Dual Regime Continuous Coagulation and Fragmentation Equations

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Abstract

The objective within this thesis is the bringing to bear of functional analytic techniques to the rigorous analysis of fragmentation and coagulation-fragmentation models involving a mass cut-off. In the models we examine, particles which have a mass exceeding the cut-off are able to fragment in the usual way. However, particles with mass below the cut-off are unable to break up any further. This results in a dual regime model, with one regime consisting of the above cut-off particles, and another containing the sub cut-off particles.

Initially we consider the case of pure fragmentation. The model setup leads to a system of two integro-differential equations. These equations are reformulated as two abstract differential equations within the setting of physically relevant function spaces. The two separate equations are then combined to form a single abstract Cauchy problem involving a 2×2 operator matrix acting on the product of our function spaces. This operator matrix is then shown to generate a strongly continuous semigroup on the product space, providing us with a unique set of strongly differentiable solutions. The solutions given by this semigroup are shown to preserve positivity and conserve mass.

Having considered the case of pure fragmentation, we then introduce a coagulation process to the dual regime model. The analysis of this combined model is most readily carried out in a revised pair of function spaces, and the first task is to establish that the fragmentation system still generates a semigroup in the revised product space. The additional coagulation terms are then treated as a nonlinear perturbation of the pure fragmentation system. Our choice of space allows us to establish the desired Lipschitz and Fréchet differentiability properties of the nonlinear coagulation operator, giving us the existence of a local in time, strongly differentiable solution. This solution is shown to preserve positivity over its maximal interval of existence. Finally, using a Gronwall type inequality, we demonstrate that the solution does not blow up in finite time, and hence is a strongly differentiable global solution.

Contents

Chapter 1

Introduction

Coagulation and fragmentation processes are commonly observed phenomena amongst physical systems, arising in areas such as colloidal chemistry, polymer science and astrophysics. Mathematical models of such processes generally characterise the particles within the system by some physical state variable, for example their volume or mass. The aim is then to determine the dynamical behaviour of the system with respect to this variable as time progresses. Models can be classified as either discrete or continuous depending on the nature of the state variable of interest. In this thesis we shall be examining continuous models exclusively.

1.1 Pure Fragmentation Equations

In a pure fragmentation process, particles can break up into smaller pieces but are unable to coalesce to form larger particles. A commonly studied model of such processes is the multiple fragmentation equation

$$\frac{\partial u(x,t)}{\partial t} = -u(x,t)\int_0^x \frac{y}{x}\gamma(x,y)\,dy + \int_x^\infty \gamma(y,x)u(y,t)dy,\;\; x>0,\; t>0. \quad (1.1)$$

This description of fragmentation was first formulated by Blatz and Tobolsky [10] and was subsequently analysed by Melzak [28]. The function $u(x,t)$ represents the density of particles of mass x at time t, so that $u(x,t)dx$ is the average number of particles with mass in the interval $(x, x+dx)$ at time t. The multiple fragmentation kernel $\gamma(x,y)$, $0 \le y \le x < \infty$, provides the rate at which particles of mass y are produced due to the break-up of a particle of mass x. The first term on the right-hand side of equation (1.1) is a loss term, accounting for the particles of mass x lost due to them fragmenting. The second term is a gain term which accounts for the gain in particles of mass x arising from the break-up of larger particles.

A special case of fragmentation is that of binary fragmentation. In binary fragmentation, each fragmentation event produces exactly two particles. We can adapt (1.1) to model such a process by setting

$$\gamma(x, y) = F(x - y, y), \quad 0 \le y \le x < \infty.$$

The symmetric function $F(x, y)$ is the binary fragmentation kernel which provides the rate at which particles of mass $x + y$ fragment into particles of mass x and y. If we replace $\gamma(x, y)$ with $F(x - y, y)$ in the first integral of (1.1), and make the substitution $y' = x - y$, we obtain

$$\int_0^x \frac{y}{x}\gamma(x, y)\, dy = \int_0^x \frac{y}{x} F(x - y, y)\, dy = \int_0^x \frac{x - y'}{x} F(y', x - y')\, dy'.$$

Utilising the symmetry of F, we have

$$\int_0^x \frac{y}{x} F(x - y, y)\, dy = \frac{1}{2}\int_0^x F(x - y, y)\, dy.$$

Making these changes in (1.1) we obtain the binary fragmentation equation

$$\frac{\partial u(x,t)}{\partial t} = -\frac{1}{2}u(x,t)\int_0^x F(x - y, y)\, dy + \int_0^\infty F(x, y)u(x + y, t)dy. \qquad (1.2)$$

The application of semigroup theory to the study of fragmentation equations was pioneered by Aizenman and Bak [1]. They considered the binary fragmentation equation (1.2) in the case of F being a constant. Their approach involved examining the problem with x restricted to a sequence of truncated intervals. They then established that each truncated problem gave rise to a semigroup, and that this sequence of semigroups converged to a limit semigroup which provided a solution to the full problem. Using a similar approach, McLaughlin *et al.* [26] were able to establish the existence of a unique non-negative, mass-conserving solution to the multiple fragmentation equation (1.1) under the constraint

$$\int_0^x \frac{y}{x}\gamma(x, y)\, dy \le C_n < \infty \quad \text{for all } x \in (0, n], \; n > 0,$$

where the sequence $\{C_n\}$ may be unbounded.

An alternative formulation for a continuous fragmentation equation is provided by

$$\frac{\partial u(x,t)}{\partial t} = -a(x)u(x,t) + \int_x^\infty a(y)b(x|y)u(y,t)dy, \; x > 0, \; t > 0. \qquad (1.3)$$

Here $a(x)$ represents the fragmentation rate for a particle of mass x, $b(x|y)$ gives the distribution of particles of mass x resulting from a break-up of a particle of mass y. The first term on the right-hand side is the loss term, which takes account of the particles of mass x lost due to them breaking up. The second term, involving the integral, is the gain term and gives the gain in particles of mass x due to the break-up of larger particles. Since each individual particle resulting from a fragmentation event cannot have mass exceeding that of the original particle, we require that $b(x|y) = 0$ for $x > y$. Also, for mass to be conserved during fragmentation events we must impose

$$\int_0^y xb(x|y)dx = y \text{ for } y > 0.$$

The expected number of particles resulting from the fragmentation of a particle of mass y is given by the integral

$$n(y) = \int_0^y b(x|y)dx.$$

We can recover our original multiple fragmentation equation, that is (1.1), by setting

$$a(x) = \int_0^x \frac{y}{x}\gamma(x,y)\,dy \quad \text{and} \quad b(x|y) = \frac{\gamma(y,x)}{a(y)},$$

and the binary fragmentation equation, (1.2), by setting

$$a(x) = \frac{1}{2}\int_0^x F(x-y,y)\,dy \quad \text{and} \quad b(x|y) = \frac{F(y-x,x)}{a(y)}.$$

A setup such as (1.3) was first presented by McGrady and Ziff in [25], where explicit solutions were sought in the case of a and b having the following power-law forms

$$a(x) = x^\alpha, \quad \alpha \in \mathbb{R}, \quad \text{and} \quad b(x|y) = (\nu+2)\frac{x^\nu}{y^{\nu+1}}, \quad -2 < \nu \leq 0. \tag{1.4}$$

Solutions were found to be mass-conserving for $\alpha > 0$. However, for $\alpha < 0$ there was an unaccounted for loss of mass due to a phenomenon known as 'shattering'. Using semigroup theory, and in particular a perturbation theorem derived from work by Voigt [38], Banasiak, in [2], was able to confirm these results rigorously for the case $\nu = 0$. This approach has proved fruitful, having been used by Lamb [22] and Banasiak and Arlotti [4] to establish the existence of unique mass-conserving positive solutions to equation (1.3) under a range of constraints on the fragmentation rate a. In [22], the semigroup obtained using the aforementioned perturbation result is shown to be the very semigroup that is obtained in [26] using the truncation/limit approach. The perturbation approach has also been successfully applied to variations of the standard fragmentation equation (1.3). For example, in [5] Banasiak and Lamb apply the method to a fragmentation model with built-in mass loss.

When the fragmentation rate blows up at zero we encounter unexpected mass loss due to the aforementioned shattering. The unbounded fragmentation rate results in a runaway fragmentation process and the creation of infinitesimally small 'dust' particles which carry positive mass. In [18], Huang *et al.* suggest that such runaway fragmentation is unphysical and that there must be some point at which particles are unable to break up any further. They propose the introduction of a cut-off mass x_c, above which particles are able to fragment as usual, but once a particle's mass drops below x_c it ceases to fragment, becoming dormant. Such a setup produces a dual regime model. Particles of mass $x > x_c$ form a 'fragmentation regime'. The density of particles in this regime is denoted by $u_F(x,t)$, the evolution of which is governed by the standard equation

$$\frac{\partial u_F(x,t)}{\partial t} = -a(x)u_F(x,t) + \int_x^\infty a(y)b(x|y)u_F(y,t)dy, \ \ x > x_c, \ t > 0. \tag{1.5}$$

Particles of mass $0 < x \leq x_c$ form a 'dust regime'; we denote by $u_D(x,t)$ the density of particles in this regime. The dust regime density is governed by the equation

$$\frac{\partial u_D(x,t)}{\partial t} = \int_{x_c}^\infty a(y)b(x|y)u_F(y,t)dy, \ \ 0 < x \leq x_c, \ t > 0. \tag{1.6}$$

In [18], equations (1.5) and (1.6) were examined in the particular case that a and b are as given in (1.4). Explicit solutions were provided in terms of the confluent hypergeometric function.

1.2 Coagulation–Fragmentation Equations

Introducing a coagulation process to the model leads to integro-differential equations of the form

$$\begin{aligned}\frac{\partial u(x,t)}{\partial t} = (\mathcal{F}u)(x,t) &+ \frac{1}{2}\int_0^x k(x-y,y)u(x-y,t)u(y,t)dy \\ &- u(x,t)\int_0^\infty k(x,y)u(y,t)dy, \end{aligned} \tag{1.7}$$

where $\mathcal{F}u$ is one of the fragmentation models from (1.1), (1.2) or (1.3). The function $k(x,y)$ is the coagulation kernel and provides the rate at which particles of mass x and mass y come together to form a particle of mass $x+y$. Physical constraints indicate that k should be non-negative and symmetric. The first of the coagulation terms is a gain term, accounting for the new particles of mass x created when two smaller particles of combined mass x join together. The factor of $\frac{1}{2}$ is included so as to avoid double counting due to symmetry. The second coagulation term is a loss term, which takes account of the loss of particles of mass

x due to such particles joining with other particles to create still larger particles. It was Smoluchowski in [35] who first modelled coagulation in this manner, albeit with a discrete model with summations in place of integrals. The pure coagulation equation (which is obtained from (1.7) by removing the term $\mathcal{F}u$) is known as the Smoluchowski coagulation equation.

When the fragmentation component of (1.7) is provided by (1.1) we get the coagulation and multiple fragmentation equation, introduced by Blatz and Tobolsky [10]. This equation was later analysed by Melzak [28], who, by assuming that both kernels γ and k were bounded and by seeking solutions in the form of a series, established existence and uniqueness of solutions.

With $\mathcal{F}u$ being given by (1.2), equation (1.7) becomes the coagulation and binary fragmentation equation. In [1], Aizenman and Bak apply a semigroup method in the case that both F and k are constants. The coagulation and binary fragmentation equation is the subject of investigation in both [36] and [37], where the kernels F and k are allowed to be unbounded, but must satisfy certain growth conditions.

The semigroup work of McLaughlin *et al.* in [26] was extended in [27] to consider the coagulation and multiple fragmentation equation under the assumptions that the coagulation kernel k is constant and the multiple fragmentation kernel γ is bounded.

In [22], Lamb considers equation (1.7) with fragmentation terms from (1.3). The equation is formulated as a semilinear abstract Cauchy problem by treating the coagulation terms as a nonlinear perturbation of the pure fragmentation equation. Using a truncation/limit technique, Lamb was able to establish global existence and uniqueness of solutions under the constraints that the fragmentation rate a satisfies a linear growth bound, $n(y)$ is equal to a finite constant and the fragmentation kernel k is bounded.

1.3 Outline

The purpose of this thesis is to apply functional analytic techniques from the theory of semigroups and operator matrices to the rigorous analysis of dual regime fragmentation and coagulation-fragmentation models. The body of this thesis is formed by four chapters.

In Chapter 2, we introduce the required preliminary material on operator semigroups, operator matrices and their application to the solution of equations in abstract spaces.

In Chapter 3, we apply the results and techniques introduced in Chapter 2 to the dual regime fragmentation model described earlier. Using an operator matrix, the system is cast as a linear abstract Cauchy problem in a suitable product space. Under certain restrictions on the fragmentation rate a, we prove the existence of a unique positive solution which conserves mass between the two regimes.

In Chapter 4, a coagulation process is introduced to our dual regime model. Treating the coagulation terms as a perturbation to the pure fragmentation system, we form a semilinear abstract Cauchy problem in a revised product space. Under a combined linear growth condition on the fragmentation rate $a(y)$ and the quantity $n(y)$, in addition to assuming that the coagulation kernel k is bounded, we prove the existence of a unique positive, strongly differentiable, global solution.

In Chapter 5, we examine the growth condition on $a(y)$ and $n(y)$, introduced in Chapter 4. We compare this condition with the standard constraint imposed in the literature, under a selection of forms for the kernel $b(x|y)$. We are able to show that for a certain choice of b, our condition permits a wider range of fragmentation rates a, than is allowed under the standard conditions.

Chapter 2

Preliminaries – Spaces, Operators and Semigroups

In this chapter we provide a summary of the terminology and theory which shall be employed in the later chapters when we come to examine our problems of interest. We are assuming a familiarity with the basic concepts and results of functional analysis covered in an introductory course, such as that provided by [21, Chapters 1 and 2].

2.1 Spaces

The analysis of the later chapters will be carried out within the setting of various function spaces. In this section we introduce the class of spaces we shall be working with and detail the key properties for our interests.

Definition 2.1.1. (Lebesgue spaces) Let (Ω, μ) be a measure space with positive measure μ. For $1 \leq p < \infty$, we denote by $L_p(\Omega, \mu)$ the set of (equivalence classes of) μ-measurable (real-valued) functions f defined almost everywhere on Ω, such that

$$\|f\|_p = \left\{ \int_\Omega |f(x)|^p \, d\mu(x) \right\}^{\frac{1}{p}} < \infty. \tag{2.1}$$

With (2.1) as a norm, $L_p(\Omega, \mu)$ $(1 \leq p < \infty)$ forms a Banach space, [33, Theorem 3.11]. By $L_\infty(\Omega, \mu)$ we denote the set of (equivalence classes of) μ-measurable (real-valued) functions f defined almost everywhere on Ω, for which there exists a finite constant M such that

$$|f(x)| \leq M \text{ for almost all } x \in \Omega. \tag{2.2}$$

We can define a norm on $L_\infty(\Omega, \mu)$ as follows

$$\|f\|_\infty = \inf \{M : (2.2) \text{ holds}\}.$$

Under this norm, $L_\infty(\Omega, \mu)$ forms a Banach space [33, Theorem 3.11].

Definition 2.1.2. For $1 \leq p < \infty$ ($p = \infty$) we denote by $L_{p,loc}(\Omega)$ the set of functions f which satisfy the condition (2.1) ((2.2)) on all compact subsets $K \subset \Omega$.

Definition 2.1.3. Let X be a vector space of real-valued functions defined on a domain Ω. The set X_+, defined as

$$X_+ = \{f \in X : f(x) \geq 0 \text{ for all } x \in \Omega\},$$

is called the *positive cone* of X.

- The positive cone is closed under addition and multiplication by non-negative scalars.
- If the space X is an L_p space then the condition is replaced by $f(x) \geq 0$ for almost all $x \in \Omega$.
- The positive cone of an L_p space forms a closed subset.
- The concept of a (positive) cone can be generalised in the case that X is a general real Banach space; see [19, Definition 8.3.1].

Definition 2.1.4. Let X_1 and X_2 be two vector spaces. The *product space* $X = X_1 \times X_2$, consists of the set of ordered pairs

$$X_1 \times X_2 = \{(f, g) : f \in X_1, g \in X_2\}.$$

- If $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are both Banach spaces, then so is the product space X, with norm

  $$\|(f, g)\| = \|f\|_1 + \|g\|_2 \text{ for } f \in X_1, g \in X_2;$$

  see [8, Lemma 1.62].
- When X_1 and X_2 are vector spaces of real-valued functions, then we define the positive cone of the product space X as $X_+ = X_{1+} \times X_{2+}$.

Definition 2.1.5. Let X be a vector space and let both $\|\cdot\|$ and $\|\cdot\|_0$ be norms defined on X. We say that the norms are *equivalent* if there exist positive constants a and b such that

$$a\|f\| \leq \|f\|_0 \leq b\|f\| \text{ for all } f \in X.$$

Such norms define the same topology on X; hence notions of convergence with respect to both norms are equivalent.

2.2 Calculus of Vector-Valued Functions

In this section we shall consider functions which map from some interval I in $\mathbb{R}$ into a normed vector space $(X, \|\cdot\|)$. We will generalise a number of concepts from standard real-valued calculus. This generally involves simply substituting a norm in place of the modulus in the standard definitions.

Definition 2.2.1. Let $u : I \to X$. Then we say that u is *strongly continuous at* $c \in I$ if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|u(t) - u(c)\| < \varepsilon \text{ whenever } t \in I \text{ and } |t - c| < \delta.$$

- The function u is *strongly continuous on* I if u is strongly continuous at each $c \in I$.
- The set of all $u : I \to X$ which are continuous on I is denoted by $C(I, X)$.
- In the case that X is a Banach space and $I = [\alpha, \beta]$ with $-\infty < \alpha < \beta < \infty$, then $C([\alpha, \beta], X)$ forms a Banach space under the norm

  $$\|u\|_\infty = \sup \{\|u(t)\|, t \in [\alpha, \beta]\};$$

  see [8, Theorem 1.39].

Definition 2.2.2. The function $u : I \to X$ is said to be *strongly differentiable at* $c \in I$ if there exists an element $v \in X$ such that, for all $\varepsilon > 0$, we can find a corresponding $\delta > 0$ such that

$$\left\| \frac{u(c+h) - u(c)}{h} - v \right\| < \varepsilon \text{ whenever } c + h \in I \text{ and } 0 < |h| < \delta.$$

The element v is referred to as the *strong derivative* of u at c and is denoted by $u'(c)$.

- The function u is *strongly differentiable on* I if u is strongly differentiable at each $c \in I$.

Definition 2.2.3. Suppose that $u : [\alpha, \beta] \to X$ where $-\infty < \alpha < \beta < \infty$. Let us denote by P_n the following partition of $[\alpha, \beta]$

$$P_n : \quad \alpha = t_0 < t_1 < t_2 < \ldots < t_n = \beta,$$

and define

$$\|P_n\| = \max_{1 \le m \le n} (t_m - t_{m-1}).$$

Just as with real-valued functions we can form a Riemann sum based on this partition and hence we define

$$S(u; P_n) = \sum_{m=1}^{n} u(\overline{t_m})(t_m - t_{m-1}),$$

where $\overline{t_m}$ is an arbitrary point chosen from $[t_{m-1}, t_m]$. If $S(u; P_n)$ converges in X as $\|P_n\| \to 0$, and if the limit is independent of the manner in which $\|P_n\| \to 0$, then we denote the limit by $\int_\alpha^\beta u(t)\, dt$ and refer to it as the *strong Riemann integral* of u on $[\alpha, \beta]$.

Theorem 2.2.4. *Let X be a real Banach space and let $u : [\alpha, \beta] \to X$ be strongly continuous. Then the strong Riemann integral $\int_\alpha^\beta u(t)\,dt$ exists as an element of X.*

Proof. See [8, Theorem 1.43]. □

Theorem 2.2.5. *Let X be a real Banach space and let $u : [\alpha, \beta] \to X$ be strongly continuous. Then*

$$\left\| \int_\alpha^\beta u(t)\,dt \right\| \leq \int_\alpha^\beta \|u(t)\|\,dt,$$

where the right-hand integral is a standard real-valued Riemann integral.

Proof. See [8, Theorem 1.44]. □

Improper integrals on infinite or semi-infinite intervals can be handled in the usual way. For example, if $u : [\alpha, \infty) \to X$ is continuous, then $\int_\alpha^\infty u(t)\,dt$ is defined as

$$\int_\alpha^\infty u(t)\,dt = \lim_{\tau\to\infty} \int_\alpha^\tau u(t)\,dt, \quad \text{if this limit exists.}$$

Theorem 2.2.6. *Let X be a real Banach space and let $u : [\alpha, \infty) \to X$ be strongly continuous. If the real integral $\int_\alpha^\infty \|u(t)\|\,dt$ exists, then $\int_\alpha^\infty u(t)\,dt$ exists in X and we have*

$$\left\| \int_\alpha^\infty u(t)\,dt \right\| \leq \int_\alpha^\infty \|u(t)\|\,dt.$$

Proof. See [8, Theorem 1.45]. □

Theorem 2.2.7. (Fundamental Theorem of Calculus) *Let X be a real Banach space and let $u : [\alpha, \beta] \to X$ be strongly continuous. Then, for each $t \in [\alpha, \beta]$, the integral $\int_\alpha^t u(s)\,ds$ exists in X and*

$$\frac{d}{dt} \int_\alpha^t u(s)\,ds = u(t).$$

Proof. See [23, Page 340]. □

2.3 Operators

By way of background, we assume a familiarity with basic concepts such as linearity and boundedness. In this section we provide an overview of some of the required concepts relating to operators which may be considered less standard.

Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be (real) normed vector spaces.

Definition 2.3.1. The operator $A : D(A) \subseteq X_1 \to X_2$ satisfies a *Lipschitz condition* on $D(A)$ if there exists a positive constant M such that

$$\|Af - Ag\|_2 \leq M\|f - g\|_1 \text{ for all } f, g \in D(A).$$

Definition 2.3.2. The operator $A : D(A) \subseteq X_1 \to X_2$ is *closed* if whenever $\{f_n\}_{n=1}^{\infty} \subset D(A)$ is such that $f_n \to f$ in X_1 and $Af_n \to g$ in X_2, then we have $f \in D(A)$ and $Af = g$.

Definition 2.3.3. The *graph* of the operator $A : D(A) \subseteq X_1 \to X_2$ is the subset $G(A)$ of $X_1 \times X_2$ defined by

$$G(A) = \{(f, g) : f \in D(A), g = Af\}.$$

Lemma 2.3.4. *The operator $A : D(A) \subseteq X_1 \to X_2$ is closed if and only if its graph $G(A)$ is a closed subset of $X_1 \times X_2$.*

Proof. See [8, Theorem 1.64]. □

Definition 2.3.5. Let $A : D(A) \subseteq X_1 \to X_2$. A sequence $\{f_n\}_{n=1}^{\infty} \subset D(A)$ is said to be A-*convergent* to $f \in X_1$ if $\{f_n\}_{n=1}^{\infty}$ converges in X_1 to f and $\{Af_n\}_{n=1}^{\infty}$ is a Cauchy sequence in X_2.

Definition 2.3.6. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces. We say that the linear operator $A : D(A) \subseteq X_1 \to X_2$ is *closable* if it has a closed extension. When A is closable we call its smallest closed extension the *closure* of A, which we denote by $\overline{A}$. An element $f \in X_1$ is in $D(\overline{A})$ if and only if there exists a sequence $\{f_n\}_{n=1}^{\infty}$ in $D(A)$ which is A-convergent to f. In this case $\overline{A}f = \lim_{n\to\infty} Af_n$; see [20, page 166].

Definition 2.3.7. Suppose that $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ are real Banach spaces equipped with positive cones X_{1+} and X_{2+}, respectively. We say that the operator $A : D(A) \subseteq X_1 \to X_2$ is a *positive operator* if it maps $D(A)_+ = D(A) \cap X_{1+}$ into X_{2+}.

Definition 2.3.8. Let X be a Banach space and $A : D(A) \subseteq X \to X$ a linear operator. Then the *resolvent set*, $\rho(A)$, of A is defined by

$$\rho(A) = \left\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in B(X)\right\}.$$

The *spectrum* of A, $\sigma(A)$, is defined as the complement of the resolvent set, $\rho(A)$, in $\mathbb{C}$, that is

$$\sigma(A) = \mathbb{C}\backslash\rho(A) = \{\lambda \in \mathbb{C} : \lambda \notin \rho(A)\}.$$

For $\lambda \in \rho(A)$ the *resolvent operator* of A is

$$R(\lambda, A) = (\lambda I - A)^{-1}.$$

Definition 2.3.9. Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces and let $A : D(A) \subseteq X_1 \to X_1$ and $B : D(B) \subseteq X_1 \to X_2$ be operators with $D(A) \subseteq D(B)$. We say that B is *A-bounded* (or B is *relatively A-bounded*) if there exist non-negative constants a and b such that

$$\|Bf\|_2 \leq a\|Af\|_1 + b\|f\|_1 \text{ for all } f \in D(A).$$

- The infimum of the values of a for which such a bound exists is known as the *A-bound of B*.

Lemma 2.3.10. *Let $(X_1, \|\cdot\|_1)$ and $(X_2, \|\cdot\|_2)$ be Banach spaces and let the operators $A : D(A) \subseteq X_1 \to X_1$ and $B : D(B) \subseteq X_1 \to X_2$ be linear, with $D(A) \subseteq D(B)$ and $\rho(A) \neq \emptyset$. Then B is A-bounded if and only if $BR(\lambda, A) \in B(X_1, X_2)$ for some $\lambda \in \rho(A)$.*

This is a more general version of the result given in [4, Lemma 4.1], therefore we include the following proof.

Proof. Suppose that B is A-bounded and let $g \in X_1$. Then for $\lambda \in \rho(A)$ we have $R(\lambda, A)g \in D(A)$. Hence

$$\|BR(\lambda, A)g\|_2 \leq a\|AR(\lambda, A)g\|_1 + b\|R(\lambda, A)g\|_1.$$

Since we can write

$$AR(\lambda, A)g = (A - \lambda I)R(\lambda, A)g + \lambda R(\lambda, A)g = \lambda R(\lambda, A)g - g,$$

we have

$$\begin{aligned}\|BR(\lambda, A)g\|_2 &\leq a\|\lambda R(\lambda, A)g - g\|_1 + b\|R(\lambda, A)g\|_1 \\ &\leq (a|\lambda| + b)\|R(\lambda, A)g\|_1 + a\|g\|_1.\end{aligned}$$

As $\lambda \in \rho(A)$, there exists a constant $M \geq 0$ such that $\|R(\lambda, A)g\|_1 \leq M\|g\|_1$, for any $g \in X_1$. Therefore

$$\|BR(\lambda, A)g\|_2 \leq (Ma|\lambda| + Mb + a)\|g\|_1.$$

Hence $BR(\lambda, A) \in B(X_1, X_2)$.

Now let us suppose that $BR(\lambda, A) \in B(X_1, X_2)$ for some $\lambda \in \rho(A)$, with $\|BR(\lambda, A)\| = K$. If $f \in D(A)$, then $f = R(\lambda, A)g$ for some $g \in X_1$. Therefore

$$\|Bf\|_2 = \|BR(\lambda, A)g\|_2 \leq K\|g\|_1 = K\|(\lambda I - A)f\|_1 \leq K\|Af\|_1 + |\lambda|K\|f\|_1.$$

Hence B is A-bounded, as defined in Definition 2.3.9. □

Lemma 2.3.11. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ and $B : D(B) \subseteq X \to X$ be such that A is closed and B is A-bounded with A-bound strictly less than 1. Then $(A + B, D(A))$ is a closed operator.*

Proof. See [14, Chapter 3, Lemma 2.4]. □

Definition 2.3.12. Let X be a Banach space and let $A : D(A) \subseteq X \to X$. Suppose that D_0 is an open subset of $D(A)$ and that $f, f+\delta \in D_0$. If there exists an operator $A_f \in B(X)$ such that

$$A(f+\delta) = Af + A_f\delta + G(f,\delta),$$

where the remainder term G satisfies

$$\lim_{\|\delta\|\to 0} \left\{ \frac{\|G(f,\delta)\|}{\|\delta\|} \right\} = 0,$$

then we say that A is *Fréchet differentiable* at f with A_f being the *Fréchet derivative* of A at f. If the operator A is Fréchet differentiable at each $f \in D_0$, then we say it is Fréchet differentiable on D_0.

2.4 Semigroups

The main tools we shall use in the following chapters are provided by the theory of semigroups. This section gives an account of the key results in this area and how they can be used to tackle problems of the type we shall be considering.

2.4.1 Introduction to Semigroups

In order to motivate what follows let us consider a dynamical system evolving with time. Suppose that we can represent the state of the system at time t by an element $u(t)$ from a Banach space X. We can then think of the evolution of the system as defining a family of transition operators $(T(s))_{s\geq 0}$, such that applying $T(s)$ has the effect of advancing the system state through a time interval of length s. If $u_0 = u(0)$ denotes the initial state of the system, then the state of the system at time $t \geq 0$ would be given by

$$u(t) = T(t)u_0, \quad (t \geq 0).$$

Let us now consider some of the properties that the operators $(T(s))_{s\geq 0}$ should possess. Since no transition can take place in zero time, application of $T(0)$ must leave the system state unchanged and hence $T(0) = I$, where I is the identity operator on X. If the system evolves for a period of length t, before evolving for a further period of length s, then we would expect it to arrive at the same state as it would, had it simply evolved over an interval of length $s+t$. We therefore require that $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$.

2.4.2 Strongly Continuous Semigroups

Definition 2.4.1. Let X be a Banach space. Then a family of operators $(T(t))_{t\geq 0} \subset B(X)$ forms a C_0*-semigroup* (*strongly continuous semigroup*) of operators on X, if it satisfies the following conditions

(*i*) $T(0) = I$ where I is the identity operator on X;

(*ii*) $T(s+t) = T(s)T(t)$ for all $s, t \geq 0$;

(*iii*) $\|T(t)f - f\|_X \to 0$ as $t \to 0^+$ for any fixed $f \in X$.

Conditions (*i*) and (*ii*) are motivated by the above consideration of a system evolving with time. Having $T(t) \in B(X)$ for all $t \geq 0$ ensures that errors in initial states remain under control, since

$$\|T(t)u_0 - T(t)v_0\| \leq \|T(t)\|\|u_0 - v_0\|.$$

The continuity condition (*iii*) may appear light. However, in conjunction with conditions (*i*) and (*ii*) it can be used to establish further results. Having said this, we may consider replacing condition (*iii*) with the alternative

$$\|T(t) - I\|_{B(X)} \to 0 \quad \text{as} \quad t \to 0^+.$$

The resulting semigroup is then known as a uniformly continuous semigroup. This condition is stronger than (*iii*) above, with each uniformly continuous semigroup automatically forming a C_0-semigroup, whilst the converse is not, in general, true. However, for most applications this condition is too strong and so we restrict our attention to strongly continuous semigroups and their properties.

Theorem 2.4.2. *Let X be a Banach space and $(T(t))_{t\geq 0}$ a C_0-semigroup on X. Then there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that*

$$\|T(t)\| \leq Me^{\omega t} \quad \text{for} \quad t \geq 0. \tag{2.3}$$

Proof. See [24, Theorem 2.16]. □

In the special case of $M = 1$ and $\omega = 0$, we say that we have a semigroup of contractions. The infimum of the values of ω for which such a bound can be formed, is called the *growth bound* of the semigroup.

Definition 2.4.3. Let X denote a Banach space of the type $L_1(\Omega, \mu)$ with positive cone X_+. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. We say that $(T(t))_{t\geq 0}$ is a *substochastic* semigroup if, for each $t \geq 0$, $\|T(t)\| \leq 1$ and $T(t)f \in X_+$ for all $f \in X_+$. If additionally $\|T(t)f\| = \|f\|$ for all $t \geq 0$ when $f \in X_+$, then we say that $(T(t))_{t\geq 0}$ is a *stochastic* semigroup.

Lemma 2.4.4. *Let X be a Banach space and let $(T(t))_{t\geq 0}$ be a C_0-semigroup on X. Then for each fixed $f \in X$, the mapping $\phi_f : [0,\infty) \to X$, defined by*

$$\phi_f(t) = T(t)f \quad (t \geq 0),$$

is continuous on the non-negative real line.

Proof. See [8, Lemma 2.4]. □

2.4.3 Generators

Suppose that the function $\phi : [0,\infty) \to \mathbb{C}$ satisfies the following properties

- $\phi(0) = 1$;
- $\phi(s+t) = \phi(s)\phi(t)$ for all $s, t \geq 0$;
- ϕ is continuous on $[0,\infty)$.

Then, as was suggested by Cauchy in [12], the function ϕ has the form e^{ta}, where a is any complex constant. Examining the above conditions we cannot help but notice their resemblance to the conditions for a C_0-semigroup together with Lemma 2.4.4. Therefore, it is perhaps not unreasonable to expect that, given a C_0-semigroup $(T(t))_{t\geq 0}$, the operators have the form $T(t) = \exp(tA)$, where A is some operator.

Theorem 2.4.5. *Let X be a Banach space and suppose that the power series $\phi(z) = \sum_{n=0}^{\infty} a_n z^n$ has radius of convergence $r > 0$. Then for $A \in B(X)$ such that $\|A\| < r$, the series*

$$\phi(A) = \sum_{n=0}^{\infty} a_n A^n \tag{2.4}$$

converges in $B(X)$, where A^n signifies composition of the operator A applied n times. Further, if the coefficients $(a_0, a_1, a_2, ...)$ are real and non-negative, then $\|\phi(A)\| \leq \phi(\|A\|)$.

Proof. See [8, Theorem 1.86]. □

Recall that the exponential function $\exp(z)$ has the power series representation $\sum_{n=0}^{\infty} z^n/n!$ with infinite radius of convergence. Hence if $A \in B(X)$, then $tA \in B(X)$ for each fixed $t \geq 0$, and therefore

$$\exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}$$

converges in $B(X)$. When $A \in B(X)$, then the family of operators $(T(t))_{t\geq 0}$, where $T(t) = \exp(tA)$, forms a uniformly continuous semigroup on X. In fact, it may be shown that every uniformly continuous semigroup on a Banach space takes this form; see [14, Chapter 1, Theorem 3.7]. We note that given this semigroup, we can differentiate term-by-term to obtain

$$\frac{d}{dt}\{T(t)\} = A + \sum_{n=1}^{\infty} \frac{t^n A^{n+1}}{n!} \qquad (t \geq 0).$$

If we evaluate this at $t = 0$ then we get

$$T'(0) = \left[\frac{d}{dt}\{T(t)\}\right]_{t=0} = A.$$

Therefore we have recovered the operator A by taking the (right) derivative of the semigroup at $t = 0$. In the preceding discussions we assumed that $A \in B(X)$. However in general this will not be the case. We now proceed to consider the more general case where A need not be bounded, using what has gone before as a guide.

Definition 2.4.6. Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Then the (*infinitesimal*) *generator* of $(T(t))_{t\geq 0}$ is the operator $A : D(A) \subseteq X \to X$ defined as follows.

For $t > 0$ and $f \in X$, let $A_t f = \{T(t)f - f\}/t$. Then

$$\left.\begin{aligned} D(A) &= \{f \in X : \lim_{t\to 0^+} A_t f \text{ exists in } X\} \\ Af &= \lim_{t\to 0^+} A_t f \text{ for } f \in D(A) \end{aligned}\right\}.$$

Here we have taken our lead from the uniformly continuous case, where the generator is given by the derivative of the semigroup at $t = 0$. However, in keeping with the continuity conditions, we have moved from the operator space $B(X)$ to the Banach space X. It is easily shown that $D(A)$ is a vector subspace of X and that A is a linear operator.

We shall now consider the generator further and establish some more of its properties. This will lead us to the most significant results concerning C_0-semigroups and their generators, namely the Hille–Yosida Theorem and its generalisation. These theorems provide both necessary and sufficient conditions for an operator to generate a strongly continuous semigroup.

Definition 2.4.7. Let X be a Banach space. Then for real numbers $M \geq 1$ and $\omega \in \mathbb{R}$, $\mathcal{G}(M, \omega; X)$ is the set of operators A which generate a C_0-semigroup on X that satisfies the bound (2.3).

Theorem 2.4.8. (Rescaled Semigroups) *Let A be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$, which satisfies the bound* (2.3). *Then for any $\mu \in \mathbb{R}$, $(S(t))_{t\geq 0} := (e^{\mu t}T(t))_{t\geq 0}$ provides a C_0-semigroup which satisfies*

$$\|S(t)\| \leq Me^{(\omega+\mu)t} \quad \text{for} \quad t \geq 0.$$

The generator of this new semigroup is $B = A + \mu I$, with $D(B) = D(A)$.

Although this result is stated in [14, page 60], no explanation is given and therefore we include the following proof.

Proof. It is easily shown that $(S(t))_{t\geq 0}$ satisfies conditions (i)-(iii) from Definition 2.4.1. Firstly,

$$S(0) = e^{\mu 0}T(0) = e^0 I = I,$$

giving us (i). For all $s, t \geq 0$, we have (ii) as follows

$$S(s+t) = e^{\mu(s+t)}T(s+t) = e^{\mu s}e^{\mu t}T(s)T(t) = e^{\mu s}T(s)e^{\mu t}T(t) = S(s)S(t).$$

Finally, for all $f \in X$, we get

$$\begin{aligned} \|S(t)f - f\| &= \left\|e^{\mu t}T(t)f - f\right\| = \left\|e^{\mu t}T(t)f - e^{\mu t}f + e^{\mu t}f - f\right\| \\ &\leq \left\|e^{\mu t}T(t)f - e^{\mu t}f\right\| + \left\|e^{\mu t}f - f\right\| \\ &= e^{\mu t}\|T(t)f - f\| + |e^{\mu t} - 1|\,\|f\| \\ &\to 0 \quad \text{as} \quad t \to 0^+. \end{aligned}$$

Hence condition (iii) is satisfied, completing the requirements for a C_0-semigroup. The bound given above for $\|S(t)\|$ is easily established, as is shown below:

$$\|S(t)\| = \left\|e^{\mu t}T(t)\right\| = e^{\mu t}\|T(t)\| \leq e^{\mu t}Me^{\omega t} = Me^{(\omega+\mu)t} \quad \text{for} \quad t \geq 0.$$

We now aim to determine the generator of this semigroup, which we shall denote by B. Recalling the explanation given in Definition 2.4.6, for any $f \in D(A)$, we have

$$\begin{aligned} \frac{S(t) - I}{t}f = \frac{e^{\mu t}T(t) - I}{t}f &= e^{\mu t}\frac{(T(t) - I)}{t}f + \frac{e^{\mu t} - 1}{t}f \\ &\to Af + \mu f = (A + \mu I)f \quad \text{as} \quad t \to 0^+, \end{aligned}$$

where we have used L'Hôpital's rule to obtain the limit of the second term. Therefore $D(A) \subseteq D(B)$ and $Bf = (A + \mu I)f$ for $f \in D(A)$. Now, if $f \notin D(A)$, then the limit of the right-hand side does not exist and hence neither does the limit of the left-hand side, implying that $f \notin D(B)$. Therefore the semigroup $(S(t))_{t\geq 0}$, has generator $B = A + \mu I$ with domain $D(B) = D(A)$. □

Theorem 2.4.9. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X and let A be the generator of this semigroup. Then, for all $f \in D(A)$, we have*

$$T(t)f \in D(A) \text{ and } A(T(t)f) = T(t)(Af) \text{ for all } t \geq 0,$$

$$\frac{d}{dt}\{T(t)f\} = A(T(t)f) = T(t)(Af),$$

where the derivative is the strong derivative with respect to the norm on X, two-sided for $t > 0$ and right-sided at $t = 0$.

Proof. See [8, Theorem 2.12]. □

Theorem 2.4.10. *Let $A : D(A) \subseteq X \to X$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then A is a closed operator with domain $D(A)$ which is dense in X.*

Proof. See [8, Theorem 2.13]. □

Lemma 2.4.11. *Let $A : D(A) \subseteq X \to X$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$ on a Banach space X. Then for all $t \geq 0$ and $f \in X$, we have*

$$\int_0^t T(s)f\, ds \in D(A),$$

with

$$T(t)f - f = A\int_0^t T(s)f\, ds.$$

Moreover, if $f \in D(A)$, we have

$$T(t)f - f = A\int_0^t T(s)f\, ds = \int_0^t T(s)Af\, ds.$$

Proof. See [14, Chapter 2, Lemma 1.3]. □

Theorem 2.4.12. *Let $(T(t))_{t\geq 0}$ and $(S(t))_{t\geq 0}$ be two C_0-semigroups on the Banach space X. Supposing that these two semigroups have the same generator A, then they are in fact the same semigroup, that is $T(t)f = S(t)f$ for all $t \geq 0$ and $f \in X$.*

Proof. See [31, Theorem 2.6]. □

Theorem 2.4.13. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup on a Banach space X. Suppose that the semigroup satisfies the bound (2.3) and has generator A. If $\lambda > \omega$, then $\lambda \in \rho(A)$, the resolvent operator is given by*

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda s}T(s)f ds \text{ for all } f \in X,$$

and we have

$$\|R(\lambda, A)\| \leq \frac{M}{\lambda - \omega}.$$

Proof. See [14, Chapter 2, Theorem 1.10]. □

Definition 2.4.14. Let $A : D(A) \subseteq X \to X$ be the generator of a C_0-semigroup. Then, for sufficiently large $\lambda > 0$, we define the *Yosida approximation* $\mathcal{A}_\lambda$ of A by

$$\mathcal{A}_\lambda = \lambda^2 R(\lambda, A) - \lambda I = \lambda A R(\lambda, A).$$

- For large enough λ, we have $\lambda \in \rho(A)$. Therefore $R(\lambda, A) \in B(X)$ and hence $\mathcal{A}_\lambda \in B(X)$ for such values of λ.
- The equality above is justified as follows.

$$\begin{aligned} \lambda^2 R(\lambda, A) - \lambda I &= \lambda(\lambda I - A)R(\lambda, A) + \lambda A R(\lambda, A) - \lambda I \\ &= \lambda I + \lambda A R(\lambda, A) - \lambda I \\ &= \lambda A R(\lambda, A). \end{aligned}$$

- The numerical analogue of the Yosida approximation is

$$\frac{\lambda^2}{\lambda - a} - \lambda = \frac{\lambda^2 - \lambda(\lambda - a)}{\lambda - a} = \frac{\lambda a}{\lambda - a} \to a \text{ as } \lambda \to \infty.$$

These points suggest that the Yosida approximations may provide bounded linear approximations to our generator A and that these approximations improve as λ increases. In the following lemma we shall consider this further.

Lemma 2.4.15. *Let $A : D(A) \subseteq X \to X$ be a closed linear operator with $D(A)$ dense in the Banach space X. Suppose there exist $\omega \in \mathbb{R}$ and $M > 0$ such that $[\omega, \infty) \subset \rho(A)$ and $\|\lambda R(\lambda, A)\| \leq M$ for all $\lambda \geq \omega$. Then the following statements hold as $\lambda \to \infty$:*

(*i*) *$\lambda R(\lambda, A)f \to f$ for all $f \in X$;*

(*ii*) *$\lambda A R(\lambda, A)f = \lambda R(\lambda, A)Af \to Af$ for all $f \in D(A)$.*

Proof. See [14, Chapter 2, Lemma 3.4]. □

To recap, if the operator $A : D(A) \subseteq X \to X$ is the generator of a C_0-semigroup, then it must be a closed linear operator with domain $D(A)$ which is a dense vector subspace of X. Also, it must satisfy the conditions set out in Theorem 2.4.13. We have now reached the point where we shall introduce the Hille–Yosida Theorem. As mentioned previously, this theorem gives not only sufficient but also necessary conditions for A to generate a strongly continuous contraction semigroup.

Theorem 2.4.16. (Contraction Semigroup; Hille [16], Yosida [39]) *The operator A generates a C_0-semigroup of contractions $(T(t))_{t\geq 0}$, that is $A \in \mathcal{G}(1, 0; X)$, if and only if* (*i*) *A is a closed linear operator with domain $D(A)$ which is dense in X;*

(*ii*) *for all real numbers $\lambda > 0$, we have $\lambda \in \rho(A)$ and*

$$\|R(\lambda, A)\| \le \frac{1}{\lambda}.$$

Proof. See [14, Chapter 2, Theorem 3.5] □

Theorem 2.4.17. (General Semigroup; Feller [15], Miyadera [29], Philips [32]) *Let X be a Banach space. Then $A \in \mathcal{G}(M, \omega; X)$ if and only if*

(*i*) *A is a closed linear operator with domain $D(A)$ which is dense in X;*

(*ii*) *for all real numbers $\lambda > \omega$, we have $\lambda \in \rho(A)$ and for $n = 1, 2, \ldots$*

$$\|(R(\lambda, A))^n\| \le \frac{M}{(\lambda - \omega)^n}.$$

Proof. In order to prove the general case, one can apply a rescaling technique and introduce a new norm under which our general semigroup becomes a contraction semigroup. We are then able to apply Theorem 2.4.16. For details, see [14, Chapter 2, Theorem 3.8]. □

In the general case, determining whether or not an operator A satisfies the requirements of Theorem 2.4.17 is not straightforward. In particular, establishing condition (*ii*) often proves difficult. However in the case of contraction semigroups the task is achievable. Such semigroups are significant as they arise regularly in practice. Therefore we shall now spend some time considering them and their generators further. It is possible to characterise the generators of contraction semigroups without recourse to the resolvent operator. However before we do so, it is necessary to introduce some additional terminology.

Definition 2.4.18. Let X be a Banach space and $A : D(A) \subseteq X \to X$ a linear operator.

- A is *dissipative* if
$$\|(\lambda I - A) f\| \ge \lambda \|f\| \text{ for all } \lambda > 0 \text{ and } f \in D(A).$$
- A is *m-dissipative* if it is dissipative and
$$Rg\,(\lambda I - A) = X \text{ for all } \lambda > 0.$$

Theorem 2.4.19. (The Lumer-Philips Characterisation of Generators of Contraction Semigroups) *The linear operator* A *generates a C_0-semigroup of contractions ($A \in \mathcal{G}(1, 0; X)$) if and only if*

(*i*) *A is a closed operator with domain $D(A)$ which is dense in X;*

(*ii*) *A is m-dissipative.*

Proof. See [31, Chapter 1, Theorem 4.3]. □

2.4.4 Constructing the Semigroup

Supposing we have an operator A which generates a C_0-semigroup, we are then faced with the matter of actually constructing the semigroup. If our generator $A \in B(X)$, then the semigroup $(T(t))_{t\geq 0}$ is simply given by

$$T(t) = \exp(tA) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!}. \tag{2.5}$$

However, for an unbounded operator A, we cannot guarantee the convergence of this series. Therefore we must seek alternative construction methods. The first method we shall consider was proposed by Hille, and takes as its inspiration an alternative definition of the standard scalar exponential function. For a real or complex scalar a it is well known that

$$\exp(ta) = (\exp(-ta))^{-1} = \left\{ \lim_{n\to\infty} \left(1 - \frac{t}{n}a\right)^n \right\}^{-1} = \lim_{n\to\infty} \left\{ \left(1 - \frac{t}{n}a\right)^{-1} \right\}^n.$$

If we replace the scalar a with the operator A, then we might expect

$$\exp(tA)f = \lim_{n\to\infty} \left\{ \left[\left(I - \frac{t}{n}A\right)^{-1} \right]^n \right\} f = \lim_{n\to\infty} \left\{ \left[\frac{n}{t} \left(\frac{n}{t}I - A\right)^{-1} \right]^n \right\} f,$$

for $f \in X$. Let us note the appearance of the resolvent operator within the last expression on the right.

Theorem 2.4.20. *Let X be a Banach space and let $A \in \mathcal{G}(M, \omega; X)$ generate the C_0-semigroup $(T(t))_{t\geq 0}$. Then for all $f \in X$*

$$T(t)f = \lim_{n\to\infty} \left\{ \left[\frac{n}{t} R\left(\frac{n}{t}, A\right) \right]^n \right\} f,$$

where convergence is uniform with respect to t on $[0, t_0]$, for any $t_0 > 0$.

Proof. See [31, Chapter 1, Theorem 8.3]. □

The second approach we shall consider was proposed by Yosida. In this method, we approximate our generator A by the family of Yosida approximations $\{\mathcal{A}_\lambda\}$. As bounded operators, each $\mathcal{A}_\lambda$ generates a C_0-semigroup via (2.5). It is our hope that by taking these semigroups and letting $\lambda \to \infty$, we shall obtain our desired semigroup.

Theorem 2.4.21. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then for all $f \in D(A)$*

$$\lim_{\lambda\to\infty} \mathcal{A}_\lambda f = Af,$$

and

$$T(t)f = \lim_{\lambda\to\infty} \exp(t\mathcal{A}_\lambda) f \text{ for all } t \geq 0 \text{ and } f \in X.$$

Proof. The case when $A \in \mathcal{G}(1, 0; X)$ is covered in the standard proof of the Hille–Yosida Theorem; see [31, Chapter 1, Lemma 3.3]. The general case when $A \in \mathcal{G}(M, \omega; X)$ is handled by renorming and rescaling; see [31, Chapter 1, Theorem 5.5]. □

2.4.5 Abstract Cauchy Problem

Definition 2.4.22. Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be some linear operator. Then the homogenous abstract Cauchy problem (ACP) associated with this operator is

$$\frac{d}{dt}u(t) = Au(t) \quad (t > 0)\,; \quad u(0) = u_0, \tag{2.6}$$

where u_0 is some given fixed element of X.

Definition 2.4.23. By a strong solution to this problem, we mean a function $u : [0, \infty) \to X$ such that

(*i*) u is (strongly) continuous on $[0, \infty)$;

(*ii*) u is (strongly) differentiable on $(0, \infty)$;

(*iii*) $u(t) \in D(A)$ for all $t > 0$;

(*iv*) $\frac{d}{dt}[u(t)] = A[u(t)]$ for $t > 0$ and $u(0) = u_0$.

Theorem 2.4.24. *Let X be a Banach space and let $A : D(A) \subseteq X \to X$ be the generator of a C_0-semigroup $(T(t))_{t\geq 0}$. Then if $u_0 \in D(A)$, the homogenous ACP associated with A has the unique strong solution*

$$u(t) = T(t)u_0 \quad (t \geq 0)\,.$$

Proof. See [8, Theorem 2.40 and Theorem 2.41]. □

Theorem 2.4.25. *Let $(T(t))_{t\geq 0}$ be a C_0-semigroup with generator $A \in \mathcal{G}(M, \omega; X)$. Then the ACP (2.6) is uniformly well posed on finite time intervals. That is, if $\{g_n\}_{n=1}^{\infty}$ is a sequence in $D(A)$ such that $g_n \to g \in D(A)$, and if $\{v_n\}_{n=1}^{\infty}$ and v are, respectively, the unique solutions of the ACP (2.6) when $u_0 = g_n$ $(n \in \mathbb{N})$ and $u_0 = g$, then for any $t_0 > 0$ we have*

$$\sup_{0\leq t\leq t_0} \|v_n(t) - v(t)\| \to 0 \text{ as } n \to \infty.$$

Proof. See [8, Theorem 2.42]. □

This result tells us that if $\{g_n\}_{n=1}^{\infty}$ is a sequence of increasingly accurate approximations to the true initial condition g, then the discrepancy between the approximate solution v_n, and the true solution v, gets uniformly small as $n \to \infty$. That is, small perturbations in our initial conditions do not lead to extremely differing solutions in finite time.

2.4.6 Systems of Equations

The particular problems we shall be considering involve systems of evolution equations. These equations are reformulated as systems of abstract differential equations within the setting of the appropriate function spaces. Just as we can express a linear system of n scalar differential equations as a single equation in $\mathbb{R}^n$ using matrix notation, so we can transform our system of abstract equations into a single equation of the form (2.6). The space X becomes a product space and the operator A is a matrix whose entries are themselves operators which map from and to the relevant spaces.

In our case we obtain a 2×2 operator matrix of upper triangular form. The following result gives sufficient conditions for such an operator to be a generator, as well as giving the semigroup generated.

Theorem 2.4.26. *Let X and Y be Banach spaces. Consider the operator matrix*

$$\boldsymbol{A} = \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \quad \text{with domain } D(\boldsymbol{A}) = D(A) \times D(D) \subseteq X \times Y.$$

Suppose that the following hold:

- *(i)* $A : D(A) \subseteq X \to X$ *generates a C_0-semigroup $(T(t))_{t\geq 0}$ on X;*
- *(ii)* $D : D(D) \subseteq Y \to Y$ *generates a C_0-semigroup $(S(t))_{t\geq 0}$ on Y;*
- *(iii)* $B : D(B) \subseteq Y \to X$ *is relatively D-bounded;*
- *(iv)* $(\boldsymbol{A}, D(\boldsymbol{A}))$ *is a closed operator;*
- *(v) the operator $\tilde{R}(t) : D(D) \subseteq Y \to X$ given by $\tilde{R}(t)f = \int_0^t T(t-s)BS(s)f\,ds$, has a unique extension $R(t) \in B(Y, X)$ which is uniformly bounded as $t \searrow 0$.*

Then $\boldsymbol{A}$ generates a strongly continuous semigroup $(\boldsymbol{T}(t))_{t\geq 0}$ on the product space $X \times Y$. Moreover, this semigroup is given by

$$\boldsymbol{T}(t) := \begin{pmatrix} T(t) & R(t) \\ 0 & S(t) \end{pmatrix}, \quad t \geq 0.$$

Proof. See [30, Proposition 3.1]. □

2.4.7 Perturbations

In applications, when formulating our equation of interest as an abstract Cauchy problem, it is often the case that the terms involved are more naturally expressed as separate operators, leading to an equation of the form.

$$\frac{d}{dt}u(t) = Au(t) + Bu(t) \quad (t > 0)\,; \quad u(0) = u_0.$$

Usually checking the conditions of Theorem 2.4.17 directly for $A + B$ is too complicated. In this situation, it is often easier to consider the operators A and B individually, making use of a set of theorems known as *perturbation results*. The aim of such results is to answer the following question. Supposing that A generates a C_0-semigroup, under what conditions does the combined operator $A + B$ (or some related operator) form a generator?

Perhaps the most obvious example is where the operator B is bounded, and such a case is covered in the following theorem.

Theorem 2.4.27. *Let $(A, D(A))$ generate a C_0-semigroup $(T(t))_{t\geq 0}$ which satisfies bound* (2.3) *on a Banach space X. If $B \in B(X)$, then $A+B$ with $D(A+B) = D(A)$ generates a C_0-semigroup, $(S(t))_{t\geq 0}$, satisfying*

$$\|S(t)\| \leq Me^{(\omega+M\|B\|)t} \quad \text{for} \quad t \geq 0.$$

Proof. See [14, Chapter 3, Theorem 1.3]. □

For the applications we shall be considering, the condition that B is bounded turns out to be too strong. We therefore require an alternative perturbation result, namely the Kato–Voigt Perturbation Theorem. This result does not require that B be bounded. However, in giving this up we lose $A + B$ as a generator. Instead we have that some extension of $A + B$ is a generator.

Theorem 2.4.28. (Kato–Voigt Perturbation Theorem) *Let $X = L_1(\Omega, \mu)$ and suppose the operators A and B satisfy the conditions*

(*i*) *$(A, D(A))$ generates a substochastic semigroup $(G_A(t))_{t\geq 0}$ on X;*

(*ii*) *B is a positive linear operator with $D(A) \subseteq D(B)$;*

(*iii*) *for all $f \in D(A)_+$*

$$\int_\Omega (Af + Bf)\, d\mu \leq 0.$$

Then there exists some extension $(K, D(K))$ of $(A + B, D(A))$ which generates a substochastic semigroup $(G_K(t))_{t\geq 0}$.

Proof. See [4, Corollary 5.17]. □

We note that condition (iii) above can always be expressed as

$$\int_\Omega (A+B)f\,d\mu = -c(f) \text{ for } f \in D(A)_+, \tag{2.7}$$

where c is some non-negative functional defined on $D(A)$.

This result gives only the existence of a generator K and provides no indication of how this operator relates to $A+B$. The nature of the generator K is closely related to the concept of semigroup honesty, as defined in [4, Definition 6.4].

Definition 2.4.29. The positive semigroup $(G_K(t))_{t\geq 0}$, generated by the extension K of $A+B$, is honest if the functional c, given by (2.7), extends to $D(K)$, and for all $u_0 \in D(K)_+$, the solution $u(t) = G_K(t)u_0$ to

$$\frac{d}{dt}u(t) = Ku(t), \quad t > 0; \quad u(0) = u_0,$$

satisfies

$$\frac{d}{dt}\int_\Omega u(t)\,d\mu = \frac{d}{dt}\|u(t)\| = -c(u(t)).$$

Theorem 2.4.30. *The semigroup* $(G_K(t))_{t\geq 0}$ *is honest if and only if* $K = \overline{A+B}$.

Proof. See [4, Theorem 6.13]. □

2.4.8 Semilinear Abstract Cauchy Problem

In the previous section our generator A was perturbed by a linear operator B. We now consider the case where the perturbation is nonlinear.

Definition 2.4.31. Let X be a Banach space, $A \in \mathcal{G}(M,\omega;X)$ and $N : D \to X$ be some (nonlinear) mapping from D into X, where D is a subset of X such that $D(A) \cap D \neq \emptyset$. Then the equation

$$\frac{d}{dt}u(t) = A[u(t)] + N[u(t)], \quad t > 0; \quad u(0) = u_0 \in D(A) \cap D, \tag{2.8}$$

is known as a *semilinear* abstract Cauchy problem.

Definition 2.4.32. By a *strong solution* on $[0,t_0)$ of (2.8), we mean a function $u : [0,t_0) \to X$ such that

(i) u is (strongly) continuous on $[0,t_0)$;

(ii) u is (strongly) differentiable on $(0,t_0)$;

(*iii*) $u(t) \in D(A) \cap D$ for all $t \in [0, t_0)$;

(*iv*) $u(t)$ satisfies (2.8) for $0 \leq t < t_0$.

Suppose that $u(t)$ is a strong solution of (2.8) and that $N[u(t)]$ is a strongly continuous function of $t \in [0, t_0)$. Then, by following [7, pages 109-110], $u(t)$ can be shown to satisfy the integral equation

$$u(t) = T(t)u_0 + \int_0^t T(t-s)N[u(s)]\,ds, \quad 0 \leq t < t_0, \tag{2.9}$$

where $(T(t))_{t\geq 0}$ is the semigroup generated by A.

Definition 2.4.33. A *mild solution* on $[0, t_0)$ of (2.8) is a function $u : [0, t_0) \to X$ such that

(*i*) u is (strongly) continuous on $[0, t_0)$;

(*ii*) $u(t) \in D$ for all $t \in [0, t_0)$;

(*iii*) $u(t)$ satisfies (2.9).

Having defined a semilinear abstract Cauchy problem, as well as both strong and mild solutions to such problems, we now consider the conditions under which such solutions exist.

Theorem 2.4.34. *Let A be a generator of class $\mathcal{G}(M, \omega; X)$ and let N satisfy a local Lipschitz condition on the closed ball $\bar{B}(u_0, r) \subset D$. Then the semilinear abstract Cauchy problem* (2.8) *has a unique, local in time, mild solution.*

Proof. See [8, Theorem 3.22]. □

Theorem 2.4.35. *Suppose that A and N satisfy the conditions of Theorem 2.4.34. Additionally, suppose*

(*i*) *N is Fréchet differentiable at each $f \in B(u_0, r)$, with Fréchet derivative N_f such that $\|N_f g\| \leq k\|g\|$ for all $f \in B(u_0, r)$ and $g \in X$, where k is a positive constant independent of f and g;*

(*ii*) *the Fréchet derivative N_f is continuous with respect to $f \in B(u_0, r)$, that is*

$$\|N_f g - N_{f'} g\| \to 0 \;\; as \;\; \|f - f'\| \to 0 \quad (f, f' \in B(u_0, r)),$$

for all $g \in X$;

(*iii*) $u_0 \in D(A)$.

Then equation (2.8) *has a unique, local in time, strong solution.*

Proof. See [8, Theorem 3.30 and Theorem 3.32]. □

Theorem 2.4.36. *Let A and N satisfy the conditions of Theorem 2.4.35 and let $u(t)$ be the solution to (2.8) provided by Theorem 2.4.35. Suppose that $[0, \hat{T})$ is the maximal interval of existence of $u(t)$, if $\hat{T} < \infty$ then we have*

$$\lim_{t \nearrow \hat{T}} \|u(t)\| = \infty.$$

Proof. See [31, Chapter 6, Theorem 1.4]. □

Chapter 3

Dual Regime Continuous Fragmentation

3.1 Dual Regime Fragmentation Model

We are considering a model of fragmentation which involves a cut-off particle mass $x_c > 0$, whereby only those particles with mass exceeding this cut-off are able to fragment. A model of this nature was examined in [18]. Such a setup produces two regimes; a fragmentation regime for particles of mass $x > x_c$, in which particles may break into smaller pieces, and a dust regime for particles of mass $0 < x \leq x_c$, in which particles are unable to break up any further.

By $u_F(x,t)$ we denote the particle mass density within the fragmentation regime, the evolution of which is described by the following rate equation

$$\frac{\partial u_F(x,t)}{\partial t} = -a(x)u_F(x,t) + \int_x^\infty a(y)b(x|y)u_F(y,t)dy, \quad x > x_c, t > 0, \tag{3.1}$$
$$u_F(x,0) = f_0(x).$$

Here $a(x)$ represents the fragmentation rate for a particle of mass x, $b(x|y)$ gives the distribution of particles of mass x resulting from a break-up of a particle of mass y and $f_0(x)$ gives the initial mass distribution within the fragmentation regime. The first term on the right-hand side of equation (3.1) is the loss term, which gives the rate at which we lose particles of mass x due to their fragmentation into smaller particles. The second term, involving the integral, is the gain term and gives the rate at which we gain particles of mass x due to the break-up of larger particles. We assume that the functions a and b are non-negative, measurable functions defined on (x_c, ∞) and $(0, \infty) \times (x_c, \infty)$, respectively.

We also have for b that

$$b(x|y) = 0 \text{ for } x > y \text{ and } \int_0^y xb(x|y)dx = y \text{ for } y > x_c. \tag{3.2}$$

These conditions reflect the fact that after a fragmentation event, resulting particles cannot have a mass exceeding that of the initial particle, and the total mass of all resulting particles must equal the mass of the initial particle.

Turning now to the dust regime. If $u_D(x,t)$ denotes the particle mass density within this regime, then its evolution is governed by the equation

$$\begin{aligned} \frac{\partial u_D(x,t)}{\partial t} &= \int_{x_c}^{\infty} a(y)b(x|y)u_F(y,t)dy, \quad 0 < x \le x_c, t > 0, \\ u_D(x,0) &= d_0(x), \end{aligned} \tag{3.3}$$

where a and b are as before and $d_0(x)$ is the initial mass distribution within the dust regime. The integral on the right-hand side of equation (3.3) provides the rate at which particles in the fragmentation regime break and produce particles of mass $x \le x_c$.

In order to facilitate the use of semigroup techniques we must select spaces in which to study equations (3.1) and (3.3). Therefore we introduce the spaces $X_D = L_1((0,x_c], xdx)$ and $X_F = L_1((x_c,\infty), xdx)$, corresponding to the dust and fragmentation regimes respectively. These are natural spaces in which to study the problem as their norms, when applied to the particle mass densities, provide a measure of mass.

3.2 Fragmentation Regime

Looking initially at equation (3.1), we introduce the following expressions

$$(\mathcal{A}f)(x) = -a(x)f(x) \text{ and } (\mathcal{B}f)(x) = \int_x^{\infty} a(y)b(x|y)f(y)dy \text{ for } x > x_c.$$

From these expressions we form the maximal operator, K_{max}, as $\mathcal{A} + \mathcal{B}$ defined on $D_{max} = \{f \in X_F : \mathcal{A}f + \mathcal{B}f \in X_F\}$. We also define the individual operators A and B as follows

$$\begin{aligned} (Af)(x) &= (\mathcal{A}f)(x), \qquad D(A) = \{f \in X_F : Af \in X_F\}, \\ (Bf)(x) &= (\mathcal{B}f)(x), \qquad D(B) = D(A). \end{aligned}$$

Lemma 3.2.1. *The operator B with domain $D(A)$ is well-defined. Moreover, $\|Bf\|_{X_F} \leq \|Af\|_{X_F}$ for $f \in D(A)$.*

Proof. Let $f \in D(A)$. Then

$$\begin{aligned} \|Bf\|_{X_F} &= \int_{x_c}^{\infty} \left| \int_{x}^{\infty} a(y)b(x|y)f(y)dy \right| xdx \\ &\leq \int_{x_c}^{\infty} \left(\int_{x}^{\infty} a(y)b(x|y)\,|f(y)|\,dy \right) xdx \\ &= \int_{x_c}^{\infty} a(y)\,|f(y)| \left(\int_{x_c}^{y} xb(x|y)dx \right) dy \\ &\leq \int_{x_c}^{\infty} a(y)\,|f(y)|\,ydy = \|Af\|_{X_F}\,. \end{aligned} \tag{3.4}$$

We are permitted to swap the order of integration in the second step by Fubinis's Theorem, [33, Theorem 7.8], since the integrand is non-negative. The final inequality follows since $\int_{x_c}^{y} xb(x|y)dx \leq y$, because of (3.2). This reflects the fact that upon fragmentation of a particle of mass y, the mass remaining within the fragmentation regime cannot exceed y. □

This allows us to form the minimal operator, K_{min}, as $A + B$ defined on $D(A)$. Equation (3.1) is then reformulated in the setting of X_F as the abstract Cauchy problem

$$\frac{d}{dt}u_F(t) = K[u_F(t)], \quad t > 0; \quad u_F(0) = f_0, \tag{3.5}$$

where K is some operator lying between the minimal and maximal operators. The following result gives the existence of such an operator K which generates a substochastic semigroup.

Theorem 3.2.2. *There exists an extension $(K, D(K))$ of $(A + B, D(A))$ which generates a substochastic semigroup $(G_K(t))_{t\geq 0}$ on X_F.*

Proof. To establish this result we show that the operators A and B satisfy the conditions of Theorem 2.4.28.

- (i) It is clear that $(A, D(A))$ generates a substochastic semigroup $(G_A(t))_{t\geq 0}$ on X_F, where $(G_A(t)f)(x) = \exp(-a(x)t)f(x)$.
- (ii) By definition, we trivially have $D(A) \subseteq D(B)$. The non-negativity of a and b make B a positive operator, so that $Bf \in X_{F+}$ for all $f \in D(B)_+$.

(iii) For all $f \in D(A)_+$ we have that

$$\int_{x_c}^{\infty} (Af + Bf)\, xdx = \int_{x_c}^{\infty} \left(-a(x)f(x) + \int_x^{\infty} a(y)b(x|y)f(y)dy \right) xdx$$
$$= -\int_{x_c}^{\infty} a(x)f(x)xdx + \int_{x_c}^{\infty} \left(\int_x^{\infty} a(y)b(x|y)f(y)dy \right) xdx$$
$$= -\int_{x_c}^{\infty} a(x)f(x)xdx + \int_{x_c}^{\infty} a(y)f(y) \left(\int_{x_c}^{y} xb(x|y)dx \right) dy$$
$$= -\int_{x_c}^{\infty} \left(x - \int_{x_c}^{x} yb(y|x)dy \right) a(x)f(x)dx =: -c(f) \leq 0. \tag{3.6}$$

The change of order of integration in the third line can be justified as before. We have introduced the notation c to represent the final integral expression, and this functional appears regularly in what follows. The non-negativity of c comes as a result of the earlier statement regarding $\int_{x_c}^{x} yb(y|x)dy \leq x$. □

This result gives only the existence of a generator K and provides no indication of where this operator lies within the range between K_{min} and K_{max}. The nature of the generator K is closely related to the concept of semigroup honesty and has implications for the behaviour of solutions provided by the semigroup. This matter is examined in [3], where various possibilities for K are considered.

Recalling Definition 2.4.29, we define semigroup honesty within the current context. The positive semigroup $(G_K(t))_{t\geq 0}$, generated by the extension K, is honest if the functional c, given by (3.6), extends to $D(K)$, and for all $u_0 \in D(K)_+$, the solution $u(t) = G_K(t)u_0$ to (3.5) satisfies

$$\frac{d}{dt}\int_{x_c}^{\infty} u(t)xdx = \frac{d}{dt}\|u(t)\|_{X_F} = -c(u(t)).$$

In order to establish the honesty of our semigroup $(G_K(t))_{t\geq 0}$, we make use of Theorem 2.4.30. To show that $K = \overline{A+B}$, we adopt the approach taken in [4, Section 6.3]. Let E denote the set of measurable functions defined on (x_c, ∞) which take values in the extended reals $[-\infty, \infty]$. By E_f, we denote the subspace of E consisting of functions which are finite almost everywhere. We also introduce $\mathsf{F} \subset \mathsf{E}$, defined such that $f \in \mathsf{F}$ if and only if for any non-negative, non-decreasing sequence $\{f_n\}_{n=1}^{\infty}$ with $\sup_{n\in\mathbb{N}} f_n = |f|$, we have $\sup_{n\in\mathbb{N}} (I-A)^{-1} f_n \in X_F$. We also make the following assumptions regarding the operator B and its domain $D(B)$,

$$f \in D(B) \text{ if and only if } f_+, f_- \in D(B), \tag{3.7}$$

where $f_+ = \max\{f, 0\}$ and $f_- = -\min\{f, 0\}$. For non-decreasing sequences $\{f_n\}_{n=1}^{\infty}$ and $\{g_n\}_{n=1}^{\infty}$ of functions in $D(B)_+$, we have

$$\sup_{n\in\mathbb{N}} f_n = \sup_{n\in\mathbb{N}} g_n \quad \text{implies} \quad \sup_{n\in\mathbb{N}} Bf_n = \sup_{n\in\mathbb{N}} Bg_n. \tag{3.8}$$

Lemma 3.2.3. *The operator* $(B, D(A))$ *satisfies* (3.7) *and* (3.8)*.*

Proof. Initially let us assume that both $f_+, f_- \in D(A) = D(B)$. Then since we can write $f = f_+ - f_-$, we obtain

$$\|Af\|_{X_F} = \|Af_+ - Af_-\|_{X_F} \leq \|Af_+\|_{X_F} + \|Af_-\|_{X_F}.$$

Therefore $f \in D(A)$ when $f_+, f_- \in D(A)$. Now suppose that $f \in D(A)$. Since $0 \leq f_\pm \leq |f|$, we have that

$$\|Af_\pm\|_{X_F} = \int_{x_c}^{\infty} a(y) f_\pm(y) y dy \leq \int_{x_c}^{\infty} a(y)|f(y)| y dy = \|Af\|_{X_F}.$$

Hence if $f \in D(A)$, then $f_+, f_- \in D(A)$. Taken together, these two results give us (3.7). The second condition, (3.8), follows from Lebesgue's monotone convergence theorem, [33, Theorem 1.26], which gives us

$$\sup_{n\in\mathbb{N}} Bf_n = \mathcal{B} \sup_{n\in\mathbb{N}} f_n = \mathcal{B} \sup_{n\in\mathbb{N}} g_n = \sup_{n\in\mathbb{N}} Bg_n.$$

□

We also define the set $\mathsf{G} \subset \mathsf{E}$ to be the set of functions $f \in X_F$ such that if $\{f_n\}_{n=1}^{\infty}$ is a non-negative, non-decreasing sequence of functions in $D(B)$ such that $\sup_{n\in\mathbb{N}} f_n = |f|$, then $\sup_{n\in\mathbb{N}} Bf_n < \infty$ almost everywhere.

Finally, we introduce the mappings $\mathsf{B} : D(\mathsf{B})_+ \to \mathsf{E}_{f,+}$, with $D(\mathsf{B}) = \mathsf{G}$, and $\mathsf{L} : \mathsf{F}_+ \to X_{F+}$ by

$$\begin{aligned} \mathsf{B}f &:= \sup_{n\in\mathbb{N}} Bf_n, \qquad && f \in D(\mathsf{B})_+, \\ \mathsf{L}f &:= \sup_{n\in\mathbb{N}} R(1, A) f_n, \qquad && f \in \mathsf{F}_+, \end{aligned}$$

where $0 \leq f_n \leq f_{n+1}$ for all $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} f_n = f$. These mappings can be extended to positive linear operators on all of $D(\mathsf{B})$ and F, respectively, via [4, Theorem 2.64].

Theorem 3.2.4. *Let* c *be the functional defined by* (3.6)*. If for all* $f \in \mathsf{F}_+$ *such that* $-f + \mathsf{BL}f \in X_F$ *and* $c(\mathsf{L}f)$ *exists, we have*

$$\int_{x_c}^{\infty} \mathsf{L}f x dx + \int_{x_c}^{\infty} (-f + \mathsf{BL}f)\, x dx \geq -c\,(\mathsf{L}f)\,, \tag{3.9}$$

then $K = \overline{A + B}$*.*

Proof. See [4, Theorem 6.22]. □

Theorem 3.2.5. *If the fragmentation rate, $a(x)$, is such that*

$$\limsup_{x\to x_c^+} a(x) < \infty \quad \text{and} \quad a \in L_{\infty,loc}(x_c,\infty),$$

then $(G_K(t))_{t\geq 0}$ is honest.

Proof. The proof of this result follows closely that of [4, Theorem 8.5] and makes use of Theorem 3.2.5. Since $Af = -af$, as in [2, Corollary 3.1], we have $\mathsf{F} = \{f \in \mathsf{E} : (1+a)^{-1}f \in X_F\}$ and $\mathsf{L}f = (1+a)^{-1}f$. Also, by Lebesgue's monotone convergence theorem, the operator B is given by the integral expression $\mathcal{B}$. For $f \in \mathsf{F}_+$, we set $g(x) = (\mathsf{L}f)(x) = (1+a(x))^{-1}f(x) \in X_{F+}$. Then we see that (3.9) is satisfied if for all $g \in X_{F+}$ such that $-ag + \mathcal{B}g \in X_F$ and $c(g)$ exists, we have

$$\int_{x_c}^{\infty} \left(-a(x)g(x) + (\mathcal{B}g)(x)\right) x dx \geq -c(g). \tag{3.10}$$

By our assumptions regarding a, we have $ag \in L_1((x_c,R], xdx)$ for any $R \in (x_c,\infty)$, which along with $-ag + \mathcal{B}g \in X_F$ gives us $\mathcal{B}g \in L_1((x_c,R], xdx)$. We may write the left-hand side of (3.10) as

$$\int_{x_c}^{\infty} \left(-a(x)g(x) + (\mathcal{B}g)(x)\right) x dx = \lim_{R\to\infty} \int_{x_c}^{R} \left(-a(x)g(x) + (\mathcal{B}g)(x)\right) x dx$$
$$= \lim_{R\to\infty} \left\{ -\int_{x_c}^{R} a(x)g(x)xdx + \int_{x_c}^{R} \left(\int_{x}^{\infty} a(y)b(x|y)g(y)dy \right) xdx \right\}. \tag{3.11}$$

If we change the order of integration in the second term, which is justified by Fubini's Theorem, we get

$$\begin{aligned} &-\int_{x_c}^{R} a(x)g(x)xdx + \int_{x_c}^{R} \left(\int_{x}^{\infty} a(y)b(x|y)g(y)dy \right) xdx \\ &= -\int_{x_c}^{R} a(y)g(y)ydy + \int_{x_c}^{R} a(y)g(y) \left(\int_{x_c}^{y} xb(x|y)dx \right) dy \\ &+ \int_{R}^{\infty} a(y)g(y) \left(\int_{x_c}^{R} xb(x|y)dx \right) dy. \end{aligned}$$

Replacing this within (3.11) yields

$$\begin{aligned}
&\int_{x_c}^{\infty} \left(-a(x)g(x) + (\mathcal{B}g)(x)\right) x dx \\
= &- \lim_{R\to\infty} \int_{x_c}^{R} \left(y - \int_{x_c}^{y} x b(x|y) dx \right) a(y) g(y) dy \\
&+ \lim_{R\to\infty} \int_{R}^{\infty} a(y) g(y) \left(\int_{x_c}^{R} x b(x|y) dx \right) dy \\
= &- c(g) + \lim_{R\to\infty} \int_{R}^{\infty} a(y) g(y) \left(\int_{x_c}^{R} x b(x|y) dx \right) dy.
\end{aligned}$$

The non-negativity of the final term gives us (3.10), and with it the honesty of the semigroup $(G_K(t))_{t\geq 0}$. □

3.3 Dust Regime

We now consider the dust regime and equation (3.3). With the aim of recasting (3.3) in our abstract setting, we introduce the operator $C : D(C) \subseteq X_F \to X_D$ defined by

$$\begin{aligned}
(Cf)(x) &= \int_{x_c}^{\infty} a(y) b(x|y) f(y) dy, \quad 0 < x \leq x_c, \\
D(C) &= \{ f \in X_F : Cf \in X_D \}.
\end{aligned}$$

Equation (3.3) is then reformulated as

$$\frac{d}{dt} u_D(t) = C[u_F(t)], \quad t > 0; \quad u_D(0) = d_0. \tag{3.12}$$

Lemma 3.3.1. *The operator C is defined on $D(A)$, with $\|Cf\|_{X_D} \leq \|Af\|_{X_F}$ for $f \in D(A)$.*

Proof. Let $f \in D(A)$. Then

$$\begin{aligned}
\|Cf\|_{X_D} &= \int_0^{x_c} \left| \int_{x_c}^{\infty} a(y) b(x|y) f(y) dy \right| x dx \\
&\leq \int_0^{x_c} \left(\int_{x_c}^{\infty} a(y) b(x|y) |f(y)| dy \right) x dx \\
&= \int_{x_c}^{\infty} a(y) |f(y)| \left(\int_0^{x_c} x b(x|y) dx \right) dy \qquad (3.13) \\
&\leq \int_{x_c}^{\infty} a(y) |f(y)| y dy = \|Af\|_{X_F},
\end{aligned}$$

where once again the change of integration order is justified by Fubini's Theorem. The final step is similar to that in Lemma 3.2.1; however, the integral now represents the mass lost from the fragmentation regime to the dust regime during a fragmentation event. □

Lemma 3.3.2. *The operator C is $(A+B)$-bounded on $D(A)$.*

Proof. By (3.4) and (3.13), for $f \in D(A)$ we have

$$\begin{aligned}
\|Bf\|_{X_F} + \|Cf\|_{X_D} &\leq \int_{x_c}^{\infty} a(y)\,|f(y)| \left(\int_{x_c}^{y} xb(x|y)dx \right) dy \\
&\quad + \int_{x_c}^{\infty} a(y)\,|f(y)| \left(\int_{0}^{x_c} xb(x|y)dx \right) dy \\
&= \int_{x_c}^{\infty} a(y)\,|f(y)| \left(\int_{0}^{y} xb(x|y)dx \right) dy \\
&= \int_{x_c}^{\infty} a(y)\,|f(y)|\, ydy = \|Af\|_{X_F}.
\end{aligned}$$

Subtracting $\|Bf\|_{X_F}$ from both sides we get

$$\begin{aligned}
\|Cf\|_{X_D} &\leq \|Af\|_{X_F} - \|Bf\|_{X_F} = \|Af\|_{X_F} - \|-Bf\|_{X_F} \\
&\leq \|Af - (-Bf)\|_{X_F} = \|(A+B)f\|_{X_F}. \qquad (3.14)
\end{aligned}$$

□

Lemma 3.3.3. *The operator C can be extended to $D(K)$, with this extension being K-bounded on $D(K)$, satisfying*

$$\|Cf\|_{X_D} \leq \|Kf\|_{X_F} \quad \textit{for } f \in D(K).$$

Proof. Let $f \in D(K)$. Then there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset D(A)$ such that $f_n \to f$ and $(A+B)f_n \to Kf$ in X_F. By linearity of the operators and Lemma 3.3.2, we have that

$$\|Cf_m - Cf_n\|_{X_D} \leq \|(A+B)f_m - (A+B)f_n\|_{X_F}.$$

Since the sequence $\{(A+B)f_n\}_{n=1}^{\infty}$ converges, it must be a Cauchy sequence. As a result $\{Cf_n\}_{n=1}^{\infty}$ must also be Cauchy. Since we are working in the Banach space X_D, the sequence $\{Cf_n\}_{n=1}^{\infty}$ must therefore converge to a limit, which we denote by Cf, with this limit being independent of the sequence $\{f_n\}_{n=1}^{\infty}$. To establish the latter, suppose that $\{g_n\}_{n=1}^{\infty} \subset D(A)$ shares the attributes of $\{f_n\}_{n=1}^{\infty}$. Then we can write

$$\begin{aligned}\|Cg_n - Cf\|_{X_D} &\leq \|Cg_n - Cf_n\|_{X_D} + \|Cf_n - Cf\|_{X_D} \\ &\leq \|(A+B)g_n - (A+B)f_n\|_{X_F} + \|Cf_n - Cf\|_{X_D} \\ &\leq \|(A+B)g_n - Kf\|_{X_F} + \|Kf - (A+B)f_n\|_{X_F} \\ &\quad + \|Cf_n - Cf\|_{X_D}.\end{aligned}$$

From this we can establish that $\{Cg_n\}_{n=1}^{\infty}$ also converges to Cf. The K-boundedness of C follows from its $(A+B)$-boundedness simply by applying (3.14) with the sequence $\{f_n\}_{n=1}^{\infty}$, and then passing the limit through the norms. □

3.4 Full Fragmentation System

We now combine equations (3.1) and (3.3), writing them as the following abstract Cauchy problem on the product space $X = X_D \times X_F$:

$$\frac{d}{dt}u(t) = \boldsymbol{A}[u(t)], \quad t > 0; \quad u(0) = u_0 \in D(\boldsymbol{A}), \tag{3.15}$$

where

$$u(t) = \begin{pmatrix} u_D(t) \\ u_F(t) \end{pmatrix}, \quad \boldsymbol{A} = \begin{pmatrix} 0_{DD} & C \\ 0_{DF} & K \end{pmatrix} \text{ and } u_0 = \begin{pmatrix} d_0 \\ f_0 \end{pmatrix} \in D(\boldsymbol{A}) = X_D \times D(K).$$

The subscripts on the zero operators indicate the spaces they map from and to; for example 0_{DF} maps from X_D into X_F. It is now our aim to show that the operator $\boldsymbol{A}$ is a generator on the space X. Before doing so, however, we shall establish the following lemma.

Lemma 3.4.1. *The operator $(\boldsymbol{A}, D(\boldsymbol{A}))$ is closed in X.*

Proof. To show that $(\boldsymbol{A}, D(\boldsymbol{A}))$ is closed, we express it as the sum of two operators $(\boldsymbol{K}, D(\boldsymbol{A}))$ and $(\boldsymbol{C}, D(\boldsymbol{A}))$, as follows

$$\boldsymbol{A} = \underbrace{\begin{pmatrix} 0_{DD} & 0_{FD} \\ 0_{DF} & K \end{pmatrix}}_{\boldsymbol{K}} + \underbrace{\begin{pmatrix} 0_{DD} & C \\ 0_{DF} & 0_{FF} \end{pmatrix}}_{\boldsymbol{C}}.$$

As the generator of a C_0-semigroup, $(K, D(K))$ is closed, and this is easily shown to carry over to $(\boldsymbol{K}, D(\boldsymbol{A}))$.

We now equip the space X with the norm $\|f\|_\alpha = \alpha\|f_1\|_{X_D} + \|f_2\|_{X_F}$, where $f = \binom{f_1}{f_2} \in X$ and $0 < \alpha < 1$. This new norm is equivalent to the standard norm on X. Hence the closedness of $(\boldsymbol{K}, D(\boldsymbol{A}))$ is maintained. Now for any $f = \binom{f_1}{f_2} \in D(\boldsymbol{A})$, Lemma 3.3.3 gives us

$$\|\boldsymbol{C}f\|_\alpha = \alpha\|Cf_2\|_{X_D} \le \alpha\|Kf_2\|_{X_F} = \alpha\|\boldsymbol{K}f\|_\alpha.$$

Hence $(\boldsymbol{C}, D(\boldsymbol{A}))$ is $\boldsymbol{K}$-bounded with $\boldsymbol{K}$-bound less than 1. By Lemma 2.3.11, $(\boldsymbol{A}, D(\boldsymbol{A}))$ is closed with respect to the norm $\|\cdot\|_\alpha$, and therefore is closed in X with respect to the standard norm, due to the equivalence of these two norms. □

Theorem 3.4.2. *The operator $\boldsymbol{A}$, as given above, generates a C_0-semigroup $(\boldsymbol{T}(t))_{t\ge 0}$ on X.*

Proof. In order to establish that $\boldsymbol{A}$ is a generator, we make use of Theorem 2.4.26. Conditions (i) to (iv) of Theorem 2.4.26 are easily verified in this case.

(i) The operator 0_{DD} generates the semigroup consisting solely of the identity operator, I_D, on X_D.

(ii) We know that the operator K generates the substochastic semigroup $(G_K(t))_{t\ge 0}$ on X_F.

(iii) We have shown in Lemma 3.3.3 that C is K-bounded.

(iv) The operator $(\boldsymbol{A}, D(\boldsymbol{A}))$ is closed in X, as was shown in Lemma 3.4.1.

We now verify that condition (v) of Theorem 2.4.26 is satisfied for this example. Consider $\tilde{R}(t) : D(K) \subseteq X_F \to X_D$ defined by $\tilde{R}(t)f = \int_0^t CG_K(s)f\,ds$. For $f \in D(K)$ we can write

$$\begin{aligned}
\tilde{R}(t)f &= \int_0^t CG_K(s)f\,ds \\
&= \int_0^t C(\lambda I_F - K)^{-1}(\lambda I_F - K)G_K(s)f\,ds \quad (\lambda > 0) \\
&= C(\lambda I_F - K)^{-1}\int_0^t (\lambda I_F - K)G_K(s)f\,ds \\
&= C(\lambda I_F - K)^{-1}\left\{\lambda\int_0^t G_K(s)f\,ds - \int_0^t KG_K(s)f\,ds\right\} \\
&= C(\lambda I_F - K)^{-1}\left\{\lambda\int_0^t G_K(s)f\,ds - \int_0^t G_K(s)Kf\,ds\right\} \\
&= C(\lambda I_F - K)^{-1}\left\{\lambda\int_0^t G_K(s)f\,ds - G_K(t)f + f\right\}.
\end{aligned}$$

We are permitted to take $C(\lambda I_F - K)^{-1}$ outside the integral, as it is a bounded operator by Lemma 2.3.10. The switching of the generator K and the semigroup operator $G_K(t)$ is allowed by Theorem 2.4.9. The final step is as a consequence of Lemma 2.4.11. From this, recalling that $(G_K(t))_{t\geq 0}$ consists of contractions, we obtain

$$\begin{aligned}\|\tilde{R}(t)f\|_{X_D} &\leq \|C(\lambda I_F - K)^{-1}\| \left\{\lambda \int_0^t \|G_K(s)f\|_{X_F} ds + \|G_K(t)f\|_{X_F} + \|f\|_{X_F}\right\} \\ &\leq \|C(\lambda I_F - K)^{-1}\| \left(\lambda t + 2\right) \|f\|_{X_F}. \end{aligned} \tag{3.16}$$

Therefore $\tilde{R}(t)$ is bounded on $D(K)$. As a densely-defined, bounded linear operator, $\tilde{R}(t)$ can be uniquely extended to a bounded linear operator $R(t)$ in $B(X_F, X_D)$. Further, (3.16) holds for all $u \in X_F$ with $\tilde{R}(t)$ replaced by $R(t)$. As a consequence we see that $R(t)$ is uniformly bounded as $t \searrow 0$. By Theorem 2.4.26, $\boldsymbol{A}$ generates a C_0-semigroup $(\boldsymbol{T}(t))_{t\geq 0}$ on the product space X. Furthermore, this semigroup is given by

$$\boldsymbol{T}(t) := \begin{pmatrix} I_D & R(t) \\ 0_{DF} & G_K(t) \end{pmatrix}, \quad t \geq 0.$$

□

The existence of the semigroup $(\boldsymbol{T}(t))_{t\geq 0}$, provides us with a unique strong solution, $u(t) = \boldsymbol{T}(t)u_0$, to the abstract Cauchy problem (3.15). For this solution to be physically relevant we would expect it to display a number of properties. In particular, we would hope that it preserves positivity and provides conservation of mass.

Lemma 3.4.3. *Given initial data* $u_0 = \binom{d_0}{f_0} \in D(\boldsymbol{A})_+ = X_{D+} \times D(K)_+$, *the solution* $u(t) = \boldsymbol{T}(t)u_0$, *emanating from* u_0, *remains positive, that is* $u(t) \in D(\boldsymbol{A})_+$ *for all* $t \geq 0$.

Proof. Let $u_0 \in D(\boldsymbol{A})_+$. The components of $u(t)$ are given by

$$\begin{aligned} u_D(t) &= d_0 + \int_0^t CG_K(s)f_0\, ds = d_0 + \int_0^t Cu_F(s)\, ds, \\ u_F(t) &= G_K(t)f_0. \end{aligned}$$

Since $(G_K(t))_{t\geq 0}$ is a substochastic semigroup, it preserves positivity. Therefore $G_K(t)f_0 \in D(K)_+$ for all $t \geq 0$, hence $u_F(t) \in D(K)_+$ for all $t \geq 0$.

As an integral operator with a non-negative kernel, the operator C is a positive operator. Hence $Cu_F(t) \in X_{D+}$ for all $t \geq 0$. The integral $\int_0^t Cu_F(s)\, ds$ is the limit of a Riemann sum of elements from X_{D+}. Given that the positive cone is closed under addition and multiplication by positive scalars, any such Riemann sum will itself be an element in X_{D+}. Since the space X_D is a Lebesgue space, its positive cone is a closed subset. Hence $\int_0^t Cu_F(s)\,ds \in X_{D+}$. As mentioned, the positive cone is closed under addition and therefore $u_D(t) \in X_{D+}$. Together $u_D(t) \in X_{D+}$ and $u_F(t) \in D(K)_+$ give us that $u(t) \in D(\boldsymbol{A})_+$. □

Let $u(t) = \binom{u_D(t)}{u_F(t)}$ be the solution to the abstract Cauchy problem (3.15) with positive initial data. The masses in the dust and fragmentation regimes, at time t, are given by

$$M_D(t) = \int_0^{x_c} (u_D(t))\,(x)\,xdx \;\text{ and }\; M_F(t) = \int_{x_c}^{\infty} (u_F(t))\,(x)\,xdx,$$

respectively, with the total mass in the system being $M(t) = M_D(t) + M_F(t)$. In any fragmentation event, mass is simply redistributed from the larger particle to the smaller resulting particles, but the total mass involved should be conserved. Therefore, although $M_D(t)$ may increase and $M_F(t)$ decrease as mass is transferred from the fragmentation regime to the dust regime, we would expect the total mass, $M(t)$, to remain constant.

Lemma 3.4.4. *The total mass within the system is conserved, that is $M(t)$ remains constant.*

Proof. Since $(G_K(t))_{t\geq 0}$ is an honest semigroup, by Definition 2.4.29 we have

$$\begin{aligned}\frac{d}{dt}M_F(t) &= \frac{d}{dt}\int_{x_c}^{\infty} (u_F(t))\,(x)\,xdx = -c\,(u_F(t)) \\ &= -\int_{x_c}^{\infty}\left(x - \int_{x_c}^{x} yb(y|x)dy\right) a(x)\,(u_F(t))\,(x)dx. \end{aligned} \tag{3.17}$$

The solution $u(t)$ of the abstract Cauchy problem (3.15) is strongly differentiable, and this property is inherited by the component $u_D(t)$. Therefore we can take the time-derivative through the integral in $M_D(t)$, which gives us

$$\begin{aligned}\frac{d}{dt}M_D(t) &= \int_0^{x_c} \frac{d}{dt}[(u_D(t))\,(x)]\,xdx \\ &= \int_0^{x_c} (Cu_F(t))\,(x)\,xdx \\ &= \int_0^{x_c}\left(\int_{x_c}^{\infty} a(y)b(x|y)\,(u_F(t))\,(y)dy\right) xdx \\ &= \int_{x_c}^{\infty} a(y)\,(u_F(t))\,(y)\left(\int_0^{x_c} b(x|y)xdx\right) dy. \end{aligned} \tag{3.18}$$

The positivity of the integrand permits the change of integration order in the final step. The rate of change of the total mass $M(t)$ is given by the addition of (3.17) and (3.18), which by (3.2) yields

$$\begin{aligned}
\frac{d}{dt}M(t) &= \int_{x_c}^{\infty} a(y)\left(u_F(t)\right)(y)\left(\int_0^{x_c} b(x|y)x dx\right) dy \\
&\quad - \int_{x_c}^{\infty} a(y)\left(u_F(t)\right)(y)\left(y - \int_{x_c}^{y} b(x|y)x dx\right) dy \\
&= \int_{x_c}^{\infty} a(y)\left(u_F(t)\right)(y)\left(\int_0^{y} b(x|y)x dx\right) dy \\
&\quad - \int_{x_c}^{\infty} a(y)\left(u_F(t)\right)(y) y dy \\
&= 0.
\end{aligned}$$

This confirms that $M(t)$ is a constant and so mass is conserved. □

Chapter 4

Dual Regime Continuous Coagulation–Fragmentation

4.1 Dual Regime Coagulation–Fragmentation Model

Having considered a model involving only fragmentation in Chapter 3, we now introduce a coagulation process whereby pairs of particles can join to form a larger particle. As before, u_D and u_F denote the particle mass density within the dust and fragmentation regimes, respectively. The coagulation kernel $k(x, y)$ provides the rate at which a particle of mass x joins with one of mass y. The function k is defined on $(0, \infty) \times (0, \infty)$ and should be non-negative and symmetric.

Within the fragmentation regime, the change in the particle mass density, u_F, due to coagulation is given by

$$
\begin{aligned}
&(N_F(u_D, u_F))(x,t) = \frac{\chi_I(x)}{2} \left\{ \int_0^{x-x_c} k(x-y,y)u_F(x-y,t)u_D(y,t)dy \right. \\
&\left. + \int_{x-x_c}^{x_c} k(x-y,y)u_D(x-y,t)u_D(y,t)dy + \int_{x_c}^{x} k(x-y,y)u_D(x-y,t)u_F(y,t)dy \right\} \\
&+ \frac{\chi_J(x)}{2} \left\{ \int_0^{x_c} k(x-y,y)u_F(x-y,t)u_D(y,t)dy \right. \\
&\left. + \int_{x_c}^{x-x_c} k(x-y,y)u_F(x-y,t)u_F(y,t)dy + \int_{x-x_c}^{x} k(x-y,y)u_D(x-y,t)u_F(y,t)dy \right\} \\
&- u_F(x,t) \left\{ \int_0^{x_c} k(x,y)u_D(y,t)dy + \int_{x_c}^{\infty} k(x,y)u_F(y,t)dy \right\},
\end{aligned}
$$

where $I = (x_c, 2x_c]$ and $J = (2x_c, \infty)$. Those terms involving the characteristic functions χ_I and χ_J are gain terms and account for the increase in particles of mass x due to the joining of two smaller particles whose combined mass is x. The requirement for multiple gain terms arises since the joining particles may come from either regime, and we must consider all possible combinations which result in a particle of mass x. For a fuller explanation of these terms, see Appendix A. The last two terms are loss terms, corresponding to the particles of mass x which are lost when such a particle joins with a particle from the dust or fragmentation regimes, respectively. The evolution of the density u_F is then governed by the following rate equation

$$\begin{aligned} \frac{\partial u_F(x,t)}{\partial t} &= (\mathcal{F}_F u_F)(x,t) + (N_F(u_D,u_F))(x,t), \quad x > x_c, t > 0, \\ u_F(x,0) &= f_0(x), \end{aligned} \tag{4.1}$$

where $\mathcal{F}_F u_F$ is a fragmentation term which is given by the right-hand side of equation (3.1).

We now consider the effect of coagulation on the dust regime. The change in the particle mass density, u_D, due to coagulation can be expressed as

$$\begin{aligned} (N_D(u_D,u_F))(x,t) = {} & \frac{1}{2}\int_0^x k(x-y,y)u_D(x-y,t)u_D(y,t)dy \\ & - u_D(x,t)\left\{\int_0^{x_c} k(x,y)u_D(y,t)dy + \int_{x_c}^{\infty} k(x,y)u_F(y,t)dy\right\}. \end{aligned}$$

Similar to before, the first term is the gain term, accounting for the particles of mass x created from the joining of two smaller particles. However, here we only require one gain term, as any particles which join together to form a dust particle must themselves be dust particles. The final two terms are again loss terms, accounting for the loss of particles of mass x due to them joining with another particle. The change in the particle density u_D is then described by the equation

$$\begin{aligned} \frac{\partial u_D(x,t)}{\partial t} &= (\mathcal{F}_D u_F)(x,t) + (N_D(u_D,u_F))(x,t), \quad 0 < x \le x_c, t > 0, \\ u_D(x,0) &= d_0(x), \end{aligned} \tag{4.2}$$

where the fragmentation term $\mathcal{F}_D u_F$ is provided by the right-hand side of equation (3.3).

Having carried out our analysis of the pure fragmentation system within the weighted L_1 spaces X_D and X_F, we now relabel these spaces as well as introduce the new spaces in which we shall analyse the combined coagulation and fragmentation system. For $k = 0, 1$, let $Y_{D,k} = L_1\left((0,x_c], x^k dx\right)$ and $Y_{F,k} = L_1\left((x_c,\infty), x^k dx\right)$, so that $Y_{D,1}$ and $Y_{F,1}$ correspond to our original spaces. We also define the intersection spaces $Y_D = Y_{D,0} \cap Y_{D,1} = L_1((0,x_c],(1+x)dx)$, corresponding to the dust regime, and $Y_F = Y_{F,0} \cap Y_{F,1} = L_1((x_c, \infty), (1+x)dx)$, for the fragmentation regime. Additionally we define the product space $Y = Y_D \times Y_F$. By working in the intersection spaces Y_D and Y_F, the norms now take account of the total number of particles in addition to the total mass. The choice of spaces Y_D and Y_F allows us to establish the required Lipschitz continuity and Fréchet differentiability properties of the nonlinear coagulation operator.

4.1 Fragmentation System

The first step in our analysis is to verify that the fragmentation system which provided a strongly continuous semigroup in $X = X_D \times X_F$, also produces a semigroup in the new space Y.

Lemma 4.1.1. *As sets, $Y_F = X_F$, with the norms on these spaces being equivalent.*

Proof. By definition $Y_F = Y_{F,0} \cap Y_{F,1}$, where $Y_{F,1} = X_F$. Hence $Y_F \subset X_F$. Also, for all $f \in Y_F$ we have $\|f\|_{X_F} = \|f\|_{Y_{F,1}} \leq \|f\|_{Y_{F,0}} + \|f\|_{Y_{F,1}} = \|f\|_{Y_F}$. Now suppose that $f \in X_F$. Then

$$\begin{aligned}\|f\|_{Y_F} &= \|f\|_{Y_{F,0}} + \|f\|_{Y_{F,1}} \\ &\leq \frac{1}{x_c}\int_{x_c}^{\infty} |f(x)|x dx + \|f\|_{Y_{F,1}} \\ &= \left(\frac{1}{x_c} + 1\right)\|f\|_{X_F} < \infty.\end{aligned}$$

Therefore $f \in Y_F$, so that $X_F \subset Y_F$, which along with the previous statement gives us $X_F = Y_F$ and

$$\|f\|_{X_F} \leq \|f\|_{Y_F} \leq \left(\frac{1}{x_c} + 1\right)\|f\|_{X_F} \quad \text{for all } f \in X_F = Y_F. \tag{4.3}$$

□

As a result, the operators A, B and K, and their domains, carry over to the space Y_F, as does the C_0-semigroup $(G_K(t))_{t\geq 0}$ generated by K, under the conditions of the previous chapter.

We have shown that the spaces Y_F and X_F are equivalent. However the same cannot be said of Y_D and X_D, as our interval domain is no longer bounded away from zero, although we do have $Y_D \subset X_D$. Therefore we must restrict the operator C, as defined in the previous chapter, and so introduce the restricted operator $C_|$.

Definition 4.1.2. We define the restriction $C_|$ of the operator C by

$$\begin{aligned} C_| f &:= Cf \quad f \in D(C_|), \\ D(C_|) &:= \{f \in D(C) : Cf \in Y_D\}. \end{aligned}$$

Definition 4.1.3. For $y > x_c$, we define $n(y)$ by

$$n(y) = \int_0^y b(x|y)\,dx.$$

The quantity $n(y)$ represents the expected number of daughter particles resulting from the fragmentation of a particle of mass y.

Lemma 4.1.4. *If a and b are such that*

$$a(x)\,(n(x) - 1) \leq \beta(x + 1), \tag{4.4}$$

for $x > x_c$, where β is a positive constant, then the restricted operator $C_|$ is defined on $D(A)$.

Proof. Let $f \in D(A)$. Then

$$\begin{aligned} \|C_| f\|_{Y_D} &\leq \int_0^{x_c} \left(\int_{x_c}^{\infty} a(y)b(x|y)|f(y)|\,dy \right) (1+x)dx \\ &= \int_{x_c}^{\infty} a(y)|f(y)| \left(\int_0^{x_c} b(x|y)(1+x)dx \right) dy \\ &\leq \int_{x_c}^{\infty} a(y)|f(y)|\,(n(y) + y)\ dy \\ &= \int_{x_c}^{\infty} a(y)|f(y)|\,(n(y) - 1)\ dy + \int_{x_c}^{\infty} a(y)|f(y)|\,(y+1)\ dy \\ &\leq \beta \int_{x_c}^{\infty} |f(y)|\,(y+1)\ dy + \int_{x_c}^{\infty} a(y)|f(y)|\,(y+1)\ dy \\ &= \|Af\|_{Y_F} + \beta\|f\|_{Y_F} < \infty. \end{aligned} \tag{4.5}$$

Therefore $D(A)$ is contained within $D(C_|)$. □

The condition (4.4) is satisfied, if for example, a is bounded by a linear polynomial with positive coefficients and $n(y)$ is bounded by a positive constant. This matter is considered further in the following chapter.

Lemma 4.1.5. *If a and b satisfy condition (4.4), then the restricted operator $C_|$ is K-bounded on $D(K)$.*

Proof. The first step is to show that $C_|$ is $(A+B)$-bounded on $D(A)$. Let $f \in D(A)$. Then

$$\begin{aligned} \|Bf\|_{Y_F} &\leq \int_{x_c}^{\infty} \left(\int_x^{\infty} a(y)b(x|y)|f(y)|\,dy \right) (1+x)dx \\ &= \int_{x_c}^{\infty} a(y)|f(y)| \left(\int_{x_c}^{y} b(x|y)(1+x)dx \right) dy. \end{aligned}$$

Combining this with (4.5), we obtain

$$\begin{aligned}\|C_{|}f\|_{Y_D} + \|Bf\|_{Y_F} &\le \int_{x_c}^{\infty} a(y)|f(y)| \left(\int_0^{x_c} b(x|y)(1+x)dx\right) dy \\ &\quad + \int_{x_c}^{\infty} a(y)|f(y)| \left(\int_{x_c}^{y} b(x|y)(1+x)dx\right) dy \\ &= \int_{x_c}^{\infty} a(y)|f(y)| \left(\int_0^{y} b(x|y)(1+x)dx\right) dy \\ &= \int_{x_c}^{\infty} a(y)|f(y)| \left(n(y)+y\right) dy \\ &= \int_{x_c}^{\infty} a(y)|f(y)| \left(y+1\right) dy \\ &\quad + \int_{x_c}^{\infty} a(y)|f(y)| \left(n(y)-1\right) dy \\ &\le \|Af\|_{Y_F} + \beta\|f\|_{Y_F}.\end{aligned}$$

Subtracting $\|Bu\|_{Y_F}$ from both sides we get

$$\begin{aligned}\|C_{|}f\|_{Y_D} &\le \|Af\|_{Y_F} - \|Bf\|_{Y_F} + \beta\|f\|_{Y_F} = \|Af\|_{Y_F} - \|-Bf\|_{Y_F} + \beta\|f\|_{Y_F} \\ &\le \|Af - (-Bf)\|_{Y_F} + \beta\|f\|_{Y_F} = \|(A+B)f\|_{Y_F} + \beta\|f\|_{Y_F}.\end{aligned}$$

By a similar approach to that taken in Lemma 3.3.3, we can extend $C_{|}$ to $D(K)$, with $C_{|}$ being K-bounded on $D(K)$, satisfying

$$\|C_{|}f\|_{Y_D} \le \|Kf\|_{Y_F} + \beta\|f\|_{Y_F}, \text{ for } f \in D(K). \tag{4.6}$$

□

The fragmentation system is then cast as the following abstract Cauchy problem on the product space $Y = Y_D \times Y_F$

$$\frac{d}{dt}u(t) = \boldsymbol{A}_{|}[u(t)], \quad t > 0; \quad u(0) = u_0 \in D(\boldsymbol{A}_{|}), \tag{4.7}$$

where

$$u(t) = \begin{pmatrix} u_D(t) \\ u_F(t) \end{pmatrix}, \ \boldsymbol{A}_{|} = \begin{pmatrix} 0_{DD} & C_{|} \\ 0_{DF} & K \end{pmatrix} \text{ and } u_0 = \begin{pmatrix} d_0 \\ f_0 \end{pmatrix} \in D(\boldsymbol{A}_{|}) = Y_D \times D(K).$$

The subscripts on the zero operators indicate the spaces they map from and to; for example 0_{DF} maps from Y_D into Y_F. To verify that $\boldsymbol{A}_{|}$ is a generator on the space Y, we invoke Theorem 2.4.26.

Theorem 4.1.6. *Assuming that a and b satisfy condition* (4.4)*, the operator* $\boldsymbol{A}_|$*, as given above, generates a* C_0*-semigroup* $\left(\boldsymbol{T}_|(t)\right)_{t\geq 0}$ *on* Y*.*

Proof. By employing the same argument as used in Lemma 3.4.1, and utilising (4.6), we can show that $(\boldsymbol{A}_|, D(\boldsymbol{A}_|))$ is a closed operator on Y. Conditions (i)-(iv) of Theorem 2.4.26 are then easily shown to be satisfied.

We now look to verify that condition (v) of Theorem 2.4.26 holds. Consider the operator $\tilde{R}_|(t) : D(K) \subseteq Y_F \to Y_D$ defined by $\tilde{R}_|(t)f = \int_0^t C_| G_K(s) f\, ds$. Since $C_|$ is K-bounded on $D(K)$, following the steps in Theorem 3.4.2, for $f \in D(K)$ we can write

$$\tilde{R}_|(t)f = C_|(\lambda I_F - K)^{-1}\left\{\lambda \int_0^t G_K(s)f\, ds - G_K(t)f + f\right\}.$$

Then, since $(G_K(t))_{t\geq 0}$ consists of contractions on the space X_F, by the equivalence condition (4.3), we get

$$\begin{aligned}
\|\tilde{R}_|(t)f\|_{Y_D} &\leq \|C_|(\lambda I_F - K)^{-1}\| \left\{\lambda \int_0^t \|G_K(s)f\|_{Y_F} ds + \|G_K(t)f\|_{Y_F} + \|f\|_{Y_F}\right\} \\
&\leq \|C_|(\lambda I_F - K)^{-1}\| \left(\frac{1}{x_c} + 1\right) \left\{\lambda \int_0^t \|G_K(s)f\|_{X_F} ds + \|G_K(t)f\|_{X_F} + \|f\|_{X_F}\right\} \\
&\leq \|C_|(\lambda I_F - K)^{-1}\| \left(\frac{1}{x_c} + 1\right) (\lambda t + 2)\, \|f\|_{X_F} \\
&\leq \|C_|(\lambda I_F - K)^{-1}\| \left(\frac{1}{x_c} + 1\right) (\lambda t + 2)\, \|f\|_{Y_F}.
\end{aligned} \tag{4.8}$$

This shows that $\tilde{R}_|(t)$ is bounded on $D(K)$. As a densely-defined, bounded linear operator, $\tilde{R}_|(t)$ can be uniquely extended to a bounded linear operator $R_|(t)$ in $B(Y_F, Y_D)$. Further, (4.8) holds for all $f \in Y_F$ with $\tilde{R}_|(t)$ replaced by $R_|(t)$. As a consequence, we see that $R_|(t)$ is uniformly bounded as $t \searrow 0$. By Theorem 2.4.26, $\boldsymbol{A}_|$ generates a C_0-semigroup $\left(\boldsymbol{T}_|(t)\right)_{t\geq 0}$ on the product space Y, with the semigroup being given by

$$\boldsymbol{T}_|(t) := \begin{pmatrix} I_D & R_|(t) \\ 0_{DF} & G_K(t) \end{pmatrix}, \quad t \geq 0.$$

□

Employing the same arguments as in Lemma 3.4.3, the semigroup $\left(\boldsymbol{T}_|(t)\right)_{t\geq 0}$ can be shown to preserve positivity.

4.2 Combined Coagulation–Fragmentation System

We now introduce the nonlinear coagulation operator N, which is defined on Y by

$$Nf = \begin{pmatrix} N_D(f_1, f_2) \\ N_F(f_1, f_2) \end{pmatrix}, \text{ for } f = \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} \in Y.$$

The combined coagulation fragmentation system from equations (4.1) and (4.2) is then expressed as the semilinear abstract Cauchy problem

$$\frac{d}{dt}u(t) = \boldsymbol{A}_|[u(t)] + N[u(t)], \quad t > 0; \quad u(0) = u_0 \in D(\boldsymbol{A}_|), \tag{4.9}$$

where, as before

$$u(t) = \begin{pmatrix} u_D(t) \\ u_F(t) \end{pmatrix}, \ \boldsymbol{A}_| = \begin{pmatrix} 0_{DD} & C_| \\ 0_{DF} & K \end{pmatrix} \text{ and } u_0 = \begin{pmatrix} d_0 \\ f_0 \end{pmatrix} \in D(\boldsymbol{A}_|) = Y_D \times D(K).$$

To more easily enable our analysis of the coagulation operator we introduce the following bilinear operators which act on the space $Y \times Y$. For $f = \binom{f_1}{f_2}$ and $g = \binom{g_1}{g_2}$ in Y, let

$$\mathcal{N}_1(f,g) = \begin{pmatrix} \frac{1}{2}\int_0^x k(x-y,y)f_1(x-y)g_1(y)dy \\ 0 \end{pmatrix},$$

$$\mathcal{N}_2(f,g) = \begin{pmatrix} f_1(x)\int_0^{x_c} k(x,y)g_1(y)dy \\ 0 \end{pmatrix},$$

$$\mathcal{N}_3(f,g) = \begin{pmatrix} f_1(x)\int_{x_c}^{\infty} k(x,y)g_2(y)dy \\ 0 \end{pmatrix},$$

$$\mathcal{N}_4(f,g) = \begin{pmatrix} 0 \\ \frac{\chi_I(x)}{2}\int_0^{x-x_c} k(x-y,y)f_2(x-y)g_1(y)dy \\ +\frac{\chi_J(x)}{2}\int_0^{x_c} k(x-y,y)f_2(x-y)g_1(y)dy \end{pmatrix},$$

$$\mathcal{N}_5(f,g) = \begin{pmatrix} 0 \\ \frac{\chi_I(x)}{2}\int_{x-x_c}^{x_c} k(x-y,y)f_1(x-y)g_1(y)dy \end{pmatrix},$$

$$\mathcal{N}_6(f,g) = \begin{pmatrix} 0 \\ \frac{\chi_I(x)}{2}\int_{x_c}^{x} k(x-y,y)f_1(x-y)g_2(y)dy \\ +\frac{\chi_J(x)}{2}\int_{x-x_c}^{x} k(x-y,y)f_1(x-y)g_2(y)dy \end{pmatrix},$$

$$\mathcal{N}_7(f,g) = \begin{pmatrix} 0 \\ \frac{\chi_J(x)}{2}\int_{x_c}^{x-x_c} k(x-y,y)f_2(x-y)g_2(y)dy \end{pmatrix},$$

$$\mathcal{N}_8(f,g) = \begin{pmatrix} 0 \\ f_2(x)\int_0^{x_c} k(x,y)g_1(y)dy \end{pmatrix},$$

$$\mathcal{N}_9(f,g) = \begin{pmatrix} 0 \\ f_2(x)\int_{x_c}^{\infty} k(x,y)g_2(y)dy \end{pmatrix}.$$

In addition to the individual terms introduced above, we also define the following combined bilinear operator, $\mathcal{N}$, on $Y \times Y$.

$$\begin{aligned}\mathcal{N}(f,g) =& \mathcal{N}_1(f,g) - \mathcal{N}_2(f,g) - \mathcal{N}_3(f,g) + \mathcal{N}_4(f,g) + \mathcal{N}_5(f,g) \\ &+ \mathcal{N}_6(f,g) + \mathcal{N}_7(f,g) - \mathcal{N}_8(f,g) - \mathcal{N}_9(f,g).\end{aligned}$$

The coagulation operator N is then given by

$$Nf = \mathcal{N}(f,f), \text{ for } f \in Y.$$

If we make the assumption

$$k \in L_\infty\left((0,\infty)\times(0,\infty)\right), \tag{4.10}$$

then the terms $\mathcal{N}_1$ to $\mathcal{N}_9$, introduced above, satisfy the following norm bounds.

$$\begin{aligned}
\|\mathcal{N}_1(f,g)\|_Y &\le \frac{1}{2}\int_0^{x_c}\left(\int_0^x k(x-y,y)|f_1(x-y)||g_1(y)|dy\right)(1+x)dx \\
&= \frac{1}{2}\int_0^{x_c}|g_1(y)|\left(\int_y^{x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy \\
&= \frac{1}{2}\int_0^{x_c}|g_1(y)|\left(\int_0^{x_c-y} k(x,y)|f_1(x)|(1+x+y)dx\right)dy \\
&\le \frac{1}{2}\|k\|_\infty\int_0^{x_c}|g_1(y)|\left(\int_0^{x_c-y}|f_1(x)|(1+x+y)dx\right)dy \\
&\le \frac{1}{2}\|k\|_\infty\left(\|f_1\|_{Y_D}\|g_1\|_{Y_{D,0}} + \|f_1\|_{Y_{D,0}}\|g_1\|_{Y_{D,1}}\right) \\
&\le \|k\|_\infty\|f_1\|_{Y_D}\|g_1\|_{Y_D} \le \|k\|_\infty\|f\|_Y\|g\|_Y,
\end{aligned} \tag{4.11}$$

$$
\begin{aligned}
\|\mathcal{N}_2(f,g)\|_Y &\le \int_0^{x_c} |f_1(x)| \left(\int_0^{x_c} k(x,y)|g_1(y)|dy \right) (1+x)dx \\
&= \int_0^{x_c} |g_1(y)| \left(\int_0^{x_c} k(x,y)|f_1(x)|(1+x)dx \right) dy \\
&\le \|k\|_\infty \int_0^{x_c} |g_1(y)| \left(\int_0^{x_c} |f_1(x)|(1+x)dx \right) dy \\
&= \|k\|_\infty \|f_1\|_{Y_D} \|g_1\|_{Y_{D,0}} \le \|k\|_\infty \|f_1\|_{Y_D} \|g_1\|_{Y_D} \\
&\le \|k\|_\infty \|f\|_Y \|g\|_Y ,
\end{aligned} \tag{4.12}
$$

$$
\begin{aligned}
\|\mathcal{N}_3(f,g)\|_Y &\le \int_0^{x_c} |f_1(x)| \left(\int_{x_c}^{\infty} k(x,y)|g_2(y)|dy \right) (1+x)dx \\
&= \int_{x_c}^{\infty} |g_2(y)| \left(\int_0^{x_c} k(x,y)|f_1(x)|(1+x)dx \right) dy \\
&\le \|k\|_\infty \int_{x_c}^{\infty} |g_2(y)| \left(\int_0^{x_c} |f_1(x)|(1+x)dx \right) dy \\
&= \|k\|_\infty \|f_1\|_{Y_D} \|g_2\|_{Y_{F,0}} \le \|k\|_\infty \|f_1\|_{Y_D} \|g_2\|_{Y_F} \\
&\le \|k\|_\infty \|f\|_Y \|g\|_Y ,
\end{aligned} \tag{4.13}
$$

$$
\begin{aligned}
\|\mathcal{N}_4(f,g)\|_Y &\le \frac{1}{2} \int_{x_c}^{2x_c} \left(\int_0^{x-x_c} k(x-y,y)|f_2(x-y)||g_1(y)|dy \right) (1+x)dx \\
&\quad + \frac{1}{2} \int_{2x_c}^{\infty} \left(\int_0^{x_c} k(x-y,y)|f_2(x-y)||g_1(y)|dy \right) (1+x)dx \\
&= \frac{1}{2} \int_0^{x_c} |g_1(y)| \left(\int_{y+x_c}^{2x_c} k(x-y,y)|f_2(x-y)|(1+x)dx \right) dy \\
&\quad + \frac{1}{2} \int_0^{x_c} |g_1(y)| \left(\int_{2x_c}^{\infty} k(x-y,y)|f_2(x-y)|(1+x)dx \right) dy \\
&= \frac{1}{2} \int_0^{x_c} |g_1(y)| \left(\int_{y+x_c}^{\infty} k(x-y,y)|f_2(x-y)|(1+x)dx \right) dy \\
&= \frac{1}{2} \int_0^{x_c} |g_1(y)| \left(\int_{x_c}^{\infty} k(x,y)|f_2(x)|(1+x+y)dx \right) dy \\
&\le \frac{1}{2} \|k\|_\infty \int_0^{x_c} |g_1(y)| \left(\int_{x_c}^{\infty} |f_2(x)|(1+x+y)dx \right) dy \\
&= \frac{1}{2} \|k\|_\infty \left(\|f_2\|_{Y_F} \|g_1\|_{Y_{D,0}} + \|f_2\|_{Y_{F,0}} \|g_1\|_{Y_{D,1}} \right) \\
&\le \|k\|_\infty \|f_2\|_{Y_F} \|g_1\|_{Y_D} \le \|k\|_\infty \|f\|_Y \|g\|_Y ,
\end{aligned} \tag{4.14}
$$

$$\begin{aligned}
\|\mathcal{N}_5(f,g)\|_Y &\le \frac{1}{2}\int_{x_c}^{2x_c}\left(\int_{x-x_c}^{x_c} k(x-y,y)|f_1(x-y)||g_1(y)|dy\right)(1+x)dx\\
&= \frac{1}{2}\int_0^{x_c}|g_1(y)|\left(\int_{x_c}^{y+x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy\\
&= \frac{1}{2}\int_0^{x_c}|g_1(y)|\left(\int_{x_c-y}^{x_c} k(x,y)|f_1(x)|(1+x+y)dx\right)dy \qquad (4.15)\\
&\le \frac{1}{2}\|k\|_\infty\int_0^{x_c}|g_1(y)|\left(\int_{x_c-y}^{x_c}|f_1(x)|(1+x+y)dx\right)dy\\
&\le \frac{1}{2}\|k\|_\infty\left(\|f_1\|_{Y_D}\|g_1\|_{Y_{D,0}}+\|f_1\|_{Y_{D,0}}\|g_1\|_{Y_{D,1}}\right)\\
&\le \|k\|_\infty\|f_1\|_{Y_D}\|g_1\|_{Y_D}\le\|k\|_\infty\|f\|_Y\|g\|_Y,
\end{aligned}$$

$$\begin{aligned}
\|\mathcal{N}_6(f,g)\|_Y &\le \frac{1}{2}\int_{x_c}^{2x_c}\left(\int_{x_c}^{x} k(x-y,y)|f_1(x-y)||g_2(y)|dy\right)(1+x)dx\\
&\quad + \frac{1}{2}\int_{2x_c}^{\infty}\left(\int_{x-x_c}^{x} k(x-y,y)|f_1(x-y)||g_2(y)|dy\right)(1+x)dx\\
&= \frac{1}{2}\int_{x_c}^{2x_c}|g_2(y)|\left(\int_{y}^{2x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy\\
&\quad + \frac{1}{2}\int_{x_c}^{2x_c}|g_2(y)|\left(\int_{2x_c}^{y+x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy\\
&\quad + \frac{1}{2}\int_{2x_c}^{\infty}|g_2(y)|\left(\int_{y}^{y+x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy\\
&= \frac{1}{2}\int_{x_c}^{2x_c}|g_2(y)|\left(\int_{y}^{y+x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy\\
&\quad + \frac{1}{2}\int_{2x_c}^{\infty}|g_2(y)|\left(\int_{y}^{y+x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy\\
&= \frac{1}{2}\int_{x_c}^{\infty}|g_2(y)|\left(\int_{y}^{y+x_c} k(x-y,y)|f_1(x-y)|(1+x)dx\right)dy\\
&= \frac{1}{2}\int_{x_c}^{\infty}|g_2(y)|\left(\int_{0}^{x_c} k(x,y)|f_1(x)|(1+x+y)dx\right)dy \qquad (4.16)\\
&\le \frac{1}{2}\|k\|_\infty\int_{x_c}^{\infty}|g_2(y)|\left(\int_{0}^{x_c}|f_1(x)|(1+x+y)dx\right)dy\\
&= \frac{1}{2}\|k\|_\infty\left(\|f_1\|_{Y_D}\|g_2\|_{Y_{F,0}}+\|f_1\|_{Y_{D,0}}\|g_2\|_{Y_{F,1}}\right)\\
&\le \|k\|_\infty\|f_1\|_{Y_D}\|g_2\|_{Y_F}\le\|k\|_\infty\|f\|_Y\|g\|_Y,
\end{aligned}$$

$$
\begin{aligned}
&\|\mathcal{N}_7(f,g)\|_Y \le \frac{1}{2}\int_{2x_c}^{\infty}\left(\int_{x_c}^{x-x_c} k(x-y,y)|f_2(x-y)||g_2(y)|dy\right)(1+x)dx\\
&= \frac{1}{2}\int_{x_c}^{\infty}|g_2(y)|\left(\int_{y+x_c}^{\infty} k(x-y,y)|f_2(x-y)|(1+x)dx\right)dy\\
&= \frac{1}{2}\int_{x_c}^{\infty}|g_2(y)|\left(\int_{x_c}^{\infty} k(x,y)|f_2(x)|(1+x+y)dx\right)dy \qquad (4.17)\\
&= \frac{1}{2}\|k\|_\infty\int_{x_c}^{\infty}|g_2(y)|\left(\int_{x_c}^{\infty} |f_2(x)|(1+x+y)dx\right)dy\\
&= \frac{1}{2}\|k\|_\infty\left(\|f_2\|_{Y_F}\|g_2\|_{Y_{F,0}} + \|f_2\|_{Y_{F,0}}\|g_2\|_{Y_{F,1}}\right)\\
&\le \|k\|_\infty\|f_2\|_{Y_F}\|g_2\|_{Y_F} \le \|k\|_\infty\|f\|_Y\|g\|_Y\,,
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathcal{N}_8(f,g)\|_Y \le \int_{x_c}^{\infty}|f_2(x)|\left(\int_0^{x_c} k(x,y)|g_1(y)|dy\right)(1+x)dx\\
&= \int_0^{x_c}|g_1(y)|\left(\int_{x_c}^{\infty} k(x,y)|f_2(x)|(1+x)dx\right)dy \qquad (4.18)\\
&\le \|k\|_\infty\int_0^{x_c}|g_1(y)|\left(\int_{x_c}^{\infty} |f_2(x)|(1+x)dx\right)dy\\
&= \|k\|_\infty\|f_2\|_{Y_F}\|g_1\|_{Y_{D,0}} \le \|k\|_\infty\|f_2\|_{Y_F}\|g_1\|_{Y_D}\\
&\le \|k\|_\infty\|f\|_Y\|g\|_Y\,,
\end{aligned}
$$

$$
\begin{aligned}
&\|\mathcal{N}_9(f,g)\|_Y \le \int_{x_c}^{\infty}|f_2(x)|\left(\int_{x_c}^{\infty} k(x,y)|g_2(y)|dy\right)(1+x)dx\\
&= \int_{x_c}^{\infty}|g_2(y)|\left(\int_{x_c}^{\infty} k(x,y)|f_2(x)|(1+x)dx\right)dy \qquad (4.19)\\
&\le \|k\|_\infty\int_{x_c}^{\infty}|g_2(y)|\left(\int_{x_c}^{\infty} |f_2(x)|(1+x)dx\right)dy\\
&= \|k\|_\infty\|f_2\|_{Y_F}\|g_2\|_{Y_{F,0}} \le \|k\|_\infty\|f_2\|_{Y_F}\|g_2\|_{Y_F}\\
&\le \|k\|_\infty\|f\|_Y\|g\|_Y\,.
\end{aligned}
$$

From these estimates, using an extended triangle inequality, we have for all f and g in Y that

$$\|\mathcal{N}(f,g)\|_Y \leq 9\,\|k\|_\infty\,\|f\|_Y\,\|g\|_Y\,. \tag{4.20}$$

Hence for all $f \in Y$

$$\|Nf\|_Y = \|\mathcal{N}(f,f)\|_Y \leq 9\,\|k\|_\infty\,\|f\|_Y^2\,. \tag{4.21}$$

Therefore $N(Y) \subset Y$, with $\|Nf\|_Y \leq 9\,\|k\|_\infty\,\|f\|_Y^2$ for all $f \in Y$.

Lemma 4.2.1. *The operator N satisfies a local Lipschitz condition on the closed ball $\bar{B}(u_0,\rho)$, where $u_0 \in Y$ and $\rho > 0$.*

Proof. Let f and g be two elements from Y. Then, exploiting the bilinear structure of $\mathcal{N}$ and the bound (4.20), we have

$$\begin{aligned}\|Nf - Ng\|_Y &= \|\mathcal{N}(f,f) - \mathcal{N}(g,g)\|_Y\\ &= \|\mathcal{N}(f,f-g) + \mathcal{N}(f-g,g)\|_Y\\ &\leq \|\mathcal{N}(f,f-g)\|_Y + \|\mathcal{N}(f-g,g)\|_Y\\ &\leq 9\,\|k\|_\infty\,(\|f\|_Y + \|g\|_Y)\,\|f-g\|_Y\,.\end{aligned}$$

Now suppose that both $f, g \in \bar{B}(u_0,\rho)$, then

$$\|f - u_0\|_Y \leq \rho \Rightarrow \|f\|_Y - \|u_0\|_Y \leq \rho \Rightarrow \|f\|_Y \leq \|u_0\|_Y + \rho.$$

Similarly, $\|g\|_Y \leq \|u_0\|_Y + \rho$. Therefore $\|f\|_Y + \|g\|_Y \leq 2\,(\|u_0\|_Y + \rho)$, hence

$$\begin{aligned}\|Nf - Ng\|_Y &\leq 18\,\|k\|_\infty\,(\|u_0\|_Y + \rho)\,\|f-g\|_Y\\ &= q_{\rho,u_0}\|f-g\|_Y,\end{aligned}$$

where $q_{\rho,u_0} = 18\|k\|_\infty(\|u_0\|_Y + \rho)$. □

From Theorem 2.4.34, since $\boldsymbol{A}_|$ is a generator and N is locally Lipschitz, the semilinear ACP (4.9) has a unique (local in time) mild solution, $u(t)$, on $[0,t_0)$, such that $u(t) \in B(u_0,\rho_0)$ for all $t \in [0,t_0)$, where t_0 and ρ_0 are suitably chosen.

Lemma 4.2.2. *The operator N is Fréchet differentiable at each $f \in Y$.*

Proof. Let $f, \delta \in Y$, then

$$\begin{aligned}N(f+\delta) &= \mathcal{N}(f+\delta, f+\delta)\\ &= \mathcal{N}(f,f) + \mathcal{N}(f,\delta) + \mathcal{N}(\delta,f) + \mathcal{N}(\delta,\delta)\\ &= Nf + N_f\delta + R(f,\delta),\end{aligned} \tag{4.22}$$

where

$$N_f\delta = \mathcal{N}(f,\delta) + \mathcal{N}(\delta,f) \;\text{ and }\; R(f,\delta) = \mathcal{N}(\delta,\delta) = N\delta.$$

The bilinear structure of $\mathcal{N}$ means that, for fixed f, N_f is a linear operator. Further, from the definition of $N_f\delta$ and the bound (4.20), we have that

$$\begin{aligned} \|N_f\delta\|_Y &= \|\mathcal{N}(f,\delta) + \mathcal{N}(\delta,f)\|_Y \\ &\le \|\mathcal{N}(f,\delta)\|_Y + \|\mathcal{N}(\delta,f)\|_Y \\ &\le 18\,\|k\|_\infty\,\|f\|_Y\,\|\delta\|_Y\,. \end{aligned} \tag{4.23}$$

Therefore N_f is a bounded linear operator on Y. Also, the remainder term $R(f,\delta)$ from (4.22), using (4.21), satisfies

$$\frac{\|R(f,\delta)\|_Y}{\|\delta\|_Y} = \frac{\|N\delta\|_Y}{\|\delta\|_Y} \le \frac{9\,\|k\|_\infty\,\|\delta\|_Y^2}{\|\delta\|_Y} = 9\,\|k\|_\infty\,\|\delta\|_Y \to 0 \text{ as } \|\delta\|_Y \to 0.$$

Hence the coagulation operator N is Fréchet differentiable at each $f \in Y$, with Fréchet derivative N_f. We note that when $f \in \bar{B}(u_0,\rho)$, then (4.23) implies $\|N_f\delta\|_Y \le q_{\rho,u_0}\,\|\delta\|_Y$. □

Lemma 4.2.3. *The Fréchet derivative, N_f, of N is continuous with respect to f.*

Proof. Let f, g and δ be elements from Y. Then, the bilinear structure of $\mathcal{N}$ and (4.20) give us

$$\begin{aligned} \|N_f\delta - N_g\delta\|_Y &= \|\mathcal{N}(f,\delta) + \mathcal{N}(\delta,f) - \mathcal{N}(g,\delta) - \mathcal{N}(\delta,g)\|_Y \\ &= \|\mathcal{N}(f-g,\delta) + \mathcal{N}(\delta,f-g)\|_Y \\ &\le \|\mathcal{N}(f-g,\delta)\|_Y + \|\mathcal{N}(\delta,f-g)\|_Y \\ &\le 18\,\|k\|_\infty\,\|\delta\|_Y\,\|f-g\|_Y\,. \end{aligned}$$

Therefore N_f is continuous with respect to f. □

The results established in Lemmas 4.2.2 and 4.2.3 mean that the local in time mild solution, $u(t)$, satisfies the conditions of Theorem 2.4.35, and hence is, in fact, a local in time strong solution of (4.9).

Having established the existence of the local in time (strong) solution, $u(t)$, of equation (4.9), we now show that when the initial condition is positive, that is $u_0 \in D(\boldsymbol{A}_|)_+ = Y_{D+} \times D(K)_+$, then the solution belongs to Y_+ for all $t \in [0, t_0)$. The approach adopted follows the ideas presented in [6].

Suppose that $u(t)$ is the solution on $[0, t_0)$ of (4.9), with positive initial condition $u_0 \in D(\boldsymbol{A}_|)_+$. Then, we note that $u(t)$ is also the unique strongly differentiable solution of

$$\frac{d}{dt}u(t) = \big(\boldsymbol{A}_|[u(t)] - \alpha u(t)\big) + N^\alpha[u(t)], \quad \alpha \in \mathbb{R}, \tag{4.24}$$

where N^α denotes $N + \alpha I$. Therefore $u(t)$ is also the unique solution of the integral equation

$$u(t) = e^{-\alpha t}\boldsymbol{T}_|(t)u_0 + \int_0^t e^{-\alpha(t-s)}\boldsymbol{T}_|(t-s)N^\alpha[u(s)]\,ds, \quad 0 \le t < t_0. \tag{4.25}$$

Lemma 4.2.4. *Let* $u_0 \in Y$ *and* $\rho_0 > 0$ *be arbitrary. If we select* $\alpha \ge \|k\|_\infty \left(\|u_0\|_Y + \rho_0\right)$, *then* $N^\alpha f \in Y_+$ *for all* $f \in B(u_0, \rho_0) \cap Y_+$.

Proof. First we note that, if $f = \binom{f_1}{f_2} \in B(u_0, \rho_0)$, then

$$\|f - u_0\|_Y < \rho_0 \Rightarrow \|f\|_Y - \|u_0\|_Y < \rho_0 \Rightarrow \|f\|_Y < \|u_0\|_Y + \rho_0.$$

Expressing $N^\alpha f$ in terms of its constituent operators gives

$$\begin{aligned} N^\alpha f = \alpha f &+ \mathcal{N}_1(f,f) - \mathcal{N}_2(f,f) - \mathcal{N}_3(f,f) + \mathcal{N}_4(f,f) + \mathcal{N}_5(f,f) \\ &+ \mathcal{N}_6(f,f) + \mathcal{N}_7(f,f) - \mathcal{N}_8(f,f) - \mathcal{N}_9(f,f). \end{aligned}$$

If $f \in Y_+$, so that the components $f_1(x)$ and $f_2(x)$ are non-negative a.e. on $(0, x_c]$ and (x_c, ∞) respectively, then it is clear that $\mathcal{N}_1(f,f) + \mathcal{N}_4(f,f) + \mathcal{N}_5(f,f) + \mathcal{N}_6(f,f) + \mathcal{N}_7(f,f) \in Y_+$. Considering the remaining terms and expressing them fully, we obtain

$$\begin{aligned} &\alpha f - \mathcal{N}_2(f,f) - \mathcal{N}_3(f,f) - \mathcal{N}_8(f,f) - \mathcal{N}_9(f,f) \\ &= \begin{pmatrix} \alpha f_1(x) - f_1(x)\Big\{\int_0^{x_c} k(x,y)f_1(y)dy + \int_{x_c}^\infty k(x,y)f_2(y)dy\Big\} \\ \alpha f_2(x) - f_2(x)\Big\{\int_0^{x_c} k(x,y)f_1(y)dy + \int_{x_c}^\infty k(x,y)f_2(y)dy\Big\} \end{pmatrix}. \end{aligned} \tag{4.26}$$

Looking more closely at the integral expressions appearing above, we see that

$$\begin{aligned} &\int_0^{x_c} k(x,y)f_1(y)dy + \int_{x_c}^\infty k(x,y)f_2(y)dy \\ &\le \|k\|_\infty \left(\|f_1\|_{Y_{D,0}} + \|f_2\|_{Y_{F,0}}\right) \le \|k\|_\infty \|f\|_Y \\ &\le \|k\|_\infty \left(\|u_0\|_Y + \rho_0\right). \end{aligned}$$

Therefore if we select $\alpha \ge \|k\|_\infty \left(\|u_0\|_Y + \rho_0\right)$, then both components of (4.26) are non-negative. Hence $N^\alpha f \in Y_+$. □

Theorem 4.2.5. *Let $u_0 \in D(\boldsymbol{A}_|)_+$, and suppose that $u : [0, t_0) \to B(u_0, \rho_0)$ is the unique strong solution of equation* (4.9). *Then there exists $t_1 \in (0, t_0]$ such that $u(t) \in Y_+$ for all $t \in [0, t_1)$.*

Proof. Let us define the space $Z := C([0, t_1], Y)$ and equip it with the norm $\|v\|_Z := \max\{\|v(t)\|_Y : 0 \le t \le t_1\}$, where $t_1 \in (0, t_0]$ will be chosen later. Also, we introduce the set $\Sigma := \{v \in Z : v(t) \in \overline{B}(u_0, \rho_1) \cap Y_+ \ \forall t \in [0, t_1]\}$, where $0 < \rho_1 < \rho_0$.

As the positive cones of Lebesgue spaces, both Y_{D+} and Y_{F+} are closed. Therefore the product $Y_+ = Y_{D+} \times Y_{F+}$ is also closed. Clearly the ball $\overline{B}(u_0, \rho_1)$ is closed. Hence the intersection $\overline{B}(u_0, \rho_1) \cap Y_+$ is a closed set. Now, suppose that the sequence $\{\nu_n\}_{n=1}^{\infty} \subset \Sigma$ converges to ν in Z. Then, for all $\epsilon > 0$, there exists $M \in \mathbb{N}$ such that

$$\max_{0 \le t \le t_1} \|\nu_n(t) - \nu(t)\|_Y < \epsilon \text{ for } n \ge M,$$
$$\Rightarrow \|\nu_n(t) - \nu(t)\|_Y < \epsilon, \ \forall t \in [0, t_1] \text{ for } n \ge M.$$

Hence for each fixed $t \in [0, t_1]$, the sequence $\{\nu_n(t)\}_{n=1}^{\infty} \subset \overline{B}(u_0, \rho_1) \cap Y_+$ converges to $\nu(t)$ in Y. Since $\overline{B}(u_0, \rho_1) \cap Y_+$ is closed, the limit $\nu(t)$ must also belong to $\overline{B}(u_0, \rho_1) \cap Y_+$ for all $t \in [0, t_1]$. Therefore $\nu \in Z$ is such that $\nu(t) \in \overline{B}(u_0, \rho_1) \cap Y_+$ for all $t \in [0, t_1]$, that is $\nu \in \Sigma$. Hence Σ is a closed subset of Z.

Having established that Σ is closed, we define the mapping Q on Σ by

$$(Qv)(t) := e^{-\alpha t} \boldsymbol{T}_|(t) u_0 + \int_0^t e^{-\alpha(t-s)} \boldsymbol{T}_|(t-s) N^\alpha[v(s)] ds, \ 0 \le t \le t_1, \ D(Q) := \Sigma,$$

where $\alpha \ge \|k\|_\infty (\|u_0\|_Y + \rho_0)$. As $v \in \Sigma$, the local Lipschitz condition on N implies that $N^\alpha[v(s)]$ is strongly continuous on $[0, t_1]$. In combination with Lemma 2.4.4, this gives us continuity of the above integrand on $[0, t_1]$. Therefore, by Theorem 2.2.4, the integral term appearing in Q is an element in Z. It is now easily seen that $Q(\Sigma) \subset Z$. The positivity of $v(s)$, along with Lemma 4.2.4 and the positivity of the semigroup $\left(\boldsymbol{T}_|(t)\right)_{t \ge 0}$, mean that the integrand above is positive. Following from the discussion in Lemma 3.4.3, this implies that the integral term from Q is a positive element of Y. If we recall that the initial condition u_0 is positive, then we have $(Qv)(t) \in Y_+$ for all $t \in [0, t_1]$. Let $v, w \in \Sigma$. By the local Lipschitz condition on N, we have for all $s \in [0, t_1]$ that

$$\begin{aligned} \|N^\alpha[v(s)] - N^\alpha[w(s)]\|_Y &= \|\alpha v(s) + N[v(s)] - \alpha w(s) - N[w(s)]\|_Y \\ &\le \alpha \|v(s) - w(s)\|_Y + \|N[v(s)] - N[w(s)]\|_Y \\ &\le (q_{\rho_1, u_0} + \alpha) \|v(s) - w(s)\|_Y, \end{aligned}$$

where $q_{\rho_1,u_0} = 18\|k\|_\infty(\|u_0\|_Y + \rho_1)$. By Theorem 2.4.2, there exists $M \geq 1$ and $\omega \geq 0$ such that

$$\|\boldsymbol{T}_|(t)\| \leq Me^{\omega t} \text{ for all } t \geq 0.$$

Therefore for $t \in [0, t_1]$ we obtain

$$\begin{aligned}\|(Qv)(t) - (Qw)(t)\|_Y &\leq \int_0^t e^{-\alpha(t-s)}\|\boldsymbol{T}_|(t-s)\|\|N^\alpha[v(s)] - N^\alpha[w(s)]\|_Y\, ds \\ &\leq M\left(q_{\rho_1,u_0} + \alpha\right)\int_0^t e^{(\omega-\alpha)(t-s)}\|v(s) - w(s)\|_Y\, ds \\ &\leq M\left(q_{\rho_1,u_0} + \alpha\right) t_1 e^{\omega t_1}\|v - w\|_Z.\end{aligned}$$

Hence

$$\|Qv - Qw\|_Z \leq M\left(q_{\rho_1,u_0} + \alpha\right) t_1 e^{\omega t_1}\|v - w\|_Z. \tag{4.27}$$

Similarly to before, using the Lipschitz condition for N in addition to the fact that $v \in \Sigma$, we have for all $s \in [0, t_1]$ that

$$\begin{aligned}\|N^\alpha[v(s)]\|_Y &= \|N^\alpha[v(s)] - N^\alpha u_0 + N^\alpha u_0\|_Y \\ &\leq \|N^\alpha[v(s)] - N^\alpha u_0\|_Y + \|N^\alpha u_0\|_Y \\ &\leq \alpha\|v(s) - u_0\|_Y + \|N[v(s)] - Nu_0\|_Y + \|N^\alpha u_0\|_Y \\ &\leq (q_{\rho_1,u_0} + \alpha)\rho_1 + \|N^\alpha u_0\|_Y.\end{aligned}$$

This gives us the following inequality for $t \in [0, t_1]$:

$$\begin{aligned}\|(Qv)(t) - u_0\|_Y &\leq \|e^{-\alpha t}\boldsymbol{T}_|(t)u_0 - u_0\|_Y + \int_0^t e^{-\alpha(t-s)}\|\boldsymbol{T}_|(t-s)\|\|N^\alpha[v(s)]\|_Y ds \\ &\leq \|e^{-\alpha t}\boldsymbol{T}_|(t)u_0 - u_0\|_Y + Mt_1e^{\omega t_1}\left((q_{\rho_1,u_0} + \alpha)\rho_1 + \|N^\alpha u_0\|_Y\right).\end{aligned} \tag{4.28}$$

Now let us define

$$\zeta(t_1) := \frac{1}{\rho_1}\max_{0\leq t\leq t_1}\|e^{-\alpha t}\boldsymbol{T}_|(t)u_0 - u_0\|_Y + \frac{1}{\rho_1}Mt_1e^{\omega t_1}((q_{\rho_1,u_0} + \alpha)\rho_1 + \|N^\alpha u_0\|_Y).$$

Then (4.27) implies that

$$\|Qv - Qw\|_Z \leq \zeta(t_1)\|v - w\|_Z, \tag{4.29}$$

and (4.28) implies that

$$\|(Qv)(t) - u_0\|_Y \leq \rho_1\zeta(t_1) \text{ for all } 0 \leq t \leq t_1. \tag{4.30}$$

Since $\zeta(t_1) \to 0^+$ as $t_1 \to 0^+$, we can select t_1 such that $0 < \zeta(t_1) < 1$. Then (4.30) implies that $Qv \in \overline{B}(u_0, \rho_1)$ for all $0 \leq t \leq t_1$. Having already noted that $(Qv)(t) \in Y_+$ for all such t, this gives us that $Qv \in \Sigma$, and hence $Q(\Sigma) \subset \Sigma$. We have shown that Σ is a closed subset of the Banach space Z, hence it is complete. If we have $0 < \zeta(t_1) < 1$, then (4.29) shows that Q is a contraction on Σ. Therefore, by the Banach fixed point theorem, [21, Theorem 5.1-2], there exists a unique fixed point $u \in \Sigma$ such that $u = Qu$, and so equation (4.25) has a unique solution $u \in C([0, t_1], Y_+)$. □

Having shown that the solution of equation (4.9) with positive initial data remains positive for a non-zero period of time, we now extend this and show that the solution is positive on the entirety of its maximal interval of existence.

Lemma 4.2.6. *Let $[0, \hat{T})$ be the maximal interval of existence for the solution, $u(t)$, to the equation (4.9). Then $u(t) \in Y_+$ for all $t \in [0, \hat{T})$.*

Proof. Fix $T_0 \in (0, \hat{T})$ arbitrarily and define

$$\tau_{\max} := \sup\left\{0 < \tau \leq T_0 : u(t) \in Y_+ \text{ for all } t \in [0, \tau]\right\}.$$

Now suppose that $\tau_{\max} < T_0$ and consider the equation

$$\begin{aligned} \frac{d}{dt}v(t) &= \boldsymbol{A}_{|}[v(t)] + N[v(t)], \quad t > 0, \\ v(0) &= u(\tau_{\max}). \end{aligned} \tag{4.31}$$

The solution to this equation on $[0, T_0 - \tau_{\max}]$ is given by $v(t) = u(t + \tau_{\max})$. We can construct a sequence in Y_+ which converges to $u(\tau_{\max})$, which, since Y_+ is closed, implies that $u(\tau_{\max}) \in Y_+$. Therefore v is the solution to equation (4.31) with positive initial data. By Theorem 4.2.5, $v(t) \in Y_+$ for sufficiently small but non-zero t, and hence $u(t + \tau_{\max}) \in Y_+$ for such t. However, this contradicts the maximal property of $\tau_{\max}$, proving that we must have $\tau_{\max} = T_0$. Since $T_0 \in (0, \hat{T})$ was arbitrary, we obtain the required result. □

We have shown the existence of a local in time, strong solution to equation (4.9), which, provided the initial condition is positive, remains positive over its maximal interval of existence. In order to prove the existence of a global in time, strong, non-negative solution, we apply Gronwall's inequality to show that the local solution does not blow up in finite time. However, prior to this, it is necessary to establish the following two lemmas.

Lemma 4.2.7. *Suppose $f = \binom{f_1}{f_2} \in Y_+$. Then*

$$\int_0^{x_c} (N_D(f_1, f_2))(x)(1+x)dx + \int_{x_c}^{\infty} (N_F(f_1, f_2))(x)(1+x)dx \leq 0.$$

Proof. The first integral is given by

$$\begin{aligned}
&\int_0^{x_c} (N_D(f_1, f_2))(x)(1+x)dx \\
&= \int_0^{x_c} \left(\frac{1}{2} \int_0^x k(x-y, y) f_1(x-y) f_1(y) dy \right) (1+x) dx \\
&- \int_0^{x_c} \left(f_1(x) \int_0^{x_c} k(x, y) f_1(y) dy \right) (1+x) dx \\
&- \int_0^{x_c} \left(f_1(x) \int_{x_c}^{\infty} k(x, y) f_2(y) dy \right) (1+x) dx.
\end{aligned}$$

Performing essentially the same manipulations as those carried out in (4.11), (4.12) and (4.13), gives us

$$\begin{aligned}
&\int_0^{x_c} (N_D(f_1, f_2))(x)(1+x)dx \\
&= \frac{1}{2} \int_0^{x_c} f_1(y) \left(\int_0^{x_c - y} k(x, y) f_1(x)(1+x+y) dx \right) dy \\
&- \int_0^{x_c} f_1(y) \left(\int_0^{x_c} k(x, y) f_1(x)(1+x) dx \right) dy \\
&- \int_{x_c}^{\infty} f_2(y) \left(\int_0^{x_c} k(x, y) f_1(x)(1+x) dx \right) dy. \qquad (4.32)
\end{aligned}$$

The second integral is given by

$$\begin{aligned}
&\int_{x_c}^{\infty} (N_F(f_1, f_2))(x)(1+x)dx \\
&= \int_{x_c}^{2x_c} \left(\frac{1}{2} \int_0^{x - x_c} k(x-y, y) f_2(x-y) f_1(y) dy \right) (1+x) dx \\
&+ \int_{2x_c}^{\infty} \left(\frac{1}{2} \int_0^{x_c} k(x-y, y) f_2(x-y) f_1(y) dy \right) (1+x) dx \\
&+ \int_{x_c}^{2x_c} \left(\frac{1}{2} \int_{x - x_c}^{x_c} k(x-y, y) f_1(x-y) f_1(y) dy \right) (1+x) dx \\
&+ \int_{2x_c}^{\infty} \left(\frac{1}{2} \int_{x_c}^{x - x_c} k(x-y, y) f_2(x-y) f_2(y) dy \right) (1+x) dx
\end{aligned}$$

$$+\int_{x_c}^{2x_c}\left(\frac{1}{2}\int_{x_c}^{x}k(x-y,y)f_1(x-y)f_2(y)dy\right)(1+x)dx$$
$$+\int_{2x_c}^{\infty}\left(\frac{1}{2}\int_{x-x_c}^{x}k(x-y,y)f_1(x-y)f_2(y)dy\right)(1+x)dx$$
$$-\int_{x_c}^{\infty}\left(f_2(x)\int_{0}^{x_c}k(x,y)f_1(y)dy\right)(1+x)dx$$
$$-\int_{x_c}^{\infty}\left(f_2(x)\int_{x_c}^{\infty}k(x,y)f_2(y)dy\right)(1+x)dx.$$

If we carry out the same manipulations as have been used to derive (4.14)-(4.19), then we obtain

$$\int_{x_c}^{\infty}(N_F(f_1,f_2))(x)(1+x)dx$$
$$=\frac{1}{2}\int_{0}^{x_c}f_1(y)\left(\int_{x_c}^{\infty}k(x,y)f_2(x)(1+x+y)dx\right)dy$$
$$+\frac{1}{2}\int_{0}^{x_c}f_1(y)\left(\int_{x_c-y}^{x_c}k(x,y)f_1(x)(1+x+y)dx\right)dy$$
$$+\frac{1}{2}\int_{x_c}^{\infty}f_2(y)\left(\int_{x_c}^{\infty}k(x,y)f_2(x)(1+x+y)dx\right)dy$$
$$+\frac{1}{2}\int_{x_c}^{\infty}f_2(y)\left(\int_{0}^{x_c}k(x,y)f_1(x)(1+x+y)dx\right)dy$$
$$-\int_{0}^{x_c}f_1(y)\left(\int_{x_c}^{\infty}k(x,y)f_2(x)(1+x)dx\right)dy$$
$$-\int_{x_c}^{\infty}f_2(y)\left(\int_{x_c}^{\infty}k(x,y)f_2(x)(1+x)dx\right)dy. \qquad (4.33)$$

Addition of (4.32) and (4.33) then gives us

$$\int_{0}^{x_c}(N_D(f_1,f_2))(x)(1+x)dx+\int_{x_c}^{\infty}(N_F(f_1,f_2))(x)(1+x)dx$$
$$=\frac{1}{2}\int_{0}^{x_c}f_1(y)\left(\int_{x_c}^{\infty}k(x,y)f_2(x)(1+x+y)dx\right)dy$$
$$+\frac{1}{2}\int_{0}^{x_c}f_1(y)\left(\int_{x_c-y}^{x_c}k(x,y)f_1(x)(1+x+y)dx\right)dy$$
$$+\frac{1}{2}\int_{x_c}^{\infty}f_2(y)\left(\int_{x_c}^{\infty}k(x,y)f_2(x)(1+x+y)dx\right)dy$$

$$
\begin{aligned}
&+\frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x+y)dx\right)dy\\
&-\int_0^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy\\
&-\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy\\
&+\frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_0^{x_c-y} k(x,y)f_1(x)(1+x+y)dx\right)dy\\
&-\int_0^{x_c} f_1(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy\\
&-\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy.
\end{aligned}
$$

The second and seventh terms combine and we get

$$
\begin{aligned}
&\int_0^{x_c}(N_D(f_1,f_2))(x)(1+x)dx+\int_{x_c}^{\infty}(N_F(f_1,f_2))(x)(1+x)dx\\
&=\frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x+y)dx\right)dy\\
&+\frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x+y)dx\right)dy\\
&+\frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x+y)dx\right)dy\\
&+\frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x+y)dx\right)dy\\
&-\int_0^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy\\
&-\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy\\
&-\int_0^{x_c} f_1(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy\\
&-\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy
\end{aligned}
$$

$$
\begin{aligned}
&= -\frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy \\
&- \frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy \\
&- \frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy \\
&- \frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy \\
&+ \frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)ydx\right)dy \\
&+ \frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_0^{x_c} k(x,y)f_1(x)ydx\right)dy \\
&+ \frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)ydx\right)dy \\
&+ \frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)ydx\right)dy.
\end{aligned}
$$

If we change order of integration and switch the variables x and y in the final four terms, noting that k is symmetric, we get

$$
\begin{aligned}
&\int_0^{x_c} (N_D(f_1,f_2))(x)(1+x)dx + \int_{x_c}^{\infty} (N_F(f_1,f_2))(x)(1+x)dx \\
&= -\frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy \\
&- \frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy \\
&- \frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)(1+x)dx\right)dy \\
&- \frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)(1+x)dx\right)dy \\
&+ \frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_0^{x_c} k(x,y)f_1(x)xdx\right)dy \\
&+ \frac{1}{2}\int_0^{x_c} f_1(y)\left(\int_0^{x_c} k(x,y)f_1(x)xdx\right)dy
\end{aligned}
$$

$$+\frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)xdx\right)dy$$
$$+\frac{1}{2}\int_{0}^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)xdx\right)dy$$

$$=-\frac{1}{2}\int_{0}^{x_c} f_1(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)dx\right)dy$$
$$-\frac{1}{2}\int_{0}^{x_c} f_1(y)\left(\int_{0}^{x_c} k(x,y)f_1(x)dx\right)dy$$
$$-\frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_{x_c}^{\infty} k(x,y)f_2(x)dx\right)dy$$
$$-\frac{1}{2}\int_{x_c}^{\infty} f_2(y)\left(\int_{0}^{x_c} k(x,y)f_1(x)dx\right)dy$$

$$\leq 0.$$

□

Lemma 4.2.8. *For $f = \binom{f_1}{f_2} \in Y$ with $f_2 \in D(K)_+$, we have*

$$\int_0^{x_c} \left(C_{|}f_2\right)(x)(1+x)dx + \int_{x_c}^{\infty} \left(Kf_2\right)(x)(1+x)dx \leq \beta\|f\|_Y,$$

where β comes from (4.4).

Proof. Let us define the operator $(\tilde{A}, D(\tilde{A})) = (A - \beta I_F, D(A))$. For $h \in D(\tilde{A})_+$, we have

$$\int_{x_c}^{\infty}\left(\tilde{A}h + Bh\right)(1+x)dx = \int_{x_c}^{\infty}\left(Ah - \beta h + Bh\right)(1+x)dx$$
$$=\int_{x_c}^{\infty}\left(-a(x)h(x) - \beta h(x) + \int_x^{\infty} a(y)b(x|y)h(y)dy\right)(1+x)dx$$
$$=-\int_{x_c}^{\infty}\left(a(x)+\beta\right)h(x)(1+x)dx + \int_{x_c}^{\infty}\left(\int_x^{\infty} a(y)b(x|y)h(y)dy\right)(1+x)dx$$
$$=-\int_{x_c}^{\infty}\left(a(x)+\beta\right)h(x)(1+x)dx + \int_{x_c}^{\infty} a(y)h(y)\left(\int_{x_c}^{y} b(x|y)(1+x)dx\right)dy$$
$$=-\int_{x_c}^{\infty}\left\{\left(y+1-\int_{x_c}^{y} b(x|y)(1+x)dx\right)a(y) + \beta(y+1)\right\}h(y)dy =: -\tilde{c}(h).$$

Considering the expression appearing within the braces, condition (4.4) gives us

$$\begin{aligned}
&\left(y+1-\int_{x_c}^{y} b(x|y)(1+x)dx\right) a(y)+\beta(y+1)\\
&\geq (y+1-n(y)-y)\, a(y)+\beta(y+1)\\
&=\beta(y+1)-a(y)\,(n(y)-1)\geq 0,
\end{aligned}$$

for $y > x_c$. Therefore the functional $\tilde{c}$ given above is non-negative. It is now easily verified that $\tilde{A}$ and B satisfy the conditions of Theorem 2.4.28. Hence there exists an extension T of $\tilde{A}+B$ which generates a substochastic semigroup $(G_T(t))_{t\geq 0}$ on Y_F. Let $\tilde{a}(x)=a(x)+\beta$. If we follow the calculations of Theorem 3.2.5, with $\tilde{a}(x)$ replacing $a(x)$, the space Y_F in place of X_F, and the functional $\tilde{c}$ as given above, then we obtain

$$\begin{aligned}
&\int_{x_c}^{\infty}(-\tilde{a}(x)g(x)+(\mathcal{B}g)(x))\,(1+x)dx=\lim_{R\to\infty}\int_{x_c}^{R}(-\tilde{a}(x)g(x)+(\mathcal{B}g)(x))\,(1+x)dx\\
&=\lim_{R\to\infty}\left\{-\int_{x_c}^{R}\tilde{a}(x)g(x)(1+x)dx+\int_{x_c}^{R}\left(\int_{x}^{\infty}a(y)b(x|y)g(y)dy\right)(1+x)dx\right\}.
\end{aligned} \tag{4.34}$$

Looking at the terms contained within the braces above, we have

$$\begin{aligned}
&-\int_{x_c}^{R}\tilde{a}(x)g(x)(1+x)dx+\int_{x_c}^{R}\left(\int_{x}^{\infty}a(y)b(x|y)g(y)dy\right)(1+x)dx\\
&=-\left\{\int_{x_c}^{R}\tilde{a}(y)g(y)(1+y)dy-\int_{x_c}^{R}a(y)g(y)\left(\int_{x_c}^{y}b(x|y)(1+x)dx\right)dy\right\}\\
&\quad+\int_{R}^{\infty}a(y)g(y)\left(\int_{x_c}^{R}b(x|y)(1+x)dx\right)dy\\
&\geq-\int_{x_c}^{R}\left\{\tilde{a}(y)(1+y)-a(y)\left(\int_{x_c}^{y}b(x|y)(1+x)dx\right)\right\}g(y)\,dy.
\end{aligned}$$

Replacing this in (4.34), we get

$$\begin{aligned}
&\int_{x_c}^{\infty}(-\tilde{a}(x)g(x)+(\mathcal{B}g)(x))\,(1+x)dx\\
&\geq-\lim_{R\to\infty}\int_{x_c}^{R}\left\{\tilde{a}(y)(1+y)-a(y)\left(\int_{x_c}^{y}b(x|y)(1+x)dx\right)\right\}g(y)\,dy=-\tilde{c}(g).
\end{aligned}$$

Therefore, by [4, Theorem 6.22], we have that $T=\overline{\tilde{A}+B}$. By following the argument presented in [4, Theorem 6.13], for $v\in D(T)$ we have

$$\int_{x_c}^{\infty}(Tv)(x)(1+x)dx=-\tilde{c}(v(x)). \tag{4.35}$$

Let $v \in D(K)$. Since $K = \overline{A+B}$, there exists a sequence $\{v_n\}_{n=1}^{\infty} \subseteq D(A) = D(\tilde{A})$ such that $v_n \to v$ and $(A+B)v_n \to Kv$ in Y_F as $n \to \infty$. We then have

$$(\tilde{A}+B)v_n = (A+B)v_n - \beta v_n \to Kv - \beta v.$$

Therefore $\{v_n\}_{n=1}^{\infty}$ is $(\tilde{A}+B)$-convergent to v. Hence $v \in D(T)$ with $Tv = Kv - \beta v$. This proves that T is an extension of $K - \beta I_F$. By reversing this argument, we can show that $K - \beta I_F$ is an extension of T, which together with the previous statement gives us $(T, D(T)) = (K - \beta I_F, D(K))$.

Let $v \in D(K)$. From (4.35) we obtain

$$\begin{aligned}
&\int_{x_c}^{\infty} (Kv)(x)(1+x)dx = \int_{x_c}^{\infty} (Tv)(x)(1+x)dx + \beta \int_{x_c}^{\infty} v(x)(1+x)dx \\
&= -\int_{x_c}^{\infty} \left\{ \left(y+1 - \int_{x_c}^{y} b(x|y)(1+x)dx \right) a(y) + \beta(y+1) \right\} v(y)dy \\
&\quad + \beta \int_{x_c}^{\infty} v(x)(1+x)dx \\
&= -\int_{x_c}^{\infty} \left(y+1 - \int_{x_c}^{y} b(x|y)(1+x)dx \right) a(y)v(y)dy.
\end{aligned}$$

Therefore if $f = \binom{f_1}{f_2} \in Y$ with $f_2 \in D(K)_+$, we have

$$\begin{aligned}
&\int_0^{x_c} \left(C_| f_2\right)(x)(1+x)dx + \int_{x_c}^{\infty} \left(K f_2\right)(x)(1+x)dx \\
&= \int_0^{x_c} \left(\int_{x_c}^{\infty} a(y)b(x|y)f_2(y)dy \right)(1+x)dx \\
&\quad - \int_{x_c}^{\infty} \left(y+1 - \int_{x_c}^{y} b(x|y)(1+x)dx \right) a(y)f_2(y)dy \\
&= \int_{x_c}^{\infty} \left(\int_0^{x_c} b(x|y)(1+x)dx \right) a(y)f_2(y)dy \\
&\quad - \int_{x_c}^{\infty} \left(y+1 - \int_{x_c}^{y} b(x|y)(1+x)dx \right) a(y)f_2(y)dy \\
&= \int_{x_c}^{\infty} \left(\int_0^{y} b(x|y)(1+x)dx - y - 1 \right) a(y)f_2(y)dy \\
&= \int_{x_c}^{\infty} a(y)f_2(y)\left(n(y)-1\right)dy \le \beta \int_{x_c}^{\infty} f_2(y)\left(y+1\right)dy \\
&= \beta \|f_2\|_{Y_F} \le \beta \|f\|_Y.
\end{aligned}$$

□

Theorem 4.2.9. *The strong non-negative local solution, $u(t) = \binom{u_D(t)}{u_F(t)}$, to equation* (4.9) *does not blow up in finite time and hence is a global solution.*

Proof. The solution $u(t)$ is strongly differentiable and this property is inherited by the two components $u_D(t)$ and $u_F(t)$. This allows us to take a time derivative through the integrals which appear in our norms, which along with Lemmas 4.2.7 and 4.2.8 gives us

$$\begin{aligned}
&\frac{d}{dt}\|u(t)\|_Y = \frac{d}{dt}\|u_D(t)\|_{Y_D} + \frac{d}{dt}\|u_F(t)\|_{Y_F} \\
&= \frac{d}{dt}\int_0^{x_c} (u_D(t))(x)(1+x)dx + \frac{d}{dt}\int_{x_c}^{\infty} (u_F(t))(x)(1+x)dx \\
&= \int_0^{x_c} \left(C_| u_F(t)\right)(x)(1+x)dx + \int_0^{x_c} (N_D(u_D(t), u_F(t)))(x)(1+x)dx \\
&\quad + \int_{x_c}^{\infty} (K u_F(t))(x)(1+x)dx + \int_{x_c}^{\infty} (N_F(u_D(t), u_F(t)))(x)(1+x)dx \\
&\leq \beta\|u(t)\|_Y.
\end{aligned}$$

Applying Gronwall's inequality, [34, Lemma D.3], yields $\|u(t)\|_Y \leq \|u_0\|_Y e^{\beta t}$ for all $t \in [0, \hat{T})$. Therefore $\|u(t)\|_Y$ does not blow up in finite time, hence by Theorem 2.4.36, $u(t)$ is a global solution.

□

Chapter 5

Constraint on Fragmentation Kernels

In Chapter 4 we imposed the constraint on a and b that

$$a(y)\,(n(y)-1) \leq \beta(y+1), \tag{5.1}$$

for some positive constant β, where $n(y)$ is the expected number of particles resulting from the fragmentation of a particle of mass y, given by

$$n(y) = \int_0^y b(x|y)dx.$$

Physically, $a(y)$ is the rate at which particles of mass y fragment, whilst $(n(y)-1)$ gives the expected gain in particles when such a fragmentation event occurs. Therefore, condition (5.1) acts as a restriction on the rate of increase in the number of particles.

As was mentioned, this condition is satisfied if a is bounded by a linear polynomial with positive coefficients and $n(y)$ is bounded by a positive constant. These two demands are common in the study of coagulation–fragmentation equations; appearing, for example, in [6], [9] and [22]. In this chapter we shall investigate whether condition (5.1) allows a wider choice for a and b than is the case with the standard restrictions.

Within [4, Section 8.2], three possible forms of the function $b(x|y)$ are introduced. We shall now consider each of these forms in turn and examine the implications for condition (5.1).

5.1 Power Law Case

The first form considered for $b(x|y)$ is the power law case, given by

$$b(x|y) = x^\nu f(y),$$

for some function $f(y)$. The mass conservation condition (3.2) demands that $f(y) = (\nu+2)/y^{\nu+1}$, with $\nu > -2$. Now, if we compute the quantity $n(y)$ under these conditions, we get

$$n(y) = \int_0^y b(x|y)\,dx = \int_0^y \frac{(\nu+2)x^\nu}{y^{\nu+1}}\,dx = \frac{\nu+2}{\nu+1}.$$

We must make the restriction $\nu > -1$ in order that the above integral exists, in which case $n(y)$ is equal to some constant. The condition (5.1) then requires that $a(y)$ be linearly bounded. As such, (5.1) is equivalent to the standard constraints.

5.2 Homogenous Case

The second form proposed in [4] is the homogenous case, in which $b(x|y)$ takes the form

$$b(x|y) = \frac{1}{y} h\left(\frac{x}{y}\right),$$

for some function h. Considering the mass conservation condition (3.2), making the change of variable $t = x/y$ gives us

$$y = \int_0^y x b(x|y)\,dx = \int_0^y \frac{x}{y} h\left(\frac{x}{y}\right)\,dx = y \int_0^1 t h(t)\,dt.$$

Therefore, we must have $\int_0^1 t h(t)\,dt = 1$. With a kernel b of this form, we get

$$n(y) = \int_0^y b(x|y)\,dx = \int_0^y \frac{1}{y} h\left(\frac{x}{y}\right)\,dx = \int_0^1 h(t)\,dt,$$

where again the substitution $t = x/y$ has been made. Hence $n(y)$ is again a constant, and so, as before, the bound (5.1) is equivalent to the standard conditions for a and b.

5.3 Separable Case

The final form for the kernel $b(x|y)$, covered in [4], is the separable form given by

$$b(x|y) = \frac{yB(x)}{\int_0^y B(s)s\,ds}, \qquad x \le y,$$

for some function B. It is easily verified that any b of this form satisfies the usual mass conservation condition (3.2). If we select $B(x) = e^x$ then, using integration by parts on the denominator, it is simple to show that

$$b(x|y) = \frac{ye^x}{ye^y - e^y + 1},$$

and hence

$$n(y) = \int_0^y \frac{ye^x}{ye^y - e^y + 1}dx = \frac{y\int_0^y e^x dx}{ye^y - e^y + 1} = \frac{ye^y - y}{ye^y - e^y + 1}.$$

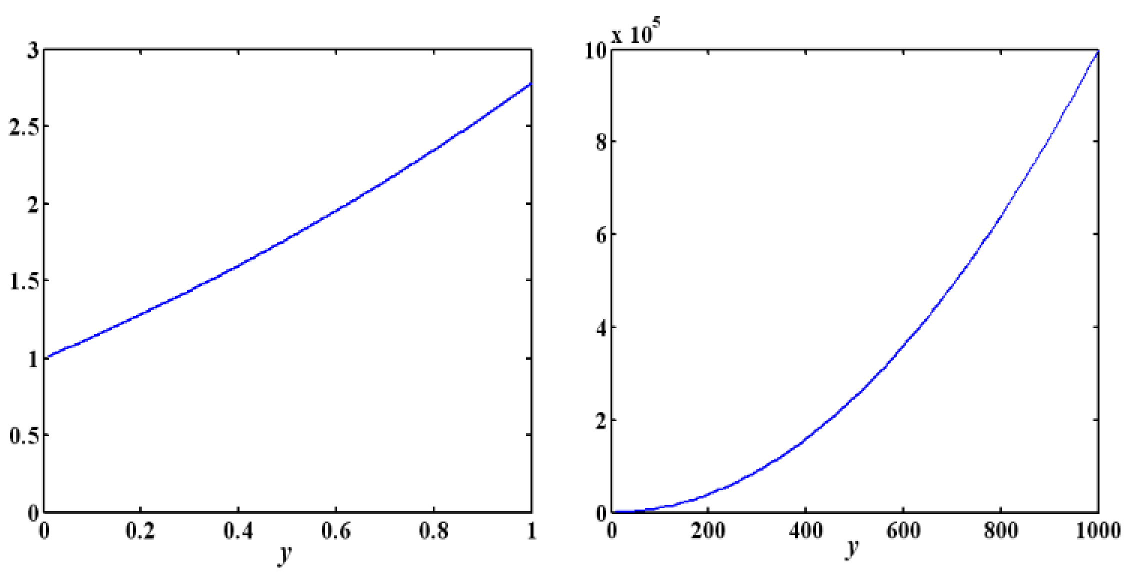
Our constraint for $a(y)$ then becomes

$$a(y)\frac{e^y - y - 1}{ye^y - e^y + 1} \leq \beta(y + 1).$$

Assuming that the factor multiplying $a(y)$ is positive (true for large y; also the constraint is satisfied automatically if the factor is negative), we can divide through to get

$$a(y) \leq \beta\frac{(y + 1)(ye^y - e^y + 1)}{e^y - y - 1} = \beta\frac{(y + 1)(y - 1 + e^{-y})}{1 - ye^{-y} - e^{-y}}. \tag{5.2}$$

Therefore $a(y)$ is allowed to grow quadratically as y increases. Hence, in this case, condition (5.1) represents a weaker restriction than the standard constraints. The figure below shows plots of the bound in (5.2), with $\beta = 1$, over a range of values for y.



Although this example may not be physically relevant, giving $n(y) \to 1$ as $y \to \infty$, where we expect $n(y) \geq 2$, it does demonstrate the potential of the new condition.

Chapter 6

Conclusion and Further Work

The goal of this work was the rigorous analysis of fragmentation and coagulation–fragmentation equations involving a mass cut-off. In Chapter 3 we examined a dual regime pure fragmentation model as introduced by Huang *et al.* [18]. By using the theory of semigroups and operator matrices, we were able to establish the existence of a unique strong solution to the system. This solution was shown to preserve positivity and also to conserve mass, confirming the assertion made in [18]. As an aim of further study, it would be interesting to obtain the explicit solution produced by the semigroup for the case of power law fragmentation kernels, as given by (1.4). Once obtained, this solution could be compared with that provided in [18], with the analysis of [9] suggesting agreement between the solutions.

As an addition to the pure fragmentation system, a mass loss mechanism could be introduced to the model. In the case of fragmentation without a mass cut-off, such a process was first modelled by Edwards *et al.* [11, 13, 17], who introduced the following equation

$$\frac{\partial u(x,t)}{\partial t} = -a(x)u(x,t) + \int_x^\infty a(y)b(x|y)u(y,t)dy + \frac{\partial}{\partial x}[r(x)u(x,t)].$$

This differs from the mass conserving equation (1.3), by the addition of the continuous mass loss term $\frac{\partial}{\partial x}[r(x)u(x,t)]$, where r is the continuous mass loss rate. Further, the usual mass conservation condition can be replaced by

$$\int_0^y xb(x|y)dx = y(1-\lambda(y)),\ \ 0 \leq \lambda(y) \leq 1,$$

allowing for discrete mass loss to occur during fragmentation. This equation was rigorously analysed, using the theory of semigroups, by Banasiak and Lamb in [5]. In the case of fragmentation with a mass cut-off, the article [18] by Huang *et al.* considers a dual regime model with continuous mass loss. The authors provide explicit solutions in the specific case that a, b and r are given by power laws; however, the methods applied in this study are not fully rigorous. The rigorous analysis of the general dual regime fragmentation model with mass loss remains to be carried out.

In Chapter 4 we introduced a coagulation process to the dual regime model. As far as we are aware, this represents the first example of the derivation and analysis of a full coagulation–fragmentation model with fragmentation mass cut-off. The main effort of this task involved the setting up of the coagulation terms in the model. Once the model was in place, using the analysis of the standard case as a guide, we were able to prove the existence of a unique, strongly differentiable, global in time, positive solution.

Also in Chapter 4, we introduced an alternative constraint, (4.4), on the fragmentation kernels a and b. The appeal of this condition lies both in the physical interpretation that can be attached to it, and also in the way in which it allows the boundedness to be spread between $a(y)$ and $n(y)$. In Chapter 5 we were able to show that the constraint (4.4) permits a wider choice for the functions a and b, than is the case with the standard conditions. In terms of further study on this matter, it would be pleasing to find an admissible form for b which gives $n(y)$ proportional to y^{α} for $\alpha \in (0, 1)$, which in turn would allow $a(y)$ to grow like $y^{1-\alpha}$. Such an example would further illustrate the advantages of condition (4.4), providing us with an unbounded $n(y)$, where the standard constraints require $n(y)$ be bounded.

Appendix

A - Explanation of Coagulation Terms

Recall the coagulation operator for the fragmentation regime given in Chapter 4. It is our aim here to clarify the construction of this expression. First consider the terms involving the characteristic function $\chi_I(x)$. If $x_c < x \leq 2x_c$, then subtracting x_c throughout we get $0 < x - x_c \leq x_c$ $(< x)$. In the diagram below we consider how the quantities y and $x - y$ change as the variable y moves through the range of integration

$$y : 0 \underset{D}{\longrightarrow} x - x_c \underset{D}{\longrightarrow} x_c \underset{F}{\longrightarrow} x$$

$$x - y : x \underset{F}{\longrightarrow} x_c \underset{D}{\longrightarrow} x - x_c \underset{D}{\longrightarrow} 0.$$

The subscripts D and F indicate the regime in which the values y and $x - y$ lie, motivating the choice of functions appearing within the first three integrals. Turning now to the terms involving the characteristic function $\chi_J(x)$. Suppose $2x_c < x$, subtracting x_c on both sides gives us $(0 <)$ $x_c < x - x_c$ $(< x)$. This provides the ordering in the diagram below

$$y : 0 \underset{D}{\longrightarrow} x_c \underset{F}{\longrightarrow} x - x_c \underset{F}{\longrightarrow} x$$

$$x - y : x \underset{F}{\longrightarrow} x - x_c \underset{F}{\longrightarrow} x_c \underset{D}{\longrightarrow} 0.$$

Again D and F indicate to which regime the variables belong, guiding us in our function selection for the second three integrals.

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