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Bifibrational Parametricity: from zero to two dimensions

by

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Declaration of the authorship

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

Signed:

Date:

Abstract

In this thesis we use bifibrations in order to study relational parametricity. There are three main contributions in this thesis. First, through the lenses of bifibrations, we give a new framework for models of parametricity. This allows us to make some of the underlying categorical structure in Reynolds' original work clearer.

Using the same approach we then give a universal property for the interpretation of forall types: they are characterized as terminal objects in a certain category. The universal property permits us to prove both Reynolds' Identity Extension Lemma and Abstraction Theorem.

The third contribution consists in defining two-dimensional parametricity. The insight derived from the bifibrational approach leads to a generalization of parametricity to proof-relevant relations, incorporating higher-dimensional relations between relations. We call the resulting theory two-dimensional parametricity.

Acknowledgements

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Contents

Chapter 1

Introduction

In this work, we start by introducing the notion of parametricity. We then recall and abstract Reynolds' fibrational model using bifibrations, which yields a framework for models of System F. Finally, we reach two dimensions by defining two-dimensional parametricity. In this chapter we give an introduction and an overview on the contents of this thesis.

1.1 Polymorphic functions

Polymorphic functions are functions defined for all types. The idea is that a polymorphic function computes terms whose types depend on the types given as input, e.g. for every input type they compute a term. We write $f\colon \forall X.T(X)$ for a polymorphic function. If A is a type, we have that $fA\colon T[X \mapsto A]$, e.g. fA is a term of type $T[X \mapsto A]$ obtained by substituting the type A for the free occurrences of X.

Polymorphic functions can be divided into two families: *ad hoc* and *parametric*. The ad hoc polymorphic functions behave differently according to the input type, while the behaviour of the parametric ones is uniform in the input type.

Consider for example a function $\mathbf{sum}\colon \forall X.X \to X \to X$ which is defined for natural numbers by $\mathbf{sum}(\mathsf{nat})(n, m) = n + m$ and for lists by the concatenation of lists $\mathbf{sum}(\mathtt{list}(A))([x_0, \ldots, x_n], [y_0, \ldots, y_m]) = [x_0, \ldots, x_n, y_0, \ldots, y_m]$. This function is ad hoc, because it inspects the input type in order to return the result: the sum for natural numbers or concatenation for lists.

On the other hand consider the polymorphic function $\mathbf{reverse}\colon \forall X.\mathtt{list}(X) \to \mathtt{list}(X)$ which reverses the elements of a list. It is clear that the function does not depend on the input type: the function "reverse a list of natural numbers" behaves the same as the function "reverse a list of characters" or "reverse a list" of every other type. This is the characteristic of parametric polymorphic functions.

In this thesis we focus on parametric polymorphic functions, or just parametric functions. If the intuitive distinction between ad hoc and parametric functions is clear, it is not evident how to formalise it. Reynolds had the great intuition to use relations in order to characterise parametric functions. The idea is to use relations in order to identify parametric functions: they are the ones preserving all the relations.

Relational parametricity has proven to be one of the key techniques for formally establishing properties of software systems, such as representation independence [ADR09, DNB12], equivalences between programs [HD11], or deriving useful theorems about programs from their type alone [Wad89].

There are several motivations at the basis of the work presented in this thesis. First of all we wanted to give a neat, simple and minimal framework for models of System F which allows us to enjoy the consequences of parametricity, something which we think was missing in the literature on parametricity. In this way we can reach the core of the theory and this clear vision simplifies the second part of our work: the definition of higher dimensional parametricity.

An additional advantage of this approach is the strength given by the possibility to use both fibrational and opfibrational properties. This will be central in some of the results.

1.2 Attention! Impredicativity!

We have to warn the readers that in this thesis we will use impredicativity. Impredicativity means the possibility to define objects by quantifying over a range which includes the object to be defined itself. This operation is delicate and it can lead to paradoxes. One of the most well-known paradoxes deriving from impredicativity is Russell's paradox from set theory. The paradox arises when we try to define the set $R = \{x \in \mathbf{Set} | x \notin x\}$, e.g. the set of sets which are not elements of themselves. It follows that if $R \notin R$, then actually R should be an element of itself, and if $R \in R$, it should not be an element of itself.

Another well known paradox based on impredicativity is the liar paradox deriving from the sentence “this sentence is false”. If this sentence was true, then it affirms that the sentence is false and vice versa.

There are situations in which is possible to use impredicativity safely. For example consider a group G and a subset $S \subset G$ of elements of G. A priori this is not a subgroup and if we want to find the smallest subgroup generated by S we can take the intersection of all the subgroups $G' \subset G$ such that $S \subset G'$.

The impredicativity of polymorphic functions is due to the fact that polymorphic functions quantify over types living at the same level where they live. Reynolds proved in [Rey84] that parametrric polymorphism is incompatible with classical set theory.

Further work showed how to deal with this. The material presented in this thesis makes sense in the (intuitionistic) internal language of a topos [Pit87], or in the Calculus of Constructions with impredicative **Set**. A fully formalised relational model of System F in this last setting has been constructed in [Atk09]. For this reason every time that we use set-theoretic notation, we are actually working with the latter meta theory.

1.3 Structure of the thesis

The thesis is divided into three parts: in the first one we present some background material and some preliminary results. In the second part we present the work on parametricity: the model and a characterisation of the interpretation of forall types via universal properties. Finally in the third part we present the work on two-dimensional parametricity.

Part I: Background, Notation and Preliminaries

In Chapter 2 we introduce a notion central to this work: bifibrations. We start the chapter by recalling the basics of category theory. This is also useful in order to fix the notation. We continue with the definition of bifibrations and some of their properties. This is all well-known from the literature.

Chapter 3 is devoted to the notion of fibrations of relations. We use this fibrational definition of relations in order to construct relational models of parametricity. We see that relations over sets (where a relation is a subset $R \subseteq A \times B$) is a particular instantiation of fibration of relations. When we have full comprehension, we can recover some of the usual intuitive behaviours of relations from the fibrational notion. Moreover the fibrational structure permits us to define two particular classes of relations: equality relations and graph relations. They correspond, in the case of relations over sets, to the relations $\{(a, b) \in A \times A \text{ such that } a = b\}$ and $\{(a, b) \in A \times B \text{ such that } fa = b\}$ for a map $f\colon A \to B$ respectively.

In Chapter 4 we focus on the type theory. We start by recalling how cartesian closed categories are models of simply typed lambda calculus. We need this result because System F is an extension of the simply typed lambda calculus, and, similarly, the framework of models of System F we define extends the one for the simply typed lambda calculus. We then introduce System F and recall a way to model it using category theory: $\lambda 2$-fibrations. We use the $\lambda 2$-fibrational structure in order to construct our framework of models of System F based on bifibrations. The internal language in a $\lambda 2$-fibration provides a powerful tool to reason about categories and we use it to prove some properties which follow from our framework of models of System F. For this reason we recall the internal language for System F in a $\lambda 2$-fibration at the end of the chapter.

Part II: Bifibrational Parametricity

We present our framework of models of System F in Chapter 5. We first give the model based on bifibrations. We show how to interpret types and terms, we prove that they satisfy Identity Extension Lemma and Abstraction Theorem, and that all the given structure forms a $\lambda 2$-fibration, i.e., a model of polymorphism.

Chapter 6 is a sanity check: we show that some of the well known consequences of parametricity follow from the model we define: existence of initial algebras and final coalgebras and dinaturality.

In Chapter 7 we focus on the most delicate point in parametric models: the interpretation of forall types. We present a way to interpret forall types using universal properties. This approach distinguishes itself from others because it allows us to prove the Identity Extension Lemma and the Abstraction Theorem as a consequence of the axioms. In many other frameworks interpretation of forall types comes with strong conditions which “bake-in” the Identity Extension Lemma.

Part III: Two-Dimensional Parametricity

Part III is devoted to the study of two-dimensional parametricity: not only there are parametric terms, but also the proofs can be parametric. We start with Chapter 8 in which we present intensional Martin-Löf type theory. In fact Martin-Löf type theory provides a suitable meta language for studying two-dimensional parametricity, and in particular it allows us to speak about two-dimensional equalities.

In Chapter 9, using the machinery introduced in Chapter 8, we define two-dimensional parametricity. The critical point is to interpret forall types and write the right logical relation which allow us to prove the Identity Extension Lemma and the Abstraction Theorem.

We conclude this part with Chapter 10 in which we show two applications of two-dimensional parametricity. First we use two-dimensional parametricity to prove coherence of proofs, in particular the coherence of the naturality proofs used in Chapter 5 to show that parametricity implies dinaturality. The second result is that two-dimensional parametricity implies 2-naturality.

1.4 Related literature

There is a rich literature on relational parametricity. Hermida, Reddy, and Robinson [HRR14] give a good introduction.

Since category theory underpins and informs many of the key ideas underlying modern programming languages, it is natural to ask whether it can provide a useful perspective on parametricity as well. Ma and Reynolds [MR92] developed the first categorical formulation of relational parametricity using the framework of PL-categories (see [See87]). Their models allow to show isomorphisms involving closed types, but nothing can be said about types containing free variables. Moreover, Birkedal and Møgelberg discovered that not all expected consequences of parametricity necessarily hold in their models (see [BM05] and [RR94]).

Another line of work, which was begun by O'Hearn and Tennent [OT95] and Robinson and Rosolini [RR94], and later refined by Dunphy and Reddy [DR04], uses reflexive graphs to model relations and functors between reflexive graph categories to model types. This is the state of the art for functorial semantics for parametric polymorphism. Interpreting types as functors is conceptually elegant and Dunphy and Reddy show that this framework is powerful enough to prove expected results, such as the existence of initial algebras for strictly positive type expressions [BB85]. However, working with reflexive graph categories means to work with internal categories in a particular presheaf category, so that all of the internal category theory applies automatically. We propose to instead to avoid internalisation by taking the fibrational view of logic from the outset, and thus to analyse parametricity through the powerful lens of categorical type theory [Jac99].

In doing so, we follow an extensive line of work by Hermida [Her93,Her06] and Birkedal and Møgelberg [BM05], who use fibrations to construct sophisticated categorical models not only of parametricity, but also of its logical structure in terms of Abadi-Plotkin logic [PA93]. Abadi-Plotkin logic is a formal logic for parametric polymorphism that includes predicate logic and a polymorphic lambda calculus, and thus requires significant machinery to handle. Using this machinery, Birkedal and Møgelberg are able to go beyond Dunphy and Reddy's results and, for instance, prove that all positive type expressions — not just the strictly positive ones as for Dunphy and Reddy — have initial algebras. However, these impressive results come at the price of the complexity of the notions involved. Our aim is to achieve the same results in a simpler setting, closer to Dunphy and Reddy's functorial semantics, where with functorial semantics we mean that the interpretation of types is given by functors. We end up with a notion of model in which each type is interpreted as an equality preserving fibred functor and each term is interpreted as a fibred natural transformation. This is quite similar to the models produced by the parametric completion process of

Robinson and Rosolini [RR94] (see also Birkedal and Møgelberg [BM05]) and to Mitchell and Scedrov's relator model [MS93], but with a more general notion of relation given by a fibration. We thus combine the generality of Birkedal and Møgelberg's fibrational models with the simplicity of Dunphy and Reddy's functorial semantics.

In our work is central the use of bifibrations to achieve this goal in the study of parametricity. This is not necessary for the definition of our framework, for which Lawvere equality [Law70] (i.e., opreindexing along diagonals only) suffices, but it helps considerably with both the concrete interpretation of ∀-types in Chapter 7 and the handling of graph relations. At a technical level, our strongest result is to use our simpler framework to recover all the expected consequences of parametricity that Birkedal and Møgelberg [BM05] prove using Abadi-Plotkin logic. In particular, we go beyond Dunphy and Reddy's result by deriving initial algebras for all positive type expressions, rather than just for strictly positive ones. Note that in a recent work Ghani, Nordvall Forsberg and Simpson (see [GNFS16]) achieved initial algebras (and final coalgebras) for all positive type expressions by using a fibrational generalisation of reflexive-graph categories.

Our approach to parametricity by universal property differs from the ones present in the literature. In fact none of the cited papers tackles the question we tackle in our work. Indeed, many follow the modern trend to bake in Identity Extension into their framework. In contrast, we prove the identity extension property from more primitive assumptions. Since our work on universal parametricity requires bifibrations, this part builds on the bifibrational model presented in Chapter 5.

The work on two-dimensional parametricity complements the more proof-theoretic work on internal parametricity in proof-relevant frameworks [BCM15, BJP12, Pol15]. Relevant is also the work on parametricity for dependent types in general [AGJ14, KD13], assuming proof-irrelevance.

This part requires to work with cubical structures [BHS11]. Cubical relations, but in a proof-irrelevant presentation, appear in [Gra09]. We adapt such definition for proof-relevant relations. Interestingly cubical techniques also arises in the semantics of Homotopy Type Theory [BCH14].

1.5 Contributions

The contributions appearing in this document are based on the papers that I wrote: [GJNF+15] with Neil Ghani, Patricia Johann, Fredrik Nordvall Forsberg and Tim Revell, [GNFO15] with Neil Ghani and Fredrik Nordvall Forsberg and [GNFO16] with Neil Ghani and Fredrik Nordvall Forsberg. I collaborated to the general development of all the material present in these papers. In particular my specific tasks were the following:

- Proof of the Graph Lemma 3.12.
- I developed the categorical formulation of Reynolds' relational parametricity presented in Chapter 5.
- I proved that the proposed interpretation of System F via bifibrations forms a $\lambda 2$-fibration (Theorem 5.24).
- I reformulated and proved that parametricity implies dinaturality using our model.
- I proved all the results in Chapter 7 on universal parametricity.
- I gave the interpretation of types in the 2 dimensional parametricity (Subsection 9.1 with proofs).
- I defined the graph functor for 2 dimensional parametricity and I proved the results in Section 10.2.

1.6 Notation

We use the notation $\mathcal{C}, \mathcal{D}, \mathcal{E}, \ldots$ for categories.

Given a category $\mathcal{C}$, we denote by $|\mathcal{C}|$ the discrete category of $\mathcal{C}$, e.g. the class (or set) of objects in $\mathcal{C}$.

Given two objects A and B in a category $\mathcal{C}$, $\mathrm{Hom}_{\mathcal{C}}(A, B)$ is the collection of morphisms from A to B in $\mathcal{C}$, possibly avoiding the subscript $\mathcal{C}$ when the category is clear from the context.

With $\mathcal{C}^{\mathrm{op}}$ we denote the opposite category of $\mathcal{C}$, e.g. the category whose objects are the objects of $\mathcal{C}$ and $\mathrm{Hom}_{\mathcal{C}^{\mathrm{op}}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A)$.

The category of sets and morphisms between them will be denoted **Set**, while **Cat** consists in the category of small categories and functors between them.

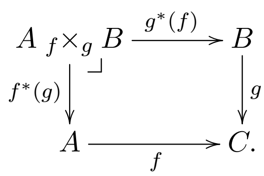
Composition of morphisms, functors and natural transformations is expressed by $\circ$ or juxtaposition, so that, given two morphisms $f\colon A \to B$ and $g\colon B \to C$, $g \circ f$ or gf denote their composite $A \to C$.

Given two parallel functors $F, G\colon \mathcal{C} \to \mathcal{D}$, a natural transformation α from F to G will be denoted as $\alpha\colon F \Rightarrow G\colon \mathcal{C} \to \mathcal{D}$.

For A and B objects in a category $\mathcal{C}$, we denote by $A \times B$ their product, and the projections are $\pi_1\colon A \times B \to A$ and $\pi_2\colon A \times B \to B$. Given $f\colon C \to A$ and $g\colon C \to B$, we denote by $\langle f, g\rangle\colon C \to A \times B$ the unique morphism identified by the universal property of the product.

Dually $A + B$ denotes their sum (coproduct), with $i_1\colon A \to A + B$ and $i_2\colon B \to A + B$ representing the structure maps of the sum.

Finally, given two morphisms $f\colon A \to C$ and $g\colon B \to C$, their pullback is represented by the diagram



Given $s\colon D \to A$ and $t\colon D \to B$ such that $f \circ s = g \circ t$, we denote $\langle s, t\rangle\colon D \to A\ {}_f{\times}_g\ B$ the unique mediating morphism.

Part I

Background, Notation and Preliminaries

Chapter 2

Bifibrations

In this section we recall some results from category theory, and we introduce bifibrations and their properties. We assume the reader is familiar with the basic notions of category theory in particular the definition of categories, functors, natural transformations, adjunctions. limits and colimits. The classical reference for the subject is Mac Lane [ML98]. Another nice book, in which there are examples also from logic and computer science is Awodey [Awo10].

The subject of fibrations and opfibrations is well treated in Jacobs [Jac99]. The book, beside giving a great introduction to the subject, shows how fibrations can be used to give categorical semantics for different type theories. Unless explicitly specified, we refer to it for the definitions and theorems in this chapter. Most of the fibrational theory that we use in this thesis can be found also in Hermida [Her93] and his work is strictly related to the one presented here.

Note that by (op)fibration we refer to what is sometimes called (op)cartesian (op)fibration or Grothendieck (op)fibration in literature. Since we will only refer to these kind of (op)fibrations, we will drop the adjective (op)cartesian.

2.1 Categorical notions

We first fix some terminology. Given a functor $U\colon \mathcal{E} \to \mathcal{B}$ we say that an object X in $\mathcal{E}$ is **over** or **above** an object I in $\mathcal{B}$ if $UX = I$. Similarly a morphism f in $\mathcal{E}$ is over or above a morphism u in $\mathcal{B}$ if $Uf = u$. A morphism in $\mathcal{E}$ is called **vertical** if it is above an identity morphism in $\mathcal{B}$.

We now recall some notions and properties we need.

2.1.1 Properties of adjoint functors

We assume the reader is familiar with the definition of adjoint functors, the unit and counit, triangular identities and natural isomorphism between Hom-sets. It is well known that every left adjoint preserves colimits, while every right adjoint preserves limits.

We will often use the following result.

Lemma 2.1. Let $L \dashv R$ be a pair of adjoint functors

- R is full and faithful if and only if the counit of the adjunction is a natural isomorphism;
- L is full and faithful if and only if the unit of the adjunction is a natural isomorphism.

2.1.2 The Frobenius and Beck-Chevalley conditions

We recall two conditions on adjoint functors: the Frobenius condition and the Beck-Chevalley condition.

Consider a pair of adjoint functors $L \dashv R$. The Frobenius condition, which sometimes is also called Frobenius reciprocity, is about the canonical morphism

$$L(C \times R(B)) \xrightarrow{\langle L\pi_1, L\pi_2 \rangle} L(C) \times L(R(B)) \xrightarrow{\mathsf{id} \times \epsilon_B} L(C) \times B$$

where $\epsilon\colon L \circ R \to \mathsf{Id}$ is the counit of the adjunction $L \dashv R$.

Definition 2.2. We say that a pair of adjoint functors $L \dashv R$ satisfies the **Frobenius condition** (Frobenius reciprocity) if the canonical morphism $L(C \times R(B)) \to L(C) \times B$ is an isomorphism.

If both the domain and codomain categories are cartesian closed, this condition is equivalent to requiring that the right adjoint preserves exponentiation. In fact, consider a pair of adjoint functors $L \dashv R$, with $L\colon \mathcal{C} \to \mathcal{D}$, and $\mathcal{C}$ and $\mathcal{D}$ are cartesian closed. Let C be an object in $\mathcal{C}$, A and B be two objects in $\mathcal{D}$. If the Frobenius condition holds for $L \dashv R$, then we have the series of isomorphisms

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(C, R(A)^{R(B)}) &\cong \mathrm{Hom}_{\mathcal{C}}(C \times R(B), R(A)) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(C \times R(B)), A) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(C) \times B, A) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(L(C), A^B) \\
&\cong \mathrm{Hom}_{\mathcal{C}}(C, R(A^B))
\end{aligned}$$

and by Yoneda $R(A)^{R(B)} \cong R(A^B)$.

For the Beck-Chevalley condition consider a commutative (up to isomorphism) square of functors

$$\begin{array}{ccc}
\mathcal{C} & \xrightarrow{F^*} & \mathcal{C}' \\
{\scriptstyle G^*}\downarrow & & \downarrow{\scriptstyle K^*} \\
\mathcal{D} & \xrightarrow[H^*]{} & \mathcal{D}'
\end{array} \tag{2.3}$$

where F^* and H^* have left adjoints $F_!$ and $H_!$, respectively. In this situation there is a canonical morphism

$$H_! \circ K^* \xrightarrow{H_! \circ K^* \eta_F} H_! \circ K^* \circ F^* \circ F_! \circ \xrightarrow{\cong} H_! \circ H^* \circ G^* \circ F_! \xrightarrow{\epsilon_H G^* \circ F_!} G^* \circ F_!, \tag{2.4}$$

where η_F is the unit of the adjunction $F_! \dashv F^*$ and ϵ_H is the counit of the adjunction $H_! \dashv H^*$.

Definition 2.5. We say that (2.3) satisfies the **Beck-Chevalley condition** if the canonical morphism (2.4) $H_! \circ K^* \to G^* \circ F_!$ is an isomorphism.

Analogous conditions arise when G^* and K^* have left adjoints or considering the dual notion where the functors have right adjoints instead of the left ones.

Example 2.6. The diagonal functor $\Delta\colon \mathcal{C} \to \mathcal{C} \times \mathcal{C}$ acts on objects by $X \mapsto (X, X)$ and on morphisms by $f \mapsto (f, f)$. If $\mathcal{C}$ is a category with products, the product functor $\times\colon \mathcal{C} \times \mathcal{C}$ is defined on objects by $(X, Y) \mapsto X \times Y$, and on morphisms by $(f, g) \mapsto f \times g$. It is well known that Δ is the left adjoint of $\times$. For every functor $F\colon \mathcal{C} \to \mathcal{D}$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{C} & \xrightarrow{\Delta} & \mathcal{C} \times \mathcal{C} \\ {\scriptstyle F}\downarrow & & \downarrow{\scriptstyle F\times F} \\ \mathcal{D} & \xrightarrow[\Delta]{} & \mathcal{D} \times \mathcal{D}. \end{array}$$

For such a square, requiring that the Beck-Chevalley condition holds is equivalent to asking that F preserves products: the Beck-Chevalley condition translates into the isomorphism $F \circ (\times) \xrightarrow{\cong} (\times) \circ (F \times F)$, which, when applied to any pair of objects (X, Y), gives $F(X \times Y) \cong F(X) \times F(Y)$.

2.2 Fibrations and opfibrations

There are two structures captured by the notion of fibration. The first one is that fibrations generalise collections of sets $(X)_{i \in I}$ indexed over a set I, and the second one is substitution: given a function $f : I \to J$ and a J-indexed collection $(X_j)_{j \in J}$, we can form an I-indexed collection whose i-th component is X_{fi}.

(I) Collection. There are two equivalent way to present collections of sets. They are represented in Figure 2.1.

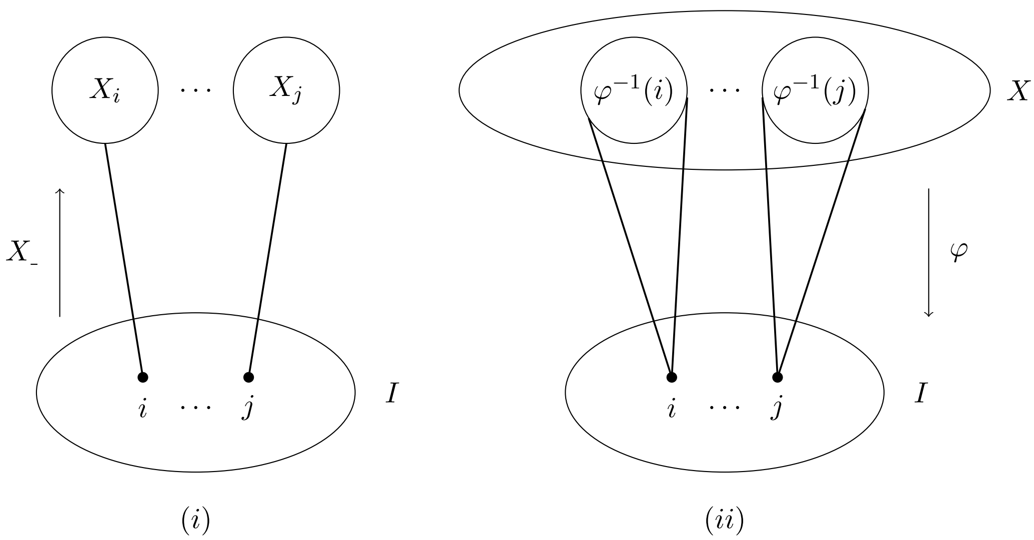


Figure 2.1: Different representations of a family of sets

(i) **Pointwise** (or **split**) **indexing**: $(X_i)_{i \in I}$ is thought as a collection where each X_i is a set. We can think it as a function $I \to \mathbf{Set}$ which maps $i \mapsto X_i$.

(ii) **Display indexing**: a family is a map $\varphi\colon X \to I$. The sets in the family are identified as the pre-image sets $\varphi^{-1}(i) = \{x \in X \mid \varphi(x) = i\}$ for $i \in I$.

It is not difficult to see that these notions are equivalent (see [Jac99]). Similarly there are two representations also for fibrations, and there is a standard method, called the Grothendieck construction, to go from the pointwise representation to the display one.

(II) Reindexing. Substitution along a function between indexes as seen above is modeled abstractly by reindexing. We see the example of the codomain functor $\mathsf{cod}\colon \mathbf{Set}^{\to} \to \mathbf{Set}$, but we first need some terminology.

Definition 2.7. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor and let D be an object in $\mathcal{D}$. The **fibre** or **fibre category** over D is the category $\mathcal{C}_D$ with

$\mathcal{C}_D$	**objects**	C in $\mathcal{C}$ such that $F(C) = D$.
	morphisms	$f\colon C_1 \to C_2$ such that $F(f) = id_D$.

For every category $\mathcal{C}$, there is the **arrow category** $\mathcal{C}^{\to}$ which consists of

$\mathcal{C}^{\to}$	**objects**	morphisms $f\colon X \to I$ in $\mathcal{C}$.
	morphisms	$(\alpha, \beta)\colon (f\colon X \to I) \to (g\colon Y \to J)$ is a pair of morphisms $\alpha\colon X \to Y$, $\beta\colon I \to J$ in $\mathcal{C}$ such that the following diagram commutes

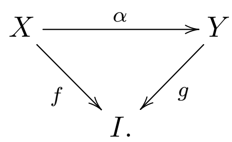
$$\begin{array}{ccc} X & \xrightarrow{\alpha} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ I & \xrightarrow[\beta]{} & J. \end{array}$$

In the case of $\mathcal{C} = \mathbf{Set}$, the objects of $\mathbf{Set}^{\to}$ are families of sets, presented by display indexing, which are indexed over every possible set.

The **codomain functor** $\mathsf{cod}\colon \mathbf{Set}^{\to} \to \mathbf{Set}$ is defined by $\mathsf{cod}(f\colon X \to I) = I$ and

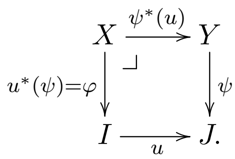
$\mathsf{cod}(\alpha, \beta) = \beta$. The fibres of this functor are the slice categories $\mathcal{C}/I$, where I is an object in $\mathcal{C}$. The slice category is defined as follows:

$\mathcal{C}/I$	**objects**	morphisms $f\colon X \to I$ with fixed codomain I.
	morphisms	$\alpha\colon (f\colon X \to I) \to (g\colon Y \to I)$ is a morphism $\alpha\colon X \to Y$ in $\mathcal{C}$ such that the following diagram commutes



The objects of $\mathbf{Set}/I$ are families of sets indexed over I.

For a morphism $u\colon I \to J$ in $\mathbf{Set}$, substitution along u consists of an operation which takes a family indexed over J and returns a family over I. In terms of cod, this operation is a map from the fibre over J to the fibre over I. Specifically, consider a family of sets $\psi\colon Y \to J$ indexed over J and let $u\colon I \to J$ be a morphism in $\mathbf{Set}$. By substitution one obtains the family $(Y_{u(i)})_{i\in I}$. The corresponding display representation can be obtained by pullback

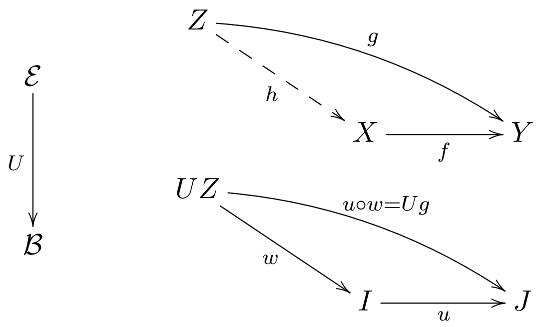


The set X is of the form $X = \{(i, y) \in I \times Y \mid u(i) = \psi(y)\}$ with obvious projection morphisms. The family $\varphi\colon X \to I$ corresponds exactly to $(Y_{u(i)})_{i\in I}$ as shown by the equalities

$$X_i = \varphi^{-1}(i) \cong \{y \in Y \mid \psi(y) = u(i)\} = \psi^{-1}(u(i)) = Y_{u(i)}.$$

By construction it is clear that it satisfies a universal property which is captured by the categorical notion of cartesian morphism.

Definition 2.8. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $f\colon X \to Y$ in $\mathcal{E}$ is **cartesian over** $u\colon I \to J$ in $\mathcal{B}$ if $Uf = u$ and if for every $g\colon Z \to Y$ in $\mathcal{E}$ whose image factorises as $U(g) = u \circ w$ for some $w\colon U(Z) \to I$, there exists a unique $h\colon Z \to X$ in $\mathcal{E}$ over w such that $f \circ h = g$. The condition is expressed by the diagram



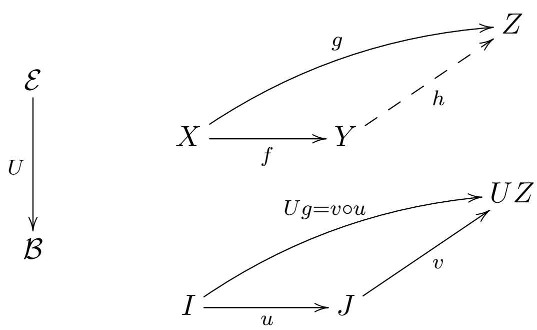
If f is cartesian over u we also say that f is a **cartesian lifting** of u.

When every map has a cartesian lifting we have a fibration.

Definition 2.9. A functor $U\colon \mathcal{E} \to \mathcal{B}$ is a **fibration** if for every Y in $\mathcal{E}$ and $u\colon I \to UY$ in $\mathcal{B}$, there is a cartesian morphism $f\colon X \to Y$ above u.

The dual notion of fibration is that of an opfibration. A functor $U\colon \mathcal{E} \to \mathcal{B}$ is an opfibration if the opposite functor $U^{op}\colon \mathcal{E}^{op} \to \mathcal{B}^{op}$ between the opposite categories is a fibration. It is useful to unwind this definition, showing the universal lifting property for opfibrations.

Definition 2.10. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $f\colon X \to Y$ in $\mathcal{E}$ is **opcartesian over** $u\colon I \to J$ in $\mathcal{B}$ if $Uf = u$ and if for every $g\colon X \to Z$ in $\mathcal{E}$ whose image factorises as $Ug = v \circ u$ for some $v\colon J \to UZ$, there exists a unique $h\colon Y \to Z$ in $\mathcal{E}$ over v such that $h \circ f = g$. The condition is expressed by the diagram



If f is opcartesian over u we also say that f is an **opcartesian lifting** of u.

Definition 2.11. A functor $U\colon \mathcal{E} \to \mathcal{B}$ is an **opfibration** if $U^{op}\colon \mathcal{E}^{op} \to \mathcal{B}^{op}$ is a fibration. Equivalently, if above every morphism $UX \to J$ in $\mathcal{B}$, there is an opcartesian morphism $X \to Y$ in $\mathcal{E}$.

Finally we can define bifibrations.

Definition 2.12. A **bifibration** is a functor which is at the same time a fibration and an opfibration.

If $U\colon \mathcal{E} \to \mathcal{B}$ is a fibration or opfibration, the category $\mathcal{E}$ is called the **total category** and $\mathcal{B}$ is called the **base category**.

Given two cartesian liftings $f\colon X \to Y$ and $f'\colon X' \to Y$ of u with the same codomain, there exists a unique vertical isomorphism $\varphi\colon X \xrightarrow{\sim} X'$ with $f' \circ \varphi = f$. The isomorphism and its inverse can be found using the universal property of the cartesian morphisms applied to the commuting diagrams $Uf \circ \mathsf{id} = Uf'$ and $Uf = Uf' \circ \mathsf{id}$ in the base category. Dually, for two opcartesian liftings $g\colon X \to Y$ and $g'\colon X \to Y'$ of u with the same domain, there exists a unique vertical isomorphism $\psi\colon Y \xrightarrow{\sim} Y'$ with $\psi \circ g = g'$. In fact, it holds:

Proposition 2.13. Cartesian (opcartesian) liftings over the same morphism and with the same codomain (domain) are unique up to unique vertical isomorphism.

Given Proposition 2.13 it is useful to be able to speak about *the* cartesian lifting and *the* opcartesian lifting. In order to do that we need to fix cartesian and opcartesian morphisms for fixed codomain and domain and this often requires the use of the axiom of choice.

Definition 2.14. A fibration $U\colon \mathcal{E} \to \mathcal{B}$ is called **cloven** if it comes equipped with a choice of cartesian lifting for each object Y of $\mathcal{E}$ and map $f\colon I \to UY$.

Dually there is the notion of cloven opfibration.

Definition 2.15. An opfibration $U\colon \mathcal{E} \to \mathcal{B}$ is called **cloven** if it comes equipped with a choice of opcartesian lifting for each object X of $\mathcal{E}$ and map $f\colon UX \to J$.

In the rest of this thesis all the (op)fibrations that we consider will be assumed to be cloven. We call the choice of (op)cartesian morphisms in a cloven (op)fibration a **cleavage** .

Consider a cloven bifibration $U\colon \mathcal{E} \to \mathcal{B}$. We fix the notation for the cartesian and opcartesian morphisms in the cleavage. Given $u\colon I \to J$ in $\mathcal{B}$, we denote the cartesian lifting of u with codomain Y as $u^{\S}_Y\colon u^*Y \to Y$ and we say that u^*Y is the **reindexing** of Y along u. We denote the opcartesian lifting of u with domain X as $u^X_{\S}\colon X \to \Sigma_u X$ and we say that $\Sigma_u X$ is the **opreindexing** of X along u. When clear from the context we drop the Y in $u^{\S}_Y$ or the X in $u^X_{\S}$.

We conclude this section with some properties of cartesian morphisms, but we first need some more notation for the statement of the lemma. Given a fibration $U\colon \mathcal{E} \to \mathcal{B}$, a morphism $u\colon I \to J$ in $\mathcal{B}$ and two objects X, Y in $\mathcal{E}$ over, respectively, I and J, we denote by $\mathcal{E}_u(X,Y)$ the morphisms from X to Y over u.

We express the following properties for cartesian morphisms: the dual properties hold for opcartesian morphisms.

Lemma 2.16. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration.

- the cartesian lifting of an isomorphism is an isomorphism;
- all isomorphisms in $\mathcal{E}$ are cartesian;
- for every $u\colon I \to J$ in $\mathcal{B}$ and X, Y in $\mathcal{E}$ over, respectively, I and J, $\mathcal{E}_u(X,Y) \cong \mathrm{Hom}_{\mathcal{E}_I}(X, u^*Y)$;
- for $f\colon X \to Y$ and $g\colon Y \to Z$ in $\mathcal{E}$, if f and g are cartesian, so is $g \circ f$. If g and $g \circ f$ are cartesian, so is f.

2.3 Examples of fibrations

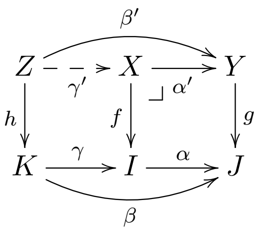
In this section we give some important examples of fibrations, opfibrations and bifibrations with some of their properties.

Example 2.17. The **identity fibration** $\mathsf{Id}\colon \mathcal{B} \to \mathcal{B}$.

For every category $\mathcal{B}$, the identity functor over itself is a fibration. Every fibre has only one object $|\mathcal{B}_I| = \{I\}$, and the cartesian lifting of any morphism f in the base is f itself. It is also an opfibration with the opcartesian lifting of f given, again, by f itself, and hence it is a bifibration.

Example 2.18. The **codomain fibration** $\mathsf{cod}\colon \mathcal{C}^{\to} \to \mathcal{C}$.

When the category $\mathcal{C}$ has pullbacks, the codomain functor is a fibration. To see this, consider the diagram

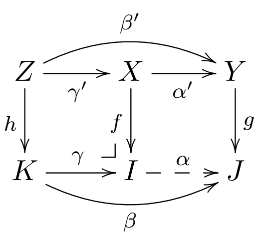


representing the universal property for the cartesian morphism which is formed by pullback. The bottom row of the diagram can be thought of as living in the base category $\mathcal{C}$. The morphism $g\colon Y \to J$ is an object in $\mathcal{C}^{\to}$ living over J, and the cartesian morphism over α is given by (α', α). In fact, given a morphism $(\beta', \beta)\colon h \to g$ for which, in the base category, there is a morphism $\gamma\colon K \to I$ such that $\beta = \alpha \circ \gamma$, there is a unique $(\gamma', \gamma)\colon h \to f$ such that $(\beta', \beta) = (\alpha', \alpha) \circ (\gamma', \gamma)$. The universal property of cartesian morphisms is exactly the universal property of the pullback since $g \circ \beta' = \alpha \circ \gamma \circ h$.

In general every functor $\mathsf{cod}\colon \mathcal{C}^{\to} \to \mathcal{C}$ is an opfibration. In fact it is not difficult to check that given $f\colon X \to I$, the morphism $(\mathsf{id}, \alpha)\colon f \to \alpha \circ f$ is opcartesian over $\alpha\colon I \to J$.

Example 2.19. The **domain opfibration** $\mathsf{dom}\colon \mathcal{C}^{\to} \to \mathcal{C}$.

This example is the dual of the codomain fibration. Consider the functor which projects on the domain. It is defined as $\mathsf{dom}(f\colon X \to I) = X$ and $\mathsf{dom}(\alpha', \alpha) = \alpha'$. If the base category has pushouts, the following diagram describes how to find the opcartesian morphisms



where (γ', γ) is opcartesian over γ. It is similar to the case of the codomain fibration, with the difference that, in this case, the upper row lives in the base category.

In general every functor $\mathsf{dom}\colon \mathcal{C}^{\to} \to \mathcal{C}$ is a fibration. In fact it is not difficult to check that given $g\colon Y \to J$, the morphism $(\alpha, \mathsf{id})\colon g \circ \alpha \to g$ is cartesian over $\alpha\colon X \to Y$.

Example 2.20. The **subset fibration** $\mathsf{sub}\colon \mathrm{Sub}(\mathbf{Set}) \to \mathbf{Set}$.

We denote by $\mathrm{Sub}(\mathbf{Set})$ the category

$\mathrm{Sub}(\mathbf{Set})$	**objects**	subsets $X \subseteq I$.
	morphisms	$f\colon (X \subseteq I) \to (Y \subseteq J)$ is a morphism $f\colon I \to J$ in $\mathbf{Set}$ such that for every $x \in X$, $f(x) \in Y$, e.g. morphisms preserving subsets.

The functor $\mathsf{sub}\colon \mathrm{Sub}(\mathbf{Set}) \to \mathbf{Set}$ which sends an object $(X \subseteq I) \mapsto I$ and a morphism $f \mapsto f$, is a fibration. Given a morphism $u\colon I \to J$ in $\mathbf{Set}$, the reindexing along u of $Y \subseteq J$ is given by the object $u^*Y \subseteq I$ defined by $u^*Y = \{i \in I \mid u(i) \in Y\}$. The cartesian lifting u with codomain $Y \subseteq J$ is u itself. It is obvious that u preserves the subset u^*Y.

The subset functor is also an opfibration. Given $u\colon I \to J$ in $\mathbf{Set}$, the opreindexing along u of $X \subseteq I$ consists of the object $\Sigma_u X \subseteq J$ defined by $\Sigma_u X = \{j \in J \mid \exists x \in X$ such that $ux = j\}$. The opcartesian lifting of u with domain $X \subseteq I$ is again u itself.

The subset fibration is a particular case of the following more general fibration.

Example 2.21. The **subobject fibration** $\mathsf{sub}\colon \mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$.

Let $\mathcal{C}$ be a category. There is an equivalence relation $\sim_{\mathtt{sub}}$ over monomorphisms defined by $(m\colon X \to I) \sim_{\mathtt{sub}} (m'\colon Y \to J)$ if and only if $I = J$ and there is an isomorphism $\phi\colon X \to Y$ such that $m' \circ \phi = m$. We denote by $\mathrm{Sub}(\mathcal{C})$ the category

$\mathrm{Sub}(\mathcal{C})$	**objects**	equivalence classes $[m]$ of monomorphisms with respect to $\sim_{\mathtt{sub}}$.

morphisms $\alpha\colon [m\colon X \to I] \to [n\colon Y \to J]$ is a morphism $\alpha\colon I \to J$ for which it exists a morphism $\alpha'\colon X \to Y$ in $\mathcal{C}$ such that the following diagram commutes

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & Y \\ {\scriptstyle m}\downarrow & & \downarrow{\scriptstyle n} \\ I & \xrightarrow[\alpha]{} & J. \end{array}$$

Note that morphisms are well defined. In fact, if α' exists it is unique since n is a monomorphism.

The subobject functor $\mathsf{sub}\colon \mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$ sends $\mathsf{sub}([m\colon X \to I]) = I$ and $\mathsf{sub}(\alpha) = \alpha$. It is well defined since I and α are independent of the choice of the representative. A morphism $\alpha\colon [m] \to [n]$ as in the diagram

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & Y \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ I & \xrightarrow[\alpha]{} & J \end{array}$$

is cartesian if and only if the diagram is a pullback square

$$\begin{array}{ccc} X & \xrightarrow{\alpha'} & Y \\ {\scriptstyle f}\downarrow & \lrcorner & \downarrow{\scriptstyle g} \\ I & \xrightarrow[\alpha]{} & J, \end{array}$$

i.e. $\mathsf{sub}\colon \mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$ is a fibration if and only if $\mathcal{C}$ has pullbacks of monomorphisms along arbitrary maps.

When $\mathcal{C}$ has an epi-mono factorisation system (E, M) (see [Jac99] for the definition of factorisation systems), the functor $\mathsf{sub}\colon \mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$ is also an opfibration. In fact consider a morphism $u\colon I \to J$ in $\mathcal{C}$. For every object $i\colon X \rightarrowtail I$ in the fibre over I, the composition $u \circ i$ factorises as $m \circ e$ with m mono and e epi. The opreindexing of i along u is $\Sigma_u i = m$ and the cartesian lifting of u is (e, u)

$$\begin{array}{ccc} X & \xrightarrow{e} & Y \\ {\scriptstyle i}\downarrow & & \downarrow{\scriptstyle m} \\ I & \xrightarrow[u]{} & J \end{array}$$

The universal property of opcartesian morphisms follows from uniqueness of the diagonal morphism in the left lifting property.

Example 2.22. The **families of sets fibration** $\mathsf{fam}\colon \mathbf{Fam}(\mathbf{Set}) \to \mathbf{Set}$.

The category $\mathrm{Fam}(\mathbf{Set})$ consists of

$\mathrm{Fam}(\mathbf{Set})$	**objects**	pairs (I, P) with I in **Set** and $P\colon I \to \mathbf{Set}$ a function which associates a set to every element of I.
	morphisms	$(u, \alpha)\colon (I, P) \to (J, Q)$ is a function $u\colon I \to J$ in **Set** and a family of morphisms $\alpha_x\colon Px \to Q(ux)$ in **Set** for $x \in I$.

The functor fam sends $\mathsf{fam}(I, P) = I$ and $\mathsf{fam}(u, \alpha) = u$.

If $u\colon I \to J$ is a morphism in **Set**, the reindexing of (J, Q) along u is defined by $u^*(J, Q) = (I, Q \circ u)$. The cartesian lifting $u^{\S}_{(J,Q)}$ is given by the morphism (u, id).

This functor is also an opfibration and thus a bifibration. Given $u\colon I \to J$ in **Set**, the opreindexing of (I, P) along u is given by $\Sigma_u(I, P) = (J, y \mapsto \coprod_{\{x \in I | ux = y\}} Px)$, where $\coprod$ is the disjoint union of sets. The opcartesian lifting $u_{\S}^{(I,P)}$ is given by the morphism (u, α), where $\alpha_x(p) = (x, p)$ is well defined since $(x, p) \in \Sigma_u P(u\,x) = \coprod_{\{x' \in I | ux' = ux\}} Px'$. Categorically it is the structure map of the coproduct $i_{Px}\colon Px \to \coprod_{\{z \in I | uz = ux\}} Pz$.

Example 2.22 generalises.

Example 2.23. The **family fibration** $\mathsf{fam}\colon \mathrm{Fam}(\mathcal{C}) \to \mathbf{Set}$.

For every category $\mathcal{C}$ we define the category $\mathrm{Fam}(\mathcal{C})$ as follows:

$\mathrm{Fam}(\mathcal{C})$	**objects**	pairs (I, P) with I in **Set** is a set of indices and $P\colon I \to \mathcal{C}$ a function which associates an object of $\mathcal{C}$ to every element of I.
	morphisms	$(u, \alpha)\colon (I, P) \to (J, Q)$ is a function $u\colon I \to J$ in **Set** and a family of morphisms $\alpha_x\colon Px \to Q(ux)$ in $\mathcal{C}$ for $x \in I$.

The functor $\mathsf{fam}\colon \mathrm{Fam}(\mathcal{C}) \to \mathbf{Set}$ acting on objects as $\mathsf{fam}(I, P) = I$ and on morphisms as $\mathsf{fam}(u, \alpha) = u$ is a fibration.

For every morphism $u\colon I \to J$ in **Set**, the the reindexing of (J, Q) along u gives $u^*(J, Q) = (I, Q \circ u)$. The cartesian lifting $u^{\S}_{(J,Q)}\colon u^*(J, Q) \to (J, Q)$ is defined by the pair (u, id).

In the case that $\mathcal{C}$ has set-indexed coproducts $\coprod$, the family fibration of $\mathcal{C}$ is an opfibration, and hence a bifibration. Given $u\colon I \to J$ morphism in the base category **Set**, the reindexing of (I, P) along u gives $\Sigma_u(I, P) = (J, y \mapsto \coprod_{\{x\in I \mid ux=y\}} Px)$. The opcartesian lifting $u_{\S}^{(I,P)}\colon (I, P) \to \Sigma_u(I, P)$ is defined by the pair (u, α) where α is given by the injections of the components Pz in the coproduct $\coprod_{\{x\in I \mid ux=uz\}} Px$, for every $z \in I$.

Example 2.24. The **fibration of relations** over **Set** $\mathsf{rel}\colon \mathbf{Rel} \to \mathbf{Set} \times \mathbf{Set}$.

We consider the category **Rel** of relations over sets, and morphisms between them preserving relations.

Rel	**objects**	triples $(A, B, R \subseteq A \times B)$ with A and B in **Set**, while $R \subseteq A \times B$ is an object in Sub(**Set**). We write (A, B, R) for $(A, B, R \subseteq A \times B)$.
	morphisms	pairs $(f, g)\colon (A, B, R) \to (A', B', R')$ with $f\colon A \to A'$ and $g\colon B \to B'$ in **Set**, such that $f \times g(R) \subseteq R'$ in Sub(**Set**).

The functor $\mathsf{rel}\colon \mathbf{Rel} \to \mathbf{Set} \times \mathbf{Set}$ which sends an object $(A, B, R) \mapsto (A, B)$ and a morphism $(f, g,) \mapsto (f, g)$ is a fibration. Given $(f, g)\colon (A, B) \to (A', B')$ in $\mathbf{Set} \times \mathbf{Set}$, the reindexing of (A', B', R') along (f, g) is $(A, B, (f, g)^*R')$ where $(f, g)^*R' = \{(a, b) \in A \times B \mid (fa, gb) \in R'\}$. This coincides with the reindexing along $f \times g$ for the subobject fibration $\mathsf{sub}\colon \mathrm{Sub}(\mathbf{Set}) \to \mathbf{Set}$.

The functor rel is also an opfibration. The opreindexing along $(f, g)\colon (A, B) \to (A', B')$ maps $(A, B, R) \mapsto (A', B', \Sigma_{(f,g)}R)$ where

$$\Sigma_{(f,g)}R = \{(a', b') \in A' \times B' \mid \exists (a, b) \in R \text{ such that } (fa, gb) = (a', b')\}.$$

2.4 Reindexing and opreindexing functorially

If a fibration $U\colon \mathcal{E} \to \mathcal{B}$ has a cloven structure, cartesian and opcartesian morphisms induce functors between fibres.

Definition 2.25. Given a cloven bifibration $U\colon \mathcal{E} \to \mathcal{B}$, every morphism $u\colon I \to J$ in $\mathcal{B}$ induces a functor $u^*\colon \mathcal{E}_J \to \mathcal{E}_I$ and a functor $\Sigma_u\colon \mathcal{E}_I \to \mathcal{E}_J$ defined as

$u^*\colon \mathcal{E}_J \to \mathcal{E}_I$ **objects** u^*Y is the domain of the cartesian morphism $u^{\S}_Y$.

morphisms the morphism $f\colon Y \to Y'$ in $\mathcal{E}_J$ is mapped to the unique morphism

$$\begin{array}{ccc} u^*Y & \xrightarrow{u^{\S}_Y} & Y \\ {\scriptstyle u^*f}\downarrow & & \downarrow{\scriptstyle f} \\ u^*Y' & \xrightarrow[u^{\S}_{Y'}]{} & Y' \end{array}$$

which exists and is unique over id by the universal property of $u^{\S}_{Y'}$.

$\Sigma_u\colon \mathcal{E}_I \to \mathcal{E}_J$ **objects** $\Sigma_u X$ is the codomain of the opcartesian morphism $u^X_{\S}$.

morphisms the morphism $f\colon X \to X'$ in $\mathcal{E}_I$ is mapped to the unique morphism

$$\begin{array}{ccc} X & \xrightarrow{u^X_{\S}} & \Sigma_u X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle \Sigma_u f} \\ X' & \xrightarrow[u^{X'}_{\S}]{} & \Sigma_u X' \end{array}$$

which exists and is unique over id by the universal property of $u^X_{\S}$.

We call u^* the **reindexing functor** along u and Σ_u the **opreindexing functor** along u.

Note that, in general, the operation mapping a morphism u to u^* is only (contravariantly) pseudofunctorial. We have indeed the following commuting diagrams

$$\begin{array}{ccc} X & & \\ {\scriptstyle \cong}\downarrow & \searrow^{\mathsf{id}} & \\ \mathsf{id}^*X & \xrightarrow[\mathsf{id}^{\S}_X]{} & X \end{array} \qquad \begin{array}{ccc} u^*v^*(X) & \xrightarrow{u^{\S}_{v^*X}} & v^*(X) \\ {\scriptstyle \cong}\downarrow & & \searrow^{v^{\S}_X} \\ (v\circ u)^*(X) & \xrightarrow[(v\circ u)^{\S}_X]{} & X \end{array} \tag{2.26}$$

where the isomorphism $X \cong \mathsf{id}^*X$ follows from the fact that $\mathsf{id}^{\S}_X$ is the cartesian morphism over the isomorphism id, hence an isomorphism by Lemma 2.16 and the isomorphism $u^*v^*(X) \cong (v\circ u)^*(X)$ arises from the following standard argument for fibrations. Both $(v\circ u)^{\S}_X$ and $v^{\S}_{u^*X} \circ u^{\S}_X$ are cartesian morphisms (composition of cartesian morphisms is a cartesian morphism, again Lemma 2.16) and both of them are over $v \circ u$. Of course, the following diagram in the base commutes:

$$\begin{array}{ccc} I & & \\ \| & \searrow^{v \circ u} & \\ I & \xrightarrow[v \circ u]{} & J. \end{array}$$

Using the universal property of $(v \circ u)^{\S}_X$ first, and $v^{\S}_{u^*X} \circ u^{\S}_X$ after, there are two morphisms $f\colon u^*v^*(X) \to (v \circ u)^*(X)$ and $g\colon (v \circ u)^*(X) \to u^*v^*(X)$ which are unique over id and make the diagram commute. We have that $f \circ g\colon (v \circ u)^*(X) \to (v \circ u)^*(X)$ is the unique morphism over the identity such that $(v \circ u)^{\S}_X \circ f \circ g = (v \circ u)^{\S}_X$ and since id satisfies such property, using uniqueness $f \circ g = \mathsf{id}$. The symmetric reasoning holds for the proof of $g \circ f = \mathsf{id}$ and it is not hard to check that the isomorphisms in diagram (2.26) yield natural isomorphisms $\mathsf{id} \Rightarrow \mathsf{id}^*$ and $u^*v^* \Rightarrow (v \circ u)^*$.

In some cases the isomorphisms $u^*v^*(X) \xrightarrow{\cong} (v \circ u)^*$ and $X \xrightarrow{\cong} \mathrm{id} * (X)$ are identities.

Definition 2.27. A **split fibration** is a cloven fibration for which the isomorphisms in diagram (2.26) are identities $\mathsf{id} \xrightarrow{=} \mathsf{id}^*$ and $u^*v^* \xrightarrow{=} (v \circ u)^*$. The cleavage involved is then called a **splitting**.

We finish this section with the definition of the Beck-Chevalley condition for bifibrations.

We first show that, in a bifibration, reindexing and opreindexing are adjoint functors. It is known (see [Jac99]) that this adjunction characterises bifibrations as follows:

Lemma 2.28. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration. Then U is a bifibration if and only if for every morphism $f\colon I \to J$ in $\mathcal{B}$, f^* has left adjoint Σ_f.

The Beck-Chevalley condition in the case of bifibrations assumes the following form.

Definition 2.29. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a bifibration. We say that U satisfies the **Beck-**

Chevalley condition if, for every pullback square

$$\begin{array}{ccc} A & \xrightarrow{t} & B \\ {\scriptstyle s}\downarrow & \lrcorner & \downarrow{\scriptstyle f} \\ C & \xrightarrow[g]{} & D \end{array}$$

in $\mathcal{B}$, the canonical natural transformation $\Sigma_s \circ t^* \to g^* \circ \Sigma_f$ defined as

$$\Sigma_s \circ t^* \xrightarrow{\Sigma_s \circ t^* \eta^f} \Sigma_s \circ t^* \circ f^* \circ \Sigma_f \xrightarrow{\cong} \Sigma_s \circ s^* \circ g^* \circ \Sigma_f \xrightarrow{\epsilon^s g^* \circ \Sigma_f} g^* \Sigma_f$$

is an isomorphism, with η^f the unit of the adjunction $\Sigma_f \dashv f^*$, and ϵ^s the counit of the adjunction $\Sigma_s \dashv s^*$. Note that the isomorphism $\Sigma_s \circ t^* \circ f^* \circ \Sigma_f \xrightarrow{\cong} \Sigma_s \circ s^* \circ g^* \circ \Sigma_f$ derives from the fact that reindexing and opreindexing are defined up to isomorphism.

We are interested in the following consequence of the Beck-Chevalley condition.

Lemma 2.30. Let $U \colon \mathcal{E} \to \mathcal{B}$ be a bifibration that satisfies the Beck-Chevalley condition. For every monomorphism $f \colon I \to J$ in $\mathcal{B}$ and every object X over I, the unit $\eta_X \colon X \to f^*\Sigma_f X$ is an isomorphism. Equivalently, for every monomorphism $f \colon I \to J$ in $\mathcal{B}$, the functor $\Sigma_f \colon \mathcal{E}_I \to \mathcal{E}_J$ is full and faithful.

Proof. The square $\begin{array}{ccc} I & \xrightarrow{\mathsf{id}} & I \\ {\scriptstyle \mathsf{id}}\downarrow & & \downarrow{\scriptstyle f} \\ I & \xrightarrow[f]{} & J \end{array}$ is a pullback since $f \colon I \to J$ is a monomorphism. The Beck-Chevalley condition implies that the composition

$$\Sigma_{\mathsf{id}} \mathsf{id}^* \xrightarrow{\Sigma_{\mathsf{id}} \mathsf{id}^* \eta^f} \Sigma_{\mathsf{id}} \mathsf{id}^* f^* \Sigma_f \xrightarrow{=} \Sigma_{\mathsf{id}} \mathsf{id}^* f^* \Sigma_f \xrightarrow{\epsilon^{\mathsf{id}} f^* \Sigma_f} f^* \Sigma_f$$

is an isomorphism. Using the fact that the (op)cartesian lifting of an isomorphism is an isomorphism, the functors id^* and Σ_{id} are full and faithful, hence the counit ϵ^{id} is an isomorphism. By the two-out-of-three for composition of isomorphisms, we have that $\Sigma_{\mathsf{id}} \mathsf{id}^* \eta^f$ is an isomorphism, and since Σ_{id} and id^* are full and faithful, the unit η^f is an isomorphism, making Σ_f full and faithful. $\square$

2.5 Constructing new fibrations from old

We now study how to obtain new fibrations starting from given ones.

Lemma 2.31. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration and $F\colon \mathcal{B}' \to \mathcal{B}$ a functor. The pullback of U along F in **Cat**

$$\begin{array}{ccc} \mathcal{B}'\,{}_F\!\times_U \mathcal{E} & \xrightarrow{U^*(F)} & \mathcal{E} \\ {\scriptstyle F^*(U)}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{B}' & \xrightarrow[F]{} & \mathcal{B} \end{array}$$

defines a new fibration $F^*(U)\colon \mathcal{B}'\,{}_F\!\times_U \mathcal{E} \to \mathcal{B}'$. We say that $F^*(U)$ is obtained via **change of base** from U along F.

It is possible to describe concretely the fibration $F^*(U)$ as follows. The objects of $\mathcal{B}'\,{}_F\!\times_U \mathcal{E}$ are pairs (I, X), where I is an object in $\mathcal{B}'$, while X is an object in $\mathcal{E}$, and such that $F(I) = U(X)$. The morphisms are pairs (u, f), with u in $\mathcal{B}'$, f in $\mathcal{E}$, and such that $F(u) = U(f)$. The functor $F^*(U)$ projects out the first component. The cartesian lifting of $u\colon I \to J$ in $\mathcal{B}'$ with codomain (J, Y), is given by $(u, (Fu)^{\S}_{Y})\colon (I, (Fu)^*Y) \to (J, Y)$, where the second component is cartesian with respect to the fibration U. This is well defined since $F(J) = U(Y)$ and then $F(u)\colon F(I) \to U(Y)$. It is immediate to verify that this satisfies the universal property of cartesian morphisms.

Example 2.32. The fibration of relations over sets in Example 2.24 arises from the subset fibration via change of base along the product functor

$$\begin{array}{ccc} \mathbf{Rel} & \longrightarrow & \mathrm{Sub}(\mathbf{Set}) \\ {\scriptstyle \mathsf{rel}}\downarrow & \lrcorner & \downarrow{\scriptstyle \mathsf{sub}} \\ \mathbf{Set} \times \mathbf{Set} & \xrightarrow[-\times-]{} & \mathbf{Set}, \end{array}$$

where the product functor sends the pair of objects (A, B) to $A \times B$ and the pair of morphisms (f, g) to $f \times g$.

By duality it also holds

Lemma 2.33. Change of base preserves opfibrations, i.e. if $U\colon \mathcal{E} \to \mathcal{B}$ is an opfibration and $F\colon \mathcal{B}' \to \mathcal{B}$ a functor, then $F^*(U)$ is an opfibration.

Combining Lemmas 2.31 and 2.33 it follows

Corollary 2.34. Change of base preserves bifibrations, i.e. if $U\colon \mathcal{E} \to \mathcal{B}$ is a bifibration and $F\colon \mathcal{B}' \to \mathcal{B}$ a functor, then $F^*(U)$ is a bifibration.

Composition of fibrations is a fibration:

Lemma 2.35. If $U\colon \mathcal{E} \to \mathcal{B}$ and $U'\colon \mathcal{E}' \to \mathcal{E}$ are fibrations, then so is the composite $U \circ U'\colon \mathcal{E}' \to \mathcal{B}$.

Proof. The cartesian lifting is obtained by iterating the lifting along the two fibrations. In detail, consider a morphism $u\colon I \to J$ in $\mathcal{B}$ and let Y be an object in $\mathcal{E}'$ over J respect to $U \circ U'$, i.e. $(U \circ U')Y = J$. We first compute the cartesian morphism $u^{\S}_{UY}$ with codomain UY with respect to the fibration U. Then we iterate and compute the cartesian lifting $(u^{\S}_{UY})^{\S}_{Y}$ with respect to the fibration U' which corresponds to the cartesian lifting of u with codomain Y with respect to the fibration $U \circ U'$. $\square$

It is not difficult to see that the dual holds for opfibrations

Lemma 2.36. If $U\colon \mathcal{E} \to \mathcal{B}$ and $U'\colon \mathcal{E}' \to \mathcal{E}$ are opfibrations, then so is the composite $U \circ U'\colon \mathcal{E}' \to \mathcal{B}$.

As a direct consequence of Lemmas 2.35 and 2.36 it follows:

Corollary 2.37. If $U\colon \mathcal{E} \to \mathcal{B}$ and $U'\colon \mathcal{E}' \to \mathcal{E}$ are bifibrations, then so is the composite $U \circ U'\colon \mathcal{E}' \to \mathcal{B}$.

There is a trivial way to obtain a bifibration from every functor $F\colon \mathcal{E} \to \mathcal{B}$. This consists of the construction of the discrete functor $|F|\colon |\mathcal{E}| \to |\mathcal{B}|$, where we write $|\mathcal{E}|$ for the discrete category of $\mathcal{E}$. The discrete functor $|F|$ acts like F on the objects and trivially on the morphisms.

Lemma 2.38. For every functor $F\colon \mathcal{E} \to \mathcal{B}$, the discrete functor $|F|\colon |\mathcal{E}| \to |\mathcal{B}|$ is a bifibration.

Proof. This is obvious since the only morphisms in the base and in the total category are identities and the identities in the total category over the identities in the base category are the (op)cartesian lifting satisfying, in this case, a vacuous universal property. □

Lemma 2.39. Let $U_1 : \mathcal{E}_1 \to \mathcal{B}_1, \ldots, U_n : \mathcal{E}_n \to \mathcal{B}_n$ be (op/bi)fibrations. Then $U_1 \times \ldots \times U_n$ is a (op/bi)fibration.

Proof. The cartesian and opcartesian morphisms are obtained componentwise. □

In particular, from the above Lemma follows that if $U : \mathcal{E} \to \mathcal{B}$ is a (op/bi)fibration, so is the n-fold product U^n.

2.6 Fibred category theory

We now consider the fibred structure. We start with the notion of fibred functors.

Definition 2.40. Let $U : \mathcal{E} \to \mathcal{B}$ and $U' : \mathcal{E}' \to \mathcal{B}'$ be two fibrations. A **fibred functor** from U to U' consists of a pair of functors $G : \mathcal{B} \to \mathcal{B}'$ and $F : \mathcal{E} \to \mathcal{E}'$ such that $U' \circ F = G \circ U$ and F maps cartesian morphisms to cartesian morphisms.

A fibred functor will be denoted $(F, G) : U \to U'$, and often represented as a commuting square

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U'} \\ \mathcal{B} & \xrightarrow[G]{} & \mathcal{B}'. \end{array}$$

Example 2.41. For every fibration $U : \mathcal{E} \to \mathcal{B}$ and every functor $F : \mathcal{B}' \to \mathcal{B}$, the pair of functors $(U^*(F), F)$ obtained via change of base

$$\begin{array}{ccc} \mathcal{B}' {}_F\!\times_U \mathcal{B} & \xrightarrow{U^*(F)} & \mathcal{E} \\ {\scriptstyle F^*(U)}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{B}' & \xrightarrow[F]{} & \mathcal{B} \end{array}$$

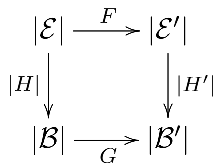
defines a fibred functor $(U^*(F), F) : F^*(U) \to U$.

Every fibred functor induces functors between fibres

Lemma 2.42. Let $U\colon \mathcal{E} \to \mathcal{B}$ and $U'\colon \mathcal{E}' \to \mathcal{B}'$ be two fibrations and let I be an object in $\mathcal{B}$. Every fibred functor $(F, G)\colon U \to U'$ induces a functor $F_I\colon \mathcal{E}_I \to \mathcal{E}'_{GI}$ between the fibres.

The functor F_I is the restriction of F to the fibre — this is well defined, since $U' \circ F = G \circ U$.

If $H\colon \mathcal{E} \to \mathcal{B}$ and $H'\colon \mathcal{E}' \to \mathcal{B}'$ are two functors, then every commuting diagram

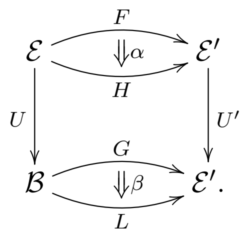


defines a fibred functor $(F, G)\colon |H| \to |H'|$ because the preservation of cartesian morphisms is trivially satisfied. When the preservation of cartesian morphisms is vacuous, we can also call such functors **lifted functors**.

The next step is to define fibred natural transformations.

Definition 2.43. Given two parallel fibred functors $(F, G), (H, L)\colon U \to U'$, where $U\colon \mathcal{E} \to \mathcal{B}$ and $U'\colon \mathcal{E}' \to \mathcal{B}'$, a **fibred natural transformation** consists of a pair of natural transformations $\alpha\colon F \Rightarrow H$ and $\beta\colon G \Rightarrow L$ such that every component α_X is over the component β_{UX}.

Given two parallel fibred functors $(F, G), (H, L)\colon U \to U'$, we write the fibred natural transformation as $(\alpha, \beta)\colon (F, G) \Rightarrow (H, L)$, and the data is represented in the diagram



The composite of fibred functors is a fibred functor and it is easy to see that the identity functor is a fibred functor. In this way fibrations and fibred functors form a category.

Definition 2.44. The category **Fib** has

Fib	**objects**	small fibrations $U \colon \mathcal{E} \to \mathcal{B}$.
	morphisms	fibred functors $(F, G) \colon U \to U'$

$$\begin{array}{ccc} \mathcal{E}' & \xrightarrow{F} & \mathcal{E} \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U'} \\ \mathcal{B}' & \xrightarrow[G]{} & \mathcal{B}. \end{array}$$

Example 2.45. We call $\mathsf{fib} \colon \mathbf{Fib} \to \mathbf{Cat}$ the functor which forgets the fibrational structure. In particular it sends a fibration $U \colon \mathcal{E} \to \mathcal{B}$ to the base category $\mathcal{B}$ and a fibred functor $(F, G) \colon U \to U'$ to G.

The functor fib is a large fibration, where the reindexing of $U \colon \mathcal{E} \to \mathcal{B}$ along $G \colon \mathcal{B}' \to \mathcal{B}$ consists of the fibration $G^*(U)$ obtained via change of base

$$\begin{array}{ccc} \mathcal{B}' {}_F{\times}_U \mathcal{E} & \xrightarrow{U^*(G)} & \mathcal{E} \\ {\scriptstyle G^*(U)}\downarrow & \lrcorner & \downarrow{\scriptstyle U} \\ \mathcal{B}' & \xrightarrow[G]{} & \mathcal{B} \end{array}$$

and the cartesian lifting of G is $(U^*(G), G)$.

We have now all the ingredients to introduce the notion of fibred adjunction.

Definition 2.46. Given two fibred functors $(F, G) \colon U \to U'$ and $(H, L) \colon U' \to U$, we say that (F, G) is **fibred left adjoint** of (H, L) if F is left adjoint to H, G is left adjoint to L and the unit and the counit of the adjunction $F \dashv H$ are over the unit and the counit of the adjunction $G \dashv L$.

A particular class of fibred functors is given by fibred functors of the form $(F, \mathsf{Id}) \colon U \to U'$, where $U \colon \mathcal{E} \to \mathcal{B}$ and $U' \colon \mathcal{E}' \to \mathcal{B}$ are over the same base $\mathcal{B}$. In this case a fibred functor from U to U' is simply a functor $F \colon \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes and F preserves cartesian morphisms

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{F} & \mathcal{E}' \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle U'} & \\ & \mathcal{B}. & \end{array}$$

We say that the functor F is fibred above $\mathcal{B}$, or simply fibred if $\mathcal{B}$ is clear from the context.

This construction determines the category $\mathbf{Fib}_{\mathcal{B}}$ of fibrations with base category $\mathcal{B}$ and fibred functors above the identity on $\mathcal{B}$. In fact $\mathbf{Fib}_{\mathcal{B}}$ is the fibre above $\mathcal{B}$ with respect to the fibration $\mathsf{fib}\colon \mathbf{Fib} \to \mathbf{Cat}$ described in Example 2.45. Notice then that every fibration $U\colon \mathcal{E} \to \mathcal{B}$ is also a fibred functor

$$\begin{array}{ccc} \mathcal{E} & \xrightarrow{U} & \mathcal{B} \\ & {\scriptstyle U}\searrow \quad \swarrow{\scriptstyle \mathsf{Id}_{\mathcal{B}}} & \\ & \mathcal{B} & \end{array}$$

and this makes the identity on $\mathcal{B}$ the terminal object in $\mathbf{Fib}_{\mathcal{B}}$.

If we restrict ourselves to the fibred functors in $\mathbf{Fib}_{\mathcal{B}}$, the notions of fibred natural transformations and fibred adjunctions simplify. Thus a fibred natural transformation over $\mathcal{B}$ is a natural transformation between fibred functors above $\mathcal{B}$ whose components are vertical. A fibred adjunction above $\mathcal{B}$ is an adjunction between fibred functors above $\mathcal{B}$ such that the unit and the counit are vertical.

2.7 Fibrewise structure

In this section we study the structure inside the fibres. In ordinary categories it is possible to define notions like product or coproduct via, for example, universal properties. In the case of fibrations and opfibrations we want that every fibre has some distinguished property, and we want it to behave well with respect to the fibrational structure, e.g. that reindexing and opreindexing preserve it.

Definition 2.47. Let ♠ be some categorical property or structure (for example some limit or colimit). We say that a fibration has **fibred ♠**'s if all the fibres have ♠'s and reindexing functors preserve ♠'s.

It is useful to distinguish between two strengths for the structure ♠. Such structure can be specified. An example is the specified binary product which assigns to every pair of objects X, Y a specified product cone $(X \times Y, \pi_1, \pi_2)$. In this case, for definition 2.47, we ask that the reindexing functor preserves the specified structure on the nose. On the other side, the weaker version is given when the structures are simply preserved up to isomorphism. For example, in the case of products, a product cone is mapped to a product cone. When we work with split fibrations we will implicitly use the stronger version, while the weaker version is the natural one in a non-split context.

Sometimes, when it is clear from the context that we are speaking about fibred properties, we drop the adjective fibred.

Definition 2.48. Let $(F, G)\colon U \to U'$ be a fibred functor. We say that (F, G) **preserves** fibred ♠'s if for each object I in $\mathcal{B}$ the functor $F_I\colon \mathcal{E}_I \to \mathcal{E}'_{GI}$ preserves ♠'s.

A notion which we will often use is the following

Definition 2.49. A **fibred CC fibration** or **fibred cartesian closed fibration** is a fibration with fibred finite products and fibred exponential objects.

Change of base preserves fibred structures.

Lemma 2.50. Let ♠ be as in Definition 2.47. If a fibration $U\colon \mathcal{E} \to \mathcal{B}$ has fibred ♠'s, then so has the fibration $F^*(U)$ obtained via change of base along a functor $F\colon \mathcal{B}' \to \mathcal{B}$. Moreover the associated morphism of fibrations $F^*(U) \to U$ preserves ♠'s.

Proof. The fibre of $F^*(U)$ over I is isomorphic to the fibre of U over FI. If one of them has ♠'s then so does the other one. They are preserved under reindexing, since the reindexing functors of $F^*(U)$ are obtained from those of U. □

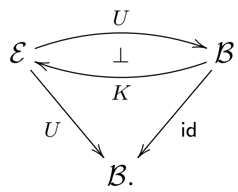
2.8 The truth functor

The truth functor is defined as follows

Definition 2.51. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration. If the fibred functor (U, Id) has a fibred right adjoint (K, Id), then we call $K\colon \mathcal{B} \to \mathcal{E}$ the **truth functor**.

The truth functor can be characterised by fibred terminal objects as shown by the following result (see [Jac99]).

Lemma 2.52. A fibration $U\colon \mathcal{E} \to \mathcal{B}$ has fibred terminal objects if and only if the fibred functor $(U, \mathsf{Id})\colon U \to \mathsf{Id}$ has a fibred right adjoint $(K, \mathsf{Id})\colon \mathsf{Id} \to U$



This functor will play an important role in this thesis. It is evident from the commuting diagram that $U \circ K = \mathsf{Id}$.

Given a fibration $U\colon \mathcal{E} \to \mathcal{B}$, when K exists, it sends every object I in the base to the fibred terminal object in $\mathcal{E}_I$. Given a morphism $u\colon I \to J$ in the base category, there is an isomorphism $u^*KJ \xrightarrow{\cong} KI$ since fibred terminal objects are preserved by reindexing functors. We then obtain $Ku\colon KI \to KJ$ as the composite $KI \xrightarrow{\cong} u^*KJ \xrightarrow{u^\S_{KJ}} KJ$.

Lemma 2.53. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration with truth functor $K\colon \mathcal{B} \to \mathcal{E}$. The truth functor is full and faithful.

Proof. The vertical morphisms in the fibration id are only the identities. It follows that the components of the counit of the adjunction $U \dashv K$ are identities. Using Lemma 2.1 we can conclude that K is full and faithful. $\square$

The following examples justify the terminology.

Example 2.54. Given a category $\mathcal{C}$ with pullbacks, the codomain fibration, see Example 2.18, has a truth functor. It sends an object I in $\mathcal{C}$ to the identity morphism id_I which represents the constantly true predicate.

Example 2.55. The truth functor for the identity fibration $\mathsf{Id}\colon \mathcal{B} \to \mathcal{B}$ is given by the identity functor again. In fact the fibre over every object X in $\mathcal{B}$ consists of in the object X itself and the identity for X. The object X is trivially terminal and reindexing trivially preserves terminal objects in the fibres.

Example 2.56. In the subset fibration, see Example 2.20, there is a truth functor and it sends a set I to $I \subseteq I$. In general every subobject fibration, see Example 2.21, $\mathsf{sub}\colon \mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$ has a truth functor which sends an object I in $\mathcal{C}$ to the class of monomorphisms of the identity monomorphism id_I which represent the constantly true predicates.

Example 2.57. In the families of sets fibration, see Example 2.22, the truth functor K sends a set I to $(I, i \mapsto 1)$, where 1 is the set with only one element.

Example 2.58. In the family fibration over $\mathcal{C}$, see Example 2.23, if $\mathcal{C}$ has terminal object 1, the truth functor exists and it sends a set I to $KI = (I, i \mapsto 1)$.

Example 2.59. In the fibration of relations over sets, see Example 2.24, the truth functor sends (A, B) to $(A, B, A \times B \subseteq A \times B)$, e.g. the truth relation where every element is related with every other element.

Example 2.60. The fibration fib, see example 2.45, has truth functor which sends a category $\mathcal{B}$ to the identity functor $\mathsf{Id}_{\mathcal{B}}$.

2.9 Structure between fibres

We now focus on the structure between fibres. We start by defining the weakening functor.

Definition 2.61. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration where $\mathcal{B}$ has cartesian products. For every projection $\pi_{I,J}\colon I \times J \to I$, the reindexing functors $\pi^*_{I,J}\colon \mathcal{E}_I \to \mathcal{E}_{I\times J}$ are called the **weakening functors**.

Definition 2.62. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration, I an object in $\mathcal{B}$, and $\mathcal{B}$ has cartesian products. We say that U has **simple I-products** if

- for every object J in $\mathcal{B}$, every weakening functor $\pi^*_{J,I}\colon \mathcal{E}_J \to \mathcal{E}_{J\times I}$ has a right adjoint $\prod_{(J,I)}$;

- the adjunction satisfies the Beck-Chevalley condition: for every morphism $u\colon K \to J$ in $\mathcal{B}$ and I in $\mathcal{B}$, the canonical natural transformation $u^* \circ \prod_{(J,I)} \to \prod_{(K,I)} \circ (u \times \mathsf{id})^*$ is an isomorphism.

$$\begin{array}{ccc}
\mathcal{E}_J & \xrightarrow{\quad u^* \quad} & \mathcal{E}_K \\
\pi^*_{J,I} \downarrow \uparrow \textstyle\prod_{(J,I)} & & \pi^*_{K,I} \downarrow \uparrow \textstyle\prod_{(K,I)} \\
\mathcal{E}_{J\times I} & \xrightarrow[\quad (u\times \mathsf{id})^* \quad]{} & \mathcal{E}_{K\times I}
\end{array}$$

We say that U has **simple products** if it has simple I-products for every object I in $\mathcal{B}$.

Chapter 3

Fibrations of relations

Relations play a central role in this thesis. We saw in Example 2.24 that relations over sets are fibred over **Set** × **Set**. In this chapter we describe a general notion of relations arising via change of base along the product functor. We call the fibrations obtained in this way fibrations of relations. The fibrational structure lets us define some fundamental relations: truth relations (every element is related to every other element), equality relations (every element is related only with itself) and graph relations (the ones arising from the graph of a function). We prove properties of such relations which justify their names. Finally we show a link between relations thought of as subobjects and the fibrations of relations. This is obtained by using the comprehension functor and motivates the name of the fibrations of relations.

3.1 Relations fibrationally

In Example 2.32 we saw that the fibration $\mathsf{rel}\colon \mathbf{Rel} \to \mathbf{Set} \times \mathbf{Set}$ arises via change of base along the functor $- \times -$. More generally, a binary relation can be thought as a triple (A, B, R), where R represents a relation between A and B. This point of view underlies the following definition.

Definition 3.1 (see [Jac99])**.** Given a fibration $U\colon \mathcal{E} \to \mathcal{B}$, where $\mathcal{B}$ has binary products, the fibration $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ obtained via change of base along the product functor $- \times -$

$$\begin{array}{ccc} \mathbf{Rel}(\mathcal{E}) & \xrightarrow{J} & \mathcal{E} \\ {\scriptstyle \mathsf{rel}(U)}\big\downarrow & \lrcorner & \big\downarrow{\scriptstyle U} \\ \mathcal{B} \times \mathcal{B} & \xrightarrow[- \times -]{} & \mathcal{B} \end{array} \tag{3.2}$$

is called **fibration of relations** obtained from U.

The objects in the category $\mathbf{Rel}(\mathcal{E})$ consist of triples (A, B, R) where A and B are objects in $\mathcal{B}$, while R is an object in $\mathcal{E}$ such that $UR = A \times B$, i.e. R is over $A \times B$. Note that there are two closely related fibrations $\mathsf{rel}(U)$ and U involved: objects (A, B, R) in the total category of $\mathsf{rel}(U)$ over (A, B) and objects R in the total category of U over $A \times B$. For this reason we will sometimes say that an object R is over A and B. We can abstractly think of the objects of the total category as predicates over the objects in the base category, and binary relations are predicates over a product. Clearly this can be generalised to n-ary relations considering n-ary products.

Example 3.3. If we start from a subobject fibration $\mathsf{sub}\colon \mathrm{Sub}(\mathcal{C}) \to \mathcal{C}$, where $\mathcal{C}$ has binary products and pullbacks, the fibration of relations $\mathsf{rel}(\mathsf{sub})\colon \mathbf{Rel}(\mathrm{Sub}(\mathcal{C})) \to \mathcal{C} \times \mathcal{C}$ consists of the category $\mathbf{Rel}(\mathrm{Sub}(\mathcal{C}))$ whose objects are triples $(A, B, [m\colon R \rightarrowtail A \times B])$ and the functor $\mathsf{rel}(\mathsf{sub})$ which sends $(A, B, [m])$ to (A, B). This represents the intuitive notion of a binary relation R over A and B as a subset of the product $A \times B$.

Example 3.4. Consider the codomain fibration $\mathsf{cod}\colon \mathcal{B}^{\to} \to \mathcal{B}$, where $\mathcal{B}$ has pullbacks and binary products. The fibration of relations $\mathsf{rel}(\mathsf{cod})\colon \mathbf{Rel}(\mathcal{B}^{\to}) \to \mathcal{B} \times \mathcal{B}$ consists of the category $\mathbf{Rel}(\mathcal{B}^{\to})$ whose objects are $(A, B, f\colon R \to A \times B)$ and the functor $\mathsf{rel}(\mathsf{cod})$ sends (A, B, f) to (A, B).

Example 3.5. Consider the families of sets fibration $\mathsf{fam}\colon \mathbf{Fam}(\mathbf{Set}) \to \mathbf{Set}$. The fibration of relations $\mathsf{rel}(\mathsf{fam})\colon \mathbf{Rel}(\mathbf{Fam}(\mathbf{Set})) \to \mathbf{Set} \times \mathbf{Set}$ obtained from it consists of the category $\mathbf{Rel}(\mathbf{Fam}(\mathbf{Set}))$ whose objects are triples (A, B, P), where $P\colon A \times B \to \mathbf{Set}$, and the functor $\mathsf{rel}(\mathsf{fam})$ sends (A, B, P) to (A, B).

The relations in Example 3.3 are **proof-irrelevant relations**, i.e. they are relations which only say whether there exists a proof relating two elements. In fact, in the case of $\mathbf{Rel}$, given a relation $(A, B, R \subseteq A \times B)$ the only thing we can say is that $a \in A$ and $b \in B$ are related in R or not. Note that the proof-irrelevant notion is strictly related to the fact that the associated fibration is faithful. In order to see this consider the Example 3.3. Given a morphism $(f, g)\colon (A, B) \to (A', B')$ in the base category $\mathcal{B} \times \mathcal{B}$, and given two relations

$(A, B, [m: R \rightarrowtail A \times B])$ and $(A', B', [m': R' \rightarrowtail A' \times B'])$ above, respectively, (A, B) and (A', B'), there is at most one morphism $(f, g, \alpha): (A, B, [m]) \to (A', B', [m'])$ above (f, g). In fact morphisms preserve relations by sending related elements to related elements and if there is no choice on the proofs relating two elements, we have faithfulness.

The relations in the Examples 3.4 and 3.5 are **proof-relevant relations**, i.e. they admit possibly more than one proof relating two elements. In fact, consider Example 3.4 with the category $\mathcal{C} = \mathbf{Set}$ and a relation $(A, B, f: R \to A \times B)$ in $\mathbf{Rel}(\mathbf{Set}^{\to})$. Two elements $a \in A$ and $b \in B$ are related if there is an element $r \in R$ such that $f(r) = (a, b)$, and in general there could be more than one such element. We can write the set of elements relating a and b as $R(a, b) = f^{-1}(a, b) = \{r \in R \,|\, fr = (a, b)\}$. More generally, if the morphism $f: R \to A \times B$ is not a monomorphism, it can be thought of as a proof-relevant relation. Similarly, in Example 3.5, consider a relation $(A, B, (A \times B, P))$ in $\mathsf{rel}(\mathbf{Fam}(\mathbf{Set}))$. Two elements $a \in A$ and $b \in B$ are related if there is an element in $P(a, b)$. More generally, the set $P(a, b)$ is the set of proofs relating a and b. Note that these possibilities of multiple proofs mean that the previous two fibrations are not faithful.

3.2 The equality functor

For each set A it is possible to define the **equality relation** over it by $\mathsf{Eq}\, A = \{(a, a) \mid a \in A\} \cong A$. Formally $\mathsf{Eq}\, A = (A, A, \{(a, a) \mid a \in A\} \subseteq A \times A)$ or, thinking of subsets as equivalence classes of monomorphisms, $\mathsf{Eq}\, A = (A, A, [\delta: A \to A \times A])$, where δ is the diagonal morphism $\langle \mathsf{id}_A, \mathsf{id}_A \rangle: A \to A \times A$. This map, in the case of fibration of relations over sets (Example 2.24), extends to a functor $\mathsf{Eq}: \mathbf{Set} \to \mathbf{Rel}$ by sending every morphism $f: A \to B$ to $\mathsf{Eq}(f) \coloneqq (f, f, f): \mathsf{Eq}(A) \to \mathsf{Eq}(B)$.

The construction of equality functors is standard in any bifibration with the necessary infrastructure [Jac99]. We first describe the process for the bifibration in Example 2.24 and then generalise it. As we noted in Example 2.32, the bifibration rel arises via change of base from the subset bifibration sub. We first consider the truth functor K with respect to sub, which sends a set A to $A \subseteq A$. We then opreindex KA along the diagonal morphism $\delta_A: A \to A \times A$. In this way we obtain $\Sigma_{\delta_A} KA = \{(a, a) \mid a \in A\} \subseteq A \times A$ which is exactly the equality relation. This leads us to the general definition of the equality functor for a bifibration of relations $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$.

Definition 3.6. Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with fibred terminal objects. If $\mathcal{B}$ has products, the **equality functor** $\mathsf{Eq} : \mathcal{B} \to \mathbf{Rel}(\mathcal{E})$ for $\mathsf{rel}(U)$ is the functor induced by the map $X \mapsto \Sigma_{\delta_X} KX$, where δ_X is the diagonal morphism $\delta_X : X \to X \times X$.

The action of Eq on the morphisms is given by universal property of opcartesian morphisms as shown by the following diagram:

$$\begin{array}{ccc} KX & \xrightarrow{(\delta_X)_\S} & \mathsf{Eq}X \\ {\scriptstyle Kf}\downarrow & & \downarrow{\scriptstyle \mathsf{Eq}(f)} \\ KY & \xrightarrow[(\delta_Y)_\S]{} & \mathsf{Eq}Y \end{array} \quad \text{over} \quad \begin{array}{ccc} KX & \xrightarrow{\delta_X} & X \times X \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f\times f} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y \end{array}$$

For this definition, it is enough to ask for opreindexing along diagonals δ_X only (see e.g. Birkedal and Møgelberg [BM05]). Graph relations (Section 3.3), will require the use of all the opfibrational structure to opreindex along arbitrary morphisms.

Lemma 3.7. The equality functor $\mathsf{Eq}\colon \mathcal{B} \to \mathbf{Rel}(\mathcal{E})$ is faithful.

Proof. Given two morphisms $f, g\colon X \to Y$ in $\mathcal{B}$ with $\mathsf{Eq}(f) = \mathsf{Eq}(g)$, we have $(f, f) = \mathsf{rel}(U)(\mathsf{Eq}(f)) = \mathsf{rel}(U)(\mathsf{Eq}(g)) = (g, g)$ and then $f = g$. ☐

In general the truth functor is not full as shown by the following counterexample.

Non-example 3.8. In the identity fibration $\mathsf{Id}\colon \mathcal{B} \to \mathcal{B}$, the equality functor sends an object X to $X \times X$ and a morphism $f\colon X \to Y$ to $f \times f\colon X \times X \to Y \times Y$. The functor is not full, since morphisms from $X \times X$ to $Y \times Y$ are not always of the form $f \times f$.

3.3 The graph functor

Every morphism $f : A \to B$ in **Set** defines a graph relation $\langle f \rangle = \{(a, b) \mid fa = b\} \subseteq A \times B$. These relations can be characterised by using the fibrational structure. In fact consider the fibration $\mathsf{rel}\colon \mathbf{Rel} \to \mathbf{Set} \times \mathbf{Set}$ and a morphism $(f, \mathsf{id})\colon (A, B) \to (B, B)$ in $\mathbf{Set} \times \mathbf{Set}$. The reindexing of $\mathsf{Eq}(B)$ along (f, id) results in $(A, B, (f, \mathsf{id})^*\mathsf{Eq}(B))$ where $(f, \mathsf{id})^*\mathsf{Eq}(B) = \{(a, b) \mid fa = b\}$ which is exactly the graph relation resulting from f. Clearly this operation can be reproduced in any fibration of relations with an equality functor.

Definition 3.9. Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with fibred terminal objects and products in $\mathcal{B}$. The **graph** of $f : X \to Y$ in $\mathcal{B}$ is $\langle f \rangle = (f, \mathsf{id}_Y)^*(\mathsf{Eq}\, Y)$ in $\mathbf{Rel}(\mathcal{E})$.

It is not difficult to show that the graph of the identity morphism is the equality relation, as one would expect. In fact, since reindexing preserves identities, $\langle \mathsf{id}_X \rangle = (\mathsf{id}_X, \mathsf{id}_X)^*(\mathsf{Eq}\, X) = \mathsf{Eq}\ X$ for every object X of $\mathcal{B}$.

Recall that rel is also an opfibration. It is possible to characterise the graph relations in $\mathbf{Rel}$ using the opcartesian structure as well. Let $f\colon A \to B$ be a morphism in $\mathbf{Set}$, and consider the morphism $(\mathsf{id}, f)\colon (A, A) \to (A, B)$ in $\mathbf{Set} \times \mathbf{Set}$. The opreindexing of $\mathsf{Eq}(A)$ along (id, f) is $(A, B, \Sigma_{(\mathsf{id},f)}\mathsf{Eq}(A))$ where $\Sigma_{(\mathsf{id},f)}\mathsf{Eq}(A) = \{(a, b) \mid fa = b\}$ which, again, is the graph relation associated to f. In the general case of bifibrations, the two constructions are equivalent if the fibration satisfies the Beck-Chevalley condition.

Lemma 3.10 (Lawvere [Law70])**.** Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with fibred terminal objects and products in $\mathcal{B}$. If U satisfies the Beck-Chevalley condition, then the graph of $f : X \to Y$ can also be described by $\langle f \rangle = \Sigma_{(\mathsf{id}_X, f)}(\mathsf{Eq}\, X)$.

Proof. For every morphism $f\colon X \to Y$ in $\mathcal{B}$ the following diagram is a pullback

$$\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
{\scriptstyle \delta_X}\downarrow & & \\
X \times X & & \downarrow{\scriptstyle \delta_Y} \\
{\scriptstyle \mathsf{id}\times f}\downarrow & & \\
X \times Y & \xrightarrow[f\times\mathsf{id}]{} & Y \times Y.
\end{array}$$

In fact consider an object Z and two morphisms $h\colon Z \to Y$ and $t\colon Z \to X \times Y$ such that $\delta_Y \circ h = (f \times \mathsf{id}) \circ t$. By universal property of the product $t = \langle t_1, t_2 \rangle$ and we have $\langle f \circ t_1, t_2 \rangle = (f \times \mathsf{id}) \circ t = \delta_Y \circ h = \langle h, h \rangle$. We then have that $t_2 = h = f \circ t_1$ and the unique morphism satisfying the pullback condition is $t_1 \colon Z \to X$. In fact if there was another $t' \colon Z \to X$ satisfying pullback condition, we would have $(\mathsf{id} \times f) \circ \delta_X \circ t' = \langle t', f \circ t' \rangle = \langle t_1, t_2 \rangle$ and then $t' = t_1$.

By using the Beck-Chevalley condition for the above pullback we have

$$(f \times \mathsf{id})^* \circ \Sigma_{\delta_Y} \cong \Sigma_{\mathsf{id} \times f} \circ \Sigma_{\delta_X} \circ f^*.$$

Using this equivalence, and that fibred terminal objects are preserved by reindexing, we have:

$$\begin{aligned} (f \times \mathsf{id})^*(\mathsf{Eq}Y) &= ((f \times \mathsf{id})^* \circ \Sigma_{\delta_Y})(KY) \\ &\cong (\Sigma_{\mathsf{id} \times f} \circ \Sigma_{\delta_X} \circ f^*)(KY) \\ &\cong \Sigma_{\mathsf{id} \times f}(\Sigma_{\delta_X}(KX)) \\ &= \Sigma_{\mathsf{id} \times f}(\mathsf{Eq}X) \end{aligned}$$

which proves that the two definitions of the graph relations are equivalent. □

Being able to describe graph relations in terms of either reindexing or opreindexing lets us use both of their universal properties when proving theorems about them.

The **graph functor** for $\mathbf{Rel}(U) : \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ is the functor $\langle \_ \rangle : \mathcal{B}^{\to} \to \mathbf{Rel}(\mathcal{E})$ mapping $f : X \to Y$ in $\mathcal{B}$ to $\langle f \rangle$ in $\mathbf{Rel}(\mathcal{E})$. To see how $\langle \_ \rangle$ acts on morphisms, recall that if $f : X \to Y$ and $f' : X' \to Y'$ are objects of $\mathcal{B}^{\to}$, then a morphism from f to f' is a pair of morphisms $g : X \to X'$ and $h : Y \to Y'$ such that $f' \circ g = h \circ f$

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ Y & \xrightarrow[h]{} & Y'. \end{array}$$

The universal property of cartesian morphisms in $\mathbf{Rel}(U)$ guarantees the existence of a unique morphism $\langle g,h\rangle : \langle f\rangle \to \langle f'\rangle$ over (g,h) such that the following diagram commutes:

$$\begin{array}{ccc} \langle f\rangle & \xrightarrow{(f,\mathsf{id})^{\S}} & \mathsf{Eq}\,Y \\ {\scriptstyle \exists!\langle g,h\rangle}\downarrow & & \downarrow{\scriptstyle \mathsf{Eq}\,h} \\ \langle f'\rangle & \xrightarrow[(f',\mathsf{id})^{\S}]{} & \mathsf{Eq}\,Y' \end{array}$$

over

$$\begin{array}{ccc} (X,Y) & \xrightarrow{(f,\mathsf{id})} & (Y,Y) \\ {\scriptstyle (g,h)}\downarrow & & \downarrow{\scriptstyle (h,h)} \\ (X',Y') & \xrightarrow[(f',\mathsf{id})]{} & (Y',Y'). \end{array}$$

It is possible to define a similar action on the morphisms also for the equivalent definition of for the graph relations based on the opcartesian morphisms. Naturally in this case we use the opcartesian property

$$\begin{array}{ccc} \mathsf{Eq}\,X & \xrightarrow{(\mathsf{id},f)_{\S}} & \langle f\rangle \\ {\scriptstyle \mathsf{Eq}\,g}\downarrow & & \downarrow{\scriptstyle \exists!\langle g,h\rangle} \\ \mathsf{Eq}\,Y' & \xrightarrow[(\mathsf{id},f')_{\S}]{} & \langle f'\rangle \end{array}$$

over

$$\begin{array}{ccc} (X,X) & \xrightarrow{(\mathsf{id},f)} & (X,Y) \\ {\scriptstyle (g,g)}\downarrow & & \downarrow{\scriptstyle (g,h)} \\ (X',X') & \xrightarrow[(\mathsf{id},f')]{} & (X',Y'). \end{array}$$

The two actions on the morphisms are equivalent. In fact consider the diagram

$$\begin{array}{ccccc} \mathsf{Eq}\,X & \xrightarrow{(\mathsf{id},f)_{\S}} & \langle f\rangle & \xrightarrow{(f,\mathsf{id})^{\S}} & \mathsf{Eq}\,Y \\ {\scriptstyle \mathsf{Eq}\,g}\downarrow & & \downarrow{\scriptstyle \exists!\langle g,h\rangle} & & \downarrow{\scriptstyle \mathsf{Eq}\,h} \\ \mathsf{Eq}\,Y' & \xrightarrow[(\mathsf{id},f')_{\S}]{} & \langle f'\rangle & \xrightarrow[(f',\mathsf{id})^{\S}]{} & \mathsf{Eq}\,Y', \end{array}$$

where the outer diagram commutes. If $\langle g,h\rangle$ is found using the cartesian property of $(f',\mathsf{id})^{\S}$, using again the cartesian property of $(f',\mathsf{id})^{\S}$ we derive that $\langle g,h\rangle \circ (\mathsf{id},f)_{\S} = (\mathsf{id},f')_{\S} \circ \mathsf{Eq}\,g$, and then $\langle g,h\rangle$ is the unique morphism which can be found using the opcartesian property of $(\mathsf{id},f)_{\S}$ as well. The dual argument holds applied to the opcartesian morphism $(\mathsf{id},f)_{\S}$.

Lemma 3.11. If the underlying bifibration satisfies the Beck-Chevalley condition, then $\langle\_\rangle : \mathcal{B}^{\rightarrow} \rightarrow \mathbf{Rel}(\mathcal{E})$ is full and faithful if and only if $\mathsf{Eq} : \mathcal{B} \rightarrow \mathbf{Rel}(\mathcal{E})$ is.

Proof. If the graph functor is full and faithful, the equality functor is full and faithful because it is a particular instantiation of the graph functor.

Let $f : X \rightarrow Y$ and $f' : X' \rightarrow Y'$ be two objects in $\mathcal{B}^{\rightarrow}$. Given a morphism $(g, h) : f \rightarrow f'$ in $\mathcal{B}^{\rightarrow}$, the morphism $\langle g, h\rangle : \langle f\rangle \rightarrow \langle f'\rangle$ is defined via the universal property of $\langle f\rangle = (f, \mathsf{id})^{*}\mathsf{Eq}B$. This ensures that $\langle g, h\rangle$ is over (g, h), and thus that the graph functor is faithful. For fullness, consider $\alpha : \langle f\rangle \rightarrow \langle f'\rangle$ over (α_1, α_2). The opcartesian definition of $\langle f\rangle$ – given by Lemma 3.10 since the Beck-Chevalley condition is satisfied by assumption – and the cartesian definition of $\langle f'\rangle$ give a map $\beta\colon \mathsf{Eq}X \rightarrow \mathsf{Eq}Y'$ as shown in the following diagram

$$\mathsf{Eq}X \xrightarrow{(\mathsf{id},f)_{\S}} \langle f\rangle \xrightarrow{\alpha} \langle f'\rangle \xrightarrow{(f',\mathsf{id})^{\S}} \mathsf{Eq}Y'.$$

By fullness of Eq, we get a map $t\colon X \rightarrow Y'$ such that $\mathsf{Eq}\, t = \beta$ and thus $(t, t) = U(\mathsf{Eq}\, t) = U(\beta) = (f' \circ \alpha_1, \alpha_2 \circ f)$ from which we can derive that $U\alpha\colon f \rightarrow f'$ in $\mathcal{B}^{\rightarrow}$ since $f' \circ \alpha_1 = t = \alpha_2 \circ f$. The cartesian morphism over (g, id) can then be used to show that α satisfies the universal property defining $\langle U\alpha\rangle$ and thus $\alpha = \langle U\alpha\rangle$ proving fullness. □

The proof uses the opfibrational characterisation of the graph functor from Lemma 3.10. The main tool we will use for deriving consequences of parametricity in Chapter 5 is the Graph Lemma, which relates the graph of the action of a functor on a morphism with the relational action of the functor on the graph of the morphism. Note that in the statement of the lemma we restrict to functors which preserve equality. We will see that this is a central feature in Reynolds' relational model which corresponds to the Identity Extension Lemma.

Theorem 3.12 (Graph Lemma)**.** Let $U\colon \mathcal{E} \rightarrow \mathcal{B}$ be a bifibration satisfying the Beck-Chevalley condition and let $F_1\colon \mathbf{Rel}(\mathcal{E}) \rightarrow \mathbf{Rel}(\mathcal{E})$ and $F_0\colon \mathcal{B} \rightarrow \mathcal{B}$ be two functors such that $\mathbf{Rel}(U) \circ F_1 = (F_0 \times F_0) \circ \mathbf{Rel}(U)$ and $F_1 \circ \mathsf{Eq} = \mathsf{Eq} \circ F_0$. For every morphism $h : X \rightarrow Y$ in $\mathcal{B}$, there are vertical morphisms $\phi_h : \langle F_0 h\rangle \rightarrow F_1\langle h\rangle$ and $\psi_h : F_1\langle h\rangle \rightarrow \langle F_0 h\rangle$ in $\mathbf{Rel}(\mathcal{E})$.

Proof. The definitions of $\langle F_0 h\rangle$ and $\langle h\rangle$ give morphisms $(\mathsf{id}_{F_0}, F_0 h)_\S : \mathsf{Eq}\,(F_0 X) \to \langle F_0 h\rangle$ and $F_1((\mathsf{id}_X, h)_\S) : F_1(\mathsf{Eq}\ X) \to F_1\langle h\rangle$. The following diagram commutes:

$$\begin{array}{ccc}
(F_0X, F_0X) & \xrightarrow{(\mathsf{id}_{F_0X}, F_0h)} & (F_0X, F_0Y) \\
{\scriptstyle (\mathsf{id}_{F_oX}, \mathsf{id}_{F_oX})}\uparrow & & \uparrow{\scriptstyle (\mathsf{id}_{F_0}, \mathsf{id}_{F_0})} \\
(F_0X, F_0X) & \xrightarrow{(\mathsf{id}_{F_0X}, F_0h)} & (F_0X, F_0Y)
\end{array}$$

Thus, by the universal property of the opcartesian map $(\mathsf{id}_{F_0X}, F_0h)_\S$, there is a unique morphism $\phi_h : \langle F_0h\rangle \to F_1\langle h\rangle$ such that the following diagram commutes:

$$\begin{array}{ccc}
F_1\,(\mathsf{Eq}\,X) & \xrightarrow{F_1((\mathsf{id}_X, h)_\S)} & F_1\langle h\rangle \\
{\scriptstyle =}\uparrow & & \uparrow{\scriptstyle \exists!\,\phi_h} \\
\mathsf{Eq}\,(F_0X) & \xrightarrow{(\mathsf{id}_{F_0X}, F_0h)_\S} & \langle F_0h\rangle
\end{array}$$

Moreover, ϕ_h is over $(\mathsf{id}_{F_0X}, \mathsf{id}_{F_0Y})$ and thus vertical. A similar argument using the universal property of $(\mathsf{id}_{F_0X}, F_0h)^\S$ gives the existence of a unique vertical morphism $\psi_h : F_1\langle h\rangle \to \langle F_0h\rangle$. □

3.4 Comprehension

We have seen how relations can be treated abstractly as objects (A, B, R) in the total category of a fibration of relations. A more concrete — but less general — approach to relations is to consider them to be spans.

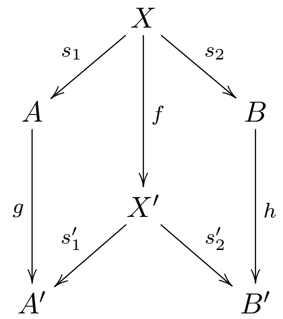
Definition 3.13. Let $\mathcal{B}$ be a category. The category $\mathsf{Span}(\mathcal{B})$ of spans in $\mathcal{B}$ has as objects pairs of morphisms with the same domain

$$\begin{array}{ccccc}
 & & X & & \\
 & {\scriptstyle s_1}\swarrow & & \searrow{\scriptstyle s_2} & \\
A & & & & B.
\end{array}$$

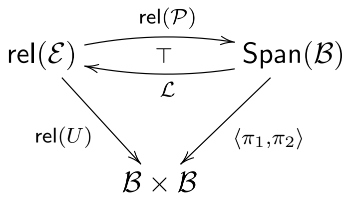
A morphism between two spans consists of a triple

$$(f, g, h) : (X, s_1, A, s_2, B) \to (X', s_1', A', s_2', B')$$

where $f\colon X \to X'$, $g\colon A \to A'$ and $h\colon B \to B'$ are morphisms in $\mathcal{B}$ such that $s_1' \circ f = g \circ s_1$ and $s_2' \circ f = h \circ s_2$

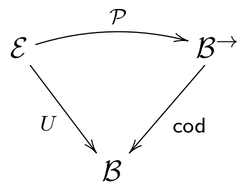


A natural question is whether the abstract, fibrational notion of relation behaves sufficiently like the concrete, span-based relations so that theorems concerning the latter can be generalised to the former. The answer is that, yes, this is possible if more structure is present. This structure is known as comprehension and is used widely in categorical logic [Jac99]. In particular, it guarantees the existence of the functor $\mathsf{rel}(\mathcal{P})$ below – if furthermore the comprehension is full, then $\mathsf{rel}(\mathcal{P})$ has a fibred left adjoint $\mathcal{L}$:



(3.14)

where $\langle \pi_1, \pi_2 \rangle$ is the obvious functor $\langle \pi_1, \pi_2 \rangle\colon \mathsf{Span}(\mathcal{B}) \to \mathcal{B} \times \mathcal{B}$ sending an object $(X, s_1, A, s_2, B) \mapsto (A, B)$ and a morphism $(f, g, h) \mapsto (f, g)$. This adjunction between $\mathsf{rel}(\mathcal{E})$ and $\mathsf{Span}(\mathcal{B})$ allows the transfer of results mentioned above. We note in passing that much of this structure arises from applying $\mathsf{rel} : \mathbf{Fib}_{\mathcal{B}} \to \mathbf{Fib}_{\mathcal{B}\times\mathcal{B}}$ (defined by pullback along $\_ \times \_$) to the diagram



up to the isomorphism $\mathsf{rel}(\mathcal{B}^{\to}) \cong \mathsf{Span}(\mathcal{B})$. Here, comprehension again guarantees the existence of $\mathcal{P}$, and full comprehension implies that $\mathcal{P}$ has a left adjoint $\mathcal{B}^{\to} \to \mathcal{E}$. In the remainder of this section, we introduce comprehension and full comprehension, and show that the above relationship (3.14) between relations and spans holds in this setting.

Definition 3.15 ([Ehr88])**.** Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration with truth functor K. The fibration U admits **comprehension** if K has a right adjoint $\{\_\}\colon \mathcal{E} \to \mathcal{B}$, called the **comprehension functor**.

The following well-known result shows how comprehension allows objects in the total category of a fibration to be seen as morphisms in the base category. We will use this technique in Chapter 7 in order to show how objects in the total category of a fibration of relations can be seen as spans in the base category.

Lemma 3.16 ([Jac93])**.** Let $U : \mathcal{E} \to \mathcal{B}$ be a fibration with $K \dashv \{\_\}$. Comprehension induces a functor $\mathcal{P} : \mathcal{E} \to \mathcal{B}^{\to}$ defined by $\mathcal{P}(X) = U(\epsilon_X)$ where ϵ is the unit of the adjunction $K \dashv \{\}$. For $f : X \to Y$ in $\mathcal{E}$, $\mathcal{P}$ is defined by $\mathcal{P}(f) = (\{f\}, U(f))$. Furthermore, the assignment $\pi_X \coloneqq U(\epsilon_X)$ is a natural transformation $\pi\colon \{\_\} \to U$.

Proof. Note that the construction of π and $\mathcal{P}$ relies on $U \circ K = \mathsf{Id}$. In fact the naturality of π follows by applying U to the naturality condition of ϵ. Using the same argument we define $\mathcal{P}$ as shown in the diagram

$$\begin{array}{ccc}
X & \xrightarrow{\quad f \quad} & Y \\
 & \big\downarrow & \\
U(K\{X\}) & \xrightarrow{\{f\}} & U(K\{Y\}) \\
{\scriptstyle U(\epsilon_X)}\big\downarrow & & \big\downarrow{\scriptstyle U(\epsilon_Y)} \\
UX & \xrightarrow[U(f)]{} & U(Y).
\end{array}$$

□

We can adapt the above lemma to the setting of relations and spans as follows:

Lemma 3.17. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a fibration with $K \dashv \{\_\}$. The map sending an object (A, B, R) of $\mathsf{rel}(\mathcal{E})$ to $\pi_R\colon \{R\} \to A \times B$ extends to a functor $\mathsf{rel}(\mathcal{P}) : \mathsf{rel}(\mathcal{E}) \to \mathsf{Span}(\mathcal{B})$.

Proof. The component R of an object (A, B, R) of $\mathsf{rel}(\mathcal{E})$ is by definition an object R of $\mathcal{E}_{A\times B}$. The action of $\mathcal{P}$ gives a morphism $\pi_R : \{R\} \to A \times B$ which is an object of $\mathsf{Span}(\mathcal{B})$.

The action of $\mathsf{rel}(\mathcal{P})$ on morphisms is defined similarly recalling that all the morphisms in $\mathbf{Rel}(\mathcal{E})$ are of the form (f, g, h) with $U(h) = f \times g$ and we can isolate the two components obtaining $(\{h\}, f, g)$. □

We now construct the candidate left adjoint $\mathcal{L} : \mathsf{Span}(\mathcal{B}) \to \mathsf{rel}(\mathcal{E})$. In order to do so we need opfibrational structure.

Lemma 3.18. Let $U \colon \mathcal{E} \to \mathcal{B}$ be an opfibration with truth functor K. The map which sends (X, s_1, A, s_2, B) in $\mathsf{Span}(\mathcal{B})$ to $(A, B, \Sigma_{\langle s_1, s_2 \rangle} KX)$ in $\mathbf{Rel}(\mathcal{E})$ extends to a functor $\mathcal{L} : \mathsf{Span}(\mathcal{B}) \to \mathbf{Rel}(\mathcal{E})$.

Proof. Given a morphism $(f, g, h) : (X, s_1, A, s_2, B) \to (X', s_1', A', s_2', B')$ in $\mathsf{Span}(\mathcal{B})$, $\mathcal{L}(f, g, h)$ is defined using the universal property of $\langle s_1, s_2 \rangle^{\S}_{KX}$ in the following diagram

$$\begin{array}{ccc} \Sigma_{\langle s_1, s_2 \rangle} KX & \overset{\mathcal{L}(f,g,h)}{\dashrightarrow} & \Sigma_{\langle s_1', s_2' \rangle} KX' \\ \uparrow{\scriptstyle \langle s_1, s_2 \rangle^{\S}_{KX}} & & \uparrow{\scriptstyle \langle s_1', s_2' \rangle^{\S}_{KX'}} \\ KX & \underset{Kf}{\longrightarrow} & KX' \end{array}$$

over the diagram

$$\begin{array}{ccc} A \times B & \overset{g \times h}{\longrightarrow} & A' \times B' \\ \uparrow{\scriptstyle \langle s_1, s_2 \rangle} & & \uparrow{\scriptstyle \langle s_1', s_2' \rangle} \\ X & \underset{f}{\longrightarrow} & X'. \end{array}$$

in $\mathcal{B}$, which commutes since (f, g, h) is a morphism in $\mathsf{Span}(\mathcal{B})$. □

We need the following result in order to prove that $\mathcal{L}$ and $\mathsf{rel}(\mathcal{P})$ are adjoint functors.

Lemma 3.19. Let $U \colon \mathcal{E} \to \mathcal{B}$ be a fibration with $K \dashv \{\_\}$. For every object A in $\mathcal{B}$, the morphism π_{KA} is an isomorphism $\pi_{KA} \colon \{KA\} \cong A$.

Proof. Since K is full and faithful, the unit η of the adjunction $K \dashv \{\_\}$ is a natural isomorphism. Using the triangle identity $\epsilon_{KA} \circ K(\eta_A) = \mathsf{id}$ we have $\epsilon_{KA} = K(\eta_A)^{-1}$. Finally $\pi_{KA} = U(\epsilon_{KA}) \colon \{KA\} \cong A$. □

In order to show that $\mathcal{L}$ is actually left adjoint to $\mathsf{rel}(\mathcal{P})$, a little extra structure is required:

Definition 3.20 ([Jac93])**.** We say that a fibration $U\colon \mathcal{E} \to \mathcal{B}$ admits **full comprehension** if U admits comprehension, and the functor $\mathcal{P} : \mathcal{E} \to \mathcal{B}^{\to}$ induced by $\{\_\}$ from Lemma 3.16 is full and faithful.

Note that, just like [Jac93], we require $\mathcal{P}$ to be both full and faithful in order for U to admit full comprehension.

Lemma 3.21. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a faithful bifibration with truth functor K and full comprehension. Then $\mathcal{L} \dashv \mathsf{rel}(\mathcal{P})$.

Proof. We describe a natural isomorphism

$$\begin{aligned}\mathrm{Hom}_{\mathsf{rel}(\mathcal{E})}(\mathcal{L}(X, s_1, A, s_2, B), (A', B', R')) \quad &\cong \\ &\mathrm{Hom}_{\mathsf{Span}}(\mathcal{B})((X, s_1, A, s_2, B), \mathsf{rel}(\mathcal{P})(A', B', R')).\end{aligned}$$

Given $(f, g, \alpha)\colon (A, B, \Sigma_{\langle s_1, s_2\rangle} KX) \to (A', B', R')$ in $\mathsf{rel}(\mathcal{E})$, consider the composition $\alpha \circ \langle s_1, s_2\rangle^{\S}\colon KX \to R'$ in $\mathcal{E}$. By applying $\mathcal{P}$, we obtain the commuting diagram

$$\begin{array}{ccccc} \{KX\} & \xrightarrow{\{\langle s_1,s_2\rangle^{\S}\}} & \{\Sigma_{\langle s_1,s_2\rangle} KX\} & \xrightarrow{\{\alpha\}} & \{R'\} \\ \downarrow{\scriptstyle \pi_{KX}} & & \downarrow{\scriptstyle \pi_{\Sigma_{\langle s_1,s_2\rangle} KX}} & & \downarrow{\scriptstyle \pi_{R'}} \\ X & \xrightarrow[\langle s_1,s_2\rangle]{} & A \times B & \xrightarrow[f\times g]{} & A' \times B'. \end{array}$$

and since π_{KX} is an isomorphism by Lemma 3.19, we can send the morphism (f, g, α) to $(\{\alpha \circ \langle s_1, s_2\rangle^{\S}\} \circ \pi_{KX}^{-1}, f, g)$ which is a morphism from $\langle s_1, s_2\rangle : X \to A \times B$ to $\pi_{R'} : \{R'\} \to A' \times B'$ in $\mathsf{Span}(\mathcal{B})$.

In the other direction, consider $(f, f_0, f_1)\colon \langle s_1, s_2\rangle \to \pi_{R'}$ in $\mathsf{Span}(\mathcal{B})$. Using the universal property of $\langle s_1, s_2\rangle_\S$ we obtain

$$\begin{array}{ccccc} KX & \xrightarrow{Kf} & K\{R'\} & \xrightarrow{\epsilon_{R'}} & R' \\ & {\scriptstyle\langle s_1,s_2\rangle_\S}\searrow & & \nearrow{\scriptstyle f^{\#}} & \\ & & \Sigma_{\langle s_1,s_2\rangle} KX & & \end{array}$$

over the diagram

$$\begin{array}{ccccc} X & \xrightarrow{f} & \{R'\} & \xrightarrow{\pi_{R'}} & A' \times B' \\ & {\scriptstyle\langle s_1,s_2\rangle}\searrow & & \nearrow{\scriptstyle f_0\times f_1} & \\ & & A \times B & & \end{array}$$

in $\mathcal{B}$, which commutes since (f, f_0, f_1) is a morphism in $\mathsf{Span}(\mathcal{B})$. We map (f, f_0, f_1) to $(f_0, f_1, f^{\#})$.

Direct calculation using faithfulness of the fibration and fullness of $\mathcal{P}$ proves that these maps are inverse of the one another, as well as naturality. $\square$

Chapter 4

System F and categorical models

In this chapter we introduce System F and the notion of the λ2-fibration and we show how a λ2-fibration constitutes a model of System F.

System F, also known as the (Girard–Reynolds) polymorphic lambda calculus or the second-order lambda calculus, is an extension of the simply typed lambda calculus which has quantification over types. It was introduced independently by Girard [Gir71] as a logical system and by Reynolds [Rey74] in computer science. A good reference available online is given by Girard, Taylor and Lafont [GTL89].

We start by introducing the simply typed lambda calculus and then we extend it to System F. At the same time we present categorical interpretations for the two systems. We conclude the chapter by recalling the notion of the internal language of a model.

We recall some definitions and properties of type theory in order to fix the notation, but in general we assume that the reader is familiar with the basic notions of type theory like terms, types, contexts, free variables, type substitution, α, β and η-equivalences and so on. The reader less familiar with type theory can find a good introduction to the subject in Pierce [Pie02].

4.1 Type theory

In this section we introduce the notion of type theory with the terminology and the notation we use. Type theory is a class of formal systems. At the basis of each of these formal systems there are different rules. The choice of different rules results in systems with very different features.

In a type system there are **terms** and **types**. We use (possibly indexed) lower case latin letters $a, t, x, t_1, x_n \ldots$ for terms, where the x and indexed x's are for term variables, and we use (possibly indexed) capital latin letters $A, T, A_1, T_n, \ldots$ for types. We write $a\colon A$ for a has type A. Often types are thought of as propositions. From this point of view, a term $a\colon A$ is a proof of A.

A **context** consists of a collection of given data (usually terms or type variables). We use Greek capital letters $\Gamma, \Delta, \ldots$ for contexts.

In a type system, in order to state what is well defined, we have **judgments**. In this work we use four different judgments:

- **context judgments**: Γ ctx meaning "the context Γ is well formed";
- **type judgments**: $\Gamma \vdash A$ type meaning "the type A is well formed in the context Γ";
- **term judgments**: $\Gamma \vdash a\colon A$ meaning "the term a has type A in the context Γ";
- **equality judgments**: $\Gamma \vdash a \equiv a'\colon A$ meaning "the terms a and a' of type A are equal in the context Γ". We call these equalities **judgmental equalities** in order to differentiate them from a different notion of equality we will introduce later.

Judgments are specified by providing **inference rules**. A typical inference rule has the form

$$\frac{\mathcal{I}_1 \quad \ldots \quad \mathcal{I}_n}{\mathcal{I}}$$

It says that we may derive the conclusion $\mathcal{I}$, provided that we have already derived the hypothesis $\mathcal{I}_1, \ldots, \mathcal{I}_n$. There may be extra side conditions that need to be checked before the rule is applicable. A **derivation** of a judgment is a tree constructed from such inference rules with the judgment at the root of the tree. It has shape of the form

$$\cfrac{\cfrac{\cfrac{\ldots}{\mathcal{I}_{11}}\,\mathrm{C}_1 \quad \ldots \quad \cfrac{\ldots}{\mathcal{I}_{n1}}\,\mathrm{C}_2}{\mathcal{I}_1}\,\mathrm{B}_1 \quad \ldots \quad \cfrac{\cfrac{\ldots}{\mathcal{I}_{1j}}\,\mathrm{C}_3 \quad \ldots \quad \cfrac{\ldots}{\mathcal{I}_{ij}}\,\mathrm{C}_4}{\mathcal{I}_j}\,\mathrm{B}_2}{\mathcal{I}}\,\mathrm{A}$$

4.2 The simply typed lambda calculus

Traditionally the **simply typed lambda calculus**, denoted by $\lambda^{\to}$, is presented as a type theory with only one type constructor $\to$ for arrow types. We extend the system with the constructor $\_ \times \_$ for products. This is justified by the fact that we model the calculus in cartesian closed categories which naturally come with products. For the same reason we could also add the unit type. However, in this work, we will only need the product types and then, for simplicity, we will consider only them.

A context Γ in the simply typed lambda calculus consists of a list $x_1 : T_1, ..., x_m : T_m$ of distinct **term variables** x_i with type T_i. Term constants $\Delta \vdash c : T$ and base types can be added if desired. We have term variables for which there is the following rule:

$$\frac{x_i : T_i \in \Gamma}{\Gamma \vdash x_i : T_i}$$

This says that if $x_i : T_i$ is a term variable in context Γ, then we can form the term $x_i : T_i$ in context Γ.

Given two types T_1 and T_2, we can form the type $T_1 \times T_2$ which consists of the product of T_1 and T_2. We can introduce terms of product type using the following rule

$$\frac{\Delta \vdash t_1 : T_1 \quad \Delta \vdash t_2 : T_2}{\Delta \vdash (t_1, t_2) : T_1 \times T_2.}$$

This says that given two terms $t_1 \colon T_1$ and $t_2 \colon T_2$ we can form the pair (t_1, t_2). Dually we have projections

$$\frac{\Delta \vdash t \colon T_1 \times T_2}{\Delta \vdash \pi_1\, t : T_1} \qquad \frac{\Delta \vdash t \colon T_1 \times T_2}{\Delta \vdash \pi_2\, t : T_2}$$

There are additional rules describing the behaviour of terms of product type:

- **First projection of a pair**: if we project out the first component of a pair (t_1, t_2) we obtain t_1:

$$\frac{\Delta \vdash (t_1, t_2) \colon T_1 \times T_2}{\Delta \vdash \pi_1\, (t_1, t_2) \equiv t_1 : T_1}$$

- **Second projection of a pair**: if we project out the second component of a pair (t_1, t_2) we obtain t_2:

$$\frac{\Delta \vdash (t_1, t_2) \colon T_1 \times T_2}{\Delta \vdash \pi_2\,(t_1, t_2) \equiv t_2 : T_2}$$

- **Surjective pairing**: a term $t \colon T_1 \times T_2$ is equal to the pairing of its projections:

$$\frac{\Delta \vdash t \colon T_1 \times T_2}{\Delta \vdash t \equiv (\pi_1\, t, \pi_2\, t) : T_1 \times T_2}$$

- **Congruence of pairing**: if we pair equal terms the pairs are equal

$$\frac{\Delta \vdash t_1 \equiv s_1 : T_1 \qquad \Delta \vdash t_2 \equiv s_2 : T_2}{\Delta \vdash (t_1, t_2) \equiv (s_1, s_2) : T_1 \times T_2}$$

- **Congruence of projections**: the projections of equal pairs give equal terms

$$\frac{\Delta \vdash t \equiv s : T_1 \times T_2}{\Delta \vdash \pi_1\, t \equiv \pi_1\, s : T_1}$$

$$\frac{\Delta \vdash t \equiv s : T_1 \times T_2}{\Delta \vdash \pi_2\, t \equiv \pi_2\, s : T_2}$$

Next we focus on arrow types. Given two types T_1 and T_2, we can form the type $T_1 \to T_2$ which consists of the functions from T_1 to T_2. The rule used to introduce terms for arrow types is called **λ-abstraction**:

$$\frac{\Delta, x : T_1 \vdash t : T_2}{\Delta \vdash \lambda x.t : T_1 \to T_2.}$$

This says that given $t \colon T_2$ where t might depend on $x \colon T_1$, we can abstract over x (or bind x in t), obtaining the function $\lambda x.t$. The λ notation explicitly emphasises that the function depends on $x \colon T_1$.

We call **term application**, or λ-application the rule used to eliminate terms of arrow types given by the rule

$$\frac{\Delta \vdash f : T_1 \to T_2 \quad \Delta \vdash t : T_1}{\Delta \vdash f\, t : T_2}$$

This says that if t_2 is a function from T_1 to T_2 and t_1 is a term of type T_1, the application of t_2 to t_1 gives, as a result, a term of type T_2.

Let $\Gamma, x\colon A \vdash t\colon T$ and $\Gamma \vdash a\colon A$ be two term judgments. We denote by $t[x \mapsto a]$ the term obtained by substituting a in every free occurrence of x (see [Pie02]). Substitution can be iterated: if we have the term judgment $\Gamma, x\colon A, y\colon B \vdash t\colon T$, the term judgment $\Gamma, x\colon A \vdash b\colon B$ and the term judgment $\Gamma \vdash a\colon A$, we can derive $\Gamma \vdash t[y \mapsto b][x \mapsto a]\colon T$. We write $\sigma = ([y \mapsto b], [x \mapsto a])$ and we denote the substitution as $\sigma\colon (\Gamma, x\colon A, y\colon B) \mapsto \Gamma$. Moreover we use the notation $\Gamma \vdash t[\sigma]\colon T$ for the result of the substitution.

Finally, in the simply typed lambda calculus there are additional rules describing the behaviour of terms of arrow type:

- **α-equivalence**: terms are the same up to renaming of bound variables

$$\frac{}{\Delta \vdash \lambda x.\, t \equiv \lambda y.\, t[x \mapsto y] : T_1 \to T_2}$$

- **β-equivalence**: term application of a λ-term is given by substituting the argument into the body of the term:

$$\frac{}{\Delta \vdash (\lambda x.\, t)\, s \equiv t[x \mapsto s] : T_2}$$

- **η-equivalence**: abstracting and immediately applying has no effect:

$$\frac{x \notin FV(t)}{\Delta \vdash t \equiv \lambda x.\, t\, x : T_1 \to T_2}$$

  where $FV(t)$ is the set of free variables in t.

- **Congruence of term application**: if we apply two equal functions t_1 and t_2 to two equal terms s_1 and s_2, the results are the same

$$\frac{\Delta \vdash t_1 \equiv t_2 : T_1 \to T_2 \qquad \Delta \vdash s_1 \equiv s_2 : T_1}{\Delta \vdash t_1\, s_1 \equiv t_2\, s_2 : T_2}$$

- **Congruence of λ-abstraction**: two λ-terms are equal if their bodies are equal:

$$\frac{\Delta, x : T_1 \vdash t_1 \equiv t_2 : T_2}{\Delta \vdash \lambda x.\, t_1 \equiv \lambda x.\, t_2 : T_1 \to T_2}$$

  This rule is also called the ξ-rule.

Moreover there are three more rules which assures that the judgmental equality $\equiv$ is an equivalence relation

- **Reflexivity**:

$$\frac{}{\Delta \vdash t \equiv t : T}$$

- **Symmetry**:

$$\frac{\Delta \vdash s \equiv t : T}{\Delta \vdash t \equiv s : T}$$

- **Transitivity**:

$$\frac{\Delta \vdash t \equiv s : T \qquad \Delta \vdash s \equiv u : T}{\Delta \vdash t \equiv u : T}$$

4.3 Interpreting the simply typed lambda calculus in a CCC

We now give the details of the interpretation of the simply typed lambda calculus in a cartesian closed category. In order to settle the notation, recall that a category $\mathcal{C}$ with finite products is cartesian closed if the functor $- \times A : \mathcal{C} \to \mathcal{C}$ has a right adjoint $A \Rightarrow \_$ for each object A, i.e., for each A, B, there is an object $A \Rightarrow B$ and an isomorphism

$$\theta : \mathsf{Hom}_{\mathcal{C}}(\Delta \times A, B) \cong \mathsf{Hom}_{\mathcal{C}}(\Delta, A \Rightarrow B)$$

natural in Δ and B. We denote the evaluation map of the exponential objects by $\mathsf{ev}_{A,B} \colon (A \Rightarrow B) \times A \to B$.

The simply typed λ-calculus is interpreted in a cartesian closed category by interpreting each type A as an object $[\![A]\!]$, contexts $\Delta = x_1 : A_1, \ldots, x_n : A_n$ as $[\![\Delta]\!] = [\![A_1]\!] \times \ldots \times [\![A_n]\!]$, and each term $x_1 : A_1, \ldots, x_n : A_n \vdash t : B$ as a morphism $[\![x_1 : A_1, \ldots, x_n : A_n \vdash t : B]\!]$:

$[\![A_1]\!] \times \ldots \times [\![A_n]\!] \to [\![B]\!]$, which, when $x_1 : A_1, \ldots, x_n : A_n$ and B are clear from the context or they are not relevant, we denote just by $[\![t]\!]$ for brevity.

We interpret types inductively by mapping every base type A to an object $[\![A]\!]$ in the category, and then, using the cartesian closed structure, we interpret $[\![A_1 \times A_2]\!] = [\![A_1]\!] \times [\![A_2]\!]$ and $[\![A_1 \to A_2]\!] = [\![A_1]\!] \Rightarrow [\![A_2]\!]$.

The term interpretation is given inductively as follows:

$$[\![x_1 : A_1, \ldots, x_n : A_n \vdash x_i : A_i]\!] = \pi_i : A_1 \times \ldots \times A_n \to A_i$$
$$[\![\Delta \vdash (t_1, t_2) : A_1 \times A_2]\!] = \langle [\![\Delta \vdash t_1 \colon A_1]\!], [\![\Delta \vdash t_2 \colon A_2]\!] \rangle$$
$$[\![\Delta \vdash \pi_1\, t : A_1]\!] = \pi_1 \circ [\![\Delta \vdash t \colon A_1 \times A_2]\!]$$
$$[\![\Delta \vdash \pi_2\, t : A_2]\!] = \pi_2 \circ [\![\Delta \vdash t \colon A_1 \times A_2]\!]$$
$$[\![\Delta \vdash \lambda x\,.\,t : A \to B]\!] = \theta([\![\Delta, x : A \vdash t : B]\!])$$
$$[\![\Delta \vdash f\,t : B]\!] = \mathsf{ev}_{A,B} \circ \langle [\![\Delta \vdash f : A \to B]\!], [\![\Delta \vdash t : A]\!] \rangle$$

Each substitution $\sigma : \Delta \mapsto \Gamma$ gives rise to a morphism $[\![\sigma]\!] : [\![\Gamma]\!] \to [\![\Delta]\!]$, and one can prove that $[\![\Gamma \vdash t[\sigma]]\!] = [\![\Delta \vdash t]\!] \circ [\![\sigma]\!]$. In particular $[x \mapsto a] : (\Gamma, x : A) \mapsto \Gamma$ is given by $\langle \mathsf{id}, [\![a]\!] \rangle : \Gamma \to \Gamma \times A$. It is easily checked that $[\![(\lambda x\,.\,t)\,a]\!] = [\![t[x \mapsto a]]\!]$.

This model is sound:

Theorem 4.1. If $\Delta \vdash t \equiv s \colon A$ in simply typed lambda calculus, then $[\![t]\!] = [\![s]\!] \colon [\![\Delta]\!] \to [\![A]\!]$ in the cartesian closed category. □

We will often implicitly use the following lemma:

Lemma 4.2. We have the following:

1. $\theta^{-1}([\![\vdash \lambda x : A\,.\,x]\!]) = \mathsf{id}_{[\![A]\!]}$.

2. For all $\Delta \vdash f : B \to C$ and $\Delta \vdash g : A \to B$, we have $\theta^{-1}([\![\vdash \lambda x\,.\,f(g(x))]\!]) = \theta^{-1}([\![\Delta \vdash f]\!]) \circ \langle \pi_1, \theta^{-1}([\![\Delta \vdash g]\!]) \rangle$.

3. In particular, if Δ is empty then $\theta^{-1}([\![\vdash \lambda x\,.\,f(g(x))]\!]) = \theta^{-1}([\![\vdash f]\!]) \circ \theta^{-1}([\![\vdash g]\!])$ up to the isomorphism $\mathbf{1} \times B \cong B$.

Proof. 1. $\theta^{-1}([\![\vdash \lambda x : A \,.\, x]\!]) = \theta^{-1}(\theta([\![x : A \vdash x]\!])) = \mathsf{id}_{[\![A]\!]}$.

2. For simplicity we write $[\![f]\!]$ and $[\![g]\!]$ instead of, respectively, $[\![\Delta \vdash f]\!]$ and $[\![\Delta \vdash g]\!]$. We have

$$\begin{aligned}
\theta^{-1}([\![\Delta \vdash \lambda x \,.\, f(g(x)) : A \to C]\!]) &= \theta^{-1}(\theta([\![\Delta, x : A \vdash f(g(x)) : C]\!])) \\
&= \mathsf{ev} \circ \langle [\![f]\!] \circ \pi_1, \mathsf{ev} \circ \langle [\![g]\!] \circ \pi_1, \pi_2 \rangle \rangle \\
&= \mathsf{ev} \circ \langle [\![f]\!] \circ \pi_1, \mathsf{ev} \circ ([\![g]\!] \times \mathsf{id}) \rangle \\
&= \mathsf{ev} \circ \langle [\![f]\!] \circ \pi_1, \theta^{-1}([\![g]\!]) \rangle \\
&= \mathsf{ev} \circ ([\![f]\!] \times \mathsf{id}) \circ \langle \pi_1, \theta^{-1}([\![g]\!]) \rangle \\
&= \theta^{-1}([\![f]\!]) \circ \langle \pi_1, \theta^{-1}([\![g]\!]) \rangle
\end{aligned}$$

3. One part of the isomorphism $\mathbf{1} \times B \cong B$ is given by $\langle !_B, \mathsf{id} \rangle$, where $!_B : B \to \mathbf{1}$ is the unique morphism from B into $\mathbf{1}$. By uniqueness of this morphism, we have $!_{\mathbf{1} \times A} = \pi_1 = !_B \circ \theta^{-1}([\![g]\!])$, so that $\theta^{-1}([\![\vdash \lambda x \,.\, f(g(x))]\!]) = \theta^{-1}([\![\vdash f]\!]) \circ \langle !_B, \mathsf{id} \rangle \circ \theta^{-1}([\![\vdash g]\!]) \cong \theta^{-1}([\![\vdash f]\!]) \circ \theta^{-1}([\![\vdash g]\!])$ simply by instantiating (2).

□

4.4 System F

System F extends the simply typed lambda calculus by permitting abstraction not only over terms, but also over types. For this reason there are two different classes of variables:

- **type variables** of the form $X_1, ..., X_n$,
- **term variables** of the form $x_1 : T_1, ..., x_m : T_m$.

The two different classes of variables form two different contexts:

- the **type context** consists of a list of type variables $X_1, ..., X_n$, and we typically denote it by Γ,
- the **term context** consists of a list of term variables $x_1 : T_1, ..., x_m : T_m$, which we typically denote by Δ.

Type judgments

$$\frac{X_i \in \Gamma}{\Gamma \vdash X_i \ \mathsf{type}} \qquad \frac{\Gamma \vdash T_1 \ \mathsf{type} \quad \Gamma \vdash T_2 \ \mathsf{type}}{\Gamma \vdash T_1 \to T_2 \ \mathsf{type}} \qquad \frac{\Gamma \vdash T_1 \ \mathsf{type} \quad \Gamma \vdash T_2 \ \mathsf{type}}{\Gamma \vdash T_1 \times T_2 \ \mathsf{type}}$$

Term judgments

$$\frac{x_i : T_i \in \Delta}{\Gamma; \Delta \vdash x_i : T_i} \qquad \frac{\Gamma; \Delta \vdash t \colon T_1 \times T_2}{\Gamma; \Delta \vdash \pi_1\, t : T_1} \qquad \frac{\Gamma; \Delta \vdash t \colon T_1 \times T_2}{\Gamma; \Delta \vdash \pi_2\, t : T_2}$$

$$\frac{\Gamma; \Delta \vdash t_1 : T_1 \quad \Delta \vdash t_2 : T_2}{\Gamma; \Delta \vdash (t_1, t_2) : T_1 \times T_2} \qquad \frac{\Gamma; \Delta, x : T_1 \vdash t : T_2}{\Gamma; \Delta \vdash \lambda x.t : T_1 \to T_2}$$

$$\frac{\Gamma; \Delta \vdash f : T_1 \to T_2 \quad \Gamma; \Delta \vdash t : T_1}{\Gamma; \Delta \vdash f\, t : T_2}$$

Figure 4.1: The simply typed lambda calculus with type variables

A context in system F consists of both a type context Γ and a term context Δ such that every type T_i in Δ is well formed according to Γ, i.e. there are type judgments $\Gamma \vdash T_i$ type for every T_i in $\Delta = x_1 : T_1, ..., x_m : T_m$. When we write $\Gamma \vdash \Delta$ it means that the previous condition is satisfied for every type T_i in Δ and we can form the context $\Gamma; \Delta = X_1, ..., X_n; x_1 : T_1, ..., x_m : T_m$.

Traditionally there are three classes of types: type variables, arrow types and forall types, but, like in the case of simply typed lambda calculus, we also add product types. We consider α-convertible types equivalent. If desired, base types or other type constants $\Gamma \vdash C$ type can also be added to the system. It is possible to add term constants $\Gamma; \Delta \vdash c : T$ as well. We still have term substitution as in the simply typed lambda calculus, and there is also substitution for types.

System F is an extension of the simply typed lambda calculus. We recall the rules for the simply typed lambda calculus with type variables in Figure 4.1, where we omit the congruence relations and the rules making $\equiv$ an equivalence relation. Compared to section 4.2 the only difference is that we also have a context of type variables.

The crucial type judgment for system F is the one which forms forall types:

$$\frac{\Gamma, X \vdash T \ \mathsf{type}}{\Gamma \vdash \forall X.T \ \mathsf{type}}$$

This says that if a type T is formed assuming a type variable X, it is possible to abstract over X obtaining the type $\forall X.T$. Note that, once X is bound in $\forall X.T$, the type variable X does not appear anymore in the type context. This structure corresponds to the structure of second-order logic, where one can quantify over properties (see Girard [Gir71]).

The introduction rule is given by **Λ-abstraction**:

$$\frac{\Gamma, X; \Delta \vdash t : T}{\Gamma; \Delta \vdash \Lambda X.t : \forall X.T} \ (X \notin \mathrm{FTV}(\Delta))$$

where X is not a free type variable in Δ. This rule permits to abstract over types.

The elimination rule is **type application**:

$$\frac{\Gamma; \Delta \vdash t : \forall X.T \quad \Gamma \vdash A \ \mathrm{Type}}{\Gamma; \Delta \vdash t\,A : T[X \mapsto A]}$$

which says that applying a type A to a term t of type $\forall X.T$, we obtain a term $t\,A$ of type $T[X \mapsto A]$, where $T[X \mapsto A]$ denotes the capture-free substitution of the type A for the free occurrences of X in the type T.

The judgmental equalities for terms of forall type are:

- **α-equivalence**:

$$\frac{}{\Gamma; \Delta \vdash \Lambda X.\, t \equiv \Lambda Y.\, t[X \mapsto Y] : \forall X.T}$$

- **β-equivalence**:

$$\frac{}{\Gamma; \Delta \vdash (\Lambda X.\, t)A \equiv t : T[X \mapsto A]}$$

- **η-equivalence**:

$$\frac{X \notin FTV(t)}{\Gamma; \Delta \vdash t \equiv \Lambda X.\, t\,X : \forall X.T}$$

- **Congruence of type application**:

$$\frac{\Gamma; \Delta, \vdash t_1 \equiv t_2 : \forall X.T}{\Gamma; \Delta \vdash t_1\, A \equiv t_2\, A : T[X \mapsto A]}$$

- **Congruence of Λ-abstraction** (or ξ-rule):

$$\frac{\Gamma, X; \Delta \vdash t_1 \equiv t_2 : T}{\Gamma; \Delta \vdash \Lambda X.\, t_1 \equiv \Lambda X.\, t_2 : \forall X.T}$$

Extensionality Because of the η-equivalence and the congruence of λ and Λ abstraction, it is easy to derive extensionality for both functions and type abstractions:

Proposition 4.3.

1. $\Gamma; \Delta \vdash t \equiv s : T_1 \to T_2$ iff $\Gamma; \Delta, x : T_1 \vdash t\, x \equiv s\, x : T_2$.
2. $\Gamma; \Delta \vdash t \equiv s : \forall X.T$ iff $\Gamma, X; \Delta \vdash t\, X \equiv s\, X : T$.

Proof. The left-to-right directions are just the congruence rules. For the other direction, we have

$$t \stackrel{\eta}{\equiv} \lambda x.t\, x \stackrel{\xi}{\equiv} \lambda x.s\, x \stackrel{\eta}{\equiv} s$$

and similarly for type abstractions. □

4.5 Categorical models of System F – λ2-fibrations

λ2-fibrations are standard categorical models of System F (see [See87, Jac99]). The point of Chapter 5 is to construct a λ2-fibration based on bifibrations for modelling parametricity.

We start with some definitions which will lead to the notion of λ2-fibration as given in [Jac99].

Definition 4.4. Consider a fibration $p \colon \mathcal{E} \to \mathcal{B}$ and an object T in $\mathcal{E}$. We say that T is a **generic object** if for every object X in $\mathcal{E}$ there exists a morphism $u \colon pX \to pT$ and a cartesian morphism $f \colon X \to T$ over u.

Equivalently T is a generic object if for every object X in $\mathcal{E}$ there exists a unique morphism $u\colon pX \to pT$ and a vertical isomorphism $f\colon u^*(T) \to X$.

In the case of split fibrations, the definition simplifies to the notion of split generic object.

Definition 4.5. A split fibration $p\colon \mathcal{E} \to \mathcal{B}$ has a **split generic object** if there is an object Ω in $\mathcal{B}$ together with a collection of isomorphisms $\xi_I\colon \mathcal{B}(I,\Omega) \xrightarrow{\cong} |\mathcal{E}_I|$ natural in I with respect to reindexing; that is, for $v\colon J \to I$, the following diagram commutes

$$\begin{array}{ccc} \mathcal{B}(I,\Omega) & \xrightarrow{\xi_I} & |\mathcal{E}_I| \\ {\scriptstyle \_\circ v}\downarrow & & \downarrow{\scriptstyle v^*} \\ \mathcal{B}(J,\Omega) & \xrightarrow[\xi_J]{} & |\mathcal{E}_I|. \end{array}$$

Definitions 4.4 and 4.5 are linked by the following lemma (see [Jac99]).

Lemma 4.6. A split fibration $p\colon \mathcal{E} \to \mathcal{B}$ has a split generic object if and only if there is an object T in $\mathcal{E}$ with the property that for every object X in $\mathcal{E}$, there exists a unique morphism $u\colon pX \to pT$ such that $u^*T = X$.

Proof. If there is a split generic object (Ω,ξ), take $T = \xi_\Omega(\mathsf{id}_\Omega)$. Then for every $X \in \mathcal{E}_I$ we have that $\xi_I^{-1}(X)\colon I \to \Omega$ satisfies

$$\xi_I^{-1}(X)^*(T) = \xi_I^{-1}(X)^*(\xi_\Omega(\mathsf{id}_\Omega)) = \xi_I(\mathsf{id}_\Omega \circ \xi_I^{-1}(X)) = X,$$

where the second equality holds because of the naturality of ξ. It is easy to see that $\xi_I^{-1}(X)$ is the unique morphism u satisfying this property: we have $X = u^*T = \xi_I(u)$ where we use again naturality of ξ for the second equality. Since ξ_I is an isomorphism, we can conclude that $\xi_I(\xi_I^{-1}(X)) = X = \xi_I(u)$, hence $\xi_I^{-1}(X) = u$.

In the reverse direction assume T in $\mathcal{E}$ as in the statement of the lemma, and write $\Omega = pT$ in $\mathcal{B}$. For every I in $\mathcal{B}$ and $u\colon I \to \Omega$ let $\xi_I(u) = u^*T$. This is clearly a bijection. Moreover $\xi_I(u \circ v) = (u \circ v)^*T = v^*(u^*T) = v^*(\xi_I(T))$, for every $v\colon J \to I$. □

Definition 4.7. We say that a fibration $p\colon \mathcal{E} \to \mathcal{B}$ is

- a $\lambda \to$ -fibration if it is a fibred CC fibration with a generic object T (we write $\Omega = pT$) and $\mathcal{B}$ has finite products;
- a $\lambda 2$-fibration if it is a $\lambda \to$ -fibration with simple Ω-products, where $\Omega = pT$.

A $\lambda \to$ -fibration is split when the fibration itself is split, the cartesian closed structure in the fibres is preserved on the nose by reindexing and the generic object is split. A $\lambda 2$-fibration is split, if the underlying $\lambda \to$ -fibration is split and moreover the Beck-Chevalley condition for simple products holds in the form that the canonical isomorphism is the identity.

We can now interpret simply typed lambda calculus with type variables in a $\lambda \to$ -fibration $p\colon \mathcal{E} \to \mathcal{B}$ which we can assume to be split (see [See87]). We use the objects in the base category of the $\lambda \to$ -fibration to model type contexts. The cartesian closed structure in the fibres allows us to interpret arrow types and term contexts exactly as for the case of simply typed lambda calculus with cartesian closed categories.

We start from a type judgment $\Gamma \vdash T$ type. We interpret $[\![\Gamma]\!]$ as an object in $\mathcal{B}$ and $[\![T]\!]$ is an object in $\mathcal{E}_{[\![\Gamma]\!]}$, i.e. an object in $\mathcal{E}$ which lives in the fibre over $[\![\Gamma]\!]$. Every type context Γ is of the form $\Gamma = X_1, \ldots, X_m$ and we interpret it as $[\![\Gamma]\!] = \underbrace{\Omega \times \ldots \times \Omega}_{m\text{-times}}$. The interpretation of types is done by induction on type judgments. A base type $\vdash A$ type is interpreted as an object $[\![A]\!]$ in $\mathcal{E}$ living over the terminal object $\mathbf{1}$ of $\mathcal{B}$ (the terminal object exists since it is the empty product and in $\mathcal{B}$ there are finite products). A type variable $X \vdash X$ type is interpreted as $\xi_\Omega(\mathsf{id}_\Omega)$. When we have a judgment for type variables like $X_1, \ldots, X_n \vdash X_i$ type, with $i \in \{1, \ldots, n\}$, the interpretation is given by $\pi_i^* \xi_\Omega(\mathsf{id}_\Omega)$, where $\pi_i\colon \underbrace{\Omega \times \ldots \times \Omega}_{n\text{-times}} \to \Omega$ in $\mathcal{B}$ is the projection on the i-th component. This operation corresponds to the weakening of the context. Note that, by the definition of the generic object, we have $\pi_i^* \xi_\Omega(\mathsf{id}_\Omega) \neq \pi_j^* \xi_\Omega(\mathsf{id}_\Omega)$ if $i \neq j$.

The interpretation of product types $\Gamma \vdash U \times V$ type and arrow types $\Gamma \vdash U \to V$ type is given by the cartesian closed structure in the fibres. In fact both $[\![U]\!]$ and $[\![V]\!]$ live in the fibre over $[\![\Gamma]\!]$, and we can define $[\![U \times V]\!] \coloneqq [\![U]\!] \times [\![V]\!]$ and $[\![U \to V]\!] \coloneqq [\![U]\!] \Rightarrow [\![V]\!]$ in the fibre over $[\![\Gamma]\!]$.

If we want to model System F we need to add another feature: the interpretation of forall types. λ2-fibrations allow us to interpret forall types by using simple Ω-products, where Ω is the generic object. Given $\Gamma \vdash \forall X.T$, we define $[\![\forall X.T]\!] := \forall_{[\![\Gamma]\!]}([\![T]\!])$, where $\forall_{[\![\Gamma]\!]}$ is the right adjoint to the reindexing functor $\pi_1^*\colon \mathcal{E}_{[\![\Gamma]\!]} \to \mathcal{E}_{[\![\Gamma]\!]\times\Omega}$ which exists because there are simple Ω-products. When $[\![\Gamma]\!]$ is clear from the context, we write only $\forall$. Note that this is well defined since $[\![T]\!]$ lives in the fibre over $[\![\Gamma, X]\!]$, which, by definition, is $[\![\Gamma]\!] \times \Omega$.

The next step is the interpretation of term contexts and terms. Given a term context $\Delta = x_1\colon T_1, \ldots, x_n\colon T_n$, its interpretation is given by the product $[\![\Delta]\!] = [\![T_1]\!] \times \ldots \times [\![T_n]\!]$. A term judgment $\Gamma; \Delta \vdash t\colon T$ is interpreted as a vertical morphism $[\![t]\!]\colon [\![\Delta]\!] \to [\![T]\!]$. Since the morphism is vertical it lives in the fibre over $[\![\Gamma]\!]$. In order to show how to interpret terms, we first need to model type substitution, which is fundamental in the interpretation of terms of forall types. Each substitution $\sigma\colon \Gamma \mapsto \Gamma'$ gives rise to a morphisms $[\![\sigma]\!]\colon [\![\Gamma']\!] \to [\![\Gamma]\!]$. In particular consider the substitution $\sigma = [X \mapsto A]\colon \Gamma, X \mapsto \Gamma$, we saw that by iterating we can cover also substitutions of the form $[X_1 \mapsto A_1, \ldots, X_n \mapsto A_n]$. It is possible to prove that $[\![\sigma]\!]\colon [\![\Gamma]\!] \to [\![\Gamma]\!] \times \Omega$ in $\mathcal{B}$ is given by $[\![\sigma]\!] := \langle \mathsf{id}, \xi_\Gamma^{-1}([\![A]\!])\rangle$. The following lemma shows that substitution is given by reindexing:

Lemma 4.8. Let $\Gamma, X \vdash T$ type and $\Gamma \vdash A$ type be two type judgments. The interpretation of $\Gamma \vdash T[X \mapsto A]$ type is given by $[\![T[X \mapsto A]]\!] = \langle \mathsf{id}, \xi_\Gamma^{-1}([\![A]\!])\rangle^*([\![T]\!])$.

Proof. The proof is done by induction on type judgments.

Type variables. Let $\Gamma, X \vdash X$ type be a type judgment for a type variable X. Recall that $X[X \mapsto A] = A$ and $X[Y \mapsto A] = X$ if $Y \neq X$. For the first case we simplify and consider the judgment $X \vdash X$ type, the general case follows by weakening. We have that

$$\begin{aligned} \xi_\Omega^{-1}([\![A]\!])^*([\![X]\!]) &= \xi_\Omega^{-1}([\![A]\!])^*(\xi_\Omega(\mathsf{id}_\Omega)) \\ &= [\![A]\!] \end{aligned}$$

where the second equality uses the same argument as in the proof of Lemma 4.6. In the other case note that $\pi_{[\![X]\!]} \circ \langle \mathsf{id}, \xi_\Gamma^{-1}([\![A]\!])\rangle\colon \Gamma \to \Omega$ is equal to $\pi_{[\![X]\!]}$, where, by abuse of notation, we denote by $\pi_{[\![X]\!]}$ the morphisms projecting the component related to X both with domain Γ and $\Gamma \times \Omega$, since it is clear from the context to which one we refer. Since the fibration is split, we have

$$
\begin{aligned}
\langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle^* \circ \pi^*_{\llbracket X \rrbracket} &= (\pi_{\llbracket X \rrbracket} \circ \langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle)^* \\
&= \pi^*_{\llbracket X \rrbracket}
\end{aligned}
$$

and, by definition of weakening, we conclude $\pi^*_{\llbracket X \rrbracket}(\xi_\Omega(\mathsf{id}_\Omega)) = \llbracket X \rrbracket$.

Product types. Let $\Gamma, X \vdash U \times V$ type and $\Gamma \vdash A$ type be two type judgments. Using the induction hypothesis and the fact that reindexing preserves fibred products we have the following derivation:

$$
\begin{aligned}
\llbracket (U \times V)[X \mapsto A] \rrbracket &= \llbracket U[X \mapsto A] \times V[X \mapsto A] \rrbracket \\
&= \llbracket U[X \mapsto A] \rrbracket \times \llbracket V[X \mapsto A] \rrbracket \\
&= \langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle^* \llbracket U \rrbracket \times \langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle^* \llbracket V \rrbracket \\
&= \langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle^* (\llbracket U \times V \rrbracket)
\end{aligned}
$$

Arrow types. Let $\Gamma, X \vdash U \to V$ type and $\Gamma \vdash A$ type be two type judgments. Using inductive hypothesis and the fact that reindexing preserves fibred exponentials we have the following derivation:

$$
\begin{aligned}
\llbracket (U \to V)[X \mapsto A] \rrbracket &= \llbracket U[X \mapsto A] \to V[X \mapsto A] \rrbracket \\
&= \llbracket U[X \mapsto A] \rrbracket \Rightarrow \llbracket V[X \mapsto A] \rrbracket \\
&= \langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle^* \llbracket U \rrbracket \Rightarrow \langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle^* \llbracket V \rrbracket \\
&= \langle \mathsf{id}, \xi_\Gamma^{-1}(\llbracket A \rrbracket)\rangle^* (\llbracket U \to V \rrbracket)
\end{aligned}
$$

Forall types. Let $\Gamma, X \vdash \forall Y.T$ type and $\Gamma \vdash A$ type be two type judgments. If we have $(\forall Y.T)[Y \mapsto A] = \forall Y.T$, this formula come from some weakening and then the thesis follows similarly to the case of type variables $X[Y \mapsto A] = X$. Otherwise consider the following derivation in which we use Beck-Chevalley condition to swap $\forall$ with reindexing and the induction hypothesis:

$$\begin{aligned}
[\![(\forall Y.T)[Z \mapsto A]]\!] &= [\![\forall Y.(T[Z \mapsto A])]\!] \\
&= \forall_{[\![\Gamma]\!]}((\langle \mathsf{id}, \xi_\Gamma^{-1}([\![A]\!])\rangle \times \mathsf{id}_\Omega)^* [\![T]\!]) \\
&= \langle \mathsf{id}, \xi_\Gamma^{-1}([\![A]\!])\rangle^* (\forall_{[\![\Gamma]\!]\times\Omega}([\![T]\!])).
\end{aligned}$$

Note that the second equality holds because $[\![T]\!]$ lives over the object $[\![\Gamma]\!] \times \Omega \times \Omega$, but we have that $\langle \mathsf{id}, \xi^{-1}(A)\rangle^* \colon \mathcal{E}_{[\![\Gamma]\!]\times\Omega} \to \mathcal{E}_{[\![\Gamma]\!]}$. We are then in a weaken case and it holds that $(\langle \mathsf{id}, \xi^{-1}(A)\rangle \circ \pi)^* = (\pi \circ (\langle \mathsf{id}, \xi^{-1}(A)\rangle \times \mathsf{id}))^*$.

□

We now describe how to interpret terms, by induction on the term. Let $|\Gamma| = n$ and $\Delta = x_1 \colon T_1, \ldots, x_m \colon T_m$.

Term variables. The interpretation of a term variable $\Gamma, \Delta \vdash x_i \colon T_i$ is given by the projection: $[\![x_i]\!] \coloneqq \pi_i \colon [\![T_1]\!] \times \ldots [\![T_m]\!] \to [\![T_i]\!]$.

Product pairing. Consider $\Gamma; \Delta \vdash (u, v) \colon U \times V$. By the induction hypothesis, we have that $[\![u]\!] \colon [\![\Delta]\!] \to [\![U]\!]$ and $[\![v]\!] \colon [\![\Delta]\!] \to [\![V]\!]$. We define $[\![(u, v)]\!] \colon [\![\Delta]\!] \to [\![U \times V]\!]$ as $\langle [\![u]\!], [\![v]\!] \rangle$.

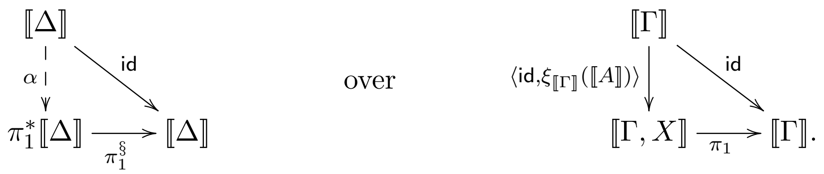
Product projections. Consider $\Gamma; \Delta \vdash \pi_i\, t \colon U_i$, where $i \in \{1, 2\}$. By the induction hypothesis we have that $[\![t]\!] \colon [\![\Delta]\!] \to [\![U_1]\!] \times [\![U_2]\!]$, and we define $[\![\pi_i\, t]\!] \colon [\![\Delta]\!] \to [\![U_i]\!]$ as $\pi_i \circ [\![t]\!]$ where π_i is the projection morphism in the category.

Function terms. Consider $\Gamma; \Delta \vdash \lambda x.v \colon U \to V$. By the induction hypothesis, we have that $[\![v]\!] \colon [\![\Delta]\!] \times [\![U]\!] \to [\![V]\!]$. We define $[\![\lambda x.v]\!] \colon [\![\Delta]\!] \to [\![U \to V]\!]$ by $\theta([\![v]\!])$.

Term application. Consider $\Gamma; \Delta \vdash t \colon U \to V$ and $\Gamma; \Delta \vdash u \colon U$, we want to interpret $\Gamma; \Delta \vdash tu \colon V$. Its interpretation is given by postcomposing with ev (the evaluation map for exponential objects). In detail we define $[\![tu]\!] \colon = \mathsf{ev} \circ \langle [\![t]\!], [\![u]\!] \rangle$, where $[\![t]\!]$ and $[\![u]\!]$ are given by the induction hypothesis.

Type abstraction. Given $\Gamma, X; \Delta \vdash t\colon T$ we want to interpret $\Gamma; \Delta \vdash \Lambda X.t\colon \forall X.T$. We denote by $\nu\colon \mathrm{Hom}(\pi^*_{\llbracket\Gamma\rrbracket,\Omega}\llbracket\Delta\rrbracket, \llbracket T\rrbracket) \cong \mathrm{Hom}(\llbracket\Delta\rrbracket, \llbracket\forall X.T\rrbracket)$ the isomorphism given by the adjunction. Using the induction hypothesis we have $\llbracket t\rrbracket\colon \llbracket\Delta\rrbracket \to \llbracket T\rrbracket$ and we consider $\llbracket t\rrbracket \circ \pi^{\S}_{\llbracket\Gamma\rrbracket}\colon \pi^*_{\llbracket\Gamma\rrbracket,\Omega}\llbracket\Delta\rrbracket \to \llbracket T\rrbracket$. We then define $\llbracket\Lambda X.t\rrbracket := \nu(\llbracket t\rrbracket \circ \pi^{\S}_{\llbracket\Gamma\rrbracket})$.

Type application. The last term judgment we want to interpret is $\Gamma; \Delta \vdash tA\colon T[X \mapsto A]$ assuming $\Gamma; \Delta \vdash t\colon \forall X.T$ and $\Gamma \vdash A$ type. We use the universal property of cartesian morphisms in order to interpret tA. First, using the universal property of $\pi^{\S}_1$, we find the following morphism



Next, using the universal property of $\langle \mathsf{id}, \xi^{-1}_\Gamma(\llbracket A\rrbracket)\rangle^{\S}$, we can find $\llbracket tA\rrbracket$ as follows

This model is sound:

Theorem 4.9. If $\Delta \vdash t \equiv s\colon A$ in System F, then $\llbracket t\rrbracket = \llbracket s\rrbracket\colon \llbracket\Delta\rrbracket \to \llbracket A\rrbracket$. □

As a consequence of soundness we have that, given two term judgments $\Gamma; \Delta \vdash f\colon A \to B$ and $\Gamma; \Delta \vdash g\colon A \to B$, if $\llbracket\Gamma; \Delta, x\colon A \vdash fx\rrbracket = \llbracket\Gamma; \Delta, x\colon A \vdash gx\rrbracket$, then $\llbracket\Gamma; \Delta \vdash f\rrbracket = \llbracket\Gamma; \Delta \vdash g\rrbracket$. Similarly, given two term judgments $\Gamma; \Delta \vdash t\colon \forall X.A$ and $\Gamma; \Delta \vdash s\colon \forall X.A$, if $\llbracket\Gamma, X; \Delta \vdash tX\rrbracket = \llbracket\Gamma, X; \Delta \vdash sX\rrbracket$ then $\llbracket\Gamma; \Delta \vdash t\rrbracket = \llbracket\Gamma; \Delta \vdash s\rrbracket$.

4.6 Internal language

The internal language allows us to use type theory to reason about categories. In Chapter 5 the use of internal language will allow us to prove properties of the categorical model. We reach the internal language of a $\lambda 2$-fibration in two steps: we first focus on the cartesian closed structure in the fibres and present the internal language given by the simply typed lambda calculus, and then we extend it to the $\lambda 2$-fibration as System F. There is a comprehensive presentation of internal language in Taylor [Tay99].

4.6.1 Cartesian closed categories

Let $\mathcal{C}$ be a cartesian closed category. In this subsection we show that we can use simply typed lambda calculus in order to reason about $\mathcal{C}$.

The idea is to use the fact that if we specify all the type constants and term constants of a particular simply typed lambda calculus, then we completely identify it. What we do is taking a cartesian closed category $\mathcal{C}$ and from this category we derive the type and term constants which form a simply typed lambda calculus. In this way, we can reason about the category using expressions in simply typed lambda calculus. In order to make this useful we will need to add some morphisms which will allow us to work with the internal language, like for example to treat exponential objects of $\mathcal{C}$ as arrow types in the type system or products in $\mathcal{C}$ as products of types in the type system.

We describe a map $\mathsf{internal}\colon |\mathbf{CCC}| \to \{\lambda^{\to}\text{-calculi}\}$ where CCC is the category of cartesian closed categories (small cartesian closed categories with specified structure and functors preserving the structure on the nose), while the codomain is the collection of all the different simply typed lambda calculi.

Type constants

Every object A in $\mathcal{C}$ defines a type constant $\vdash \underline{A}\ \mathsf{type}$.

Term constants

Every morphism $f\colon A \to B$ defines a term constant $\vdash \underline{f}\colon \underline{A} \to \underline{B}$.

Duplication isomorphisms: we need terms in order to identify $\underline{\mathbf{1}}$ with the unit type, $\underline{A_1 \times A_2}$ with $\underline{A_1} \times \underline{A_2}$ and $\underline{A_1 \to A_2}$ with $\underline{A_1} \Rightarrow \underline{A_2}$. These terms are:

- unit type:

$$\vdash \star \colon \underline{\mathbf{1}}$$

- product types:

$$\vdash \mathsf{prod}_{A,B} : (\underline{A} \times \underline{B}) \to \underline{A \times B}$$

- arrow types:

$$\vdash \mathsf{lam}_{\mathsf{A,B}} : (\underline{A} \to \underline{B}) \to \underline{A \Rightarrow B}$$

Term equalities

First of all we want to think of morphisms $f \colon \mathbf{1} \to A$ as terms of type $\underline{A}$. In order to do that we add an equality making $\underline{\mathbf{1}}$ the unit type:

$$x \colon \underline{\mathbf{1}} \vdash x \equiv \star \colon \underline{\mathbf{1}}.$$

In this way, given $f \colon \mathbf{1} \to A$ in the category, we have $\underline{f} \star \colon \underline{A}$ in the syntax.

We add another rule which identifies term application and substitution in the following way: for every pair of morphisms $f \colon \mathbf{1} \to A$ and $g \colon A \to B$ in the category we have

$$\vdash \underline{g}\,(\underline{f}\,\star) \equiv (\underline{g \circ f})\,\star \colon \underline{B}$$

For the product types we add the equations

$$t \colon \underline{A \times B} \vdash \mathsf{prod}_{A,B}(\underline{\pi_1}\,t, \underline{\pi_2}\,t) \equiv t \colon \underline{A \times B}$$

$$t \colon \underline{A} \times \underline{B} \vdash (\underline{\pi_1}(\mathsf{prod}_{A,B}\,t), \underline{\pi_2}(\mathsf{prod}_{A,B}\,t)) \equiv t \colon \underline{A} \times \underline{B}$$

Here $\underline{\pi_1} \colon \underline{A \times B} \to \underline{A}$ and $\underline{\pi_2} \colon \underline{A \times B} \to \underline{B}$ are the internal terms corresponding to the external projections $\pi_1 \colon A \times B \to A$ and $\pi_2 \colon A \times B \to B$. The equations above thus state that $\underline{A \times B} \cong (\underline{A} \times \underline{B})$. Semantically, we interpret $\mathsf{prod}_{A,B}$ as the identity morphism. We will often abuse notation and treat $\underline{A \times B}$ and $(\underline{A} \times \underline{B})$ as identical.

Next we add the following equations for function types:

$$x : \underline{A \Rightarrow B} \vdash \mathsf{lam}_{A,B}\,(\lambda a\,.\,\underline{\mathsf{ev}_{A,B}}\,(x, a)) = x : \underline{A \Rightarrow B}$$

$$y : \underline{A} \to \underline{B} \vdash \lambda a\,.\,\underline{\mathsf{ev}_{A,B}}\,(\mathsf{lam}_{A,B}\,y, a) = y : \underline{A} \to \underline{B}$$

Here, $\underline{\mathsf{ev}_{A,B}} \colon (\underline{A \Rightarrow B}) \times \underline{A} \to \underline{B}$ is the internal term corresponding to the external evaluation map $\mathsf{ev}_{\mathsf{A,B}} : (A \Rightarrow B) \times A \to B$ (up to the isomorphism $(\underline{A \Rightarrow B}) \times \underline{A} \cong \underline{(A \Rightarrow B) \times A}$). The equations above thus state that $\underline{A \Rightarrow B} \cong (\underline{A} \to \underline{B})$. Semantically, we interpret $\mathsf{lam}_{A,B}$ as the identity morphism. We will often abuse notation and treat $\underline{A \Rightarrow B}$ and $(\underline{A} \to \underline{B})$ as identical.

We now extend this internal language to $\lambda 2$-fibrations.

4.6.2 $\lambda 2$-fibrations

Let $p \colon \mathcal{E} \to \mathcal{B}$ be a $\lambda 2$-fibration. In this subsection we show that we can use polymorphic lambda calculus in order to reason about p.

We use the same idea as in the case of cartesian closed categories and simply typed lambda calculi: we specify all the type constants and term constants of a particular polymorphic lambda calculus.

We describe a map $\mathsf{internal} \colon \{\lambda 2\text{-fibrations}\} \to \{\lambda 2\text{-calculi}\}$ where $\{\lambda 2\text{-fibrations}\}$ is the collection of $\lambda 2$-fibrations, while the codomain is the collection of all the different polymorphic lambda calculi. Recall that a type context is of the form $\Gamma = X_1, \ldots, X_n$ and we interpreted it as $[\![\Gamma]\!] = \underbrace{\Omega \times \ldots \times \Omega}_{n\text{-times}}$. In a $\lambda 2$-fibration the interpretation lives in the base category, but, thanks the previous observation, only objects of the form $\underbrace{\Omega \times \ldots \times \Omega}_{n\text{-times}}$ correspond to type contexts.

Type constants

For every type context Γ and object A in $\mathcal{E}_{[\![\Gamma]\!]}$, add a constant $\Gamma \vdash \underline{A}(\Gamma)\ \mathsf{type}$, where $\Gamma = X_1, \ldots, X_n$ and $\underline{A}(\Gamma) = \underline{A}(X_1, \ldots, X_n)$ is given with its type variables as arguments.

Term constants

For every context Γ and vertical morphism $f\colon A \to B$ in $\mathcal{E}_{[\![\Gamma]\!]}$, add a term constant $\Gamma \vdash \underline{f}\colon \underline{A}(\Gamma) \to \underline{B}(\Gamma)$.

Duplication isomorphisms: in this case we need to identify also $\underline{\forall A}(\Gamma)$ and $\forall X.\underline{A}(\Gamma, X)$, so that we have the following terms:

- unit type:

$$\vdash \star\colon \underline{\mathbf{1}}()$$

- product types:

$$\Gamma \vdash \mathsf{prod}_{A,B} : (\underline{A}(\Gamma) \times \underline{B}(\Gamma)) \to \underline{A \times B}(\Gamma)$$

- arrow types:

$$\Gamma \vdash \mathsf{lam}_{\mathsf{A,B}} : (\underline{A}(\Gamma) \to \underline{B}(\Gamma)) \to \underline{A \Rightarrow B}(\Gamma)$$

- forall types:

$$\Gamma \vdash \mathsf{Lam}_{\mathsf{A}} : (\forall X.\underline{A}(\Gamma, X)) \to \underline{\forall A}(\Gamma)$$

Term equalities

We add an equality making $\underline{\mathbf{1}}$ the unit type:

$$\lambda x\colon \underline{\mathbf{1}}() \vdash x \equiv \star\colon \underline{\mathbf{1}}(),$$

and again we identify term application and substitution: for every pair of morphisms $f\colon \mathbf{1} \to A$ and $g\colon A \to B$ in the category we have

$$\vdash \underline{g}\,(\underline{f}\,\star) \equiv (\underline{g \circ f})\,\star\colon \underline{B}()$$

We have the following equations for product and arrow types:

$$\Gamma; t \colon \underline{A \times B}(\Gamma) \vdash \mathsf{prod}_{A,B}((\underline{\pi_1})_{A,B}\, t, (\underline{\pi_2})_{A,B}\, t) \equiv t \colon \underline{A \times B}(\Gamma)$$

$$\Gamma; t \colon \underline{A}(\Gamma) \times \underline{B}(\Gamma) \vdash ((\underline{\pi_1})_{A,B}(\mathsf{prod}_{A,B}\, t), ((\underline{\pi_2})_{A,B}(\mathsf{prod}_{A,B}\, t)) \equiv t \colon \underline{A}(\Gamma) \times \underline{B}(\Gamma)$$

$$\Gamma; x : \underline{A \Rightarrow B}(\Gamma) \vdash \mathsf{lam}_{A,B}\, (\lambda a \,.\, \underline{\mathsf{ev}_{A,B}}\, (x, a)) = x : \underline{A \Rightarrow B}(\Gamma)$$

$$\Gamma; y : \underline{A}(\Gamma) \to \underline{B}(\Gamma) \vdash \lambda a \,.\, \underline{\mathsf{ev}_{A,B}}\, (\mathsf{lam}_{A,B}\, y, a) = y : \underline{A}(\Gamma) \to \underline{B}(\Gamma)$$

For every object A in the total category of a $\lambda 2$-fibration and substitution $\sigma : \Gamma \mapsto \Gamma'$ in the type system, we define $\underline{A}(\Gamma)[\sigma] = \underline{[\![\sigma]\!]^* A}(\Gamma')$.

Finally for forall types we have the equations

$$\Gamma; x : \underline{\forall A}(\Gamma) \vdash \mathsf{Lam}_A\, (\Lambda X. \underline{\epsilon_A}\, x) = x : \underline{\forall A}(\Gamma)$$

$$\Gamma; y : \forall X. \underline{A}(\Gamma, X) \vdash (\Lambda X. \underline{\epsilon_A}\, (\mathsf{Lam}_A y)) = y : \forall X. \underline{A}(\Gamma, X)$$

Here $\underline{\epsilon_A}$ is the term corresponding to the counit $\epsilon : \pi^* \forall \to \mathsf{Id}$ of the adjunction $\pi^* \dashv \forall$. The equations above thus state that $\underline{\forall A}(\Gamma) \cong \forall X. \underline{A}(\Gamma, X)$. Semantically, we interpret Lam_A as the identity morphism. We will often abuse notation and treat $\underline{\forall A}(\Gamma)$ and $(\forall X. \underline{A})(\Gamma, X)$ as identical.

Part II

Bifibrational Parametricity

Chapter 5

Bifibrational parametricity

In this chapter we recall Reynolds' relational model for System F and in particular the two properties which permit to restrict to parametric polymorphic functions: the Identity Extension Lemma and the Abstraction Theorem. We show that Reynolds' model forms a $\lambda 2$-fibration where the objects in the total category have a fibrational structure. The $\lambda 2$-fibration constructed in this way is part of a more general framework of models of System F in which we can express the Identity Extension Lemma and the Abstraction Theorem. As a sanity check we conclude this chapter by showing that the expected properties of parametricity follow. This chapter is based on our work [GJNF$^+$15].

Note that since we want to cast everything within the unifying framework of fibrations, we will use the language of fibrations also if, in some case like Theorem 5.3 or Theorem 5.6, the fibrational properties are unnecessary. In any case the fibrational structure will be essential in Chapter 6.

We fix some notation for this chapter. Consider a bifibration $U\colon \mathcal{E} \to \mathcal{B}$ and the associated fibration of relations $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$. Objects in the base category $\mathcal{B}$ will be denoted by A and B, possibly indexed like for example A_i and B_i, and relations by R and indexed R_i, where R is a relation between A and B, and R_i is a relation between A_i and B_i. With $\bar{A}$ and $\bar{B}$ we denote n-tuples of objects $(A_1, \dots A_n)$ and $(B_1, \dots, B_n)$ in $\mathcal{B}$, while $\bar{R}$ is an n-tuple $(R_1, \dots, R_n)$ of relations between $\bar{A}$ and $\bar{B}$ living in $\mathbf{Rel}(\mathcal{E})$. Note that an alternative way to say that R is a relation between A and B, is to say that R is in $\mathbf{Rel}(\mathcal{E})_{(A,B)}$, i.e. the fibre over (A, B) with respect to $\mathsf{rel}\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$.

5.1 Reynolds' relational model

In this section we recall Reynolds' set-theoretic model based on [Rey83]. We remind that, in order to have a set theoretical model, we need to work in the (intuitionistic) internal language of a topos [Pit87], or in the Calculus of Constructions with impredicative **Set** because, as Reynolds discovered, there are no set-theoretic models if the meta-theory is classical logic [Rey84].

We skip most of the proofs in this section because the results are a particular instantiation of more general statements appearing later in this chapter.

In this section and the following we restrict to the case of the fibration of relation $\mathsf{rel}\colon \mathbf{Rel} \to \mathbf{Set} \times \mathbf{Set}$. For this reason $\bar{A}$ or $\bar{B}$ will be n-tuples of sets, $\bar{R}$ will be an n-tuple of relations over sets, and so on.

5.1.1 Semantics of types

Reynolds presents two "parallel" semantics for System F: a standard set-based semantics $[\![-]\!]_0$, and a relational semantics $[\![-]\!]_1$. Types are interpreted as maps which take (respectively) an n-tuple of sets (relations) to a set (relation). We represent such maps as functors with discrete domain since, in these cases, a functor is just a map between objects. Given $\Gamma \vdash T$ type, where the type context Γ contains $|\Gamma| = n$ type variables, Reynolds defines interpretations $[\![T]\!]_0 : |\mathbf{Set}|^n \to \mathbf{Set}$ and $[\![T]\!]_1 : |\mathbf{Rel}|^n_{(\bar{A},\bar{B})} \to \mathbf{Rel}_{([\![T]\!]_0\bar{A},[\![T]\!]_0\bar{B})}$ by structural induction on type judgments as follows:

Type variables. The interpretation of type variables is given by projection maps: $[\![X_i]\!]_0\bar{A} = A_i$ and $[\![X_i]\!]_1\bar{R} = R_i$

Product types. The interpretation of product types is given by using cartesian products:

$$[\![U \times V]\!]_0\bar{A} = [\![U]\!]_0\bar{A} \times [\![V]\!]_0\bar{A}$$

$$[\![U \times V]\!]_1\bar{R} = [\![U]\!]_1\bar{R} \times [\![V]\!]_1\bar{R}.$$

Arrow types. The interpretation of arrow types is given by using exponential objects:

$$[\![U \to V]\!]_0 \bar{A} = [\![U]\!]_0 \bar{A} \Rightarrow [\![V]\!]_0 \bar{A}$$

$$[\![U \to V]\!]_1 \bar{R} = [\![U]\!]_1 \bar{R} \Rightarrow [\![V]\!]_1 \bar{R}$$

where the exponentiation in the category of relations is defined by $R \Rightarrow S = \{(f, g) \mid (a, b) \in R \text{ implies } (fa, gb) \in S\}$, i.e. the set of pairs of maps preserving the relations.

Forall types. The interpretation of forall types is based on maps $f\colon \prod_{S:\mathbf{Set}} [\![T]\!]_0(\bar{A}, S)$ which take a set S as input and compute an element of the set $[\![T]\!]_0(\bar{A}, S)$:

$$[\![\forall X.T]\!]_0 \bar{A} \;= \{f : \prod_{S:\mathbf{Set}} [\![T]\!]_0(\bar{A}, S) \mid \forall R' \in \mathbf{Rel}_{(A',B')} \,.\, (fA', fB') \in [\![T]\!]_1(\mathsf{Eq}^n\ \bar{A}, R')\}$$

$$[\![\forall X.T]\!]_1 \bar{R} \;= \{(f, g) \mid \forall R' \in \mathbf{Rel}_{(A',B')} \,.\, (fA', gB') \in [\![T]\!]_1(\bar{R}, R')\}$$

We call the condition $\forall R' \in \mathbf{Rel}_{(A',B')} \,.\, (fA', fB') \in [\![T]\!]_1(\mathsf{Eq}^n\ \bar{A}, R')$ the **parametricity condition** because it is the condition which cuts down all the polymorphic functions to the parametric ones.

The definitions of $[\![\forall X.T]\!]_0$ and $[\![\forall X.T]\!]_1$ depend crucially on one another. Thus, there are not really two semantics – one based on **Set** and one based on **Rel** – but rather a single semantics based on the fibration $\mathsf{rel} : \mathbf{Rel} \to \mathbf{Set} \times \mathbf{Set}$.

The two level semantics is such that the relational interpretation of every type preserves equality relations[1] as stated by the Identity Extension Lemma:

Lemma 5.1 (Identity Extension Lemma)**.** If $\Gamma \vdash T$ type then $[\![T]\!]_1 \circ |\mathsf{Eq}|^{|\Gamma|} = \mathsf{Eq} \circ [\![T]\!]_0$. □

The Identity Extension Lemma is key for many applications of parametricity.

[1] Reynolds' approach also handles "identity relations" that are not equality relations, such as the information order on domains. In this work, like many others [BFSS90, BM05, Her06, PA93], we only treat equality relations.

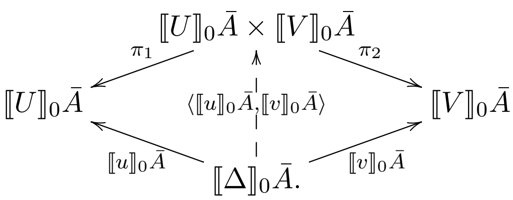
5.1.2 Semantics of terms

Reynolds' main result is his Abstraction Theorem, stating that all terms send related environments to related values. Reynolds first gives set-valued and relational interpretations of term contexts $\Delta = x_1 : T_1, \ldots, x_n : T_n$ by defining $[\![\Delta]\!]_0 = [\![T_1]\!]_0 \times \cdots \times [\![T_n]\!]_0$ and $[\![\Delta]\!]_1 = [\![T_1]\!]_1 \times \cdots \times [\![T_n]\!]_1$. He then interprets each judgment $\Gamma; \Delta \vdash t : T$ as a family of functions $[\![t]\!]_0\bar{A} : [\![\Delta]\!]_0\bar{A} \to [\![T]\!]_0\bar{A}$ by induction on term structure. We denote by $(a_1, \ldots, a_n)$ an n-tuple of elements in $[\![\Delta]\!]_0\bar{A}$ where $n = |\Delta|$.

Term variables. The interpretation of a term variable $\Gamma, x_1 \colon T_1, \ldots, x_m \colon T_m \vdash x_i \colon T_i$, with $1 \le i \le m$, is given by

$$[\![x_i]\!]_0\bar{A}(a_1, \ldots, a_n) = a_i.$$

Product pairing. Consider $\Gamma; \Delta \vdash (u, v) \colon U \times V$. By the induction hypothesis, we have the two families of morphisms $[\![u]\!]_0 \colon [\![\Delta]\!]_0 \to [\![U]\!]_0$ and $[\![v]\!]_0 \colon [\![\Delta]\!]_0 \to [\![V]\!]_0$. For every $\bar{A}$ in $|\mathbf{Set}|^{|\Gamma|}$ and every element $t \in [\![\Delta]\!]_0\bar{A}$, we define $[\![(u, v)]\!]_0\bar{A}\, t \coloneqq \langle [\![u]\!]\bar{A}, [\![v]\!]\bar{A} \rangle\, t$



Product projections. Consider $\Gamma; \Delta \vdash \pi_i\, t \colon U_i$, where $i \in \{1, 2\}$. By the induction hypothesis we have the family of morphisms $[\![t]\!]_0 \colon [\![\Delta]\!]_0 \to [\![U_1]\!]_0 \times [\![U_2]\!]_0$, and we define $[\![\pi_i\, t]\!]_0 \coloneqq \pi_i \circ [\![t]\!]_0$ where π_i is the projection morphism in $\mathbf{Set}$.

Function terms. Assume $\Gamma; \Delta \vdash \lambda x.v \colon U \to V$ and let u be an element in $[\![U]\!]_0\bar{A}$. The interpretation $[\![\lambda x.v]\!]_0\bar{A}(a_1, \ldots, a_n)$ is the map $[\![U]\!]_0\bar{A} \to [\![V]\!]_0\bar{A}$ given by

$$[\![\lambda x.v]\!]_0\bar{A}(a_1, \ldots, a_n)u \coloneqq [\![v]\!]_0(a_1, \ldots, a_n, u).$$

It is well defined since, by induction, $[\![v]\!]_0 \colon [\![\Delta]\!]_0 \times [\![U]\!]_0 \to [\![V]\!]_0$.

Term application. Let $\Gamma; \Delta \vdash t \colon U \to V$ and $\Gamma; \Delta \vdash u \colon U$ be two term judgments. By induction

$$[\![t]\!]_0 \bar{A}(a_1, \dots, a_n) \colon [\![U]\!]_0 \bar{A} \to [\![V]\!]_0 \bar{A} \quad \text{and} \quad [\![u]\!]_0 \bar{A}(a_1, \dots, a_n) \in [\![U]\!]_0 \bar{A}$$

and then we can define

$$[\![tu]\!]_0 \bar{A}(a_1, \dots, a_n) \coloneqq [\![t]\!]_0 \bar{A}(a_1, \dots, a_n)([\![u]\!]_0 \bar{A}(a_1, \dots, a_n)).$$

Type abstraction. Consider $\Gamma, \Delta \vdash \Lambda X.t \colon \forall X.T$. By the induction hypothesis we have a family of morphisms $[\![t]\!]_0 \colon [\![\Delta]\!]_0 \to [\![T]\!]_0$. We use this family to define the element $[\![\Lambda X.t]\!]_0 \bar{A}(a_1, \dots, a_n) \colon \Pi_{S \colon \mathbf{Set}} [\![T]\!]_0(\bar{A}, S)$ as follows

$$[\![\Lambda X.t]\!]_0 \bar{A}(a_1, \dots, a_n) S \coloneqq [\![t]\!]_0(\bar{A}, S)(a_1, \dots, a_n).$$

It is well defined since, in the type abstraction rule, X does not appear in Δ and for this reason $(a_1, \dots, a_n) \in [\![\Delta]\!]_0 \bar{A} = [\![\Delta]\!]_0(\bar{A}, S)$. Note that, a priori, this element does not satisfy the parametricity conditions. The parametricity conditions will follow from the Abstraction Theorem as we will show in the proof of Theorem 5.2. Formally, this means that we have to define the interpretation of terms and prove the Abstraction Theorem simultaneously.

Type application. Consider the term judgment $\Gamma; \Delta \vdash tV \colon T[X \mapsto V]$. By the induction hypothesis $[\![t]\!]_0 \bar{A}(a_1, \dots, a_n) \in [\![\forall X.T]\!]_0 \bar{A}$. The interpretation is given by

$$[\![tV]\!]_0 \bar{A}(a_1, \dots, a_n) \coloneqq [\![t]\!]_0 \bar{A}(a_1, \dots, a_n)[\![V]\!]_0 \bar{A}.$$

The Abstraction Theorem shows how to interpret terms at the relational level: it states that the set level interpretation of terms naturally extends to the relational level.

Theorem 5.2 (Abstraction Theorem)**.** Consider the term judgment $\Gamma, \Delta \vdash t \colon T$. For every n-tuple of relations $\bar{R}$, if (a, b) are related in $[\![\Delta]\!]_1 \bar{R}$, then $([\![t]\!]_0 \bar{A} a, [\![t]\!]_0 \bar{B} b)$ are related in

$[\![T]\!]_1\bar{R}$.

Proof. This result is a particular case of the more general Theorem 5.26. Anyway, it is interesting to see the case of type abstraction because it shows how this theorem is related to the parametricity condition.

The interpretation for type abstraction is given by

$$[\![\Lambda X.t]\!]_0\bar{A}(a_1,\dots,a_n)S = [\![t]\!]_0(\bar{A},S)(a_1,\dots,a_n).$$

Note that $([\![\Lambda X.t]\!]_0\bar{A}\,a\,A, [\![\Lambda X.t]\!]_0\bar{B}\,b\,B)$ are related in $[\![T]\!]_1(\bar{R},R)$ for every relation R. In fact, by definition of $[\![\Lambda X.t]\!]_0$, it is equivalent to prove that $([\![t]\!]_0(\bar{A},A)\,a, [\![t]\!]_0(\bar{B},B)\,b)$ are related in $[\![T]\!]_1(\bar{R},R)$ for every relation R, which follows by the induction hypothesis and the observation that, since X does not appear in Δ, we have $[\![\Delta]\!]_1(\bar{R}) = [\![\Delta]\!]_1(\bar{R},R)$ for every relation R and in particular $(a,b) \in [\![\Delta]\!]_1(\bar{R}) = [\![\Delta]\!]_1(\bar{R},R)$.

It follows that if $[\![\Lambda X.t]\!]_0\bar{A}\,a$ is in $[\![\forall X.T]\!]_0\bar{A}$ for every $\bar{A}$ and $a \in [\![\Delta]\!]_0\bar{A}$, then the Abstraction Theorem holds for $[\![\Lambda X.t]\!]_0$. Note that using the Identity Extension Lemma we have $(a,a) \in [\![\Delta]\!]_1\mathsf{Eq}(\bar{A}) = \mathsf{Eq}([\![\Delta]\!]_0\bar{A})$ and then $([\![\Lambda X.t]\!]_0\bar{A}\,a\,A, [\![\Lambda X.t]\!]_0\bar{A}\,a\,B)$ are related in $[\![T]\!]_1(\mathsf{Eq}(\bar{A}),R)$ for every R. This latter condition is in particular the parametricity condition and then we have the thesis. □

5.2 Reynolds' model as a λ2-fibration

In this section we restructure Reynolds' model presented in Section 5.1 in order to produce a λ2-fibration.

5.2.1 Reynolds' model, restructured

The interpretation of types in Subsection 5.1.1 comes with the fibrational structure $\mathsf{rel}\colon \mathbf{Rel} \to \mathbf{Set}\times\mathbf{Set}$. Reynolds' definitions of $[\![-]\!]_0$ and $[\![-]\!]_1$ entail the following theorem:

Theorem 5.3 (Fibrational Semantics of Types)**.** Every judgment $\Gamma \vdash T$ induces a fibred functor $[\![T]\!] : |\mathsf{rel}|^{|\Gamma|} \to \mathsf{rel}$.

$$\begin{CD}
|\mathbf{Rel}|^{|\Gamma|} @>{[\![T]\!]_1}>> \mathbf{Rel}\\
@V{|\mathsf{rel}|^{|\Gamma|}}VV @VV{\mathsf{rel}}V\\
|\mathsf{Set}|^{|\Gamma|} \times |\mathsf{Set}|^{|\Gamma|} @>>{[\![T]\!]_0 \times [\![T]\!]_0}> \mathsf{Set} \times \mathsf{Set}
\end{CD}$$

□

Reynolds does not give a functorial action of types on morphisms. This is reflected in the appearance of discrete categories in Theorem 5.3. As a result, Reynolds' pointwise interpretation of function spaces is the exponential in the functor category $|\mathsf{rel}|^{|\Gamma|} \to \mathsf{rel}$ [Rob94]. In fact instead of acting on morphisms, the interpretation of types acts on graph relations induced by morphisms. For now, we simply note that the use of discrete domains does not take us out of the fibrational framework; Lemmas 2.38 and 2.39 ensures that $[\![T]\!]$ is a functor between fibrations.

We use the following definition to restate the Identity Extension Lemma from a fibrational perspective.

Definition 5.4. We say that a fibred functor $(F_1, F_0 \times F_0)\colon |\mathsf{rel}|^n \to \mathsf{rel}$ is **equality preserving** if $F_1 \circ |\mathsf{Eq}|^n = \mathsf{Eq} \circ F_0$.

Within fibrational language it is possible to state the Identity Extension Lemma exactly as

Theorem 5.5 (Identity Extension Lemma)**.** Let $\Gamma \vdash T$ type be a type judgment. Then $[\![T]\!]$ is equality preserving, i.e. the following diagram commutes

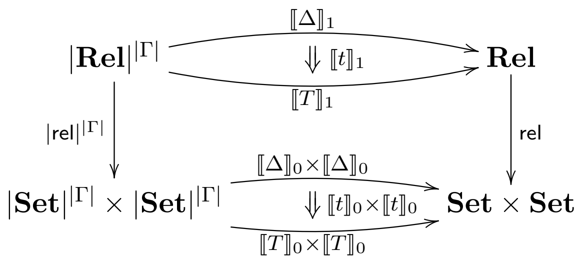
$$\begin{CD}
|\mathbf{Rel}|^{|\Gamma|} @>{[\![T]\!]_1}>> \mathbf{Rel}\\
@A{|\mathsf{Eq}|^{|\Gamma|}}AA @AA{\mathsf{Eq}}A\\
|\mathbf{Set}|^{|\Gamma|} @>>{[\![T]\!]_0}> \mathbf{Set}
\end{CD}$$

□

Note that when we say equality reserving, we mean to map the discretisation $|\mathsf{Eq}|^{|\Gamma|}$ of equality on the left to equality on the right. The left-hand discretisation is not the equality functor in the discrete fibration.

Using fibrational language we can restate the Abstraction Theorem in the following way:

Theorem 5.6 (Abstraction Theorem)**.** Let $\bar{A}, \bar{B} \in \mathbf{Set}^{|\Gamma|}$ and $\bar{R} \in \mathbf{Rel}^{|\Gamma|}(\bar{A}, \bar{B})$. We can interpret every judgment $\Gamma; \Delta \vdash t : T$ as a fibred natural transformation $(\llbracket t \rrbracket_1, \llbracket t \rrbracket_0 \times \llbracket t \rrbracket_0) : \llbracket \Delta \rrbracket \to \llbracket T \rrbracket$.



Proof. We have that $(\llbracket t \rrbracket_1, \llbracket t \rrbracket_0 \times \llbracket t \rrbracket_0)$ is a fibred natural transformation if and only if $\llbracket t \rrbracket_1$ and $\llbracket t \rrbracket_0$ are families of morphisms with $\llbracket t \rrbracket_1$ living over $\llbracket t \rrbracket_0 \times \llbracket t \rrbracket_0$. In fact the condition to be fibred, i.e. preservation of (op)cartesian morphisms, is trivially satisfied since the domain is discrete. Using Theorem 5.2 we can interpret $\llbracket t \rrbracket_1 = (\llbracket t \rrbracket_0, \llbracket t \rrbracket_0, \llbracket t \rrbracket_0 \times \llbracket t \rrbracket_0)$. □

The fibrational version makes it clear that the Abstraction Theorem states the existence of additional algebraic structure to $\llbracket t \rrbracket_0$ given by $\llbracket t \rrbracket_1$ and, more generally, the interpretation of terms as fibred natural transformations.

5.2.2 The λ2-fibration

We can now present the λ2-fibration associated to Reynolds' model based on the fibrational presentation just given. The base category $\mathcal{N}^{\mathsf{Eq}}$ is defined as

$\mathcal{N}^{\mathsf{Eq}}$	**objects**	natural numbers $n \in \mathbb{N}$.
	morphisms	a morphism from n to m consists of m-tuples of equality preserving fibred functors $H_i : \lvert\mathsf{rel}\rvert^n \to \mathsf{rel}$.

Note that a morphism $h = (H_1, \dots, H_m)\colon n \to m$ in $\mathcal{N}^{\mathsf{Eq}}$ gives rise to a functor

$$(H_1(-), \dots, H_m(-))\colon |\mathsf{rel}(U)|^n \to |\mathsf{rel}(U)|^m$$

defined by

$$(H_1(-), \dots, H_m(-))_j(X_1, \dots, X_m) = ((H_1)_j(X_1, \dots, X_m), \dots, (H_1)_j(X_1, \dots, X_m))$$

where $j \in \{1, 0\}$ and every X_i lives in the correct category.

The total category $\mathcal{F}^{\mathsf{Eq}}$ is defined as

$\mathcal{F}^{\mathsf{Eq}}$ **objects** equality preserving fibred functors from $|\mathsf{rel}|^n$ to rel for some $n \in \mathbb{N}$.

morphisms a morphism $(h, \eta)\colon F \to G$, where $F\colon |\mathsf{rel}|^n \to \mathsf{rel}$ and $G\colon |\mathsf{rel}|^m \to \mathsf{rel}$, consists of a morphism $h = (H_1, \dots, H_m) : n \to m$ in $\mathcal{N}^{\mathsf{Eq}}$ and a natural transformation $\eta : F \to G \circ (H_1(-), \dots, H_m(-))$.

The $\lambda 2$-fibration is given by the functor $p\colon \mathcal{F}^{\mathsf{Eq}} \to \mathcal{N}^{\mathsf{Eq}}$ defined by

$p\colon \mathcal{F}^{\mathsf{Eq}} \to \mathcal{N}^{\mathsf{Eq}}$ **objects** $p(F\colon |\mathsf{rel}|^n \to \mathsf{rel}) = n.$

morphisms $p(h, \eta) = h.$

That p is a $\lambda 2$-fibration will follow as an instantiation of the general framework we present in the next section.

5.3 Bifibrational relational parametricity

In this section we want to show that the $\lambda 2$-fibration $p\colon \mathcal{F}^{\mathsf{Eq}} \to \mathcal{N}^{\mathsf{Eq}}$ is a particular instantiation of a more general framework for parametric models of System F. For the rest of the chapter we consider a fibration of relations $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ arising via change of base from a fibration $U\colon \mathcal{E} \to \mathcal{B}$. For brevity we denote by $T = (T_1, T_0)\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$

a fibred functor $(T_1, T_0 \times T_0)$ as shown in the following diagram

$$\begin{array}{ccc} |\mathsf{rel}(\mathcal{E})|^n & \xrightarrow{T_1} & \mathsf{rel}(\mathcal{E}) \\ {\scriptstyle |\mathsf{rel}(U)|^n}\downarrow & & \downarrow{\scriptstyle \mathsf{rel}(U)} \\ |\mathcal{B}|^n \times |\mathcal{B}|^n & \xrightarrow[T_0 \times T_0]{} & \mathcal{B} \times \mathcal{B}. \end{array}$$

We generalise Definition 5.4 to a general fibration of relations $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$.

Definition 5.7. Let $\mathsf{rel}(U)\colon \mathsf{rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ be a fibration of relations with equality functor $\mathsf{Eq}\colon \mathcal{B} \to \mathsf{rel}(\mathcal{E})$. We say that a fibred functor $(F_1, F_0)\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ is **equality preserving** if $F_1 \circ |\mathsf{Eq}|^n \cong \mathsf{Eq} \circ F_0$.

We generalise $\mathcal{N}^{\mathsf{Eq}}$ in the following way

Definition 5.8. We define the category $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ as

$\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$	**objects**	natural numbers $n \in \mathbb{N}$.
	morphisms	a morphism $h\colon n \to m$ consists of m-tuples of equality preserving fibred functors $H_i : |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$.

Again every morphism $h = (H_1, \ldots, H_m)\colon n \to m$ in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ gives rise to a functor

$$(H_1(-), \ldots, H_m(-))\colon |\mathsf{rel}(U)|^n \to |\mathsf{rel}(U)|^m$$

defined by

$$(H_1(-), \ldots, H_m(-))_j(X_1, \ldots, X_m) = ((H_1)_j(X_1, \ldots, X_m), \ldots, (H_1)_j(X_1, \ldots, X_m))$$

where $j \in \{1, 0\}$ and every X_i lives in the correct category.

The generalisation of $\mathcal{F}^{\mathsf{Eq}}$ is the following one:

Definition 5.9. The category $\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ is given by

$\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$	**objects**	equality preserving fibred functors from $|\mathsf{rel}(U)|^n$ to $\mathsf{rel}(U)$ for some $n \in \mathbb{N}$.

morphisms a morphism $(h, \eta)\colon F \to G$, where $F\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ and $G\colon |\mathsf{rel}(U)|^m \to \mathsf{rel}(U)$, consists of a morphism $h = (H_1, \ldots, H_m) : n \to m$ in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ and a fibred natural transformation $\eta : F \to G \circ (H_1(-), \ldots, H_m(-))$.

Definition 5.10. We define the functor $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ as

$p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ **objects** $p(F\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)) = n$

morphisms $p(f, \eta) = f$

In the rest of this section we are going to prove that $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ is a $\lambda 2$-fibration.

5.3.1 The functor p is a fibration with generic object

We start by showing that the functor p is a fibration, actually a split one.

Proposition 5.11. The functor $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ is a split fibration.

Proof. Let $h = (H_1, \ldots, H_m)\colon n \to m$ be a morphism in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ and $G\colon |\mathsf{rel}(U)|^m \to \mathsf{rel}(U)$ be an object in $\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ over m. We denote by $H = (H_1(-), \ldots, H_m(-))$. The domain of the cartesian morphism over h is given by $G \circ H$ and the cartesian morphism is (h, id). Let $F\colon |\mathsf{rel}(U)|^l \to \mathsf{rel}(U)$ be an equality preserving fibred functor and $(q, \eta)\colon F \to G$ be a morphism in $\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ for which there is a morphism $k\colon l \to n$ in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ such that $h \circ k = q$. We want to show that (h, id) has the cartesian property. The unique morphism from F to $H \circ G$ over k is given by (k, η), where $\eta\colon F \Rightarrow G \circ Q = G \circ H \circ K$. It is clear that $(q, \eta) = (h, \mathsf{id}) \circ (k, \eta)$. For the uniqueness notice that the first component k is fixed, and if there was another η' such that $\mathsf{id} \circ \eta' = \eta$, then $\eta' = \eta$. □

The fibration p has a split generic object.

Lemma 5.12. The split fibration $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ has split generic object $\Omega = 1$.

Proof. Since it is a split fibration for the generic object we need an object Ω in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ such that $\mathrm{Hom}_{\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}}(n, \Omega) \cong |(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n|$ for every object n in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$. It is immediate to see that the morphisms from n to 1 in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ are equality preserving fibred functors $F\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ which are exactly the objects in $|(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n|$. □

The next step is to prove that p is fibred cartesian closed.

5.3.2 The fibration p is fibred cartesian closed

The fibres $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$ are given by

$(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$	**objects**	equality preserving fibred functors from $|\mathsf{rel}(U)|^n$ to $\mathsf{rel}(U)$.
	morphisms	if $F\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ and $G\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$, a morphism $(\mathsf{id}, \eta)\colon F \to G$ consists of a fibred natural transformation $\eta : F \to G$.

We first study the general case in which the objects of $\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ are fibred functors, not necessarily equality preserving. In detail let $\mathcal{F}_{\mathsf{rel}(U)}$ be the category

$\mathcal{F}_{\mathsf{rel}(U)}$	**objects**	fibred functors from $|\mathsf{rel}(U)|^n$ to $\mathsf{rel}(U)$ for some $n \in \mathbb{N}$.
	morphisms	if $F\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ and $G\colon |\mathsf{rel}(U)|^m \to \mathsf{rel}(U)$, a morphism $(h, \eta)\colon F \to G$ consists of a morphism $h = (H_1, \ldots, H_m) : n \to m$ in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ and a fibred natural transformation $\eta : F \to G \circ (H_1(-), \ldots, H_m(-))$.

Clearly $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ extends to a functor $p\colon \mathcal{F}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ defined as

$p\colon \mathcal{F}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$	**objects**	$p(F\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)) = n$
	morphisms	$p(f, \eta) = f$

The fibres for $\mathcal{F}_{\mathsf{rel}(U)}$ are given by

$(\mathcal{F}_{\mathsf{rel}(U)})_n$	**objects**	fibred functors from $|\mathsf{rel}(U)|^n$ to $\mathsf{rel}(U)$.

morphisms if $F\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ and $G\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ a morphism $(\mathsf{id}, \eta)\colon F \to G$ consists of a fibred natural transformation $\eta : F \to G$.

Then we use the following lemma in order to restrict to the case of $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$:

Lemma 5.13. Let $\mathcal{D}$ be a cartesian closed category and $\mathcal{C}$ be a full subcategory of $\mathcal{D}$. If $\mathcal{C}$ is closed under the cartesian closed structure from $\mathcal{D}$, i.e. $\mathbf{1} \in \mathcal{C}$, and if X, Y are objects in $\mathcal{C}$ then $X \times Y$, $X \Rightarrow Y$ are in $\mathcal{C}$, then $\mathcal{C}$ is cartesian closed, inheriting the cartesian closed structure from $\mathcal{D}$.

Proof. Using full and faithfulness it is clear that $\mathbf{1}$ is terminal in $\mathcal{C}$ and that if X and Y are objects of $\mathcal{C}$ then $X \times Y$ in $\mathcal{D}$ is also the product in $\mathcal{C}$.

For the exponential we have that

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{C}}(X \times Y, Z) &= \mathrm{Hom}_{\mathcal{D}}(X \times Y, Z) \\
&\cong \mathrm{Hom}_{\mathcal{D}}(X, Y \Rightarrow Z) \\
&= \mathrm{Hom}_{\mathcal{C}}(X, Y \Rightarrow Z)
\end{aligned}$$

which is the natural isomorphism of the adjunction $\_ \times Y \dashv Y \Rightarrow \_$. $\square$

Lemma 5.13 applies for $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$ which trivially injects into $(\mathcal{F}_{\mathsf{rel}(U)})_n$ for every $n \in \mathbb{N}$ and the injection is full.

Note that we are in the situation where the objects in the fibres are fibred functors $T\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ like in the case of Reynolds' model

$$\begin{array}{ccc}
|\mathbf{Rel}(\mathcal{E})|^n & \xrightarrow{T_1} & \mathbf{Rel}(\mathcal{E}) \\
{\scriptstyle |\mathsf{rel}(U)|^n}\downarrow & & \downarrow{\scriptstyle |\mathsf{rel}(U)|^n} \\
|\mathcal{B}|^n \times |\mathcal{B}|^n & \xrightarrow[T_0 \times T_0]{} & \mathcal{B} \times \mathcal{B}
\end{array}$$

and the morphisms are fibred natural transformations $\eta\colon T \to G$

$$\begin{array}{ccc}
|\mathbf{Rel}(\mathcal{E})|^n & \overset{T_1}{\underset{G_1}{\rightrightarrows}} \ \Downarrow \eta_1 & \mathbf{Rel}(\mathcal{E}) \\
\downarrow {\scriptstyle |\mathsf{rel}(U)|^n} & & \downarrow {\scriptstyle \mathsf{rel}(U)} \\
|\mathcal{B}|^{|\Gamma|} \times |\mathcal{B}|^n & \overset{T_0 \times T_0}{\underset{G_0 \times G_0}{\rightrightarrows}} \ \Downarrow \eta_0 \times \eta_0 & \mathcal{B} \times \mathcal{B}.
\end{array}$$

The condition that p is fibred cartesian closed is essential for interpreting term contexts $\Delta = x_1 \colon T_1, \ldots, x_n \colon T_n$ as $[\![\Delta]\!] = [\![T_1]\!] \times \ldots \times [\![T_n]\!]$, the product types using the cartesian product and the arrow types $U \to V$ as exponential objects $[\![U]\!] \Rightarrow [\![V]\!]$. In order to see how it works fibrationally, we first start by studying the simpler case of the category $|\mathcal{C}|^n \to \mathcal{C}$ whose objects are functors $F \colon |\mathcal{C}|^n \to \mathcal{C}$ for some fixed n, and a morphism $\eta \colon F \to G$ is a natural transformation. We will then extend the result to fibred functors.

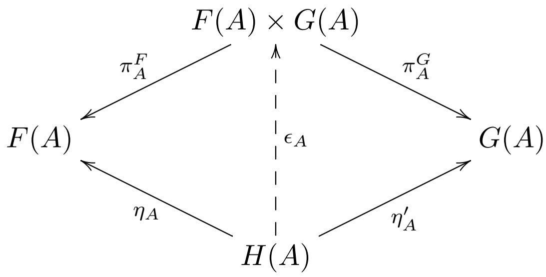
Lemma 5.14. If $\mathcal{C}$ is cartesian closed, then $|\mathcal{C}|^n \to \mathcal{C}$ is cartesian closed.

Proof. Limits are computed pointwise. Moreover, since the domain of the functors are discrete categories, in order to have a natural transformation it is enough to provide a family of morphisms since the condition to be natural is trivially satisfied.

The terminal object in $|\mathcal{C}|^n \to \mathcal{C}$ is the constant functor $K_{\mathbf{1}}$ sending every object to $\mathbf{1}$, the terminal object in $\mathcal{C}$. Note in fact that for every object F in $|\mathcal{C}|^n \to \mathcal{C}$ there is a unique natural transformation $! \colon F \to K_{\mathbf{1}}$ whose components are the unique maps $!_A \colon F(A) \to \mathbf{1}$.

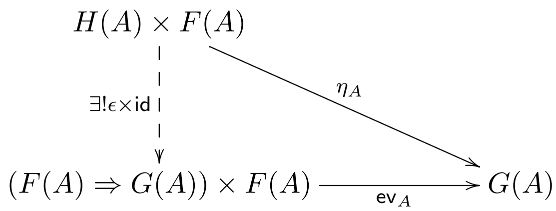
The product of two functors $F, G \colon |\mathcal{C}|^n \to \mathcal{C}$ is given componentwise by the formula $(F \times G)(A) = F(A) \times G(A)$. In fact, given a third functor $H \colon |\mathcal{C}|^n \to \mathcal{C}$, and two natural transformations $\eta \colon H \to F$ and $\eta' \colon H \to G$, the unique natural transformation

$\epsilon\colon H \to F \times G$ has components



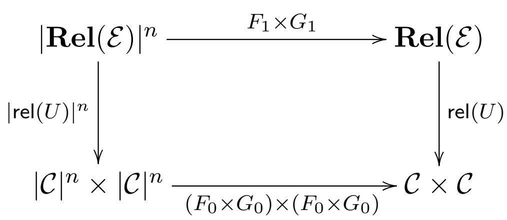
identified by the universal property of the product $F(A) \times G(A)$.

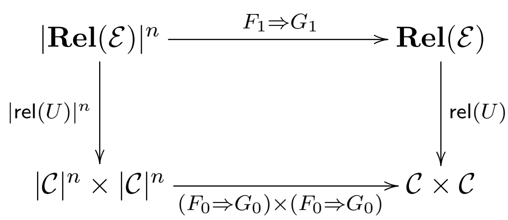
Finally also the exponentiation is given componentwise. Let $F, G, H\colon |\mathcal{C}|^n \to \mathcal{C}$ be three functors and consider a natural transformation $\eta\colon H \times F \to G$. The exponentiation $F \Rightarrow G$ is given by the formula $(F \Rightarrow G)(A) = F(A) \Rightarrow G(A)$, and the universal property is given pointwise by



which is the universal property for the exponential object $F(A) \Rightarrow G(A)$ in $\mathcal{C}$. □

In the fibre $(\mathcal{F}_{\mathsf{rel}(U)})_n$ objects are fibred functors from $|\mathsf{rel}(U)|^n$ to $\mathsf{rel}(U)$. If $\mathbf{Rel}(\mathcal{E})$ and $\mathcal{B}$ are cartesian closed, then both $|\mathbf{Rel}(\mathcal{E})|^n \to \mathbf{Rel}(\mathcal{E})$ and $|\mathcal{B}|^n \times |\mathcal{B}|^n \to \mathcal{B} \times \mathcal{B}$ are cartesian closed. This is not enough for two reasons. First in order to have a $\lambda 2$-fibration we want that reindexing functors preserve structure in the fibres. We will see that reindexing functors are given by precomposition and it is not difficult to check that precomposition preserves the needed structure on the nose. The second reason is that we want fibred functors, which means that we want the following diagrams to commute





for every two fibred functors $F, G\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$, i.e. the cartesian closed structure should be preserved by the fibration. For this reason we introduce the following definition:

Definition 5.15. A fibration $U : \mathcal{E} \to \mathcal{B}$ is an **arrow fibration** if it admits truth functor, both $\mathcal{E}$ and $\mathcal{B}$ are CCCs, and U preserves the cartesian closed structure.

Change of base preserves arrow fibrations.

Lemma 5.16. If $U\colon \mathcal{E} \to \mathcal{B}$ is an arrow fibration, so it is $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$.

Proof. This is straightforward to check. □

As a direct consequence of Lemma 5.14 and Lemma 5.16 we have the two following results:

Lemma 5.17. If $U\colon \mathcal{E} \to \mathcal{B}$ is an arrow fibration and $F, G\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ are two fibred functors, then $F \times G = (F_1 \times G_1, F_0 \times G_0)$ and $F \Rightarrow G = (F_1 \Rightarrow G_1, F_0 \Rightarrow G_0)$ are fibred functors.

Proof. It follows from Lemma 5.14, Lemma 5.16 and the definition of arrow fibrations. □

Corollary 5.18. If $U\colon \mathcal{E} \to \mathcal{B}$ is an arrow fibration, then $p\colon \mathcal{F}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ is fibred cartesian closed.

Proof. By Lemma 5.17, we have exponential objects and products. The terminal fibred functor is given by $K_{\mathbf{1}}$ sending every relation R to $(\mathbf{1}, \mathbf{1}, \mathbf{1})$ and every object A in $\mathcal{B}$ to $\mathbf{1}$. Since $U\colon \mathcal{E} \to \mathcal{B}$ is an arrow fibration, so it is $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ which means that terminal object is preserved and $K_{\mathbf{1}}$ is a fibred functor. □

The following results from [Jac99] can be used to find arrow fibrations.

Lemma 5.19. Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with fibred terminal objects and $\mathcal{B}$ be a CCC. If $U : \mathcal{E} \to \mathcal{B}$ is a fibred CCC and has simple products, then $\mathcal{E}$ is a CCC and U preserves the cartesian closed structure. □

Change of base preserves simple products and fibred structure, so $\mathsf{rel}(U)$ is a fibred CCC with simple products if U is. Moreover, $\mathcal{B} \times \mathcal{B}$ is a CCC if $\mathcal{B}$ is. Lemma 5.19 thus derives structure in $\mathsf{rel}(U)$ from structure in U.

Finally we restrict to $\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ and prove that also $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ is fibred cartesian closed. As shown by Reynolds in [Rey83], exponential objects in the case of rel preserve equality. In general we need the following definition:

Definition 5.20. A fibration of relations $\mathbf{Rel}(U)$ is an **equality preserving arrow fibration** if it is an arrow fibration and $\mathsf{Eq} : \mathcal{B} \to \mathbf{Rel}(\mathcal{E})$ preserves exponentials and products.

Lemma 5.21. Let $U\colon \mathcal{E} \to \mathcal{B}$ be an equality preserving arrow fibration, then the fibration $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ is fibred cartesian closed.

Proof. By Lemma 5.13 and Lemma 5.14, we need only to prove that every fibre $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$ is closed under products, exponentials and has terminal object. This means that we need to prove that if $F, G\colon |\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ are two equality preserving fibred functors, then $F \times G$, $F \Rightarrow G$, and $K_{\mathbf{1}}$ are equality preserving.

For the equality preservation of $K_{\mathbf{1}}$ note that, by definition, the truth functor K is a right adjoint. In particular, it preserves the terminal object. By definition $\mathsf{Eq}(\mathbf{1}) = \Sigma_{\delta_{\mathbf{1}}} K(\mathbf{1})$ and $\delta_{\mathbf{1}}\colon \mathbf{1} \to \mathbf{1} \times \mathbf{1}$ is an isomorphism. Hence $(\delta_{\mathbf{1}})_{\S}\colon 1 \cong K(1) \to \mathsf{Eq}(1)$ is an isomorphism by Lemma 2.16.

Using the fact that we are in a equality preserving arrow fibration and that both F and G are equality preserving, we have the following derivations:

$$\begin{aligned} \mathsf{Eq}(F_0(\bar{A}) \times G_0(\bar{A})) &\cong \mathsf{Eq}(F_0(\bar{A})) \times \mathsf{Eq}(F_0(\bar{A})) \\ &\cong F_1(\mathsf{Eq}^n(\bar{A})) \times G_1(\mathsf{Eq}^n(\bar{A})), \end{aligned}$$

and

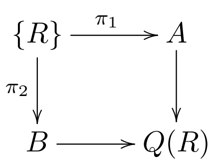
$$\begin{aligned} \mathsf{Eq}(F_0(\bar{A}) \Rightarrow G_0(\bar{A})) &\cong \mathsf{Eq}(F_0(\bar{A})) \Rightarrow \mathsf{Eq}(F_0(\bar{A})) \\ &\cong F_1(\mathsf{Eq}^n(\bar{A})) \Rightarrow G_1(\mathsf{Eq}^n(\bar{A})). \end{aligned}$$

This gives the thesis. □

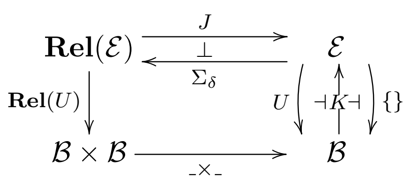
The following result helps in determining when a fibration of relations is an equality preserving arrow fibration:

Lemma 5.22. Let $U : \mathcal{E} \to \mathcal{B}$ be a bifibration with fibred terminal objects and $\mathcal{B}$ be a CCC. If $\mathsf{Eq} : \mathcal{B} \to \mathbf{Rel}(\mathcal{E})$ has a left adjoint Q, then Eq preserves products. Moreover Eq preserves exponentials iff Q satisfies the Frobenius property. Such a Q exists if $U : \mathcal{E} \to \mathcal{B}$ has full comprehension, $\mathsf{Eq} : \mathcal{B} \to \mathbf{Rel}(\mathcal{E})$ is full and $\mathcal{B}$ has pushouts.

Proof. We have that Eq preserves products because it is a right adjoint. The part on the preservation of exponentials is Proposition 6.2 in Hermida and Jacobs [HJ98]. They study the case of homogeneous relations (relations on one set), but the same proof is applicable here because the proof's structure uses only functorial arguments. For the last part, if we have comprehension and pushouts, then Q can be defined as mapping a relation R over (A, B) to the pushout of π_1 and π_2, where $\langle \pi_1, \pi_2 \rangle : \{R\} \to A \times B$ is the canonical map:



To see that Q is left adjoint to $\mathsf{Eq} = \Sigma_\delta \circ K$, first note that $\mathsf{Eq} \dashv \{\_\} \circ J$, where $J : \mathbf{Rel}(\mathcal{E}) \to \mathcal{E}$ arises from the pullback construction of $\mathbf{Rel}(\mathcal{E})$:



Since Eq is always faithful and is full by assumption, the unit of the adjunction $\mathsf{Eq} \dashv \{\_\} \circ J$ is a natural isomorphism, i.e., $\{\mathsf{Eq}(A)\} \cong A$ for all A. By the universal property of the pushout, morphisms $f : Q(R) \to C$ are in bijective correspondence with pairs of morphisms $f_1 : A \to C$ and $f_2 : B \to C$ such that $f_1 \circ \pi_1 = f_2 \circ \pi_2$. This is only the case if the following diagram commutes:

$$\begin{array}{ccc} \{R\} & \xrightarrow{f_1 \circ \pi_1} & \{\mathsf{Eq}(C)\} \\ {\scriptstyle \langle \pi_1, \pi_2 \rangle}\downarrow & & \downarrow{\scriptstyle \delta} \\ A \times B & \xrightarrow[f_1 \times f_2]{} & C \times C \end{array}$$

But this is exactly the diagram from the definition of full comprehension. Thus, by full and faithfulness of Eq, this diagram commutes iff there is a morphism $g : R \to \mathsf{Eq}(C)$ such that $\pi_1 \circ f_1 = \{g\}$. In other words, morphisms from $Q(R)$ to C are in bijective correspondence with morphisms from R to $\mathsf{Eq}(C)$, as required.

For the Frobenius property, we need to show that $Q(R \times \mathsf{Eq}\, C) = Q(R) \times C$, i.e., that $Q(R) \times C$ is the pushout

$$\begin{array}{ccc} \{R\} \times C & \xrightarrow{\pi_1 \times \mathsf{id}} & A \times C \\ {\scriptstyle \pi_2 \times \mathsf{id}}\downarrow & & \downarrow \\ B \times C & \longrightarrow & Q(R) \times C \end{array}$$

Here, we have used the facts that $\{\mathsf{Eq}(C)\} = C$ and that $\{\}$ is a right adjoint and thus preserves products. However, since $\mathcal{B}$ is a CCC, we have that $\_ \times C$ is a left adjoint and thus preserves colimits. □

5.3.3 Simple Ω-products

In order to interpret forall types we need simple Ω-products.

We saw that the fibres $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$ have equality preserving fibred functors $|\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$ as objects and fibred natural transformations as morphisms. Recall that the generic object Ω is given by 1 in $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$, and the product is the sum of natural numbers.

The projection $\pi_n \colon n + 1 \to n$ consists of the family of functors $(\pi_n)_i \colon |\mathbf{Rel}(U)|^{n+1} \to \mathbf{Rel}(U)$ for $1 \leq i \leq n$, where $(\pi_n)_i$ projects out the i-th component. We denote with the same notation π_n the functor $\pi_n \colon |\mathbf{Rel}(U)|^{n+1} \to |\mathbf{Rel}(U)|^n$ which projects out the first n components. In the proof of Lemma 5.11 we saw that reindexing is given by precomposition and that weakening $\pi_n^* \colon (|\mathbf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathbf{Rel}(U)) \to (|\mathbf{Rel}(U)|^{n+1} \to_{\mathsf{Eq}} \mathbf{Rel}(U))$ is given by precomposition $\_ \circ \pi_n$.

The previous observations lead us to the following definition:

Definition 5.23. $\mathbf{Rel}(U)$ is a **$\forall$-fibration** if, for every projection $\pi_n : |\mathbf{Rel}(U)|^{n+1} \to |\mathbf{Rel}(U)|^n$, the functor $\_ \circ \pi_n : (|\mathbf{Rel}(U)|^n \to_{\mathsf{Eq}} \mathbf{Rel}(U)) \to (|\mathbf{Rel}(U)|^{n+1} \to_{\mathsf{Eq}} \mathbf{Rel}(U))$ has a right adjoint $\forall_n$ and this family of adjunctions is natural in n.

Definition 5.23 is equivalent to asking that p has simple Ω-products. We write $\forall$ for $\forall_n$ when n can be inferred. This definition follows, e.g., Dunphy and Reddy [DR04] by "baking" the Identity Extension Lemma into the definition of forall types — in the sense that the very existence of $\forall$ requires that if F is equality preserving then so is $\forall F$ — rather than relegating it to a result to be proved post facto. If U is faithful, then Definition 5.23 can be reformulated in terms of more basic concepts using its opfibrational structure. The Identity Extension Lemma then becomes a consequence of the definition, rather than an intrinsic part of it. We will come back later to this in Chapter 7.

5.3.4 Fibred functors with discrete domains form a parametric model

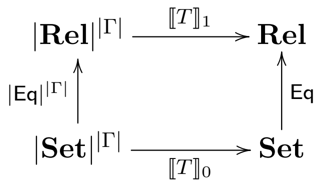
All the structure considered in this section is assumed to be split. We sum up the results obtained so far in the following theorem:

Theorem 5.24. If $\mathbf{Rel}(U)$ is an equality preserving arrow fibration and a $\forall$-fibration, then $p \colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ as in Proposition 5.11 is a $\lambda 2$-fibration in which types $\Gamma \vdash T\ \mathsf{type}$ are interpreted as equality preserving fibred functors $[\![T]\!] : |\mathbf{Rel}(U)|^{|\Gamma|} \to \mathbf{Rel}(U)$ and terms $\Gamma; \Delta \vdash t : T$ are interpreted as fibred natural transformations $[\![t]\!] : [\![\Delta]\!] \to [\![T]\!]$.

Proof. By showing that p is a $\lambda 2$-fibration we can model System F as we described in Section 4.5. It naturally follows that we interpret types as equality preserving fibred functors and terms as fibred natural transformations since they are the objects and the morphisms in the fibres of p. By Proposition 5.11, the functor p a fibration. The base category $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ has finite products given by natural number addition, and 1 is a generic object Ω for $\mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ as shown in Section 5.3.1. By Corollary 5.18 the fibres are cartesian closed. Finally, p has simple Ω-products since $\mathbf{Rel}(U)$ is a $\forall$-fibration. Thus, p is a $\lambda 2$-fibration. □

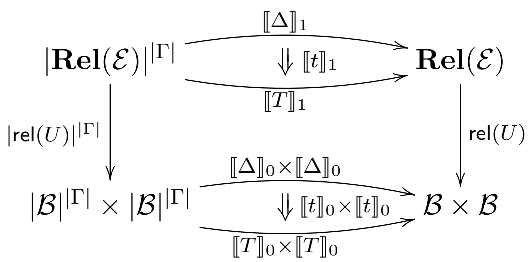
It easily follows that Identity Extension Lemma and Abstraction Theorem hold:

Corollary 5.25 (Identity Extension Lemma)**.** Let $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ be an equality preserving arrow fibration and $\forall$-fibration, and let $\Gamma \vdash T$ type be a type judgment. Then $[\![T]\!]_1 \circ |\mathsf{Eq}|^{|\Gamma|} = \mathsf{Eq} \circ [\![T]\!]_0$, i.e. the following diagram commutes



□

Corollary 5.26 (Abstraction Theorem)**.** Let $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ be an equality preserving arrow fibration and a $\forall$-fibration. We can interpret every term judgment $\Gamma; \Delta \vdash t : T$ as a fibred natural transformation $([\![t]\!]_1, [\![t]\!]_0) : [\![\Delta]\!] \to [\![T]\!]$.



□

Unwinding the interpretation of System F in $\lambda 2$-fibrations as shown in Section 4.5, we see that, for every fibration $U : \mathcal{E} \to \mathcal{B}$ satisfying the hypotheses of the theorem, we get the following: for every System F type $\Gamma \vdash T$ and term $\Gamma; \Delta \vdash t : T$, we get

1. a standard interpretation of $\Gamma \vdash T$ as a functor $[\![T]\!]_0 : |\mathcal{B}|^{|\Gamma|} \to \mathcal{B}$;

2. a relational interpretation of $\Gamma \vdash T$ as a functor $[\![T]\!]_1 : |\mathbf{Rel}(\mathcal{E})|^{|\Gamma|} \to \mathbf{Rel}(\mathcal{E})$;

3. the standard and relational interpretation of $\Gamma \vdash T$ form a fibred functor $([\![T]\!]_1, [\![T]\!]_0) : |\mathsf{rel}(U)|^{|\Gamma|} \to \mathsf{rel}(U)$;

4. the Identity Extension Lemma in the form of Corollary 5.25, i.e., a proof that $[\![T]\!]$ is equality preserving;

5. a standard interpretation of $\Gamma; \Delta \vdash t : T$ as a natural transformation $[\![t]\!]_0 : [\![\Delta]\!]_0 \to [\![T]\!]_0$; and

6. the Abstraction Theorem in the form of Corollary 5.26, i.e., a proof that $\Gamma; \Delta \vdash t : T$ has a relational interpretation as a natural transformation $[\![t]\!]_1 : [\![\Delta]\!]_1 \to [\![T]\!]_1$ over $[\![t]\!]_0 \times [\![t]\!]_0$.

Theorem 5.24 also gives a powerful internal language [Jac99], where base types in type context Γ are given by fibred functors $|\mathbf{Rel}(U)|^{|\Gamma|} \to_{\mathsf{Eq}} \mathbf{Rel}(U)$, and base term constants in term context Δ are given by fibred natural transformations $[\![\Delta]\!] \to [\![T]\!]$. Thus, we can use this language to reason about our models using System F. This will be used in the proofs of Theorems 6.3 and 6.7 below.

Chapter 6

Consequences of parametricity

In this chapter we use the framework that we introduced in Chapter 5 in order to derive expected consequences of parametricity: existence of initial algebras, existence of final coalgebras and dinaturality.

6.1 Existence of initial algebras

Let $F : \mathcal{C} \to \mathcal{C}$ be an endofunctor. An **F-algebra** is a pair (A, k_A) with A an object of $\mathcal{C}$ and $k_A : FA \to A$ a morphism. We call A the **carrier** of the F-algebra and k_A its **structure map**. A morphism $h : A \to B$ in $\mathcal{C}$ is an **F-algebra homomorphism** $h : (A, k_A) \to (B, k_B)$ if $k_B \circ F(h) = h \circ k_A$. An F-algebra (Z, in) is **weakly initial** if, for any F-algebra (A, k_A), there exists a mediating F-algebra homomorphism $\mathit{fold}[A, k_A] : (Z, in) \to (A, k_A)$. It is an **initial F-algebra** if $\mathit{fold}[A, k_A]$ is unique.

Let $F = (F_1, F_0) : \mathsf{rel}(U) \to \mathsf{rel}(U)$ be an equality preserving lifted functor (note that the domain of F is not discrete and that F need not preserve cartesian morphisms) with a **strength** $t = (t_1, t_0)$, i.e., a family of morphisms $(t_0)_{A,B} : A \Rightarrow B \to F_0A \Rightarrow F_0B$ and $(t_1)_{R,S} : R \Rightarrow S \to F_1R \Rightarrow F_1S$ with $(t_1)_{R,S}$ over $((t_0)_{A,B}, (t_0)_{C,D})$ if R is over (A, B) and S is over (C, D), such that t preserves identity and composition. A functor with a strength is said to be **strong**. Because of the discrete domains, t is a natural transformation from $\_ \Rightarrow \_$ to $F\_ \Rightarrow F\_$ in $|\mathsf{rel}(U)|^2 \to_{\mathsf{Eq}} \mathsf{rel}(U)$, which is the full subcategory of $|\mathsf{rel}(U)|^2 \to \mathsf{rel}(U)$ whose objects are the equality preserving fibred functors, and thus $\alpha, \beta; \cdot \vdash \underline{t} : (\alpha \to \beta) \to (\underline{F}[\alpha] \to \underline{F}[\beta])$ represents the action of $\underline{F}$ on morphisms in the internal language (see Section 4.6), and where $\underline{F}$ acts on the objects as $\underline{F}(\underline{A}) = \underline{F(A)}$. All type expressions with one free type variable occurring only positively give rise to strong functors, but there are further examples of such functors, for instance if the model contains non-System F type constructions with natural functorial (and relational) interpretations — for example, those of dependent types in **Set**. We will show that an initial F_0-algebra exists. For this, we first construct a weak initial F_0-algebra, which can be done in any $\lambda 2$-fibration. Using the internal language, we define Z by $(Z_1, Z_0) = [\![\forall X.(\underline{F}X \to X) \to X]\!]$.

Lemma 6.1. Z_0 is the carrier of a weak initial F_0-algebra (Z_0, in_0) with mediating morphism $fold_0[A, k]$ and Z_1 is the carrier of a weak initial F_1-algebra (Z_1, in_1) with mediating morphism $fold_1[A, k]$.

Proof. Using the internal language, we first define the term $fold = \Lambda A.\lambda k : \underline{F}A \to A.\lambda z.\, z\, A\, k$. We then define $fold_i[A, k] = \theta^{-1}([\![fold\ \underline{A}\ \underline{k}]\!]_i)$, where θ is the natural bijection $\theta : \mathrm{Hom}_{\mathcal{C}}(\Gamma \times A, B) \cong \mathrm{Hom}_{\mathcal{C}}(\Gamma, A \Rightarrow B)$ of the adjunction $- \times A \dashv A \Rightarrow -$, and $\underline{A}$ and $\underline{k}$ are the internal expressions corresponding to the components of another F_0- or F_1-algebra (A, k), as appropriate. We further define $(in_1, in_0) = \theta^{-1}([\![\lambda x.\, \Lambda X.\, \lambda k.\, k\, (\underline{t}\, (fold\, X\, k)\, x)]\!])$, where $\underline{t}$ is the internal representation of the strength of F. By equational reasoning in System F, $fold_0$ and $fold_1$ are algebra homomorphisms:

$$\begin{aligned}
fold[A,k] \circ in &= \theta^{-1}([\![\lambda z.\, z\, \underline{A}\, \underline{k}]\!]) \circ \theta^{-1}([\![\lambda x.\, \Lambda X.\, \lambda k.\, k\, (\underline{t}\, (fold\, X\, k)\, x)]\!]) \\
&= \theta^{-1}([\![\lambda x.((\lambda z.\, z\, \underline{A}\, \underline{k})(\, \Lambda X.\, \lambda k.\, k\, (\underline{t}\, (fold\, X\, k)\, x)))]\!]) \\
&= \theta^{-1}([\![\lambda x.(\underline{k}\, (\underline{t}\, (fold\, \underline{A}\, \underline{k})\, x))]\!]) \\
&= \theta^{-1}([\![\underline{k}]\!]) \circ \theta^{-1}([\![\underline{t}\, (fold\, \underline{A}\, \underline{k})]\!]) \\
&= k \circ F(fold[A,k])
\end{aligned}$$

□

To show that $fold_0$ is unique, we use the graph relations from Section 3.3. Recall that a category with a terminal object $\mathbf{1}$ is **well-pointed** if, for any $f, g : A \to B$, we have $f = g$ iff $f \circ e = g \circ e$ for all $e : \mathbf{1} \to A$. We only consider well-pointed base categories; well-pointedness is used to convert internal language reasoning in non-empty contexts to closed contexts, so that we can apply semantic techniques such as Theorem 3.12. In detail, if we have a term judgment $\Gamma; \Delta \vdash t \colon T$, its interpretation is given by the fibred natural transformation $[\![t]\!] \colon [\![\Delta]\!] \to [\![T]\!]$ and we compose it with a natural transformation

$k\colon \mathbf{1} \to [\![\Delta]\!]$ obtaining $[\![t]\!] \circ k\colon \mathbf{1} \to [\![T]\!]$. We will often only say that we use well pointedness and leave implicit the morphism k writing only $[\![t]\!]$.

The next results use the map $\psi_h\colon F_1\langle h\rangle \to \langle F_0 h\rangle$ given in the Graph Lemma 3.12.

Lemma 6.2. Assume that the underlying bifibration satisfies the Beck-Chevalley condition, and that Eq is full.

1. If $\mathcal{B}$ is well-pointed, then $\mathit{fold}_0[Z_0, \mathit{in}_0] = \mathsf{id}_Z$.
2. For every F_0-algebra homomorphism $h : (Z_0, \mathit{in}_0) \to (A, k_A)$, we have that $h \circ \mathit{fold}_0[Z_0, \mathit{in}_0] = \mathit{fold}_0[A, k_A]$.

Proof. 1. We want to show $[\![\vdash \mathit{fold}\ \underline{Z_0}\ \underline{\mathit{in}_0}]\!]_0 = [\![\vdash \lambda z.z]\!]_0$. By the ξ- and η-rules, which are valid in all $\lambda 2$-fibrations by soundness Theorem 4.9, it suffices to show that

$$[\![X; k : \underline{F_0}X \to X \vdash \lambda z\,.\,(\mathit{fold}\ \underline{Z_0}\ \underline{\mathit{in}_0})\, z\, X\, k]\!]_0 = [\![X; k : \underline{F_0}X \to X \vdash \lambda z\,.\, z\, X\, k]\!]_0$$

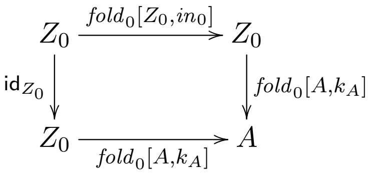
By well-pointedness, this reduces to showing $[\![\lambda z.\,(\mathit{fold}\ \underline{Z_0}\ \underline{\mathit{in}_0})\, z\, \underline{A}\, \underline{k_A}]\!]_0 = [\![\lambda z.\, z\, \underline{A}\, \underline{k_A}]\!]_0$ for any natural transformation $k : F_0 \to \mathsf{Id}$. We first prove $\mathit{fold}_0[A, k_A] \circ \mathit{fold}_0[Z_0, \mathit{in}_0] = \mathit{fold}_0[A, k_A]$. The following diagram commutes by weak initiality:

$$\begin{array}{ccc} F_0 Z_0 & \xrightarrow{F_0\,(\mathit{fold}_0[A,k_A])} & F_0 A \\ {\scriptstyle \mathit{in}_0}\downarrow & & \downarrow{\scriptstyle k_A} \\ Z_0 & \xrightarrow[\mathit{fold}_0[A,k_A]]{} & A \end{array}$$

so $(\mathit{in}_0, k_A) : F_0\,(\mathit{fold}_0[A, k_A]) \to \mathit{fold}_0[A, k_A]$ is a morphism in $\mathcal{B}^{\to}$. The graph functor then gives a morphism $\langle \mathit{in}_0, k_A\rangle : \langle F_0\,(\mathit{fold}_0[A, k_A])\rangle \to \langle \mathit{fold}_0[A, k_A]\rangle$, and thus a F_1-algebra k_1 defined by $\langle \mathit{in}_0, k_A\rangle \circ \psi_{\mathit{fold}_0[A,k_A]} : F_1\langle \mathit{fold}_0[A, k_A]\rangle \to \langle \mathit{fold}_0[A, k_A]\rangle$. By weak initiality of Z_1, the following diagram commutes:

$$\begin{array}{ccc} F_1 Z_1 & \xrightarrow{F_1\,(\mathit{fold}_1[\langle \mathit{fold}_0[A,k_A]\rangle,k_1])} & F_1\langle \mathit{fold}_0[A, k_A]\rangle \\ {\scriptstyle \mathit{in}_1}\downarrow & & \downarrow{\scriptstyle k_1} \\ Z_1 & \xrightarrow[\mathit{fold}_1[\langle \mathit{fold}_0[A,k_A]\rangle,k_1]]{} & \langle \mathit{fold}_0 A, k_A]\rangle \end{array}$$

By the Identity Extension Lemma, $Z_1 = \mathsf{Eq}\, Z_0 = \langle \mathsf{id}_{Z_0} \rangle$, so $\mathit{fold}_1[\langle \mathit{fold}_0[A, k_A] \rangle, k_1]$ is actually a morphism from $\langle \mathsf{id}_{Z_0} \rangle$ to $\langle \mathit{fold}_0[A, k_A] \rangle$. Since $\mathit{fold}_1[\langle \mathit{fold}_0[A, k_A] \rangle, k_1]$ is over $(\mathit{fold}_0[Z_0, \mathit{in}_0], \mathit{fold}_0[A, k_A])$, and Eq is always faithful and is full by assumption, Lemma 3.11 gives that $\mathit{fold}_1[\langle \mathit{fold}_0[A, k_A] \rangle, k_1] = \langle \mathit{fold}_0[Z_0, \mathit{in}_0], \mathit{fold}_0[A, k_A] \rangle$ and the following diagram commutes:

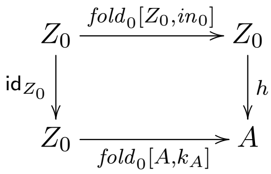


By the definition of *fold*,

$$\begin{aligned} [\![\lambda z.\, (\mathit{fold}\, \underline{Z_0}\, \underline{\mathit{in}_0})\, z\, \underline{A}\, \underline{k_A}]\!]_0 &= [\![\lambda z.\, (\mathit{fold} \underline{A}\, \underline{k_A})\, ((\mathit{fold}\, \underline{Z_0}\, \underline{\mathit{in}_0})\, z)]\!]_0 \\ &= [\![\lambda z.\, (\mathit{fold} \underline{A}\, \underline{k_A})\, z]\!]_0 \\ &= [\![\lambda z.\, z\, \underline{A}\, \underline{k_A}]\!]_0 \end{aligned}$$

Thus $\mathit{fold}_0[Z_0, \mathit{in}_0] = \mathsf{id}_{Z_0}$ as required.

2. Let $h : (Z_0, \mathit{in}_0) \to (A, k_A)$. The definition of $\langle \_ \rangle$ and the Graph Lemma give a unique morphism $k_1 = \langle \mathit{in}_0, k_A \rangle \circ \psi_h : F_1 \langle h \rangle \to \langle h \rangle$, and by weak initiality of Z_1 we have $\mathit{in}_1 \circ \mathit{fold}_1[\langle h \rangle, k_1] = k_1 \circ F_1\, (\mathit{fold}_1[\langle h \rangle, k_1])$. By the Identity Extension Lemma, $Z_1 = \mathsf{Eq}\, Z_0 = \langle \mathsf{id}_{Z_0} \rangle$, so that $\mathit{fold}_1[\langle h \rangle, k_1]$ is a morphism from $\langle \mathsf{id}_{Z_0} \rangle$ to $\langle h \rangle$. Since $\mathit{fold}_1[\langle h \rangle, k_1]$ is over $(\mathit{fold}_0[Z_0, \mathit{in}_0], \mathit{fold}_0[A, k_A])$, Lemma 3.11 gives that $\mathit{fold}_1[\langle h \rangle, k_1] = \langle \mathit{fold}_0[Z_0, \mathit{in}_0], \mathit{fold}_0[A, k_A] \rangle$ and the following diagram also commutes by fullness of the graph functor:



Thus, $h \circ \mathit{fold}_0[Z_0, \mathit{in}_0] = \mathit{fold}_0[A, k_A,]$. □

The proofs of the two parts of Lemma 6.2 are similar: both use the graph functor to map commuting diagrams in $\mathcal{B}$ to morphisms in $\mathbf{Rel}(U)$, and then use the Graph Lemma to see that these morphisms are F_1-algebras. Lemma 6.1 and Lemma 6.2 together now immediately imply the main result.

Theorem 6.3. If the underlying bifibration satisfies the Beck-Chevalley condition, Eq is full, and $\mathcal{B}$ is well-pointed, then (Z_0, in_0) is an initial F_0-algebra.

Proof. By Lemma 6.1, we know that (Z_0, in_0) is a weak initial F_0-algebra. We must show that $h = fold_0[A, k_A]$ for any $k_A : F_0 A \to A$ and any F_0-algebra morphism $h : (Z_0, in_0) \to (A, k_A)$. By Lemma 6.2(2), $fold_0[A, k_A] = h \circ fold_0[Z_0, in_0]$ and since $fold_0[Z_0, in_0] = \mathsf{id}_{Z_0}$ by Lemma 6.2(1), we have $h = fold_0[A, k_A]$, as required. □

One may wonder if the above result can be strengthened to get not only an initial F_0-algebra, but also an initial F_1-algebra. Certainly this is possible for the relations fibration $\mathsf{rel} \colon \mathbf{Rel} \to \mathbf{Set}$, since relations in $\mathbf{Rel}$ are proof irrelevant: maps either preserve relatedness or not. This translates in the axiomatic bifibrational setting to requiring the fibration $\mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ to be faithful. When it is, the weak initial F_1-algebra is, in fact, initial: faithfulness implies the required uniqueness.

6.2 Existence of final coalgebras

We can also dualise the proof from the previous section to show the existence of final coalgebras in the usual manner [Has94]. This requires us to first encode existential types in System F.

We encode existential types by $\exists X.T = \forall Y.(\forall X.(T \to Y)) \to Y$. We can support the introduction and elimination rules and they are the following

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma; \Delta \vdash u : T[X \mapsto A]}{\Gamma; \Delta \vdash \langle A, u \rangle : \exists X.T(X)} \qquad \frac{\Gamma; \Delta \vdash t : \exists X.T \quad \Gamma, Z; \Delta, y : T[X \mapsto Z] \vdash s : S}{\Gamma; \Delta \vdash (\mathsf{open}\, t \,\mathsf{as}\, \langle Z, y \rangle \,\mathsf{in}\, s) : S}$$

with the conversion $\mathsf{open}\, \langle A, t \rangle \,\mathsf{as}\, \langle Z, y \rangle \,\mathsf{in}\, s = s[X \mapsto A, y \mapsto t]$ by defining $\langle A, t \rangle = \Lambda Y.\lambda f.f\, A\, t$ and $\mathsf{open}\, t \,\mathsf{as}\, \langle Z, y \rangle \,\mathsf{in}\, s = t\, S\, (\Lambda Z.\lambda y.s)$. Using parametricity we can prove the following commutation property and η-rule for existential types:

Lemma 6.4. Assume the underlying bifibration satisfies the Beck-Chevalley condition, and that Eq is full.

1. Let $\Gamma; \Delta \vdash t : \exists X.T$, let $\Gamma, Z; \Delta, u : T[X \mapsto Z] \vdash s : S$ and let $\Gamma; \Delta \vdash f : S \to S'$ with Δ being empty and for a closed type S'. Then $[\![f(\mathsf{open}\, t \,\mathsf{as}\, \langle Z, u\rangle \,\mathsf{in}\, s)]\!]_0 = [\![\mathsf{open}\, t \,\mathsf{as}\, \langle Z, u\rangle \,\mathsf{in}\, f(s)]\!]_0$.

2. If $\Gamma; \Delta \vdash t : \exists X.T$ with Δ empty, then $[\![\mathsf{open}\, t \,\mathsf{as}\, \langle Z, u\rangle \,\mathsf{in}\, \langle Z, u\rangle]\!]_0 = [\![t]\!]_0$.

Proof. 1. We first define a morphism $\alpha : \mathsf{Eq}(\mathbf{1}) \to \forall([\![T]\!]_1 \Rightarrow \langle [\![f]\!]_0\rangle)$ in $\mathbf{Rel}(\mathcal{E})^{|\Gamma|}$ over (h, h'), where $h = [\![\Lambda X.\lambda y.s]\!]_0$ and $h' = [\![\Lambda X.\lambda y.f(s)]\!]_0$. By adjointness of $\forall$ and exponentials, this is equivalent to a morphism $\mathsf{Eq}(\mathbf{1}) \times [\![T]\!]_1 \to \langle [\![f]\!]_0\rangle$ over $\tilde{h} = [\![s]\!]_0$ and $\tilde{h'} = [\![f(s)]\!]_0$. Since S' is closed, $[\![S']\!]_1 = \mathsf{Eq}([\![S']\!]_0)$ by the Identity Extension Lemma, and $[\![f(s)]\!]_1 : \mathsf{Eq}(\mathbf{1}) \times [\![T]\!]_1 \to \mathsf{Eq}([\![S']\!]_0)$ is over $(\tilde{h'}, \tilde{h'})$ (recall that $\mathsf{Eq}(\mathbf{1}) \times [\![T]\!]_1 \cong [\![T]\!]_1$). The following triangle commutes:

$$\begin{array}{ccc}
(\mathbf{1} \times [\![T]\!]_0, \mathbf{1} \times [\![T]\!]_0) & & \\
\downarrow{\scriptstyle (\tilde{h}, \tilde{h'})} & \searrow{\scriptstyle (\tilde{h'}, \tilde{h'})} & \\
(\mathbf{1} \times [\![S]\!]_0, [\![S']\!]_0) & \xrightarrow[([\![f]\!]_0, \mathsf{id})]{} & ([\![S']\!]_0, [\![S']\!]_0)
\end{array}$$

Thus, the cartesian property of $\langle [\![f]\!]_0\rangle$ gives a unique $\alpha : \mathsf{Eq}(\mathbf{1}) \times [\![T]\!]_1 \to \langle [\![f]\!]_0\rangle$ over $(\tilde{h}, \tilde{h'})$, or equivalently, $\widehat{\alpha} : \mathsf{Eq}(\mathbf{1}) \to \forall([\![T]\!]_1 \Rightarrow \langle [\![f]\!]_0\rangle)$ over (h, h'). From this, we construct the morphism $\mathsf{ev} \circ \langle [\![t]\!]_{\langle f\rangle}, \widehat{\alpha}\rangle : [\![\Delta]\!]_1 \to \langle [\![f]\!]_0\rangle$ over $([\![t\, S\, h]\!]_0, [\![t\, S'\, h']\!]_0)$. This is in fact a morphism between graph relations, and fullness of the graph functor gives that the following diagram commutes:

$$\begin{array}{ccc}
[\![\Delta]\!]_0 & \xrightarrow{[\![t\, S\, h]\!]_0} & S \\
\downarrow{\scriptstyle \mathsf{id}} & & \downarrow{\scriptstyle [\![f]\!]_0} \\
[\![\Delta]\!]_0 & \xrightarrow[{[\![t\, S'\, h']\!]_0}]{} & S'
\end{array}$$

By unfolding our encoding of $\mathsf{open}\, t\, \mathsf{as}\, \langle Z, u\rangle\, \mathsf{in}\, s = t\, S\, (\Lambda Z.\, \lambda u\, .\, s)$, this exactly says that $[\![f(\mathsf{open}\, t\, \mathsf{as}\, \langle Z, u\rangle\, \mathsf{in}\, s)]\!]_0 = [\![\mathsf{open}\, t\, \mathsf{as}\, \langle Z, u\rangle\, \mathsf{in}\, f(s)]\!]_0$.

2. By function extensionality, it is enough to show that for every type variable Z and term variable y, we have $[\![Z; y : \forall X.T \to Z \vdash (\mathsf{open}\, t\, \mathsf{as}\, \langle X, u\rangle\, \mathsf{in}\, \langle X, u\rangle)\, Z\, y]\!] = [\![Z; y : \forall X.T \to Z \vdash t\, Z\, y]\!]$. In this context, define $f : \exists X.T \to \exists X.T$ by $f\ p = p\, Z\, y$. We have

$$
\begin{aligned}
[\![(\mathsf{open}\, t\, \mathsf{as}\, \langle X, u\rangle\, \mathsf{in}\, \langle X, u\rangle)\, Z\, y]\!]_0 &= [\![f(\mathsf{open}\, t\, \mathsf{as}\, \langle X, u\rangle\, \mathsf{in}\, \langle X, u\rangle)]\!]_0 \\
&= [\![\mathsf{open}\, t\, \mathsf{as}\, \langle X, u\rangle\, \mathsf{in}\, f(\langle X, u\rangle)]\!]_0 \\
&= [\![t\, Z\, y]\!]_0
\end{aligned}
$$

Here, the first equality is by the definition of f, the second by (1), and the last one by the definition of open and $\langle \_, \_ \rangle$. □

If $F : \mathcal{C} \to \mathcal{C}$ is an endofunctor, an F**-coalgebra** is a pair (A, k_A) with A an object of $\mathcal{C}$ and $k_A : A \to FA$ a morphism. We call A the **carrier** of the F-coalgebra and k_A its *structure map*. A morphism $h : A \to B$ in $\mathcal{C}$ is an F**-coalgebra homomorphism** $h : (A, k_A) \to (B, k_B)$ if $k_B \circ h = Fh \circ k_A$. An F-coalgebra (W, out) is **weakly final** if, for any F-coalgebra (A, k_A), there exists a mediating F-coalgebra homomorphism $unfold[A, k_A] : (A, k_A) \to (W, out)$. It is a **final** F**-coalgebra** if $unfold[A, k_A]$ is unique.

Let $F = (F_1, F_0)\colon \mathbf{Rel}(U) \to \mathbf{Rel}(U)$ be an equality preserving lifted functor with a strength t. We show that the final F_0-coalgebra exists. Again, we first construct a weakly final coalgebra by defining $W = (W_1, W_0) = [\![\exists X.(X \to \underline{F}(X)) \times X]\!]$.

Lemma 6.5. W_0 is the carrier of a weakly final F_0-coalgebra (W_0, out_0) with mediating morphism $unfold_0[A, k]$ and W_1 is the carrier of a weakly final F_1-coalgebra (W_1, out_1) with mediating morphism $unfold_1[A, k]$.

Proof. The structure of the proof is similar to the proof of Lemma 6.1. We first construct the term $unfold = \Lambda A.\lambda w : A \to \underline{F}A.\lambda x.\langle A, (w, x)\rangle$, and define $unfold_i[A, k] = \theta^{-1}[\![unfold\, \underline{A}\, \underline{k}]\!]_i$, where θ is the bijection corresponding to the adjunction $- \times X \dashv X \Rightarrow -$, and $(\underline{A}, \underline{k})$ are the internal expressions corresponding to the F_0- or F_1-coalgebra (A, k), as appropriate. We define the structure maps (out_1, out_0) by

$$out_i = \theta^{-1}[\![\lambda z.\mathsf{open}\ z\ \mathsf{as}\ \langle Z, v\rangle\ \mathsf{in}\ (\underline{t}\,(unfold\ Z\ (\pi_1 v)))(\pi_1\, v\,(\pi_2\, v))]\!]_i,$$

where t is the strength of F. By equational reasoning in System F, $unfold_0$ and $unfold_1$ are coalgebra morphisms:

$$\begin{aligned} out \circ unfold[A,k] &= \theta^{-1}[\![\lambda x.\mathsf{open}\ \langle \underline{A}, (\underline{k}, x)\rangle\ \mathsf{as}\ \langle Z, u\rangle\ \mathsf{in}\ \big(\underline{t}(unfold\ Z\,(\pi_1\, u))\,(\pi_1\, u\,(\pi_2\, u))\big)]\!] \\ &= \theta^{-1}[\![\lambda x.\underline{t}(unfold\ \underline{A}\ \underline{k})\,(\underline{k}\, x)]\!] \\ &= \theta^{-1}[\![\underline{t}(unfold\ \underline{A}\ \underline{k})]\!] \circ \theta^{-1}[\![\underline{k}]\!] \\ &= F(unfold[A,k]) \circ k \end{aligned}$$

which proves the thesis. □

We proceed similarly to Lemma 6.2. This time, we use the opfibrational part of the Graph Lemma which gives the map $\phi_h \colon \langle F_0 h\rangle \to F_1\langle h\rangle$ to construct F_1-coalgebras.

Lemma 6.6. Assume the underlying bifibration satisfies the Beck-Chevalley condition, wellpointedness and that Eq is full.

1. For every F_0-coalgebra morphism $h : (A, k_A) \to (B, k_B)$ we have $unfold_0[B, k_B] \circ h = unfold_0[A, k_A]$.

2. $unfold_0[W_0, out_0] = \mathsf{id}_{W_0}$.

Proof. 1. Since h is a coalgebra morphism, $k_B \circ h = F_0 h \circ k_A$. Applying the graph functor, we obtain the morphism $\langle k_A, k_B\rangle : \langle h\rangle \to \langle F_0 h\rangle$, which by composing with the morphism of the Graph Lemma gives an F_1-coalgebra $\phi_h \circ \langle k_A, k_B\rangle : \langle h\rangle \to F_1\langle h\rangle$. By weak finality, we have a F_1-coalgebra morphism $unfold_1[\langle h\rangle, \psi_h \circ \langle k_A, k_B\rangle] : (\langle h\rangle, \psi_h \circ \langle (k_A, k_B)\rangle) \to (W_1, out_1)$. Since $W_1 = Eq(W_0) = \langle \mathsf{id}_{W_0}\rangle$, and since the graph functor is full by Lemma 3.11, we see that $(unfold_0[A, k_A], unfold_0[B, k_B]) : h \to \mathsf{id}$ in $\mathcal{B}^{\to}$, i.e., $unfold_0[A, k_A] = unfold_0[B, k_B] \circ h$.

2. By function extensionality, it is enough to prove $[\![\mathit{unfold}\,\underline{W_0}\,\underline{\mathit{out}}\,x]\!]_0 = [\![x]\!]_0$ for a fresh variable $x : \underline{W_0}$. We first note that by (1), $\mathit{unfold}_0[A, k] = \mathit{unfold}_0[W_0, \mathit{out}] \circ \mathit{unfold}_0[A, k]$ for any A, $k : A \to F(A)$, i.e., by the definition of $\mathit{unfold}_0[A, k]$, for any type X, and terms $h : X \to F(X)$, and $y : X$ we have $[\![\langle X, (h, y)\rangle]\!] = [\![\mathit{unfold}\,\underline{W_0}\,\underline{\mathit{out}}\,\langle X, (h, y)\rangle]\!]$.

$$\begin{aligned}
[\![x]\!] &= [\![\mathsf{open}\,x\,\mathsf{as}\,\langle Z, u\rangle\,\mathsf{in}\,\langle Z, u\rangle]\!] \\
&= [\![\mathsf{open}\,x\,\mathsf{as}\,\langle Z, u\rangle\,\mathsf{in}\,\langle Z, (\pi_1\,u, \pi_2\,u)\rangle]\!] \\
&= [\![\mathsf{open}\,x\,\mathsf{as}\,\langle Z, u\rangle\,\mathsf{in}\,(\mathit{unfold}\,\underline{W_0}\,\underline{\mathit{out}}\,\langle Z, (\pi_1\,u, \pi_2\,u)\rangle)]\!] \\
&= [\![\mathit{unfold}\,\underline{W_0}\,\underline{\mathit{out}}(\mathsf{open}\,x\,\mathsf{as}\,\langle Z, u\rangle\,\mathsf{in}\,\langle Z, (\pi_1\,u, \pi_2\,u)\rangle)]\!] \\
&= [\![\mathit{unfold}\,\underline{W_0}\,\underline{\mathit{out}}(\mathsf{open}\,x\,\mathsf{as}\,\langle Z, u\rangle\,\mathsf{in}\,\langle Z, u\rangle)]\!] \\
&= [\![\mathit{unfold}\,\underline{W_0}\,\underline{\mathit{out}}\,x]\!]
\end{aligned}$$

Here, the first equality comes from Lemma 6.4(2), the second one from surjective pairing, the third from the observation above, the fourth from Lemma 6.4(1), and the fifth and sixth respectively from surjective pairing and Lemma 6.4(2) again. □

Putting things together, we have constructed a final coalgebra.

Theorem 6.7. If the underlying bifibration satisfies the Beck-Chevalley condition, and if Eq is full, then (W_0, out_0) is a final F_0-coalgebra.

Proof. By Lemma 6.5, (W_0, out_d) is weakly final. We must show that $h = \mathit{unfold}_0[A, k_A]$ for any $k_A : A \to F_0 A$ and any F_0-coalgebra morphism $h : (A, k_A) \to (W_0, \mathit{out}_0)$. By Lemma 6.6(1), $\mathit{unfold}_0[A, k_A] = \mathit{unfold}_0[W_0, \mathit{out}_0] \circ h$ and since $\mathit{unfold}_0[W_0, \mathit{out}_0] = \mathsf{id}_{W_0}$, by Lemma 6.6(2), we have $h = \mathit{unfold}_0[A, k_A]$, as required. □

6.3 Parametricity implies dinaturality

We show that the axiomatic foundations for parametricity can be used to prove that dinaturality can be deduced from parametricity. First, the definition of dinaturality:

Definition 6.8. If $F, G : \mathcal{B}^{op} \times \mathcal{B} \to \mathcal{B}$ are mixed variant functors, then a dinatural transformation $t : F \to G$ is a collection of morphisms $t_X : FXX \to GXX$ indexed by objects X of $\mathcal{B}$ such that, for every morphism $g : X \to Y$ of $\mathcal{B}$, the following hexagonal diagram commutes:

$$\xymatrix{ & F\,X\,X \ar[r]^{t_X} & G\,X\,X \ar[dr]^{G(\mathsf{id}_X,g)} & \\ F\,Y\,X \ar[ur]^{F(g,\mathsf{id}_X)} \ar[dr]_{F(\mathsf{id}_Y,g)} & & & G\,X\,Y \\ & F\,Y\,Y \ar[r]_{t_Y} & G\,Y\,Y \ar[ur]_{G(g,\mathsf{id}_Y)} & }$$

We note that the proof applies to all mixed variant functors with equality preserving liftings, not just strong such functors.

Theorem 6.9. Let $(F_1, F_0), (G_1, G_0) : \mathsf{rel}(U)^{op} \times \mathsf{rel}(U) \to \mathsf{rel}(U)$ be equality preserving lifted functors. Further, let $t^0_A : F_0AA \to G_0AA$ be a family indexed by objects A of $\mathcal{B}$, and $t^1_R : F_1RR \to G_1RR$ be a family indexed by objects R of $\mathbf{Rel}(\mathcal{E})$ such that if R is over (A, B), then t^1_R is over (t^0_A, t^0_B). Then t^0 is a dinatural transformation from F_0 to G_0.

Proof. Let $g : A \to B$ be a morphism in $\mathcal{B}$. Let $\phi : \mathsf{Eq}\,A \to \langle g \rangle$ and $\psi : \langle g \rangle \to \mathsf{Eq}\,B$ be the maps associated to the opreindexing and reindexing definitions of $\langle g \rangle$. The morphism

$$F_1(\mathsf{Eq}\,B)(\mathsf{Eq}\,A) \xrightarrow{F_1\psi\phi} F_1\langle g \rangle\langle g \rangle \xrightarrow{t^1_{\langle g \rangle}} G_1\langle g \rangle\langle g \rangle \xrightarrow{G_1\,\phi\,\psi} G_1(\mathsf{Eq}\,A)(\mathsf{Eq}\,B)$$

is such that $F_1\,\psi\,\phi$ is over $(F_0(g, \mathsf{id}_A), F_0(\mathsf{id}_B, g))$, $t^1_{\langle g \rangle}$ is over (t^0_A, t^0_B), and $G_1\,\phi\,\psi$ is over $(G_0(\mathsf{id}_A, g), G_0(g, \mathsf{id}_B))$. Since F_1 and G_1 are equality preserving, $F_1(\mathsf{Eq}\,B)(\mathsf{Eq}\,A) = \mathsf{Eq}\,(F_0BA) = \langle \mathsf{id}_{F_0BA} \rangle$ and $G_1(\mathsf{Eq}\,A)(\mathsf{Eq}\,B) = \mathsf{Eq}\,(G_0AB) = \langle \mathsf{id}_{G_0AB} \rangle$. Finally, by the fullness and faithfulness of the graph functor we have

$$\begin{array}{ccccccc} F_0\,B\,A & \xrightarrow{F_0(g,\mathsf{id}_A)} & F_0\,A\,A & \xrightarrow{t^0_A} & G_0\,A\,A & \xrightarrow{G_0(\mathsf{id}_A,g)} & G_0\,A\,B \\ \downarrow{\scriptstyle \mathsf{id}} & & & & & & \downarrow{\scriptstyle \mathsf{id}} \\ F_0\,B\,A & \xrightarrow[F_0(\mathsf{id}_B,g)]{} & F_0\,B\,B & \xrightarrow[t^0_B]{} & G_0\,B\,B & \xrightarrow[G_0(g,\mathsf{id}_B)]{} & G_0\,A\,B \end{array}$$

This proves the required hexagon commutes. □

Theorem 6.9 applies in particular to the interpretation of terms $t : \forall X.FXX \to GXX$ where F and G are type expressions with one free type variable. As is well known, dinaturality reduces to naturality when F and G are covariant.

Note that the results on naturality are possible because System F has limited expressiveness. If one moves to dependent types, not every function of type $\forall X.FX - > GX$ is natural anymore (see [ml15]).

Chapter 7

Universal parametricity

We saw in the previous chapter that the interpretation of product types and arrow types is supported by universal properties in our framework. For example the arrow types are interpreted, both at the relational and the base level, as exponential objects in their respective categories. Using this universal property we saw that it is possible to derive the Identity Extension Lemma and the Abstraction Theorem. The situation is less clear for forall types. For example, the solution adopted in the previous bifibrational model has baked the Identity Extension Lemma into the definition. In this section we study a universal property underpinning the interpretation of forall types which permits to prove the Identity Extension Lemma and the Abstraction Theorem in an axiomatic manner. The result holds for a large class of models axiomatically built from faithful bifibrations which admit full comprehension – this includes, for instance, subobject bifibrations.

The material presented in this chapter is based on [GNFO15], and we use the same notation as in Chapter 5 for objects in the base and total category of a fibration of relations $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$.

7.1 Reynolds' parametrically polymorphic functions

In this first section we analyze the universal property behind Reynolds' set theoretical interpretation of parametric forall types in the setting of the relations of fibrations

$\mathsf{rel}\colon \mathbf{Rel} \to \mathbf{Set} \times \mathbf{Set}$, which we recall is given by

$$\llbracket \forall X.T \rrbracket_0 \bar{A} = \{f : \prod_{X:\mathbf{Set}} \llbracket T \rrbracket_0(\bar{A}, X) \mid \forall R \in \mathbf{Rel}_{(A,B)}.\ (fA, fB) \in \llbracket T \rrbracket_1(\mathsf{Eq}\ \bar{A}, R)\}$$

$$(f, g) \in \llbracket \forall X.T \rrbracket_1 \bar{R} \text{ iff } \forall R \in \mathbf{Rel}_{(A,B)}.\ (fA, gB) \in \llbracket T \rrbracket_1(\bar{R}, R). \tag{7.1}$$

Note that if we were only to consider ad-hoc polymorphic functions, i.e. the collection

$$\prod_{A:\mathbf{Set}} \llbracket T \rrbracket_0(\bar{A}, A)$$

then we could characterise this collection as the product of the functor $\llbracket T \rrbracket_0(\bar{A}, -) : \mathbf{Set} \to \mathbf{Set}$ (naïvely assuming the product exists), that is, as the terminal $\llbracket T \rrbracket_0(\bar{A}, -)$-cone. Including Reynolds' condition that a parametrically polymorphic function $f : \prod_{S:\mathbf{Set}} \llbracket T \rrbracket_0(\bar{A}, S)$ is one where for every relation $R \in \mathbf{Rel}_{(A,B)}$ we have that $(fA, fB) \in \llbracket T \rrbracket_1(\mathsf{Eq}\bar{A}, R)$ cuts down the the number of ad-hoc polymorphic functions. Now the key bit. Define $\nu_A : \llbracket \forall X.T \rrbracket_0 \bar{A} \to \llbracket T \rrbracket_0(\bar{A}, A)$ to be type application, i.e. $\nu_A f = fA$. Then Reynolds' parametricity condition that for all $R \in \mathbf{Rel}_{(A,B)}$, if $f : \llbracket \forall X.T \rrbracket_0 \bar{A}$, then $(fA, fB) \in \llbracket T \rrbracket_1(\mathsf{Eq}\bar{A}, R)$ is equivalent to a morphism $\mathsf{Eq}\ (\llbracket \forall X.T \rrbracket_0 \bar{A}) \to \llbracket T \rrbracket_1(\mathsf{Eq}\bar{A}, R)$ over ν_A and ν_B. Generalizing, we have:

Definition 7.2. Let $F = (F_1, F_0)$ be a pair of functors with $F_0 : |\mathbf{Set}| \to \mathbf{Set}$ and $F_1 : |\mathbf{Rel}| \to \mathbf{Rel}$ such that F_1 is over $F_0 \times F_0$. An F-eqcone is an F_0-cone (A, ν) such that there is a (necessarily unique since rel is faithful) F_1-cone with vertex $\mathsf{Eq}A$ over (ν, ν). The category of such cones is the full subcategory of F_0-cones whose objects are F-eqcones.

Our axiomatic definition is linked to Reynolds' definition in the following way:

Theorem 7.3. Assume $\Gamma, X \vdash T$ type. For every tuple $\bar{A}$, Reynolds' set of parametrically polymorphic functions $\llbracket \forall X.T \rrbracket_0 \bar{A}$ from (7.1) is the terminal F-eqcone for the pair of functors $F = (\llbracket T \rrbracket_1(\mathsf{Eq}\bar{A}, -), \llbracket T \rrbracket_0(\bar{A}, -))$.

Proof. Application at A, defined by $\nu_A f = fA$, makes $\llbracket \forall X.T \rrbracket_0 \bar{A}$ a vertex of a $\llbracket T \rrbracket_0(\bar{A}, -)$-cone. The uniformity condition on elements of $\llbracket \forall X.T \rrbracket_0 \bar{A}$ ensures this cone is an F-eqcone.

To see that this is the terminal such, consider any other F-eqcone (A, η). As this is a $[\![T]\!]_0(\bar{A}, -)$-cone, there is a unique map $\bar{\eta}$ of such cones into $\prod_{A:\mathbf{Set}}[\![T]\!]_0(\bar{A}, A)$. However, the fact that (A, η) is an F-eqcone means the image of this mediating map lies within $[\![\forall X.T]\!]_0\bar{A}$. Hence we have a morphism of F-eqcones $A \to [\![\forall X.T]\!]_0\bar{A}$. The uniqueness of this mediating morphism follows from the uniqueness of $\bar{\eta}$. □

We can also give a universal property to characterise $[\![\forall X.T]\!]_1\bar{R}$.

Definition 7.4. Let $F = (F_1, F_0)$ and $G = (G_1, G_0)$ be pairs of functors $|\mathbf{Set}| \to \mathbf{Set}$ and $|\mathbf{Rel}| \to \mathbf{Rel}$ with F_1 over $F_0 \times F_0$, G_1 over $G_0 \times G_0$, and let $H : |\mathbf{Rel}| \to \mathbf{Rel}$ with H over $F_0 \times G_0$. A fibred (F, G, H)-eqcone consists of an F-eqcone (A, ν), a G-eqcone (B, μ) and a H-cone (Q, γ) over (ν, μ). The category of such cones has as morphisms triples (f, g, h), where f is a morphism between the underlying F-eqcones, g is a morphism between the underlying G-eqcones and h is a (again necessarily unique) morphism of H-cones above (f, g).

The above definition can be understood as follows. For every relation $R \in \mathbf{Rel}_{(A,B)}$ we need two things to be related, which is forced by γ. That the related things are instances of polymorphic functions is reflected by the fact that γ_R is over (ν_A, μ_B). This intuition can be formalised via the following theorem:

Theorem 7.5. Assume $\Gamma, X \vdash T$ type. For every relation $\bar{R} \in \mathbf{Rel}^{|\Gamma|}_{(\bar{A},\bar{B})}$, we have that the relation $[\![\forall X.T]\!]_1\bar{R}$ from (7.1) is the terminal fibred (F, G, H)-eqcone for the functors $F = ([\![T]\!]_1(\mathsf{Eq}\bar{A}, -), [\![T]\!]_0(\bar{A}, -))$, $G = ([\![T]\!]_1(\mathsf{Eq}\bar{B}, -), [\![T]\!]_0(\bar{B}, -))$ and $H = [\![T]\!]_1(\bar{R}, -)$.

Proof. We have the F-eqcone $([\![\forall X.T]\!]_0\bar{A}, \nu)$ and the G-eqcone $([\![\forall X.T]\!]_0\bar{B}, \mu)$, where $\nu_A f = fA$ and $\mu_A g = gA$ are given by type application. The object $[\![\forall X.T]\!]_1\bar{R}$ lives over $([\![\forall X.T]\!]_0\bar{A}, [\![\forall X.T]\!]_0\bar{B})$, and $([\![\forall X.T]\!]_1\bar{R}, (\nu, \mu))$ is an H-cone over (ν, μ). Given any other fibred (F, G, H)-eqcone (A, ρ), (B, τ), $(Q, (\rho, \tau))$, by terminality there are two unique morphisms $h_0 \colon A \to [\![\forall X.T]\!]_0\bar{A}$ and $h_1 \colon B \to [\![\forall X.T]\!]_0\bar{B}$. The pair of morphisms (h_0, h_1) defines a morphism $Q \to [\![\forall X.T]\!]_1\bar{R}$ which is unique by faithfulness of the fibration. □

In Theorems 7.9 and 7.17 below, we will see that this gives another proof that Reynolds concrete model satisfies the Identity Extension Lemma and the Abstraction Theorem.

Next step is to turn to the universal property we will use to define the object of parametrically polymorphic functions in our axiomatic framework. We carefully formulated the definitions of this section so that they seamlessly generalise to the fibrational setting. We assume for the rest of the chapter that $U\colon \mathcal{E} \to \mathcal{B}$ is faithful fibration.

7.2 Eqcones and fibred eqcones

We start by generalizing the definition of eqcones for a faithful fibration $U\colon \mathcal{E} \to \mathcal{B}$ and the fibration of relations $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ obtained via change of base from U.

Definition 7.6. Let $F = (F_1, F_0) : |\mathsf{rel}(U)| \to \mathsf{rel}(U)$ be a fibred functor. An F-eqcone is an F_0-cone (A, ν) such that there is a (necessarily unique since U is faithful) F_1-cone with vertex $\mathsf{Eq}A$ over (ν, ν). The category of such cones is the full subcategory of F_0-cones whose objects are F-eqcones. We denote by $\forall_0 F$ the terminal object of this category, if it exists. We denote by $\forall_0 F$ also the vertex of the terminal cone, when it exists. It will be clear from the context to which one we refer.

The universal property defining the relational interpretation of parametrically polymorphic functions smoothly generalises also to the fibrational setting:

Definition 7.7. Let $F = (F_1, F_0)$ and $G = (G_1, G_0)$ be fibred functors $|\mathsf{rel}(U)| \to \mathsf{rel}(U)$ and let $H : |\mathbf{Rel}(\mathcal{E})| \to \mathbf{Rel}(\mathcal{E})$ be over $F_0 \times G_0$. A fibred (F, G, H)-eqcone consists of an F-eqcone (A, ν), a G-eqcone (B, μ) and an H-cone (Q, γ) such that Q is over $A \times B$, and γ is over (ν, μ). A morphism $(A, \nu, B, \mu, Q, \gamma) \to (A', \nu', B', \mu', Q', \gamma')$ in the category of such cones consists of triples (f, g, h) where $f : (A, \nu) \to (A', \nu')$, $g : (B, \mu) \to (B', \mu')$, and h is a (again necessarily unique) morphism of H-cones above (f, g). We denote the terminal object of this category by $\forall_1(F, G, H)$, if it exists. By abuse of notation, we denote by $\forall_1(F, G, H)$ also the vertex of the H-cone in $\forall_1(F, G, H)$; it will always be clear from context to which one we refer.

We next show how to interpret forall types using our axiomatic definition. We show that they support:

1. **A fibred semantics:** our axiomatic definitions do not by definition guarantee that if $\bar{R} \in \mathbf{Rel}(U)^n_{(\bar{A},\bar{B})}$, then $[\![\forall X.T]\!]_1 \bar{R}$ is a relation between $[\![\forall X.T]\!]_0 \bar{A}$ and $[\![\forall X.T]\!]_0 \bar{B}$, so we prove this axiomatically.

2. **Equality preservation:** since we do not restrict to the case of equality preserving fibred functors, the preservation of equalities does not come for free. Again, we prove it from the axiomatic definition.

3. **The interpretation of terms:** we need to prove that, in the axiomatic setting, we can interpret terms of forall types as fibred natural transformations. In order to do so we again construct a model of System F in the form of a $\lambda 2$-fibration.

We prove equality preservation (item 2) for subobject fibrations first, and then generalise it to the case of faithful fibrations which admit full comprehension. The proofs of the fibred semantics (item 1) and of the interpretation of terms (item 3) do not require an instantiation to any particular fibration. These proofs are general enough that they can be derived uniformly from Definitions 7.6 and 7.7.

7.3 A fibred semantics

The proof that if $\bar{R} \in \mathbf{Rel}^n_{(\bar{A},\bar{B})}$, then $[\![\forall X.T]\!]_1 \bar{R}$ is a relation between $[\![\forall X.T]\!]_0 \bar{A}$ and $[\![\forall X.T]\!]_0 \bar{B}$ crucially requires opfibrational structure.

Lemma 7.8. Consider an opfibration $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$. Let $F = (F_1, F_0)$ and $G = (G_1, G_0)$ be fibred functors $|\mathsf{rel}(U)| \to \mathsf{rel}(U)$ and assume H is over $F_0 \times G_0$. Then $\forall_1(F, G, H)$ is over $\forall_0 F \times \forall_0 G$.

Proof. The forgetful functor which maps a fibred (F, G, H)-eqcone to its pair of underlying F-eqcones and G-eqcones is an opfibration, since it inherits the opfibrational structure of $\mathsf{rel}(U) : \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$. Any opfibration $V : \mathcal{D} \to \mathcal{C}$ which has terminal objects $1_\mathcal{C}$ in the base and $1_\mathcal{D}$ in the total category, also has a terminal object $\Sigma_! 1_\mathcal{D}$ in the total category over the terminal object $1_\mathcal{C}$ in the base, where $! : V(1_\mathcal{D}) \to 1_\mathcal{C}$ is the unique morphism from

$V(1_{\mathcal{D}})$ to the terminal object. Since terminal objects are defined up to isomorphism, we can take $\forall_1(F, G, H)$ to be over $\forall_0 F \times \forall_0 G$. □

This lemma, when taken with the usual treatment of function spaces (and assuming $\forall_0$ and $\forall_1$ exist), ensures that we have replicated Reynolds' original fibred semantics within our axiomatic framework. That is, for all judgments $\Gamma \vdash T$ type, $(\llbracket T \rrbracket_1, \llbracket T \rrbracket_0)$ forms a fibred functor $|\mathbf{Rel}(U)|^n \to \mathbf{Rel}(U)$.

7.4 Equality preservation

We saw in Section 7.1 that the axiomatisation gives the right interpretation of forall types for relations over sets. We now show that the Identity Extension Lemma can be proven instantiating the axiomatisation with any subobject bifibration in Section 7.4.1. We then generalise this further to bifibrations with full comprehension in Section 7.4.2.

7.4.1 Subobject fibrations

In a subobject fibration, the functor $\mathsf{Eq} : \mathcal{B} \to \mathbf{Rel}(\mathcal{E})$ maps an object X to the mono $\langle \mathsf{id}_X, \mathsf{id}_X \rangle : X \hookrightarrow X \times X$. Thus to show $\mathsf{Eq}(\forall_0 F(A)) = \forall_1(F, F, F_1)(\mathsf{Eq}\, A)$, we need to show:

Lemma 7.9. Let U be a subobject bifibration and $F = (F_1, F_0) : |\mathsf{rel}(U)| \to \mathsf{rel}(U)$ be a fibred functor which is equality preserving. If the terminal fibred (F, F, F_1)-eqcone exists and it is given by the monomorphism $\langle v_1, v_2 \rangle : \forall_1(F, F, F_1) \hookrightarrow \forall_0 F \times \forall_0 F$, then we have $\langle v_1, v_2 \rangle = \langle \mathsf{id}, \mathsf{id} \rangle : \forall_0 F \hookrightarrow \forall_0 F \times \forall_0 F$.

Proof. The heart of the proof is to show that $v_1 = v_2$. To see this, let $\pi_X : \forall_0 F \to F_0 X$ be the natural transformation of the terminal F-eqcone $\forall_0 F$, and let $\gamma_R : \forall_1(F, F, F_1) \to F_1 R$ be the natural transformation of the F_1-cone in $\forall_1(F, F, F_1)$. By Lemma 7.8, for every X, $\gamma_{\mathsf{Eq} X} : \forall_1(F, F, F_1) \to F_1(\mathsf{Eq} X) = \mathsf{Eq}(F_0 X) = F_0 X$ is over $(\pi_X \times \pi_X)$. By the definition of the equality functor in a subobject fibration, we have

$$\begin{array}{ccc}
\forall_1(F,F,F_1) & \xrightarrow{\gamma_{\mathsf{Eq}X}} & F_1(\mathsf{Eq}X) = \mathsf{Eq}(F_0X) = F_0X \\
\downarrow{\scriptstyle \langle v_1,v_2\rangle} & & \downarrow{\scriptstyle \langle \mathsf{id},\mathsf{id}\rangle} \\
\forall_0 F \times \forall_0 F & \xrightarrow[\pi_X\times\pi_X]{} & F_0X \times F_0X
\end{array}$$

Thus $\pi_X v_1 = \gamma_{\mathsf{Eq}(X)} = \pi_X v_2$ and $(\forall_1(F,F,F_1), \gamma_{\mathsf{Eq}(-)})$ is a F-eqcone with F_1-cone given by γ_R. Hence both v_1 and v_2 are mediating morphisms into the terminal F-eqcone, and thus $v_1 = v_2$. Furthermore, they are vertical since $\langle \mathsf{id}, \mathsf{id}\rangle \circ v_i = \langle v_1, v_2\rangle$. We can now show that $\mathsf{Eq}(\forall_0 F)$ is isomorphic to $\forall_1(F,F,F_1)$. In one direction, $\mathsf{Eq}(\forall_0 F)$ is easily seen to be a fibred (F,F,F_1)-eqcone and hence there is a map of subobjects $\mathsf{Eq}(\forall_0 F) \to \forall_1(F,F,F_1)$. In the other direction, v_1 is a map of subobjects since $v_1 = v_2$. These maps are mutually inverse, as they are both vertical and the fibration is faithful. □

The Identity Extension Lemma for fibred functors $(T_1, T_0) : |\mathsf{rel}(U)|^{n+1} \to \mathsf{rel}(U)$ immediately follows by instantiating $F_0 = T_0(\bar{A}, -)$ and $F_1 = T_1(\mathsf{Eq}\bar{A}, -)$.

Corollary 7.10. Let U be a subobject bifibration and $(T_1, T_0)\colon |\mathsf{rel}(U)|^{n+1} \to \mathsf{rel}(U)$ be an equality preserving fibred functor. Then

$$\mathsf{Eq}(\forall_0 T(\bar{A}, \_)) \cong \forall_1((T_0(\bar{A}, \_), T_1(\mathsf{Eq}\bar{A}, \_)), (T_0(\bar{A}, \_), T_1(\mathsf{Eq}\bar{A}, \_)), T_1(\mathsf{Eq}\bar{A}, \_))$$

for every n-tuple $\bar{A}$ of objects in the base category $\mathcal{B}$.

This lemma shows that in the axiomatic setting instantiated to subobject fibrations, all type expressions are interpreted not just as fibred functors $|\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$, but as equality preserving fibred functors.

7.4.2 Faithful bifibrations which admit full comprehension

We now generalise the previous section to faithful bifibrations which admit full comprehension. In order to do so, we first prove some properties of the comprehension functor. Consider an opfibration $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ obtained via change of base as in (2.31). We define the functor $S\colon \mathcal{E} \to \mathbf{Rel}(\mathcal{E})$ as follows:

Lemma 7.11. Let $\mathsf{rel}(U)\colon \mathbf{Rel}(\mathcal{E}) \to \mathcal{B} \times \mathcal{B}$ be an opfibration. The map on objects $P \mapsto \Sigma_{\delta_{UP}} P$ extends to a functor $S\colon \mathcal{E} \to \mathbf{Rel}(\mathcal{E})$.

Proof. The functor S acts on a morphism $f\colon P \to Q$ in $\mathcal{E}$ via the universal property of $(\delta_X)_\S$ as follows:

$$\begin{array}{ccc} P & \xrightarrow{(\delta_X)_\S} & \Sigma_{\delta_X} P \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle S(f)} \\ Q & \xrightarrow[(\delta_Y)_\S]{} & \Sigma_{\delta_Y} Q \end{array}$$

over the diagram

$$\begin{array}{ccc} X & \xrightarrow{\delta_X} & X \times X \\ {\scriptstyle U(f)}\downarrow & & \downarrow{\scriptstyle U(f)\times U(f)} \\ Y & \xrightarrow[\delta_Y]{} & Y \times Y. \end{array}$$

in $\mathcal{B}$ which obviously commutes. □

Lemma 7.12. The functor $S : \mathcal{E} \to \mathbf{Rel}(\mathcal{E})$ is left adjoint to the projection functor $J : \mathbf{Rel}(\mathcal{E}) \to \mathcal{E}$ defined as in Diagram (3.2).

Proof. We explicitly give the unit η and counit ϵ of the adjunction since we will use them later. The unit component $\eta_P\colon P \to \Sigma_{\delta_{UP}} P$, is given by $\eta_P \coloneqq (\delta_{UP})_\S$. The counit component $\epsilon_{(A,B,R)}\colon (A \times B, A \times B, \Sigma_{\delta_{A\times B}} R) \to (A, B, R)$, is given by $\epsilon_{(A,B,R)} \coloneqq (\pi_1, \pi_2, p_{A,B})$, where π_1 and π_2 are, respectively, the first and second projections, while $p_{A,B}$ is given by the universal property of $(\delta_{A\times B})_\S$ with respect to the diagrams

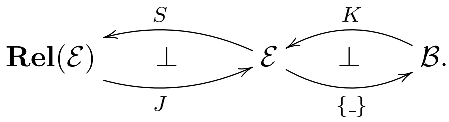
$$\begin{array}{ccc} R & \xrightarrow{(\delta_{A\times B})_\S} & \Sigma_{\delta_{A\times B}} R \\ {\scriptstyle \mathsf{id}}\downarrow & \swarrow{\scriptstyle p_{A,B}} & \\ R & & \end{array}$$

over the commuting diagram

$$\begin{array}{ccc} A \times B & \xrightarrow{\delta_{A\times B}} & (A \times B) \times (A \times B) \\ {\scriptstyle \mathsf{id}}\downarrow & \swarrow{\scriptstyle \pi_1\times\pi_2} & \\ A \times B. & & \end{array}$$

in $\mathcal{B}$. Naturality and triangle identities follow from direct calculation and opfibrational properties. $\square$

Hence we have a string of adjunctions

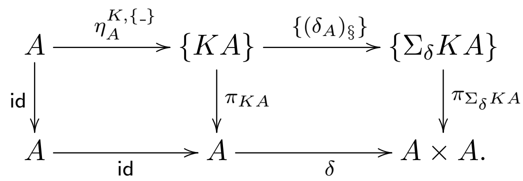


Note that $\mathsf{Eq} = S \circ K$. By composing the adjunctions, we obtain $\mathsf{Eq} \dashv \{J(\_)\}$. From now on, we will assume that the equality functor is full and faithful. As we saw in Chapter 5, this assumption is essential in order to derive the usual consequences of parametricity in the bifibrational axiomatisation.

Before proving the Identity Extension Lemma, we need one more result about the comprehension functor.

Lemma 7.13. Let $\pi \colon \{\_\} \to U$ be a natural transformation as in Lemma 3.16 and Eq be full and faithful. The unit of the adjunction $\mathsf{Eq} \dashv \{J(\_)\}$ is an isomorphism $\eta_A^{\mathsf{Eq},\{J\}} \colon A \cong \{\Sigma_\delta KA\}$ since Eq is full and faithful and it holds $\pi_{\mathsf{Eq}(A)} = \delta \circ \eta^{\mathsf{Eq},\{J\}}$.

Proof. Consider the following diagram:



The left square commutes since it arises as the application of U to the triangle identity $\epsilon_{KA}^{K,\{\text{-}\}} \circ K\eta_A^{K,\{\text{-}\}} = \mathsf{id}$. The right square commutes since it is given by $\mathcal{P}((\delta_A)_\S)$ (where $\mathcal{P}$ is defined in Lemma 3.16). The morphism $(\delta_A)_\S$ is the unit of the adjunction $S \dashv J$, and the composition $\{(\delta_A)_\S\} \circ \eta_A^{K,\{\text{-}\}}$ is the unit $\eta_A^{\mathsf{Eq},\{J\}}$ of the adjunction $\mathsf{Eq} \dashv \{J(-)\}$, which is an isomorphism since Eq is full and faithful by assumption. Hence $\pi_{\mathsf{Eq}(A)} = \delta \circ (\eta_A^{\mathsf{Eq},\{J\}})^{-1}$. $\square$

Again, the key point in the proof of the Identity Extension Lemma is in showing that, given $\pi_{\forall_1 F_1} = \langle v_1, v_2 \rangle \colon \{\forall_1(F, F, F_1)\} \to \forall_0 F \times \forall_0 F$, we have $v_1 = v_2$.

Lemma 7.14. Let U be a faithful bifibration which admits full comprehension and $F = (F_1, F_0) : |\mathsf{rel}(U)| \to \mathsf{rel}(U)$ be a fibred functor which is equality preserving. Then the morphism $\pi_{\forall_1 F_1} = \langle v_1, v_2 \rangle \colon \{\forall_1(F, F, F_1)\} \to \forall_0 F \times \forall_0 F$ is such that $v_1 = v_2$.

Proof. The terminal fibred eqcone comes, for every object A in $\mathcal{B}$, with a morphism $\gamma_{\mathsf{Eq}(A)}$ over $\nu_A \times \nu_A$, which, by Lemma 7.13, is sent, up to isomorphism, by $\mathcal{P}$ to the commuting diagram

$$\begin{array}{ccc} \{\forall_1(F, F, F_1)\} & \xrightarrow{\{\gamma_{\mathsf{Eq}(A)}\}} & F_0(A) \\ {\scriptstyle \langle v_1, v_2 \rangle}\downarrow & & \downarrow{\scriptstyle \delta} \\ \forall_0 F \times \forall_0 F & \xrightarrow[\nu_A \times \nu_A]{} & F_0(A) \times F_0(A) \end{array}$$

from which we conclude $\nu \circ v_1 = \nu \circ v_2$.

The system $(\{\forall_1(F, F, F_1)\}, \nu \circ v_1)$ defines an F-eqcone, where the equality cone part is given by precomposing the terminal F-eqcone $(\forall_0 F_0, \nu)$ with $\mathsf{Eq}(v_1) \colon \mathsf{Eq}(\{\forall_1 F_1\}) \to \mathsf{Eq}(\forall_0 F)$. It follows that both v_1 and v_2 define an eqcone morphism from $(\{\forall_1 F_1\}, \nu \circ v_1)$ to the terminal eqcone $(\forall_0 F, \nu)$, hence $v_1 = v_2$ by uniqueness. $\square$

Lemma 7.15. Let U be a faithful bifibration which admits full comprehension and $F = (F_1, F_0) : |\mathsf{rel}(U)| \to \mathsf{rel}(U)$ be an equality preserving fibred functor. The vertex of the F_1-cone in $\forall_1(F, F, F_1)$ is isomorphic to $\mathsf{Eq}(\forall_0 F)$, if it exists.

Proof. We give two vertical morphisms $h \colon \mathsf{Eq}(\forall_0 F) \to \forall_1(F, F, F_1)$ and $s \colon \forall_1(F, F, F_1) \to \mathsf{Eq}(\forall_0 F_0)$ and since the fibration is faithful, their compositions are necessarily identity morphisms.

The terminal F-eqcone $(\forall_0 F, \nu)$ defines a fibred (F, F, F_1)-eqcone whose F_1-cone vertex is $\mathsf{Eq}(\forall_0 F)$ and the vertices of the F-eqcones are $\forall_0 F$. There is a unique morphism h from this cone to $\forall_1(F, F, F_1)$ and it is vertical since both the fibred (F, F, F_1)-eqcones are over $(\forall_0 F, \forall_0 F)$. In fact, being $(\forall_0 F, \forall_0 F)$ terminal, the unique morphism $(\forall_0 F, \forall_0 F) \to (\forall_0 F, \forall_0 F)$ is the identity.

For the other morphism, by Lemma 7.14, we have a commuting diagram

$$\begin{array}{ccc} \{\forall_1 F_1\} & \xrightarrow{\ v_1\ } & \forall_0 F \\ {\scriptstyle \langle v_1, v_1\rangle}\big\downarrow & & \big\downarrow{\scriptstyle \delta} \\ \forall_0 F \times \forall_0 F & \xrightarrow[\ \mathsf{id}\ ]{} & \forall_0 F \times \forall_0 F. \end{array}$$

and using fullness and faithfulness of $\mathcal{P}$, there is a unique morphism $s\colon \forall_1 F_1 \to \mathsf{Eq}(\forall_0 F)$ such that $\{s\} = v_1$ and $U(s) = \mathsf{id}$. □

Again, the Identity Extension Lemma for fibred functors $(T_1, T_0) : \mathsf{rel}(U)|^{n+1} \to \mathsf{rel}(U)$ immediately follows by instantiating Lemma 7.15 with $F_0 = T_0(\bar{A}, -)$ and $F_1 = T_1(\mathsf{Eq}\bar{A}, -)$.

Corollary 7.16. Let U be a subobject bifibration and $(T_1, T_0)\colon |\mathsf{rel}(U)|^{n+1} \to \mathsf{rel}(U)$ be an equality preserving fibred functor. Then

$$\mathsf{Eq}(\forall_0 T(\bar{A}, \_)) \cong \forall_1((T_0(\bar{A}, \_), T_1(\mathsf{Eq}\bar{A}, \_)), (T_0(\bar{A}, \_), T_1(\mathsf{Eq}\bar{A}, \_)), T_1(\mathsf{Eq}\bar{A}, \_))$$

for every n-tuple of objects in the base category $\mathcal{E}$.

These lemmas show that once again in our axiomatic setting, all type expressions are interpreted not just as fibred functors $|\mathsf{rel}(U)|^n \to \mathsf{rel}(U)$, but as equality preserving fibred functors. We now turn to the construction of a model exploiting this fact.

7.5 ∀-fibrational structure

Consider the functor $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ as in Definition 5.10. In this section we finally show that if the fibre $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_\Omega$ over the generic object Ω has terminal fibred (F, G, H)-eqcones then p has simple Ω-products. In this way, we find a condition for being a ∀-fibration. If p is also an equality preserving arrow fibration, we get a $\lambda 2$-fibration where we can interpret terms of System F.

Lemma 7.17. Let $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ be the functor as in Definition 5.10. If $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_\Omega$ has terminal fibred (F, G, H)-eqcones for every pair of fibred functors $F, G\colon |\mathsf{rel}(U)| \to$

$\mathsf{rel}(U)$ and $H\colon \mathbf{Rel}(\mathcal{E}) \to \mathbf{Rel}(\mathcal{E})$ over $F_0 \times G_0$, then p has simple Ω-products. In particular the right adjoint to $\pi^*\colon (\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n \to (\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_{n+1}$ is given by the functor $\Pi = (\Pi_0, \Pi_1)\colon (\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_{n+1} \to (\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$ which sends an equality preserving fibred functor $F\colon |\mathsf{rel}(U)|^{n+1} \to \mathsf{rel}(U)$ to the fibred functor whose components are

$$(\Pi_0 F)\bar{A} = \forall_0(F_0(\bar{A}, -), F_1(\mathsf{Eq}\bar{A}, -))$$

and

$$(\Pi_1 F)\bar{R} = \forall_1((F_0(\bar{A}, -), F_1(\mathsf{Eq}\bar{A}, -)), (F_0(\bar{B}, -), F_1(\mathsf{Eq}\bar{B}, -)), F_1(\bar{R}, -)).$$

Proof. We show that there is a natural isomorphism between $\pi^*G \to F$ and $G \to \Pi F$, where F is in $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_{n+1}$ and G is in in $(\mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)})_n$. Consider a fibred natural transformation $(\tau, \xi) : \pi^*G \to F$. Note that $(\pi^*G)_0(\bar{A}, X) = G_0(\bar{A})$ for every X. Hence $\tau_{\bar{A},-}$ and $\xi_{\mathsf{Eq}(\bar{A}),-}$ define an $(F_0(\bar{A}, -), F_1(\mathsf{Eq}(\bar{A}), -))$-eqcone with vertex $G_0(\bar{A})$, so that there is a a map $\rho_0 : G_0(\bar{A}) \to (\Pi_0 F)\bar{A}$ into the terminal such. Similarly $\xi_{\bar{R},-}$ over $(\tau_{\bar{A},-}, \tau_{\bar{B},-})$ defines a fibred $((F_0(\bar{A}, -), F_1(\mathsf{Eq}(\bar{A}), \_)), (F_0(\bar{B}, \_), F_1(\mathsf{Eq}(\bar{B}), -), F_1(\bar{R}, \_))$-eqcone and there is a unique morphism $\rho_1 : G_1(\bar{R}) \to (\Pi_1 F)\bar{R}$ which together with ρ_0 makes up a fibred natural transformation $G \to \Pi F$. In the other direction, composition with the projections (ν_A, ν_B, γ_R) turns natural transformations $G \to \Pi F$ into natural transformations $\pi^*G \to F$. By the universal property of terminal fibred eqcones, these correspondences are mutually inverse. The Beck-Chevalley condition boils down to the fact that both $(f^* \circ \Pi_m)F\bar{A}$ and $(\Pi_n \circ (f \times \mathsf{id})^*)F\bar{A}$ are defined to be the terminal eqcone for the same functor $F(f\bar{A}, -)$. □

Recall that using Lemma 5.22 we showed how the equality preserving arrow fibration structure arises from standard structure, e.g. that our original fibration $U : \mathcal{E} \to \mathcal{B}$ is cartesian closed, and that the functor Eq has a left adjoint satisfying Frobenius. To summarise, in this section we have proven:

Theorem 7.18. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a faithful bifibration which admits full comprehension and U is cartesian closed. Assume that the functor Eq has a left adjoint satisfying Frobenius, and the terminal fibred eqcones exist. Then the functor $p\colon \mathcal{F}^{\mathsf{Eq}}_{\mathsf{rel}(U)} \to \mathcal{N}^{\mathsf{Eq}}_{\mathsf{rel}(U)}$ as in Definition

5.10 is a $\lambda 2$-fibration and thus a model of System F where the Identity Extension Lemma and the Abstraction Theorem hold in the sense of Corollaries 5.25 and 5.26. □

From this theorem it follows that we have the same model as in Chapter 5. Hence all the usual expected consequences of parametricity shown in Chapter 5 follow.

Part III

Two-Dimensional Parametricity

Chapter 8

Proof-relevant relations

Throughout the previous part of the thesis, we worked with faithful fibrations. Faithfulness is a reasonable assumption, as it corresponds to proof-irrelevant relations, which is a standing assumption in the literature. In this part of the thesis, based on the paper [GNFO16], we start to study what happens when we remove the faithfulness assumption and we work with proof-relevant relations. We will see that, in order to recover parametricity, we need another layer of relations: relations between relations which we call 2-relations.

The work that we present here is only the first step of a wider project. We introduce two-dimensional parametricity in a concrete way, like Reynolds' set theoretical model that we presented in Section 5.1.

Since we are working with proof relevant relations, we need a notion of proof relevant equality and this is provided by intensional Martin-Löf type theory. Note that we are still studying System F: the use of Martin-Löf type theory is to provide a meta-mathematics that we need for the proof relevant framework. A standard reference for Martin-Löf type theory, also if not in its intensional form, is [ML84], but also [Uni13] has good introductory chapters.

In this chapter we introduce intensional Martin-Löf type theory and we give the definition of two-dimensional relations in intensional Martin-Löf type theory.

8.1 Intensional Martin-Löf type theory

Intensional Martin-Löf type theory is an extension of the simply typed lambda calculus, different from System F, where the two important features are the dependent types and the identity types. We refer to the intensional Martin-Löf type theory as MLTT, since we will only use the intensional version.

8.1.1 Dependent types

So far we did not give an environment in which types live: we just assumed as primitive the notion of types. We want now to collect together types, like in a type of types.

Every type A lives in the universe of (small) types, and we formally express it by writing $A\colon \mathsf{Type}$, where Type is the universe of (small) types. Note that we need that $\mathsf{Type} \notin \mathsf{Type}$, in order to avoid situations similar to Russel's paradox for set theory.

A **dependent type** is a type which depends on a term variable. We write $x\colon A \vdash B$ type for a type B which depends on the term variable $x\colon A$. A dependent type is sometimes called **family of types** and denoted by $B\colon A \to \mathsf{Type}$. In fact a dependent type consists of a collection of types $B(a)\colon \mathsf{Type}$ indexed over terms $a\colon A$.

Example 8.1. An example of a dependent type is given by the type of vectors $\mathsf{Vec}_T(n)$ parametrised by their length, i.e. $\mathsf{Vec}_T(n)$ is the type of vectors with n components. Equivalently, we may write $n\colon \mathsf{nat} \vdash \mathsf{Vec}_T$ type.

The dependence of types from terms requires new rules to construct contexts. A context in Martin-Löf type theory consists of a list of term variables $x_1\colon A_1, \dots x_n\colon A_n$ where $x_i \neq x_j$ if $i \neq j$. The list may be empty. We use the Greek letters Γ or Δ for the context. In a context each type must be well-formed in the context composed of the previous variables of the list. The judgment Γ ctx formally expresses the fact that Γ is a well-formed context. The formal rules for the context are the following

$$\frac{}{\_\ \mathsf{ctx}} \qquad \frac{x_1\colon A_1, \dots, x_{n-1}\colon A_{n-1} \vdash A_n\ \mathsf{type}}{(x_1\colon A_1, \dots, x_n\colon A_n)\ \mathsf{ctx}}$$

where, in the second rule, the variable x_n must be distinct from the variables $x_1, \dots, x_{n-1}$.

We have the usual rule for term variables

$$\frac{x_i\colon A_i \dots, x_n\colon A_n\ \mathsf{ctx}}{x_1\colon A_1, \dots, x_n\colon A_n \vdash x_i\colon A_i}\ i \in \{1, \dots, n\}$$

8.1.2 Dependent functions (Π-types)

Dependent functions, sometimes called **dependent products** or **Π-types**, are a more general version of arrow types. The elements of a dependent product are functions whose codomain type changes depending on the input term. The following rule shows how to form dependent function types:

$$\frac{\Gamma \vdash A \text{ type} \quad \Gamma, x\colon A \vdash B \text{ type}}{\Gamma \vdash (\Pi x\colon A)B(x) \text{ type}}$$

Example 8.2. Consider $\mathsf{Vec}_{\mathsf{nat}}$ as in Example 8.1. The dependent function $\mathsf{vec}_{\mathsf{zero}}$ mapping n to the vector $[0, \dots, 0]$ of length n has type $\mathsf{vec}_{\mathsf{zero}}\colon (\Pi n\colon \mathsf{nat})\mathsf{Vec}_{\mathsf{nat}}(n)$.

We can introduce terms of dependent function type using the following rule

$$\frac{\Gamma, x\colon A \vdash b\colon B}{\Gamma \vdash \lambda(x\colon A).b\colon (\Pi x\colon A)B}$$

This is clearly reminiscent of the introduction rule for arrow types, but with dependencies taken into account. Similarly we have the other rules:

- Term application
$$\frac{\Gamma \vdash f\colon (\Pi x\colon A)B \quad \Gamma \vdash a\colon A}{\Gamma \vdash f(a)\colon B[a/x]}$$

- β-equivalence
$$\frac{\Gamma, x\colon A \vdash b\colon B \quad \Gamma \vdash a\colon A}{\Gamma \vdash (\lambda(x\colon A).b)(a) \equiv b[a/x]\colon B[a/x]}$$

- η-equivalence
$$\frac{\Gamma \vdash f\colon (\Pi x\colon A)B}{\Gamma \vdash f \equiv (\lambda x.f(x))\colon (\Pi x\colon A)B}$$

where $B[a/x]$ is the type obtained by substituting a in x.

When $x\colon A$ does not occur freely in B, this means that B does not depend on terms of type A, and, as a special case, we obtain the ordinary arrow types $A \to B \coloneqq (\Pi x\colon A)B$.

We will abbreviate the expression $\lambda(x\colon A).b$ as $\lambda x.b$, with the understanding that the omitted type A should be filled in appropriately before type-checking.

8.1.3 Dependent pair types (Σ-types)

Just as dependent functions generalise arrow types, **dependent pairs** generalise products. Dependent pairs are also known as **dependent sums** or **Σ-types**.

We can form a dependent pair type using the following rule:

$$\frac{\Gamma \vdash A \ \mathsf{type} \quad \Gamma, x\colon A \vdash B \ \mathsf{type}}{\Gamma \vdash (\Sigma x\colon A)B\colon \ \mathsf{type}}$$

Example 8.3. Consider again Vec_T as in Example 8.1. The terms of type $(\Sigma n\colon \mathsf{nat})\mathsf{Vec}_T$ consist of pairs $(n, [x_1, \dots, x_n])$ where n is a natural number and $[x_1, \dots, x_n]$ is a vector of length n, with $x_i\colon T$ for every $i \in \{1, \dots, n\}$.

The way to introduce terms of dependent pair type is by pairing, as shown by the following rule:

$$\frac{\Gamma, x\colon A \vdash B \ \mathsf{type} \quad \Gamma \vdash a\colon A \quad \Gamma \vdash b\colon B[a/x]}{\Gamma \vdash (a, b)\colon (\Sigma x\colon A)B}$$

We can eliminate terms of dependent pair type using the following rule:

$$\frac{\Gamma, z\colon (\Sigma x\colon A)B \vdash C \ \mathsf{type} \quad \Gamma, x\colon A, y\colon B \vdash g\colon C[(x, y)/z] \quad \Gamma \vdash p\colon (\Sigma x\colon A)B}{\Sigma \vdash \mathsf{ind}_{(\Sigma x\colon A)B}(C, g, p)\colon C[p/z]}$$

The computation rule is a judgmental equality explaining what happens when elimination rules are applied to results of introduction rules. In the case of dependent pair types the computational rule is

$$\frac{\begin{array}{cc}\Gamma, z\colon (\Sigma x\colon A)B \vdash C \ \mathsf{type} & \Gamma, x\colon A, y\colon B \vdash g\colon C[(x, y)/z] \\ \Gamma \vdash a\colon A & \Gamma \vdash b\colon B[a/x]\end{array}}{\Gamma \vdash \mathsf{ind}_{(\Sigma x\colon A)B}(C, g, (a, b)) \equiv g[(a/x, b/y]\colon C[(a, b)/z]}$$

which says that elimination applied to a pair (a, b) reduces exactly to $g(a, b)$.

When B does not contain free occurrences of $x\colon A$, as a special case we obtain the cartesian product $A \times B \coloneqq (\Sigma x\colon A)B$.

8.1.4 The identity type

The way MLTT handles with identities is the second distinguishing feature of the system. Given a type A and two terms $a, b\colon A$, we have the **identity type** $\mathsf{Id}_A(a,b)$. Terms $p\colon \mathsf{Id}_A(a,b)$ are proofs that a is equal to b. Given a type $A\colon \mathsf{Type}$, we have the family of types $\mathsf{Id}_A\colon A \to A \to \mathsf{Type}$. It is convenient to use the standard symbol of equality $a = b$ for $\mathsf{Id}_A(a,b)$, and, for clarity, we may also write $a =_A b$. If we have a term of type $a =_A b$ we say that a and b are equal, or sometimes **propositionally equal** if we want to emphasise that this is different from the judgmental equality $a \equiv b$.

The following rule shows how to form identity types:

$$\frac{\Gamma \vdash A\colon \mathsf{Type} \quad \Gamma \vdash a\colon A \quad \Gamma \vdash b\colon A}{\Gamma \vdash a =_A b\colon \mathsf{Type}}$$

We expect that there should be a way to construct a term of type $a =_A a$ for every term a of type A. In fact there is the following rule

$$\frac{\Gamma \vdash A\colon \mathsf{Type} \quad \Gamma \vdash a\colon A}{\Gamma \vdash \mathsf{refl}_a\colon a =_A a}$$

Thus, we have a dependent function

$$\mathsf{refl}\colon (\Pi a\colon A)(a =_A a)$$

which is called **reflexivity**, and it introduces a term of identity type which is a proof that every term of type A is equal to itself (in a specific way). In particular, this means that if a and b are judgmentally equal ($a \equiv b$) then we also have an element $\mathsf{refl}_a\colon a =_A b$. This is well typed because $a \equiv b$ means that also the type $a =_A b$ is judgmentally equal to $a =_A a$, which is the type of refl_a.

The induction rule for identity types is very important. It can be seen as stating that the family of identity types is freely generated by the elements of the form $\mathsf{refl}_x\colon x = x$. Due to its importance, we first analyze the induction rule which is as follows: given a family of types

$$C\colon (\Pi x, y\colon A)(x =_A y) \to \mathsf{Type}$$

and a function

$$c\colon (\Pi x\colon A)C(x, x, \mathsf{refl}_x),$$

then there is a function

$$f\colon (\Pi x, y\colon A)(\Pi p\colon x =_A y)C(x, y, p)$$

which satisfies the computation rule

$$f(x, x, \mathsf{refl}_x) \equiv c(x).$$

Formally the rule is

$$\frac{\begin{array}{c}\Gamma, x\colon A, y\colon A, p\colon x =_A y \vdash C\colon \mathsf{Type} \\ \Gamma, z\colon A \vdash c\colon C(z, z, \mathsf{refl}_z) \quad \Gamma \vdash a\colon A \quad \Gamma \vdash b\colon A \quad \Gamma \vdash p'\colon a =_A b\end{array}}{\Gamma \vdash \mathsf{ind}_{=_A}(C, c, a, b, p')\colon C(a, b, p')}$$

and the computation rule is

$$\frac{\Gamma, x\colon A, y\colon A, p\colon x =_A y \vdash C\colon \mathsf{Type} \quad \Gamma, z\colon A \vdash c\colon C(z, z, \mathsf{refl}_z) \quad \Gamma \vdash a\colon A}{\Gamma \vdash \mathsf{ind}_{=_A}(C, c, a, a, \mathsf{refl}_a) \equiv c[a/z]\colon C(a, a, \mathsf{refl}_a)}$$

We use the same notation as in [Uni13] and we write $\mathsf{ind}_{=_A}$, but traditionally $\mathsf{ind}_{=_A}$ is known as J.

For the identity on function types we rely on the following axiom:

Axiom 8.4 (Function extensionality)**.** The function

$$\mathsf{happly} : \mathsf{Id}_{(\Pi x:A)B(x)}(f, g) \to (\Pi x : A)\mathsf{Id}_{B(x)}(f(x), g(x))$$

defined using path induction in the obvious way (see [Uni13]) is an equivalence in the sense that there exists an inverse function and the two compositions are identities. In particular, we have an inverse

$$\mathsf{ext} : (\Pi x : A)\mathsf{Id}_{B(x)}(f(x), g(x)) \to \mathsf{Id}_{(\Pi x:A)B(x)}(f, g)$$

This axiom is justified by models of Type Theory in intuitionistic set theory. It also follows from Voevodsky's Univalence Axiom [Voe10], which we do not assume in this thesis. We will use function extensionality in order to derive the Identity Extension Lemma for arrow types, as in e.g. [Wad07].

The corresponding statement for sigma types requires more tools. In fact given two terms (a, p) and (b, q) of type $(\Sigma a\colon A)B(a)$, one would expect them to be equal if they are equal componentwise. This is not possible to do straightforward because $p\colon B(a)$ and $q\colon B(b)$ live in different types. We will see in Section 8.2.4 how to do that.

8.2 Homotopy interpretation of MLTT

In this section we show some properties of MLTT and sketch how one can think of MLTT from the point of view of homotopy theory.

The key idea in homotopy type theory is that an identity type $\mathsf{Id}_A(a, b)$ can be thought of as the type of paths from a to b. In the rest of this thesis, we will use terminology and intuitions of homotopy type theory. For example, we follow the practice of calling the induction on identity types **path induction**.

The rest of this section develops the algebra of the identity types. We first construct a map which sends every path to its inverse.

Lemma 8.5. For every type A and every $x, y\colon A$ there is a function

$$\_^{-1}\colon (x =_A y) \to (y =_A x)$$

such that $\mathsf{refl}_x^{-1} \equiv \mathsf{refl}_x$ for every $x\colon A$. We call p^{-1} the **inverse** of p.

Proof. By path induction it is enough to define the map for every x and refl_x. We define $\mathsf{refl}_x^{-1} \equiv \mathsf{refl}_x$. ☐

The composition of paths corresponds to transitivity of equality.

Lemma 8.6. For every type A and every $x, y, z \colon A$ there is a function

$$\_ \cdot \_ \colon (x =_A y) \to (y =_A z) \to (x =_A z)$$

such that $\mathsf{refl}_x \cdot \mathsf{refl}_x \equiv \mathsf{refl}_x$ for every $x \colon A$. We call $p \cdot q$ the **concatenation** or **composite** of p and q.

Proof. We apply path induction two times: The first to p and the second to q and we reduce to the case in which we have only x and refl_x. We define $\mathsf{refl}_x \cdot \mathsf{refl}_x \equiv \mathsf{refl}_x$. ☐

The following lemma shows the behaviour of refl, inverses and composition.

Lemma 8.7. Suppose $A \colon \mathsf{Type}$, $x, y, z, w \colon A$, $p \colon x =_A y$, $q \colon y =_A z$ and $r \colon z =_A w$. We have the following:

1. $p = p \cdot \mathsf{refl}_y$ and $p = \mathsf{refl}_x \cdot p$
2. $p^{-1} \cdot p = \mathsf{refl}_y$ and $p \cdot p^{-1} = \mathsf{refl}_x$
3. $(p^{-1})^{-1} = p$
4. $p \cdot (q \cdot r) = (p \cdot q) \cdot r$.

Proof. The proof of each of the points uses induction on paths.

1. By induction on p it reduces to $\mathsf{refl}_x \cdot \mathsf{refl}_x = \mathsf{refl}_x$.
2. By induction on p it reduces to $\mathsf{refl}_x^{-1} \cdot \mathsf{refl}_x = \mathsf{refl}_x$.
3. By induction on p it reduces to $\mathsf{refl}_x^{-1^{-1}} = \mathsf{refl}_x$
4. By induction on p, q and r it reduces to $\mathsf{refl}_x \cdot (\mathsf{refl}_x \cdot \mathsf{refl}_x) = \mathsf{refl}_x = (\mathsf{refl}_x \cdot \mathsf{refl}_x) \cdot \mathsf{refl}_x$.

□

Recall that a **groupoid** is a category where every morphism is invertible. Thus, so far in this section, we proved that the identity types $\mathsf{Id}_A(a,b)$ have the structure of a groupoid, up to propositional equality. In fact, by considering $\mathsf{Id}_{\mathsf{Id}_{\dots}}$, one can show that each type comes equipped with the structure of an ∞-groupoid (see [LL10, vdBG12]).

8.2.1 Functions are functors

Since types are groupoids, it is natural to study if functions between types have functorial behaviour.

Lemma 8.8. Suppose $f\colon A \to B$ is a function. Then for any $x, y\colon A$ there is an operation

$$\mathsf{ap}(f)\colon (x =_A y) \to (f(x) =_B f(y)).$$

Moreover, for every $x\colon A$ we have $\mathsf{ap}(f)(\mathsf{refl}_x) \equiv \mathsf{refl}_{f(x)}$.

Proof. By induction, it suffices to assume p is refl_x. In this case we can define $\mathsf{ap}(f)(\mathsf{refl}_x) \equiv \mathsf{refl}_{f(x)}\colon f(x) = f(x)$. □

The map $\mathsf{ap}(\_)$ behaves functorially.

Lemma 8.9. For functions $f\colon A \to B$ and $g\colon B \to C$, and for paths $p\colon x =_A y$ and $q\colon y =_A z$ we have

- $\mathsf{ap}(f)(p \cdot q) = \mathsf{ap}(f)(p) \cdot \mathsf{ap}(f)(q)$.
- $\mathsf{ap}(f)(p^{-1}) = \mathsf{ap}(f)(p)^{-1}$.
- $\mathsf{ap}(g)(\mathsf{ap}(f)(p)) = \mathsf{ap}(g \circ f)(p)$.
- $\mathsf{ap}(\mathsf{id}_A)(p) = p$.

Proof. By path induction. □

8.2.2 Homotopies and equivalences

We define a homotopy between functions as follows:

Definition 8.10. Let $f, g\colon (\Pi x\colon A)P(x)$ be two dependent functions with codomains the type family $P\colon A \to \mathsf{Type}$. A **homotopy** from f to g is a dependent function of type

$$(f \sim g) :\equiv (\Pi x\colon A)(f(x) =_{P(x)} g(x)).$$

Lemma 8.11. Homotopy is an equivalence relation. That is, we have terms of types

$$(\Pi f\colon (\Pi x\colon A)P(x))(f \sim f)$$

$$(\Pi f, g\colon (\Pi x\colon A)P(x))(f \sim g) \to (g \sim f)$$

$$(\Pi f, g, h\colon (\Pi x\colon A)P(x))(f \sim g) \to (g \sim h) \to (f \sim h)$$

The following lemma expresses a naturality condition for homotopies:

Lemma 8.12. Suppose $H\colon f \sim g$ is an homotopy between two functions $f, g\colon A \to B$ and let $p\colon x =_A y$. Then we have $H(x) \cdot \mathsf{ap}(g)(p) =_{f(x) =_B g(y)} \mathsf{ap}(f)(p) \cdot H(y)$ and we can draw this condition as a commutative diagram:

$$\begin{array}{ccc} f(x) & \overset{\mathsf{ap}(f)(p)}{=\!=\!=\!=} & f(y) \\ {\scriptstyle H(x)}\Big\| & & \Big\|{\scriptstyle H(y)} \\ g(x) & \underset{\mathsf{ap}(g)(p)}{=\!=\!=\!=} & g(y). \end{array}$$

A function $f : A \to B$ is said to be an **equivalence** if it has homotopy left and right inverses and we write

$$\mathsf{isequiv}(f) :\equiv ((\Sigma g\colon B \to A)(f \circ g \sim \mathsf{id}_B)) \times ((\Sigma h\colon B \to A)(h \circ f \sim \mathsf{id}_A)).$$

If there exists an equivalence between two types A and B, we write $A \cong B$ and the type of equivalences between two types is denoted by

$$(A \cong B) :\equiv (\Sigma f \colon A \to B)(\mathsf{isequiv}(f)).$$

8.2.3 n-Types

In this thesis we restrict the attention to types where identity proofs of identity proofs are unique, i.e. to types A where $\mathsf{Id}_{\mathsf{Id}A(x,y)}(p,q)$ is trivial. Garner in [Gar09] has investigated the semantics of Type Theory where all types are of this form. They are particular cases of n-types which are defined inductively as follows:

$$\begin{aligned}
\mathsf{isContr}(A) &\coloneqq (\Sigma a \colon A)(\Pi x : A)\mathsf{Id}_A(a,x) & \mathsf{Contr} &\coloneqq (\Sigma X : \mathsf{Type})(\mathsf{isContr}(X))\\
\mathsf{isProp}(A) &\coloneqq (\Pi x, y : A)\mathsf{isContr}(\mathsf{Id}_A(x,y)) & \mathsf{Prop} &\coloneqq (\Sigma X : \mathsf{Type})(\mathsf{isProp}(X))\\
\mathsf{isSet}(A) &\coloneqq (\Pi x, y : A)\mathsf{isProp}(\mathsf{Id}_A(x,y)) & \mathsf{Set} &\coloneqq (\Sigma X : \mathsf{Type})(\mathsf{isSet}(X))\\
\mathsf{is\text{-}1\text{-}Type}(A) &\coloneqq (\Pi x, y : A)\mathsf{isSet}(\mathsf{Id}_A(x,y)) & \mathsf{1\text{-}Type} &\coloneqq (\Sigma X : \mathsf{Type})(\mathsf{is\text{-}1\text{-}Type}(X))\\
&\vdots & &\vdots\\
\mathsf{is\text{-}n\text{-}Type}(A) &\coloneqq (\Pi x, y : A)\mathsf{is\text{-}(n\text{-}1)\text{-}Type}(\mathsf{Id}A(x,y)) & \mathsf{n\text{-}Type} &\coloneqq (\Sigma X : \mathsf{Type})(\mathsf{is\text{-}n\text{-}Type}(X))\\
&\vdots & &\vdots
\end{aligned}$$

The types isContr, isProp and isSet are also called, respectively, is-(-2)-Type, is-(-1)-Type and is-0-Type. For our purposes, it is enough to restrict to the first four levels. Here Prop is the collection of **propositions**, i.e. types with at most one inhabitant up to identity, while Set is the collection of **sets**, i.e. types whose identity types in turn are propositional. Finally, we are interested in 1-Type which is the collection of 1-types, i.e. types whose identity types are sets. Furthermore, all three of Prop, Set and 1-Type are closed under Π- and Σ-types. The witness that a type is in some of these classes is itself a proposition, and so we will abuse notation and leave it implicit – if there is a proof, it is unique up to identity.

If $P : A \to \mathsf{Prop}$, then we write $\{x : A \,|\, P(x)\}$ for $(\Sigma x : A)P(x)$. Since $P(x)$ is a proposition for each $x : A$, we have that $\mathsf{Id}_{\{x:A\,|\,P(x)\}}((a,p),(b,q)) \cong \mathsf{Id}_A(a,b)$. For this reason, we will often leave the proof $p : P(a)$ implicit when talking about an element (a, p) of $\{x : A \mid P(x)\}$. We also suggestively write $a \in P$ for $P(a)$.

We recall some of the properties of n-types. We refer to [Uni13] for the proofs.

Lemma 8.13. For every type A, the following are equivalent:

- A is contractible;
- A is equivalent to the singleton $\mathbf{1}$.

Lemma 8.14. For any type A and any $a\colon A$, the type $(\Sigma x\colon A)\mathsf{Id}_A(a, x)$ is contractible.

Lemma 8.15. The hierarchy of n-types is cumulative in the following sense: given a number $n \geq -2$, if X is an n-type, then it is also an $(n+1)$-type.

Lemma 8.16. Let $n \geq -2$, and let $A\colon \mathsf{Type}$ and $B\colon A \to \mathsf{Type}$. If A is an n-type and for all $a\colon A$, $B(a)$ is an n-type, then so is $(\Sigma x\colon A)B(x)$.

Lemma 8.17. Let $n \geq -2$, and let $A\colon \mathsf{Type}$ and $B\colon A \to \mathsf{Type}$. If A is an n-type and for all $a\colon A$, $B(a)$ is an n-type, then so is $(\Pi x\colon A)B(x)$.

8.2.4 Transport

Let $P\colon A \to \mathsf{Type}$ be a family of types and $p\colon a =_A b$ be a path. Intuitively, we would expect that if a and b are the same, then $P(a)$ and $P(b)$ should be related. This is the case and the following lemma shows how they are related:

Lemma 8.18. Suppose P is a family of types over A and that $p\colon x =_A y$. Then there is a function $p_*\colon P(x) \to P(y)$.

Proof. By induction we can assume p is refl_x. In this case we can take $(\mathsf{refl}_x)_*\colon P(x) \to P(x)$ to be the identity function. □

This operation gives rise to a **transport** term of type

$$\mathsf{tr} : (P : A \to \mathsf{Type}) \to \mathsf{Id}_A(x, y) \to P(x) \to P(y)$$

where $\mathsf{tr}(P, p) \equiv p_*$. Given a type family $P \colon A \to \mathsf{Type}$, two terms $x, y \colon A$, a path $p \colon \mathsf{Id}_A(x, y)$, and a term $u \colon P(x)$ when convenient we use the more concise notation $\mathsf{tr}(p)u$ instead of $\mathsf{tr}(P, p)u$ when it is possible to infer P.

Using transport we have the following characterisation of equality in Σ-types (see [Uni13]):

Lemma 8.19. $\mathsf{Id}_{(\Sigma x:A)B(x)}((x, y), (x', y')) \cong (\Sigma p : \mathsf{Id}_A(x, x'))\mathsf{Id}_{B(x')}(\mathsf{tr}(B, p)y, y')$.

Some useful results on transport are the following:

Lemma 8.20. Given $P \colon A \to \mathsf{Type}$, $p \colon x =_A y$, $q \colon y =_A z$ and $u \colon P(x)$, we have

$$q_*(p_*(u)) = (p \cdot q)_*(u).$$

Lemma 8.21. Let $f \colon A \to B$ be a function and consider a type family $P \colon B \to \mathsf{Type}$, $p \colon x =_A y$ and $u \colon P(f(x))$. We have

$$\mathsf{tr}(P \circ f, p)u = \mathsf{tr}(P, \mathsf{ap}(f)p)u.$$

Lemma 8.22. Let $P, Q \colon A \to \mathsf{Type}$ be two families of types and $f \colon (\Pi x \colon A)P(x) \to Q(x)$ a family of functions. If $p \colon x =_A y$ and $u \colon P(x)$, then we have

$$\mathsf{tr}(Q, p)f(x, u) = f(y, \mathsf{tr}(P, p)u).$$

When $P \colon A \to \mathsf{Type}$ is a family of types of the form $P(x) = \mathsf{Id}_A(a, x)$, $P(x) = \mathsf{Id}_A(x, a)$ or $P(x) = \mathsf{Id}(x, x)$ where $a \colon A$, the transport is given by composition of paths as specified by the following lemma:

Lemma 8.23. For any type A, $P \colon A \to \mathsf{Type}$ and $p \colon \mathsf{Id}_A(x_1, x_2)$, we have

- if $P(x) = \mathsf{Id}_A(a, x)$ and $q \colon \mathsf{Id}_A(a, x_1)$, then $\mathsf{tr}(P, p)q = q \cdot p$;
- if $P(x) = \mathsf{Id}_A(x, a)$ and $q \colon \mathsf{Id}_A(x_1, a)$, then $\mathsf{tr}(P, p)q = p^{-1} \cdot q$;
- if $P(x) = \mathsf{Id}_A(x, x)$ and $q \colon \mathsf{Id}_A(x_1, x_1)$, then $\mathsf{tr}(P, p)q = p^{-1} \cdot q \cdot p$.

Thanks to the transport we can generalise the functorial behaviour of functions to dependent functions. In general, given a dependent function $f\colon (\Pi x\colon A)P(x)$, if $p\colon x = y$, it makes no sense to ask if $f\,x$ is equal to $f\,y$ because they live in two different types $P(x)$ and $P(y)$. Using the transport we can move between the two types and we have:

Lemma 8.24. Suppose $f\colon (\Pi x\colon A)P(x)$ is a dependent function. Then we have a map

$$\mathsf{apd}(f)\colon (\Pi p\colon x =_A y)(p_*(f(x)) =_{P(y)} f(y)).$$

Proof. By induction, it suffices to assume p is refl_x. It follows that the desired equation is $(\mathsf{refl}_x)_*(f(x)) = f(x)$ which clearly holds. □

8.3 Impredicativity and MLTT

In order to make proof-relevant relations precise, we work in the constructive framework of impredicative intensional Martin-Löf Type Theory [ML72]. Impredicativity allows us to quantify with Π-types over all types of sort Type in order to construct a new object of sort Type. Following [Atk12], we will use impredicative quantification in the meta-theory to interpret impredicative quantification in the object theory:

$$\frac{X\colon \mathsf{Type} \vdash B\ \mathsf{type}}{\vdash (\Pi X\colon \mathsf{Type})B\ \mathsf{type}}$$

This simplifies the presentation, and allows us to focus on the proof-relevant aspects of the logical relations.

We stress that we are not assuming Uniqueness of Identity Proofs, as that would in effect result in proof-irrelevance once again. From now on, we will however restrict attention to types where identity proofs of identity proofs are unique, i.e. to types A where $\mathsf{Id}_{\mathsf{Id}_A(x,y)}(p, q)$ is trivial. We work with Prop, Set and $1\text{-}\mathsf{Type}$ as defined in Subsection 8.2.3.

8.4 Proof-relevant relations

We now define proof-relevant relations:

Definition 8.25. The collection of proof-relevant relations is denoted **PrRel** and consists of triples (R_0, R_1, R), where $R_0, R_1 : \mathsf{1\text{-}Type}$ and $R : R_0 \times R_1 \to \mathsf{Set}$. The 1–type of morphisms from (R_0, R_1, R) to (R_0', R_1', R') is

$$(\Sigma f_0 : R_0 \to R_0')(\Sigma f_1 : R_1 \to R_1')(\Pi a : R_0, b : R_1) R(a, b) \to R'(fa, gb).$$

Note that **PrRel** has a natural categorical structure. In the rest of this chapter we take relation to mean proof-relevant relation. The above definition means that morphisms between relations have a proof-relevant equality and, thus, showing morphisms are equal involves constructing explicit proofs to that effect. Indeed, the equality of morphisms is given by

$$\mathsf{Id}((f_0, f_1, f), (f_0', f_1', f')) \cong (\Sigma \phi : \mathsf{Id}(f_0, f_0'), \psi : \mathsf{Id}(f_1, f_1'))\ \mathsf{Id}(\mathsf{tr}(\phi, \psi) f, f').$$

However, since $R : R_0 \times R_1 \to \mathsf{Set}$ has codomain Set, while R_0 and R_1 are 1–types, the complexity of R compared to R_0 and R_1 has decreased. This means relations between proof-relevant relations are in fact proof-irrelevant (see Section 8.5).

When not differently specified, when we say "consider a relation R" we mean the triple (R_o, R_1, R), and we write $R : \mathbf{PrRel}(R_0, R_1)$, or $R : R_0 \leftrightarrow R_1$, and call R a relation between R_0 and R_1. Similarly, a morphism $f \colon R \to R'$ consists of a triple (f_0, f_1, f). Given $f \colon R \to R'$ and $r \colon R(a, b)$, formally we should write $f\,a\,b\,r$ for the application of f to r. However, when a and b can be inferred, we simply write $f\,r$. If we still want to emphasise the elements a and b, we may write $f_{(a,b)}\,r$.

If $R : \mathbf{PrRel}(R_0, R_1)$ and $P : \mathbf{PrRel}(P_0, P_1)$, then we have $\mathbf{1} : \mathbf{PrRel}(\mathbf{1}, \mathbf{1})$, $P \times R : \mathbf{PrRel}(P_0 \times R_0, P_1 \times R_1)$ and $R \Rightarrow P : \mathbf{PrRel}(R_0 \to P_0, R_1 \to P_1)$ defined by

$$\begin{aligned} \mathbf{1}(x, y) &:= \mathbf{1} \\ (R \times P)((x, y), (x', y')) &:= R(x, x') \times P(y, y') \\ (R \Rightarrow P)(f, g) &:= (\Pi x : R_0, y : R_1)(R(x, y) \to P(fx, gy)) \end{aligned}$$

These exponentials have the right universal property:

Lemma 8.26. Let $R : \mathbf{PrRel}(R_0, R_1)$, $R' : \mathbf{PrRel}(R'_0, R'_1)$, and $R'' : \mathbf{PrRel}(R''_0, R''_1)$. There is an equivalence $\mathsf{abs} : (R \times R' \to R'') \to (R \to (R' \Rightarrow R''))$ with inverse $\mathsf{app} : (R \to (R' \Rightarrow R'')) \to (R \times R' \to R'')$.

Proof. The function abs is defined as

$$\mathsf{abs} = \lambda t.(\lambda a.\lambda b.t_0(a,b), \lambda a'.\lambda b'.t_1(a',b'), \lambda a.\lambda b.\lambda p.\lambda a'.\lambda b'.\lambda p'.t(a,a')(b,b')(p,p'))$$

and the function app is defined as

$$\begin{aligned}\mathsf{app} = \lambda t.(&\lambda a.t_0(\pi_1 a)(\pi_2 a), \lambda b.t_1(\pi_1 b)(\pi_2 b),\\ &\lambda a.\lambda b.\lambda p.t(\pi_1 a)(\pi_1 b)(\pi_1 p)(t_0(\pi_1 a), t_1(\pi_1 b))(\pi_2 a)(\pi_2 b)(\pi_2 p)).\end{aligned}$$

By using function extensionality it is not difficult to show that $\mathsf{abs} \circ \mathsf{app} = \mathsf{id}$ and $\mathsf{app} \circ \mathsf{abs} = \mathsf{id}$. □

We will also make use of the **equality relation** $\mathsf{Eq}(A)$ for each 1-type A:

Definition 8.27. Equality $\mathsf{Eq} : \mathsf{1\text{-}Type} \to \mathbf{PrRel}$ is defined by $\mathsf{Eq}(A) = (A, A, \mathsf{Id}_A)$ on objects and $\mathsf{Eq}(f) = (f, f, \mathsf{ap}(f))$ on morphisms.

Proposition 8.28. Eq is full and faithful in that $(\mathsf{Eq}X \to \mathsf{Eq}Y) \cong X \to Y$.

Proof. By function extensionality and Lemmas 8.14 and 8.13, we have

$$\begin{aligned}(\mathsf{Eq}X \to \mathsf{Eq}Y) &= (\Sigma f : X \to Y)(\Sigma g : X \to Y)(\Pi x x')\mathsf{Id}_X(x,x') \to \mathsf{Id}_Y(fx, gx')\\ &\cong (\Sigma f : X \to Y)(\Sigma g : X \to Y)\mathsf{Id}_{X \to Y}(f,g)\\ &\cong (\Sigma f : X \to Y)\mathbf{1} \cong X \to Y \quad .\end{aligned}$$ □

Similarly, the exponential of equality relations is an equality relation. Here, we abuse notation and use the same symbol for equivalence of types and isomorphisms of relations:

Proposition 8.29. For all $X, Y : \mathsf{1\text{-}Type}$, we have $(\mathsf{Eq}X \Rightarrow \mathsf{Eq}Y) \cong \mathsf{Eq}(X \to Y)$.

Proof. By extensionality it is enough to show

$$((\Pi x, x' : X)\mathsf{Id}(x, x') \to \mathsf{Id}(fx, gx')) \cong (\Pi x : X)\mathsf{Id}(fx, gx)$$

for every $f, g : X \to Y$. Functions can easily be constructed in both directions and proved inverse using extensionality and path induction. □

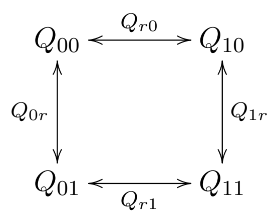
8.5 Relations between relations

Intuitively, 2-relations should relate proofs of relatedness in proof-relevant relations. Although conceptually simple, formalizing 2-relations is non-trivial as various choices arise. For instance, if R and R' are proof-relevant relations, one may consider 2-relations between them as being given by functions

$$Q : (\Pi a : R_0, a' : R'_0, b : R_1, b' : R'_1)\ (R(a, b) \times R'(a', b')) \to \mathsf{Prop}$$

with the intuition of $(p, p') \in Q(a, a', b, b')$ being that Q relates the proof p to the proof p'. However, the natural arrow type of such 2-relations does not preserve equality. The problem is that, while a is related to b, and a' is related to b', there is no relationship between a and a' and b and b'. Thus, we were led to the following definition which seems to originate with Grandis (see e.g. [Gra09]):

Definition 8.30. A 2-relation consists of the following 1-types and proof-relevant relations between them



together with a predicate

$$Q \quad : \quad (\Pi a : Q_{00}, b : Q_{10}, c : Q_{01}, d : Q_{11}) \\ Q_{r0}(a, b) \times Q_{0r}(a, c) \times Q_{r1}(c, d) \times Q_{1r}(b, d) \to \mathsf{Prop}$$

A morphism of 2-relations consists of 4 functions between each corresponding node, 4 maps of relations such that each is over the appropriate pair of morphisms of 1-types, and a predicate stating that proofs related in one 2-relation are mapped to proofs which are related in the other 2-relation.

Thus a 2-relation is a 9-tuple and, even worse, a morphism of 2-relations is a 27-tuple! This combinatorial complexity is enough to scupper any noble mathematical intentions. We therefore develop a more abstract treatment beginning with the indices in a 2-relation. This extends the notion of reflexive graphs [RR94, OT95, DR04] to a second level of 2-relations; this notion, in turn, is just the first few levels of the notion of a cubical set [BH81].

Definition 8.31. Let I_0 be the type with elements $\{00, 01, 10, 11\}$ of indices for 1-types, and I_1 the type with elements $\{0r, r0, 1r, r1\}$ of indices for proof-relevant relations. Define the source and target function $@ : I_1 \times \mathbf{Bool} \to I_0$ where $w@i$ replaces the occurrence of r in w by i. We write $w@i$ as wi.

8.5.1 I_0-types

Next we develop algebra for the types contained in 2-relations.

Definition 8.32. An I_0-type is a function $X\colon I_0 \to 1\text{-}\mathsf{Type}$. To increase legibility we write X_w for Xw. The collection of maps between two I_0-types is defined by

$$X \to X' \coloneqq (\Pi w : I_0) X_w \to X'_w$$

We define the following operations on I_0-types:

$$\begin{aligned} \mathbf{1} &:= \lambda w.\mathbf{1} \\ X \times X' &:= \lambda w.X_w \times X'_w \\ X \Rightarrow X' &:= \lambda w.X_w \to X'_w \end{aligned}$$

If X is an I_0-type, define its elements $\mathsf{El}X = (\Pi w : I_0)X_w$. The natural extension of this action to morphisms $f : X \to X'$ is denoted $\mathsf{El}\, f : \mathsf{El}X \to \mathsf{El}X'$.

Note that elements deserve that name as $\mathsf{El}X \cong \mathbf{1} \to X$. The construction of elements preserves structure as the following lemma shows:

Lemma 8.33. Let X and X' be I_0-types. Then

$$\begin{aligned} \mathsf{El}\,\mathbf{1} &\cong \mathbf{1} \\ \mathsf{El}(X \times X') &\cong \mathsf{El}X \times \mathsf{El}X' \\ \mathsf{El}(X \Rightarrow X') &= (\Pi w : I_0)X_w \to X'_w \end{aligned}$$

Proof. For the first equivalence it is clear that there is only one map of type $(\Pi w\colon I_0)\mathbf{1}$.

For the second equivalence, on one side we use the universal property of the product and we have

$$(\Pi w\colon I_0)(X_w \times X'_w) \cong (X_{00} \times X'_{00}) \times (X_{01} \times X'_{01}) \times (X_{10} \times X'_{10}) \times (X_{11} \times X'_{11}).$$

On the other side we have

$$(\Pi w\colon I_0)X_w \times (\Pi w\colon I_0)X'_w \cong (X_{00} \times X_{01} \times X_{10} \times X_{11}) \times (X'_{00} \times X'_{01} \times X'_{10} \times X'_{11}).$$

The two are equivalent up to reordering.

The third equality holds by the definition of El. □

Finally, we show how to interpret abstraction and application over I_0-types:

Lemma 8.34. Let X, X' and X'' be I_0-types. The function

$$\mathsf{abs} = \lambda f.\, \lambda w.\, \lambda x.\, \lambda x'.\, f\, w\, (x, x') : (X \times X' \to X'') \to (X \to (X' \Rightarrow X''))$$

is an equivalence with inverse

$$\mathsf{app} = \lambda f.\, \lambda w.\, \lambda y.\, f\, w\, (\pi_1 y)\, (\pi_2 y).$$

Proof. The proof is just straightforward calculation using function extensionality. □

8.5.2 I_1-Relations

Next we develop algebra for the relations contained in 2-relations.

Definition 8.35. An I_1-relation is a pair (X, R) where X is a I_0-type and R is a function of type $R : (\Pi w : I_1)\mathbf{PrRel}(X_{w0}, X_{w1})$. The collection of maps between two I_1-relations is defined by

$$(X, R) \to (X', R') := (\Sigma f : X \to X')(\Pi w : I_1)(R_w \Rightarrow R'_w)(f_{w0}, f_{w1})$$

We can represent a I_1-relation (X, R) using the following picture:

$$\begin{array}{ccc} X_{00} & \overset{R_{r0}}{\longleftrightarrow} & X_{10} \\ {\scriptstyle R_{0r}}\updownarrow & & \updownarrow{\scriptstyle R_{1r}} \\ X_{01} & \underset{R_{r1}}{\longleftrightarrow} & X_{11}. \end{array}$$

We define the following operations on I_1-relations:

$$\begin{aligned} \mathbf{1} \quad &:= \quad (\mathbf{1}, \lambda w.\mathbf{1}) \\ (X, R) \times (X', R') \quad &:= \quad (X \times X', \lambda w.R_w \times R'_w) \\ (X, R) \Rightarrow (X', R') \quad &:= \quad (X \Rightarrow X', \lambda w.R_w \Rightarrow R'_w) \end{aligned}$$

If (X, R) is an I_1-relation, define its elements

$$\mathsf{El}(X, R) = (\Sigma x : \mathsf{El}X)(\Pi w : I_1)R_w(x_{w0}, x_{w1})$$

The natural extension of El to morphisms $(f, g) : (X, R) \to (X', R')$ is denoted $\mathsf{El}(f, g) : \mathsf{El}(X, R) \to \mathsf{El}(X', R')$.

We can represent an element (x, p) of an I_1-relation (X, R) using the following picture

$$\begin{array}{ccc} x_{00} & \overset{p_{r0}}{\longleftrightarrow} & x_{10} \\ {\scriptstyle p_{0r}}\updownarrow & & \updownarrow{\scriptstyle p_{1r}} \\ x_{01} & \underset{p_{r1}}{\longleftrightarrow} & x_{11} \end{array}$$

where $p_{r0} \colon R_{r0}(x_{00}, x_{10})$, $p_{0r} \colon R_{0r}(x_{00}, x_{01})$, $p_{r1} \colon R_{r1}(x_{01}, x_{11})$ and $p_{1r} \colon R_{1r}(x_{10}, x_{11})$.

Note that elements deserve that name as $\mathsf{El}(X, R) \cong \mathbf{1} \to (X, R)$. The construction of elements preserves structure as the following lemma shows:

Lemma 8.36. Let (X, R) and (X', R') be I_1-relations. Then

$$\begin{aligned} \mathsf{El}\,\mathbf{1} &\cong \mathbf{1} \\ \mathsf{El}((X, R) \times (X', R')) &\cong \mathsf{El}(X, R) \times \mathsf{El}(X', R') \\ \mathsf{El}((X, R) \Rightarrow (X', R')) &= (\Sigma f : \mathsf{El}(X \Rightarrow X'))(\Pi w : I_1)(R_w \Rightarrow R'_w)(f_{w0}, f_{w1}) \end{aligned}$$

Proof. For the first equivalence we have that, by definition, $\mathsf{El}\mathbf{1} = (\Sigma x \colon \mathsf{El}\mathbf{1})(\Pi w \colon I_1)\mathbf{1}(\star, \star)$ which is clearly equivalent to $\mathbf{1}$.

By definition of El, for the second equivalence we have that

$$\begin{aligned} \mathsf{El}((X, R) \times (X', R')) &= \mathsf{El}(X \times X', \lambda w.R_w \times R'_w) \\ &= (\Sigma x \colon \mathsf{El}(X \times X'))(\Pi w \colon I_1)(R_w \times R'_w)(x_{w0}, x_{w_1}). \end{aligned}$$

Using Lemma 8.33 we can consider x being a pair (z, z') and rewrite

$$\begin{aligned}(\Sigma x\colon \mathsf{El}(X \times X'))(\Pi w\colon I_1)(R_w \times R'_w)(x_{w0}, x_{w_1}) \;\cong&\\ (\Sigma (z,z')\colon \mathsf{El}(X) \times \mathsf{El}(X'))(R_{r0}(z_{00}, z_{10}) \times R'_{r0}(z'_{00}, z'_{10}) \times&\\ R_{0r}(z_{00}, z_{01}) \times R'_{0r}(z'_{00}, z'_{01}) \times&\\ R_{r1}(z_{01}, z_{11}) \times R'_{r1}(z'_{01}, z'_{11}) \times&\\ R_{1r}(z_{10}, z_{11}) \times R'_{1r}(z'_{10}, z'_{11})).&\end{aligned}$$

We can be reorder the above equation obtaining

$$\begin{gathered}(\Sigma z\colon \mathsf{El}(X))(R_{r0}(z_{00}, z_{10}) \times R_{0r}(z_{00}, z_{01}) R_{r1}(z_{01}, z_{11}) \times R_{1r}(z_{10}, z_{11})) \times\\ (\Sigma z'\colon \mathsf{El}(X'))(R'_{r0}(z'_{00}, z'_{10}) \times R'_{0r}(z'_{00}, z'_{01}) \times R'_{r1}(z'_{01}, z'_{11}) \times R'_{1r}(z'_{10}, z'_{11}))\end{gathered}$$

which, by unwinding the definition, is exactly $\mathsf{El}(X, R) \times \mathsf{El}(X', R')$.

The last point holds by definition. □

Finally, we show how to interpret abstraction and application over I_0-types:

Lemma 8.37. Let $(X, R), (X', R')$ and (X'', R'') be I_1-relations. There is an equivalence $\mathsf{abs} : ((X, R) \times (X', R') \to (X'', R'')) \to ((X, R) \to ((X', R') \Rightarrow (X'', R'')))$ with inverse given by $\mathsf{app} : ((X, R) \to ((X', R') \Rightarrow (X'', R''))) \to ((X, R) \times (X', R') \to (X'', R''))$.

Proof. The proof is similar to the proof of Lemma 8.26, but rests crucially on the fact that $R \Rightarrow P : \mathbf{PrRel}(R_0 \to P_0, R_1 \to P_1)$. □

8.5.3 2-Relations

Finally, we develop the same algebra for 2-relations.

Definition 8.38. The collection of 2-relations is denoted **2Rel** and consists of pairs $((X, R), Q)$ where (X, R) is an I_1-relation and Q is a function $Q : \mathsf{El}(X, R) \to \mathsf{Prop}$. The collection of maps between two 2-relations is defined by

$$
\begin{aligned}
((X,R),Q) \to ((X',R'),Q') \quad := \quad & (\Sigma(f,g) : (X,R) \to (X',R')) \\
& (\Pi(x,p) : \mathsf{El}(X,R))(p \in Q(x) \to (\mathsf{El}\ g\ p) \in Q'(\mathsf{El}\ f\ x))
\end{aligned}
$$

Note that **2Rel** has a natural categorical structure with 2-relations $((X,R),Q)$ as objects and a morphism $(f,g)\colon ((X,R),Q) \to ((X',R'),Q')$ is a morphism $(f,g)\colon (X,R) \to (X',R')$ of I_1-relations satisfying the condition in Definition 8.38.

We write Q for $((X,R),Q)$ and given $(x,p)\colon \mathsf{El}(X,R)$ such that $p \in Q(x)$, we just write $p \in Q(x)$ leaving implicit that $(x,p)\colon \mathsf{El}(X,R)$. We represent $p \in Q(x)$ using the following picture:

$$
\begin{array}{ccc}
x_{00} & \overset{p_{r0}}{\longleftrightarrow} & x_{10} \\
{\scriptstyle p_{0r}}\updownarrow & Q & \updownarrow{\scriptstyle p_{1r}} \\
x_{01} & \underset{p_{r1}}{\longleftrightarrow} & x_{11}
\end{array}
$$

where $p_{r0} \in R_{r0}(x_{00}, x_{10})$, $p_{0r} \in R_{0r}(x_{00}, x_{01})$, $p_{r1} \in R_{r1}(x_{01}, x_{11})$ and $p_{1r} \in R_{1r}(x_{10}, x_{11})$. We use this graphical representation when convenient.

Similarly we can picture a 2-relation Q as

$$
\begin{array}{ccc}
Q_{00} & \overset{Q_{r0}}{\longleftrightarrow} & Q_{10} \\
{\scriptstyle Q_{0r}}\updownarrow & Q & \updownarrow{\scriptstyle Q_{1r}} \\
Q_{01} & \underset{Q_{r1}}{\longleftrightarrow} & Q_{11}
\end{array}
$$

and we say that Q is indexed by the above diagram. Note that the subscript by convention determines that e.g. Q_{00} is in 1-Type, Q_{r0} is in **PrRel** and Q in **2Rel**.

Finally we write $f\colon Q \to Q'$ for a morphism between two 2-relations Q and Q' with components $f = (f_{00}, f_{10}, f_{01}, f_{11}, f_{r0}, f_{0r}, f_{r1}, f_{1r}, f)$, where $f_{ij}\colon Q_{ij} \to Q'_{ij}$ in 1-Type or **PrRel** depending on the index, and f is a proof that if $(p,q,p',q') \in Q(a,b,c,d)$ then $(f_{r0}\,p, f_{0r}\,q, f_{r1}\,p', f_{1r}q') \in Q'(f_{00}\,a, f_{10}\,b, f_{01}\,c, f_{11}\,d)$. Note that f, when it exists, is unique because it is a condition on propositions. For this reason when we need to specify a morphism between two 2-relations sometimes we leave f implicit and we do not write it if we know that it exists.

We define the following operations on 2-relations

$$\begin{aligned}
\mathbf{1} &= (\mathbf{1}, \lambda_.\mathbf{1}) \\
((X,R),Q) \times ((X',R)',Q') &= ((X,R) \times (X',R'), \\
&\qquad \lambda(x,y)\lambda(p,q).p \in Q(x) \wedge q \in Q'(y)) \\
((X,R),Q) \Rightarrow ((X',R'),Q') &= ((X,R) \Rightarrow (X',R'), \\
&\qquad \lambda(f,g).(\Pi(x,p) : \mathsf{El}(X,R))p \in Q(x) \Rightarrow (\mathsf{El}\ g\ p) \in Q'(\mathsf{El}\ f\ x))
\end{aligned}$$

Lemma 8.39. Let $((X,R),Q), ((X',R'),Q')$ and $((X'',R''),Q'')$ be 2-relations. There is an equivalence

$$\begin{aligned}
\mathsf{abs} : (((X,R),Q) \times ((X',R'),Q') \to ((X'',R''),Q'')) &\cong \\
(((X,R),Q) \to (((X',R'),Q') \Rightarrow ((X'',R''),Q'')))
\end{aligned}$$

with inverse app.

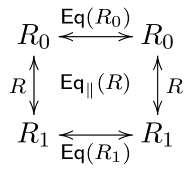
Proof. Note that if X, X' and X'' are I_0-types, and if $f : X \times X' \to X''$ then $\mathsf{abs} f : X \to (X' \Rightarrow X'')$, and for any $x : \mathsf{El}X, x' : \mathsf{El}X'$, and $w : I_0$

$$(\mathsf{El} f)\ (x,x')\ w = (\mathsf{El}\ (\mathsf{abs} f)\ x\ w)(x'\ w)$$

Similar results hold for app and for the analogous lemmas for I_1-sets. This, together with Lemma 8.37, extensionality and direct calculation gives the result. □

As in cubical and simplicial settings, there is more than one "degenerate" relation in two-dimensional relations. For example, we can duplicate a relation vertically or horizontally. These operations induce two functors defined by the following lemmas:

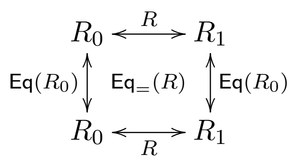
Lemma 8.40. The map sending a relation R to the 2-relation $\mathsf{Eq}_{\|}(R)$ indexed by



and defined by $(p, q, p', q') \in \mathsf{Eq}_{\|}(R)(a, b, c, d)$ if and only if $\mathsf{tr}(p, p')q =_{R(b,d)} q'$ extends to a functor $\mathsf{Eq}_{\|} \colon \mathbf{PrRel} \to \mathbf{2Rel}$.

Proof. Consider a morphism $(f, g, t) \colon (A, B, R) \to (A', B', R')$. The functor $\mathsf{Eq}_{\|}$ sends (f, g, t) to $(f, f, g, g, \mathsf{ap}(f), t, \mathsf{ap}(g), t)$. We need to check that if $(p, q, p', q') \in \mathsf{Eq}_{\|}(R)(a, b, c, d)$ then $(\mathsf{ap}(f)\, p, t\, q, \mathsf{ap}(g)\, p', t\, q') \in \mathsf{Eq}_{\|}(R')(f\,a, f\,b, g\,c, g\,d)$, and, by definition, this is true if and only if $\mathsf{tr}(\mathsf{ap}(f)p, \mathsf{ap}(g)p')t\, q =_{R(f\,b, g\,d)} t\, q'$. We know that $\mathsf{tr}(p, p')q =_{R(b,d)} q'$. We use path induction and we can assume $a \equiv b$ and $c \equiv d$, while $p \equiv \mathsf{refl}_a$ and $p' \equiv \mathsf{refl}_c$. We then have $\mathsf{tr}(\mathsf{refl}_a, \mathsf{refl}_c)q =_{R(b,d)} q'$, that is $q =_{R(b,d)} q'$ and then we can conclude $\mathsf{tr}(\mathsf{refl}_{f\,a}, \mathsf{refl}_{g\,c})t\, q =_{R(f\,b, g\,d)} t\, q'$. □

Lemma 8.41. The map sending a relation R to the 2-relation $\mathsf{Eq}_{=}(R)$ indexed by



and defined by $(p, q, p', q') \in \mathsf{Eq}_{=}(R)(a, b, c, d)$ if and only if $\mathsf{tr}(q, q')p =_{R(c,d)} p'$ extends to a functor $\mathsf{Eq}_{=} \colon \mathbf{PrRel} \to \mathbf{2Rel}$.

Proof. Consider a morphism $(f, g, t) \colon (A, B, R) \to (A', B', R')$. The functor $\mathsf{Eq}_{=}$ sends (f, g, t) to $(f, g, f, g, t, \mathsf{ap}(f), t, \mathsf{ap}(g))$. The proof that if $(p, q, p', q') \in \mathsf{Eq}_{=}(R)(a, b, c, d)$ then $(t\, p, \mathsf{ap}(f)\, q, t\, p', \mathsf{ap}(g)\, q') \in \mathsf{Eq}_{=}(R')(f\,a, g\,b, f\,c, g\,d)$ uses the same argument used in the proof of Lemma 8.40. □

There is another different kind of degeneracy, which in the cubical set context is called connection [BHS11], and it is defined as follows:

Lemma 8.42. The map sending a relation R to the 2-relation $\mathbf{C}R$ indexed by

$$\begin{array}{ccc} R_0 & \overset{\mathsf{Eq}(R_0)}{\longleftrightarrow} & R_0 \\ {\scriptstyle \mathsf{Eq}(R_0)}\updownarrow & \mathbf{C}R & \updownarrow{\scriptstyle R} \\ R_0 & \underset{R}{\longleftrightarrow} & R_1 \end{array}$$

and defined by $(p, q, p', q') \in \mathbf{C}(R)(a, b, c, d)$ if and only if $\mathsf{tr}(q^{-1} \cdot p)p' =_{R(b,d)} q'$ extends to a functor $\mathbf{C} \colon \mathbf{PrRel} \to \mathbf{2Rel}$.

Proof. Consider a morphism $(f, g, t) \colon (A, B, R) \to (A', B', R')$. The functor $\mathbf{C}$ sends (f, g, t) to $(f, f, f, g, \mathsf{ap}(f), \mathsf{ap}(f), t, t)$. The proof that if $(p, q, p', q') \in \mathbf{C}(R)(a, b, c, d)$ then $(\mathsf{ap}(f)\, p, \mathsf{ap}(f)\, q, t\, p', t\, q') \in \mathbf{C}(R')(f\, a, f\, b, f\, c, g\, d)$ uses the same argument used in the proof of Lemma 8.40. $\square$

There is of course also a symmetric version which swaps the role of $\mathsf{Eq}(R_0)$ and R, but we will not make us of this in the current thesis.

Note that all the compositions $\mathsf{Eq}_{||} \circ \mathsf{Eq}$, $\mathsf{Eq}_{=} \circ \mathsf{Eq}$ and $\mathbf{C} \circ \mathsf{Eq}$ give the same functor.

Lemma 8.43. We have that $\mathsf{Eq}_{||} \circ \mathsf{Eq} = \mathsf{Eq}_{=} \circ \mathsf{Eq} = \mathbf{C} \circ \mathsf{Eq}$.

Proof. All the three functors goes from 1-Type to **2Rel** and if we apply them to a 1-type A all the three resulting 2-relations are indexed by the same diagram

$$\begin{array}{ccc} A & \overset{\mathsf{Eq}(A)}{\longleftrightarrow} & A \\ {\scriptstyle \mathsf{Eq}(A)}\updownarrow & & \updownarrow{\scriptstyle \mathsf{Eq}(A)} \\ A & \underset{\mathsf{Eq}(A)}{\longleftrightarrow} & A. \end{array}$$

Given the elements $p \in \mathsf{Eq}(A)(a, b)$, $q \in \mathsf{Eq}(A)(a, c)$, $p' \in \mathsf{Eq}(A)(c, d)$ and $q' \in \mathsf{Eq}(A)(b, d)$, we have that $(p, q, p', q') \in \mathsf{Eq}_{||} \circ \mathsf{Eq}(A)(a, b, c, d)$ if and only if $\mathsf{tr}(p, p')q =_{\mathsf{Eq}(A)(b,d)} q'$, $(p, q, p', q') \in \mathsf{Eq}_{=} \circ \mathsf{Eq}(A)(a, b, c, d)$ if and only if $\mathsf{tr}(q, q')p =_{\mathsf{Eq}(A)(c,d)} p'$ and $(p, q, p', q') \in \mathbf{C} \circ \mathsf{Eq}(A)(a, b, c, d)$ or if and only if $\mathsf{tr}(q^{-1} \cdot p)p' =_{\mathsf{Eq}(A)(b,d)} q'$. These three conditions are equivalent. In fact all the elements p, q, p', q' are identity proofs between terms of the same type, and, in this case, transport is given by composition of paths as shown by Lemma

8.23. Unwinding the first condition $\mathsf{tr}(p,p')q =_{\mathsf{Eq}(A)(b,d)} q'$ we obtain $p^{-1} \cdot q \cdot p' =_{\mathsf{Eq}(A)(b,d)} q'$ which corresponds also to the condition $\mathsf{tr}(q^{-1} \cdot p)p' =_{\mathsf{Eq}(A)(b,d)} q'$. By precomposing with inverses we obtain $q^{-1} \cdot p \cdot q' =_{\mathsf{Eq}(A)(c,d)} p'$ which is exactly $\mathsf{tr}(q,q')p =_{\mathsf{Eq}(A)(c,d)} p'$.

For the action on the morphisms note that all the three compositions send a morphism $f\colon A \to A'$ to the morphism $(f,f,f,f,\mathsf{ap}(f),\mathsf{ap}(f),\mathsf{ap}(f),\mathsf{ap}(f))$. □

Definition 8.44. We call Eq_2 the functor resulting from the composite $\mathsf{Eq}_{||} \circ \mathsf{Eq}$, which, by Lemma 8.43, is the same as the composites $\mathsf{Eq}_= \circ \mathsf{Eq}$ and $\mathbf{C} \circ \mathsf{Eq}$.

Proposition 8.45. The functor $\mathsf{Eq}_{||}$ is full and faithful.

Proof. Let $(f_0,f_1,f),(g_0,g_1,g)\colon (A,B,R) \to (A',B',R')$ be two morphisms in **PrRel** such that $\mathsf{Eq}_{||}(f_0,f_1,f) = \mathsf{Eq}_{||}(g_0,g_1,g)$, which, by unwinding the definition of $\mathsf{Eq}_{||}$, means that $(f_0,f_0,f_1,f_1,\mathsf{ap}(f_0),f,\mathsf{ap}(f_1),f) = (g_0,g_0,g_1,g_1,\mathsf{ap}(g_0),g,\mathsf{ap}(g_1),g)$. It immediately follows that $(f_0,f_1,f) = (g_0,g_1,g)$ and we have the faithfulness.

For fullness, consider a morphism $f\colon \mathsf{Eq}_{||}(R) \to \mathsf{Eq}_{||}(R')$. Using the fullness of Eq, we have that the morphism $(f_{00},f_{10},f_{r0})\colon \mathsf{Eq}(R_0) \to \mathsf{Eq}(R_0')$ has components $f_{00} = f_{10}$ and $f_{r0} = \mathsf{ap}(f_{00})$. Similarly the morphism $(f_{01},f_{11},f_{r1})\colon \mathsf{Eq}(R_1) \to \mathsf{Eq}(R_1')$ has components $f_{01} = f_{11}$ and $f_{r1} = \mathsf{ap}(f_{01})$. Finally, in order to prove that $f_{0r} = f_{1r}$, consider the elements $(a,b) \in R_0 \times R_1$ and $r \in R(a,b)$. Clearly $(\mathsf{refl}_a, r, \mathsf{refl}_b, r) \in \mathsf{Eq}_{||}(R)$ and then its image $(\mathsf{ap}(f_{00})\mathsf{refl}_a, (f_{0r})_{a,b}r, \mathsf{ap}(f_{01})\mathsf{refl}_b, (f_{1r})_{a,b}r) = (\mathsf{refl}_{f_{00}a}, (f_{0r})_{a,b}r, \mathsf{refl}_{f_{01}b}, (f_{1r})_{a,b}r) \in \mathsf{Eq}_{||}(R')$ which, by definition of $\mathsf{Eq}_{||}$, means that $\mathsf{tr}(\mathsf{refl}_{f_{00}a}, \mathsf{refl}_{f_{01}b})(f_{0r})_{a,b}r = (f_{1r})_{a,b}r$, i.e. $f_{0r} = f_{1r}$. We then have $f = \mathsf{Eq}_{||}(f_{00},f_{01},f_{0r})$. □

Again, we can prove that exponentiation preserves all the degeneracies and the connection:

Proposition 8.46. For all R, R' : **PrRel**, we have

1. an equivalence $\mathsf{Eq}_{||}R \Rightarrow \mathsf{Eq}_{||}R' \cong \mathsf{Eq}_{||}(R \Rightarrow R')$
2. an equivalence $\mathsf{Eq}_{=}R \Rightarrow \mathsf{Eq}_{=}R' \cong \mathsf{Eq}_{=}(R \Rightarrow R')$
3. an equivalence $\mathbf{C}R \Rightarrow \mathbf{C}R' \cong \mathbf{C}(R \Rightarrow R')$.

Proof. For the point (1) note that $\mathsf{Eq}_{||}R \Rightarrow \mathsf{Eq}_{||}R'$ is indexed over

$$\begin{array}{ccc}
(R_0 \to R_0') & \xleftrightarrow{\mathsf{Eq}R_0 \Rightarrow \mathsf{Eq}R_0'} & (R_0 \to R_0') \\
{\scriptstyle R \Rightarrow R'}\updownarrow & \mathsf{Eq}_{||}R \Rightarrow \mathsf{Eq}_{||}R' & \updownarrow{\scriptstyle R \Rightarrow R'} \\
(R_1 \to R_1') & \xleftrightarrow[\mathsf{Eq}R_1 \Rightarrow \mathsf{Eq}R_1']{} & (R_1 \to R_1')
\end{array}$$

while $\mathsf{Eq}_{||}(R \Rightarrow R')$ is indexed over

$$\begin{array}{ccc}
(R_0 \to R_0') & \xleftrightarrow{\mathsf{Eq}(R_0 \to R_0')} & (R_0 \to R_0') \\
{\scriptstyle R \Rightarrow R'}\updownarrow & \mathsf{Eq}_{||}(R \Rightarrow R') & \updownarrow{\scriptstyle R \Rightarrow R'} \\
(R_1 \to R_1') & \xleftrightarrow[\mathsf{Eq}(R_1 \to R_1')]{} & (R_1 \to R_1')
\end{array}$$

and the two are equivalent up to the equivalences $\phi\colon \mathsf{Eq}R_0 \Rightarrow \mathsf{Eq}R_0' \cong \mathsf{Eq}(R_0 \to R_0')$ and $\phi'\colon \mathsf{Eq}R_1 \Rightarrow \mathsf{Eq}R_1' \cong \mathsf{Eq}(R_1 \to R_1')$.

The proof consists in first proving that

$$\text{if } (p, q, p', q') \in (\mathsf{Eq}_{||}R \Rightarrow \mathsf{Eq}_{||}R')(f, g, f', g') \\ \text{then } (\phi p, q, \phi' p', q') \in \mathsf{Eq}_{||}(R \Rightarrow R')(f, g, f', g'),$$

and it follows by unwinding the definition of the 2-relations and the definition of the equivalence ϕ.

Next we need to prove the opposite direction of the implication:

$$\text{if } (t, q, t', q') \in \mathsf{Eq}_{||}(R \Rightarrow R')(f, g, f', g') \\ \text{then } (\phi^{-1} t, q, \phi'^{-1} t', q') \in (\mathsf{Eq}_{||}R \Rightarrow \mathsf{Eq}_{||}R')(f, g, f', g'),$$

which, in this case, follows by unwinding the definitions and by path induction. Since the 2-relations have values in Prop, the maps back and forth are enough to define an equivalence.

The proofs of item (2) and item (3) follow the same structure of the proof of item (1). The difference is that the permutation in the shape over the 2-relations are indexed requires to adjust the argument. □

Chapter 9

Two-dimensional parametricity

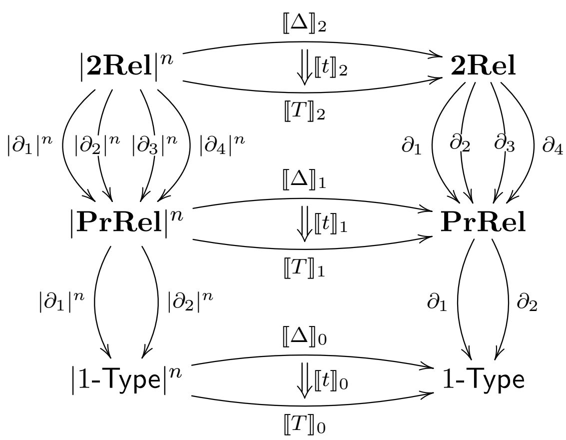
We now have the structure needed to define a two-dimensional, proof-relevant model of System F. Each type judgment $\Gamma \vdash T$ type, with $|\Gamma| = n$, will be interpreted in the semantics as

$$\llbracket T \rrbracket_0 : |\mathsf{1\text{-}Type}|^n \to \mathsf{1\text{-}Type}$$

$$\llbracket T \rrbracket_1 : |\mathbf{PrRel}|^n \to \mathbf{PrRel}$$

$$\llbracket T \rrbracket_2 : |\mathbf{2Rel}|^n \to \mathbf{2Rel}$$

by induction on type judgments with $\llbracket T \rrbracket_1$ over $\llbracket T \rrbracket_0 \times \llbracket T \rrbracket_0$, and $\llbracket T \rrbracket_2$ over $\llbracket T \rrbracket_1 \times \llbracket T \rrbracket_1 \times \llbracket T \rrbracket_1 \times \llbracket T \rrbracket_1$. This is similar to the previous work on bifibrational functorial models of (proof-irrelevant) parametricity presented in Chapter 5, but with an additional 2-relational level. To give an idea, the picture of bifibrational parametricity presented in Theorem 5.6 generalises in the two-dimensional setting to the following



$$[\![X_0, \ldots, X_n \vdash X_k \ \mathsf{type}]\!]_i \bar{Y} = Y_k \qquad [\![S \to T]\!]_i \bar{Y} = [\![S]\!]_i \bar{Y} \Rightarrow_i [\![T]\!]_i \bar{Y}$$

$$\begin{aligned}
[\![\forall X.T]\!]_0 \bar{A} = \{\, & f_0 : (\Pi A : \mathsf{1\text{-}Type})[\![T]\!]_0(\bar{A}, A), \\
& f_1 : (\Pi R : \mathbf{PrRel})[\![T]\!]_1(\mathsf{Eq}(\bar{A}), R)(f_0 R_0, f_0 R_1) \mid \\
& (\forall Q\colon \mathbf{2Rel}) \ \ (f_1 Q_{r0}, f_1 Q_{0r}, f_1 Q_{r1}, f_1 Q_{1r}) \in \\
& \quad [\![T]\!]_2(\mathsf{Eq}_2(\bar{A}), Q)(f_0 Q_{00}, f_0 Q_{10}, f_0 Q_{01}, f_0 Q_{11}) \qquad \text{(A0.1)} \\
& \wedge (\forall A\colon \mathsf{1\text{-}Type}) \ \ \Theta_{T,0}(f_1 \mathsf{Eq}(A)) = \mathsf{refl}_{f_0\, A} \}. \qquad \text{(A0.2)}
\end{aligned}$$

$$\begin{aligned}
([\![\forall X.T]\!]_1 \bar{R})((f_0, f_1), (g_0, g_1)) = \{\, & \phi : (\Pi R : \mathbf{PrRel})[\![T]\!]_1(\bar{R}, R)(f_0 R_0, g_0 R_1) \ \mid \\
& (\forall Q\colon \mathbf{2Rel}) \\
& \big((f_1 Q_{r0}, \phi Q_{0r}, g_1 Q_{r1}, \phi Q_{1r}) \in \\
& \quad [\![T]\!]_2(\mathsf{Eq}_{||}(\bar{R}), Q)(f_0 Q_{00}, f_0 Q_{10}, g_0 Q_{01}, g_0 Q_{11}) \qquad \text{(A1.1)} \\
& \wedge (\phi Q_{r0}, f_1 Q_{0r}, \phi Q_{r1}, g_1 Q_{1r}) \in \\
& \quad [\![T]\!]_2(\mathsf{Eq}_{=}(\bar{R}), Q)(f_0 Q_{00}, f_0 Q_{10}, g_0 Q_{01}, g_0 Q_{11}) \qquad \text{(A1.2)} \\
& \wedge (f_1 Q_{r0}, f_1 Q_{0r}, \phi Q_{r1}, \phi Q_{1r}) \in \\
& \quad [\![T]\!]_2(\mathbf{C}\bar{R}, Q)(f_0 Q_{00}, f_0 Q_{10}, f_0 Q_{01}, g_0 Q_{11})\big)\} \qquad \text{(A1.3)}
\end{aligned}$$

$$\begin{aligned}
(\phi_0, \phi_1, \phi_2, \phi_3) \in ([\![\forall X.T]\!]_2 \bar{Q})(f, g, h, l) \ \text{iff} \\
(\forall Q\colon \mathbf{2Rel}) \ \ (\phi_0 Q_{r0}, \phi_1 Q_{0r}, \phi_2 Q_{r1}, \phi_3 Q_{1r}) \in \\
[\![T]\!]_2(\bar{Q}, Q)(f_0 Q_{00}, g_0 Q_{10}, h_0 Q_{01}, l_0 Q_{11})
\end{aligned}$$

Figure 9.1: Interpretation of types

where the ∂'s from **2Rel** to **PrRel** are the maps projecting out the proof-relevant components of the 2-relations, while the ones from **PrRel** to 1-Type project out the 1-types of the proof-relevant relations. Finally in this chapter we consider System F without product types which, in any case, can be added since there are products at every level and products at one level live over products at the level below.

9.1 Interpretation of System F types in two-dimensional parametricity

The full interpretation of types can be found in Fig. 9.1. For type variables and arrow types, we just use projections and exponentials at each level. Elements of $[\![\forall X.T]\!]_0\bar{A}$ consist of an ad-hoc polymorphic function f_0, a proof f_1 that f_0 is suitably uniform, and finally (unique) proofs (A0.1) and (A0.2) that also the proof f_1 is uniform. The $\Theta_{T,0}$ appearing in condition (A0.2) derives from the Identity Extension Lemma 9.2, which means that this lemma needs to be proven simultaneously with the definition of the interpretation. In other words, this is an inductive-recursive definition (see [DS99]). Condition (A0.2) says that, assuming the Identity Extension Lemma i.e. $\Theta_{T,0}\colon [\![T]\!]_1 \circ \mathsf{Eq} \cong \mathsf{Eq} \circ [\![T]\!]_0$, the only way to define $f_1\mathsf{Eq}(A)$ without looking at the type — i.e. in a uniform way — is to pick refl. We do not know if condition (A0.2) follows from the others or not.

Elements of $([\![\forall X.T]\!]_1\bar{R})(f,g)$ are proofs ϕ that are suitably uniform in relation to f and g, both with respect to equalities (conditions A1.1 and A1.2) and connections (condition A1.3). The presence of connections might surprise. The idea behind the use of connections is the following. Let T_2 be a polymorphic 2-relations indexed by

$$\begin{array}{ccc} T_0 & \overset{T_1}{\longleftrightarrow} & T_0 \\ {\scriptstyle T_1}\updownarrow & T_2 & \updownarrow{\scriptstyle T_1} \\ T_0 & \underset{T_1}{\longleftrightarrow} & T_0, \end{array}$$

and consider $(p,q,p',q') \in T_2(f,g,l,h)$. For T_2 being polymorphic means that for every 2-relation Q we have the 2-relation $T_2(Q)$ indexed by

$$\begin{array}{ccc} T_0\,Q_{00} & \overset{T_1\,Q_{r0}}{\longleftrightarrow} & T_0\,Q_{10} \\ {\scriptstyle T_1\,Q_{0r}}\updownarrow & T_2(Q) & \updownarrow{\scriptstyle T_1\,Q_{1r}} \\ T_0\,Q_{01} & \underset{T_1\,Q_{r1}}{\longleftrightarrow} & T_0\,Q_{11}, \end{array}$$

and the following elements are related in $T_2(Q)$

$$\begin{array}{ccc} f\,Q_{00} & \xleftrightarrow{p\,Q_{r0}} & g\,Q_{10} \\ {\scriptstyle q\,Q_{0r}}\updownarrow & T_2(Q) & \updownarrow{\scriptstyle q'\,Q_{1r}} \\ h\,Q_{01} & \xleftrightarrow[p'\,Q_{r1}]{} & l\,Q_{11}. \end{array}$$

We want to prove that p, q, p' and q' are the same (up to transport) and we try to apply the same technique used in relational parametricity: we use the Identity Extension Lemma. We expect that the two-dimensional generalisation of the Identity Extension Lemma is given by $T_2(\mathsf{Eq}_{||}) \cong \mathsf{Eq}_{||}(T_1)$ and $T_2(\mathsf{Eq}_{=}) \cong \mathsf{Eq}_{=}(T_1)$, from which we can derive that for every relation R we have the following related elements:

$$\begin{array}{ccc} f\,R_0 & \xleftrightarrow{p\,\mathsf{Eq}(R_0)} & g\,R_0 \\ {\scriptstyle q\,R}\updownarrow & \mathsf{Eq}_{||}(T_1 R) & \updownarrow{\scriptstyle q'\,R} \\ h\,R_1 & \xleftrightarrow[p'\,\mathsf{Eq}(R_1)]{} & l\,R_1 \end{array} \qquad \begin{array}{ccc} f\,R_0 & \xleftrightarrow{p\,R} & g\,R_1 \\ {\scriptstyle q\,\mathsf{Eq}(R_0)}\updownarrow & \mathsf{Eq}_{=}(T_1 R) & \updownarrow{\scriptstyle q'\,\mathsf{Eq}(R_1)} \\ h\,R_0 & \xleftrightarrow[p'\,R]{} & l\,R_1. \end{array}$$

The left square proves that $\mathsf{tr}(\lambda X.p\,\mathsf{Eq}(X), \lambda Y.p'\,\mathsf{Eq}(Y))q = q'$, while the right one that $\mathsf{tr}(\lambda x.q\,\mathsf{Eq}(X), \lambda Y.q'\,\mathsf{Eq}(Y))p = p'$, but we cannot find any relation between p and q or p' and q' and so on. For this reason we use the connection $\mathbf{C}$ and we extend the Identity Extension Lemma to connections as well: $T_2(\mathbf{C}) \cong \mathbf{C}(T_1)$. In this way we obtain

$$\begin{array}{ccc} f\,R_0 & \xleftrightarrow{p\,\mathsf{Eq}(R_0)} & g\,R_0 \\ {\scriptstyle q\,\mathsf{Eq}(R_0)}\updownarrow & \mathbf{C}(T_1 R) & \updownarrow{\scriptstyle q'\,R} \\ h\,R_0 & \xleftrightarrow[p'\,R]{} & l\,R_1 \end{array}$$

and we can relate all the four edges together. Note that the two kinds of equalities and the one kind of connection are enough to link together all the four edges, and since we want to keep the logical relation minimal, we do not include the "symmetric" connection.

We want to prove the two-dimensional version of the Identity Extension Lemma in the setting of two-dimensional parametricity for $\mathsf{Eq}_{||}$, $\mathsf{Eq}_{=}$ and $\mathbf{C}$. In order to do so we first characterise equality in the interpretation of $\forall$-types in the following way (note that

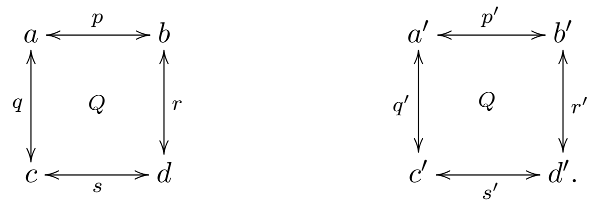
$\mathsf{Id}_{(\llbracket \forall X.T \rrbracket_2 \bar{Q})\bar{f}}(\bar{\phi}, \bar{\psi})$ is trivial by assumption, since $(\llbracket \forall X.T \rrbracket_2 \bar{Q})\bar{f}$ is a proposition):

Lemma 9.1. For all $f, g : \llbracket \forall X.T \rrbracket_0 \bar{A}$, we have the equivalence

$$\begin{aligned} \varphi\colon \mathsf{Id}_{\llbracket \forall X.T \rrbracket_0 \bar{A}}(f,g) \cong \{\, \tau : (\Pi A : 1\text{-}\mathsf{Type})\mathsf{Id}_{\llbracket T \rrbracket_0(\bar{A},A)}(f_0 A, g_0 A) \mid \\ (\forall R : \mathbf{PrRel})\ \ (f_1 R, \tau R_0, g_1 R, \tau R_1) \in \\ \mathsf{Eq}_{=}(\llbracket T \rrbracket_1(\mathsf{Eq}(\bar{A}), R))(f_0 R_0, f_0 R_1, g_0 R_0, g_0 R_1) \} \end{aligned}$$

Proof. It follows by unwinding the definition of identity types and using Lemma 8.19 and function extensionality. □

From now on when we write $\tau\colon \mathsf{Id}_{\llbracket \forall X.T \rrbracket_0 \bar{A}}(f,g)$, we actually think it living in the image of φ. One last step before proving the two-dimensional Identity Extension Lemma: we introduce a notation which will make it easier to read the proofs in the rest of this chapter. Let Q be a 2-relation and consider the following elements which are related in Q:



The type of the 8-tuples (a,b,c,d,p,q,r,s) and $(a',b',c',d',p',q',r',s')$ is the Σ-type

$$(\Sigma(a,b,c,d)\colon Q_{00} \times Q_{10} \times Q_{01} \times Q_{11})(p,q,r,s)\colon Q_{r0}(a,b) \times Q_{0r}(a,c) \times Q_{r1}(c,d) \times Q_{r1}(b,d),$$

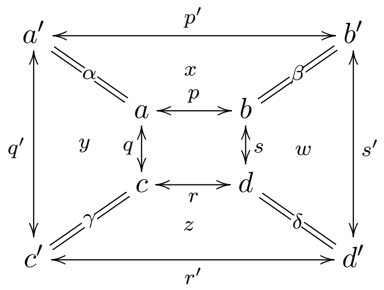
and for this reason if we want to prove that $(a,b,c,d,p,q,r,s) = (a',b',c',d',p',q',r',s')$ we first need four proofs

$$\alpha\colon a = a' \qquad \beta\colon b = b' \qquad \gamma\colon c = c' \qquad \delta\colon d = d',$$

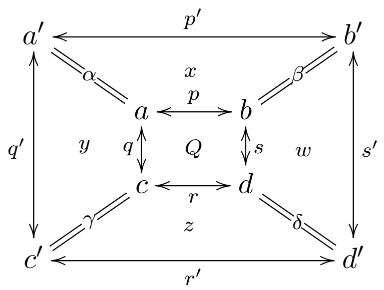
and then transport (p,q,r,s) along $(\alpha,\beta,\gamma,\delta)$ and prove that the result is equal to (p',q',r',s'), i.e. $\mathsf{tr}(\alpha,\beta,\gamma,\delta)(p,q,r,s) = (p',q',r',s')$. In order to do that we need four more proofs

$$x\colon \mathsf{tr}(\alpha,\beta)\,p = p' \qquad y\colon \mathsf{tr}(\alpha,\gamma)\,q = q' \qquad z\colon \mathsf{tr}(\gamma,\delta)\,r = r' \qquad w\colon \mathsf{tr}(\beta,\delta)\,s = s'.$$

An efficient way to represent all these data is by using the following picture



We will often use the following argument: if $(p,q,r,s) \in Q(a,b,c,d)$ and $(a,b,c,d,p,q,r,s) = (a',b',c',d',p',q',r',s')$, then $(p',q',r',s') \in Q(a',b',c',d')$. We represent this argument in a similar picture as before, but we fill the middle of the inner square:



We can now prove the two-dimensional Identity Extension Lemma (note that since in this chapter we only use the two-dimensional version of the Identity Extension Lemma, we drop the adjective "two-dimensional")

Theorem 9.2 (Identity Extension Lemma)**.** For every type judgment $\Gamma \vdash T$ type, we have

1. an equivalence $\Theta_{T,0} : [\![T]\!]_1 \circ \mathsf{Eq} \cong \mathsf{Eq} \circ [\![T]\!]_0$,

2. an equivalence $\Theta_{T,\|} : [\![T]\!]_2 \circ \mathsf{Eq}_\| \cong \mathsf{Eq}_\| \circ [\![T]\!]_1$ over $\Theta_{T,0}$,

3. an equivalence $\Theta_{T,=} : [\![T]\!]_2 \circ \mathsf{Eq}_= \cong \mathsf{Eq}_= \circ [\![T]\!]_1$ over $\Theta_{T,0}$, and

4. an equivalence $\Theta_{T,\mathbf{C}} : [\![T]\!]_2 \circ \mathbf{C} \cong \mathbf{C} \circ [\![T]\!]_1$ over $\Theta_{T,0}$.

Proof. The proof is done by induction on type judgments. For type variables, all statements are trivial. For arrow types, this is Propositions 8.29 and 8.46. It is left to prove it for forall types.

For the point 1 we define the maps

$$\Theta_{\forall X.T,0} : [\![\forall X.T]\!]_1 \mathsf{Eq}(\bar{A})(f,g) \to \mathsf{Eq}([\![\forall X.T]\!]_0 \bar{A})(f,g)$$
$$\Theta^{-1}_{\forall X.T,0} : \mathsf{Eq}([\![\forall X.T]\!]_0 \bar{A})(f,g) \to [\![\forall X.T]\!]_1 \mathsf{Eq}(\bar{A})(f,g)$$

for all f, g and show that they are inverses. Using the induction hypothesis, we first define

$$\Theta_{\forall X.T,0}(\phi) \coloneqq \lambda(A : 1\text{-}\mathsf{Type}).\, \Theta_{T,0}(\phi\, \mathsf{Eq}(A))$$

where $\Theta_{T,0}(\phi\, \mathsf{Eq}(A))$ goes from $[\![T]\!]_1(\mathsf{Eq}(\bar{A}, A))(f_0\, A, g_0\, A) \to \mathsf{Eq}([\![T]\!]_0(\bar{A}, A))(f_0\, A, g_0\, A)$. The condition from Lemma 9.1 is satisfied by (A1.1) together with the induction hypothesis. In fact the axiom (A1.1) instantiated with $\mathsf{Eq}_=(R)$ gives

$$\begin{array}{ccc} f_0\, R_0 & \xleftrightarrow{\;f_1\, R\;} & f_0\, R_1 \\ {\scriptstyle \phi\, \mathsf{Eq}(R_0)}\updownarrow & {\scriptstyle [\![T]\!]_2(\mathsf{Eq}_{||}(\mathsf{Eq}(\bar{A})), \mathsf{Eq}_=(R))} & \updownarrow{\scriptstyle \phi\, \mathsf{Eq}(R_1)} \\ g_0\, R_0 & \xleftrightarrow[\;g_1\, R\;]{} & g_0\, R_1. \end{array}$$

Using the induction hypothesis plus the fact that $\mathsf{Eq}_{||} \circ \mathsf{Eq} = \mathsf{Eq}_= \circ \mathsf{Eq}$ we have that

$$[\![T]\!]_2(\mathsf{Eq}_{||}(\mathsf{Eq}(\bar{A})), \mathsf{Eq}_=(R)) \cong \mathsf{Eq}_=([\![T]\!]_1(\mathsf{Eq}(\bar{A}), R))$$

and we obtain

$$\begin{array}{ccc} f_0\, R_0 & \xleftrightarrow{\;f_1\, R\;} & f_0\, R_1 \\ {\scriptstyle \Theta_{T,0}(\phi\, \mathsf{Eq}(R_0))}\updownarrow & {\scriptstyle \mathsf{Eq}_=([\![T]\!]_1(\mathsf{Eq}(\bar{A}), R))} & \updownarrow{\scriptstyle \Theta_{T,0}(\phi\, \mathsf{Eq}(R_1))} \\ g_0\, R_0 & \xleftrightarrow[\;g_1\, R\;]{} & g_0\, R_1 \end{array}$$

and hence $\lambda A.\Theta_{T,0}(\phi\, \mathsf{Eq}(A))$ is a proof that $f = g$ by Lemma 9.1.

We define the inverse morphism as $\Theta^{-1}_{\forall X.T,0}(\tau) \coloneqq \lambda R\colon \mathbf{PrRel}.\, \mathsf{tr}(\mathsf{refl}, \tau\, R_1) f_1 R$. We need to check that conditions (A1.1), (A1.2) and (A1.3) are satisfied. We verify (A1.1) in detail, (A1.2) and (A1.3) follow analogously and we only sketch their proof. If $Q : \mathbf{2Rel}$, using axiom (A0.1) we obtain

$$(f_1 Q_{r0}, f_1 Q_{0r}, f_1 Q_{r1}, f_1 Q_{1r}) \in [\![T]\!]_2(\mathsf{Eq}_2(\bar{A}), Q)(f_0 Q_{00}, f_0 Q_{10}, f_0 Q_{01}, f_0 Q_{11})$$

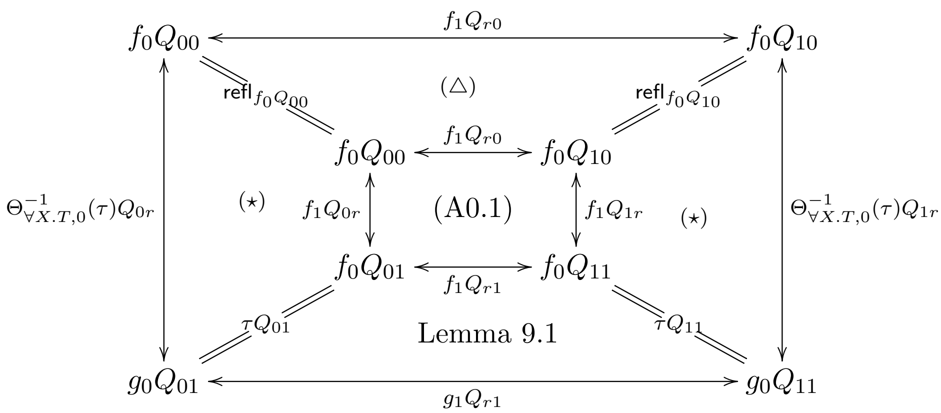
while we want to prove

$$\begin{aligned}(f_1 Q_{r0}, \Theta^{-1}_{\forall X.T,0}(\tau) Q_{0r}, g_1 Q_{r1}, \Theta^{-1}_{\forall X.T,0}(\tau) Q_{1r})\\ \in [\![T]\!]_2(\mathsf{Eq}_{||}(\mathsf{Eq}(\bar{A})), Q)(f_0 Q_{00}, f_0 Q_{10}, g_0 Q_{01}, g_0 Q_{11})\end{aligned}$$

and in order to do that, we prove that

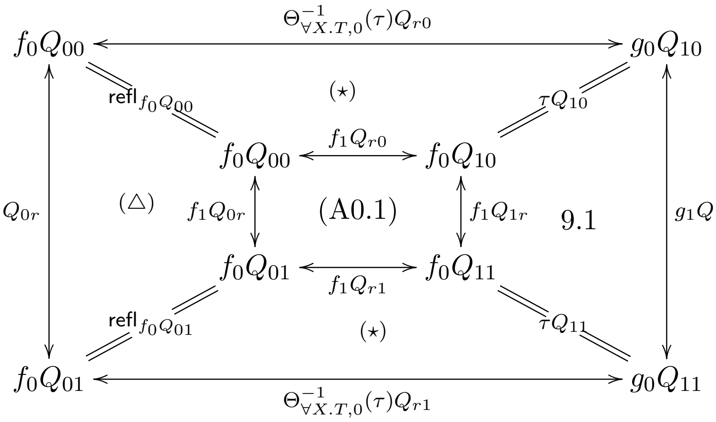
$$\begin{aligned}(f_0 Q_{00}, f_0 Q_{10}, f_0 Q_{01}, f_0 Q_{11}, f_1 Q_{r0}, f_1 Q_{0r}, f_1 Q_{r1}, f_1 Q_{1r}) \quad =\\ = (f_0 Q_{00}, f_0 Q_{10}, g_0 Q_{01}, g_0 Q_{11}, f_1 Q_{r0}, \Theta^{-1}_{\forall X.T,0}(\tau) Q_{0r}, g_1 Q_{r1}, \Theta^{-1}_{\forall X.T,0}(\tau) Q_{1r})\end{aligned}$$

by considering the following picture:

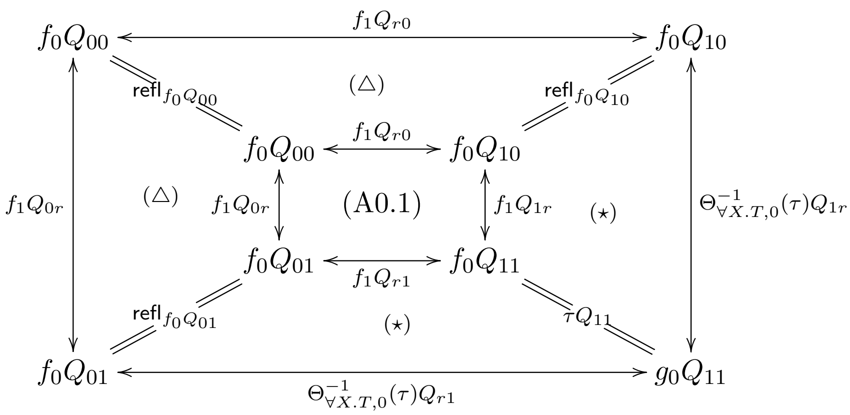


where the squares $(\star)$ follow from the definition of $\Theta^{-1}_{\forall X.T,0}(\tau)$, the top condition $(\triangle)$ trivially follows from transporting along refl, and the last square derives from Lemma 9.1.

Similarly condition (A1.2) follows from the picture



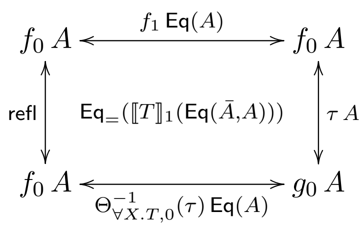
and condition (A1.3) from



We now check that $\Theta_{\forall X.T,0} \circ \Theta^{-1}_{\forall X.T,0} = \mathsf{id}$ and $\Theta^{-1}_{\forall X.T,0} \circ \Theta_{\forall X.T,0} = \mathsf{id}$. One way round

$$\Theta_{\forall X.T,0}(\Theta^{-1}_{\forall X.T,0}(\tau))(A) = \Theta_{T,0}(\mathsf{tr}(\mathsf{refl}, \tau A) f_1 \mathsf{Eq}(A)).$$

If we consider the related elements



and we apply the induction hypothesis $\mathsf{Eq}_=(\llbracket T\rrbracket_1(\mathsf{Eq}(\bar{A}, A))) \cong \mathsf{Eq}_2(\llbracket T\rrbracket_0(\bar{A}, A))$, we obtain

$$\Theta_{T,0}(\Theta^{-1}_{\forall X.T,0}(\tau)\,\mathsf{Eq}(A)) = \mathsf{tr}(\mathsf{refl}, \tau\, A)\Theta_{T,0}(f_1\,\mathsf{Eq}(A)).$$

In this way we obtain the thesis from the following equalities

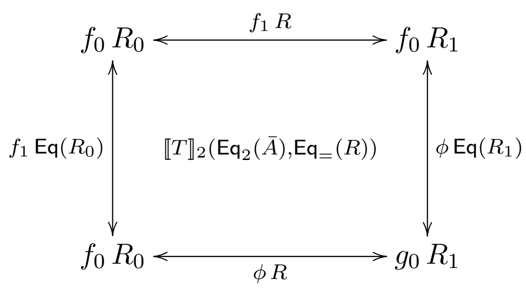
$$\begin{aligned}\Theta_{T,0}(\Theta^{-1}_{\forall X.T,0}(\tau)\,\mathsf{Eq}(A)) &= (\mathsf{tr}(\mathsf{refl}, \tau\, A)\Theta_{T,0}(f_1\,\mathsf{Eq}(A)))\\ &= \Theta_{T,0}(f_1\,\mathsf{Eq}(A)) \bullet \tau\, A\\ &= \tau\, A,\end{aligned}$$

where for the last equality we used condition (A0.2).

The other way round is

$$\Theta^{-1}_{\forall X.T,0}(\Theta_{\forall X.T,0}(\phi))(R) = \mathsf{tr}(\mathsf{refl}, \Theta_{T,0}(\phi\,\mathsf{Eq}(R_1)))f_1 R.$$

Instantiating condition A1.3 with $\mathsf{Eq}_=(R)$ we have the related elements



and using the induction hypothesis $\llbracket T\rrbracket_2(\mathsf{Eq}_2(\bar{A}), \mathsf{Eq}_=(R)) \cong \mathsf{Eq}_=(\llbracket T\rrbracket_1(\mathsf{Eq}(\bar{A}), R))$, we obtain

$$\mathsf{tr}(\Theta_{T,0}(f_1\,\mathsf{Eq}(R_0)), \Theta_{T,0}(\phi\,\mathsf{Eq}(R_1)))f_1\, R = \phi\, R,$$

which is the thesis.

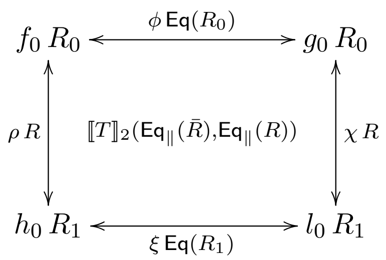
The proofs of item 2 and item 3 are similar. They require just to adjust the argument to the permutation of the orientation of the 2-relations $\mathsf{Eq}_{||}$ and $\mathsf{Eq}_=$. We see in detail the proof of item 2. We first show that if

$$(\phi, \rho, \xi, \chi) \in [\![\forall X.T]\!]_2 \mathsf{Eq}_{||}(\bar{R})(f, g, h, l) \qquad \text{then}$$
$$(\Theta_{\forall X.T,0}(\phi), \rho, \Theta_{\forall X.T,0}(\xi), \chi) \in \mathsf{Eq}_{||}([\![\forall X.T]\!]_1 \bar{R})(f, g, h, l),$$

which means to prove that $\mathsf{tr}(\Theta_{\forall X.T,0}(\phi)R_0, \Theta_{\forall X.T,0}(\xi)R_1)\rho\, R = \chi\, R$ for every relation R. By unfolding the definition of $\Theta_{\forall X.T,0}$, we obtain

$$\mathsf{tr}(\Theta_{\forall X.T,0}(\phi)R_0, \Theta_{\forall X.T,0}(\xi)R_1)\rho\, R = \mathsf{tr}(\Theta_{T,0}(\phi\, \mathsf{Eq}(R_0)), \Theta_{T,0}(\xi\, \mathsf{Eq}(R_1)))\rho\, R.$$

By definition, if we instantiate $[\![\forall X.T]\!]_2 \mathsf{Eq}_{||}(\bar{R})$ with $\mathsf{Eq}_{||}(R)$ we obtain the following related elements



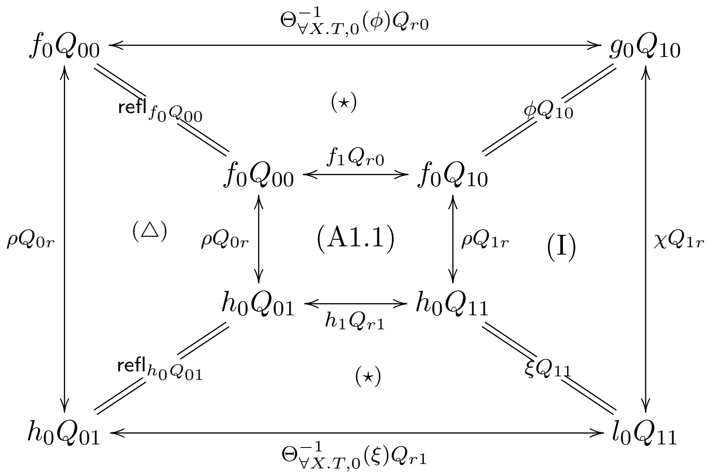
and using the induction hypothesis $[\![T]\!]_2(\mathsf{Eq}_{||}(\bar{R}, R)) \cong \mathsf{Eq}_{||}([\![T]\!]_1(\bar{R}, R))$, we obtain the thesis: $\mathsf{tr}(\Theta_{T,0}(\phi\, \mathsf{Eq}(R_0)), \Theta_{T,0}(\xi\, \mathsf{Eq}(R_1)))\rho\, R = \chi\, R$.

In the opposite direction we want to prove that if

$$(\phi, \rho, \xi, \chi) \in \mathsf{Eq}_{||}([\![\forall X.T]\!]_1 \bar{R})(f, g, h, l) \qquad \text{then}$$
$$(\Theta^{-1}_{\forall X.T,0}(\phi), \rho, \Theta^{-1}_{\forall X.T,0}(\xi), \chi) \in [\![\forall X.T]\!]_2(\mathsf{Eq}_{||} \bar{R})(f, g, h, l),$$

which, by definition of $[\![\forall X.T]\!]_2(\mathsf{Eq}_{||}(\bar{R}))$, means to prove that for every 2-relation Q we have $(\Theta^{-1}_{\forall X.T,0}(\phi)Q_{r0}, \rho\, Q_{0r}, \Theta^{-1}_{\forall X.T,0}(\xi)Q_{r1}, \chi\, Q_{1r}) \in [\![T]\!]_2(\mathsf{Eq}_{||}(\bar{R}), Q)(f\, Q_{00}, g\, Q_{10}, h\, Q_{01}, l\, Q_{11})$.

The thesis follows from the following picture:



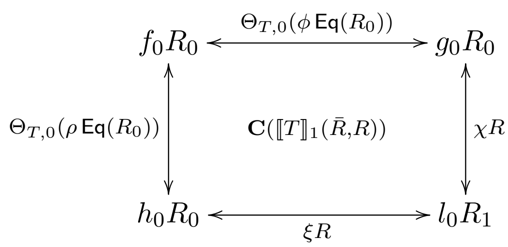
where the middle square A1.1 is the axiom applied to $\rho \in [\![\forall X.T]\!]_1(\bar{R})(f,h)$, the top and bottom squares $(\star)$ are the definition of $\Theta^{-1}_{\forall X.T,0}$, $(\triangle)$ is given by transport along refl and (I) is the hypothesis $(\phi,\rho,\xi,\chi) \in \mathsf{Eq}_{||}([\![\forall X.T]\!]_1\bar{R})(f,g,h,l)$ which, applied to Q_{1r}, implies that $\mathsf{tr}(\phi\, Q_{10}, \xi\, Q_{11})\rho\, Q_{1r} = \chi\, Q_{1r}$. As we said the proof of item 3 is similar.

What is left to prove is item 4. First we show that if

$$(\phi,\rho,\xi,\chi) \in [\![\forall X.T]\!]_2\mathbf{C}(\bar{R})(f,g,h,l) \qquad \text{then}$$

$$(\Theta_{\forall X.T,0}(\phi), \Theta_{\forall X.T,0}(\rho), \xi, \chi) \in \mathbf{C}([\![\forall X.T]\!]_1\bar{R})(f,g,h,l),$$

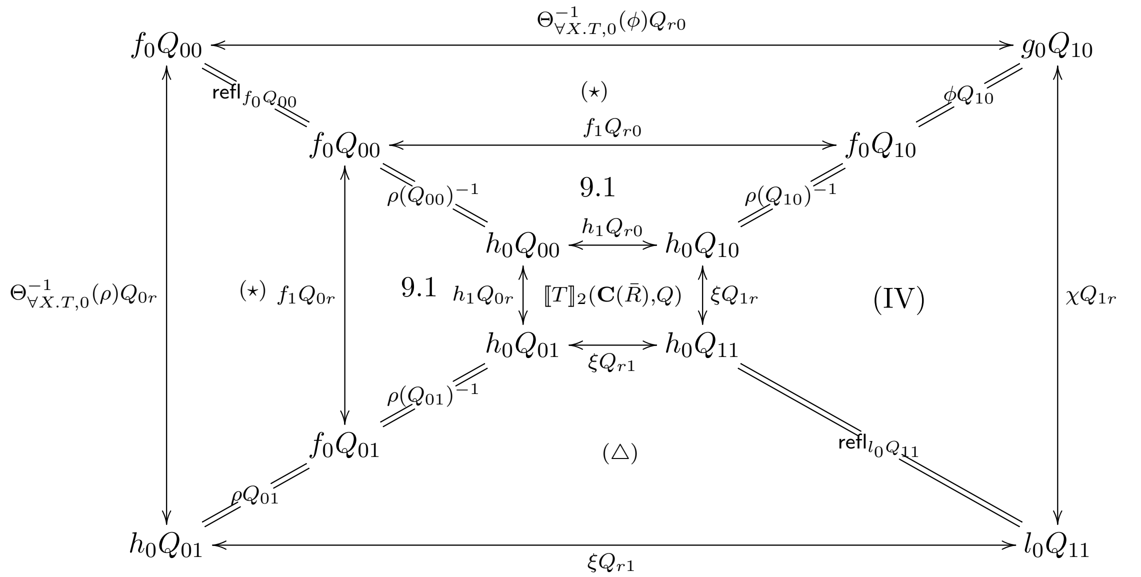
which, by unfolding the definition of $\Theta_{\forall X.T,0}$, means to prove that for every relation R we have that $(\Theta_{T,0}(\phi\,\mathsf{Eq}(R_0)), \Theta_{T,0}(\rho\,\mathsf{Eq}(R_0)), \xi\, R, \chi\, R) \in \mathbf{C}([\![\forall X.T]\!]_1\bar{R})(f,g,h,l)$. We obtain the thesis by instantiating $[\![\forall X.T]\!]_2\mathbf{C}(\bar{R})$ with $\mathbf{C}(R)$ and using the induction hypothesis $[\![T]\!]_2\mathbf{C}(\bar{R},R) \cong \mathbf{C}([\![T]\!]_1(\bar{R},R))$:



Finally we want to prove the other direction: if

$$(\phi, \rho, \xi, \chi) \in \mathbf{C}([\![\forall X.T]\!]_1 \bar{R})(f, g, h, l) \qquad \text{then}$$
$$(\Theta^{-1}_{\forall X.T}(\phi), \Theta^{-1}_{\forall X.T}(\rho), \xi, \chi) \in [\![\forall X.T]\!]_2(\mathbf{C}\bar{R})(f, g, h, l),$$

which means that we need to prove that $(\Theta^{-1}_{\forall X.T}(\phi)\, Q_{r0}, \Theta^{-1}_{\forall X.T}(\rho)\, Q_{0r}, \xi\, Q_{r1}, \chi\, Q_{1r}) \in [\![T]\!]_2(\mathbf{C}\bar{R}, Q)(f\, Q_{00}, g\, Q_{10}, h\, Q_{01}, l\, Q_{11})$ for every 2-relation Q. In order to do that we consider the following picture



where it should be at this point clear where every square comes from but (IV), which comes from the hypothesis $(\phi, \rho, \xi, \chi) \in \mathbf{C}([\![\forall X.T]\!]_1 \bar{R})(f, g, h, l)$ applied to the relation Q_{1r}. $\square$

9.2 Interpretation of terms

We next show how to interpret terms. A term $\Gamma; \Delta \vdash t : T$, with $|\Gamma| = n$, will give a "standard" interpretation

$$[\![t]\!]_0 \bar{A} \colon [\![\Delta]\!]_0 \bar{A} \to [\![T]\!]_0 \bar{A},$$

for every $\bar{A} : 1\text{-}\mathsf{Type}^n$, a relational interpretation

$$([\![t]\!]_0 \bar{R}_0, [\![t]\!]_0 \bar{R}_1, [\![t]\!]_1 \bar{R}) \colon [\![\Delta]\!]_1 \bar{R} \to [\![T]\!]_1 \bar{R},$$

for every $\bar{R} : \mathbf{PrRel}^n$, and finally a 2-relational interpretation

$$((\llbracket t \rrbracket_0 \bar{Q}_-, \llbracket t \rrbracket_1 \bar{Q}_-), \llbracket t \rrbracket_2 \bar{Q})\colon \llbracket \Delta \rrbracket_2 \bar{Q} \to \llbracket T \rrbracket_2 \bar{Q}$$

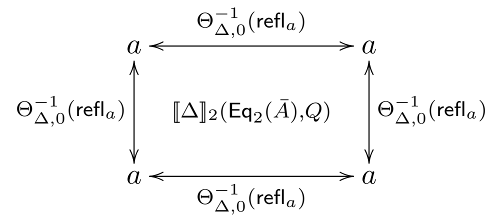
for every $\bar{Q} : \mathbf{2Rel}^n$, where we have written e.g. $\llbracket t \rrbracket_0 \bar{Q}_-$ for the map of I_0-types with components $(\llbracket t \rrbracket_0 \bar{Q}_-)w = \llbracket t \rrbracket_0 \bar{Q}_w : \llbracket \Delta \rrbracket_0 \bar{Q}_w \to \llbracket T \rrbracket_0 \bar{Q}_w$ and similarly for $\llbracket t \rrbracket_1 \bar{Q}_-$. At each level, $\Delta = x_1 : T_1, \ldots, x_m : T_m$ is interpreted as the product

$$\llbracket x_1 : T_1, \ldots, x_m : T_m \rrbracket_i = \llbracket T_1 \rrbracket_i \times \ldots \times \llbracket T_m \rrbracket_i \ .$$

The full interpretation is given in Fig. 9.2. Variables, term abstraction and term application are again given by projections and the exponential structure at each level. For type abstraction and type application, we use the same concepts at the meta-level, but in the case of type abstraction we also have to prove that the resulting term satisfies the uniformity conditions (A0.1), (A1.1), (A1.2) and (A1.3).

Lemma 9.3. The interpretation in Fig. 9.2 is well-defined.

Proof. The interpretation of $\Gamma; \Delta \vdash \Lambda X.t : \forall X.T$ is type-correct, since Δ is weakened with respect to X in $\Gamma, X; \Delta \vdash t : T$ which means that $\llbracket \Gamma, X \vdash \Delta \rrbracket = \llbracket \Gamma \vdash \Delta \rrbracket \circ \pi_1$. For this reason we write just $\llbracket \Delta \rrbracket$, and it will be clear from the arity of the input to which one we refer. The uniformity conditions (A0.1), (A1.1), (A1.2) and (A1.3) can all be proven using $\llbracket t \rrbracket_2$. In fact, in order to prove (A0.1), we apply $\llbracket t \rrbracket_2(\mathsf{Eq}_2(\bar{A}), Q)\colon \llbracket \Delta \rrbracket_2(\mathsf{Eq}_2(\bar{A}), Q) \to \llbracket T \rrbracket_2(\mathsf{Eq}_2(\bar{A}), Q)$ to the following related elements



which are related because they are obtained by applying the string of equivalences $\mathsf{Eq}_2(\llbracket \Delta \rrbracket_0 \bar{A}) \cong \mathsf{Eq}_{||}(\llbracket \Delta \rrbracket_1(\mathsf{Eq}(\bar{A}))) \cong \llbracket \Delta \rrbracket_2(\mathsf{Eq}_2(\bar{A}))$ which derives from the Identity Ex-

$$\llbracket x_0 : T_0, \dots, x_n : T_n \vdash x_k : T_k \rrbracket_i \bar{X} = \pi_k \qquad \llbracket \Delta, x : T \vdash t : T \rrbracket_i = \llbracket \Delta \vdash t : T \rrbracket_i \circ \pi_1$$

$$\llbracket \Delta \vdash \lambda x.\, t : S \to T \rrbracket_0 \bar{A}(\gamma) = \lambda s.\, \llbracket \Delta, x : S \vdash t : T \rrbracket_0 \bar{A}(\gamma, s)$$

$$\llbracket \Delta \vdash \lambda x.\, t : S \to T \rrbracket_1 \bar{R}(\overline{\gamma}) = \lambda s_0.\, \lambda s_1.\, \lambda s.\, \llbracket \Delta, x : S \vdash t : T \rrbracket_1 \bar{R}((\gamma_0, s_0), (\gamma_1, s_1), (\gamma, s))$$

$$\llbracket \Delta \vdash \lambda x.\, t : S \to T \rrbracket_2 \bar{Q}((\overline{x}, \overline{p}), \overline{\gamma}) = \lambda((x, p), \gamma).\, \llbracket \Delta, x : S \vdash t : T \rrbracket_2 \bar{Q}((\overline{x}, x), (\overline{p}, p))(\overline{\gamma}, \gamma)$$

$$\llbracket f\, t \rrbracket_0 \bar{A}(\gamma) = \llbracket f \rrbracket_0 \bar{A}(\gamma)\, (\llbracket t \rrbracket_0 \bar{A}(\gamma))$$

$$\llbracket f\, t \rrbracket_1 \bar{R}(\gamma_0, \gamma_1, \gamma) = \llbracket f \rrbracket_1 \bar{R}(\gamma_0, \gamma_1, \gamma, \llbracket t \rrbracket_0 \bar{R}_0(\gamma_0), \llbracket t \rrbracket_0 \bar{R}_1(\gamma_1), \llbracket t \rrbracket_1 \bar{R}(\gamma_0, \gamma_1, \gamma))$$

$$\llbracket f\, t \rrbracket_2 \bar{Q}((\overline{x}, \overline{p}), \overline{\gamma}) = \llbracket f \rrbracket_2 \bar{Q}((\overline{x}, \overline{p}), \overline{\gamma}, \llbracket t \rrbracket_0 \bar{Q}_{\overline{i}}(\overline{x}), \llbracket t \rrbracket_1 \bar{Q}_{\overline{j}}(\overline{p}), \llbracket t \rrbracket_2 \bar{Q}((\overline{x}, \overline{p}), \overline{\gamma}))$$

$$\llbracket \Lambda X.t \rrbracket_0 \bar{A}(\gamma) = (\lambda A.\, \llbracket t \rrbracket_0(\bar{A}, A)\gamma, \lambda R.\, \llbracket t \rrbracket_1(\mathsf{Eq}(\bar{A}), R)\Theta^{-1}_{\Delta,0}(\mathsf{refl}_\gamma))$$

$$\llbracket \Lambda X.t \rrbracket_1 \bar{R}(\gamma_0, \gamma_1, \gamma) = \lambda R.\, (\llbracket t \rrbracket_1(\bar{R}, R))(\gamma_0, \gamma_1, \gamma)$$

$$\llbracket \Delta \vdash \Lambda X.t : \forall X.T \rrbracket_2 \bar{Q}((\overline{x}, \overline{p}), \overline{\gamma}) = \lambda Q.\, \llbracket t \rrbracket_2(\bar{Q}, Q)((\overline{x}, \overline{p}), \overline{\gamma})$$

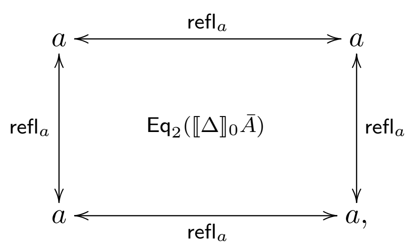
$$\llbracket \Delta \vdash t[S] : T[S \mapsto X] \rrbracket_0 \bar{A}(\gamma) = \mathsf{fst}(\llbracket t \rrbracket_0 \bar{A}(\gamma))(\llbracket S \rrbracket_0 \bar{A})$$

$$\llbracket \Delta \vdash t[S] : T[S \mapsto X] \rrbracket_1 \bar{R}(\gamma_0, \gamma_1, \gamma) = \llbracket t \rrbracket_1 \bar{R}(\gamma_0, \gamma_1, \gamma)(\llbracket S \rrbracket_1 \bar{R})$$

$$\llbracket \Delta \vdash t[S] : T[S \mapsto X] \rrbracket_2 \bar{Q}((\overline{x}, \overline{p}), \overline{\gamma}) = \llbracket t \rrbracket_2 \bar{Q}((\overline{x}, \overline{p}), \overline{\gamma})(\llbracket S \rrbracket_2 \bar{Q})$$

Figure 9.2: Interpretation of terms

tension Lemma, to the related elements



and using the fact that $\llbracket \Delta \rrbracket_2(\mathsf{Eq}_2(\bar{A}), Q) = \llbracket \Delta \rrbracket_2 \mathsf{Eq}_2(\bar{A})$. The image of $\llbracket t \rrbracket_2(\mathsf{Eq}_2(\bar{A}), Q)$

applied to the previous related elements is exactly the Axiom (A0.1):

$$\begin{array}{ccc}
[\![t]\!]_0(\bar{A}, Q_{00})a & \xleftrightarrow{[\![t]\!]_1(\mathsf{Eq}(\bar{A}),Q_{r0})\Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)} & [\![t]\!]_0(\bar{A}, Q_{10})a \\
{\scriptstyle [\![t]\!]_1(\mathsf{Eq}(\bar{A}),Q_{r0})\Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)}\updownarrow & [\![T]\!]_2(\mathsf{Eq}_2(\bar{A}),Q) & \updownarrow{\scriptstyle [\![t]\!]_1(\mathsf{Eq}(\bar{A}),Q_{r0})\Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)} \\
[\![t]\!]_0(\bar{A}, Q_{01})a & \xleftrightarrow[{[\![t]\!]_1(\mathsf{Eq}(\bar{A}),Q_{r0})\Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)}]{} & [\![t]\!]_0(\bar{A}, Q_{11})a.
\end{array}$$

For conditions (A1.1), (A1.2) and (A1.3) we use the same argument but applied to different elements and two-relations. In order we use

$$\begin{array}{ccc}
a & \xleftrightarrow{\Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)} & a \\
{\scriptstyle r}\updownarrow & [\![\Delta]\!]_2(\mathsf{Eq}_{\|}(\bar{A}),Q) & \updownarrow{\scriptstyle r} \\
b & \xleftrightarrow[\Theta^{-1}_{\Delta,0}(\mathsf{refl}_b)]{} & b
\end{array}
\qquad
\begin{array}{ccc}
a & \xleftrightarrow{r} & b \\
{\scriptstyle \Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)}\updownarrow & [\![\Delta]\!]_2(\mathsf{Eq}_{=}(\bar{A})Q) & \updownarrow{\scriptstyle \Theta^{-1}_{\Delta,0}(\mathsf{refl}_b)} \\
a & \xleftrightarrow[r]{} & b
\end{array}$$

$$\begin{array}{ccc}
a & \xleftrightarrow{\Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)} & a \\
{\scriptstyle \Theta^{-1}_{\Delta,0}(\mathsf{refl}_a)}\updownarrow & [\![\Delta]\!]_2(\mathbf{C}(\bar{A}),Q) & \updownarrow{\scriptstyle r} \\
a & \xleftrightarrow[r]{} & b
\end{array}$$

□

In order to prove soundness we need the following lemma:

Lemma 9.4. Let $\Gamma; \Delta \vdash t \colon T$ be a term judgment. Consider

$$\Theta_{T,0} \circ [\![t]\!]_1 \mathsf{Eq}(\bar{A}) \circ \Theta^{-1}_{\Delta,0}(a, b) \colon \mathsf{Id}_{[\![\Delta]\!]_0 \bar{A}}(a, b) \to \mathsf{Id}_{[\![T]\!]_0 \bar{A}}([\![t]\!]_0 \bar{A}\, a, [\![t]\!]_0 \bar{A}\, b).$$

We have that $\Theta_{T,0} \circ [\![t]\!]_1 \mathsf{Eq}(\bar{A}) \circ \Theta^{-1}_{\Delta,0} = \mathsf{ap}([\![t]\!]_0 \bar{A})$.

Proof. By induction on terms and unwinding the definition of $\Theta_{T,0}$. □

Theorem 9.5. The interpretation defined in Fig. 9.2 is sound, i.e. if $\Gamma; \Delta \vdash s = t : T$, then there is $p_{\bar{A}} : \mathsf{Id}_{[\![T]\!]_0 \bar{A}}([\![s]\!]_0, [\![t]\!]_0)$ and $q_{\bar{R}} : \mathsf{Id}_{[\![T]\!]_1 \bar{R}}(\mathsf{tr}(p_{\bar{R}_0})([\![s]\!]_1), [\![t]\!]_1)$. (We automatically have $\mathsf{tr}(p, q)[\![s]\!]_2 \equiv [\![t]\!]_2$ by proof-irrelevance of 2-relations.)

Proof. We need to check that the β- and η-rules for both term and type abstraction are respected. For term abstraction, this follows from Lemmas 8.34 and 8.37.

That the β-rule for type abstraction is respected follows from direct calculation, while we need a little work in the case of the η-rule for type abstraction. Let $\Delta;\Gamma \vdash t : \forall X.T$ be given and let $[\![t]\!]_0\bar{A}\gamma = (f_0, f_1)$. Showing $[\![\Lambda X.t[X]]\!]_0 = [\![t]\!]_0$ means giving $p_0 : \mathsf{Id}(\lambda A.f_0 A, f_0)$ and $p_1 : \mathsf{Id}(\lambda R.([\![t]\!]_1(\mathsf{Eq}(\bar{A}), R)\Theta^{-1}_{\Delta,0}(\mathsf{refl}_\gamma)), \mathsf{snd}([\![t]\!]_0\bar{A}\gamma))$. For p_0, we choose $p_0 = \mathsf{refl}$. In order to give p_1, note first that we have the following morphism

$$\lambda p\colon \mathsf{Eq}([\![\Delta]\!]_0\bar{A})(a,b).\Theta_{\forall X.T,0}(\lambda R.([\![t]\!]_1(\mathsf{Eq}(\bar{A}), R)\Theta^{-1}_{\Delta,0}\,p))\colon \\ \mathsf{Eq}([\![\Delta]\!]_0\bar{A}) \to \mathsf{Eq}([\![\forall X.T]\!]_0\bar{A})$$

between two equality relations. Such morphism satisfies the hypothesis of Lemma 9.4 and then using Lemma we have: $\Theta_{\forall X.T,0}(\lambda R.([\![t]\!]_1(\mathsf{Eq}(\bar{A}), R)\Theta^{-1}_{\Delta,0}(\mathsf{refl}_\gamma)) = \mathsf{refl}_{[\![t]\!]_0\bar{A}\gamma}$. For this reason, if we instantiate the above equation with a relation R, we obtain:

$$\begin{aligned}
[\![t]\!]_1(\mathsf{Eq}(\bar{A}), R)\Theta^{-1}_{\Delta,0}(\mathsf{refl}_\gamma) &= \Theta^{-1}_{\forall X.T,0}(\mathsf{refl}_{[\![t]\!]_0\bar{A}\gamma})R \\
&= \mathsf{tr}(\mathsf{refl},\mathsf{refl}) f_1 R \\
&= f_1\, R \\
&= \mathsf{snd}([\![t]\!]_0\bar{A}\gamma).
\end{aligned}$$

Finally things are exactly lined up to make $\mathsf{tr}((p_0, p_1))([\![\Lambda X.t[X]]\!]_1) = [\![t]\!]_1$ trivial. □

This model reveals hidden uniformity not only in the "standard" interpretation of terms as functions, but also in the canonical proofs of this uniformity via Reynolds relational interpretation of terms. In more detail: consider a term $\Gamma;\Delta \vdash t : T$ with $|\Gamma| = n$. By construction, our model shows that if $\bar{R} : \mathbf{PrRel}^n$, $a : [\![\Delta]\!]_0\bar{R}_0$, $b : [\![\Delta]\!]_0\bar{R}_1$ and $p : [\![\Delta]\!]_1\bar{R}(a,b)$, then $[\![t]\!]_1\bar{R}\,p : [\![T]\!]_1\bar{R}([\![t]\!]_0\bar{R}_0\,a, [\![t]\!]_0\bar{R}_1\,b)$, i.e. $[\![t]\!]_1\bar{R}\,p$ is a proof that $[\![t]\!]_0\bar{R}_0\,a$ and $[\![t]\!]_0\bar{R}_1\,b$ are related at $[\![T]\!]_1\bar{R}$. This is a proof-relevant version of Reynolds' Abstraction Theorem. Furthermore, if $\bar{Q} : \mathbf{2Rel}^n$, $(a,b,c,d) : [\![\Delta]\!]_0\bar{Q}_{00} \times [\![\Delta]\!]_0\bar{Q}_{10} \times [\![\Delta]\!]_0\bar{Q}_{01} \times [\![\Delta]\!]_0\bar{Q}_{11}$

and $(p, q, r, s) \in [\![\Delta]\!]_2 \bar{Q}(a, b, c, d)$, then

$$([\![t]\!]_1 \bar{Q}_{r0}\ p, [\![t]\!]_1 \bar{Q}_{0r}\ q, [\![t]\!]_1 \bar{Q}_{r1}\ r, [\![t]\!]_1 \bar{Q}_{1r}\ s) \in \\ [\![T]\!]_2 \bar{Q}([\![t]\!]_0 \bar{Q}_{00}\ a, [\![t]\!]_0 \bar{Q}_{10}\ b, [\![t]\!]_0 \bar{Q}_{10}\ c, [\![t]\!]_0 \bar{Q}_{11}\ d)$$

This is the Abstraction Theorem “one level up” for the proofs $[\![t]\!]_1$, which we will put to use in the next chapter.

Chapter 10

Applications of two-dimensional parametricity

In this chapter we show applications of two-dimensional parametricity. We generalise some of the techniques and results obtained in Chapter 6. In order to do that we work in a 2-categorical setting.

We start the chapter by recalling the two-dimensional categorical notions that we need. We then equip 1-Type, **PrRel** and **2Rel** with a 2-categorical structure which allows us to define the graph functor for proof-relevant relations. In particular we need a notion of fibration between 2-categories in order to supply the proof relevant version of the graph lemma. The graph lemma is essential in order to give the applications of two-dimensional parametricity, but it is not enough. In fact we need to generalise the graph functor and the graph lemma to 2-relations. This operation leads to the definition of the 2-graph functor and to the 2-relational graph lemma.

At this point we will have all the tools that we need in order to give some applications. We first show a coherence condition for the naturality proofs defined in Section 6.3. We then conclude the chapter by showing that two-dimensional naturality implies 2-naturality.

10.1 Two-dimensional categorical structure

In order to define the graph relations and graph 2-relations for, respectively, **PrRel** and **2Rel** we need the higher dimensional structure of such categories and the right definition of cartesian morphisms in this setting.

We start by recalling the definition of a 2-category as in [Bor94]:

Definition 10.1. A 2-category $\mathcal{C}$ consists of

- a class of objects $\mathcal{C}$,
- for each pair A, B of elements of $\mathcal{C}$, a small category $\mathcal{C}(A, B)$,
- for each triple A, B, C of elements of $\mathcal{C}$, a bifunctor

$$c_{A,B,C} \colon \mathcal{C}(A, B) \times \mathcal{C}(B, C) \to \mathcal{C}(A, C),$$

- for each element A of $\mathcal{C}$, a functor

$$u_A \colon \mathbf{1} \to \mathcal{C}(A, A)$$

  where $\mathbf{1}$ is the discrete category with just one object.

Such a system should satisfy the following axioms

- **associativity axiom**: for A, B, C and D elements in $\mathcal{C}$, the following equality holds:

$$c_{A,C,D} \circ (c_{A,B,C} \times \mathsf{id}) = c_{A,B,D} \circ (\mathsf{id} \times c_{B,C,D}),$$

- **unit axiom**: for A and B objects in $\mathcal{C}$, the following equalities hold:

$$c_{A,A,B} \circ (u_A \times \mathsf{id}) \circ i_r = \mathsf{id} = c_{A,B,B} \circ (\mathsf{id} \times u_B) \circ i_l,$$

  where $i_r \colon \mathcal{C}(A, B) \cong 1 \times \mathcal{C}(A, B)$ and $i_l \colon \mathcal{C}(A, B) \cong \mathcal{C}(A, B) \times 1$.

The elements of $\mathcal{C}$ are called objects or 0-cells, the objects in the category $\mathcal{C}(A, B)$ are called 1-cells or morphisms and we denote them as $f \colon A \to B$, and finally we call 2-cells the morphisms in $\mathcal{C}(A, B)$ and we denote them by $\alpha \colon f \Rightarrow g$.

The bifunctor c gives the composition of morphisms. Given $f \in \mathcal{C}(A, B)$ and $g \in \mathcal{C}(B, C)$, we denote the composition $c(f, g) \coloneqq g \circ f$. With this composition, objects and 1-cells form a category where unit and associativity of composition are given by the unit and associativity axiom in Definition 10.1. We denote the category of objects and 1-cells by $\mathcal{C}_0$.

Note that the functor c equips also the 2-cells with a composition structure which we call horizontal composition (of 2-cells). Given two 2-cells $\alpha\colon f \Rightarrow g$ and $\beta\colon h \Rightarrow k$, where $f, g\colon A \to B$ and $h, k\colon B \to C$, we denote the horizontal composition $\beta * \alpha := c(\alpha, \beta)\colon h \circ f \Rightarrow k \circ g$.

The 2-cells inherit another composition, which we call vertical composition (of 2-cells), from the categorical structure of $\mathcal{C}(A, B)$. Given $\alpha\colon f \Rightarrow g$ and $\beta\colon g \Rightarrow h$, the vertical composition is denoted by $\beta \circ \alpha\colon f \Rightarrow h$.

All these compositions have identity. The functor $u_A : \mathbf{1} \to \mathcal{C}(A, A)$ identifies the identity morphism for the object A respect to the compositions defined by c. The categorical structure on $\mathcal{C}(A, B)$ furnishes the identity for 1-cells with respect to the vertical composition.

Example 10.2. A standard example of a 2-category is obtained by choosing small categories as objects, functors as 1-cells and natural transformations as 2-cells.

Example 10.3. Every ordinary category $\mathcal{C}$ can be viewed as a 2-category with the trivial 2-cells, i.e. each category $\mathcal{C}(A, B)$ is discrete.

The notion of functors generalises to 2-functors:

Definition 10.4. Given two 2-categories $\mathcal{C}$ and $\mathcal{D}$, a 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ consists of

- for every object A in $\mathcal{C}$, an object $F(A)$ in $\mathcal{D}$;
- for every two objects A, A' in $\mathcal{C}$, a functor

$$F_{A,A'}\colon \mathcal{C}(A, A') \to \mathcal{D}(F(A), F(A')),$$

which satisfy the axioms

- **compatibility for composition**: for A, A' and A'' objects in $\mathcal{C}$

$$F_{A,A'} \circ c_{A,A',A''} = c_{F(A),F(A'),F(A'')} \circ (F_{A,A'} \times F_{A',A''}),$$

- **unit**: for every object A in C

$$F_{AA} \circ u_A = u_{F(A)}.$$

For the sake of brevity we will often write F instead of $F_{A,A'}$. Any 2-functor $F\colon \mathcal{C} \to \mathcal{D}$ induces a functor on the category of objects and 1-cells which we denote by $F_0\colon \mathcal{C}_0 \to \mathcal{D}_0$.

There is also the notion of 2-natural transformations. In order to define it, we first need two functors. Let A and $f\colon A' \to A''$ be, respectively, an object and a 1-cell in a 2-category $\mathcal{C}$. We define the functor $\mathcal{C}(A, f)\colon \mathcal{C}(A, A') \to \mathcal{C}(A, A'')$ as

$$\mathcal{C}(A, f)(g) = f \circ g \qquad \mathcal{C}(A, f)(\alpha) = \mathsf{id}_f * \alpha.$$

Similarly, the functor $\mathcal{C}(f, A)\colon \mathcal{C}(A'', A) \to \mathsf{A}(A', A)$ is defined as

$$\mathcal{C}(f, A)(g) = g \circ f \qquad \mathcal{C}(f, A)(\alpha) = \alpha * \mathsf{id}_f.$$

A 2-natural transformation between two parallel 2-functors is defined as follows:

Definition 10.5. Let $\mathcal{C}$ and $\mathcal{D}$ be two 2-categories and $F, G\colon \mathcal{C} \to \mathcal{D}$ be two 2-functors. A 2-natural transformation $\theta\colon F \Rightarrow G$ consists of a morphism $\theta_A\colon F(A) \to G(A)$ for every objects A in $\mathcal{C}$ which, for every two elements A, A' in $\mathcal{C}$, satisfies

$$\mathcal{D}(F(A), \theta_{A'}) \circ F_{A,A'} = \mathcal{D}(\theta_A, G(A')) \circ G_{A,A'}.$$

In order to understand better the definition, we unwind the equality. Let $\alpha\colon f \Rightarrow g$ be a 2-cell, with $f, g\colon A \to A'$. The left hand side gives $\mathcal{D}(F(A), \theta_{A'}) \circ F_{A,A'}(\alpha) = \mathsf{id}_{\theta_{A'}} * F(\alpha)\colon \theta_{A'} \circ F(f) \Rightarrow \theta_{A'} \circ F(g)$. The other side, similarly, gives $\mathcal{D}(\theta_A, G(A')) \circ G_{A,A'}(\alpha) = G(\alpha) * \mathsf{id}_{\theta_A}\colon G(f) \circ \theta_A \Rightarrow G(g) \circ \theta_A$. From the equality of these two 2-cells we obtain $\theta_{A'} \circ F(f) = G(f) \circ \theta_A$, which is the usual naturality condition, and $\mathsf{id}_{\theta A'} * \alpha = \alpha * \mathsf{id}_{\theta_A}$.

We now show that 1-Type, **PrRel** and **2Rel** have a 2-categorical structure:

1-Type	**objects**	1-types A.

	1-cells	$f\colon A \to B$ morphism of 1-types.
	2-cells	$u\colon f \Rightarrow g$ is a proof $u\colon f = g$.
PrRel	**objects**	proof-relevant relations $R = (R_0, R_1, R)$.
	1-cells	$f = (f_0, f_1, f)\colon R \to R'$ is a morphism of relations.
	2-cells	$u = (u_0, u_1, u)\colon f \Rightarrow g$ is a proof that $f = g$, i.e. $u_0\colon f_0 = g_0$, $u_1\colon f_1 = g_1$ and $u\colon \mathsf{tr}(u_0, u_1) f = g$.
2Rel	**objects**	2-relations $Q = (Q_{00}, Q_{10}, Q_{01}, Q_{11}, Q_{r0}, Q_{0r}, Q_{r1}, Q_{1r}, Q)$.
	1-cells	$f = (f_{00}, f_{10}, f_{01}, f_{11}, f_{r0}, f_{0r}, f_{r1}, f_{1r}, f)\colon Q \to Q'$ is a morphism of 2-relations.
	2-cells	$u = (u_{00}, u_{10}, u_{01}, u_{11}, u_{r0}, u_{0r}, u_{r1}, u_{1r}, u)\colon f = g$, where • $u_j\colon f_j = g_j$ with $j \in (\{0,1\} \times \{0,1\})$, • $u_i\colon \mathsf{tr}(u_{i0}, u_{i1}) f_i = g_i$ with $i \in (\{r\} \times \{0,1\} \cup \{0,1\} \times \{r\})$ and $i0$ substitutes r with 0 and $i1$ substitutes r with 1, • $u\colon \mathsf{tr}(u_{00}, u_{10}, u_{01}, u_{11}, u_{r0}, u_{0r}, u_{r1}, u_{1r}) f = g$, which is trivial because it acts on propositions.

For each one of the above data is clear that objects and 1-cells form a category. We need to define the horizontal and vertical composition of 2-cells. Note that, also if the 2-cells are given by equalities, the 2-categorical structure is not trivial since equalities are proof-relevant.

We start from the horizontal composition of 2-cells. We define the horizontal composition $v * u\colon h \circ f \Rightarrow k \circ g$ where $u\colon f \Rightarrow g$, and $v\colon h \Rightarrow k$, $f, g\colon X \to Y$ and $h, k\colon Y \to Z$ are, respectively, two 2-cells and two 1-cells in

- 1-Type: $v * u\colon h \circ f = k \circ g$ is defined by the composition of paths

$$\mathsf{ap}(h) u\, a \cdot v(g\, a)\colon (h \circ f)\, a = (k \circ g)\, a$$

- **PrRel**: $v * u\colon \mathsf{tr}(v_0 * u_0, v_1 * u_1)h \circ f = k \circ g$ is defined by first using

$$\mathsf{apd}(h)\, u\colon \mathsf{tr}(\mathsf{ap}(h_0)u_0, \mathsf{ap}(h_1)u_1)(h \circ f) = h \circ g$$

  and then by composing with

$$v(g\,\_)\colon \mathsf{tr}(v_0, v_1)(h \circ g) = k \circ g.$$

- **2Rel**: in this case $v * u$ has 9 components: 4 living in 1-Type, 4 living in **PrRel**, and for these 8 we have already defined the horizontal composition, and the last one has trivial composition because it is a proof that two maps between two propositions are equal.

Next we define the vertical composition $v \circ u\colon f \Rightarrow h$ where $f, g, h\colon A \to B$, $u\colon f \Rightarrow g$ and $v\colon g \Rightarrow h$ are 1-cells and 2-cells in

- 1-Type: $v \circ u\colon f \Rightarrow h$ is defined by $u \cdot v$.
- **PrRel**: $v \circ u\colon f \Rightarrow h$ is defined by $\mathsf{apd}(\mathsf{tr}(v_0, v_1))u \cdot v\colon \mathsf{tr}(v_0, v_1)(\mathsf{tr}(u_0, u_1)f) = h$.
- **2Rel**: again u has 9 components: the first 8 live in 1-Type and **PrRel** and we have already defined the composition for them, while the last one has trivial composition because it is a proof that two maps between two propositions are equal.

It is straightforward to check that both vertical and horizontal composition have unit given by refl and the associativity of the compositions derives from the associativity of path composition.

There are two 2-functors in which we are interested:

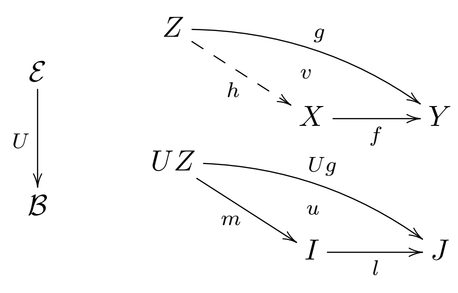
$\mathsf{pr\text{-}rel}\colon \mathbf{PrRel} \to$ 1-Type $\times$ 1-Type	**objects**	$R \mapsto (R_0, R_1)$.
	1-cells	$(f_0, f_1, f) \mapsto (f_0, f_1)$.
	2-cells	$(u_0, u_1, u) \mapsto (u_0, u_1)$.

$\mathsf{2rel}\colon \mathbf{2Rel} \to \mathbf{PrRel} \times \mathbf{PrRel}$	**objects**	$Q \mapsto (Q_{0r}, Q_{1r})$.
	1-cells	$f \mapsto ((f_{00}, f_{01}, f_{0r}), (f_{10}, f_{11}, f_{1r}))$.
	2-cells	$u \mapsto ((u_{00}, u_{01}, u_{0r}), (u_{10}, u_{11}, u_{1r}))$.

Since they are projections, it is clear that they are both functorial. Moreover the domains are well defined: the product of two 2-categories has a 2-categorical structure which is given componentwise.

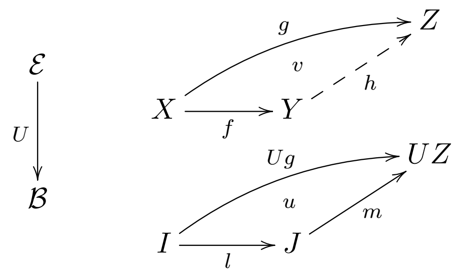
We conclude this section with the notion of proof-relevant cartesian morphisms. There are different definitions of cartesian morphisms with respect to higher dimensional functors, see for example Hermida [Her99], Street [Str80] or Lurie [Lur09]. The proof-relevant cartesian morphisms that we use are, in particular, cartesian morphisms in the sense of the definition presented in [Lur09].

Definition 10.6. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a 2-functor. We say that a morphism $f\colon X \to Y$ in $\mathcal{E}$ is **proof-relevant cartesian over** $l\colon I \to J$ in $\mathcal{B}$ with respect to the functor U if $Uf = l$ and if for every $g\colon Z \to Y$ in $\mathcal{E}$ and 2-cell $u\colon l \circ m \Rightarrow U(g)$ for some $m\colon U(Z) \to I$, there exists a morphism $h\colon Z \to X$ in $\mathcal{E}$ over m and a 2-cell $v\colon f \circ h \Rightarrow g$ in $\mathcal{E}$ over u such that h is unique up to 2-cells. That is if there is another morphism $h'\colon Z \to X$ in $\mathcal{E}$ over m and another 2-cell $v'\colon f \circ h' \Rightarrow g$ in $\mathcal{E}$ over u, then there is a 2-cell $w\colon h \Rightarrow h'$. The condition is expressed by the diagram



Dually we have the definition of proof-relevant opcartesian morphisms:

Definition 10.7. Let $U\colon \mathcal{E} \to \mathcal{B}$ be a 2-functor. We say that a morphism $f\colon X \to Y$ in $\mathcal{E}$ is **proof-relevant opcartesian over** $l\colon I \to J$ in $\mathcal{B}$ with respect to the functor U if $Uf = l$ and if for every $g\colon X \to Z$ in $\mathcal{E}$ and 2-cell $u\colon m \circ l \Rightarrow U(g)$ for some $m\colon J \to UZ$, there exists a morphism $h\colon Y \to Z$ in $\mathcal{E}$ over m and a 2-cell $v\colon h \circ f \Rightarrow g$ in $\mathcal{E}$ over u such that h is unique up to 2-cells. That is if there is another morphism $h'\colon Y \to Z$ in $\mathcal{E}$ over m and another 2-cell $v'\colon h' \circ f \Rightarrow g$ in $\mathcal{E}$ over u, then there is a 2-cell $w\colon h \Rightarrow h'$. The condition is expressed by the diagram



Clearly the above definitions are a generalisation of the cartesian and opcartesian morphisms defined in Chapter 2, taking in consideration that we want to lift also the 2-cells. In fact, since we are working in a proof-relevant framework, it is not enough to say that some diagram commutes, but we need to provide the proof that it commutes, i.e. we need a 2-cell filling the diagram.

We will just write cartesian and opcartesian morphism since we are working in a proof-relevant framework and it will be clear that we refer to Definition 10.6 and Definition 10.7.

10.2 Graph relations and graph 2-relations

Relations representing graphs of functions are key to many applications of parametricity.

Definition 10.8. Let $f : A \to B$ in $\mathsf{1\text{-}Type}$. We define the **graph** of f as the relation $\langle f \rangle := (A, B, \lambda a.\, \lambda b.\, \mathsf{Id}_B(fa, b))$ in **PrRel**.

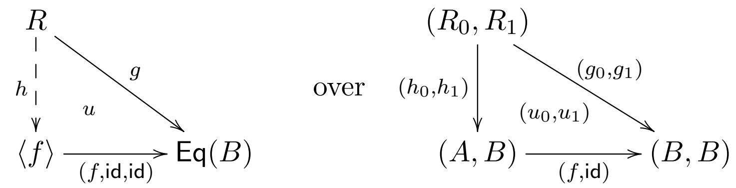
The following result shows that, similarly to the proof-irrelevant case in Section 3.3, it is possible to characterise graph relations using proof-relevant cartesian and opcartesian morphisms:

Lemma 10.9. Let $f\colon A \to B$ be a morphism in $\mathsf{1\text{-}Type}$. We have that:

1. the morphism $(f, \mathsf{id}, \mathsf{id})\colon \langle f\rangle \to \mathsf{Eq}(B)$ in **PrRel** is proof-relevant cartesian over (f, id) with respect to the 2-functor pr-rel;

2. the morphism $(\mathsf{id}, f, \mathsf{ap}(f))\colon \mathsf{Eq}(A) \to \langle f\rangle$ in **PrRel** is proof-relevant opcartesian over (id, f) with respect to the 2-functor pr-rel.

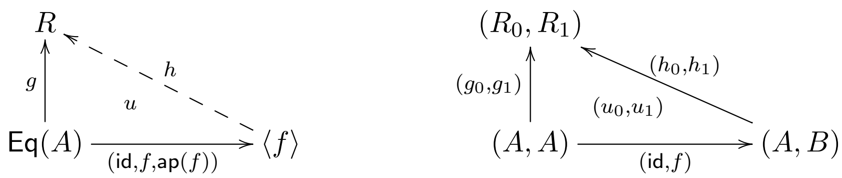
Proof. It is not difficult to see that the morphisms $(f, \mathsf{id}, \mathsf{id})$ and $(\mathsf{id}, f, \mathsf{ap}(f))$ are well defined. For the cartesian property consider the following diagram:



where the 2-cell (u_0, u_1) of the right diagram is given by hypothesis. In order to define h we need to define a term $h_{(r_0,r_1)}r\colon \mathsf{Id}((f \circ h_0)r_0, h_1\, r_1)$ for every $r \in R(r_0, r_1)$. We define it as $h_{(r_0,r_1)}r = (u_0\, r_0) \cdot (g_{r_0,r_1}\, r) \cdot (u_1\, r_1)^{-1}$. The 2-cell $u\colon \mathsf{tr}(u_0, u_1)\, h = g$ is defined using Lemma 8.7. In fact, by unwinding the definition of h and transport, u should be a proof that $(u_0\, r_0)^{-1} \cdot (u_0\, r_0) \cdot (g_{r_0,r_1}\, r) \cdot (u_1\, r_1)^{-1} \cdot (u_1\, r_1) = g_{r_0,r_1}\, r$ for every $r \in R(r_0, r_1)$.

The morphism h is unique up to 2-cells since, if there was another morphism h' over (h_0, h_1) and $u'\colon (f, \mathsf{id}, \mathsf{id}) \circ h' = g$ over (u_0, u_1), we would have $\mathsf{tr}(u_0, u_1)\, h = g = \mathsf{tr}(u_0, u_1)\, h'$ and then, by composing with $\mathsf{tr}(u_0, u_1)^{-1}$, we obtain $h = h'$. Note that the first two components are fixed because of the condition of living over (h_0, h_1).

For the opcartesian morphism we consider this other diagram



where the right 2-cells is given by hypothesis, and h is defined for every $r\colon f\, a = b$ by

$$h\, r = \mathsf{tr}((u_0\, a)^{-1}, (u_1\, a)^{-1} \cdot (\mathsf{ap}(h_1)r))\, g\, \mathsf{refl}_a.$$

In order to define u we use path induction obtaining the equality:

$$g_{(a,a)}\mathsf{refl}_a = \mathsf{tr}(u_0\,a, u_1\,a)(\mathsf{tr}((u_0\,a)^{-1}, (u_1\,a)^{-1} \cdot (\mathsf{ap}(h_1)\mathsf{refl}_a))\, g_{(a,a)}\,\mathsf{refl}_a.$$

Finally we want to prove that if there is another $h' \colon \langle f \rangle \to R$ over (h_0, h_1) and $u' \colon h' \circ (\mathsf{id}, f, \mathsf{ap}(f)) = g$ over u, then $h = h'$. By composing u and $(u')^{-1}$ we obtain the equality $\mathsf{tr}(u_0, u_1)(h \circ \mathsf{ap}(f)) = g = \mathsf{tr}(u_0, u_1)(h' \circ \mathsf{ap}(f))$ and then we can derive $h \circ \mathsf{ap}(f) = h' \circ \mathsf{ap}(f)$ which is not enough because what we need to prove is that for every $p \colon f\,a = b$ we have $h_{(f\,a,b)}p = h'_{(f\,a,b)}p$. In order to do so consider $(\mathsf{refl}_a, p) \colon (a, f\,a) = (a, b)$, and using Lemma 8.22 we have:

$$\begin{aligned}
h_{(a,b)}p &= h_{(a,b)}(\mathsf{tr}(\mathsf{refl}_{fa}, p)\mathsf{refl}_{f\,a}) \\
&= h_{(a,b)}(\mathsf{tr}(\mathsf{refl}_{fa}, p)(\mathsf{ap}(f)\mathsf{refl}_a)) \\
&= \mathsf{tr}(\mathsf{refl}_a, p)(h_{(a,fa)}(\mathsf{ap}(f)\mathsf{refl}_a)) \\
&= \mathsf{tr}(\mathsf{refl}_a, p)(h'_{(a,fa)}(\mathsf{ap}(f)\mathsf{refl}_a)) \\
&= h'_{(a,b)}(\mathsf{tr}(\mathsf{refl}_{fa}, p)\mathsf{refl}_{f\,a}) \\
&= h'_{(a,b)}p
\end{aligned}$$

which proves the thesis. □

The map $\langle - \rangle$ extends to a functor $\langle - \rangle \colon \mathsf{1\text{-}Type}^{\to} \to \mathbf{PrRel}$ where $\mathsf{1\text{-}Type}^{\to}$ has morphisms $f \colon A \to B$ in $\mathsf{1\text{-}Type}$ as objects and a morphism $(\alpha, \beta, p) \colon f \to g$, where $f \colon A \to B$ and $g \colon A' \to B'$, consists of two morphisms $\alpha \colon A \to A'$ and $\beta \colon B \to B'$ in $\mathsf{1\text{-}Type}$, and a proof $p : (\Pi x : A)\, \mathsf{Id}_{B'}(g(\alpha(a)), \beta(f(a)))$ that the following square commutes

$$\begin{array}{ccc}
A & \xrightarrow{\alpha} & A' \\
{\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\
B & \xrightarrow[\beta]{} & B'.
\end{array}$$

We define the action of $\langle - \rangle$ on the morphisms in the following way:

$$\langle \alpha, \beta, p \rangle = (\alpha, \beta, \lambda a.\, \lambda b.\, \lambda(r : fa = b).\, p(a) \cdot \mathsf{ap}(\beta)(r)) : \langle f \rangle \to \langle g \rangle.$$

It is straightforward to see that the morphism $\langle \alpha, \beta, p \rangle$ completes the following diagram

$$\begin{array}{ccc} \langle f \rangle & \xrightarrow{(f,\mathsf{id},\mathsf{id})} & \mathsf{Eq}(B) \\ {\scriptstyle \langle \alpha,\beta,p \rangle} \downarrow & & \downarrow {\scriptstyle \mathsf{Eq}(\beta)} \\ \langle g \rangle & \xrightarrow[(g,\mathsf{id},\mathsf{id})]{} & \mathsf{Eq}(B') \end{array} \qquad \text{over} \qquad \begin{array}{ccc} (A,B) & \xrightarrow{(f,\mathsf{id})} & (B,B) \\ {\scriptstyle (\alpha,\beta)} \downarrow & (p,\mathsf{refl}) & \downarrow {\scriptstyle (\beta,\beta)} \\ (A',B') & \xrightarrow[(g,\mathsf{id})]{} & (B',B'). \end{array}$$

which represents the universal property of the cartesian morphism $(g, \mathsf{id}, \mathsf{id})\colon \langle g \rangle \to \mathsf{Eq}(B')$.

Just like Eq is full and faithful, so is $\langle - \rangle : \text{1-Type}^{\to} \to \mathbf{PrRel}$:

Lemma 10.10. The graph functor $\langle - \rangle\colon \text{1-Type}^{\to} \to \mathbf{PrRel}$ is full and faithful.

Proof. For the faithfulness consider $\langle \alpha, \beta, p \rangle = \langle \alpha', \beta', p' \rangle$. It is immediate to derive that $\alpha = \alpha'$ and $\beta = \beta'$. The equality of the third component gives that for every $r\colon f\,a = b$ it holds $p(a) \cdot \mathsf{ap}(\beta)(r) = p'(a) \cdot \mathsf{ap}(\beta')(r)$. Since β and β' are equal, we can compose both the sides with $\mathsf{ap}(\beta)(r)^{-1}$ and obtain $p(a) = p'(a)$ for every a.

For the fullness note that, by Proposition 8.28, Eq is full and faithful and we have both the cartesian and opcartesian properties for the graph functor. For this reason we can use the same argument that we used in the proof of Lemma 3.11. $\square$

The main tool for deriving consequences of parametricity is the Graph Lemma, which relates the graph of the action of a functor on a morphism with its relational action on the graph of the morphism.

Theorem 10.11. Let F_0 : 1-Type $\to$ 1-Type and $F_1 : \mathbf{PrRel} \to \mathbf{PrRel}$ over $F_0 \times F_0$ be functorial. If $F_1(\mathsf{Eq}(A)) \cong \mathsf{Eq}(F_0 A)$ for all A, then for any $f : A \to B$, there are morphisms $(\mathsf{id}, \mathsf{id}, \phi_{F,f}) : \langle F_0 f \rangle \to F_1 \langle f \rangle$ and $(\mathsf{id}, \mathsf{id}, \psi_{F,f}) : F_1 \langle f \rangle \to \langle F_0 f \rangle$.

Proof. Using the cartesian and opcartesian property of the graph functor, we can reproduce the same argument that we used in the proof of Theorem 3.12. $\square$

Note that in our proof-relevant setting, this theorem does not construct an equivalence $\langle F_0 f \rangle \cong F_1 \langle f \rangle$. Instead, we only have a logical equivalence, i.e. maps in both directions, and that seems to be enough for all known consequences of parametricity. (In a proof-irrelevant setting, the constructed logical equivalence would automatically be an equivalence.)

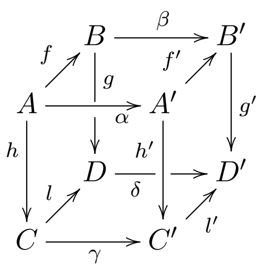
Next, we consider also graph 2-relations. Since we have multiple "equality 2-relations", one could expect also multiple graph 2-relations, but for the application we have in mind, one suffices. Given functions f, g, l and h, we write $\square(f, g, l, h)$ for the 1-type of proofs that the square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[l]{} & D \end{array}$$

commutes, i.e. $\square(f, g, l, h) = (\Pi x : A)\mathsf{Id}_D(g(fx), l(hx))$. We define the category $(\mathsf{1\text{-}Type}^{\rightarrow})^{\rightarrow}$ as the one with commuting squares (f, g, l, h, p) as objects and a morphism

$$(\alpha, \beta, \gamma, \delta, q, r, q', r', t)\colon (f, g, l, h, p) \to (f', g', l', h', p')$$

in $(\mathsf{1\text{-}Type}^{\rightarrow})^{\rightarrow}$ consists of four morphisms $\alpha : A \to A'$, $\beta : B \to B'$, $\gamma : C \to C'$ and $\delta : D \to D'$, and four proofs $q : \square(\alpha, f', \beta, f)$, $r : \square(\beta, g', \delta, g)$, $q' : \square(\gamma, l', \delta, l)$ and $r' : \square(\alpha, h', \gamma, h)$ such that they form a "commuting cube"



i.e. such that $t\colon p' \star r' \star q' = q \star r \star p$, where $p' \star r' \star q'$ and $q \star r \star p$ are pastings of the squares that proves that both ways from one corner of the cube to the opposite one commutes.

We unwind the previous condition and formally explain what we mean by a "commuting cube". We start first from $l' \circ h' \circ \alpha$. The pasting $r' \star q'\colon l' \circ h' \circ \alpha = \delta \circ l \circ h$ is formally represented by

$$l' \circ h' \circ \alpha \overset{\mathsf{ap}(l')r'}{=\!=\!=\!=} l' \circ \gamma \circ h \overset{q'(h(\_))}{=\!=\!=\!=} \delta \circ l \circ h,$$

and the pasting $q \star r \colon g' \circ f' \circ \alpha = \delta \circ g \circ f$ is defined in a similar way:

$$g' \circ f' \circ \alpha \overset{\mathsf{ap}(g')q}{=\!=\!=} g' \circ \beta \circ f \overset{r(f(\_))}{=\!=\!=} \delta \circ g \circ f.$$

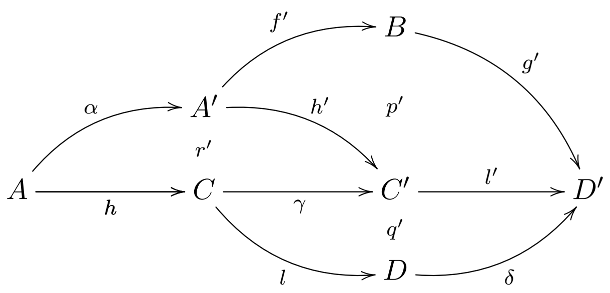
They form two 2-cells, but in order to compare the two of them we need that they agree on the boundary, and we can use p' and p to fill the gap:

$$g' \circ f' \circ \alpha \overset{p'(\alpha(\_))}{=\!=\!=} l' \circ h' \circ \alpha \overset{r' \star q'}{=\!=\!=} \delta \circ l \circ h \overset{\mathsf{ap}(\delta)(p^{-1})}{=\!=\!=} \delta \circ g \circ f.$$

The "commuting cube" condition $p' \star r' \star q' = q \star r \star p$ follows by composing with p both the sides of the equality $p' \star r' \star q' \star p^{-1} = q \star r$, and explicitly is a proof of type

$$\lambda a.p'(\alpha\, a) \cdot \mathsf{ap}(l')(r'\, a) \cdot q'(h\, a) = \lambda a.\mathsf{ap}(g')(q\, a) \cdot r(f\, a) \cdot \mathsf{ap}(\delta)(p\, a).$$

Naturally the commuting cube condition can be expressed using the 2-categorical structure of 1-Type as well. For example the pasting $p' \star r' \star q'$ is represented by the following diagram:



and using the 2-categorical notation we have $p' * \mathsf{id}_\alpha \colon g' \circ f' \circ \alpha = l' \circ h' \circ \alpha$, $\mathsf{id}_{l'} * r' \colon l' \circ h' \circ \alpha = l' \circ \gamma \circ h$, and $q' * \mathsf{id}_h \colon l' \circ \gamma \circ h = \delta \circ l \circ h$. It is not difficult to check that $p' \star r' \star q' = (q' * \mathsf{id}_h) \circ (\mathsf{id}_{l'} * r') \circ (p' * \mathsf{id}_\alpha)$. Similarly we can rewrite also $q \star r \star p$ using the 2-categorical notation.

Lemma 10.12. Let p' and r' be two 2-cells as in the above picture. If $h = \mathsf{id}$, $h' = \mathsf{id}$, and $g' = \mathsf{id}$, we have that $p' \star r' = p' * r'$.

Proof. The equality follows from straightforward calculation and Lemma 8.12. □

We can now define

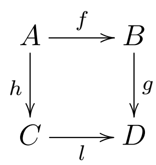
Definition 10.13. Let (f, g, l, h, p) be an object in $(1\text{-}\mathsf{Type}^{\rightarrow})^{\rightarrow}$. We define the **2-graph** of (f, g, l, h, p) as the 2-relation

$$\langle f, g, l, h, p\rangle_2 = (\langle f\rangle, \langle h\rangle, \langle l\rangle, \langle g\rangle, \lambda(a, b, c, d).\, \lambda(q, r, s, t).\, \mathsf{ap}(g)q \cdot t = p(a) \cdot \mathsf{ap}(l)r \cdot s).$$

The 2-graph 2-relation $\langle f, g, l, h, p\rangle_2$ says that the two ways to prove $h(l(a)) = d$ using p, q, r, s, t are in fact equal.

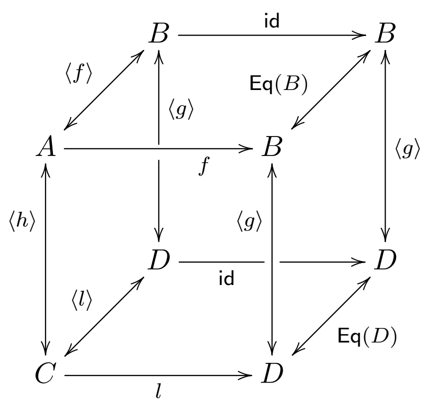
Again, more abstractly, we can see the 2-graph relations being domains of proof-relevant cartesian morphisms or codomains of proof-relevant opcartesian morphisms with respect to the 2-functor $\mathsf{2rel}\colon \mathbf{2Rel} \to \mathbf{PrRel} \times \mathbf{PrRel}$.

Lemma 10.14. Let (f, g, l, h, p) be a commuting square



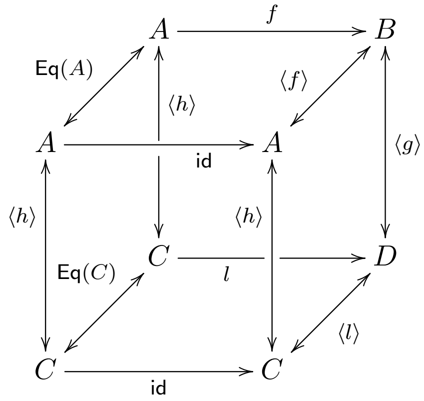
i.e. an object in $(1\text{-}\mathsf{Type}^{\rightarrow})^{\rightarrow}$. We have that

1. $(f, \mathsf{id}, l, \mathsf{id}, \mathsf{id}, \langle f, l, p\rangle, \mathsf{id}, \mathsf{id})\colon \langle f, g, l, h, p\rangle_2 \to \mathsf{Eq}_{||}(\langle g\rangle)$ in $\mathbf{2Rel}$ is proof-relevant cartesian over $((f, l, \langle f, l, p\rangle), (\mathsf{id}, \mathsf{id}, \mathsf{id}))\colon (\langle h\rangle, \langle g\rangle) \to (\langle g\rangle, \langle g\rangle)$ with respect to the 2-functor $\mathsf{2rel}$.

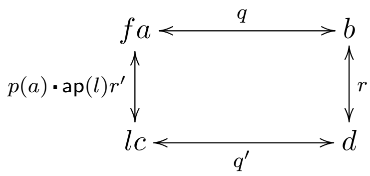


2. $(\mathsf{id}, f, \mathsf{id}, l, \mathsf{ap}(f), \mathsf{id}, \mathsf{ap}(l), \langle f, l, p\rangle)\colon \mathsf{Eq}_{||}(\langle h\rangle) \to \langle f, g, l, h, p\rangle_2$ in $\mathbf{2Rel}$ is proof-relevant opcartesian over $((\mathsf{id}, \mathsf{id}, \mathsf{id}), (f, l, \langle f, l, p\rangle))\colon (\langle h\rangle, \langle h\rangle) \to (\langle h\rangle, \langle g\rangle)$ with respect to the

2-functor 2rel.



Proof. We first need to check that the two morphisms are well defined. In the first case consider the related elements $(q, r', q', r) \in \langle f, g, l, h, p\rangle_2(a, b, c, d)$ and we want to show that their images

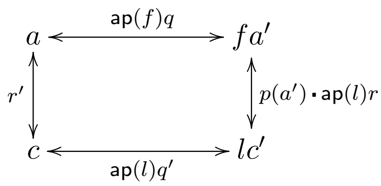


are related in $\mathsf{Eq}_{||}(\langle g\rangle)(fa, b, lc, d)$, i.e. $\mathsf{tr}(q, q')(p(a) \cdot \mathsf{ap}(l)r') = r$. By definition of transport and Lemma 8.21 first and then by composing both sides with $\mathsf{ap}(f)q$, this is equivalent to:

$$
\begin{aligned}
\mathsf{tr}(q, q')(p(a) \cdot \mathsf{ap}(l)r') &= r \\
\Rightarrow (\mathsf{ap}(f)(q))^{-1} \cdot p(a) \cdot \mathsf{ap}(l)r' \cdot q' &= r \\
\Rightarrow \qquad p(a) \cdot \mathsf{ap}(l)r' \cdot q' &= \mathsf{ap}(f)(q) \cdot r
\end{aligned}
$$

which is exactly the hypothesis condition for (q, r', q', r) being in $\langle f, g, l, h, p\rangle_2(a, b, c, d)$.

For the second morphism consider the related elements $(q, r', q', r) \in \mathsf{Eq}_{||}(\langle h\rangle)(a, a', c, c')$, that is $\mathsf{tr}(q, q')r' = r$ and, by definition of transport plus Lemma 8.21, we rewrite it as $(\mathsf{ap}(h)q)^{-1} \cdot r' \cdot q' = r$. We want to show that their images



are related in $\langle f, g, l, h, p\rangle_2$, i.e. that $\mathsf{ap}(g)(\mathsf{ap}(f)q) \cdot p(a') \cdot \mathsf{ap}(l)r = p(a) \cdot \mathsf{ap}(l)r' \cdot \mathsf{ap}(l)q'$. Using Lemma 8.12, we have $\mathsf{ap}(g\circ f)q \cdot p(a') = p(a)\cdot\mathsf{ap}(l\circ h)q$, and from $(\mathsf{ap}(h)q)^{-1}\cdot r'\cdot q' = r$ we derive $\mathsf{ap}(h)q = r' \cdot q' \cdot r^{-1}$. Using these equalities we can rewrite

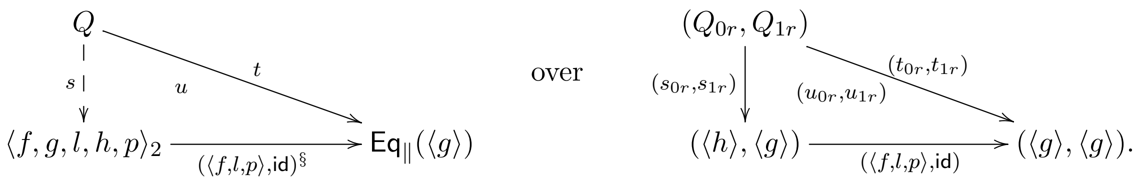
$$\mathsf{ap}(g)(\mathsf{ap}(f)q) \cdot p(a') \cdot \mathsf{ap}(l)r = p(a) \cdot \mathsf{ap}(l)r' \cdot \mathsf{ap}(l)q'$$

$$p(a) \cdot \mathsf{ap}(l)(\mathsf{ap}(h)q) \cdot \mathsf{ap}(l)r = p(a) \cdot \mathsf{ap}(l)r' \cdot \mathsf{ap}(l)q'$$

$$p(a) \cdot \mathsf{ap}(l)(r' \cdot q' \cdot r^{-1}) \cdot \mathsf{ap}(l)r = p(a) \cdot \mathsf{ap}(l)r' \cdot \mathsf{ap}(l)q'$$

and we can simplify everything obtaining $\mathsf{refl} = \mathsf{refl}$.

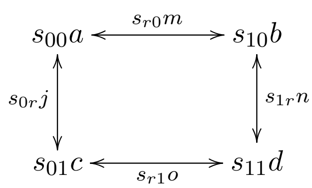
We now prove that $(f, \mathsf{id}, l, \mathsf{id}, \mathsf{id}, \langle f, l, p\rangle, \mathsf{id}, \mathsf{id})$, which we denote by $(\langle f, l, p\rangle, \mathsf{id})^{\S}$, is cartesian over $(\langle f, l, p\rangle, \mathsf{id})\colon (\langle h\rangle, \langle g\rangle) \to (\langle g\rangle, \langle g\rangle)$. In order to do that consider the following diagram:



In order to complete the triangle on the left we need to find s_{r0}, s_{r1}, s, u_{r0}, u_{r1} and u because the other components are given since they live over the 2-cell on the right hand side.

The components s_{r0}, s_{r1}, u_{r0} and u_{r1} are defined using the universal property of the cartesian morphism $(f, \mathsf{id}, \mathsf{id})\colon \langle f\rangle \to \mathsf{Eq}(B)$ with $t_{r0}\colon Q_{r0} \to \mathsf{Eq}(B)$ over the 2-cell (u_{00}, u_{10}) and the cartesian morphism $(l, \mathsf{id}, \mathsf{id})\colon \langle l\rangle \to \mathsf{Eq}(D)$ with $t_{r1}\colon Q_{r_1} \to \mathsf{Eq}(D)$ over the 2-cell (u_{01}, u_{11}). In particular we recall the definitions of s_{r0} and s_{r1}: if $m \in Q_{r0}(a, b)$, then $(s_{r0})_{(a,b)}m = u_{00}a \cdot (t_{r0})_{(a,b)}m \cdot (u_{10}b)^{-1}$, and if $o \in Q_{r1}(c, d)$, then $(s_{r1})_{(c,d)}o = u_{01}c \cdot (t_{r1})_{(c,d)}o \cdot (u_{11}b)^{-1}$.

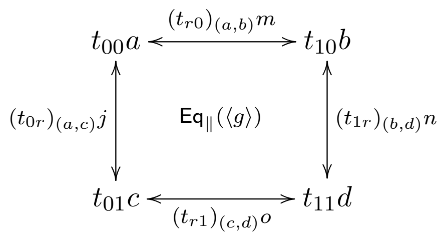
The components s and u act on propositions and then it is enough to prove that if $(m, j, o, n) \in Q(a, b, c, d)$ then their images



are related in $\langle f, g, l, h, p\rangle_2$, that is we need to show that

$$\mathsf{ap}(g)(u_{00}a \cdot (t_{r0})_{(a,b)}m \cdot (u_{10}b)^{-1}) \cdot (s_{1r})_{(b,d)}n = \\ p(s_{00}a) \cdot \mathsf{ap}(l)(s_{0r})_{(a,c)}j \cdot u_{01}c \cdot (t_{r1})_{(b,c)}o \cdot (u_{11}d)^{-1}.$$

Since t is a morphism between 2-relations, the following elements are related



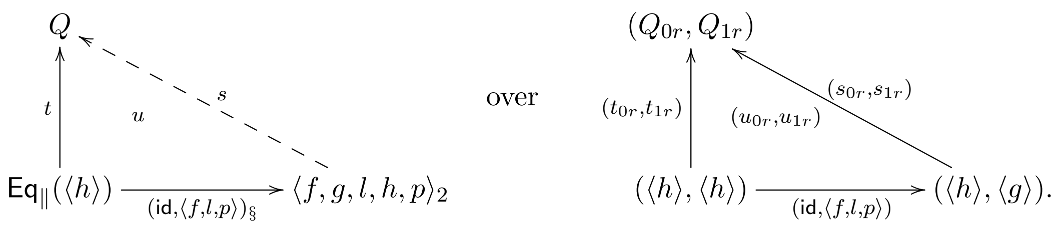
which means that $\mathsf{tr}((t_{r0})_{(a,b)}m, (t_{r1})_{(c,d)}o)(t_{0r})_{(a,c)}j = (t_{1r})_{(b,d)}n$. By using the proofs u_{ij} we can rewrite the previous equality as

$$(s_{1r})_{(b,d)}n = \\ \mathsf{tr}(u_{00}a \cdot (t_{r0})_{(a,b)}m \cdot (u_{10}b)^{-1}, u_{01}c \cdot (t_{r1})_{(b,c)}o \cdot (u_{11}d)^{-1})p(s_{00}a) \cdot \mathsf{ap}(l)(s_{0r})_{(a,c)}j$$

which, by definition of transport together with Lemma 8.21 and moving elements between the two sides, gives the equality that we wanted to prove.

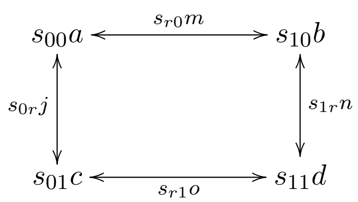
For the uniqueness note that most of the components are fixed by living over (s_{0r}, s_{1r}), s_{r0} and s_{r1} are unique up to 2-cells since they are defined using the cartesian property of the graph functor, and s is unique because it is a map between propositions.

It is left to prove that $(\mathsf{id}, f, \mathsf{id}, l, \mathsf{ap}(f), \mathsf{id}, \mathsf{ap}(l), \langle f, l, p\rangle)$, which we denote by $(\mathsf{id}, \langle f, l, p\rangle)_{\S}$, is opcartesian over $(\mathsf{id}, \langle f, l, p\rangle)$. In order to do that, consider the following diagram:

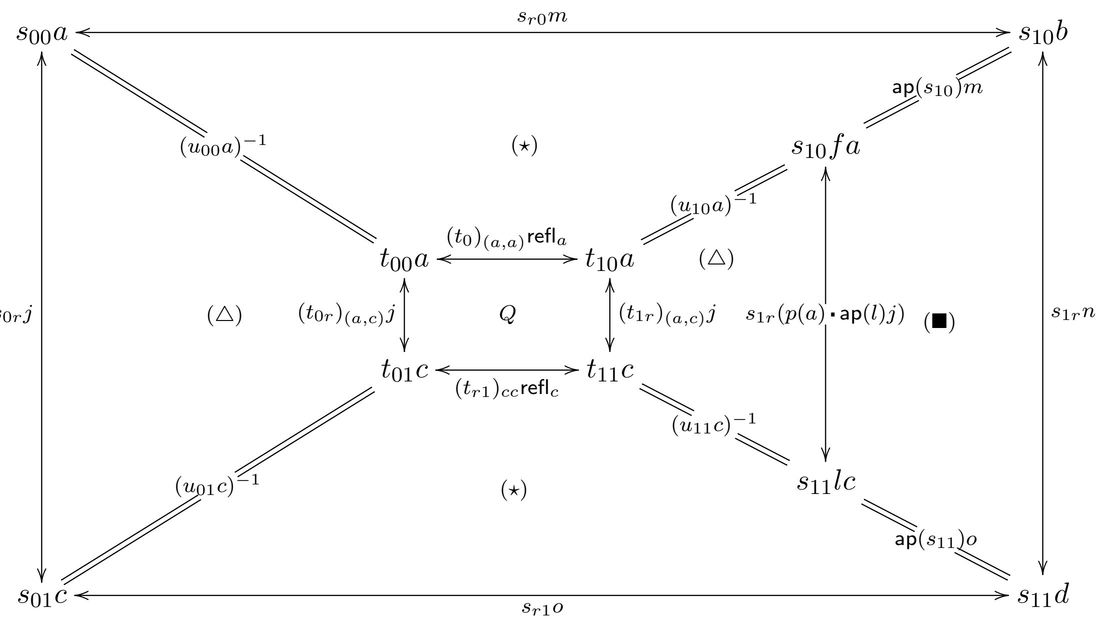


We define s_{r0}, s_{r1}, u_{r0} and u_{r1} using the universal property of the opcartesian morphism $(\mathsf{id}, f, \mathsf{ap}(f))\colon \mathsf{Eq}(A) \to \langle f \rangle$ with $t_{r0}\colon \mathsf{Eq}(A) \to Q_{r0}$ over the 2-cell (u_{00}, u_{10}) and the opcartesian morphism $(\mathsf{id}, l, \mathsf{ap}(l))\colon \mathsf{Eq}(C) \to \langle l \rangle$ with $t_{r1}\colon \mathsf{Eq}(C) \to Q_{r_1}$ over the 2-cell (u_{01}, u_{11}). In particular we recall the definitions of s_{r0} and s_{r1}: if $m \in \langle f \rangle(a, b)$, $(s_{r0})_{(a,b)}m = \mathsf{tr}((u_{00}\, a)^{-1}, (u_{10}\, b)^{-1} \cdot (\mathsf{ap}(s_{10})m))(t_{r0})_{(a,a)}\, \mathsf{refl}_a$ and if $o \in \langle l \rangle(c, d)$, then $(s_{r1})_{(c,d)}o = \mathsf{tr}((u_{01}\, c)^{-1}, (u_{11}\, d)^{-1} \cdot (\mathsf{ap}(s_{11})o))(t_{r1})_{(c,c)}\, \mathsf{refl}_c$.

Since s and u act at on propositions, in order to define them it is enough to prove that for every $(m, j, o, n) \in \langle f, g, l, h, p \rangle_2(a, b, c, d)$, their images

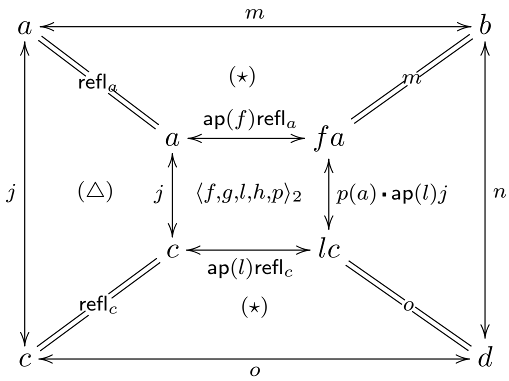


are related in Q. Consider the following picture:



where the middle square is the image of t applied to $(\mathsf{refl}_a, j, \mathsf{refl}_c, j) \in \mathsf{Eq}_{\|}(\langle h \rangle)(a, a, c, c)$, the two squares with $(\star)$ are given by definition of s_{r0} and s_{r1}, and the two squares with $(\triangle)$ are given by $u_{0r}\colon \mathsf{tr}(u_{00}, u_{01})s_{0r} = t_{0r}$ and $u_{1r}\colon \mathsf{tr}(u_{10}, u_{11})s_{1r} = t_{1r}$ up to moving the transport on the other side of the equality. The square $(\blacksquare)$ requires a little more work.

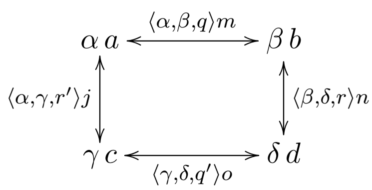
Consider the following picture



where the middle square is the application of $(\mathsf{id}, \langle f, l, p\rangle)_{\S}$ to $(\mathsf{refl}_a, j, \mathsf{refl}_c, j)$, the two squares with $(\star)$ are given by path composition, the $(\triangle)$ is trivially obtained by transporting along refl, and the remaining one derives from the fact that n is the unique element which completes the outer square. In fact, by definition of $\langle f, g, l, h, p\rangle_2$, the right most element should be equal to $(\mathsf{ap}(g)m)^{-1} \cdot p(a) \cdot \mathsf{ap}(l)j \cdot o$ which, using the fact that $(m, j, o, n) \in \langle f, g, l, h, p\rangle_2(a, b, c, d)$, is n. We then have $\mathsf{tr}(m, o)(p(a) \cdot \mathsf{ap}(l)j) = n$, and using $\mathsf{apd}(s_{1r})$ we obtain exactly $\mathsf{tr}(\mathsf{ap}(s_{10})m, \mathsf{ap}(s_{11})o)(s_{1r}(p(a) \cdot \mathsf{ap}(l)j)) = s_{1r}n$ which is $(\blacksquare)$.

Finally the uniqueness of the morphism follows like before: most of the components are fixed because they live over (s_{0r}, s_{1r}), the morphisms s_{r0} and s_{r1} are unique up to 2-cells because of the opcartesian property of the graph functor, and the remaining component is unique because it acts on propositions. □

The 2-graph map extends to a functor $\langle - \rangle_2 \colon (\text{1-Type}^{\to})^{\to} \to \mathbf{2Rel}$ by sending a morphism $(\alpha, \beta, \gamma, \delta, q, r, q', r', t) \colon (f, g, l, h, p) \to (f', g', l', h', p')$ in $(\text{1-Type}^{\to})^{\to}$ to the morphism $\langle \alpha, \beta, \gamma, \delta, q, r, q', r', t\rangle_2 = (\alpha, \beta, \gamma, \delta, \langle \alpha, \beta, q\rangle, \langle \alpha, \gamma, r'\rangle, \langle \gamma, \delta, q'\rangle, \langle \beta, \delta, r\rangle, \chi)$, where we have already seen how each component is defined but χ. Since the component χ acts on propositions, we only need to show that given $(m, j, o, n) \in \langle f, g, l, h, p\rangle_2(a, b, c, d)$, their images



are related in $\langle f', g', l', h', p'\rangle_2$. By unwinding the definition of $\langle - \rangle$ and $\langle - \rangle_2$ we need to show that

$$\begin{aligned}&\mathsf{ap}(g')(q(a) \cdot \mathsf{ap}(\beta)m) \cdot r(b) \cdot \mathsf{ap}(\delta)n = \\ &\qquad = p'(\alpha\, a) \cdot \mathsf{ap}(l')(r'(a) \cdot \mathsf{ap}(\gamma)j) \cdot q'(c) \cdot \mathsf{ap}(\delta)o.\end{aligned}$$

Using Lemma 8.12 and Lemma 8.9 we can rewrite the left hand side as

$$\begin{aligned}&\mathsf{ap}(g')(q(a) \cdot \mathsf{ap}(\beta)m) \cdot r(b) \cdot \mathsf{ap}(\delta)n = \\ &\qquad \mathsf{ap}(g')(q\, a) \cdot r\,(f\, a) \cdot \mathsf{ap}(\delta)((\mathsf{ap}(g)m) \cdot n)\end{aligned}$$

and the right hand side as

$$\begin{aligned}&p'(\alpha\, a) \cdot \mathsf{ap}(l')(r'(a) \cdot \mathsf{ap}(\gamma)j) \cdot q'(c) \cdot \mathsf{ap}(\delta)o = \\ &\qquad p'(\alpha\, a) \cdot \mathsf{ap}(l')(r'\, a) \cdot q'(h\, a) \cdot \mathsf{ap}(\delta)((\mathsf{ap}(l)j) \cdot o).\end{aligned}$$

Since $(m, j, o, n) \in \langle f, g, l, h, p\rangle_2(a, b, c, d)$, we have

$$\mathsf{ap}(\delta)((\mathsf{ap}(g)m) \cdot n) = \mathsf{ap}(\delta)(p(a) \cdot (\mathsf{ap}(l)j) \cdot o).$$

When then reduced to prove

$$\begin{aligned}&\mathsf{ap}(g')(q\, a) \cdot r\,(f\, a) \cdot \mathsf{ap}(\delta)(p(a)) \cdot \mathsf{ap}(\delta)((\mathsf{ap}(l)j) \cdot o) = \\ &\qquad = p'(\alpha\, a) \cdot \mathsf{ap}(l')(r'\, a) \cdot q'(h\, a) \cdot \mathsf{ap}(\delta)((\mathsf{ap}(l)j) \cdot o)\end{aligned}$$

and we see that the last part $\mathsf{ap}(\delta)((\mathsf{ap}(l)j) \cdot o)$ is the same, the first part of the left hand side is $q \star r \star p$, while the first part of the right hand side is $p' \star r' \star q'$, and we have t which proves that they are the same.

The morphism $\langle f, g, l, h, p\rangle_2$ completes the following diagram

$$\begin{array}{ccc}
\langle f, g, l, h, p\rangle_2 & \xrightarrow{(\langle f,l,p\rangle,\mathsf{id})^{\S}} & \mathsf{Eq}_{\|}(\langle g\rangle) \\
\Big\downarrow {\scriptstyle \langle \alpha,\beta,\gamma,\delta,q,r,q',r',t\rangle_2} & & \Big\downarrow {\scriptstyle \mathsf{Eq}_{\|}(\langle \beta,\delta,r\rangle)} \\
\langle f', g', l', h', p'\rangle_2 & \xrightarrow[(\langle f',l',p'\rangle,\mathsf{id})^{\S}]{} & \mathsf{Eq}_{\|}(\langle g'\rangle)
\end{array}$$

over

$$\begin{array}{ccc}
(\langle h\rangle, \langle g\rangle) & \xrightarrow{(\langle f,l,p\rangle,\mathsf{id})} & (\langle g\rangle, \langle g\rangle) \\
{\scriptstyle (\langle \alpha,\gamma,r'\rangle,\langle \beta,\delta,r\rangle)} \Big\downarrow & {\scriptstyle ((q^{-1},q'^{-1},\xi),(\mathsf{refl},\mathsf{refl},\mathsf{refl}))} & \Big\downarrow {\scriptstyle (\langle \beta,\delta,r\rangle,\langle \beta,\delta,r\rangle)} \\
(\langle h'\rangle, \langle g'\rangle) & \xrightarrow[(\langle f',l',p'\rangle,\mathsf{id})]{} & (\langle g'\rangle, \langle g'\rangle)
\end{array}$$

which represents the cartesian property of the morphism $(\langle f', l', p'\rangle, \mathsf{id})^{\S} : \langle f', g', l', h', p'\rangle_2 \to \mathsf{Eq}_{\|}(\langle g'\rangle)$. Every component in the diagram has already been defined but ξ. The component ξ should be a proof that $\mathsf{tr}(q^{-1}, q'^{-1})(\langle \beta, \delta, r\rangle \circ \langle f, l, p\rangle) = \langle f', l', p'\rangle \circ \langle \alpha, \gamma, r'\rangle$. By unwinding the previous equality and using the same techniques that we used so far, we obtain that we need to prove

$$\mathsf{ap}(g')(q\,a) \bullet r(f\,a) \bullet \mathsf{ap}(\delta)(p(a) \bullet \mathsf{ap}(l)w) \bullet q'^{-1}(c) = \\ p'(\alpha\,a) \bullet \mathsf{ap}(l')(r'(a) \bullet \mathsf{ap}(\gamma)w)$$

for every $w : h\,a = c$ in $\langle h\rangle$. By composing both sides with $q'(c)$ and using Lemma 8.9 we can rewrite the previous equality as

$$\mathsf{ap}(g')(q\,a) \bullet r(f\,a) \bullet \mathsf{ap}(\delta)(p\,a) \bullet \mathsf{ap}(\delta)(\mathsf{ap}(l)w) = \\ p'(\alpha\,a) \bullet \mathsf{ap}(l')(r'\,a) \bullet \mathsf{ap}(l')(\mathsf{ap}(\gamma)\,w) \bullet q'(c).$$

Using Lemma 8.12 we obtain the equality $\mathsf{ap}(l')(\mathsf{ap}(\gamma)w) \bullet q'(c) = q'(h\,a) \bullet \mathsf{ap}(\delta)(\mathsf{ap}(l)w)$ and then

$$\mathsf{ap}(g')(q\,a) \bullet r(f\,a) \bullet \mathsf{ap}(\delta)(p\,a) \bullet \mathsf{ap}(\delta)(\mathsf{ap}(l)w) = \\ p'(\alpha\,a) \bullet \mathsf{ap}(l')(r'\,a) \bullet q'(h\,a) \bullet \mathsf{ap}(\delta)(\mathsf{ap}(l)w)$$

and simplifying both sides finally we have

$$\mathsf{ap}(g')(q\,a) \cdot r(f\,a) \cdot \mathsf{ap}(\delta)(p\,a) =$$
$$p'(\alpha\,a) \cdot \mathsf{ap}(l')(r'\,a) \cdot q'(h\,a)$$

and t is exactly a proof of such equality. Finally it is straightforward to check that $\langle f, g, l, h, p\rangle_2$ satisfies the cartesian property using that most of the components are fixed, two of them are defined using the cartesian characterisation of the graph functor, and the last component is unique because it acts on propositions.

Lemma 10.15. The 2-graph functor $\langle - \rangle_2 \colon (\text{1-}\mathsf{Type}^{\to})^{\to} \to \mathbf{2Rel}$ is full and faithful.

Proof. The faithfulness can be derived componentwise from the faithfulness of the graph functor.

We have that $\mathsf{Eq}_{||}$ is full and faithful by Proposition 8.45, and we have both the cartesian and opcartesian properties for the 2-graph functor. For this reason we can use the same argument that we used in the proof of Lemma 3.11. □

This lemma can be used to prove a 2-relational version of the Graph Lemma:

Theorem 10.16 (2-relational Graph Lemma)**.** Let $F_2 : \mathbf{2Rel} \to \mathbf{2Rel}$ be functorial, and over (F_0, F_1) where F_0 and F_1 are as in Theorem 10.11. If $F_2(\mathsf{Eq}_{||}R) \cong \mathsf{Eq}_{||}(F_1 R)$ for all R, then for any (f, g, h, l, p) in $(\text{1-}\mathsf{Type}^{\to})^{\to}$, there are two morphisms

$$\phi_2 : \langle F_0 f, F_0 g, F_0 h, F_0 l, \mathsf{ap}(F_0)p\rangle_2 \to F_2\langle f, g, h, l, p\rangle_2$$

$$\psi_2 : F_2\langle f, g, h, l, p\rangle_2 \to \langle F_0 f, F_0 g, F_0 h, F_0 l, \mathsf{ap}(F_0)p\rangle$$

in $\mathbf{2Rel}$ over (ϕ, ϕ) and (ψ, ψ) from Theorem 10.11.

Proof. We can use the same argument that we used in the proof of Theorem 3.12. □

10.3 Two dimensional naturality

As we showed in Chapter 6, a very well known result about parametricity is that all System F terms of the right type are natural (see e.g. [Rey83, PA93]). We can extend this result to 2-naturality thanks to two-dimensional parametricity. We start by recalling the standard theorem that holds also with proof-irrelevant parametricity and show that we need a coherence condition for naturality proofs in the case of two-dimensional parametricity.

Theorem 10.17 (Parametric terms are natural)**.** Let $F(X)$ and $G(X)$ be functorial type expressions in the free type variable X in some type context Γ. Every term $\Gamma; - \vdash t : \forall X.F(X) \to G(X)$ gives rise to a natural transformation $[\![F]\!]_0 \to [\![G]\!]_0$, i.e. if $f : A \to B$ then there is $\mathsf{nat}(f) : \mathsf{Id}([\![G]\!]_0(f) \circ [\![t]\!]_0\, A, [\![t]\!]_0\, B \circ [\![F]\!]_0(f))$.

Proof. We construct $\mathsf{nat}(f)$ using the relational interpretation of t: by construction, $[\![t]\!]_1 \langle f \rangle : [\![F]\!]_1(\langle f \rangle) \to [\![G]\!]_1(\langle f \rangle)$, hence using Theorem 10.11,

$$\psi_{G,f} \circ [\![t]\!]_1 \langle f \rangle \circ \phi_{F,f} : (\Pi xy)\, \langle [\![F]\!]_0 f \rangle(x, y) \to \langle [\![G]\!]_0 f \rangle([\![t]\!]_0\, Ax, [\![t]\!]_0\, B\, y)$$

and since $\mathsf{refl} : \langle [\![F]\!]_0 f \rangle(a, ([\![F]\!]_0 f)a)$ for each $a : [\![F]\!]_0 A$, we can define

$$\mathsf{nat}(f) :\equiv \mathsf{ext}(\lambda a.\, (\psi_{G,f} \circ [\![t]\!]_1 \langle f \rangle \circ \phi_{F,f})\, a\, (([\![F]\!]_0 f)a)\, \mathsf{refl}). \qquad \square$$

In two-dimensional parametricity naturality alone is not enough anymore. In order to see that, consider the equivalence (see [Rey83]) $[\![A]\!] \cong [\![\forall X.(A \to X) \to X]\!]$ which holds for all types A. From a categorical perspective this as an instance of the Yoneda Lemma (see e.g. [ML98]) for the identity functor. The right hand side of the equation consists of natural transformations from the Hom functor $A \to X$ to the identity functor X.

In a more expressive theory such as (impredicative) Martin-Löf Type Theory with proof-irrelevant identity types and function extensionality, we can go further even without a relational interpretation, as pointed out by Steve Awodey (personal communication). Taking inspiration from the Yoneda Lemma once again, we can show

$$A \cong (\Sigma t : (\Pi X : \mathsf{Set})(A \to X) \to X)\, \mathsf{isNat}(t) \tag{10.18}$$

where

$$\mathsf{isNat}(t) :\equiv (\Pi X, Y : \mathsf{Set})(\Pi f : X \to Y)\mathsf{Id}_{(A\to X)\to Y}(f \circ t_X, t_Y \circ (f \circ \_))$$

expresses that t is a natural transformation.

The above isomorphism (10.18) relies on A being a set, i.e. that A has no non-trivial higher structure. If we instead consider $A : \mathsf{1\text{-}Type}$, the isomorphism (10.18) fails; instead we have

$$A \cong (\Sigma t : (\Pi X : \mathsf{1\text{-}Type})(A \to X) \to X)(\Sigma p : \mathsf{isNat}(t))\,\mathsf{isCoh}(p) \tag{10.19}$$

where

$$\mathsf{isCoh}(p) :\equiv (\Pi X, Y, Z : \mathsf{1\text{-}Type})(\Pi f : X \to Y)(\Pi g : Y \to Z)$$
$$(p\,X\,Z\,(g \circ f)) = (p\,Y\,Z\,g) \star (p\,X\,Y\,f)$$

expresses that the proof p is suitably coherent. Proof-irrelevant parametricity can not ensure this coherence conditionan extension of the usual naturality argument to proof-relevant parametricity guarantees this extra uniformity of the proof as well:

Theorem 10.20 (Naturality proofs are coherent)**.** Let F, G and t be as in Theorem 10.17. The proof $\mathsf{nat} : \mathsf{isNat}([\![t]\!]_0)$ is coherent, i.e. for all $f : A \to B$ and $g : B \to C$, there is a proof $\mathsf{coh}(f, g) : \mathsf{Id}(\mathsf{nat}(g \circ f), \mathsf{nat}(g) \star \mathsf{nat}(f))$.

Proof. We construct $\mathsf{coh}(f, g)$ using the 2-relational interpretation of t. By construction, $[\![t]\!]_2\langle f, g, g \circ f, \mathsf{id}, \mathsf{refl}\rangle_2 : [\![F]\!]_2\langle f, g, g \circ f, \mathsf{id}, \mathsf{refl}\rangle_2 \to [\![G]\!]_2\langle f, g, g \circ f, \mathsf{id}, \mathsf{refl}\rangle_2$, hence using Theorem 10.16,

$$\phi_2 \circ [\![t]\!]_2\langle f, g, g \circ f, \mathsf{id}, \mathsf{refl}\rangle_2 \circ \psi_2 :$$
$$(\Pi(\bar{x}, \bar{r}))\big(\bar{r} \in \langle F_0 f, F_0 g, F_0(g \circ f), \mathsf{id}, \mathsf{ap}(F_0)p\rangle_2 \bar{x}$$
$$\to ([\![t]\!]_1 \bar{r}) \in \langle G_0 f, G_0 g, G_0(g \circ f), \mathsf{id}, \mathsf{ap}(G_0)p\rangle_2([\![t]\!]_0 \bar{x})\big)$$

We define

$$\mathsf{coh}(f,g) := \mathsf{ext}(\lambda a.\,(\phi_2 \circ [\![t]\!]_2\langle f, g, g\circ f, \mathsf{id}, \mathsf{refl}\rangle_2 \circ \psi_2)\,(a, (F_0 f)a, F_0(g\circ f)a, a)\,\bar{\mathsf{refl}})$$

— this works, since ϕ_2 and ψ_2 are over (ϕ,ϕ) and (ψ,ψ) respectively, since $\mathsf{nat}(h)$ is defined to be $(\phi \circ [\![t]\!]_1 \circ \psi)\mathsf{refl}$, and since the 2-relation $\langle G_0 f, G_0 g, G_0(g\circ f), \mathsf{id}, \mathsf{ap}(G_0)p\rangle_2$ exactly says that pasting the two diagrams produces the third in this case. □

Finally we can prove that we have 2-naturality:

Theorem 10.21 (2-naturality)**.** Let $F(X)$, $G(X)$ be 2-functorial type expressions in the free type variable X in some context Γ. Every term $\Gamma; - \vdash t : \forall X.F(X) \to G(X)$ gives rise to a 2-natural transformation $[\![F]\!]_0 \Rightarrow [\![G]\!]_0$, i.e. if $f, g\colon A \to B$ and $\alpha\colon f \Rightarrow g$, then there is a proof

$$(\mathsf{nat}(f), \mathsf{nat}(g), \mathsf{nat}_2(f,g))\colon \mathsf{refl}_{[\![t]\!]_0\,B} * F(\alpha) = G(\alpha) * \mathsf{refl}_{[\![t]\!]_0\,A}.$$

Proof. Note that since $\mathsf{refl}_{[\![t]\!]_0 B} * F(\alpha)\colon [\![t]\!]_0 B \circ F(f) = [\![t]\!]_0 B \circ F(g)$ and $G(\alpha) * \mathsf{refl}_{[\![t]\!]_0 A}\colon G(f) \circ [\![t]\!]_0 A = G(g) \circ [\![t]\!]_0 A$, we need to use $\mathsf{nat}(f)$ and $\mathsf{nat}(g)$ in order to identify the boundaries of the two 2-cells. For this reason what we prove is that $\mathsf{tr}(\mathsf{nat}(f), \mathsf{nat}(g))(G(\alpha) * \mathsf{refl}_{[\![t]\!]_0 A}) = \mathsf{refl}_{[\![t]\!]_0 B} * F(\alpha)$, which, by unwinding the definition of transport, is

$$\mathsf{nat}(f)^{-1} \cdot (G(\alpha) * \mathsf{refl}_{[\![t]\!]_0 A}) \cdot \mathsf{nat}(g) = \mathsf{refl}_{[\![t]\!]_0 B} * F(\alpha).$$

We can think to the proofs $G(\alpha)$, $F(\alpha)$, $\mathsf{refl}_{[\![t]\!]_0 A}$ and $\mathsf{refl}_{[\![t]\!]_0 B}$ as squares by adding identities: for example $G(\alpha)\colon \mathsf{id} \circ f = g \circ \mathsf{id}$. In this way we can use Lemma 10.12 and by definition of $\star$, we can rewrite the previous equality as

$$\mathsf{nat}(f)^{-1} \star G(\alpha) \star \mathsf{refl}_{[\![t]\!]_0 A} \star \mathsf{nat}(g) = \mathsf{refl}_{[\![t]\!]_0 B} \star F(\alpha)$$

and by composing both the sides with $\mathsf{nat}(f)$ we find that the lemma holds if we can prove

$$G(\alpha) \star \mathsf{refl}_{[\![t]\!]_0 A} \star \mathsf{nat}(g) = \mathsf{nat}(f) \star \mathsf{refl}_{[\![t]\!]_0 B} \star F(\alpha).$$

In order to show that the above equality holds, we use the same technique that we used in the proof of Theorem 10.20, but in this case we consider the morphism $[\![t]\!]_2\langle f, \mathsf{id}, g, \mathsf{id}, \alpha\rangle_2 : [\![F]\!]_2\langle f, \mathsf{id}, g, \mathsf{id}, \alpha\rangle_2 \to [\![G]\!]_2\langle f, \mathsf{id}, g, \mathsf{id}, \alpha\rangle_2$. $\square$

Chapter 11

Conclusion and future work

This thesis is just the first step towards the study of higher dimensional parametricity. The work presented in Chapters 5 and 7 allowed us to grab the essence of parametricity and present it in a suitable way for the generalisation to higher dimensions. The results presented in those two chapters suggest that, in order to study higher dimensional parametricity, we should leave the world of faithful fibrations and for this reason we worked with proof-relevant relations in Chapter 9.

The first difficulty we met was in the choice of the shape of higher dimensional relations. The cubical definition of higher dimensional relations in Grandis [Gra09] worked well for us. It might be interesting to study two-dimensional parametricity using other different shapes for 2-relations: globular (see [Lei04]), simplicial (see [Rie08] or [Fri08]), etc. We have pointed out that our first approach using globular relations does not work, but this might be due to the fact that we related proofs p and p' defined over different elements, i.e. $p \in R(a, b)$ and $p' \in R(a', b')$. Nonetheless, this approach might work defining globular 2-relations only for proofs with fixed boundaries. Papers from topological algebra show that there is a link between the different approaches (see for example [AABS02]) which bring different advantages (see discussions [of09] and [of17], or the comparision of the simplicial and cubical sets models of Martin-Löf Type Theory in Bezem, Coquand and Huber [BCH14]).

Another observation on Chapter 9 is that we presented a concrete model and not a general framework. We tried to introduce the material in such a way that it exposes the categorical structure underlying it. We expect that it is possible to prove that the model we defined forms a $\lambda 2$-fibration p, and generalises the work of Chapter 5. In particular, the objects in the total category of p should be fibred functors on three layers corresponding to **2Rel**, **PrRel** and 1-Type. This is work in progress and there are two big questions which immediately arise. First we need to find the right definition of fibrations for proof-relevant relations. In this thesis we saw 1-Type, **PrRel** and **2Rel** as 2-categories and defined proof-relevant cartesian morphisms. This definition worked well for the concrete case that we studied, but for the generalisation we might need a stricter definition. For example it is possible to strengthen the notion of proof-relevant cartesian morphisms by asking uniqueness also for 2-cells. There are papers on parametricity which do not require the opfibrational structure, but only some opcartesian liftings (see [HRR14]). In this thesis we took a similar approach by asking only for the needed cartesian and opcartesian liftings. In Chapters 5, 6 and 7 we showed clear advantages of having both the fibrational and opfibrational structure, for this reason it would be nice to have them.

There is a second question on the nature of 2-relations. If we want to keep track of all the four relations, considering **2Rel** over $\mathbf{PrRel}^4$ is the most natural choice. On the other hand we can think of a 2-relation as a relation between two other relations, i.e. considering **2Rel** over $\mathbf{PrRel}^2$. In this case we must choose an orientation: we can display relations vertically and 2-relations horizontally or vice-versa. One would expect the two orientations to be equivalent.

These two questions assume more relevance if we think of the natural evolution of two-dimensional parametricity into ω-parametricity, i.e. infinite-dimensional parametricity. In fact, in the infinite-dimensional case where the complexity increases exponentially, it is important to use the best approach. We have some work in progress on logical relations for ω-parametricity which assume that, also if relations are proof-relevant, the equality relations are proof-irrelevant at each level. We use a cubical approach which extends the work on two-dimensional parametricity presented in Chapter 9 and we managed to prove the Identity Extension Lemma at each level. We expect that, using techniques similar to the ones used in Chapter 9, the Identity Extension Lemma generalises to the case where equalities are proof-relevant. The next step will be to prove that the Abstraction Theorem holds and that we have a model of System F.

Another interesting point to study is the generalisation of the results of Chapter 7 to two-dimensional parametricity. Since the world of parametricity is relational, the universal property was based on cones living at two different levels: one at the set level and one at the relational level. We then expect that the generalisation uses three different levels of cones: one for sets, one for relations and one for 2-relations.

Finally, since this work is only the first step, there is plenty of work to do in showing the behaviour of two-dimensional parametricity — the applications presented in Chapter 10 are only the tip of the iceberg.

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