry and today liquid crystals are used on a daily basis in millions of devices. Research is still ongoing to try and provide better displays with faster switching times in

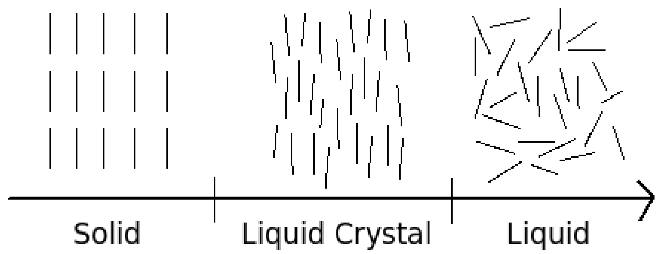


Figure 1.1: The amount of order present in solids, liquids and liquid crystals. Temperature increases from left to right.

an attempt to ever improve the picture quality. However, in the last 20 years or so there has also been a lot of important research conducted into the role of liquid crystals in a biological setting. Cell membranes can be thought of as a type of smectic A liquid crystal [34], and so the study of liquid crystals can also be applied to membrane problems, including the case of cell ruptures. Some of the most interesting and cutting edge applications include targeted drug delivery systems and biosensors.

1.2 Background

Many substances in nature can occur in more than one state of matter, with the three main states being solid, liquid and gas. The state which a substance is in depends on many factors, including temperature, pressure, and various attributes of the substance itself. A simple example to consider is water, which is solid below 0 degrees, is liquid between 0 and 100 degrees, and is a gas above the boiling point of 100 degrees. There are multiple differences between these three states, but one of the most important is the amount of order present in the molecules. When water is in the liquid state it is said to be isotropic, meaning that the position and orientation of the molecules are uniform in all directions. This is in stark contrast to when water is in the frozen state, where the molecules are packed tightly together and form a strict lattice structure. However, there are more than just these three phases of matter present in nature. There also exist *mesophases* that lie between the classic phases, with one such example being the liquid crystal phase, see Figure 1.1. Substances that are in the liquid crystal phase exhibit behaviour and characteristics of both liquids and solids: they flow like a liquid but have orientational order like a solid. The exact characteristics are highly dependent on the type of liquid crystal being considered. The point at which a substance changes from a solid to a liquid crystal is called the *melting point* and the point it changes from a liquid crystal to an isotropic liquid is known as the *clearing point*.

1.3 Classification

The molecules that make up a liquid crystal have complex structures but, as a simplified model, can be thought of as being elongated rod-like molecules. These molecules are free to move within the sample, but they all align in the same average direction, with this general direction denoted by the unit vector $\mathbf{n}$. Since we assume that liquid crystals have no polarity, we have that $\mathbf{n}$ and $-\mathbf{n}$ are indistinguishable.

In nematics, the molecules can move freely anywhere within the sample, and the director $\mathbf{n}$, which they orientate along, is often referred to as the anisotropic axis. In this phase, we say that the molecules posses orientational order but no positional order.

Smectic liquid crystals are very similar to nematics except they also posses positional order, meaning that the molecules align themselves in layers. The molecules can move freely within each layer, and there is even the possibility for molecules to move between layers, but they always maintain a general layered structure. There are various types of smectic liquid crystals, but in this thesis we will focus only on smectic A and smectic C. In smectic A (SmA) liquid crystals, the average alignment of the molecules is again described by the vector $\mathbf{n}$, but there is the additional unit vector, $\mathbf{a}$, that denotes the normal to each layer. In the smectic C (SmC) phase the orientation of the molecules is denoted by the vector $\mathbf{c}$. This is because, in the SmC phase, the molecules tilt away from $\mathbf{a}$ at a fixed angle θ, called the smectic tilt angle or smectic cone angle, and so the director $\mathbf{n}$ can be seen to lie on a fictitious cone of angle θ. The vector $\mathbf{c}$ is then given by the unit orthogonal projection of $\mathbf{n}$ onto the smectic planes, and the

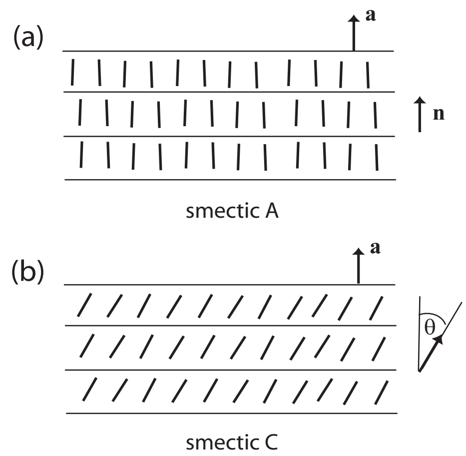


Figure 1.2: Graphical representation of (a) the smectic A phase, and (b) the smectic C phase. In the SmA phase the director **n** is perpendicular to the layers, while in the SmC phase the director makes an angle θ with the unit layer normal.

unit vector $\mathbf{b} = \mathbf{a} \times \mathbf{c}$ is introduced for convenience. The smectic tilt angle is usually considered to be temperature dependent, but in this thesis we assume that the temperature is constant in any given sample. The SmA and SmC phases are shown in Figure 1.2, with a more detailed diagram of the SmC phase shown in Figure 1.3.

In this thesis we will also consider chiral smectic C (SmC*) liquid crystals. SmC* liquid crystals have a twist axis which is perpendicular to the usual SmC layers, and they are known to be ferroelectric, meaning that they posses a spontaneous polarisation **P**. This polarisation is always perpendicular to both **n** and **a**, and it rotates relative to the smectic layers.

1.4 Outline of Thesis

Chapter 2 introduces the notation we will be using throughout this thesis and some preliminary results and equations that will be utilised. Chapter 3 considers the case of static SmA liquid crystals in two experimental geometries and investigates the stability of each using the Ritz method. As well as determining stability, we develop analytical solutions for one of the geometries for which only numerical results were previously known. The remainder of the thesis then fo-

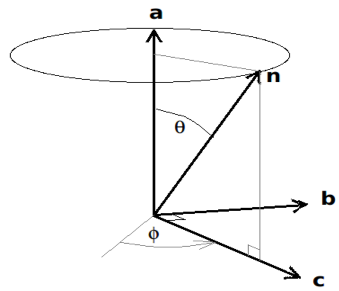


Figure 1.3: A schematic diagram of the vectors used the describe the structure of a SmC liquid crystal. The vector $\mathbf{c}$ is given by the unit orthogonal projection of $\mathbf{n}$ onto the smectic planes, and $\mathbf{b} = \mathbf{a} \times \mathbf{c}$ is introduced for convenience.

cuses on dynamic problems. Chapters 4 and 5 deal with oscillatory shear flow in SmA and SmC liquid crystals, respectively, with analytical solutions given for a SmA sample in finite and semi-infinite domains. An investigation into solutions for the SmC case is then carried out. Chapter 6 builds on work already published in relation to the experimentally observed 'pumping phenomenon' in SmC* liquid crystals. The problem proposed in the literature is extended to include more effects, with a related problem then discussed. We conclude with Chapter 7 which gives a brief and elementary investigation into the role played by surface tension on the displacement of lipid bilayers.

Chapter 2

Preliminaries

2.1 Notation

Throughout this thesis we will be using standard index notation and the summation convention. That is, for any vector $\mathbf{a} = (a_1, a_2, a_3)$, with $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ representing the usual basis vectors in $\mathbb{R}^3$, we write

$$\mathbf{a} = a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \sum_{i=1}^{3} a_i\mathbf{e}_i. \tag{2.1}$$

Using the summation notation, (2.1) can be written in the more compact form

$$\mathbf{a} = a_i\mathbf{e}_i, \tag{2.2}$$

where it is assumed that any index that appears twice, and only twice, in a single term is summed over all possible values of that index. We also assume that an index preceded by a comma will denote differentiation with respect to that index. For example, $a_{i,j}$ denotes the derivative of the ith component of $\mathbf{a}$ with respect to the jth variable, where the total derivative is defined to be [49, Eqn. (2.123)]

$$f_{,i} \equiv \frac{\partial f}{\partial x_i} + \frac{\partial f}{\partial u_j} u_{j,i} + \frac{\partial f}{\partial u_{j,k}} u_{j,ki}, \tag{2.3}$$

where $f = f(\mathbf{x}, \mathbf{u}(\mathbf{x}), \nabla \mathbf{u}(\mathbf{x}))$. We will also make use of the Kronecker delta, δ_{ij}, defined to be

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \neq j, \end{cases} \tag{2.4}$$

and the alternator ϵ_{ijk} given by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } i,\ j \text{ and } k \text{ are unequal and in cyclic order,} \\ -1 & \text{if } i,\ j \text{ and } k \text{ are unequal and in non-cyclic order,} \\ 0 & \text{if any two of } i,\ j \text{ or } k \text{ are equal.} \end{cases} \tag{2.5}$$

The scalar product of the vectors $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$ is defined as

$$\mathbf{a} \cdot \mathbf{b} = a_i b_i, \tag{2.6}$$

and the vector product is defined to be

$$\mathbf{a} \times \mathbf{b} = \epsilon_{ijk} a_j b_k \mathbf{e}_i, \tag{2.7}$$

where $\mathbf{e}_i$ is a unit vector in the i-direction. The magnitude of the vector $\mathbf{a}$ is given by $|\mathbf{a}| = \sqrt{\mathbf{a} \cdot \mathbf{a}}$, and $\mathbf{a}$ and $\mathbf{b}$ are said to be orthogonal if and only if $\mathbf{a} \cdot \mathbf{b} = 0$. We define the scalar triple product of the vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to be

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = a_i \epsilon_{ijk} b_j c_k. \tag{2.8}$$

The gradient of the scalar quantity p is defined to be

$$\nabla p = p_{,i}\mathbf{e}_i. \tag{2.9}$$

The divergence of the vector $\mathbf{a}$ is given by

$$\nabla\cdot \mathbf{a} = a_{i,i}, \tag{2.10}$$

and its curl is defined as

$$\nabla \times \mathbf{a} = \epsilon_{ijk}a_{k,j}\mathbf{e}_i. \tag{2.11}$$

For any second order tensor T_{ij}, we define its divergence to be $T_{ij,j}$. This is in line with the notation adopted by Leigh [24].

2.2 Energy Density

The average alignment of molecules in a nematic liquid crystal at a point $\mathbf{x}$ is denoted by the unit vector $\mathbf{n}$. This is written as

$$\mathbf{n} = \mathbf{n}(\mathbf{x}), \quad \mathbf{n}\cdot \mathbf{n} = 1. \tag{2.12}$$

Liquid crystals have an elastic component to them which, when all outside influences are removed, allows them to return to a natural, lowest energy configuration. Therefore, any distortion of a liquid crystal sample has associated with it a free energy density represented in general by

$$w = w(\mathbf{n}, \nabla\mathbf{n}). \tag{2.13}$$

To find the total bulk energy for the system we simply integrate the energy density over the volume, i.e.

$$W = \int_V w(\mathbf{n}, \nabla\mathbf{n})\, dV, \tag{2.14}$$

where V is the volume of the sample. The total energy is defined up to an additive constant, and since we will search for solutions that provide the lowest energy state, we choose this constant such that $w = 0$ for any natural orientation, that is the state in which the liquid crystal returns to when all outside influences are removed. It is then supposed that any state imposed on the sample will produce an energy that is greater than or equal to the energy of the natural state. Since liquid crystals lack polarity, it is assumed that the vectors $\mathbf{n}$ and $-\mathbf{n}$ are indistinguishable, and so we require that

$$w = w(\mathbf{n}, \nabla\mathbf{n}) = w(-\mathbf{n}, -\nabla\mathbf{n}). \tag{2.15}$$

The free energy per unit volume must also be frame indifferent, that is, it must be the same when described in any frame of reference. If we consider the proper orthogonal matrix Q (where $\det Q = 1$), and require that the energy density be invariant under arbitrary rigid body rotations, then we require

$$w(\mathbf{n}, \nabla\mathbf{n}) = w(Q\mathbf{n}, Q\nabla\mathbf{n}Q^T), \tag{2.16}$$

where Q^T is the transpose of Q. It can be shown [49] that the Frank-Oseen elastic energy for a general anisotropic medium is given by

$$\begin{aligned} w = {} & \frac{1}{2}k_{11}(\nabla\cdot\mathbf{n} - s_0)^2 + \frac{1}{2}k_{22}(\mathbf{n}\cdot\nabla\times\mathbf{n} + t_0)^2 + \frac{1}{2}k_{33}(\mathbf{n}\times\nabla\times\mathbf{n}) \\ & -k_{12}(\nabla\cdot\mathbf{n})(\mathbf{n}\cdot\nabla\times\mathbf{n}) + \frac{1}{2}(k_{22}+k_{24})\nabla\cdot[(\mathbf{n}\cdot\nabla)\mathbf{n} - (\nabla\cdot\mathbf{n})^2]. \end{aligned} \tag{2.17}$$

For nematics, it can be shown that $s_0 = t_0 = k_{12} = 0$, and it is common to set $K_1 = k_{11}$, $K_2 = k_{22}$, $K_3 = k_{33}$ and $K_4 = k_{24}$, where the K_i are known as the Frank elastic constants. Here, s_0 is a spontaneous splay constant, and t_0 is a twist constant related to chirality. Hence the free energy density for nematics can be written as

$$\begin{aligned} w = \frac{1}{2}K_1(\nabla\cdot \mathbf{n})^2 + \frac{1}{2}K_2(\mathbf{n}\cdot \nabla\times\mathbf{n})^2 + \frac{1}{2}K_3(\mathbf{n}\times\nabla\times\mathbf{n})^2 \\ + \frac{1}{2}(K_2 + K_4)\nabla\cdot[(\mathbf{n}\cdot\nabla)\mathbf{n} - (\nabla\cdot\mathbf{n})\mathbf{n}]. \end{aligned} \tag{2.18}$$

The constants K_1, K_2 and K_3 are the splay, twist and bend constants, respectively, and the combination $(K_2 + K_4)$ is known as the saddle-splay constant.

In smectics there is an energy density associated with the distortions of **a** and **c**. Similar to the nematic case, the energy density is of the form

$$w = w(\mathbf{a}, \mathbf{c}, \nabla\mathbf{a}, \nabla\mathbf{c}). \tag{2.19}$$

The energy density for a smectic must also be frame indifferent, so we require

$$w(\mathbf{a}, \mathbf{c}, \nabla\mathbf{a}, \nabla\mathbf{c}) = w(Q\mathbf{a}, Q\mathbf{c}, Q\nabla\mathbf{a}Q^T, Q\nabla\mathbf{c}Q^T), \tag{2.20}$$

where Q is a proper orthogonal matrix. From the work of Leslie *at el* [27], it can be shown that for a non-chiral SmC liquid crystal, the general energy density is of the form

$$\begin{aligned} w = {} & \frac{1}{2}K_1(\nabla\cdot\mathbf{a})^2 + \frac{1}{2}K_2(\nabla\cdot\mathbf{c})^2 + \frac{1}{2}K_3(\mathbf{a}\cdot\nabla\times\mathbf{c})^2 + \frac{1}{2}K_4(\mathbf{c}\cdot\nabla\times\mathbf{c})^2 \\ & + \frac{1}{2}K_5(\mathbf{b}\cdot\nabla\times\mathbf{c})^2 + K_6(\nabla\cdot\mathbf{a})(\mathbf{b}\cdot\nabla\times\mathbf{c}) + \frac{1}{2}K_7(\mathbf{a}\cdot\nabla\times\mathbf{c})(\mathbf{c}\cdot\nabla\times\mathbf{c}) \\ & + K_8(\nabla\cdot\mathbf{c})(\mathbf{b}\cdot\nabla\times\mathbf{c}) + K_9(\nabla\cdot\mathbf{a})(\nabla\cdot\mathbf{c}), \end{aligned} \tag{2.21}$$

where the K_i are elastic constants and $\mathbf{b} = \mathbf{a} \times \mathbf{c}$. Most dynamic contiuum theories before 2007 supposed that $\mathbf{n}$ and $\mathbf{a}$ always coincided. However, it was discovered in experiments that, due to the complex structure of SmA liquid crystals, this was not always the case. This work is documented by Auernhammer *et al.* [3] and Soddemann [47] in the case of a shear applied to a sample of SmA. The paper by Ribotta and Durand [39] states that combining a tilt of both the molecules and the layer results in a strain comprising of a compression and dilation contribution. When the compression term is subjected to a dilative stress, a layer undulation instability is incurred, and when the dilation term is subjected to a compressive stress, it is expected that a molecular tilt instability will occur inside the layers. This tilt inside the layers leads to a buckling of the director, and so it is important to allow some freedom for the director and layer normal to decouple. Hence, in Chapters 3–6, we use the energy density based on the initial modelling work of Auernhammer *et al.* [3], Ribotta and Durand [39], and Stewart [51], which has the form

$$w = \frac{1}{2}K_1^n(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_1^a(\nabla\cdot\mathbf{a})^2 + \frac{1}{2}B_0(|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2)^2 + \frac{1}{2}B_1(1 - (\mathbf{n}\cdot\mathbf{a})^2). \quad (2.22)$$

This form of the energy density allows for the decoupling of $\mathbf{n}$ and $\mathbf{a}$, while enabling them to coincide when it is natural to do so. We have introduced the notation Φ as a way of describing the local planar layer structure. The scalar function Φ is related to the unit layer normal through the identity

$$\mathbf{a} = \frac{\nabla\Phi}{|\nabla\Phi|}. \quad (2.23)$$

In the energy density (2.22), the first term is the traditional splay deformation of the director $\mathbf{n}$, and the second term is associated with the bending of the smectic layers. The quantities K_1^n and K_1^a are positive elastic constants with units of newtons (N) and which are generally of a similar size. The third term deals with the compression of the smectic layers and the fourth term measures the strength of coupling between the director and layer normal. The constants B_0 and B_1

are both positive constants, with units of Nm^{-2}, and, in general, the magnitude of B_1 should be comparable to B_0 or smaller [39]. We note that the energy density is minimised when $\mathbf{n}\cdot\mathbf{a} = 1$, that is when the director and unit normal are parallel, which is what we would expect from a physical interpretation. We also note that the energy density in (2.22) is invariant to simultaneous changes in the sign of $\mathbf{n}$ and $\mathbf{a}$. The form stated in (2.22) is widely used in the literature but it is known that the compression term $\frac{B_0}{2}(1 - |\nabla\Phi|)^2$ is not entirely suitable since it is a quadratic approximation. A more appropriate formulation is given by $\frac{B_0}{2}|\nabla\Phi|^{-2}(1 - |\nabla\Phi|)^2$, as stated in Capriz and Napoli [5], and is based on Hooke's Law. Both terms coincide up to quadratic order, but when dealing with nonlinear equations the results obtained may differ. In Chapter 7, we use the second formulation for the compression term due to the nature of the problem being studied and the work previously carried out in this area. Hence in Chapter 7 we adopt the alternative form [56]

$$w = \frac{1}{2}K_1^n(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_1^a(\nabla\cdot\mathbf{a})^2 + \frac{1}{2}\frac{B_0}{|\nabla\Phi|^2}(1 - |\nabla\Phi|)^2 + \frac{1}{2}B_1(1 - (\mathbf{n}\cdot\mathbf{a})^2). \quad (2.24)$$

2.3 Electric Fields

One of the most interesting aspects of liquid crystals is their response to external forces, in particular those due to electric and magnetic fields. Even applying a relatively small voltage to a sample of liquid crystal can hugely alter the structure of the sample. We consider an electric field $\mathbf{E}$ which, as a basic preliminary approximation, is subject to the constraints of the static field Maxwell's equations when the charge density is neglected:

$$\nabla\cdot\mathbf{E} = 0, \quad \nabla\times\mathbf{E} = \mathbf{0}. \quad (2.25)$$

The general dielectric energy density for liquid crystals is given by [49, p28]

$$w_{elec} = -\frac{1}{2}\epsilon_0\epsilon_a(\mathbf{n}\cdot\mathbf{E})^2, \tag{2.26}$$

where ϵ_0 is the permittivity of free space and ϵ_a is the dielectric anistropy of the liquid crystal. In SI units, $\epsilon_0 \simeq 8.854 \times 10^{-12}$ F m^{-1} [49], where the abbreviation F represents a farad, and ϵ_a is unitless since it is measured relative to ϵ_0. The electric energy density in a SmC* liquid crystal has an additional electrical energy contribution given by [49]

$$w_{pol} = -\mathbf{P}\cdot\mathbf{E}, \tag{2.27}$$

where $\mathbf{P}$ is the spontaneous polarisation of the SmC* liquid crystal, and can be written as a vector parallel to the unit vector $\mathbf{b}$ such that

$$\mathbf{P} = P_0\mathbf{b}, \quad P_0 > 0. \tag{2.28}$$

This is the reason why it is a convenient convention to adopt the vector $\mathbf{b}$ such that $\mathbf{b} = \mathbf{a} \times \mathbf{c}$ when considering SmC* liquid crystals. This then defines the 'positive' direction of the polarisation of the liquid crystal when $P_0 > 0$. There are also SmC* materials for which $P_0 < 0$.

2.4 Euler-Lagrange Equations

In general, when minimising a functional with respect to n variables, y_n, the Euler-Lagrange equations take the form [42]

$$\frac{d}{dz}f_{,y'_k}(z,\mathbf{y},\mathbf{y}') - f_{,y_k}(z,\mathbf{y},\mathbf{y}') = 0, \quad k = 1, 2, ..., n, \tag{2.29}$$

where $\mathbf{y} = [y_1, y_2, ...y_n]$. In Chapter 3, we will be searching for equilibrium solutions that minimise the total energy of a static configuration. To find these equilibrium solutions, we must minimise the energy W, given by (2.14), with respect to the tilt angle of the director, $\theta(z)$. When we consider a sample homeotropically planar aligned SmA, θ is defined to be the angle that the director makes with respect to the z-axis, while in a bookshelf aligned sample, θ is the angle that the director makes with respect to the x-axis. In this particular case, we will be considering only 2d-orientations, with the tilt angle θ solely dependent on one variable. A necessary condition for the minimisation of $\theta(z)$ is that $\theta(z)$ satisfies the Euler-Lagrange equation [42]. So, in particular, when minimising (2.22) with respect to $\theta(z)$ in Sections 3.1 and 3.2 we must have, for example in the $n = 1$ case,

$$\frac{d}{dz}\left(\frac{\partial w}{\partial \theta'}\right) - \frac{\partial w}{\partial \theta} = 0, \tag{2.30}$$

while in Section 3.3 we have the additional angle of $\delta(z)$ to incorporate into the problem, so the angles θ and δ must satisfy the coupled Euler-Lagrange equations

$$\frac{d}{dz}\left(\frac{\partial w}{\partial \theta'}\right) - \frac{\partial w}{\partial \theta} = 0, \tag{2.31}$$

$$\frac{d}{dz}\left(\frac{\partial w}{\partial \delta'}\right) - \frac{\partial w}{\partial \delta} = 0. \tag{2.32}$$

2.5 Dynamic Equations

In the dynamic sections of this thesis, we will make use of the Navier-Stokes equations in specifically simplified problems, in addition to the equations for the balance of angular momentum, to derive governing equations. In the case of the dynamic SmA problem, we will also require the permeation equation. The Navier-Stokes equations for an incompressible isotropic Newtonian fluid are given by [2]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{v} + \mathbf{g}, \tag{2.33}$$

$$\nabla\cdot\mathbf{v} = 0, \tag{2.34}$$

where ν is the kinematic viscosity, ρ is the density of the fluid, ∇^2 is the Laplacian operator, $\mathbf{g}$ denotes the gravitational force, $\mathbf{v}(x, y, z, t) = (v_1, v_2, v_3)$ is the velocity and the pressure is denoted by $p = p(x, y, z, t)$. The Leslie-Nakagawa dynamic equations for smectic liquid crystals are derived from the equations for the balance of angular and linear momentum, and differ depending on the type of liquid crystal being considered. First let us consider a SmA sample.

2.5.1 Dynamic Smectic A Theory

In a SmA liquid crystal, the vectors $\mathbf{n}$ and $\mathbf{a}$ are subject to the constraints

$$\mathbf{n}\cdot\mathbf{n} = 1, \quad \mathbf{a}\cdot\mathbf{a} = 1, \quad \nabla\times\mathbf{a} = \mathbf{0}. \tag{2.35}$$

The third constraint in (2.35), known as the Oseen condition, is satisfied in general in the absence of defects or singularities in the smectic layer. However, when the SmA dynamic theory introduced by Stewart [51] is used, this condition need not necessarily hold, as is frequently the case in dynamics away from equilibrium. If we consider a sample of SmA liquid crystal, free from defects, with unit layer normal $\mathbf{a}$ and interlayer distance a, then an integral along an arbitrary closed loop Γ, as shown in Figure 2.1, must vanish [49]. This must be true since the number of layers crossed in one direction is the same as those crossed in the opposite direction. We can write this mathematically as

$$\oint_\Gamma \mathbf{a}\cdot d\mathbf{x} = 0. \tag{2.36}$$

Since $\mathbf{a}$ is differentiable in the absence of defects, as in this case, we can use Stokes' theorem to write

$$\oint_\Gamma \mathbf{a}\cdot d\mathbf{x} = \int_S (\nabla\times\mathbf{a})\cdot\boldsymbol{\nu}\,dS = 0, \tag{2.37}$$

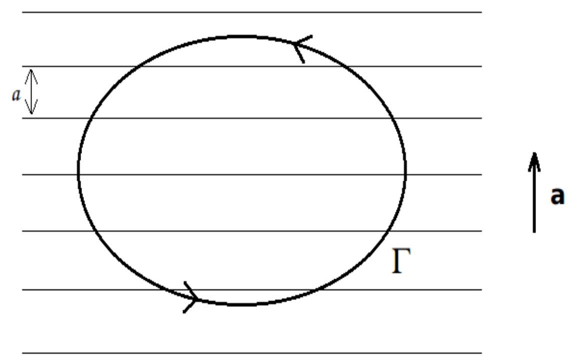


Figure 2.1: A closed loop Γ in a SmA liquid crystal with unit layer normal $\mathbf{a}$ and interlayer distance a. An integral over the arbitrary loop Γ must vanish which leads to the constraint $\nabla \times \mathbf{a} = \mathbf{0}$.

where $\boldsymbol{\nu}$ is the outward unit normal of the area S enclosed by Γ. Since S is arbitrary, it follows from (2.37) that we require $\nabla \times \mathbf{a} = \mathbf{0}$.

When we allow this condition to be violated, we have the equation for the balance of linear momentum given by [51]

$$\rho \dot{v}_i = \rho F_i - \tilde{p}_{,i} + \tilde{g}_j n_{j,i} + G_j n_{j,i} + |\nabla\Phi| a_i J_{j,j} + \tilde{t}_{ij,j}, \tag{2.38}$$

where ρ is the density of the fluid, F_i is the external body force per unit mass, $\tilde{p}$ is the pressure, $\tilde{g}_i$ is a dynamic contribution, G_j is the generalised body force related to the external body moment per unit mass, J_j is the phase flux term, and $\tilde{t}_{ij}$ is the viscous stress tensor. The dynamic contribution $\tilde{g}_i$ takes the form [51]

$$\tilde{g}_i = -\gamma_1 N_i - \gamma_2 A_{ip} n_p - 2\kappa_1 A_{ip} a_p, \tag{2.39}$$

where $N_i = \dot{n}_i - W_{ij} n_j$ is the co-rotational time flux of $\mathbf{n}$, $A_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ is the rate of strain tensor and $W_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i})$ is the vorticity tensor, both common tensors in continuum mechanics. The quantities $\gamma_1 = \alpha_3 - \alpha_2$, $\gamma_2 = \alpha_3 + \alpha_2$ and

κ_1 are viscosity coefficients. The full viscous stress tensor is given by

$$\begin{aligned}
\tilde{t}_{ij} =& \alpha_1(n_k A_{kp} n_p) n_i n_j + \alpha_2 N_i n_j + \alpha_3 n_i N_j + \alpha_4 A_{ij} + \alpha_5(n_j A_{ip} n_p + n_i A_{jp} n_p) \\
&+ (\alpha_2 + \alpha_3) n_i A_{jp} n_p + \tau_1(a_k A_{kp} a_p) a_i a_j + \tau_2(a_j A_{ip} a_p + a_i A_{jp} a_p) \\
&+ \kappa_1(a_i N_j + a_j N_i + n_i A_{jp} a_p - n_j A_{ip} a_p) + \kappa_2(n_k A_{kp} a_p)(n_i a_j + a_i n_j) \\
&+ \kappa_3[(n_k A_{kp} n_p) a_i a_j + (a_k A_{kp} a_p) n_i n_j] \\
&+ \kappa_4[2(n_k A_{kp} a_p) n_i n_j + (n_k A_{kp} n_p)(a_i n_j + n_i a_j)] \\
&+ \kappa_5[2(n_k A_{kp} a_p) a_i a_j + (a_k A_{kp} a_p)(n_i a_j + a_i n_j)] \\
&+ \kappa_6(n_j A_{ip} a_p + n_i A_{jp} a_p + a_i A_{jp} n_p + a_j A_{ip} n_p).
\end{aligned} \tag{2.40}$$

If all terms containing the director $\mathbf{n}$ are neglected, this reduces to

$$\tilde{t}_{ij} = \alpha_4 A_{ij} + \tau_1(a_k A_{kp} a_p) a_i a_j + \tau_2(a_i A_{jp} a_p + a_j A_{ip} a_p), \tag{2.41}$$

which is the form proposed by Martin *et al.* [31], de Gennes and Prost [7] and E [11] for an incompressible SmA liquid crystal.

The equation for the balance of angular momentum is given by

$$\left(\frac{\partial w}{\partial n_{i,j}}\right)_{,j} - \frac{\partial w}{\partial n_i} + \tilde{g}_i + G_i = \mu n_i, \tag{2.42}$$

where w is the energy density, $\tilde{g}_i$ is the dynamic contribution given by (2.39), G_i is the generalised body force related to the external body moment per unit mass, and μ is a Lagrange multiplier. In general, the Lagrange multiplier can be evaluated explicitly or eliminated from the problem.

Starting with the second law of thermodynamics, a dissipation inequality involving the scalar function Φ, as defined in equation (2.23), can be derived. To ensure the positivity of this dissipation function, it can be shown that Φ must be a negative multiple of the divergence of the phase flux term, J_i [51]. Hence, the permeation equation, which describes the amount of flow through the layers, is defined to be

$$\dot{\Phi} = -\lambda_p J_{i,i}, \tag{2.43}$$

where λ_p is the positive permeation constant [51] based on work originally carried out by Helfrich [15]. The superposed dot represents the material time derivative, and so the right hand side of (2.43) can be written explicitly as

$$\dot{\Phi} = \frac{\partial \Phi}{\partial t} + \mathbf{v}\cdot \nabla \Phi. \tag{2.44}$$

The phase flux term, J_i, introduced above is defined to be [51]

$$J_i = -\frac{\partial w}{\partial \Phi_{,i}} + \frac{1}{|\nabla \Phi|}\left[\left(\frac{\partial w}{\partial a_{p,k}}\right)_{,k} - \frac{\partial w}{\partial a_p}\right](\delta_{pi} - a_p a_i). \tag{2.45}$$

The first term in (2.45) is related to the permeation in the layers, while the second term can be thought of as a projection of the Euler-Lagrange equations on the smectic layers. The projector term $(\delta_{pi} - a_p a_i)$ is required since there is no permeation in an equilibrium state. We note that the phase flux term is solely dependent on the layer normal, $\mathbf{a}$, since it only contributes to the permeation in the layers. Now we consider the dynamic equations for a SmC* liquid crystal as introduced by Leslie *et al.* [28].

2.5.2 Dynamic Smectic C* Theory

In the case of a SmC* liquid crystal, the vectors $\mathbf{a}$ and $\mathbf{c}$ are subject to the constraints

$$\mathbf{a}\cdot \mathbf{a} = 1, \quad \mathbf{c}\cdot \mathbf{c} = 1, \quad \mathbf{a}\cdot \mathbf{c} = 0, \quad \nabla \times \mathbf{a} = \mathbf{0}, \tag{2.46}$$

with no relaxation of the Oseen condition $\nabla \times \mathbf{a} = \mathbf{0}$. The equation for the balance of linear momentum is given by [49]

$$\rho \dot{v}_i = \rho F_i - \tilde{p}_{,i} + G_k^a a_{k,i} + G_k^c c_{k,i} + \tilde{g}_k^a a_{k,i} + \tilde{g}_k^c c_{k,i} + \tilde{t}_{ij,j}, \tag{2.47}$$

and the equation for the balance of angular momentum reduces to the coupled equations [49]

$$\left(\frac{\partial w}{\partial a_{i,j}}\right)_{,j} - \frac{\partial w}{\partial a_i} + \tilde{g}_i^a + G_i^a + \mu c_i + \gamma a_i + \epsilon_{ijk}\beta_{k,j} = 0, \tag{2.48}$$

$$\left(\frac{\partial w}{\partial c_{i,j}}\right)_{,j} - \frac{\partial w}{\partial c_i} + \tilde{g}_i^c + G_i^c + \tau c_i + \mu a_i = 0, \tag{2.49}$$

where F_i is the external body force per unit mass, $\tilde{p}$ is a pressure, $\tilde{t}_{ij}$ is the viscous stress tensor, G_i^a and G_i^c are generalised external body forces related to $\mathbf{a}$ and $\mathbf{c}$, respectively, and the scalar functions μ, γ and τ and the vector function $\boldsymbol{\beta}$ are Lagrange multipliers. The 'a-equations' (2.48) are coupled to the 'c-equations' (2.49) via the Lagrange multiplier μ. The dynamic contributions $\tilde{g}_i^a$ and $\tilde{g}_i^c$ are given by

$$\begin{aligned} \tilde{g}_i^a = -2(&\lambda_1 D_i^a + \lambda_3 c_i c_p D_p^a + \lambda_4 A_i + \lambda_6 c_i c_p A_p + \tau_2 D_i^c \\ &+ \tau_3 c_i a_p D_p^a + \tau_4 c_i c_p D_p^c + \tau_5 C_i), \end{aligned} \tag{2.50}$$

$$\tilde{g}_i^c = -2(\lambda_2 D_i^c + \lambda_5 C_i + \tau_1 D_i^a + \tau_5 A_i). \tag{2.51}$$

The viscous stress tensor $\tilde{t}_{ij}$ for a SmC* liquid crystal is given by

$$\tilde{t}_{ij} = \tilde{t}_{ij}^s + \tilde{t}_{ij}^{ss}, \tag{2.52}$$

where $\tilde{t}_{ij}^s$ and $\tilde{t}_{ij}^{ss}$ are the symmetric and skew-symmetric parts of the viscous stress, respectively, and are given by [49]

$$
\begin{aligned}
\tilde{t}_{ij} &= \mu_0 D_{ij} + \mu_1 a_p D^a_p a_i a_j + \mu_2 (D^a_i a_j + D^a_j a_i) + \mu_3 c_p D^c_p c_i c_j + \mu_4 (D^c_i c_j + D^c_j c_i) \\
&\quad + \mu_5 c_p D^a_p (a_i c_j + a_j c_i) + \lambda_1 (A_i a_j + A_j a_i) + \lambda_2 (C_i c_j + C_j c_i) + \lambda_3 c_p A_p (a_i c_j + a_j c_i) \\
&\quad + \kappa_1 (D^a_i c_j + D^a_j c_i + D^c_i a_j + D^c_j a_i) + \kappa_2 (a_p D^a_p (a_i c_j + a_j c_i) + 2 a_p D^c_p a_i a_j) \\
&\quad + \kappa_3 (c_p D^c_p (a_i c_j + a_j c_i) + 2 a_p D^c_p c_i c_j) + \tau_1 (C_i a_j + C_j a_i) + \tau_2 (A_i c_j + A_j c_i) \\
&\quad + 2\tau_3 c_p A_p a_i a_j + 2\tau_4 c_p A_p c_i c_j, \qquad (2.53)
\end{aligned}
$$

and

$$
\begin{aligned}
\tilde{t}^{ss}_{ij} &= \lambda_1 (D^a_j a_i - D^a_i a_j) + \lambda_2 (D^c_j c_i - D^c_i c_j) + \lambda_3 c_p D^a_p (a_i c_j - a_j c_i) + \lambda_4 (A_j a_i - A_i a_j) \\
&\quad + \lambda_5 (C_j c_i - C_i c_j) + \lambda_6 c_p A_p (a_i c_j - a_j c_i) + \tau_1 (D^a_j c_i - D^a_i c_j) + \tau_2 (D^c_j a_i - D^c_i a_j) \\
&\quad + \tau_3 a_p D^a_p (a_i c_j - a_j c_i) + \tau_4 c_p D^c_p (a_i c_j - a_j c_i) + \tau_5 (A_j c_i - A_i c_j + C_j a_i - C_i a_j). \qquad (2.54)
\end{aligned}
$$

The viscosity μ_0 is independent of **a** and **c** and corresponds to the usual isotropic part of the viscous stress. The viscosities μ_1, μ_2, λ_1 and λ_4 are dependent only on contributions from **a**, and so are said to be SmA-like in nature, with the SmC-like viscosities given by μ_3, μ_4, λ_2 and λ_5. Finally, there are eleven viscosities that are considered to be coupling terms due to their dependence on both **a** and **c**. They are μ_5, λ_3, λ_6, κ_1, κ_2, κ_3, τ_1, τ_2, τ_3, τ_4 and τ_5. The viscosities that are dependent on the c-director can be related to the smectic tilt angle θ as follows [49]:

$$\mu_3 = \hat{\mu}_3 \theta^4, \quad \mu_4 = \hat{\mu}_4 \theta^2, \quad \mu_5 = \hat{\mu}_5 \theta^2, \qquad (2.55)$$

$$\lambda_2 = \hat{\lambda}_2 \theta^2, \quad \lambda_3 = \hat{\lambda}_3 \theta^2, \quad \lambda_4 = \hat{\lambda}_4 \theta^2, \quad \lambda_5 = \hat{\lambda}_5 \theta^2, \qquad (2.56)$$

$$\kappa_1 = \hat{\kappa}_1 \theta, \quad \kappa_2 = \hat{\kappa}_2 \theta, \quad \kappa_3 = \hat{\kappa}_3 \theta^2, \qquad (2.57)$$

$$\tau_1 = \hat{\tau}_1 \theta, \quad \tau_2 = \hat{\tau}_2 \theta, \quad \tau_3 = \hat{\tau}_3 \theta, \quad \tau_4 = \hat{\tau}_4 \theta^3, \quad \tau_5 = \hat{\tau}_5 \theta, \qquad (2.58)$$

where the constants $\hat{\mu}_i$, $\hat{\lambda}_i$, $\hat{\kappa}_i$ and $\hat{\tau}_i$ are assumed to be only weakly dependent on temperature. These classifications can be used to help identify the dominant viscosities in theoretical investigations of SmC* liquid crystals. It can be shown

[49] that these viscosities satisfy certain inequalities, in particular

$$\mu_0 + \mu_4 + \lambda_5 \pm 2\lambda_2 \geq 0. \tag{2.59}$$

This inequality comes from the dissipation function, which has restraints placed upon it to ensure that the liquid crystal loses energy when expected. For a more in depth discussion we refer the reader to Stewart [49].

Chapter 3

Stability of Static Smectic A Liquid Crystals

Some particularly novel static solutions of smectic A liquid crystals were found by Stewart [50] with interesting results involving a parameter B, a dimensionless measure of the strength of coupling between **n** and **a** in relation to the compression present in the layers. In the case of planar aligned SmA, it was found that the behaviour of the solutions changed in response to this parameter at a critical value B*. We investigate the stability of these solutions, paying specific interest to the parameter B, and we also consider the stability of a bookshelf aligned sample in the variable layer case.

3.1 Planar Homeotropically Aligned Smectic A

We first consider a sample of planar homeotropically aligned SmA liquid crystal confined between two parallel plates at $z = 0$ and $z = d$, arranged in a fixed layer. In this geometrical set-up we have the director, unit normal and scalar function Φ to be of the form

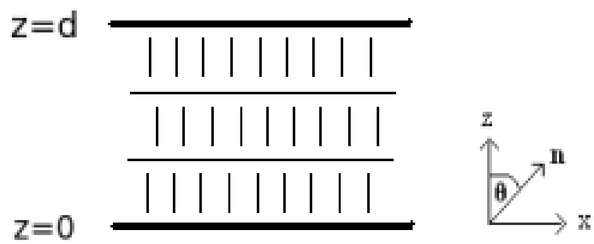


Figure 3.1: Planar homeotropically aligned SmA liquid crystal confined between two parallel plates a distance d apart, where θ denotes the angle that the director makes with the z-axis.

$$\mathbf{n} = (\sin\theta(z), 0, \cos\theta(z)), \tag{3.1}$$

$$\mathbf{a} = (0, 0, 1), \tag{3.2}$$

$$\Phi = z, \tag{3.3}$$

where the strong anchoring boundary conditions are given by

$$\theta(0) = \theta(d) = \theta_0 > 0. \tag{3.4}$$

We choose the director to be dependent only on z since experimental data show that the behaviour of the liquid crystal is uniform in the x-y plane, and the only spatial dependence recorded is in the z-direction. Inserting $\mathbf{n}$, $\mathbf{a}$ and Φ into (2.22) gives the specific energy density for this problem, namely

$$w = \frac{1}{2}K_1^n(\theta')^2\sin^2\theta + \frac{1}{2}B_0(1-\cos\theta)^2 + \frac{1}{2}B_1\sin^2\theta. \tag{3.5}$$

We can write

$$\frac{\partial w}{\partial \theta'} = K_1^n \theta' \sin^2\theta, \tag{3.6}$$
$$\frac{\partial w}{\partial \theta} = K_1^n (\theta')^2 \sin\theta\cos\theta + B_0(1-\cos\theta)\sin\theta + B_1\sin\theta\cos\theta, \tag{3.7}$$

and so inserting (3.5) into the Euler-Lagrange equation (2.30) yields

$$K_1^n\left[\theta''\sin^2\theta + (\theta')^2\sin\theta\cos\theta\right] - B_0\sin\theta(1-\cos\theta) - B_1\sin\theta\cos\theta = 0. \tag{3.8}$$

We can non-dimensionlise equation (3.8) by adopting the rescaled variables

$$\mu = \sqrt{\frac{K_1^n}{B_0}}, \qquad B = \frac{B_1}{B_0}, \qquad \bar{z} = \frac{z}{\mu}, \qquad \bar{d} = \frac{d}{\mu}, \tag{3.9}$$

where μ is a typical length scale and B is a dimensionless parameter that plays a key role in the stability of the sample. Hence we arrive at the non-dimensionalised second order differential equation

$$\theta''\sin^2\theta + (\theta')^2\sin\theta\cos\theta - \sin\theta(1-\cos\theta) - B\sin\theta\cos\theta = 0. \tag{3.10}$$

3.1.1 Known Equilibrium Solutions

It can be shown that equation (3.10) has the solution given implicitly by [50]

$$\bar{z} = \begin{cases} \frac{1}{\sqrt{B-1}}\{\sin^{-1}(g(\theta,\theta_m)) - \sin^{-1}(g(\theta_0,\theta_m))\} & \text{if } B > 1, \\ \sqrt{2}\{(\cos\theta_m - \cos\theta_0)^{1/2} - (\cos\theta_m - \cos\theta)^{1/2}\} & \text{if } B = 1, \\ \frac{1}{\sqrt{1-B}}\{\ln|h(\theta_0,\theta_m)| - \ln|h(\theta,\theta_m)|\} & \text{if } 0 < B < 1, \end{cases} \tag{3.11}$$

where

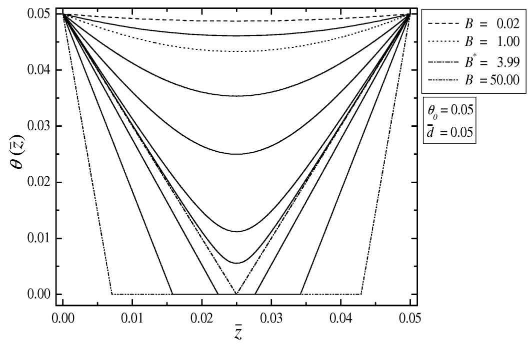


Figure 3.2: The solutions for $\theta(\bar{z})$ given by the implicit equations (3.11) for increasing values of B. For values of $B < B^*$ the solutions are smooth and continuous but as B is increased beyond the value B^*, the solutions develop corner points and the smoothness is lost.

$$g(\theta, \theta_m) = \frac{1 + (B-1)\cos\theta}{1 + (B-1)\cos\theta_m}, \tag{3.12}$$

$$h(\theta, \theta_m) = \left\{ \left(\frac{1}{1-B} - \cos\theta \right)^2 - \left(\frac{1}{1-B} - \cos\theta_m \right)^2 \right\}^{1/2} + \frac{1}{1-B} - \cos\theta, \tag{3.13}$$

and θ_m is the minimum angle that $\theta(z)$ attains in the sample. These solutions are plotted in Figure 3.2 [50], where we note that there are two types of solution present. For the values $B < B^*$ we have extrema with no 'corners' but as soon as we reach a critical value $B^* \approx 3.99$, corner points are induced into the solution. This change in behaviour at the point $B = B^*$ is of great interest and we will see later that it proves to be a defining feature of the solutions. The value of B^* given here was found numerically and is dependent on other parameters in the problem, such as the tilt angle. A full derivation of, and discussion about, these solutions can be found in Stewart [50], but here we now turn our attention to their stability.

3.1.2 Stability of Solutions

We first perturb the known solution by setting

$$\theta(\bar{z}, t) = \bar{\theta}(\bar{z}) + \epsilon(\bar{z})e^{-\lambda t} \tag{3.14}$$

where $\bar{\theta}(\bar{z})$ is a known equilibrium solution given by (3.11), λ is an eigenvalue of the system, and $\epsilon(\bar{z})$ is a small perturbation such that $|\epsilon(\bar{z})| \ll 1$. We shall neglect any flow effects at this small order approximation: we therefore need only concern ourselves with the angular momentum equations and ignore the linear momentum (due to the presumed absence of any flow). Hence the director is now given by

$$\mathbf{n} = (\sin(\bar{\theta} + \epsilon e^{-\lambda t}), 0, \cos(\bar{\theta} + \epsilon e^{-\lambda t})). \tag{3.15}$$

We use a time dependent perturbation here even though we are considering a static problem. Although the known solution $\bar{\theta}$ is dependent on $\bar{z}$ only, the perturbation is dependent on both $\bar{z}$ and t. We have chosen the perturbation to be of the form $\epsilon e^{-\lambda t}$ because we want to set up an eigenvalue problem and this form gives us a natural way into the problem via a linear analysis, but more generally the perturbation is of the form $\epsilon(\bar{z}, t)$. To study the stability we must consider the appropriate dynamic equations for the system which are given by (2.42), namely,

$$\left(\frac{\partial w}{\partial n_{i,j}}\right)_{,j} - \frac{\partial w}{\partial n_i} + \tilde{g}_i + G_i = \mu n_i. \tag{3.16}$$

The quantity $\tilde{g}_i$ is given in general by (2.39), but in the absence of flow this simplifies to

$$\tilde{g}_i = -(\alpha_3 - \alpha_2)\frac{\partial n_i}{\partial t}, \tag{3.17}$$

where α_2 and α_3 are viscosity coefficients, and satisfy the relation $\gamma_1 = \alpha_3 - \alpha_2 > 0$, due to constraints of the dissipation function. This is a 'rotational viscosity' effect and so is positive. We have set $G_i \equiv 0$ since we are assuming that there are no outside forces acting on the sample. Equations (3.16) can be written in the more compact form [49]

$$\mathbf{\Pi}^n + \tilde{\mathbf{g}} = \mu\mathbf{n}, \tag{3.18}$$

where $\Pi_i^n = K_1^n n_{j,ji} - B_0(|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2)a_i + B_1(\mathbf{n}\cdot\mathbf{a})a_i$. Since the second components of both the director and unit normal are zero we have $\Pi_2^n = 0$ and $\tilde{g}_2 = 0$, hence we can write (3.18) explicitly as

$$\Pi_1^n + \tilde{g}_1 = \mu n_1, \tag{3.19}$$

$$\Pi_3^n + \tilde{g}_3 = \mu n_3. \tag{3.20}$$

We can eliminate the Lagrange multiplier from the problem by multiplying equation (3.19) by n_3, multiplying equation (3.20) by n_1 and subtracting the two resulting equations. Hence we obtain

$$\begin{aligned}
\Pi_1^n n_3 - \Pi_3^n n_1 &= \tilde{g}_3 n_1 - \tilde{g}_1 n_3 \\
&= -(\alpha_3 - \alpha_2)\Big[\frac{\partial n_3}{\partial t}n_1 - \frac{\partial n_1}{\partial t}n_3\Big] \\
&= -(\alpha_3 - \alpha_2)\big[\sin^2(\bar{\theta} + \epsilon e^{-\lambda t}) + \cos^2(\bar{\theta} + \epsilon e^{-\lambda t})\big]\lambda\epsilon e^{-\lambda t} \\
&= -\lambda(\alpha_3 - \alpha_2)\epsilon e^{-\lambda t}.
\end{aligned} \tag{3.21}$$

Now, we find Π_1^n and Π_3^n explicitly to be

$$\Pi_1^n = 0, \tag{3.22}$$

$$\begin{aligned}\Pi_3^n =& - K_1^n\big[\cos(\bar\theta+\epsilon e^{-\lambda t})\big(\bar\theta\,'+\epsilon' e^{-\lambda t}\big)^2 - \sin(\bar\theta+\epsilon e^{-\lambda t})\big(\bar\theta\,''+\epsilon'' e^{-\lambda t}\big)\big]\\ &- B_0[\cos(\bar\theta+\epsilon e^{-\lambda t})-1] + B_1\cos(\bar\theta+\epsilon e^{-\lambda t}).\end{aligned} \tag{3.23}$$

Since we are only perturbing the solution by a small amount we assume that $|\epsilon(\bar z)| \ll 1$, and so we can linearise each term in ϵ to obtain

$$\sin(\bar\theta+\epsilon e^{-\lambda t}) \approx \sin\bar\theta + \cos\bar\theta e^{-\lambda t}\epsilon, \tag{3.24}$$

$$\cos(\bar\theta+\epsilon e^{-\lambda t}) \approx \cos\bar\theta - \sin\bar\theta e^{-\lambda t}\epsilon, \tag{3.25}$$

$$\sin(\bar\theta+\epsilon e^{-\lambda t})\cos(\bar\theta+\epsilon e^{-\lambda t}) \approx \sin\bar\theta\cos\bar\theta + e^{-\lambda t}(\cos^2\bar\theta - \sin^2\bar\theta)\epsilon, \tag{3.26}$$

$$\sin^2(\bar\theta+\epsilon e^{-\lambda t}) \approx \sin^2\bar\theta + 2\sin\bar\theta\cos\bar\theta e^{-\lambda t}\epsilon, \tag{3.27}$$

and hence we can write

$$\begin{aligned}\Pi_1^n n_3 - \Pi_3^n n_1 = K_1^n\Big[&(\bar\theta\,')^2\sin\bar\theta\cos\bar\theta + 2\bar\theta\,'\sin\bar\theta\cos\bar\theta\epsilon' e^{-\lambda t} + \bar\theta\,^2(\cos^2\bar\theta - \sin^2\bar\theta)\epsilon e^{-\lambda t}\\ &+ \bar\theta\,''\sin^2\bar\theta + \sin^2\bar\theta\epsilon'' e^{-\lambda t} + 2\bar\theta\,''\sin\bar\theta\cos\bar\theta\epsilon e^{-\lambda t}\Big]\\ +B_0\Big[&\sin\bar\theta\cos\bar\theta + (\cos^2\bar\theta - \sin^2\bar\theta)\epsilon e^{-\lambda t} - \sin\bar\theta - \cos\bar\theta\epsilon e^{-\lambda t}\Big]\\ -B_1\Big[&\sin\bar\theta\cos\bar\theta + (\cos^2\bar\theta - \sin^2\bar\theta)\epsilon e^{-\lambda t}\Big].\end{aligned} \tag{3.28}$$

Equating (3.28) with (3.21) allows the dynamic equation (3.21) to be written as

$$
\begin{aligned}
-\lambda(\alpha_3-\alpha_2)\epsilon e^{-\lambda t} = K_1^n\Big[&2\bar{\theta}\,'\sin\bar{\theta}\cos\bar{\theta}\epsilon' e^{-\lambda t}+\bar{\theta}\,^2(\cos^2\bar{\theta}-\sin^2\bar{\theta})\epsilon e^{-\lambda t} \\
&+\sin^2\bar{\theta}\epsilon'' e^{-\lambda t}+2\bar{\theta}\,''\sin\bar{\theta}\cos\bar{\theta}\epsilon e^{-\lambda t}\Big] \\
&+B_0\Big[(\cos^2\bar{\theta}-\sin^2\bar{\theta})\epsilon e^{-\lambda t}-\cos\bar{\theta}\epsilon e^{-\lambda t}\Big] \\
&-B_1\Big[(\cos^2\bar{\theta}-\sin^2\bar{\theta})\epsilon e^{-\lambda t}\Big] \\
&+\Big\{K_1^n(\bar{\theta}\,')^2\sin\bar{\theta}\cos\bar{\theta}+K_1^n\bar{\theta}\,''\sin^2\bar{\theta}+B_0\sin\bar{\theta}\cos\bar{\theta} \\
&-B_0\sin\bar{\theta}-B_1\sin\bar{\theta}\cos\bar{\theta}\Big\}. \qquad (3.29)
\end{aligned}
$$

We can simplify equation (3.29) by setting the terms in the curly brackets to be zero since these terms satisfy the equilibrium equation (3.8). Now, dividing (3.29) throughout by $B_0 e^{-\lambda t}$, non-dimensionalising as before, and collecting terms in powers of ϵ, yields the dynamic equation for the system, namely

$$
\epsilon''\sin^2\bar{\theta}+2\bar{\theta}'\sin\bar{\theta}\cos\bar{\theta}\epsilon'-\epsilon q_1(\bar{z}) = -\frac{\lambda}{B_0}(\alpha_3-\alpha_2)\epsilon, \qquad (3.30)
$$

where $q_1(\bar{z}) = \cos\bar{\theta}-(1-B+(\bar{\theta}')^2)(\cos^2\bar{\theta}-\sin^2\bar{\theta})-2\bar{\theta}''\cos\bar{\theta}\sin\bar{\theta}$, and λ is an eigenvalue of the system. Multiplying equation (3.30) throughout by ϵ then integrating with respect to $\bar{z}$ from 0 to $\bar{d}$ yields

$$
\int_0^{\bar{d}}\Big[\epsilon''\epsilon\sin^2\bar{\theta}+2\bar{\theta}\,'\sin\bar{\theta}\cos\bar{\theta}\epsilon'\epsilon-\epsilon^2 q_1(\bar{z})\Big]d\bar{z} = -\int_0^{\bar{d}}\frac{\lambda}{B_0}(\alpha_3-\alpha_2)\epsilon^2 d\bar{z}. \qquad (3.31)
$$

Taking the first term in (3.31) and integrating by parts gives

$$
\begin{aligned}
\int_0^{\bar{d}}\epsilon''\epsilon\sin^2\bar{\theta}\,d\bar{z} &= [\epsilon'\epsilon\sin^2\bar{\theta}]_0^{\bar{d}}-\int_0^{\bar{d}}\epsilon'(\epsilon\sin^2\bar{\theta}\,)'\,d\bar{z} \\
&= -\int_0^{\bar{d}}\epsilon'(\epsilon\sin^2\bar{\theta})'\,d\bar{z} \\
&= -\int_0^{\bar{d}}\Big[(\epsilon')^2\sin^2\bar{\theta}+2\epsilon'\epsilon\,\bar{\theta}\,'\sin\bar{\theta}\cos\bar{\theta}\Big]d\bar{z}, \qquad (3.32)
\end{aligned}
$$

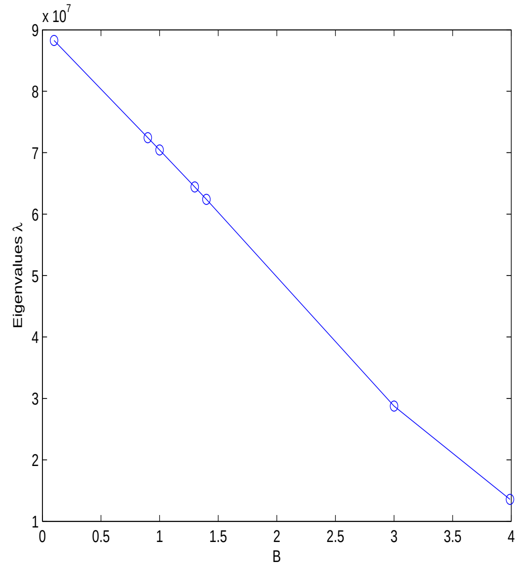


Figure 3.3: The eigenvalues plotted against the dimensionless parameter B for a planar aligned SmA liquid crystal.

since we set $\epsilon(\bar{z})$ to be zero on the boundaries. Hence (3.31) can be written as

$$\int_0^{\bar{d}} \left\{(\epsilon')^2 \sin^2 \bar{\theta} + \epsilon^2 q_1(\bar{z})\right\} d\bar{z} = \int_0^{\bar{d}} \frac{\lambda}{B_0}(\alpha_3 - \alpha_2)\epsilon^2 d\bar{z}. \tag{3.33}$$

Equation (3.33) is then in a suitable form to apply the Ritz method to it. Implementing the Ritz method in MAPLE12 [30] allows us to find the eigenvalues of the system for different values of B, with the results being shown in Figure 3.3, where we have used the parameters estimates $\gamma_1 = \alpha_3 - \alpha_2 = 0.0776$ Pa s and $B_0 = 8.95 \times 10^7$ N m^{-2}, where Pa denotes pascals. We have taken $N = 8$ in the summation of terms for the unknown function $\epsilon(\bar{z})$, see Appendix C for details, which may seem like very few terms, however, having experimented with various values of N we concluded that increasing N beyond the value of 8 did not greatly improve the accuracy of the results. For example, increasing N to 20 did not improve the accuracy at all to four significant figures, and increasing N to 50 increases the accuracy only in the sixth significant figure.

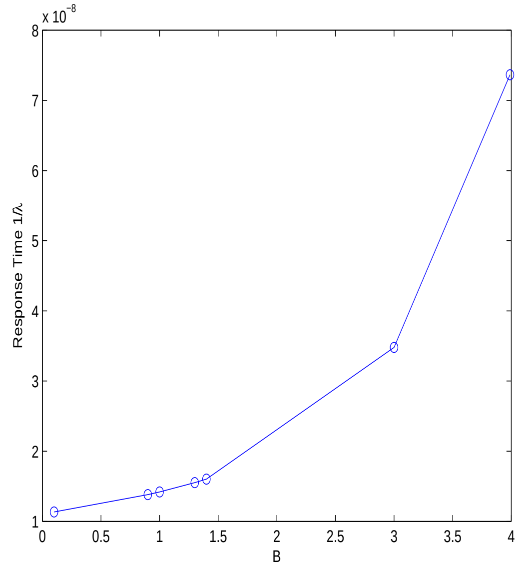


Figure 3.4: The response time plotted against the dimensionless parameter B for a planar aligned SmA liquid crystal.

3.1.3 Conclusions

In the case of planar homeotropically aligned SmA, we were able to obtain results that proved stability for $B < B^* \approx 3.99$, but were unable to generate results for $B \geq B^*$. The Ritz method is a well used and researched method that is known to generate the eigenvalues of a system, so the fact that it does not work for values of B greater than the critical quantity B^* suggests that there is a problem in the system, most probably an instability. This ties in with the fact that the extrema in Figure 3.2 undergo a change in behaviour at this point. The fact that the extrema become broken at $B = B^*$, coupled with the break down of the Ritz method, strongly suggests an instability. However, this is simply speculation and is not enough to prove instability, so we would need to employ other methods to prove or disprove this theory. One possibility would be to use bifurcation theory. It is possible that there is a bifurcation point at $B = B^*$, in a similar manner to the case of the Freedericksz transition in nematics where there is a bifurcation point at a critical magnetic field strength H_C. At this critical point two things happen: firstly, additional solutions for the director profile come into existence and, secondly, the original zero solution changes from a stable branch

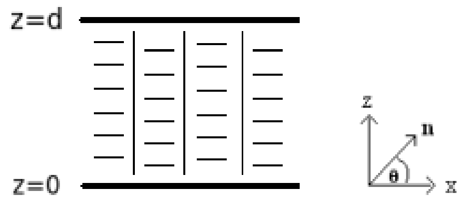


Figure 3.5: Finite sample of bookshelf aligned SmA liquid crystal bounded between two parallel plates a distance d apart, where θ denotes the angle that the director makes with the x-axis.

to an unstable one. We may have a similar situation here but more investigation would be needed before any conclusions are drawn. Another possible explanation for the breakdown of the Ritz method is the fact that this method uses functions for ϵ that are smooth and continuous. We can see from Figure 3.2 that after the point B^* the solutions are no longer smooth. They are still continuous at each point in $z \in [0, \bar{d}]$ but the non-smoothness may be providing a problem that the Ritz method cannot overcome.

We note that the value for B^* was obtained numerically as it is impossible, using the tools at our disposal, to calculate this critical value analytically. The value found for B^* is specific to the parameter values used in this case, such as sample depth and tilt angle. It is possible that this critical value will change depending on which material parameters are chosen, but we surmise that the numerical changes will be small and the critcal value will always occur in the analysis.

3.2 Bookshelf Aligned Smectic A

We now consider the case of a bookshelf aligned liquid crystal confined between two parallel plates a distance d apart. In this geometry we have

$$\mathbf{n} = (\cos\theta(z), 0, \sin\theta(z)), \tag{3.34}$$

$$\mathbf{a} = (1, 0, 0), \tag{3.35}$$

$$\Phi = x, \tag{3.36}$$

with the boundary conditions again given by

$$\theta(0) = \theta(d) = \theta_0 > 0. \tag{3.37}$$

The energy density (2.22) and dynamic equation (3.16) are exactly the same as in the planar aligned case. We can follow the same procedure as the previous section by inserting w into the Euler-Lagrange equation (2.30) to obtain the second order governing equation

$$\theta'' \cos^2\theta - (\theta')^2 \sin\theta\cos\theta - \sin\theta(1 - \cos\theta) - B\sin\theta\cos\theta = 0, \tag{3.38}$$

where we have again non-dimensionalised the problem using the same scalings as (3.9).

3.2.1 Known Equilibrium Solutions

The solution to equation (3.38) is slightly more complicated than (3.11), so we merely state the solution for $B = 1$ here, and direct the reader to Stewart [50] for full details of the cases $B < 1$ and $B > 1$. For $B = 1$ we have the solution to equation (3.38) given by

$$\bar{z} = \frac{\bar{d}}{2} - F(\gamma(\theta), \sqrt{n}) - (\cos\theta_m - 1)\Pi(\gamma(\theta), n, \sqrt{n}), \tag{3.39}$$

where

$$\gamma(\theta) = \arcsin\sqrt{\frac{2(\cos\theta_m - \cos\theta)}{(1+\cos\theta_m)(1-\cos\theta)}}, \tag{3.40}$$

$$n = \frac{1}{2}(1+\cos\theta_m), \tag{3.41}$$

$$k = \frac{1}{2}\sqrt{\frac{(1+\cos\theta_m)(B+1+(B-1)\cos\theta_m)}{(1+(B-1)\cos\theta_m)}}. \tag{3.42}$$

The functions F and Π are the incomplete elliptic integrals of the first and third kind, respectively, where k represents the elliptic modulus, and θ_m represents the minimum angle that θ attains. From Stewart [50], we have the additional requirement that $\theta(z)$ achieves its minimum value, θ_m, at the point $z = d/2$, for $\theta_m \leq \theta(z) \leq \theta_0$. Hence, the value of θ_m can be found by substituting $z = d/2$ into equation (3.39) to obtain

$$\frac{\bar{d}}{2} = F(\gamma(\theta_0), \sqrt{n}) - (\cos\theta_m - 1)\Pi(\gamma(\theta_0), n, \sqrt{n}). \tag{3.43}$$

This then provides an impicit equation to solve for θ_m, given the parameters θ_0, $\bar{d}$ and B. The full solutions of equation (3.38) for various values of B are plotted in Figure 3.6 [50]. In this case we notice immediately that the solutions for all values of B are smooth and continuous, in stark contrast to the planar aligned geometry. Now we investigate the stability of these solutions.

3.2.2 Stability of Solutions

We begin again by perturbing the solution by setting

$$\theta(\bar{z}, t) = \bar{\theta}(\bar{z}) + \epsilon(\bar{z})e^{-\lambda t}, \tag{3.44}$$

and ignore any flow effects. Then we write equations (3.16) in the compact form [49]

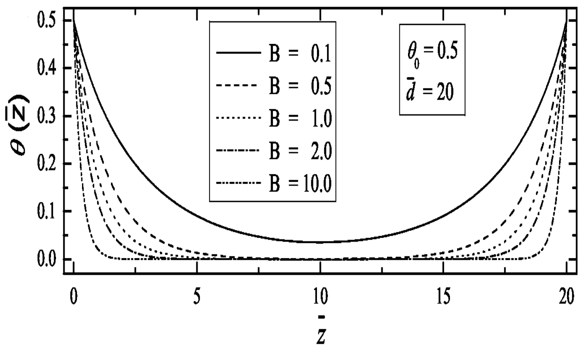


Figure 3.6: The solutions for $\theta(\bar{z})$, for example given by the implicit equation (3.39) in the $B = 1$ case, for increasing values of B. In this geometry, the solutions are smooth and continuous for all values of B.

$$\mathbf{\Pi}^n + \tilde{\mathbf{g}} = \mu \mathbf{n}, \tag{3.45}$$

where $\Pi_i^n = K_1^n n_{j,ji} - B_0(|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2)a_i + B_1(\mathbf{n}\cdot\mathbf{a})a_i$. As in Section 3.1, the second components of both the director and unit normal are zero and so we have $\Pi_2^n = 0$ and $\tilde{g}_2 = 0$, hence (3.45) becomes

$$\Pi_1^n + \tilde{g}_1 = \mu n_1, \tag{3.46}$$
$$\Pi_3^n + \tilde{g}_3 = \mu n_3. \tag{3.47}$$

Multiplying equation (3.46) by n_3, then multiplying equation (3.47) by n_1 and subtracting the two resulting equations yields

$$\begin{aligned}\Pi_1^n n_3 - \Pi_3^n n_1 &= \tilde{g}_3 n_1 - \tilde{g}_1 n_3 \\ &= \lambda(\alpha_3 - \alpha_2)\epsilon e^{-\lambda t}.\end{aligned} \tag{3.48}$$

Now we can compute Π_1^n and Π_3^n explicitly to be

$$
\begin{aligned}
\Pi_1^n =& -B_0(\cos(\bar{\theta}+\epsilon e^{-\lambda t})-1)+B_1\cos(\bar{\theta}+\epsilon e^{-\lambda t}), && (3.49)\\
\Pi_3^n =& -K_1^n\big[\cos(\bar{\theta}+\epsilon e^{-\lambda t})(\bar{\theta}''+\epsilon'' e^{-\lambda t})-\sin(\bar{\theta}+\epsilon e^{-\lambda t})(\bar{\theta}''+\epsilon' e^{-\lambda t})^2\big], && (3.50)
\end{aligned}
$$

and hence we can write

$$
\begin{aligned}
\Pi_1^n n_3-\Pi_3^n n_1 = {}& \sin(\bar{\theta}+\epsilon e^{-\lambda t})[B_1\cos(\bar{\theta}+\epsilon e^{-\lambda t})-B_0(\cos(\bar{\theta}+\epsilon e^{-\lambda t})-1)]\\
&+K_1^n\big[+\sin(\bar{\theta}+\epsilon e^{-\lambda t})\cos(\bar{\theta}+\epsilon e^{-\lambda t})(\bar{\theta}''+\epsilon' e^{-\lambda t})^2\\
&-\cos^2(\bar{\theta}+\epsilon e^{-\lambda t})(\bar{\theta}''+\epsilon'' e^{-\lambda t})\big]. \qquad (3.51)
\end{aligned}
$$

Linearising (3.51) in ϵ yields

$$
\begin{aligned}
\Pi_1^n n_3-\Pi_3^n n_1 = {}& K_1^n\Big[2\bar{\theta}''\epsilon\cos\bar{\theta}\sin\bar{\theta}e^{-\lambda t}+2\bar{\theta}'\epsilon' e^{-\lambda t}\sin\bar{\theta}\cos\bar{\theta}-\epsilon''\cos^2\bar{\theta}e^{-\lambda t}\Big]\\
&+B_0\cos\bar{\theta}\epsilon e^{-\lambda t}+\big(K_1^n(\bar{\theta}')^2+B_1-B_0\big)(\cos^2\bar{\theta}-\sin^2\bar{\theta})\epsilon e^{-\lambda t}\\
&+\Big\{K_1^n(\bar{\theta}')^2\sin\bar{\theta}\cos\bar{\theta}-K_1^n\bar{\theta}''\cos^2\bar{\theta}+B_0\sin\bar{\theta}+(B_1-B_0)\sin\bar{\theta}\cos\bar{\theta}\Big\}, \qquad (3.52)
\end{aligned}
$$

where the term in the curly brackets is zero since it satisfies the equilibrium equation (3.38). Then equating (3.52) with (3.48) allows the dynamic equation (3.48) to be written as

$$
\begin{aligned}
\lambda(\alpha_3-\alpha_2)\epsilon e^{-\lambda t} = {}& K_1^n\Big[2\bar{\theta}''\epsilon\cos\bar{\theta}\sin\bar{\theta}e^{-\lambda t}+2\bar{\theta}'\epsilon' e^{-\lambda t}\sin\bar{\theta}\cos\bar{\theta}-\epsilon''\cos^2\bar{\theta}e^{-\lambda t}\Big]\\
&+B_0\cos\bar{\theta}\epsilon e^{-\lambda t}+\big(K_1^n(\bar{\theta}')^2+B_1-B_0\big)(\cos^2\bar{\theta}-\sin^2\bar{\theta})\epsilon e^{-\lambda t}. \qquad (3.53)
\end{aligned}
$$

Dividing equation (3.53) throughout by $B_0 e^{-\lambda t}$, and non-dimensionalising, produces the dynamic equation

$$-\epsilon'' \cos^2 \bar{\theta} + 2\bar{\theta}' \sin \bar{\theta} \cos \bar{\theta} \epsilon' + \epsilon q_2(\bar{z}) = \frac{\lambda}{B_0}(\alpha_3 - \alpha_2)\epsilon, \tag{3.54}$$

where $q_2(\bar{z}) = (B - 1 + (\bar{\theta}')^2)(\cos^2 \bar{\theta} - \sin^2 \bar{\theta}) + 2\bar{\theta}'' \cos \bar{\theta} \sin \bar{\theta} + \cos \bar{\theta}$. Multiplying equation (3.54) throughout by ϵ and integrating with respect to $\bar{z}$ from 0 to $\bar{d}$ yields

$$\int_0^{\bar{d}} \Big[- \epsilon'' \epsilon \cos^2 \bar{\theta} + 2\bar{\theta}' \sin \bar{\theta} \cos \bar{\theta} \epsilon' \epsilon + \epsilon^2 q_2(\bar{z}) \Big] d\bar{z} = \int_0^{\bar{d}} \frac{\lambda}{B_0}(\alpha_3 - \alpha_2)\epsilon^2 d\bar{z}. \tag{3.55}$$

Performing integration by parts on the first term on the right hand side of (3.55), then simplifying, results in the equation

$$\int_0^{\bar{d}} \Big[(\epsilon')^2 \cos^2 \bar{\theta} + \epsilon^2 q_2(\bar{z}) \Big] d\bar{z} = \int_0^{\bar{d}} \frac{\lambda}{B_0}(\alpha_3 - \alpha_2)\epsilon^2 d\bar{z}. \tag{3.56}$$

The Ritz method was again used on (3.56) to compute the eigenvalues of the system for varying values of B, with the results being shown in Figure 3.7. We can see in this case that the eigenvalues are positive for all values of B up to and including 10. In physical terms, this is bordering on a realistic value for B, but we advanced the computations to include this case for completeness. We note here that if we take the linear approximation of $\bar{\theta}$ in (3.56), we obtain

$$\int_0^{\bar{d}} ((\epsilon')^2 + B\epsilon^2) d\bar{z} = \int_0^{\bar{d}} \frac{\lambda}{B_0}(\alpha_3 - \alpha_2)\epsilon^2 d\bar{z}, \tag{3.57}$$

which is stable for all $B > 0$ since $\lambda > 0$, due to the positive nature of the other terms in (3.57).

3.2.3 Conclusions

We showed that bookshelf aligned SmA is stable for all values of B from zero up to and including $B = 10$. For all physical relevance this is more than enough to prove complete stability. Ribotta and Durand [39] showed that $B < 1$ is

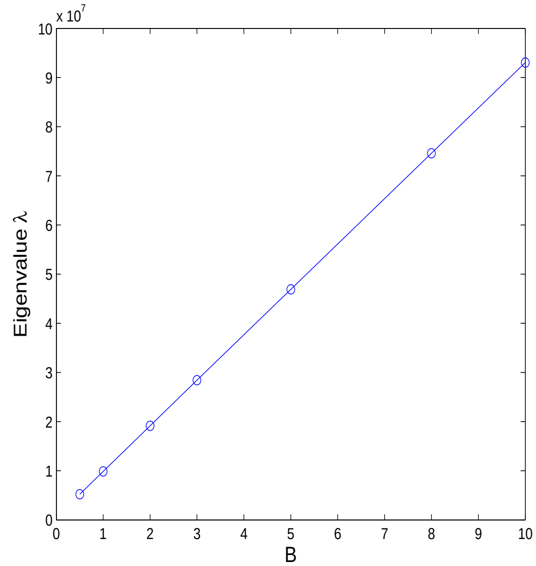


Figure 3.7: The eigenvalues plotted against the dimensionless parameter B for a bookshelf aligned SmA liquid crystal.

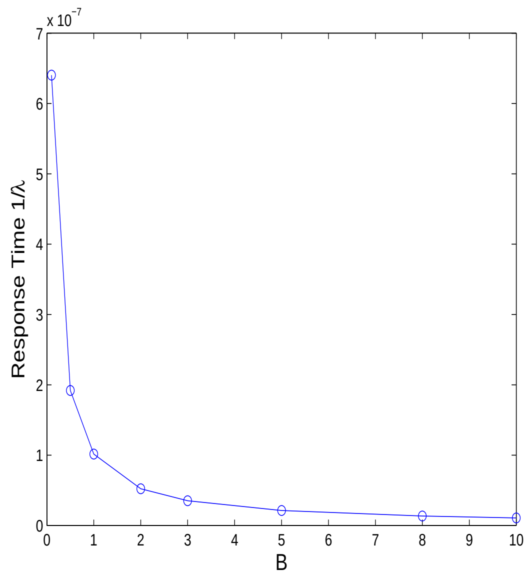


Figure 3.8: The response time plotted against the dimensionless parameter B for a bookshelf aligned SmA liquid crystal.

a particularly significant case since, in general, the magnitude of B_1 should be smaller than or comparable to that of B_0. However, values of B up to about 10 are physically relevant.

3.3 Bookshelf Aligned Smectic A with Variable Smectic Layers

Now we consider the nonlinear problem of a bookshelf alignment where the director $\mathbf{n}$ varies as before, but with the smectic layer normal $\mathbf{a}$ also being allowed to vary. This means that $\mathbf{n}$ and $\mathbf{a}$ can decouple at any point in the sample where it is energetically favourable for them to do so. We set the orientation angles of $\mathbf{n}$ and $\mathbf{a}$ to be θ and δ, respectively, and we assume that the alignment of the director and smectic layer normal are uniform in the x and y directions so that θ and δ are functions of z only. Strong anchoring of the director is supposed so we set θ to be the fixed angle θ_b at the lower boundary $z = 0$, and $-\theta_b$ at the upper boundary $z = d$. We also assume that the smectic layer normal is fixed at an angle δ_b at $z = 0$ and $-\delta_b$ at $z = d$ by symmetry. The director $\mathbf{n}$ and scalar function Φ take the forms [59]

$$\mathbf{n} = (\cos\theta(z), 0, \sin\theta(z)), \tag{3.58}$$

$$\Phi(x, z) = x + \int_{z_0}^{z} \tan\delta(t)dt, \tag{3.59}$$

where z_0 is an arbitrary constant, and the unit layer normal $\mathbf{a}$ is given by

$$\mathbf{a} \equiv \frac{\nabla\Phi}{|\nabla\Phi|}. \tag{3.60}$$

We note that this identification for Φ will only be valid if δ depends on only one spatial variable. Inserting the form of Φ into (3.60) allows $\mathbf{a}$ to be written

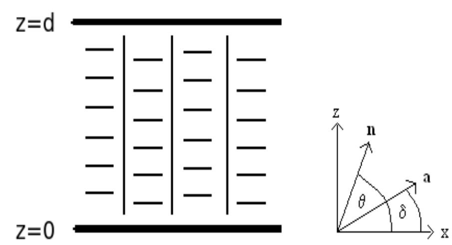


Figure 3.9: Bookshelf aligned SmA liquid crystal confined between two parallel plates a distance d apart, where the director and unit normal are allowed to separate.

explicitly as

$$\mathbf{a} = (\cos\delta(z), 0, \sin\delta(z)). \tag{3.61}$$

Also from (3.59) we can write

$$\nabla\Phi = (1, 0, \tan\delta(z)), \tag{3.62}$$

and hence $|\nabla\Phi| = \sec\delta(z)$. Finally we have the inner product of the director and unit normal given by $\mathbf{n}\cdot\mathbf{a} = \cos(\theta(z) - \delta(z))$. Inserting $\mathbf{n}$, $\mathbf{a}$ and Φ into the energy density (2.22) yields

$$w = \frac{1}{2}K_1^n(\theta')^2\cos^2\theta + \frac{1}{2}K_1^a(\delta')^2\cos^2\delta + \frac{1}{2}B_0(\sec\delta + \cos(\theta-\delta) - 2)^2 + \frac{1}{2}B_1\sin^2(\theta-\delta). \tag{3.63}$$

For the functions $\theta(z)$ and $\delta(z)$ to minimise the energy of the system, they need to satisfy the coupled Euler-Lagrange equations (2.31) and (2.32). Using the energy density (3.63) we can write

$$
\begin{aligned}
\frac{\partial w}{\partial \theta'} &= K_1^n \theta' \cos^2\theta, && (3.64)\\
\frac{\partial w}{\partial \theta} &= -\, K_1^n (\theta')^2 \cos\theta \sin\theta - B_0(\sec\delta + \cos(\theta-\delta) - 2)\sin(\theta-\delta)\\
&\quad + B_1 \sin(\theta-\delta)\cos(\theta-\delta), && (3.65)\\
\frac{\partial w}{\partial \delta'} &= K_1^a \delta' \cos^2\delta, && (3.66)\\
\frac{\partial w}{\partial \delta} &= -\, K_1^a (\delta')^2 \cos\delta \sin\delta + B_0(\sec\delta + \cos(\theta-\delta) - 2)(\sec\delta\tan\delta + \sin(\theta-\delta))\\
&\quad - B_1 \sin(\theta-\delta)\cos(\theta-\delta). && (3.67)
\end{aligned}
$$

Hence inserting w into the coupled Euler-Lagrange equations (2.31) and (2.32), then non-dimensionalising using the scalings adopted in (3.9), yields the equilibrium equations

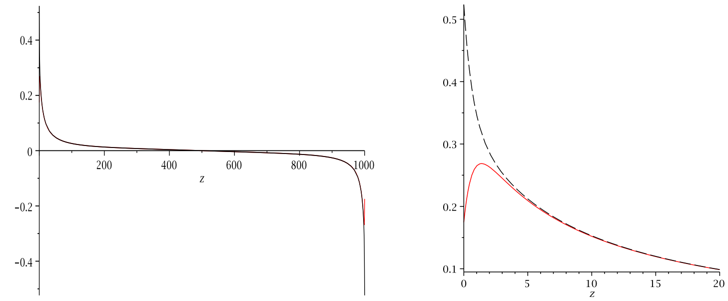
$$
\begin{aligned}
&\theta'' \cos^2\theta - (\theta')^2 \sin\theta\cos\theta + [\sec\delta + \cos(\theta-\delta) - 2]\sin(\theta-\delta)\\
&\qquad -B\cos(\theta-\delta)\sin(\theta-\delta) = 0, && (3.68)\\
&\kappa[\delta''\cos^2\delta - (\delta')^2\sin\delta\cos\delta] - [\sec\delta + \cos(\theta-\delta) - 2][\sec\delta\tan\delta + \sin(\theta-\delta)]\\
&\qquad +B\cos(\theta-\delta)\sin(\theta-\delta) = 0, && (3.69)
\end{aligned}
$$

where $\kappa = K_1^a/K_1^n$ is a measure of the anisotropy in the elastic constants. The functions θ and δ in (3.68) and (3.69) are now functions of the dimensionless variable $\bar{z}$.

3.3.1 Known Equilibrium Solutions

A numerical solution has been found for this problem [50] which satisfies the relations

$$
\begin{aligned}
\theta(\bar{z}) &= -\theta(\bar{d} - \bar{z}), && (3.70)\\
\delta(\bar{z}) &= -\delta(\bar{d} - \bar{z}), && (3.71)
\end{aligned}
$$



(a) (b)

Figure 3.10: Full non-linear solution (a) between $\bar{z} = 0$ and $\bar{z} = \bar{d} = 1000$, and (b) between $\bar{z} = 0$ and $\bar{z} = 20$, where the black dashed line represents the angle $\theta(\bar{z})$ and the red line represents $\delta(\bar{z})$. We have set the boundary values to be $\theta_0 = \pi/6$ and $\delta_0 = \pi/18$.

for $0 \leq \bar{z} \leq \bar{d}$. Due to the symmetry of the solutions it was only necessary to look for solutions for the values $0 \leq \bar{z} \leq \bar{d}/2$ and the solutions for $\bar{d}/2 \leq \bar{z} \leq \bar{d}$ can be obtained by the symmetry requirements stated above. Equations (3.68) and (3.69) can be solved numerically using a standard differential equation solver in MAPLE12 [30], with the results shown in Figure 3.10. Although a numerical solution is known, we are interested in trying to find an analytical solution to provide more detailed insight into the behaviour of the director. Now we try to find an approximation to the solutions by the use of a linear approximation and inner and outer solutions.

3.3.2 Linearised Equations

We can obtain some insight into the behaviour of the solutions for small angles by linearising the governing equations, hence providing a simpler system to solve analytically. Linearising equations (3.68) and (3.69) in θ and δ yields

$$\theta'' = B(\theta - \delta), \tag{3.72}$$

$$\delta'' = -B(\theta - \delta). \tag{3.73}$$

We can write these equations in the form $A\mathbf{x} = \mathbf{x}'$ by setting $x_1 = \theta$, $x_2 = \theta'$, $x_3 = \delta$ and $x_4 = \delta'$, and hence we have the matrix system

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ B & 0 & -B & 0 \\ 0 & 0 & 0 & 1 \\ -B & 0 & B & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1' \\ x_2' \\ x_3' \\ x_4' \end{bmatrix}. \tag{3.74}$$

If we label the matrix in this system as A, then the characteristic equation of this matrix is given by $\det(A - \lambda I) = 0$, i.e. $\lambda^2(\lambda^2 - 2B) = 0$, and so the eigenvalues are given by $\lambda = 0$ (twice) and $\lambda = \pm\sqrt{2B}$. We can find the eigenvectors associated with each eigenvalue by solving the equation [23]

$$(A - \lambda I)\mathbf{x}_0 = \mathbf{0}, \tag{3.75}$$

for each λ, where $\mathbf{x}_0$ is the unknown eigenvector $[a, b, c, d]^T$. In the case of the repeated eigenvalue $\lambda_{1,2} = 0$, we find the eigenvector by solving equation (3.75) for $\mathbf{x}_0$, then the second linearly independent solution, $\mathbf{x}_1$, associated with the repeated eigenvalue is found by solving

$$(A - \lambda I)\mathbf{x}_1 = \mathbf{x}_0. \tag{3.76}$$

Solving equations (3.75) and (3.76) for the four vectors yields

$$\mathbf{x}_0 = [1, 0, 1, 0]^T, \tag{3.77}$$

$$\mathbf{x}_1 = [1, 1, 1, 1]^T, \tag{3.78}$$

$$\mathbf{x}_2 = [1, \sqrt{2B}, -1, -\sqrt{2B}]^T, \tag{3.79}$$

$$\mathbf{x}_3 = [-1, \sqrt{2B}, 1, -\sqrt{2B}]^T. \tag{3.80}$$

Hence the matrix system (3.74) has solution [23]

$$\begin{aligned}\mathbf{x} &= c_0 e^{\lambda_0 \bar{z}}\mathbf{x}_0 + c_1 \bar{z} e^{\lambda_1 \bar{z}}\mathbf{x}_1 + c_2 e^{\lambda_2 \bar{z}}\mathbf{x}_2 + c_3 e^{\lambda_3 \bar{z}}\mathbf{x}_3 \\ &= c_0 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c_1 \bar{z} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} + c_2 e^{\sqrt{2B}\bar{z}} \begin{bmatrix} 1 \\ \sqrt{2B} \\ -1 \\ -\sqrt{2B} \end{bmatrix} + c_3 e^{-\sqrt{2B}\bar{z}} \begin{bmatrix} -1 \\ \sqrt{2B} \\ 1 \\ -\sqrt{2B} \end{bmatrix}, \end{aligned} \tag{3.81}$$

where $\mathbf{x} = [x_1, x_2, x_3, x_4]^T = [\theta, \theta', \delta, \delta']^T$. We have boundary conditions $\theta(0) = \theta_b$, $\theta(d) = -\theta_b$, $\delta(0) = \delta_b$ and $\delta(d) = -\delta_b$, and so the constants in solution (3.81) can be found to be

$$c_0 = \frac{1}{2}(\theta_b - \delta_b), \tag{3.82}$$

$$c_1 = -\frac{1}{d}(\theta_b - \delta_b), \tag{3.83}$$

$$c_2 = -\frac{(1 + e^{-\sqrt{2B}d})}{4\sinh(\sqrt{2B}d)}(\theta_b - \delta_b), \tag{3.84}$$

$$c_3 = -\frac{(1 + e^{\sqrt{2B}d})}{4\sinh(\sqrt{2B}d)}(\theta_b - \delta_b). \tag{3.85}$$

Hence we can write the angles $\theta(\bar{z})$ and $\delta(\bar{z})$ as

$$\theta(\bar{z}) = c_0 + c_1 \bar{z} + c_2 e^{\sqrt{2B}\bar{z}} - c_3 e^{-\sqrt{2B}\bar{z}}, \tag{3.86}$$

$$\delta(\bar{z}) = c_0 + c_1 \bar{z} - c_2 e^{\sqrt{2B}\bar{z}} + c_3 e^{-\sqrt{2B}\bar{z}}, \tag{3.87}$$

with the constants c_i as given by (3.82)-(3.85). The characteristic length, or boundary layer, is given by $l_c = 1/\sqrt{2B}$, and should be closely related to the characteristic length scale of the nonlinear problem. Boundary effects of the same magnitude should occur in the nonlinear problem since all functions involved in the problem are bounded.

3.3.3 Outer Solution

When the governing equations of a system are too complex to solve exactly using analytical techniques, approximation techniques are used to find a solution. There are many approximation techniques available but here we consider perturbation methods [29]. Perturbation methods allow an approximate solution of the problem to be found when some terms in the equations are small, with these small terms having a coefficient parameter, in our case Δ, that is small. Approximations can be made in the bulk of the sample, leading to an *outer solution*, but these approximations are often too crude to use close to the boundaries. In this boundary layer, the equations must often be rescaled to accomodate the small lengthscales, and in this case we find an *inner solution*. The inner and outer solutions are then 'matched' at the edge of the boundary layer. Now we aim to find an inner and outer approximation to the governing equations in this case.

First we look for an outer solution through the use of Taylor expansions. Dividing equations (3.68) and (3.69) throughout by B yields

$$\begin{aligned}\Delta[\theta''\cos^2\theta-(\theta')^2\sin\theta\cos\theta]+\Delta[\sec\delta+\cos(\theta-\delta)-2]\sin(\theta-\delta)\\ -\cos(\theta-\delta)\sin(\theta-\delta)=0, \quad (3.88)\\ \Delta[\delta''\cos^2\delta-(\delta')^2\sin\delta\cos\delta]-\Delta[\sec\delta+\cos(\theta-\delta)-2][\sec\delta\tan\delta+\sin(\theta-\delta)]\\ +\cos(\theta-\delta)\sin(\theta-\delta)=0, \quad (3.89)\end{aligned}$$

where we have set $\Delta = 1/B$, and have taken $\kappa = 1$. Now we set

$$\theta(\Delta,\bar{z}) = \theta_0(\bar{z}) + \Delta\theta_1(\bar{z}) + O(\Delta^2), \quad (3.90)$$
$$\delta(\Delta,\bar{z}) = \delta_0(\bar{z}) + \Delta\delta_1(\bar{z}) + O(\Delta^2), \quad (3.91)$$

and use Taylor expansions with respect to Δ, keeping $\bar{z}$ fixed, to find approximations for each term in the two equilibrium equations. So, for example, we have

$$\cos^2\theta = \cos^2(\theta_0(\bar{z}) + \Delta\theta_1(\bar{z})) \simeq \cos^2\theta_0 - 2\cos\theta_0\sin\theta_0\theta_1\Delta, \tag{3.92}$$

$$\sec\delta = \sec(\delta_0(\bar{z}) + \Delta\delta_1(\bar{z})) \simeq \sec\delta_0 + 2\sec\delta_0\tan\delta_0\delta_1\Delta. \tag{3.93}$$

Inserting the Taylor expansions into equilibrium equations (3.88) and (3.89), truncating at the Δ^1 term, gives four equations to solve for θ_0, θ_1, δ_0 and δ_1. The Δ^0 term from both equilibrium equations yields

$$\cos(\theta_0 - \delta_0)\sin(\theta_0 - \delta_0) = 0, \tag{3.94}$$

the Δ^1 term from (3.88) gives

$$\theta_0''\cos^2\theta_0 - (\theta_0')^2\sin\theta_0\cos\theta_0 + (\sec\delta_0 + \cos(\theta_0 - \delta_0) - 2)\sin(\theta_0 - \delta_0) \\ -(\cos^2(\theta_0 - \delta_0) - \sin^2(\theta_0 - \delta_0))(\theta_1 - \delta_1) = 0, \tag{3.95}$$

and from the Δ^1 term in (3.89) we obtain

$$\delta_0''\cos^2\delta_0 - (\delta_0')^2\sin\delta_0\cos\delta_0 - (\sec\delta_0 + \cos(\theta_0 - \delta_0) - 2)(\sin(\theta_0 - \delta_0) + \sec\delta_0\tan\delta_0 \\ +(\cos^2(\theta_0 - \delta_0) - \sin^2(\theta_0 - \delta_0))(\theta_1 - \delta_1) = 0. \tag{3.96}$$

Equation (3.94) implies that the difference between θ_0 and δ_0 is equal to $n\pi$ or $\frac{1}{2}(2n+1)\pi$. Taking the simplest case when $n = 0$ leads to the conclusion that θ_0 and δ_0 either coincide or are a $\pi/2$ multiple apart. Due to the physical constraints of the liquid crystal, it is unlikely that the director and unit normal would be at right angles to each other, hence we conclude that $\theta_0 = \delta_0$. Inserting this constraint into equations (3.95) and (3.96) yields

$$\theta_0'' \cos^2\theta_0 - (\theta_0')^2 \sin\theta_0 \cos\theta_0 - (\theta_1 - \delta_1) = 0, \quad (3.97)$$

$$\theta_0'' \cos^2\theta_0 - (\theta_0')^2 \sin\theta_0 \cos\theta_0 - (\sec\theta_0 - 1)\sec\theta_0 \tan\theta_0 + (\theta_1 - \delta_1) = 0, \quad (3.98)$$

then adding equation (3.98) to (3.97) produces the equation

$$2\theta_0'' \cos^2\theta_0 - 2(\theta_0')^2 \sin\theta_0 \cos\theta_0 - (\sec\theta_0 - 1)\sec\theta_0 \tan\theta_0 = 0. \quad (3.99)$$

Multiplying (3.99) throughout by θ_0' allows this equation to be written in the form

$$\frac{d}{d\bar{z}}\Big((\theta_0')^2 \cos^2\theta_0\Big) + \frac{d}{d\bar{z}}\Big(-\frac{1}{2}\sec^2\theta_0 + \sec\theta_0\Big) = 0, \quad (3.100)$$

which can be integrated with respect to $\bar{z}$ to yield

$$(\theta_0')^2 = \frac{1}{2\cos^4\theta_0} - \frac{1}{\cos^3\theta_0} + \frac{c_1}{\cos^2\theta_0}, \quad (3.101)$$

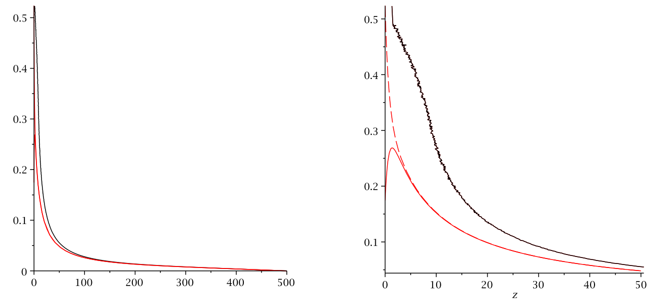
where c_1 is an arbitrary constant of integration. We know that at the midpoint of the sample $\theta_0(\bar{d}/2) = 0$, and we set $\theta_0'(\bar{d}/2) = -\hat{\theta}_0 \neq 0$, where $0 < \hat{\theta}_0 \ll 1$. Substituting $\bar{z} = \bar{d}/2$ into (3.101) yields

$$c_1 = \frac{1}{2} + \hat{\theta}_0^2. \quad (3.102)$$

Now inserting c_1 back into (3.101) gives

$$\frac{d\theta_0}{d\bar{z}} = -\frac{1}{\sqrt{2}}\left(\frac{1}{\cos^4\theta} - \frac{2}{\cos^3\theta} + \frac{1 + 2\hat{\theta}_0}{\cos^2\theta}\right)^{1/2}, \quad (3.103)$$

where we have taken the negative square root since the function is decreasing (as



(a) (b)

Figure 3.11: Outer solution θ_0 represented by the black line plotted (a) between $\bar{z} = 0$ and $\bar{z} = 500$, and (b) between $\bar{z} = 0$ and $\bar{z} = 50$, where the dashed red line represents the angle $\theta(\bar{z})$, the solid red line represents the angle $\delta(\bar{z})$.

seen in Figure 3.10). Separating the variables in (3.103) and integrating yields

$$\sqrt{2}\int_0^{\theta_0}\left(\frac{1}{\cos^4\theta} - \frac{2}{\cos^3\theta} + \frac{1+2\hat{\theta}_0}{\cos^2\theta}\right)^{-1/2} d\theta = \int_{\bar{z}}^{\bar{d}/2} d\bar{z} = \frac{\bar{d}}{2} - \bar{z}. \tag{3.104}$$

Equation (3.104) provides an implicit equation to solve for the first approximation to the angles of the director and unit normal. We can evaluate the value of $\hat{\theta}_0$ by inserting $\bar{z} = 0$ into equation (3.104), hence we obtain

$$\sqrt{2}\int_0^{\theta_b}\left(\frac{1}{\cos^4\theta} - \frac{2}{\cos^3\theta} + \frac{1+2\hat{\theta}_0}{\cos^2\theta}\right)^{-1/2} d\theta = \frac{\bar{d}}{2}. \tag{3.105}$$

Since all the quantities in (3.105) are known except for $\hat{\theta}_0$, where we have set the boundary angle to be $\theta_b = \pi/6$, we have an implicit equation to solve for this value. We solved equation (3.105) for $\hat{\theta}_0$ using Simpson's rule in MAPLE12 [30], then inserted this value into (3.104) to give an equation in $\bar{z}$ and $\theta_0(\bar{z})$ only. We also solved this equation using Simpson's rule and plotted the results in Figure 3.11. A higher order approximation can be obtained by considering equation (3.97) and the equation arising from the Δ^2 term. However, the manipulation involved is tedious, and the additional term in the expansion does not add to the accuracy of the solution. Now we consider the inner solution.

3.3.4 Inner Solution

To find an inner expansion of the solution we first introduce a new stretched variable ζ such that [29]

$$\zeta = \frac{\bar{z}}{\chi(\Delta)}, \tag{3.106}$$

where χ is a function to be determined, then equations (3.88) and (3.89) become

$$\begin{aligned}\frac{\Delta}{\chi^2}\left[\theta''(\zeta\chi)\cos^2\big(\theta(\zeta\chi)\big) - \big(\theta'(\zeta\chi)\big)^2 \sin\big(\theta(\zeta\chi)\big)\cos\big(\theta(\zeta\chi)\big)\right] \\ +\Delta\Big[\sec\big(\delta(\zeta\chi)\big) + \cos\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big) - 2\Big]\sin\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big) \\ -\cos\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big)\sin\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big) = 0,\end{aligned} \tag{3.107}$$

$$\begin{aligned}\frac{\Delta}{\chi^2}\left[\delta''(\zeta\chi)\cos^2\big(\delta(\zeta\chi)\big) - \big(\delta'(\zeta\chi)\big)^2 \sin\big(\delta(\zeta\chi)\big)\cos\big(\delta(\zeta\chi)\big)\right] \\ -\Delta\Big[\sec\big(\delta(\zeta\chi)\big) + \cos\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big) - 2\Big]\Big[\sec\big(\delta(\zeta\chi)\big)\tan\big(\delta(\zeta\chi)\big) + \sin\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big)\Big] \\ +\cos\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big)\sin\big(\theta(\zeta\chi) - \delta(\zeta\chi)\big) = 0.\end{aligned} \tag{3.108}$$

We now consider equation (3.107) and look at the behaviour of each coefficient in the limiting case $\Delta \to 0$ and $\bar{z} \to 0$. Hence the four coefficients behave like

$$\frac{\Delta}{\chi^2}\cos\big(\theta(0)\big) = \mathcal{O}\left(\frac{\Delta}{\chi^2}\right), \tag{3.109}$$

$$\frac{\Delta}{\chi^2}\cos\big(\theta(0)\big)\sin\big(\theta(0)\big) = \mathcal{O}\left(\frac{\Delta}{\chi^2}\right), \tag{3.110}$$

$$\Delta\Big[\sec\big(\theta(0)\big) + \cos\big(\theta(0) - \delta(0)\big) - 2\Big]\sin\big(\theta(0) - \delta(0)\big) = \mathcal{O}(\Delta), \tag{3.111}$$

$$\cos\big(\theta(0) - \delta(0)\big)\sin\big(\theta(0) - \delta(0)\big) = \mathcal{O}(1), \tag{3.112}$$

since $\theta(0)$ and $\delta(0)$ are constant and so are of order 1. Matching the first and second terms allows a free choice of $\chi(\Delta)$ but since we want to make the other two terms small in comparison we choose $\chi = \Delta$. Hence equation (3.107) becomes

$$\begin{aligned}
&\cos^2\big(\theta(\Delta\zeta)\big)\theta''(\Delta\zeta) - \sin\big(\theta(\Delta\zeta)\big)\cos\big(\theta(\Delta\zeta)\big)\big(\theta'(\Delta\zeta)\big)^2 \\
&\quad +\Delta^2\Big[\sec\big(\delta(\Delta\zeta)\big) + \cos\big(\theta(\Delta\zeta) - \delta(\Delta\zeta)\big) - 2\Big]\sin\big(\theta(\Delta\zeta) - \delta(\Delta\zeta)\big) \\
&\quad -\Delta\sin\big(\theta(\Delta\zeta) - \delta(\Delta\zeta)\big)\cos\big(\theta(\Delta\zeta) - \delta(\Delta\zeta)\big) = 0,
\end{aligned} \tag{3.113}$$

and so from the Δ^0 term we obtain

$$\theta''\cos^2\theta - \sin\theta\cos\theta(\theta')^2 = 0. \tag{3.114}$$

Multiplying equation (3.114) throughout by θ' allows this equation to be written in the form

$$\frac{1}{2}\frac{d}{d\zeta}\Big((\theta')^2\cos^2\theta\Big) = 0, \tag{3.115}$$

which can be integrated with respect to ζ to give

$$(\theta')^2 = \frac{c_1}{\cos^2\theta}. \tag{3.116}$$

From the boundary conditions we have $\theta(0) = \theta_b$ and $\theta'(0) = \Theta_b < 0$, so the constant of integration is given by $c_1 = \Theta_b^2\cos^2\theta_b$. Then (3.116) becomes

$$\frac{d\theta}{d\zeta} = \frac{\Theta_b\cos\theta_b}{\cos\theta}. \tag{3.117}$$

We have taken the positive square root because we have $\Theta_b < 0$ and we need the derivative to be negative. Then separating the variables in (3.117) and integrating

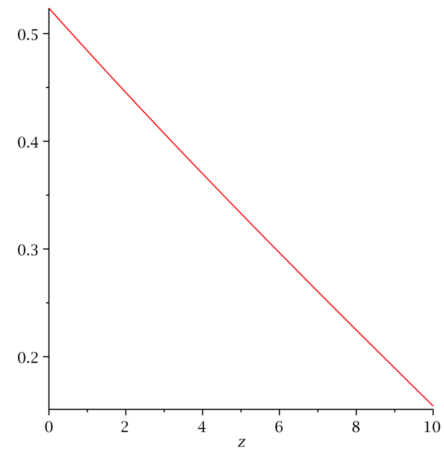


Figure 3.12: Inner solution for $\theta(\bar{z})$, as found in (3.119), plotted against $\bar{z}$.

yields

$$\theta(\zeta) = \sin^{-1}\left(\sin\theta_b + \zeta\Theta_b\cos\theta_b\right). \tag{3.118}$$

The solution shown in (3.118) is in terms of the stretched variable ζ, so now we substitute back for the original variable $\bar{z}$ to attain the solution

$$\theta(\bar{z}) = \sin^{-1}\left(\sin\theta_b + \frac{\bar{z}}{\Delta}\Theta_b\cos\theta_b\right), \tag{3.119}$$

where the unknown constant Θ_b can be determined by matching the outer and inner solutions. We take the boundary region to be one tenth of the sample, i.e. at $\bar{z} = 10$. Inserting $\bar{z} = 10$ into the outer solution (3.105) and the inner solution (3.119) then equating these functions gives the unknown constant to be $\Theta_b = -0.028$. We have plotted the inner solution (3.118) in Figure 3.12, where we can clearly see the inaccuracy of the result. The correct general profile is attained such that the director begins at an angle θ_b then decreases as $\bar{z}$ increases, however, the decrease is not sharp enough and so the outer solution gives quite a poor representation of the director. The outer solution for the angle δ is also obtained from equation (3.114) but with θ replaced by δ, hence the solution plotted in Figure 3.12 is even less accurate when describing the unit normal as it doesn't capture the 'hump' in the solution. The inaccuracy of the inner solution could be a result of the method chosen, since the approximations in Sections 3.3.3 and 3.3.4 only work for large values of B. For the solutions found in these two sections, we choose $B = 10$ which, as discussed previously, is on the limit of physically relevant values. These approximations will not be suitable for all values of B, however, they are the first approximation to a very complex nonlinear problem. Now we consider the stability of the variable layer case.

3.3.5 Stability of the Variable Layer

To study the stability of the variable layer, we consider the equation of angular momentum and the permeation equation, viz.

$$\left(\frac{\partial w}{\partial n_{i,j}}\right)_{,j} - \frac{\partial w}{\partial n_i} + \tilde{g}_i + G_i = \mu n_i, \tag{3.120}$$

$$\dot{\Phi} = -\lambda_p J_{i,i}, \tag{3.121}$$

where λ_p is the (positive) permeation constant. At this level of approximation we have again neglected the effects of flow and concentrate now on permeation. The director, unit normal and the local planar layer function Φ are as given in (3.58), (3.61) and (3.59), respectively. The quantity J_i is the phase flux term and is defined to be

$$J_i = -\frac{\partial w}{\partial \Phi_{,i}} + \frac{1}{|\nabla\Phi|}\left[\left(\frac{\partial w}{\partial a_{p,k}}\right)_{,k} - \frac{\partial w}{\partial a_p}\right](\delta_{pi} - a_p a_i). \tag{3.122}$$

Inserting the values for $\mathbf{n}$, $\mathbf{a}$ and Φ into equations (3.120) yields

$$K_1^n(\cos\theta\,\theta')_{,i} - B_0(\sec\delta + \cos(\theta-\delta) - 2)a_i + B_1\cos(\theta-\delta)a_i = \mu n_i, \tag{3.123}$$

which can be written explicitly as

$$-B_0[\sec\delta + \cos(\theta-\delta) - 2]\cos\delta + B_1\cos(\theta-\delta)\cos\delta = \mu\cos\theta, \tag{3.124}$$

$$K_1^n(\cos\theta\,\theta')_{,z} - B_0(\sec\delta + \cos(\theta-\delta) - 2)\sin\delta + B_1\cos(\theta-\delta)\sin\delta = \mu\sin\theta. \tag{3.125}$$

Multiplying equation (3.124) by $\sin\theta$ and multiplying equation (3.125) by $\cos\theta$, then subtracting the two resulting equations yields the governing equation

$$\begin{aligned}-B_0[\sec\delta + \cos(\theta-\delta) - 2]\sin(\theta-\delta) + B_1\cos(\theta-\delta)\sin(\theta-\delta)\\ -K_1^n[\cos\theta\,\theta'' - \sin\theta(\theta')^2]\cos\delta = 0.\end{aligned} \tag{3.126}$$

Now we consider the permeation equation. Since the phase flux term J_i is only dependent on z, the derivatives of this term with respect to x and y is zero. Hence we are only required to find the J_3 term, which is given by

$$\begin{aligned}J_3 &= B_0[\sec\delta + \cos(\theta-\delta) - 2][-\sin\theta\cos\delta - \sin\delta + \cos(\theta-\delta)\sin\delta\cos\delta]\\ &\quad + B_1[\cos\delta\sin\theta\cos(\theta-\delta) - \cos^2(\theta-\delta)\sin\delta\cos\delta]\\ &\quad + K_1^a[\cos\delta\,\delta'' - \sin\delta(\delta')^2]\cos^3\delta.\end{aligned} \tag{3.127}$$

Hence the divergence of $\mathbf{J}$ is given by

$$
\begin{aligned}
J_{i,i} &= J_{3,z} \\
&= B_0\Big[\big\{-\cos\theta\cos\delta\,\theta' + \sin\theta\sin\delta\,\delta' - \cos\delta\delta' - \sin(\theta-\delta)(\theta'-\delta')\sin\delta\cos\delta \\
&\qquad + \cos(\theta-\delta)(\cos^2\delta - \sin^2\delta)\delta'\big\}(\sec\delta + \cos(\theta-\delta) - 2) \\
&\qquad + \big\{-\sin\theta\cos\delta - \sin\delta + \cos(\theta-\delta)\sin\delta\cos\delta\big\}(\sec\delta\tan\delta\,\delta' - \sin(\theta-\delta)(\theta'-\delta'))\Big] \\
&+ B_1\Big[-\sin\delta\sin\theta\cos(\theta-\delta)\delta' + \cos\delta\cos\theta\cos(\theta-\delta)\theta' - \cos\delta\sin\theta\sin(\theta-\delta)(\theta'-\delta') \\
&\qquad + 2\cos(\theta-\delta)\sin(\theta-\delta)\sin\delta\cos\delta(\theta'-\delta') - 2\cos^2(\theta-\delta)\cos^2\delta\,\delta' + \cos^2(\theta-\delta)\delta'\Big] \\
&+ K_1^a\Big[\cos^4\delta\,\delta''' - \cos^4\delta(\delta')^3 - 6\cos^3\delta\sin\delta\,\delta'\,\delta'' + 3\sin^2\delta\cos^2\delta\,(\delta')^3\Big].
\end{aligned}
\tag{3.128}
$$

We can write the term on the left hand side of equation (3.121) as

$$
\begin{aligned}
\dot{\Phi} &= \frac{\partial}{\partial t}\Big(x + \int_{z_0}^{z} \tan\big(\delta(\hat{z},t)\big)d\hat{z}\Big) + \mathbf{v}\cdot\nabla\Phi \\
&= \int_{z_0}^{z} \sec^2\big(\delta(\hat{z},t)\big)\delta_t(\hat{z},t)d\hat{z},
\end{aligned}
\tag{3.129}
$$

since the velocity is zero due to the static nature of the problem, and where the subscript t to represents differentiation with respect to time. Hence the permeation equation can be written as

$$
\int_{z_0}^{z} \sec^2\big(\delta(\hat{z},t)\big)\delta_t(\hat{z},t)d\hat{z} = -\lambda_p J_{3,z}. \tag{3.130}
$$

In their current form, governing equations (3.126) and (3.130) are too complicated to solve analytically so we simplify the problem by linearising in θ and δ which yields the more tractable system

$$K_1^n\theta'' = B_1(\theta - \delta), \tag{3.131}$$

$$\int_{z_0}^{z} \delta_t(\hat{z}, t)d\hat{z} = -\lambda_p(K_1^a\delta''' + B_1(\theta' - \delta')), \tag{3.132}$$

where the superscripts represent differentiation with respect to z. We can obtain a more precise form for the permeation equation by performing the integration in (3.132), hence we arrive at the system of governing equations

$$K_1^n\theta'' = B_1(\theta - \delta), \tag{3.133}$$

$$\delta_t = -\lambda_p(K_1^a\delta^{(iv)} + B_1(\theta'' - \delta'')). \tag{3.134}$$

We search for non-zero solutions for this system of the form [51]

$$\theta = \theta_0 \exp(\omega t + iqz), \tag{3.135}$$

$$\delta = \delta_0 \exp(\omega t + iqz), \tag{3.136}$$

where θ_0, δ_0 are small constants and q is a wave number. These forms for the solutions are typical in standard wave perturbation analysis because a disturbance to a physical system will generally have an oscillatory component in space and a decaying term in time. The result of inserting these forms into equations (3.133) and (3.134) can be written in the matrix system

$$\begin{bmatrix} K_1^n q^2 + B_1 & -B_1 \\ -B_1 q^2 \lambda_p & \omega + \lambda_p K_1^a q^4 + B_1 \lambda_p q^2 \end{bmatrix} \begin{bmatrix} \theta_0 \\ \delta_0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{3.137}$$

To obtain non-zero solutions for θ_0 and δ_0, we require the determinant of the matrix on the left hand side of (3.137) to be equal to zero. Taking the determinant of this matrix and rearranging for ω yields

$$\omega = -\frac{\lambda_p q^4[K_1^n K_1^a q^2 + B_1(\kappa+1)/K_1^n]}{K_1^n q^2 + B_1}.$$

Due to the positive nature of the material parameters present in the expression for ω, we can clearly see that $\omega < 0$ for all wave numbers q. Hence, the initial bookshelf alignment of SmA is linearly asymptotically stable to small perturbations.

3.3.6 Conclusions

For the variable layer case, we were able to find an accurate outer solution which matched the numerical solution almost perfectly in the bulk of the sample. However this solution, and the inner solution we obtained by rescaling the variable $\bar{z}$, were not at all accurate in the boundary region of the sample. We had hoped to be able to accurately describe the curves for the director and unit normal and, in the case of the unit normal, be able to pinpoint exactly the location of the turning point in terms of the material parameters of the liquid crystal. Although we have so far been unable to accomplish this, we believe it is a worthy future project as it would give great insight into which of the material parameters drive most strongly the behaviour of the layer normal in this case. In terms of stability, we showed that this geometry is stable for small perturbations around the steady state for all parameter values. This is not entirely surprising since the case where **a** is a constant is always stable, and so it is not too surprising that small perturbations away from this state are also always stable.

3.4 Discussion

One of the physically relevant aspects of studying liquid crystals is determining the switching times involved. Clearly if a liquid crystal is being used in an optical device, for example, a key requirement is the ability to switch from one state to another as quickly as possible, hence providing a faster switching image on the device. If we consider the switching times in the case of planar aligned SmA, we find that as B increases, λ decreases, hence increasing the response time. This means that as B is becoming larger the system is becoming more sluggish. There is also a link between response time and stability. If the response time is fast, it means that the system is settling to an equilibrium quickly after being perturbed, hence reducing the likelihood of instability. However, if the system is varying a lot before settling to a equilibrium state, and taking longer and longer to do this as a parameter is increased, it leads to a greater risk of instability. The fact that the planar aligned sample exhibits a longer settling period, combined with the fact that the Ritz method fails to return a solution after the point B^*, suggests that there is an instability in the system.

There are two reasons for B becoming larger: the first is that B_1 is increasing, i.e. $\mathbf{n}$ and $\mathbf{a}$ becoming more strongly coupled, the the second is that B_0 is decreasing. If we consider the physical interpretation of the first case we see that as $\mathbf{n}$ and $\mathbf{a}$ separate more, the system exhibits more flexibility and so takes longer to settle. In the second case of B_0 decreasing, the layer compression is decreasing and so the layers are becoming more flexible, hence increasing the response time.

For the bookshelf alignment we find the opposite is true: as B increases so does λ, hence decreasing the response time. The stark contrast in behaviour is due to the effect strong anchoring has on the bookshelf geometry. The strong anchoring encourages less flexibility in the director, so if $\mathbf{a}$ is increasingly connected to $\mathbf{n}$ (i.e. B increasing), then $\mathbf{n}$ dominates the behaviour and the bookshelf becomes more rigid. Therefore the response time will become faster in this case.

In regards to the form chosen for Φ in (3.59), we made the remark that this only holds for δ dependent on one spatial variable. It is also the case that the form given in (3.59) is not unique. An equally viable form would be

$$\Phi = z + \int_{x_0}^{x} \cot\delta(\bar{x})d\bar{x}, \tag{3.138}$$

where x_0 is an arbitrary constant. These different forms are available due to the fact the Φ is constant, and so x and z are related through (3.59) or (3.138). We chose to use the form in (3.59) instead of (3.138) because if we had of used the latter we would have attained $\nabla\Phi = (\cot\delta(x), 0, 1)$, and so the unit layer normal would be represented by

$$\mathbf{a} \equiv \frac{\nabla\Phi}{|\nabla\Phi|} = (\cos\delta(x), 0, \sin\delta(x)). \tag{3.139}$$

Therefore in this formulation we have the layer normal angle δ in terms of x, but due to the way we set up the problem it is more natural for δ to be a function of the height, z. Hence, for the work in this thesis we chose the more natural form of Φ given by (3.59).

Chapter 4

Oscillatory Shear Flow in Smectic A Liquid Crystals

Motivated by the work of Moir [33], who studied oscillatory shear flow of SmC liquid crystals, and Stewart [48], who studied oscillatory shear flow of planar aligned SmA liquid crystals, we study the previously unexamined case of bookshelf aligned SmA liquid crystals. We consider both finite and semi-infinite domains when one of the plates is oscillated and the other is fixed, and we study this geometrical set-up using the continuum dynamic theory of Stewart [51]. There are two possible set-ups in this geometry. The first is somewhat counter-intuitive but is supported by experimental results [3] and consists of setting the director to be $\mathbf{n} = (1, 0, \theta(z,t))$, which allows for the possibility of transverse flow. Transverse flow is flow which occurs in a direction perpendicular to that of the induced oscillation. The second option, more intuitively, is to set the director to be $\mathbf{n} = (1, \theta(z,t), 0)$, which assumes that there will be no transverse flow. We set up the problem for both cases and examine the consequent phenomena.

4.1 Geometrical Set-up – Transverse Flow

Consider a sample of SmA liquid crystal on a glass plate positioned at $z = 0$. We consider this sample to be infinite in the x and y directions, and we will consider

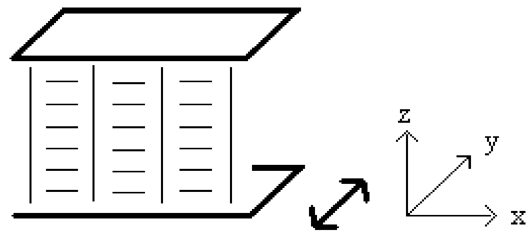


Figure 4.1: Bookshelf aligned SmA liquid crystal confined between two parallel plates a distance d apart. The top plate remains fixed while an oscillatory shear parallel to the bottom plate, with an amplitude A and frequency ω, is applied.

the cases when the sample is finite and semi-infinite in the z-direction. We apply an oscillatory shear parallel to the bottom plate along the y-axis, with an amplitude A and frequency ω. The vector $\mathbf{x} = (0, y(t), 0)$ gives the displacement of the plate, i.e. at $z = 0$, where $y(t) = A\sin\omega t$. Thus the velocity on the bottom plate is given by

$$v_2(0,t) = \frac{dy}{dt} = A\omega\cos\omega t. \tag{4.1}$$

We linearise this set-up with respect to the tilt angle $\theta(z,t)$, assuming $|\theta| \ll 1$, to obtain

$$\mathbf{n} = (1, 0, \theta(z,t)), \tag{4.2}$$

$$\mathbf{a} = (1, 0, 0), \tag{4.3}$$

$$\mathbf{v} = (0, v_2(z,t), v_3(z,t)), \tag{4.4}$$

$$\Phi = x, \tag{4.5}$$

where θ is the orientation of the director $\mathbf{n}$ in relation to the x-axis, $\mathbf{a}$ is the unit layer normal, and v_2 and v_3 are the velocities in the y and z directions respectively. We consider the system to have dependence on z and t only, and we have assumed zero velocity in the x-direction, i.e. $v_1 \equiv 0$. The usual constraints $\mathbf{n}\cdot\mathbf{n} = 1$ and $\mathbf{a}\cdot\mathbf{a} = 1$ are both satisfied in the linearised system, and the incompressibility condition $\nabla \cdot \mathbf{v} = 0$ induces the constraint $v_{3,z} = 0$ onto the z-direction velocity. Since $v_{3,z} = 0$ and we set $v_3(0,t) = 0$, it follows that the z-direction velocity is always zero. Now we consider the governing equations for the system.

The Balance of Linear Momentum

The equation for the balance of linear momentum is given by

$$\rho \dot{v}_i = \rho F_i - \tilde{p}_{,i} + \tilde{g}_j n_{j,i} + G_j n_{j,i} + |\nabla\Phi| a_i J_{j,j} + \tilde{t}_{ij,j}, \tag{4.6}$$

where ρ is the density of the fluid, F_i is the external body force per unit mass, $\tilde{p}$ is the pressure, $\tilde{g}_i$ is the dynamic contribution, G_i is the generalised body force related to the external body moment per unit mass, J_i is the phase flux term, and $\tilde{t}_{ij}$ is the viscous stress tensor. In general, the pressure is of the form $\tilde{p} = p + w$, where w is the energy density. However we are only considering the linearised system, and since w is of quadratic order, we set $\tilde{p} = p$. We take the viscous stress to be of the form proposed by Martin *et al.* [31], de Gennes and Prost [7], and E [11] for an incompressible SmA liquid crystal, namely

$$\tilde{t}_{ij} = \alpha_4 A_{ij} + \tau_1 (a_k A_{kp} a_p) a_i a_j + \tau_2 (a_i A_{jp} a_p + a_j A_{ip} a_p), \tag{4.7}$$

where τ_1,τ_2 are SmA-like viscosities, α_4 is twice the usual Newtonian viscosity, and $A_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ is the rate of strain tensor. This coincides with a restricted version of the viscous stress introduced by Stewart [51]. The individual components of the stress tensor are given by

$$\tilde{t}_{1j} = \alpha_4 A_{1j} + \tau_1 (a_k A_{kp} a_p) a_j + \tau_2 (A_{jp} a_p + a_j A_{1p} a_p), \tag{4.8}$$

$$\tilde{t}_{2j} = \alpha_4 A_{2j} + \tau_2 a_j A_{2p} a_p, \tag{4.9}$$

$$\tilde{t}_{3j} = \alpha_4 A_{3j} + \tau_2 a_j A_{3p} a_p, \tag{4.10}$$

$$\tag{4.11}$$

and so taking the divergence of the stress tensor yields

$$\tilde{t}_{1j,j} = \alpha_4 A_{1j,j} + \tau_1 a_k A_{kp,j} a_p a_j + \tau_2 (A_{jp,j} a_j + a_p A_{1p,j} a_p), \tag{4.12}$$
$$\tilde{t}_{2j,j} = \alpha_4 A_{2j,j} + \tau_2 a_j A_{2p,j} a_p, \tag{4.13}$$
$$\tilde{t}_{3j,j} = \alpha_4 A_{3j,j} + \tau_2 a_j A_{3p,j} a_p. \tag{4.14}$$

By definition we have $A_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i})$ and so we can write the divergence of the stress tensor in terms of the velocity components as

$$\tilde{t}_{1j,j} = 0, \quad \tilde{t}_{2j,j} = \frac{\alpha_4}{2}\frac{\partial^2 v_2}{\partial z^2}, \quad \tilde{t}_{3j,j} = 0. \tag{4.15}$$

Next we consider the dynamic contribution $\tilde{g}_i$, which takes the general form [49]

$$\tilde{g}_i = -\gamma_1 N_i - \gamma_2 A_{ip} n_p - 2\kappa_i A_{ip} a_p, \tag{4.16}$$

and hence we can write

$$\tilde{g}_1 = 0, \tag{4.17}$$
$$\tilde{g}_2 = \frac{1}{2}(\gamma_1 - \gamma_2)\frac{\partial v_2}{\partial z}\theta, \tag{4.18}$$
$$\tilde{g}_3 = -\gamma_1 \frac{\partial \theta}{\partial t}, \tag{4.19}$$

and, after some manipulations, we find that $\tilde{g}_i n_i = 0$ for the linearised case. Lastly, we consider the phase flux term J_i, which is given by

$$J_i = -\frac{\partial w}{\partial \Phi_{,i}} + \frac{1}{|\nabla \Phi|}\left[\left(\frac{\partial w}{\partial a_{p,k}}\right)_{,k} - \frac{\partial w}{\partial a_p}\right](\delta_{pi} - a_p a_i). \tag{4.20}$$

To find the divergence of $\boldsymbol{J}$ we need to only consider the term J_3. This is because $\boldsymbol{J}$ is only dependent on z and t, and so $J_{1,1} = J_{2,2} = 0$. For $i = 3$ we have

$$
\begin{aligned}
J_3 &= \frac{1}{|\nabla\Phi|}\left[\left(\frac{\partial w}{\partial a_{p,k}}\right)_{,k} - \frac{\partial w}{\partial a_p}\right](\delta_{p3} - a_p a_3) - \frac{\partial w}{\partial \Phi_{,z}} \\
&= \frac{1}{|\nabla\Phi|}\left[\left(\frac{\partial w}{\partial a_{3,k}}\right)_{,k} - \frac{\partial w}{\partial a_3}\right] \\
&= 0, \qquad (4.21)
\end{aligned}
$$

since $a_3 = 0$. Therefore $J_3 = 0$ and so $\nabla \cdot \mathbf{J} = 0$. Hence we can now write the balance of linear momentum equations (4.6) explicitly as

$$0 = -\,p_{,x}, \qquad (4.22)$$

$$\rho\frac{\partial v_2}{\partial t} = -\,p_{,y} + \frac{\alpha_4}{2}\frac{\partial^2 v_2}{\partial z^2}, \qquad (4.23)$$

$$0 = -\,p_{,z}. \qquad (4.24)$$

The Balance of Angular Momentum

The equation for the balance of angular momentum is given by

$$\left(\frac{\partial w}{\partial n_{i,j}}\right)_{,j} - \frac{\partial w}{\partial n_i} + \tilde{g}_i + G_i = \mu n_i, \qquad (4.25)$$

and we take the energy density to be of the form stated in (2.22), namely

$$w = \frac{1}{2}K_1^n(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_1^a(\nabla\cdot\mathbf{a})^2 + \frac{1}{2}B_0(|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2)^2 + \frac{1}{2}B_1[1 - (\mathbf{n}\cdot\mathbf{a})^2]. \qquad (4.26)$$

Substituting the energy density (4.26) into equation (4.25) yields

$$K_1^n n_{i,jj} + B_1(\mathbf{n}\cdot\mathbf{a})a_i - B_0(|\nabla\Phi| + \mathbf{n}\cdot\mathbf{a} - 2)a_i + \tilde{g}_i + \mu n_i = 0, \qquad (4.27)$$

where we have set $G_i \equiv 0$ since we are not applying any external forces. To find the Lagrange multiplier μ, we simply take the inner product of (4.27) with $\mathbf{n}$, hence we obtain

$$K_1^n n_{i,jj} n_i + B_1 + \tilde{g}_i n_i + \mu = 0. \tag{4.28}$$

Equation (4.28) can be simplified to obtain $\mu = -B_1$, noting that $\tilde{g}_i n_i = 0$ and $n_{i,jj} n_i = 0$ in the linearised geometry. Hence we have obtained the Lagrange multiplier explicitly in terms of a material parameter. Equation (4.27) is automatically satisfied for $i = 1, 2$, and so our final governing equation comes from the $i = 3$ equation in (4.27), hence we have

$$K_1^n \frac{\partial^2 \theta}{\partial z^2} - \gamma_1 \frac{\partial \theta}{\partial t} - B_1 \theta = 0. \tag{4.29}$$

Combining the equations from the balance of linear momentum and the balance of angular momentum yields the system of governing equations

$$
\begin{align}
0 &= -\, p_{,x}, \tag{4.30}\\
\rho \frac{\partial v_2}{\partial t} &= -\, p_{,y} + \frac{\alpha_4}{2} \frac{\partial^2 v_2}{\partial z^2}, \tag{4.31}\\
0 &= -\, p_{,z}, \tag{4.32}\\
K_1^n \frac{\partial^2 \theta}{\partial z^2} &= \gamma_1 \frac{\partial \theta}{\partial t} + B_1 \theta. \tag{4.33}
\end{align}
$$

This system provides four equations in four unknowns: θ, v_2, v_3, and p. We note here that the B_0 term is absent from the governing equations. This is not uncommon in a linearised system since, in this particular context, compression is of a higher order effect. Now we aim to find the pressure p in a form that will simplify equations (4.30) – (4.33). From (4.30) and (4.32) we clearly see that p is independent of x and z, respectively, so we have $p = p(y, t)$. Hence, we can write the pressure as $p = f_1(y, t)$. If we differentiate this form of the pressure with respect to y and substitute into (4.31) we obtain

$$\frac{\partial f_1}{\partial y} = \frac{\alpha_4}{2}\frac{\partial^2 v_2}{\partial z^2} - \rho\frac{\partial v_2}{\partial t}. \tag{4.34}$$

The left hand side of (4.34) is dependent on y and t only, while the right hand side is dependent on z and t, so we have $LHS = RHS = c_1(t)$. Hence we can write the unknown function $f_1(y,t)$ in the form

$$f_1(y,t) = c_1(t)y + c_2(t), \tag{4.35}$$

and hence the pressure can be written in the form

$$p(y,z,t) = c_1(t)y + c_2(t), \tag{4.36}$$

where $c_1(t)$ and $c_2(t)$ are arbitrary functions in time. We note that $\nabla p = (0, c_1(t), 0)$ and so in the absence of any pressure gradient $c_1(t) = 0$. The form of the pressure given in (4.36) clearly satisfies equation (4.30), and substituting (4.36) into (4.32) we find that (4.32) is automatically satisfied. The two governing equations for the system are then given by (4.33) and (4.31), with the general form of the pressure having been inserted from (4.36) (assuming the absence of applied pressure gradients). Hence the governing equations are

$$\rho\frac{\partial v_2}{\partial t} = \frac{\alpha_4}{2}\frac{\partial^2 v_2}{\partial z^2}, \tag{4.37}$$

$$K_1^n\frac{\partial^2 \theta}{\partial z^2} = \gamma_1\frac{\partial \theta}{\partial t} + B_1\theta. \tag{4.38}$$

We note that the equations we have obtained for the velocity and director profile are an uncoupled set. This is in direct contrast to the work of Stewart [48] where the velocity played a key role in the behaviour of the director profile. Clearly the geometry has had a significant impact in allowing the decoupling of the equations.

Now we solve (4.37) and (4.38) to obtain a solution for the y-direction velocity $v_2(z,t)$ and the director $\theta(z,t)$.

4.2 Finite Domain

We now solve the governing equations derived in Section 4.1. First considering the case where we have a sample of SmA liquid crystal confined between two parallel plates a distance d apart. As stated above, we oscillate the bottom plate with a frequency ω and amplitude A.

4.2.1 Velocity Profile (Long Time Behaviour)

The velocity in the y-direction is governed by the diffusion equation

$$\frac{\partial v_2}{\partial t} = \kappa \frac{\partial^2 v_2}{\partial z^2}, \tag{4.39}$$

where $\kappa = \alpha_4/2\rho$, with boundary conditions $v_2(0,t) = A\omega\cos\omega t$, $v_2(d,t) = 0$ and initial condition $v_2(z,0) = f(z)$. We seek a solution of the form [21]

$$v_2(z,t) = \Re\Big(H(z)e^{i\omega t}\Big), \tag{4.40}$$

where $H(z)$ is an unknown function which we aim to find. Substituting $v_2(z,t) = H(z)e^{i\omega t}$ into (4.39) yields

$$H''(z) - \frac{i\omega}{\kappa} H(z) = 0. \tag{4.41}$$

This has the general solution

$$H(z) = c_1 e^{\lambda(1+i)z} + c_2 e^{-\lambda(1+i)z}, \tag{4.42}$$

where $\lambda = \sqrt{\omega/2\kappa}$, and hence we can write (4.40) as

$$v_2 = \Re\Big(c_1 e^{\lambda z} e^{i(\omega t + \lambda z)} + c_2 e^{-\lambda z} e^{i(\omega t - \lambda z)}\Big). \tag{4.43}$$

We can obtain expressions for the constants c_1 and c_2 by considering the boundary conditions. From the bottom plate at $z = 0$ we find

$$(c_1 + c_2)(\cos \omega t + i \sin \omega t) = A\omega \cos \omega t, \tag{4.44}$$

which, after equating the real parts, implies that

$$c_1 + c_2 = A\omega, \tag{4.45}$$

and from the top plate at $z = d$ we have

$$c_1 + c_2 e^{-2\lambda(1+i)d} = 0. \tag{4.46}$$

Solving equations (4.45) and (4.46) for c_1 and c_2 yields

$$c_1 = -\frac{A\omega e^{-\lambda(1+i)d}}{e^{\lambda(1+i)d} - e^{-\lambda(1+i)d}}, \tag{4.47}$$

$$c_2 = \frac{A\omega e^{\lambda(1+i)d}}{e^{\lambda(1+i)d} - e^{-\lambda(1+i)d}}, \tag{4.48}$$

and so $H(z)$ is given by

$$H(z) = A\omega\left(\frac{e^{\lambda(1+i)(d-z)} - e^{-\lambda(1+i)(d-z)}}{e^{\lambda(1+i)d} - e^{-\lambda(1+i)d}}\right). \tag{4.49}$$

We would like to eliminate the complex terms in the denominator and we can achieve this by multiplying $H(z)$ by its complex conjugate. This leads to

$$\begin{aligned}H(z) &= A\omega\left(\frac{e^{\lambda(1+i)(d-z)} - e^{-\lambda(1+i)(d-z)}}{e^{\lambda(1+i)d} - e^{-\lambda(1+i)d}}\right) \times \left(\frac{e^{\lambda(1-i)d} - e^{-\lambda(1-i)d}}{e^{\lambda(1-i)d} - e^{-\lambda(1-i)d}}\right)\\ &= A\omega\left(\frac{e^{-\lambda(z-2d)}e^{-iz\lambda} - e^{\lambda z}e^{i\lambda(z-2d)} - e^{-\lambda z}e^{-i\lambda(z-2d)} + e^{\lambda(z-2d)}e^{i\lambda z}}{2(\cosh(2\lambda d) - \cos(2\lambda d))}\right),\end{aligned} \tag{4.50}$$

and hence we can write the velocity $v_2(z,t)$ as

$$\begin{aligned}v_2(z,t) =&\Re(H(z)e^{i\omega t})\\ =&\frac{A\omega}{2(\cosh(2\lambda d) - \cos(2\lambda d))}\Big[- e^{\lambda z}\cos(z\lambda - 2\lambda d + \omega t) - e^{\lambda(z-2d)}\cos(z\lambda + \omega t)\\ &- e^{-\lambda z}\cos(z\lambda - 2\lambda d - \omega t) + e^{-\lambda(z-2d)}\cos(z\lambda - \omega t)\Big].\end{aligned} \tag{4.51}$$

Due to the assumption we made for the form of the solution in (4.40), solution (4.51) gives no information about the velocity at an initial time $t = 0$. To obtain a solution that gives more information for small times we must use a slightly more subtle method, which we detail next.

4.2.2 Velocity Profile (All Time)

The problem we wish to solve for the unknown velocity $v_2(z,t)$ has inhomogeneous boundary conditions, so we first obtain a problem with homogeneous boundary conditions by setting a new variable $u(z,t)$ to be

$$u(z,t) = v_2(z,t) - \left(1 - \frac{z}{d}\right)A\omega\cos(\omega t). \tag{4.52}$$

This will allow us to track any transient behaviour in the problem. The change of variable in (4.52) allows us to transform the problem to

$$u_{,t} - \kappa u_{,zz} = F(z,t), \tag{4.53}$$

$$u(0,t) = u(d,t) = 0, \tag{4.54}$$

$$u(z,0) = g(z), \tag{4.55}$$

where $F(z,t) = (1 - z/d)A\omega^2 \sin \omega t$ and $g(z) = f(z) - (1 - z/d)A\omega$. Now we assume a solution of the form [9]

$$u(z,t) = \sum_{n=1}^{\infty} u_n(t) \sin\left(\frac{n\pi z}{d}\right), \tag{4.56}$$

where $u_n(t)$ is an unknown function that we will solve for, and we write

$$F(z,t) = \sum_{n=1}^{\infty} F_n(t) \sin\left(\frac{n\pi z}{d}\right), \tag{4.57}$$

$$g(z) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi z}{d}\right), \tag{4.58}$$

where $F_n(t)$ and f_n are the standard Fourier coefficients [9]. Substituting (4.56) into (4.53) yields

$$\sum_{n=1}^{\infty} \left[u_n'(t) + \kappa\left(\frac{n\pi}{d}\right)^2 u_n(t) - F_n(t)\right] \sin\left(\frac{n\pi z}{d}\right) = 0, \tag{4.59}$$

and the initial condition (4.55) becomes

$$\sum_{n=1}^{\infty} \left[u_n(0) - f_n\right] \sin\left(\frac{n\pi z}{d}\right) = 0. \tag{4.60}$$

Since $\{\sin(n\pi z/d)\}$ is a complete orthogonal family, (4.59) and (4.60) are equivalent to

$$u_n'(t) + \kappa\left(\frac{n\pi}{d}\right)^2 u_n(t) = F_n(t), \tag{4.61}$$

$$u_n(0) = f_n, \tag{4.62}$$

for $n = 1, 2, 3, \ldots$ [23]. Taking Laplace Transforms of (4.61) yields

$$s\hat{u}_n(s) - u_n(0) + \kappa\left(\frac{n\pi}{d}\right)^2 \hat{u}_n(s) - \hat{F}_n(s) = 0, \tag{4.63}$$

which can be rearranged to give

$$\hat{u}_n(s) = \frac{f_n}{s + \kappa(\frac{n\pi}{d})^2} + \frac{\hat{F}_n(s)}{s + \kappa(\frac{n\pi}{d})^2}, \tag{4.64}$$

where $\hat{u}_n(s)$ and $\hat{F}_n(s)$ are the Laplacian transforms of $u_n(t)$ and $F_n(t)$ respectively, and s is the transform variable. Inverting $\hat{u}_n(s)$ yields the solution [37]

$$u_n(t) = f_n e^{-\kappa t(n\pi/d)^2} + \int_0^t e^{-\kappa(t-\tau)(n\pi/d)^2} F_n(\tau)d\tau. \tag{4.65}$$

Hence (4.56) becomes

$$\begin{aligned} u(z,t) = &\sum_{n=1}^{\infty} f_n e^{-\kappa t(n\pi/d)^2} \sin\left(\frac{n\pi z}{d}\right) \\ &+ \sum_{n=1}^{\infty} \left[\int_0^t e^{-\kappa(t-\tau)(n\pi/d)^2} F_n(\tau)d\tau\right] \sin\left(\frac{n\pi z}{d}\right). \end{aligned} \tag{4.66}$$

Finally, substituting for the original function $v_2(z,t)$ yields the solution

Parameter	Typical Value
d	10μm
ρ	1020 kg m^{-3}
α_4	0.0652 Pa s
A	10^{-6} m
ω	160π Hz

Table 4.1: Parameter estimates.

$$\begin{aligned} v_2(z,t) = &\sum_{n=1}^{\infty} f_n e^{-\kappa(n\pi/d)^2 t} \sin\left(\frac{n\pi z}{d}\right) \\ &+ \sum_{n=1}^{\infty}\left[\int_0^t e^{-\kappa(n\pi/d)^2(t-\tau)} F_n(\tau) d\tau\right] \sin\left(\frac{n\pi z}{d}\right) \\ &+ \left(1-\frac{z}{d}\right) A\omega\cos(\omega t), \end{aligned} \tag{4.67}$$

where

$$F_n(t) = \frac{2}{d}\int_0^d \left(1-\frac{z}{d}\right) A\omega^2 \cos(\omega t)\sin\left(\frac{n\pi z}{d}\right) dz, \tag{4.68}$$

$$f_n = \frac{2}{d}\int_0^d \left[f_1(z) - \left(1-\frac{z}{d}\right)A\omega\right]\sin\left(\frac{n\pi z}{d}\right) dz. \tag{4.69}$$

Using the parameter estimates shown in Table 4.1, we can plot the velocity profile, given by (4.67), as a function of time for various sample depths, as shown in Figure 4.2. We can see that $v_2(z,t)$ is periodic in time, with period $2\pi/\omega$. The highest velocity is at $z = 0$ and steadily decreases over the sample before decaying to zero at $z = d$. This is as we would expect due to the restrictions we place at the boundaries. The first sum in (4.67) is transient in nature (and so negligible for large time) and the 'true' response to the driven oscillations is given by

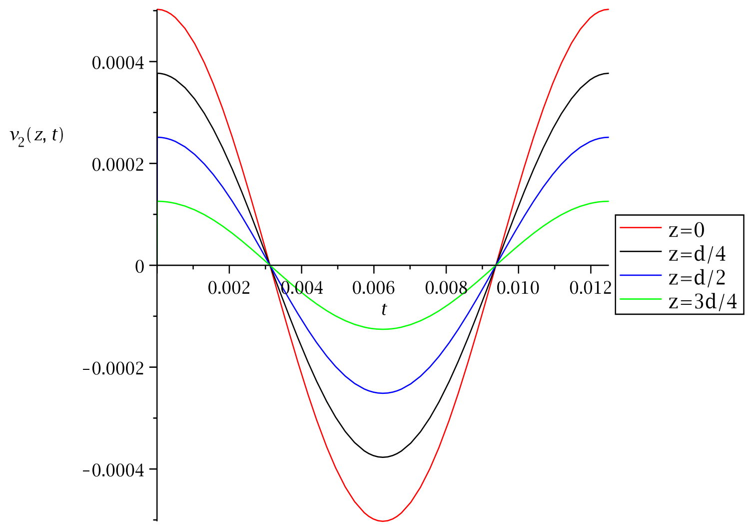


Figure 4.2: Velocity (all time) $v_2(z,t)$ plotted against time for various sample depths. The veloctiy $v_2(z,t)$ is measured in metres per second (ms^{-1}) and time is measured in seconds.

$$v_2(z,t) = \sum_{n=1}^{\infty}\left[\int_0^t e^{-\kappa(n\pi/d)^2(t-\tau)}F_n(\tau)d\tau\right]\sin\left(\frac{n\pi z}{d}\right) + \left(1-\frac{z}{d}\right)A\omega\cos(\omega t). \tag{4.70}$$

The transient response time can be calculated from the exponent of the exponential to yield

$$\tau_1 = \frac{1}{\kappa}\left(\frac{d}{\pi}\right)^2 \approx 3.17\times 10^{-7}\text{s}. \tag{4.71}$$

Force on the top plate

The reason for studying the oscillating plate is motivated by the experimental work of Jákli and Saupe [20]. In this paper, the authors discuss the effects of electric fields on SmA and SmC liquid crystals. By way of a device known as an

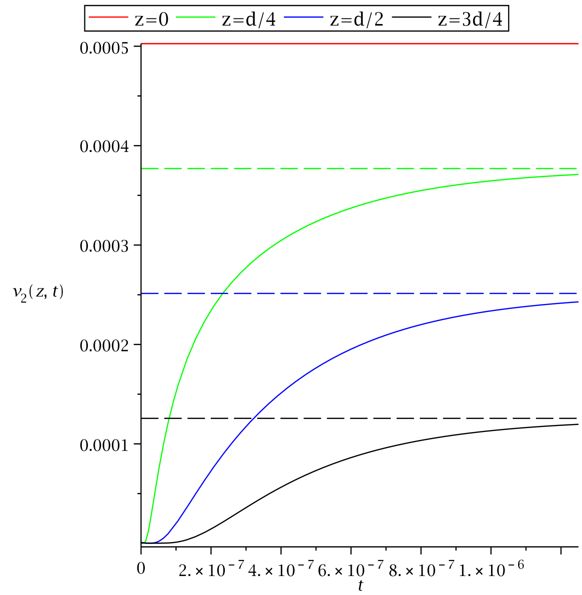


Figure 4.3: Velocity $v_2(z,t)$ plotted against time for various sample depths, comparing long time (dashed lines) and all time (solid lines) solutions. The veloctiy $v_2(z,t)$ is measured in ms^{-1} and time is measured in seconds.

accelerometer, which is placed on the top plate, one can measure the stress that these field effects cause on the glass plates that confine the sample, see Figure 4.4. Although we are not studying the effects of fields in this chapter, the same method for measuring the force on the plates can be used. In an experimental setting, the energy being put into the system would be known and so if we can mathematically compute the energy being exerted, then equate that with the data being produced by the accelerometer, we would theoretically have a way of measuring the material parameters present in the sample. To obtain the force exerted on the top plate of the sample we need to take the product of the stress tensor with the inward unit normal of the plate, i.e. $t_{ij}\nu_j$. The inward unit normal is $\boldsymbol{\nu} = (0,0,-1)$ and the stress tensor is given by

$$t_{ij} = -p\delta_{ij} + t^0_{ij} + \tilde{t}_{ij}, \tag{4.72}$$

where δ_{ij} is the standard Kronecker delta, $\tilde{t}_{ij}$ is the viscous stress tensor given by (4.7) and t^0_{ij} is given by

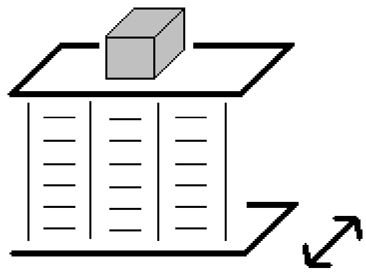


Figure 4.4: Experimental set-up of an accelerometer- a sensitive device for detecting accelerations (and thereby forces) that is placed on the top plate of a sample.

$$
\begin{aligned}
t^0_{ij} &= |\nabla\Phi| a_i J_j - \frac{\partial w}{\partial n_{p,j}} n_{p,i} - \frac{\partial w}{\partial a_{p,j}} a_{p,i} \\
&= a_i J_j - K^n_1 (n_{j,j})^2. \qquad (4.73)
\end{aligned}
$$

Obtaining t^0_{ij} explicitly for $i = 1, 2, 3$, yields

$$
t^0_{1j} = J_j, \quad t^0_{2j} = 0, \quad t^0_{3j} = 0. \qquad (4.74)
$$

We can calculate the flux term $\boldsymbol{J}$ from equation (4.20) explicitly for this problem, and so we have

$$
J_j = \frac{1}{|\nabla\Phi|}\left[\left(\frac{\partial w}{\partial a_{p,k}}\right)_{,k} - \frac{\partial w}{\partial a_p}\right](\delta_{pj} - a_p a_j) = 0, \qquad (4.75)
$$

since $\delta_{pj} - a_p a_j$ is only non-zero if $j = p \neq 1$, but $a_p = 0$ if $j = p \neq 1$. Hence we have shown that $t^0_{ij} = 0$ for $i = 1, 2, 3$. Since ν_1 and ν_2 are both zero, we only need to compute the $\tilde{t}_{i3}$ components, which are given by

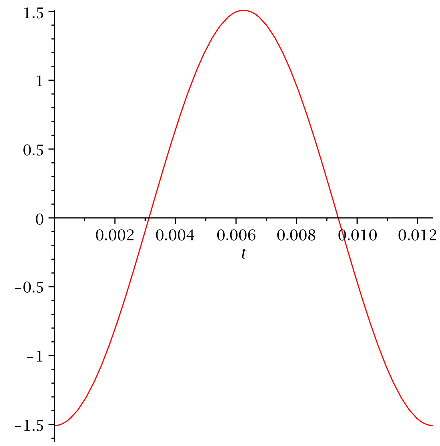


Figure 4.5: Force per unit area on top plate plotted against time.

$$\tilde{t}_{13} = \frac{1}{2}\alpha_4(v_{1,z} + v_{3,x}) + \frac{1}{2}\tau_2(v_{3,x} + v_{1,z}) = 0, \tag{4.76}$$

$$\tilde{t}_{23} = \frac{1}{2}\alpha_4 v_{2,z}, \tag{4.77}$$

$$\tilde{t}_{33} = \frac{1}{2}\alpha_4(v_{3,z} + v_{3,z}) = 0. \tag{4.78}$$

We see from (4.76)-(4.78) that the only non-zero component of the stress tensor is the $\tilde{t}_{23}$ component, which means that the only force being exerted on the top plate is in the y-direction. This is not surprising since the oscillations we are imposing on the bottom plate are in the y-direction only. From (4.67) we have an expression for v_2, and so inserting (4.67) into (4.77) we can obtain the force explicitly. Hence the only non-zero component of the stress tensor (4.72) is given by

$$t_{23} = \tilde{t}_{23} = \frac{1}{2}\alpha_4\frac{\partial v_2}{\partial z}. \tag{4.79}$$

Inserting the parameter values from Table 4.1, we can plot the surface force (4.79) as a function of time, as shown in Figure 4.5.

4.2.3 Director Profile

From the equation for the balance of angular momentum we derived equation (4.29) to model the director profile. Including the boundary and initial conditions we have

$$K_1^n \frac{\partial^2 \theta}{\partial z^2} - \gamma_1 \frac{\partial \theta}{\partial t} - B_1 \theta = 0, \tag{4.80}$$

$$\theta(0,t) = \theta_0, \tag{4.81}$$

$$\theta(d,t) = \theta_d, \tag{4.82}$$

$$\theta(z,0) = h(z). \tag{4.83}$$

We now solve equation (4.80) with boundary conditions (4.81) and (4.82), and initial condition (4.83), for the angle $\theta(z,t)$. We begin by transforming (4.80)-(4.83) into a problem with homogeneous boundary conditions by setting a new variable

$$\hat{\theta}(z,t) = \theta(z,t) - \left(1 - \frac{z}{d}\right)\theta_0 - \frac{z}{d}\theta_d. \tag{4.84}$$

So the differential equation (4.80) becomes

$$K_1^n \frac{\partial^2 \hat{\theta}}{\partial z^2} - \gamma_1 \frac{\partial \hat{\theta}}{\partial t} - B_1 \hat{\theta} = K_1^n F(z), \tag{4.85}$$

where

$$F(z) = \frac{B_1}{K_1^n}\left(\left(1 - \frac{z}{d}\right)\theta_0 + \frac{z}{d}\theta_d\right). \tag{4.86}$$

The boundary conditions are now given by $\hat{\theta}(0,t) = \hat{\theta}(d,t) = 0$, with initial condition $\hat{\theta}(z,0) = k(z)$. Making the substitution

$$\tau = \frac{K_1^n}{\gamma_1} t, \tag{4.87}$$

enables equation (4.85) to be written in the form

$$\frac{\partial \hat{\theta}}{\partial \tau} - \frac{\partial^2 \hat{\theta}}{\partial z^2} + c\hat{\theta} = F(z), \tag{4.88}$$

where we have set $c = \frac{B_1}{K_1^n}$. Now using the canonical transformation [49]

$$\hat{\theta}(z, \tau) = \Theta(z, \tau) e^{-c\tau}, \tag{4.89}$$

transforms equation (4.88) and its associated boundary and initial conditions to

$$\frac{\partial \Theta}{\partial \tau} - \frac{\partial^2 \Theta}{\partial z^2} = \hat{F}(z, \tau), \tag{4.90}$$

$$\Theta(0, \tau) = \Theta(d, \tau) = 0, \tag{4.91}$$

$$\Theta(z, 0) = \hat{k}(z), \tag{4.92}$$

where $\hat{F}(z, \tau) = F(z) e^{c\tau}$. Now we assume a solution of the form [9]

$$\Theta(z, \tau) = \sum_{n=1}^{\infty} \theta_n(\tau) \sin\left(\frac{n\pi z}{d}\right), \tag{4.93}$$

where $\theta_n(t)$ is an unknown function that we will solve for, and we write

$$\hat{F}(z, \tau) = \sum_{n=1}^{\infty} \hat{F}_n(\tau) \sin\left(\frac{n\pi z}{d}\right), \tag{4.94}$$

$$\hat{k}(z) = \sum_{n=1}^{\infty} f_n \sin\left(\frac{n\pi z}{d}\right). \tag{4.95}$$

The functions $\hat{F}_n(\tau)$ and f_n are again the standard Fourier coefficients. Substituting (4.93) into (4.90) allows this differential equation to be written as

$$\sum_{n=1}^{\infty}\left[\theta_n'(\tau)+\beta\theta_n(\tau)-\hat{F}_n(\tau)\right]\sin\left(\frac{n\pi z}{d}\right)=0, \tag{4.96}$$

where we have set $\beta=(n\pi/d)^2$, and the initial condition (4.92) becomes

$$\sum_{n=1}^{\infty}\left[\theta_n(0)-f_n\right]\sin\left(\frac{n\pi z}{d}\right)=0. \tag{4.97}$$

Since $\{\sin(n\pi z/d)\}$ is a complete orthogonal family, (4.96) and (4.97) are equivalent to [23], for $n=1,2,3,\ldots$,

$$\theta_n'(\tau)+\beta\theta_n(\tau)=\hat{F}_n(\tau), \tag{4.98}$$

$$\theta_n(0)=f_n. \tag{4.99}$$

Now taking Laplace Transforms of (4.98) and rearranging yields

$$\hat{\theta}_n(s)=\frac{f_n}{s+\beta}+\frac{\hat{\hat{F}}_n(s)}{s+\beta}, \tag{4.100}$$

where $\hat{\theta}_n(s)$ and $\hat{\hat{F}}_n(s)$ are the Laplacian transforms of $\theta_n(\tau)$ and $\hat{F}_n(\tau)$ respectively, and s is again the transform variable. Inverting $\hat{\theta}_n(s)$ using standard results produces the solution [37]

$$\theta_n(\tau)=f_ne^{-\beta\tau}+\int_0^{\tau}e^{-\beta(\tau-u)}\hat{F}_n(u)du. \tag{4.101}$$

Hence (4.93) becomes

$$\Theta(z,\tau) = \sum_{n=1}^{\infty} f_n e^{-\beta\tau} \sin\left(\frac{n\pi z}{d}\right) + \sum_{n=1}^{\infty} \left[\int_0^{\tau} e^{-\beta(\tau-u)} \hat{F}_n(u)du\right] \sin\left(\frac{n\pi z}{d}\right), \tag{4.102}$$

and so from (4.89) we obtain

$$\begin{aligned}
\hat{\theta}(z,\tau) =&\Theta(z,\tau)e^{-c\tau} \\
=&\sum_{n=1}^{\infty} f_n e^{-(\beta+c)\tau} \sin\left(\frac{n\pi z}{d}\right) + \sum_{n=1}^{\infty} e^{-(\beta+c)\tau} \left[\int_0^{\tau} e^{\beta u} \hat{F}_n(u)du\right] \sin\left(\frac{n\pi z}{d}\right) \\
=&\sum_{n=1}^{\infty} f_n e^{-(\beta+c)\tau} \sin\left(\frac{n\pi z}{d}\right) + \sum_{n=1}^{\infty} \frac{\hat{F}_n}{\beta+c}(1-e^{-(\beta+c)\tau}) \sin\left(\frac{n\pi z}{d}\right).
\end{aligned} \tag{4.103}$$

Finally, substituting back for the original variable t and original function $\theta(z,t)$, we obtain the solution

$$\begin{aligned}
\theta(z,t) =& \sum_{n=1}^{\infty} f_n \exp\left(-\frac{K_1^n}{\gamma_1}\left(\frac{B_1}{K_1^n} + \left(\frac{n\pi}{d}\right)^2\right)t\right) \sin\left(\frac{n\pi z}{d}\right) \\
&+ \sum_{n=1}^{\infty} \frac{\hat{F}_n}{\frac{B_1}{K_1^n} + (\frac{n\pi}{d})^2}\left[1 - \exp\left(-\frac{K_1^n}{\gamma_1}\left(\frac{B_1}{K_1^n} + \left(\frac{n\pi}{d}\right)^2\right)t\right)\right] \sin\left(\frac{n\pi z}{d}\right) \\
&+ \left(1-\frac{z}{d}\right)\theta_0 + \frac{z}{d}\theta_d.
\end{aligned} \tag{4.104}$$

We can see from Figures 4.6 and 4.7 that the director profile is close to zero in the bulk of the sample, with some oscillations occuring close to the boundaries as the director settles from its predetermined angle of $\pi/6$. This is intuitively what one would expect, with the oscillations settling to zero after a boundary layer of approximately one tenth of the sample.

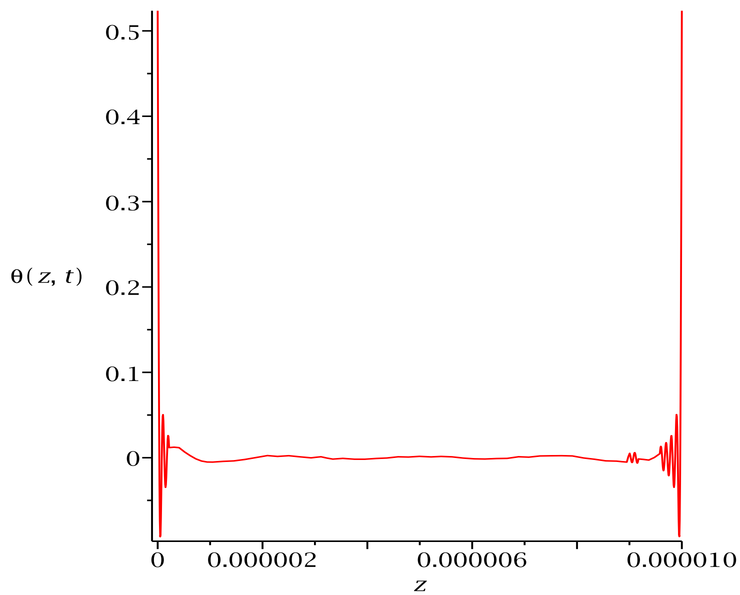


Figure 4.6: Director profile $\theta(z,t)$ plotted against sample depth between $z = 0$ and $z = d = 10^{-5}$ at time $t = 0.1$, with $\theta_0 = \pi/6$.

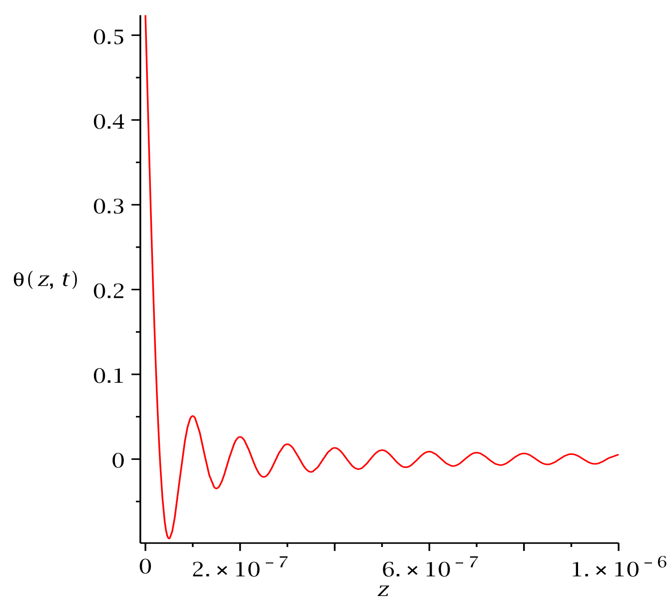


Figure 4.7: Director profile $\theta(z,t)$ plotted against sample depth between $z = 0$ and $z = 10^{-6}$, a tenth of the total sample depth, at time $t = 0.1$, with $\theta_0 = \pi/6$.

4.3 Semi-infinite Domain

Now we consider the same geometrical set-up as before but evaluate the problem over a semi-infinite domain. This means we are using the same governing equations as the previous section, given by (4.37) and (4.38), but here the boundary and initial values for the velocity equation are given simply by $v_2(0,t) = A\omega\cos\omega t$ and $v_2(z,0) = v_0 e^{-\lambda_1 z}$, $\lambda_1 > 0$, and $\theta(0,t) = \theta_0$ and $\theta(z,0) = \theta_0 e^{-\lambda_2 z}$, $\lambda_2 > 0$, for the θ equation. We have set the boundary and initial conditions on the velocity to be of these forms since we expect the velocity in the fluid to decrease, and eventually become zero, as the height, z, from the boundary increases. We also expect the director profile to settle to an equilibrium value far from the oscillating boundary. In Section 4.3.1, we solve the equations using analytical Fourier Sine Transforms and in Section 4.3.2 we use Discrete Fourier Transforms.

4.3.1 Analytical Velocity Profile

We consider the diffusion equation

$$\frac{\partial v_2}{\partial t} = \kappa \frac{\partial^2 v_2}{\partial z^2}, \tag{4.105}$$

for the y-direction velocity, where $\kappa = \alpha_4/2\rho$. Multiplying both sides of equation (4.105) by $\sqrt{\frac{2}{\pi}}\sin\alpha z$ then integrating with respect to z between 0 and ∞ yields

$$\sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial v_2}{\partial t} \sin\alpha z \, dz = \kappa \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial^2 v_2}{\partial z^2} \sin\alpha z \, dz. \tag{4.106}$$

The integral on the left hand side of (4.106) can be written as

$$\int_0^\infty \frac{\partial v_2}{\partial t} \sin\alpha z \, dz = \frac{\partial}{\partial t} \int_0^\infty v_2(z,t) \sin\alpha z \, dz, \tag{4.107}$$

and we can perform integration by parts on the integral on the right hand side of (4.106) to obtain

$$\int_0^\infty \frac{\partial^2 v_2}{\partial z^2} \sin \alpha z \, dz = \alpha \, v_2(0,t) - \alpha^2 \int_0^\infty v_2(z,t) \sin \alpha z \, dz. \tag{4.108}$$

When perfoming integration by parts in (4.108), there is also a term involving the velocity evaluated at infinity. However, we have excluded this term on physical grounds as we expect the velocity to be finite, and so we assume that the velocity tends to zero as z tends to infinity. Hence (4.106) becomes

$$\frac{\partial \bar{V}}{\partial t} + \kappa \alpha^2 \bar{V} = \sqrt{\frac{2}{\pi}} \kappa \omega \alpha A \cos \omega t, \tag{4.109}$$

where we have set $\bar{V} = \bar{V}(\alpha, t)$ to be the Fourier Sine Transform of $v_2(z,t)$, which is defined to be [46]

$$\bar{V}(\alpha, t) = \sqrt{\frac{2}{\pi}} \int_0^\infty v_2(z,t) \sin \alpha z \, dz. \tag{4.110}$$

Applying the Fourier Sine Transform to the initial condition yields

$$\begin{aligned} \bar{V}(\alpha, 0) &= \sqrt{\frac{2}{\pi}} \int_0^\infty v_2(z,0) \sin \alpha z \, dz \\ &= \sqrt{\frac{2}{\pi}} \int_0^\infty e^{-\lambda_1 z} \sin \alpha z \, dz \\ &= \sqrt{\frac{2}{\pi}} \frac{\alpha}{\lambda_1^2 + \alpha^2}. \end{aligned} \tag{4.111}$$

Using the integrating factor method, we can solve (4.109) for $\bar{V}(\alpha, t)$ to obtain the solution

$$\bar{V}(\alpha, t) = \bar{V}(\alpha, 0) e^{-\kappa \alpha^2 t} + \sqrt{\frac{2}{\pi}} \kappa \omega \alpha A \int_0^t e^{-\kappa \alpha^2 (\bar{t} - t)} \cos \omega \bar{t} \, d\bar{t}. \tag{4.112}$$

From Gradshteyn and Ryzhik [13, Eqn. (2.663.3)] we have

$$\int e^{ax}\cos(bx)dx = \frac{e^{ax}(a\cos(bx)+b\sin(bx))}{a^2+b^2}, \tag{4.113}$$

so the integral in (4.112) becomes

$$\begin{aligned}
&\int_0^t e^{\kappa\alpha^2(\bar{t}-t)}\cos(\omega\bar{t})\,d\bar{t}\\
=&e^{-\kappa\alpha^2 t}\left[\frac{e^{\kappa\alpha^2\bar{t}}(\kappa\alpha^2\cos\omega\bar{t}+\omega\sin\omega\bar{t})}{(\kappa\alpha^2)^2+\omega^2}\right]\Bigg|_{\bar{t}=0}^{\bar{t}=t}\\
=&e^{-\kappa\alpha^2 t}\left[\frac{e^{\kappa\alpha^2 t}(\kappa\alpha^2\cos\omega t+\omega\sin\omega t)-\kappa\alpha^2}{(\kappa\alpha^2)^2+\omega^2}\right].
\end{aligned} \tag{4.114}$$

Hence we can write (4.112) explicitly as

$$\begin{aligned}
\bar{V}(\alpha,t) = \bar{V}(\alpha,0)e^{-\kappa\alpha^2 t} &- \sqrt{\frac{2}{\pi}}\frac{\kappa^2\alpha^3\omega A e^{-\kappa\alpha^2 t}}{\kappa^2\alpha^4+\omega^2}\\
&+\sqrt{\frac{2}{\pi}}\frac{\kappa^2\alpha^3\omega A\cos\omega t}{\kappa^2\alpha^4+\omega^2}+\sqrt{\frac{2}{\pi}}\frac{\kappa\alpha\omega^2 A\sin\omega t}{\kappa^2\alpha^4+\omega^2}.
\end{aligned} \tag{4.115}$$

Equation (4.115) provides the transform of the original velocity $v_2(z,t)$. To transform back to z and t space we simply multiply $\bar{V}(\alpha,t)$ by $\sin\alpha z$ and integrate from 0 to ∞ with respect to α, i.e.

$$v_2(z,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \bar{V}(\alpha,t)\sin\alpha z\,d\alpha. \tag{4.116}$$

Hence the velocity solution is given by

$$v_2(z,t) = \sqrt{\frac{2}{\pi}}\int_0^\infty \left[\bar{V}(\alpha,0)e^{-\kappa\alpha^2 t} + B_2(t)e^{-\kappa\alpha^2 t}\frac{\alpha^3}{\alpha^4+c} + B_3(t)\frac{\alpha^3}{\alpha^4+c} + B_4(t)\frac{\alpha}{\alpha^4+c}\right]\sin\alpha z\,d\alpha, \tag{4.117}$$

where $B_2(t)$, $B_3(t)$, and $B_4(t)$ are functions of time that will be specified further on in this chapter. There are numerous integration methods that could be used to obtain a numerical solution for $v_2(z,t)$ given by (4.117), however, we would like to find an analytical solution for $v_2(z,t)$ that would give some insight into key aspects of the solution, such as the phase lag. Due to their simplicity, we first consider terms three and four before moving onto terms one and two. The third term in (4.115) is given by

$$f_3(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{\kappa^2\alpha^3\omega A\cos\omega t}{\kappa^2\alpha^4+\omega^2} = B_3(t)\frac{\alpha^3}{\alpha^4+c}, \tag{4.118}$$

where $B_3(t) = \sqrt{\frac{2}{\pi}}\,\omega A\cos\omega t$ and $c = (\omega/\kappa)^2$. To invert this term we need to multiply by $\sin\alpha z$ and integrate from 0 to ∞ with respect to α. So the integral we need to consider is

$$\begin{aligned} I_3 &= \sqrt{\frac{2}{\pi}}\int_0^\infty B_3(t)\frac{\alpha^3}{\alpha^4+c}\sin\alpha z\,d\alpha \\ &= \sqrt{\frac{\pi}{2}}B_3(t)\exp\left(\frac{-zc^{1/4}}{\sqrt{2}}\right)\cos\left(\frac{zc^{1/4}}{\sqrt{2}}\right) \\ &= A\omega\cos\omega t\exp\left(-\sqrt{\frac{\omega}{2\kappa}}z\right)\cos\left(\sqrt{\frac{\omega}{2\kappa}}z\right), \end{aligned} \tag{4.119}$$

where we have evaluated the integral using Gradshteyn and Ryzhik [13, Eqn. (3.727.10)]. The fourth term is given by

$$f_4(\alpha,t) = \sqrt{\frac{2}{\pi}}\frac{\kappa\alpha\omega^2 A\sin\omega t}{\kappa^2\alpha^4+\omega^2} = B_4(t)\frac{\alpha}{\alpha^4+c}, \tag{4.120}$$

where $B_4(t) = \sqrt{\frac{2}{\pi}}\,\frac{\omega^2}{\kappa}A\sin\omega t$ and $c = (\omega/\kappa)^2$. For this term the integral we need to consider is

$$
\begin{aligned}
I_4 &= \sqrt{\frac{2}{\pi}} \int_0^\infty B_4(t) \frac{\alpha}{\alpha^4 + c} \sin \alpha z \, d\alpha \\
&= \sqrt{\frac{\pi}{2}} \frac{1}{c^{1/2}} B_4(t) \exp\left(\frac{-zc^{1/4}}{\sqrt{2}}\right) \sin\left(\frac{zc^{1/4}}{\sqrt{2}}\right) \\
&= A\omega \sin \omega t \exp\left(-\sqrt{\frac{\omega}{2\kappa}} z\right) \sin\left(\sqrt{\frac{\omega}{2\kappa}} z\right),
\end{aligned} \tag{4.121}
$$

where we have evaluated the integral using Gradshteyn and Ryzhik [13, Eqn. (3.727.4)]. Next we evaluate the first term in (4.115), which is given by

$$
f_1(\alpha, t) = \bar{V}(\alpha, 0) e^{-\kappa \alpha^2 t}. \tag{4.122}
$$

From Haberman [14], we have

$$
\mathcal{F}_S^{-1}\left[\mathcal{F}_S[f(z)]\mathcal{F}_C[g(z)]\right] = \frac{1}{\pi} \int_0^\infty f(\bar{z})[g(z+\bar{z}) - g(z+\bar{z})] d\bar{z}, \tag{4.123}
$$

where $\mathcal{F}_S$ is the Fourier Sine Transform and $\mathcal{F}_C$ is the Fourier Cosine Transform [46], and we know that

$$
\mathcal{F}_C^{-1}\left[\frac{1}{\sqrt{2a}} e^{-z^2/4a}\right] = e^{-a\alpha^2}, \tag{4.124}
$$

for $a > 0$. Now we set $f(z) = v_2(z, 0)$, $g(z) = \frac{1}{\sqrt{2a}} e^{-z^2/4a}$, and $a = kt$ in (4.123), then inverting $f_1(\alpha, t)$ yields

$$
\begin{aligned}
I_1 = \mathcal{F}_S^{-1}[f_1(\alpha, t)] &= \mathcal{F}_S^{-1}\left[\mathcal{F}_S\left[v_2(z, 0)\right]\mathcal{F}_C\left[\frac{1}{\sqrt{2a}} e^{-\alpha^2/4a}\right]\right] \\
&= \frac{1}{\pi} \int_0^\infty \frac{v_2(\bar{z}, 0)}{\sqrt{2a}} [e^{-(z-\bar{z})^2/4a} - e^{-(z+\bar{z})^2/4a}] \, d\bar{z}.
\end{aligned} \tag{4.125}
$$

We have previously set $v_2(z, 0) = e^{-\lambda_1 z}$, so (4.125) becomes

$$
\begin{aligned}
I_1 =&\frac{1}{\pi}\int_0^\infty \frac{e^{-\lambda_1\bar{z}}}{\sqrt{2a}}[e^{-(z-\bar{z})^2/4a}-e^{-(z+\bar{z})^2/4a}]\,d\bar{z}\\
=&\frac{e^{\lambda_1^2 a}}{\sqrt{2\pi}}\bigg[-2\sinh(\lambda_1 z)-e^{-\lambda_1 z}\operatorname{erf}\left(\frac{2\lambda_1 a-z}{2\sqrt{a}}\right)+e^{\lambda_1 z}\operatorname{erf}\left(\frac{2\lambda_1 a+z}{2\sqrt{a}}\right)\bigg],
\end{aligned}
\tag{4.126}
$$

where erf is the error function given by

$$
\operatorname{erf}(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-\bar{x}^2}\,d\bar{x}. \tag{4.127}
$$

The second and final term, which is clearly part of the transient response, is given by

$$
f_2(\alpha,t)=-\sqrt{\frac{2}{\pi}}\frac{\kappa^2\alpha^3\omega A\cos\omega t e^{-\kappa\alpha^2 t}}{\kappa^2\alpha^4+\omega^2}=B_2(t)e^{-\kappa\alpha^2 t}\frac{\alpha^3}{\alpha^4+c}, \tag{4.128}
$$

where $B_2(t)=-\sqrt{\frac{2}{\pi}}\,\omega A$ and $c=(\omega/\kappa)^2$. Unlike the previous three terms, the inverse transform $\mathcal{F}^{-1}(f_2(\alpha,t))$ is not in any Fourier transform tables, and there is no obvious way to simplify it to something that is easily computable. We tried to use one of the formulae in Gradshteyn and Ryzhik [13] to perform an integration on the product $\frac{b_1\alpha^3}{b_2\alpha^4+b_3}e^{-at}\sin\alpha z$ but unfortunately, to the best of our knowledge, no such formula exists for an integral of this type. Since we have been unable to fully invert $\bar{V}(\alpha,t)$ analytically we will consider the long time solution only. When taking the limit as $t\to\infty$ in (4.117), terms one and two vanish and we are left with terms three and four only, both of which are invertible. Hence the long time velocity profile is given by

$$
\begin{aligned}
v_\infty(z,t)=A\omega\cos\omega t\exp\left(-\sqrt{\frac{\omega}{2\kappa}}z\right)\cos\left(\sqrt{\frac{\omega}{2\kappa}}z\right)\\
+A\omega\sin\omega t\exp\left(-\sqrt{\frac{\omega}{2\kappa}}z\right)\sin\left(\sqrt{\frac{\omega}{2\kappa}}z\right).
\end{aligned}
\tag{4.129}
$$

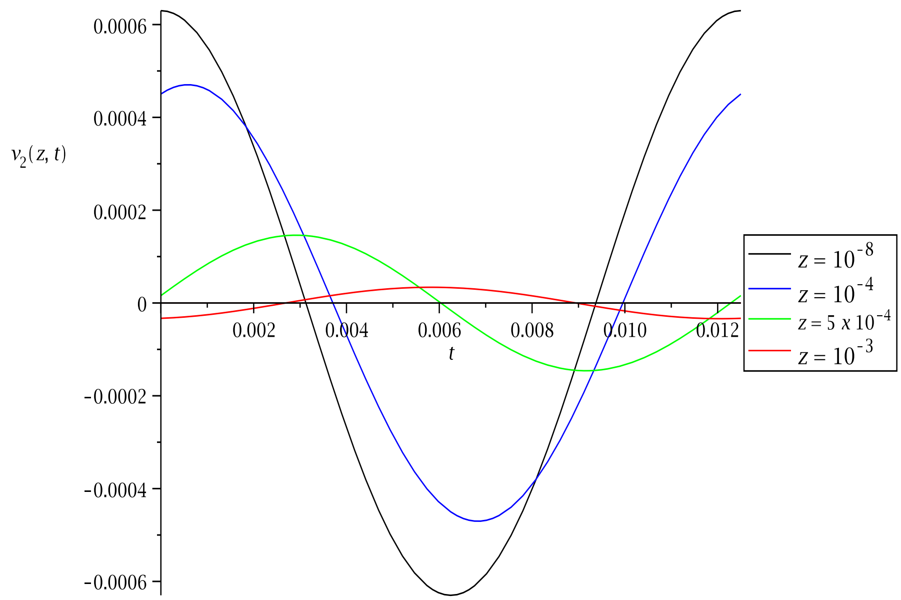


Figure 4.8: The velocity profile $v_2(z,t)$ plotted over the period $t = 2\pi/\omega$ for various sample depths, where the phase shift present in the solution can clearly be seen. The veloctiy $v_2(z,t)$ is measured in ms^{-1} and time is measured in seconds.

Using the trigonometric identity $A\cos(\omega t) + B\sin(\omega t) = \sqrt{A^2+B^2}\cos(\omega t - \tan^{-1}(B/A))$, the expression for $v_\infty(z,t)$ in (4.129) can be simplified to

$$v_\infty(z,t) = A\omega \exp\left(-\sqrt{\frac{\omega}{2\kappa}}z\right)\cos\left(\omega t - \sqrt{\frac{\omega}{2\kappa}}z\right). \tag{4.130}$$

From (4.130) we can see the penetration depth is given by $d_p = \sqrt{\frac{2\kappa}{\omega}}$ and the phase shift is given by $\Omega = \sqrt{\frac{\omega}{2\kappa}}z$. The phase shift can be seen very clearly in Figure 4.8, as well as the decay present in the sample over time. For the values used in Table 4.1, the penetration depth is given by $d_p \approx 2.6 \times 10^{-4}$m.

4.3.2 Numerically Derived Velocity Profile

Since we have been unable to fully invert the analytical solution for $v_2(z,t)$, we now aim to solve this problem discretely. We now use Discrete Fourier Transforms (DFTs) and follow the approach used by Le Bail [4] for diffusion problems, as described in Appendix B, to approximate our problem in a finite domain, then we extend it to a semi-infinite domain. Beginning from the system (4.53)-(4.55) we have

$$u_{,t} - \kappa u_{,zz} = F(z,t), \tag{4.131}$$

$$u(0,t) = u(d,t) = 0, \tag{4.132}$$

$$u(z,0) = u_0(z), \tag{4.133}$$

where $F(z,t) = (1 - z/d)A\omega^2 \sin \omega t$ and $u_0(z) = v_0(z) - (1 - z/d)A\omega$. We impose a grid split up into M points in the z-direction and N points in the t-direction, indexed with I and J respectively. The value for M is restricted to discrete values of the form $M = 2^Q$, where Q is a positive integer. Now we rescale the z and t variables by setting $z = \hat{z}d/M$ and $t = \hat{t}T/N$, so equation (4.131) becomes

$$\frac{\partial^2 u}{\partial \hat{z}^2} + a\frac{\partial u}{\partial \hat{t}} = F(\hat{z},\hat{t}), \tag{4.134}$$

where

$$F(\hat{z},\hat{t}) = -\left(1 - \frac{\hat{z}}{M}\right) A\omega^2 \sin\left(\frac{\omega T}{N}\hat{t}\right), \tag{4.135}$$

and

$$a = -\frac{Nd^2}{\kappa T M^2}. \tag{4.136}$$

Now we set

$$u(I,J) = \sum_{S=1}^{M-1} U_S(J) \sin\left(\frac{\pi IS}{M}\right), \tag{4.137}$$

$$F(I,J) = \sum_{S=1}^{M-1} F_S(J) \sin\left(\frac{\pi IS}{M}\right), \tag{4.138}$$

and following the same method as described in Appendix B, we arrive at the recurrence relation

$$U_S(J)\left[2\cos\left(\frac{\pi S}{M}\right) + E_1\right] + U_S(J-1)\left[2\cos\left(\frac{\pi S}{M}\right) + E_2\right] = [F_S(J) + F_S(J-1)](\Delta\hat{z})^2, \tag{4.139}$$

where we have defined E_1 and E_2 to be

$$E_1 = -2 + 2\left(\frac{Nd^2}{\kappa TM^2}\right)\left(\frac{\Delta\hat{z}}{\Delta\hat{t}}\right)\Delta\hat{z}, \tag{4.140}$$

$$E_2 = -2 - 2\left(\frac{Nd^2}{\kappa TM^2}\right)\left(\frac{\Delta\hat{z}}{\Delta\hat{t}}\right)\Delta\hat{z}. \tag{4.141}$$

We can write equation (4.139) in the form

$$U_S(J) + a_1 U_S(J-1) = a_2[F_S(J) + F_S(J-1)](\Delta\hat{z})^2, \tag{4.142}$$

where

$$a_1 = \frac{2\cos\left(\frac{\pi S}{M}\right) + E_2}{2\cos\left(\frac{\pi S}{M}\right) + E_1}, \tag{4.143}$$

$$a_2 = \frac{1}{2\cos\left(\frac{\pi S}{M}\right) + E_1}, \tag{4.144}$$

and we have set $\Delta\hat{z} = \Delta\hat{t} = 1$. Solving equation (4.142) for $U_S(J)$, following the method as described in Appendix B, yields the solution

$$U(S,J) = c_1(S)(-a_1)^J + \sum_{J_0=1}^{J} a_2(-a_1)^{J-J_0} F(S,J_0) + \sum_{J_0=1}^{J} a_2(-a_1)^{J-J_0} F(S,J_0-1), \tag{4.145}$$

where the function $c_1(S)$ is related to the initial condition and is defined to be

$$c_1(S) = \frac{2}{M} \sum_{I=1}^{M-1} U(I,0) \sin\left(\frac{\pi I S}{M}\right). \tag{4.146}$$

Substituting $U(S,J)$ from (4.145) into (4.137) provides the solution for the transformed homogeneous solution, so the solution to the original problem (4.39) is given by

$$v_2(I,J) = u(I,J) + \left(1 - \frac{I}{d}\right) A\omega \cos(\omega t). \tag{4.147}$$

Figure 4.9 shows the results of the discrete solution for a finite domain plotted against the exact solution found in Section 4.2.2. We can clearly see that the results from using the DFTs are remarkably accurate, especially since only 16 grid points were used. This method works perfectly for a finite domain sample but it is not possible to extend the method to a truly semi-infinite domain due to the nature of the sum in the solution. However, we can set d to be large, i.e. $d = 5$, which, in terms of a SmA liquid crystal sample of average depth 10μm, should be sufficiently large to replicate the behaviour of a semi-infinite sample. The results of the 'semi-infinite' domain are shown in Figure 4.10. We can see that the solution, for all z values other than zero, is not periodic in the interval $t \in [0, 2\pi/\omega]$, but if we allow the solution to evolve over time it attains periodicity. This is because the solution is made up of a complementary function and particular integral which are transient and oscillatory in nature, respectively.

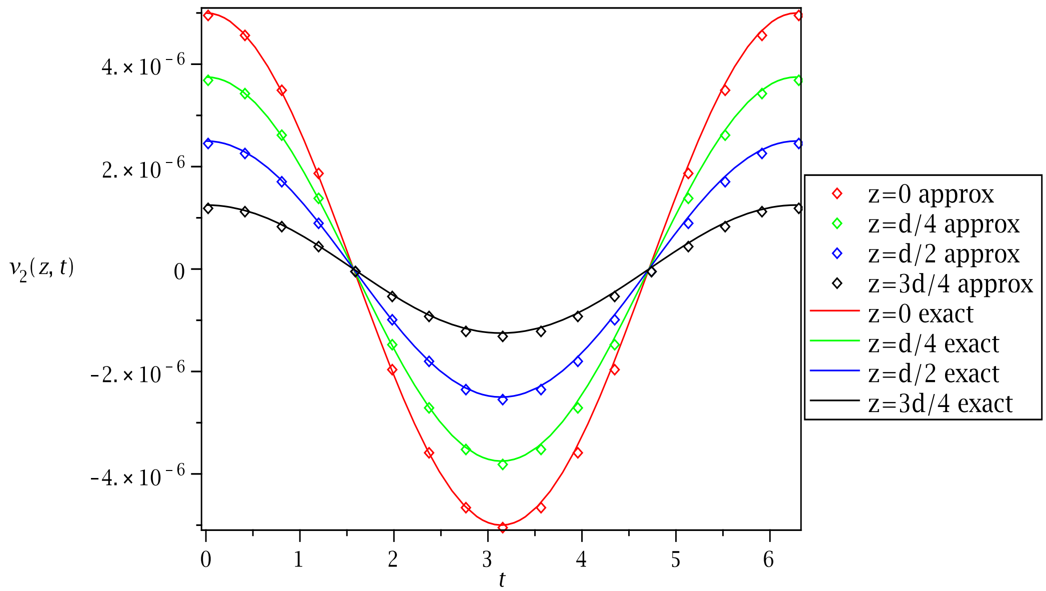


Figure 4.9: A comparison of the approximate DFT solution (4.147) and the exact solution (4.67) to the oscillating plate problem as described in (4.39). We have taken a finite domain of height $d = 10^{-5}$.

For small time, the complementary function dominates the behaviour and, clearly, it takes longer than one time period to diminish in stature. The particular integral part of the solution has the same period as the driven oscillation but is phase lagged: an effect that can clearly be seen in Figure 4.10.

4.3.3 Director Profile

Consider the equation

$$K_1^n \frac{\partial^2 \theta}{\partial z^2} - \gamma_1 \frac{\partial \theta}{\partial t} - B_1 \theta = 0, \tag{4.148}$$

with boundary condition $\theta(0,t) = \theta_0$ and initial condition $\theta(z,0) = \theta_0 e^{-\lambda_2 z}$, $\lambda_2 > 0$. Multiplying (4.148) throughout by $\sqrt{\frac{2}{\pi}} \sin \alpha z$ and integrating from 0 to ∞ with respect to z yields

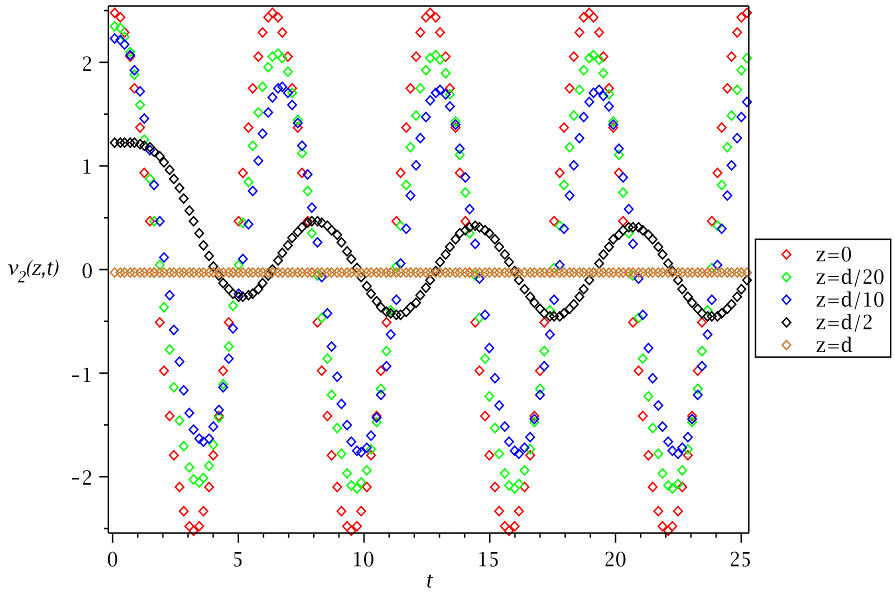


Figure 4.10: Approximate DFT solution (4.147) of the oscillating plate problem (4.39) for a 'semi-infinite' domain with $d = 5$.

$$\sqrt{\frac{2}{\pi}}\beta_1 \int_0^\infty \frac{\partial^2\theta}{\partial z^2} \sin\alpha z \, dz - \sqrt{\frac{2}{\pi}} \int_0^\infty \frac{\partial\theta}{\partial t} \sin\alpha z \, dz - \sqrt{\frac{2}{\pi}}\beta_2 \int_0^\infty \theta \sin\alpha z \, dz = 0, \tag{4.149}$$

where $\beta_1 = \frac{K_1^n}{\gamma_1}$ and $\beta_2 = \frac{B_1}{\gamma_1}$. Using integration by parts to simplify the first term in (4.149) allows us to express this equation as

$$\sqrt{\frac{2}{\pi}}\beta_1\alpha\theta_0 - \beta_1\alpha^2\Theta_S(\alpha,t) - \frac{\partial\Theta_S}{\partial t} - \beta_2\Theta_S(\alpha,t) = 0, \tag{4.150}$$

where we have set

$$\Theta_S(\alpha,t) = \sqrt{\frac{2}{\pi}} \int_0^\infty \theta(z,t) \sin\alpha z \, dz, \tag{4.151}$$

to be the Fourier Sine transform [46] of $\theta(z,t)$, with α as the transform variable. Using the integrating factor method, equation (4.150) can be written as

$$\frac{\partial}{\partial t}\left(\Theta_S(\alpha,t)e^{(\beta_2+\beta_1\alpha^2)t}\right) = \sqrt{\frac{2}{\pi}}\beta_1\alpha\theta_0 e^{(\beta_2+\beta_1\alpha^2)t}, \tag{4.152}$$

which has the solution

$$\Theta_S(\alpha,t) = \Theta_S(\alpha,0)e^{-(\beta_2+\beta_1\alpha^2)t} + \sqrt{\frac{2}{\pi}}\frac{\beta_1\alpha\theta_0}{(\beta_2+\beta_1\alpha^2)}[1-e^{-(\beta_2+\beta_1\alpha^2)t}]. \tag{4.153}$$

To obtain a solution in the original variables z and t, we must invert the solution $\Theta_S(\alpha,t)$ by multiplying (4.153) throughout by $\sin\alpha z$ and integrating with respect to α from 0 to ∞. First we consider the second term as this is the most straightforward to invert. The second term is given by

$$h_2 = \sqrt{\frac{2}{\pi}}\frac{\alpha\theta_0}{\alpha^2+\beta^2}, \tag{4.154}$$

where $\beta^2 = \frac{\beta_2}{\beta_1}$. Inverting h_2 requires us to compute

$$J_2 = \sqrt{\frac{2}{\pi}}\int_0^\infty \sqrt{\frac{2}{\pi}}\frac{\alpha\theta_0}{\alpha^2+\beta^2}\sin\alpha z\, d\alpha = \theta_0 e^{-\beta z}, \tag{4.155}$$

where we have evaluated the integral using Gradshteyn and Ryzhik [13, Eqn. (3.723.3)]. Next we will invert term 3, which is given by

$$h_3 = -C_3\frac{\alpha}{\alpha^2+\beta^2}e^{-\gamma\alpha^2}, \tag{4.156}$$

where $C_3 = \sqrt{\frac{2}{\pi}}\theta_0 e^{-\beta_2 t}$, $\beta^2 = \frac{\beta_2}{\beta_1}$ and $\gamma = \beta_1 t$. The integral that we need to consider is

$$
\begin{aligned}
J_3 &= \sqrt{\frac{2}{\pi}} \int_0^\infty C_3 \frac{\alpha}{\alpha^2+\beta^2} e^{-\gamma\alpha^2} \sin \alpha z \, d\alpha \\
&= \frac{\theta_0}{2} \left[2\sinh(\beta z) + e^{-\beta z} \operatorname{erf}\left(\beta\sqrt{\gamma} - \frac{z}{2\sqrt{\gamma}}\right) - e^{\beta z} \operatorname{erf}\left(\beta\sqrt{\gamma} + \frac{z}{2\sqrt{\gamma}}\right)\right],
\end{aligned}
\tag{4.157}
$$

where erf is the error function as defined in (4.127), and we have computed the integral in (4.157) from Gradshteyn and Ryzhik [13, Eqn. (3.954.1)]. The square bracket in (4.157) can be written as

$$
L_1 = 2\sinh(\beta z) - \left[e^{\beta z} \frac{2}{\sqrt{\pi}} \int_0^{\beta\sqrt{\gamma}+\frac{z}{2\sqrt{\gamma}}} e^{-\xi^2} d\xi - e^{-\beta z} \frac{2}{\sqrt{\pi}} \int_0^{\beta\sqrt{\gamma}-\frac{z}{2\sqrt{\gamma}}} e^{-\xi^2} d\xi \right].
\tag{4.158}
$$

For large time, $\gamma \to \infty$, so we have

$$
\begin{aligned}
L_1 &\approx 2\sinh(\beta z) - \left[e^{\beta z} \frac{2}{\sqrt{\pi}} \int_0^{\beta\sqrt{\gamma}} e^{-\xi^2} d\xi - e^{-\beta z} \frac{2}{\sqrt{\pi}} \int_0^{\beta\sqrt{\gamma}} e^{-\xi^2} d\xi \right] \\
&= 2\sinh(\beta z) - 2\sinh(\beta z) \frac{2}{\sqrt{\pi}} \int_0^{\beta\sqrt{\gamma}} e^{-\xi^2} d\xi.
\end{aligned}
\tag{4.159}
$$

The integral in (4.159) can be written as the sum of integrals

$$
\begin{aligned}
\int_0^{\beta\sqrt{\gamma}} e^{-\xi^2} d\xi &= \int_0^\infty e^{-\xi^2} d\xi - \int_{\beta\sqrt{\gamma}}^\infty e^{-\xi^2} d\xi \\
&= 1 - \operatorname{erf}_c(\beta\sqrt{\gamma}),
\end{aligned}
\tag{4.160}
$$

where $\operatorname{erf}_c(x)$ is the complementary error function defined to be

$$
\operatorname{erf}_c(x) = \int_x^\infty e^{-\xi^2} d\xi.
\tag{4.161}
$$

From Abramowitz and Stegun [1, Eqn. (7.1.23)], we have that for large x the complementary error function behaves like

$$\operatorname{erf}_c(x) \approx \frac{1}{x^2\sqrt{\pi}} e^{-x^2}, \tag{4.162}$$

and so for large time, i.e. large γ, equation (4.160) behaves like

$$\int_0^{\beta\sqrt{\gamma}} e^{-\xi^2} d\xi \approx 1 - \frac{1}{\beta^2\gamma\sqrt{\pi}} e^{-\beta^2\gamma}. \tag{4.163}$$

Hence we obtain

$$\begin{aligned} L_1 &\approx 2\sinh(\beta z) - 2\sinh(\beta z)\left[1 - \frac{1}{\beta^2\gamma\sqrt{\pi}} e^{-\beta^2\gamma}\right] \\ &= \frac{2\sinh(\beta z)}{\beta^2\gamma\sqrt{\pi}} e^{-\beta^2\gamma}. \end{aligned} \tag{4.164}$$

For large time, we can write the term J_3 as

$$\begin{aligned} J_3 &= \sqrt{\frac{\pi}{2}} \frac{C_3}{2} e^{\gamma\beta^2} L_1 \\ &\approx \sqrt{\frac{\pi}{2}} \frac{C_3}{2} e^{\gamma\beta^2} \frac{2\sinh(\beta z)}{\beta^2\gamma\sqrt{\pi}} e^{-\beta^2\gamma} \\ &= \frac{C_3 \sinh(\beta z)}{\beta^2\gamma\sqrt{2}}, \end{aligned} \tag{4.165}$$

and so we clearly have $J_3 \to 0$ as $t \to \infty$. Next we consider the first term, which is given by

$$h_1 = \Theta_S(\alpha, 0) e^{-\beta_2 t} e^{-\gamma\alpha^2}, \tag{4.166}$$

where $\Theta_S(\alpha,0)$ is the Fourier Sine Transform of the initial condition $\theta(z,0)$. We can treat $e^{-\beta_2 t}$ as a constant during the α integration in the transform, so we only need to consider

$$\mathcal{F}^{-1}(\Theta_S(\alpha,0)e^{-\gamma\alpha^2}). \tag{4.167}$$

From Haberman [14], we have

$$\mathcal{F}_S\left[\frac{1}{\pi}\int_0^\infty f(\bar{x})[g(x-\bar{x})-g(x+\bar{x})]d\bar{x}\right]=\mathcal{F}_S[f(x)]\mathcal{F}_C[g(x)]. \tag{4.168}$$

We know that $\mathcal{F}_C^{-1}\left[\frac{1}{\sqrt{2\gamma}}e^{-\alpha^2/4\gamma}\right]=e^{-\gamma z^2}$, for $\gamma>0$ and real, and we can write (4.168) as

$$\mathcal{F}_S^{-1}\left[\mathcal{F}_S[f(z)]\mathcal{F}_C[g(z)]\right]=\frac{1}{\pi}\int_0^\infty f(\bar{z})[g(z-\bar{z})-g(z+\bar{z})]d\bar{z}, \tag{4.169}$$

where $f(z)=\theta(z,0)$, $g(z)=\frac{1}{\sqrt{2\gamma}}e^{-z^2/4\gamma}$, and $\gamma=\beta_1 t$ in this problem. Using (4.169) allows us to write

$$\begin{aligned}
J_1 &= e^{-\beta_2 t}\mathcal{F}_S^{-1}\left[\mathcal{F}_S[f(z)]\mathcal{F}_C[g(z)]\right]\\
&= \frac{e^{-\beta_2 t}}{\pi}\int_0^\infty \frac{\theta(\bar{z},0)}{\sqrt{2\gamma}}\left[e^{-(z-\bar{z})^2/4\gamma}-e^{-(z+\bar{z})^2/4\gamma}\right]d\bar{z}\\
&= \frac{e^{-\beta_2 t}}{\pi}\int_0^\infty \frac{\theta_0 e^{-\lambda_2\bar{z}}}{\sqrt{2\gamma}}\left[e^{-(z-\bar{z})^2/4\gamma}-e^{-(z+\bar{z})^2/4\gamma}\right]d\bar{z}\\
&= \theta_0 e^{-(\beta_2-\lambda_2^2\beta_1)t}\left[-2\sinh(\lambda_2 z)-e^{-\lambda_2 z}\operatorname{erf}\left(\lambda_2\sqrt{\gamma}-\frac{z}{\sqrt{\gamma}}\right)+e^{\lambda_2 z}\operatorname{erf}\left(\lambda_2\sqrt{\gamma}+\frac{z}{\sqrt{\gamma}}\right)\right],
\end{aligned} \tag{4.170}$$

assuming $\theta(z,0)=\theta_0 e^{-\lambda_2 z}$. We can find the long time behaviour of J_1 by following a similar approach to the one considered for J_3, hence we can write for large time

$$J_1 \approx -\frac{\theta_0 e^{-(\beta_2-\lambda_2^2\beta_1)t}\sinh(\lambda_2 z)}{\beta_1 t}. \tag{4.171}$$

Hence, the full director profile is given by

$$\begin{aligned}\theta(z,t) =& J_1 + J_2 + J_3\\ =& \theta_0 e^{-(\beta_2-\lambda_2^2\beta_1)t}\Bigg[-2\sinh(\lambda_2 z) - e^{-\lambda_2 z}\operatorname{erf}\left(\lambda_2\sqrt{\beta_1 t} - \frac{z}{\sqrt{\beta_1 t}}\right) + e^{\lambda_2 z}\operatorname{erf}\left(\lambda_2\sqrt{\beta_1 t} + \frac{z}{\sqrt{\beta_1 t}}\right)\Bigg]\\ &+\theta_0 e^{-\beta z} + \frac{\theta_0}{2}\Bigg[2\sinh(\beta z) + e^{-\beta z}\operatorname{erf}\left(\beta\sqrt{\beta_1 t} - \frac{z}{2\sqrt{\beta_1 t}}\right) - e^{\beta z}\operatorname{erf}\left(\beta\sqrt{\beta_1 t} + \frac{z}{2\sqrt{\beta_1 t}}\right)\Bigg],\end{aligned} \tag{4.172}$$

where $\beta^2 = \frac{\beta_2}{\beta_1}$, and λ_2 is a positive constant. The director profile found in (4.172) is valid for $t > 0$ due to the constraint placed on the Fourier Cosine Transform. We can write the director in the alternative form

$$\theta(z,t) = \theta_0 e^{-\beta z} + \sinh(\lambda z)\, O\left(\frac{e^{-(\beta_2-\lambda_2^2\beta_1)t}}{t}\right) + \sinh(\beta z)\, O\left(\frac{1}{t}\right). \tag{4.173}$$

Hence, the long time solution is given by

$$\theta_\infty(z,t) = \theta_0 e^{-\beta z}. \tag{4.174}$$

Clearly (4.174) satisfies the original differential equation (4.148) and boundary condition given by $\theta(z,0) = \theta_0$. We can find the penetration depth of this solution by taking the reciprocal of the exponential exponent, hence the penetration depth is given by $d_p = 1/\beta = \sqrt{K_1^n/B_1} \approx 3.5 \times 10^{-10}$m.

4.4 Geometrical Set-up – No Transverse Flow

In the introduction to this section, we discussed the possibility for two different geometrical set-ups, depending on whether or not there was a chance of transverse flow. We considered the equations for the first case $\mathbf{n} = (1, 0, \theta(z,t))$ in Section 4.1, and now we consider the case when no transverse flow is assumed. The general set-up with the oscillating plate at $z = 0$ is exactly the same as Section 4.1, as is the orientation of the layer normal, with the only difference being the assumed orientation of the director. Hence the linearised set-up for this problem is

$$\mathbf{n} = (1, \theta(z,t), 0), \tag{4.175}$$

$$\mathbf{a} = (1, 0, 0), \tag{4.176}$$

$$\mathbf{v} = (0, v_2(z,t), v_3(z,t)), \tag{4.177}$$

$$\Phi = x, \tag{4.178}$$

where we have again assumed that there will be no flow through layers of the sample. Working through the equations for the balance of linear and angular momentum as before we arrive at the system of governing equations

$$0 = -\, p_{,x}, \tag{4.179}$$

$$\rho \frac{\partial v_2}{\partial t} = -\, p_{,y} + \frac{\alpha_4}{2} \frac{\partial^2 v_2}{\partial z^2}, \tag{4.180}$$

$$0 = -\, p_{,z}, \tag{4.181}$$

$$0 = \gamma_1 \frac{\partial \theta}{\partial t} + B_1 \theta. \tag{4.182}$$

Equations (4.179)–(4.181) are exactly the same as equations (4.30)–(4.32) from the previous section, and equation (4.182) only differs from (4.33) by a single term. The term $K_1^n \theta_{,zz}$ is excluded from the director profile equation in this set-up due to the form chosen for the director. This term comes directly from the $n_{i,i}$ term in the equation for the balance of linear momentum. Since the director is only dependent on z and t it is clearly impossible to pick up this term in the case of no transverse flow. Equation (4.182) can be solved to yield the solution

$$\theta(z,t) = c_1(z)e^{-\frac{B_1}{\gamma_1}t}, \tag{4.183}$$

where $c_1(z)$ in an arbitrary function in z. The decay rate of this solution is given by $\kappa = \gamma_1/B_1 \approx 2 \times 10^{-9}\mathrm{s}^{-1}$, which was a feature that we did not observe in the case where transverse flow may occur. In that case, we found the penetration depth to be $d_p = \sqrt{K_1^n/B_1}$, but here we have discovered unknown information about the decay time instead. We note here that the decay rate, κ, found in the assumption of no transverse flow is the same decay rate as found in Stewart [48] in the case of planar aligned SmA.

We stated at the beginning of this section that both geometrical set-ups are plausible and now we see that, in the linearised case at least, both set-ups are governed by almost identical equations of motion.

4.5 Nonlinear Equations

So far in this chapter we have only been concerned with the linear equations of the system but for completeness we now state the fully nonlinear equations. In the assumption of transverse flow we set the director to be $\mathbf{n} = (\cos\theta(z,t), 0, \sin\theta(z,t))$, with $\mathbf{a}$, $\mathbf{v}$ and Φ as before. Working through the equations for the balance of linear and angular momentum, we arrive at the nonlinear governing equations given by

$$\begin{aligned}
0 &= -\, p_{,x}, && (4.184)\\
\rho\frac{\partial v_2}{\partial t} &= -\, p_{,y} + \frac{\alpha_4}{2}\frac{\partial^2 v_2}{\partial z^2}, && (4.185)\\
0 &= -\, p_{,z} - K_1^n \cos\theta\frac{\partial\theta}{\partial z}\Big[\cos\theta\frac{\partial^2\theta}{\partial z^2} - \sin\theta\Big(\frac{\partial\theta}{\partial z}\Big)^2\Big] + B_0(\cos\theta - 1)\sin\theta\frac{\partial\theta}{\partial z} \\
&\quad - B_1\sin\theta\cos\theta\frac{\partial\theta}{\partial z} - \gamma_1\frac{\partial\theta}{\partial z}\frac{\partial\theta}{\partial t}, && (4.186)
\end{aligned}$$

$$(B_1+\mu)\cos\theta - B_0(\cos\theta-1) + \gamma_1 \sin\theta\frac{\partial\theta}{\partial t} = 0, \tag{4.187}$$

$$\frac{1}{2}(\gamma_1-\gamma_2)\frac{\partial v_2}{\partial z}\sin\theta = 0, \tag{4.188}$$

$$\mu\sin\theta + K_1^n\Big(\cos\theta\frac{\partial^2\theta}{\partial z^2} - \sin\theta\frac{\partial\theta}{\partial z}\Big) - \gamma_1\frac{\partial\theta}{\partial t}\cos\theta = 0, \tag{4.189}$$

where

$$\mu = -K_1^n\sin\theta\Big(\cos^2\theta\frac{\partial^2\theta}{\partial z^2} - \sin\theta\frac{\partial\theta}{\partial z}\Big) - B_1\cos^2\theta + B_0(\cos\theta-1)\cos\theta. \tag{4.190}$$

From equation (4.187) we require that either $\frac{\partial v_2}{\partial z} = 0$ or $\sin\theta = 0$. Clearly, from the nature of the oscillating plate problem, we cannot have the velocity purely as a function of time, it must also depend on the distance from the top plate, so we need $\sin\theta = 0$. Hence, in the simplest case, we get $\theta \equiv 0$ and therefore $\mathbf{n} = (1,0,0)$. Since $\mathbf{n}$ is forced to be of this specific form, it follows that the equation for the balance of angular momentum vanishes and we have a constant director profile. This is similar to work carried out by Stewart [49, §5.5.1]. In the assumption that there is no transverse flow, we obtain the following nonlinear system of equations

$$0 = -\,p_{,x}, \tag{4.191}$$

$$\rho\frac{\partial v_2}{\partial t} = -\,p_{,y} + \frac{\alpha_4}{2}\frac{\partial^2 v_2}{\partial z^2}, \tag{4.192}$$

$$0 = -\,p_{,z} + B_0(\cos\theta-1)\sin\theta\frac{\partial\theta}{\partial z} - B_1\cos\theta\sin\theta\frac{\partial\theta}{\partial z}, \tag{4.193}$$

$$(B_1+\mu)\cos\theta - B_0(\cos\theta-1) + \gamma_1\sin\theta\frac{\partial\theta}{\partial t} = 0, \tag{4.194}$$

$$\mu\sin\theta - \gamma_1\cos\theta\frac{\partial\theta}{\partial t} = 0, \tag{4.195}$$

where $\mu = B_0(\cos\theta - 1)\cos\theta - B_1\cos^2\theta$. In this case we are not forced into making the director constant since the equation that caused a problem in the previous case is now automatically satisfied.

4.6 Conclusions and Discussion

When deriving the equations, we found that in the bookshelf geometry the governing equations for the velocity and director profile were a decoupled system. This was in direct contrast to the planar aligned geometry in which the velocity played a role in the director alignment [48]. Clearly the type of geometry being studied is a key factor on the interactions between the director and velocity in the liquid crystal. However, there are some similarities between the two geometries. For example, when considering the case of no transverse flow we obtained the same decay rate, $\kappa = \gamma_1/B_1$, as is observed in the planar case.

For the semi-infinite case, we solved the problem using Discrete Fourier Transforms (DFTs) in a pseudo-semi-infinite domain. The difficulty in analytically inverting the Fourier Transform meant we were unable to obtain a fully analytical profile, but the DFT approximation showed that as we move further away from the oscillating plate, the velocity tends to zero as we would expect.

Chapter 5

Oscillatory Shear Flow in Smectic C Liquid Crystals

We consider the method of the generalised separation of variables, as detailed in Polyanin [36], to first solve the classical backflow equations in nematics, then adapt the method to solve an oscillating flow problem for a SmC liquid crystal as proposed by Moir [33]. We begin by briefly deriving a special case for nematics before moving on to the smectic case.

5.1 Backflow Equations in Nematics

When considering the dynamics of the the Freedericksz Transition in the splay geometry, Pierianski *et al.* [35] found that the case of the planar to homeotropic transition involves a 'backflow effect'. Flow in a nematic liquid crystal can have dramatic effects on the orientation of the director, with the realignment of the director itself inducing flow in certain cases. In particular, it has been shown that the flow induced at the onset of the Freedericksz transition in the splay and bend geometries actually enhances the switch on time since the flow lowers the effective viscosity of the liquid crystal [49]. This type of flow is called 'backflow'. We consider backflow in the planar to homeotropic transition splay geometry, as described in Stewart [49, pg. 224] which has the geometrical set-up given by

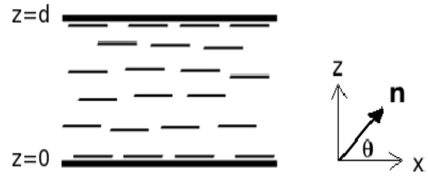


Figure 5.1: A sample of nematic liquid crystal bounded between two parallel plates a distance d apart, where θ denotes the angle that the director makes with the x-axis.

$$\mathbf{n} = (\cos\theta(z,t), 0, \sin\theta(z,t)), \tag{5.1}$$

$$\mathbf{v} = (v(z,t), 0, 0), \tag{5.2}$$

and it can be shown [49] that the linearised governing equations for the problem are given by

$$\xi^2\theta_{,zz} + \theta - \lambda\theta_{,t} - \lambda_1 v_{,z} = 0, \tag{5.3}$$

$$\eta_1 v_{,zz} + \alpha_3\theta_{,zt} = 0, \tag{5.4}$$

where

$$\xi^2 = \frac{K_1}{\chi_a H^2}, \quad \lambda = \frac{\gamma_1}{\chi_a H^2}, \quad \lambda_1 = \frac{\alpha_3}{\chi_a H^2}, \tag{5.5}$$

and $\eta_1 = \frac{1}{2}(\alpha_3 + \alpha_4 + \alpha_6)$ is a viscosity. Solutions to equations (5.3) and (5.4) were stated by Pierianski *et al.* [35] but without a rigorous derivation. Here, for completeness and to highlight the generalised separation of variables technique, which is introduced in Appendix A, we will solve these particular backflow equations from first principles. We find that solving equations (5.3) and (5.4) using the standard separation of variables technique results in finding the zero solution for both unknown functions, so we immediately have to adapt our approach to involve the generalised separation of variables. First we set

$$\theta(z,t) = \varphi_1(z)\psi_1(t) + \chi_1(t), \tag{5.6}$$
$$v(z,t) = \varphi_2(z)\psi_2(t) + \chi_2(t), \tag{5.7}$$

then we can write equations (5.3) and (5.4) in the form

$$\xi^2\varphi_1''\psi_1 + \varphi_1\psi_1 + \chi_1 - \lambda\varphi_1\dot{\psi}_1 - \lambda\dot{\chi}_1 - \lambda_1\varphi_2'\psi_2 = 0, \tag{5.8}$$
$$\eta_1\varphi_2''\psi_2 + \alpha_3\varphi_1'\dot{\psi}_1 = 0, \tag{5.9}$$

where the dot represents differentiation with respect to time and the dash represents differentiation with respect to space. Collecting terms in z and t dependencies allows us to write these equations in the bilinear form

$$\Phi_1(z)\Psi_1(t) + \Phi_2(z)\Psi_2(t) + \Phi_3(z)\Psi_3(t) + \Phi_4(z)\Psi_4(t) = 0, \tag{5.10}$$
$$\Phi_5(z)\Psi_5(t) + \Phi_6(z)\Psi_6(t) = 0, \tag{5.11}$$

where we have set

$$\Phi_1(z) = \varphi_1 + \xi^2\varphi_1'', \qquad \Psi_1(t) = \psi_1, \tag{5.12}$$
$$\Phi_2(z) = \varphi_1, \qquad \Psi_2(t) = \psi_1 - \lambda\dot{\psi}_1, \tag{5.13}$$
$$\Phi_3(z) = \varphi_2', \qquad \Psi_3(t) = -\lambda_1\psi_2, \tag{5.14}$$
$$\Phi_4(z) = 1, \qquad \Psi_4(t) = \chi_1 - \lambda\dot{\chi}_1, \tag{5.15}$$
$$\Phi_5(z) = \varphi_2'', \qquad \Psi_5(t) = \eta_1\psi_2, \tag{5.16}$$
$$\Phi_6(z) = \varphi_1', \qquad \Psi_6(t) = \alpha_3\dot{\psi}_1. \tag{5.17}$$

In general, the bilinear equation

$$\Phi_1(z)\Psi_1(t) + \Phi_2(z)\Psi_2(t) + \cdots + \Phi_k(z)\Psi_k(t) = 0, \tag{5.18}$$

has $k-1$ solutions of the form [36]

$$\Phi_i(z) = C_{i,1}\Phi_{m+1}(z) + C_{i,2}\Phi_{m+2}(z) + \cdots + C_{i,k-m}\Phi_k(z), \tag{5.19}$$
$$\Psi_{m+j}(t) = -C_{1,j}\Psi_1(t) - C_{2,j}\Psi_2(t)\cdots - C_{m,j}\Psi_m(t), \tag{5.20}$$

for $i = 1, ..., m,\ j = 1, ..., k-m$ and $m = 1, 2, ..., k-1$, and where the $C_{i,j}$ are arbitrary constants. To solve equations (5.10) and (5.11), we choose solutions of the form

$$\Phi_1 = A_1\Phi_3 + A_2\Phi_4, \tag{5.21}$$
$$\Phi_2 = A_3\Phi_3 + A_4\Phi_4, \tag{5.22}$$
$$\Phi_5 = A_5\Phi_6, \tag{5.23}$$
$$\Psi_3 = -A_1\Psi_1 - A_3\Psi_2, \tag{5.24}$$
$$\Psi_4 = -A_2\Psi_1 - A_4\Psi_2, \tag{5.25}$$
$$\Psi_6 = -A_5\Psi_5, \tag{5.26}$$

where the A_i are arbitrary constants. Substituting the expressions for Φ_i and Ψ_i into these equations yields the system of ordinary differential equations

$$\varphi_1 + \xi^2\varphi_1'' = A_1\varphi_2' + A_2, \tag{5.27}$$
$$\varphi_1 = A_3\varphi_2' + A_4, \tag{5.28}$$
$$\varphi_2'' = A_5\varphi_1', \tag{5.29}$$
$$-\lambda_1\psi_2 = -A_1\psi_1 - A_3(\psi_1 - \lambda\dot{\psi}_1), \tag{5.30}$$
$$\chi_1 - \lambda\dot{\chi}_1 = -A_2\psi_1 - A_4(\psi_1 - \lambda\dot{\psi}_1), \tag{5.31}$$
$$\alpha_3\dot{\psi}_1 = -A_5\eta_1\psi_2. \tag{5.32}$$

Rearranging (5.28) to obtain an expression for φ_2', then substituting into (5.27) yields

$$\xi^2\varphi_1'' + \left(1 - \frac{A_1}{A_3}\right)\varphi_1 = A_2 - \frac{A_1 A_4}{A_3}. \tag{5.33}$$

This provides an ordinary differential equation which can be solved to find φ_1. To find φ_2 explicitly, we can substitute φ_1 into equation (5.28) to obtain the differential equation

$$\varphi_2' = \frac{\varphi_1 - A_4}{A_3}. \tag{5.34}$$

Next, we can obtain an expression for ψ_2 from (5.32), and then substitute this into (5.30) to obtain a single equation in ψ_1, namely

$$\dot{\psi}_1 - \frac{(A_1 + A_3)A_5\eta_1}{\eta_1\lambda A_3 A_5 - \lambda_1\alpha_3}\psi_1 = 0. \tag{5.35}$$

We can solve this equation for ψ_1, then substitute the resulting function into (5.32) to obtain the following equation for ψ_2

$$-A_5\eta_1\psi_2 = \alpha_3\dot{\psi}_1 = \frac{\alpha_3 A_5\eta_1(A_1 + A_3)}{\eta_1\lambda A_3 A_5 - \lambda_1\alpha_3}\psi_1. \tag{5.36}$$

We can also substitute ψ_1 into (5.31) to obtain an equation in χ_1, hence we have to solve

$$\lambda\dot{\chi}_1 - \chi_1 = (A_2 + A_4)\psi_1 - A_4\lambda\dot{\psi}_1. \tag{5.37}$$

In summary, we reduce the set of equations (5.27)-(5.32) to the following five equations

$$\varphi_1'' + b_1\varphi_1 = b_2, \tag{5.38}$$
$$\varphi_2' = b_3\varphi_1 + b_4, \tag{5.39}$$
$$\dot{\psi}_1 + b_5\psi_1 = 0, \tag{5.40}$$
$$\psi_2 = b_6\psi_1, \tag{5.41}$$
$$\dot{\chi}_1 + b_7\chi_1 = b_8\psi_1 + b_9\dot{\psi}_1, \tag{5.42}$$

where the b_i are constants dependent on material parameters and the constants A_i. The unknown functions effectively have now been separated ready for solution. Solving equations (5.38)-(5.42) yields the general solutions

$$\varphi_1(z) = c_1\cos(\sqrt{b_1}z) + c_2\sin(\sqrt{b_1}z) + \frac{b_2}{b_1}, \tag{5.43}$$
$$\varphi_2(z) = c_1\frac{b_3}{\sqrt{b_1}}\sin(\sqrt{b_1}z) - c_2\frac{b_3}{\sqrt{b_1}}\cos(\sqrt{b_1}z) + \left(\frac{b_2b_3 - b_1b_4}{b_1}\right)z + c_3, \tag{5.44}$$
$$\psi_1(t) = c_4e^{-b_5t}, \tag{5.45}$$
$$\psi_2(t) = c_4b_6e^{-b_5t}, \tag{5.46}$$
$$\chi_1(t) = c_4\left(\frac{b_5b_9 - b_8}{b_5 - b_7}\right)e^{-b_5t} + c_5e^{-b_7t}, \tag{5.47}$$

where the c_i are constants of integration. Since equations (5.3) and (5.4) only contain z derivatives of the velocity, it follows that the arbitrary function $\chi_2(t)$ has not been determined. So, without loss of generality, we choose to set $\chi_2(t) = 0$ in the absence of any criteria to do otherwise. Now, the director profile is given by

$$\begin{aligned}\theta(z,t) &= \varphi_1(z)\psi_1(t) + \chi_1(t)\\ &= \left[c_1\cos(\sqrt{b_1}z) + c_2\sin(\sqrt{b_1}z) + c_6\right]c_4e^{-b_5t} + c_5e^{-b_7t},\end{aligned} \tag{5.48}$$

and the velocity profile is represented by

$$
\begin{aligned}
v(z,t) =&\varphi_2(z)\psi_2(t) + \chi_2(t)\\
=&\left[c_1\frac{b_3}{\sqrt{b_1}}\sin(\sqrt{b_1}z) - c_2\frac{b_3}{\sqrt{b_1}}\cos(\sqrt{b_1}z) + \left(\frac{b_2b_3+b_1b_4}{b_1}\right)z + c_3\right]c_4b_6e^{-b_5t},
\end{aligned}
\tag{5.49}
$$

where we have set

$$
c_6 = \frac{b_2}{b_1} + \left(\frac{b_5b_9 - b_8}{b_5 - b_7}\right). \tag{5.50}
$$

The boundary conditions are given by $\theta(\pm d/2,t) = 0$ and $v(\pm d/2,t) = 0$. Since we need these boundary conditions to hold for all time, from (5.48) we deduce that $c_5 = 0$. We can write the director boundary conditions as $\varphi_1(\pm d/2)\psi_1(t)+\chi_1(t) = 0$, for all time, hence we require

$$
c_1\cos\left(\pm\frac{d}{2}\sqrt{b_1}\right) + c_2\sin\left(\pm\frac{d}{2}\sqrt{b_1}\right) + c_6 = 0, \tag{5.51}
$$

which can be simplified to

$$
\sqrt{c_1^2+c_2^2}\cos\left(\pm\frac{d}{2}\sqrt{b_1} - \tan^{-1}\left(\frac{c_2}{c_1}\right)\right) + c_6 = 0, \tag{5.52}
$$

using a trigonometric identity. For this relation to hold for both $z = d/2$ and $z = -d/2$ we require $\tan^{-1}(c_2/c_1) = 0$, which implies that $c_2 = 0$. Setting $c_2 = 0$ in the expression for $\theta(z,t)$ then inserting the boundary condition again yields

$$
c_6 = -c_1\cos\left(\frac{d}{2}\sqrt{b_1}\right), \tag{5.53}
$$

which allows us to write the function $\theta(z,t)$ in the form

$$\theta(z,t) = c_1 c_4 \Big[\cos(\sqrt{b_1}z) - \cos\Big(\frac{d}{2}\sqrt{b_1}\Big)\Big] e^{-b_5 t}. \tag{5.54}$$

Similarly, we have $v(\pm d/2, t) = 0$ so we require $\varphi_2(\pm d/2) = 0$. This can be written explicitly as

$$c_1 \frac{b_3}{\sqrt{b_1}} \sin\Big(\frac{d}{2}\sqrt{b_1}\Big) + \Big(\frac{b_2 b_3 - b_1 b_4}{b_1 b_4}\Big)\frac{d}{2} + c_3 = 0, \tag{5.55}$$

$$c_1 \frac{b_3}{\sqrt{b_1}} \sin\Big(-\frac{d}{2}\sqrt{b_1}\Big) - \Big(\frac{b_2 b_3 - b_1 b_4}{b_1 b_4}\Big)\frac{d}{2} + c_3 = 0. \tag{5.56}$$

Now, adding (5.55) to (5.56) gives the constant in the solution to be zero, i.e. $c_3 = 0$. Setting $c_3 = 0$ in the function $\varphi_2(z)$ then inserting the boundary condition again yields

$$\frac{b_2 b_3 - b_1 b_4}{b_1 b_4} = -c_1 \frac{b_3}{\sqrt{b_1}} \frac{2}{d} \sin\Big(\frac{d}{2}\sqrt{b_1}\Big). \tag{5.57}$$

Hence we can write the velocity profile as

$$v(z,t) = \frac{2}{d}\frac{c_1 c_4 b_3}{\sqrt{b_1}} \Big[\frac{d}{2}\sin(\sqrt{b_1}z) - \sin\Big(\frac{d}{2}\sqrt{b_1}\Big) z\Big] e^{-b_5 t}. \tag{5.58}$$

Now we relabel the constants in (5.54) and (5.58) by setting

$$\theta_0 = c_1 c_4, \tag{5.59}$$

$$v_0 = \frac{2}{d}\frac{c_1 c_4 b_3}{\sqrt{b_1}}, \tag{5.60}$$

$$b_5 = -\frac{1}{\tau}, \tag{5.61}$$

$$\sqrt{b_1} = \frac{2q}{d}, \tag{5.62}$$

and so we can write the solutions as

$$\theta(z,t) = \theta_0 \left[\cos\left(\frac{2q}{d}z\right) - \cos q \right] e^{t/\tau}, \tag{5.63}$$

$$v(z,t) = v_0 \left[\frac{d}{2} \sin\left(\frac{2q}{d}z\right) - z \sin q \right] e^{t/\tau}, \tag{5.64}$$

which are the exact solutions as stated by Pierianski *et al.* [35]. The values θ_0 and v_0 are constants, and q and τ are to be determined.

Equations (5.63) and (5.64) have been exploited extensively in the literature [49] but not established by any means other than ansatzes. Inserting (5.63) and (5.64) into equation (5.4) yields a relationship between the constants v_0 and θ_0, namely

$$v_0 = -\frac{\theta_0 \alpha_3}{q \eta_1 \tau}. \tag{5.65}$$

It can also be found that the values of q are restricted such that $q_0 < q < q_1$ where q_0 and q_1 are the roots of

$$\tan(q) - \frac{q}{1-\alpha} = 0, \tag{5.66}$$

$$\tan(q) - q = 0, \tag{5.67}$$

respectively, where $\alpha = 1 - \frac{\alpha_1^2}{\gamma_1 \eta_1}$. A full derivation from equation (5.63) onwards can be found in Stewart [49], with a more detailed analysis of the backflow equations.

5.2 Smectic C Equations

Now we examine work that was originally proposed in the thesis of Moir [33]. We consider the case of a planar aligned SmC liquid crystal, confined between two parallel plates a distance d apart, under the influence of an oscillating bound-

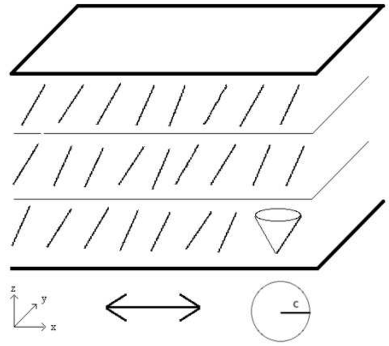


Figure 5.2: Finite sample of SmC liquid crystal with the director prealignment parallel to the plate oscillations.

ary. We consider the two separate cases of parallel and perpendicular director alignment on the boundaries.

5.2.1 Parallel Prealignment

We first consider the case of a SmC liquid crystal orientated with parallel prealignment, as shown in Figure 5.2. In the thesis of Moir [33], it was stated, without proof, that since the y-direction velocity $v_2(z,t)$ and director alignment $\theta(z,t)$ were both set to be zero on the top and bottom plate that the solutions in fact remained zero throughout the sample. The thesis of Stewart [48] used Fourier Transforms to establish this result. However, this method was complicated to use and required some very tedious and extensive working. Here we use generalised separation of variables to prove that both the velocity and director fields are identically zero throughout the sample. These equations have been derived by Moir and we simply state them here. We begin with the linearised governing equations of the system which are given by [33]

$$\eta_1 v_{2,zz} - \rho v_{2,t} - \tau\theta_{,zt} = 0, \tag{5.68}$$

$$K\theta_{,zz} - \lambda\theta_{,t} + \tau v_{2,z} = 0, \tag{5.69}$$

where we have set

$$K = K_2^c, \quad \lambda = 2\lambda_5, \quad \tau = \tau_5 - \tau_1, \tag{5.70}$$

with boundary conditions $v_2(0,t) = v_2(d,t) = 0$ and $\theta(0,t) = \theta(d,t) = 0$. We now apply the technique for the nematic equations from the previous section in order to derive solutions in the smectic case. First, we set

$$\theta(z,t) = \varphi_1(z)\psi_1(t) + \chi_1(t), \tag{5.71}$$

$$v_2(z,t) = \varphi_2(z)\psi_2(t) + \chi_2(t), \tag{5.72}$$

then substituting these forms into equations (5.68) and (5.69) yields

$$\eta_1\varphi_2''\psi_2 - \rho\varphi_2\dot{\psi}_2 - \rho\dot{\chi}_2 - \tau\varphi_1'\dot{\psi}_1 = 0, \tag{5.73}$$

$$K\varphi_1''\psi_1 - \lambda\varphi_1\dot{\psi}_1 - \lambda\dot{\chi}_1 + \tau\varphi_2'\psi_2 = 0. \tag{5.74}$$

Collecting terms in z and t dependencies allows us to write these equations in the bilinear form

$$\Phi_1(z)\Psi_1(t) + \Phi_2(z)\Psi_2(t) + \Phi_3(z)\Psi_3(t) + \Phi_4(z)\Psi_4(t) = 0, \tag{5.75}$$

$$\Phi_5(z)\Psi_5(t) + \Phi_6(z)\Psi_6(t) + \Phi_7(z)\Psi_7(t) + \Phi_8(z)\Psi_8(t) = 0, \tag{5.76}$$

where we have set

$$\Phi_1(z) = \eta_1 \varphi_2'', \qquad \Psi_1(t) = \psi_2, \tag{5.77}$$
$$\Phi_2(z) = -\rho\varphi_2, \qquad \Psi_2(t) = \dot{\psi}_2, \tag{5.78}$$
$$\Phi_3(z) = -\rho, \qquad \Psi_3(t) = \dot{\chi}_2, \tag{5.79}$$
$$\Phi_4(z) = -\tau\varphi_1', \qquad \Psi_4(t) = \dot{\psi}_1, \tag{5.80}$$
$$\Phi_5(z) = K\varphi_1'', \qquad \Psi_5(t) = \psi_1, \tag{5.81}$$
$$\Phi_6(z) = -\lambda\varphi_1, \qquad \Psi_6(t) = \dot{\psi}_1, \tag{5.82}$$
$$\Phi_7(z) = -\lambda, \qquad \Psi_7(t) = \dot{\chi}_1, \tag{5.83}$$
$$\Phi_8(z) = \tau\varphi_2', \qquad \Psi_8(t) = \psi_2. \tag{5.84}$$

Equations (5.75) and (5.76) have solutions of the form [36]

$$\begin{aligned}
\Phi_1 &= A_1\Phi_3 + A_2\Phi_4, & \Psi_3 &= -A_1\Psi_1 - A_3\Psi_2,\\
\Phi_2 &= A_3\Phi_3 + A_4\Phi_4, & \Psi_4 &= -A_2\Psi_1 - A_4\Psi_2,\\
\Phi_5 &= A_5\Phi_7 + A_6\Phi_8, & \Psi_7 &= -A_5\Psi_5 - A_7\Psi_6,\\
\Phi_6 &= A_7\Phi_7 + A_8\Phi_8, & \Psi_8 &= -A_6\Psi_5 - A_8\Psi_6,
\end{aligned} \tag{5.85}$$

where the A_i are again arbitrary constants. Substituting the expressions for Φ_i and Ψ_i from (5.77)-(5.84) into equations (5.85) yields a system of ordinary differential equations given by

$$\eta_1 \varphi_2'' = -A_1 \rho - A_2 \tau \varphi_1', \tag{5.86}$$

$$-\rho \varphi_2 = -A_3 \rho - A_4 \tau \varphi_1', \tag{5.87}$$

$$K \varphi_1'' = -A_5 \lambda + A_6 \tau \varphi_2', \tag{5.88}$$

$$-\lambda \varphi_1 = -A_7 \lambda + A_8 \tau \varphi_2', \tag{5.89}$$

$$\dot{\chi}_2 = -A_1 \psi_2 - A_3 \dot{\psi}_2, \tag{5.90}$$

$$\dot{\psi}_1 = -A_2 \psi_2 - A_4 \dot{\psi}_2, \tag{5.91}$$

$$\dot{\chi}_1 = -A_5 \psi_1 - A_7 \dot{\psi}_1, \tag{5.92}$$

$$\psi_2 = -A_6 \psi_1 - A_8 \dot{\psi}_1. \tag{5.93}$$

Rearranging these equations, and solving in the correct order, provides us with six ordinary differential equations to solve for the six unknown functions. Hence, we require to solve

$$\varphi_2'' + \hat{b}_1 \varphi_2 = \hat{b}_2, \tag{5.94}$$

$$\varphi_1' = \hat{b}_3 \varphi_2 + \hat{b}_4, \tag{5.95}$$

$$\ddot{\psi}_1 + \hat{b}_5 \dot{\psi}_1 + \hat{b}_6 \psi_1 = 0, \tag{5.96}$$

$$\psi_2 = \hat{b}_7 \psi_1 + \hat{b}_8 \dot{\psi}_1, \tag{5.97}$$

$$\dot{\chi}_1 = \hat{b}_9 \psi_1 + \hat{b}_{10} \dot{\psi}_1, \tag{5.98}$$

$$\dot{\chi}_2 = \hat{b}_{11} \psi_1 + \hat{b}_{12} \dot{\psi}_1, \tag{5.99}$$

where the $\hat{b}_i$ are positive constants depending on the A_i and material parameters. In general, the signs of the $\hat{b}_i$ are arbitrary but due to the nature of the boundary conditions in this case, we have selected them to be positive, without loss of generality. Solving these equations, and dropping the hats for convenience, yields the solutions

$$\varphi_1(z) = c_1\frac{b_3}{\sqrt{b_1}}\sin(\sqrt{b_1}z) - c_2\frac{b_3}{\sqrt{b_1}}\cos(\sqrt{b_1}z) + b_4 z + c_3, \tag{5.100}$$

$$\varphi_2(z) = c_1\cos(\sqrt{b_1}z) + c_2\sin(\sqrt{b_1}z) + \frac{b_2}{b_1}, \tag{5.101}$$

$$\psi_1(t) = c_4 e^{-\delta_1 t} + c_5 e^{-\delta_2 t}, \tag{5.102}$$

$$\psi_2(t) = c_4(b_7 - b_8\delta_1)e^{-\delta_1 t} + c_5(b_7 - b_8\delta_2)e^{-\delta_2 t}, \tag{5.103}$$

$$\chi_1(t) = \frac{c_4}{\delta_1}(b_{10}\delta_1 - b_9)e^{-\delta_1 t} + \frac{c_5}{\delta_2}(b_{10}\delta_2 - b_9)e^{-\delta_2 t} + c_6, \tag{5.104}$$

$$\chi_2(t) = \frac{c_4}{\delta_1}[\delta_1(b_7 b_{12} + b_8 b_{12}\sqrt{b_5^2 - 4b_6} + b_{11}b_8) - 2b_{11}b_7]e^{-\delta_1 t} \tag{5.105}$$

$$+ \frac{c_5}{\delta_2}[\delta_2(b_7 b_{12} + b_8 b_{12}\sqrt{b_5^2 - 4b_6} + b_{11}b_8) - 2b_{11}b_7]e^{-\delta_2 t} + c_7, \tag{5.106}$$

where the c_i are constants of integration, and δ_1 and δ_2 are given by

$$\delta_1 = \frac{1}{2}\left(b_5 - \sqrt{b_5^2 - 4b_6}\right), \tag{5.107}$$

$$\delta_2 = \frac{1}{2}\left(b_5 + \sqrt{b_5^2 - 4b_6}\right). \tag{5.108}$$

From the boundary conditions we have that $\theta(0, t) = 0$, and so applying this condition to the expression for $\theta(z, t)$, given in (5.71), yields

$$\left(-c_2\frac{b_3}{\sqrt{b_1}} + c_3\right)(c_4 e^{-\delta_1 t} + c_5 e^{-\delta_2 t}) + \frac{c_4}{\delta_1}(b_{10}\delta_1 - b_9)e^{-\delta_1 t} + \frac{c_5}{\delta_2}(b_{10}\delta_2 - b_9)e^{-\delta_2 t} + c_6 = 0, \tag{5.109}$$

which can be simplified to

$$c_4\left[-c_2\frac{b_3}{\sqrt{b_1}} + c_3\frac{1}{\delta_1}(b_{10}\delta_1 - b_9)\right]e^{-\delta_1 t} + c_5\left[-c_2\frac{b_3}{\sqrt{b_1}} + c_3\frac{1}{\delta_2}(b_{10}\delta_2 - b_9)\right]e^{-\delta_2 t} + c_6 = 0. \tag{5.110}$$

Since the functions in (5.110) are linearly independent, each coefficient must be equal to zero, so we have that $c_6 = 0$ and

$$-c_2\frac{b_3}{\sqrt{b_1}} + c_3\frac{1}{\delta_1}(b_{10}\delta_1 - b_9) = 0, \tag{5.111}$$

$$-c_2\frac{b_3}{\sqrt{b_1}} + c_3\frac{1}{\delta_2}(b_{10}\delta_2 - b_9) = 0. \tag{5.112}$$

For equations (5.111) and (5.112) to both be simultaneously true, we require $\delta_1 = \delta_2$ for all values of b_9 and b_{10}, which further implies that $\sqrt{b_5^2 - 4b_6} = 0$. Setting $\delta_1 = \delta_2$ above reduces the time dependent functions to

$$\psi_1(t) = (c_4 + c_5)e^{-\delta_1 t}, \tag{5.113}$$

$$\psi_2(t) = (c_4 + c_5)(b_7 - b_8\delta_1)e^{-\delta_1 t}, \tag{5.114}$$

$$\chi_1(t) = \frac{(c_4 + c_5)}{\delta_1}(b_{10}\delta_1 - b_9)e^{-\delta_1 t}, \tag{5.115}$$

$$\chi_2(t) = \frac{(c_4 + c_5)}{\delta_1}[\delta_1(b_7 b_{12} + b_{11}b_8) - 2b_{11}b_7]e^{-\delta_1 t} + c_7. \tag{5.116}$$

Now applying the zero initial condition for the director profile yields

$$(c_4 + c_5)\left[c_1\frac{b_3}{\sqrt{b_1}}\sin(\sqrt{b_1}z) - c_2\frac{b_3}{\sqrt{b_1}}\cos(\sqrt{b_1}z) + b_4 z + c_3 + \frac{1}{\delta_1}(b_{10}\delta_1 - b_9)\right] = 0. \tag{5.117}$$

We require (5.117) to be true for all z and the only way we can guarantee this is if the first bracketed term equals zero, i.e. $c_4 + c_5 = 0$. Clearly if we set $c_4 = -c_5$ in (5.113)-(5.116) we obtain $v_2(z,t) \equiv 0$ and $\theta(z,t) \equiv 0$, as stated in Moir [33]. We have therefore rigorously confirmed the result that was merely stated by Moir.

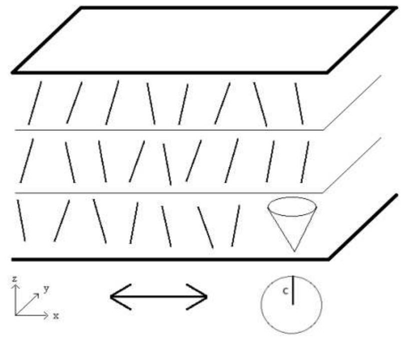


Figure 5.3: Finite sample of SmC liquid crystal with the director prealignment perpendicular to the plate oscillations.

5.2.2 Perpendicular Prealignment

Now we consider the case of perpendicular director prealignment to the oscillating plate, as shown in Figure 5.3. Here we aim to solve for the director alignment $\theta(z,t)$ and the x-direction velocity $v_1(z,t)$. This geometrical set-up has been partially solved for an asymptotic solution in the thesis of Moir [33], but we will attempt to find a full analytical solution by using the generalised separation of variables technique.

5.2.2.1 Inhomogeneous Boundary Conditions

The governing equations for this geometry are given by [33]

$$\eta_1 v_{1,zz} - \rho v_{1,t} + \tau\theta_{,zt} = 0, \tag{5.118}$$

$$K\theta_{,zz} - \lambda\theta_{,t} - \tau v_{1,z} = 0, \tag{5.119}$$

where K, λ and τ are as described in (5.70), and the boundary conditions are given by $v_1(0,t) = 0$, $v_1(d,t) = -\alpha\omega\sin(\omega t)$ and $\theta(0,t) = \theta(d,t) = \pi/2$. Since the equations for $v_1(z,t)$ and $\theta(z,t)$ in this set-up are very similar to the equations for $v_2(z,t)$ and $\theta(z,t)$ in Section 5.2.1, we obtain the same system of ordinary differential equations as in the parallel alignment but with different arbitrary constants given by the $\bar{b}_i$. Hence, for perpendicular alignment we have to solve

$$\varphi_1'' + \bar{b}_1\varphi_1 = \bar{b}_2, \tag{5.120}$$
$$\varphi_2' = \bar{b}_3\varphi_1 + \bar{b}_4, \tag{5.121}$$
$$\ddot{\psi}_1 + \bar{b}_5\dot{\psi}_1 + \bar{b}_6\psi_1 = 0, \tag{5.122}$$
$$\psi_2 = \bar{b}_7\psi_1 + \bar{b}_8\dot{\psi}_1, \tag{5.123}$$
$$\dot{\chi}_1 = \bar{b}_9\psi_1 + \bar{b}_{10}\dot{\psi}_1, \tag{5.124}$$
$$\dot{\chi}_2 = \bar{b}_{11}\psi_1 + \bar{b}_{12}\dot{\psi}_1, \tag{5.125}$$

with boundary conditions as stated previously. Writing the boundary conditions in terms of the functions φ_i, ψ_i and χ_i, for $i = 1, 2$, we obtain

$$\varphi_1(0)\psi_1(t) + \chi_1(t) = \pi/2, \tag{5.126}$$
$$\varphi_1(d)\psi_1(t) + \chi_1(t) = \pi/2, \tag{5.127}$$
$$\varphi_2(0)\psi_2(t) + \chi_2(t) = 0, \tag{5.128}$$
$$\varphi_2(d)\psi_2(t) + \chi_2(t) = i\omega\alpha e^{i\omega t}. \tag{5.129}$$

Now solving the system of differential equations, and dropping the bars for convenience, yields the same solutions of the previous section, namely

$$\varphi_1(z) = c_1\cos(\sqrt{b_1}z) + c_2\sin(\sqrt{b_1}z) + \frac{b_2}{b_1}, \tag{5.130}$$
$$\varphi_2(z) = c_1\frac{b_3}{\sqrt{b_1}}\sin(\sqrt{b_1}z) - c_2\frac{b_3}{\sqrt{b_1}}\cos(\sqrt{b_1}z) + b_4z + c_3, \tag{5.131}$$
$$\psi_1(t) = c_4e^{-\delta_1 t} + c_5e^{-\delta_2 t}, \tag{5.132}$$
$$\psi_2(t) = c_4(b_7 - b_8\delta_1)e^{-\delta_1 t} + c_5(b_7 - b_8\delta_2)e^{-\delta_2 t}, \tag{5.133}$$
$$\chi_1(t) = \frac{c_4}{\delta_1}(b_{10}\delta_1 - b_9)e^{-\delta_1 t} + \frac{c_5}{\delta_2}(b_{10}\delta_2 - b_9)e^{-\delta_2 t} + c_6, \tag{5.134}$$
$$\chi_2(t) = \frac{c_4}{\delta_1}[\delta_1(b_7b_{12} + b_8b_{12}\sqrt{b_5^2 - 4b_6} + b_{11}b_8) - 2b_{11}b_7]e^{-\delta_1 t} \tag{5.135}$$
$$+ \frac{c_5}{\delta_2}[\delta_2(b_7b_{12} + b_8b_{12}\sqrt{b_5^2 - 4b_6} + b_{11}b_8) - 2b_{11}b_7]e^{-\delta_2 t} + c_7, \tag{5.136}$$

where the c_i are arbitrary constants and δ_1 and δ_2 are again given by

$$\delta_1 = \frac{1}{2}\left(b_5 - \sqrt{b_5^2 - 4b_6}\right), \tag{5.137}$$

$$\delta_2 = \frac{1}{2}\left(b_5 + \sqrt{b_5^2 - 4b_6}\right). \tag{5.138}$$

Applying the boundary conditions to these solutions allows us to write the director profile as

$$\theta(z,t) = \hat{c}e^{i\omega t}\sqrt{c_1^2 + c_2^2}\left[\cos\left(\sqrt{b_1}z - \tan^{-1}\left(\frac{c_2}{c_1}\right)\right) - \cos\left(\tan^{-1}\left(\frac{c_2}{c_1}\right)\right)\right] + \frac{\pi}{2}. \tag{5.139}$$

and the velocity profile in the form

$$\begin{aligned} v_1(z,t) = i\omega b_8\hat{c}e^{i\omega t}\Bigg[&-\frac{b_3}{\sqrt{b_1}}\sqrt{c_1^2 + c_2^2}\cos\left(\sqrt{b_1}z + \tan^{-1}\left(\frac{c_1}{c_2}\right)\right) + c_2\frac{b_3}{\sqrt{b_1}} \\ &+ z\left(\frac{\alpha}{db_8\hat{c}} - c_1\frac{b_3}{\sqrt{b_1}d}\sin(\sqrt{b_1}d) + c_2\frac{b_3}{\sqrt{b_1}d}\cos(\sqrt{b_1}d) - c_2\frac{b_3}{\sqrt{b_1}d}\right)\Bigg]. \end{aligned} \tag{5.140}$$

Now, in a similar way to Section 5.1, we set new constants q_1 and q_2 to be

$$q_1 = \frac{d}{2}\sqrt{b_1}, \tag{5.141}$$

$$q_2 = \tan^{-1}\left(\frac{c_1}{c_2}\right), \tag{5.142}$$

which allows us to write $\sqrt{c_1^2 + c_2^2} = c_2\sec(q_2)$ and

$$\tan^{-1}\left(\frac{c_2}{c_1}\right) = \tan^{-1}(\cot(q_2)) \tag{5.143}$$

$$= \frac{\pi}{2} - q_2. \tag{5.144}$$

Hence the solutions for the velocity and director become

$$v_1(z,t) = \frac{i\omega d}{2q_1} ABe^{i\omega t}\Big[\cos(q_2) - \cos\left(\frac{2q_1 z}{d} + q_2\right)\Big]$$
$$+ z\Big(\frac{\alpha q_1}{dAB} - \frac{1}{2}\sin(2q_1) - \frac{1}{2}\cos(q_2) + \frac{1}{2}\cos(2q_1)\cos(q_2)\Big), \tag{5.145}$$

$$\theta(z,t) = Ae^{i\omega t}\Big[\sin\left(\frac{2q_1 z}{d} + q_2\right) - \sin(q_2)\Big] + \frac{\pi}{2}, \tag{5.146}$$

where we have set $A = \hat{c}c_2\sec(q_2)$ and $B = b_8 b_3$. Now inserting $v_1(z,t)$ and $\theta(z,t)$ into equations (5.118) and (5.119), and collecting terms, yields the compatibility conditions

$$f_1\cos\left(\frac{2q_1 z}{d}\right) + f_2\sin\left(\frac{2q_1 z}{d}\right) + f_3 z + f_4\cos(q_2) = 0, \tag{5.147}$$

$$f_5\cos\left(\frac{2q_1 z}{d}\right) + f_6\sin\left(\frac{2q_1 z}{d}\right) + f_7\sin(q_2) = 0, \tag{5.148}$$

where the f_i are complex functions involving material parameters and the unknown constants in the solutions. For the above equations to hold, we require each of the functions f_i to be equal to zero independently since the z-dependent functions and constants are all linearly independent. Setting each f_i to be zero and working through the algebra, we return the condition $\cos(q_2) = 0$. However, the constant multiplying both θ and v_1 involves the term $\sec(q_2)$, so clearly these particular solutions do not satisfy the original differential equations. To try to overcome this problem, we set $\omega = i\delta + \beta$. In this case, we determine that $q_2 = n\pi$, which is compatible with the solutions since we are no longer dividing by zero. However, after some manipulation we also find that we require $\beta = 0$ to satisfy one of the conditions. Due to the nature of the boundary conditions, we cannot have ω as a purely imaginary quantity as it leads to an inconsistency. It is possible that some other constants in the solution are complex. However, upon inspection, we find that setting b_3, b_8 or q_1 to be a complex constant always results in the requirement of one of the key constants of the solution being zero. Even combinations of real and imaginary constants does not seem to overcome the problem. One possible reason for finding functions $\theta(z,t)$ and $v_1(z,t)$ that are not solutions to the original system of differential equations is that solutions (5.85) of the bilinear equation (5.75) are not unique, as the general solutions are given by (5.19) and (5.20). We therefore deploy the generalised separation of variables technique again and look at the consequences. It will turn out that there will be unexpected problems which lead to an unsatisfactory position, and the problem will not be fully resolved here.

We have three possible solutions to equations (5.19) and (5.20): the first solution (Solution 1), as we used previously, is

$$
\begin{aligned}
\Phi_1 &= A_1\Phi_3 + A_2\Phi_4, \\
\Phi_2 &= A_3\Phi_3 + A_4\Phi_4, \\
\Psi_3 &= -A_1\Psi_1 - A_3\Psi_2, \\
\Psi_4 &= -A_2\Psi_1 - A_4\Psi_2.
\end{aligned}
\tag{5.149}
$$

The second solution (Solution 2) is given by

$$
\begin{aligned}
\Phi_4 &= -A_1\Phi_1 - A_2\Phi_2 - A_3\Phi_3, \\
\Psi_1 &= A_1\Psi_4, \\
\Psi_2 &= A_2\Psi_4, \\
\Psi_3 &= A_3\Psi_4,
\end{aligned}
\tag{5.150}
$$

and the third solution (Solution 3) may be represented by

$$\begin{aligned}
\Phi_1 &= A_1\Phi_4,\\
\Phi_2 &= A_2\Phi_4,\\
\Phi_3 &= A_3\Phi_4,\\
\Psi_4 &= -A_1\Psi_1 - A_2\Psi_2 - A_3\Psi_3.
\end{aligned} \tag{5.151}$$

Using solutions (5.149) or (5.150) to solve both bilinear equations (5.75) and (5.76) is an option, but we are unable to use solution (5.151) to solve both equations as we would have only two equations for the four unknown $\psi_i(t)$ functions. There are also many variations of how to solve the two bilinear equations using combinations of the three solutions. Now we use Solutions 2 and 1 to solve the bilinear equations (5.75) and (5.76), respectively, hence we have the system of bilinear equations

$$\begin{aligned}
\Phi_4 &= -A_1\Phi_1 - A_2\Phi_2 - A_3\Phi_3,\\
\Phi_5 &= A_5\Phi_7 + A_6\Phi_8,\\
\Phi_6 &= A_7\Phi_7 + A_8\Phi_8,\\
\Psi_1 &= A_1\Psi_4,\\
\Psi_2 &= A_2\Psi_4,\\
\Psi_3 &= A_3\Psi_4,\\
\Psi_7 &= -A_5\Psi_5 - A_7\Psi_6,\\
\Psi_8 &= -A_6\Psi_5 - A_8\Psi_6.
\end{aligned} \tag{5.152}$$

Solving these equations yields the general solutions

$$v_1(z,t) = \Big[-c_1 \frac{Kb_1}{\tau A_6} \sin(b_1 z) + c_2 \frac{Kb_1}{\tau A_6} \cos(b_1 z) + b_3 z + c_3 + b_4 \Big] c_4 e^{\delta t} + c_6, \tag{5.153}$$

$$\theta(z,t) = \big[c_1 \cos(b_1 z) + c_2 \sin(b_1 z) + b_2\big] \Big[\frac{c_4}{A_2} e^{\delta t} + c_5\Big] - \Big[\frac{A_5}{\delta} - A_7\Big] \frac{c_4}{A_2} e^{\delta t} - A_5 c_5 t + c_7. \tag{5.154}$$

To make the solution as general as possible, we set $b_1 = p_1 + ip_2$, then insert (5.153) and (5.154) into the original governing equations to obtain the constraints

$$c_2 = 0, \tag{5.155}$$

$$p_1 = -p_2, \tag{5.156}$$

$$A_6 = \frac{\lambda \eta_1}{\tau^2} - 1, \tag{5.157}$$

$$p_1^2 = \frac{\omega}{2K} \left(\lambda - \frac{\tau^2}{\eta_1} \right), \tag{5.158}$$

with c_1 remaining arbitrary. The form of the constant A_6 is very similar to that of α in equation (5.477) in Stewart [49]. In the case of α it is possible to show, through the implementation of well established inequalities for viscosities, that $0 < \alpha < 1$. Due to the viscosities involved in this problem, we cannot conclude such strict bounds for A_6, however, we can make an estimate about the size of A_6 from experimental measurements on the combination of certain viscosities. From Stewart [49], we have

$$\lambda = 2\lambda_5 = 0.0600 \quad \text{Pa s}, \tag{5.159}$$

$$|\tau| = |\tau_5 - \tau_1| = 0.0273 \quad \text{Pa s}, \tag{5.160}$$

$$\eta_1 = \frac{1}{2}(\mu_0 + \mu_2 - 2\lambda_1 + \lambda_4) = 0.0377 \quad \text{Pa s} \tag{5.161}$$

and we know that $K = K_2^c \simeq 5 \times 10^{-10}$, hence we obtain

$$A_6 = \frac{2\lambda_5\eta_1}{(\tau_5 - \tau_1)^2} - 1 \simeq 0.5175. \tag{5.162}$$

Although we cannot provide an exact estimate or specific bounds for the constant A_6, we have shown that it is positive in this specific case and so is in line with the theory of nematics. Now, if we insert constraints (5.155)-(5.158) into solutions (5.153) and (5.154), then consider only the real part of the solutions, we obtain

$$\begin{aligned} v_1(z,t) = & \left(-\frac{BKp_1 \, \sin(p_1 \, z) \cosh(p_1 \, z)}{\tau \, A_6} + \frac{BKp_1 \, \cos(p_1 \, z) \sinh(p_1 \, z)}{\tau \, A_6} \right) \cos(\omega \, t) \\ & - \left(\frac{BKp_1 \, \sin(p_1 \, z) \cosh(p_1 \, z)}{\tau \, A_6} + \frac{BKp_1 \, \cos(p_1 \, z) \sinh(p_1 \, z)}{\tau \, A_6} \right) \sin(\omega \, t)\,, \end{aligned} \tag{5.163}$$

$$\theta(z,t) = B\cos(p_1 \, z)\cosh(p_1 \, z)\cos(\omega \, t) - B\sin(p_1 \, z)\sinh(p_1 \, z)\sin(\omega \, t) + c_7, \tag{5.164}$$

where we have set $c_6 = 0$ due to the condition on v_1 at the lower boundary. These solutions look promising as they are of the form we expect, however, if we apply the boundary condition $\theta(0,t) = \pi/2$ as before we obtain

$$B\cos(\omega t) + \frac{\pi}{2} = \frac{\pi}{2}, \tag{5.165}$$

which implies $B = 0$, again producing the zero solution. We have set $c_7 = \pi/2$ in the formulation of θ above to satisfy the boundary conditions imposed. If, however, we leave the director free at the boundaries and apply the boundary condition at $z = d$ to the velocity instead, we find that

$p_1 d$	ω	B
3.9266	153.3	-1537α
7.0686	496.8	119.6α
10.2102	1036.5	-7.462α
13.3518	1772.5	0.4217α
16.4934	2704.7	-0.0225α

Table 5.1: The first five positive roots of equation (5.166), with the corresponding ω and B values.

$$-\frac{BKp_1\ \sin\left(p_1\, d\right)\cosh\left(p_1\, d\right)}{\tau\, A_6}+\frac{BKp_1\ \cos\left(p_1\, d\right)\sinh\left(p_1\, d\right)}{\tau\, A_6}=0, \tag{5.166}$$

$$-\left(\frac{BKp_1\ \sin\left(p_1\, d\right)\cosh\left(p_1\, d\right)}{\tau\, A_6}+\frac{BKp_1\ \cos\left(p_1\, d\right)\sinh\left(p_1\, d\right)}{\tau\, A_6}\right)=-\alpha\omega, \tag{5.167}$$

then (5.166) yields the condition $p_1 d \approx \pm 3.9266$. This presents a problem because from the constraints given in (5.156) we found p_1 to be dependent only on ω, but now that we have p_1 to be a fixed value, we have also fixed ω. Inserting the value of p_1 obtained from (5.166) into the constraint in (5.156) yields $\omega \approx 153.3$ Hz. This is an unwelcome development as we would like to have a solution that is valid for any oscillation, but for the moment we will continue with a fixed ω. The first five positive roots of equation (5.166) are given in Table 5.1, with the corresponding ω and B values. Now, from equation (5.167) we have

$$-\frac{2BKp_1}{\tau A_6}\sin(p_1 d)\cosh(p_1 d)=-\alpha\omega, \tag{5.168}$$

which implies $B \approx -1537\alpha$. Using a trigonometric identity, we can write the director profile as

$$\theta(z,t) = B\sqrt{\cos^2(p_1 z)\cosh^2(p_1 z) + \sin^2(p_1 z)\sinh^2(p_1 z)} \times \cos\left(\omega t + \tan^{-1}\left(\tan(p_1 z)\tanh(p_1 z)\right)\right) + c_7, \tag{5.169}$$

and inserting the boundary values into (5.169) yields

$$\theta(0,t) = B\cos(\omega t) + c_7, \tag{5.170}$$

$$\theta(d,t) \approx 25.367B\cos(\omega t + 0.7846) + c_7. \tag{5.171}$$

From (5.170) and (5.171) it appears that the director on the lower, stationary plate is in phase with the oscillations of the upper plate, but that the director at the top plate is actually out of phase with these oscillations. This is somewhat counter-intuitive as one would expect the phase lag in the director to be present at the lower plate. It is also worth noting that since B is dependent on the amplitude α, it follows that the director profile at the boundaries is also strongly dependent on this amplitude.

5.2.2.2 Discussion

To gain some insight into the complex structure discussed in Section 5.2.2.1, consider the special case when the constants in equations (5.118) and (5.119) are set to unity. Hence we have

$$v_{1,zz} - v_{1,t} + \theta_{,zt} = 0, \tag{5.172}$$

$$\theta_{,zzz} - \theta_{,tz} - v_{1,zz} = 0, \tag{5.173}$$

with the latter equation being the z-derivative of (5.119). Adding equations (5.172) and (5.173) yields the single equation

$$v_{1,t} = \theta_{,zzz}. \tag{5.174}$$

If we now consider the solutions given by (5.145) and (5.146) and substitute these solutions into equation (5.174), we obtain

$$\frac{\omega^2 d}{2q_1} ABe^{i\omega t}\left[\cos(q_2) - \cos\left(\frac{2q_1 z}{d} + q_2\right)\right] = Ae^{i\omega t}\left(\frac{2q_1}{d}\right)^3 \cos\left(\frac{2q_1 z}{d} + q_2\right). \tag{5.175}$$

For equation (5.175) to be consistent, we require $\cos(q_2) = 0$ which, as we have seen previously, cannot be true due to the way we have defined the constant A. This basic analysis of the solutions shows that the ansatz chosen are giving solutions that lead to an inconsistency. This seems to indicate that the generalised separation of variables technique may be inappropriate to this particular coupled system.

5.2.2.3 Homogeneous Boundary Conditions

In Section 5.2.2.1, we found a partial solution for this problem but we would like a complete solution valid for all parameter values. There is one final option to make this method tractable for the system of differential equations. For the first solution we obtained using this method, problems arose with the separation of the real and imaginary parts of the solution. The imaginary constants in solutions (5.145) and (5.146) are induced due to the imaginary boundary condition at $z = d$ for the velocity. In Section 5.1 we used the generalised separation of variables method to obtain exact, correct solutions for a backflow problem in nematics. In that case we dealt with zero boundary conditions and since the difficulties are arising in the problem with the implementation of the boundary conditions, we try to overcome the difficulties by taking the original problem and transforming to a problem with homogeneous boundary conditions. First we set

$$\hat{u}(z,t) = v_1(z,t) + \frac{z}{d}\,\alpha\omega\sin(\omega t), \tag{5.176}$$

then equations (5.118) and (5.119) become

$$\eta_1\hat{u}_{,zz} - \rho\hat{u}_{,t} + \tau\theta_{,zt} + z\frac{\rho\alpha\omega^2}{d}\cos(\omega t) = 0, \tag{5.177}$$
$$K\theta_{,zz} - \lambda\theta_{,t} - \tau\hat{u}_{,z} + \frac{\tau\alpha\omega}{d}\sin(\omega t) = 0. \tag{5.178}$$

Working through the process as detailed above, first by setting $\theta(z,t) = \varphi_1(z)\psi_1(t) + \chi_1(t)$ and $\hat{u}(z,t) = \varphi_2(z)\psi_2(t) + \chi_2(t)$, then formulating and solving the bilinear equations similar to (5.75), we arrive at the following system of equations to solve,

$$\eta_1\varphi_2'' = -A_1\rho - A_2\tau\varphi_1 + A_3 z, \tag{5.179}$$
$$-\rho\varphi_2 = -A_4\rho - A_5\tau\varphi_1 + A_6 z, \tag{5.180}$$
$$K\varphi_1'' = A_7 - A_8\tau\varphi_2', \tag{5.181}$$
$$-\lambda\varphi_1' = A_9 - A_{10}\tau\varphi_2', \tag{5.182}$$
$$\dot{\chi}_2 = -A_1\psi_2 - A_4\dot{\psi}_2, \tag{5.183}$$
$$\dot{\psi}_1 = -A_2\psi_2 - A_5\dot{\psi}_2, \tag{5.184}$$
$$\frac{\rho\alpha\omega^2}{d}\cos(\omega t) = -A_3\psi_2 - A_6\dot{\psi}_2, \tag{5.185}$$
$$-\lambda\dot{\chi}_1 + \frac{\tau\alpha\omega}{d} = -A_7\psi_1 - A_9\dot{\psi}_1, \tag{5.186}$$
$$\psi_2 = -A_8\psi_1 - A_{10}\dot{\psi}_1. \tag{5.187}$$

Following similar techniques to the previous examples, these equations have solutions

$$\varphi_1 = c_1 b_1 b_4 \cos(\sqrt{b_1} z) + c_2 b_1 b_4 \sin(\sqrt{b_1} z) + b_6 z - b_5, \tag{5.188}$$

$$\varphi_2 = c_1 \sin(\sqrt{b_1} z) + c_2 \cos(\sqrt{b_1} z) + b_2 z - b_3, \tag{5.189}$$

$$\psi_1 = A_1 \cos(\omega t) + A_2 \sin(\omega t) + A_3 e^{-b_7 t}, \tag{5.190}$$

$$\psi_2 = A_4 \cos(\omega t) + A_5 \sin(\omega t) + c_3 e^{-b_7 t}, \tag{5.191}$$

$$\chi_1 = A_6 \cos(\omega t) + A_7 \sin(\omega t) + A_8 e^{-b_7 t} + b_{15} \frac{\tau \alpha \omega}{d} t + c_4, \tag{5.192}$$

$$\chi_2 = A_9 \cos(\omega t) + A_{10} \sin(\omega t) + A_{11} e^{-b_7 t} + c_5, \tag{5.193}$$

where the A_i are dependent on constants of integration and material parameters. Using the boundary condition $\theta(0,t) = \frac{\pi}{2}$ implies $c_4 = b_{15} = \frac{\pi}{2}$ and enables us to write

$$A_6 = -\big(c_1 b_1 b_4 - b_5\big) A_1, \tag{5.194}$$

$$A_7 = -\big(c_1 b_1 b_4 - b_5\big) A_2, \tag{5.195}$$

$$A_8 = -\big(c_1 b_1 b_4 - b_5\big) A_3, \tag{5.196}$$

and the boundary condition $\hat{u}(0,t) = 0$ implies

$$A_9 = -\big(c_1 - b_3\big) A_4, \tag{5.197}$$

$$A_{10} = -\big(c_1 - b_3\big) A_5, \tag{5.198}$$

$$A_{11} = -\big(c_1 - b_3\big) c_3, \tag{5.199}$$

and $c_5 = 0$. Applying these constraints allows us to write the original solutions as

$$
\begin{aligned}
v_1(z,t) =& \left[c_1\cos(\sqrt{b_1}z) + c_2\sin(\sqrt{b_1}z) + b_2 z - c_1\right] \\
&\times\left[A_4\cos(\omega t) + A_5\sin(\omega t) + c_3 e^{-b_7 t}\right] - \frac{z}{d}\,\alpha\omega\sin(\omega t), \qquad (5.200)\\
\theta(z,t) =& \left[c_1 b_1 b_4\cos(\sqrt{b_1}z) + c_2 b_1 b_4\sin(\sqrt{b_1}z) + b_6 z - c_1 b_1 b_4\right] \\
&\times\left[A_1\cos(\omega t) + A_2\sin(\omega t) + A_3 e^{-b_7 t}\right] + \frac{\pi}{2}, \qquad (5.201)
\end{aligned}
$$

where

$$
\begin{aligned}
A_1 &= \frac{b_8\rho\alpha\omega^2(b_7 b_{10} - b_9)}{d(b_7^2+\omega^2)}, & (5.202)\\
A_2 &= \frac{b_8\rho\alpha\omega(b_7 b_9 + b_{10})}{d(b_7^2+\omega^2)}, & (5.203)\\
A_3 &= \frac{c_3 b_7^2 d(b_9 - b_{10}b_7) - b_7 b_{10} c_3 d\omega^2 + b_9 - c_3 d\omega^2}{d b_7(b_7^2+\omega^2)}, & (5.204)\\
A_4 &= -\frac{b_7 b_8\rho\alpha\omega^2}{d(b_7^2+\omega^2)}, & (5.205)\\
A_5 &= -\frac{b_8\rho\alpha\omega^3}{d(b_7^2+\omega^2)}. & (5.206)
\end{aligned}
$$

Inserting $v_1(z,t)$ and $\theta(z,t)$ from (5.200) and (5.201) into governing equation (5.118) yields the twelve constraints

$$b_2 A_5 \omega - \frac{\alpha\omega^2}{d} = 0, \tag{5.207}$$

$$\rho b_2 c_3 b_7 = 0, \tag{5.208}$$

$$\rho b_2 A_4 \omega = 0, \tag{5.209}$$

$$-\eta_1 c_1 b_1 A_4 - \rho c_1 A_5 \omega + \tau c_2 b_1^{3/2} b_4 A_2 \omega = 0, \tag{5.210}$$

$$-\eta_1 c_1 b_1 c_3 + \rho c_1 c_3 b_7 - \tau c_2 b_1^{3/2} b_4 A_3 b_7 = 0, \tag{5.211}$$

$$-\eta_1 c_1 b_1 A_5 + \rho c_1 A_4 \omega - \tau c_2 b_1^{3/2} b_4 A_1 \omega = 0, \tag{5.212}$$

$$-\eta_1 c_2 b_1 A_4 - \rho c_2 A_5 \omega - \tau c_1 b_1^{3/2} b_4 A_2 \omega = 0, \tag{5.213}$$

$$\rho c_1 A_5 \omega + \tau b_6 A_2 \omega = 0, \tag{5.214}$$

$$-\eta_1 c_2 b_1 c_3 + \rho c_2 c_3 b_7 + \tau c_1 b_1^{3/2} b_4 A_3 b_7 = 0, \tag{5.215}$$

$$\rho c_1 c_3 b_7 + \tau b_6 A_3 b_7 = 0, \tag{5.216}$$

$$-\eta_1 c_2 b_1 A_5 + \rho c_2 A_4 \omega + \tau c_1 b_1^{3/2} b_4 A_1 \omega = 0, \tag{5.217}$$

$$\rho c_1 A_4 \omega + \tau b_6 A_1 \omega = 0. \tag{5.218}$$

From condition (5.207), we require $b_2 A_5 = \alpha\omega/d$, and since we are dealing with a non-zero amplitude and frequency this implies that $b_2 \neq 0$ and $A_5 \neq 0$. From condition (5.208) we have that either $b_2 = 0$ or $c_3 = 0$, but from condition (5.207) we know b_2 is non-zero, hence $c_3 = 0$. Similarly from condition (5.208) we have $b_2 A_4 = 0$, and hence $A_4 = 0$. For condition (5.216) to be satisfied we require $\tau b_6 b_7 A_3 = 0$. Since we are diving by b_7 in (5.204), we require it to be non-zero, which implies that $b_6 = 0$ or $A_3 = 0$. Similarly, from (5.218) we obtain the condition $b_6 = 0$ or $A_1 = 0$. If we assume that $b_6 = 0$ to fulfil both of these requirements, then condition (5.214) implies $c_1 = 0$ (since we have A_5 non-zero), and condition (5.213) further implies that $c_2 = 0$ (again since $A_5 \neq 0$). If we follow this line of reasoning from the condition that $b_6 = 0$, then we have to set each of the main constants, c_1 and c_2, in the z-dependent part of the solutions to be zero. This in itself may not be a big issue, however, if we set $c_1 = c_2 = c_3 = b_6 = A_4 = 0$ in solutions (5.200) and (5.201), we arrive at the solutions

$$v_1(z,t) = b_2 A_5 z \sin \omega t - \frac{\alpha\omega}{d} z \sin \omega t, \tag{5.219}$$

$$\theta(z,t) = \frac{\pi}{2}. \tag{5.220}$$

However, from condition (5.207) we found that $b_2 A_5 = \alpha\omega/d$, and so $v_1(z,t) \equiv 0$. So we have again recovered the zero solution for the velocity. If we now return to conditions (5.216) and (5.218) and assume that b_6 is non-zero, then we have that $A_1 = A_3 = 0$. In this case, substituting $A_1 = A_3 = 0$ into condition (5.210) yields $c_1 = 0$, and condition (5.217) returns $c_2 = 0$ or $b_1 = 0$. Since b_1 is key to the behaviour of the sinusoidal terms we require it to be non-zero. Hence we again arrive at the requirement $c_1 = c_2 = 0$. By substituting these conditions into (5.200), we obtain the equation

$$\left[-\rho\left(b_2 A_5 \omega - \frac{\alpha\omega^2}{d}\right)\cos(\omega t) + \rho\omega b_2 A_4 \sin(\omega t)\right] z + \tau\omega b_6 A_2 \cos(\omega t) = 0, \tag{5.221}$$

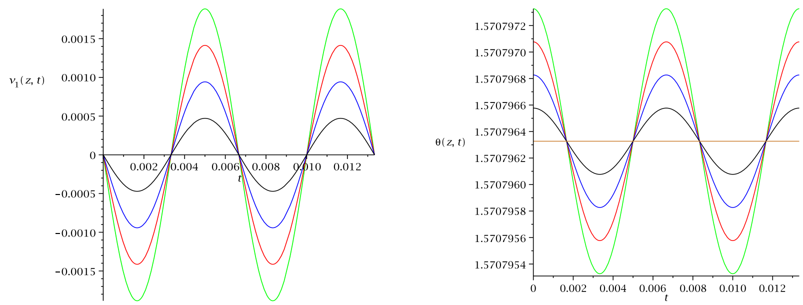
which must be true for all t, and so we require $A_2 = A_4 = 0$. Setting $c_1 = c_2 = c_3 = b_6 = A_2 = A_4 = 0$ in solutions (5.200) and (5.201) returns the same results as (5.219) and (5.220).

5.2.3 Conclusions and Discussion

In this chapter we have found it very difficult to obtain exact solutions to the equations governing shear flow in a SmC liquid crystal. Asymptotic solutions were already known for this geometry [33], but we had hoped to obtain full solutions. Since the generalised separation of variables technique solved the nematic backflow equations, we were confident that it would also be appropriate for the SmC problem. Unfortunately we were only able to obtain partial results. We note here that in Appendix A, we solved the classical fluid mechanics problems using two different methods. However, in the case of the SmC equations, only Method Two is detailed in Section 5.2.2. This is because the algebra involved with Method One becomes intractable after only one iteration of the method, and the differential equations become more complex as we proceed.

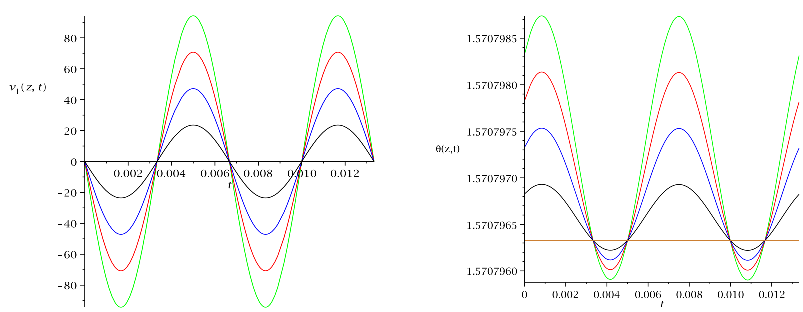
We can see from Appendix A that the generalised separation of variables technique works perfectly in two of the classical cases, and even in the case of the backflow equations in nematics we can find exact solutions. In the nematic equations considered in Section 5.1, we were dealing with homogeneous boundary conditions which made the computations more tractable. When we tried to implement the same method for the SmC equations in Section 5.2.2 we ran into various difficulties, with the problem appearing to be founded in the non-zero boundary condition. However, the method worked when applied to the classical oscillating boundary problem in Appendix A, and even when we transformed the equations to have homogeneous boundary conditions, we encountered more problems. To investigate the nature of the solutions further we will look at the solutions given by (5.140) and (5.139) for the inhomogeneous boundary conditions and solutions (5.200) and (5.201) for the transformed problem.

For a qualitative discussion on the results, we set all the constants in the problem to unity and plot the solutions for various fixed values of z over the time interval 0 to $4\pi/\omega$ (twice the period). Clearly, we can see from Figure 5.4 and Figure 5.5 that the solutions are acting in a way that we would expect. They are periodic in time, with the highest velocity close to the moving boundary, and the amplitude of the velocity decaying with distance from the boundary. These graphs by no means prove that the general solution to the problem is correct, but it certainly is of the anticipated form. It is interesting that the solution is of the expected form but we have been unable to satisfy the governing equations with the rigorously correct restrictions on the constants.



(a) (b)

Figure 5.4: Qualitative SmC solutions for (a) the velocity from (5.140) and (b) for the director profile from (5.139), with all constants set to unity.



(a) (b)

Figure 5.5: Qualitative SmC solutions for (a) the velocity from (5.200) and (b) for the director profile from (5.201), with all constants set to unity.

Chapter 6

The Pumping Phenomenon in a Smectic C* Liquid Crystal

Experiments have shown that in the case of an electric field applied to bookshelf aligned smectic C* liquid crystals a pumping phenomenon can be seen. This occurs when one plate is fixed with the other free to move in small amounts; the electric field is instantly reversed to reveal a 'pumping' effect. This pumping phenomenon was shown experimentally in Jákli *et al.* [18], and Jákli and Saupe [19], and was investigated theoretically by Stewart [52]. Stewart's theoretical work neglected the effect of elasticity, hence in Section 6.1 we first extend the work carried out by Stewart to include a term dealing with the elasticity of the SmC* liquid crystal. Then in Section 6.2 we look at a different but related problem where one plate is fixed and the other has the freedom to move.

6.1 Pumping Phenomenon Adapted to Include Elastic Effects

To derive the governing equations of this system we will require the equations for the balance of linear and angular momentum for SmC* liquid crystals given, respectively, by [49, 52]

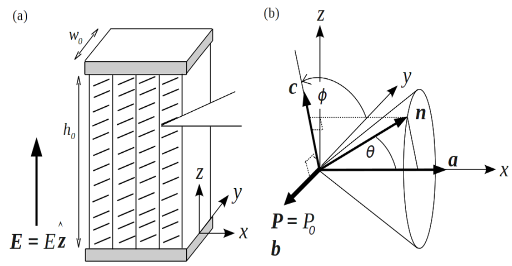


Figure 6.1: (a) A sample of bookshelf aligned SmC* liquid crystal where the short, bold lines represent the angle of the director, and (b) a schematic view of the structure of the director **n**, the unit layer normal **a**, the vector **c** and the spontaneous polarisation **P**.

$$\rho\dot{v}_i = \rho F_i - \tilde{p}_{,i} + G^a_k a_{k,i} + G^c_k c_{k,i} + \tilde{g}^a_k a_{k,i} + \tilde{g}^c_k c_{k,i} + \tilde{t}_{ij,j}, \tag{6.1}$$

$$\left(\frac{\partial w}{\partial c_{i,j}}\right)_{,j} - \frac{\partial w}{\partial c_i} + \tilde{g}^c_i + G^c_i + \tau c_i + \mu a_i = 0, \tag{6.2}$$

where F_i is the external body force per unit mass, $\tilde{g}^a_k$ and $\tilde{g}^c_k$ are dynamic contributions, $\tilde{p}$ is the pressure and $\tilde{t}_{ij}$ is the viscous stress tensor. In general, the pressure term is given by $\tilde{p} = p+w$, where w is the energy density. In the paper by Stewart [52], the term $\tilde{p}$ was reduced to p since the c-director was dependent only on time, which meant that the gradients of the c-director were zero and hence $w = 0$. However, in this adapted problem the director is no longer solely time dependent and so we must take the full form of the pressure, namely $\tilde{p} = p + w$. The term G^c_k is used to account for the effects of the electric field and is defined to be

$$G^c_k = \frac{\partial \Psi}{\partial c_k}, \tag{6.3}$$

where Ψ is the potential of the electric field due to spontaneous polarisation and is given by

$$\Psi = \mathbf{P} \cdot \mathbf{E}. \tag{6.4}$$

The dielectric contribution will be neglected because it has magnitude proportional to E^2, which can be ignored for small field effects. This is a common approach in simplified problems [49, p316]. Since the electric field energy has been considered in the G_k^c term, the regular energy density, w, consists solely of the elastic terms. From Stewart [49, Eqn. (6.13)], we take a simplified model of the energy density to be

$$w = \frac{1}{2}B_1(\mathbf{a} \cdot \nabla \times \mathbf{c})^2 + \frac{1}{2}B_2(\nabla \cdot \mathbf{c})^2 + \frac{1}{2}B_3(\mathbf{c} \cdot \nabla \times \mathbf{c} + q_0)^2, \tag{6.5}$$

where q_0 is the pitch, and we have set $B_1 = K_3$, $B_2 = K_2$ and $B_3 = K_4$ in line with the theory, and have neglected the other elastic terms. For this set-up we have

$$\mathbf{a} = (1, 0, 0), \tag{6.6}$$

$$\mathbf{c} = (0, \cos\phi, \sin\phi), \tag{6.7}$$

$$\mathbf{b} = (0, -\sin\phi, \cos\phi), \tag{6.8}$$

$$\mathbf{E} = (0, 0, 1), \tag{6.9}$$

$$\mathbf{v} = (0, k(t)y, -k(t)z), \tag{6.10}$$

where $\phi = \phi(z,t)$, $\mathbf{a}$ is the unit layer normal, $\mathbf{c}$ is the c-director, $\mathbf{b}$ is a unit vector such that $\mathbf{b} = \mathbf{a} \times \mathbf{c}$, $\mathbf{E}$ is the electric field, $\mathbf{v}$ is the velocity and $k(t)$ is the shear rate. The velocity ansatz is motivated by the review by Leslie [26], and satisfies the incompressibility condition $\nabla \cdot \mathbf{v} = 0$, as well as obeying the symmetry requirements that fit with the geometrical description of the model. For example, the flow in the y-direction is an odd function of y, which is what you would expect if both vertical boundaries are expanded or contracted at the same rate. For more information on the velocity ansatz, we direct the reader to

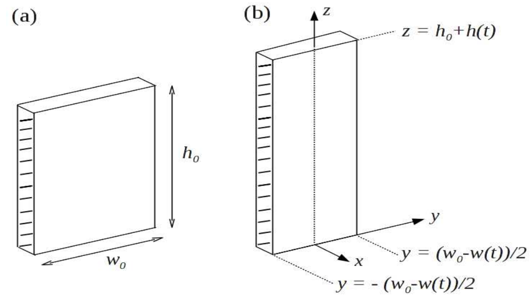


Figure 6.2: The description of a single layer of SmC* liquid crystal (a) at $t = 0$ where the initial height and width are represented by h_0 and w_0, respectively, and (b) under a field reversal at some time $t > 0$. For the layer to maintain a fixed volume, any increase in height must correspond to a decrease in width such that the relation in (6.12) holds true.

Stewart [52]. The spontaneous polarisation, $\mathbf{P}$, is written as a vector parallel to the vector $\mathbf{b}$ such that

$$\mathbf{P} = P_0\mathbf{b}, \tag{6.11}$$

where P_0 is taken to be positive. Let $h(t)$ be the vertical height displacement of the upper boundary plate from its initial height h_0, and let w_0 be the initial width of the sample. From Stewart [52], we have the area preserving relations

$$w(t) = \frac{w_0 h(t)}{h_0 + h(t)}, \tag{6.12}$$

$$k(t) = -\frac{dh}{dt}[h_0 + h(t)]^{-1}, \tag{6.13}$$

which arise from an assumed incompressibility of the smectic layers such that $(w_0 - w(t))(h_0 + h(t)) = w_0 h_0$. This will enable us to simplify the problem in order to solve only for the two functions $\phi(t)$ and $h(t)$, then substitute back to find the width $w(t)$ and shear rate $k(t)$. Now we will use the equation for the balance of angular momentum to derive the first governing equation of the system. First, by inserting the expressions for $\mathbf{a}$ and $\mathbf{c}$ into the energy density (6.5), we obtain the expression

$$w = \frac{1}{2}(B_1 \sin^2\phi + B_2 \cos^2\phi)\Big(\frac{\partial\phi}{\partial z}\Big)^2, \tag{6.14}$$

ignoring constant contributions to the energy. To evaluate the first terms in the equation of angular momentum, we write them in the more compact form

$$\Pi_i^c = \left(\frac{\partial w}{\partial c_{i,q}}\right)_{,q} - \frac{\partial w}{\partial c_i}, \tag{6.15}$$

which can be written as [49]

$$\begin{aligned}\mathbf{\Pi}^c &= B_2\nabla(\nabla\cdot\mathbf{c}) - B_1\nabla\times\{(\mathbf{a}\cdot\nabla\times\mathbf{c})\mathbf{a}\} - B_3\big(\nabla\times\{\mathbf{c}\cdot\nabla\times\mathbf{c}\} + (\mathbf{c}\cdot\nabla\times\mathbf{c})(\nabla\times\mathbf{c})\big)\\ &= -B_1\left(\frac{\partial^2\phi}{\partial z^2}\sin\phi + \Big(\frac{\partial\phi}{\partial z}\Big)^2\cos\phi\right)\mathbf{j} + B_2\left(\frac{\partial^2\phi}{\partial z^2}\cos\phi - \Big(\frac{\partial\phi}{\partial z}\Big)^2\sin\phi\right)\mathbf{k}.\end{aligned} \tag{6.16}$$

Next, we evaluate the dynamic term, which is given by (2.51), i.e.

$$\tilde{g}_i^c = 2(\lambda_2 D_i^c + \lambda_5 C_i + \tau_1 D_i^a - \tau_5 A_i), \tag{6.17}$$

where λ_2, λ_5, τ_1 and τ_5 are viscosity coefficients. The quantities $\mathbf{A}$ and $\mathbf{C}$ are the co-rotational time fluxes for $\mathbf{a}$ and $\mathbf{c}$, respectively, and are defined to be

$$A_i = \dot{a}_i - W_{ik}a_k, \quad C_i = \dot{c}_i - W_{ik}c_k, \tag{6.18}$$

where $\dot{a}_i$ and $\dot{c}_i$ are material time derivatives, D_{ij} is the rate of strain tensor, and W_{ij} is the vorticity tensor, which are given by

$$D_{ij} = \frac{1}{2}(v_{i,j} + v_{j,i}), \quad W_{ij} = \frac{1}{2}(v_{i,j} - v_{j,i}). \tag{6.19}$$

We note that for $\mathbf{v}$ given by (6.10), $W_{ij} = 0$ for all i and j. We also require the quantities D_i^a and D_i^c which are written as

$$D_i^a = D_{ij}a_j, \quad D_i^c = D_{ij}c_j. \tag{6.20}$$

By inserting $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{v}$ into (6.18), (6.19) and (6.20) we obtain

$$\mathbf{D}^a = \mathbf{0}, \quad \mathbf{D}^c = (0, \cos\phi, -\sin\phi)k(t), \tag{6.21}$$

$$\mathbf{A} = \mathbf{0}, \quad \mathbf{C} = (0, -\sin\phi, \cos\phi)\frac{\partial\phi}{\partial t}, \tag{6.22}$$

and hence the dynamic contribution (6.17) can be written as

$$\tilde{g}_1^c = 0, \tag{6.23}$$

$$\tilde{g}_2^c = -2\Big(\lambda_2 \cos\phi k(t) - \lambda_5 \sin\phi\frac{\partial\phi}{\partial t}\Big), \tag{6.24}$$

$$\tilde{g}_3^c = -2\Big(\lambda_2 \sin\phi k(t) - \lambda_5 \cos\phi\frac{\partial\phi}{\partial t}\Big). \tag{6.25}$$

The electric potential is defined in terms of Ψ, where Ψ is defined to be

$$\Psi = \mathbf{P}\cdot\mathbf{E} = P_0\,\mathbf{E}\cdot(\mathbf{a}\times\mathbf{c}) = P_0 E_i \epsilon_{ijk} a_j c_k, \tag{6.26}$$

so the electric term can be written as

$$G_i^c = \frac{\partial\Psi}{\partial c_i} = P_0 E \epsilon_{3ji} a_j = P_0 E \epsilon_{312} a_1. \tag{6.27}$$

From the first of the equations of angular momentum we derive the first Lagrange multiplier to be $\mu = 0$, and from the second and third respectively we obtain

$$-B_1\left(\frac{\partial^2\phi}{\partial z^2}\sin\phi+\left(\frac{\partial\phi}{\partial z}\right)^2\cos\phi\right)+P_0E-2\lambda_2\cos\phi\,k(t)+2\lambda_5\sin\phi\frac{\partial\phi}{\partial t}$$
$$+\tau\cos\phi=0, \tag{6.28}$$

$$B_2\left(\frac{\partial^2\phi}{\partial z^2}\cos\phi-\left(\frac{\partial\phi}{\partial z}\right)^2\sin\phi\right)+2\lambda_2\sin\phi\,k(t)-2\lambda_5\cos\phi\frac{\partial\phi}{\partial t}+\tau\sin\phi=0. \tag{6.29}$$

We can eliminate the Lagrange multiplier τ by multiplying equation (6.28) by $\sin\phi$, multiplying equation (6.29) by $\cos\phi$ then subtracting the two resulting equations. Hence we obtain

$$-\frac{\partial^2\phi}{\partial z^2}(B_1\sin^2\phi+B_2\cos^2\phi)-\left(\frac{\partial\phi}{\partial z}\right)^2\cos\phi\sin\phi(B_1-B_2)+P_0E\sin\phi$$
$$+4\lambda_2\cos\phi\sin\phi\,(h_0+h(t))^{-1}\frac{dh}{dt}+2\lambda_5\frac{\partial\phi}{\partial t}=0, \tag{6.30}$$

where we have simplified the expression using equation (6.13). In the case of the one constant approximation, we set $B_1=B_2=B$ to obtain the equation

$$2\lambda_5\frac{\partial\phi}{\partial t}=B\frac{\partial^2\phi}{\partial z^2}-P_0E\sin\phi-4\lambda_2\cos\phi\sin\phi\,(h_0+h(t))^{-1}\frac{dh}{dt}. \tag{6.31}$$

Now we consider the equation for the balance of linear momentum which is defined to be

$$\rho\dot{v}_i=\rho F_i-\tilde{p}_{,i}+G^a_k a_{k,i}+G^c_k c_{k,i}+\tilde{g}^a_k a_{k,i}+\tilde{g}^c_k c_{k,i}+\tilde{t}_{ij,j}, \tag{6.32}$$

which can be written explicitly as

$$\tilde{p}_{,x} = \tilde{t}_{1j,j}, \tag{6.33}$$

$$\tilde{p}_{,y} = \tilde{t}_{2j,j} - \rho y \left(k^2(t) + \frac{dk}{dt} \right), \tag{6.34}$$

$$\tilde{p}_{,z} = \tilde{t}_{3j,j} - \rho z \left(k^2(t) - \frac{dk}{dt} \right) - \rho g - P_0 E \sin\phi \frac{\partial \phi}{\partial z} \\ - 2\lambda_5 \frac{\partial \phi}{\partial z}\frac{\partial \phi}{\partial t} + 4\lambda_2 k(t) \sin\phi \cos\phi \frac{\partial \phi}{\partial z}. \tag{6.35}$$

Now the viscous stress tensor $\tilde{t}_{ij}$ for the SmC* phase can be written in the form

$$\tilde{t}_{ij} = \tilde{t}^{s}_{ij} + \tilde{t}^{ss}_{ij}, \tag{6.36}$$

where $\tilde{t}^{s}_{ij}$ and $\tilde{t}^{ss}_{ij}$ are the symmetric and skew-symmetric components of the viscous stress given by (2.53) and (2.54), respectively. Since we are taking the divergence of the stress tensor and all the functions are dependent only on z and t, we only need to calculate the terms $\tilde{t}_{13}$, $\tilde{t}_{23}$ and $\tilde{t}_{33}$. After some manipulations we find

$$\tilde{t}_{13} = \mu_0 D_{13} + (\tau_1 + \tau_5) C_3 + (\tau_2 + \kappa_1) D^c_3 + (\tau_4 + \kappa_3) c_p D^c_p c_3, \tag{6.37}$$

$$\tilde{t}_{23} = \mu_0 D_{23} + \mu_3 c_p D^c_p c_2 c_3 + \mu_4 (D^c_2 c_3 + D^c_3 c_2) + \lambda_2 (C_2 c_3 + C_3 c_2 + D^c_3 c_2 - D^c_2 c_3) \\ + \lambda_5 (C_3 c_2 - C_2 c_3), \tag{6.38}$$

$$\tilde{t}_{33} = \mu_0 D_{33} + \mu_3 c_p D^c_p c_3^2 + 2\mu_4 D^c_3 c_3 + 2\lambda_2 C_3 c_3 + 2\kappa_3 a_p D^c_p c_3^2. \tag{6.39}$$

Now using the values for $\mathbf{c}$, $\mathbf{C}$ and D_{ij} allows us to calculate the divergence of the stress tensor explicitly, which is given by

$$
\begin{aligned}
\tilde{t}_{13,z} =& k(t)\cos\phi\frac{\partial\phi}{\partial z}\Big((\kappa_3+\tau_4)(\cos^2\phi-5\sin^2\phi)-\kappa_1-\tau_2\Big)\\
&+(\tau_1+\tau_5)\bigg(\cos\phi\frac{\partial^2\phi}{\partial z\partial t}-\sin\phi\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial t}\bigg), \qquad (6.40)\\
\tilde{t}_{23,z} =& -2\lambda_2 k(t)\frac{\partial\phi}{\partial z}(\cos^2\phi-\sin^2\phi)+\lambda_2\bigg((\cos^2\phi-\sin^2\phi)\frac{\partial^2\phi}{\partial z\partial t}-4\cos\phi\sin\phi\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial t}\bigg)\\
&+\mu_3\bigg((\cos^2\phi-\sin^2\phi)\frac{\partial\phi}{\partial z}-4\cos^2\phi\sin^2\phi\frac{\partial\phi}{\partial z}\bigg)+\lambda_5\frac{\partial^2\phi}{\partial z\partial t}, \qquad (6.41)\\
\tilde{t}_{33,z} =& \mu_3\sin\phi\cos\phi\frac{\partial\phi}{\partial z}k(t)(2\cos^2\phi-6\sin^2\phi)+2\lambda_2(\cos^2\phi-\sin^2\phi)\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial t}\\
&+2\sin\phi\cos\phi\bigg(\lambda_2\frac{\partial^2\phi}{\partial z\partial t}-2\mu_4 k(t)\frac{\partial\phi}{\partial z}\bigg). \qquad (6.42)
\end{aligned}
$$

These quantities can then be substituted back into equations (6.33)-(6.35) to give the complete balance of linear momentum equations. These equations are very complex and are impossible to solve analytically, however, we will now aim to find a general form for the pressure. We first write equations (6.33)-(6.35) in the form

$$
\begin{aligned}
\tilde{p}_{,x} &= f_1(z,t), &(6.43)\\
\tilde{p}_{,y} &= f_2(z,t)-\rho y\Big(k^2(t)+\frac{dk}{dt}\Big), &(6.44)\\
\tilde{p}_{,z} &= f_3(z,t)-\rho z\Big(k^2(t)-\frac{dk}{dt}\Big)-\rho g. &(6.45)
\end{aligned}
$$

Integrating equation (6.43) with respect to x, equation (6.44) with respect to y and equation (6.45) with respect to z yields, respectively,

$$
\begin{aligned}
\tilde{p} =& f_1(z,t)x+g_1(y,z,t), &(6.46)\\
\tilde{p} =& f_2(z,t)y+g_2(x,z,t)-\frac{1}{2}\rho y^2\Big(k^2(t)+\frac{dk}{dt}\Big), &(6.47)\\
\tilde{p} =& \int_{h_0}^{z} f_3(\hat{z},t)d\hat{z}+g_3(x,y,t)-\frac{1}{2}\rho z^2\Big(k^2(t)-\frac{dk}{dt}\Big)+\rho g(h_0-z)+p_0. &(6.48)
\end{aligned}
$$

Hence we can write the pressure in the form

$$
\begin{aligned}
\tilde{p} =& f_1(z,t)x + f_2(z,t)y + \int_{h_0}^{z} f_3(\hat{z},t)d\hat{z} + \rho g(h_0 - z) + p_0 \\
&-\frac{1}{2}\rho y^2\left(k^2(t) + \frac{dk}{dt}\right) - \frac{1}{2}\rho z^2\left(k^2(t) - \frac{dk}{dt}\right).
\end{aligned} \tag{6.49}
$$

It can be verified directly that the form of the pressure chosen in (6.49) solves equations (6.43)–(6.45). However, if we consider equation (6.43) differentiated with respect to z, and (6.45) differentiated with respect to x, then equate the two results we find that we are led to the restriction $\partial f_1/\partial z = 0$. This implies that the function $f_1(z,t) = \tilde{t}_{13,z}$ is independent of z, which in turn forces the director angle ϕ to be solely dependent on time. If ϕ is independent of z then we have collapsed back to the case considered in Stewart [52]. This is an unexpected development which means that the extension developed in this chapter is not tractable in its current format: this highlights the need for a more sophisticated approach. One way of overcoming this problem is to fully extend the director alignment to be three dimensional. Setting $\phi = \phi(x,y,z,t)$ allows the equations for the balance of linear momentum to be written as

$$
\tilde{p}_{,x} = f_1(x,y,z,t), \tag{6.50}
$$

$$
\tilde{p}_{,y} = f_2(x,y,z,t), \tag{6.51}
$$

$$
\tilde{p}_{,z} = f_3(x,y,z,t), \tag{6.52}
$$

for some functions f_1, f_2 and f_3, and since the functions on the right hand side are now dependent on x, y, and z, we should not be forced into setting ϕ to be only a function of time. The work required to obtain corresponding expressions for the viscous stress tensor $\tilde{t}_{ij}$ in the case of $\phi = \phi(x,y,z,t)$ would be extensive, and the resulting equations for the balance of linear and angular momentum equations would be impossible to solve without making many simplifications.

It is also possible that in this extended case, the ansatz for the velocity given by (6.10) is no longer adequate. It is possible that by extending the director alignment to include space variables, we must extend the velocity profile to also be space dependent. For our future work on the pumping phenomenon with elastic effects, we would first extend the director angle to be fully three dimensional before studying the applicability of the velocity ansatz in this case.

6.1.1 Discussion

In the paper by Stewart [52], the theoretical investigation into the pumping phenomenon was exceptionally close to the real life effects observed during experiments. The theoretical results almost coincide with the experimental results with the only discrepancy being the magnitude of the acceleration of the upper boundary under a field reversal. There is some discussion at the end of this paper as to why this could be the case. An argument is made that the discrepancy between the theory and experimental results is due to the fact that the elastic terms have been neglected in the original formulation, which we have looked at in Section 6.1. There was also the possibility that the model could be extended by considering the case of a variable layer structure. We considered this case but, at this moment, have found it intractable. The first problem encountered involves the form of the c-director. In the original formulation, this quantity was given by $\mathbf{c} = (0, \cos\phi, \sin\phi)$ when dealing with a fixed layer. If we now allow the layer normal to vary with an angle δ, then the c-director will also rotate in some way that is related to δ. Hence, the form that the c-director should take is not obvious.

Another problem that occurs when allowing $\mathbf{a}$ to vary is how to satisfy the constraint $\mathbf{a}\cdot\mathbf{c} = 0$. One way to satisfy this constraint would be to keep the layer normal $\mathbf{a}$ constant but allow the scalar function Φ to vary. If we allow Φ to vary then we need to include the permeation equation in our analysis. From Shalaginov *et al.* [44], we have that the permeation equation is given by

$$J_i = \delta_{ix}\left[B\left(\frac{1}{2}u_{,x}^2 - \epsilon\right)u_{,x} - Ku_{,xxx}\right], \tag{6.53}$$

where $u(x, z, t)$ is the layer displacement, ϵ is a measure of the layer strain, K is the bend elastic constant, δ_{ij} is the usual dirac delta, and B is a measure of layer compression. This paper cites J_i for SmA liquid crystal but close to the SmA-SmC boundary. The reason we have cited this and not a form that deals solely with SmC* liquid crystal is because there are no sources for the quantity $\mathbf{J}$ in the literature for general SmC* equations in the Ericksen-Leslie type of formulation. It can be seen from (6.53) that when elastic and compression effects are neglected we get $\mathbf{J} \equiv \mathbf{0}$. Hence we can gain no more information that allows the possibility for Φ to vary when we neglect these two effects.

There is another possibility for allowing the expansion of the layers. Instead of trying to find Φ as an unknown of the problem we could instead build the change in depth of the layers into the original set up. So we begin with height h_0, width w_0 and depth d_0, then after a time t the new height, width and depth are given by $h_0 + h(t)$, $w_0 - w(t)$ and $d_0 - d(t)$, respectively. Then in a similar way to the derivation of equation (2) in Stewart [52], we have

$$(h_0 + h(t))(w_0 - w(t))(d_0 - d(t)) = h_0\, w_0\, d_0, \tag{6.54}$$

which can be used to eliminate one of the time dependent functions from the problem. However, it does complicate the nature of the problem as we would now need to derive an additional equation to solve for the extra unknown function. This complication on its own would not be too much to handle as it would simply mean solving three coupled equations for three functions rather than two. However, the main problem arises when we consider the velocity in this case. For the original problem, the velocity ansatz was given by

$$\mathbf{v} = (0, k(t)y, -k(t)z), \tag{6.55}$$

where the velocity in the x-direction was set to be zero. In this case we are assuming that the layers are compressing and so the velocity in the x-direction would be non-zero. However, the velocity was chosen in such a way to satisfy the incompressibility condition $\nabla\cdot \mathbf{v} = 0$. Other than working with the full dynamic equations with $\mathbf{v}$ to be determined, it is not obvious how to adapt the velocity in such a way that it keeps the main features of flow from the original problem, but exhibits non-zero flow in the x-direction while still satisfying the incompressibility condition. Dealing with the full dynamic equations is a very difficult approach that would normally require numerical methods.

6.2 Dewetting of an Isotropic Fluid Between Two Parallel Separating Plates

Now we consider a different but related problem. In the previous section we looked at the effect of an electric field on a liquid crystal sample without being fully able to solve the problem. From Stewart [52], we know that when an electric field is applied, the top plate moves away from the bottom plate in a pumping fashion. We have searched in the literature and have found it extremely difficult to find previous work that helps us to better understand this problem and adjust our approach appropriately to solve the problem in Section 6.1. However, we did find a research paper [57] that deals with the behaviour of a fluid between two parallel plates that are then separated. We now consider this problem and attempt to adapt it to our frame of reference.

6.2.1 Fluid Adhesive Strip

We consider the case of a general viscous adhesive film between two parallel plates, one stationary and one that is free to move. There is a weight F attached to the bottom plate and the instantaneous distance between the two plates is denoted by $h(t)$. In Section 6.1, we made use of the fact that the surface area of the smectic layer was constant, but in this case we have a volume conserving ratio given by

$$V_0 = a(t)h(t) = a_0 h_0, \tag{6.56}$$

where $a(t)$ is the instantaneous area of the film. In the paper by Thamida *et al.* [57], Darcy's Law is used to set up the problem then the governing equations in

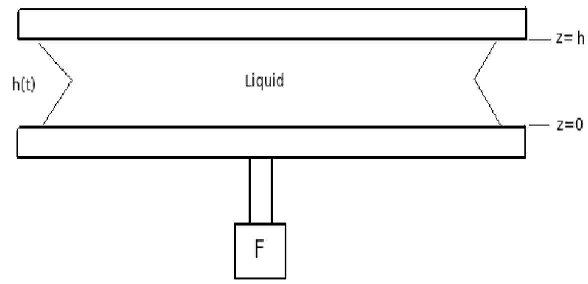


Figure 6.3: Geometrical set-up of a general viscous adhesive film between two parallel plates initially a distance h_0 apart. The top plate at $z = h_0$ is fixed while the bottom plate is free to move. There is a weight F attached to the bottom plate and the instantaneous distance between the two plates is denoted by $h(t)$.

the mathematical discussion are stated without any derivation. Here, we neglect Darcy's Law and instead derive the governing equations from the full Navier-Stokes equations. The Navier-Stokes equations for an incompressible isotropic Newtonian fluid are given by [2]

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u}.\nabla)\mathbf{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} + \mathbf{g}, \tag{6.57}$$

$$\nabla.\mathbf{u} = 0, \tag{6.58}$$

where ν is the kinematic viscosity, ρ is the density of the fluid, ∇^2 is the Laplacian operator, $\mathbf{g}$ is the gravitational term and $p = p(x, y, z, t)$ is the pressure. To use the same notation as the original paper by Thaminda *et al.* [57], we set the velocity to be $\mathbf{u}(x, y, z, t) = (u, v, w)$. For this problem, the left hand side of (6.57) is assumed to be zero and the term involving gravity is neglected to simplify these equations to

$$-\frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u} = 0, \tag{6.59}$$

which are known as the Stokes equations. Due to the geometry of the problem, it is clear that the length scale in the z-direction is much shorter than in the x and y directions, which in turn makes the second derivative with respect to z much larger than the other two second derivatives. Hence, taking the thin film approximation reduces equations (6.59) to the simpler form

$$0 = -p_{,x} + \mu u_{,zz}, \tag{6.60}$$

$$0 = -p_{,y} + \mu v_{,zz}, \tag{6.61}$$

$$0 = -p_{,z}, \tag{6.62}$$

where μ is the dynamic viscosity and is given by $\mu = \nu\rho$. Equation (6.62) involves only a pressure term because the velocity perpendicular to the plane is negligible and so the $w_{,zz}$ term can be omitted. Next we define [10]

$$\bar{u} = \frac{1}{h}\int_0^h u\, dz, \tag{6.63}$$

$$\bar{v} = \frac{1}{h}\int_0^h v\, dz, \tag{6.64}$$

$$\bar{w} = \frac{1}{h}\int_0^h w\, dz. \tag{6.65}$$

The governing equations for the system are given by (6.58) and (6.60)-(6.62), and we have the kinematic condition at the free surface given by [2]

$$\frac{\partial h}{\partial t} + u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial y} = w, \quad \text{at } z = h(t), \tag{6.66}$$

which can be simplified to $w = h_t$ on $z = h$ since h is a function of time only. We also have the boundary conditions $u = v = w = 0$ on $z = 0$ and $z = h$.

Integrating equation (6.60) twice with respect to z gives

$$u = -\frac{p_{,x}}{2\mu}(hz - z^2), \tag{6.67}$$

then substituting for the quantity $\bar{u}$ yields

$$\bar{u} = -\frac{p_{,x}}{2\mu h}\left[h\frac{z^2}{2} - \frac{z^3}{3}\right]_{z=0}^{z=h} = -\frac{p_{,x}h^2}{12\mu}, \tag{6.68}$$

and, similarly, integrating equation (6.61) twice with respect to z and then substituting for $\bar{v}$ gives

$$\bar{v} = -\frac{p_{,y}h^2}{12\mu}. \tag{6.69}$$

By considering the quantities (6.63)-(6.65) we observe that

$$\begin{aligned}
\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} + \frac{\partial \bar{w}}{\partial y} &= \frac{\partial}{\partial x}\left(\frac{1}{h}\int_0^h u dz\right) + \frac{\partial}{\partial y}\left(\frac{1}{h}\int_0^h v dz\right) + \frac{\partial}{\partial z}\left(\frac{1}{h}\int_0^h w dz\right) \\
&= \frac{1}{h}\int_0^h \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) dz \\
&= 0,
\end{aligned} \tag{6.70}$$

which shows that if $\mathbf{u}$ satisfies the continuity equation, then $\bar{\mathbf{u}}$ also satisfies this equation. Also, we can write

$$\begin{aligned}
\frac{\partial \bar{w}}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{1}{h}\int_0^h w dz\right) \\
&= \frac{1}{h}\int_0^h \frac{\partial w}{\partial z} dz \\
&= \frac{1}{h} w|_{z=h},
\end{aligned} \tag{6.71}$$

which, when combined with kinematic condition (6.66), leads to

$$\frac{\partial \bar{w}}{\partial z} = \frac{1}{h}\frac{\partial h}{\partial t}. \tag{6.72}$$

Now, inserting (6.68), (6.69) and (6.72) into (6.70) yields

$$-\left(\frac{p_{,x}h^2}{12\mu}\right)_{,x} - \left(\frac{p_{,y}h^2}{12\mu}\right)_{,y} + \frac{1}{h}\frac{dh}{dt} = 0. \tag{6.73}$$

From the conservation of volume principle, we can write $h(t) = a_0 h_0 / a(t)$ and so (6.73) simplifies to

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = -\frac{12\mu}{a_0^2 h_0^2}\, a\frac{da}{dt}, \tag{6.74}$$

which provides the governing equation of the system. We also observe that the pressure is constrained by the equation

$$\int_{\Omega(t)} p\; dx\; dy = -F, \tag{6.75}$$

where $\Omega(t)$ is the time dependent surface area domain of the film. This means that the net force of separation of the plates is a constant at any given time. Now we aim to solve equation (6.74) for the pressure $p(x, y, t)$, which we will then substitute into (6.75) to find an expression for the area. In Thaminda *et al.* [57], the problem is solved using polar coordinates, but since we are looking to extend this work to include liquid crystals we will solve the problem in cartesian coordinates. We consider the domain to be of the form

$$\Omega(t) = \{(x, y) : 0 \leq x \leq w_c(t), 0 \leq y \leq w_c(t)\}, \tag{6.76}$$

where we have taken $\Omega(t)$ to be a perfect square and will assume that the domain in the x and y directions changes at the same rate. Now we solve the problem by seeking a Navier-type solution [58].

6.2.2 Navier-type Solution

First we consider equation (6.74) and write

$$-\frac{12\mu}{a_0^2 h_0^2}\, a\frac{da}{dt} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} B_{mn}(t)\, \sin\left(\frac{m\pi x}{w_c}\right) \sin\left(\frac{n\pi y}{w_c}\right), \tag{6.77}$$

where

$$\begin{aligned} B_{mn}(t) &= -\frac{4}{w_c^2}\int_0^{w_c}\int_0^{w_c} \frac{12\mu}{a_0^2 h_0^2}\, a\frac{da}{dt} \sin\left(\frac{m\pi x}{w_c}\right) \sin\left(\frac{n\pi y}{w_c}\right) dx dy \\ &= -\frac{48\mu(1-(-1)^m)(1-(-1)^n)}{\pi^2 a_0^2 h_0^2 mn}\, a\frac{da}{dt}. \end{aligned} \tag{6.78}$$

Now, the pressure can also be written in the form

$$p(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} A_{mn}(t)\, \sin\left(\frac{m\pi x}{w_c}\right) \sin\left(\frac{n\pi y}{w_c}\right), \tag{6.79}$$

which is then substituted, along with $B_{mn}(t)$, into equation (6.74) to return

$$-\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left[\left\{\left(\frac{m\pi}{w_c}\right)^2 + \left(\frac{n\pi}{w_c}\right)^2\right\} A_{mn}(t) + B_{mn}(t)\right] \sin\left(\frac{m\pi x}{w_c}\right) \sin\left(\frac{n\pi y}{w_c}\right) = 0. \tag{6.80}$$

Since $\sin(m\pi x/w_c)$ and $\sin(n\pi y/w_c)$ are a complete orthogonal set of functions [23], we can set the bracketed term to zero to obtain the relation

$$A_{mn}(t) = -\frac{B_{mn}(t)\left(\frac{w_c}{\pi}\right)^2}{m^2+n^2}. \tag{6.81}$$

Hence we can write the pressure as

$$\begin{aligned}
p(x,y,t) &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{B_{mn}(t)\left(\frac{w_c}{\pi}\right)^2}{-(m^2+n^2)} \sin\left(\frac{m\pi x}{w_c}\right)\sin\left(\frac{n\pi y}{w_c}\right)\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{48\mu w_c^2 a\frac{da}{dt}(1-(-1)^m)(1-(-1)^n)}{\pi^4 a_0^2 h_0^2 mn(m^2+n^2)} \sin\left(\frac{m\pi x}{w_c}\right)\sin\left(\frac{n\pi y}{w_c}\right).
\end{aligned} \tag{6.82}$$

We have that the area is given in terms of width by the relation $a(t) = w_c^2(t)$, and so substituting this, as well as the pressure $p(x,y,t)$, into (6.75) yields

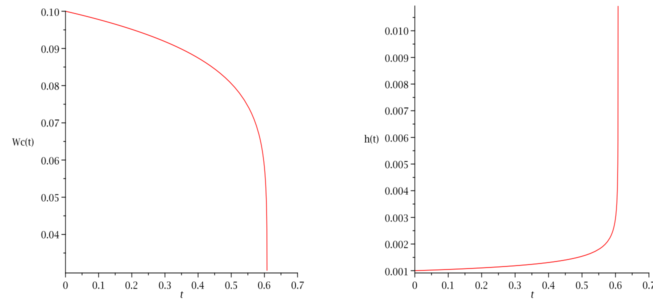
$$\begin{aligned}
-F &= \int_0^{w_c}\int_0^{w_c} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{96\mu w_c^5(1-(-1)^m)(1-(-1)^n)}{\pi^4 a_0^2 h_0^2 mn(m^2+n^2)} \sin\left(\frac{m\pi x}{w_c}\right)\sin\left(\frac{n\pi y}{w_c}\right)dxdy\\
&= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{96\mu w_c^7(1-(-1)^m)^2(1-(-1)^n)^2}{\pi^4 a_0^2 h_0^2 mn(m^2+n^2)}.
\end{aligned} \tag{6.83}$$

Now we write

$$F = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}\sin\left(\frac{m\pi x}{w_c}\right)\sin\left(\frac{n\pi y}{w_c}\right), \tag{6.84}$$

where

$$\begin{aligned}
C_{mn} &= -\frac{4}{w_c^2}\int_0^{w_c}\int_0^{w_c} F\sin\left(\frac{m\pi x}{w_c}\right)\sin\left(\frac{n\pi y}{w_c}\right)dxdy\\
&= \frac{4F}{\pi^2 mn}(1-(-1)^m)(1-(-1)^n),
\end{aligned} \tag{6.85}$$



(a) (b)

Figure 6.4: The first mode approximation of a fluid adhesive strip in a square domain showing (a) the width $w_c(t)$ of the fluid, and (b) the instantaneous separation of the plates $h(t)$.

for F independent of x and y. Substituting (6.84) into (6.83) yields

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{96\mu w_c^7(1-(-1)^m)^2(1-(-1)^n)^2}{\pi^4 a_0^2 h_0^2 mn(m^2+n^2)}+\frac{4F}{\pi^2 mn}(1-(-1)^m)(1-(-1)^n)\right) \times \sin\left(\frac{m\pi x}{w_c}\right)\sin\left(\frac{n\pi y}{w_c}\right)=0, \tag{6.86}$$

and hence setting the bracketed term to be zero produces a differential equation for $w_c(t)$, namely

$$w_c^7(t)\,\frac{dw_c}{dt}=-\frac{F\pi^2 a_0^2 h_0^2 mn(m^2+n^2)}{24\mu(1-(-1)^m)(1-(-1)^n)}. \tag{6.87}$$

Separating the variables in equation (6.87) and then integrating gives the solution

$$w_c(t)=\left[w_0^8-\frac{F\pi^2 a_0^2 h_0^2 mn(m^2+n^2)}{24\mu(1-(-1)^m)(1-(-1)^n)}\right]^{1/8}, \tag{6.88}$$

and since we know the area in terms of the width we can write

$$a_c(t) = \left[w_0^8 - \frac{F\pi^2 a_0^2 h_0^2 mn(m^2+n^2)}{24\mu(1-(-1)^m)(1-(-1)^n)} \right]^{1/4}. \tag{6.89}$$

When analysing these solutions, we find that the first mode approximation to the solutions is quite accurate and is in line with the solution found in the case of radial coordinates [57]. However, on closer inspection we see that as m and n are increased the solution diverges. Clearly this is not correctly capturing the nature of the solutions since, for large enough m and n, we will attain instantaneous separation of the plates. The explanation for this could be in the type of solution chosen. We set the pressure to be a double Fourier series in x and y but this might not be completely appropriate. The 'true' solution could be something approximating a Fourier Series, but not a true series. Take for example the case of the Freedericksz transition. In Stewart [49], we see that the full solution is given by an Elliptic Integral but this solution can be closely approximated by $\sin(x)$ to the first degree. However, if one attempts to approximate to a higher degree in a series of sinusoidal terms then the solution becomes inaccurate. It is possible that something similar is happening in this case, but a more intricate analysis would have to be carried out to make firm conclusions.

The solution obtained in (6.88) is accurate in the first mode approximation but we would like to have a full, complete solution. Given this impasse to a basic series solution, we proceed to solve the problem using Green's functions.

6.2.3 Green's Function Solution

It can be shown that [41] the general Poisson equation

$$\nabla^2 w + \Phi(x) = 0, \tag{6.90}$$

with the condition that $w = 0$ on the boundaries $x = 0, \alpha$ and $y = 0, \beta$, has a solution given by

$$w(x,y) = \int_0^\alpha \int_0^\beta \Phi(\xi,\eta) G(x,y,\xi,\eta)\, d\eta\, d\xi, \tag{6.91}$$

where

$$G(x,y,\xi,\eta) = \frac{2}{\alpha} \sum_{n=1}^{\infty} \frac{\sin(p_n x)\sin(p_n \xi)}{p_n \sinh(p_n \beta)} H_n(y,\eta), \tag{6.92}$$

$$H_n(y,\eta) = \begin{cases} \sinh(p_n\eta)\sinh[p_n(\beta-y)], & 0 \le \eta < y \le \beta, \\ \sinh(p_n y)\sinh[p_n(\beta-\eta)], & 0 \le y < \eta \le \beta, \end{cases} \tag{6.93}$$

and $p_n = \frac{n\pi}{\alpha}$. For simplicity, and due to the nature of the problem to be solved here, we set $\Phi(\xi,\eta) \equiv 1$. Now, since the function $H_n(y,\eta)$ is split up depending on where η is in relation to (a fixed) y, we have to split the integral dependent on η. Hence the second integral in (6.91) becomes

$$\int_0^\beta G\, d\eta = \int_0^y G^- d\eta + \int_y^\beta G^+ d\eta, \tag{6.94}$$

where G^- is the Green's function dependent on H_n^-, i.e. the function in the region $0 \le \eta < y \le \beta$, and G^+ is the function dependent on H_n^+, i.e. in the region $0 \le y < \eta \le \beta$. Now we consider each of the integrals in (6.94) separately. First we have

$$\begin{aligned} \int_0^y G^- d\eta &= \frac{2}{\alpha} \int_0^y \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{\alpha}\right)\sin\left(\frac{n\pi\xi}{\alpha}\right)}{\frac{n\pi}{\alpha}\sinh\left(\frac{n\pi\beta}{\alpha}\right)} \sinh\left(\frac{n\pi\eta}{\alpha}\right)\sinh\left(\frac{n\pi}{\alpha}(\beta-y)\right) d\eta \\ &= \sum_{n=1}^{\infty} \frac{2\alpha\sin\left(\frac{n\pi x}{\alpha}\right)\sin\left(\frac{n\pi\xi}{\alpha}\right)}{n^2\pi^2\sinh\left(\frac{n\pi\beta}{\alpha}\right)} \left[\cosh\left(\frac{n\pi y}{\alpha}\right) - 1\right] \sinh\left(\frac{n\pi}{\alpha}(\beta-y)\right), \end{aligned} \tag{6.95}$$

and similarly

$$\int_y^\beta G^+ d\eta = \frac{2}{\alpha}\int_y^\beta \sum_{n=1}^{\infty} \frac{\sin\left(\frac{n\pi x}{\alpha}\right)\sin\left(\frac{n\pi\xi}{\alpha}\right)}{\frac{n\pi}{\alpha}\sinh\left(\frac{n\pi\beta}{\alpha}\right)} \sinh\left(\frac{n\pi y}{\alpha}\right)\sinh\left(\frac{n\pi}{\alpha}(\beta-\eta)\right)d\eta$$
$$= \sum_{n=1}^{\infty} \frac{2\alpha\sin\left(\frac{n\pi x}{\alpha}\right)\sin\left(\frac{n\pi\xi}{\alpha}\right)}{n^2\pi^2\sinh\left(\frac{n\pi\beta}{\alpha}\right)}\left[\cosh\left(\frac{n\pi}{\alpha}(\beta-y)\right)-1\right]\sinh\left(\frac{n\pi y}{\alpha}\right). \tag{6.96}$$

Hence we can write

$$w(x,y) = \int_0^\alpha \sum_{n=1}^{\infty} \frac{2\alpha\sin\left(\frac{n\pi x}{\alpha}\right)\sin\left(\frac{n\pi\xi}{\alpha}\right)}{n^2\pi^2\sinh\left(\frac{n\pi\beta}{\alpha}\right)}\left[\left\{\cosh\left(\frac{n\pi}{\alpha}(\beta-y)\right)-1\right\}\sinh\left(\frac{n\pi y}{\alpha}\right)\right.$$
$$\left.+\left\{\cosh\left(\frac{n\pi y}{\alpha}\right)-1\right\}\sinh\left(\frac{n\pi}{\alpha}(\beta-y)\right)\right]d\xi, \tag{6.97}$$

which, after performing the integration over ξ, becomes

$$w(x,y) = \sum_{n=1}^{\infty} \frac{2\alpha^2(1-(-1)^n)\sin\left(\frac{n\pi x}{\alpha}\right)}{n^3\pi^3\sinh\left(\frac{n\pi\beta}{\alpha}\right)}\left[\left\{\cosh\left(\frac{n\pi}{\alpha}(\beta-y)\right)-1\right\}\sinh\left(\frac{n\pi y}{\alpha}\right)\right.$$
$$\left.+\left\{\cosh\left(\frac{n\pi y}{\alpha}\right)-1\right\}\sinh\left(\frac{n\pi}{\alpha}(\beta-y)\right)\right]. \tag{6.98}$$

This is the working for the general case but it is easy to adapt it for our problem in which we have to solve the equation

$$\nabla^2 p + \frac{12\mu}{a_0^2 h_0^2} a\frac{da}{dt} = 0, \tag{6.99}$$

in the domain $0 \le x \le w_c(t)$, $0 \le y \le w_c(t)$. Mapping from the general solution we have

$$\Phi(x,y,t) = \frac{12\mu}{a_0^2 h_0^2}\, a\frac{da}{dt}, \tag{6.100}$$

$$\alpha = \beta = w_c(t), \tag{6.101}$$

and so equation (6.99) has a solution given by

$$\begin{aligned}
p(x,y,t) &= \int_0^{w_c(t)} \int_0^{w_c(t)} \frac{12\mu}{a_0^2 h_0^2} a\frac{da}{dt}\, G(x,y,\xi,\eta)\, d\eta\, d\xi \\
&= \frac{12\mu}{a_0^2 h_0^2} a\frac{da}{dt} \sum_{n=1}^{\infty} \frac{w_c^2(1-(-1)^n)}{n^3\pi^3 \sinh(n\pi)} \sin\left(\frac{n\pi x}{w_c}\right) \\
&\times\left[\left\{ \cosh\left(\frac{n\pi}{w_c}(w_c - y)\right) - 1\right\} \sinh\left(\frac{n\pi y}{w_c}\right)\right. \\
&\left. + \left\{ \cosh\left(\frac{n\pi y}{w_c}\right) - 1\right\} \sinh\left(\frac{n\pi}{w_c}(w_c - y)\right)\right].
\end{aligned} \tag{6.102}$$

Now substituting this expression for the pressure into (6.75) yields

$$\begin{aligned}
-F &= \int_{\Omega(t)} p\, dxdy \\
&= \frac{12\mu w_c^2(t)}{a_0^2 h_0^2} a\frac{da}{dt} \sum_{n=1}^{\infty} \frac{(1-(-1)^n)}{n^3\pi^3 \sinh(n\pi)}\, I_1(n),
\end{aligned} \tag{6.103}$$

where $I_1(n)$ is the integral given by

$$\begin{aligned}
I_1(n) &= \int_0^{w_c(t)} \int_0^{w_c(t)} \sin\left(\frac{n\pi x}{w_c}\right) \left[\left\{ \cosh\left(\frac{n\pi}{w_c}(w_c - y)\right) - 1\right\} \sinh\left(\frac{n\pi y}{w_c}\right)\right. \\
&\qquad \left. + \left\{ \cosh\left(\frac{n\pi y}{w_c}\right) - 1\right\} \sinh\left(\frac{n\pi}{w_c}(w_c - y)\right)\right] dxdy \\
&= \frac{w_c^2}{n^2\pi^2}(1-(-1)^n)\big[-1 + 2e^{n\pi} - e^{2n\pi} + n\pi e^{n\pi}\sinh(n\pi)\big]e^{-n\pi}.
\end{aligned} \tag{6.104}$$

Now we set $a(t) = w_c^2(t)$ and substitute this, along with $I_1(n)$, into equation (6.103) to obtain the differential equation

$$\frac{24\mu}{a_0^2 h_0^2} S\, w_c^7(t)\, \frac{dw_c}{dt} = -F, \tag{6.105}$$

where

$$S = \sum_{n=1}^{\infty} \frac{e^{-n\pi}(1-(-1)^n)^2}{n^5\pi^5 \sinh(n\pi)} \left[2e^{n\pi} - e^{2n\pi} - 1 + n\pi e^{n\pi}\sinh(n\pi)\right]. \tag{6.106}$$

Separating the variables in (6.105) and integrating yields

$$\int w_c^7\, dw_c = -\int \frac{F}{S}\frac{a_0^2 h_0^2}{24\mu}\, dt, \tag{6.107}$$

which can be integrated to obtain the solution

$$w_c(t) = \left(w_0^8 - \frac{a_0^2 h_0^2 F}{3\mu S}\, t\right)^{1/8}. \tag{6.108}$$

This solution does not encounter the intrinsic methodological problems that the previous one did. As m and n are increased, the solution tends to a steady state and we do not encounter the problem of the plates separating at an instantaneous time. We can, however, find the critical time at which the solution breaks down, i.e. at the point where the plates separate, by setting $w_c(t) = 0$ which gives the separation time to be

$$t_d = \frac{6\mu S w_0^8}{a_0^2 h_0^2 F}. \tag{6.109}$$

The separation time found by Thaminda *et al.* [57] is defined to be

$$\hat{t}_d^* = \frac{3\pi\mu r_0^4}{8Fh_0^2}. \tag{6.110}$$

Using the parameters stated in Thaminda *et al.* [57], we set

$$r_0 = 6\,\mathrm{cm}, \quad h_0 = 0.22\,\mathrm{mm}, \quad F = 13.7\,\mathrm{N}, \quad \mu = 0.8\,\mathrm{Nsm}^{-2}, \tag{6.111}$$

and so the separation time found in this chapter is $t_d = 0.00024$ and the separation time found by Thaminda *et al.* is $\hat{t}_d^* = 0.00021$, with both times given to five decimal places. We believe that the small discrepency in the time is due to the summation involved in the separation time in this thesis. The terms in (6.106) should be summed to infinity, but we have taken an approximation of ten terms. Including more terms in the summation should bring the two separation times closer together.

In the paper by Thaminda *et al.* [57], the authors then proceeded to numerically simulate the phenomenon of the separating plates with an emphasis on looking at the fractal patterns produced by the fluid adhesive strip. Their main focus was then the consequent 'shielding time' and 'shielding distance' associated with the receding fractal fingers, properties that are highly dependent upon many material and numerical values. These authors also obtained a theoretical estimate for the cumulative node number density, which is the number of nodes within a radius r divided by the total number of nodes, where a node is defined to be the point at which a finger stops and others continue to expand. Although this part of the work is interesting and adds an extra dimension to the purely analytical solution, it is outwith the scope of this thesis, so we mention it here briefly just for completeness.

6.3 The Liquid Crystal Problem

We can now extend the work done in the previous section to the case of a liquid crystal. The general set up is the same, except that the viscous stress tensor now takes a more complicated form. For the case of an isotropic liquid, equation (6.59) can be written in the alternative form

$$-\tilde{p}_{,i} + \tilde{t}_{ij,j} = 0, \tag{6.112}$$

where $\tilde{t}_{ij} = \mu(v_{i,j} + v_{j,i})$, $\tilde{p} = p + w$ and w is an energy density. In the case of a SmC* liquid crystal, the stress tensor is given by

$$\tilde{t}_{ij} = \tilde{t}^s_{ij} + \tilde{t}^{ss}_{ij}, \tag{6.113}$$

where $\tilde{t}^s_{ij}$ and $\tilde{t}^{ss}_{ij}$ are the symmetric and skew-symmetric parts of the stress tensor, respectively. The stress tensor we will work with is given by

$$\tilde{t}^s_{ij} = \mu_0 D_{ij} + \mu_4(D^c_i c_j + D^c_j c_i) + \lambda_2(C_i c_j + C_j c_i), \tag{6.114}$$

$$\tilde{t}^{ss}_{ij} = \lambda_2(D^c_j c_i - D^c_i c_j) + \lambda_5(C_j c_i - C_i c_j). \tag{6.115}$$

This is not the full stress tensor, which can be found in (2.53) and (2.54), but in order to make the computations manageable we suppose the problem to be dependent on only four key viscosities, instead of the full twenty. These key viscosities have been selected as being physically important; their selection is based upon known results from other model problems in the SmC* literature [49]. The viscosities we select are μ_0, λ_2, λ_5 and μ_4. The first viscosity, μ_0, is related to the usual Newtonian isotropic viscosity, λ_5 is the SmC rotational viscosity related to the c-director, and λ_2 and μ_4 are two viscosities closely related via standard inequalities that arise from the positivity of the dissipation function. The three viscosities λ_2, λ_5 and μ_4, are all nematic-like in their behaviour. For more information on these viscosities we direct the reader to Stewart [49]. In this set-up we have that the unit normal and c-director are given by

$$\mathbf{c} = (0, \cos\phi, \sin\phi), \tag{6.116}$$

$$\mathbf{a} = (1, 0, 0), \tag{6.117}$$

where $\phi = \phi(z)$, and the velocity is given by $\mathbf{u}(x, y, z, t) = (u, v, w)$. We again make use of the quantities defined in equations (6.18), (6.19) and (6.20), so inserting $\mathbf{a}$, $\mathbf{c}$ and $\mathbf{v}$ into these quantities yields the rate of strain tensor

$$D_{11} = \frac{\partial u}{\partial x}, \quad D_{22} = \frac{\partial v}{\partial y}, \quad D_{33} = \frac{\partial w}{\partial z}, \tag{6.118}$$

$$D_{12} = D_{21} = \frac{1}{2}\Big(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\Big), \tag{6.119}$$

$$D_{13} = D_{31} = \frac{1}{2}\Big(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\Big), \tag{6.120}$$

$$D_{32} = D_{23} = \frac{1}{2}\Big(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\Big), \tag{6.121}$$

the vorticity tensor

$$W_{11} = W_{22} = W_{33} = 0, \tag{6.122}$$

$$W_{12} = -W_{21} = \frac{1}{2}\Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big), \tag{6.123}$$

$$W_{13} = -W_{31} = \frac{1}{2}\Big(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\Big), \tag{6.124}$$

$$W_{32} = -W_{23} = \frac{1}{2}\Big(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\Big), \tag{6.125}$$

the co-rotational time flux for $\mathbf{a}$

$$A_1 = 0, \tag{6.126}$$

$$A_2 = \frac{1}{2}\Big(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\Big), \tag{6.127}$$

$$A_3 = \frac{1}{2}\Big(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\Big), \tag{6.128}$$

the co-rotational time flux for $\mathbf{c}$

$$C_1 = -\frac{1}{2}\left(\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)\cos\phi + \left(\frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}\right)\sin\phi\right), \tag{6.129}$$

$$C_2 = -\frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)\sin\phi, \tag{6.130}$$

$$C_3 = \frac{1}{2}\left(\frac{\partial v}{\partial z} - \frac{\partial w}{\partial y}\right)\cos\phi, \tag{6.131}$$

and the quantities D_i^a and D_i^c

$$\mathbf{D}^a = \frac{1}{2}\left(2\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right), \tag{6.132}$$

$$\begin{aligned}\mathbf{D}^c = \frac{1}{2}\Bigg(&\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)\cos\phi + \left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right)\sin\phi,\\ &2\frac{\partial v}{\partial y}\cos\phi + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\sin\phi,\\ &2\frac{\partial w}{\partial z}\sin\phi + \left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right)\cos\phi\Bigg).\end{aligned} \tag{6.133}$$

After extensive manipulations, we find the individual components of the divergence of the symmetric part of the stress tensor to be the following:

$$
\begin{aligned}
\tilde{t}^s_{11,x} &= \mu_0 u_{,xx}, &(6.134)\\
\tilde{t}^s_{21,x} &= \frac{1}{2}u_{,xy}\big(\mu_0 + (\mu_4 - \lambda_2)\cos^2\phi\big) + \frac{1}{2}v_{,xx}\big(\mu_0 + (\mu_4 + \lambda_2)\cos^2\phi\big)\\
&\quad + \frac{1}{2}u_{,xz}(\mu_4 - \lambda_2)\cos\phi\sin\phi + \frac{1}{2}w_{,xx}(\mu_4 + \lambda_2)\cos\phi\sin\phi, &(6.135)\\
\tilde{t}^s_{31,x} &= \frac{1}{2}u_{,xz}\big(\mu_0 + (\mu_4 + \lambda_2)\sin^2\phi\big) + \frac{1}{2}w_{,xx}\big(\mu_0 + (\mu_4 - \lambda_2)\sin^2\phi\big)\\
&\quad + \frac{1}{2}v_{,xx}(\mu_4 - \lambda_2)\cos\phi\sin\phi + \frac{1}{2}u_{,xy}(\mu_4 + \lambda_2)\cos\phi\sin\phi, &(6.136)\\
\tilde{t}^s_{12,y} &= \frac{1}{2}u_{,yy}\big(\mu_0 + (\mu_4 - \lambda_2)\cos^2\phi\big) + \frac{1}{2}v_{,xy}\big(\mu_0 + (\mu_4 + \lambda_2)\cos^2\phi\big)\\
&\quad + \frac{1}{2}u_{,yz}(\mu_4 - \lambda_2)\cos\phi\sin\phi + \frac{1}{2}w_{,xy}(\mu_4 + \lambda_2)\cos\phi\sin\phi, &(6.137)\\
\tilde{t}^s_{22,y} &= v_{,yy}(\mu_0 - \mu_4\cos^2\phi) + v_{,yz}(\mu_4 - \lambda_2)\cos\phi\sin\phi + w_{,yy}(\mu_4 + \lambda_2)\cos\phi\sin\phi, &(6.138)\\
\tilde{t}^s_{32,y} &= v_{,yy}\mu_4\cos\phi\sin\phi + w_{,yz}\mu_4\sin\phi(\cos\phi + \sin\phi)\\
&\quad + \frac{1}{2}w_{,yy}\big(\mu_0 + \mu_4 - \lambda_2(\cos^2\phi - \sin^2\phi)\big) + \frac{1}{2}v_{,yz}\big(\mu_0 + \mu_4 + \lambda_2(\cos^2\phi - \sin^2\phi)\big), &(6.139)\\
\tilde{t}^s_{13,z} &= \frac{1}{2}u_{,y}(\mu_4 - \lambda_2)(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz} + \frac{1}{2}v_{,x}(\mu_4 + \lambda_2)(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz}\\
&\quad + u_{,z}(\mu_4 - \lambda_2)\cos\phi\sin\phi\frac{d\phi}{dz} + w_{,x}(\mu_4 + \lambda_2)\cos\phi\sin\phi\frac{d\phi}{dz} + \frac{1}{2}u_{,zz}\big(\mu_0 + (\mu_4 - \lambda_2)\sin^2\phi\big)\\
&\quad + \frac{1}{2}w_{,yz}\big(\mu_0 + (\mu_4 + \lambda_2)\sin^2\phi\big) + \frac{1}{2}u_{,yz}(\mu_4 - \lambda_2)\cos\phi\sin\phi + \frac{1}{2}v_{,xz}(\mu_4 + \lambda_2)\cos\phi\sin\phi, &(6.140)\\
\tilde{t}^s_{23,z} &= v_{,yy}\mu_4\cos\phi\sin\phi + w_{,zz}\mu_4\cos\phi\sin\phi + v_{,y}\mu_4(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz}\\
&\quad + w_{,z}\mu_4(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz} - 2v_{,z}\lambda_2\cos\phi\sin\phi\frac{d\phi}{dz} + 2w_{,y}\lambda_2\cos\phi\sin\phi\frac{d\phi}{dz}\\
&\quad + \frac{1}{2}v_{,zz}\big(\mu_0 + (\mu_4 - \lambda_2)\sin^2\phi + (\mu_4 + \lambda_2)\cos^2\phi\big)\\
&\quad + \frac{1}{2}w_{,yz}\big(\mu_0 + (\mu_4 + \lambda_2)\sin^2\phi + (\mu_4 - \lambda_2)\cos^2\phi\big), &(6.141)\\
\tilde{t}^s_{33,z} &= v_{,z}(\mu_4 + \lambda_2)(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz} + w_{,y}(\mu_4 - \lambda_2)(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz}\\
&\quad + 2w_{,z}\mu_4\cos\phi\sin\phi\frac{d\phi}{dz} + w_{,zz}(\mu_0 + 2\mu_4\sin^2\phi)\\
&\quad + v_{,zz}(\mu_4 + \lambda_2)\cos\phi\sin\phi + w_{,yz}(\mu_4 - \lambda_2)\cos\phi\sin\phi. &(6.142)
\end{aligned}
$$

We also have that the components of the divergence of the skew-symmetric part of the stress tensor are given by

$$\tilde{t}^{ss}_{11,x} = \tilde{t}^{ss}_{22,y} = \tilde{t}^{ss}_{33,z} = 0, \tag{6.143}$$

$$\tilde{t}^{ss}_{21,x} = \frac{1}{2}v_{,xx}(\lambda_2+\lambda_5)\cos^2\phi + \frac{1}{2}u_{,xy}(\lambda_2-\lambda_5)\cos^2\phi + \frac{1}{2}u_{,xz}(\lambda_2-\lambda_5)\cos\phi\sin\phi + \frac{1}{2}w_{,xx}(\lambda_2+\lambda_5)\cos\phi\sin\phi, \tag{6.144}$$

$$\tilde{t}^{ss}_{31,x} = \frac{1}{2}w_{,xx}(\lambda_2+\lambda_5)\sin^2\phi + \frac{1}{2}u_{,xz}(\lambda_2-\lambda_5)\sin^2\phi + \frac{1}{2}u_{,xy}(\lambda_2-\lambda_5)\cos\phi\sin\phi + \frac{1}{2}v_{,xx}(\lambda_2+\lambda_5)\cos\phi\sin\phi, \tag{6.145}$$

$$\tilde{t}^{ss}_{12,y} = \frac{1}{2}u_{,yy}(\lambda_5-\lambda_2)\cos^2\phi - \frac{1}{2}v_{,xy}(\lambda_5+\lambda_2)\cos^2\phi + \frac{1}{2}u_{,yz}(\lambda_5-\lambda_2)\cos\phi\sin\phi - \frac{1}{2}w_{,xy}(\lambda_5+\lambda_2)\cos\phi\sin\phi, \tag{6.146}$$

$$\tilde{t}^{ss}_{32,y} = v_{,yy}\lambda_2\cos\phi\sin\phi - w_{,yz}\lambda_2\cos\phi\sin\phi + \frac{1}{2}v_{,yz}\big(\lambda_2(\sin^2\phi-\cos^2\phi)-\lambda_5\big) + \frac{1}{2}w_{,yy}\big(\lambda_2(\sin^2\phi-\cos^2\phi)+\lambda_5\big), \tag{6.147}$$

$$\begin{aligned}\tilde{t}^{ss}_{13,z} &= \frac{1}{2}u_{,yz}(\lambda_5-\lambda_2)\cos\phi\sin\phi - \frac{1}{2}v_{,xy}(\lambda_5+\lambda_2)\cos\phi\sin\phi + \frac{1}{2}u_{,zz}(\lambda_5-\lambda_2)\sin^2\phi \\ &\quad - \frac{1}{2}w_{,xz}(\lambda_5+\lambda_2)\sin^2\phi + \frac{1}{2}u_{,y}(\lambda_5-\lambda_2)(\cos^2\phi-\sin^2\phi)\frac{d\phi}{dz} \\ &\quad - \frac{1}{2}v_{,x}(\lambda_5+\lambda_2)(\cos^2\phi-\sin^2\phi)\frac{d\phi}{dz} + u_{,z}(\lambda_5-\lambda_2)(\cos\phi\sin\phi)\frac{d\phi}{dz} \\ &\quad - w_{,x}(\lambda_5+\lambda_2)(\cos\phi\sin\phi)\frac{d\phi}{dz},\end{aligned} \tag{6.148}$$

$$\begin{aligned}\tilde{t}^{ss}_{23,z} &= v_{,y}\lambda_2(\sin^2\phi-\cos^2\phi)\frac{d\phi}{dz} - w_{,z}\lambda_2(\sin^2\phi-\cos^2\phi)\frac{d\phi}{dz} - 2v_{,z}\lambda_2\cos\phi\sin\phi\frac{d\phi}{dz} \\ &\quad - 2w_{,y}\lambda_2\cos\phi\sin\phi\frac{d\phi}{dz} + w_{,zz}\lambda_2\cos\phi\sin\phi - v_{,yz}\lambda_2\cos\phi\sin\phi \\ &\quad + \frac{1}{2}v_{,zz}\big(\lambda_2(\cos^2\phi-\sin^2\phi)+\lambda_5\big) + \frac{1}{2}w_{,yz}\big(\lambda_2(\cos^2\phi-\sin^2\phi)-\lambda_5\big).\end{aligned} \tag{6.149}$$

Clearly, if we insert the above evaluated stress tensor into governing equations (6.112) we obtain a very complex nonlinear system. Even though we simplified the problem by including only four viscosities, it is still impossible to analytically solve the equations in their present form. Hence we simplify further by including only the first and second derivatives in z. This is a reasonable and justifiable thing to do since the z-direction is much smaller than the other two directions and so the z derivatives will be much larger. So, if we include only the terms involving $\frac{\partial}{\partial z}$ and $\frac{\partial^2}{\partial z^2}$, we obtain the simplified expressions

$$\tilde{t}_{11,x} = \tilde{t}_{21,x} = \tilde{t}_{31,x} = 0, \tag{6.150}$$

$$\tilde{t}_{12,y} = \tilde{t}_{22,y} = \tilde{t}_{32,y} = 0, \tag{6.151}$$

$$\tilde{t}_{13,z} = u_{,z} \cos\phi \sin\phi \frac{d\phi}{dz}(\mu_4 + \lambda_5 - 2\lambda_2) + \frac{1}{2} u_{,zz}\big(\mu_0 + (\mu_4 + \lambda_5 - 2\lambda_2)\sin^2\phi\big), \tag{6.152}$$

$$\begin{aligned}\tilde{t}_{23,z} &= w_{,z}\lambda_2(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz} + w_{,zz}\lambda_2\cos\phi\sin\phi - 4v_{,z}\lambda_2\cos\phi\sin\phi\frac{d\phi}{dz} \\ &\quad + \frac{1}{2}v_{,zz}\big(\mu_0 + \lambda_5 + (\mu_4 - 2\lambda_2)\sin^2\phi + (\mu_4 + 2\lambda_2)\cos^2\phi\big),\end{aligned} \tag{6.153}$$

$$\begin{aligned}\tilde{t}_{33,z} &= 2w_{,z}\mu_4\cos\phi\sin\phi\frac{d\phi}{dz} + w_{,zz}(\mu_0 + 2\mu_4\sin^2\phi) + v_{,z}(\mu_4 + \lambda_2)(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz} \\ &\quad + v_{,zz}(\mu_4 + \lambda_2)\cos\phi\sin\phi.\end{aligned} \tag{6.154}$$

The governing equations in (6.112) can now be written explicitly as

$$-\tilde{p}_{,x} + \tilde{t}_{13,z} = 0, \tag{6.155}$$

$$-\tilde{p}_{,y} + \tilde{t}_{23,z} = 0, \tag{6.156}$$

$$-\tilde{p}_{,z} + \tilde{t}_{33,z} = 0, \tag{6.157}$$

with the $\tilde{t}_{i3,z}$ as shown in (6.152)-(6.154). Even with the further simplified terms, this an extremely difficult system of equations to solve. As a first approach, and to compare with the case of an isotropic fluid adhesive strip, we linearise equations (6.155)-(6.157) in ϕ but not in the velocity. We also exclude the terms involving derivatives in w since the velocity in the z-direction is anticipated to be negligible. Hence, linearising the stress tensor in ϕ produces the governing equations

$$
\begin{aligned}
-\tilde{p}_{,x} + \frac{1}{2}\frac{\partial^2 u}{\partial z^2}\mu_0 =&0, && (6.158)\\
-\tilde{p}_{,y} + \frac{1}{2}\frac{\partial^2 v}{\partial z^2}(\mu_0 + \mu_4 + \lambda_5 + 2\lambda_2) =&0, && (6.159)\\
-\tilde{p}_{,z} + (\mu_4 + \lambda_2)\left[\frac{\partial v}{\partial z}\frac{d\phi}{dz} + \frac{\partial^2 v}{\partial z^2}\phi\right] =&0. && (6.160)
\end{aligned}
$$

We can write equations (6.158) and (6.159) in the more compact form

$$
\begin{aligned}
\frac{\partial^2 u}{\partial z^2} &= \frac{2}{\mu_0}\tilde{p}_{,x}, && (6.161)\\
\frac{\partial^2 v}{\partial z^2} &= \frac{2}{\zeta_0}\tilde{p}_{,y}, && (6.162)
\end{aligned}
$$

where $\zeta_0 = \mu_0 + \mu_4 + \lambda_5 + 2\lambda_2$, then integrating equation (6.161) with respect to z twice yields

$$
u = \frac{\tilde{p}_{,x}}{\mu_0}z^2 + f_1(x, y, t)z + f_2(x, y, t), \qquad (6.163)
$$

which, after applying the boundary conditions $u = 0$ at $z = 0$ and $z = h$, reduces to

$$
u = -\frac{\tilde{p}_{,x}}{\mu_0}(hz - z^2). \qquad (6.164)
$$

Similarly, we can integrate equations (6.162) twice with respect to z and apply the equivalent boundary conditions on v to obtain

$$
v = -\frac{\tilde{p}_{,y}}{\zeta_0}(hz - z^2). \qquad (6.165)
$$

Now averaging the variables u and v as in equations (6.63)–(6.65) we find

$$\bar{u} = -\frac{1}{h}\int_0^h \frac{\tilde{p}_{,x}}{\mu_0}(hz - z^2)dz = -\frac{\tilde{p}_{,x}h^2}{6\mu_0}, \tag{6.166}$$

and similarly

$$\bar{v} = -\frac{\tilde{p}_{,y}h^2}{6\zeta_0}. \tag{6.167}$$

Equation (6.72) also holds as in the previous section, and inserting this, along with (6.166) and (6.167), into equation (6.70) yields

$$\frac{\partial^2 \tilde{p}}{\partial x^2} + \zeta_1 \frac{\partial^2 \tilde{p}}{\partial y^2} = -\frac{6\mu_0}{a_0^2 h_0^2} a \frac{da}{dt}, \tag{6.168}$$

where the dimensionless anisotropic control parameter ζ_1 is defined to be $\zeta_1 = \mu_0/\zeta_0$. We know that ζ_1 is always positive due to constraint (2.59). If we consider the term ζ_1, we see that it is given by

$$\begin{aligned}\zeta_1 &= \frac{\mu_0}{\mu_0 + \mu_4 + \lambda_5 + 2\lambda_2} \\ &= \frac{\mu_0}{\mu_0 + \bar{\mu}_4\theta^2 + \bar{\lambda}_5\theta^2 + 2\bar{\lambda}_2\theta^2},\end{aligned} \tag{6.169}$$

where we have included the dependence on the tilt angle (see equations (2.55)–(2.58) for details). Clearly, this collapses to $\zeta_1 = 1$ in the linearised case. Now we rescale the y variable such that $y = \sqrt{\zeta_1}Y$, then equation (6.168) becomes

$$\frac{\partial^2 \tilde{p}}{\partial x^2} + \frac{\partial^2 \tilde{p}}{\partial Y^2} = -\frac{6\mu_0}{a_0^2 h_0^2} a \frac{da}{dt}, \tag{6.170}$$

and equation (6.75) becomes

$$
\begin{aligned}
-F &= \int_0^{w_c(t)} \int_0^{w_c(t)} \tilde{p}(x, y, t)\, dx\, dy \\
&= \int_0^{\frac{w_c(t)}{\sqrt{\zeta_1}}} \int_0^{w_c(t)} \sqrt{\zeta_1}\, \tilde{p}(x, Y, t)\, dx\, dy. \qquad (6.171)
\end{aligned}
$$

We can obtain the particular solution of equation (6.170) from equation (6.98) by setting $\alpha = w_c$ and $\beta = w_c/\sqrt{\zeta_1}$. Hence the solution of (6.170) is given by

$$
\begin{aligned}
\tilde{p}(x, Y, t) =& \frac{12\mu_0}{a_0^2 h_0^2} a \frac{da}{dt} \sum_{n=1}^{\infty} \frac{w_c^2 (1 - (-1)^n) \sin\left(\frac{n\pi x}{w_c}\right)}{n^3 \pi^3 \sinh\left(\frac{n\pi}{\sqrt{\zeta_1}}\right)} \\
&\times \Bigg[\Big\{ \cosh\left(\frac{n\pi}{w_c}\left(\frac{w_c}{\sqrt{\zeta_1}} - Y\right)\right) - 1 \Big\} \sinh\left(\frac{n\pi Y}{w_c}\right) \\
&+ \Big\{ \cosh\left(\frac{n\pi Y}{w_c}\right) - 1 \Big\} \sinh\left(\frac{n\pi}{w_c}\left(\frac{w_c}{\sqrt{\zeta_1}} - Y\right)\right) \Bigg]. \qquad (6.172)
\end{aligned}
$$

Inserting (6.172) into the pressure constraint (6.171) yields

$$
\begin{aligned}
-F &= \int_0^{\frac{w_c(t)}{\sqrt{\zeta_1}}} \int_0^{w_c(t)} \sqrt{\zeta_1}\, \tilde{p}(x, Y, t)\, dx\, dY \\
&= \frac{12\mu_0 \sqrt{\zeta_1}}{a_0^2 h_0^2} a \frac{da}{dt} w_c^2(t) \sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n^3 \pi^3 \sinh\left(\frac{n\pi}{\sqrt{\zeta_1}}\right)} I_2(n), \qquad (6.173)
\end{aligned}
$$

where

$$
\begin{aligned}
I_2(n) =& \int_0^{\frac{w_c(t)}{\sqrt{\zeta_1}}} \int_0^{w_c(t)} \sin\left(\frac{n\pi x}{w_c}\right) \Bigg[\Big\{ \cosh\left(\frac{n\pi}{w_c}\left(\frac{w_c}{\sqrt{\zeta_1}} - Y\right)\right) - 1 \Big\} \sinh\left(\frac{n\pi Y}{w_c}\right) \\
&+ \Big\{ \cosh\left(\frac{n\pi Y}{w_c}\right) - 1 \Big\} \sinh\left(\frac{n\pi}{w_c}\left(\frac{w_c}{\sqrt{\zeta_1}} - Y\right)\right) \Bigg] dx\, dy \\
=& \frac{w_c(t)(1 - (-1)^n) e^{-n\pi/\sqrt{\zeta_1}}}{n^2 \pi^2} \left[-1 + 2e^{n\pi/\sqrt{\zeta_1}} - e^{2n\pi/\sqrt{\zeta_1}} + \frac{n\pi}{\sqrt{\zeta_1}} \sinh\left(\frac{n\pi}{\sqrt{\zeta_1}}\right) e^{-n\pi/\sqrt{\zeta_1}} \right]. \qquad (6.174)
\end{aligned}
$$

Hence equation (6.173) becomes

$$\frac{24\mu_0\sqrt{\zeta_1}}{a_0^2h_0^2}w_c^7(t)\tilde{S} = -F, \tag{6.175}$$

where we have written $a(t)$ in terms of $w_c(t)$ and have defined the sum $\tilde{S}_n$ to be

$$\tilde{S} = \sum_{n=1}^{\infty} \frac{(1-(-1)^n)^2 e^{-n\pi/\sqrt{\zeta_1}}}{n^5\pi^5 \sinh\left(\frac{n\pi}{\sqrt{\zeta_1}}\right)} \left[-1 + 2e^{n\pi/\sqrt{\zeta_1}} - e^{2n\pi/\sqrt{\zeta_1}} + \frac{n\pi}{\sqrt{\zeta_1}} \sinh\left(\frac{n\pi}{\sqrt{\zeta_1}}\right) e^{-n\pi/\sqrt{\zeta_1}} \right]. \tag{6.176}$$

Separating the variables in (6.175) and integrating yields the solution

$$w_c(t) = \left(w_0^8 - \frac{Fa_0^2h_0^2}{3\tilde{S}\mu_0\sqrt{\zeta_1}}\, t \right)^{1/8}. \tag{6.177}$$

We see that if we linearise in θ, i.e. set $\zeta_1 = 1$, the function in (6.177) collapses to the solution we found in (6.108) for the fluid adhesive strip. We can also find the separation time of the plates in terms of the control parameter ζ_1, which is given by

$$\tilde{t}_d = \frac{3\mu_0\sqrt{\zeta_1}w_0^8\tilde{S}}{a_0^2h_0^2F}. \tag{6.178}$$

This, again, collapses to the result found in the anisotropic case when the solution for θ is linearised. To obtain a solution for the pressure we used governing equations (6.158) and (6.159), but so far we have not used equation (6.160). If we consider this equation now, we find that, since the pressure is defined to be independent of z, it can be simplified to

$$(\mu_4 + \lambda_2)\left[\frac{\partial v}{\partial z}\frac{d\phi}{dz} + \frac{\partial^2 v}{\partial z^2}\phi\right] = 0. \tag{6.179}$$

Since the combination of viscosities $\mu_4 + \lambda_2 \neq 0$, we can divide throughout by it and, noting that the bracketed term can be written as the derivative of a product, equation (6.179) reduces to

$$(v_{,z}\phi)_{,z} = 0, \tag{6.180}$$

which can be solved directly to find the solution for $\phi(z)$ given by

$$\phi(z) = -\frac{c_1\zeta_0}{\tilde{p}_{,y}(h - 2z)}, \tag{6.181}$$

when we substitute $v(x, y, z, t)$ from (6.165). From the solution we have obtained for $\phi(z)$, we can see that is is possible for the solution to become infinite for a finite value of z. This means that the c-director given by $\mathbf{c} = (0, \cos\phi, \sin\phi)$ begins to oscillate infinitely at the point $z = h/2$. We were not expecting to find a singularity in this problem and the fact that we have suggests that we possibly omitted too much information when we linearised equation (6.160), so now we consider a nonlinear approach.

6.3.1 Nonlinear Approach

We now consider equation (6.157) again, but with the nonlinearised term for $\tilde{t}_{33,z}$. Hence we have

$$v_{,z}(\mu_4 + \lambda_2)(\cos^2\phi - \sin^2\phi)\frac{d\phi}{dz} + v_{,zz}(\mu_4 + \lambda_2)\cos\phi\sin\phi = 0, \tag{6.182}$$

where we have again omitted the terms involving w for the reasons stated earlier. Inserting the known expression for v yields a first order differential equation in ϕ given by

$$\frac{\tilde{p}_{,y}}{\zeta_0}(\cos^2\phi - \sin^2\phi)(h-2z)\frac{d\phi}{dz} - 2\frac{\tilde{p}_{,y}}{\zeta_0}\cos\phi\sin\phi = 0, \tag{6.183}$$

which can be written as

$$\cos(2\phi)(h-2z)\frac{d\phi}{dz} - \sin(2\phi) = 0. \tag{6.184}$$

Separating the variables in (6.184) gives

$$\int \frac{\cos 2\phi}{\sin 2\phi}\, d\phi = \int \frac{dz}{h-2z}, \tag{6.185}$$

and this can be integrated to obtain the solution

$$\phi(z) = \frac{1}{2}\sin^{-1}\left(\frac{c_2}{h-2z}\right), \tag{6.186}$$

where c_2 is a constant of integration. We note here that the series expansion for the inverse sine function is given by

$$\sin^{-1}(x) = \sum_{n=0}^{\infty} \frac{(2n)!}{2^{2n}(n!)^2(2n+1)} x^{2n+1} = x + \frac{1}{6}x^3 + \frac{3}{40}x^5 + \cdots, \tag{6.187}$$

as described in Gradshteyn and Ryzhik [13, Eqn. (1.641.1)], and so the solution (6.186) collapses to the solution in (6.181) when considering the linearised case.

6.3.2 Alternative Approach

We now remark on an alternative approach that uses conventional fluid mechanics methods [6]. This method allows the elimination of the pressure to obtain equations for ϕ, u and v explicitly, with the pressure being identifiable in special cases. We first eliminate the pressure from equations (6.158)-(6.160) by taking the curl of these equations. If we write (6.112) in the form

$$\nabla \mathbf{p} + \nabla .\, \mathbf{T} = 0, \tag{6.188}$$

where $\mathbf{T}$ is a tensor representing the stresses $\tilde{t}_{ij}$, then taking the curl of this equation yields

$$\begin{aligned} 0 &= \nabla \times (\nabla \mathbf{p} + \nabla .\, \mathbf{T}) \\ &= \nabla \times (\nabla \mathbf{p}) + \nabla \times (\nabla \cdot \mathbf{T}) \\ &= \nabla \times (\nabla \cdot \mathbf{T}), \end{aligned} \tag{6.189}$$

since $\nabla \times (\nabla \mathbf{p}) = 0$. In the linearised set-up, the quantity $\nabla \cdot \mathbf{T}$ is defined to be

$$\nabla \cdot \mathbf{T} = \Big[\frac{\mu_0}{2} u_{,zz}, \frac{\zeta_0}{2} v_{,zz}, (\mu_4 + \lambda_2)(v_{,z}\phi)_{,z}\Big], \tag{6.190}$$

then taking the curl yields

$$\begin{aligned} \nabla \times (\nabla \cdot \mathbf{T}) = \Big[&(\mu_4 + \lambda_2)(v_{,z}\phi)_{,zy} - \frac{\zeta_0}{2} v_{,zzz}, \\ &\frac{\mu_0}{2} u_{,zzz} - (\mu_4 + \lambda_2)(v_{,z}\phi)_{,zx}, \\ &\frac{\zeta_0}{2} v_{,zzz} - \frac{\mu_0}{2} u_{,zzz}\Big]. \end{aligned} \tag{6.191}$$

We require that $\nabla \times (\nabla \cdot \mathbf{T}) = 0$, so we must satisfy the three equations

$$(\mu_4 + \lambda_2)(v_{,z}\phi)_{,zy} - \frac{\zeta_0}{2} v_{,zzz} = 0, \tag{6.192}$$

$$(\mu_4 + \lambda_2)(v_{,z}\phi)_{,zx} - \frac{\mu_0}{2} u_{,zzz} = 0, \tag{6.193}$$

$$\frac{\zeta_0}{2} v_{,zzz} - \frac{\mu_0}{2} u_{,zzz} = 0. \tag{6.194}$$

First we differentiate equation (6.193) with respect to y, differentiate equation (6.192) with respect to x, and then subtract the resulting equations to obtain

$$\frac{\zeta_0}{2}v_{,zzzx} - \frac{\mu_0}{2}u_{,zzzy} = 0. \tag{6.195}$$

Equation (6.194) can be rearranged to give $u_{,zzz} = \frac{\zeta_0}{\mu_0}v_{,zzz}$ which, when substituted into (6.195), yields

$$0 = \frac{\zeta_0}{2}v_{,zzzx} - \frac{\mu_0}{2}\left(\frac{\zeta_0}{\mu_0}v_{,zzz}\right)_{,y} = \frac{\zeta_0}{2}\Big(v_{,zzzx} - v_{,zzzy}\Big). \tag{6.196}$$

Equation (6.196) has a solution given by

$$v(x,y,z,t) = G_1(x,y,t)z^2 + G_2(x,y,t)z + G_3(x,y,t) + G_4(x+y,z,t), \tag{6.197}$$

where the G_i are arbitrary functions. Applying the boundary condition at $z = 0$ gives $G_4 = -G_3$ and applying the boundary condition at $z = h$ gives $G_2 = -G_1 h$. Hence the velocity v can be written as

$$v(x,y,z,t) = -G_1(x,y,t)(hz - z^2). \tag{6.198}$$

The form of the solution given in (6.198) is consistent with the solution for v found in (6.165), however, in the current method it is more difficult to determine the unknown function G_1 since we eliminated the pressure at the beginning of this section. In a similar way we can also find the velocity u to be

$$u(x,y,z,t) = -F_1(x,y,t)(hz - z^2). \tag{6.199}$$

Inserting v from (6.198) into equation (6.192) yields

$$(\mu_4 + \lambda_2)(G_{1,y}(h - 2z)\phi_{,z} + 2G_{1,y}\phi) = 0, \tag{6.200}$$

which implies that

$$\frac{d\phi}{dz}(h - 2z) - 2\phi = 0, \tag{6.201}$$

or

$$\frac{\partial}{\partial y}\big(G_1(x, y, t)\big) = 0. \tag{6.202}$$

The solution to equation (6.201) is given by

$$\phi(z) = \frac{c_3}{h - 2z}, \tag{6.203}$$

where c_3 is an arbitrary constant of integration. The form of $\phi(z)$ in (6.203) is exactly the same as that found in (6.181). If we assume that (6.202) holds true then this implies that $G_1 = G_1(x, t)$, i.e. the y-direction velocity is independent of y. Although this seems counter-intuitive, it is a reasonable option. For example, in the case of a simple shear flow in the x-direction, the velocity profile may be approximated by $\mathbf{v} = (kz, 0, 0)$, for some constant k, hence the flow in the x-direction is independent of x. Hence, it is possible to have $G_1 = G_1(x, t)$, but we are unable to obtain any further information about the exact form of G_1.

If we now use this method again but with the nonlinearised forms of $\tilde{t}_{13,z}$, $\tilde{t}_{23,z}$ and $\tilde{t}_{33,z}$, we have

$$\nabla\cdot\mathbf{T} = \Big[\zeta_2 u_{,z}\cos\phi\sin\phi\frac{d\phi}{dz} + \frac{1}{2}u_{,zz}(\mu_0+\zeta_2\sin^2\phi),$$
$$-4\lambda_2 v_{,z}\cos\phi\sin\phi\frac{d\phi}{dz} + \frac{1}{2}v_{,zz}(\mu_0+\mu_4+\lambda_5+2\lambda_2(\cos^2\phi-\sin^2\phi)),$$
$$v_{,z}(\mu_4+\lambda_2)(\cos^2\phi-\sin^2\phi)\frac{d\phi}{dz} + v_{,zz}(\mu_4+\lambda_2)\cos\phi\sin\phi\Big], \tag{6.204}$$

and so the curl of this expression is given by

$$\nabla\times(\nabla\cdot\mathbf{T}) = \Big[v_{,zy}(\mu_4+\lambda_2)(\cos^2\phi-\sin^2\phi)\frac{d\phi}{dz} + v_{,zzy}(\mu_4+\lambda_2)\cos\phi\sin\phi$$
$$+\Big[4\lambda_2 v_{,z}\cos\phi\sin\phi\frac{d\phi}{dz}\Big]_{,z} - \frac{1}{2}\Big[v_{,zz}(\mu_0+\mu_4+\lambda_5+2\lambda_2(\cos^2\phi-\sin^2\phi))\Big]_{,z},$$
$$v_{,zx}(\mu_4+\lambda_2)(\cos^2\phi-\sin^2\phi)\frac{d\phi}{dz} + v_{,zzx}(\mu_4+\lambda_2)\cos\phi\sin\phi$$
$$-\Big[\zeta_2 u_{,z}\cos\phi\sin\phi\frac{d\phi}{dz}\Big]_{,z} - \frac{1}{2}\Big[u_{,zz}(\mu_0+\zeta_2\sin^2\phi)\Big]_{,z},$$
$$-4\lambda_2 v_{,zx}\cos\phi\sin\phi\frac{d\phi}{dz} + \frac{1}{2}v_{,zzx}(\mu_0+\mu_4+\lambda_5+2\lambda_2(\cos^2\phi-\sin^2\phi))$$
$$-\zeta_2 u_{,zy}\cos\phi\sin\phi\frac{d\phi}{dz} - \frac{1}{2}u_{,zzy}(\mu_0+\zeta_2\sin^2\phi)\Big]. \tag{6.205}$$

Equating the three terms of $\nabla\times(\nabla\cdot\mathbf{T})$ to be zero yields the three differential equations

$$4\lambda_2 v_{,z}\cos\phi\sin\phi\frac{d^2\phi}{dz^2}+4\lambda_2 v_{,z}(\cos^2\phi-\sin^2\phi)\left(\frac{d\phi}{dz}\right)^2+v_{,zzy}(\mu_4+\lambda_2)\cos\phi\sin\phi$$
$$+\frac{d\phi}{dz}\left[v_{,zy}(\mu_4+\lambda_2)(\cos^2\phi-\sin^2\phi)+8\lambda_2 v_{,zz}\cos\phi\sin\phi\right]-\frac{1}{2}v_{,zzz}(\mu_0+\mu_4+\lambda_5)=0, \tag{6.206}$$

$$\zeta_3 u_{,z}\cos\phi\sin\phi\frac{d^2\phi}{dz^2}-\zeta_3 u_{,z}(\cos^2\phi-\sin^2\phi)\left(\frac{d\phi}{dz}\right)^2-v_{,zzx}(\mu_4+\lambda_2)\cos\phi\sin\phi$$
$$-\frac{d\phi}{dz}\left[v_{,zx}(\mu_4+\lambda_2)(\cos^2\phi-\sin^2\phi)-2\zeta_3 u_{,zz}\cos\phi\sin\phi\right]+\frac{1}{2}u_{,zzz}(\mu_0+\zeta_3\sin^2\phi)=0, \tag{6.207}$$

$$\frac{d\phi}{dz}(4\lambda_2 v_{,zx}+\zeta_3 u_{,zy})\cos\phi\sin\phi-\frac{1}{2}v_{,zzx}(\mu_0+\mu_4+\lambda_5+2\lambda_2(\cos^2\phi-\sin^2\phi))$$
$$+\frac{1}{2}u_{,zzy}(\mu_0+\zeta_3\sin^2\phi)=0. \tag{6.208}$$

We had hoped that by considering the nonlinear case we would be able to obtain a little more information about the director ϕ, as we did with the previous method. However, this set of nonlinear equations is extremely complex and impossible to solve analytically, hence we simply state them here for completeness. It would be interesting to solve these equations numerically to see if they provide any more information about the director profile than has already been obtained.

6.4 Conclusions and Discussion

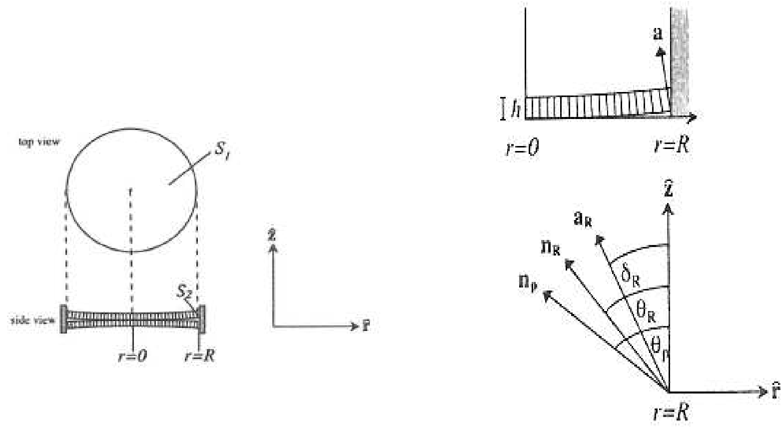
We extended the theoretical work already carried out into the 'pumping' effect in SmC* liquid crystals to include elastic effects. In this case, we found that the adjustments led to an inconsistency that meant the extended form of the director angle ϕ collapsed back to the form given in Stewart [52]. Clearly, the extended model in its current form is not viable, but could be further extended to overcome the problems encountered in this chapter.

We then considered the related problem of the separating plates and adapted work previously carried out in the literature [57] to a liquid crystal framework. We found a solution to the liquid crystal problem in terms of a dimensionless anisotropic parameter ζ_1. In the linearised case, this parameter is set to be one, and the solutions reduce to those of the isotropic case.

Chapter 7

Lipid Bilayers

The role of liquid crystals in biological research has become increasingly important in the last few years. Two subjects that seemed completely unrelated 20 years ago are now so closely linked that some of the most exciting and most important breakthroughs in biology are coming from research carried out in the study of SmA liquid crystals. Two of the most exciting new areas are targeted drug delivery systems and biosensors. One of the most popular continuum models that has been introduced to study the behaviour of lipid bilayers is the *spontaneous-curvature* model proposed by Helfrich [16]. This model is derived from the Frank energy for liquid crystals and works on the assumption that the layer normal, $\mathbf{a}$, and the director, $\mathbf{n}$, coincide, but does not take into account the tilt of the lipid molecules. The tilt of the molecules was included in a model proposed by May [32] that extended the original work of Helfrich. Although this updated model included molecular tilt, as well as different types of energy contributions, it did not allow for nonlinear contributions in the tilt and layer displacements. The most recent extension to the Helfrich model was proposed by De Vita and Stewart [55], which assumes that the tilt and displacement of the lipid bilayers are fully nonlinear functions. This model also allows for the decoupling of $\mathbf{n}$ and $\mathbf{a}$ when it is energetically favourable for them to do so, while taking into account the effects of bending, compression, splay, and tension of the lipid bilayer. We further extend the model of De Vita and Stewart [55] to include an additional anchoring term and briefly examine the role that surface anchoring plays in the case of circular membranes.



(a) (b)

Figure 7.1: (a) The top view and side view of a planar lipid bilayer suspended across a circular pore , and (b) a single lipid layer with height h and layer normal $\mathbf{a}$ and the proposed coordinate system. The vector $\mathbf{n}_R$ represents the director fixed at the boundary at an angle θ_R, $\mathbf{n}_P$ represents the preferred alignment of the director at the boundary with corresponding angle θ_R, and $\mathbf{a}_R$ represents the layer normal fixed at the boundary at an angle δ_R [55].

There are also other continuum models that have been developed to describe the behaviour of a lipid bilayer. One of these is the *bilayer coupling* model proposed by Svetina and Zeks [53, 54] that assumes the layers are coupled at a fixed distance apart but that there will be no exchange of molecules between layers. This model represents a Legendre transformation of the spontaneous-curvature model by Helfrich. We simply mention this alternative model for completeness and direct the reader to Svetina and Zeks [53] for more information. Now we focus on the extended Helfrich model as proposed by De Vita and Stewart [55].

7.1 Planar Lipid Bilayers

We look at the work by De Vita and Stewart [55] and expand it by using a weak anchoring energy density with an additional term. In this set-up with have the director and unit normal given by

$$\mathbf{n}(r) = -\sin\theta\hat{\mathbf{r}} + \cos\theta\hat{\mathbf{z}}, \tag{7.1}$$

$$\mathbf{a}(r) = -\sin\delta\hat{\mathbf{r}} + \cos\delta\hat{\mathbf{z}}, \tag{7.2}$$

respectively, where $\theta(r)$ and $\delta(r)$ are the angles formed by the director and unit normal with the $\hat{\mathbf{z}}$-axis. Then the scalar function $\Phi(r,z)$, describing the local planar layer structure, can be found by writing

$$\mathbf{a} = \frac{\nabla\Phi}{|\nabla\Phi|}. \tag{7.3}$$

We note that in the paper by De Vita and Stewart [55], the scalar function is denoted by $\eta(r,z)$, but we use $\Phi(r,z)$ in this chapter to be consistent with the other chapters. Equation (7.3) provides a linear partial equation to solve for Φ which has solution [59]

$$\Phi(r,z) = c(z - u(r)), \tag{7.4}$$

where c is a dimensionless constant and $u(r)$ is the displacement of the layer and is given by

$$u(r) = \int_0^r \tan(\delta(t))dt. \tag{7.5}$$

The layer displacement $u(r)$ is a fully nonlinear function since no restrictions were placed on the magnitude of this displacement when solving equation (7.3). In this problem, we use the adapted energy density model as given by (2.24), namely

$$w_A = \frac{1}{2}K_1^n(\nabla\cdot\mathbf{n})^2 + \frac{1}{2}K_1^a(\nabla\cdot\mathbf{a})^2 + \frac{1}{2}\frac{B_0}{|\nabla\Phi|^2}(|\nabla\Phi|-1)^2 + \frac{1}{2}B_1(1-(\mathbf{n}\cdot\mathbf{a})^2). \tag{7.6}$$

In De Vita and Stewart [55], the anchoring energy density used is the one employed by Rapini and Papoular [38], namely

$$w_s = \frac{1}{2}\tau_0(1 - (\mathbf{n}_R \cdot \mathbf{n}_P)^2) = \frac{1}{2}\tau_0 \sin^2(\mathbf{n}_R - \mathbf{n}_P), \tag{7.7}$$

where $\mathbf{n}_R = \mathbf{n}(R)$ and $\mathbf{n}_P$ is the preferred alignment of the director at the boundary surface. As a comparison, we now take the energy density used in Yokoyama and Van Sprang [60], given by

$$w_s = \frac{1}{2}\tau_0 \sin^2(\mathbf{n}_R - \mathbf{n}_P) + \frac{1}{4}\tau_1 \sin^4(\mathbf{n}_R - \mathbf{n}_P). \tag{7.8}$$

The total energy W is then defined to be

$$W = \int_\nu w_A d\nu + \int_{S_1} 2\gamma dS_1 + \int_{S_2} w_s dS_2, \tag{7.9}$$

where S_1 is the top or bottom circular surface of a lipid bilayer with radius R, S_2 is the radial surface, ν is the volume and $2h$ is the thickness of the bilayer. The energetic contribution of the surface tension is defined to be γ and is assumed to be finite. In equation (7.9), the quantities dS_1, dS_2 and $d\nu$ are

$$dS_1 = |\nabla\Phi|\, r\, dr\, d\phi = c \sec\delta\, r\, dr\, d\phi, \tag{7.10}$$

$$dS_2 = R\, dz\, d\phi, \tag{7.11}$$

$$d\nu = r\, dr\, d\phi\, dz, \tag{7.12}$$

where $|\nabla\Phi|$ is used to account for the curvature at the boundaries. Hence, evaluating the integral in (7.9) yields the total energy [55]

$$W = 4\pi\left\{ \int_0^R (hrw_A + c\gamma r \sec\delta)dr + hRw_s \right\}. \tag{7.13}$$

From De Vita and Stewart [55], we have that for the total energy W to have an extremum, the Euler-Lagrange equations

$$\frac{\partial \bar{w}_A}{\partial \theta} - \frac{d}{dr}\left(\frac{\partial \bar{w}_A}{\partial \theta'}\right) = 0, \tag{7.14}$$

$$\frac{\partial \bar{w}_A}{\partial \delta} - \frac{d}{dr}\left(\frac{\partial \bar{w}_A}{\partial \delta'}\right) + \frac{c\gamma}{a}\sec\delta\tan\delta r = 0, \tag{7.15}$$

must be satisfied along with the boundary conditions

$$\frac{\partial \bar{w}_A}{\partial \theta'} + R\frac{\partial \bar{w}_S}{\partial \theta} = 0, \quad \text{at } r = R, \tag{7.16}$$

$$\frac{\partial \bar{w}_A}{\partial \delta'} = 0, \quad \text{at } r = R, \tag{7.17}$$

$$\frac{\partial \bar{w}_A}{\partial \theta'} = 0, \quad \text{at } r = 0, \tag{7.18}$$

$$\frac{\partial \bar{w}_A}{\partial \delta'} = 0, \quad \text{at } r = 0, \tag{7.19}$$

where $\bar{w}_A = rw_A$ and $\bar{w}_S = rw_S$. Since we are only changing the weak anchoring energy density w_S, only boundary condition (7.16) is altered. Inserting $\mathbf{n}$ and $\mathbf{a}$ into $\bar{w}_A$ yields

$$\bar{w}_A = \frac{r}{2}\left[K_1^n \frac{1}{r^2}\left[\frac{d}{dr}(r\sin\theta)\right]^2 + K_1^a \frac{1}{r^2}\left[\frac{d}{dr}(r\sin\delta)\right]^2 + B_0\left(1 - \frac{\cos\delta}{c}\right)^2 + B_1\sin^2(\theta - \delta)\right], \tag{7.20}$$

and so we can write

$$\frac{\partial \bar{w}_A}{\partial \theta'} = K_1^n \cos\theta_R[\sin\theta_R + R\cos\theta_R\theta'(R)].$$

We can differentiate the anchoring energy density (7.8) with respect to θ to obtain

$$
\begin{aligned}
\frac{\partial w_S}{\partial \theta} &= \tau_0 \sin(\theta_R - \theta_P)\cos(\theta_R - \theta_P) + \tau_1 \sin^3(\theta_R - \theta_P)\cos(\theta_R - \theta_P) \\
&= \frac{1}{2}\tau_0 \sin\big(2(\theta_R - \theta_P)\big) - \frac{1}{2}\tau_1 \sin\big(4(\theta_R - \theta_P)\big) + \frac{1}{4}\tau_1 \sin\big(2(\theta_R - \theta_P)\big) \\
&= \frac{1}{2}\Big(\tau_0 + \frac{1}{2}\tau_1\Big)\sin\big(2(\theta_R - \theta_P)\big) - \frac{1}{8}\tau_1 \sin\big(4(\theta_R - \theta_P)\big).
\end{aligned}
$$

Hence we can write boundary condition (7.16) as

$$
2K_1^n \cos\theta_R\left[\frac{\sin\theta_R}{R} + \cos\theta_R\theta'(R)\right] + \Big(\tau_a + \frac{1}{2}\tau_b\Big)\sin\big(2(\theta_R - \theta_P)\big) - \frac{1}{4}\tau_b \sin\big(4(\theta_R - \theta_P)\big) = 0. \tag{7.21}
$$

Now, in order to obtain solutions for this problem we use the same governing equations as in de Vita and Stewart [55], but with the altered boundary condition stated in (7.21). We can non-dimensionalise the equilibrium equations and boundary conditions by adopting the rescaled variables

$$
\lambda = \sqrt{\frac{K_1^n}{B_0}}, \quad B = \frac{B_1}{B_0}, \quad \kappa = \frac{K_1^a}{K_1^n}, \quad \tau_a = \frac{\lambda\tau_0}{K_1^n}, \quad \tau_b = \frac{\lambda\tau_1}{K_1^n},
$$
$$
\bar{r} = \frac{r}{h}, \quad \alpha = \frac{c\gamma h}{K_1^n}, \quad \bar{R} = \frac{R}{h}, \quad m = \frac{h}{\lambda},
$$

where λ is a typical lengthscale. Hence, we now need to solve the non-dimensionalised equilibrium equations given by

$$
\cos^2\theta(\bar{r}\theta'' + \theta') - \frac{1}{\bar{r}}\sin\theta\cos\theta(1 + \bar{r}^2\theta'^2) - m^2 B\bar{r}\sin(\theta - \delta)\cos(\theta - \delta) = 0, \tag{7.22}
$$

$$
\begin{aligned}
\cos^2\delta(\bar{r}\delta'' + \delta') - \frac{1}{\bar{r}}\sin\delta\cos\delta(1 + \bar{r}^2\delta'^2) + m^2 B\bar{r}\sin(\theta - \delta)\cos(\theta - \delta)& \\
-m^2\bar{r}\sin\delta\Big(1 - \frac{\cos\delta}{c}\Big) - \alpha\bar{r}\sec\delta\tan\delta = 0,&
\end{aligned} \tag{7.23}
$$

with boundary conditions

$$2\cos\theta_R\left[\frac{\sin\theta_R}{\bar{R}}+\cos\theta_R\theta'(\bar{R})\right]+m\Big(\tau_a+\frac{1}{2}\tau_b\Big)\sin\big(2(\theta_R-\theta_P)\big)$$
$$-\frac{m}{4}\tau_b\sin\big(4(\theta_R-\theta_P)\big)=0, \tag{7.24}$$
$$\delta'(\bar{R})+\frac{1}{\bar{R}}\tan\delta_R=0, \tag{7.25}$$
$$\theta(0)=0, \tag{7.26}$$
$$\delta(0)=0. \tag{7.27}$$

Finally, the layer displacement given by (7.5) can be non-dimensionalised by setting

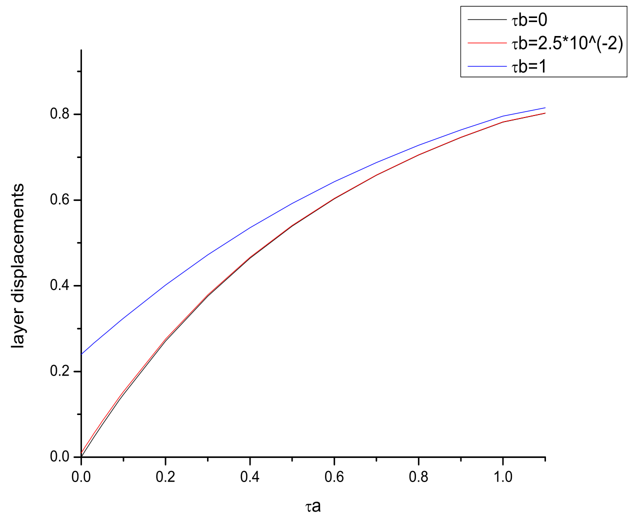
$$\hat{u}(\bar{r})=\frac{u(r)}{h},$$

where $\hat{u}(\bar{r})$ is now given by

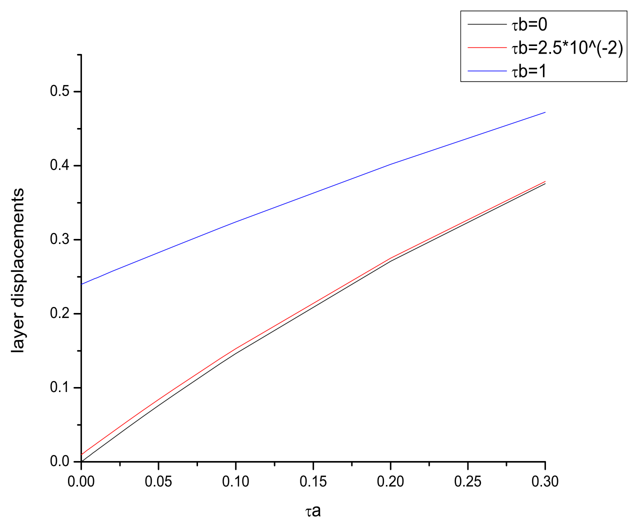
$$\hat{u}(\bar{r})=\int_0^{\bar{r}}\tan\delta(ht)\,dt. \tag{7.28}$$

Equilibrium equations (7.22) and (7.23), with boundary conditions (7.24)-(7.27), can be solved numerically using a standard differential equation sovlver in MAPLE12 [30], with the results shown in the graphs below. In all the computations we have set $\kappa=m=c=1$.

When we consider the effect that the alternative anchoring has on the lipid bilayer, we see from Figure 7.2 (a) that the only case in which the layer displacement is affected greatly is in the extreme value of $\tau_b=1$. This is an extreme value and unrealistic in practice, however, it does highlight the possibility that the layer displacement is susceptible to changes in the surface anchoring. Figure 7.2 (b), showing small values of τ_a, reinforces this conclusion. We can see that there is a slight difference in the layer displacement caused by the addition of τ_b, however, the maximum difference is approximately 0.01. The effect of surface tension on the layer displacement can be seen in Figure 7.3 (a). When we employ the two-term anchoring energy, the change in layer displacement due to surface tension is very similar to that in De Vita and Stewart [55]. The profile of the graph is almost identical, with the only difference being a slight increase in the size of the layer displacement at the starting value of $\alpha = 10^{-5}$. Figure 7.3 (b) shows the change in boundary values of the director and unit normal as the surface tension is increased. Again, we can see a very similar profile for the change in contact angles to those in De Vita and Stewart [55]. With the inclusion of the new anchoring term, the angles θ_R and δ_R are slightly larger than in the original case, and the difference between the two angles is greater in this formulation than that of the single anchoring term formulation in De Vita and Stewart [55].

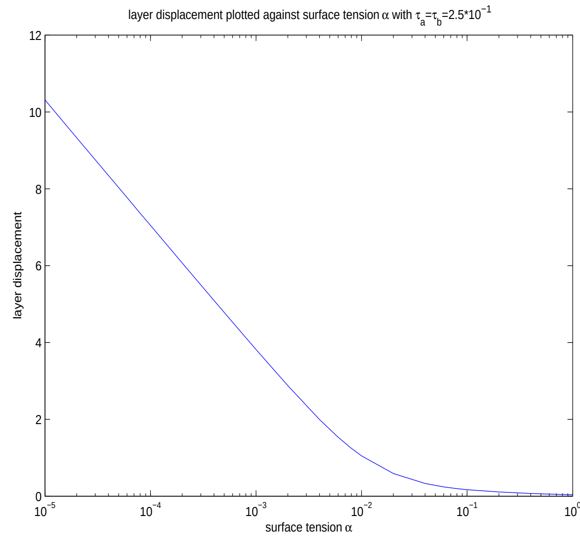


(a)

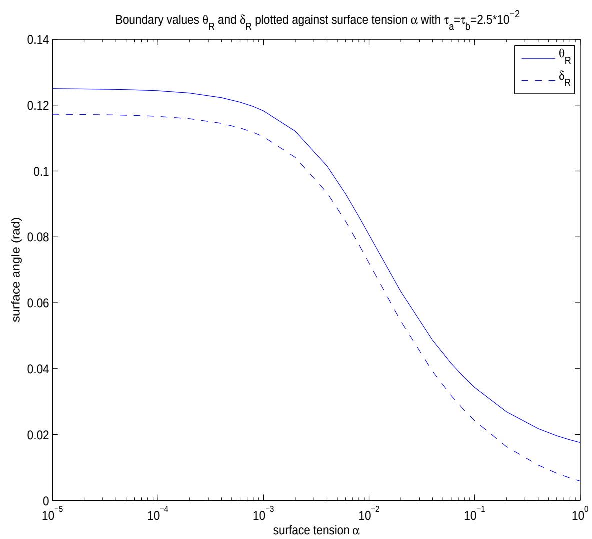


(b)

Figure 7.2: (a) Layer displacements plotted against τ_a between 0 and 1 for various τ_b values, and (b) Layer displacements plotted against τ_a between 0 and 0.3 for various τ_b values

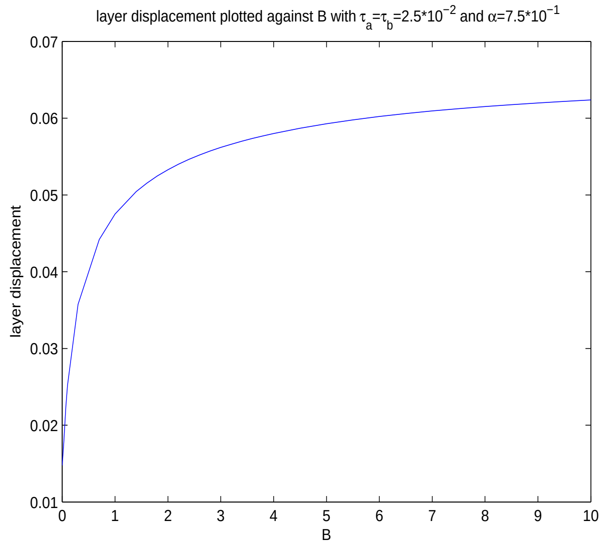


(a)

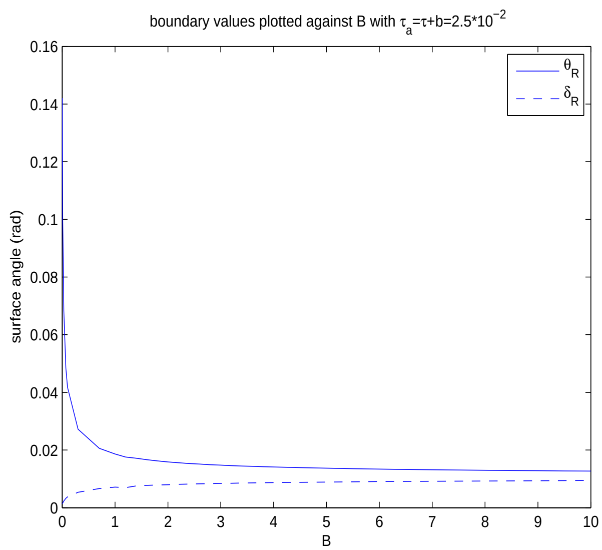


(b)

Figure 7.3: (a) Layer displacements plotted against α with fixed τ_a and τ_b, and (b) Surface angles plotted against α with fixed τ_a and τ_b.



(a)



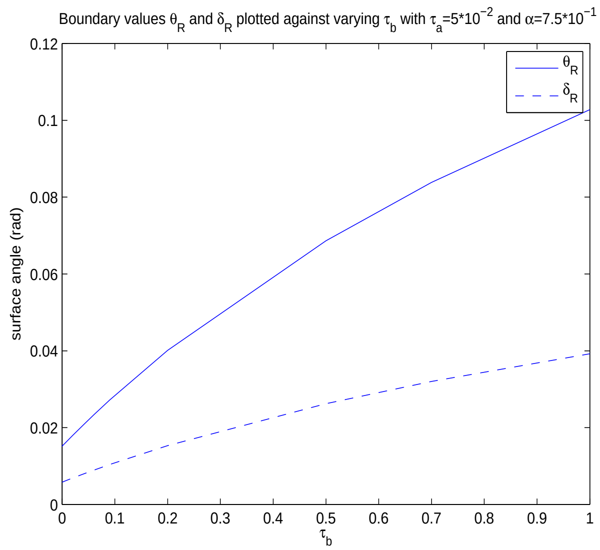
(b)

Figure 7.4: (a) Layer displacements plotted against B with fixed τ_a and τ_b, and (b) Surface angles plotted against B with fixed τ_a and τ_b.

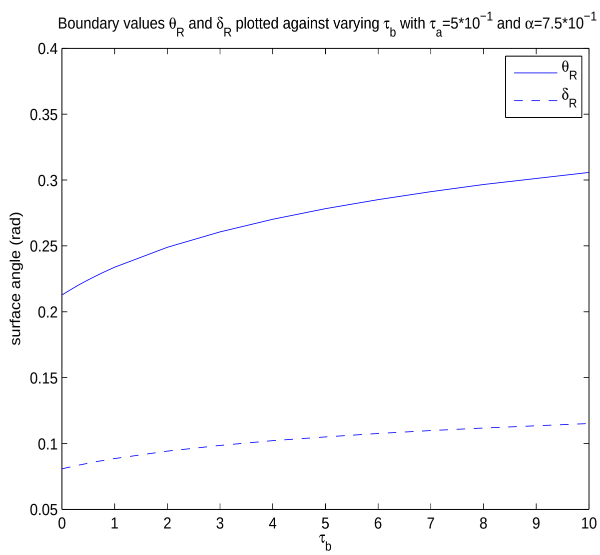
Figure 7.4 shows the effect that the dimensionless parameter B has on layer displacement and surface angles. The profile of the change in layer displacement is similar to that in De Vita and Stewart [55], however, the magnitude of the displacement is much greater in this case. For the extreme value of $B = 10$, the magnitude of layer displacement in the two term energy is more than double that of the single term. Figure 7.5 shows the effect that τ_a and τ_b have on the surface angles. When τ_a is increased by a factor of 10, the surface angles also increase by a factor of 10. We note that, in the case of small τ_a values, there is a sharper increase in the surface angles as τ_b is increased. In a similar comparison, Figure 7.6 shows the effect that τ_a has on layer displacement. From Figure 7.7 we can see that for small τ_b, increasing τ_a by a factor of 10 also increases the layer displacement by a factor of 10. However, for larger τ_b, increasing τ_a by a factor of 10 results in the layer displacement approximately doubling in magnitude. Also, the change is layer displacement is more abrupt when we consider smaller values of τ_a.

For the graphs in this chapter, we have only considered positive values of τ_b. However, from Yokoyama and Van Sprang [60] we have the possibility for τ_b to be negative, with the value given in the paper to be $\tau_b = -1.8 \times 10^{-5}$. In Table 7.1, we compare the values obtained for layer displacement and surface angles, θ_R and δ_R, for various values of τ_b, both negative and positive. We can see from this table that the layer behaves in a very similar way for the corresponding positive and negative values of τ_b, provided τ_b is sufficiently small. When the negative value of τ_b becomes too large, the numerical routine breaks down and does not return an answer, suggesting that large negative values are inadmissable for this problem.

In conclusion, the graphs we considered in this chapter involving the two term anchoring energy are very similar to those in De Vita and Stewart [55]. All the graphs have the same profile in both cases, with only slight variations in extreme parameter values. Clearly the additional term is influencing the solutions slightly, but perhaps not enough to warrant inclusion.

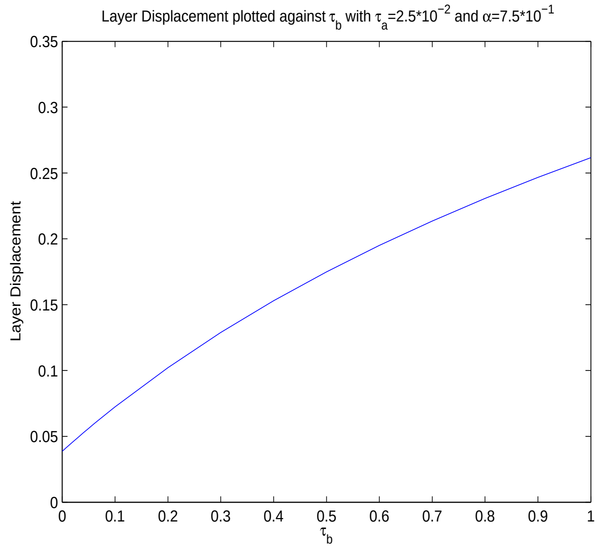


(a)

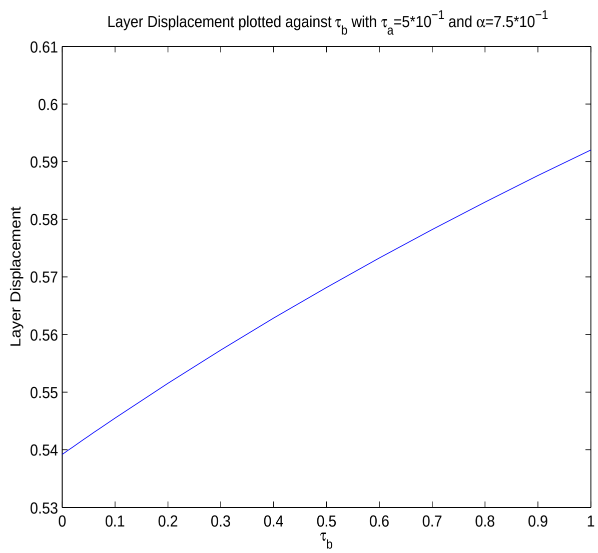


(b)

Figure 7.5: (a) Surface angles plotted against τ_b with fixed $\tau_a = 5 \times 10^{-2}$, and (b) Surface angles plotted against τ_b with fixed $\tau_a = 5 \times 10^{-1}$



(a)



(b)

Figure 7.6: (a) Layer displacement plotted against τ_b with fixed $\tau_a = 2.5 \times 10^{-2}$, and (b) layer displacement plotted against τ_b with fixed $\tau_a = 5 \times 10^{-1}$, both with surface tension given by $\alpha = 7.5 \times 10^{-1}$

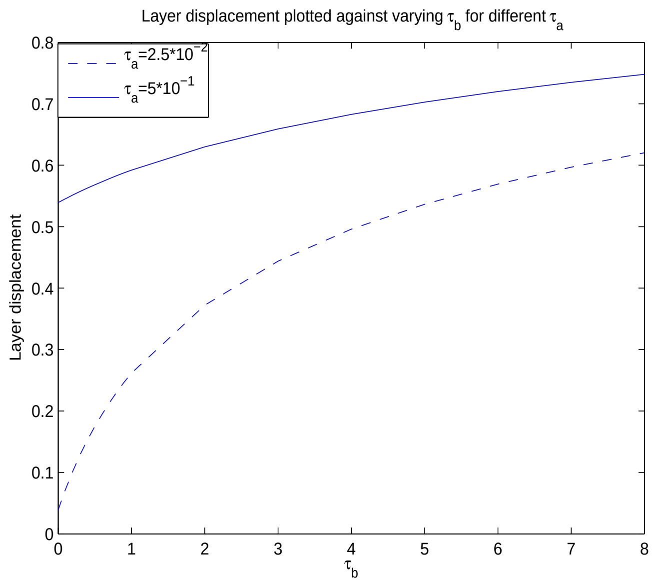


Figure 7.7: Layer displacements plotted against τ_b with various fixed τ_a values.

τ_b	Layer Displacement	θ_R	δ_R
1.8×10^{-5}	0.03867	0.01517	0.00580
-1.8×10^{-5}	0.03865	0.01517	0.00580
1.8×10^{-4}	0.03872	0.01520	0.00581
-1.8×10^{-4}	0.03860	0.01514	0.00580
1.8×10^{-3}	0.03931	0.01543	0.00590
-1.8×10^{-3}	0.03801	0.01492	0.00570
1.8×10^{-2}	0.04506	0.01768	0.00676
-1.8×10^{-2}	0.03210	0.01260	0.00482
1.8×10^{-1}	0.09650	0.03787	0.01447
-1.8×10^{-1}	*undefined*	*undefined*	*undefined*
1	0.26164	0.10278	0.03923
-1	*undefined*	*undefined*	*undefined*

Table 7.1: The layer displacement and surface angles, θ_R and δ_R, for various values of τ_b, both positive and negative.

Appendix A

Generalised Separation of Variables for Classical Problems

We demonstrate the use of the generalised separation of variables technique for the classical diffusion equation first with zero boundary conditions, as considered in Stewart [49], and then for the case where the bottom plate is being oscillated while the top plate is being held stationary, as discussed by Drazin and Riley [8]. First we consider the case of zero boundary conditions. As described in Polyanin, once we obtain an equation for the unknown functions in z and t, there are two ways to proceed. Method 1 involves dividing the governing equation throughout by a function of one variable, and then differentiating with respect to that variable in order to eliminate one of the terms from the equation. This process is carried out repeatedly until the n term equation is reduced to a two term equation, wherein the standard separation of variables technique can be implemented. Method Two involves organising the differential equation into a bilinear form then solving for this form using solutions stated in Polyanin [36].

A.1 Background

Separation of variables is a technique for solving partial differential equations without the need for specifying a particular form of the solution. In the tradition technique we search for solutions of the form $u(z,t) = F(z)G(t)$, where F and

G are functions to be determined. This technique can be generalised to include different forms of function. The generalised separation of variables has the form

$$u(z,t) = \varphi(z)\psi(t) + \chi(t). \tag{A.1}$$

We note that the same idea can be generalised further for the functional separation of variables technique which includes solutions of the form

$$u(z,t) = \varphi(z) + \psi(t), \tag{A.2}$$

$$u(z,t) = F(x), \quad x = \psi_1(t)z + \psi_2(t), \tag{A.3}$$

$$u(z,t) = F(x), \quad x = \psi_1(t)z^2 + \psi_2(t). \tag{A.4}$$

In this thesis we will be seeking solutions of the form given by (A.1). We define a general nonlinear partial differential equation to be of the form

$$f_1(z)g_1(t)\Pi_1[u] + f_2(z)g_2(t)\Pi_2[u] + \cdots + f_n(z)g_n(t)\Pi_n[u] = 0, \tag{A.5}$$

where the $\Pi_i[u]$ are differential forms of powers of $u(z,t)$ and its derivatives. Then substituting the general form of the solution $u(z,t)$ from (A.1) into equation (A.5) produces the functional-differential equation

$$\Phi_1(Z)\Psi_1(T) + \Phi_2(Z)\Psi_2(T) + \cdots + \Phi_k(Z)\Psi_k(T) = 0, \tag{A.6}$$

where the $\Phi_j(Z)$ are dependent on z, and the functions φ_i and their derivatives, and the $\Psi_j(T)$ are dependent on t, and the functions ψ_i and their derivatives. Let us first solve equation (A.6) by considering Method One. We divide equation (A.6) throughout by $\Psi_k(T)$ (assuming $\Psi_k(T) \neq 0$) and then differentiate the resulting equation with respect to t. This produces another functional-differential equation of the same form but with fewer terms. This process is repeated until we arrive at a separable two-term equation given by

$$\tilde{\Phi}_1(Z)\tilde{\Psi}_1(T) + \tilde{\Phi}_2(Z)\tilde{\Psi}_2(T) = 0. \tag{A.7}$$

Equation (A.7) is then solved for the functions $\tilde{\Phi}_i(Z)$ and $\tilde{\Psi}_i(T)$, and hence for the functions $\varphi_i(z)$ and $\psi_i(t)$.

Method Two involves solving the functional-differential equation directly. It can be shown that equation (A.6) has $k-1$ solutions given by [36]

$$\Phi_i(z) = C_{i,1}\Phi_{m+1}(z) + C_{i,2}\Phi_{m+2}(z) + \cdots + C_{i,k-m}\Phi_k(z), \tag{A.8}$$

$$\Psi_{m+j}(t) = -C_{1,j}\Psi_1(t) - C_{2,j}\Psi_2(t) \cdots - C_{m,j}\Psi_m(t), \tag{A.9}$$

for $i = 1, ..., m$, $j = 1, ..., k-m$ and $m = 1, 2, ..., k-1$, and where the $C_{i,j}$ are arbitrary constants. For example, the functional equation

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 = 0, \tag{A.10}$$

has two solutions given by

$$\Phi_1 = A_1\Phi_3, \quad \Phi_2 = A_2\Phi_3, \quad \Psi_3 = -A_1\Psi_1 - A_2\Psi_2, \tag{A.11}$$

and

$$\Psi_1 = A_1\Psi_3, \quad \Psi_2 = A_2\Psi_3, \quad \Phi_3 = -A_1\Phi_1 - A_2\Phi_2. \tag{A.12}$$

Now we highlight the generalised separation of variables method by solving two classical fluid mechanics problems using Methods One and Two.

A.2 The Classical Diffusion Problem With Homogeneous Boundary Conditions

The classical diffusion problem

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2}, \tag{A.13}$$

$$u(0,t) = u(d,t) = 0, \tag{A.14}$$

$$u(z,0) = u_0(z), \tag{A.15}$$

has solution [49]

$$u(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi z}{d}\right) \exp\left[-\left(\frac{n\pi z}{d}\right)^2 t\right], \tag{A.16}$$

where A_n is dependent on the summing index n and is defined to be

$$A_n = \frac{2}{d}\int_0^d u_0(z)\sin\left(\frac{n\pi z}{d}\right)dz. \tag{A.17}$$

Now we aim to solve this problem using the generalised separation of variables technique, Method One. First we set $u(z,t) = \varphi(z)\psi(t) + \chi(t)$ and substitute this into (A.13) to obtain

$$\varphi''\psi - \varphi\dot{\psi} - \dot{\chi} = 0. \tag{A.18}$$

Differentiating with respect to z and rearranging yields

$$\frac{\varphi'''}{\varphi'} = \frac{\dot{\psi}}{\psi} = -\lambda, \tag{A.19}$$

which provides two ordinary differential equations to solve for the unknown functions $\varphi(z)$ and $\psi(t)$, where λ is a positive constant. Once we have found these solutions we substitute these into (A.18) to find $\chi(t)$. Hence we have to solve

$$\varphi'''(z) + \lambda\varphi'(z) = 0, \tag{A.20}$$

$$\dot{\psi}(t) + \lambda\psi(t) = 0, \tag{A.21}$$

$$\dot{\chi} = \varphi''\psi - \varphi\dot{\psi}, \tag{A.22}$$

which have solutions

$$\varphi(z) = c_1 \cos(\sqrt{\lambda}z) + c_2 \sin(\sqrt{\lambda}z) + c_3, \tag{A.23}$$

$$\psi(t) = c_4 e^{-\lambda t}, \tag{A.24}$$

$$\chi(t) = c_5 - c_3 c_4 e^{-\lambda t}. \tag{A.25}$$

Now we can write $u(z,t)$ in the general form

$$u(z,t) = [c_1 \cos(\sqrt{\lambda}z) + c_2 \sin(\sqrt{\lambda}z)]c_4 e^{-\lambda t} + c_5. \tag{A.26}$$

For the boundary condition at $z = 0$ to be satisfied, we require $c_1 = c_5 = 0$ for non-zero solutions, and the boundary condition at $z = d$ implies that $\sqrt{\lambda} = \frac{n\pi}{d}$. Hence the solution becomes

$$u(z,t) = c_2 c_4 \sin\left(\frac{n\pi z}{d}\right) e^{-\left(\frac{n\pi}{d}\right)^2 t}. \tag{A.27}$$

Although we have set c_2 and c_4 to be simple constants, they are in fact indexed with n, so we set $c_2^n c_4^n = A_n$, and we have recovered the solution from Stewart [49]. We can also solve this problem using Method Two. We can write equation (A.18) in the bilinear form

$$\Phi_1\Psi_1 + \Phi_2\Psi_2 + \Phi_3\Psi_3 = 0, \tag{A.28}$$

where $\Phi_i = \Phi_i(z)$, $\Psi_i = \Psi_i(t)$, and we write

$$\Phi_1 = \varphi'', \quad \Psi_1 = \psi, \tag{A.29}$$
$$\Phi_2 = -\varphi, \quad \Psi_2 = \dot{\psi}, \tag{A.30}$$
$$\Phi_3 = 1, \quad \Psi_3 = -\dot{\chi}. \tag{A.31}$$

One solution of equation (A.28) is given by

$$\Phi_3 = -A_1\Phi_1 - A_2\Phi_2, \tag{A.32}$$
$$\Psi_1 = A_1\Psi_3, \tag{A.33}$$
$$\Psi_2 = A_2\Psi_3, \tag{A.34}$$

where the A_i are arbitrary constants. Inserting the expressions for Φ_i and Ψ_i into the above equations yields

$$1 = -A_1\nu\varphi'' + A_2\varphi, \tag{A.35}$$
$$\psi = -A_1\dot{\chi}, \tag{A.36}$$
$$\dot{\psi} = -A_2\dot{\chi}. \tag{A.37}$$

Solving equations (A.36) and (A.37) for $\psi(t)$, and equation (A.35) for $\varphi(z)$ yields the solutions

$$\varphi(z) = c_1 \cos\left(\sqrt{\frac{B_1}{\nu}}z\right) + c_2 \sin\left(\sqrt{\frac{B_1}{\nu}}z\right) - \frac{1}{A_2}, \tag{A.38}$$
$$\psi(t) = c_3 e^{-B_1 t}, \tag{A.39}$$

where $B_1 = -\frac{A_2}{A_1}$. Substituting $\psi(t)$ into (A.37) and solving for $\chi(t)$ yields

$$\chi(t) = \frac{c_3}{A_2} e^{-B_1 t} + c_4, \tag{A.40}$$

and so we can write

$$u(z,t) = \left[c_1 \cos\left(\sqrt{\frac{B_1}{\nu}} z \right) + c_2 \sin\left(\sqrt{\frac{B_1}{\nu}} z \right) \right] c_3 e^{-\frac{B_1}{\nu} t} + c_4. \tag{A.41}$$

Applying the boundary conditions requires $c_1 = c_4 = 0$ for non-zero solutions, and $\sqrt{\frac{B_1}{\nu}} = \frac{n\pi}{d}$. Hence we obtain the solution

$$u(z,t) = c_2 c_3 \sin\left(\frac{n\pi z}{d} \right) e^{-\left(\frac{n\pi}{d}\right)^2 t}, \tag{A.42}$$

as before.

A.3 Oscillating Boundary

Now we consider the case of the classical diffusion problem where one boundary is oscillated while the other remains fixed. This is given by

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial z^2}, \tag{A.43}$$

$$u(0,t) = A\omega \cos(\omega t) \tag{A.44}$$

$$u(d,t) = 0, \tag{A.45}$$

$$u(z,0) = u_0(z). \tag{A.46}$$

This problem has been solved by Drazin and Riley [8] using the standard separation of variables technique. Now we solve this problem using generalised separation of variables technique, Method One. From Section A.2, we have that the general solution to equation (A.43) is given by

$$u(z,t) = [c_1 \cos(\sqrt{\lambda} z) + c_2 \sin(\sqrt{\lambda} z)] c_4 e^{-\lambda t} + c_5. \tag{A.47}$$

Applying the boundary condition at the bottom plate, i.e. $u(0,t) = A\omega e^{-i\omega t}$, yields the constraint

$$c_1 c_4 e^{-\lambda t} + c_5 = A\omega e^{-i\omega t}. \tag{A.48}$$

The only way that condition (A.48) can be satisfied is if λ is complex, so we set $\lambda = \lambda_1 + \lambda_2 i$. Hence the boundary condition at the lower plate becomes

$$c_1 c_4 [e^{-\lambda_1 t} \cos(\lambda_2 t) - i e^{-\lambda_1 t} \sin(\lambda_2 t)] + c_5 = A\omega[\cos(\omega t) - i \sin(\omega t)], \tag{A.49}$$

which implies that

$$\lambda_1 = 0, \tag{A.50}$$

$$\lambda_2 = \omega, \tag{A.51}$$

$$c_1 c_4 = A\omega, \tag{A.52}$$

$$c_5 = 0. \tag{A.53}$$

Applying the boundary condition at the stationary top plate yields a relationship between the two unknown constants, c_1 and c_2, such that

$$c_2 = -c_1 \cot\left(\sqrt{\frac{i\omega}{\nu}} d\right), \tag{A.54}$$

and hence we can write

$$u(z,t) = A\omega\left[\cos\left(\sqrt{\frac{i\omega}{\nu}}z\right) - \cot\left(\sqrt{\frac{i\omega}{\nu}}d\right)\sin\left(\sqrt{\frac{i\omega}{\nu}}z\right)\right]e^{-i\omega t}. \tag{A.55}$$

We only require the real part of the solution, and so our final solution is given by

$$\Re[u(z,t)] = \frac{A\omega}{2\cosh(2\lambda d) - 2\cos(2\lambda d)}\Big[e^{-\lambda(z-2d)}\cos(\omega t - \lambda z) - e^{-\lambda z}\cos(\omega t + 2\lambda d - \lambda z) \tag{A.56}$$

$$+e^{\lambda(z-2d)}\cos(\omega t + \lambda z) - e^{\lambda z}\cos(\omega t - 2\lambda d + \lambda z)\Big], \tag{A.57}$$

where $\lambda = \sqrt{\frac{\omega}{2\nu}}$. This is the exact solution found in Drazin and Riley [8].

Appendix B

Discrete Fourier Transforms

From work carried out by Le Bail [4], we have that the general partial differential equation

$$\frac{\partial^2 \varphi}{\partial x^2} + a(y)\frac{\partial^2 \varphi}{\partial y^2} + b(y)\frac{\partial \varphi}{\partial y} + c(y)\varphi = \rho(x, y), \tag{B.1}$$

where $\varphi = \varphi(x, y)$, may be approximated by the symmetric nine-point finite difference formula

$$\sum_{k=-1}^{1} \Big\{ \alpha_k(J)[\varphi(I-1, J+k) + \varphi(I+1, J+k)] + \beta_k(J)\varphi(I, J+k) \Big\} = \rho(I, J), \tag{B.2}$$

for all interior mesh points (I, J) of a grid split up into M points in the x-direction and N points in the y-direction, indexed with I and J respectively. The value for M is restricted to discrete values of the form $M = 2^Q$, where Q is a positive integer. When we are dealing with a parabolic equation, for example the diffusion equation, we can set $a(y) = 0$ in (B.1), which simplifies the finite difference formula we need to use. In the parabolic case, finite difference equation (B.2) is replaced by the six-point formula

$$\sum_{k=-1}^{0} \Big\{ \alpha_k(J)[\varphi(I-1, J+k) + \varphi(I+1, J+k)] + \beta_k(J)\varphi(I, J+k) \Big\} = \rho(I, J). \tag{B.3}$$

When the α_k and β_k are independent of J, that is equation (B.1) has constant coefficients, finite difference equation (B.3) can be written explicitly as the six-point stencil equation

$$\begin{array}{|c|c|c|} \hline 1 & -2+2b\alpha\Delta X + c(\Delta X)^2 & 1 \\ \hline 1 & -2-2b\alpha\Delta X + c(\Delta X)^2 & 1 \\ \hline \end{array} \; \varphi(*,*) = \begin{array}{|c|} \hline 1 \\ \hline 1 \\ \hline \end{array} \; (\Delta X)^2 \rho(*,*), \tag{B.4}$$

where $\alpha = \Delta X/\Delta Y$. For simplicity in the equations to follow, we write $E_1 = -2 + 2b\alpha\Delta X + c(\Delta X)^2$ and $E_2 = -2 - 2b\alpha\Delta X + c(\Delta X)^2$. Hence equation (B.4) can be written explicitly as

$$\begin{aligned} \varphi(I-1, J) + E_1\varphi(I, J) + \varphi(I+1, J) &+ \varphi(I-1, J-1) \\ + E_2\varphi(I, J-1) + \varphi(I+1, J-1) &= (\Delta X)^2[\rho(I, J) + \rho(I, J-1)]. \end{aligned} \tag{B.5}$$

Now we set

$$\varphi(I, J) = \sum_{S=1}^{M-1} \varphi_S(J) \sin\left(\frac{\pi I S}{M}\right), \tag{B.6}$$

$$\Upsilon(I, J) = \sum_{S=1}^{M-1} \Gamma_S(J) \sin\left(\frac{\pi I S}{M}\right), \tag{B.7}$$

where $\Upsilon(I, J) = (\Delta X)^2\rho(I, J)$, for $1 \le I \le M-1$. Substituting (B.6) and (B.7) into (B.5) yields the equation

$$\sum_{S=1}^{M-1}\left\{\varphi_S(J)\sin\left(\frac{\pi(I-1)S}{M}\right)+E_1\varphi_S(J)\sin\left(\frac{\pi IS}{M}\right)+\varphi_S(J)\sin\left(\frac{\pi(I+1)S}{M}\right)\right.$$
$$+\varphi_S(J-1)\sin\left(\frac{\pi(I-1)S}{M}\right)+E_2\varphi_S(J-1)\sin\left(\frac{\pi IS}{M}\right)$$
$$\left.+\varphi_S(J-1)\sin\left(\frac{\pi(I+1)S}{M}\right)\right\}=\sum_{S=1}^{M-1}\left\{\Gamma_S(J)\sin\left(\frac{\pi IS}{M}\right)+\Gamma_S(J-1)\sin\left(\frac{\pi IS}{M}\right)\right\}, \tag{B.8}$$

which simplifies to

$$\sum_{S=1}^{M-1}\left\{\varphi_S(J)\left[2\cos\left(\frac{\pi S}{M}\right)+E_1\right]+\varphi_S(J-1)\left[2\cos\left(\frac{\pi S}{M}\right)+E_2\right]\right\}\sin\left(\frac{\pi IS}{M}\right)$$
$$=\sum_{S=1}^{M-1}\left\{\Gamma_S(J)+\Gamma_S(J-1)\right\}\sin\left(\frac{\pi IS}{M}\right), \tag{B.9}$$

by implementing the trigonometric identity

$$\sin\left(\frac{\pi(I-1)S}{M}\right)+\sin\left(\frac{\pi(I+1)S}{M}\right)=2\cos\left(\frac{\pi S}{M}\right)\sin\left(\frac{\pi IS}{M}\right). \tag{B.10}$$

Since $\sin(\pi IS/M)$ is a complete orthonormal set [23], equation (B.9) is equivalent to the system

$$\varphi_S(J)\left[2\cos\left(\frac{\pi S}{M}\right)+E_1\right]+\varphi_S(J-1)\left[2\cos\left(\frac{\pi S}{M}\right)+E_2\right]=\Gamma_S(J)+\Gamma_S(J-1), \tag{B.11}$$

for $1 \le S \le M-1$. Now we rewrite equation (B.11) in the form

$$\varphi_S(J)+a_1\varphi_S(J-1)=a_2[\Gamma_S(J)+\Gamma_S(J-1)], \tag{B.12}$$

where we have set

$$a_1 = \frac{2\cos\left(\frac{\pi S}{M}\right) + E_2}{2\cos\left(\frac{\pi S}{M}\right) + E_1},$$
$$a_2 = \frac{1}{2\cos\left(\frac{\pi S}{M}\right) + E_1}.$$

Equation (B.12) provides a recurrence relation to solve for the unknown function $\varphi_S(J)$. From Rade [40], we have for a recurrence relation of the form

$$x(n+N) + a_{N-1}x(n+N-1) + \cdots + a_0 x(n) = 0, \tag{B.13}$$

where N is the order of the equation, the characteristic equation is given by

$$r^N + a_{N-1}r^{N-1} + \cdots + a_0 = 0, \tag{B.14}$$

with roots $r_1, r_2, .., r_k$ of multiplicity $m_1, m_2, .., m_k$, respectively. If all roots are simple (i.e. not repeated) then we have that the solution $x(n)$ is given by

$$x(n) = c_1 r_1^n + \cdots + c_N r_N^n, \tag{B.15}$$

where the c_j are arbitrary constants. Now let N=1, then recurrence relation (B.13) becomes

$$x(n+1) + a_0 x(n) = 0, \tag{B.16}$$

which has characteristic equation $r + a_0 = 0$. The solution to equation (B.16) is then given by

$$x(n) = c_1 r_1^n = c_1(-a_0)^n, \tag{B.17}$$

which clearly satisfies recurrence relation (B.13). Hence (B.12) can be solved exactly in terms of a general $\Gamma_S(J)$, with the solution given by

$$\varphi(S,J) = c_1(S)(-a_1)^J + \sum_{J_0=1}^{J} a_2(-a_1)^{J-J_0}\Gamma(S,J_0) + \sum_{J_0=1}^{J} a_2(-a_1)^{J-J_0}\Gamma(S,J_0-1), \tag{B.18}$$

where the function $c_1(S)$ is related to the initial condition and is given by

$$c_1(S) = \frac{2}{M}\sum_{I=1}^{M-1}\varphi(I,0)\sin\left(\frac{\pi IS}{M}\right). \tag{B.19}$$

Substituting (B.18) into (B.6) yields the final solution to our original equation. Now we consider the example of the diffusion equation. From Stewart [49, p220], we have that the problem

$$\frac{\partial\Phi}{\partial t} = \frac{\partial^2\Phi}{\partial t^2}, \tag{B.20}$$

$$\Phi(z,0) = \phi_0(z), \tag{B.21}$$

$$\Phi(0,t) = \Phi(d,t) = 0, \tag{B.22}$$

where $0 \leq z \leq d$ and $t \geq 0$, can be solved using the standard separation of variables technique to yield the solution

$$\Phi(z,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi z}{d}\right)\exp\left[-\left(\frac{n\pi}{d}\right)^2 t\right], \tag{B.23}$$

where A_n is given by

$$A_n = \frac{2}{d}\int_0^d \phi_0(z)\sin\left(\frac{n\pi z}{d}\right)dz. \tag{B.24}$$

Now we will solve (B.20) using Discrete Fourier Transforms (DFTs), and compare the accuracy of the discrete solution to the exact solution given in (B.23). When using DFTs for this problem there is no forcing term so $\rho \equiv 0$. Now we set

$$\varphi(I,J) = \sum_{S=1}^{M-1} \varphi_S(J)\sin\left(\frac{\pi IS}{M}\right). \tag{B.25}$$

In the notation of Le Bail [4], we have $a = c = 0$ and $b = -1$, which allows us to explicitly evaluate the terms E_1 and E_2. Substituting (B.25) into (B.5), with $\rho \equiv 0$, $E_1 = -4$ and $E_2 = 0$, gives

$$\begin{aligned}\sum_{S=1}^{M-1}\Bigg\{\varphi_S(J)\sin\left(\frac{\pi(I-1)S}{M}\right) - 4\varphi_S(J)\sin\left(\frac{\pi IS}{M}\right) + \varphi_S(J)\sin\left(\frac{\pi(I+1)S}{M}\right)\\ +\varphi_S(J-1)\sin\left(\frac{\pi(I-1)S}{M}\right) + \varphi_S(J-1)\sin\left(\frac{\pi(I+1)S}{M}\right)\Bigg\} = 0,\end{aligned} \tag{B.26}$$

which can be simplified to

$$\sum_{S=1}^{M-1}\left\{\varphi_S(J)\left[2\cos\left(\frac{\pi S}{M}\right) - 4\right] + 2\varphi_S(J-1)\cos\left(\frac{\pi S}{M}\right)\right\}\sin\left(\frac{\pi IS}{M}\right) = 0, \tag{B.27}$$

using identity (B.10). Equation (B.27) is equivalent to

$$\varphi_S(J)\left[2\cos\left(\frac{\pi S}{M}\right) - 4\right] + 2\varphi_S(J-1)\cos\left(\frac{\pi S}{M}\right) = 0, \tag{B.28}$$

for $1 \leq S \leq M-1$, with equation (B.28) providing a recurrence relation to solve for the unknown function $\varphi_S(J)$. Rearranging (B.28) allows us to write the recurrence relation in the form

$$\varphi_S(J) + a_0 \varphi_S(J-1) = 0, \tag{B.29}$$

where $a_0 = \frac{2\cos(\pi S/M)}{2\cos(\pi S/M)-4}$, which has the general solution

$$\varphi_S(J) = c_1(S)\left(\frac{\cos\left(\frac{\pi S}{M}\right)}{2-\cos\left(\frac{\pi S}{M}\right)}\right)^J, \tag{B.30}$$

where the function $c_1(S)$ is given by

$$c_1(S) = \frac{2}{M}\sum_{I=1}^{M-1} \varphi(I,0)\sin\left(\frac{\pi I S}{M}\right). \tag{B.31}$$

Hence we can write the full solution of the diffusion equation as

$$\begin{aligned} \varphi(I,J) &= \sum_{S=1}^{M-1} \varphi_S(J)\sin\left(\frac{\pi I S}{M}\right) \\ &= \sum_{S=1}^{M-1} \frac{\left[1-(-1)^S\right]}{S}\left(\frac{\cos\left(\frac{\pi S}{M}\right)}{2-\cos\left(\frac{\pi S}{M}\right)}\right)^J \sin\left(\frac{\pi I S}{M}\right), \end{aligned} \tag{B.32}$$

where we have taken $\varphi(I,0) = \pi/2$. We compare the exact solution (B.23) with the DFT approximation (B.32) in Figure B.1.

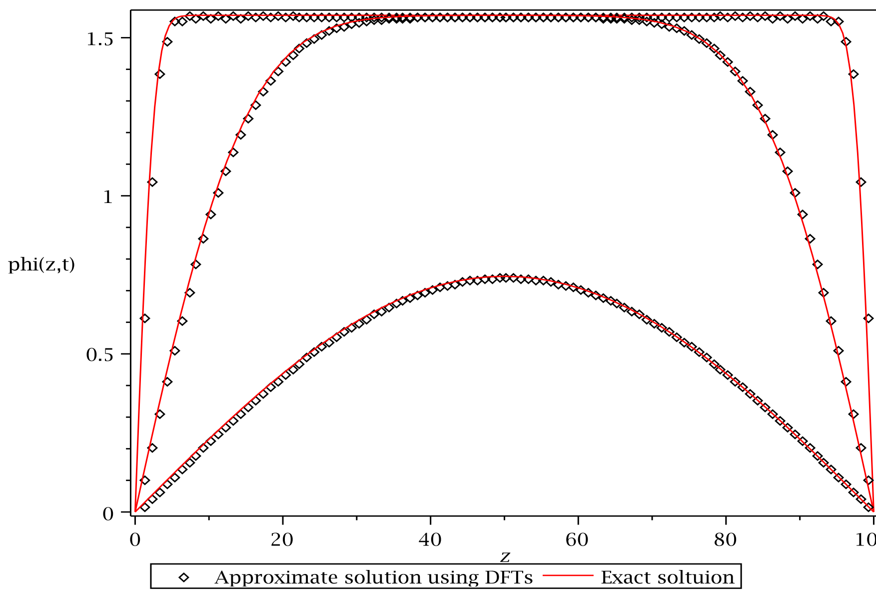


Figure B.1: Approximate DFT solution $\varphi(z,t)$ of the diffusion equation (B.20) compared to exact solution $\Phi(z,t)$.

Appendix C

Sturm-Liouville Eigenvalue Problem

A basic approach to solving a variational problem would be to reduce this problem to one involving a differential equation, however, this approach is quite complex and is not always successful [12]. The difficulties in this approach led to the development of direct methods which are used to calculate a solution from the variational problem. Here, we introduce the classical Sturm-Liouville problem and describe one of the most popular direct methods, the Ritz method.

C.1 The Sturm-Liouville Problem

Let us consider the Sturm-Liouville equation

$$-(P(z)y'(z))' + Q(z)y(z) = \lambda y(z), \tag{C.1}$$

subject to the boundary conditions

$$y(a) = 0, \quad y(b) = 0, \tag{C.2}$$

where $P(z) > 0$ and $Q(z)$ are two known functions, and λ is an eigenvalue. It can be proven that the Sturm-Liouville problem given by (C.1) and (C.2) has an infinite sequence of eigenvalues λ_1, λ_2,..., with each eigenvalue, λ_n, having a corresponding eigenfunction, y_n, which is unique up to a multiplicative constant [12]. It can be shown that the eigenvalues satisfy

$$\lambda_1 < \lambda_2 < \cdots < \lambda_n, \tag{C.3}$$

for all n. This means that if we can show λ_1 is positive, then all subsequent λ_n are also positive, which is of key importance when proving stability. For example, in Chapter 3 we analyse the stability of different geometrical set-ups through the use of a linear stability analysis. In this analysis we perturb the equilibrium system using a perturbation of the form $e^{-\lambda t}$. For the system to be stable we require this perturbation to decrease to zero as t tends to infinity, and so we need $\lambda > 0$. Hence, if we can prove in Chapter 3 that the smallest eigenvalue, λ_1, is positive then we can prove, using condition (C.3), that all eigenvalues are positive, and so the system is stable. Equation (C.1) is in fact the Euler equation corresponding to the problem of finding an extremum of the quadratic functional

$$J[y] = \int_a^b (P(z)(y')^2 + Q(z)y^2)dz, \tag{C.4}$$

subject to boundary conditions (C.2) and the additional constraint

$$\int_a^b y^2(z)dz = 1. \tag{C.5}$$

Hence, if $y(z)$ is a solution of the variational problem, then it is also a solution to the differential equation. It also follows from condition (C.5) that $y(z)$ is not identically zero. In practice, it is often more straightforward to solve the variational problem rather than to find the solution from the differential equation, which is what led to the development of several direct variational methods. We highlight the Ritz method next.

C.2 The Ritz Method

The Ritz method is a direct variational method used to find the eigenvalues and eigenvectors of a boundary value problem [12]. First we set

$$\epsilon(z) = \sum_{k=1}^{N} \alpha_k \sin\left(\frac{k\pi z}{d}\right), \tag{C.6}$$

and aim to minimise the functional

$$J[\epsilon] = \int_0^d \left\{P(z)(\epsilon')^2 + Q(z)\epsilon^2 - \lambda\epsilon^2\right\} dz, \tag{C.7}$$

with respect to α_k (and λ), over the domain $z \in [0, d]$, subject to the normalising condition

$$\frac{d}{2}\sum_{k=1}^{N} \alpha_k^2 = 1, \tag{C.8}$$

where we have set λ to be the smallest eigenvalue λ_1. We have chosen sinusoidal test functions for ϵ for the problems solved in Chapter 3 because the functions $P(z)$ and $Q(z)$ are dependent on trigonometric functions and so in (C.6) they are natural choices for ϵ. However, in general, other test functions can be used such as polynomials [12]. Minimising the functional $J[\epsilon]$ with respect to each α_k (and λ), subject to the constraint (C.8), provides $N+1$ equations to solve for the $N+1$ unknowns in the problem. Solving this system of equations provides an approximation to the eigenvalue of the problem λ, and hence allows us to prove whether or not the geometry is stable. It can be shown that the accuracy of λ is improved monotonically as N increases and that it converges to the actual value of λ as N$\rightarrow \infty$ [12].

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