Advances in the Euler-Maruyama method for stochastic differential equations with locally Lipschitz coefficients

Yiyi TANG
Department of Mathematics and Statistics
University of Strathclyde
Glasgow, UK

December 29, 2024

This thesis is submitted to the University of Strathclyde for the degree of Doctor of Philosophy in the Faculty of Science.

Declaration

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

Yiyi TANG

December 29, 2024

Acknowledgements

I deeply appreciate my supervisor Professor Xuerong MAO FRSE for his support. Without his kind guidance and encouragement, it is impossible to conduct my research. Some parts of this thesis have been published as research papers. During the process of these papers, I have benefited significantly from the journal referees for their useful suggestions to improve my research papers. I would therefore like to thank them. Finally, I sincerely acknowledge my parents for their support.

Abstract

My PhD research is devoted to enriching the strong convergence theory of modified Euler-Maruyama methods for stochastic differential equation with locally Lipschitz coefficients. In this PhD thesis, we will introduce several modified Euler-Maruyama methods and establish their strong convergence theory. First, we will use new numerical analysis techniques to improve strong convergence results of the truncated Euler-Maruyama method. We then combine analysis techniques for polynomially growing coefficients and concave coefficients to extend the truncated EM method for multi-dimensional SDEs with polynomially growing drift and concave diffusion coefficients satisfying the Osgood condition.

Then we will pay attention to scalar SDEs with locally Lipschitz coefficients. We will start with improving strong convergence results of the logarithmic truncated Euler-Maruyama method. To be concrete, we will use new numerical analysis techniques and further extend them for the constant elasticity of variance model and the Aït-Sahalia model with almost full parameter ranges. We will prove that the logarithmic truncated Euler-Maruyama method is strongly convergent with order one half in general $\mathcal{L}^p$-norm.

In the rest of this thesis, we will focus on the projected Euler-Maruyama method. It has good convergence properties for scalar SDEs with locally Lipschitz coefficients. For example, it is strong $\mathcal{L}^p$-convergent with order one half for the Cox-Ingersoll-Ross model with a wide parameter ranges. In particular, we will introduce a novel numerical analysis technique to prove that the projected Euler-Maruyama method may have finite inverse moments, which other modified Euler-Maruyama methods generally do not have. We will use finite inverse moments to prove that the projected Euler-Maruyama

method is strong $\mathcal{L}^p$-convergent with order one for many useful scalar SDE models, e.g., the constant elasticity of variance model, the Aït-Sahalia model, the Heston-3/2 volatility model, the Wright-Fisher model and so on.

Contents

List of Figures

List of Tables

Notations

positive	:	> 0.
negative	:	< 0.
nonnegative	:	$\geqslant 0$.
$a.s.$	:	almost surely, or $\Pr$-almost surely, or with probability 1.
$A := B$	:	A is defined by B or A is denoted by B.
$\emptyset$	:	the empty set.
$I_{\mathcal{S}}$	:	the indicator function of a set $\mathcal{S}$, i.e., $I_{\mathcal{S}}(x) = 1$ if $x \in \mathcal{S}$ or otherwise 0.
A^c	:	the complement of A in Ω, i.e., $A^c = \Omega - A$.
$A \subset B$	:	$A \cap B^c = \emptyset$.
$A \subset B$ $a.s.$	:	$\Pr(A \cap B^c) = 0$.
$a \vee b$	:	the maximum of a and b.
$a \wedge b$	:	the minimum of a and b.
$\lfloor a \rfloor$	:	the integer part of real number a.
$f : A \to B$	:	the mapping f from A to B.
$\mathbb{R}^m$	:	the m-dimensional Euclidean space.
$\mathbb{R}$	:	the real line.

$\mathbb{R}_+$	:	the set of all positive real numbers.
$\bar{\mathbb{R}}_+$	:	the set of all nonnegative real numbers.
$\mathbb{R}^m$	:	the m-dimensionsal Euclidean space.
$\mathbb{R}^{m\times n}$	:	the space of real $m \times n$-matrices
$\mathbb{N}$	:	the set of natural numbers.
$\mathbb{N}_+$	:	the set of positive natural numbers.
$\mathcal{B}^m$	:	the Borel-σ-algebra on $\mathbb{R}^m$
$\lvert x\rvert$	:	the Euclidean norm of a vector x.
A^T	:	the transpose of a vector or matrix A.
$tr(A)$	:	the trace of a square matrix $A = (a_{ij})_{d\times d}$, i.e. ,$tr(A) = \sum_{i=1}^{d} a_{ii}$.
$\lvert A\rvert$	:	the trace norm of a matrix A, i.e. $\lvert A\rvert = \sqrt{tr(A^T A)}$.
Δ	:	the step size of the Euler-Maruyama method, and its value is between 0 and 1.
t_k	:	$t_k = k\Delta$, for $k \in \mathbb{N}_+$
$(\Omega, \mathcal{F}, \Pr)$	:	a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions, i.e., this filtration is right continuous, increasing and $\mathcal{F}_0$ contains all $\Pr$-null sets.
$\mathbb{E}$	:	the expectation corresponding to $\Pr$.
$B(t)$	:	a Brownian motion, but its dimension varies in different sections.

$\mathcal{L}^p(\Omega;\mathbb{R}^m)$: the family of $\mathbb{R}^m$-valued random variables X with $\mathbb{E}|X|^p < \infty$.

$\mathcal{L}^p([a,b];\mathbb{R}^m)$: the family of $\mathbb{R}^m$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{t\in[a,b]}$ such that $\int_a^b |f(t)|^p dt < \infty$ $a.s.$

$\mathcal{M}^p([a,b];\mathbb{R}^m)$: the family of processes $\{f(t)\}_{t\in[a,b]}$ in $\mathcal{L}^p([a,b];\mathbb{R}^m)$ such that $\mathbb{E}\int_a^b |f(t)|^p dt < \infty$.

$\mathcal{L}^p(\mathbb{R}_+;\mathbb{R}^m)$: the family of processes $\{f(t)\}_{t\geqslant 0}$ such that for every $T>0$, $\{f(t)\}_{t\in[0,T]} \in \mathcal{L}^p([a,b];\mathbb{R}^m)$.

$\mathcal{M}^p(\mathbb{R}_+;\mathbb{R}^m)$: the family of processes $\{f(t)\}_{t\geqslant 0}$ such that for every $T>0$, $\{f(t)\}_{t\in[0,T]} \in \mathcal{M}^p([a,b];\mathbb{R}^m)$.

Chapter 1

Introduction

1.1 background

Let $B(t)$ be a Brownian motion. Then an m-dimensional stochastic differential equation (SDE) can be expressed as:

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t),$$

where f and g are called the drift and diffusion coefficients, respectively. SDEs are very useful to describe natural phenomena and real life activities. For example, the geometric Brownian motion is used to model stock prices in the Black-Scholes model. The Cox-Ingersoll-Ross model models the evolution of interest rates. Except for mathematical finance models, there are also many famous SDE models in physics, biology, engineering and so on (see Table 1.1.1 for more examples). However, most of SDEs do not have analytical solutions. That is, we generally have to use numerical approximation methods to simulate SDEs in practice.

The classical Euler-Maruyama (EM) method is one of most useful numerical approximation methods. Its strong convergence theory is well established for SDEs with globally Lipschitz coefficients (e.g., the GBM model in Table 1.1.1), i.e, there exists a

Model	Drift coefficient	Diffusion coefficient
Geometric Brownian motion (GBM) model	αx	σx
Scalar stochastic Ginzburg-Landau equation	$(\alpha x - \beta x^3)$	σx
Cox-Ingersoll-Ross (CIR) model	$\lambda(\mu - x)$	$\sigma x^{1/2}$
Aït-Sahalia model	$(a_{-1}x^{-1} - a_0 + a_1 x - a_2 x^{\theta})$	σx^{ρ}
Constant elasticity of variance (CEV) model	$\lambda(\mu - x)$	$\sigma x^{1/2+\theta}$
Lamperti-transformed CEV model	$(1/2-\theta)\left(\lambda\mu x^{-\frac{1+2\theta}{1-2\theta}} - \frac{2\theta+1}{4}\sigma^2 x^{-1} - \lambda x\right)$	$(1/2-\theta)\sigma$
Wright-Fisher (WF) mode	$(\alpha - \beta x)$	$\sigma\sqrt{\lvert x(1-x)\rvert}$
Lamperti-transformed WF model	$(\alpha - \sigma^2/4)\cot(x/2) - (\beta - \alpha - \sigma^2/4)\tan(x/2)$	σ

Table 1.1.1: A selection of important SDE models

constant $K > 0$ such that

$$|f(u,t) - f(v,t)| \vee |g(u,t) - g(v,t)| \leqslant K|u - v|$$

for all $u, v \in \mathbb{R}^m$. However, there are also many useful SDE models with locally Lipschitz coefficients (e.g., the CEV model in Table 1.1.1). Then numerical analysis methods for globally Lipschitz coefficients will fail, and the classical strong convergence theory fails. In this thesis, we will develop modified EM methods and new numerical analysis methods. We will focus on three types of locally Lipschitz coefficients:

i. polynomially growing coefficients (e.g., the scalar stochastic Ginzburg-Landau equation in Table 1.1.1);

ii. have reciprocal parts (e.g., the Aït-Sahalia model in Table 1.1.1);

iii. Hölder continuous near some points (e.g., the CEV model in Table 1.1.1).

To make our introduction easier to read, detailed background, challenges and previous works for each type will be systematically introduced in corresponding chapters.

This thesis is organized as follows. First, Chapter 2 provides basic mathematical background and useful inequalities. In particular, we will briefly introduce the classical EM method and its strong convergence theory for SDEs with globally Lipschitz coefficients. We will point out why the classical strong convergence theory fails for locally Lipschitz coefficients. We will also introduce useful indices to judge the EM method, which we will frequently use in next chapters.

Then we will develop modified EM methods for different types of locally Lipschitz coefficients. In Chapter 3, we are concerned with multi-dimensional SDEs with polynomially growing drift and concave diffusion coefficients satisfying the Osgood condition. We will introduce a modified EM method, called the the truncated EM method, for SDEs with polynomially growing coefficients. Then we will extend it for multi-dimensional SDEs with polynomially growing drift and concave diffusion coefficients satisfying the Osgood condition.

In Chapter 4, we will focus on modified EM methods for the CEV model and the Aït-Sahalia model. We will introduce the logarithmic truncated EM method, which can preserve positivity of numerical solutions. Then we will introduce new numerical analysis techniques and use weaker assumptions to prove finite inverse moments of the logarithmic truncated EM numerical solution, which is necessary to establish the strong convergence theory. In addition, we will show that our new numerical analysis methods can improve strong convergence results of the truncated EM method. We will prove that the logarithmic truncated EM method is strongly convergent with order one half in general $\mathcal{L}^p$-norm for almost all parameter settings.

The strong convergence theory of the logarithmic truncated EM method is now valid for more parameter settings. However, it only works for the CEV model and the Aït-Sahalia model. Then the projected EM method is developed to cover more SDE models. It is valid for the CIR model, the CEV model, the Aït-Sahalia model, the Heston-3/2 volatility model, the epidemic SIS model and so on. Nevertheless, concrete numerical analysis for each model is different. Therefore, Chapters 5-7 are devoted for different SDE models and different numerical analysis methods. In Chapter 5, we will focus on the CIR model at first. We will invoke Cozma and Reisinger's numerical analysis technique for the full truncated EM method, and prove that the projected EM method is also $\mathcal{L}^p$-strongly convergent with order one half but for more parameter settings. In Chapter 6, we are concerned with SDEs whose coefficients are polynomially growing and have reciprocal parts (e.g., the Aït-Sahalia model in Table 1.1.1). It is worth noting that the CEV model, the Heston-3/2 volatility model and the epidemic SIS model will be covered after applying the Lamperti transformation (e.g., see the Lamperti-transformed

CEV model in Table 1.1.1). We will introduce a new numerical analysis technique and prove that the projected EM method has finite inverse moments, which many modified EM methods do not have. With this good property, we can then further prove that the projected EM method is strong $\mathcal{L}^p$-convergent with order one. Finally, we extend numerical analysis in Chapter 6 for SDEs whose coefficients are locally Lipschitz near two finite points (e.g., the drift coefficient of the Lamperti-transformed WF model in Table 1.1.1). We will show that the projected EM method is also strong $\mathcal{L}^p$-convergent with order one for the WF model in Chapter 7.

Please note that the materials in Chapters 3 and 4 have been published in Journal of Computational and Applied Mathematics and Applied Numerical Mathematics, respectively (see [1] and [2]).

Chapter 2

Preliminaries

First, we will introduce some basic mathematical background to make our thesis self-contained. However, for the sake of simplicity, we only offer necessary introduction. We recommend [3] for further readings. In addition, we will introduce the classical EM method and establish its strong convergence theory for SDEs with linear growth condition and globally Lipschitz condition. Some analysis methods and concepts about EM methods will also be introduced and will be frequently referred in this thesis.

2.1 Random variables

Probability space

Let $(\Omega, \mathcal{F})$ be a measurable space. Then a measurable function $\Pr : \mathcal{F} \to [0, 1]$ on $(\Omega, \mathcal{F})$ is a probability measure, if it satisfies

i. $\Pr(\Omega) = 1$;

ii. for any disjoint sequence $\{A_i\}_{i=1}^{\infty} \subset \mathcal{F}$, we have $\Pr(\cup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \Pr(A_i)$.

The triple $(\Omega, \mathcal{F}, \Pr)$ is then called a probability space.

Let $\bar{\mathcal{F}} = \{A \subset \Omega : \text{there exist } B, C \in \mathcal{F} \text{ such that } \Pr(B) = \Pr(C) \text{ and } B \subset A \subset C\}$. $\bar{\mathcal{F}}$ is a σ-algebra and is called the completion of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, then $(\Omega, \mathcal{F}, \Pr)$ is said to be complete. In this thesis, we always let $(\Omega, \mathcal{F}, \Pr)$ be a complete probability space.

In addition, if $A \in \mathcal{F}$ with $\Pr(A) = 1$, then it is said to happen almost surely. If $\Pr(X \neq Y) = 0$, it is reasonable think they are same, since they are only different on a null set which happens with probability zero.

Random variables

The measurable mapping $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^m, \mathcal{B}^m)$ is called a $\mathbb{R}^m$-valued random variable. For sake of convenience, we simply call X a random variable in this thesis.

The cumulative distribution function of X is given by $F_X(u) = \Pr(X \leqslant u)$.

Example 2.1.1. Normal distribution The cumulative distribution function of a normal distribution $N(\mu, \sigma^2)$ is given by

$$F(u) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{u} e^{-\frac{(v-\mu)^2}{2\sigma^2}} dv,$$

where $\mu, \sigma \in \mathbb{R}_+$.

Independence

Let I be an index set. A family of sets $\{A_i : i \in I\} \subset \mathcal{F}$ is said to be independent, if

$$\Pr\left(A_{i_1} \cap \ldots \cap A_{i_k}\right) = \Pr(A_{i_1}) \ldots \Pr(A_{i_k}),$$

for all possible choices of indices $i_1, \cdots, i_k \in I$. Let $X : (\Omega, \mathcal{F}) \to (\mathbb{R}^m, \mathcal{B}^m)$ and $Y : (\Omega, \mathcal{F}) \to (\mathbb{R}^n, \mathcal{B}^n)$ be two random variables. If

$$\Pr\left(\omega \in \Omega : X(\omega) \in A, Y(\omega) \in B\right) = \Pr\left(\omega \in \Omega : X(\omega) \in A\right) \Pr\left(\omega \in \Omega : Y(\omega) \in B\right),$$

for all $A \in \mathcal{B}^m$, $B \in \mathcal{B}^n$, then X and Y are independent.

Expectation

Let X be a random variable and is integrable with respect to Pr, then

$$\mathbb{E}(X) = \int_{\Omega} X(\omega) d \Pr(\omega)$$

is called the expectation of X. We also call $Var(X) = \mathbb{E}\left((X - \mathbb{E}(X))^2\right)$ the variance of X. If Y is also an integrable random variable but independent with X, then XY is also integrable and $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

Let $p > 0$. Let X be an $\mathbb{R}^m$-valued random variable and let $\mathbb{E}(X) = (\mathbb{E}(X_1), \cdots, \mathbb{E}(X_m))$. Then $\mathbb{E}|X|^p$ is said to be the p-th moment of X. More useful inequalities for the expectation can be found in later section.

Example 2.1.2. Normal distribution The mean and variance of the normal distribution $N(\mu, \sigma^2)$ are μ and σ^2.

Conditional expectation

Let X be a random variable in $\mathcal{L}^p(\Omega; \mathbb{R}^m)$ and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. No matter whether X is $\mathcal{G}$-measurable, by the Radon-Nikodym theorem, there always exists an integrable $\mathcal{G}$-measurable almost surely unique random variable Y such that

$$\mathbb{E}(I_G Y) = \mathbb{E}(I_G X), \quad \text{i.e.,} \int_G Y(\omega) d\,\mathrm{Pr}(\omega) = \int_G X(\omega) d\,\mathrm{Pr}(\omega), \quad \text{for all } G \in \mathcal{G}.$$

Y is then called the conditional expectation of X under the condition $\mathcal{G}$, and we write $Y = \mathbb{E}(X|\mathcal{G})$.

The conditional expectation has some properties:

i. $\mathbb{E}(\mathbb{E}(X|\mathcal{G})) = \mathbb{E}(X)$;

ii. if X is $\mathcal{G}$-measurable, then $\mathbb{E}(X|\mathcal{G}) = X$;

iii. if X is a constant c, then $\mathbb{E}(X|\mathcal{G}) = c$;

iv. if $X \geqslant 0$, then $\mathbb{E}(X|\mathcal{G}) \geqslant 0$;

v. $|\mathbb{E}(X|\mathcal{G})| \leqslant \mathbb{E}(|X||\mathcal{G})$;

vi. if X is $\mathcal{G}$-measurable, then $\mathbb{E}(XY|\mathcal{G}) = X\mathbb{E}(Y|\mathcal{G})$;

vii. $\mathbb{E}(aX + bY|\mathcal{G}) = a\mathbb{E}(X|\mathcal{G}) + b\mathbb{E}(Y|\mathcal{G})$, for any $a, b \in \mathbb{R}$;

viii. $\sigma(X)$, $\mathcal{G}$ are independent, then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}(X)$;

ix. let $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F}$, then $\mathbb{E}\left(\mathbb{E}(X|\mathcal{G}_2)|\mathcal{G}_1\right) = \mathbb{E}(X|\mathcal{G}_1)$.

2.2 Stochastic process

Stochastic process

Let $(\Omega, \mathcal{F}, \Pr)$ be a complete probability space. A filtration is a family $\{\mathcal{F}_t\}_{t\geqslant 0}$ of increasing sub-σ-algebras of $\mathcal{F}$, i.e., $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$, for all $0 \leqslant s < t < \infty$. If $\mathcal{F}_t = \cap_{s>t}\mathcal{F}_s$ for all $t \geqslant 0$, the filtration is said to be right continuous. If $\{\mathcal{F}_t\}_{t\geqslant 0}$ is right continuous and $\mathcal{F}_0$ contains all Pr-null sets, the filtration is said to satisfy the usual conditions. In this thesis, we always work on a given complete probability space $(\Omega, \mathcal{F}, \Pr)$ with a filtration satisfying the usual conditions.

A set of random variables $\{X_t\}_{t\geqslant 0}$ defined on $(\Omega, \mathcal{F}, \Pr)$ is said to be a stochastic process. Given a $t \geqslant 0$, we have a $\mathbb{R}^m$-valued random variable $X_t(\omega)$. Given a $\omega \in \Omega$, we have a function $X_t(\omega) : \bar{\mathbb{R}}_+ \to \mathbb{R}^m$, which is called a sample path of the stochastic process. For sake of convenience, $\{X_t\}_{t\geqslant 0}$ will usually be simply denoted by X_t or $X(t)$.

$\{X_t\}_{t\geqslant 0}$ is said to be continuous, if for for almost all $\omega \in \Omega$, function $X_t(\omega)$ is continuous on $t \geqslant 0$. It is integrable, if for any $t \geqslant 0$, $X(t)$ is an integrable random variable. If for any $t \geqslant 0$, $X(t)$ is $\mathcal{F}_t$-measurable, then it is said to be adapted. If $\mathbb{E}|X_t|^2 < \infty$ for every $t \in \bar{\mathbb{R}}_+$, then it is said to be square-integrable.

Stopping time

A random variable $\tau : \Omega \to [0, \infty]$ is called a stopping time, if for any $t \geqslant 0$

$$\{\omega \in \Omega : \tau(\omega) \leqslant t\} \in \mathcal{F}_t.$$

If τ and θ are stopping times, then $\tau \wedge \theta$ and $\tau \vee \theta$ are also stopping times. We also define

$$\mathcal{F}_\tau = \left\{A \in \mathcal{F} : A \cap \{\omega \in \Omega : \tau(\omega) \leqslant t\} \in \mathcal{F}_t,\ \text{for } t \geqslant 0\right\},$$

which is a sub-σ-algebra of $\mathcal{F}$. Then we have the next theorem.

Theorem 2.2.1. *If* $\{X_t\}_{t\geqslant 0}$ *is a progressively measurable process and* τ *is a stopping time, then* $X_\tau I_{\{\tau<\infty\}}$ *is* $\mathcal{F}_\tau$*-measurable. In particular, if* τ *is finite, then* X_τ *is* $\mathcal{F}_\tau$*-measurable.*

Martingale

An $\mathbb{R}^m$-valued, adapted, integrable process $\{M_t\}_{t\geqslant 0}$ is called a martingale, if

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \quad a.s. \quad \text{for all } 0 \leqslant s < t < \infty.$$

Let $X = \{X_t\}_{t\geqslant 0}$ be a progressively measurable process and let τ be a stopping time, then $X^\tau = \{X_{t\wedge\tau}\}_{t\geqslant 0}$ is called a stopped process of X.

Theorem 2.2.2. (**Doob stopping theorem**) *Let* $\{M_t\}_{t\geqslant 0}$ *be an* $\mathbb{R}^m$*-valued martingale, and let* τ*,* θ *two finite stopping times. Then*

$$\mathbb{E}(M_\theta|\mathcal{F}_\tau) = M_{\theta\wedge\tau} \quad a.s.$$

In particular, the stopped process $M^\tau = \{M_{t\wedge\tau}\}$ *is a martingale.*

2.3 Stochastic differential equation

Brownian motion

A one-dimensional Brownian motion is a real-valued, continuous, adapted process $\{B(t)\}_{t\geqslant 0}$ with:

i. $B(0) = 0$ a.s.;

ii. for $0 \leqslant s < t < \infty$, the increment $B(t) - B(s)$ is normally distributed with mean 0 and variance $t - s$;

iii. for $0 \leqslant s < t < \infty$, the increment $B(t) - B(s)$ is independent of $\mathcal{F}_s$.

In particular, $\{B(t)\}_{t\geqslant 0}$ is a martingale.

An m-dimensional process $\{(B_1(t), \dots, B_m(t))\}_{t \geqslant 0}$ is called an m-dimensional Brownian motion if every $\{B_i(t)\}_{t \geqslant 0}$ is a one-dimensional Brownian motion, and $\{B_i(t)\}_{t \geqslant 0}$ are independent.

Itô integral

We now introduce the Itô integral. Let $f \in \mathcal{M}^2([0,T];\mathbb{R})$, i.e., $\mathbb{E}\int_0^T |f(s)|^2 ds < \infty$. Then we can define a random variable, called the Itô integral of f with respect to $\{B(t)\}$, and denote it by $\int_0^T f(t)dB(t)$. Let $0 \leqslant \tau \leqslant \theta \leqslant T$ be two stopping times, then we define

$$\int_0^\tau f(s)dB(s) = \int_0^T f(s)I_{\{t \leqslant \tau\}}dB(s)$$

and

$$\int_\tau^\theta f(s)dB(s) = \int_0^\theta f(s)dB(s) - \int_0^\tau f(s)dB(s).$$

The Itô integral has some nice properties. Let $a, b \in \mathbb{R}$ and $f, g \in \mathcal{M}^2([a,b];\mathbb{R})$, we then have

i. $\int_a^b f(s)dB(s)$ is $\mathcal{F}_b$-measurable;

ii. $\mathbb{E}\int_a^b f(s)dB(s) = 0$;

iii. $\mathbb{E}\left|\int_a^b f(s)dB(s)\right|^2 = \mathbb{E}\int_a^b |f(s)|^2 ds$;

iv. $\int_a^b (c_1 f(s) + c_2 g(s))\, dB(s) = c_1 \int_a^b f(s)dB(s) + c_2 \int_a^b g(s)dB(s)$;

v. $\mathbb{E}\left(\int_a^b f(s)dB(s)|\mathcal{F}_a\right) = 0$;

vi. $\mathbb{E}\left(\left|\int_a^b f(s)dB(s)\right|^2 |\mathcal{F}_a\right) = \int_a^b \mathbb{E}\left(|f(s)|^2 |\mathcal{F}_a\right) ds$;

vii. $\mathbb{E}\int_\tau^\theta f(s)dB(s) = 0$;

viii. $\mathbb{E}\left|\int_\tau^\theta f(s)dB(s)\right|^2 = \mathbb{E}\int_\tau^\theta |f(s)|^2 ds$;

ix. $\mathbb{E}\left(\int_\tau^\theta f(s)dB(s)|\mathcal{F}_\tau\right) = 0$;

x. $\mathbb{E}\left(\left|\int_\tau^\theta f(s)dB(s)\right|^2 |\mathcal{F}_\tau\right) = \mathbb{E}\left(\int_\tau^\theta |f(s)|^2 ds|\mathcal{F}_\tau\right)$.

Let $f \in \mathcal{M}^2([0,T];\mathbb{R})$. Moreover, we can define a continuous stochastic process $\{I(t)\}_{0\leqslant t\leqslant T}$ by

$$I(t) = \int_0^t f(s)dB(s).$$

It is a square-integrable martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geqslant 0}$. If $0 \leqslant \tau \leqslant T$, then

$$I(\tau) = \int_0^\tau f(s)dB(s).$$

Finally, we consider multi-dimensional cases. Let $f \in \mathcal{M}^2([0,T];\mathbb{R}^{m\times n})$. Then the multi-dimensional indefinite Itô integral is defined by

$$\int_0^t f(s)dB(s) = \int_0^t \begin{pmatrix} f_{11}(s) & \cdots & f_{1n}(s) \\ \vdots & & \vdots \\ f_{m1}(s) & \cdots & f_{mn}(s) \end{pmatrix} \begin{pmatrix} dB_1(s) \\ \vdots \\ dB_n(s) \end{pmatrix}.$$

It is an m-column-vector-valued process, and the i-th component is the sum of one-dimensional Itô integrals: $\sum_{j=1}^n \int_0^t f_{ij}(s)dB_j(s)$. Similarly, we have

i. $\mathbb{E}\left(\int_\tau^\theta f(s)dB(s)|\mathcal{F}_\tau\right) = 0$;

ii. $\mathbb{E}\left(\left|\int_\tau^\theta f(s)dB(s)\right|^2 |\mathcal{F}_\tau\right) = \mathbb{E}\left(\int_\tau^\theta |f(s)|^2 ds|\mathcal{F}_\tau\right)$,

for two arbitrary stopping times $0 \leqslant \tau \leqslant \theta \leqslant T$.

Itô formula

We now introduce the Itô formula, which can be considered as the stochastic version of chain rule for the Itô integral. It will be frequently used in this thesis.

Let $\{B(t)\}_{t\geqslant 0}$ be an m-dimensional Brownian motion. An m-dimensional Itô process is an $\mathbb{R}^m$-valued, continuous, adapted process $(x_1(t), \cdots, x_d(t))^T$ on $\bar{\mathbb{R}}_+$, of the form

$$x(t) = x(0) + \int_0^t f(s)ds + \int_0^t g(s)dB(s),$$

where $f = (f_1, \cdots, f_m)^T \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^m)$ and $g = (g_{ij})_{m\times n} \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{m\times n})$. We shall say that $x(t)$ has an Itô differential $dx(t)$ on $\bar{\mathbb{R}}_+$, which is given by

$$dx(t) = f(t)dt + g(t)dB(t).$$

Let $V(x,t) \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}_+; \mathbb{R})$, i.e., the family of all real-valued functions defined on $\mathbb{R}^m \times \bar{\mathbb{R}}_+$ such that they are continuously twice differentiable in x and once in t with

$$V_t = \frac{\partial V}{\partial t}, \quad V_x = \left(\frac{\partial V}{\partial x_1}, \ldots, \frac{\partial V}{\partial x_m}\right), \quad V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{m\times m}.$$

Theorem 2.3.1. (Itô formula) *Let $x(t)$ be an m-dimensional Itô process on $\bar{\mathbb{R}}_+$ with the Itô differential*

$$dx(t) = f(t)dt + g(t)dB(t).$$

Let $V(x,t) \in C^{2,1}(\mathbb{R}^m \times \mathbb{R}_+; \mathbb{R})$. Then $V(x(t),t)$ is a real-valued Itô process with Itô differential

$$dV(x(t),t) = \left(V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}tr\left(g^T(t)V_{xx}(x(t),t)g(t)\right)\right)dt + V_x(x(t),t)g(t)dB(t),$$

where $tr(A)$ is the trace of a square matrix A.

Stochastic differential equation

Let $\left\{B(t) = (B_1(t), \ldots, B_n(t))^T\right\}_{t\geqslant 0}$ be an n-dimensional Brownian motion on this space. Let $0 \leqslant t_0 < T < \infty$ and let x_{t_0} be an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^m$-valued random variable such that $\mathbb{E}|x_{t_0}|^2 < \infty$. Let $f : \mathbb{R}^m \times [t_0, T] \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times [t_0, T] \to \mathbb{R}^{m\times n}$ both be Borel measurable.

If an $\mathbb{R}^m$-valued stochastic process $\{x(t)\}_{t\in[t_0,T]}$ has the following properties:

i. $\{x(t)\}_{t\in[t_0,T]}$ is continuous and $\mathcal{F}_t$-adapted;

ii. $\{f(x(t),t)\} \in \mathcal{L}^1([t_0,T];\mathbb{R}^m)$ and $\{g(x(t),t)\} \in \mathcal{L}^2([t_0,T];\mathbb{R}^{m\times n})$;

iii. $x(t) = x_{t_0} + \int_{t_0}^t f(x(s),s)\,ds + \int_{t_0}^t g(x(s),s)\,dB(s)$, for all $t \in [t_0,T]$ with probability one,

then it is called a solution of the m-dimensional stochastic differential equation of Itô type

$$dx(t) = f(x(t),t)\,dt + g(x(t),t)\,dB(t)$$

with initial value x_{t_0}. In particular, a solution $\{x(t)\}$ is said to be unique if any other solution $\{\bar{x}(t)\}$ is indistinguishable from $\{x(t)\}$, that is

$$\Pr(\{\omega : x(t) = \bar{x}(t),\ \text{for } t \in [t_0,T]\}) = 1.$$

As an example, we give two classical assumptions to guarantee the existence and the uniqueness of solutions here.

Assumption 2.3.1. (globally Lipschitz condition) Assume that there exists a constant $K > 0$ such that

$$|f(u,t) - f(v,t)| \vee |g(u,t) - g(v,t)| \leqslant K|u - v|,\ \text{for } u,v \in \mathbb{R}^m, t \in [t_0,T].$$

Assumption 2.3.2. (Linear growth condition) Assume that there exists a constant $\bar{K} > 0$ such that

$$|f(u,t)| \vee |g(u,t)| \leqslant \bar{K}(1 + |u|),\ \text{for } u \in \mathbb{R}^m, t \in [t_0,T].$$

Theorem 2.3.2. Existence and uniqueness *Assume that Assumptions 2.3.1 and 2.3.2 hold. Then there exists a unique solution $x(t)$, and it belongs to $\mathcal{M}^2([t_0,T];\mathbb{R}^m)$.*

For some special criteria to ensure existence and uniqueness, we recommend [3–5] for further readings.

2.4 Useful inequalities

First, we list some useful inequalities for moments. Let $0 < q \leqslant p < \infty$. Let X, Y be two $\mathbb{R}^m$-valued random variable with $\mathbb{E}|X|^p < \infty$ and $\mathbb{E}|Y|^q < \infty$. Then we have

1. **(Hölder's inequality)** let $p, q > 1$ and $1/p + 1/q = 1$, then we have

$$\mathbb{E}(X^T Y) \leqslant (\mathbb{E}|X|^p)^{\frac{1}{p}} (\mathbb{E}|Y|^q)^{\frac{1}{q}};$$

2. **(Lyapunov's inequality)** let $0 < q < p < \infty$, then we have

$$(\mathbb{E}|X|^q)^{\frac{1}{q}} \leqslant (\mathbb{E}|X|^p)^{\frac{1}{p}};$$

3. **(Minkowski's inequality)** let $p = q > 1$, then we have

$$(\mathbb{E}|X+Y|^p)^{\frac{1}{p}} \leqslant (\mathbb{E}|X|^p)^{\frac{1}{p}} + (\mathbb{E}|Y|^p)^{\frac{1}{p}};$$

4. **(Chebyshev's inequality)** let $p > 0$ and $c > 0$, then we have

$$\Pr\left(\{\omega : |X(\omega)| \geqslant c\}\right) \leqslant \frac{\mathbb{E}|X|^p}{c^p}.$$

We will also frequently use the Young inequality. Let $a, b > 0$ and $p, q > 1$ with $1/p + 1/q = 1$, we then have

$$ab \leqslant \varepsilon a^p + \frac{(p\varepsilon)^{-q/p}}{q} b^q.$$

Then we introduce the Burkholder-Davis-Gundy inequality.

Theorem 2.4.1. (The Burkholder-Davis-Gundy inequality) *Let $g \in \mathcal{L}^2([0,T]; \mathbb{R}^{m\times n})$ and let $p > 0$ be arbitrary. Define*

$$x(t) = \int_0^t g(s) dB(s) \quad \text{and} \quad A(t) = \int_0^t |g(s)|^2 \, ds,$$

for all $t \in [0,T]$. *Then there exist universal positive constants* c_p, C_p, *only depending on p, such that*

$$c_p \mathbb{E}|A(t)|^{\frac{p}{2}} \leqslant \mathbb{E}\left(\sup_{s\in[0,t]} |x(s)|^p\right) \leqslant C_p \mathbb{E}|A(t)|^{\frac{p}{2}},$$

for all $t \in [0,T]$. *In particular, we may take*

$$\begin{aligned}
&c_p = (p/2)^p, && C_p = (32/p)^{p/2}, && p \in (0,2);\\
&c_p = 1, && C_p = 4, && p = 2;\\
&c_p = (2p)^{-p/2}, && C_p = \left(p^{p+1}/2(p-1)^{p-1}\right)^{p/2}, && p > 2.
\end{aligned}$$

We also have another upper bound estimation theorem.

Theorem 2.4.2. *Let* $p \geqslant 2$ *be arbitrary and let* $g \in \mathcal{L}^2([0,T];\mathbb{R}^{m\times n})$ *with*

$$\mathbb{E}\int_0^T |g(s)|^p ds < \infty.$$

Then we have

$$\mathbb{E}\left|\int_0^T g(s)dB(s)\right|^p \leqslant \left(\frac{p(p-1)}{2}\right)^{p/2} T^{p/2-1}\mathbb{E}\int_0^T |g(s)|^p ds.$$

In particular, the equality holds for $p = 2$.

Now we introduce two useful inequalities, which will be frequently used in this thesis.

Theorem 2.4.3. (The Bihari inequality) *Let* $T > 0$ *and* $c \geqslant 0$. *Let* $u(t)$ *be a Borel measurable bounded nonnegative function on* $[0,T]$, *and let* $v(t)$ *be a nonnegative integrable function on* $[0,T]$. *Let* $K : \bar{\mathbb{R}}_+ \to \bar{\mathbb{R}}_+$ *be a continuous non-decreasing function such that* $K(t) > 0$ *for all* $t > 0$. *Let*

$$G(r) = \int_1^r \frac{ds}{K(s)}, \quad \text{for } r > 0.$$

If

$$u(t) \leqslant c + \int_0^t v(s)K(u(s))ds, \quad \textit{for all } t \in [0, T],$$

then we have

$$u(t) \leqslant G^{-1}\left(G(c) + \int_0^t v(s)ds\right),$$

for all $t \in [0, T]$ *such that*

$$G(c) + \int_0^t v(s)ds < G(\infty).$$

Especially, if $K(x) = x$, we have the Gronwall inequality.

Theorem 2.4.4. (The Gronwall inequality) *Let* $T > 0$ *and* $c \geqslant 0$. *Let* $u(t)$ *be a Borel measurable bounded nonnegative function on* $[0, T]$, *and let* $v(t)$ *be a nonnegative integrable function on* $[0, T]$. *If*

$$u(t) \leqslant c + \int_0^t v(s)u(s)ds, \quad \textit{for all } t \in [0, T],$$

then we have

$$u(t) \leqslant c \exp\left(\int_0^t v(s)ds\right), \quad \textit{for all } t \in [0, T].$$

2.5 Classical Euler-Maruyama method

Basic introduction

Let $B(t)$ be an n-dimensional Brownian motion on this space. Let $0 \leqslant t_0 < T < \infty$ and let x_{t_0} be an $\mathcal{F}_{t_0}$-measurable $\mathbb{R}^m$-valued random variable such that $\mathbb{E}|x_{t_0}|^2 < \infty$. Let $f : \mathbb{R}^m \times [t_0, T] \to \mathbb{R}^m$ and $g : \mathbb{R}^m \times [t_0, T] \to \mathbb{R}^{m \times n}$ both be Borel measurable. Then we consider the SDE:

$$dx(t) = f(x(t), t)dt + g(x(t), t)dB(t), \quad t \in [t_0, T]$$

with the initial x_{t_0}. Moreover, we assume that Assumptions 2.3.1 and 2.3.2 hold.

However, an SDE may not have an analytical solution even under these two simple assumptions. Therefore, it is necessary to develop numerical approximation methods to approximate the exact solution $x(t)$. The classical EM method is one of useful numerical methods. In this section, we will introduce the classical EM method and establish its strong convergence theory for SDEs with globally Lipschitz coefficients.

Let $\Delta \in \{(T-t_0)/k : k \in \mathbb{N}_+\}$ be a step size. We first define the classical EM numerical solution on the $[k\Delta, (k+1)\Delta]$. The classical EM numerical solution $x_\Delta(t)$ is defined by starting from $x(t_0)$ and computing the recursion

$$x_\Delta(t) = x_\Delta(k\Delta) + \int_{k\Delta}^{t} f(x_\Delta(k\Delta), s)ds + \int_{k\Delta}^{t} g(x_\Delta(k\Delta), s)dB(s),$$

for $t \in [k\Delta, (k+1)\Delta]$ and $k \in \mathbb{N}$. It gives an expression for the continuous version of the scheme over a single step. Then we define

$$\bar{x}_\Delta(t) = \sum_{i=0}^{(T-t_0)/\Delta-1} x_\Delta(i\Delta) I_{\{t\in[i\Delta,(i+1)\Delta)\}}.$$

In addition, we have

$$x_\Delta(t) = x_\Delta(t_0) + \int_{t_0}^{t} f(\bar{x}_\Delta(s), s)dt + \int_{t_0}^{t} g(\bar{x}_\Delta(s), s)dB(s),$$

for all $t \in [t_0, T]$. It gives an expression for the continuous version of the scheme over the full time set.

Now we establish the strong convergence theory of the classical EM method. We will use C to stand for generic positive real numbers which are dependent on T, t_0, K, $\bar{K}$ and p, but independent of Δ and its values may change between occurrences. We first establish two necessary lemmas.

Lemma 2.5.1. (Finite p-moments of $x_\Delta(t)$) *Let $p \geqslant 2$. There then exists a constant*

$C > 0$ *such that*

$$\sup_{\Delta\in(0,1]} \sup_{t\in[t_0,T]} \mathbb{E}|x_\Delta(t)|^p < C.$$

Proof. Given a $k \in \mathbb{N}_+$, we define a stopping time $\tau_k = \inf\{t \in [t_0, T] : |x_\Delta(t)| > k\}$. In particular, we set $\inf \emptyset = \infty$, where $\emptyset$ is an empty set. Using the Itô formula and taking expectations on both sides, we have

$$\begin{aligned}\mathbb{E}|x_\Delta(t\wedge\tau_k)|^p =&\mathbb{E}|x_\Delta(t_0)|^p + p\mathbb{E}\int_{t_0}^{t\wedge\tau_k} |x_\Delta(s)|^{p-2} x_\Delta(s)^T f(\bar{x}_\Delta(s), s)ds \\ &+ \frac{p(p-2)}{2}\mathbb{E}\int_{t_0}^{t\wedge\tau_k} |x_\Delta(s)|^{p-4}|g^T(\bar{x}_\Delta(s), s)x_\Delta(s)|^2 ds \\ &+ \frac{p}{2}\mathbb{E}\int_{t_0}^{t\wedge\tau_k} |x_\Delta(s)|^{p-2}|g(\bar{x}_\Delta(s), s)|^2 ds \\ &+ p\mathbb{E}\int_{t_0}^{t\wedge\tau_k} |x_\Delta(s)|^{p-2} x_\Delta(s)^T g(\bar{x}_\Delta(s), s)dB(s),\end{aligned}$$

for all $t \in [t_0, T]$.

Since $|x_\Delta(t)| \leqslant k$ for $t \in [t_0, T\wedge\tau_k]$, each component of $|x_\Delta(t)|^{p-2}x_\Delta(t)^T g(x_\Delta(t), t)$ is bounded for $t \in [t_0, T\wedge\tau_k]$. Therefore,

$$\left\{|x_\Delta(t)|^{p-2}x_\Delta(t)^T g(\bar{x}_\Delta(t), t) I_{\{t\in[T\wedge\tau_k]\}}\right\}_{t\in[t_0,T]} \in \mathcal{M}^2([t_0, T]; \mathbb{R}^n).$$

It then implies that

$$\mathbb{E}\int_{t_0}^{t\wedge\tau_k} |x_\Delta(s)|^{p-2}x_\Delta(s)^T g(\bar{x}_\Delta(s), s)dB(s) = 0,$$

for $t \in [t_0, T]$.

Using the linear growth condition and the Young inequality, we have

$$\begin{aligned}
\mathbb{E}|x_\Delta(t\wedge\tau_k)|^p \leqslant& \mathbb{E}|x_\Delta(t_0)|^p + pK\mathbb{E}\int_{t_0}^{t\wedge\tau_k} |x_\Delta(s)|^{p-1}(1+|\bar{x}_\Delta(s)|)ds \\
&+ \frac{p(p-1)\bar{K}^2}{2}\mathbb{E}\int_{t_0}^{t\wedge\tau_k} |x_\Delta(s)|^{p-2}(1+|\bar{x}_\Delta(s)|)^2 ds \\
\leqslant& \mathbb{E}|x_\Delta(t_0)|^p + K\mathbb{E}\int_{t_0}^{t\wedge\tau_k} \left((p-1)|x_\Delta(s)|^p + 2^{p-1}(1+|\bar{x}_\Delta(s)|^p)\right) ds \\
&+ \frac{(p-1)\bar{K}^2}{2}\mathbb{E}\int_{t_0}^{t\wedge\tau_k} ((p-2)|x_\Delta(s)|^p + 2^p(1+|\bar{x}_\Delta(s)|^p)) ds \\
\leqslant& \mathbb{E}|x_\Delta(t_0)|^p + C\mathbb{E}\int_{t_0}^{t\wedge\tau_k} (1+|x_\Delta(s)|^p + |\bar{x}_\Delta(s)|^p) ds,
\end{aligned}$$

for $t \in [t_0, T]$.

Using the Fubini theorem, we then have

$$\begin{aligned}
&\mathbb{E}\int_{t_0}^{t\wedge\tau_k} (1+|x_\Delta(s)|^p + |\bar{x}_\Delta(s)|^p) ds \\
\leqslant& \mathbb{E}\int_{t_0}^{t} (1+|x_\Delta(s\wedge\tau_k)|^p + |\bar{x}_\Delta(s\wedge\tau_k)|^p) ds \\
=& \int_{t_0}^{t} \mathbb{E}\,(1+|x_\Delta(s\wedge\tau_k)|^p + |\bar{x}_\Delta(s\wedge\tau_k)|^p) ds \\
\leqslant& \int_{t_0}^{t} \left(1 + 2\sup_{u\in[t_0,s]} \mathbb{E}|x_\Delta(u\wedge\tau_k)|^p\right) ds.
\end{aligned}$$

Using the Fubini theorem

$$\sup_{u\in[t_0,t]} \mathbb{E}|x_\Delta(u\wedge\tau_k)|^p \leqslant (C + \mathbb{E}|x_\Delta(t_0)|^p) + C\int_{t_0}^{t} \sup_{u\in[t_0,s]} \mathbb{E}|x_\Delta(u\wedge\tau_k)|^p ds,$$

for $t \in [t_0, T]$. Then the Gronwall inequality implies that

$$\sup_{u\in[t_0,T]} \mathbb{E}|x_\Delta(u\wedge\tau_k)|^p < C.$$

Letting $k \to \infty$, we then have the conclusion. □

By similar arguments, we have the next lemma for the exact solution $x(t)$.

Lemma 2.5.2. (Finite p-moments of $x(t)$) *Let $p \geqslant 2$. There exists a constant $C > 0$*

such that

$$\sup_{t\in[t_0,T]} \mathbb{E}|x(t)|^p < C.$$

Lemma 2.5.3. *Let $p \geqslant 2$. There exists a constant $C > 0$ such that*

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leqslant C\Delta^{p/2},$$

for all $t \in [t_0, T]$ and $\Delta \in (0, 1]$.

Proof. Using the Hölder inequality and Theorem 2.4.2, we have

$$\begin{aligned}
&\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \\
=&\mathbb{E}\left|\int_{\lfloor t/\Delta\rfloor\Delta}^t f(\bar{x}_\Delta(s), s)ds + \int_{\lfloor t/\Delta\rfloor\Delta}^t g(\bar{x}_\Delta(s), s)dB(s)\right|^p \\
\leqslant&2^{p-1}\mathbb{E}\left(\left|\int_{\lfloor t/\Delta\rfloor\Delta}^t f(\bar{x}_\Delta(s), s)ds\right|^p + \left|\int_{\lfloor t/\Delta\rfloor\Delta}^t g(\bar{x}_\Delta(s), s)dB(s)\right|^p\right) \\
\leqslant&2^{p-1}\Delta^{p-1}\mathbb{E}\int_{\lfloor t/\Delta\rfloor\Delta}^t |f(\bar{x}_\Delta(s), s)|^p ds + 2^{p-1}\Delta^{p/2-1}\mathbb{E}\int_{\lfloor t/\Delta\rfloor\Delta}^t |g(\bar{x}_\Delta(s), s)|^p\, ds.
\end{aligned}$$

Using the linear growth condition and Lemma 2.5.1, we have

$$\begin{aligned}
&\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \\
\leqslant&2^{p-1}\Delta^{p-1}\mathbb{E}\int_{\lfloor t/\Delta\rfloor\Delta}^t \bar{K}^p(1 + |\bar{x}_\Delta(s)|)^p ds + 2^{p-1}\Delta^{p/2-1}\mathbb{E}\int_{\lfloor t/\Delta\rfloor\Delta}^t \bar{K}^p(1 + |\bar{x}_\Delta(s)|)^p ds \\
\leqslant&2^{2p-2}\Delta^{p/2-1}\left(\Delta^{p/2} + 1\right)\bar{K}^p\int_{\lfloor t/\Delta\rfloor\Delta}^t (1 + \mathbb{E}|\bar{x}_\Delta(s)|^p)ds \\
\leqslant&C\Delta^{p/2}\left(\Delta^{p/2} + 1\right) \\
\leqslant&C\Delta^{p/2}.
\end{aligned}$$

□

Definition 2.5.1. If we have

$$\lim_{\Delta \to 0} \mathbb{E}|x(T) - x_\Delta(T)|^p = 0,$$

then the classical EM method is said to be $\mathcal{L}^p$-strongly convergent (at time T). If there further exist positive real numbers C and δ such that

$$\mathbb{E}|x(T) - x_\Delta(T)|^p \leqslant C\Delta^{p\delta},$$

for every $\Delta \in (0, 1]$, then the classical EM method is said to be $\mathcal{L}^p$-strongly convergent with order δ.

Theorem 2.5.1. *Let $p \geqslant 2$ The classical EM method is $\mathcal{L}^p$-strongly convergent with order one half. That is, there exists a constant C such that*

$$\mathbb{E}|x(T) - x_\Delta(T)|^p \leqslant C\Delta^{p/2},$$

for all Δ.

Proof. Using the Itô formula and taking expectations on both sides, we have

$$\begin{aligned}
&\mathbb{E}|x(t) - x_\Delta(t)|^p \\
=&p\mathbb{E}\int_{t_0}^{t} |x(s) - x_\Delta(s)|^{p-2} \left(x(s) - x_\Delta(s)\right)^T \left(f(x(s), s) - f(\bar{x}_\Delta(s), s)\right) ds \\
&+ \frac{p(p-2)}{2}\mathbb{E}\int_{t_0}^{t} |x(s) - x_\Delta(s)|^{p-4} | \left(g(x(s), s) - g(\bar{x}_\Delta(s), s)\right)^T \left(x(s) - x_\Delta(s)\right) |^2 ds \\
&+ \frac{p}{2}\mathbb{E}\int_{t_0}^{t} |x(s) - x_\Delta(s)|^{p-2} |g(x(s), s) - g(\bar{x}_\Delta(s), s)|^2 ds \\
&+ p\mathbb{E}\int_{t_0}^{t} |x(s) - x_\Delta(s)|^{p-2} \left(x(s) - x_\Delta(s)\right)^T \left(g(x(s), s) - g(\bar{x}_\Delta(s), s)\right) dB(s),
\end{aligned}$$

for all $t \in [t_0, T]$.

Using Lemmas 2.5.1 and 2.5.2, we have

$$p\mathbb{E}\int_{t_0}^{t} |x(s) - x_\Delta(s)|^{p-2} \left(x(s) - x_\Delta(s)\right)^T \left(g(x(s), s) - g(\bar{x}_\Delta(s), s)\right) dB(s) = 0.$$

Using Lemma 2.5.3, the globally Lipschitz condition and the Young inequality, we then have

$$\begin{aligned}
&\mathbb{E}|x(t)-x_\Delta(t)|^p\\
=&p\mathbb{E}\int_{t_0}^t |x(s)-x_\Delta(s)|^{p-2}\left(x(s)-x_\Delta(s)\right)^T\left(f(x(s),s)-f(\bar{x}_\Delta(s),s)\right)ds\\
&+\frac{p(p-2)}{2}\mathbb{E}\int_{t_0}^t |x(s)-x_\Delta(s)|^{p-4}|\left(g(x(s),s)-g(\bar{x}_\Delta(s),s)\right)^T\left(x(s)-x_\Delta(s)\right)|^2 ds\\
&+\frac{p}{2}\mathbb{E}\int_{t_0}^t |x(s)-x_\Delta(s)|^{p-2}|g(x(s),s)-g(\bar{x}_\Delta(s),s)|^2 ds\\
\leqslant& pK\mathbb{E}\int_{t_0}^t |x(s)-x_\Delta(s)|^{p-1}\left(|x(s)-x_\Delta(s)|+|x_\Delta(s)-\bar{x}_\Delta(s)|\right)ds\\
&+p(p-1)\bar{K}^2\mathbb{E}\int_{t_0}^t |x(s)-x_\Delta(s)|^{p-2}|\left(|x(s)-x_\Delta(s)|^2+|x_\Delta(s)-\bar{x}_\Delta(s)|^2\right)ds\\
\leqslant&\left((2p-1)K+2p(p-1)^2\bar{K}^2\right)\mathbb{E}\int_{t_0}^t |x(s)-x_\Delta(s)|^p ds\\
&+\left(K+2(p-1)\bar{K}^2\right)\int_{t_0}^t \mathbb{E}|x_\Delta(s)-\bar{x}_\Delta(s)|^p ds\\
\leqslant& C\Delta^{p/2}+C\int_{t_0}^t \mathbb{E}|x(s)-x_\Delta(s)|^p ds,
\end{aligned}$$

for all $t\in[t_0,T]$. Finally, the Gronwall inequality implies the conclusion. □

Under Assumptions 2.3.1 and 2.3.2, the strong convergence theory of the classical EM method for SDEs is established. It has some properties.

i. It is an explicit numerical method, i.e., there exists a function F such that $x_\Delta((k+1)\Delta)=F(x_\Delta(k\Delta))$. On the other hand, some numerical methods require solving an equation $\bar{F}(x_\Delta((k+1)\Delta),x_\Delta(k\Delta))=0$.

ii. It is $\mathcal{L}^p$-strongly convergent with order one half;

iii. Its numerical solution takes values in the whole of the Euclidean space.

Challenges

Now we consider three types of locally Lipschitz coefficients:

i. polynomially growing coefficients;

ii. have reciprocal parts;

iii. Hölder continuous near some points.

First, we consider polynomially growing coefficients. The drift coefficient of the scalar stochastic Ginzburg-Landau equation is $\alpha x - \beta x^3$, where $\alpha, \beta > 0$. It does not satisfy the linear growth condition. Therefore, numerical solutions may fail to have finite p-moments. As a concrete example, Hutzenthaler, Jentzen and Kloeden [6] showed the moments of the EM numerical method may diverge to infinity within a finite time even when the moments of the exact solution are finite. It is then impossible to give an upper bound for $\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p$, which is necessary in Theorem 2.5.1. In addition, we have

$$\left(\alpha u - \beta u^3\right) - \left(\alpha v - \beta v^3\right) \leqslant \left(\alpha - \beta(u^2 + uv + v^2)\right)(u - v).$$

Then we have

$$\begin{aligned}
&|x(s) - x_\Delta(s)|^{p-1}|f(x(s), s) - f(x_\Delta(s), s)| \\
\leqslant &\left(\alpha - \beta(x(s)^2 + x(s)x_\Delta(s) + x_\Delta(s)^2)\right)|x(s) - x_\Delta(s)|^p
\end{aligned}$$

in Theorem 2.5.1, which means that we cannot apply the Gronwall inequality here.

However, we also notice that

$$\begin{aligned}
&|x(s) - x_\Delta(s)|^{p-2}(x(s) - x_\Delta(s))(f(x(s), s) - f(x_\Delta(s), s)) \\
\leqslant &|x(s) - x_\Delta(s)|^{p-2}\left(\alpha(x(s) - x_\Delta(s))^2\right) \\
= &\alpha|x(s) - x_\Delta(s)|^p.
\end{aligned}$$

That is because the drift coefficient of the scalar stochastic Ginzburg-Landau equation is one-sided Lipschitz. With this relaxed coefficient condition, we will improve the classical EM numerical analysis methods. These new numerical analysis techniques will be introduced in Chapter 3, and the strong convergence theory will be established.

For the second case, similar problems arise. Moreover, the classical EM numerical solutions take values in the whole of the Euclidean space, the Brownian motion takes values in the whole of the Euclidean space. For example, the classical EM numerical solutions to the Aït-Sahalia model always generate negative approximations. However, the exact solution to the Aït-Sahalia model only takes value to positive real numbers, i.e., the classical EM is not boundary preserving. Furthermore, the classical EM numerical solutions do not have inverse moments, i.e., it is even impossible to define $\mathbb{E}\left(x_\Delta(t)^{-1}\right)$. However, finite inverse moments are necessary to estimate the upper bound for $\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p$. Therefore, additional corrections and related numerical analysis techniques are needed. We will address these problems in two ways in Chapters 4, 6 and 7.

Finally, we will consider SDEs with Hölder continuous coefficients. To be precise, we are concerned with the CEV model and the CIR model. There are similar problems for these two SDE models, e.g., the exact solution only take values in positive real numbers and the derivative of the diffusion coefficient is reciprocal. Some numerical analysis techniques in the previous chapters can be applied for the CEV model, but fail to work for the CIR model. In Chapter 5, we will introduce a new EM method and slightly improve proven convergence results for the CIR model.

Chapter 3

The modified truncated EM method for stochastic differential equations with concave diffusion coefficients

3.1 Background

In 2002, Higham, Mao and Stuart [7] proved the strong convergence theory under the condition that the exact solution and the numerical solution both have finite p-th moments. They then introduced the split-step backward EM method, which is computed by

$$
\begin{aligned}
x_{\Delta}^{k}(t_{k+1}) =& x_{\Delta}(t_k) + f(x_{\Delta}^{k}(t_{k+1}))\Delta, \\
x_{\Delta}(t_{k+1}) =& x_{\Delta}^{k}(t_{k+1}) + g(x_{\Delta}^{k}(t_{k+1}))\left(B(t_{k+1}) - B(t_k)\right),
\end{aligned}
$$

where $t_k = k\Delta$ and Δ is the step size. They then proved finite p-th moments for exact solutions and split-step backward EM numerical solutions to SDEs with the one-side Lipschitz continuous drift coefficients and globally Lipschitz continuous diffusion coefficients. However, we have to solve an implicit function to compute $x^k_\Delta(t_{k+1})$. That is, expensive computational cost is required for implementation of this implicit EM method. In addition, they did not prove a concrete convergence rate order.

Many explicit numerical methods for polynomially growing coefficients were also developed in recent years. For example, Hutzenthaler, Jentzen and Kloeden [8] proposed the tamed EM method. Sabanis [9, 10] then further developed the strong convergence theory of the tamed EM method. Liu and Mao [11] developed the stopped EM method. Especially, inspired by [7], Mao [12, 13] established the truncated EM method. Mao also proved that the truncated EM method has a concrete convergence rate order under appropriate assumptions. Li, Mao and Yin [14] then used several truncation methods and extended the truncated EM method.

However, diffusion coefficients in the above articles both are globally Lipschitz continuous, which exclude some important SDE models. For example, Malliavin [15] studied the right invariant canonic horizontal diffusion and deduced a relevant diffusion coefficient $-x(t)\ln^{1/2}(|x(t)|)$ which is not globally Lipschitz continuous. There are some papers which are concerned with this type of diffusion coefficient (e.g., see [16–30]). Nevertheless, both of them are concerned with the constant elasticity of variance model model or the Cox-Ingersoll-Ross model, whose drift coefficients are globally Lipschitz continuous. In [31–33], researchers developed different modified EM methods and established their strong convergence for SDEs with polynomially growing drift coefficients and Hölder continuous diffusion coefficients. These three papers both use the Yamada and Watanabe's analysis method, so they can establish the strong convergence theory only for one-dimensional SDEs.

This chapter is extracted from [1]. In this chapter, we will establish the strong convergence theory of the truncated EM for multi-dimensional SDEs with polynomially growing drift coefficients and concave diffusion coefficients. Section 2 first introduces assumptions and establishes some useful lemmas. Then section 3 investigates the convergence of the modified truncated EM method at a given time T. Moreover, we study the convergence of the modified truncated EM method over a finite time interval in section 4. In section 5, we present an example and conduct simulations to support our theoretical results. Finally, we make a brief conclusion in section 6.

3.2 Preliminaries and assumptions

Let $B(t) = (B_1(t), B_2(t), \dots, B_n(t))^T$ be an n-dimensional Brownian motion defined on this space. Let $f : \mathbb{R}^m \to \mathbb{R}^m$ and $g : \mathbb{R}^m \to \mathbb{R}^{m\times n}$ both be Borel measurable. In this chapter, we will use C to stand for generic positive real numbers which are dependent on T, γ, L, H_1, etc., but independent of Δ and its values may change between occurrences. We also let $\inf \emptyset = \infty$.

In this chapter, we consider an m-dimensional SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t), \tag{3.2.1}$$

on $0 \leqslant t \leqslant T$ with the initial value $x(0) = x_0 \in \mathbb{R}^m$, where $T \in (0, \infty)$ is fixed. We impose the following standing hypotheses in this chapter.

Assumption 3.2.1. Assume that there is a pair of positive constants γ and L such that

$$|f(u) - f(v)| \leqslant L(1 + |u|^\gamma + |v|^\gamma)|u - v|,$$

for all $u, v \in \mathbb{R}^m$.

Assumption 3.2.2. Assume that there exists a continuous non-decreasing concave function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that

$$\int_{0^+} \frac{du}{\kappa(u)} = \infty, \tag{3.2.2}$$

and we have

$$|g(u) - g(v)|^2 \leqslant \kappa(|u - v|^2),$$

for all $u \neq v$, where $u, v \in \mathbb{R}^m$.

Assumption 3.2.3. Assume that there exists a positive constant H_1 such that

$$(u-v)^T(f(u)-f(v)) \leqslant H_1|u-v|^2,$$

for all $u, v \in \mathbb{R}^m$.

Example 3.2.1. If $\kappa_1(u) = Ku$ with $K > 0$, then it satisfies (3.2.2). In this case, Assumption 3.2.2 reduces to the globally Lipschitz condition.

Let $u^* \in (0, 0.5e^{-1})$, we define

$$\kappa_2(u) = \begin{cases} -u \ln u, & 0 < u \leqslant u^*, \\ -u^* \ln u^* - (1+\ln u^*)(u-u^*), & u > u^*. \end{cases}$$

(3.2.2) is satisfied.

Let $u^* \in (0, e^{-3})$, we define

$$\kappa_3(u) = \begin{cases} -u \ln u \ln(-\ln u), & 0 < u \leqslant u^*, \\ -u^* \ln u^* \ln(-\ln u^*) + \kappa_3'(u^*)(u-u^*), & u > u^*. \end{cases}$$

(3.2.2) is satisfied.

Remark 3.2.1. If Assumption 3.2.2 holds, then concavity implies that there exists a positive constant C such that

$$|g(u)-g(v)|^2 \leqslant C(1+|u-v|^2),$$

and therefore

$$|g(u)|^2 \leqslant 2|g(u)-g(\mathbf{0})|^2 + 2|g(\mathbf{0})|^2 \leqslant C(1+|u|^2).$$

Combining this with Assumption 3.2.3, we have

$$u^T f(u) = (u-\mathbf{0})^T \left(f(u)-f(\mathbf{0})\right) + u^T f(\mathbf{0}) \leqslant C(1+|u|^2).$$

Then we derive the Khasminskii-type condition: there exists a positive constant $C(p)$ depending on p such that

$$u^T f(u) + \frac{p-1}{2}|g(u)|^2 \leqslant C(p)(1+|u|^2), \tag{3.2.3}$$

for all $u \in \mathbb{R}^m$ and $p \geqslant 2$.

To study the strong convergence theory, we first introduce a variant of κ which also satisfies Assumption 3.2.2.

Lemma 3.2.1. *Let $p \geqslant 2$. There exists a continuous non-decreasing concave function $\hat{\kappa}(u) = \kappa(u) + (\kappa(1)+1)u$ such that*

$$\hat{\kappa}(u) \geqslant u^{\frac{p-2}{p}} \kappa(u^{\frac{2}{p}}) \vee u,$$

and $\hat{\kappa}(u)$ satisfies (3.2.2).

Proof. To satisfy Assumption 3.2.2, it is clear that $\lim_{u\to 0^+} \kappa(u) = 0$. Now we define $\kappa(0) = 0$. If $m > 1$, then

$$m\kappa(u) \geqslant m\left(\frac{1}{m}\kappa(mu) + (1-\frac{1}{m})\kappa(0)\right) = \kappa(mu),$$

for any $u \in \bar{\mathbb{R}}_+$. Therefore,

$$\frac{v}{u}\kappa(u) \geqslant \kappa(v),$$

for $0 < u < v$.

Now we let $p \geqslant 2$ and set $\hat{\kappa}(u) = \kappa(u) + (\kappa(1)+1)u$. Since $\kappa(u)/u$ is decreasing, we have

$$u^{\frac{p-2}{p}} \kappa(u^{\frac{2}{p}}) = u\frac{\kappa(u^{\frac{2}{p}})}{u^{\frac{2}{p}}} \leqslant u\frac{\kappa(u)}{u} = \kappa(u) \leqslant \hat{\kappa}(u),$$

for $0 < u < 1$. Besides, we have

$$u^{\frac{p-2}{p}}\kappa(u^{\frac{2}{p}}) = u\frac{\kappa(u^{\frac{2}{p}})}{u^{\frac{2}{p}}} \leqslant u\frac{\kappa(1)}{1} = \kappa(1)u \leqslant \hat{\kappa}(u),$$

for $1 \leqslant u$. Then we have $\hat{\kappa}(u)$ is concave and $\hat{\kappa}(u) \geqslant u^{\frac{p-2}{p}}\kappa(u^{\frac{2}{p}}) \vee u$, for all $u > 0$.

If $\kappa(u) \leqslant (\kappa(1)+1)u$, for all $u > 0$, then we have $\frac{1}{2(\kappa(1)+1)u} < \frac{1}{\hat{\kappa}(u)}$. Therefore $\hat{\kappa}(u)$ satisfies (3.2.2). If there exists a $u^* > 0$ such that

$$\kappa(u) > (\kappa(1)+1)u,$$

then we have

$$\kappa(u) \geqslant \frac{u}{u^*}\kappa(u^*) > (\kappa(1)+1)u,$$

for $0 < u < u^*$. It follows that, for $0 < u < u^*$, $\frac{1}{2\kappa(u)} < \frac{1}{\hat{\kappa}(u)}$ and therefore $\hat{\kappa}(u)$ satisfies (3.2.2). □

Remark 3.2.2. Since $\lim_{u\to 0^+}\kappa(u) = 0$, we can find a $u^* > 0$ such that $\kappa(u^*) < 1$. Since $\kappa(u)/u$ is decreasing, we have

$$\kappa(u) \leqslant \frac{\kappa(u^*)}{u^*}u, \quad u^* \leqslant u.$$

Therefore, for $p \geqslant 2$, it is clear that

$$\kappa(u)^{\frac{p}{2}} = \kappa(u)^{\frac{p}{2}}I_{\{u\leqslant u^*\}} + \kappa(u)^{\frac{p}{2}}I_{\{u>u^*\}} \leqslant \kappa(u) + \left(\frac{\kappa(u^*)}{u^*}\right)^{\frac{p}{2}}u^{\frac{p}{2}}.$$

If $\kappa(u) \leqslant u$, we directly have $\kappa(u)^{\frac{p}{2}} \leqslant u^{\frac{p}{2}}$.

Now we cite Theorem 1 in Yamada [34] as an auxiliary lemma.

Lemma 3.2.2. *Assume that*

$$|f(u)-f(v)|^2 \vee |g(u)-g(v)|^2 \leqslant \nu(|u-v|^2),$$

where ν satisfies (3.2.2)*. Then SDE* (3.2.1) *has a unique solution on* $[0,T]$*, and it can be constructed through the Picard iteration method.*

Theorem 3.2.1. *Under Assumptions 3.2.1, 3.2.2 and 3.2.3, the SDE* (3.2.1) *has a unique solution* $x(t)$ *on* $[0,T]$*. Moreover, we have*

$$\sup_{t\in[0,T]} \mathbb{E}|x(t)|^p < \infty \quad \textit{and} \quad \mathbb{E}\left(\sup_{t\in[0,T]} |x(t)|^p\right) < \infty,$$

for all $p \geqslant 2$*.*

Proof. We divide the whole proof into three parts.

(i) Existence

For each positive integer $k \in \mathbb{N}_+$ and $u \in \mathbb{R}^m$, we define

$$\pi_k(u) = \frac{k}{|u|}u \quad \text{and} \quad f_k(u) = f(\pi_k(u)),$$

where we set $u/|u| = 0$ when $u = 0$. Therefore, $f_k(u)$ is globally Lipschitz continuous and

$$dx_k(t) = f_k(x_k(t))dt + g(x_k(t))dB(t)$$

has a unique solution on $[0,T]$ by Lemma 3.2.2. Now we define the stopping time

$$\theta_k = \inf\{t \in [0,T] : |x_k(t)| \geqslant k\},$$

for all positive integer k. It is clear that $x_k(t) = x_j(t)$, for $0 \leqslant t \leqslant \theta_k \wedge T$, where $j > k$. Then θ_k is non-decreasing, and we then let $\theta_\infty = \lim_{k\to\infty} \theta_k$.

Let $\omega \in \Omega$. Let $t < \theta_\infty(\omega)$ be arbitrary, then there exists a $k(\omega) > 0$ such that $t < \theta_k(\omega) \leqslant \theta_\infty(\omega)$. Then we define $x(t,\omega) = x_k(t,\omega)$, and it is well-defined by the above arguments. For each $k \in \mathbb{N}_+$, we have

$$\begin{aligned} x(t \wedge \theta_k) =& x_k(t \wedge \theta_k) \\ =& x_0 + \int_0^{t \wedge \theta_k} f_k(x_k(s))ds + \int_0^{t \wedge \theta_k} g(x_k(s))dB(s) \\ =& x_0 + \int_0^{t \wedge \theta_k} f(x(s))ds + \int_0^{t \wedge \theta_k} g(x(s))dB(s), \end{aligned}$$

for $t \in [0, T]$.

Now we prove $\theta_\infty = \infty$. By the Itô formula, we have

$$\begin{aligned} |x(t \wedge \theta_k)|^2 =& |x_0|^2 + 2\int_0^{t \wedge \theta_k} \left(x(s)^T f(x(s)) + |g(x(s))|^2 \right) ds \\ &+ 2\int_0^{t \wedge \theta_k} x(s)^T g(x(s))dB(s), \end{aligned}$$

for all $t \in [0, T]$. Using (3.2.3), there exists a constant C such that

$$\mathbb{E}|x(t \wedge \theta_k)|^2 \leqslant C + C\mathbb{E}\int_0^t |x(s \wedge \theta_k)|^2 ds.$$

Then the Gronwall inequality implies that there exists a constant C such that

$$\mathbb{E}|x(T \wedge \theta_k)|^2 \leqslant C.$$

If $\Pr(\theta_\infty < \infty) = \Pr(\theta_\infty \leqslant T) > 0$, then

$$\mathbb{E}|x(T \wedge \theta_k)|^2 \geqslant k^2 \Pr(\theta_k \leqslant T) \geqslant k^2 \Pr(\theta_\infty \leqslant T),$$

which is unbounded by letting $k \to \infty$. This is a contradiction and hence $\theta_\infty = \infty$. In other words, $x(t)$ is a solution on $[0, T]$.

(ii) Uniqueness

Let $x(t)$ and $\bar{x}(t)$ be two solutions of SDE (3.2.1). We define the stopping times

$$\tau_k = \inf\{t \in [0, T] : |x(t)| \geqslant k\} \quad \text{and} \quad \bar{\tau}_k = \inf\{t \in [0, T] : |\bar{x}(t)| \geqslant k\}.$$

Clearly, $x(t \wedge \tau_k \wedge \bar{\tau}_k)$ and $\bar{x}(t \wedge \tau_k \wedge \bar{\tau}_k)$ are solutions of

$$dx_k(t) = f_k(x_k(t))I_{\{t \leqslant \tau_k \wedge \bar{\tau}_k\}}dt + g(x_k(t))I_{\{t \leqslant \tau_k \wedge \bar{\tau}_k\}}dB(t)$$

and

$$d\bar{x}_k(t) = f_k(\bar{x}_k(t))I_{\{t \leqslant \tau_k \wedge \bar{\tau}_k\}}dt + g(\bar{x}_k(t))I_{\{t \leqslant \tau_k \wedge \bar{\tau}_k\}}dB(t),$$

respectively.

Since $f_k(u)$ is globally Lipschitz continuous, it satisfies Assumption 3.2.2. Using Lemma 3.2.2,

$$dx_k(t) = f_k(x_k(t))I_{\{t \leqslant \tau_k \wedge \bar{\tau}_k\}}dt + g(x_k(t))I_{\{t \leqslant \tau_k \wedge \bar{\tau}_k\}}dB(t)$$

has a unique solution on $[0, T]$. Therefore, we have $x_k(t) = \bar{x}_k(t)$, for all $t \in [0, T]$. Then we have $x(t \wedge \tau_k \wedge \bar{\tau}_k) = \bar{x}(t \wedge \tau_k \wedge \bar{\tau}_k)$, for all $t \in [0, T]$. Letting $k \to \infty$, we then have $x(t) = \bar{x}(t)$, for all $t \in [0, T]$.

(iii) Finite Moment

By Remark 3.2.1, SDE (3.2.1) satisfies the Khasminskii-type condition and then its finite moments are known results (e.g. see [3, 12]). □

Now we construct the modified truncated EM numerical solutions by borrowing the truncation method from [14] and [33] instead of using the classical truncation method in [12], [13] and [32]. Using Assumption 3.2.1 and the triangle inequality, we have

$$\begin{aligned} |f(u)| \leqslant & |f(u) - f(\mathbf{0})| + |f(\mathbf{0})| \\ \leqslant & L(1 + |u|^{\gamma})|u| + |f(\mathbf{0})| \\ \leqslant & (L + |f(\mathbf{0})|)(1 + |u|^{\gamma})(1 + |u|) \\ \leqslant & \varphi(|u|)(1 + |u|), \end{aligned} \tag{3.2.4}$$

where $\bar{L} = L + |f(\mathbf{0})|$ and $\varphi(r) = \bar{L}(1 + |r|^{\gamma})$, for $r \in \bar{\mathbb{R}}_+$. Using (3.2.4), we have

$$\sup_{|u| \leqslant r} \frac{|f(u)|}{1+|u|} \leqslant \varphi(r),$$

for all $r > 0$. Denote the inverse function of φ by φ^{-1} and obviously $\varphi^{-1} : \left[\bar{L}, \infty\right) \to \bar{\mathbb{R}}_+$ is a strictly increasing continuous function. Given a stepsize $\Delta \in (0, 1]$, let us define the truncation mapping $\pi_\Delta : \mathbb{R}^m \to \mathbb{R}^m$ by

$$\pi_\Delta(u) = \left(|u| \wedge \varphi^{-1}\left(K\Delta^{-\frac{1}{2}}\right)\right) \frac{u}{|u|},$$

where $K = \varphi(|x_0|)$. We then define the truncated function

$$f_\Delta(u) = f(\pi_\Delta(u)),$$

for all $u \in \mathbb{R}^m$. We then have

$$|f_\Delta(u)| \leqslant K\Delta^{-\frac{1}{2}}(1 + |\pi_\Delta(u)|) \leqslant K\Delta^{-\frac{1}{2}}(1 + |u|),$$

for all $u \in \mathbb{R}^m$.

The discrete-time truncated EM numerical solutions $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ are defined by starting from $X_\Delta(0) = x_0$ and computing

$$X_\Delta(t_{k+1}) = X_\Delta(t_k) + f_\Delta\left(X_\Delta(t_k)\right)\Delta + g\left(X_\Delta(t_k)\right)\Delta B_k,$$

for $k \in \mathbb{N}$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Now we form two versions of the continuous-time truncated EM solutions. The first one is defined by

$$\bar{x}_\Delta(t) = \sum_{k=0}^{\infty} X_\Delta(t_k) I_{[t_k, t_{k+1})}(t),$$

for $t \geqslant 0$. Clearly, it is a simple step process and its sample paths are simple functions.

The continuous version is defined by

$$x_\Delta(t) = x_0 + \int_0^t f_\Delta\left(\bar{x}_\Delta(s)\right) ds + \int_0^t g\left(\bar{x}_\Delta(s)\right) dB(s),$$

for $t \geqslant 0$. It is easy to observe that $x_\Delta(t_k) = \bar{x}_\Delta(t_k) = X_\Delta(t_k)$, for all $k \geqslant 0$. Moreover, $x_\Delta(t)$ is an Itô process with its Itô differential

$$dx_\Delta(t) = f_\Delta\left(\bar{x}_\Delta(t)\right) dt + g\left(\bar{x}_\Delta(t)\right) dB(t).$$

This modified truncated EM solutions for SDEs with concave diffusion coefficients have a number of nice properties which are similar to those established in [12, 33].

Lemma 3.2.3. *Under Assumptions 3.2.1, 3.2.2 and 3.2.3, there exists a constant C such that*

$$u^T f_\Delta(u) + \frac{p-1}{2}|g(u)|^2 \leqslant C(1+|u|^2),$$

for all $u \in \mathbb{R}^m$, $p \geqslant 2$ and stepsize $\Delta \in (0,1]$.

Proof. Note that

$$\begin{aligned}
u^T f_\Delta(u) &= \frac{|u|}{|\pi_\Delta(u)|}\pi_\Delta(u)^T f(\pi_\Delta(u)) \\
&= \frac{|u|}{|\pi_\Delta(u)|}\left((\pi_\Delta(u) - \mathbf{0})^T \left(f(\pi_\Delta(u)) - f(\mathbf{0})\right) + \pi_\Delta(u)^T f(\mathbf{0})\right) \\
&\leqslant H_1|u||\pi_\Delta(u)| + |u||f(\mathbf{0})| \\
&\leqslant \left((H_1 + 0.5) \vee 0.5|f(\mathbf{0})|^2\right)(1+|u|^2),
\end{aligned}$$

for $|\pi_\Delta(u)| > 0$ and the inequality also holds when $|\pi_\Delta(u)| = 0$. By the similar arguments in Remark 3.2.1, the result is obvious. □

Theorem 3.2.2. *Let $p \geqslant 2$. Under Assumptions 3.2.1, 3.2.2 and 3.2.3, there exist constants C_1 and C_2, depending on x_0, p, T, etc. but independent of Δ, such that*

$$\sup_{\Delta\in(0,1]} \mathbb{E}(\sup_{t\in[0,T]} |x_\Delta(t)|^p) \leqslant C_1,$$

and

$$\sup_{t\in[0,T]} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leqslant C_2\Delta^{\frac{p}{2}}.$$

Proof. The proof is similar to that of Theorem 3.2.1 in [33]. □

3.3 Strong convergence at a finite time T

In the following, we set $e_\Delta(t) = x(t) - x_\Delta(t)$ and let $R > |x_0|$ be a real number. We also define two stopping times,

$$\tau_R = \inf\{t \in [0,T] : |x(t)| \geqslant R\} \quad \text{and} \quad \tau_R^\Delta = \inf\{t \in [0,T] : |x_\Delta(t)| \geqslant R\}.$$

In addition, we set $\tau = \tau_R \wedge \tau_R^\Delta$. From now on, we use C to stand for generic positive real constants depending on x_0, T, etc. but independent of Δ and R. Besides, its values may change between occurrences.

Lemma 3.3.1. *Let Assumptions 3.2.1, 3.2.2 and 3.2.3 hold and fix a $R > 0$. Let $p \geqslant 2$. Let $\Delta \in (0,1]$ be sufficiently small such that $\varphi^{-1}(K\Delta^{-\frac{1}{2}}) \geqslant R$ for given R. Then we have*

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t\wedge\tau)|^p \leqslant G^{-1}\left(G\left(C\kappa(\Delta) + C\Delta^{\frac{p}{2}}\right) + CT\right),$$

where $G(r) = \int_1^r \frac{du}{\hat{\kappa}(u)}$, for $r > 0$, and G^{-1} is the inverse function of G.

Proof. Before the proof, we observe that $|x_\Delta(s)| \leqslant R$ for $s \in [0, T\wedge\tau]$. Since $\varphi^{-1}(K\Delta^{-\frac{1}{2}}) \geqslant R$, we have $f_\Delta(x_\Delta(s)) = f(x_\Delta(s))$ for $s \in [0, T\wedge\tau]$.

Under Assumption 3.2.3, we use the Itô formula to derive

$$\begin{aligned}
|e_\Delta(t\wedge\tau)|^p \leqslant & p\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-2} e_\Delta(s)^T \left(f(x(s)) - f(\bar{x}_\Delta(s))\right) ds \\
& + p\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-2} e_\Delta(s)^T \left(g(x(s)) - g(\bar{x}_\Delta(s))\right) dB(s) \\
& + \frac{p(p-1)}{2}\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-2} |g(x(s)) - g(\bar{x}_\Delta(s))|^2 ds,
\end{aligned}$$

for all $t \in [0, T]$.

Using the Young inequality and Assumption 3.2.1, we have

$$
\begin{aligned}
&\mathbb{E}|e_\Delta(t\wedge\tau)|^p \\
\leqslant & p\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-2} e_\Delta(s)^T \left(f(x(s)) - f(x_\Delta(s))\right) ds \\
&+ p\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-1} \left|f(x_\Delta(s)) - f(\bar{x}_\Delta(s))\right| ds \\
&+ p(p-1)\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-2} |g(x(s)) - g(x_\Delta(s))|^2 ds \\
&+ p(p-1)\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-2} |g(x_\Delta(s)) - g(\bar{x}_\Delta(s))|^2 ds, \\
\leqslant & \left((p-1)^2 + pH_1\right)\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^p ds + \mathbb{E}\int_0^{t\wedge\tau} \left|f(x_\Delta(s)) - f(\bar{x}_\Delta(s))\right|^p ds \\
&+ p(p-1)\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{p-2} |g(x(s)) - g(x_\Delta(s))|^2 ds \\
&+ 2(p-1)\mathbb{E}\int_0^{t\wedge\tau} |g(x_\Delta(s)) - g(\bar{x}_\Delta(s))|^p ds,
\end{aligned}
$$

for $t \in [0, T]$. Using Remark 3.2.2, we have

$$
|g(x(s)) - g(x_\Delta(s))|^p \leqslant \kappa(|x(s) - x_\Delta(s)|^2)^{p/2} \leqslant \kappa(|x(s) - x_\Delta(s)|^2) + C|x(s) - x_\Delta(s)|^p,
$$

for $s \in [0, T]$. Using Lemma 3.2.1, we have

$$
|e_\Delta(s)|^{p-2}|g(x(s)) - g(x_\Delta(s))|^2 \leqslant |e_\Delta(s)|^{p-2}\kappa(|e_\Delta(s)|^2) \leqslant \hat{\kappa}(|e_\Delta(s)|^p),
$$

and

$$
|e_\Delta(s)|^p \leqslant \hat{\kappa}(|e_\Delta(s)|^p),
$$

for $s \in [0,T]$. Using Assumption 3.2.1, we have

$$
\begin{aligned}
&\mathbb{E}|e_\Delta(t\wedge\tau)|^p\\
\leqslant &C\mathbb{E}\int_0^{t\wedge\tau}\hat{\kappa}(|e_\Delta(s)|^p)ds + C\mathbb{E}\int_0^t (1+|x_\Delta(s)|^\gamma + |\bar{x}_\Delta(s)|^\gamma)^p|x_\Delta(s)-\bar{x}_\Delta(s)|^p ds\\
&+C\mathbb{E}\int_0^t \left(\kappa(|x_\Delta(s)-\bar{x}_\Delta(s)|^2) + |x_\Delta(s)-\bar{x}_\Delta(s)|^p\right) ds,
\end{aligned}
$$

for $t \in [0,T]$. Using the Hölder inequality and the Jensen inequality, we have

$$
\begin{aligned}
&\mathbb{E}|e_\Delta(t\wedge\tau)|^p\\
\leqslant &C\int_0^{t\wedge\tau}\hat{\kappa}(\mathbb{E}|e_\Delta(s)|^p)ds + C\int_0^t \left(\kappa(\mathbb{E}|x_\Delta(s)-\bar{x}_\Delta(s)|^2) + \mathbb{E}|x_\Delta(s)-\bar{x}_\Delta(s)|^p\right) ds\\
&+C\int_0^t \left(\mathbb{E}(1+|x_\Delta(s)|^{2\gamma p} + |\bar{x}_\Delta(s)|^{2\gamma p})\right)^{1/2} \left(\mathbb{E}|x_\Delta(s)-\bar{x}_\Delta(s)|^{2p}\right)^{1/2} ds,
\end{aligned}
$$

for $t \in [0,T]$. Using Theorems 3.2.1 and 3.2.2, we finally have

$$
\mathbb{E}|e_\Delta(t\wedge\tau)|^p \leqslant C\int_0^t \hat{\kappa}(\mathbb{E}|e_\Delta(s\wedge\tau)|^p)ds + C\left(\kappa(C\Delta)+\Delta^{\frac{p}{2}}\right),
$$

for $t \in [0,T]$.

Since κ is non-decreasing and $m\kappa(u) \geqslant \kappa(mu)$, for $m > 1$, $\kappa(C\Delta) \leqslant (C\vee 1)\kappa(\Delta)$. Using Lemma 3.2.1, $\hat{\kappa}(u)$ a continuous non-decreasing positive concave function for $u > 0$. Then

$$
G(r) = \int_1^r \frac{du}{\hat{\kappa}(u)}
$$

is well-defined, for $r > 0$. Let G^{-1} be the inverse function of G, then the domain of G^{-1} is the real line. Then the Bihari inequality implies

$$
\mathbb{E}|e_\Delta(t\wedge\tau)|^p \leqslant G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right).
$$

□

3.3. Strong convergence at a finite time T

Theorem 3.3.1. *Let Assumptions 3.2.1, 3.2.2 and 3.2.3 hold. Let $p \geqslant 2$. Then*

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^p \leqslant 2G^{-1}\left(G\left(C\kappa(\Delta) + C\Delta^{\frac{p}{2}}\right) + CT\right),$$

for all $\Delta \in \left(0, \left(\frac{K}{2\bar{L}}\right)^2\right]$. In other words,

$$\lim_{\Delta\to 0} \sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^p = 0.$$

Proof. Given $\Delta \in \left(0, \left(\frac{K}{2\bar{L}}\right)^2\right]$, we let $R = \left(\frac{K\Delta^{-\frac{1}{2}}}{2\bar{L}}\right)^{\frac{1}{\gamma}}$. Then we have

$$\varphi(R) = \bar{L}(1+|R|^\gamma) = \bar{L}\left(1 + \frac{K\Delta^{-\frac{1}{2}}}{2\bar{L}}\right) \leqslant K\Delta^{-\frac{1}{2}}.$$

Now we use the Young inequality and Theorem 3.2.2 to derive

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^p I_{\{\tau\leqslant T\}}) \\
\leqslant&\frac{1}{2}\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^{2p}\Delta^{\frac{p}{2}} + \frac{1}{2}\Pr(\tau \leqslant T)\Delta^{-\frac{p}{2}}, \\
\leqslant& C\Delta^{\frac{p}{2}} + \frac{1}{2}\frac{\mathbb{E}\left(\sup_{t\in[0,T]}|x(t)|^{2p\gamma}\right) + \mathbb{E}\left(\sup_{t\in[0,T]}|x_\Delta(t)|^{2p\gamma}\right)}{R^{2p\gamma}}\Delta^{-\frac{p}{2}}, \\
\leqslant& C\Delta^{\frac{p}{2}}.
\end{aligned}$$

Using the above results and Lemma 3.3.1, we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^p \\
&= \sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^p I_{\{\tau>T\}}) + \sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^p I_{\{\tau\leqslant T\}}), \\
&\leqslant \sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t\wedge\tau)|^p + \sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^p I_{\{\tau\leqslant T\}}), \\
&\leqslant G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right)+C\Delta^{\frac{p}{2}}, \\
&= G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right)+G^{-1}\left(G(C\Delta^{\frac{p}{2}})\right), \\
&\leqslant 2G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right),
\end{aligned}$$

since G and G^{-1} is non-decreasing.

As $\Delta \to 0$, $C\kappa(\Delta)+C\Delta^{\frac{p}{2}} \to 0$. Using Lemma 3.2.1 and (3.2.2),

$$G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT \to -\infty \quad \text{as } \Delta \to 0.$$

It follows that

$$G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right) \to 0 \quad \text{as } \Delta \to 0.$$

Therefore, $\lim_{\Delta\to 0}\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^p = 0$. □

3.4 Strong convergence over a finite time interval

In this section, we establish the strong convergence theory of the modified truncated EM method over a finite time interval.

Theorem 3.4.1. *Let Assumptions 3.2.1, 3.2.2 and 3.2.3 hold. Let $p \geqslant 2$. Then*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |e_\Delta(t)|^p\right) \leqslant 2G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right),$$

for all $\Delta \in \left(0, \left(\frac{K}{2L}\right)^2\right]$. *In other words,*

$$\lim_{\Delta\to 0} \mathbb{E}\left(\sup_{t\in[0,T]} |e_\Delta(t)|^p\right) = 0.$$

Proof. Let $T_1 \in [0,T]$ be arbitrary. Let $J = \mathbb{E}\left(\int_0^{T_1\wedge\tau} |e_\Delta(s)|^{2p-2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2 ds\right)^{\frac{1}{2}}$. Using the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T_1]} \int_0^{T_1\wedge\tau} |e_\Delta(s)|^{p-2} e_\Delta(s)^T \left(g(x(s)) - g(\bar{x}_\Delta(s))\right) dB(s)\right)\\
\leqslant&\mathbb{E}\left(\int_0^{T_1\wedge\tau} |e_\Delta(s)|^{2p-2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2 ds\right)^{\frac{1}{2}},\\
=&J.
\end{aligned}$$

Using the Young inequality, we have

$$\begin{aligned}
J \leqslant&\mathbb{E}\left(\sup_{t\in[0,T_1]} |e_\Delta(t\wedge\tau)|^p \int_0^{T_1\wedge\tau} |e_\Delta(s)|^{p-2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2 ds\right)^{\frac{1}{2}},\\
\leqslant&\frac{p}{2}\mathbb{E}\int_0^{T_1\wedge\tau} |e_\Delta(s)|^{p-2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2 ds + \frac{1}{2p}\mathbb{E}\left(\sup_{t\in[0,T_1]} |e_\Delta(t\wedge\tau)|^p\right).
\end{aligned}$$

Using arguments in Lemma 3.3.1, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{t\in[0,T_1]} |e_\Delta(t\wedge\tau)|^p\right)\\
\leqslant&C\int_0^{T_1} \hat{\kappa}(\mathbb{E}|e_\Delta(s\wedge\tau)|^p)ds + C\left(\kappa(C\Delta) + \Delta^{\frac{p}{2}}\right)\\
&+ p\mathbb{E}\left(\sup_{t\in[0,T_1]} \int_0^{T_1\wedge\tau} |e_\Delta(s)|^{p-2} e_\Delta(s)^T \left(g(x(s)) - g(\bar{x}_\Delta(s))\right) dB(s)\right),\\
\leqslant&C\int_0^{T_1} \hat{\kappa}(\mathbb{E}|e_\Delta(s\wedge\tau)|^p)ds + C\left(\kappa(C\Delta) + \Delta^{\frac{p}{2}}\right)\\
&+ \frac{p^2}{2}\mathbb{E}\int_0^{T_1\wedge\tau} |e_\Delta(s)|^{p-2}|g(x(s)) - g(\bar{x}_\Delta(s))|^2 ds + \frac{1}{2}\mathbb{E}\left(\sup_{t\in[0,T_1]} |e_\Delta(t\wedge\tau)|^p\right),\\
\leqslant&C\int_0^{T_1} \hat{\kappa}\left(\mathbb{E}\left(\sup_{0\leqslant t\leqslant s} |e_\Delta(t\wedge\tau)|^p\right)\right) ds + C\left(\kappa(C\Delta) + \Delta^{\frac{p}{2}}\right) + \frac{1}{2}\mathbb{E}\left(\sup_{t\in[0,T_1]} |e_\Delta(t\wedge\tau)|^p\right).
\end{aligned}$$

Then the Bihari inequality implies

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T_1]}|e_\Delta(t\wedge\tau)|^p\right) \leqslant& G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT_1\right),\\ \leqslant& G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right).\end{aligned}$$

By similar arguments in Theorem 3.3.1, we have

$$\mathbb{E}\left(\sup_{t\in[0,T]}|e_\Delta(t)|^p\right) \leqslant 2G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right).$$

As $\Delta\to 0$, $C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\to 0$. Using Lemma 3.2.1 and (3.2.2),

$$G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\to-\infty, \quad \text{as } \Delta\to 0.$$

It follows that

$$G^{-1}\left(G\left(C\kappa(\Delta)+C\Delta^{\frac{p}{2}}\right)+CT\right)\to 0, \quad \text{as } \Delta\to 0.$$

Therefore,

$$\lim_{\Delta\to 0}\mathbb{E}\left(\sup_{t\in[0,T]}|e_\Delta(t)|^p\right)=0.$$

□

We now consider a non-linear concave function κ and derive a concrete convergence rate for fixed T by applying our new theorems (see Example 3.5.1 for concrete diffusion coefficients which satisfies Assumption 3.2.2 with this κ).

Example 3.4.1. Let $p=2$ and $\Delta\in\left(0,\left(\frac{K}{2L}\right)^2\right]$. Let

$$\kappa(u)=\begin{cases}-u\ln u, & 0\leqslant u\leqslant e^{-2},\\ u+e^{-2}, & u>e^{-2}.\end{cases}$$

Therefore, we can set $\hat{\kappa}(u) = \kappa(u)$, and we have $G(r) = -\ln(-\ln r) + 2\ln 2 - 2 - \ln(1 + e^{-2})$, for $0 < r < e^{-2}$. We now use Theorem 3.4.1 to derive

$$\mathbb{E}\left(\sup_{t\in[0,T]} |e_\Delta(t)|^2\right) \leqslant 2G^{-1}\left(G(C\kappa(\Delta)) + CT\right),$$

since $\kappa(u) > u$. It follows that

$$\begin{aligned}\mathbb{E}\left(\sup_{t\in[0,T]} |e_\Delta(t)|^2\right) &\leqslant 2G^{-1}\left(G(C\kappa(\Delta)) + CT\right),\\ &\leqslant 2G^{-1}\left(-\ln(-\ln C - \ln\kappa(\Delta)) + 2\ln 2 - 2 - \ln(1+e^{-2}) + CT\right),\\ &\leqslant C\kappa(\Delta)^{e^{-CT}}.\end{aligned}$$

Given $\varepsilon \in (0,1)$, we then have $-\ln\Delta \leqslant C\Delta^{-\varepsilon}$, for some a constant C and sufficiently small $\Delta > 0$. Therefore, we have

$$\mathbb{E}\left(\sup_{t\in[0,T]} |e_\Delta(t)|^2\right) \leqslant C\Delta^{(1-\varepsilon)e^{-CT}}.$$

The $\mathcal{L}^2$-strong convergence rate is of order $0.5(1-\varepsilon)e^{-CT}$, which is smaller than 1/2. An explicit bound on the actual rate of convergence will depend on C. It follows that

$$\lim_{\Delta\to 0}\mathbb{E}\left(\sup_{t\in[0,T]} |e_\Delta(t)|^2\right) = 0.$$

3.5 Example and simulation

Before we apply the modified truncated EM method to an example, we first state a property of the concave function $\kappa(u)$.

Remark 3.5.1. Let $\kappa(u)$ be a continuous non-decreasing concave function which satisfies (3.2.2). Since $\frac{v}{u}\kappa(u) \geqslant \kappa(v)$, for $0 \leqslant u < v$, we have

$$\kappa(u+v) \leqslant \frac{v}{v-u}\kappa(v) - \frac{u}{v-u}\kappa(u) = (\kappa(u)+\kappa(v)) + \frac{u}{v-u}\left(\kappa(v) - \frac{v}{u}\kappa(u)\right) \leqslant \kappa(u)+\kappa(v),$$

for all $0 \leqslant u < v$.

Example 3.5.1. We now consider a two-dimensional Langevin equation (see [8]) but with locally logarithmic diffusion coefficient of the form

$$dx(t) = \begin{pmatrix} x_1(t) - (x_1(t)^2 + x_2(t)^2)x_1(t) \\ x_2(t) - (x_1(t)^2 + x_2(t)^2)x_2(t) \end{pmatrix} dt + \begin{pmatrix} \kappa_1(x_1(t)) + x_2(t) \\ \kappa_2(x_2(t)) + x_1(t) \end{pmatrix} dB(t),$$

where

$$\kappa_1(u) = \begin{cases} 0.5u - 0.5e^{-1}, & u < -e^{-1}, \\ u\sqrt{-\ln|u|}, & -e^{-1} \leqslant u \leqslant e^{-1}, \\ 0.5u + 0.5e^{-1}, & u > e^{-1}, \end{cases}$$

and

$$\kappa_2(u) = \begin{cases} au - b, & u < -e^{-2}, \\ u\sqrt{-\ln u \ln(-\ln u)}, & -e^{-2} \leqslant u \leqslant e^{-2}, \\ au + b, & u > e^{-2}, \end{cases}$$

where $a = \frac{3\ln 2 - 1}{2\sqrt{2\ln 2}}$ and $b = \sqrt{2\ln 2}e^{-2}$. We also define

$$\kappa_3(u) = \begin{cases} -u\ln u, & 0 \leqslant u \leqslant e^{-2}, \\ u + e^{-2}, & u > e^{-2}, \end{cases}$$

and

$$\kappa_4(u) = \begin{cases} -u\ln u \ln(-\ln u), & 0 \leqslant u \leqslant e^{-4}, \\ (6\ln 2 - 1)(u - e^{-4}) + 8e^{-4}\ln 2, & u > e^{-4}. \end{cases}$$

For $u, v \in \mathbb{R}^2$, we let $z = u - v$. Then we have

$$\begin{aligned}
(u-v)^T(f(u)-f(v)) =& z^T\left((1-|v+z|^2)(v+z)-(1-|v|^2)v\right)\\
=&|z|^2-|v|^2|z|^2-(2v+z)^T zz^T(v+z)\\
=&|z|^2-|v|^2|z|^2-|z^T(v+z)|^2-|z^Tv|^2-v^Tzz^Tz\\
=&|z|^2-|v|^2|z|^2-|z^Tz+\frac{3}{2}z^Tv|^2+\frac{1}{4}|z^Tv|^2\\
\leqslant&|z|^2\\
=&|u-v|^2,
\end{aligned}$$

since $|z^Tv|^2 \leqslant |v|^2|z|^2$. In other words, f satisfies Assumption 3.2.3. Also, we have

$$\begin{aligned}
|f(u)-f(v)|^2 =&|(u-v)-|u|^2(u-v)-(u-v)^T(u+v)v|^2\\
\leqslant&3(|u-v|^2+|u|^4|u-v|^2+|u+v|^2|v|^2|u-v|^2)\\
\leqslant&9(1+|u|^4+|v|^4)|u-v|^2,
\end{aligned}$$

for $u, v \in \mathbb{R}^2$. In other words, f satisfies Assumptions 3.2.1.

Using Remark 3.5.1, we have

$$-\kappa_1(|u-v|) \leqslant \kappa_1(u)-\kappa_1(v) \leqslant \kappa_1(|u-v|),$$

for $u, v \geqslant 0$. Then we have

$$|\kappa_1(u)-\kappa_1(v)|^2 \leqslant \kappa_1(|u-v|)^2 \leqslant 0.5\kappa_3(|u-v|^2),$$

for $u, v \geqslant 0$. The symmetry implies that this inequality also holds for $u, v \leqslant 0$. When $v < 0 < u$ or $u < 0 < v$, we have

$$|\kappa_1(u)-\kappa_1(v)|^2 = |\kappa_1(|u|)+\kappa_1(|v|)|^2 \leqslant 4\kappa_1(|u-v|)^2 \leqslant 2\kappa_3(|u-v|^2).$$

Similarly, we have

$$|\kappa_2(u) - \kappa_2(v)|^2 \leqslant 2\kappa_4(|u-v|^2).$$

Therefore, we have

$$\begin{aligned}|g(u) - g(v)|^2 \leqslant & 2(\kappa_1(u_1) - \kappa_1(v_1))^2 + 2(u_2 - v_2)^2 + 2(\kappa_2(u_2) - \kappa_2(v_2))^2 + 2(u_1 - v_1)^2, \\ \leqslant & 2|u-v|^2 + 2\kappa_3(|u-v|^2) + 2\kappa_4(|u-v|^2),\end{aligned}$$

for $u, v \in \mathbb{R}^2$.

Here, we have

$$\kappa(u) = 2u + 2\kappa_3(u) + 2\kappa_4(u).$$

When $u \in [0, e^{-4}]$, $\ln(-\ln u) \geqslant \ln 4 > 1$. Since $(6\ln 2 - 1) > 3$, we have $\kappa_4(u) > \kappa_3(u) > u$. $\kappa_4(u)$ satisfies Example 3.2.1, and we have

$$\int_{0+} \frac{du}{\kappa_4(u)} = \infty.$$

Therefore, we have

$$\int_{0+} \frac{du}{\kappa(u)} > \int_{0+} \frac{du}{6\kappa_4(u)} = \infty$$

in this example. That is, g satisfies Assumption 3.2.2.

Let $T = 1$ and $x_0 = (1, 2)$. We now conduct numerical simulations with 1000 sample paths for stepsizes $\Delta = 2^{-10}, 2^{-9}, \ldots, 2^{-4}$. In view of the fact that there is no analytical solution for this SDE, we regard the numerical solution with stepsize $\Delta = 2^{-18}$ as the "exact" solution. Using the linear regression, the experimental errors (see Figures 3.5.1 and 3.5.2) show that the strong convergence error for the second moment have order about 1.18 and 1.12, which validate our theory.

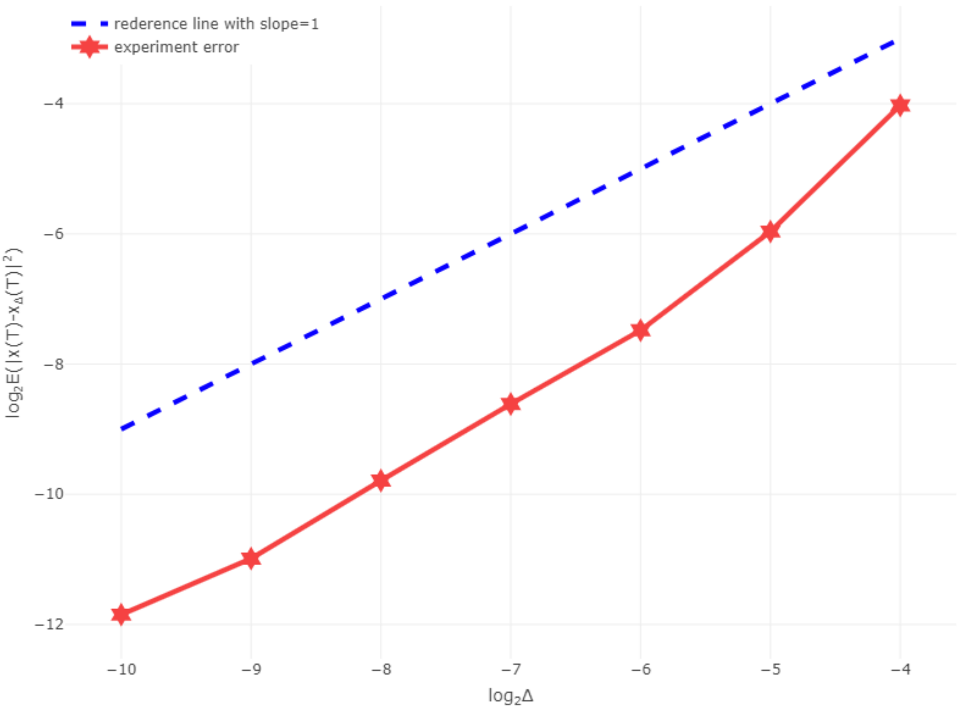

Figure 3.5.1: The strong errors of Example 3.5.1 between modified truncated EM method and “exact” solution at time T.

3.6 Conclusion

In this chapter, we study and establish the strong convergence of the modified truncated EM method for multi-dimensional SDEs with polynomially growing drift coefficients and concave diffusion coefficients satisfying the Osgood condition. We derive a concrete strong $\mathcal{L}^p$-strong convergence of the modified truncated EM method. Our result does not rely on the Yamada-Watanabe method and therefore is valid for multi-dimensional SDEs. An interesting thing is that the numerical simulations show that the exact strong convergence error may also have order 1/2 which is the same as that in the classical case. The experimental strong convergence error is better than our theoretical error and will be tackled elsewhere.

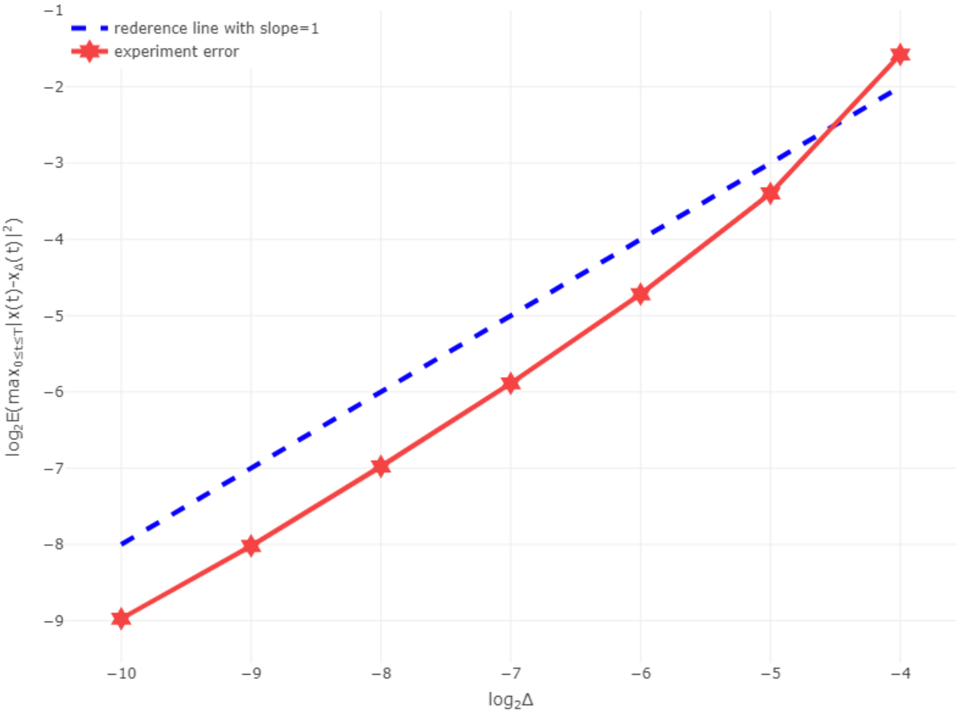

Figure 3.5.2: The strong errors of Example 3.5.1 between modified truncated EM method and "exact" solution over $[0, T]$.

Chapter 4

The logarithmic truncated EM method with weaker conditions

4.1 Background

In this chapter, we will focus on the CEV model and the Aït-Sahalia model. Coefficients of these two SDE models are not globally Lipschitz near some finite points. For example, the diffusion coefficient of the CEV model is $\sigma x^{1/2+\theta}$, which is Hölder continuous near the zero. In recent years, many researchers developed many useful modified EM methods and establish their strong convergence theory for these two models (see [16], [18–20], [22, 23] [28] and [35–37]).

In particular, Neuenkirch and Szpruch [20] established the drift-implicit EM method for a series of SDEs which take values in a given domain. Their examples include the CIR model, the Heston-3/2 volatility model, the CEV model, the Aït-Sahalia model and the Wright-Fisher model. The drift-implicit EM method is boundary preserving, e.g., the numerical solution of the Aït-Sahalia model is still positive like the exact solution is positive. In particular, it is $\mathcal{L}^p$-strongly convergent with order one, while many modified EM methods are generally $\mathcal{L}^p$-strongly convergent with order only one half. However, expensive computational cost is required since the drift-implicit EM method is an implicit numerical method.

In 2016, Chassagneux, Jacquier and Mihaylov [23] developed an explicit EM scheme, which works for these two SDE models. The domain preserving property of their numerical solutions are guaranteed by the projection technique. They proved that their EM method also are $\mathcal{L}^1$-strongly convergence with order one.

There are also some modified EM methods with strong convergence order one half. In [16], the reflected EM method is proved to be $\mathcal{L}^p$-strong convergence with order one half for the CEV model. In particular, a competitive explicit positivity preserving EM scheme, called the logarithmic truncated EM method (see [35] and [36]), is developed for scalar SDEs which take values in the positive domain. To be concrete, researchers apply the logarithmic transformation for appropriate SDEs, and then use the truncated EM method for transformed SDEs.

The logarithmic transformation will generate exponentially growing coefficients. Therefore, numerical analysis methods and assumptions in [12] and [13] cannot be used directly for transformed SDEs. In [35] and [36], authors give restricted assumptions to derive finite exponential moments for numerical solutions. They then prove that the logarithmic truncated EM method is $\mathcal{L}^p$-strongly convergent with order one half for the CEV model and the Aït-Sahalia model with appropriate parameter settings.

The main aim of this chapter is to further study the logarithmic truncated EM method. We will apply weaker assumptions (see section 4 for detailed examples) and use a new numerical analysis method to prove finite exponential moments of numerical solutions. We will prove that the logarithmic truncated EM method is $\mathcal{L}^p$-strongly convergent with order one half. Compared to results in [23], the logarithmic truncated EM method has better theoretical convergence rates for large p.

This chapter is extracted from [2] and is organized as follows. In section 2, we first introduce assumptions and establish some useful lemmas. Then we construct the logarithmic truncated EM method and investigate its convergence rates in section 3. In addition, our numerical analysis methods in section 3 can improve strong convergence results in [13] and [38]. Two examples will be presented in section 4 to illustrate that the logarithmic truncated EM method can work well for the CEV model and the Aït-Sahalia model with mild parameter settings. Finally, we make a brief conclusion in section 6.

4.2 Preliminaries and assumptions

Let $B(t) = (B_1(t), B_2(t), \ldots, B_n(t))^T$ be an n-dimensional Brownian motion defined on this space. In this chapter, we will use C to stand for generic positive real numbers which are dependent on T, K_1, K_2, α, β, H, etc., but independent of k, Δ and R (used below) and its values may change between occurrences. We also let $\inf \emptyset = \infty$.

In this chapter, we consider a scalar SDE

$$dx(t) = f(x(t))dt + g(x(t))dB(t) \tag{4.2.1}$$

on $t \in [0, T]$ with the initial value $x(0) = x_0 \in \mathbb{R}_+$, where T is a fixed positive number and $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}^n$ are Borel measurable.

We first impose three hypotheses.

Assumption 4.2.1. Assume that the drift coefficient f satisfies the locally Lipschitz condition: there exist real numbers $K_1 > 0$, $\alpha > 0$ and $\beta > 0$ such that

$$|f(u) - f(v)| \leqslant K_1(1 + u^\alpha + u^{-\beta} + v^\alpha + v^{-\beta})|u - v|,$$

for all $u, v \in \mathbb{R}_+$.

Assumption 4.2.2. Assume that there exist positive real numbers $u^* > 0$, $p^* > 1$, $q^* > 0$ and $K_2 > 0$ such that

$$\begin{cases} uf(u) - \frac{q^*+1}{2}|g(u)|^2 \geqslant 0, & u \in (0, u^*), \\ uf(u) + \frac{p^*-1}{2}|g(u)|^2 \leqslant K_2(1 + u^2), & u \in [u^*, \infty). \end{cases}$$

Assumption 4.2.3. Assume that there exists a pair of positive real numbers $r^* > 2$ and $H > 0$ such that

$$(u - v)\,(f(u) - f(v)) + \frac{r^* - 1}{2}|g(u) - g(v)|^2 \leqslant H|u - v|^2,$$

for all $u, v \in \mathbb{R}_+$.

The Aït-Sahalia model and the transformed CEV model satisfy the above assumptions (see section 4).

Remark 4.2.1. From Assumption 4.2.1, we can conclude that

$$|f(u)| \leqslant |f(u) - f(1)| + |f(1)| \leqslant 3K_1(1 + u^{\alpha} + u^{-\beta})|u - 1| + |f(1)|,$$

and therefore

$$|f(u)| \leqslant \begin{cases} 6K_1(1 + u^{\alpha})u + |f(1)|, & u > 1, \\ 6K_1(1 + u^{-\beta}) + |f(1)|, & 0 < u < 1. \end{cases}$$

Therefore, Assumption 4.2.1 implies that

$$|f(u)| \leqslant (12K_1 \vee |f(1)|)(1 + u^{\alpha+1} + u^{-\beta}),$$

for $u \in \mathbb{R}_+$.

Assumption 4.2.2 requires that

$$|g(u)|^2 \leqslant \frac{2}{q^* + 1}|u||f(u)| \leqslant \frac{2(12K_1 \vee |f(1)|)}{q^* + 1}(u + u^{\alpha+2} + u^{-\beta+1}),$$

and

$$\begin{aligned} |g(u)|^2 \leqslant & \frac{2}{p^* - 1}\left(|u||f(u)| + K_2(1 + u^2)\right), \\ \leqslant & \frac{2}{p^* - 1}\Big((12K_1 \vee |f(1)|)(u + u^{\alpha+2} + u^{-\beta+1}) + K_2(1 + u^2)\Big), \end{aligned}$$

for $u \in (0, u^*)$ and $u \in [u^*, \infty)$, respectively. Therefore, there exists a constant C such that

$$|g(u)|^2 \leqslant C(1 + u^{\alpha+2} + u^{-\beta+1}),$$

for $u \in \mathbb{R}_+$.

The following lemma shows that SDE (4.2.1) has a unique strong solution on $[0,T]$. In addition, the lemma shows that this solution takes values in the positive domain, i.e.,

$$\Pr(x(t) \in (0,\infty),\ \text{for } t \in [0,T]) = 1.$$

Therefore, as the above assumptions show, we only need to check properties of drift and diffusion coefficients for positive real numbers.

Lemma 4.2.1. *Let Assumptions 4.2.1, 4.2.2 and 4.2.3 hold with $\alpha \vee (\beta+1) \leqslant p^* + q^*$. Then SDE (4.2.1) has a unique strong solution on $[0,T]$. Moreover, there exists a constant C such that*

$$\sup_{t\in[0,T]} \mathbb{E}|x(t\wedge\theta)|^{p^*} < C \quad \textit{and} \quad \sup_{t\in[0,T]} \mathbb{E}|x(t\wedge\theta)|^{-q^*} < C,$$

where θ is an arbitrary stopping time. Furthermore, we have that

$$\Pr(x(t) \in (0,\infty),\ \textit{for } t \in [0,T]) = 1.$$

Proof. Let $k \in \mathbb{N}_+$ be a positive integer. Define

$$\pi_k(u) = k^{-1} I_{\{u<k^{-1}\}} + x I_{\{k^{-1}\leqslant u\leqslant k\}} + k I_{\{k<u\}},$$

for $u \in \mathbb{R}$. From Remark 4.2.1,

$$f_k(u) = f(\pi_k(u)) \quad \text{and} \quad g_k(u) = g(\pi_k(u))$$

are globally Lipschitz and therefore linear growing. Then the uniqueness and existence of the solution on $[0,T]$ to

$$dx_k(t) = f_k(x_k(t))dt + g_k(x_k(t))dB(t)$$

are given in Chapter 4.2.3 of [3]. Now we define the stopping time

$$\tau_k = \inf\{t \in [0,T] : x_k(t) \notin (1/k, k)\}.$$

By the uniqueness of $x_k(t)$, we have $x_j(t) = x_k(t)$, for $t \in [0, T \wedge \tau_k]$, where $j > k$ and j and k are sufficiently large. Therefore, $\tau_k \leqslant \tau_j$ for all $j > k$. We then define $\tau_\infty = \lim_{j\to\infty} \tau_j$.

Let $\omega \in \Omega$. For an arbitrary $t < \tau_\infty(\omega)$, there exists a $k(\omega) > 0$ such that $t < \tau_k(\omega) \leqslant \tau_\infty(\omega)$. Now we define $x(t,\omega) = x_k(t,\omega)$ and it is well-defined by the above arguments. Let $m \in \mathbb{N}_+$ be sufficiently large such that $u^* \in (1/m, m)$. Let $t \in [0,T]$ be arbitrary, we have

$$\begin{aligned} x(t\wedge\tau_m) =& x_m(t\wedge\tau_m) \\ =& x_0 + \int_0^{t\wedge\tau_m} f_m(x_m(s))ds + \int_0^{t\wedge\tau_m} g_m(x_m(s))dB(s) \\ =& x_0 + \int_0^{t\wedge\tau_m} f(x(s))ds + \int_0^{t\wedge\tau_m} g(x(s))dB(s). \end{aligned}$$

Using the Itô formula, we have

$$\begin{aligned} & x(t\wedge\tau_m)^{p^*} + x(t\wedge\tau_m)^{-q^*} \\ =& x_0^{p^*} + x_0^{-q^*} \\ &+ p^* \int_0^{t\wedge\tau_m} x(s)^{p^*-2}\left(x(s)f(x(s)) + \frac{p^*-1}{2}|g(x(s))|^2\right) ds \\ &+ p^* \int_0^{t\wedge\tau_m} x(s)^{p^*-1} g(x(s))dB(s) \\ &- q^* \int_0^{t\wedge\tau_m} x(s)^{-(q^*+2)}\left(x(s)f(x(s)) - \frac{q^*+1}{2}|g(x(s))|^2\right) ds \\ &- q^* \int_0^{t\wedge\tau_m} x(s)^{-(q^*+1)} g(x(s))dB(s), \end{aligned} \tag{4.2.2}$$

for all $t \in [0,T]$.

Using Assumption 4.2.2, Remark 4.2.1 and the Young inequality, we have

$$
\begin{aligned}
& x(t)^{p^*-2}\left(x(t)f(x(t))+\frac{p^*-1}{2}|g(x(t))|^2\right)\\
\leqslant & Cx(t)^{p^*-2}\left(1+x(t)+x(t)^{\alpha+2}+x(t)^{-\beta+1}\right)I_{\{x(t)\in(0,u^*)\}}\\
& +K_2x(t)^{p^*-2}\left(1+x(t)^2\right)I_{\{x(t)\in[u^*,\infty)\}},\\
\leqslant & C\left(1+x(t)^{p^*}+x(t)^{p^*-\beta-1}\right),
\end{aligned}
$$

for all $t\in[0,T\wedge\tau_m]$. Similarly, we have

$$
\begin{aligned}
& -x(t)^{-(q^*+2)}\left(x(t)f(x(t))-\frac{q^*+1}{2}|g(x(t))|^2\right)\\
\leqslant & -x(t)^{-(q^*+2)}\left(x(t)f(x(t))-\frac{q^*+1}{2}|g(x(t))|^2\right)I_{\{x(t)\in(0,u^*)\}}\\
& +x(t)^{-(q^*+2)}\left(1+x(t)+x(t)^{\alpha+2}+x(t)^{-\beta+1}\right)I_{\{x(t)\in[u^*,\infty)\}},\\
\leqslant & C\left(|u^*|^{-(q^*+2)}+|u^*|^{-(q^*+1)}+x(t)^{-q^*+\alpha}+|u^*|^{-q^*-\beta-1}\right)I_{\{x(t)\in[u^*,\infty)\}},\\
\leqslant & C\left(1+x(t)^{-q^*+\alpha}\right),
\end{aligned}
$$

for all $t\in[0,T\wedge\tau_m]$. Since $\alpha\vee(\beta+1)\leqslant p^*+q^*$, we further have

$$
C\left(1+x(t)^{p^*}+x(t)^{p^*-\beta-1}+x(t)^{-q^*+\alpha}\right)\leqslant C\left(1+x(t)^{p^*}+x(t)^{-q^*}\right),
$$

for all $t\in[0,T\wedge\tau_m]$.

Taking expectations on both sides of (4.2.2), we then have

$$
\begin{aligned}
& \mathbb{E}\left(x(t\wedge\tau_m)^{p^*}+x(t\wedge\tau_m)^{-q^*}\right)\\
\leqslant & x_0^{p^*}+x_0^{-q^*}+C\mathbb{E}\int_0^{t\wedge\tau_m}\left(1+x(s)^{p^*}+x(s)^{-q^*}\right)ds,\\
\leqslant & x_0^{p^*}+x_0^{-q^*}+C\mathbb{E}\int_0^{t}\left(1+x(s\wedge\tau_m)^{p^*}+x(s\wedge\tau_m)^{-q^*}\right)ds,
\end{aligned}
$$

for all $t\in[0,T]$.

Then the Gronwall inequality implies that there exists a constant C such that

$$\sup_{t\in[0,T]} \mathbb{E}\left(x(t\wedge\tau_m)^{p^*} + x(t\wedge\tau_m)^{-q^*}\right) < C.$$

If $\Pr(\tau_\infty \leqslant T) > 0$, then we have

$$\mathbb{E}\left(x(T\wedge\tau_m)^{p^*} + x(T\wedge\tau_m)^{-q^*}\right) \geqslant m^{p^*\wedge q^*}\Pr(\tau_\infty \leqslant T),$$

which is unbounded by letting m tend to infinity. It is a contradiction, and therefore we have $\Pr(\tau_\infty > T) = 1$. It means that SDE (4.2.1) has a unique strong solution on $[0, T]$ and

$$\Pr(x(t) \in (0,\infty),\ \text{for } t\in[0,T]) = 1.$$

By similar arguments as above, there exists a constant C such that

$$\sup_{t\in[0,T]} \mathbb{E}|x(t\wedge\theta)|^{p^*} < C \quad \text{and} \quad \sup_{t\in[0,T]} \mathbb{E}|x(t\wedge\theta)|^{-q^*} < C,$$

where θ is an arbitrary stopping time. □

4.3 The logarithmic truncated EM method

In [12, 13], Mao established the truncated EM method for SDEs with polynomially growing coefficients. The truncated EM method is an explicit EM method and it does not preserve the positivity if it is applied to the SDE (4.2.1). It follows that the truncated EM numerical solution cannot have finite inverse moments, which are critical to establish the strong convergence rate theory of the truncated EM method. However, if we use the logarithmic transformation, then transformed SDEs take values in the whole of real line. Then we only need to adjust the truncated EM method for transformed SDEs.

To define the logarithmic truncated EM numerical solutions, we first take the logarithmic transformation

$$y = \ln x, \ x \in \mathbb{R}^+.$$

Using the Itô formula, we have a new SDE:

$$y(t) = F(y(t))dt + G(y(t))dB(t),$$

where

$$F(u) = e^{-u} f(e^u) - 0.5e^{-2u}|g(e^u)|^2 \quad \text{and} \quad G(u) = e^{-u} g(e^u),$$

for $u \in \mathbb{R}$.

From Remark 4.2.1, we can conclude that

$$|F(u)| \vee |G(u)|^2 \leqslant C_0(1 + e^{\alpha u} + e^{-(\beta+1)u}),$$

for some a constant $C_0 > 1$. Now we set $\varphi(r) = C_0(2 + e^{(\alpha \vee (\beta+1))r})$, which is a strictly increasing continuous function such that

$$\sup_{|u| \leqslant r} |F(u)| \vee |G(u)|^2 \leqslant \varphi(r),$$

for $r > 0$. Denote the inverse function of φ by φ^{-1} and obviously $\varphi^{-1} : [3C_0, \infty) \to \bar{\mathbb{R}}_+$ is also a strictly increasing continuous function.

[6] showed that the classical EM numerical solution will explode for SDEs with polynomially growing coefficients, as the step size tends to zero. Similar phenomena also happen here. To avoid the explosion, we use two controlled functions $|F_\Delta(u)| \vee |G_\Delta(u)|^2$ to construct the numerical solutions. First, we define a function $h(\Delta)$ to control the value of $|F(u)| \vee |G(u)|^2$. It gives an upper bound of value of $|F(u)| \vee |G(u)|^2$ that the step size $\Delta \in (0, 1]$ can control.

Definition 4.3.1. Let $h : (0,1] \to [1,\infty)$ be a strictly decreasing function, such that

$$\lim_{\Delta\to 0} h(\Delta) = \infty, \quad \Delta h(\Delta) \leqslant 4C_0 \vee 2\varphi(|\ln x_0|) \text{ and } h(1) > 3C_0 \vee \varphi(|\ln x_0|), \tag{4.3.1}$$

for $\Delta \in (0,1]$. In Theorem 4.3.1 and Remark 4.3.2, we will give precise expressions of $h(\Delta)$, for different parameter settings.

Given a stepsize $\Delta \in (0,1]$, let us define the truncation mapping $\pi_\Delta : \mathbb{R} \to \mathbb{R}$ by

$$\pi_\Delta(u) = \big(|u| \wedge \varphi^{-1}(h(\Delta))\big)\,\frac{u}{|u|},$$

where we use the convention $\frac{u}{|u|} = 0$ when $u = 0$. We then define the truncated function

$$F_\Delta(u) = F(\pi_\Delta(u)) \quad \text{and} \quad G_\Delta(u) = G(\pi_\Delta(u)),$$

for all $u \in \mathbb{R}$ and therefore

$$|F_\Delta(u)| \vee |G_\Delta(u)|^2 \leqslant \varphi(\varphi^{-1}(h(\Delta))) = h(\Delta), \tag{4.3.2}$$

for all $u \in \mathbb{R}$. The discrete-time logarithmic truncated EM numerical solution to transformed SDEs $Y_\Delta(t_k) \approx y(t_k)$ for $t_k = k\Delta$ is defined by starting from $Y_\Delta(0) = y_0 = \ln x_0$ and computing

$$Y_\Delta(t_{k+1}) = Y_\Delta(t_k) + F_\Delta\left(Y_\Delta(t_k)\right)\Delta + G_\Delta\left(Y_\Delta(t_k)\right)\Delta B_k,$$

for $k \in \mathbb{N}$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$. Now we form two versions of the continuous-time logarithmic truncated EM solution to transformed SDEs. The first one is defined by

$$\bar{y}_\Delta(t) = \sum_{k=0}^{\infty} Y_\Delta(t_k) I_{[t_k,t_{k+1})}(t),$$

for $t \in \mathbb{R}_+$. It is a simple step process and its sample paths are simple functions. The continuous version is defined by

$$y_\Delta(t) = y_0 + \int_0^t F_\Delta\left(\bar{y}_\Delta(s)\right) ds + \int_0^t G_\Delta\left(\bar{y}_\Delta(s)\right) dB(s),$$

for $t \in \mathbb{R}_+$. We have $y_\Delta(t_k) = \bar{y}_\Delta(t_k) = Y_\Delta(t_k)$, for all $k \geqslant 0$. Moreover, $y_\Delta(t)$ is an Itô process with its Itô differential

$$dy_\Delta(t) = F_\Delta\left(\bar{y}_\Delta(t)\right) dt + G_\Delta\left(\bar{y}_\Delta(t)\right) dB(t).$$

Finally, we use the transformation $\bar{x}_\Delta(t) = e^{\bar{y}_\Delta(t)}$ and $x_\Delta(t) = e^{y_\Delta(t)}$ to derive numerical solutions $\bar{x}_\Delta(t)$ and $x_\Delta(t)$ for original SDEs.

To establish the strong convergence theory of the logarithmic truncated EM method, we first prove some necessary lemmas.

Lemma 4.3.1. *Given a real number p, there exists a constant $C_1(p)$, depending on p, such that*

$$\sup_{\Delta\in(0,1]} \sup_{t\in[0,T]} \mathbb{E}\left(\frac{x_\Delta(t)}{\bar{x}_\Delta(t)}\right)^p \leqslant C_1(p).$$

Let $p \geqslant 2$. Then there exists a constant $C_2(p)$, depending on p, such that

$$\sup_{t\in[0,T]} \mathbb{E}\left|\frac{x_\Delta(t)}{\bar{x}_\Delta(t)} - 1\right|^p \leqslant C_2(p)\Delta^{\frac{p}{2}} h(\Delta)^{\frac{p}{2}},$$

for all $\Delta \in (0,1]$, where $h(\Delta)$ is defined in Definition 4.3.1.

Proof. In this proof, we use $C_1(p)$ and $C_2(p)$ to stand for generic positive real constants which depend on p but independent of Δ and k and their values may change between occurrences. By definitions of $x_\Delta(t)$ and $y_\Delta(t)$, we have

$$x_\Delta(t) = \bar{x}_\Delta(t) \exp\left(F_\Delta\left(\bar{y}_\Delta(t)\right)(t - t_k) + G_\Delta\left(\bar{y}_\Delta(t)\right)\left(B(t) - B(t_k)\right)\right),$$

for $t \in [t_k, t_{k+1})$. Lemma 4.6 in [35] states that $\mathbb{E}\left(e^{\beta|Z|}\right) \leqslant 2e^{\frac{\beta^2\Delta}{2}}$, where $\beta > 0$ and $Z \sim N(0, \sqrt{\Delta})$ is a one dimensional normal random variable. Let p be an arbitrary real number. We have

$$
\begin{aligned}
\mathbb{E}\left(\frac{x_\Delta(t)}{\bar{x}_\Delta(t)}\right)^p =& \mathbb{E}\left(\exp\left(pF_\Delta\left(\bar{y}_\Delta(t)\right)(t-t_k)+pG_\Delta\left(\bar{y}_\Delta(t)\right)\left(B(t)-B(t_k)\right)\right)\right),\\
\leqslant& \mathbb{E}\left(\exp\left(|p|h(\Delta)\Delta+|p|h(\Delta)^{0.5}\left|\left(B(t)-B(t_k)\right)\right|\right)\right),\\
\leqslant& 2^n\exp\left(|p|h(\Delta)\Delta+\frac{np^2h(\Delta)\Delta}{2}\right),
\end{aligned}
$$

for $t\in[t_k,t_{k+1})$, where n is the dimension of the Brownian motion $B(t)$. Since $\Delta h(\Delta)\leqslant 4C_0\vee 2\varphi(|\ln x_0|)$, there exists a constant $C_1(p)$ depending on p such that

$$
\mathbb{E}\left(\frac{x_\Delta(t)}{\bar{x}_\Delta(t)}\right)^p \leqslant C_1(p),
$$

for all $t\in[0,T]$ and $\Delta\in(0,1]$.

Using the Itô formula for $e^{y_\Delta(t)}$, we have

$$
x_\Delta(t)=\bar{x}_\Delta(t)+\int_{t_k}^t x_\Delta(s)\left(F_\Delta(\bar{y}_\Delta(s))+0.5|G_\Delta(\bar{y}_\Delta(s))|^2\right)ds+\int_{t_k}^t x_\Delta(s)G_\Delta(\bar{y}_\Delta(s))dB(s),
$$

for $t\in[t_k,t_{k+1})$. Now we let $p\geqslant 2$. Since $x_\Delta(t),\bar{x}_\Delta(t)\in\mathbb{R}_+$, we use (4.3.1), (4.3.2), the Hölder inequality and Theorem 1.7.1 in [3] to derive

$$
\begin{aligned}
&\mathbb{E}\left|\frac{x_\Delta(t)}{\bar{x}_\Delta(t)}-1\right|^p\\
=&\mathbb{E}\left|\int_{t_k}^t\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\left(F_\Delta(\bar{y}_\Delta(s))+0.5|G_\Delta(\bar{y}_\Delta(s))|^2\right)ds+\int_{t_k}^t\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}G_\Delta(\bar{y}_\Delta(s))dB(s)\right|^p,\\
\leqslant& C_2(p)(t-t_k)^{p-1}\mathbb{E}\int_{t_k}^t\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^p\left|F_\Delta(\bar{y}_\Delta(s))+0.5|G_\Delta(\bar{y}_\Delta(s))|^2\right|^p ds\\
&+C_2(p)(t-t_k)^{\frac{p}{2}-1}\mathbb{E}\int_{t_k}^t\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^p|G_\Delta(\bar{y}_\Delta(s))|^p ds,\\
\leqslant& C_2(p)\Delta^{\frac{p}{2}-1}(\Delta^{\frac{p}{2}}h(\Delta)^p+h(\Delta)^{\frac{p}{2}})\mathbb{E}\int_{t_k}^t\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^p ds,\\
\leqslant& C_2(p)\Delta^{\frac{p}{2}}h(\Delta)^{\frac{p}{2}},
\end{aligned}
$$

for $t \in [t_k, t_{k+1})$. In other words, there exists a constant $C_2(p)$ such that

$$\mathbb{E}\left|\frac{x_\Delta(t)}{\bar{x}_\Delta(t)} - 1\right|^p \leqslant C_2(p)\Delta^{\frac{p}{2}} h(\Delta)^{\frac{p}{2}},$$

for all $t \in [0, T]$ and $\Delta \in (0, 1]$. □

Lemma 4.3.2. *Let Assumptions 4.2.1, 4.2.2 and 4.2.3 hold with $\alpha \vee (\beta+1) < p^* + q^*$. Let θ be an arbitrary stopping time. Then there exists a constant C such that*

$$\sup_{\Delta\in(0,1]} \sup_{t\in[0,T]} \mathbb{E}|x_\Delta(t\wedge\theta)|^{p^*} < C \quad \text{and} \quad \sup_{\Delta\in(0,1]} \sup_{t\in[0,T]} \mathbb{E}|x_\Delta(t\wedge\theta)|^{-q^*} < C.$$

Proof. Let $\Delta \in (0, 1]$ and $\tau_m = \inf\{t \in [0, T] : x_\Delta(t) \notin (1/m, m)\}$. Using the Itô formula, we have

$$\begin{aligned}
e^{p^* y_\Delta(t\wedge\tau_m\wedge\theta)} + e^{-q^* y_\Delta(t\wedge\tau_m\wedge\theta)} =& e^{p^* y_0} + e^{-q^* y_0} \\
&+ p^* \int_0^{t\wedge\tau_m\wedge\theta} e^{p^* y_\Delta(s)} \left(F_\Delta(\bar{y}_\Delta(s)) + \frac{p^*}{2}|G_\Delta(\bar{y}_\Delta(s))|^2 \right) ds \\
&- q^* \int_0^{t\wedge\tau_m\wedge\theta} e^{-q^* y_\Delta(s)} \left(F_\Delta(\bar{y}_\Delta(s)) - \frac{q^*}{2}|G_\Delta(\bar{y}_\Delta(s))|^2 \right) ds \\
&+ p^* \int_0^{t\wedge\tau_m\wedge\theta} e^{p^* y_\Delta(s)} G_\Delta(\bar{y}_\Delta(s)) dB(s) \\
&- q^* \int_0^{t\wedge\tau_m\wedge\theta} e^{-q^* y_\Delta(s)} G_\Delta(\bar{y}_\Delta(s)) dB(s). \quad (4.3.3)
\end{aligned}$$

Using Assumptions 4.2.3, Remark 4.2.1 and the Young inequality, we have

$$\begin{aligned}
& x_\Delta(t)^{p^*} \left(\frac{f_\Delta(\bar{x}_\Delta(t))}{\bar{x}_\Delta(t)} + \frac{p^*-1}{2} \frac{|g(\bar{x}_\Delta(t))|^2}{\bar{x}_\Delta(t)^2} \right) \\
\leqslant & C x_\Delta(t)^{p^*} \left(\frac{1 + \bar{x}_\Delta(t)^{\alpha+1} + \bar{x}_\Delta(t)^{-\beta}}{\bar{x}_\Delta(t)} + \frac{1 + \bar{x}_\Delta(t)^{\alpha+2} + \bar{x}_\Delta(t)^{-\beta+1}}{\bar{x}_\Delta(t)^2} \right) I_{\{\bar{x}_\Delta(t)\in(0,u^*)\}} \\
& + x_\Delta(t)^{p^*} \left(\frac{K_2(1 + \bar{x}_\Delta(t)^2)}{\bar{x}_\Delta(t)^2} \right) I_{\{\bar{x}_\Delta(t)\in[u^*,\infty)\}}, \\
\leqslant & C \left(\frac{x_\Delta(t)}{\bar{x}_\Delta(t)} \right)^{p^*} \bar{x}_\Delta(t)^{p^*-\beta-1} I_{\{\bar{x}_\Delta(t)\in(0,u^*)\}} + C x_\Delta(t)^{p^*},
\end{aligned}$$

and

$$
\begin{aligned}
& - x_\Delta(t)^{-q^*} \left(\frac{f_\Delta(\bar{x}_\Delta(t))}{\bar{x}_\Delta(t)} - \frac{q^*+1}{2} \frac{|g(\bar{x}_\Delta(t))|^2}{\bar{x}_\Delta(t)^2} \right) \\
\leqslant & - x_\Delta(t)^{-q^*} \left(\frac{f_\Delta(\bar{x}_\Delta(t))}{\bar{x}_\Delta(t)} - \frac{q^*+1}{2} \frac{|g(\bar{x}_\Delta(t))|^2}{\bar{x}_\Delta(t)^2} \right) I_{\{\bar{x}_\Delta(t) \in (0,u^*)\}} \\
& + C x_\Delta(t)^{-q^*} \left(\frac{1+\bar{x}_\Delta(t)^{\alpha+1} + \bar{x}_\Delta(t)^{-\beta}}{\bar{x}_\Delta(t)} + \frac{1+\bar{x}_\Delta(t)^{\alpha+2} + \bar{x}_\Delta(t)^{-\beta+1}}{\bar{x}_\Delta(t)^2} \right) I_{\{\bar{x}_\Delta(t) \in [u^*,\infty)\}}, \\
\leqslant & C \left(\frac{x_\Delta(t)}{\bar{x}_\Delta(t)} \right)^{-q^*} \bar{x}_\Delta(t)^{-q^*+\alpha} I_{\{\bar{x}_\Delta(t) \in [u^*,\infty)\}} + C x_\Delta(t)^{-q^*},
\end{aligned}
$$

for all $t \in [0, T \wedge \tau_m \wedge \theta]$.

If $p^* - \beta - 1 > 0$, then $\bar{x}_\Delta(t)^{p^*-\beta-1} I_{\{\bar{x}_\Delta(t) \in (0,u^*)\}}$ is bounded. If $-q^* + \alpha < 0$, then $\bar{x}_\Delta(t)^{-q^*+\alpha} I_{\{\bar{x}_\Delta(t) \in [u^*,\infty)\}}$ is bounded. Since $\alpha \vee (\beta+1) < p^* + q^*$, we have $p^* - \beta - 1 > -q^*$ and $-q^* + \alpha < p^*$. Let $\varepsilon > 0$ be sufficiently small such that $(1+\varepsilon)(p^* - \beta - 1) > -q^*$ and $(1+\varepsilon)(-q^*+\alpha) < p^*$, there exists a constant C such that

$$
\bar{x}_\Delta(t)^{(p^*-\beta-1)(1+\varepsilon)} I_{\{\bar{x}_\Delta(t) \in (0,u^*)\}} < \bar{x}_\Delta(t)^{-q^*} + C,
$$

and

$$
\bar{x}_\Delta(t)^{(-q^*+\alpha)(1+\varepsilon)} I_{\{\bar{x}_\Delta(t) \in [u^*,\infty)\}} < \bar{x}_\Delta(t)^{p^*} + C,
$$

for all $t \in [0, T \wedge \tau_m \wedge \theta]$.

Taking expectations on both sides of (4.3.3) and using the above arguments and the Young inequality, we have

$$\begin{aligned}&\mathbb{E}\left(x_\Delta(t\wedge\tau_m\wedge\theta)^{p^*}+x_\Delta(t\wedge\tau_m\wedge\theta)^{-q^*}\right)\\ \leqslant&C+\mathbb{E}\int_0^{t\wedge\tau_m\wedge\theta}(x_\Delta(s)^{p^*}+x_\Delta(s)^{-q^*})ds\\ &+C\mathbb{E}\int_0^{t\wedge\tau_m\wedge\theta}\left(\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)^{p^*(1+\varepsilon^{-1})}+\left(\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right)^{-q^*(1+\varepsilon^{-1})}\right)ds\\ &+C\mathbb{E}\int_0^{t\wedge\tau_m\wedge\theta}\left(1+\bar{x}_\Delta(s)^{p^*}+\bar{x}_\Delta(s)^{-q^*}\right)ds.\end{aligned}$$

Using Lemma 4.3.1, there exists a constant C such that

$$\begin{aligned}&\sup_{u\in[0,t]}\mathbb{E}\left(x_\Delta(u\wedge\tau_m\wedge\theta)^{p^*}+x_\Delta(u\wedge\tau_m\wedge\theta)^{-q^*}\right)\\ \leqslant&C+C\int_0^t\sup_{u\in[0,s]}\mathbb{E}\left(x_\Delta(u\wedge\tau_m\wedge\theta)^{p^*}+x_\Delta(u\wedge\tau_m\wedge\theta)^{-q^*}\right)ds,\end{aligned}$$

for all $t\in[0,T]$. The Gronwall inequality implies that there exists a constant C such that

$$\sup_{t\in[0,T]}\mathbb{E}\left(x_\Delta(t\wedge\tau_m\wedge\theta)^{p^*}+x_\Delta(t\wedge\tau_m\wedge\theta)^{-q^*}\right)<C.$$

Letting $m\to\infty$ to conclude conclusions. □

In the following, we set $e_\Delta(t)=x(t)-x_\Delta(t)$ and let $R>|\ln x_0|$ be a real number. Then we define two stopping times:

$$\tau_R=\inf\{t\in[0,T]:|y(t)|\geqslant R\}\quad\text{and}\quad\tau_R^\Delta=\inf\{t\in[0,T]:|y_\Delta(t)|\geqslant R\},$$

where $y(t)=\ln x(t)$. In addition, we set $\tau=\tau_R\wedge\tau_R^\Delta$.

Lemma 4.3.3. *Let Assumptions 4.2.1, 4.2.2 and 4.2.3 hold with* $0<\frac{p^*r^*}{p^*-r^*}<\frac{p^*}{\alpha+1}\wedge\frac{q^*}{\beta}$. *Given a* $R>|\ln x_0|$, *let* τ *be the stopping time defined above. Let* Δ *be sufficiently small such that* $\varphi^{-1}(h(\Delta))\geqslant R$. *Let* $2\leqslant r<r^*$, *then there exists a constant* C, *which is*

independent of stepsize Δ, such that

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t\wedge\tau)|^r < C\Delta^{\frac{r}{2}} h(\Delta)^{\frac{r}{2}},$$

where $h(\Delta)$ is defined in Definition 4.3.1.

Proof. First we observe that $|y_\Delta(s)| \leqslant R$ for $s \in [0, T\wedge\tau]$. Since we have the assumption $\varphi^{-1}(h(\Delta)) \geqslant R$, $F_\Delta(\bar{y}_\Delta(s)) = F(\bar{y}_\Delta(s))$ and $G_\Delta(\bar{y}_\Delta(s)) = G(\bar{y}_\Delta(s))$, for $s \in [0, T\wedge\tau]$.

Using the Itô formula for $e^{y_\Delta(t)}$ and $|x(t) - x_\Delta(t)|^r$, we have

$$\begin{aligned}
|e_\Delta(t\wedge\tau)|^r =& r\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-2} e_\Delta(s)\left(f(x(s)) - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)} f(\bar{x}_\Delta(s))\right) ds \\
&+ \frac{r(r-1)}{2}\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-2}\left|g(x(s)) - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)} g(\bar{x}_\Delta(s))\right|^2 ds \\
&+ r\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-2} e_\Delta(s)\left(g(x(s)) - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)} g(\bar{x}_\Delta(s))\right) dB(s).
\end{aligned}$$

Taking expectations on both sides and using the Young inequality, we then have

$$\mathbb{E}|e_\Delta(t\wedge\tau)|^r \leqslant J_1 + J_2,$$

where

$$J_1 = r\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-2}\left(e_\Delta(s)\left(f(x(s)) - f(x_\Delta(s))\right) + \frac{r^*-1}{2}\left|g(x(s)) - g(x_\Delta(s))\right|^2\right) ds,$$

and

$$\begin{aligned}
J_2 =& r\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-2} e_\Delta(s)\left(f(x_\Delta(s)) - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)} f(\bar{x}_\Delta(s))\right) ds \\
&+ \frac{r(r-1)(r^*-1)}{2(r^*-r)}\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-2}\left|g(x_\Delta(s)) - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)} g(\bar{x}_\Delta(s))\right|^2 ds.
\end{aligned}$$

Using Assumption 4.2.3, we have $J_1 \leqslant rH\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^r ds$. Using the Young inequality, we derive

$$J_2 \leqslant C\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-1}\left|f(x_\Delta(s)) - f(\bar{x}_\Delta(s)) + f(\bar{x}_\Delta(s)) - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)}f(\bar{x}_\Delta(s))\right|ds$$
$$+ C\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^{r-2}\left|g(x_\Delta(s)) - g(\bar{x}_\Delta(s)) + g(\bar{x}_\Delta(s)) - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)}g(\bar{x}_\Delta(s))\right|^2 ds,$$
$$\leqslant C\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^r ds + C\mathbb{E}\int_0^{t\wedge\tau} |f(x_\Delta(s)) - f(\bar{x}_\Delta(s))|^r ds$$
$$+ C\mathbb{E}\int_0^{t\wedge\tau} |1 - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)}|^r |f(\bar{x}_\Delta(s))|^r ds + C\mathbb{E}\int_0^{t\wedge\tau} |g(x_\Delta(s)) - g(\bar{x}_\Delta(s))|^r ds$$
$$+ C\mathbb{E}\int_0^{t\wedge\tau} |1 - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)}|^r |g(\bar{x}_\Delta(s))|^r ds.$$

Using Assumption 4.2.1, Remark 4.2.1 and the Hölder inequality, we have

$$J_2 \leqslant C\mathbb{E}\int_0^{t\wedge\tau} |e_\Delta(s)|^r ds + \int_0^{t\wedge\tau} \left(\mathbb{E}|x_\Delta(s) - \bar{x}_\Delta(s)|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} \left(J_3(s)^{\frac{1}{1+\varepsilon}} + J_4(s)^{\frac{1}{1+\varepsilon}}\right) ds$$
$$+ C\int_0^{t\wedge\tau} \left(\mathbb{E}|1 - \frac{x_\Delta(s)}{\bar{x}_\Delta(s)}|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right)^{\frac{\varepsilon}{1+\varepsilon}} \left(J_5(s)^{\frac{1}{1+\varepsilon}} + J_6(s)^{\frac{1}{1+\varepsilon}}\right) ds,$$

where

$$J_3(s) = \mathbb{E}\left(1 + x_\Delta(s)^{(1+\varepsilon)\alpha r} + x_\Delta(s)^{-(1+\varepsilon)\beta r} + \bar{x}_\Delta(s)^{(1+\varepsilon)\alpha r} + \bar{x}_\Delta(s)^{-(1+\varepsilon)\beta r}\right),$$
$$J_4(s) = \mathbb{E}\left(1 + x_\Delta(s)^{(1+\varepsilon)\alpha r/2} + x_\Delta(s)^{-(1+\varepsilon)\beta r/2} + \bar{x}_\Delta(s)^{(1+\varepsilon)\alpha r/2} + \bar{x}_\Delta(s)^{-(1+\varepsilon)\beta r/2}\right),$$
$$J_5(s) = \mathbb{E}\left(1 + \bar{x}_\Delta(s)^{(1+\varepsilon)(\alpha+1)r} + \bar{x}_\Delta(s)^{-\beta(1+\varepsilon)r}\right),$$

and

$$J_6(s) = \mathbb{E}\left(1 + \bar{x}_\Delta(s)^{(1+\varepsilon)(\alpha+2)r/2} + \bar{x}_\Delta(s)^{-(\beta-1)(1+\varepsilon)r/2}\right).$$

Under the condition $\frac{p^*r^*}{p^*-r^*} < \frac{p^*}{\alpha+1} \wedge \frac{q^*}{\beta}$, there exists a $\varepsilon > 0$ such that

$$\frac{p^*}{\alpha+1} \wedge \frac{q^*}{\beta} > (1+\varepsilon)r^* > \frac{p^*r^*}{p^*-r^*}.$$

If follows that

$$(1+\varepsilon)(\alpha+1)r^* < p^*, \quad \text{and} \quad (1+\varepsilon)\beta r^* < q^*.$$

Since $r^* > 2$, we have $\alpha \vee (\beta+1) < p^* + q^*$. In addition, we have

$$(1+\varepsilon)(p^*-r^*) > p^*, \quad \text{and therefore} \quad 2 < \frac{(1+\varepsilon)r^*}{\varepsilon} < p^*.$$

Using (4.3.1), (4.3.2), Lemma 4.3.2, the Hölder inequality and Theorem 7.1 in [3], we then have

$$\begin{aligned}
&\mathbb{E}\,|x_\Delta(t)-\bar{x}_\Delta(t)|^{\frac{(1+\varepsilon)r}{\varepsilon}} \\
=&\mathbb{E}\left|\int_{t_k}^{t} x_\Delta(s)\left(F_\Delta(\bar{y}_\Delta(s))+0.5|G_\Delta(\bar{y}_\Delta(s))|^2\right)ds+\int_{t_k}^{t} x_\Delta(s)G_\Delta(\bar{y}_\Delta(s))dB(s)\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}, \\
\leqslant&C(t-t_k)^{\frac{(1+\varepsilon)r}{\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}|x_\Delta(s)|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left|F_\Delta(\bar{y}_\Delta(s))+0.5|G_\Delta(\bar{y}_\Delta(s))|^2\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds \\
&+C(t-t_k)^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}|x_\Delta(s)|^{\frac{(1+\varepsilon)r}{\varepsilon}}|G_\Delta(\bar{y}_\Delta(s))|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds, \\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}(\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}}h(\Delta)^{\frac{(1+\varepsilon)r}{\varepsilon}}+h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}})\mathbb{E}\int_{t_k}^{t}|x_\Delta(s)|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds, \\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}}h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}.
\end{aligned}$$

Using Lemmas 4.3.1 and 4.3.2, we have $J_2 \leqslant C\mathbb{E}\int_0^{t\wedge\tau}|e_\Delta(s)|^r ds + C\Delta^{\frac{r}{2}}h(\Delta)^{\frac{r}{2}}$. Then the Gronwall inequality implies that $\sup_{t\in[0,T]}\mathbb{E}|e_\Delta(t\wedge\tau)|^r < C\Delta^{\frac{r}{2}}h(\Delta)^{\frac{r}{2}}$. □

Now we state our results on convergence rates.

Theorem 4.3.1. *Let Assumptions 4.2.1, 4.2.2 and 4.2.3 hold with* $0 < \frac{p^*r^*}{p^*-r^*} < \frac{p^*}{\alpha+1} \wedge \frac{q^*}{\beta}$. *Let* $2 \leqslant r < r^*$ *and* $\Delta \in (0,1]$, *then there exists a constant* C *such that*

$$\sup_{t\in[0,T]}\mathbb{E}|e_\Delta(t)|^r < C\Delta^{\frac{(p^*-r)(p^*\wedge q^*)r}{2(p^*-r)(p^*\wedge q^*)+(\alpha\vee(\beta+1))p^*r}},$$

by setting

$$h(\Delta) = (3.5C_0 \vee \varphi(|\ln x_0|))\,\Delta^{-\frac{(\alpha\vee(\beta+1))p^*r}{2(p^*-r)(p^*\wedge q^*)+(\alpha\vee(\beta+1))p^*r}}.$$

Proof. Let $R = \varphi^{-1}(h(\Delta))$. Since $\frac{p^*r^*}{p^*-r^*} < \frac{p^*}{\alpha+1} \wedge \frac{q^*}{\beta}$, we have $\alpha \vee (\beta+1) < p^* + q^*$. Using the Young inequality, the Chebyshev inequality and Lemma 4.3.2, then we have

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^r I_{\{\tau\leqslant T\}})\\
=&\sup_{t\in[0,T]} \mathbb{E}\left(|e_\Delta(t)|^r \delta^{\frac{r}{p^*}} I_{\{\tau\leqslant T\}}\delta^{-\frac{r}{p^*}}\right),\\
\leqslant&\frac{r}{p^*}\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^{p^*}\delta + \frac{p^*-r}{p^*}\mathbb{E}\left(I_{\{\tau\leqslant T\}}^{\frac{p^*}{p^*-r}}\right)\delta^{-\frac{r}{p^*-r}},\\
=&\frac{r}{p^*}\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^{p^*}\delta + \frac{p^*-r}{p^*}\Pr(\tau\leqslant T)\delta^{-\frac{r}{p^*-r}},\\
\leqslant&C\delta\\
&+C\left(\frac{\mathbb{E}\left(|x(T\wedge\tau)|^{p^*}\right)+\mathbb{E}\left(|x_\Delta(T\wedge\tau)|^{p^*}\right)}{e^{p^*\varphi^{-1}(h(\Delta))}} + \frac{\mathbb{E}\left(|x(T\wedge\tau)|^{-q^*}\right)+\mathbb{E}\left(|x_\Delta(T\wedge\tau)|^{-q^*}\right)}{e^{q^*\varphi^{-1}(h(\Delta))}}\right)\delta^{-\frac{r}{p^*-r}},\\
\leqslant&C\delta + Ce^{-(p^*\wedge q^*)\varphi^{-1}(h(\Delta))}\delta^{-\frac{r}{p^*-r}}.
\end{aligned}$$

Letting $\delta = e^{-\left((p^*-r)(p^*\wedge q^*)\varphi^{-1}(h(\Delta))\right)/p^*}$, then we have that

$$\begin{aligned}
&\sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^r I_{\{\tau\leqslant T\}})\\
\leqslant&Ce^{-\left((p^*-r)(p^*\wedge q^*)\varphi^{-1}(h(\Delta))\right)/p^*} + Ce^{-(p^*\wedge q^*)\varphi^{-1}(h(\Delta))}e^{\left(r(p^*\wedge q^*)\varphi^{-1}(h(\Delta))\right)/p^*},\\
\leqslant&Ce^{-\left((p^*-r)(p^*\wedge q^*)\varphi^{-1}(h(\Delta))\right)/p^*}.
\end{aligned}$$

Using the above results and Lemma 4.3.3, we have

$$\begin{aligned}
\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^r =& \sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^r I_{\{\tau>T\}}) + \sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^r I_{\{\tau\leqslant T\}}),\\
\leqslant& \sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t\wedge\tau)|^r + \sup_{t\in[0,T]} \mathbb{E}(|e_\Delta(t)|^r I_{\{\tau\leqslant T\}}),\\
\leqslant&C\Delta^{\frac{r}{2}}h(\Delta)^{\frac{r}{2}} + Ce^{-\left((p^*-r)(p^*\wedge q^*)\varphi^{-1}(h(\Delta))\right)/p^*}.
\end{aligned}$$

Since $h(\Delta) \geqslant h(1) > 3C_0$, we have

$$e^{-\left((p^*-r)(p^*\wedge q^*)\varphi^{-1}(h(\Delta))\right)/p^*} = \left(\frac{h(\Delta)}{C_0} - 2\right)^{-\frac{(p^*-r)(p^*\wedge q^*)}{(\alpha\vee(\beta+1))p^*}} \leqslant \left(\frac{h(\Delta)}{3C_0}\right)^{-\frac{(p^*-r)(p^*\wedge q^*)}{(\alpha\vee(\beta+1))p^*}}.$$

Now we set

$$h(\Delta) = (3.5C_0 \vee \varphi(|\ln x_0|))\,\Delta^{-\frac{(\alpha\vee(\beta+1))p^*r}{2(p^*-r)(p^*\wedge q^*)+(\alpha\vee(\beta+1))p^*r}}.$$

Then there exists a constant C such that

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^r < C\Delta^{\frac{(p^*-r)(p^*\wedge q^*)r}{2(p^*-r)(p^*\wedge q^*)+(\alpha\vee(\beta+1))p^*r}}.$$

□

Remark 4.3.1. In [13] and [38], authors are concerned about a SDE satisfying Assumptions 4.2.1 and 4.2.3 with $\beta = 0$. Since they only considered polynomially growing when $|x| \to \infty$, they only require that there exist positive real numbers $p^* > 0$ and $K_2 > 0$ such that

$$uf(u) + \frac{p^*-1}{2}|g(u)|^2 \leqslant K_2(1+u^2),$$

for $u \in \mathbb{R}$, which is a part of Assumption 4.2.2. In [38], authors pointed that conditions of Theorem 4.3.4 in [13] are valid only for extremely small step sizes. Therefore, their new Theorem 4.3.4 in [38] are developed for all $\Delta \in (0,1]$. Using their techniques, then results can finally be expressed as

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^r < C\Delta^{\frac{r}{2}}h(\Delta)^r + Ce^{-(2p^*-(2+\alpha)r)\varphi^{-1}(h(\Delta))/2}$$

with assuming that $p^* > (1+\alpha)r$. However, Theorem 4.3.1 shows that, using our new techniques, their results can be improved as follows:

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^r < C\Delta^{\frac{r}{2}}h(\Delta)^{\frac{r}{2}} + Ce^{-(p^*-r)\varphi^{-1}(h(\Delta))}$$

with assuming that $p^* > (\alpha+1)r$. Since $\lim_{\Delta\to 0} h(\Delta) = +\infty$ and $(p^* - r) > p^* - (1 + \alpha/2)r$, our convergence rate results are better. Moreover, in Theorem 4.3.1, we give an explicit formula $h(\Delta)$ and a more detailed convergence rate:

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^r < C\Delta^{\frac{(p^*-r)r}{2(p^*-r)+\alpha p^* r}}.$$

In particular, if $\alpha = 0$, then $\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^r < C\Delta^{\frac{r}{2}}$, which is exactly the optimal convergence rate of the classical EM method for SDEs with globally Lipschitz coefficients.

Remark 4.3.2. Now we fix $\varepsilon = 1/2$ in Lemma 4.3.3. If we further assume that $1.5r^* < \frac{p^*}{\alpha+2} \wedge \frac{q^*}{\beta+1}$, then we have

$$\frac{p^*}{\alpha+2} \wedge \frac{q^*}{\beta+1} > (1+\varepsilon)r^* = \frac{(1+\varepsilon)r^*}{2\varepsilon}.$$

If follows that

$$(1+\varepsilon)(\alpha+1)r^* < p^*, \quad (1+\varepsilon)\beta r^* < q^*, \quad \frac{(1+\varepsilon)(\alpha+2)r^*}{2\varepsilon} < p^* \quad \text{and} \quad \frac{(1+\varepsilon)(\beta+1)r^*}{2\varepsilon} < q^*.$$

Using (4.3.1), (4.3.2), Remark 4.2.1, Lemmas 4.3.1, 4.3.2, the Hölder inequality and

Theorem 7.1 in [3], we have

$$
\begin{aligned}
&\mathbb{E}\left|\frac{x_\Delta(t)}{\bar{x}_\Delta(t)}-1\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}\\
=&\mathbb{E}\left|\int_{t_k}^{t}\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\left(F_\Delta(\bar{y}_\Delta(s))+0.5|G_\Delta(\bar{y}_\Delta(s))|^2\right)ds+\int_{t_k}^{t}\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}G_\Delta(\bar{y}_\Delta(s))dB(s)\right|^{\frac{(1+\varepsilon)r}{\varepsilon}},\\
\leqslant& C(t-t_k)^{\frac{(1+\varepsilon)r}{\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left|F_\Delta(\bar{y}_\Delta(s))+0.5|G_\Delta(\bar{y}_\Delta(s))|^2\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
&+C(t-t_k)^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}G_\Delta(\bar{y}_\Delta(s))\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds,\\
\leqslant& C\Delta^{\frac{(1+\varepsilon)r}{\varepsilon}-1}h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}\mathbb{E}\int_{t_k}^{t}\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left(|F_\Delta(\bar{y}_\Delta(s))|^{\frac{(1+\varepsilon)r}{2\varepsilon}}+|G_\Delta(\bar{y}_\Delta(s))|^{\frac{(1+\varepsilon)r}{\varepsilon}}\right)ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^{t}\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}|G_\Delta(\bar{y}_\Delta(s))|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds,\\
\leqslant& C\Delta^{\frac{(1+\varepsilon)r}{\varepsilon}-1}h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}\int_{t_k}^{t}\left(\mathbb{E}\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^{\frac{(1+\varepsilon)(1+\eta)r}{\varepsilon\eta}}\right)^{\frac{\eta}{1+\eta}}\left(\mathbb{E}\left(1+\bar{x}_\Delta(s)^{\frac{\alpha(1+\varepsilon)(1+\eta)r}{2\varepsilon}}+\bar{x}_\Delta(s)^{\frac{-(\beta+1)(1+\varepsilon)(1+\eta)r}{2\varepsilon}}\right)\right)^{\frac{1}{1+\eta}}ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\int_{t_k}^{t}\left(\mathbb{E}\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^{\frac{(1+\varepsilon)(1+\eta)r}{\varepsilon\eta}}\right)^{\frac{\eta}{1+\eta}}\left(\mathbb{E}\left(1+\bar{x}_\Delta(s)^{\frac{\alpha(1+\varepsilon)(1+\eta)r}{2\varepsilon}}+\bar{x}_\Delta(s)^{\frac{-(\beta+1)(1+\varepsilon)(1+\eta)r}{2\varepsilon}}\right)\right)^{\frac{1}{1+\eta}}ds,\\
\leqslant& C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}},
\end{aligned}
$$

since

$$\frac{(1+\varepsilon)(\alpha+2)r^*}{2\varepsilon}<p^* \quad \text{and} \quad \frac{(1+\varepsilon)(\beta+1)r^*}{2\varepsilon}<q^*.$$

Similarly, we have

$$
\begin{aligned}
&\mathbb{E}\left|x_\Delta(t)-\bar{x}_\Delta(t)\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}\\
=&\mathbb{E}\left|\int_{t_k}^t x_\Delta(s)\left(F_\Delta(\bar{y}_\Delta(s))+0.5G_\Delta^2(\bar{y}_\Delta(s))\right)ds+\int_{t_k}^t x_\Delta(s)G_\Delta(\bar{y}_\Delta(s))dB(s)\right|^{\frac{(1+\varepsilon)r}{\varepsilon}},\\
\leqslant&C(t-t_k)^{\frac{(1+\varepsilon)r}{\varepsilon}-1}\mathbb{E}\int_{t_k}^t|x_\Delta(s)|^{\frac{(1+\varepsilon)r}{\varepsilon}}\left|F_\Delta(\bar{y}_\Delta(s))+0.5G_\Delta^2(\bar{y}_\Delta(s))\right|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
&+C(t-t_k)^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^t|x_\Delta(s)|^{\frac{(1+\varepsilon)r}{\varepsilon}}|G_\Delta(\bar{y}_\Delta(s))|^{\frac{(1+\varepsilon)r}{\varepsilon}}ds,\\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{\varepsilon}-1}h(\Delta)^{\frac{(1+\varepsilon)r}{2\varepsilon}}\mathbb{E}\int_{t_k}^t x_\Delta(s)^{\frac{(1+\varepsilon)r}{\varepsilon}}\left(1+\bar{x}_\Delta(s)^{\frac{\alpha(1+\varepsilon)r}{2\varepsilon}}+\bar{x}_\Delta(s)^{\frac{-(\beta+1)(1+\varepsilon)r}{2\varepsilon}}\right)ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^t x_\Delta(s)^{\frac{(1+\varepsilon)r}{\varepsilon}}\left(1+\bar{x}_\Delta(s)^{\frac{\alpha(1+\varepsilon)r}{2\varepsilon}}+\bar{x}_\Delta(s)^{\frac{-(\beta+1)(1+\varepsilon)r}{2\varepsilon}}\right)ds,\\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\mathbb{E}\int_{t_k}^t x_\Delta(s)^{\frac{(1+\varepsilon)r}{\varepsilon}}ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\int_{t_k}^t\left(\mathbb{E}|x_\Delta(s)|^{\frac{(1+\varepsilon)(\alpha+2)r}{2\varepsilon}}\right)^{\frac{2}{\alpha+2}}\left(\mathbb{E}|\bar{x}_\Delta(s)|^{\frac{(1+\varepsilon)(\alpha+2)r}{2\varepsilon}}\right)^{\frac{\alpha}{\alpha+2}}ds\\
&+C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}-1}\int_{t_k}^t\left(\mathbb{E}\left|\frac{x_\Delta(s)}{\bar{x}_\Delta(s)}\right|^{\frac{2(1+\varepsilon)q^*r}{2q^*\varepsilon-(\beta-1)(1+\varepsilon)r}}\right)^{\frac{2q^*\varepsilon-(\beta-1)(1+\varepsilon)r}{2q^*\varepsilon}}\left(\mathbb{E}|\bar{x}_\Delta(s)|^{-q^*}\right)^{\frac{(\beta-1)(1+\varepsilon)r}{2q^*\varepsilon}}ds,\\
\leqslant&C\Delta^{\frac{(1+\varepsilon)r}{2\varepsilon}}.
\end{aligned}
$$

Now we set $h(\Delta)=(4C_0\vee 2\varphi(|\ln x_0|))\,\Delta^{-1}$. By similar arguments in Lemma 4.3.3 and Theorem 4.3.1, we then have

$$
\sup_{t\in[0,T]}\mathbb{E}|e_\Delta(t)|^r<C\Delta^{\frac{r}{2}}+C\Delta^{\frac{(p^*-r)(p^*\wedge q^*)}{(\alpha\vee(\beta+1))p^*}}<C\Delta^{\frac{(3\alpha+4)(p^*\wedge q^*)r^*}{2(\beta+1)p^*}\wedge\frac{r}{2}},
$$

since

$$
p^*-r>p^*-r^*>(3\alpha+4)r^*/2.
$$

4.4 Examples

In section 3, we establish general convergence rate theorems for the logarithmic truncated EM method. The convergence rate results are complicated. In this section, we will apply the logarithmic truncated EM method for the Aït-Sahalia model and the CEV model. It can be seen that convergence rate orders are exactly one half.

Example 4.4.1 (Aït-Sahalia model)**.** The Aït-Sahalia model is given by

$$dx(t) = f(x(t))dt + g(x(t))dB(t),$$

where

$$f(u) = a_{-1}u^{-1} - a_0 + a_1 u - a_2 u^{\theta}$$

and

$$g(u) = \sigma u^{\rho}$$

with $a_{-1}, a_0, a_1, a_2, \sigma > 0$, $\rho, \theta > 1$ and $\theta + 1 > 2\rho$.

Let $r > 2$ be a positive real number. Let $0 < v < u$ be arbitrary. The mean value theorem implies that there exists a $w \in (v, u)$ such that

$$f(u) - f(v) = f'(w)(u - v).$$

It follows that

$$|f(u)-f(v)| \leqslant |f'(w)||u-v| \leqslant (a_{-1}+a_1+a_2)(1+u^{-2}+v^{-2}+u^{\theta-1}+v^{\theta-1})|u-v|,$$

since

$$f'(w) = -a_{-1}w^{-2} + a_1 - a_2\theta w^{\theta-1}.$$

Therefore, Assumption 4.2.1 is satisfied with $\alpha = \theta - 1$ and $\beta = 2$.

Let $r^* = 3r$ and $q^* = 15r$. Since $uf(u) \to a_{-1}$ and $|g(u)|^2 \to 0$ as $u \to 0$, we can always find a sufficiently small $u^* > 0$ such that

$$uf(u) - \frac{q^*+1}{2}|g(u)|^2 > 0, \quad u \in (0, u^*).$$

Let $p^* = 5(\theta+1)r$ and we have

$$uf(u) + \frac{p^*-1}{2}|g(u)|^2 = a_{-1} - a_0 u + a_1 u^2 - a_2 u^{\theta+1} + \frac{(p^*-1)\sigma^2}{2} u^{2\rho}.$$

It tends to negative infinity as $u \to \infty$ since $\theta + 1 > 2\rho$. Therefore, there always exists a $K > 0$ such that

$$uf(u) + \frac{p^*-1}{2}|g(u)|^2 \leqslant K(1+u^2), \quad u \in [u^*, \infty).$$

That is, the Aït-Sahalia model satisfies Assumption 4.2.2.

Without loss of generality, we let $v < u$. Using the mean value theorem, we have

$$a_{-1}(u-v)(u^{-1} - v^{-1}) < 0.$$

Using the Hölder inequality, we then have

$$\begin{aligned}
&(u-v)(f(u)-f(v)) + \frac{r^*-1}{2}|g(u)-g(v)|^2 \\
=&a_{-1}(u-v)(u^{-1}-v^{-1}) + a_1(u-v)^2 - a_2(u-v)(x^\theta - y^\theta) + \frac{(r^*-1)\sigma^2}{2}|u^\rho - v^\rho|^2, \\
<&a_1(u-v)^2 - a_2\theta(u-v)\int_v^u z^{\theta-1}dz + \frac{(r^*-1)\sigma^2\rho^2}{2}\left(\int_v^u z^{\rho-1}dz\right)^2, \\
\leqslant&a_1(u-v)^2 + (u-v)\int_v^u \left(-a_2\theta z^{\theta-1} + \frac{(r^*-1)\sigma^2\rho^2}{2} z^{2\rho-2}\right) dz, \\
\leqslant&C(u-v)^2,
\end{aligned}$$

since $\rho, \theta > 1$ and $\theta + 1 > 2\rho$. Therefore, drift and diffusion coefficients also satisfy Assumption 4.2.3.

We have $1.5r^* < \frac{p^*}{\alpha+2} \wedge \frac{q^*}{\beta+1}$. That is, conditions in Remark 4.3.2 are also satisfied.

We also have

$$\frac{(3\alpha+4)(p^*\wedge q^*)r^*}{2(\beta+1)p^*} = \frac{(3\theta+1)(3\wedge(\theta+1))r}{2(\theta+1)} > \frac{3r}{2},$$

for $\theta > 1$. Therefore, we have

$$\sup_{t\in[0,T]} \mathbb{E}|x(t) - x_\Delta(t)|^r < C\Delta^{\frac{r}{2}},$$

for all $\Delta \in (0,1]$.

Example 4.4.2 (CEV process)**.** The CEV process is given by

$$dx(t) = \lambda(\mu - x(t))dt + \sigma x(t)^{0.5+\theta}dB(t),$$

where $\lambda, \mu, \sigma > 0$ and $\theta \in (0, 0.5)$. Using the Lamperti transformation $y = x^{0.5-\theta}$, we have a new SDE

$$dy(t) = f(y(t))dt + g(y(t))dB(t),$$

where

$$f(u) = (0.5-\theta)\left(\lambda\mu u^{-\frac{1+2\theta}{1-2\theta}} - \lambda u - \frac{2\theta+1}{4}\sigma^2 u^{-1}\right),$$

and

$$g(u) = (0.5-\theta)\sigma.$$

Let $r > 1$ be a positive real number. Let $0 < v < u$ be arbitrary. The mean value theorem implies that there exists a $w \in (v, u)$ such that

$$f(u) - f(v) = f'(w)(u - v).$$

It follows that

$$|f(u)-f(v)| \leqslant (1-2\theta)\left(\frac{(1+2\theta)\lambda\mu}{1-2\theta}+\lambda+\frac{(2\theta+1)\sigma^2}{4}\right)(1+u^{-\frac{2}{1-2\theta}}+v^{-\frac{2}{1-2\theta}})|u-v|,$$

since $\frac{2}{1-2\theta} > 2$ and

$$f'(w) = (0.5-\theta)\left(-\frac{(1+2\theta)\lambda\mu}{1-2\theta}w^{-\frac{2}{1-2\theta}} - \lambda + \frac{2\theta+1}{4}\sigma^2 w^{-2}\right).$$

Therefore, Assumption 4.2.1 is satisfied with $\alpha = 0$ and $\beta = \frac{2}{1-2\theta}$.

Let $r^* = (\beta+2)r$ and $q^* = (1.5\beta+4)r^*$. Since $uf(u) \to \infty$ and $g(u)$ is a constant, we can always find a sufficiently small $u^* > 0$ such that

$$uf(u) - \frac{q^*+1}{2}|g(u)|^2 > 0, \quad u \in (0, u^*).$$

Let $p^* = 4r^*$ and we have

$$uf(u) + \frac{p^*-1}{2}|g(u)|^2 = (0.5-\theta)\left(\lambda\mu u^{-\frac{4\theta}{1-2\theta}} - \lambda u^2 - \frac{((2\theta-1)p^*+2)\sigma^2}{4}\right).$$

Since it tends to negative infinite as $u \to \infty$, there always exists a $K > 0$ such that

$$uf(u) + \frac{p^*-1}{2}|g(u)|^2 \leqslant K(1+u^2), \quad u \in [u^*, \infty).$$

That is, the transformed CEV process satisfies Assumption 4.2.2.

Finally, [20] shows that $f(u)$ also satisfies Assumption 4.2.3 for all $r^* > 2$.

Since $1.5r^* < \frac{p^*}{\alpha+2} \wedge \frac{q^*}{\beta+1}$, conditions in Remark 4.3.2 are satisfied. Since

$$\frac{(3\alpha+4)(p^* \wedge q^*)r^*}{2(\beta+1)p^*} = \frac{2r^*}{(\beta+1)} > r,$$

we then have

$$\sup_{t\in[0,T]} \mathbb{E}|y(t) - y_\Delta(t)|^{2r} < C\Delta^r,$$

for all $\Delta \in (0,1]$.

Using $y = x^{0.5-\theta}$ and the mean value theorem, we have

$$
\begin{aligned}
|x(t) - x_\Delta(t)| =& |y(t)^{\frac{2}{1-2\theta}} - y_\Delta(t)^{\frac{2}{1-2\theta}}|, \\
=& |\frac{2}{1-2\theta}||\xi^{\frac{1+2\theta}{1-2\theta}}||y(t) - y_\Delta(t)|, \\
\leqslant& \frac{2}{1-2\theta}|y(t)^{\frac{1+2\theta}{1-2\theta}} + y_\Delta(t)^{\frac{1+2\theta}{1-2\theta}}||y(t) - y_\Delta(t)|,
\end{aligned}
$$

where ξ is a real number between $y(t)$ and $y_\Delta(t)$. Using Lemmas 4.2.1, 4.3.2 and the Hölder inequality, we then have

$$
\begin{aligned}
& \sup_{t\in[0,T]} \mathbb{E}|x(t) - x_\Delta(t)|^r \\
\leqslant& C \sup_{t\in[0,T]} \mathbb{E}\left(|y(t)^{\frac{1+2\theta}{1-2\theta}} - y_\Delta(t)^{\frac{1+2\theta}{1-2\theta}}|^r |y(t) - y_\Delta(t)|^r\right), \\
\leqslant& C \sup_{t\in[0,T]} \left(\mathbb{E}\left(y(t)^{\frac{2(1+2\theta)r}{1-2\theta}} + y_\Delta(t)^{\frac{2(1+2\theta)r}{1-2\theta}}\right)\right)^{1/2} \sup_{t\in[0,T]} \left(\mathbb{E}|y(t) - y_\Delta(t)|^{2r}\right)^{1/2}, \\
\leqslant& C\Delta^{\frac{r}{2}},
\end{aligned}
$$

for $\Delta \in (0,1]$.

In [35] and [36], authors proved strong convergence theory only for the Aït-Sahalia model with $\theta > 4\rho - 3$ and the CEV process with $\theta \in (0.25, 0.5)$. However, our convergence rate results can be applied for the Aït-Sahalia model with $\theta > 2\rho - 1$ and the CEV process with $\theta \in (0, 0.5)$. In other words, our convergence theory is established for more parameter settings.

In addition, we prove that $\mathcal{L}^p$-strong convergence rate orders are $1/2$ for these two important SDE models. However, in [23], theoretical $\mathcal{L}^p$-strong convergence rate orders are only $1/p$, which decays when p becomes large. Therefore, compared to results in [23], the logarithmic truncated EM method has better theoretical $\mathcal{L}^p$-strong convergence rates when p is large.

4.5 Numerical simulations

In this section, we will conduct numerical simulations for the Aït-Sahalia model and the CEV model to support our theoretical results. Let $T = 1$ and $x_0 = 0.01$. We will conduct numerical simulations with 1000 sample paths for stepsizes $\Delta = 2^{-14}, 2^{-13}, 2^{-12}, 2^{-11}$. In view of the fact that there is no analytical solution for the Aït-Sahalia model and the CEV model, we regard the numerical solution with the stepsize $\Delta = 2^{-24}$ as the "exact" solution.

First we consider the Aït-Sahalia model with $a_{-1} = 9$, $a_0 = 2$, $a_1 = 1$, $a_2 = 2$, $\theta = 4$, $\rho = 2$ and $\sigma = 7$. Then we have $\alpha = 3$ and $\beta = 2$. We can then set

$$\varphi(r) = (\sum_{i=-1}^{2} a_i + \sigma^2)(2 + e^{(\alpha \vee (\beta+1))r}) = 63(2 + e^{3r}),$$

and

$$h(\Delta) = 252\Delta^{-1}.$$

Using the linear regression, the experimental error (see Figure 4.5.1) shows that the strong convergence error for the second moment has order about 1.2871, which is close to the proven result in Remark 4.3.2.

We also consider the CEV model with $\lambda = 9$, $\mu = 2$, $\theta = 0.25$ and $\sigma = 7$. Then we have $\alpha = 0$ and $\beta = 4$. We can then set

$$\varphi(r) = \left((0.5-\theta)^2\sigma^2 + (0.5-\theta)(\lambda\mu + \lambda + \frac{2\theta+1}{4}\sigma^2)\right)(2+e^{(\alpha \vee (\beta+1))r}) = \frac{461}{32}(2+e^{5r}),$$

and

$$h(\Delta) = \frac{461}{8}\Delta^{-1}.$$

Using the linear regression, the experimental error (see Figure 4.5.2) shows that the strong convergence error for the second moment has order about 1.2786, which is close to the proven result in Remark 4.3.2.

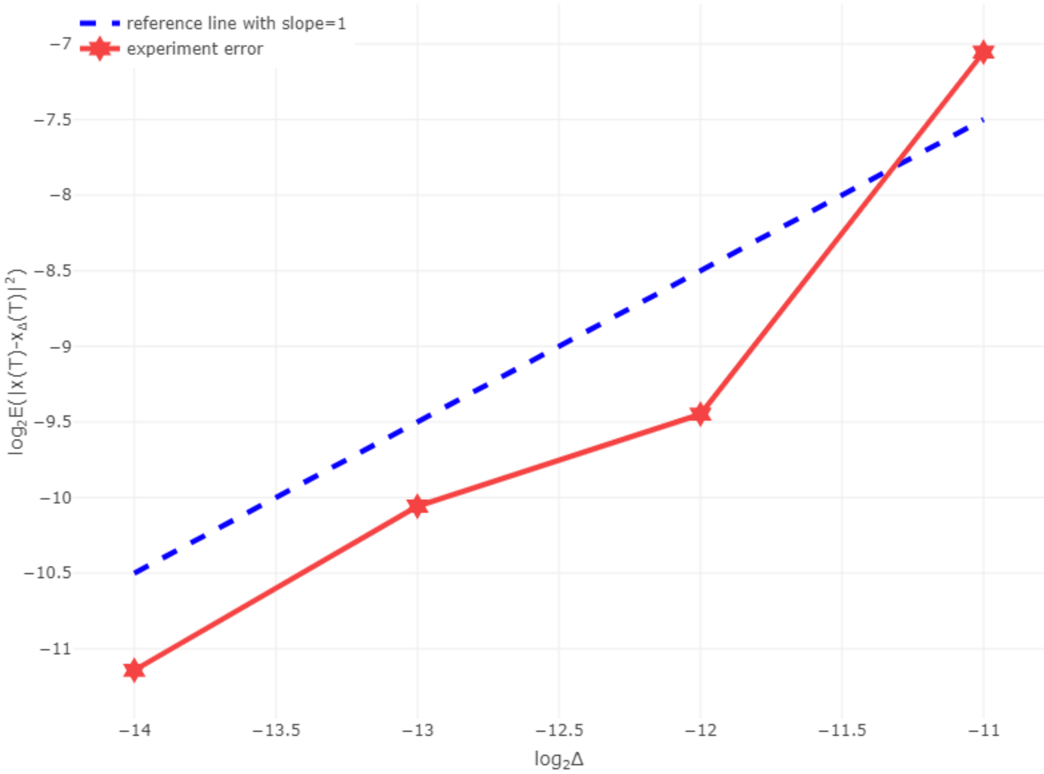

Figure 4.5.1: The $\mathcal{L}^2$-strongly convergence order of the logarithmic truncated EM method for the Aït-Sahalia model.

4.6 Conclusion

In this chapter, we further study the logarithmic truncated EM method. We use weaker assumptions so that the logarithmic truncated EM method can be applied for the Aït-Sahalia model and the CEV model with more general parameter settings. We also prove concrete $\mathcal{L}^p$-strong convergence rate of the logarithmic truncated EM method and our numerical solutions are positive. For the Aït-Sahalia model and the CEV model, convergence rate orders are half which is exactly the optimal convergence rate order of the classical EM method for SDEs with globally Lipschitz coefficients. However, our results excludes SDE models which stay in a given domain, e.g., stochastic SIS epidemic models and the Wright-Fisher model. But we trust that our techniques can be generalized for those SDE models with little modifications.

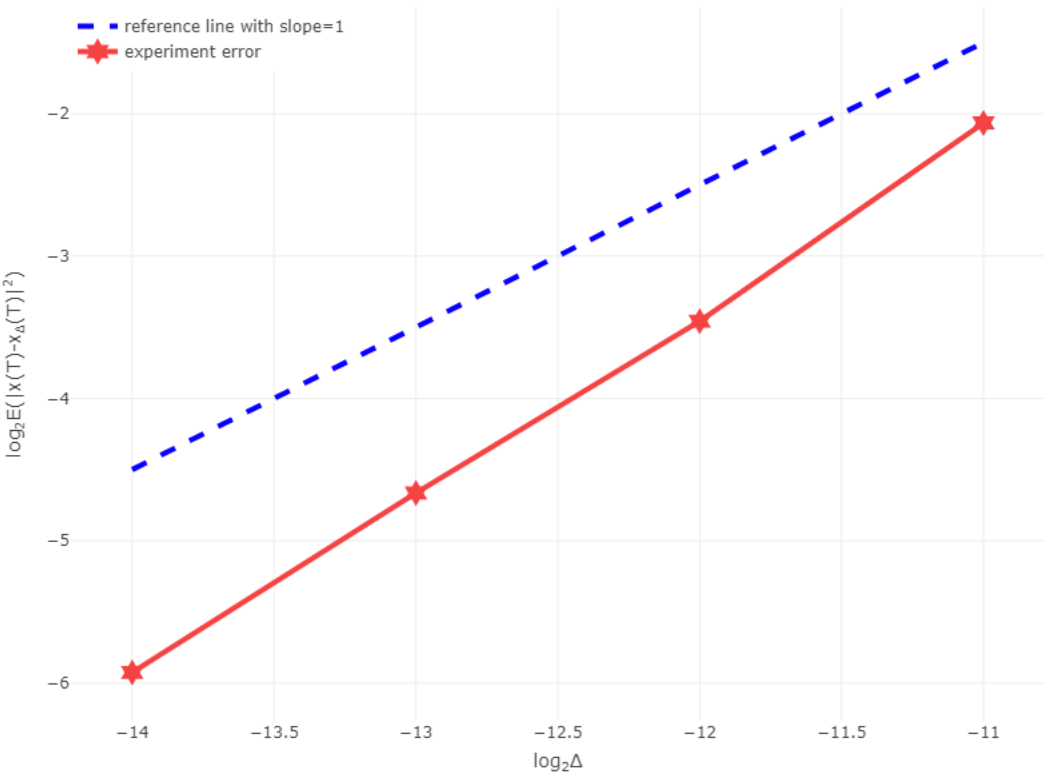

Figure 4.5.2: The $\mathcal{L}^2$-strongly convergence order of the logarithmic truncated EM method for the CEV model.

Chapter 5

Strong order 0.5 convergence of the projected EM method for the CIR model

5.1 Background

In this chapter, we are concerned with the CIR model. It is originally introduced to model the evolution of interest rates (see [39]), and daily used in the financial engineering industry. In addition, we focus on the inaccessible boundary case. To be concrete, we are concerned with the SDE of the form

$$dx(t) = \lambda(\mu - x(t))dt + \sigma x(t)^{\frac{1}{2}} dB(t), \quad x(0) = x_0 > 0, \quad t \geqslant 0, \tag{5.1.1}$$

with a scalar Brownian motion $B(t)$ and parameters $\lambda, \mu, \sigma > 0$. If $2\lambda\mu/\sigma^2 \geqslant 1$, then its solution is strictly positive by the Feller test. In this chapter, we assume that $2\lambda\mu/\sigma^2 \geqslant 1.5$, so that the boundary point zero is inaccessible.

In [40], Broadie and Kaya showed that its increments can be simulated exactly by using a noncentral chi-squared conditional distribution. However, the exact sampling method cannot perform well in some situations. As an example, the exact sampling method is computationally inefficient and potentially restrictive if the CIR model is part of a coupled system of SDEs with correlated driving Brownian motions (see [25]). It happens when the CIR model plays the role of a stochastic volatility process, as in the Heston model. Then the alternative numerical simulation methods are the EM method, the Milstein method and their variants (see [16–30]).

In the last years, the speed of convergence with regard to convergence rates of these modified EM methods and Milstein methods has been intensively studied. In Table 5.1.1, we give a summary of a selection of important EM methods and Milstein methods with their proven strong convergence rates and corresponding parameter ranges, where $\nu = \frac{2\lambda\mu}{\sigma^2}$. To the best of our knowledge, the first non-logarithmic convergence rate result was derived by [16]. [16] introduced the symmetrized EM method and proved that it is $\mathcal{L}^p$-strongly convergent with order one half. However, their parameter conditions are restrictive. In [17], [19], [20] and [21], researchers combine the Lamperti transformation and the backward EM method. Then they developed the drift implicit EM (actually it is an explicit EM method for the CIR model) and proved that it is $\mathcal{L}^p$-strongly convergent with order one for $\nu > 1.5p$. [23] introduced an explicit EM numerical method with truncations and proved its theoretical convergence rate in the $\mathcal{L}^1$-norm. In [26] and [27], Kelly and Lord combined the adaptive stepsize method with the splitting method. They developed the adaptive splitting EM method, whose convergence rate is of order $1/4$. In [22], the truncated Milstein method was proved to have polynomial convergence rates for full parameter range in the $\mathcal{L}^1$-norm. The full truncation EM method is proposed in [24], and it is widely used in practice. In [25], Cozma and Reisinger proved that the full truncation EM method is $\mathcal{L}^p$-strongly convergent with order one half for $\nu > (p+1)$.

Both of these results are valuable and make great contributions to developing efficient EM methods and Milstein methods for the CIR model. In particular, Cozma and Reisinger used a novel numerical analysis method in [25]. They studied the ratio of the difference between the exact solution and the approximation numerical solution to the value of the exact solution. In this chapter, we will combine the projection technique with their novel method to study the general $\mathcal{L}^p$-strong convergence of the projected EM method.

The projected EM method was used to approximate reflected SDEs with globally

Method	Norm	Parameter Regime	Convergence rate order
Classical EM [18], [28]	$\mathcal{L}^1$	Full parameter range	$1/\ln n$
Symmetrized EM [16]	$\mathcal{L}^p, p \geqslant 2$	$\frac{(\nu-1)^2\sigma^2}{(8p-1)\vee(2(4p-1)\sigma)^2} > 8\lambda$	$1/2$
Drift implicit EM [17], [19], [20], [21]	$\mathcal{L}^p, p \geqslant 1$	$\nu > 1.5p$	1
Truncated EM [23]	$\mathcal{L}^1$	$\nu > 2$	$\begin{cases} 1/2 - 1/(\nu+1) & \nu \in (2,3], \\ 1/2 & \nu \in (3,5], \\ 1, & \nu \in (5,\infty) \end{cases}$
Truncated Milstein [22]	$\mathcal{L}^1$	Full parameter range	$0.5 \wedge (\nu - \varepsilon)$
Full truncated EM [24], [25]	$\mathcal{L}^p, p \geqslant 2$	$\nu > (p+1)$	$1/2$
Adaptive splitting EM [26], [27]	$\mathcal{L}^1, \mathcal{L}^2$	$\nu > 2$	$1/4$
Projected EM [29], [30]	$\mathcal{L}^1$	Full parameter range	$\begin{cases} \nu/2 - \varepsilon, & \nu \in (0,1] \\ 1/2 - \varepsilon, & \nu \in (1,\infty) \end{cases}$

Table 5.1.1: Important EM and Milstein methods with their proven convergence rates and corresponding parameter requirements ($\nu = \frac{2\lambda\mu}{\sigma^2}$).

Lipschitz coefficients (see [41], [42] and [43]), and it is proved to be strongly convergent with order $1/2 - \varepsilon$. Recently, some researchers also used it to approximate SDEs and stochastic delay differential equations with superlinearly growing coefficients (see [44], [45] and [46]). Its strong convergence theory for the Wright-Fisher model is established in [47]. In [29] and [30], researchers studied the weak and $\mathcal{L}^1$-strong convergence rate of the projected EM method for the CIR model. In this chapter, we will study the strong convergence in the $\mathcal{L}^p$-norm. As a result, we prove that the projected EM method is $\mathcal{L}^p$-strongly convergent with order one half for $\nu > (p+1)/2$.

This chapter is organized as follows. In section 2, we first introduce notations and present a lemma to show the uniform moment bound of the exact solution to the CIR model. Then we construct the projected EM method and investigate its convergence rates in section 3. In section 4, we will conduct numerical simulations for the CIR model to support our theoretical results. We first conduct an experiment to validate Theorem 5.3.1. In [48] and [49], researchers showed that the $\mathcal{L}^1$-strong convergence of numerical methods using equidistant evaluations of the Brownian process is at best of order $\min(\nu, 1)$. Although not included in our theoretical numerical analysis, we will also conduct numerical experiments for $\nu \in (0, 1.5)$ to numerically show the performance of the projected EM method. We will also compare the performances of the projected EM method and the full truncation EM method. Finally, we make a brief conclusion in section 5.

5.2 Preliminaries

In this chapter, we consider the CIR model:

$$dx(t) = \lambda(\mu - x(t))dt + \sigma x(t)^{\frac{1}{2}} dB(t) \tag{5.2.1}$$

on $t \in [0, T]$ with the initial value $x(0) = x_0 > 0$, where $\lambda, \mu, \sigma, T > 0$. Moreover, we consider cases $\nu = \frac{2\lambda\mu}{\sigma^2} > 1.5$ in this chapter. Throughout this chapter, the Feller condition holds for all theoretical results.

Lemma 5.2.1. *For any $p > -\nu$,*

$$\sup_{t\in[0,T]} \mathbb{E}|x(t)|^p < \infty.$$

Proof. See Lemma 2.1 in [25]. □

5.3 The projected EM method

To define the projected EM method, we first choose a stepsize $\Delta \in (0, 1]$. Then the projected EM numerical solutions $x_\Delta(t)$ are defined by computing the recursion

$$x_\Delta^{k+1}(t) = x_\Delta(t_k) + \lambda(\mu - x_\Delta(t_k))(t - t_k) + \sigma x_\Delta(t_k)^{\frac{1}{2}}(B(t) - B(t_k)),$$
$$x_\Delta(t) = x_\Delta^{k+1}(t) \vee 0,$$

where $x_\Delta(0) = x_0$, $t_k = k\Delta$ and $t \in [t_k, t_{k+1}]$. We also define $\mathbb{N}_\Delta = \{0, 1, \ldots, \lfloor T/\Delta \rfloor\}$, where $\lfloor T/\Delta \rfloor$ is the largest integer which is smaller than T/Δ.

Lemma 5.3.1. *Let $p \geqslant 2$ be arbitrary. There exists a constant $C_1(p)$ such that*

$$\sup_{\Delta\in(0,1]} \sup_{t\in[0,T]} \mathbb{E}|x_\Delta(t)|^p < C_1(p).$$

Let $k \in \mathbb{N}_\Delta$. For any $\Delta \in (0, 1]$, there exists a constant $C_2(p)$, independent of Δ and *k, such that*

$$\sup_{t\in[t_k,t_{k+1}]} \mathbb{E}|x_\Delta^{k+1}(t) - x_\Delta(t_k)|^p < C_2(p)\Delta^{\frac{p}{2}}.$$

Proof. See Lemma 2.8 in [30]. □

Given a stepsize $\Delta \in (0,1]$, for $k \in \mathbb{N}_\Delta$, we define $e_\Delta^{k+1}(t) = x(t) - x_\Delta^{k+1}(t)$ on $t \in [0, t_{k+1}]$ and $e_\Delta(t) = x(t) - x_\Delta(t)$ pn $t \in [0,T]$.

Theorem 5.3.1. *Let $2 \leqslant p < (2\nu - 1)$, where $\nu = \frac{2\lambda\mu}{\sigma^2} > 1.5$ in this chapter. Then there exists a constant C such that, for all $\Delta \in (0,1]$,*

$$\sup_{t\in[0,T]} \mathbb{E}|e_\Delta(t)|^q \leqslant C\Delta^{q/2},$$

where $q \in (0,p)$.

Proof. Let $k \in \mathbb{N}$. Then we define the stopping time $\tau_k^n = \inf\{t \in [t_k, t_{k+1}] : x(t) < 1/n\}$ for $n \in \mathbb{N}_+$, and set $\tau_k^n = \infty$ if it is an empty set. In this proof, we use C to stand for generic positive real constants, independent of k, n and Δ, and its values may change between occurrences.

Let $\varepsilon > 0$ be sufficiently small such that $\nu > 0.5(p+1) + 2\varepsilon$. Let $\beta = 0.5(p-1) + \varepsilon$.

Using the Itô formula, we then have

$$
\begin{aligned}
& x(t\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t\wedge\tau_k^n)|^p \\
=& x(t_k)^{-\beta}|e_\Delta^{k+1}(t_k)|^p - \beta\lambda\mu\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^p ds + \beta\lambda\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta}|e_\Delta^{k+1}(s)|^p ds \\
& - p\lambda\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}e_\Delta^{k+1}(s)(x(s)-x_\Delta(t_k))ds \\
& + \frac{p(p-1)\sigma^2}{2}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})^2 ds \\
& + \frac{\beta(\beta+1)\sigma^2}{2}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^p ds \\
& - p\beta\sigma^2\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-\frac{1}{2}}|e_\Delta^{k+1}(s)|^{p-2}e_\Delta^{k+1}(s)(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})ds \\
& - \beta\sigma\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-\frac{1}{2}}|e_\Delta^{k+1}(s)|^p dB(s) \\
& + p\sigma\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}e_\Delta^{k+1}(s)(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})dB(s), \qquad (5.3.1)
\end{aligned}
$$

for all $t\in[t_k,t_{k+1}]$.

Using the Young inequality, we then have

$$
\begin{aligned}
& - p\lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}e_\Delta^{k+1}(s)(x(s)-x_\Delta(t_k)) \\
=& - p\lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}e_\Delta^{k+1}(s)(x(s)-x_\Delta^{k+1}(s)) \\
& - p\lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}e_\Delta^{k+1}(s)(x_\Delta^{k+1}(s)-x_\Delta(t_k)), \\
\leqslant& - p\lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^p + p\lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-1}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|, \\
\leqslant& - p\lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^p \\
& + (p-1)\lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^p + \lambda x(s)^{-\beta}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|^p, \\
\leqslant& - \lambda x(s)^{-\beta}|e_\Delta^{k+1}(s)|^p + \lambda x(s)^{-\beta}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|^p, \qquad (5.3.2)
\end{aligned}
$$

for all $s\in[t_k,t_{k+1}\wedge\tau_k^n]$.

Substituting (5.3.2) into (5.3.1) and taking expectations on both sides of (5.3.1), we deduce

$$
\begin{aligned}
&\mathbb{E}\left(x(t\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t\wedge\tau_k^n)|^p\right)\\
\leqslant&\mathbb{E}\left(x(t_k\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t_k\wedge\tau_k^n)|^p\right)-0.5\beta\sigma^2\left(\nu-\beta-1\right)\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n}x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^p ds\\
&+(\beta-1)\lambda\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n}x(s)^{-\beta}|e_\Delta^{k+1}(s)|^p ds\\
&+\lambda\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n}x(s)^{-\beta}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|^p ds+J_{k,n}(t),
\end{aligned}
$$

where

$$
\begin{aligned}
J_{k,n}(t)=&\frac{p(p-1)\sigma^2}{2}\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n}x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})^2 ds\\
&-p\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n}x(s)^{-\beta-\frac{1}{2}}|e_\Delta^{k+1}(s)|^{p-2}e_\Delta^{k+1}(s)(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})ds.
\end{aligned}
$$

We have

$$
\begin{aligned}
&e_\Delta^{k+1}(s)(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})\\
=&\Big((x(s)-x_\Delta(t_k))-(x_\Delta^{k+1}(s)-x_\Delta(t_k))\Big)\left(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}}\right),\\
=&(x(s)^{\frac{1}{2}}+x_\Delta(t_k)^{\frac{1}{2}})(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})^2-(x_\Delta^{k+1}(s)-x_\Delta(t_k))(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}}),\\
\geqslant&x(s)^{\frac{1}{2}}(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})^2-|x_\Delta^{k+1}(s)-x_\Delta(t_k)||x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}}|,
\end{aligned}
$$

for $s\in[t_k,t_{k+1}\wedge\tau_k^n]$.

Substituting the above formula into $J_{k,n}(t)$ and using the Young inequality, we have

$$
\begin{aligned}
&J_{k,n}(t)\\
\leqslant& I_{k,n}(t)+p\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}x(s)^{\frac{1}{2}}|e_\Delta^{k+1}(s)|^{p-2}|x_\Delta^{k+1}(s)-x_\Delta(t_k)||x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}}|ds,\\
\leqslant& I_{k,n}(t)\\
&+p\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^{p-2}|x_\Delta^{k+1}(s)-x_\Delta(t_k)||x(s)^{\frac{1}{2}}+x_\Delta(t_k)^{\frac{1}{2}}||x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}}|ds,\\
\leqslant& I_{k,n}(t)+p\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^{p-2}|x_\Delta^{k+1}(s)-x_\Delta(t_k)||x(s)-x_\Delta(t_k)|ds,\\
\leqslant& I_{k,n}(t)+p\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^{p-1}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|ds\\
&+p\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^{p-2}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|^2ds,\\
\leqslant& I_{k,n}(t)+(2p-3)\delta\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^p ds\\
&+(\delta^{-(p-1)}+2\delta^{-(p-2)/2})\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|^p ds,
\end{aligned}
$$

where $\delta>0$ and

$$
I_{k,n}(t)=\frac{p(p-2\beta-1)\sigma^2}{2}\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})^2ds.
$$

Then we have

$$
\begin{aligned}
&\mathbb{E}\left(x(t\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t\wedge\tau_k^n)|^p\right)\\
\leqslant&\mathbb{E}\left(x(t_k\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t_k\wedge\tau_k^n)|^p\right)+(\beta-1)\lambda\mathbb{E}\int_{t_k}^{t} x(s\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(s\wedge\tau_k^n)|^p ds\\
&+\left((2p-3)\delta+\frac{\beta+1-\nu}{2}\right)\beta\sigma^2\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta-1}|e_\Delta^{k+1}(s)|^p ds\\
&+\frac{p(p-2\beta-1)\sigma^2}{2}\mathbb{E}\int_{t_k}^{t\wedge\tau_k^n} x(s)^{-\beta}|e_\Delta^{k+1}(s)|^{p-2}(x(s)^{\frac{1}{2}}-x_\Delta(t_k)^{\frac{1}{2}})^2ds\\
&+\lambda\mathbb{E}\int_{t_k}^{t} x(s)^{-\beta}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|^p ds\\
&+(\delta^{-(p-1)}+2\delta^{-(p-2)/2})\beta\sigma^2\mathbb{E}\int_{t_k}^{t} x(s)^{-\beta-1}|x_\Delta^{k+1}(s)-x_\Delta(t_k)|^p ds, \qquad (5.3.3)
\end{aligned}
$$

for all $t \in [t_k, t_{k+1}]$.

Since $\nu > 0.5(p+1) + 2\varepsilon$ and $\beta = 0.5(p-1) + \varepsilon$, we have

$$(2p-3)\delta + \frac{\beta + 1 - \nu}{2} < 0 \quad \text{and} \quad p - 2\beta - 1 < 0$$

by letting $\delta > 0$ be sufficiently small. Using the Hölder inequality, Lemmas 5.2.1 and 5.3.1, we have

$$\begin{aligned}
&\mathbb{E}\left(x(s)^{-(\beta+\rho)}|x_\Delta^{k+1}(s) - x_\Delta(t_k)|^p\right) \\
\leqslant &\left(\mathbb{E}\left(x(s)^{-(\beta+1+\varepsilon)}\right)\right)^{\frac{\beta+\rho}{\beta+1+\varepsilon}} \left(\mathbb{E}\left(|x_\Delta^{k+1}(s) - x_\Delta(t_k)|^{\frac{p(\beta+1+\varepsilon)}{1+\varepsilon-\rho}}\right)\right)^{\frac{1+\varepsilon-\rho}{\beta+1+\varepsilon}}, \\
\leqslant &C\Delta^{\frac{p}{2}}, \qquad (5.3.4)
\end{aligned}$$

where $\rho \in \{0, 1\}$. Substituting (5.3.4) into (5.3.3), we have

$$\begin{aligned}
&\mathbb{E}\left(x(t \wedge \tau_k^n)^{-\beta}|e_\Delta^{k+1}(t \wedge \tau_k^n)|^p\right) \\
\leqslant &\mathbb{E}\left(x(t_k \wedge \tau_k^n)^{-\beta}|e_\Delta^{k+1}(t_k \wedge \tau_k^n)|^p\right) + (\beta - 1)\lambda\mathbb{E}\int_{t_k}^t x(s \wedge \tau_k^n)^{-\beta}|e_\Delta^{k+1}(s \wedge \tau_k^n)|^p ds + C\Delta^{\frac{p+2}{2}},
\end{aligned}$$

for all $t \in [t_k, t_{k+1}]$.

Then the Gronwall inequality implies that

$$\begin{aligned}
&\sup_{t \in [t_k, t_{k+1}]} \mathbb{E}\left(x(t \wedge \tau_k^n)^{-\beta}|e_\Delta^{k+1}(t \wedge \tau_k^n)|^p\right) \\
\leqslant &\left(\mathbb{E}\left(x(t_k \wedge \tau_k^n)^{-\beta}|e_\Delta^{k+1}(t_k \wedge \tau_k^n)|^p\right) + C\Delta^{\frac{p+2}{2}}\right) e^{((\beta-1)\lambda \vee 0)\Delta}.
\end{aligned}$$

Since $x_\Delta^{k+1}(t_k) = x_\Delta(t_k) = x_\Delta^k(t_k) \vee 0$, we have

$$|e_\Delta^{k+1}(t_k \wedge \tau_k^n)| = |e_\Delta(t_k \wedge \tau_k^n)| \leqslant |e_\Delta^k(t_k \wedge \tau_k^n)|.$$

Then we have

$$\begin{aligned}
&\sup_{t\in[t_k,t_{k+1}]} \mathbb{E}\left(x(t\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t\wedge\tau_k^n)|^p\right)\\
&\leqslant\left(\mathbb{E}\left(x(t_k\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t_k\wedge\tau_k^n)|^p\right)+C\Delta^{\frac{p+2}{2}}\right)e^{((\beta-1)\lambda\vee 0)\Delta},\\
&=\left(\mathbb{E}\left(x(t_k\wedge\tau_k^n)^{-\beta}|e_\Delta(t_k\wedge\tau_k^n)|^p\right)+C\Delta^{\frac{p+2}{2}}\right)e^{((\beta-1)\lambda\vee 0)\Delta},\\
&\leqslant\left(\mathbb{E}\left(x(t_k\wedge\tau_k^n)^{-\beta}|e_\Delta^{k}(t_k\wedge\tau_k^n)|^p\right)+C\Delta^{\frac{p+2}{2}}\right)e^{((\beta-1)\lambda\vee 0)\Delta},\\
&\leqslant\left(\sup_{t\in[t_{k-1},t_k]}\mathbb{E}\left(x(t\wedge\tau_k^n)^{-\beta}|e_\Delta^{k}(t\wedge\tau_k^n)|^p\right)+C\Delta^{\frac{p+2}{2}}\right)e^{((\beta-1)\lambda\vee 0)\Delta}.
\end{aligned}$$

By induction, we have

$$\sup_{t\in[t_k,t_{k+1}]} \mathbb{E}\left(x(t\wedge\tau_k^n)^{-\beta}|e_\Delta^{k+1}(t\wedge\tau_k^n)|^p\right)\leqslant C\left((k+1)\Delta\right)e^{((\beta-1)\lambda\vee 0)(k+1)\Delta}\Delta^{\frac{p}{2}},$$

and therefore

$$\sup_{t\in[0,T]} \mathbb{E}\left(x(t\wedge\tau_k^n)^{-\beta}|e_\Delta(t\wedge\tau_k^n)|^p\right)\leqslant C\Delta^{\frac{p}{2}}.$$

Letting $n\to\infty$, we then have

$$\sup_{t\in[0,T]} \mathbb{E}\left(x(t)^{-\beta}|e_\Delta(t)|^p\right)\leqslant C\Delta^{\frac{p}{2}}.$$

Let $q\in(0,p)$. Finally, the Hölder inequality and Lemma 5.2.1 imply that

$$\begin{aligned}
\sup_{t\in[0,T]}\mathbb{E}|e_\Delta(t)|^q&=\sup_{t\in[0,T]}\mathbb{E}|x(t)|^{\frac{\beta q}{p}}|x(t)|^{-\frac{\beta q}{p}}|e_\Delta(t)|^q\\
&\leqslant\sup_{t\in[0,T]}\left(\left(\mathbb{E}|x(t)|^{\frac{\beta q}{p-q}}\right)^{\frac{p-q}{p}}\left(\mathbb{E}\left(x(t)^{-\beta}|e_\Delta(t)|^p\right)\right)^{\frac{q}{p}}\right),\\
&\leqslant\left(\sup_{t\in[0,T]}\mathbb{E}|x(t)|^{\frac{\beta q}{p-q}}\right)^{\frac{p-q}{p}}\left(\sup_{t\in[0,T]}\mathbb{E}\left(x(t)^{-\beta}|e_\Delta(t)|^p\right)\right)^{\frac{q}{p}},\\
&\leqslant C\Delta^{\frac{q}{2}}.
\end{aligned}$$

□

5.4 Numerical simulations

In this section, we will conduct numerical simulations for the CIR model (5.2.1) to support our theoretical results. We let $T = 1$ and use the plain Monte Carlo method. First, we would like to estimate the rate of the decay of the errors. We will conduct numerical simulations with 1000 sample paths for step sizes $\Delta = 2^{-11}, 2^{-10}, 2^{-9}, 2^{-8}$. We regard the truncated Milstein numerical solution (see [22]) with the step size $\Delta = 2^{-18}$ as the "exact" solution. We will show that experimental $\mathcal{L}^p$-strong convergence errors have about order $p/2$ in Example 5.4.1. We then perform the test for $\nu \in (0, 1.5)$ in Example 5.4.2.

Example 5.4.1. In this example, we let $p = 4$, $x_0 = 0.001$, $\lambda = 3$, $\mu = 7$ and $\sigma = 4$. We have $\nu = 2.625$. Experimental errors of the projected EM method (see Figure 5.4.1) show that the $\mathcal{L}^4$-strong convergence rate has order about 2, which validate Theorem 5.3.1. The $\mathcal{L}^4$-strong convergence of the full truncation EM method has not been proved in [25]. However, the numerical experiment shows that there is almost no difference between the projected EM method and the full truncation EM method.

Example 5.4.2. In this example, we let $p = 1$, $x_0 = 0$, $\lambda = 3$, $\mu = 4$ and $\sigma = 11$. We have $\nu \approx 0.1983 < 1.5$ which is excluded in our theory. Experimental errors (see Figure 5.4.2) show that the $\mathcal{L}^4$-strong convergence rate has order about ν. Different from Example 5.4.1, the error constants of the projected EM method are now smaller than that of the full truncation EM method.

Now we conduct numerical simulations for varying ν. We still let $T = 1$ and regard the truncated Milstein numerical solution with the step size $\Delta = 2^{-20}$ as the "exact" solution. To approximate the strong convergence rate order, we use the linear regression method.

Example 5.4.3. In this example, we let $p = 1$, $x_0 = 0$, $\lambda = 3$, $\mu = 2$ and $\nu \in \{0.05, 0.1, 0.15, 0.3, \ldots, 1.5\}$. The numerical experiments show that the $\mathcal{L}^1$-strong convergence rate has order about $\min(\nu, 1)$, which validate the result in [48] and [49].

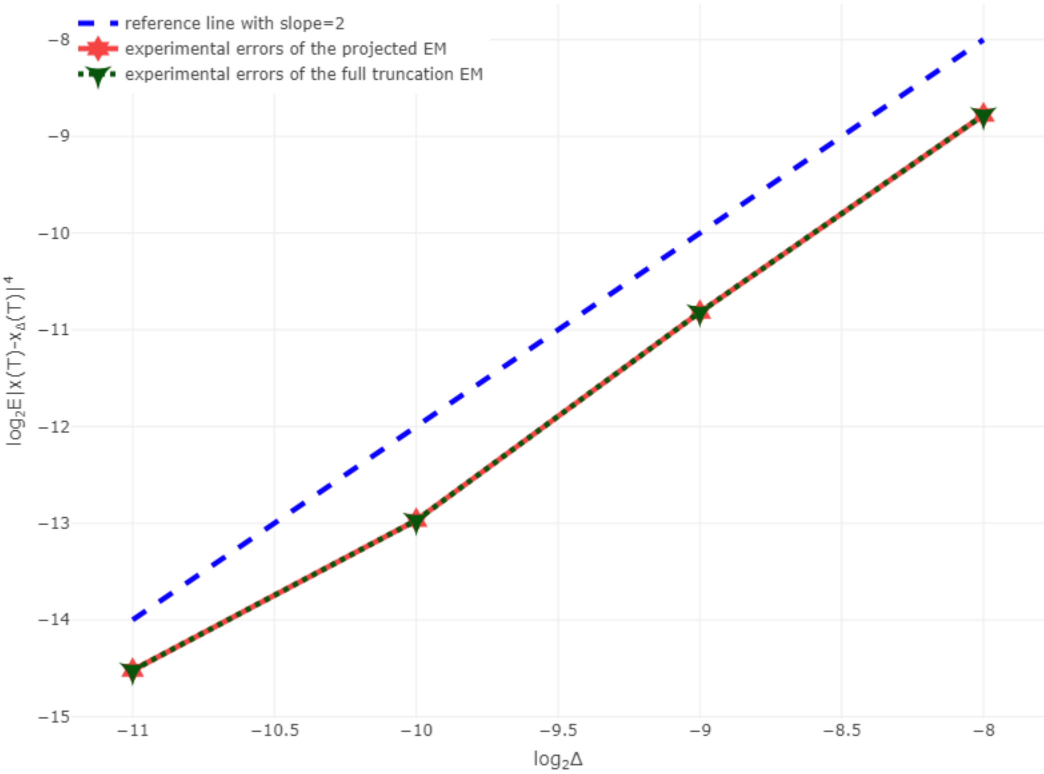

Figure 5.4.1: The $\mathcal{L}^4$-strong convergence errors of the projected EM method and the full truncation EM method.

5.5 Conclusion

In this chapter, we combine the projection technique with Cozma and Reisinger's novel numerical analysis technique to study the $\mathcal{L}^p$-strong convergence of the projected EM method for the CIR model. We show that the projected EM method is $\mathcal{L}^p$-strongly convergent with order one half for $\nu > (p+1)/2$. Compared to results in [29] and [30], our strong convergence theory is concerned with the general $\mathcal{L}^p$-strong convergence. This chapter also answered the question in the conclusion of [25]. The projection technique can relax the condition on the parameters for the strong convergence theory of the full truncation EM method without losing the convergence.

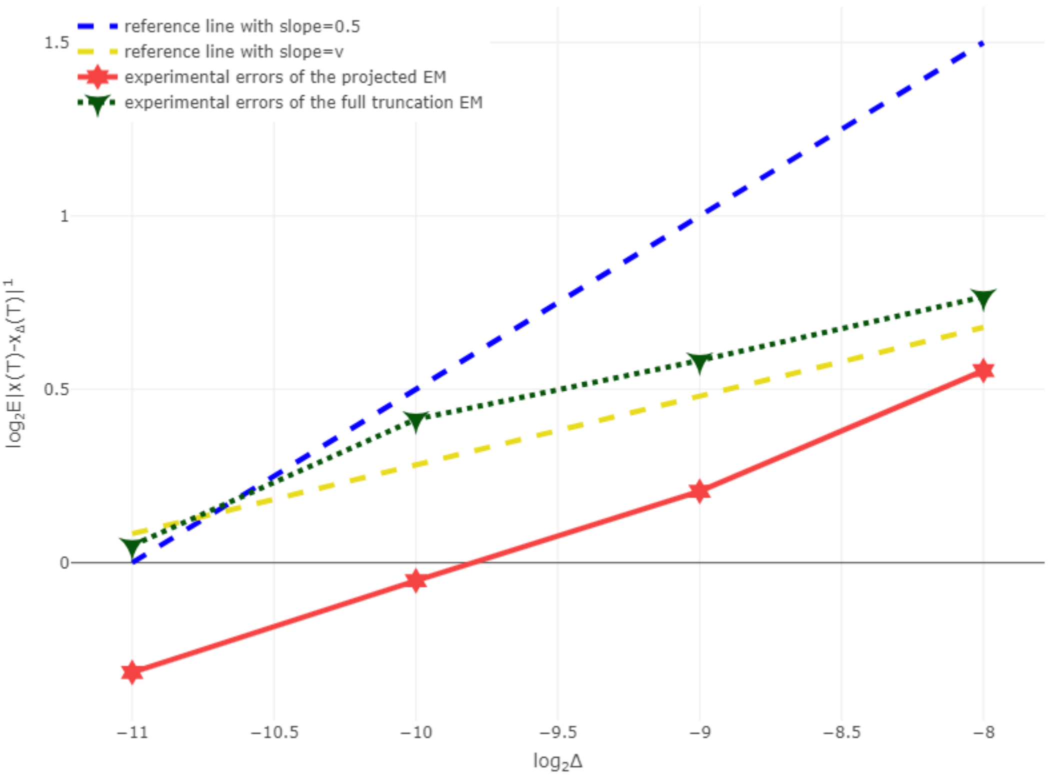

Figure 5.4.2: The $\mathcal{L}^1$-strong convergence errors of the projected EM method and the full truncation EM method.

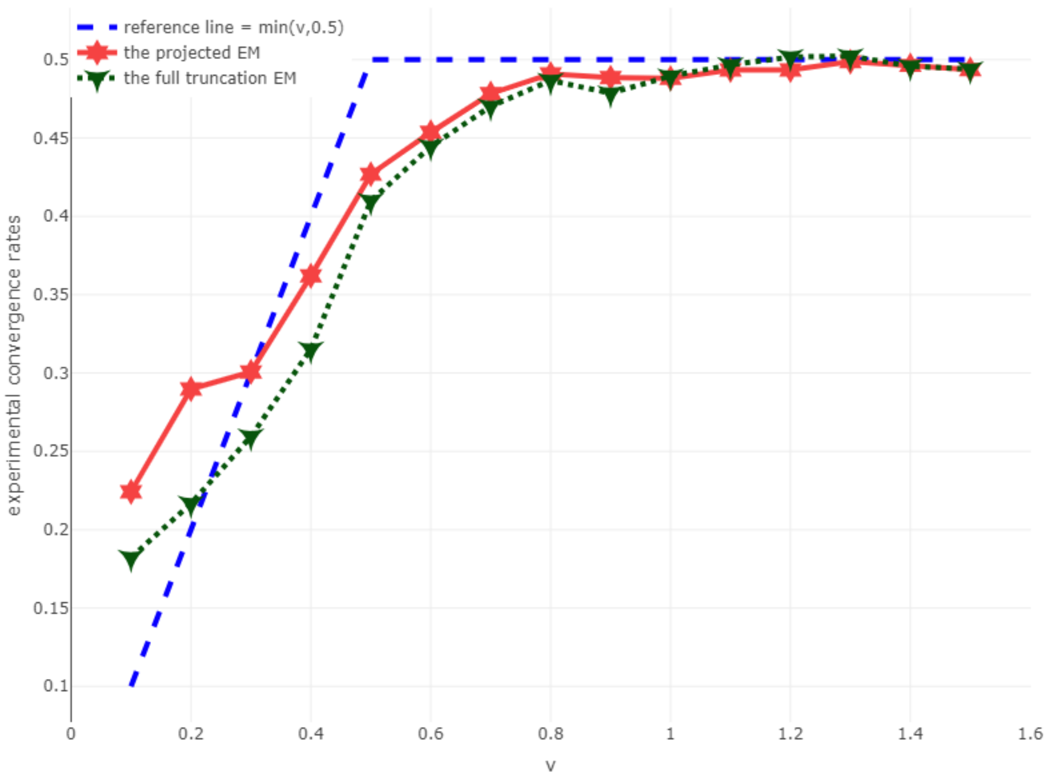

Figure 5.4.3: The $\mathcal{L}^1$-strong convergence rates for varying ν.

Chapter 6

Strong convergence order one of the projected Euler-Maruyama method for scalar SDEs defined in the positive domain

6.1 Introduction

In 2014, Neuenkirch and Szpruch established the drift-implicit EM method [20] for a series of scalar stochastic differential equations (SDEs) which take values in a domain. The drift-implicit EM method covers many important SDE models in finance or biology, e.g., the CIR, the CEV model, the WF diffusion and so on. The drift-implicit EM method has the following advantages:

i. The drift-implicit EM method is $\mathcal{L}^p$-strongly convergent with order one;

ii. The drift-implicit EM numerical solution also takes values in the same domain which the exact solution takes values in.

However, expensive computational cost is required for implementation of this implicit numerical method.

There also are many existing explicit EM methods for these SDE models (see Table 6.1.1, where detailed parameter settings are explained in section 4), but most of them are only convergent with order one half. Some of them prove strong order one convergence, but only in at most $\mathcal{L}^2$-norm (see [23], [50], [51], [52]) or for certain parameter settings (see [51, 52]). Therefore, it is still meaningful to develop an explicit EM method with convergence of order one for these SDEs. The main aim of this chapter is to introduce an explicit EM method, called the projected EM method, to replace the drift-implicit EM method to some extent. We will show that the projected EM method is also $\mathcal{L}^p$-strongly convergent with order one for those SDE models with a wide parameter range.

Model	Method	Norm	Convergence rate order	Parameter range
The Aït-Sahalia model	Lamperti truncated EM ([23])	$\mathcal{L}^1$	1	$\theta+1>2\rho$
	Truncated EM ([53], [54])	$\mathcal{L}^p$	$1/(2p)$	$\theta+1>2\rho$
	Logarithmic truncated EM ([35], [36], [2])	$\mathcal{L}^p$	1/2	$\theta+1>2\rho$
	Exponential tamed EM ([50])	$\mathcal{L}^2$	1	$\theta+1>2\rho$
	Semi-discrete EM ([51])	$\mathcal{L}^p$	1/2	$\theta+1>2\rho$
			1	$\theta=2, \rho=1.5, a_2/\sigma^2 \geqslant (2p-0.5)$
	Positivity-preserving tamed EM ([55])	$\mathcal{L}^2$	1/2	$\theta+1>2\rho$ $\theta+1=2\rho, a_2/\sigma^2 \geqslant 2\theta-0.5$
	Projected EM	$\mathcal{L}^p$	1	$\theta+1>2\rho$ $\theta+1=2\rho, \left(\frac{2\rho p}{\rho-1}+2\right) \vee 6p < \frac{2a_2/\sigma^2+1}{\rho-1}$
The CEV model	Reflected EM ([16])	$\mathcal{L}^p$	1/2	full parameter range
	Logarithmic truncated EM ([35], [36], [2])	$\mathcal{L}^p$	1/2	full parameter range
	Projected EM	$\mathcal{L}^p$	1	full parameter range
The Heston-3/2 volatility model	Lamperti truncated EM ([23])	$\mathcal{L}^1$	1	$a_1/a_3^2 > 1.5$
	Splitting Milstein-type ([52])	$\mathcal{L}^2$	1	$a_1/a_3^2 > 2.5$
	Projected EM	$\mathcal{L}^p$	1	$a_1/a_3^2 > (3p-1)/2$

Table 6.1.1: Existing explicit EM methods for the CEV model, the Aït-Sahalia model and the Heston-3/2 volatility model.

To use the projected EM method, we have to apply the Lamperti transformation to the original SDE at first (see section 3 in [20] for details). Then the transformed SDEs have constant diffusion coefficients, which is critical to prove the strong order one convergence. The drift coefficients of some transformed SDEs will contain reciprocal parts (see section 4 for examples), e.g., the CEV model, the Aït-Sahalia model and the Heston-3/2 volatility model. Therefore, finite inverse moments of the numerical solutions may be necessary to prove the strong convergence rate order one.

There have been some research papers which are concerned with explicit EM methods for the Lamperti transformed SDEs (e.g., see [23], [50], [51], [56] and [57]). In [56] and [57], researchers apply the truncated EM method [12] and [13] for the transformed stochastic SIS epidemic model. However, the transformed stochastic SIS epidemic model does not have reciprocal coefficients parts. In [23], researchers use some tricks to avoid requiring finite inverse moments of the numerical solutions. Nevertheless, they can only use those tricks to prove the strong convergence rate order one in $\mathcal{L}^1$-norm. In [23], [51] and [54], the researchers did not consider finite inverse moments of the numerical solutions either. Reciprocal parts are multiplied by an extremely small quantity to guarantee the expectation of the product is finite. Then they have to make a balance to derive an optimal convergence rate.

The projected EM has been studied in [29], [30], [41], [42], [43], [44], [45] and [46], but none of them are concerned with finite inverse moments and applications to the above SDEs. Finite inverse moments of the numerical solutions have been studied in [2], [35] and [36], but an additional logarithmic transformation will generate a non-constant diffusion coefficient. Then it may be hard to prove the strong order one convergence. Therefore, the key innovation point of this chapter is that we prove finite inverse moments of the projected EM numerical solutions. We then prove first strong order convergence in a more general $\mathcal{L}^p$-norm for the above SDEs.

This chapter is organized as follows. In section 2, we first introduce notations, assumptions and establish some useful lemmas. Then we construct the projected EM method and investigate its inverse moments and convergence rates in section 3. In section 4, we will illustrate that the projected EM method can be applied for the CEV model, the Heston-3/2 volatility model and the Aït-Sahalia model. In section 5, we then conduct numerical simulations for examples in section 4. Finally, we make a brief conclusion in section 6.

6.2 Notations and preliminaries

As before, we set $\inf \emptyset = \infty$, where $\emptyset$ is an empty set. Moreover, we will use C to stand for generic positive real numbers which are dependent on T, α, β, H, K_1, etc., but independent of Δ, t, s, k and m (used below) and its values may change between occurrences.

In this chapter, we consider a scalar SDE

$$dx(t) = f(x(t))dt + \varsigma dB(t) \tag{6.2.1}$$

on $t \in [0, T]$ with $\varsigma > 0$ and the initial value $x(0) = x_0 \in \mathbb{R}_+$, where T is a fixed positive number and $f : \mathbb{R}_+ \to \mathbb{R}$ is Borel measurable.

We first impose three hypotheses.

Assumption 6.2.1. Assume that the drift coefficient f is twice differentiable. Assume that there exist real numbers $K_1, K_2 > 0$, $\alpha > 0$ and $\beta \geqslant 2$ such that

$$|f'(x)| \leqslant K_1 \left(1 + x^{\alpha} + x^{-\beta}\right), \quad \text{and} \quad |f''(x)| \leqslant K_2 \left(1 + x^{\alpha-1} + x^{-\beta-1}\right),$$

for all $x \in \mathbb{R}_+$.

Assumption 6.2.2. Assume that there exists a positive real number $r \geqslant 1$ such that

$$\liminf_{x \downarrow 0^+} x f(x) > (3(\beta - 1)r + 0.5)\, \varsigma^2.$$

Assumption 6.2.3. Assume that there exists a positive real number $H > 0$ such that

$$(x - y)(f(x) - f(y)) \leqslant H|x - y|^2,$$

for all $x, y \in \mathbb{R}_+$.

Remark 6.2.1. First, we let $1 < x$.

$$
\begin{aligned}
f(x) =& f(1) + \int_1^x f'(z)dz, \\
\leqslant& f(1) + K_1 \int_1^x (2 + z^{\alpha})dz, \\
=& f(1) + 2K_1(x-1) + \frac{K_1(x^{\alpha+1}-1)}{\alpha+1}, \\
\leqslant& \left(|f(1)| + 2K_1 + \frac{K_1}{\alpha+1}\right)(1 + x^{\alpha+1}).
\end{aligned}
$$

Now we let $0 < x < 1$. We then have

$$
\begin{aligned}
f(x) =& f(1) - \int_x^1 f'(z)dz, \\
\leqslant& f(1) + K_1 \int_x^1 (2 + z^{-\beta})dz, \\
=& f(1) + 2K_1(1-x) + \frac{K_1(x^{-\beta+1}-1)}{\beta-1}, \\
\leqslant& \left(|f(1)| + 2K_1 + \frac{K_1}{\beta-1}\right)(1 + x^{-\beta+1}).
\end{aligned}
$$

Therefore, Assumption 6.2.1 implies

$$
|f(x)| \leqslant C_0(1 + x^{\alpha+1} + x^{-\beta+1}), \quad \forall x \in \mathbb{R}_+,
$$

where $C_0 = |f(1)| + 2K_1 + \frac{K_1}{\alpha+1} + \frac{K_1}{\beta-1}$.

Remark 6.2.2. In the rest of this chapter, we let $q = 6(\beta - 1)r$. We then have

$$
\liminf_{x \downarrow 0^+} x f(x) > 0.5(q+1)\varsigma^2.
$$

Let $\varepsilon_0 > 0$ be sufficiently small such that

$$
\liminf_{x \downarrow 0^+} x f(x) > \frac{(q+1)\varsigma^2}{2(1-\varepsilon_0)}.
$$

We also let $p > (q+2) \vee 6(\alpha+1)r$ be sufficiently large.

From Assumption 6.2.3, we have

$$xf(x) \leqslant H|x-1|^2 - f(1) + f(x) + f(1)x,$$

for $x \in \mathbb{R}_+$. When $x > 2$, we have

$$0.5f(x)/x \leqslant \left(1 - \frac{1}{x}\right)\frac{f(x)}{x} \leqslant \frac{H|x-1|^2 - f(1) + f(1)x}{x^2}.$$

It follows that

$$\limsup_{x\uparrow+\infty} f(x)/x \leqslant 2H.$$

Therefore, there exist positive real numbers $x^* \in (0,1)$ and K_3 such that

$$\begin{cases} (1-\varepsilon_0)xf(x) - (q+1)\varsigma^2/2 \geqslant 0, & x \in (0,x^*), \\ f(x) \leqslant K_3 x, & x \in [x^*,\infty). \end{cases}$$

The next lemma shows that SDE (6.2.1) has a unique strong solution on $[0,T]$. Moreover, this solution takes values in the positive domain, i.e.,

$$\Pr(x(t) \in (0,\infty), \forall t \in [0,T]) = 1.$$

Therefore, as above assumptions show, we only need to check properties of the drift coefficient for positive real numbers.

Lemma 6.2.1. *Assume that Assumptions 6.2.1, 6.2.2 and 6.2.3 hold. Then SDE* (6.2.1) *has a unique positive strong solution on* $[0,T]$ *such that*

$$\sup_{t\in[0,T]} \mathbb{E}|x(t)|^{2p} \leqslant C, \quad \text{and} \quad \sup_{t\in[0,T]} \mathbb{E}|x(t)|^{-q} \leqslant C,$$

where $r \geqslant 1$, $q = 6(\beta-1)r$ *and* $p > (q+2) \vee 6(\alpha+1)r$ *and they are fixed in Assumption 6.2.2 and Remark 6.2.2.*

Proof. Using Remark 6.2.2, we have $\alpha \vee (\beta+1) \leqslant 2p+q$. Then this is an application of Lemma 2.1 in [2]. □

We also establish a stronger lemma which will be used in section 4. In the proof, we use different ways to estimate upper bounds of

$$x(t)f(x(t)) + (p-1)\varsigma^2/2 \quad \text{and} \quad x(t)f(x(t)) - (q-1)\varsigma^2/2,$$

based on the value of $x(t)$. This technique will be frequently used in the rest of this chapter. For the sake of convenience, we simply write

$$\{x(t) \in [x^*, \infty)\} = \{\omega \in \Omega \mid x(t,\omega) \in [x^*, \infty)\},$$

and it is an $\mathcal{F}_t$-measurable subset of the probability space $(\Omega, \mathcal{F}, \Pr)$. Similar subset notations will also be frequently used in section 3.

Lemma 6.2.2. *Assume that Assumptions 6.2.1, 6.2.2 and 6.2.3 hold. Then there exists a constant C, depending on p, q, T, α, β, H, K_1, K_2 and ς, such that*

$$\mathbb{E}\left(\sup_{t\in[0,T]} |x(t)|^p\right) \leqslant C, \quad \textit{and} \quad \mathbb{E}\left(\sup_{t\in[0,T]} |x(t)|^{-q+2}\right) \leqslant C,$$

where $r \geqslant 1$, $q = 6(\beta-1)r$, $p > (q+2) \vee 6(\alpha+1)r$.

Proof. Define $\tau_k = \inf\{t \mid x(t) < 1/k\}$ for $k \in \mathbb{N}_+$. Using the Itô formula, we have

$$\begin{aligned}
|x(t\wedge\tau_k)|^p + |x(t\wedge\tau_k)|^{-q+2} = \big(&|x_0|^p + |x_0|^{-q+2}\big) \\
&+ p\int_0^{t\wedge\tau_k} |x(s)|^{p-2}\big(x(s)f(x(s)) + (p-1)\varsigma^2/2\big)\,ds \\
&+ \varsigma p\int_0^{t\wedge\tau_k} |x(s)|^{p-1}dB(s) \\
&- (q-2)\int_0^{t\wedge\tau_k} |x(s)|^{-q}\big(x(s)f(x(s)) - (q-1)\varsigma^2/2\big)\,ds \\
&- \varsigma(q-2)\int_0^{t\wedge\tau_k} |x(s)|^{-(q-1)}dB(s),
\end{aligned}$$

for all $t \in [0,T]$.

Using the Young inequality, Remarks 6.2.1 and 6.2.2, we have

$$\begin{aligned}
&|x(t)|^{p-2}\left(x(t)f(x(t))+(p-1)\varsigma^2/2\right)\\
\leqslant& C|x(t)|^{p-2}\left(1+|x(t)|^{-\beta+2}\right)I_{\{x(t)\in(0,x^*)\}}\\
&+C|x(t)|^{p-2}\left(1+|x(t)|^{2}\right)I_{\{x(t)\in[x^*,\infty)\}},\\
\leqslant& C\left(1+|x(t)|^{-\beta+2}+|x(t)|^{p}\right),
\end{aligned}$$

for all $t\in[0,T]$. Similarly, we also have

$$\begin{aligned}
&-|x(t)|^{-q}\left(x(t)f(x(t))-(q-1)\varsigma^2/2\right)\\
\leqslant& -|x(t)|^{-q}\left(x(t)f(x(t))-(q-1)\varsigma^2/2\right)I_{\{x(t)\in(0,x^*)\}}\\
&+C|x(t)|^{-q}\left(1+|x(t)|^{\alpha+2}\right)I_{\{x(t)\in[x^*,\infty)\}},\\
\leqslant& C\left(1+|x(t)|^{\alpha+2-q}\right),\\
\leqslant& C\left(1+|x(t)|^{-q}+|x(t)|^{p}\right),
\end{aligned}$$

for all $t\in[0,T]$, since $p>\alpha+1>\alpha+2-q$.

Since $x(t)$ has finite $2p$-th moment in Lemma 6.2.1, we then use the above arguments and the Burkholder-Davis-Gundy inequality to derive

$$\begin{aligned}
&\mathbb{E}\left(\sup_{u\in[0,t]}\left(|x(u\wedge\tau_k)|^{p}+|x(u\wedge\tau_k)|^{-q+2}\right)\right)\\
\leqslant&\mathbb{E}\left(|x_0|^{p}+|x_0|^{-q+2}\right)\\
&+C\mathbb{E}\int_0^t\left(1+|x(s)|^{-\beta+2}+|x(s)|^{p}\right)ds\\
&+C\mathbb{E}\int_0^t\left(1+|x(s)|^{-q}+|x(s)|^{p}\right)ds\\
&+32^{1/2}\varsigma p\mathbb{E}\left(\int_0^t|x(s)|^{2p-2}I_{\{s\in[0,t\wedge\tau_k]\}}ds\right)^{1/2}\\
&+32^{1/2}\varsigma(q-2)\mathbb{E}\left(\int_0^t|x(s)|^{-2q+2}I_{\{s\in[0,t\wedge\tau_k]\}}ds\right)^{1/2},
\end{aligned}$$

for all $t \in [0,T]$. Using the Young inequality, we have

$$
\begin{aligned}
&32^{1/2}\varsigma p\mathbb{E}\left(\int_0^t |x(s)|^{2p-2}I_{\{s\in[0,t\wedge\tau_k]\}}ds\right)^{1/2}\\
\leqslant&32^{1/2}\varsigma p\mathbb{E}\left(\sup_{u\in[0,t]}|x(u\wedge\tau_k)|^p\int_0^t |x(s)|^{p-2}ds\right)^{1/2},\\
\leqslant&0.5\mathbb{E}\left(\sup_{u\in[0,t]}|x(u\wedge\tau_k)|^p\right)+C\mathbb{E}\int_0^t |x(s)|^{p-2}ds,
\end{aligned}
$$

and

$$
\begin{aligned}
&32^{1/2}\varsigma(q-2)\mathbb{E}\left(\int_0^t |x(s)|^{-2q+2}I_{\{s\in[0,t\wedge\tau_k]\}}ds\right)^{1/2}\\
\leqslant&32^{1/2}\varsigma(q-2)\mathbb{E}\left(\sup_{u\in[0,t]}|x(u\wedge\tau_k)|^{-q+2}\int_0^t |x(s)|^{-q}ds\right)^{1/2},\\
\leqslant&0.5\mathbb{E}\left(\sup_{u\in[0,t]}|x(u\wedge\tau_k)|^{-q+2}\right)+C\mathbb{E}\int_0^t |x(s)|^{-q}ds.
\end{aligned}
$$

Finally, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{u\in[0,t]}\left(|x(u\wedge\tau_k)|^p+|x(u\wedge\tau_k)|^{-q+2}\right)\right)\\
\leqslant&\mathbb{E}\left(|x_0|^p+|x_0|^{-q+2}\right)\\
&+C\mathbb{E}\int_0^t\left(1+|x(s)|^{-\beta+2}+|x(s)|^p\right)ds\\
&+C\mathbb{E}\int_0^t\left(1+|x(s)|^{-q}+|x(s)|^p\right)ds\\
&+0.5\mathbb{E}\left(\sup_{u\in[0,t]}|x(u\wedge\tau_k)|^p\right)+C\mathbb{E}\int_0^t |x(s)|^{p-2}ds\\
&+0.5\mathbb{E}\left(\sup_{u\in[0,t]}|x(u\wedge\tau_k)|^{-q+2}\right)+C\mathbb{E}\int_0^t |x(s)|^{-q}ds,
\end{aligned}
$$

for all $t \in [0,T]$. Since $q = 6(\beta-1)r > \beta$, we further have

$$\mathbb{E}\left(\sup_{u\in[0,t]}\left(|x(u\wedge\tau_k)|^p + |x(u\wedge\tau_k)|^{-q+2}\right)\right) \leqslant 2\mathbb{E}\left(|x_0|^p + |x_0|^{-q+2}\right) + C\mathbb{E}\int_0^t \left(1+|x(s)|^{-q}+|x(s)|^p\right)ds,$$

for all $t \in [0,T]$. We then use Lemma 6.2.1 to derive

$$\mathbb{E}\left(\sup_{u\in[0,t]}\left(|x(u\wedge\tau_k)|^p + |x(u\wedge\tau_k)|^{-q+2}\right)\right) \leqslant C.$$

Finally, we let $k\to\infty$ to achieve the result. $\square$

6.3 The projected EM method

Given a step size $\Delta \in (0,1]$, we first define the truncation function by

$$\phi(\Delta) = \Delta^{\frac{1}{2(\beta-1)}-\varepsilon_1},$$

where $\varepsilon_1 \in \left(0, \frac{1}{6(\beta-1)^2+2(\beta-1)}\right)$.

Let $\Delta_0 < 1$ be sufficiently small such that

$$x_0 \wedge 0.5x^* \in (\phi(\Delta_0), \Delta^{-0.5/(\alpha+1)}).$$

Let $\Delta \in (0,\Delta_0]$ and $k \in \mathbb{N}$. Then the projected EM numerical solutions to (6.2.1) $X_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ are defined by starting from x_0 and computing

$$x_\Delta^k(t) = x_\Delta(t_k) + f\left(x_\Delta(t_k)\right)(t-t_k) + \varsigma\left(B(t)-B(t_k)\right),$$
$$x_\Delta(t) = \left(\phi(\Delta)\vee x_\Delta^k(t)\right)\wedge\Delta^{-0.5/(\alpha+1)},$$

for $t\in[t_k,t_{k+1}]$.

To establish the strong convergence theory of the projected EM solution, we first prove some necessary lemmas. In Lemma 6.3.1, we will estimate the upper bounds of probabilities of some important subsets of $(\Omega, \mathcal{F}, \Pr)$. They will be used in proving Lemmas 6.3.2 and 6.3.3. In Lemmas 6.3.2 and 6.3.3, we prove the uniform boundedness of moments of the numerical solution. In particular, Lemma 6.3.3 is devoted to proving the uniformly bounded inverse moments of the numerical solution, which is one of main contributions of this paper. Finally, we establish a stronger result in Lemma 6.3.4. We will prove

$$\sup_{\Delta\in(0,\Delta_0]} \mathbb{E}\left(\sup_{u\in[0,T]} \left(|x_\Delta(u)|^p + |x_\Delta(u)|^{-q+2}\right)\right) \leqslant C,$$

which will be used in section 4.

Lemma 6.3.1. *Let $k \in \mathbb{N}$ be arbitrary and $t \in [t_k, t_{k+1}]$. Let $\Delta \in (0, \Delta_0]$. Let*

$$\mathcal{S}^1_{\Delta,t} = \left\{\inf_{u\in[t_k,t]} x^k_\Delta(u) \leqslant x^*/2, x_\Delta(t_k) \in [x^*, \Delta^{-0.5/(\alpha+1)}]\right\},$$

and

$$\mathcal{S}^2_{\Delta,t} = \left\{\sup_{u\in[t_k,t]} |x^k_\Delta(u) - x_\Delta(t_k)| \geqslant \varepsilon_0 x_\Delta(t_k), x_\Delta(t_k) \in [\phi(\Delta), x^*)\right\},$$

where ε_0 is fixed in Remark 6.2.2. Then we have

$$\Pr\left(\mathcal{S}^1_{\Delta,t} \cup \mathcal{S}^2_{\Delta,t}\right) \leqslant C\Delta^p,$$

where $r \geqslant 1$, $q = 6(\beta-1)r$ and $p > (q+2) \vee 6(\alpha+1)r$.

Proof. Using Remark 6.2.1, we have

$$\begin{aligned}
|f(x)| \leqslant & C_0(1 + x^{\alpha+1} + x^{-\beta+1}), \\
\leqslant & C_0(1 + (\Delta^{-0.5/(\alpha+1)})^{\alpha+1} + \phi(\Delta)^{-(\beta-1)}), \\
= & C_0(1 + (\Delta^{-0.5/(\alpha+1)})^{\alpha+1} + (\Delta^{0.5/(\beta-1)-\varepsilon_1})^{-(\beta-1)}), \\
\leqslant & C_0(1 + 2\Delta^{-\frac{1}{2}}),
\end{aligned}$$

for $x \in [\phi(\Delta), \Delta^{-0.5/(\alpha+1)}]$. Then we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}\left|x_\Delta^k(u)-x_\Delta(t_k)\right|^{p/\varepsilon_1}\right)\\
\leqslant&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}\left|f(x_\Delta(t_k))(u-t_k)+\varsigma\left(B(u)-B(t_k)\right)\right|^{p/\varepsilon_1}\right),\\
\leqslant&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}\left|C_0(1+2\Delta^{-\frac{1}{2}})\Delta+\varsigma\left(B(u)-B(t_k)\right)\right|^{p/\varepsilon_1}\right),\\
\leqslant&2^{p/\varepsilon_1}\left(\mathbb{E}\left|C_0(1+2\Delta^{-\frac{1}{2}})\Delta\right|^{p/\varepsilon_1}+\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}\left|\varsigma\left(B(u)-B(t_k)\right)\right|^{p/\varepsilon_1}\right),\\
\leqslant&C\Delta^{0.5p/\varepsilon_1}.
\end{aligned}
$$

Using the Chebyshev inequality, $\varepsilon_1 < 0.5$ and $\Delta \leqslant 1$, we have

$$
\begin{aligned}
&\Pr\left(\mathcal{S}^1_{\Delta,t}\right)\\
=&\Pr\left(\inf_{u\in[t_k,t]}\left(x_\Delta^k(u)-x_\Delta(t_k)\right)\leqslant\left(x^*/2-x_\Delta(t_k)\right),x_\Delta(t_k)\in[x^*,\Delta^{-0.5/(\alpha+1)}]\right),\\
\leqslant&\Pr\left(\inf_{u\in[t_k,t]}\left(x_\Delta^k(u)-x_\Delta(t_k)\right)\leqslant -x^*/2,x_\Delta(t_k)\in[x^*,\Delta^{-0.5/(\alpha+1)}]\right),\\
\leqslant&\Pr\left(\sup_{u\in[t_k,t]}\left|x_\Delta^k(u)-x_\Delta(t_k)\right|\geqslant x^*/2,x_\Delta(t_k)\in[x^*,\Delta^{-0.5/(\alpha+1)}]\right),\\
\leqslant&\mathbb{E}\left(\sup_{u\in[t_k,t]}\left|x_\Delta^k(u)-x_\Delta(t_k)\right|^{p/\varepsilon_1}\right)/(x^*/2)^{p/\varepsilon_1},\\
\leqslant&C\Delta^{0.5p/\varepsilon_1},\\
\leqslant&C\Delta^p.
\end{aligned}
$$

Using $\beta \geqslant 2$, $\Delta \leqslant 1$, $\phi(\Delta) = \Delta^{\frac{1}{2(\beta-1)}-\varepsilon_1}$ and the Chebyshev inequality, we have

$$
\begin{aligned}
&\Pr\left(\mathcal{S}^2_{\Delta,t}\right) \\
=&\Pr\left(\sup_{u\in[t_k,t]}|x^k_\Delta(u)-x_\Delta(t_k)| \geqslant \varepsilon_0 x_\Delta(t_k), x_\Delta(t_k)\in[\phi(\Delta),x^*)\right), \\
\leqslant&\Pr\left(\sup_{u\in[t_k,t]}|x^k_\Delta(u)-x_\Delta(t_k)| \geqslant \varepsilon_0 \phi(\Delta), x_\Delta(t_k)\in[\phi(\Delta),x^*)\right), \\
\leqslant&\mathbb{E}\left(\sup_{u\in[t_k,t]}\left|x^k_\Delta(u)-x_\Delta(t_k)\right|^{p/\varepsilon_1}\right)/(\varepsilon_0\Delta^{0.5/(\beta-1)-\varepsilon_1})^{p/\varepsilon_1}, \\
\leqslant&C\Delta^{0.5p/\varepsilon_1}(\Delta^{-0.5/(\beta-1)+\varepsilon_1})^{p/\varepsilon_1}, \\
=&C\Delta^{0.5p(1-1/(\beta-1))/\varepsilon_1}\Delta^p, \\
\leqslant&C\Delta^p.
\end{aligned}
$$

□

Remark 6.3.1. Let $k \in \mathbb{N}$ and $t \in [t_k, t_{k+1}]$. First,

$$
\mathcal{S}^1_{\Delta,t} = \left\{\inf_{u\in[t_k,t]} x^k_\Delta(u) \leqslant x^*/2, x_\Delta(t_k) \in [x^*, \Delta^{-0.5/(\alpha+1)}]\right\}
$$

is $\mathcal{F}_t$-measurable for $t \in [t_k, t_{k+1}]$. Second, $I_{\mathcal{S}^1_{\Delta,t}}$ is cadlag (right continuous and left limit). Since $I_{\mathcal{S}^1_{\Delta,t}}$ is cadlag and adapted, it is measurable (see section 1.3 in [3]). Therefore, $\mathbb{E}\int_0^t I_{\mathcal{S}^1_{\Delta,s}} ds$ is well-defined and will be used in proving Lemmas 6.3.2 and 6.3.3.

Lemma 6.3.2. *Assume that Assumptions 6.2.1, 6.2.2 and 6.2.3 hold. Then there exists a constant C, depending on T, α, β, H, K_1, etc., such that*

$$
\sup_{\Delta\in(0,\Delta_0]}\sup_{t\in[0,T]}\mathbb{E}|x_\Delta(t)|^{2p} \leqslant C,
$$

where $r \geqslant 1$, $q = 6(\beta-1)r$ and $p > (q+2) \vee 6(\alpha+1)r$. In addition, we have

$$
\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}|x^k_\Delta(u)|^{2p} \leqslant C,
$$

for all $k \in \mathbb{N}$ such that $k\Delta \leqslant (T+\Delta)$.

Proof. Let $k \in \mathbb{N}$. Define $\tau_j^k = \inf\left\{t \in [t_k, t_{k+1}] \mid x_\Delta^k(t) > j\right\}$ for $j \in \mathbb{N}_+$. Let $\Delta \in (0, \Delta_0]$. Using the Itô formula, we have

$$
\begin{aligned}
|x_\Delta^k(t \wedge \tau_j^k)|^{2p} =& |x_\Delta(t_k)|^{2p} + 2p \int_{t_k}^{t\wedge\tau_j^k} |x_\Delta^k(s)|^{2p-2} \Big(x_\Delta^k(s) f(x_\Delta(t_k)) + (2p-1)\varsigma^2/2 \Big) ds \\
&+ 2\varsigma p \int_{t_k}^{t\wedge\tau_j^k} |x_\Delta^k(s)|^{2p-2} x_\Delta^k(s) dB(s),
\end{aligned}
$$

for all $t \in [t_k, t_{k+1}]$.

Using the Young inequality, Remarks 6.2.1 and 6.2.2, we have

$$
\begin{aligned}
&|x_\Delta^k(s)|^{2p-2} \left(x_\Delta^k(s) f(x_\Delta(t_k)) + (2p-1)\varsigma^2/2 \right) \\
\leqslant& C|x_\Delta^k(s)|^{2p-2} \left(1 + |x_\Delta^k(s)| x_\Delta(t_k)\right) I_{\left\{x_\Delta^k(s) > x^*/2,\, x_\Delta(t_k) \in [x^*, \Delta^{-0.5/(\alpha+1)}]\right\}} \\
&+ C|x_\Delta^k(s)|^{2p-2} \left(1 + |x_\Delta^k(s)| \left(1 + 2\Delta^{-\frac{1}{2}}\right)\right) I_{\mathcal{S}^1_{\Delta,s}} \\
&+ C \left| \frac{x_\Delta^k(s)}{x_\Delta(t_k)} \right|^{2p-1} \left(|x_\Delta(t_k)|^{2p-1} + |x_\Delta(t_k)|^{2p-\beta} \right) I_{\left\{\left(x_\Delta^k(s)/x_\Delta(t_k) - 1\right) \in (-\varepsilon_0, \varepsilon_0),\, x_\Delta(t_k) \in [\phi(\Delta), x^*)\right\}} \\
&+ C|x_\Delta^k(s)|^{2p-2} I_{\left\{\left(x_\Delta^k(s)/x_\Delta(t_k) - 1\right) \in (-\varepsilon_0, \varepsilon_0),\, x_\Delta(t_k) \in [\phi(\Delta), x^*)\right\}} \\
&+ C|x_\Delta^k(s)|^{2p-2} \left(1 + |x_\Delta^k(s)| \left(1 + 2\Delta^{-\frac{1}{2}}\right)\right) I_{\mathcal{S}^2_{\Delta,s}}, \\
\leqslant& C \left(1 + |x_\Delta^k(s)|^{2p} + |x_\Delta(t_k)|^{2p}\right) I_{\left\{x_\Delta^k(s) > x^*/2,\, x_\Delta(t_k) \in [x^*, \Delta^{-0.5/(\alpha+1)}]\right\}} \\
&+ C \left(1 + |x_\Delta^k(s)|^{2p} + \Delta^{-p}\right) I_{\mathcal{S}^1_{\Delta,s}} \\
&+ C \left(1 + |x_\Delta^k(s)|^{2p} + |x_\Delta(t_k)|^{2p}\right) I_{\left\{\left(x_\Delta^k(s)/x_\Delta(t_k) - 1\right) \in (-\varepsilon_0, \varepsilon_0),\, x_\Delta(t_k) \in [\phi(\Delta), x^*)\right\}} \\
&+ C \left(1 + |x_\Delta^k(s)|^{2p} + \Delta^{-p}\right) I_{\mathcal{S}^2_{\Delta,s}}, \\
\leqslant& C\Delta^{-p} \left(I_{\mathcal{S}^1_{\Delta,s}} + I_{\mathcal{S}^2_{\Delta,s}} \right) + C \left(1 + |x_\Delta^k(s)|^{2p} + |x_\Delta(t_k)|^{2p}\right),
\end{aligned}
$$

for all $s \in [t_k, t_{k+1} \wedge \tau_j^k]$, since $2p > 2q + 4 > 12\beta - 8 > \beta$.

Taking expectations on both sides and using above arguments, we then have

$$
\begin{aligned}
&\mathbb{E}|x_{\Delta}^{k}(t\wedge\tau_j^k)|^{2p}\\
\leqslant&\mathbb{E}|x_{\Delta}(t_k)|^{2p}+C\Delta+C\mathbb{E}\int_{t_k}^{t\wedge\tau_j^k}|x_{\Delta}(t_k)|^{2p}ds+C\mathbb{E}\int_{t_k}^{t\wedge\tau_j^k}|x_{\Delta}^{k}(s)|^{2p}ds\\
&+C\Delta^{-p}\int_{t_k}^{t}\left(\Pr(\mathcal{S}^1_{\Delta,s})+\Pr(\mathcal{S}^2_{\Delta,s})\right)ds,
\end{aligned}
$$

for all $t\in[t_k,t_{k+1}]$. Using Lemma 6.3.1, we have $\Big(\Pr(\mathcal{S}^1_{\Delta,s})+\Pr(\mathcal{S}^2_{\Delta,s})\Big)\leqslant C\Delta^p$. Then we have

$$
\sup_{u\in[t_k,t]}\mathbb{E}|x_{\Delta}^{k}(u\wedge\tau_j^k)|^{2p}\leqslant\mathbb{E}|x_{\Delta}(t_k)|^{2p}+C\Delta+C\int_{t_k}^{t}\sup_{u\in[t_k,s]}\mathbb{E}|x_{\Delta}^{k}(u\wedge\tau_j^k)|^{2p}ds,
$$

for all $t\in[t_k,t_{k+1}]$. The Gronwall inequality implies that

$$
\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}|x_{\Delta}^{k}(u\wedge\tau_j^k)|^{2p}\leqslant\left(\mathbb{E}|x_{\Delta}(t_k)|^{2p}+C\Delta\right)e^{C\Delta}.
$$

Letting $j\to\infty$, we then have

$$
\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}|x_{\Delta}^{k}(u)|^{2p}\leqslant\left(\mathbb{E}|x_{\Delta}(t_k)|^{2p}+C\Delta\right)e^{C\Delta}.
$$

Moreover, we have

$$
\begin{aligned}
&\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}|x_{\Delta}(u)|^{2p}\\
\leqslant&\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}\left(|x_{\Delta}^{k}(u)|^{2p}I_{\{x_{\Delta}^{k}(u)\in[\phi(\Delta),\infty)\}}+\phi(\Delta)^{2p}I_{\{x_{\Delta}^{k}(u)\in(-\infty,\phi(\Delta))\}}\right),\\
\leqslant&\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}|x_{\Delta}^{k}(u)|^{2p}+\Delta^{\frac{p(1-2(\beta-1)\varepsilon_1)}{\beta-1}},\\
\leqslant&\left(\mathbb{E}|x_{\Delta}(t_k)|^{2p}+C\Delta\right)e^{C\Delta}+\Delta,\\
\leqslant&\left(\mathbb{E}|x_{\Delta}(t_k)|^{2p}+C\Delta\right)e^{C\Delta},
\end{aligned}
$$

since $p > 6(\beta-1) > \frac{(\beta-1)}{1-2(\beta-1)\varepsilon_1}$. By induction, we have

$$\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x_\Delta(u)|^{2p} \leqslant \left(|x_0|^{2p} + C(k+1)\Delta\right) e^{C(k+1)\Delta}.$$

That is,

$$\sup_{t\in[0,T]} \mathbb{E}|x_\Delta(t)|^{2p} \leqslant C.$$

In addition, we have

$$\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x^k_\Delta(u)|^{2p} \leqslant C,$$

for all $k \in \mathbb{N}$ such that $k\Delta \leqslant (T+\Delta)$. □

Lemma 6.3.3. *Assume that Assumptions 6.2.1, 6.2.2 and 6.2.3 hold. Then there exists a constant C, depending on T, α, β, H, K_1, etc., such that*

$$\sup_{\Delta\in(0,\Delta_0]} \sup_{t\in[0,T]} \mathbb{E}|x_\Delta(t)|^{-q} \leqslant C,$$

where $r \geqslant 1$, $q = 6(\beta-1)r$ and $p > (q+2) \vee 6(\alpha+1)r$.

Proof. Let $k \in \mathbb{N}$ and $\Delta \in (0,\Delta_0]$. We define

$$\tau^k_\Delta = \inf\{t \in [t_k, t_{k+1}] \mid x^k_\Delta(t) < \phi(\Delta)\}.$$

Using the Itô formula, we have

$$\begin{aligned}|x^k_\Delta(t\wedge\tau^k_\Delta)|^{-q} =& |x_\Delta(t_k)|^{-q} - q\int_{t_k}^{t\wedge\tau^k_\Delta} |x^k_\Delta(s)|^{-(q+2)} \left(x^k_\Delta(s) f(x_\Delta(t_k)) - (q+1)\varsigma^2/2\right) ds \\ &- \varsigma q \int_{t_k}^{t\wedge\tau^k_\Delta} |x^k_\Delta(s)|^{-(q+1)} dB(s),\end{aligned}$$

for all $t \in [t_k, t_{k+1}]$.

Using the Young inequality, Remarks 6.2.1 and 6.2.2, we then have

$$
\begin{aligned}
& -|x^k_\Delta(s)|^{-(q+2)}\left(x^k_\Delta(s) f(x_\Delta(t_k)) - (q+1)\varsigma^2/2\right) \\
\leqslant & -|x^k_\Delta(s)|^{-(q+2)}\left((1-\varepsilon_0)x_\Delta(t_k) f(x_\Delta(t_k)) - (q+1)\varsigma^2/2\right) I_{\left\{x^k_\Delta(s)>(1-\varepsilon_0)x_\Delta(t_k),\, x_\Delta(t_k)\in[\phi(\Delta),x^*)\right\}} \\
& + C\left(\Delta^{-(q+1)/2}\left(1+2\Delta^{-\frac{1}{2}}\right)+\Delta^{-(q+2)/2}\right) I_{\left\{x^k_\Delta(s)\leqslant(1-\varepsilon_0)x_\Delta(t_k),\, x_\Delta(t_k)\in[\phi(\Delta),x^*)\right\}} \\
& + C\left(\Delta^{-(q+1)/2}\left(1+2\Delta^{-\frac{1}{2}}\right)+\Delta^{-(q+2)/2}\right) I_{\mathcal{S}^1_{\Delta,s}} \\
& + C\left(1+|x_\Delta(t_k)|^{\alpha+1}\right) I_{\left\{x^k_\Delta(s)>x^*/2,\, x_\Delta(t_k)\in[x^*,\Delta^{-0.5/(\alpha+1)}]\right\}}, \\
\leqslant & C\left(1+|x_\Delta(t_k)|^p\right) + C\Delta^{-\frac{q}{2}-1}\left(I_{\mathcal{S}^1_{\Delta,s}} + I_{\mathcal{S}^2_{\Delta,s}}\right),
\end{aligned}
$$

for all $s \in [t_k, t_{k+1}\wedge\tau^k_\Delta]$, since $p > 6(\alpha+1)r > (\alpha+1)$.

Taking expectations on both sides and using the Young inequality, Lemmas 6.3.1 and 6.3.2, we then have

$$
\begin{aligned}
& \mathbb{E}|x^k_\Delta(t\wedge\tau^k_\Delta)|^{-q} \\
= & \mathbb{E}|x_\Delta(t_k)|^{-q} - q\mathbb{E}\int_{t_k}^{t\wedge\tau^k_\Delta} |x^k_\Delta(s)|^{-(q+2)}\left(x^k_\Delta(s) f(x_\Delta(t_k)) - (q+1)\varsigma^2/2\right) ds, \\
\leqslant & \mathbb{E}|x_\Delta(t_k)|^{-q} + C\mathbb{E}\int_{t_k}^{t}\left(1+|x_\Delta(t_k)|^p\right) ds + C\Delta^{-\frac{q}{2}-1}\mathbb{E}\int_{t_k}^{t}\left(I_{\mathcal{S}^1_{\Delta,s}} + I_{\mathcal{S}^2_{\Delta,s}}\right) ds, \\
\leqslant & \mathbb{E}|x_\Delta(t_k)|^{-q} + C\Delta,
\end{aligned}
$$

for all $t \in [t_k, t_{k+1}]$.

Now we have

$$
\begin{cases}
x_\Delta(t) = x^k_\Delta(t\wedge\tau^k_\Delta) I_{\{x^k_\Delta(t\wedge\tau^k_\Delta)<\Delta^{-0.5/(\alpha+1)}\}} + \Delta^{-0.5/(\alpha+1)} I_{\{x^k_\Delta(t\wedge\tau^k_\Delta)\geqslant\Delta^{-0.5/(\alpha+1)}\}}, & t\leqslant\tau^k_\Delta, \\
x_\Delta(t) \geqslant \phi(\Delta) = x^k_\Delta(t\wedge\tau^k_\Delta), & t>\tau^k_\Delta.
\end{cases}
$$

Using Lemma 6.3.2 and the Chebyshev inequality, we then have

$$
\begin{aligned}
&\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x_\Delta(u)|^{-q}\\
\leqslant &\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q} + \sup_{u\in[t_k,t_{k+1}]} \mathbb{E}\left((\Delta^{-0.5/(\alpha+1)})^{-q} I_{\{x^k_\Delta(u\wedge\tau^k_\Delta)\geqslant\Delta^{-0.5/(\alpha+1)}\}}\right),\\
= &\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q} + (\Delta^{-0.5/(\alpha+1)})^{-q} \sup_{u\in[t_k,t_{k+1}]} \Pr\left(x^k_\Delta(u\wedge\tau^k_\Delta)\geqslant\Delta^{-0.5/(\alpha+1)}\right),\\
\leqslant &\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q} + (\Delta^{-0.5/(\alpha+1)})^{-q} \sup_{u\in[t_k,t_{k+1}]} \frac{\mathbb{E}|x^k_\Delta(u)|^{2p}}{(\Delta^{-0.5/(\alpha+1)})^{2p}},\\
= &\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q} + C(\Delta^{-0.5/(\alpha+1)})^{-(2p+q)},\\
\leqslant &\mathbb{E}|x_\Delta(t_k)|^{-q} + C\Delta + C\Delta^{0.5(2p+q)/(\alpha+1)}.
\end{aligned}
$$

Since $\Delta\leqslant\Delta_0<1$, $r\geqslant 1$ and $p>(q+2)\vee 6(\alpha+1)r$, we have

$$
\begin{aligned}
&\sup_{u\in[t_k,t_{k+1}]} \mathbb{E}|x_\Delta(u)|^{-q}\\
\leqslant &\mathbb{E}|x_\Delta(t_k)|^{-q} + C\Delta + C\Delta^{p/(\alpha+1)},\\
\leqslant &\mathbb{E}|x_\Delta(t_k)|^{-q} + C\Delta + C\Delta^{6r},\\
\leqslant &\mathbb{E}|x_\Delta(t_k)|^{-q} + C\Delta,\\
\leqslant &C.
\end{aligned}
$$

By induction, we have

$$
\sup_{t\in[0,T]} \mathbb{E}|x_\Delta(t)|^{-q}\leqslant C.
$$

□

Furthermore, we use similar arguments to derive a stronger result that the numerical solutions have finite moments over the time interval $[0,T]$. This lemma is useful in section 4.

Lemma 6.3.4. *Assume that Assumptions 6.2.1, 6.2.2 and 6.2.3 hold. Then there* *exists a constant C, depending on T, α, β, H, K_1, etc., such that*

$$\sup_{\Delta\in(0,\Delta_0]} \mathbb{E}\left(\sup_{u\in[0,T]}\left(|x_\Delta(u)|^p + |x_\Delta(u)|^{-q+2}\right)\right) \leqslant C,$$

where $r \geqslant 1$, $q = 6(\beta-1)r$ and $p > (q+2) \vee 6(\alpha+1)r$.

Proof. Let $\Delta \in (0,\Delta_0]$. This proof is simliar to that of Lemma 6.3.2. The only difference is that

$$\varsigma p\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}\int_{t_k}^{u}|x_\Delta^k(s)|^{p-2}x_\Delta^k(s)dB(s)\right) \neq 0.$$

Therefore, an additional estimate should be added. Using the Burkholder-Davis-Gundy inequality and the Young inequality, we have

$$\begin{aligned}
&\varsigma p\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}\int_{t_k}^{u}|x_\Delta^k(s)|^{p-2}x_\Delta^k(s)dB(s)\right)\\
\leqslant& C\mathbb{E}\left(\int_{t_k}^{t_{k+1}}|x_\Delta^k(s)|^{2p-2}ds\right)^{1/2},\\
\leqslant& C\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x_\Delta^k(u)|^p\int_{t_k}^{t_{k+1}}|x_\Delta^k(s)|^{p-2}ds\right)^{1/2},\\
\leqslant& 0.5\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x_\Delta^k(u)|^p\right) + C\mathbb{E}\int_{t_k}^{t_{k+1}}|x_\Delta^k(s)|^{p-2}ds.
\end{aligned}$$

Combining it with the arguments in Lemma 6.3.2, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x^k_\Delta(u)|^p\right)\\
\leqslant&\mathbb{E}|x_\Delta(t_k)|^p+C\Delta+C\mathbb{E}\int_{t_k}^{t_{k+1}}|x_\Delta(t_k)|^pds+C\mathbb{E}\int_{t_k}^{t_{k+1}}|x^k_\Delta(s)|^pds\\
&+C\Delta^{-p+1}\left(\Pr(\mathcal{S}^1_{\Delta,s})+\Pr(\mathcal{S}^2_{\Delta,s})\right)\\
&+\varsigma p\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}\left(\int_{t_k}^{t_{k+1}}|x^k_\Delta(s)|^{p-2}x^k_\Delta(s)dB(s)\right),\\
\leqslant&e^{C\Delta}\mathbb{E}|x_\Delta(t_k)|^p+C\Delta+C\mathbb{E}\int_{t_k}^{t_{k+1}}|x^k_\Delta(s)|^pds\\
&+0.5\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x^k_\Delta(u)|^p\right)+C\mathbb{E}\int_{t_k}^{t_{k+1}}|x^k_\Delta(s)|^{p-2}ds.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x_\Delta(u)|^p\right)\leqslant&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x^k_\Delta(u)|^p\right)+\phi(\Delta)^p,\\
\leqslant&C\sup_{u\in[t_k,t_{k+1}]}\mathbb{E}|x^k_\Delta(u)|^p+C\Delta+\phi(\Delta)^p,\\
\leqslant&C.
\end{aligned}
$$

That is,

$$
\mathbb{E}\left(\sup_{u\in[0,T]}|x_\Delta(u)|^p\right)\leqslant C.
$$

Using the Burkholder-Davis-Gundy inequality and the Young inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}\int_{t_k}^{u}|x^k_\Delta(s)|^{-q+1}dB(s)\right)\\
\leqslant&C\mathbb{E}\left(\int_{t_k}^{t_{k+1}\wedge\tau^k_\Delta}|x^k_\Delta(s)|^{-2q+2}ds\right)^{1/2},\\
\leqslant&0.5\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q+2}\right)+C\mathbb{E}\int_{t_k}^{t_{k+1}\wedge\tau^k_\Delta}|x^k_\Delta(s)|^{-q}ds.
\end{aligned}
$$

Using arguments in Lemma 6.3.3, we have

$$\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q+2}\right)$$
$$\leqslant\mathbb{E}|x_\Delta(t_k)|^{-q+2}+C\Delta+0.5\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q+2}\right)$$
$$+C\mathbb{E}\int_{t_k}^{t_{k+1}\wedge\tau^k_\Delta}|x^k_\Delta(s)|^{-q}ds.$$

Using arguments in Lemma 6.3.3, we have

$$\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x_\Delta(u)|^{-q+2}\right)$$
$$\leqslant\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}|x^k_\Delta(u\wedge\tau^k_\Delta)|^{-q+2}\right)+C\Delta^{0.5(2p+q)/(\alpha+1)},$$
$$\leqslant\mathbb{E}|x_\Delta(t_k)|^{-q+2}+C\Delta.$$

Therefore, we have

$$\mathbb{E}\left(\sup_{u\in[0,T]}|x_\Delta(u)|^{-q+2}\right)\leqslant C.$$

□

In the following, we define $e_k = x(t_{k+1}) - x^k_\Delta(t_{k+1})$ and $e(t) = x(t) - x_\Delta(t)$ for $t \in [0,T]$. We also let

$$\bar{\mathcal{S}}^1_{\Delta,k}=\left\{x^k_\Delta(t_{k+1})\in(-\infty,\phi(\Delta))\right\},$$
$$\bar{\mathcal{S}}^2_{\Delta,k}=\left\{x^k_\Delta(t_{k+1})\in[\phi(\Delta),\Delta^{-0.5/(\alpha+1)}]\right\},$$
$$\bar{\mathcal{S}}^3_{\Delta,k}=\left\{x^k_\Delta(t_{k+1})\in(\Delta^{-0.5/(\alpha+1)},\infty)\right\},$$
$$\bar{\mathcal{S}}^1_{k}=\left\{x(t_{k+1})\in(0,\phi(\Delta))\right\},$$
$$\bar{\mathcal{S}}^2_{k}=\left\{x(t_{k+1})\in(\Delta^{-0.5/(\alpha+1)},\infty)\right\}.$$

Theorem 6.3.1. *Assume that Assumptions 6.2.1, 6.2.2 and 6.2.3 hold. Let* $\Delta \in$

$(0, \Delta_0]$. *Let* $n = \lfloor T/\Delta \rfloor$. *Then there exists a constant* C*, depending on* T*,* α*,* β*,* H*,* K_1*, etc., but independent of* Δ*, such that*

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n} |e(t_k)|^{2r}\right) \leqslant C\Delta^{2r}.$$

Proof. Using the Itô formula for $f(x(s)) - f(x(t_k))$, we have

$$\begin{aligned}
e_k =& e(t_k) + \int_{t_k}^{t_{k+1}} \left(f(x(s)) - f(x_\Delta(t_k))\right) ds, \\
=& e(t_k) + \int_{t_k}^{t_{k+1}} \left(f(x(t_k)) - f(x_\Delta(t_k))\right) ds \\
&+ \int_{t_k}^{t_{k+1}} \left(f(x(s)) - f(x(t_k))\right) ds, \\
=& e(t_k) + \int_{t_k}^{t_{k+1}} \left(f(x(t_k)) - f(x_\Delta(t_k))\right) ds \\
&+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \left(f'(x(u))f(x(u)) + 0.5\varsigma^2 f''(x(u))\right) duds \\
&+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \varsigma f'(x(u))dB(u)ds, \\
=& e(t_k) + \left(f(x(t_k)) - f(x_\Delta(t_k))\right)\Delta + J_k,
\end{aligned}$$

where

$$\begin{aligned}
J_k =& \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \left(f'(x(u))f(x(u)) + 0.5\varsigma^2 f''(x(u))\right) duds \\
&+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \varsigma f'(x(u))dB(u)ds.
\end{aligned}$$

Using Assumption 6.2.3 and the Young inequality, we then have

$$\begin{aligned}
e_k^2 =& e(t_k)^2 + \left(f(x(t_k)) - f(x_\Delta(t_k))\right)^2\Delta^2 + J_k^2 \\
&+ 2e(t_k)\left(f(x(t_k)) - f(x_\Delta(t_k))\right)\Delta + 2e(t_k)J_k \\
&+ 2\left(f(x(t_k)) - f(x_\Delta(t_k))\right)J_k\Delta, \\
\leqslant& (1 + 2H\Delta)e(t_k)^2 + 2\left(f(x(t_k)) - f(x_\Delta(t_k))\right)^2\Delta^2 + 2J_k^2 + 2e(t_k)J_k.
\end{aligned}$$

We have

$$
\begin{aligned}
e(t_{k+1})^2 =& (\phi(\Delta) - x(t_{k+1}))^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} \\
&+ \left(x^k_\Delta(t_{k+1}) - x(t_{k+1})\right)^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} \\
&+ \left(\Delta^{-0.5/(\alpha+1)} - x(t_{k+1})\right)^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}.
\end{aligned}
$$

Then we have

$$
e(t_{k+1})^2 \leqslant \begin{cases} \phi(\Delta)^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}, & x(t_{k+1}) \in (0, \phi(\Delta)), \\ e_k^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}, & x(t_{k+1}) \in [\phi(\Delta), \Delta^{-0.5/(\alpha+1)}], \\ e_k^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} + x(t_{k+1})^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}, & x(t_{k+1}) \in (\Delta^{-0.5/(\alpha+1)}, \infty). \end{cases}
$$

In summary, we have

$$
e(t_{k+1})^2 \leqslant e_k^2 + \phi(\Delta)^2 I_{\bar{\mathcal{S}}^1_k} + x(t_{k+1})^2 I_{\bar{\mathcal{S}}^2_k}.
$$

By induction, we have

$$
\begin{aligned}
& e(t_{k+1})^2 \\
\leqslant & e^{2kH\Delta} \sum_{i=0}^{k} \left(2\left(f(x(t_i)) - f(x_\Delta(t_i))\right)^2 \Delta^2 + 2J_i^2 + \phi(\Delta)^2 I_{\bar{\mathcal{S}}^1_i} + |x(t_{i+1})|^2 I_{\bar{\mathcal{S}}^2_i}\right) \\
& + 2\sum_{i=0}^{k} (1+2H\Delta)^{k-i} e(t_i) J_i, \\
= & e^{2kH\Delta} \sum_{i=0}^{k} \left(2\left(f(x(t_i)) - f(x_\Delta(t_i))\right)^2 \Delta^2 + 2J_i^2 + \phi(\Delta)^2 I_{\bar{\mathcal{S}}^1_i} + |x(t_{i+1})|^2 I_{\bar{\mathcal{S}}^2_i}\right) \\
& + 2\sum_{i=0}^{k} (1+2H\Delta)^{k-i} e(t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left(f'(x(u)) f(x(u)) + 0.5\varsigma^2 f''(x(u))\right) du ds \\
& + 2\sum_{i=0}^{k} (1+2H\Delta)^{k-i} e(t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \varsigma f'(x(u)) dB(u) ds.
\end{aligned}
$$

Let $0 \leqslant m \leqslant n$ be an arbitrary integer. Taking expectations on both sides, we then

have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m+1} e(t_k)^{2r}\right)\\
\leqslant &C\mathbb{E}\left(\sum_{i=0}^{m}\left(2\left(f(x(t_i))-f(x_\Delta(t_i))\right)^2\Delta^2+2J_i^2+\phi(\Delta)^2 I_{\bar{\mathcal{S}}_i^1}+|x(t_{i+1})|^2 I_{\bar{\mathcal{S}}_i^2}\right)\right)^r\\
&+C\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left(f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right)duds\right|^r\right)\\
&+C\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\varsigma f'(x(u))dB(u)ds\right|^r\right).
\end{aligned}
\tag{6.3.1}
$$

Using Remark 6.2.1, Assumption 6.2.1, the mean value theorem and the Young inequality, we have

$$
\begin{aligned}
&\left(f(x(t_i))-f(x_\Delta(t_i))\right)^{2r}\Delta^{2r}\\
=&\left(f(x(t_i))+f(x_\Delta(t_i))\right)^r\left(f(x(t_i))-f(x_\Delta(t_i))\right)^r\Delta^{2r},\\
\leqslant &C\left(1+x(t_i)^{(2\alpha+1)r}+x(t_i)^{-(2\beta-1)r}+x_\Delta(t_i)^{(2\alpha+1)r}+x_\Delta(t_i)^{-(2\beta-1)r}\right)\left(x(t_i)-x_\Delta(t_i)\right)^r\Delta^{2r},\\
\leqslant &C\left(1+x(t_i)^{(2\alpha+1)r}+x(t_i)^{-(2\beta-1)r}+x_\Delta(t_i)^{(2\alpha+1)r}+x_\Delta(t_i)^{-(2\beta-1)r}\right)^2\Delta^{3r}+e(t_i)^{2r}\Delta^r,\\
\leqslant &C\left(1+x(t_i)^{2(2\alpha+1)r}+x(t_i)^{-2(2\beta-1)r}+x_\Delta(t_i)^{2(2\alpha+1)r}+x_\Delta(t_i)^{-2(2\beta-1)r}\right)\Delta^{3r}+e(t_i)^{2r}\Delta^r.
\end{aligned}
$$

Using Lemmas 6.2.1, 6.3.2 6.3.3, we then have

$$
\Delta^{2r}\mathbb{E}\left(f(x(t_i))-f(x_\Delta(t_i))\right)^{2r}\leqslant C\Delta^{3r}+e(t_i)^{2r}\Delta^r,
\tag{6.3.2}
$$

since $p>6(\alpha+1)r$ and $q\geqslant 2(2\beta-1)r$.

Using Lemma 6.2.1, Remark 6.2.1, Assumption 6.2.1 and the Hölder inequality, we

have

$$\begin{aligned}
\mathbb{E}\sum_{i=0}^{m}|J_i|^{2r} \leqslant & C\Delta^{2r-1}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\left|\int_{t_i}^{s}\left(f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right)du\right|^{2r}ds \\
& + C\Delta^{2r-1}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\left|\int_{t_i}^{s}f'(x(u))dB(u)\right|^{2r}ds, \\
\leqslant & C\Delta^{4r-2}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right|^{2r}duds \\
& + C\Delta^{3r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\mathbb{E}\int_{t_i}^{s}|f'(x(u))|^{2r}duds, \\
\leqslant & C\Delta^{4r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}\left(1+x(u)^{2(2\alpha+1)r}+x(u)^{-2(2\beta-1)r}\right)duds \\
& + C\Delta^{3r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}\left(1+x(u)^{2\alpha r}+x(u)^{-2\beta r}\right)duds, \\
\leqslant & C\Delta^{3r-1}, \qquad (6.3.3)
\end{aligned}$$

since $p > 2(2\alpha+1)r$ and $q \geqslant 2(2\beta-1)r$.

Using (6.3.2), (6.3.3), Lemma 6.2.1, the Hölder inequality and the Chebyshev inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sum_{i=0}^{m}\left(2\left(f(x(t_i))-f(x_\Delta(t_i))\right)^2\Delta^2+2J_i^2+\phi(\Delta)^2 I_{\bar{\mathcal{S}}_i^1}+|x(t_{i+1})|^2 I_{\bar{\mathcal{S}}_i^2}\right)\right)^r\\
\leqslant &Cm^{r-1}\sum_{i=0}^{m}\mathbb{E}\left(2\left(f(x(t_i))-f(x_\Delta(t_i))\right)^2\Delta^2+2J_i^2+\phi(\Delta)^2 I_{\bar{\mathcal{S}}_i^1}+|x(t_{i+1})|^2 I_{\bar{\mathcal{S}}_i^2}\right)^r,\\
\leqslant &Cm^{r-1}\sum_{i=0}^{m}\mathbb{E}\left(\left(f(x(t_i))-f(x_\Delta(t_i))\right)^{2r}\Delta^{2r}+|J_i|^{2r}\right)\\
&+Cm^{r-1}\phi(\Delta)^{2r}\sum_{i=0}^{m}\Pr\left(x(t_{i+1})\in(0,\phi(\Delta))\right)\\
&+Cm^{r-1}\sum_{i=0}^{m}\left(\mathbb{E}\left(|x(t_{i+1})|^{4r}\right)\right)^{1/2}\left(\Pr\left(x(t_{i+1})\in(\Delta^{-0.5/(\alpha+1)},\infty)\right)\right)^{1/2},\\
\leqslant &C\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r}+C\Delta^{2r}\\
&+Cm^{r-1}\phi(\Delta)^{2r}\sum_{i=0}^{m}\frac{\mathbb{E}|x(t_{i+1})|^{-q}}{\phi(\Delta)^{-q}}\\
&+Cm^{r-1}\sum_{i=0}^{m}\left(\frac{\mathbb{E}|x(t_{i+1})|^{2p}}{\Delta^{-p/(\alpha+1)}}\right)^{1/2},\\
\leqslant &C\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r}+C\Delta^{2r},
\end{aligned}\tag{6.3.4}
$$

since $p>6(\alpha+1)r$, $q\geqslant 2(2\beta-1)r$ and $\frac{(q+2r)(1-2(\beta-1)\varepsilon_1)}{2(\beta-1)}\geqslant 3r$.

Using the Hölder inequality, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left(f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right)duds\right|^r\right)\\
\leqslant &\mathbb{E}\left(\sum_{i=0}^{m}(1+2H\Delta)^{m-i}|e(t_i)|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right|duds\right)^r,\\
\leqslant &Cm^{r-1}\mathbb{E}\sum_{i=0}^{m}|e(t_i)|^r\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right|duds\right|^r.
\end{aligned}
$$

Using the Young inequality, we have

$$
\begin{aligned}
&m^{r-1}|e(t_i)|^r\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right|duds\right|^r\\
\leqslant &\Delta|e(t_i)|^{2r}+m^{2r-2}\Delta^{-1}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right|duds\right|^{2r}.
\end{aligned}
$$

Using the Hölder inequality and Lemma 6.2.1, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left(f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right)duds\right|^r\right)\\
\leqslant & C\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r}+Cm^{2r-2}\Delta^{2r-2}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\left|\int_{t_i}^{s}\left|f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right|du\right|^{2r}ds,\\
\leqslant & C\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r}+Cm^{2r-2}\Delta^{4r-3}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}\left(1+x(u)^{2(2\alpha+1)r}+x(u)^{-2(2\beta-1)r}\right)duds,\\
\leqslant & C\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r}+Cm^{2r-1}\Delta^{4r-1}.
\end{aligned}
$$

Since $m\leqslant\lfloor T/\Delta\rfloor$, we have $m\Delta\leqslant T$. Therefore, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left(f'(x(u))f(x(u))+0.5\varsigma^2 f''(x(u))\right)duds\right|^r\right)\\
\leqslant & C\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r}+C\Delta^{2r}. \qquad (6.3.5)
\end{aligned}
$$

Since

$$
\begin{aligned}
&\mathbb{E}\left(e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\varsigma f'(x(u))dB(u)ds\mid\mathcal{F}_{t_i}\right)\\
=&e(t_i)\mathbb{E}\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\varsigma f'(x(u))dB(u)ds\mid\mathcal{F}_{t_i}\right),\\
=&0,
\end{aligned}
$$

$$
\left\{\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\varsigma f'(x(u))dB(u)ds\right\}_{k=0,1,2,\ldots,m}
$$

is a martingale. Using the Burkholder-Davis-Gundy inequality, the Young inequality and the Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\varsigma f'(x(u))dB(u)ds\right|^r\right)\\
\leqslant &C\mathbb{E}\left(\sum_{i=0}^{m}|e(t_i)|^2\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}f'(x(u))dB(u)ds\right|^2\right)^{r/2},\\
\leqslant &C\mathbb{E}\left(\left(\sup_{0\leqslant k\leqslant m}|e(t_k)|^r\right)\left(\sum_{i=0}^{m}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}f'(x(u))dB(u)ds\right|^2\right)^{r/2}\right),\\
\leqslant &0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}e(t_k)^{2r}\right)+C\mathbb{E}\left(\sum_{i=0}^{m}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}f'(x(u))dB(u)ds\right|^2\right)^r,\\
\leqslant &0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}e(t_k)^{2r}\right)+Cm^{r-1}\mathbb{E}\sum_{i=0}^{m}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}f'(x(u))dB(u)ds\right|^{2r},\\
\leqslant &0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}e(t_k)^{2r}\right)+Cm^{r-1}\Delta^{2r-1}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\mathbb{E}\left|\int_{t_i}^{s}f'(x(u))dB(u)\right|^{2r}ds,\\
\leqslant &0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}e(t_k)^{2r}\right)+Cm^{r-1}\Delta^{3r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}(1+x(u)^{2\alpha r}+x(u)^{-2\beta r})duds,\\
\leqslant &0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}e(t_k)^{2r}\right)+Cm^r\Delta^{3r}.
\end{aligned}$$

Since $m\Delta\leqslant T$, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k}(1+2H\Delta)^{k-i}e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\varsigma f'(x(u))dB(u)ds\right|^r\right)\\
\leqslant &0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}e(t_k)^{2r}\right)+C\Delta^{2r}. \qquad (6.3.6)
\end{aligned}$$

Substituting (6.3.4), (6.3.5) and (6.3.6) into (6.3.1), we finally have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant m+1}e(t_k)^{2r}\right)\leqslant C\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r}+C\Delta^{2r},$$

for all $0\leqslant m\leqslant n$. Then the Gronwall inequality implies

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n+1}e(t_k)^{2r}\right)\leqslant C\Delta^{2r}.$$

□

6.4 Examples

In this section, we will apply Theorem 6.3.1 to several important SDE models. Markedly, the projected EM method can be applied for most of examples in [20].

Example 6.4.1. In this example, we let

$$f(x) = \sum_{i=1}^{k} a_i x^{b_i},$$

where $a_1 > 0 > a_k$, $b_1 < b_2 < \ldots < b_k$, $b_k > 1$ and $b_1 < -1$.

In this case, SDE (6.2.1) has the following properties.

1. We have

$$f'(x) = \sum_{i=1}^{k} a_i b_i x^{b_i - 1} \quad \text{and} \quad f''(x) = \sum_{i=1}^{k} a_i b_i (b_i - 1) x^{b_i - 2}.$$

   That is, Assumption 6.2.1 holds with

$$\alpha = b_k - 1 \quad \text{and} \quad \beta = -b_1 + 1.$$

2. Since

$$\lim_{x \downarrow 0^+} x f(x) = a_1 x^{b_1 + 1} = \infty,$$

   Assumption 6.2.2 holds for arbitrary large $r \geqslant 1$.

3. Since $a_1 > 0 > a_k$, $b_1 < b_2 < \ldots < b_k$ and $b_1 < -1$,

$$f'(x) = \sum_{i=1}^{k} a_i b_i x^{b_i - 1}$$

   is bounded from above for $x \in \mathbb{R}^+$. Then the mean value theorem implies that Assumption 6.2.3 holds.

Let $\Delta \in (0, \Delta_0]$ Then we have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n} |x(t_k) - x_\Delta(t_k)|^{2r}\right) \leqslant C\Delta^{2r},$$

for arbitrary large $r \geqslant 1$. Moreover, Lemmas 6.2.2 and 6.3.4 hold for arbitrary $p > 0$ and $q > 0$.

Example 6.4.2 (The Aït-Sahalia model)**.** The Aït-Sahalia model is given by

$$dx(t) = f(x(t))dt + g(x(t))dB(t),$$

where

$$f(x) = a_{-1}x^{-1} - a_0 + a_1 x - a_2 x^{\theta},$$

and

$$g(x) = \sigma x^{\rho}$$

with $a_{-1}, a_0, a_1, a_2, \sigma > 0$, $\rho, \theta > 1$. Using the Lamperti transformation $y = x^{1-\rho}$, we have a new SDE

$$dy(t) = f(y(t))dt + (1-\rho)\sigma dB(t),$$

where

$$f(y) = (\rho - 1)\left(a_2 y^{\frac{\rho-\theta}{\rho-1}} + \rho\sigma^2 y^{-1}/2 - a_1 y + a_0 y^{\frac{\rho}{\rho-1}} - a_{-1} y^{\frac{\rho+1}{\rho-1}}\right).$$

First, we consider the case $\theta + 1 > 2\rho$. Let $r \geqslant 1$ be arbitrary and $\Delta \in (0, \Delta_0]$. From Example 6.4.1, we have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n} |y(t_k) - y_\Delta(t_k)|^{2r}\right) \leqslant C\Delta^{2r}$$

and

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{-\frac{2\rho r}{\rho-1}}+y_\Delta(t_k)^{-\frac{2\rho r}{\rho-1}}\right)\right)\leqslant C.$$

Let $0<u<v$. Using the mean value theorem, there exists a $\xi\in(u,v)$ such that

$$|u^{-\frac{1}{\rho-1}}-v^{-\frac{1}{\rho-1}}|=\frac{1}{\rho-1}|\xi^{-\frac{\rho}{\rho-1}}||u-v|\leqslant\frac{1}{\rho-1}|u^{-\frac{\rho}{\rho-1}}+v^{-\frac{\rho}{\rho-1}}||u-v|.$$

Using the Hölder inequality, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|x(t_k)-x_\Delta(t_k)|^r\right)\\
=&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)^{-\frac{1}{\rho-1}}-y_\Delta(t_k)^{-\frac{1}{\rho-1}}|^r\right),\\
\leqslant&C\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(\left|y(t_k)^{-\frac{\rho}{\rho-1}}+y_\Delta(t_k)^{-\frac{\rho}{\rho-1}}\right|^r|(y(t_k)-y_\Delta(t_k))|^r\right)\right),\\
\leqslant&C\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{-\frac{2\rho r}{\rho-1}}+y_\Delta(t_k)^{-\frac{2\rho r}{\rho-1}}\right)\right)\right)^{1/2}\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)-y_\Delta(t_k)|^{2r}\right)\right)^{1/2},\\
\leqslant&C\Delta^r.
\end{aligned}$$

Now we focus on the critical case with $\theta+1=2\rho$ and $\frac{2a_2/\sigma^2+1}{\rho-1}>\left(\frac{2\rho}{\rho-1}\vee 4\right)+2$. In this case, we have

$$f(y)=(\rho-1)\left(\left(a_2+\rho\sigma^2/2\right)y^{-1}-a_1y+a_0y^{\frac{\rho}{\rho-1}}-a_{-1}y^{\frac{\rho+1}{\rho-1}}\right).$$

Then we have the following conclusions.

1. Assumption 6.2.1 holds with

$$\alpha=\frac{2}{\rho-1}\quad\text{and}\quad\beta=2.$$

2. Let $1 \leqslant r < \frac{2a_2/\sigma^2+1}{6(\rho-1)}$. We have

$$\begin{aligned}
\liminf_{x\downarrow 0^+} xf(x) =&(\rho-1)\left(a_2+\rho\sigma^2/2\right),\\
=&(1-\rho)^2\sigma^2\left(\frac{a_2/\sigma^2+0.5}{\rho-1}+0.5\right),\\
>&(1-\rho)^2\sigma^2\left(3r+0.5\right),\\
\\
=&(1-\rho)^2\sigma^2\left(3(\beta-1)r+0.5\right).
\end{aligned}$$

That is, Assumption 6.2.2 holds for $1 \leqslant r < \frac{2a_2/\sigma^2+1}{6(\rho-1)}$.

3.

$$f'(y) = (\rho-1)\left(-\left(a_2+\rho\sigma^2/2\right)y^{-2} - a_1 + \frac{a_0\rho}{\rho-1}y^{\frac{1}{\rho-1}} - \frac{a_{-1}(\rho+1)}{\rho-1}y^{\frac{2}{\rho-1}}\right)$$

is bounded from above for $y \in \mathbb{R}^+$. Then the mean value theorem implies that Assumption 6.2.3 holds.

Let $1 \leqslant r$ such that $\left(\frac{2\rho r}{\rho-1}+2\right) \vee 6r < \frac{2a_2/\sigma^2+1}{\rho-1}$. Let $r_0 = \frac{1}{3}\left(\frac{\rho r}{\rho-1}+1\right)$, then we have

$$\begin{aligned}
\liminf_{x\downarrow 0^+} xf(x) =&(\rho-1)\left(a_2+\rho\sigma^2/2\right),\\
=&(1-\rho)^2\sigma^2\left(\frac{a_2/\sigma^2+0.5}{\rho-1}+0.5\right),\\
>&(1-\rho)^2\sigma^2\left(\frac{\rho r}{\rho-1}+1.5\right),\\
=&(1-\rho)^2\sigma^2\left(3(\beta-1)r_0+0.5\right),
\end{aligned}$$

since $\frac{\rho r}{\rho-1} < \frac{a_2/\sigma^2+0.5}{\rho-1} - 1$. That is, Assumption 6.2.2 holds for r_0. Since $-6r_0+2 = -\frac{2\rho r}{\rho-1}$, we have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{-\frac{2\rho r}{\rho-1}} + y_\Delta(t_k)^{-\frac{2\rho r}{\rho-1}}\right)\right) \leqslant C.$$

From Theorem 6.3.1, we have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)-y_\Delta(t_k)|^{2r}\right)\leqslant C\Delta^{2r}.$$

Using the Hölder inequality, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|x(t_k)-x_\Delta(t_k)|^{r}\right)\\
=&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)^{-\frac{1}{\rho-1}}-y_\Delta(t_k)^{-\frac{1}{\rho-1}}|^{r}\right),\\
\leqslant& C\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(\left|y(t_k)^{-\frac{\rho}{\rho-1}}+y_\Delta(t_k)^{-\frac{\rho}{\rho-1}}\right|^{r}|(y(t_k)-y_\Delta(t_k))|^{r}\right)\right),\\
\leqslant& C\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{-\frac{2\rho r}{\rho-1}}+y_\Delta(t_k)^{-\frac{2\rho r}{\rho-1}}\right)\right)\right)^{1/2}\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)-y_\Delta(t_k)|^{2r}\right)\right)^{1/2},\\
\leqslant& C\Delta^{r}.
\end{aligned}$$

Example 6.4.3 (The CEV process)**.** The CEV process is given by

$$dx(t)=\lambda(\mu-x(t))dt+\sigma x(t)^{1/2+\theta}dB(t),$$

where $\lambda,\mu,\sigma>0$ and $\theta\in(0,1/2)$. Using the Lamperti transformation $y=x^{1/2-\theta}$, we have a new SDE

$$dy(t)=f(y(t))dt+(1/2-\theta)\sigma dB(t),$$

where

$$f(y)=(1/2-\theta)\left(\lambda\mu y^{-\frac{1+2\theta}{1-2\theta}}-\frac{2\theta+1}{4}\sigma^2y^{-1}-\lambda y\right).$$

Then we have $\alpha=0$ and $\beta=\frac{2}{1-2\theta}$.

Let $r\geqslant 1$ be arbitrary. Let $\Delta\in(0,\Delta_0]$. From Example 6.4.1, we have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)-y_\Delta(t_k)|^{2r}\right)\leqslant C\Delta^{2r},$$

and

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{\frac{2(1+2\theta)r}{1-2\theta}}+y_\Delta(t_k)^{\frac{2(1+2\theta)r}{1-2\theta}}\right)\right)\leqslant C.$$

Let $0<u<v$. Using the mean value theorem, there exists a $\xi\in(u,v)$ such that

$$|u^{\frac{2}{1-2\theta}}-v^{\frac{2}{1-2\theta}}|=\frac{2}{1-2\theta}|\xi^{\frac{1+2\theta}{1-2\theta}}||u-v|\leqslant\frac{2}{1-2\theta}|u^{\frac{1+2\theta}{1-2\theta}}+v^{\frac{1+2\theta}{1-2\theta}}||u-v|.$$

Using the Hölder inequality, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|x(t_k)-x_\Delta(t_k)|^r\right)\\
=&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)^{\frac{2}{1-2\theta}}-y_\Delta(t_k)^{\frac{2}{1-2\theta}}|^r\right),\\
\leqslant&C\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(\left|y(t_k)^{\frac{1+2\theta}{1-2\theta}}+y_\Delta(t_k)^{\frac{1+2\theta}{1-2\theta}}\right|^r|(y(t_k)-y_\Delta(t_k))|^r\right)\right),\\
\leqslant&C\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{\frac{2(1+2\theta)r}{1-2\theta}}+y_\Delta(t_k)^{\frac{2(1+2\theta)r}{1-2\theta}}\right)\right)\right)^{1/2}\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)-y_\Delta(t_k)|^{2r}\right)\right)^{1/2},\\
\leqslant&C\Delta^r.
\end{aligned}$$

Example 6.4.4 (The Heston-3/2 volatility model)**.** The Heston-3/2 volatility model is given by

$$dx(t)=a_1x(t)(a_2-x(t))dt+a_3x(t)^{3/2}dB(t),$$

where $a_1,a_2,a_3>0$ and $a_1/a_3^2>1$. Using the Lamperti transformation $y=x^{-1/2}$, we have a new SDE

$$dy(t)=f(y(t))dt-0.5a_3dB(t),$$

where

$$f(y)=\left(a_1/2+3a_3^2/8\right)y^{-1}-a_1a_2y/2.$$

Then we have the following conclusions.

1. Assumption 6.2.1 holds with

$$\alpha = 0 \quad \text{and} \quad \beta = 2.$$

2. Let $1 \leqslant r < \frac{2a_1/a_3^2+1}{3}$. We have

$$\begin{aligned}\liminf_{x\downarrow 0^+} xf(x) =& a_1/2 + 3a_3^2/8, \\ =& 0.25\left(2a_1 + 1.5a_3^2\right), \\ =& 0.75a_3^2\left(\frac{2a_1/a_3^2+1}{3} + \frac{1}{6}\right), \\ >& 0.25a_3^2\left(3r + 0.5\right), \\ =& 0.25a_3^2\left(3(\beta-1)r + 0.5\right).\end{aligned}$$

That is, Assumption 6.2.2 holds for $1 \leqslant r < \frac{2a_1/a_3^2+1}{3}$.

3.
$$f'(y) = -\left(a_1/2 + 3a_3^2/8\right)y^{-2} - a_1a_2/2$$

is negative for all $y \in \mathbb{R}^+$. Then the mean value theorem implies that Assumption 6.2.3 holds.

Let $1 \leqslant r$ such that $1 \leqslant r < \frac{2a_1}{3a_3^2}$. Let $r_0 = r + 1/3$, then we have

$$\begin{aligned}\liminf_{x\downarrow 0^+} xf(x) =& a_1/2 + 3a_3^2/8, \\ =& 0.25\left(2a_1 + 1.5a_3^2\right), \\ >& 0.25a_3^2\left(3r + 1.5\right), \\ =& 0.25a_3^2\left(3(\beta-1)r_0 + 0.5\right),\end{aligned}$$

since $1 \leqslant r < \frac{2a_1}{3a_3^2}$. That is, Assumption 6.2.2 holds for r_0. Since $-6r_0 + 2 = -6r$, we

have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{-6r}+y_\Delta(t_k)^{-6r}\right)\right)\leqslant C.$$

From Theorem 6.3.1, we have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)-y_\Delta(t_k)|^{2r}\right)\leqslant C\Delta^{2r}.$$

Let $0<u<v$. Using the mean value theorem, there exists a $\xi\in(u,v)$ such that

$$|u^{-2}-v^{-2}|=2|\xi^{-3}||u-v|\leqslant 2|u^{-3}+v^{-3}||u-v|.$$

Using the Hölder inequality, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|x(t_k)-x_\Delta(t_k)|^r\right)\\
=&\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)^{-2}-y_\Delta(t_k)^{-2}|^r\right),\\
\leqslant& C\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(\left|y(t_k)^{-3}+y_\Delta(t_k)^{-3}\right|^r|(y(t_k)-y_\Delta(t_k))|^r\right)\right),\\
\leqslant& C\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}\left(y(t_k)^{-6r}+y_\Delta(t_k)^{-6r}\right)\right)\right)^{1/2}\left(\mathbb{E}\left(\sup_{0\leqslant k\leqslant n}|y(t_k)-y_\Delta(t_k)|^{2r}\right)\right)^{1/2},\\
\leqslant& C\Delta^r.
\end{aligned}$$

Compared to existing explicit EM methods, the strong convergence theory of the projected EM method is established in general $\mathcal{L}^p$-norm (see Table 6.1.1). In particular, we consider the critical cases for the Aït-Sahalia model. In [51], the researchers only consider the case: $\theta=2,\rho=1.5$. In [55], the research proves strong one half order convergence for $\theta+1=2\rho, a_2/\sigma^2\geqslant 4\rho-2.5$ in $\mathcal{L}^2$-norm. Example 6.4.2 shows we require $a_2/\sigma^2>(3\rho-1.5)\vee(6\rho-6.5)$ for a mean-square convergence rate of order one. If $\rho\in(1,2]$, our parameter range is wider. If $\rho>2$, then our parameter range $(a_2/\sigma^2>(6\rho-6.5))$ is smaller. However, a better theoretical $\mathcal{L}^p$-strongly convergence rate is proved.

For the Heston-3/2 volatility model, the strong convergence theory is also established for $a_1/a_3^2 > 1.5$ in $\mathcal{L}^1$-norm. Compared to results in [52] ($a_1/a_3^2 > 2.5$), our parameter range ($a_1/a_3^2 > 3$) is a little smaller. However, a better theoretical $\mathcal{L}^p$-strongly convergence rate is proved.

6.5 Numerical simulations

In this section, we will conduct numerical simulations for examples in section 4 to support our theoretical results. In each example, we let $T = 1$. We now conduct numerical simulations with 1000 sample paths for step sizes $\Delta = 2^{-17}, 2^{-16}, 2^{-15}, 2^{-14}$. In view of the fact that there is no analytical solution for many models in section 4, we regard the numerical solution with the step size $\Delta = 2^{-24}$ as the "exact" solution.

One important contribution of this chapter is that we prove that the projected EM method is $\mathcal{L}^p$-strongly convergent with order one. Therefore, we will show that experimental p-th strong convergence errors have about order p in each example.

Example 6.5.1 (Aït-Sahalia model)**.** First we consider the Aït-Sahalia model with $x_0 = 0.01$, $a_{-1} = 0.5$, $a_0 = 2$, $a_1 = 1$, $a_2 = 2$, $\theta = 4$, $\rho = 2$, $\sigma = 1$ and $r = 8$ in Example 6.4.2. This is a non-critical case, since $\theta + 1 > 2\rho$. Using the linear regression method, the experimental error (see Figure 6.5.1) shows that the strong convergence error for the 8th moment has order about 8.8382.

Then we consider the Aït-Sahalia model with $x_0 = 0.01$, $a_{-1} = 0.1$, $a_0 = 1$, $a_1 = 2$, $a_2 = 1$, $\theta = 2$, $\rho = 1.5$, $\sigma = 0.1$ and $r = 10$. This is a critical case with $\frac{2a_2/\sigma^2+1}{\rho-1} > \frac{2\rho r}{\rho-1} \vee 6r + 2$. Using the linear regression method, the experimental error (see Figure 6.5.2) shows that the strong convergence error for the 10th moment has order about 10.1668.

Example 6.5.2 (CEV model)**.** In this example, we consider the CEV model with $x_0 = 0.01$, $\lambda = 1$, $\mu = 1$, $\theta = 0.25$, $\sigma = 1$ and $r = 6$ in Example 6.4.3. Using the linear regression method, the experimental error (see Figure 6.5.3) shows that the strong convergence error for the 6th moment has order about 5.9578.

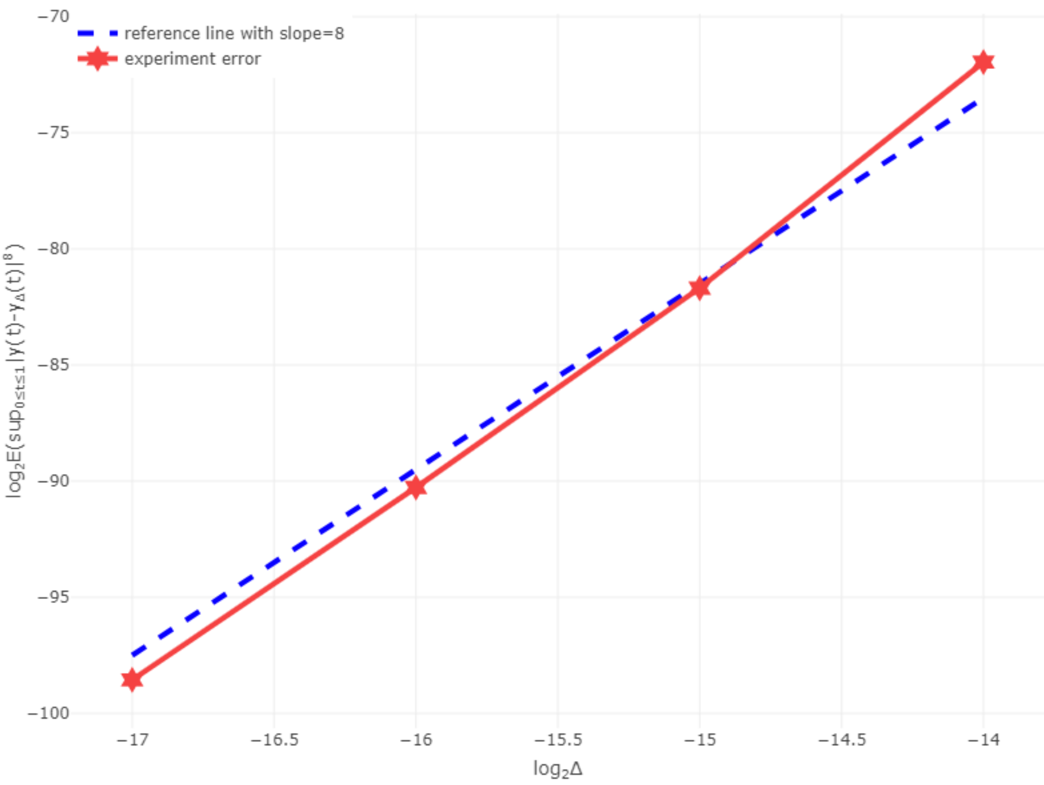

Figure 6.5.1: The $\mathcal{L}^8$-strongly convergence order of the projected EM method for the Aït-Sahalia model with non-critical parameters.

Example 6.5.3 (Heston-3/2 volatility model)**.** In this example, we consider the Heston-3/2 volatility model with $x_0 = 0.01$, $a_1 = 1$, $a_2 = 1$, $a_3 = 0.2$ and $r = 16$ in Example 6.4.4. Then we have $6r < 4a_1/a_3^2$. Using the linear regression method, the experimental error (see Figure 6.5.4) shows that the strong convergence error for the 16th moment has order about 16.0405.

6.6 Conclusion

In this chapter, we introduce a new explicit EM method, called the projected EM method, for a series of scalar positive SDEs. Compared to existing explicit EM methods, its strong convergence theory has better theoretical $\mathcal{L}^p$-strongly convergence rates for more parameter settings. In addition, we prove that the projected EM method is positivity preserving. We also conduct numerical simulations to support our theoretical

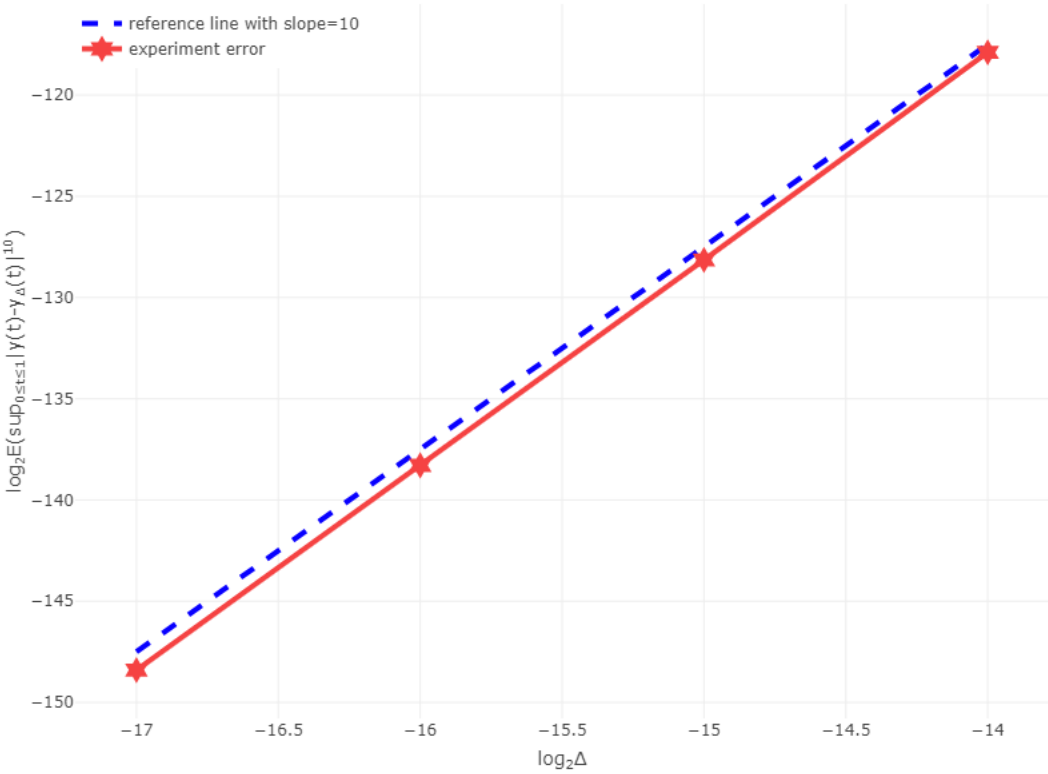

Figure 6.5.2: The $\mathcal{L}^{10}$-strongly convergence order of the projected EM method for the Aït-Sahalia model with critical parameters.

convergence rate order results. The projected EM method can be applied for many important SDE models, e.g., the Aït-Sahalia model, the CEV model and the Heston-3/2 volatility model. A pity thing is that our results exclude SDE models which stay in an interval, e.g., the Wright-Fisher model. However, we trust that our techniques can be extended for those SDE models with little modifications.

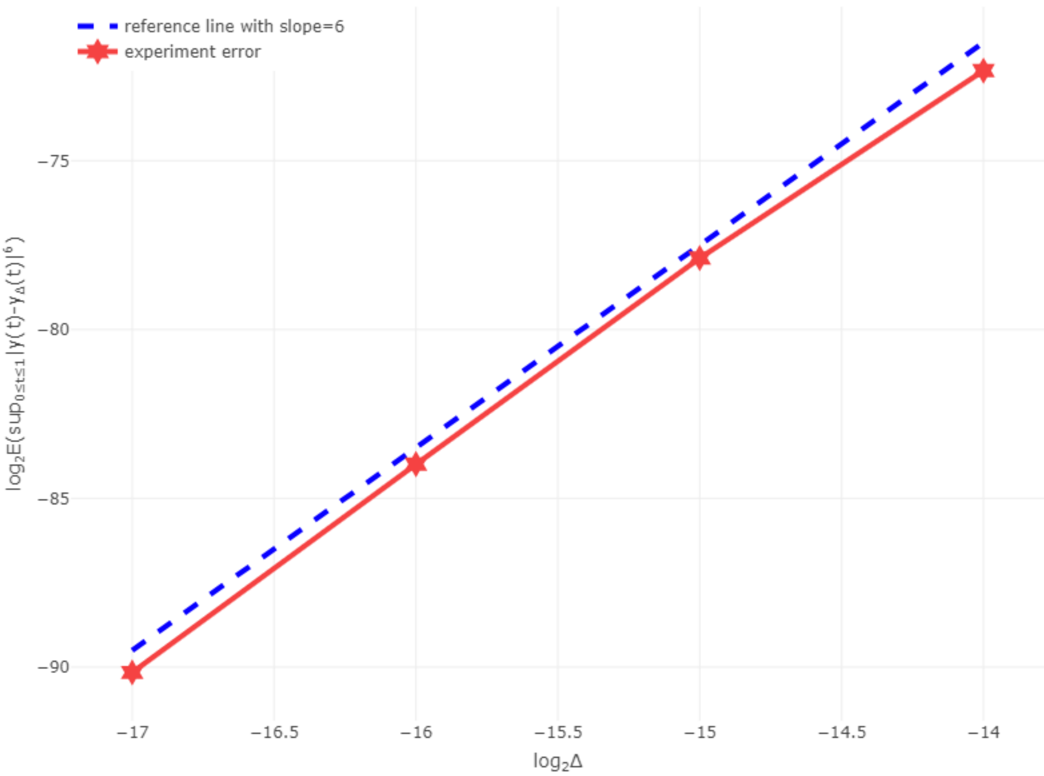

Figure 6.5.3: The $\mathcal{L}^6$-strongly convergence order of the projected EM method for the CEV model.

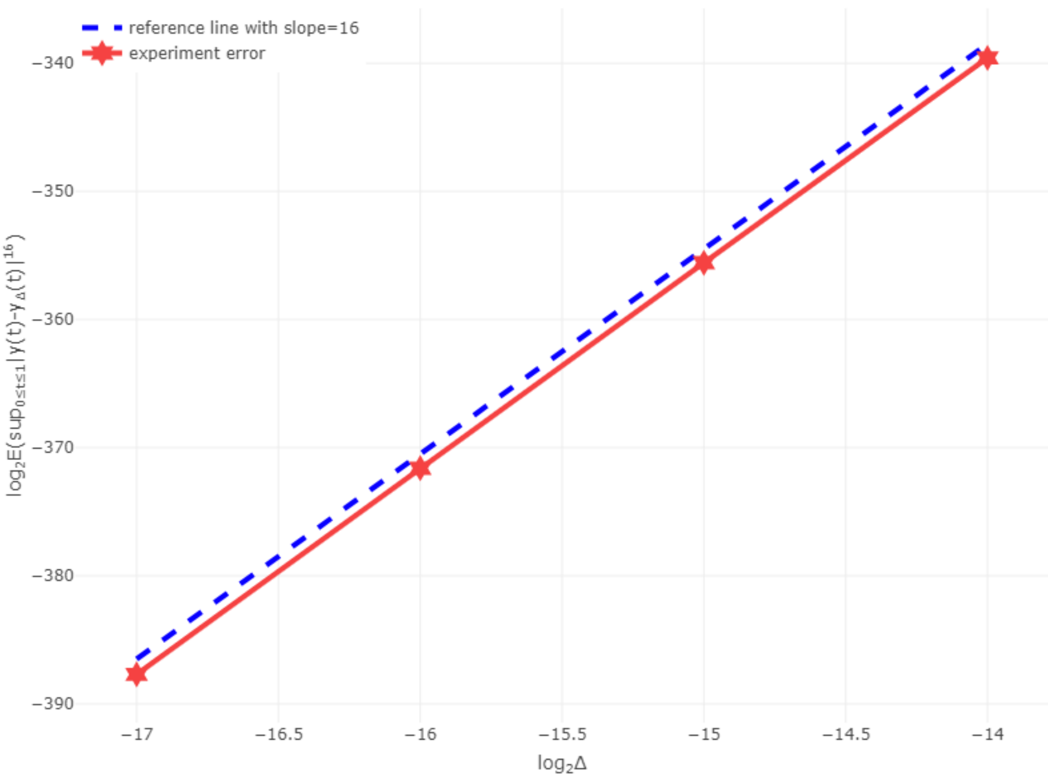

Figure 6.5.4: The $\mathcal{L}^{16}$-strongly convergence order of the projected EM method for the Heston-3/2 volatility model.

Chapter 7

Strong order one convergence of the projected EM method for the Wright-Fisher model

7.1 Background

Let $B(t)$ be a scalar Brownian motion defined on the complete probability space $(\Omega, \mathcal{F}, \Pr)$. The main aim of this chapter is to establish the strong convergence theory of the projected EM method for the Wright-Fisher (WF) model, which is defined by

$$dy(t) = (\alpha - \beta y(t))\, dt + \sigma \sqrt{|y(t)\,(1 - y(t))|} dB(t), \tag{7.1.1}$$

where $\alpha, \beta, \sigma > 0$.

The WF model has many applications in finance and biology (see [58] and [59] for detailed introductions). However, the analytical solution is inaccessible currently. In [58], the authors proposed an algorithm to exactly simulate the WF model. It should be the best numerical simulation method for the WF model, if we only need to simulate values for a small amount of grid points. However, a complete sample path over an interval may be required in some situations, e.g., evaluating discounted payoff. Then the computational cost of the exact simulation will be expensive, and an alternative numerical method is the EM method.

Therefore, an alternative effective EM method with high convergence rate order is desirable. Moreover, it can be proved that $y(t) \in (0,1)$ if $\alpha \wedge (\beta - \alpha) \geqslant \sigma^2/2$ (see Appendix A in [60]). Therefore, we also hope EM numerical approximations can stay in $(0,1)$. As we mentioned above, the drift-implicit EM method [20] can be applied for the WF model and is $\mathcal{L}^p$-strongly convergent with order one. However, expensive computational cost is also required for implementation of it, since it is an implicit EM method.

In previous chapters, we introduced many explicit EM methods which are developed for scalar SDEs whose coefficients are locally Lipschitz near 0. However, coefficients of the WF model are also locally Lipschitz near 1. Therefore, numerical analysis methods in those papers cannot be directly used for the WF model. The SIS epidemic model is defined by

$$dy(t) = y(t) \left(\beta N - \mu - \nu - \beta y(t)\right) dt + \sigma y(t) \left(N - y(t)\right) dB(t),$$

where $N > 0$ and $\mu, \nu, \beta \geqslant 0$. The exact solution of it also takes values in an interval. In [56] and [57], researchers used the Lamperti transformation and the exact solution of the Lamperti transformed model will take values in the whole real line. They then use the modified truncated EM to deal with superlinearly growing coefficients of the transformed SIS epidemic model. However, the Lamperti transformed WF model still takes values in $(0,\pi)$. Therefore, numerical analysis techniques for the transformed SIS epidemic model is not valid for the WF model.

There are also some specific explicit EM methods which are devised to simulate the WF model. Stamatiou [60] proposed a boundary preserving semi-discrete method and proved its convergence without concrete convergence rate order. The balanced implicit split step method [61] is also a boundary preserving and $\mathcal{L}^1$-strongly convergent with order one half. For appropriate parameter settings, the Lamperti smooth sloping truncation [59] is proved to be $\mathcal{L}^2$-strongly convergent with order one.

In this chapter, we will further study the strong convergence theory of the projected

EM method and extend it for WF model. Similarly, we will prove the convergence rate order one in general $\mathcal{L}^p$-norm and has better theoretical $\mathcal{L}^p$-strong convergence rate for some parameter settings. The main challenge in this chapter is to prove finite inverse moments near two endpoints, while we only consider one endpoint in Chapter 6. This chapter is organized as follows. In section 2, we first establish a useful lemma. Then we construct the projected EM method and investigate its convergence rates in section 3. In section 4, we will conduct numerical simulations for the WF model to support our theoretical results. Finally, we make a brief conclusion in section 5.

7.2 Preliminaries

As before, we set $\inf \emptyset = \infty$, where $\emptyset$ is an empty set. Moreover, we use C to stand for generic positive real numbers which are dependent on T, α, β, σ, r (used below), etc., but independent of Δ, t, k and m (used below) and its values may change between occurrences.

In this chapter, we first consider the Lamperti transformed WF model. We apply the transformation $x = 2\arcsin(\sqrt{y})$ to the SDE (7.1.1). We then have

$$dx(t) = f(x(t))dt + \sigma dB(t) \tag{7.2.1}$$

on $t \in [0, T]$ with the initial value $x(0) = x_0 = 2\arcsin(\sqrt{y_0}) \in (0, \pi)$, where

$$f(x) = \big(\alpha - \sigma^2/4\big)\cot(x/2) - \big(\beta - \alpha - \sigma^2/4\big)\tan(x/2)$$

and $\alpha, \beta, \sigma, T > 0$, $y_0 \in (0, 1)$ and $2 < \frac{(\beta-\alpha)\wedge\alpha}{\sigma^2}$. We fix $1 \leqslant r < \frac{2(\beta-\alpha)\wedge 2\alpha}{3\sigma^2} - \frac{1}{3}$ and $6r \leqslant q < \frac{4\alpha\wedge 4(\beta-\alpha)}{\sigma^2} - 2$. We also let $x_2 = 2\arctan\left(\sqrt{\frac{4\alpha-\sigma^2}{4(\beta-\alpha)-\sigma^2}}\right)$.

Proposition 7.2.1. *Using Lemma 3 in [59], we have*

$$\begin{aligned} f'(x) &\leqslant -C_0, \\ |f(x)| &\leqslant 2\pi C_0 \left(x^{-1} + (\pi - x)^{-1}\right), \\ |f'(x)| &\leqslant \pi^2 C_0 \left(x^{-2} + (\pi - x)^{-2}\right), \\ |f''(x)| &\leqslant \pi^3 C_0 \left(x^{-3} + (\pi - x)^{-3}\right), \end{aligned}$$

where $C_0 = 0.5(\beta - \sigma^2/2)$.

Since $f'(x) < 0$, *we have*

$$(x - y)\left(f(x) - f(y)\right) < 0,$$

for any $x, y \in (0, \pi)$. *In addition,* x_2 *is the unique root of* $f(x)$.

Since

$$\lim_{x \downarrow 0^+} x f(x) = 2\left(\alpha - \sigma^2/4\right) > (q+1)\sigma^2/2,$$

and

$$\lim_{x \uparrow \pi^-} (\pi - x) f(x) = -2\left(\beta - \alpha - \sigma^2/4\right) < -(q+1)\sigma^2/2,$$

there exist $0 < x_1 < x_2 < x_3 < \pi$ *and sufficiently small* $\varepsilon_0 > 0$ *such that*

$$\begin{cases} (1-\varepsilon_0) x f(x) - (q+1)\sigma^2/2 > 0, & x \in (0, x_1), \\ (1-\varepsilon_0)(\pi - x) f(x) + (q+1)\sigma^2/2 < 0, & x \in (x_3, \pi). \end{cases}$$

Then we prove finite moments of the exact solution to the Lamperti transformed WF model.

Lemma 7.2.1.

$$\sup_{t\in[0,T]} \mathbb{E}\left(x(t)^{-q} + (\pi - x(t))^{-q}\right) \leqslant C.$$

Proof. Given a $k \in \mathbb{N}_+$, we define the stopping time

$$\tau_k = \inf\{t \in [0,T] : x(t) \notin (1/k, \pi - 1/k)\}.$$

Using the Itô formula, we have

$$\begin{aligned}
&x(t\wedge\tau_k)^{-q} + (\pi - x(t\wedge\tau_k))^{-q} \\
=&x_0^{-q} + (\pi - x_0)^{-q} - q\int_0^{t\wedge\tau_k} x(s)^{-(q+2)}\left(x(s)f(x(s)) - (q+1)\sigma^2/2\right)ds \\
&+ q\int_0^{t\wedge\tau_k} (\pi - x(s))^{-(q+2)}\left((\pi - x(s))f(x(s)) + (q+1)\sigma^2/2\right)ds \\
&- q\sigma\int_0^{t\wedge\tau_k} x(s)^{-(q+1)} dB(s) \\
&+ q\sigma\int_0^{t\wedge\tau_k} (\pi - x(s))^{-(q+1)} dB(s),
\end{aligned} \tag{7.2.2}$$

for all $t \in [0,T]$.

Using Proposition 7.2.1, we have

$$\begin{aligned}
&- x(t)^{-(q+2)}\left(x(t)f(x(t)) - (q+1)\sigma^2/2\right) \\
\leqslant& - x(t)^{-(q+2)}\left(x(t)f(x(t)) - (q+1)\sigma^2/2\right) I_{\{x(t)\in(0,x_1)\}} \\
&+ x(t)^{-(q+2)}\left(2\pi C_0 x(t)\left(x(t)^{-1} + (\pi - x(t))^{-1}\right) + (q+1)\sigma^2/2\right) I_{\{x(t)\in[x_1,\pi)\}}, \\
\leqslant& C\left(1 + (\pi - x(t))^{-1}\right),
\end{aligned}$$

and

$$
\begin{aligned}
&(\pi - x(t))^{-(q+2)} \left((\pi - x(t))f(x(t)) + (q+1)\sigma^2/2\right) \\
\leqslant & C(\pi - x(t))^{-(q+2)} \left(1 + x(t)^{-1}(\pi - x(t))\right) I_{\{x(t)\in(0,x_3)\}} \\
&+ (\pi - x(t))^{-(q+2)} \left((\pi - x(t))f(x(t)) + (q+1)\sigma^2/2\right) I_{\{x(t)\in[x_3,\pi)\}}, \\
\leqslant & C\left(1 + x(t)^{-1}\right),
\end{aligned}
$$

for all $t \in [0, T \wedge \tau_k]$.

Taking expectations on both sides of (7.2.2) and using the Young inequality and the above arguments, we then have

$$
\mathbb{E}\left(x(t\wedge\tau_k)^{-q} + (\pi - x(t\wedge\tau_k))^{-q}\right) \leqslant C + C\mathbb{E}\int_0^t \left(x(s\wedge\tau_k)^{-q} + (\pi - x(s\wedge\tau_k))^{-q}\right) ds,
$$

for all $t \in [0, T]$. Then the Gronwall inequality implies that

$$
\mathbb{E}\left(x(t\wedge\tau_k)^{-q} + (\pi - x(t\wedge\tau_k))^{-q}\right) \leqslant C.
$$

Letting $k \to \infty$, we have the desired conclusion. □

7.3 The projected EM method

Given a step size $\Delta \in (0, 1]$, we first define the projection function by

$$
\phi(\Delta) = \Delta^{\frac{1}{2}-\varepsilon_1},
$$

where $\varepsilon_1 \in (0, 0.125)$. Then the projected EM numerical solutions to the Lamperti transformed WF model $x_\Delta(t_k) \approx x(t_k)$ for $t_k = k\Delta$ are defined by starting from x_0 and computing the recursion

$$
\begin{aligned}
&x_\Delta^k(t) = x_\Delta(t_k) + f\left(x_\Delta(t_k)\right)(t - t_k) + \sigma\left(B(t) - B(t_k)\right), \\
&x_\Delta(t) = \left(\phi(\Delta) \vee x_\Delta^k(t)\right) \wedge (\pi - \phi(\Delta)),
\end{aligned}
$$

for $t \in [t_k, t_{k+1}]$. Finally, we let $y_\Delta(t) = \sin^2(x_\Delta(t)/2)$ to derive projected EM solutions to the original WF model.

To establish the strong convergence theory of the projected EM solution, we first prove two useful lemmas. In Lemma 7.3.1, we will estimate upper bounds of some subsets of $(\Omega, \mathcal{F}, \Pr)$. For example, we will estimate an upper bound of the probability of

$$\mathcal{S}^1_{\Delta,t} = \left\{ \omega \in \Omega \mid \inf_{u \in [t_k,t]} x^k_\Delta(u,\omega) \leqslant (1-\varepsilon_0) x_\Delta(t_k,\omega), x_\Delta(t_k,\omega) \in [\phi(\Delta), x_1] \right\},$$

for $t \in [t_k, t_{k+1}]$. For the sake of convenience, we will simply write it as

$$\mathcal{S}^1_{\Delta,t} = \left\{ \inf_{u \in [t_k,t]} x^k_\Delta(u) \leqslant (1-\varepsilon_0) x_\Delta(t_k), x_\Delta(t_k) \in [\phi(\Delta), x_1] \right\}.$$

Similarly, we let

$$\left\{ x^k_\Delta(t_{k+1}) \in [\phi(\Delta), x_3] \right\} = \left\{ \omega \in \Omega \mid x^k_\Delta(t_{k+1},\omega) \in [\phi(\Delta), x_3] \right\}$$

in Lemma 7.3.2.

Lemma 7.3.1. *Let $\Delta_0 < 1$ be sufficiently small such that $x_0, x_1, x_2, x_3 \in (\phi(\Delta_0), \pi - \phi(\Delta_0))$. Let $\Delta \in (0, \Delta_0]$ and $k \in \mathbb{N}$ be arbitrary. Let $t \in [t_k, t_{k+1}]$. Then we have*

$$\Pr\left(\mathcal{S}^1_{\Delta,t} \cup \mathcal{S}^2_{\Delta,t} \cup \mathcal{S}^3_{\Delta,t} \cup \mathcal{S}^4_{\Delta,t}\right) \leqslant C\Delta^{q+2}, \tag{3.1}$$

where

$$\mathcal{S}^2_{\Delta,t} = \left\{ \sup_{u \in [t_k,t]} x^k_\Delta(u) \geqslant x_\Delta(t_k) + \varepsilon_0(\pi - x_\Delta(t_k)), x_\Delta(t_k) \in [x_3, \pi - \phi(\Delta)] \right\},$$

$$\mathcal{S}^3_{\Delta,t} = \left\{ \sup_{u \in [t_k,t]} x^k_\Delta(u) \geqslant 0.5\left(x_3 + \pi - \phi(\Delta_0)\right), x_\Delta(t_k) \in [\phi(\Delta), x_3] \right\}$$

and

$$\mathcal{S}^4_{\Delta,t} = \left\{ \inf_{u\in[t_k,t]} x^k_\Delta(u) \leqslant 0.5(\phi(\Delta_0) + x_1), x_\Delta(t_k) \in [x_1, \pi - \phi(\Delta)] \right\}.$$

Proof. Using Proposition 7.2.1 and the Burkholder-Davis-Gundy inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]} \left| x^k_\Delta(u) - x_\Delta(t_k) \right|^{(q+2)/\varepsilon_1} \right) \\
=&\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]} \left| f\left(x_\Delta(t_k)\right)(u - t_k) + \sigma\left(B(u) - B(t_k)\right) \right|^{(q+2)/\varepsilon_1} \right), \\
\leqslant& C\mathbb{E}\left(\left| 4\pi C_0 \phi(\Delta)^{-1}\Delta \right|^{(q+2)/\varepsilon_1} + \sigma^{(q+2)/\varepsilon_1} \sup_{u\in[t_k,t_{k+1}]} \left| B(u) - B(t_k) \right|^{(q+2)/\varepsilon_1} \right), \\
\leqslant& C\Delta^{\frac{q+2}{2\varepsilon_1}}.
\end{aligned}$$

Using the Chebyshev inequality, we then have

$$\begin{aligned}
&\Pr\left(\inf_{u\in[t_k,t]} x^k_\Delta(u) \leqslant (1-\varepsilon_0)x_\Delta(t_k), x_\Delta(t_k) \in [\phi(\Delta), x_1] \right) \\
=&\Pr\left(\inf_{u\in[t_k,t]} \left(x^k_\Delta(u) - x_\Delta(t_k) \right) \leqslant -\varepsilon_0 x_\Delta(t_k), x_\Delta(t_k) \in [\phi(\Delta), x_1] \right), \\
\leqslant&\Pr\left(\sup_{u\in[t_k,t]} |x^k_\Delta(u) - x_\Delta(t_k)| \geqslant \varepsilon_0\phi(\Delta), x_\Delta(t_k) \in [\phi(\Delta), x_1] \right), \\
\leqslant&\Pr\left(\sup_{u\in[t_k,t_{k+1}]} |x^k_\Delta(u) - x_\Delta(t_k)| \geqslant \varepsilon_0\phi(\Delta) \right), \\
\leqslant&\frac{\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]} \left| x^k_\Delta(u) - x_\Delta(t_k) \right|^{(q+2)/\varepsilon_1} \right)}{(\varepsilon_0\phi(\Delta))^{(q+2)/\varepsilon_1}}, \\
\leqslant& C\Delta^{q+2},
\end{aligned}$$

where $\phi(\Delta) = \Delta^{\frac{1}{2}-\varepsilon_1}$. Similarly, we also have

$$
\begin{aligned}
&\Pr\left(\sup_{u\in[t_k,t]} x_\Delta^k(u) \geqslant x_\Delta(t_k) + \varepsilon_0(\pi - x_\Delta(t_k)), x_\Delta(t_k) \in [x_3, \pi - \phi(\Delta)]\right)\\
=&\Pr\left(\sup_{u\in[t_k,t]} \left(x_\Delta^k(u) - x_\Delta(t_k)\right) \geqslant \varepsilon_0(\pi - x_\Delta(t_k)), x_\Delta(t_k) \in [x_3, \pi - \phi(\Delta)]\right),\\
\leqslant&\Pr\left(\sup_{u\in[t_k,t_{k+1}]} |x_\Delta^k(u) - x_\Delta(t_k)| \geqslant \varepsilon_0\phi(\Delta)\right),\\
\leqslant& C\Delta^{q+2}.
\end{aligned}
$$

Since $x_1, x_3 \in (\phi(\Delta_0), \pi - \phi(\Delta_0))$, $0.5(\pi - \phi(\Delta_0) - x_3)$ and $0.5(x_1 - \phi(\Delta_0))$ are constants. We then have

$$
\begin{aligned}
&\Pr\left(\sup_{u\in[t_k,t]} x_\Delta^k(u) \geqslant 0.5\left(x_3 + \pi - \phi(\Delta_0)\right), x_\Delta(t_k) \in [\phi(\Delta), x_3]\right)\\
=&\Pr\left(\sup_{u\in[t_k,t]} \left(x_\Delta^k(u) - x_\Delta(t_k)\right) \geqslant 0.5\left(x_3 + \pi - \phi(\Delta_0)\right) - x_\Delta(t_k), x_\Delta(t_k) \in [\phi(\Delta), x_3]\right),\\
\leqslant&\Pr\left(\sup_{u\in[t_k,t]} \left(x_\Delta^k(u) - x_\Delta(t_k)\right) \geqslant 0.5(\pi - \phi(\Delta_0) - x_3), x_\Delta(t_k) \in [\phi(\Delta), x_3]\right),\\
\leqslant&\Pr\left(\sup_{u\in[t_k,t_{k+1}]} |x_\Delta^k(u) - x_\Delta(t_k)| \geqslant 0.5(\pi - \phi(\Delta_0) - x_3), x_\Delta(t_k) \in [\phi(\Delta), x_3]\right),\\
\leqslant&\frac{\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]} \left|x_\Delta^k(u) - x_\Delta(t_k)\right|^{(q+2)/\varepsilon_1}\right)}{\left(0.5(\pi - \phi(\Delta_0) - x_3)\right)^{(q+2)/\varepsilon_1}},\\
\leqslant& C\Delta^{\frac{q+2}{2\varepsilon_1}},\\
\leqslant& C\Delta^{q+2},
\end{aligned}
$$

and

$$
\begin{aligned}
&\Pr\left(\inf_{u\in[t_k,t]} x_\Delta^k(u) \leqslant 0.5(\phi(\Delta_0)+x_1), x_\Delta(t_k)\in[x_1,\pi-\phi(\Delta)]\right)\\
=&\Pr\left(\inf_{u\in[t_k,t]} \left(x_\Delta^k(u)-x_\Delta(t_k)\right) \leqslant 0.5(\phi(\Delta_0)+x_1)-x_\Delta(t_k), x_\Delta(t_k)\in[x_1,\pi-\phi(\Delta)]\right),\\
\leqslant&\Pr\left(\inf_{u\in[t_k,t]} \left(x_\Delta^k(u)-x_\Delta(t_k)\right) \leqslant 0.5(\phi(\Delta_0)-x_1), x_\Delta(t_k)\in[x_1,\pi-\phi(\Delta)]\right),\\
\leqslant&\Pr\left(\sup_{u\in[t_k,t_{k+1}]} |x_\Delta^k(u)-x_\Delta(t_k)| \geqslant 0.5(x_1-\phi(\Delta_0)), x_\Delta(t_k)\in[x_1,\pi-\phi(\Delta)]\right),\\
\leqslant&\frac{\mathbb{E}\left(\sup_{u\in[t_k,t_{k+1}]}\left|x_\Delta^k(u)-x_\Delta(t_k)\right|^{(q+2)/\varepsilon_1}\right)}{(0.5(x_1-\phi(\Delta_0)))^{(q+2)/\varepsilon_1}},\\
\leqslant&C\Delta^{\frac{q+2}{2\varepsilon_1}},\\
\leqslant&C\Delta^{q+2},
\end{aligned}
$$

since $\varepsilon_1 \in (0, 0.125)$ and $\Delta < 1$. □

Lemma 7.3.2 is devoted to proving finite inverse moments of the projected EM numerical solution, which is critical in proving the strong convergence for $r > 1$. However, existing modified EM methods for the transformed WF model do not have this property.

Lemma 7.3.2.

$$
\sup_{\Delta\in(0,1]}\sup_{t\in[0,T]} \mathbb{E}\left(x_\Delta(t)^{-q}+(\pi-x_\Delta(t))^{-q}\right) \leqslant C.
$$

Proof. First we let $\Delta \in (0, \Delta_0]$. Given a $k \in \mathbb{N}$, we define two stopping times:

$$
\bar{\tau}_\Delta^k = \inf\{t\in[t_k,t_{k+1}] : x_\Delta^k(t) < \phi(\Delta)\},
$$

and

$$
\hat{\tau}_\Delta^k = \inf\{t\in[t_k,t_{k+1}] : x_\Delta^k(t) > \pi-\phi(\Delta)\}.
$$

Let $t \in [t_k, t_{k+1}]$. Using the Itô formula, we have

$$
\begin{aligned}
&\mathbb{E}\left(x_\Delta^k(t\wedge\bar\tau_\Delta^k\wedge\hat\tau_\Delta^k)^{-q}\right)+\mathbb{E}\left((\pi-x_\Delta^k(t\wedge\bar\tau_\Delta^k\wedge\hat\tau_\Delta^k))^{-q}\right)\\
=&\mathbb{E}\left(x_\Delta(t_k)^{-q}\right)+\mathbb{E}\left((\pi-x_\Delta(t_k))^{-q}\right)\\
&-q\mathbb{E}\int_{t_k}^{t\wedge\bar\tau_\Delta^k\wedge\hat\tau_\Delta^k} x_\Delta^k(s)^{-(q+2)}\left(x_\Delta^k(s)f(x_\Delta(t_k))-(q+1)\sigma^2/2\right)ds\\
&+q\mathbb{E}\int_{t_k}^{t\wedge\bar\tau_\Delta^k\wedge\hat\tau_\Delta^k}(\pi-x_\Delta^k(s))^{-(q+2)}\left((\pi-x_\Delta^k(s))f(x_\Delta(t_k))+(q+1)\sigma^2/2\right)ds.
\end{aligned}
$$

Using Proposition 7.2.1 and the Young inequality, we have

$$
\begin{aligned}
&-x_\Delta^k(s)^{-(q+2)}\left(x_\Delta^k(s)f(x_\Delta(t_k))-(q+1)\sigma^2/2\right)\\
\leqslant&-(1-\varepsilon_0)x_\Delta^k(s)^{-(q+2)}x_\Delta(t_k)f(x_\Delta(t_k))I_{\left\{x_\Delta^k(s)\geqslant(1-\varepsilon_0)x_\Delta(t_k),x_\Delta(t_k)\in[\phi(\Delta),x_1)\right\}}\\
&+0.5(q+1)\sigma^2x_\Delta^k(s)^{-(q+2)}I_{\left\{x_\Delta^k(s)\geqslant(1-\varepsilon_0)x_\Delta(t_k),x_\Delta(t_k)\in[\phi(\Delta),x_1)\right\}}\\
&+C\phi(\Delta)^{-(q+1)}\left(1+\phi(\Delta)^{-1}\right)I_{\left\{x_\Delta^k(s)<(1-\varepsilon_0)x_\Delta(t_k),x_\Delta(t_k)\in[\phi(\Delta),x_1)\right\}}\\
&+C\left(1+(\pi-x_\Delta(t_k))^{-1}\right)I_{\left\{x_\Delta^k(s)\geqslant0.5(\phi(\Delta_0)+x_1),x_\Delta(t_k)\in[x_1,\pi-\phi(\Delta)]\right\}}\\
&+C\phi(\Delta)^{-(q+1)}\left(1+\phi(\Delta)^{-1}\right)I_{\left\{x_\Delta^k(s)<0.5(\phi(\Delta_0)+x_1),x_\Delta(t_k)\in[x_1,\pi-\phi(\Delta)]\right\}},\\
\leqslant&C+C\phi(\Delta)^{-(q+2)}\left(I_{\mathcal{S}^1_{\Delta,s}}+I_{\mathcal{S}^4_{\Delta,s}}\right)+C(\pi-x_\Delta(t_k))^{-1},
\end{aligned}
\tag{7.3.1}
$$

and

$$
\begin{aligned}
&(\pi-x_\Delta^k(s))^{-(q+2)}\left((\pi-x_\Delta^k(s))f(x_\Delta(t_k))+(q+1)\sigma^2/2\right)\\
\leqslant&C\left(1+x_\Delta(t_k)^{-1}\right)I_{\left\{x_\Delta^k(s)\leqslant0.5(x_3+\pi-\phi(\Delta_0)),x_\Delta(t_k)\in[\phi(\Delta),x_3]\right\}}\\
&+C\phi(\Delta)^{-(q+1)}\left(1+\phi(\Delta)^{-1}\right)I_{\left\{x_\Delta^k(s)>0.5(x_3+\pi-\phi(\Delta_0)),x_\Delta(t_k)\in[\phi(\Delta),x_3]\right\}}\\
&+(1-\varepsilon_0)(\pi-x_\Delta^k(s))^{-(q+2)}(\pi-x_\Delta(t_k))f(x_\Delta(t_k))I_{\left\{x_\Delta^k(s)\leqslant x_\Delta(t_k)+\varepsilon_0(\pi-x_\Delta(t_k)),x_\Delta(t_k)\in(x_3,\pi-\phi(\Delta)]\right\}}\\
&+0.5(q+1)\sigma^2(\pi-x_\Delta^k(s))^{-(q+2)}I_{\left\{x_\Delta^k(s)\leqslant x_\Delta(t_k)+\varepsilon_0(\pi-x_\Delta(t_k)),x_\Delta(t_k)\in(x_3,\pi-\phi(\Delta)]\right\}}\\
&+C\phi(\Delta)^{-(q+1)}\left(1+\phi(\Delta)^{-1}\right)I_{\left\{x_\Delta^k(s)>x_\Delta(t_k)+\varepsilon_0(\pi-x_\Delta(t_k)),x_\Delta(t_k)\in(x_3,\pi-\phi(\Delta)]\right\}},\\
\leqslant&C+C\phi(\Delta)^{-(q+2)}\left(I_{\mathcal{S}^2_{\Delta,s}}+I_{\mathcal{S}^3_{\Delta,s}}\right)+Cx_\Delta(t_k)^{-1},
\end{aligned}
\tag{7.3.2}
$$

for all $s \in [t_k, t_{k+1}\wedge\bar\tau_\Delta^k\wedge\hat\tau_\Delta^k]$.

Using the Young inequality, (7.3.1), (7.3.2) and Lemma 7.3.1, we then have

$$
\begin{aligned}
&\mathbb{E}\left(x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k})^{-q}\right)+\mathbb{E}\left((\pi-x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k}))^{-q}\right)\\
\leqslant& C\Delta+C\phi(\Delta)^{-(q+2)}\sum_{i=1}^{4}\mathbb{E}\int_{t_k}^{t_{k+1}}I_{\mathcal{S}_{\Delta,s}^{i}}ds+(1+C\Delta)\mathbb{E}\left((\pi-x_{\Delta}(t_k))^{-q}+x_{\Delta}(t_k)^{-q}\right),\\
=& C\Delta+C\phi(\Delta)^{-(q+2)}\sum_{i=1}^{4}\int_{t_k}^{t_{k+1}}\Pr\left(\mathcal{S}_{\Delta,s}^{i}\right)ds+(1+C\Delta)\mathbb{E}\left((\pi-x_{\Delta}(t_k))^{-q}+x_{\Delta}(t_k)^{-q}\right),\\
\leqslant& e^{C\Delta}\mathbb{E}\left(x_{\Delta}(t_k)^{-q}+(\pi-x_{\Delta}(t_k))^{-q}\right)+C\Delta.
\end{aligned}
$$

For $u\in[\phi(\Delta),\pi-\phi(\Delta)]$, the function $u^{-q}+(\pi-u)^{-q}$ takes its maximum at $u=\phi(\Delta)$ and $u=\pi-\phi(\Delta)$. If $\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k}<t$, we then have

$$
\begin{aligned}
x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k})^{-q}+\left(\pi-x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k})\right)^{-q}=&\phi(\Delta)^{-q}+(\pi-\phi(\Delta))^{-q},\\
\geqslant& x_{\Delta}(t)^{-q}+(\pi-x_{\Delta}(t))^{-q},
\end{aligned}
$$

since $x_{\Delta}(t)\in[\phi(\Delta),\pi-\phi(\Delta)]$. Otherwise, we have

$$
x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k})^{-q}+\left(\pi-x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k})\right)^{-q}=x_{\Delta}(t)^{-q}+(\pi-x_{\Delta}(t))^{-q},
$$

since $t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k}=t$. In either case, we always have

$$
x_{\Delta}(t)^{-q}+(\pi-x_{\Delta}(t))^{-q}\leqslant x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k})^{-q}+\left(\pi-x_{\Delta}^{k}(t\wedge\bar{\tau}_{\Delta}^{k}\wedge\hat{\tau}_{\Delta}^{k})\right)^{-q},
$$

for all $t\in[t_k,t_{k+1}]$. Therefore, we have

$$
\sup_{t\in[t_k,t_{k+1}]}\mathbb{E}\left(x_{\Delta}(t)^{-q}+(\pi-x_{\Delta}(t))^{-q}\right)\leqslant e^{C\Delta}\mathbb{E}\left(x_{\Delta}(t_k)^{-q}+(\pi-x_{\Delta}(t_k))^{-q}\right)+C\Delta.
$$

By induction, we have

$$
\sup_{t\in[t_k,t_{k+1}]}\mathbb{E}\left(x_{\Delta}(t)^{-q}+(\pi-x_{\Delta}(t))^{-q}\right)\leqslant e^{C(k+1)\Delta}\left(\mathbb{E}\left(x_0^{-q}+(\pi-x_0)^{-q}\right)+C(k+1)\Delta\right),
$$

and therefore

$$\sup_{t\in[0,T]} \mathbb{E}\left(x_\Delta(t)^{-q} + (\pi - x_\Delta(t))^{-q}\right) \leqslant C.$$

Since

$$\sup_{t\in[0,T]} \mathbb{E}\left(x_\Delta(t)^{-q} + (\pi - x_\Delta(t))^{-q}\right) \leqslant 2\phi(\Delta_0)^{-q},$$

for all $\Delta \in (\Delta_0, 1]$, we derive the conclusion. □

Given a $k \in \mathbb{N}$, we define $e_k = x(t_{k+1}) - x_\Delta^k(t_{k+1})$ and $e(t) = x(t) - x_\Delta(t)$ for $t \in [t_k, t_{k+1}]$. We also let

$$\begin{aligned}
\bar{\mathcal{S}}^1_{\Delta,k} &= \left\{x^k_\Delta(t_{k+1}) \in (-\infty, \phi(\Delta))\right\},\\
\bar{\mathcal{S}}^2_{\Delta,k} &= \left\{x^k_\Delta(t_{k+1}) \in [\phi(\Delta), \pi - \phi(\Delta)]\right\},\\
\bar{\mathcal{S}}^3_{\Delta,k} &= \left\{x^k_\Delta(t_{k+1}) \in (\pi - \phi(\Delta), \infty)\right\},\\
\bar{\mathcal{S}}^1_{k} &= \left\{x(t_{k+1}) \in (0, \phi(\Delta))\right\},\\
\bar{\mathcal{S}}^2_{k} &= \left\{x(t_{k+1}) \in (\pi - \phi(\Delta), \pi)\right\}.
\end{aligned}$$

Now we prove the strong order one convergence of the projected EM method for the transformed WF model.

Theorem 7.3.1. *Let $\Delta \in (0,1]$. Then we have*

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant\lfloor T/\Delta\rfloor} e(t_k)^{2r}\right) \leqslant C\Delta^{2r}.$$

Proof. First, we let $\Delta \in (0, \Delta_0]$. Using the Itô formula for $f(x(s)) - f(x(t_k))$, we have

$$\begin{aligned} e_k =& e(t_k) + \int_{t_k}^{t_{k+1}} \left(f(x(s)) - f(x_\Delta(t_k))\right) ds, \\ =& e(t_k) + \int_{t_k}^{t_{k+1}} \left(f(x(t_k)) - f(x_\Delta(t_k))\right) ds \\ &+ \int_{t_k}^{t_{k+1}} \left(f(x(s)) - f(x(t_k))\right) ds, \\ =& e(t_k) + \int_{t_k}^{t_{k+1}} \left(f(x(t_k)) - f(x_\Delta(t_k))\right) ds \\ &+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \left(f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right) duds \\ &+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \sigma f'(x(u)) dB(u) ds, \\ =& e(t_k) + \left(f(x(t_k)) - f(x_\Delta(t_k))\right) \Delta + J_k, \end{aligned}$$

where

$$\begin{aligned} J_k =& \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \left(f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right) duds \\ &+ \int_{t_k}^{t_{k+1}} \int_{t_k}^{s} \sigma f'(x(u)) dB(u) ds. \end{aligned}$$

Now we have

$$\begin{aligned} e_k^2 =& e(t_k)^2 + \left(f(x(t_k)) - f(x_\Delta(t_k))\right)^2 \Delta^2 + J_k^2 \\ &+ 2e(t_k) \left(f(x(t_k)) - f(x_\Delta(t_k))\right) \Delta + 2e(t_k) J_k \\ &+ 2 \left(f(x(t_k)) - f(x_\Delta(t_k))\right) J_k \Delta. \end{aligned}$$

Using Proposition 7.2.1, we have

$$e(t_k) \left(f(x(t_k)) - f(x_\Delta(t_k))\right) < 0.$$

Using the the Young inequality, we then have

$$
\begin{aligned}
e_k^2 \leqslant & e(t_k)^2 + \left(f(x(t_k)) - f(x_\Delta(t_k))\right)^2 \Delta^2 + J_k^2 \\
& + 2e(t_k)J_k + \left(f(x(t_k)) - f(x_\Delta(t_k))\right)^2 \Delta^2 + J_k^2, \\
= & e(t_k)^2 + 2\left(f(x(t_k)) - f(x_\Delta(t_k))\right)^2 \Delta^2 + 2J_k^2 + 2e(t_k)J_k.
\end{aligned}
$$

Now we estimate $e(t_{k+1})^2$. We have

$$
e(t_{k+1})^2 = |\phi(\Delta) - x(t_{k+1})|^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} + |x^k_\Delta(t_{k+1}) - x(t_{k+1})|^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} + |\pi - \phi(\Delta) - x(t_{k+1})|^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}.
$$

Then we have

$$
e(t_{k+1})^2 \leqslant \begin{cases}
\phi(\Delta)^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}, & x(t_{k+1}) \in (0, \phi(\Delta)), \\
e_k^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}, & x(t_{k+1}) \in [\phi(\Delta), \pi - \phi(\Delta)], \\
e_k^2 I_{\bar{\mathcal{S}}^1_{\Delta,k}} + e_k^2 I_{\bar{\mathcal{S}}^2_{\Delta,k}} + \phi(\Delta)^2 I_{\bar{\mathcal{S}}^3_{\Delta,k}}, & x(t_{k+1}) \in (\pi - \phi(\Delta), \pi).
\end{cases}
$$

In summary, we have

$$
\begin{aligned}
e(t_{k+1})^2 \leqslant & e_k^2 I_{\{x(t_{k+1}) \in [\phi(\Delta), \pi - \phi(\Delta)]\}} + \left(e_k^2 + \phi(\Delta)^2\right) \left(I_{\bar{\mathcal{S}}^1_k} + I_{\bar{\mathcal{S}}^2_k}\right) \\
= & e_k^2 + \phi(\Delta)^2 \left(I_{\bar{\mathcal{S}}^1_k} + I_{\bar{\mathcal{S}}^2_k}\right).
\end{aligned}
$$

By induction, we have

$$
\begin{aligned}
e(t_{k+1})^2 \leqslant & \sum_{i=0}^{k} \left(2\left(f(x(t_i)) - f(x_\Delta(t_i))\right)^2 \Delta^2 + 2J_i^2 + \phi(\Delta)^2 \left(I_{\bar{\mathcal{S}}_i^1} + I_{\bar{\mathcal{S}}_i^2} \right) \right) \\
& + 2\sum_{i=0}^{k} e(t_i) J_i, \\
= & \sum_{i=0}^{k} \left(2\left(f(x(t_i)) - f(x_\Delta(t_i))\right)^2 \Delta^2 + 2J_i^2 + \phi(\Delta)^2 \left(I_{\bar{\mathcal{S}}_i^1} + I_{\bar{\mathcal{S}}_i^2} \right) \right) \\
& + 2\sum_{i=0}^{k} e(t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left(f'(x(u)) f(x(u)) + 0.5\sigma^2 f''(x(u)) \right) du ds \\
& + 2\sum_{i=0}^{k} e(t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \sigma f'(x(u)) dB(u) ds.
\end{aligned}
$$

Let $0 \leqslant m \leqslant \lfloor T/\Delta \rfloor$ be an arbitrary integer. Taking expectations on both sides, we then have

$$
\begin{aligned}
& \mathbb{E}\left(\sup_{0\leqslant k\leqslant m+1} e(t_k)^{2r} \right) \\
\leqslant & C\mathbb{E}\left(\sum_{i=0}^{m} \left(2\left(f(x(t_i)) - f(x_\Delta(t_i))\right)^2 \Delta^2 + 2J_i^2 + \phi(\Delta)^2 \left(I_{\bar{\mathcal{S}}_i^1} + I_{\bar{\mathcal{S}}_i^2} \right) \right) \right)^r \\
& + C\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} \left| \sum_{i=0}^{k} e(t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \left(f'(x(u)) f(x(u)) + 0.5\sigma^2 f''(x(u)) \right) du ds \right|^r \right) \\
& + C\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} \left| \sum_{i=0}^{k} e(t_i) \int_{t_i}^{t_{i+1}} \int_{t_i}^{s} \sigma f'(x(u)) dB(u) ds \right|^r \right) .. \qquad (7.3.3)
\end{aligned}
$$

Using Proposition 7.2.1, the mean value theory and the Young inequality, we have

$$
\begin{aligned}
& \left(f(x(t_i)) - f(x_\Delta(t_i))\right)^{2r} \Delta^{2r} \\
= & \left(f(x(t_i)) + f(x_\Delta(t_i))\right)^r \left(f(x(t_i)) - f(x_\Delta(t_i))\right)^r \Delta^{2r}, \\
\leqslant & C\left(x(t_i)^{-3r} + (\pi - x(t_i))^{-3r} + x_\Delta(t_i)^{-3r} + (\pi - x_\Delta(t_i))^{-3r} \right) \left(x(t_i) - x_\Delta(t_i) \right)^r \Delta^{2r}, \\
\leqslant & C\left(x(t_i)^{-3r} + (\pi - x(t_i))^{-3r} + x_\Delta(t_i)^{-3r} + (\pi - x_\Delta(t_i))^{-3r} \right)^2 \Delta^{3r} + e(t_i)^{2r} \Delta^r, \\
\leqslant & C\left(x(t_i)^{-6r} + (\pi - x(t_i))^{-6r} + x_\Delta(t_i)^{-6r} + (\pi - x_\Delta(t_i))^{-6r} \right) \Delta^{3r} + e(t_i)^{2r} \Delta^r.. \qquad (7.3.4)
\end{aligned}
$$

Using Lemma 7.2.1, Proposition 7.2.1 and the Hölder inequality, we have

$$\begin{aligned}
\mathbb{E}\sum_{i=0}^{m}|J_i|^{2r} \leqslant & C\Delta^{2r-1}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\left|\int_{t_i}^{s}\left(f'(x(u))f(x(u))+0.5\sigma^2 f''(x(u))\right)du\right|^{2r}ds\\
&+C\Delta^{2r-1}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\left|\int_{t_i}^{s}f'(x(u))dB(u)\right|^{2r}ds,\\
\leqslant & C\Delta^{4r-2}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u))+0.5\sigma^2 f''(x(u))\right|^{2r}duds\\
&+C\Delta^{3r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\mathbb{E}\int_{t_i}^{s}|f'(x(u))|^{2r}duds,\\
\leqslant & C\Delta^{4r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}\left(x(u)^{-6r}+(\pi-x(u))^{-6r}\right)duds\\
&+C\Delta^{3r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}\left(x(u)^{-4r}+(\pi-x(u))^{-4r}\right)duds,\\
\leqslant & C\Delta^{3r-1}, \qquad (7.3.5)
\end{aligned}$$

since $q \geqslant 6r$.

Using (7.3.4), (7.3.5), Lemmas 7.2.1, 7.3.2, the Young inequality, the Hölder inequality and the Chebyshev inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sum_{i=0}^{m}\left(2\left(f(x(t_i))-f(x_\Delta(t_i))\right)^2\Delta^2+2J_i^2+\phi(\Delta)^2\left(I_{\bar{\mathcal{S}}_i^1}+I_{\bar{\mathcal{S}}_i^2}\right)\right)\right)^r\\
\leqslant & Cm^{r-1}\sum_{i=0}^{m}\mathbb{E}\left(\left(f(x(t_i))-f(x_\Delta(t_i))\right)^{2r}\Delta^{2r}+|J_i|^{2r}\right)\\
&+Cm^{r-1}\phi(\Delta)^{2r}\sum_{i=0}^{m}\left(\Pr\left(\bar{\mathcal{S}}_i^1\right)+\Pr\left(\bar{\mathcal{S}}_i^2\right)\right),\\
\leqslant & Cm^{r-1}\Delta^{3r}\sum_{i=0}^{m}\mathbb{E}\left(x(t_i)^{-6r}+(\pi-x(t_i))^{-6r}+x_\Delta(t_i)^{-6r}+(\pi-x_\Delta(t_i))^{-6r}\right)\\
&+Cm^{r-1}\Delta^{3r-1}\\
&+Cm^{r-1}\phi(\Delta)^{2r}\sum_{i=0}^{m}\frac{\mathbb{E}\left(x(t_{i+1})^{-q}+(\pi-x(t_{i+1}))^{-q}\right)}{\phi(\Delta)^{-q}},\\
\leqslant & Cm^r\Delta^{3r}+Cm^{r-1}\Delta^{3r-1}+Cm^r\Delta^{4r-8\varepsilon_1 r},\\
\leqslant & Cm^r\Delta^{3r}+Cm^{r-1}\Delta^{3r-1}+Cm^r\Delta^{3r},
\end{aligned}$$

since $q \geqslant 6r$ and $\varepsilon < 0.125$. Since $0 \leqslant m \leqslant \lfloor T/\Delta \rfloor$, we have $m\Delta \leqslant T$. Then we have

$$\mathbb{E}\left(\sum_{i=0}^{m}\left(2\left(f(x(t_i)) - f(x_\Delta(t_i))\right)^2\Delta^2 + 2J_i^2 + \phi(\Delta)^2\left(I_{\bar{\mathcal{S}}_i^1} + I_{\bar{\mathcal{S}}_i^2}\right)\right)\right)^r \leqslant C\Delta^{2r}. \quad (7.3.6)$$

Using the Hölder inequality and the Young inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k} e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left(f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right)duds\right|^r\right)\\
\leqslant&\mathbb{E}\left(\sum_{i=0}^{m}|e(t_i)|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right|duds\right)^r,\\
\leqslant&m^{r-1}\mathbb{E}\sum_{i=0}^{m}|e(t_i)|^r\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right|duds\right|^r,\\
\leqslant&\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r} + Cm^{2r-2}\Delta^{-1}\mathbb{E}\sum_{i=0}^{m}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left|f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right|duds\right|^{2r},\\
\leqslant&\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r} + Cm^{2r-2}\Delta^{2r-2}\mathbb{E}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\left|\int_{t_i}^{s}\left|f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right|du\right|^{2r}ds.
\end{aligned}$$

Using Proposition 7.2.1 and Lemma 7.2.1, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k} e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left(f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right)duds\right|^r\right)\\
\leqslant&\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r} + Cm^{2r-2}\Delta^{4r-3}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}\left(x(u)^{-6r} + (\pi - x(u))^{-6r}\right)duds,\\
\leqslant&\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r} + Cm^{2r-1}\Delta^{4r-1}.
\end{aligned}$$

Since $m\Delta \leqslant T$, we finally have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k} e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\left(f'(x(u))f(x(u)) + 0.5\sigma^2 f''(x(u))\right)duds\right|^r\right)\\
\leqslant&\Delta\mathbb{E}\sum_{i=0}^{m}e(t_i)^{2r} + C\Delta^{2r}. \qquad (7.3.7)
\end{aligned}$$

Since

$$\mathbb{E}\left(e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\sigma f'(x(u))dB(u)ds \mid \mathcal{F}_{t_i}\right) = e(t_i)\mathbb{E}\left(\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\sigma f'(x(u))dB(u)ds \mid \mathcal{F}_{t_i}\right)$$
$$=0,$$

$$\left\{\sum_{i=0}^{k} e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\sigma f'(x(u))dB(u)ds\right\}_{k=0,1,2,\dots,m}$$

is a martingale. Using the Burkholder-Davis-Gundy inequality and the Young inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k} e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\sigma f'(x(u))dB(u)ds\right|^r\right)\\
\leqslant & C\mathbb{E}\left(\sum_{i=0}^{m}|e(t_i)|^2\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s} f'(x(u))dB(u)ds\right|^2\right)^{r/2},\\
\leqslant & C\mathbb{E}\left(\left(\sup_{0\leqslant k\leqslant m}|e(t_k)|^r\right)\left(\sum_{i=0}^{m}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s} f'(x(u))dB(u)ds\right|^2\right)^{r/2}\right),\\
\leqslant & 0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} e(t_k)^{2r}\right)+C\mathbb{E}\left(\sum_{i=0}^{m}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s} f'(x(u))dB(u)ds\right|^2\right)^r.
\end{aligned}$$

Using the Hölder inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k} e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\sigma f'(x(u))dB(u)ds\right|^r\right)\\
\leqslant & 0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} e(t_k)^{2r}\right)+Cm^{r-1}\mathbb{E}\sum_{i=0}^{m}\left|\int_{t_i}^{t_{i+1}}\int_{t_i}^{s} f'(x(u))dB(u)ds\right|^{2r},\\
\leqslant & 0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} e(t_k)^{2r}\right)+Cm^{r-1}\Delta^{2r-1}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\mathbb{E}\left|\int_{t_i}^{s} f'(x(u))dB(u)\right|^{2r}ds,\\
\leqslant & 0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} e(t_k)^{2r}\right)+Cm^{r-1}\Delta^{3r-2}\sum_{i=0}^{m}\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\mathbb{E}\left(x(u)^{-4r}+(\pi-x(u))^{-4r}\right)duds.
\end{aligned}$$

Using Lemma 7.2.1 and $m\Delta \leqslant T$, we finally have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant m}\left|\sum_{i=0}^{k} e(t_i)\int_{t_i}^{t_{i+1}}\int_{t_i}^{s}\sigma f'(x(u))dB(u)ds\right|^r\right)\\
\leqslant&0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} e(t_k)^{2r}\right) + Cm^r\Delta^{3r},\\
\leqslant&0.5\mathbb{E}\left(\sup_{0\leqslant k\leqslant m} e(t_k)^{2r}\right) + C\Delta^{2r}. \qquad (7.3.8)
\end{aligned}$$

Substituting (7.3.6), (7.3.7) and (7.3.8) into (7.3.3), we finally have

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant m+1} e(t_k)^{2r}\right) \leqslant C\Delta\mathbb{E}\sum_{i=0}^{m} e(t_i)^{2r} + C\Delta^{2r},$$

for all $0 \leqslant m \leqslant \lfloor T/\Delta\rfloor$. Then the Gronwall inequality implies

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant\lfloor T/\Delta\rfloor} e(t_k)^{2r}\right) \leqslant C\Delta^{2r}.$$

Finally, the conclusion clearly holds for $\Delta \in (\Delta_0, 1]$. □

Finally, we use Theorem 7.3.1 to prove the strong order one convergence of the projected EM method for the original WF model.

Theorem 7.3.2. *Let $\Delta \in (0, 1]$. Then we have*

$$\mathbb{E}\left(\sup_{0\leqslant k\leqslant\lfloor T/\Delta\rfloor} |y(t_k) - y_\Delta(t_k)|^{2r}\right) \leqslant C\Delta^{2r}.$$

Proof. Using Theorem 7.3.1, we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\leqslant k\leqslant \lfloor T/\Delta\rfloor} |y(t_k)-y_\Delta(t_k)|^{2r}\right)\\
=&\mathbb{E}\left(\sup_{0\leqslant k\leqslant \lfloor T/\Delta\rfloor} |\sin^2(x(t_k)/2)-\sin^2(x_\Delta(t_k)/2)|^{2r}\right),\\
=&\mathbb{E}\left(\sup_{0\leqslant k\leqslant \lfloor T/\Delta\rfloor} |\sin(x(t_k)/2)+\sin(x_\Delta(t_k)/2)|^{2r}|\sin(x(t_k)/2)-\sin(x_\Delta(t_k)/2)|^{2r}\right),\\
\leqslant& C\mathbb{E}\left(\sup_{0\leqslant k\leqslant \lfloor T/\Delta\rfloor} e(t_k)^{2r}\right),\\
\leqslant& C\Delta^{2r}.
\end{aligned}
$$

□

The strong convergence theory of the Lamperti smooth sloping truncation method has been established only for $\frac{(\beta-\alpha)\wedge\alpha}{\sigma^2} > 2.75$ and in the $\mathcal{L}^2$-norm (see Corollary 9 in [59]). In this section, we establish the strong convergence theory for $\frac{(\beta-\alpha)\wedge\alpha}{\sigma^2} \in (2,\infty)$ and in the general $\mathcal{L}^p$-norm.

7.4 Numerical simulations

We first conduct numerical simulations to support our theoretical results. In each example, we let $T=1$. We conduct numerical simulations with 1000 sample paths for step sizes $\Delta = 2^{-10}, 2^{-9}, 2^{-8}, 2^{-7}$. In view of the fact that there is no analytical solution for the WF model, we regard the numerical solution with the step size $\Delta = 2^{-20}$ as the "exact" solution. We will let r be different values and show that experimental $2r$-th strong convergence errors over an interval have about order $2r$ in each example.

We now conduct numerical simulations for three different parameter settings.

1. $r=2$, $y_0=0.01$, $\alpha=1$, $\beta=2$ and $\sigma=0.5$ (Figure 7.4.1);

2. $r=6$, $y_0=0.99$, $\alpha=0.1$, $\beta=0.4$ and $\sigma=0.1$ (Figure 7.4.2).

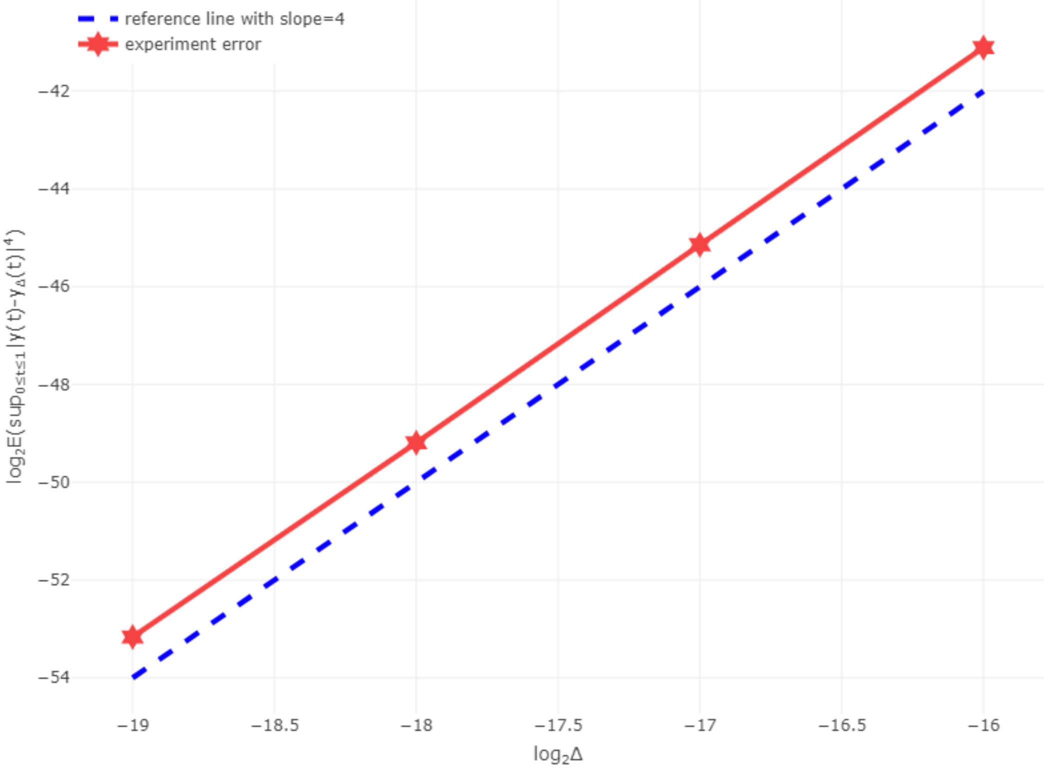

Figure 7.4.1: The $\mathcal{L}^4$-strongly convergence order of the projected EM method for the WF model with the initial value $y_0 = 0.01$.

Using the linear regression method, the experimental error (see Figures 7.4.1 and 7.4.2) shows that the strong convergence error have order about 4.0523 and 12.0535. They suggest that the strong convergence error for the $2r$-th moment has order about $2r$. Our numerical simulations show that the projected EM method works well for general $\mathcal{L}^{2r}$-norm as long as $1 \leqslant r < \frac{2(\beta-\alpha)\wedge 2\alpha}{3\sigma^2} - \frac{1}{3}$.

7.5 Conclusion

In this chapter, we study the strong convergence theory of the projected EM method for the WF model, which is a popular SDE model without an analytical solution. We extend numerical analysis techniques in Chapter 6 and prove finite inverse moments near two endpoints. Then we prove that the projected EM method is positivity preserving and $\mathcal{L}^{2r}$-strongly convergent with order one, where $1 \leqslant r < \frac{2(\beta-\alpha)\wedge 2\alpha}{3\sigma^2} - \frac{1}{3}$. Compared

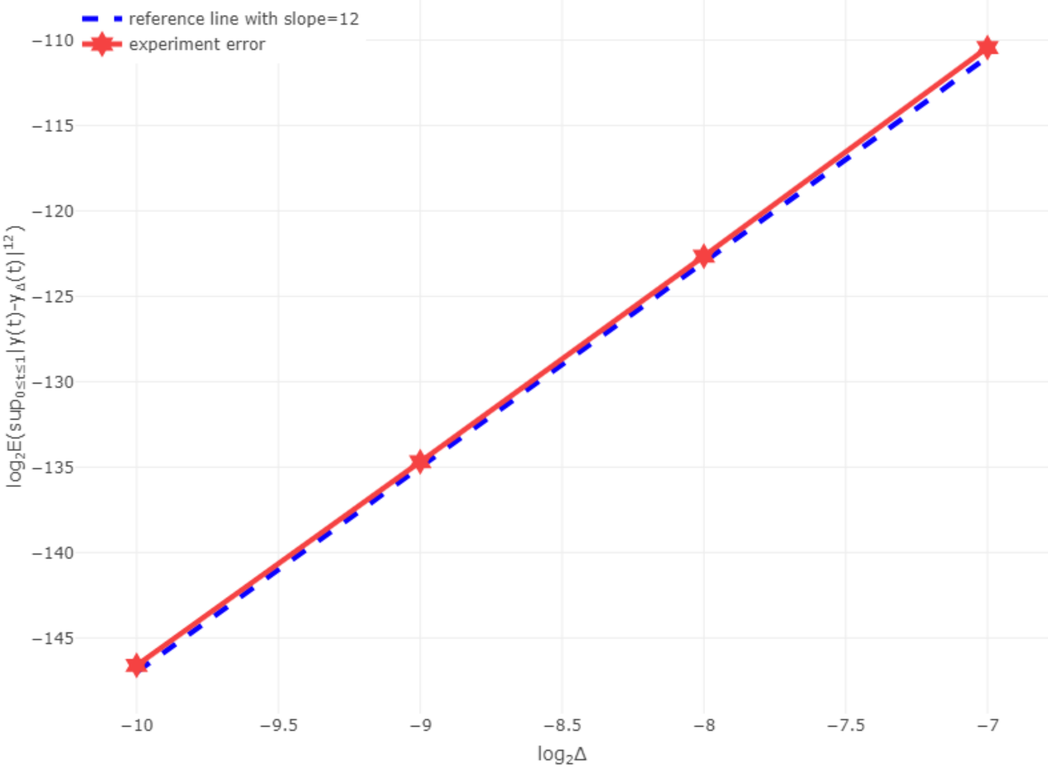

Figure 7.4.2: The $\mathcal{L}^{12}$-strongly convergence order of the projected EM method for the WF model with the initial value $y_0 = 0.99$.

to existing explicit EM methods for the WF model, the projected EM method has better proven $\mathcal{L}^p$-strong convergence rate for some parameter settings. We also conduct numerical simulations to support our theoretical results.

Chapter 8

Conclusion and Future work

In this thesis, we introduce in detail our contributions to developing modified EM methods for SDEs with locally Lipschitz coefficients. In each chapter, we also conduct numerical simulations to support our theoretical results. In Chapter 3, we extend the truncated EM method for multi-dimensional SDEs with polynomially growing drift and concave diffusion coefficients satisfying the Osgood condition. We then introduced the the logarithmic truncated EM method, and used an improved numerical analysis method to prove finite inverse moments of the logarithmic truncated EM numerical solution. In Chapter 4, we then show that the logarithmic truncated EM method is $\mathcal{L}^p$-strongly convergent with order one half for the CEV model and the Aït-Sahalia model with a wider parameter range. Our numerical analysis methods can also improve the strong convergence results of the truncated EM methods.

In Chapter 5-7, we focus on studying the strong convergence theory of the projected EM method. The projected EM method is developed to replace the drift-implicit EM method to some extent. Compared to existing EM methods, we proved that the projected EM method has better theoretical $\mathcal{L}^p$-strong convergence rate for many important SDE models, e.g., the CIR model, the CEV model, the Aït-Sahalia model, the Heston-3/2 volatility model and the WF model. In particular, many existing explicit EM methods has only at most $\mathcal{L}^2$-strongly convergence rate for some SDEs. That is because previous explicit EM methods generally do not have finite inverse moments and they have to use specific numerical analysis methods to derive concrete convergence rate. However, the projected EM method will generate approximations only in a certain range, which can guarantee finite inverse moments near some finite endpoints.

Nevertheless, the projected EM still fails to cover some boundary parameter settings. A possible way is to modify the projected EM to further capture structures of coefficients of a specific SDE model. There are also many other problems needed to be solved. For example, the strong convergence theory is established based on the one-sided Lipschitz condition. However, there are still many SDE models which fail to be covered, e.g., the stochastic population model. Novel numerical analysis techniques should be developed to study the strong convergence rate of EM methods for these SDEs. In addition, we used the Bihari inequality in Chapter 3 and only prove the $\mathcal{L}^p$-strong convergence without concrete convergence rate. We also only consider modified EM methods for SDEs, and do not involve delay functions, Poisson jumps, Markov switching and so on. Further work should be devoted into these more complicated models.

Bibliography

[1] Y. Tang, X. Mao, The modified truncated Euler-Maruyama method for stochastic differential equations with concave diffusion coefficients, Journal of Computational and Applied Mathematics 440 (2024) 115660. doi:https://doi.org/10.1016/j.cam.2023.115660.
URL https://www.sciencedirect.com/science/article/pii/S0377042723006040

[2] Y. Tang, X. Mao, The logarithmic truncated EM method with weaker conditions, Applied Numerical Mathematics 198 (2024) 258–275. doi:https://doi.org/10.1016/j.apnum.2024.01.009.
URL https://www.sciencedirect.com/science/article/pii/S0168927424000102

[3] X. Mao, Stochastic Differential Equations and Applications, Woodhead Publishing, 2007.
URL https://books.google.com.hk/books?id=l5ejAgAAQBAJ

[4] T. Yamada, S. Watanabe, On the uniqueness of solutions of stochastic differential equations, Journal of Mathematics of Kyoto University 11 (1) (1971) 155–167. doi:10.1215/kjm/1250523691.
URL https://doi.org/10.1215/kjm/1250523691

[5] S. Watanabe, T. Yamada, On the uniqueness of solutions of stochastic differential equations ii, Journal of Mathematics of Kyoto University 11 (3) (1971) 553 – 563.

doi:10.1215/kjm/1250523620.
URL https://doi.org/10.1215/kjm/1250523620

[6] M. Hutzenthaler, A. Jentzen, P. E. Kloeden, Strong and weak divergence in finite time of Euler's method for stochastic differential equations with non-globally Lipschitz continuous coefficients, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 467 (2130) (2011) 1563–1576. doi:https://doi.org/10.1098/rspa.2010.0348.

[7] D. J. Higham, X. Mao, A. M. Stuart, Strong convergence of Euler-type methods for nonlinear stochastic differential equations, SIAM Journal on Numerical Analysis 40 (3) (2003) 1041–1063. doi:10.1137/S0036142901389530.
URL http://www.jstor.org/stable/4100914

[8] M. Hutzenthaler, A. Jentzen, P. E. Kloeden, Strong convergence of an explicit numerical method for SDEs with nonglobally Lipschitz continuous coefficients, The Annals of Applied Probability 22 (4) (2012) 1611 – 1641. doi:10.1214/11-AAP803.
URL https://doi.org/10.1214/11-AAP803

[9] S. Sabanis, A note on tamed Euler approximations, Electronic Communications in Probability 18 (none) (2013) 1 – 10. doi:10.1214/ECP.v18-2824.
URL https://doi.org/10.1214/ECP.v18-2824

[10] S. Sabanis, Euler approximations with varying coefficients: The case of superlinearly growing diffusion coefficients, The Annals of Applied Probability 26 (4) (2016) 2083 – 2105. doi:10.1214/15-AAP1140.
URL https://doi.org/10.1214/15-AAP1140

[11] W. Liu, X. Mao, Strong convergence of the stopped Euler-Maruyama method for nonlinear stochastic differential equations, Applied Mathematics and Computation 223 (2013) 389–400. doi:https://doi.org/10.1016/j.amc.2013.08.023.
URL https://www.sciencedirect.com/science/article/pii/S0096300313008722

[12] X. Mao, The truncated Euler-Maruyama method for stochastic differential equations, Journal of Computational and Applied Mathematics 290 (2015) 370–384. doi:10.1016/j.cam.2015.06.002.
URL https://www.sciencedirect.com/science/article/pii/S0377042715003210

[13] X. Mao, Convergence rates of the truncated Euler-Maruyama method for stochastic differential equations, Journal of Computational and Applied Mathematics 296 (2016) 362–375. doi:10.1016/j.cam.2015.09.035.
URL https://www.sciencedirect.com/science/article/pii/S0377042715004884

[14] X. Li, X. Mao, G. Yin, Explicit numerical approximations for stochastic differential equations in finite and infinite horizons: truncation methods, convergence in pth moment and stability, IMA Journal of Numerical Analysis 39 (2) (2018) 847–892. doi:https://doi.org/10.1093/imanum/dry015.

[15] P. Malliavin, The canonic diffusion above the diffeomorphism group of the circle, Comptes Rendus de l'Académie des Sciences - Series I - Mathematics 329 (4) (1999) 325–329. doi:https://doi.org/10.1016/S0764-4442(00)88575-4.
URL https://www.sciencedirect.com/science/article/pii/S0764444200885754

[16] A. Berkaoui, M. Bossy, A. Diop, Euler scheme for SDEs with non-Lipschitz diffusion coefficient: strong convergence, ESAIM: Probability and Statistics 12 (2008) 1–11. doi:10.1051/ps:2007030.
URL http://www.numdam.org/articles/10.1051/ps:2007030/

[17] A. Alfonsi, On the discretization schemes for the CIR (and Bessel squared) processes, Monte Carlo Methods and Applications 11 (4) (2005) 355–384 [cited 2024-06-22]. doi:doi:10.1515/156939605777438569.
URL https://doi.org/10.1515/156939605777438569

[18] D. Higham, X. Mao, Convergence of Monte Carlo simulations involving the mean-reverting square root process, Journal of Computational Finance 8 (3) (2005) 35–61. doi:https://doi.org/10.21314/JCF.2005.136.

[19] A. Alfonsi, Strong order one convergence of a drift implicit Euler scheme: Application to the CIR process, Statistics & Probability Letters 83 (2) (2013) 602–607. doi:10.1016/j.spl.2012.10.034.
URL https://www.sciencedirect.com/science/article/pii/S0167715212004063

[20] A. Neuenkirch, L. Szpruch, First order strong approximations of scalar SDEs defined in a domain, Numerische Mathematik 128 (1) (2014) 103–136. doi:10.1007/s00211-014-0606-4.
URL https://doi.org/10.1007/s00211-014-0606-4

[21] S. Dereich, A. Neuenkirch, L. Szpruch, An Euler-type method for the strong approximation of the Cox-Ingersoll-Ross process, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences 468 (2140) (2012) 1105–1115. arXiv:https://royalsocietypublishing.org/doi/pdf/10.1098/rspa.2011.0505, doi:10.1098/rspa.2011.0505.
URL https://royalsocietypublishing.org/doi/abs/10.1098/rspa.2011.0505

[22] M. Hefter, A. Herzwurm, Strong convergence rates for Cox-Ingersoll-Ross processes — Full parameter range, Journal of Mathematical Analysis and Applications 459 (2) (2018) 1079–1101. doi:10.1016/j.jmaa.2017.10.076.
URL https://www.sciencedirect.com/science/article/pii/S0022247X17309897

[23] J.-F. Chassagneux, A. Jacquier, I. Mihaylov, An explicit Euler scheme with strong rate of convergence for financial SDEs with non-Lipschitz coefficients, SIAM Journal on Financial Mathematics 7 (1) (2016) 993–1021. arXiv:https:

//doi.org/10.1137/15M1017788, doi:10.1137/15M1017788.
URL https://doi.org/10.1137/15M1017788

[24] R. Lord, R. Koekkoek, D. V. Dijk, A comparison of biased simulation schemes for stochastic volatility models, Quantitative Finance 10 (2) (2010) 177–194. arXiv:https://doi.org/10.1080/14697680802392496, doi:10.1080/14697680802392496.
URL https://doi.org/10.1080/14697680802392496

[25] A. Cozma, C. Reisinger, Strong order 1/2 convergence of full truncation Euler approximations to the Cox-Ingersoll-Ross process, IMA Journal of Numerical Analysis 40 (1) (2018) 358–376. arXiv:https://academic.oup.com/imajna/article-pdf/40/1/358/31786574/dry067.pdf, doi:10.1093/imanum/dry067.
URL https://doi.org/10.1093/imanum/dry067

[26] C. Kelly, G. J. Lord, An adaptive splitting method for the Cox-Ingersoll-Ross process, Applied Numerical Mathematics 186 (2023) 252–273. doi:https://doi.org/10.1016/j.apnum.2023.01.014.
URL https://www.sciencedirect.com/science/article/pii/S0168927423000144

[27] C. Kelly, G. Lord, H. Maulana, The role of adaptivity in a numerical method for the Cox-Ingersoll-Ross model, Journal of Computational and Applied Mathematics 410 (2022) 114208. doi:https://doi.org/10.1016/j.cam.2022.114208.
URL https://www.sciencedirect.com/science/article/pii/S0377042722000760

[28] I. Gyöngy, M. Rásonyi, A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients, Stochastic Processes and their Applications 121 (10) (2011) 2189–2200. doi:10.1016/j.spa.2011.06.008.
URL https://www.sciencedirect.com/science/article/pii/S030441491100144X

[29] A. Mickel, A. Neuenkirch, The weak convergence order of two Euler-type discretization schemes for the log-Heston model, IMA Journal of Numerical Analysis 43 (6) (2022) 3326–3356. arXiv:https://academic.oup.com/imajna/article-pdf/43/6/3326/53969063/drac069.pdf, doi:10.1093/imanum/drac069.
URL https://doi.org/10.1093/imanum/drac069

[30] A. Mickel, A. Neuenkirch, Sharp L^1-approximation of the log-Heston stochastic differential equation by Euler-type methods, Journal of Computational Finance 26 (4) (2023) 67–100. doi:10.21314/JCF.2023.002.
URL https://doi.org/10.21314/JCF.2023.002

[31] H.-L. Ngo, D.-T. Luong, Strong rate of tamed Euler-Maruyama approximation for stochastic differential equations with Hölder continuous diffusion coefficient, Brazilian Journal of Probability and Statistics 31 (1) (2017) 24–40. doi:10.1214/15-BJPS301.
URL https://doi.org/10.1214/15-BJPS301

[32] H. Yang, F. Wu, P. E. Kloeden, X. Mao, The truncated Euler-Maruyama method for stochastic differential equations with Hölder diffusion coefficients, Journal of Computational and Applied Mathematics 366 (2020) 112379. doi:https://doi.org/10.1016/j.cam.2019.112379.
URL https://www.sciencedirect.com/science/article/pii/S0377042719303826

[33] H. Yang, J. Huang, Convergence and stability of modified partially truncated Euler-Maruyama method for nonlinear stochastic differential equations with hölder continuous diffusion coefficient, Journal of Computational and Applied Mathematics 404 (2022) 113895. doi:10.1016/j.cam.2021.113895.
URL https://www.sciencedirect.com/science/article/pii/S0377042721005185

[34] T. Yamada, On the successive approximation of solutions of stochastic differential equations, Journal of Mathematics of Kyoto University 21 (3) (1981) 501–515. doi:https://doi.org/10.1215/kjm/1250521975.
URL https://doi.org/10.1215/kjm/1250521975

[35] Y. Yi, Y. Hu, J. Zhao, Positivity preserving logarithmic Euler-Maruyama type scheme for stochastic differential equations, Communications in Nonlinear Science and Numerical Simulation 101 (2021) 105895. doi:https://doi.org/10.1016/j.cnsns.2021.105895.
URL https://www.sciencedirect.com/science/article/pii/S1007570421002070

[36] Z. Lei, S. Gan, Z. Chen, Strong and weak convergence rates of logarithmic transformed truncated EM methods for SDEs with positive solutions, Journal of Computational and Applied Mathematics 419 (2023) 114758. doi:https://doi.org/10.1016/j.cam.2022.114758.
URL https://www.sciencedirect.com/science/article/pii/S0377042722003867

[37] L. Szpruch, X. Mao, D. J. Higham, J. Pan, Numerical simulation of a strongly nonlinear Ait-Sahalia-type interest rate model, BIT Numerical Mathematics 51 (2) (2011) 405–425. doi:https://doi.org/10.1007/s10543-010-0288-y.
URL https://doi.org/10.1007/s10543-010-0288-y

[38] L. Hu, X. Li, X. Mao, Convergence rate and stability of the truncated Euler-Maruyama method for stochastic differential equations, Journal of Computational and Applied Mathematics 337 (2018) 274–289. doi:https://doi.org/10.1016/j.cam.2018.01.017.
URL https://www.sciencedirect.com/science/article/pii/S0377042718300463

[39] J. C. Cox, J. E. Ingersoll, S. A. Ross, A theory of the term structure of interest rates, Econometrica 53 (2) (1985) 385–407.
URL http://www.jstor.org/stable/1911242

[40] M. Broadie, O. Kaya, Exact simulation of stochastic volatility and other affine jump diffusion processes, Operations Research 54 (2) (2006) 217–231. arXiv:https://doi.org/10.1287/opre.1050.0247, doi:10.1287/opre.1050.0247.
URL https://doi.org/10.1287/opre.1050.0247

[41] R. Chitashvili, N. Lazrieva, Strong solutions of stochastic differential equations with boundary conditions, Stochastics 5 (4) (1981) 255–309. arXiv:https://doi.org/10.1080/17442508108833184, doi:10.1080/17442508108833184.
URL https://doi.org/10.1080/17442508108833184

[42] L. Słomiński, On approximation of solutions of multidimensional SDE's with reflecting boundary conditions, Stochastic Processes and their Applications 50 (2) (1994) 197–219. doi:https://doi.org/10.1016/0304-4149(94)90118-X.
URL https://www.sciencedirect.com/science/article/pii/030441499490118X

[43] D. Lépingle, Euler scheme for reflected stochastic differential equations, Mathematics and Computers in Simulation 38 (1) (1995) 119–126. doi:https://doi.org/10.1016/0378-4754(93)E0074-F.
URL https://www.sciencedirect.com/science/article/pii/0378475493E0074F

[44] W.-J. Beyn, E. Isaak, R. Kruse, Stochastic C-stability and B-consistency of explicit and implicit Euler-type schemes, Journal of Scientific Computing 67 (3) (2016) 955–987. doi:10.1007/s10915-015-0114-4.
URL https://doi.org/10.1007/s10915-015-0114-4

[45] M. Li, C. Huang, Z. Chen, Compensated projected Euler-Maruyama method for stochastic differential equations with superlinear jumps, Applied Mathematics and Computation 393 (2021) 125760. doi:https:

//doi.org/10.1016/j.amc.2020.125760.
URL https://www.sciencedirect.com/science/article/pii/S009630032030713X

[46] M. Li, C. Huang, Projected Euler-Maruyama method for stochastic delay differential equations under a global monotonicity condition, Applied Mathematics and Computation 366 (2020) 124733. doi:https://doi.org/10.1016/j.amc.2019.124733.
URL https://www.sciencedirect.com/science/article/pii/S0096300319307258

[47] Y. Tang, Strong order one convergence of the projected Euler-Maruyama method for the wright-fisher model, Communications in Nonlinear Science and Numerical Simulation 145 (2025) 108759. doi:https://doi.org/10.1016/j.cnsns.2025.108759.
URL https://www.sciencedirect.com/science/article/pii/S1007570425001704

[48] M. Hefter, A. Herzwurm, T. Müller-Gronbach, Lower error bounds for strong approximation of scalar SDEs with non-Lipschitzian coefficients, The Annals of Applied Probability 29 (1) (2019) pp. 178–216.
URL https://www.jstor.org/stable/26581788

[49] M. Hefter, A. Jentzen, On arbitrarily slow convergence rates for strong numerical approximations of Cox-Ingersoll-Ross processes and squared Bessel processes, Finance and Stochastics 23 (1) (2019) 139–172. doi:10.1007/s00780-018-0375-5.
URL https://doi.org/10.1007/s00780-018-0375-5

[50] G. Lord, M. Wang, Convergence of a exponential tamed method for a general interest rate model, Applied Mathematics and Computation 467 (2024) 128503. doi:https://doi.org/10.1016/j.amc.2023.128503.
URL https://www.sciencedirect.com/science/article/pii/S0096300323006720

[51] N. Halidias, I. S. Stamatiou, Boundary preserving explicit scheme for the Aït-Sahalia mode, Discrete and Continuous Dynamical Systems - B 28 (1) (2023) 648–664. doi:10.3934/dcdsb.2022092.
URL https://www.aimsciences.org/article/id/627e24542d80b75a35c2ba8c

[52] X. Wu, S. Gan, An explicit positivity-preserving scheme for the Heston 3/2-model with order-one strong convergence, Communications in Nonlinear Science and Numerical Simulation 140 (2025) 108372. doi:https://doi.org/10.1016/j.cnsns.2024.108372.
URL https://www.sciencedirect.com/science/article/pii/S1007570424005574

[53] C. Emmanuel, X. Mao, Truncated EM numerical method for generalised Ait-Sahalia-type interest rate model with delay, Journal of Computational and Applied Mathematics 383 (2021) 113137. doi:https://doi.org/10.1016/j.cam.2020.113137.
URL https://www.sciencedirect.com/science/article/pii/S0377042720304283

[54] S. Deng, C. Fei, W. Fei, X. Mao, Positivity-preserving truncated Euler-Maruyama method for generalised Ait-Sahalia-type interest model, BIT Numerical Mathematics 63 (4) (2023) 59. doi:10.1007/s10543-023-01000-x.
URL https://doi.org/10.1007/s10543-023-01000-x

[55] R. Liu, Y. Cao, X. Wang, Unconditionally positivity-preserving explicit Euler-type schemes for a generalized Aït-Sahalia model, Numerical Algorithmsdoi:10.1007/s11075-024-01810-2.
URL https://doi.org/10.1007/s11075-024-01810-2

[56] L. Chen, S. Gan, X. Wang, First order strong convergence of an explicit scheme for the stochastic SIS epidemic model, Journal of Computational and Applied Mathematics 392 (2021) 113482. doi:https://doi.org/10.1016/j.cam.2021.113482.

URL https://www.sciencedirect.com/science/article/pii/S0377042721001011

[57] H. Yang, J. Huang, First order strong convergence of positivity preserving logarithmic Euler-Maruyama method for the stochastic SIS epidemic model, Applied Mathematics Letters 121 (2021) 107451. doi:https://doi.org/10.1016/j.aml.2021.107451.
URL https://www.sciencedirect.com/science/article/pii/S089396592100238X

[58] P. A. Jenkins, D. Spanò, Exact simulation of the Wright-Fisher diffusion, The Annals of Applied Probability 27 (3) (2017) 1478 – 1509. doi:10.1214/16-AAP1236.
URL https://doi.org/10.1214/16-AAP1236

[59] L. Chen, S. Gan, Strong convergence and stationary distribution of an explicit scheme for the Wright-Fisher model, Journal of Computational and Applied Mathematics 424 (2023) 115017. doi:https://doi.org/10.1016/j.cam.2022.115017.
URL https://www.sciencedirect.com/science/article/pii/S037704272200615X

[60] I. Stamatiou, A boundary preserving numerical scheme for the Wright-Fisher model, Journal of Computational and Applied Mathematics 328 (2018) 132–150. doi:https://doi.org/10.1016/j.cam.2017.07.011.
URL https://www.sciencedirect.com/science/article/pii/S0377042717303527

[61] C. E. Dangerfield, D. Kay, S. MacNamara, K. Burrage, A boundary preserving numerical algorithm for the Wright-Fisher model with mutation, BIT Numerical Mathematics 52 (2) (2012) 283–304. doi:10.1007/s10543-011-0351-3.
URL https://doi.org/10.1007/s10543-011-0351-3