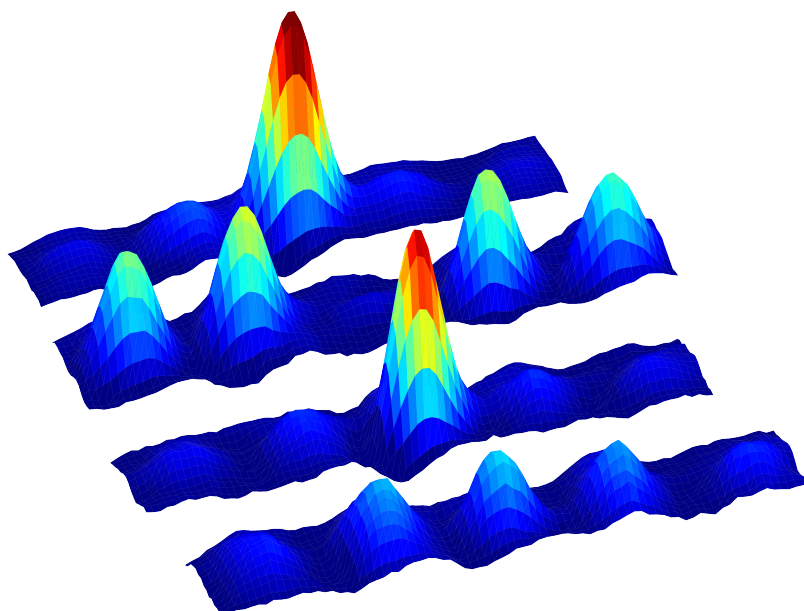


Matter-Wave Solitons and Floquet-Driven Instabilities with Ultracold Atoms in Optical Lattices



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This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

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Abstract

This thesis reports two groups of experiments using ultracold caesium atoms which are trapped in an optical lattice potential.

We first demonstrate the existence of matter-wave lattice solitons and characterise their properties. Using an optical accordion lattice with tunable spacing, we prepare atomic wavepackets across a well-defined number of lattice sites. By rapidly changing atomic interactions from repulsive to attractive, the wavepacket is found to remain localised for hundreds of milliseconds. These solitons can be classified as single-site or multisite solitons, which span one or many sites, respectively. We probe the nature of their collapse by observing the wavepacket in time, finding that for strong interactions the local density increases, leading to dynamical collapse and three-body loss.

In a second group of experiments, we investigate the role of Floquet heating in periodically driven systems. Floquet engineering has become a key technique to control quantum simulation experiments. By periodically shifting the lattice sites, variables such as the inter-site tunnelling energy can be configured. However, this driving force excites the system and induces heating and instabilities, limiting experimental times. Parametric instabilities occur when the driving frequency matches the energy of an existing phonon mode, while negative-mass instabilities occur whenever the wavepacket enters regions of the Brillouin zone associated with negative effective mass. We probe these instabilities in a range of systems including tilted, untilted, driven, and undriven systems. We measure the fraction of excited atoms for a range of parameters and create stability diagrams, finding stable parameter regimes associated with zero growth.

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Chapter 1

Introduction

Nonlinearity is a ubiquitous phenomenon in nature, appearing across many disciplines such as the physical sciences, economics and medicine [1]. It describes systems in which the output is not directly proportional to the input. As a result, such systems are often difficult to predict, and in some cases may exhibit a strong sensitivity to initial conditions, leading to complex or chaotic dynamics. The underlying cause of nonlinearity varies. For example, ocean waves can be nonlinear due to a variation in the water depth [2], while financial markets can show nonlinearity due to investor response to market shocks [3].

In condensed-matter physics, the physics of ions trapped in crystalline lattice structures is inherently nonlinear due to the interactions between the ions [4]. These lattice structures are, however, difficult to experimentally investigate due to their complexity. In their place, analogues have been utilised to facilitate investigations, such as waveguide arrays [5] or superconducting circuits [6]. One notable example is that of ultracold atoms in lattice potentials created by light, known as optical lattices [7]. In these systems, interactions arise from low-energy atomic collisions, which can be precisely controlled experimentally using Feshbach resonances [8]. The high degree of tunability of both interaction strengths and lattice parameters has made optical lattices one of the most versatile and widely used platforms for quantum simulation [7]. In addition, ultracold atoms provide a range of techniques for the preparation and detection of quantum states. For example, the momentum distribution can be measured using time-of-flight imaging, in which the trapping potentials are switched off and the atoms expand and interfere during free evolution [9]. Alternatively, the atoms' spatial distribution can be measured by taking *in situ* images, with single-atom resolution being possible through quantum gas microscope setups [10]. Some of the landmark results in quantum simulation over the last quarter century have involved optical lattices in some way, including the observation of the superfluid to Mott insulator transition [11], access to the strongly interacting Tonks-Girardeau regime [12] and the creation of an antiferromagnet [13].

Solitons

A paradigmatic manifestation of nonlinearity is the formation of dispersionless solitary waves, known as solitons [14]. They form when the dispersion of a wave is cancelled

by a nonlinear term in the system, which in the case of ultracold gases is the aforementioned atomic interactions. Solitons were first documented in water waves in a canal by John Scott Russell in 1834 [15], and have since been observed in plasmas [16], nonlinear optical media [17, 18], and biological cells [19]. In optical fibres, solitons are used in the telecommunications industry [14]. In ultracold atoms, soliton experiments have constituted an active area of research since their first observation around the start of the century [20, 21]. They have been used to create matter-wave interferometers [22], show quantum reflection off of a barrier [23], and show Bloch oscillations in the absence of a lattice [24].

Over the years, new types of solitons have been theorised and demonstrated, these include (but are not limited to) dark solitons [25–27], which are characterised by a density dip, vortex solitons [28], which exist in two- and three-dimensions with non-zero vorticity, and magnetic solitons [29, 30], which arise due to magnetic interactions between spinor condensates.

The topics of solitons and lattice physics overlap with the investigation of lattice solitons (also known as discrete solitons) [31]. Studied theoretically since the 1970s after the seminal paper by Davydov and Kislukha [32], lattice solitons are predicted to allow for rare access to condensed matter phenomena [33–36], and enable studies of topological insulators [37], such as in the Su-Schrieffer-Heeger model [38, 39]. Lattice solitons have been observed in optical waveguide arrays [40], however studies on their formation in matter has historically been sparse. In this thesis, the first observation of matter-wave lattice solitons is reported using Bose-Einstein condensates as a medium.

Floquet induced instabilities

Time-periodic driving of ultracold lattice systems has become one of the central tools in the quantum toolbox over the last two decades [41]. Periodically driven systems were discussed by Gaston Floquet in 1883 [42], whereby he showed that the system is described by a time-independent “Floquet” Hamiltonian when observed at stroboscopic time steps in multiples of the driving period. The ability to fine-tune system parameters in this Floquet Hamiltonian by changing either the strength or frequency of the drive provides an attractive technique for experimentalists. Floquet engineering in ultracold systems has been used to drive the quantum phase transitions [43], the creation of artificial gauge fields [44, 45] and the creation of topological band structures [37].

However, the presence of nonlinear interactions in Floquet driven systems leads to the growth of phonon modes on experimentally relevant timescales, which manifests as heating and a destruction of coherence in the system [41]. Common strategies to mitigate these instabilities include restricting the driving amplitude to avoid resonant excitation of existing modes, or operating at sufficiently high frequencies such that the system cannot respond to the drive. Both approaches, however, constrain the accessible parameter space, while reducing the interaction strength limits access to strongly interacting regimes of interest. A detailed understanding of the mechanisms underlying these instabilities is developed in this thesis resulting in more effective strategies to suppress them. This includes the demonstration of separate types of instabilities which often exist in the same parameter regimes, and further complicate the system. The

stability of the system is measured for a range of parameters, allowing for the creation of stability diagrams which show stable and unstable regions.

Outline

This thesis is structured as follows: Chapter 2 introduces the fundamental concepts used in the rest of the thesis, such as Bose-Einstein condensates in optical lattice potentials and the associated dynamics. Chapter 3 briefly describes the experimental setup and discusses the different steps used to create a Bose-Einstein condensate. Some general experimental techniques are discussed in Chapter 4, such as calibration of the optical lattice depth and magnetic field strength, and setup of an optical accordion lattice with associated use cases. Chapter 5 reports the experimental observation of matter-wave solitons in an optical lattice, which follows a general introduction to solitons. The last three chapters focus on the effect of Floquet-driven instabilities in ultracold atom systems. Chapter 6 provides theoretical background on such instabilities in different systems, Chapter 7 reports experimental observation and investigation of the instabilities in an optical lattice, and Chapter 8 focusses on the case of a tilted lattice. A conclusion and outlook to future extensions of the results reported in this thesis is provided in Chapter 9.

List of publications

- *Suppressing Parametric Instabilities in Driven Bosonic Lattices through Multi-tone Control.*
R. Cruickshank, S. Lellouch, M. Bukov, E. Demler, N. Goldman, E. Haller, arXiv:2606.31667 (2026).
- *Tunable optical lattices for the creation of matter-wave lattice solitons.*
R. Cruickshank, A. La Rooij, E. F. Kerr, T. Hilker, S. Kuhr, E. Haller, Opt. Express **34**, 623-637 (2026).
- *Experimental Observation of Single- and Multisite Matter-Wave Solitons in an Optical Accordion Lattice.*
R. Cruickshank, F. Lorenzi, A. La Rooij, E. F. Kerr, T. Hilker, S. Kuhr, L. Salasnich, E. Haller, Phys. Rev. Lett. **135**, 263404 (2025).
- *Phonon excitations of Floquet-driven superfluids in a tilted optical lattice.*
R. Cruickshank, A. Di Carli, M. Mitchell, A. La Rooij, S. Kuhr, C. Creffield, E. Haller, Phys. Rev. Res. **6**, 023323 (2024).
- *Instabilities of interacting matter waves in optical lattices with Floquet driving.*
A. Di Carli, R. Cruickshank, M. Mitchell, A. La Rooij, S. Kuhr, C. Creffield, E. Haller, Phys. Rev. Res. **5**, 033024 (2023).

Chapter 2

Theory of Bose-Einstein condensates in optical potentials

This chapter briefly reviews the fundamental concepts of Bose-Einstein condensates in optical potentials, as well as the key parameters used in subsequent chapters discussing the experimental results. Section 2.1 introduces the principles of trapping neutral atoms using optical potentials and the basic properties of Gaussian laser beams. Section 2.2 discusses optical lattice potentials, which are artificial crystals created with light, before going on to discuss the basic concepts of lattice physics.

Section 2.3 introduces the phenomena of Bose-Einstein condensation and the associated ground states of trapped condensates. This section begins with the non-interacting limit, corresponding to an ideal Bose gas, and is subsequently extended to include atomic interactions via s -wave scattering. Interacting condensates are effectively described by the Gross-Pitaevskii equation, which is discussed in Sec. 2.3.2. The chapter concludes with a discussion about the dynamics of superfluids in optical lattice potentials when subjected to external forces, which can give rise to phenomena such as Bloch oscillations.

2.1 Optical dipole potentials

Since the first demonstration in the 1980s [46–48], controlling atoms using light potentials has become one of the most common tools used in the quantum simulation toolbox. Together with magnetic and radio-frequency trapping [49], optical trapping forms one of the main methods to confine and control atoms. By tailoring the properties of the light field and employing lenses, mirrors, and other optomechanical components, highly controllable trapping environments can be realised.

For neutral atoms, potentials using light fields are created from the effect of the electric field $\mathbf{E}$ on the atom, creating an atomic dipole moment $\mathbf{d} = \alpha(\omega)\mathbf{E}$, which oscillates at the electric light field's frequency ω and depends on the atomic polarisability $\alpha(\omega)$ [50, 51]. The resulting trapping potential is

$$V_{\text{dip}} = -\frac{1}{2}\langle \mathbf{d}\mathbf{E} \rangle = -\frac{1}{2\epsilon_0 c}\text{Re}(\alpha)I, \quad (2.1)$$

where $\langle \cdot \rangle$ denotes the time average and $I = 2\epsilon_0 c|E|^2$ is the intensity of the light, allow-

2.1. Optical dipole potentials

ing for precise control over the trapping potential. The oscillating dipole affects the internal states of the atom while in the trap, allowing it to absorb and re-emit light at a rate

$$\Gamma_{\text{sc}} = \frac{1}{\hbar \epsilon_0 c} \text{Im}(\alpha) I. \quad (2.2)$$

Optical potentials are typically generated using light that is far detuned from internal atomic resonances in order to suppress unwanted excitations.

For detunings $|\Delta| \ll \omega_0$, where ω_0 is the resonant frequency, Eq. (2.1) and Eq. (2.2) can be approximated using the rotating-wave approximation [51], giving

$$V_{\text{dip}} = \frac{3\pi c^2}{2\omega_0^3} \frac{\Gamma}{\Delta} I, \quad (2.3)$$

$$\Gamma_{\text{sc}} = \frac{3\pi c^2}{2\hbar \omega_0^3} \left(\frac{\Gamma}{\Delta} \right)^2 I. \quad (2.4)$$

In the experiment, where caesium is the atomic species used, the relevant transitions for trapping are the D_1 ($6^2S_{1/2} \rightarrow 6^2P_{1/2}$) and D_2 ($6^2S_{1/2} \rightarrow 6^2P_{3/2}$) transitions with wavelengths of 894.59 nm and 852.35 nm respectively. Since the photon scattering rate scales as $\Gamma_{\text{sc}} \propto 1/\Delta^2$, it is advantageous to operate at large detunings in order to suppress scattering and realise a mostly conservative trapping potential. However, such large detunings require higher light intensities to achieve a given trap depth.

2.1.1 Trapping multi-level atoms

The above equations hold for a simple two-level atom, however in caesium the multiple energy levels further complicate the trapping equations. Taking the detuning Δ with respect to the midpoint between the D_1 and D_2 transitions, the trap depth can then be described by [51]

$$V_{\text{dip}}(\mathbf{r}) = \frac{3\pi c^2}{2\omega_0^3} \frac{\Gamma}{\Delta} \left(1 + \frac{1}{3} \mathcal{P} g_F m_F \frac{\Delta'_{\text{FS}}}{\Delta} \right) I(\mathbf{r}), \quad (2.5)$$

where $\mathcal{P}$ describes the polarisation of the light and is 0 for linear polarised light and ± 1 for circularly polarised light, g_F is the Landé factor, m_F the magnetic quantum number, and Δ'_{FS} is the fine structure splitting. If the laser beam is detuned with $\Delta < 0$ (red detuned) the trap is attractive and atoms sit at the point of maximum intensity. If $\Delta > 0$ (blue detuned) the atoms are repelled by intensity maxima.

2.1.2 Gaussian beams and dipole traps

In Eq. (2.3) the spatial distribution of the potential is given by the intensity profile of the laser beam. For the experiments discussed in this thesis, the beams are well described by a Gaussian intensity profile

$$I(r, z) = \frac{2P}{\pi w^2(z)} \exp \left[-2 \frac{r^2}{w^2(z)} \right], \quad (2.6)$$

2.2. Optical lattice potentials and lattice physics

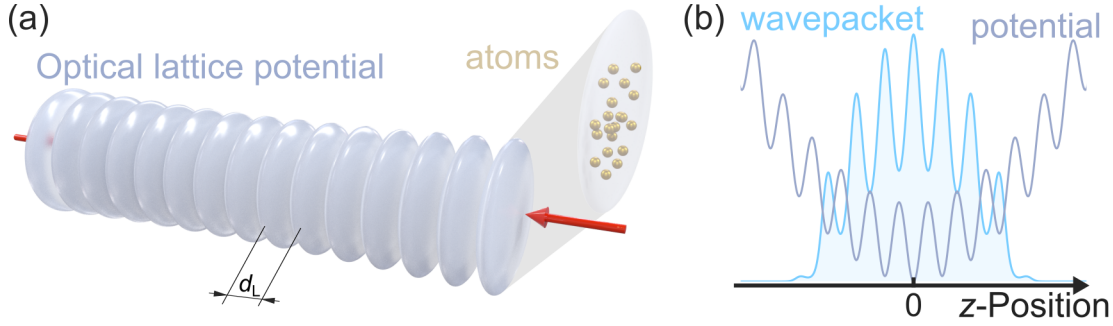


Figure 2.1: Visualisation of optical lattice potentials. (a) 1D optical lattice formed by the interference of two counter-propagating laser beams (red arrows). In this configuration the lattice sites are referred to “pancakes” in which the atoms are trapped (inset). (b) Potential energy of lattice as a function of the axial distance z (grey). Due to the finite width of the laser beams and the change in intensity of the beams, the lattice potential has an additional harmonic confinement, producing an energy offset that increases with $|z|$. The density profile (blue) of the ground state was calculated for 50,000 atoms and a scattering length of $a_s = 80 a_0$.

where P is the beam power, and the beam width is given by $w^2(z) = w_0^2 (1 + z^2/z_R^2)$ with waist w_0 and Rayleigh length $z_R = \pi w_0^2/\lambda$. This intensity distribution creates a dipole trap that is radially symmetric and can be approximated at the centre as a harmonic potential with frequencies in the radial and longitudinal directions respectively

$$\omega_r = \frac{1}{w_0} \sqrt{\frac{4V_0}{m}}, \quad \omega_z = \frac{1}{z_R} \sqrt{\frac{2V_0}{m}}, \quad (2.7)$$

where m is the atomic mass, and V_0 is the lattice depth. Typical values for experiments in this thesis are $\omega_r/(2\pi) < 100 \text{ Hz}$ for the radial trap frequency and $\omega_z/(2\pi) < 1 \text{ Hz}$ for the longitudinal frequency (assuming $w_0 \approx 50 \mu\text{m}$ and $P < 50 \text{ mW}$).

For three-dimensional confinement, a second laser beam is introduced perpendicular to the first in the $x-z$ plane. The resulting crossed-beam dipole trap confines the atoms along all directions with a trapping potential

$$V(r) = \frac{1}{2} m (\omega_x^2 x^2 + \omega_y^2 y^2 + \omega_z^2 z^2), \quad (2.8)$$

where the trap frequencies $\omega_x, \omega_y, \omega_z$ depend on the radial and longitudinal frequencies of both beams.

2.2 Optical lattice potentials and lattice physics

In solid-state systems, crystal lattice potentials arise from Coulomb interactions between ions arranged in a periodic structure. Experimental investigation of these systems is often complicated by lattice vibrations, impurities and structural defects, which can obscure or modify the underlying physics and limit control over system parameters [4]. In contrast, *optical* lattice potentials in ultracold atom experiments provide a

2.2. Optical lattice potentials and lattice physics

highly controllable macroscopic analogue, making them an ideal platform for quantum simulation.

Optical lattices are potentials (Sec. 2.1) formed by interfering laser beams. The simplest implementation of an optical lattice is a standing-wave created by retro-reflecting a laser beam with a mirror. In the case of 1D lattice potentials two interfering laser beams create the potential

$$V_{\text{lat}}(z) = V_0 \cos^2 \left(\frac{\pi}{d_L} z \right), \quad (2.9)$$

where the lattice spacing is defined as $d_L = \lambda/2$. The depth of the lattice, V_0 , can be calculated as

$$V_0 = \frac{3\pi c^2 \Gamma}{2\omega_0^3} \frac{2P}{\Delta \pi w_0^2}. \quad (2.10)$$

It is useful to measure the lattice depth relative to the recoil energy of an atom

$$E_r = \frac{(\hbar\pi)^2}{2d_L^2 m}, \quad (2.11)$$

as physical effects usually occur on timescales close to $\hbar/E_r$. Depths are typically on the order of $< 10 E_r$ for the experiments discussed.

As the laser beams have a changing Gaussian width, their intensity varies across the beam profile. This leads to an additional, slowly varying harmonic confinement superimposed on the periodic lattice potential, resulting in a small energy offset between neighbouring lattice sites [Fig. 2.1(b)]. In this geometry, the atomic distribution is often referred to as “pancakes” due to their strong axial confinement and radial extent, Fig. 2.1(a). In lattice experiments a useful energy scale to calculate is the trapping frequency in a single lattice well

$$\omega_{\text{lat}} = \frac{1}{m} \sqrt{V_0 \hbar^2 k_L^4}, \quad (2.12)$$

where $k_L = \pi/d_L$ is the lattice momentum. Even for relatively shallow lattice depths, ω_{lat} is on the order of kHz and is therefore orders of magnitude larger than the global trapping frequencies discussed above.

2.2.1 Band structure

The Hamiltonian describing N_{atom} interacting atoms in an optical lattice is given by [7]

$$H = \int \phi^\dagger(z) \left[-\frac{\hbar^2}{2m} \nabla^2 + V_{\text{ext}} \right] \phi(z) d^3x + \frac{1}{2} \frac{4\pi a_s \hbar^2}{m} \int \phi^\dagger(z) \phi^\dagger(z) \phi(z) \phi(z) d^3x, \quad (2.13)$$

where the first term describes the inter-site tunnelling [see Eq. (2.17) below] and the second describes the interactions between atoms [see Eq. (2.19) and Sec. 2.3.2]. Using Bloch’s theorem [52], the eigenstates $\phi(z)$ of Eq. (2.13) can be written as a plane wave e^{ikz} multiplied by a periodic function

$$\phi_{n,k}(z) = e^{ikz} u_{n,k}(z). \quad (2.14)$$

2.2. Optical lattice potentials and lattice physics

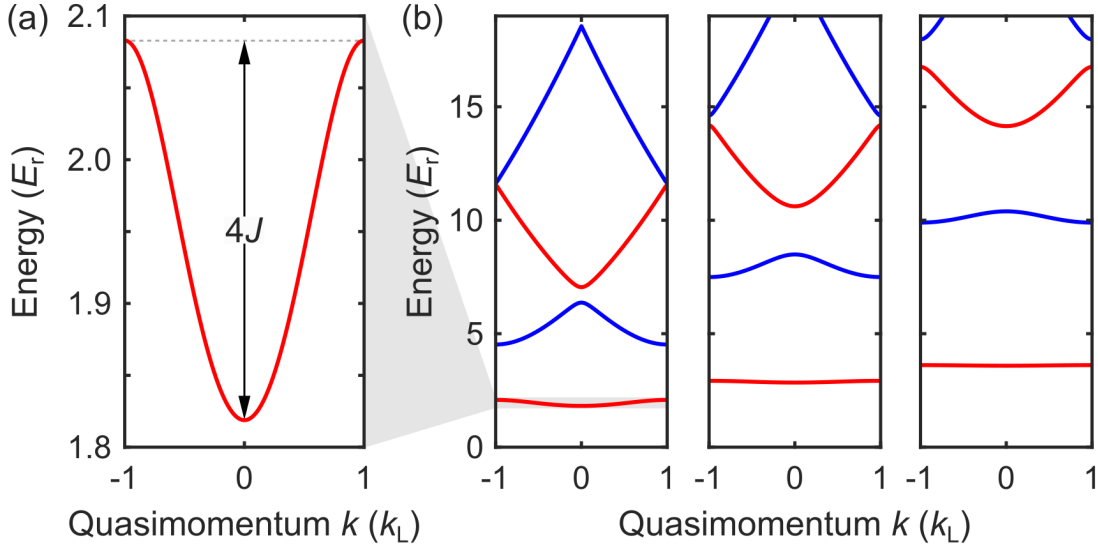


Figure 2.2: Band structure of one dimensional lattices. (a) First Brillouin zone, the lowest band takes the form of an inverted cos function of amplitude $4J$, where J is the tunnelling energy which is formally defined below. (b) The band structure of the lattice potential for various lattice depths, $V_0 = 5, 10, 15 E_r$, showing the first four bands. As the depth increases the gaps between bands grow and the bands themselves become flatter.

As the lattice is periodic, the eigenstates are also periodic $\phi_{n,k}(z + d_L) = \phi_{n,k}(z)$. The quantity $\hbar k$ is a momentum-like parameter that characterises how the phase evolves from one lattice site to the next. The wavenumber k characterises the phase evolution between neighbouring lattice sites, and is only uniquely within a Brillouin zone, $k \in [-\hbar k_L, \hbar k_L)$ due to the translational symmetry of the lattice, the corresponding momentum $\hbar k$ is called quasimomentum.

The eigenvalues of the Bloch functions Eq. (2.14) can be grouped into discrete bands in quasimomentum space [Fig. 2.2(a)]. For deep lattices, the lowest band takes the form

$$E_1(k) = -2J \cos(d_L k), \quad (2.15)$$

where J is the tunnelling matrix element, i.e. the energy associated with an atom hopping between neighbouring lattice sites [formally defined below in Eq. (2.17)]. There exists an infinite set of higher-order bands, forming the full band structure seen in Fig. 2.2(b), which are labelled by the index n in Eq. (2.14). For shallow lattices, the curvature of the bands is large and the edges of adjacent bands approach one another, as $V_0 \rightarrow 0$ the lowest band can be increasingly approximated as a parabola. Conversely as the lattice depth increases the bands become flatter, reflecting an increase in the effective mass and a corresponding suppression of tunnelling ($J \rightarrow 0$). As the depth increases, the gaps between the bands become larger and the system can eventually be defined as an array of isolated harmonic traps [rightmost plot in Fig. 2.2(b)].

For finite size systems the energy bands are not continuous but are instead made up of a discrete number of momenta states. The description in momentum space is related to real space by a Fourier transform, resulting in an equal number of momenta states as lattice sites present in real space.

2.2. Optical lattice potentials and lattice physics

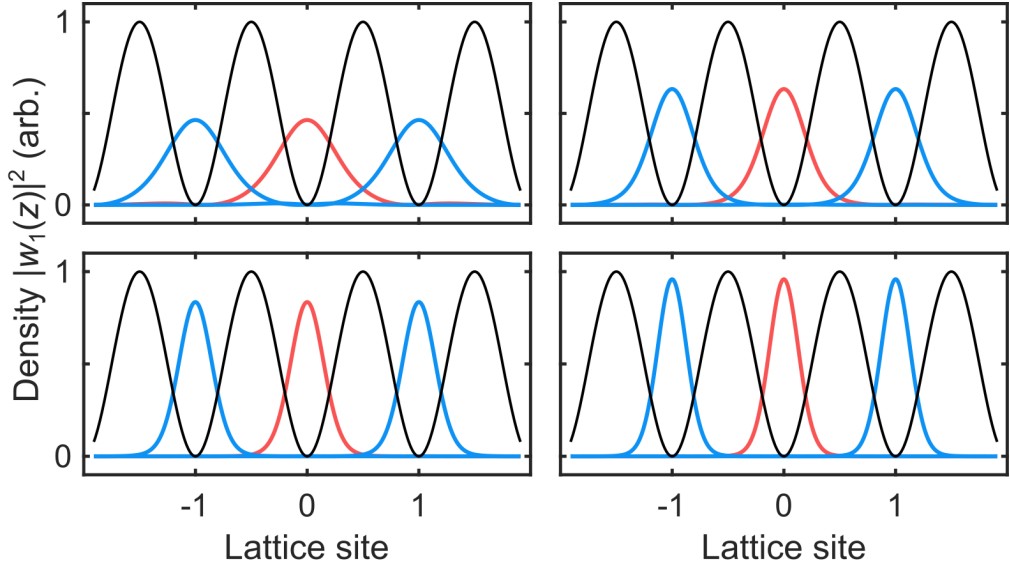


Figure 2.3: Wannier functions as a function of lattice depth. Wannier functions in the first band $w_1(z)$ for lattice depths of $V_0 = 1, 3, 11, 7 E_r$ (clockwise from top left). As the lattice depth increases a negligible portion of the wavefunction is present beyond the lattice site, known as the tight-binding regime. Colours differentiate Wannier functions at different sites.

2.2.2 Wannier functions

The Bloch functions, Eq. (2.14), describe eigenstates that are localised in momentum space but are delocalised in position space. A convenient description for wave packets that are localised in position space is obtained by introducing a second set of functions, known as Wannier functions. These are found by taking a discrete Fourier transform of the Bloch waves and for a particle in the n th band of the lattice are defined as

$$w_n(z - z_l) = \frac{1}{\sqrt{N_{\text{lat}}}} \sum_k e^{-ikz_l} \phi_{n,k}(z), \quad (2.16)$$

where $z_l = ld_L$ is the position at the l th lattice site, N_{lat} is the number of lattice sites, $\phi_{n,k}(z)$ is a Bloch wave and the sum adds all available states in the band. A Wannier function $w_n(z - z_l)$ is localised at the centre of site l and functions from different sites are orthogonal, meaning $\langle w_l | w_m \rangle = 0$.

Wannier functions in lattices are shown in Fig. 2.3 for varying depths. For small values of V_0 the Wannier functions extend significantly into neighbouring sites, introducing strong inter-site tunnelling among atoms in each well. As the depth increases this overlap decreases and the Wannier function becomes more peaked and localised (lower right panel in Fig. 2.3).

2.2.3 Tight-binding regime

For increasing lattice depth, the overlap between neighbouring Wannier functions decreases as the system approaches the tight-binding regime, in which the lattice can be treated as a discrete array of sites with particles largely confined to individual wells.

2.3. Bose-Einstein condensates and superfluids

The concept originates from solid-state physics and applies when the lowest band is well separated from higher bands. In this limit, system parameters such as the inter-site tunnelling amplitude and the on-site interaction energy can be directly calculated from the Wannier functions and the band structure.

The tunnelling matrix element J is obtained from the Hamiltonian matrix element between Wannier functions centred on neighbouring lattice sites

$$J = \int w_n(z - z_l) \left(-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V(z) \right) w_n(z - z_{l+1}) dz. \quad (2.17)$$

This quantity represents the hopping energy associated with an atom tunnelling from one lattice site to a neighbouring one. In the tight-binding limit, the Wannier functions can be approximated as Gaussian distributions and J can be found analytically using the 1D Mathieu-equation [53]

$$J = \frac{4}{\sqrt{\pi}} \left(\frac{V_0}{E_r} \right)^{3/4} e^{-2\sqrt{V_0/E_r}}. \quad (2.18)$$

However, for small lattice depths Eq. (2.18) underestimates J by an order of magnitude.

In systems with inter-atomic interactions described by the Hamiltonian Eq. (2.13), the interaction energy is calculated as

$$U = g_{3D} N \int |w(x, y, z)|^4 dx dy dz, \quad (2.19)$$

where $g_{3D} = 4\pi\hbar^2 a_s / m$, with a_s being the s -wave scattering length (see Sec. 2.3.2), and N the number of atoms in the lattice site. For a positive value the atoms are repelled from each other, while for negative values interactions are attractive.

Equation (2.19) simplifies in the 1D case by assuming the wave function is Gaussian in the radial directions, meaning the integral depends only on the z direction. The coupling constant g is then given by [54]

$$g_{1D} = 2\hbar\omega_r a_s, \quad (2.20)$$

where ω_r is the radial trapping frequency.

2.3 Bose-Einstein condensates and superfluids

Developed in the 1920s, Bose-Einstein statistics describe how a gas of particles with integer spin behave [55, 56]. Below a critical temperature, these atoms, known as bosons, all condense into a single quantum state and share the same wavefunction, forming a BEC. Direct experimental realisations of BECs came in the mid 1990s, where teams at Boulder and MIT, using rubidium and sodium atoms respectively, cooled dilute gases to ultra-low temperatures [57, 58]. Following these breakthroughs BECs were observed with other alkali species including lithium [20] and potassium [59], and finally caesium in 2003 [60, 61].

2.3. Bose-Einstein condensates and superfluids

2.3.1 Ground state of a non-interacting BEC

The total many body wavefunction can be described as the product

$$\Psi_N(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \prod_{j=1}^N \psi_0(\mathbf{r}_j), \quad (2.21)$$

of the individual wavefunctions $\psi_0(\mathbf{r})$ [62]. In an optical dipole trap like those discussed in the previous section ψ_0 is the ground state of the harmonic oscillator

$$\psi_0(\mathbf{r}) = \left(\frac{m\omega_{\text{ho}}}{\pi\hbar} \right)^{3/4} \exp \left[-\frac{m}{2\hbar} (\omega_x x^2 + \omega_y y^2 + \omega_z z^2) \right], \quad (2.22)$$

where $\omega_{\text{ho}} = (\omega_x \omega_y \omega_z)^{1/3}$ is the geometric average of the trapping potentials in three dimensions. The spatial density distribution of the atoms can be found from $n(\mathbf{r}) = |\psi_0(\mathbf{r})|^2$. Equation (2.22) corresponds to a 3D Gaussian whose size directly depends on the trapping potentials through the harmonic oscillator length

$$a_{\text{ho}} = \sqrt{\frac{\hbar}{m\omega_{\text{ho}}}}, \quad (2.23)$$

which is independent of the atom number N . Typical length scales in the experiment are on the order of $1 - 3 \mu\text{m}$.

2.3.2 Interacting BECs and the Gross-Pitaevskii equation

In ultracold atomic gases, interactions between atoms play a crucial role in determining the many-body behaviour of the system. At low temperatures, the de Broglie wavelength of the atoms becomes much larger than the range of the interatomic potential, and collisions occur at very low relative energies. For neutral atoms, the interaction is dominated by short-range forces arising from induced dipole–dipole attraction, commonly described by a van der Waals potential. Despite the complex short-range potential, ultracold collisions are generally well described by a simpler low-energy scattering theory. These types of interactions are difficult to calculate but the resulting potential only depends on the two atoms' centre of mass separation $\mathbf{r}$ and their initial kinetic energies [63]. For distances $r \gtrsim 5 \text{ \AA}$ the potential can be treated as a van der Waals interaction.

A common ansatz used for a solution to the potential takes the form of a plane wave for the incoming particle and a spherical wave for the outgoing scattered wave

$$\psi_{\mathbf{k}}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}} + f(\mathbf{k}, \theta) \frac{e^{ikr}}{r}, \quad (2.24)$$

where $k = |\mathbf{k}|$ is the modulus of the wave-number of the incoming wave. The prefactor $f(k, \theta)$ is the scattering amplitude that determines the probability that the scattered wave will be scattered into a given direction.

Using partial-wave decomposition the scattered wavefunction can be split into different components: a radial part and a set of spherical harmonic functions for each

2.3. Bose-Einstein condensates and superfluids

orbital quantum number l . The scattering amplitude can then be rewritten in terms of l and the phase shift $\delta_l(k)$ which results from a scattering process

$$f(k, \theta) = \sum_{l=0}^{\infty} \frac{2l+1}{k \cot[\delta_l(k)] - ik} P_l(\cos \theta), \quad (2.25)$$

where $P_l(\cos \theta)$ are the Legendre polynomials. The collisional cross-section is given as

$$\sigma_{\text{tot}}(k) = \frac{4\pi}{k^2} \sum_{l=0}^{\infty} (2l+1) \sin^2[\delta_l(k)], \quad (2.26)$$

which indicates the probability two atoms will interact. For bosons only even values of l contribute while for fermions only odd values contribute.

s -wave scattering length

The scattering contributions in Eq. (2.26) are labelled by the orbital angular momentum quantum number l . For $l > 0$, the atoms experience an additional centrifugal potential, which creates a barrier in the effective interaction potential. For ultracold temperatures with collision energies far below this barrier (for caesium the height is $192 \mu\text{K}$ [64]), contributions with larger orbital angular momentum are suppressed and only s -wave ($l = 0$) scattering remains relevant [65].

The resulting cross section is given by

$$\sigma_0(k) = \frac{8\pi}{k^2} \sin^2[\delta_0(k)], \quad (2.27)$$

where the so called s -wave scattering length a_s can, in the limit of a vanishing collision energy, be defined from

$$a_s = -\lim_{k \rightarrow 0} \frac{\tan[\delta_0(k)]}{k}. \quad (2.28)$$

The scattering length sets the effective range of the interaction: $a_s > 0$ corresponds to repulsive interactions, while $a_s < 0$ corresponds to attractive interactions. As it determines the macroscopic behaviour of the gas and can be tuned experimentally (see Sec. 4.3.1), it is the central parameter describing interactions in ultracold atomic systems. The scattering length is typically on the order of 10^{-11} m and as such usually given in units of Bohr's radius $a_0 = 5.292 \times 10^{-11}$ m.

The Gross-Pitaevskii equation

The low-energy two-body interaction can be incorporated using a pseudopotential characterised by the s -wave scattering length

$$V(\mathbf{r}) = \frac{4\pi\hbar^2 a_s}{m} \delta(\mathbf{r}) \frac{\partial}{\partial r}(r \cdot), \quad (2.29)$$

which reproduces the correct scattering behaviour at short distances. In the limit of small values of ka_s , Eq. (2.29) can be approximated by the delta potential and the partial derivative can be neglected

$$V(\mathbf{r}) = g\delta(\mathbf{r}). \quad (2.30)$$

2.3. Bose-Einstein condensates and superfluids

This produces the following Hamiltonian for N interacting bosons in an external potential

$$H = \sum_{i=1}^N \left[-\frac{\hbar^2}{2m} \nabla_i^2 + V_{\text{ext}}(\mathbf{r}_i) \right] + \frac{1}{2} \sum_{i=1}^N \sum_{j=1}^N V(\mathbf{r}_i - \mathbf{r}_j), \quad (2.31)$$

where $V(\mathbf{r}_i - \mathbf{r}_j)$ is a two-body potential. Solving this many-body Hamiltonian exactly is generally intractable because the wavefunction depends on all particle coordinates simultaneously. However, in a dilute Bose gas the particles predominantly occupy a single macroscopic quantum state. Using a mean-field approximation, the system can therefore be described by a single order parameter $\Phi(\mathbf{r}, t)$, leading to the Gross-Pitaevskii equation (GPE) [66, 67]

$$i\hbar \frac{\partial}{\partial t} \Phi(\mathbf{r}, t) = \left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}, t) + g|\Phi(\mathbf{r}, t)|^2 \right] \Phi(\mathbf{r}, t), \quad (2.32)$$

where $\Phi(\mathbf{r}, t) = \sqrt{N} \psi_0(\mathbf{r}, t)$ is the order parameter which is normalised by the total atom number $\int |\Phi(\mathbf{r}, t)|^2 d\mathbf{r} = N$. The GPE accurately describes the macroscopic dynamics of interacting bosons and has been successful in describing many phenomena [62, 63]. It can be split up into different parts:

$E_{\text{kin}} = -\frac{\hbar^2 \nabla^2}{2m}$	Kinetic energy of the condensate and relates to the curvature of the wavefunction
$V_{\text{ext}} = V_{\text{lat}} + V_{\text{harm}}$	External trapping potential such as optical lattice potentials or a harmonic trap
$E_{\text{int}} = g \Phi(\mathbf{r}, t) ^2$	Mean-field energy due to inter-atomic interactions. It is proportional to the condensate density, so is nonlinear

The GPE holds in the limit where interactions are weak and the condensate is dilute, more specifically the condition used is $n|a_s|^3 \ll 1$, which is typically on the order of 10^{-3} for ultra-cold atom experiments [62].

Equation (2.32) is known as the time-dependent GPE, however for static potentials a time-independent one can be formulated. By seeking solutions in the form $\Phi(\mathbf{r}, t) = \psi_0(\mathbf{r}) e^{-i\mu t/\hbar}$ the following variant of the GPE is found

$$\left[-\frac{\hbar^2 \nabla^2}{2m} + V_{\text{ext}}(\mathbf{r}) + g|\psi_0(\mathbf{r})|^2 \right] \psi_0(\mathbf{r}) = \mu \psi_0(\mathbf{r}), \quad (2.33)$$

where μ is the chemical potential. The chemical potential is related to the aforementioned energy terms through the following [62]

$$\mu = \frac{E_{\text{kin}} + E_{\text{ext}} + E_{\text{int}}}{N}. \quad (2.34)$$

In three dimensions the energies can be related through the virial equation $2E_{\text{kin}} - 2E_{\text{ext}} + 3E_{\text{int}} = 0$.

In the experiment it is possible to measure the energy of the system by quickly removing any external trapping potentials and letting the gas expand and measuring its

2.3. Bose-Einstein condensates and superfluids

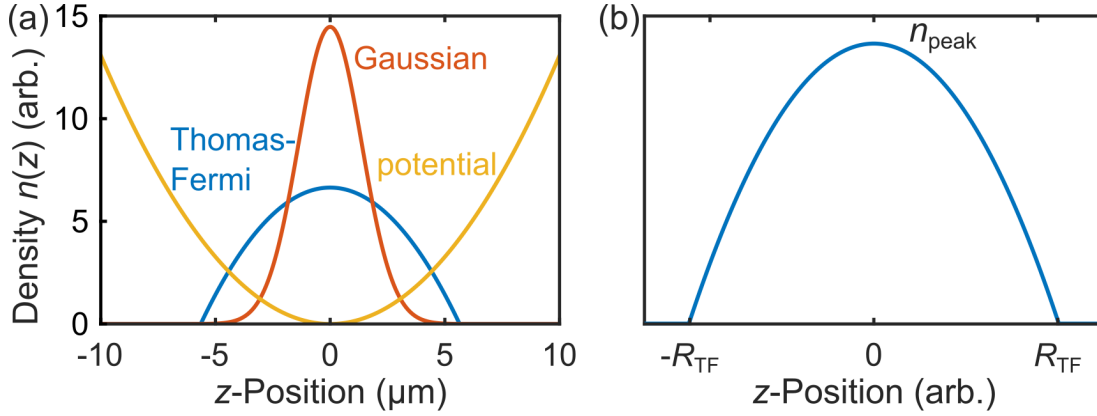


Figure 2.4: Visualisation of the ground state of a repulsive BEC using the Thomas-Fermi approximation. (a) Comparison between TF ground state (blue) and a Gaussian (red) in a parabolic trap. The TF profile takes the form of an inverted parabola. (b) The density distribution of a TF profile has a peak at $\mathbf{r} = 0$, the points at which the density is zero is known as the TF radius.

rate of expansion [68]. After release, the kinetic and interaction energies are quickly converted to a kinetic energy known as the release energy

$$E_{\text{rel}} = E_{\text{kin}} + E_{\text{int}}. \quad (2.35)$$

For non-interacting systems this is independent of the atom number N , while for interactions it scales nonlinearly with N .

2.3.3 Ground state of an interacting BEC and the Thomas-Fermi limit

Compared to the Gaussian ground state discussed in Sec. 2.3.1, the presence of atomic interactions can significantly modify the condensate ground state. For attractive interactions, the peak density can increase to the point where no stable ground state exists. In contrast, for repulsive interactions in the regime $Na_s/a_{\text{ho}} \gg 1$ the Thomas-Fermi approximation (TFA) can be used to find the new ground state. In this limit, the kinetic energy term is neglected, and the wavefunction takes the form

$$\psi_0(\mathbf{r}) = \begin{cases} \sqrt{\frac{\mu - V_{\text{ext}}(\mathbf{r})}{g}}, & \text{if } \mu > V_{\text{ext}}(\mathbf{r}) \\ 0, & \text{otherwise.} \end{cases} \quad (2.36)$$

As the external potential $V_{\text{ext}}(\mathbf{r})$ is the only parameter with a spatial dependence, the resulting density profile follows the shape of the potential, but in an inverted manner [Fig. 2.4]. For a harmonic trapping potential, the condensate density profile is therefore also parabolic, with a peak density at the trap centre given by

$$n_{\text{peak}} = \frac{\mu}{g}, \quad (2.37)$$

2.3. Bose-Einstein condensates and superfluids

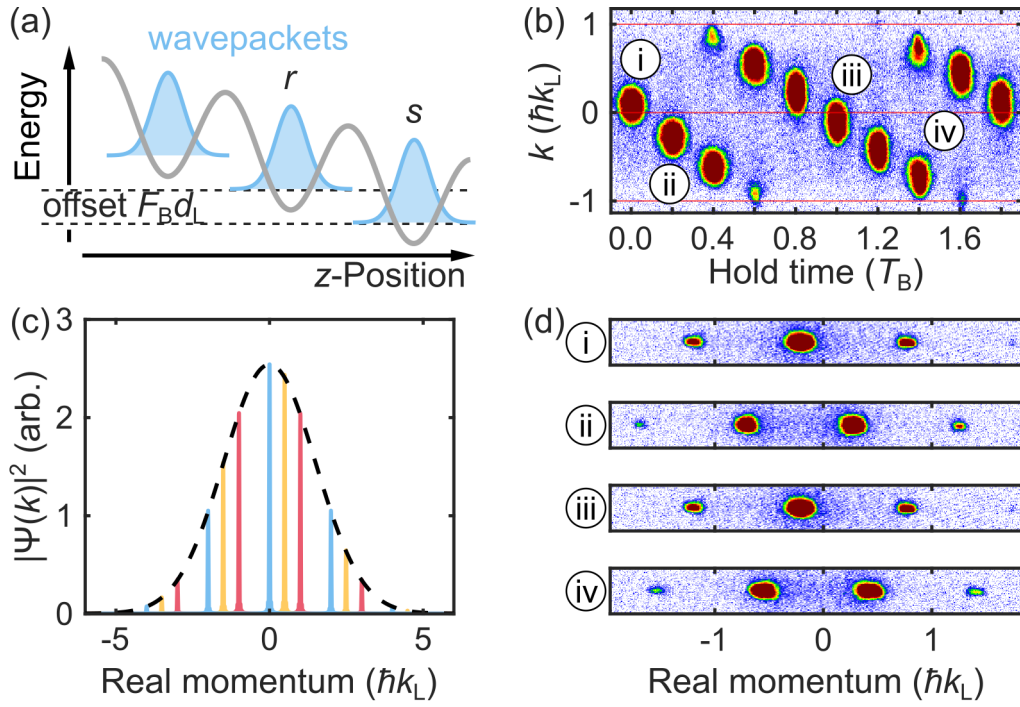


Figure 2.5: Tilted optical lattice and Bloch oscillations. (a) When a force F_B is applied to a system each lattice site is shifted by the energy $F_B d_L$ creating a Wannier-Stark ladder. (b) Averaged absorption images of Bloch oscillations with a force of $0.5mg$ ($T_B = 1.15$ ms), parameters: $V_0 = 9E_r$, $a_s = 20a_0$. Roman numerals denote points of interest shown in (d). (c) As the atoms move the momentum peaks shift in position and weight, the envelope is given by the Fourier transform of the Wannier functions while the position and weight come from the motion (blue $t = 0$ and $1 T_B$, yellow $t = 0.33 T_B$, red $t = 0.5 T_B$). (d) Absorption images of the real momentum profile at different points in the oscillation.

indicating that stronger repulsive interactions lead to a reduction in the peak density. The point at which the wave function vanishes [Fig. 2.4(b)] is known as the Thomas-Fermi radius and is defined as [62]

$$R_i = a_{ho} \left(\frac{15Na_s}{a_{ho}} \right)^{1/5} \frac{\omega_{ho}}{\omega_i}, \quad (2.38)$$

where $i = x, y, z$ for each Cartesian direction, typical experimental values are on the order of micrometres. The chemical potential in the Thomas-Fermi limit is given by

$$\mu = \frac{\hbar\omega_{ho}}{2} \left(\frac{15Na_s}{a_{ho}} \right)^{2/5}. \quad (2.39)$$

2.3.4 Terminology: BEC vs superfluid

Superfluidity is the effect of a liquid moving without viscosity. A BEC and a superfluid are closely related but indicate different concepts. A BEC is defined by the macroscopic occupation of a single quantum state and can occur even in an ideal,

2.4. Dynamics of superfluids in optical lattices

non-interacting Bose gas. Superfluidity, however, such as in ^4He and ^3He atoms, is a collective dynamical property that requires interactions between the particles. Repulsive interactions couple the atoms and give the system phase rigidity, producing a finite critical velocity below which flow is dissipationless [69]. Thus interactions convert a condensate into a superfluid [70], while sufficiently strong interactions, e.g. in a deep optical lattice, can instead destroy superfluid transport.

For the experiments discussed within this thesis where the BEC has repulsive interactions it is assumed to be in the superfluid state and as such will from now on be referred to as a superfluid.

2.4 Dynamics of superfluids in optical lattices

The merging of the fields of lattice physics and superfluids has led to some of the most seminal experiments in cold-atom physics in the past decades. Notable examples include the observation of interference from atoms tunnelling [71], the superfluid to Mott insulator transition [11], access to the strongly interacting Tonks-Girardeau regime [12], among others [44, 45, 72–74]. This section aims to discuss some of the more ubiquitous phenomena associated with superfluids in lattice potentials.

2.4.1 Bloch oscillations and Wannier-Stark localisation

First predicted by Bloch and Zener in the context of solid-state physics [52, 75], Bloch oscillations describe the counterintuitive motion of a particle in a periodic potential subjected to a constant force. The derivation can be based on the single-particle Hamiltonian, consisting of the kinetic energy, lattice potential $V(z)$, and an external force F_B [76]

$$H = \frac{p^2}{2m} + V(z) + F_B z. \quad (2.40)$$

According to the acceleration theorem, the quasimomentum k evolves in time as

$$k(t) = k(0) + \frac{F_B t}{\hbar}. \quad (2.41)$$

The quasimomentum is uniquely defined only within the first Brillouin zone. Consequently, when the particle reaches the edge of the Brillouin zone, it is mapped back to the opposite edge due to Bragg reflection [77, 78]. As a result, the quasimomentum evolves periodically in time with a sawtooth-like profile micromotion [Fig. 2.5(b)], giving rise to Bloch oscillations.

The period between the Bragg-reflections is inversely proportional to the force

$$T_B = \frac{2\pi\hbar}{d_L F_B}, \quad (2.42)$$

and is known as the Bloch period. This micromotion leads to a non-intuitive picture in position-space where instead of accelerating away the particle oscillates back and forth. Also counter-intuitive is the fact that the amplitude of this oscillation is inversely proportional to the force

$$L = \frac{\Delta_n}{F_B}, \quad (2.43)$$

2.4. Dynamics of superfluids in optical lattices

where Δ_n is the bandwidth of the n th band. This means for weak forces, both the period T_B and amplitude L are large.

The addition of the constant force has the effect of offsetting successive lattice sites with offset $F_B d_L$ [Fig. 2.5(a)], this type of potential is often referred to as a Wannier-Stark ladder or a 'washboard potential' [79]. In this setting, there is still a flow of information possible between sites as long as the offset is not greater than the bandwidth

$$F_B d_L < 4J, \quad (2.44)$$

in which case the lattice sites act as an array of isolated harmonic wells, this is known as Wannier-Stark localisation.

2.4.2 Lattice momentum distribution

Information about the state of atoms in an optical lattice can be obtained using free expansion measurements. In this method the trapping potentials are suddenly switched off, allowing the atoms to expand and the matter-waves from different lattice sites to interfere. If phase coherence exists across the lattice, a diffraction pattern appears [Fig. 2.5(d)] [7]. Typically, we observe several distinct peaks in the momentum distribution. Because quasimomentum is defined only within $\pm \hbar k_L$, the peaks are separated by $2\hbar k_L$ and correspond to equivalent momenta in neighbouring Brillouin zones [Fig. 2.5(d)].

The momentum distribution in an optical lattice can be obtained by summing the interfering contributions from all lattice sites r, s [7, 80],

$$n(k, t) = |\tilde{w}(k)|^2 \times \sum_{r,s} \langle \psi | \hat{b}_r^\dagger \hat{b}_s | \psi \rangle \exp[i(r-s)(k d_L)]. \quad (2.45)$$

Here, $\hat{b}_j^\dagger$ ($\hat{b}_j$) are the bosonic creation (annihilation) operators for a state on lattice site j and $\tilde{w}(k)$ is the Fourier transform of the Wannier function. The correlator $\langle \hat{b}_r^\dagger \hat{b}_s \rangle$ describes phase coherence between different lattice sites, while term with the exponent accounts for the phase shift between lattice sites. The envelope of the momentum profile is given by the $\tilde{w}$.

For a constant quasimomentum k , Eq. (2.45) produces momentum peaks centred at $k \pm 2n\hbar k_L$, where n is an integer. The relative peak heights are determined by the envelope function [Fig. 2.5(c)]. Under a constant external force, the peak positions evolve in time as $k(t)$ changes according to the acceleration theorem Eq. (2.41) [peaks in Fig. 2.5(c)].

2.4.3 Real momentum and quasimomentum

Two different momentum regimes can be imaged: the physical 'real' momentum of the atoms [shown in Fig. 2.5(d)] and quasimomentum [shown in Fig. 2.5(b)]. The sudden removal of the lattice potential maps the lattice momenta to real momentum, producing a distribution in the frame of the laboratory.

Alternatively, the quasimomentum of the atoms can be revealed by the process of band mapping [81, 82]. Using this technique, before imaging, the lattice is adiabatically removed (typically in 2 ms). This maps the lattice momenta of different bands to

2.5. Conclusion

specific ranges of momenta in real space, allowing for the imaging of the atom number in different bands and shows the momentum distribution in the frame of the lattice. In the experiments described in this thesis, however, the atoms are always restricted to the lowest band, meaning the quasimomentum distribution shows a single peak [horizontally concatenated images in Fig. 2.5(b)].

Generally, it is easier to use real momentum to extract information from the experiment as during the additional time needed for adiabatic transfer the atoms may be subject to additional effects such as heating. However, viewing the quasimomentum distribution can make it easier to visualise the micromotion of the atoms as the motion appears as a moving wavepacket rather than a shift in the relative weighting of real momenta peaks [Fig. 2.5(b) compared with Fig. 2.5(d)].

2.5 Conclusion

This chapter has provided a brief overview of dipole potentials, optical lattices, and BEC physics. Optical lattices provide a useful analogue to crystal lattices in solid state physics, with the benefit of system parameters being easily controlled. The results presented in this thesis involve a lattice potential in some way, with the resulting band structure discussed in Sec. 2.2.1 being instrumental in the interpretation of the results.

The GPE outlined in Sec. 2.3.2 will be used as a starting point for the theory of matter-wave solitons in BECs in Chap. 5, which can be accurately described by the GPE. Section 2.4 described a superfluid's response to a constant force, which induces Bloch oscillations. This, and the discussion about the resulting momentum distribution in Sec. 2.4.2, will be used to understand a superfluid's response to Floquet driving, which is the topic of Chap. 6-8. In particular, the system used in Chap. 8 uses a tilted lattice in conjunction with a driving force, with the combination of forces creating a complex system with many resonances.

Chapter 3

General experimental setup and BEC creation

This chapter provides a brief overview of the general experimental setup and the methods used to create a Bose-Einstein condensate of caesium atoms. For further details on the experimental apparatus the reader should refer to the theses of C. Colquhoun and A. Di Carli [83, 84].

The first half of this chapter outlines the experimental apparatus. Section 3.1 describes the vacuum setup. This is followed by a section to describe the electric coils used to generate magnetic fields used for levitation, changing scattering length and moving the atoms (Sec. 3.2) and details of the laser frequencies used (Sec. 3.3). The second half of the chapter focuses on the production of a Bose-Einstein condensate (BEC). It begins with the pre-cooling stage (Sec. 3.4.2), where atoms are trapped in magneto-optical traps (MOTs) and cooled via degenerate Raman cooling, before being transferred into dipole traps (Sec. 3.4.3). The final cooling stage to create a BEC consists of evaporative cooling, which is discussed in Sec. 3.4.4, with temperature benchmarks provided thereafter. The chapter closes with a brief overview of the imaging setup and absorption imaging (Sec. 3.5).

3.1 Vacuum setup

In cold-atom setups it is necessary to conduct the experiments in ultra-high vacuum, as collisions with the background gas can eject atoms from the trapping potential. Our vacuum setup [Fig. 3.1] consists of two glass cells, each dedicated to different stages of the experimental sequence. The glass cells are compact and offer improved optical access over stainless-steel chambers.

The starting point of the experiment is the higher-pressure section [leftmost region highlighted in Fig. 3.1], which consists of the 2D+MOT glass cell and an oven section made of steel, which contains a 5 g ampoule of caesium and a bellows, allowing atoms to diffuse into the 2D+MOT section. The flow rate of atoms can be regulated by the valve separating the oven from the glass cell. The 2D+MOT glass cell (manufactured by Japan Cell) measures 50 mm $\times$ 50 mm $\times$ 50 mm and has an anti-reflection coating for a wavelength of 852 nm on both the internal and external surfaces. The glass panels for this cell were bonded using epoxy glue. The pressure inside this cell is 10^{-8} mbar

3.2. Magnetic field coils

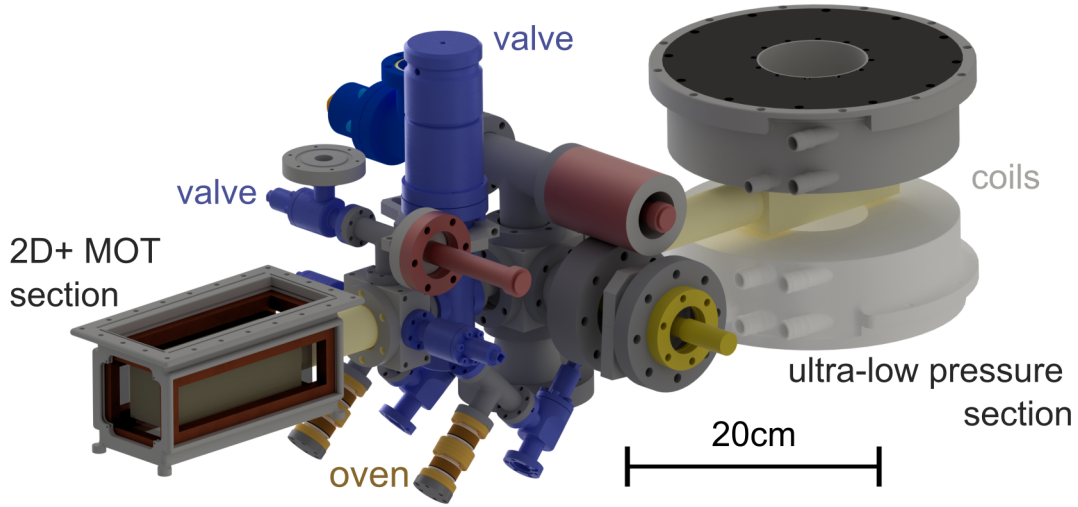


Figure 3.1: Overview of the experimental setup and configuration of magnetic field coils. The drawing shows the arrangement of the experiment. It is split into two sections: a cell in which a 2D magneto optical trap (MOT) is created and an addition cell where the 3D MOT and BEC creation occurs. These are connected by a stainless steel chamber where the caesium atoms are fed into. Ion pumps are not shown.

and is maintained by an ion pump (Agilent Vaclon 8L/s).

The second section [rightmost region highlighted in Fig. 3.1] has low-pressure of below 5×10^{-11} mbar, maintained by a second ion pump (Agilent Vaclon Plus 55) and an NEG pump. It contains an optically bonded glass cell, in which all stages of the experiment cycle are carried out following the 2D+MOT stage.

The two cells are connected by a stainless steel section with an all-metal gate valve in-between, to isolate the cells for protection during periods where the setup is not used. The pressure difference is maintained by a differential-pumping tube, with each section's pressure being monitored by the current of the ion pumps.

3.2 Magnetic field coils

Control over magnetic fields is crucial for the creation of a BEC and any experiments using them, as with other parameters the degree of control over the magnetic field strongly correlates with how long the atoms can effectively be used.

To generate the 2D quadrupole field required for the 2D+MOT section the setup has four rectangular coils mounted near the glass cell [leftmost cell in Fig. 3.1]. They are driven by a power-supply (TENMA 72-10495) of up to 1 A, generating field gradients of approximately 10 G/cm in the y and z axis. The ultra-low pressure section consists of two pairs of nested magnetic coils that each generate a magnetic field for a different purpose. For added flexibility, the coils are divided into several independently controllable sections [Fig. 3.2(a)]. During operation, the coils dissipate up to 6kW of heat and as such are cooled using water that is continuously pumped through water channels.

Due to the large mass of caesium atoms, gravitational sag occurs in the opti-

3.2. Magnetic field coils

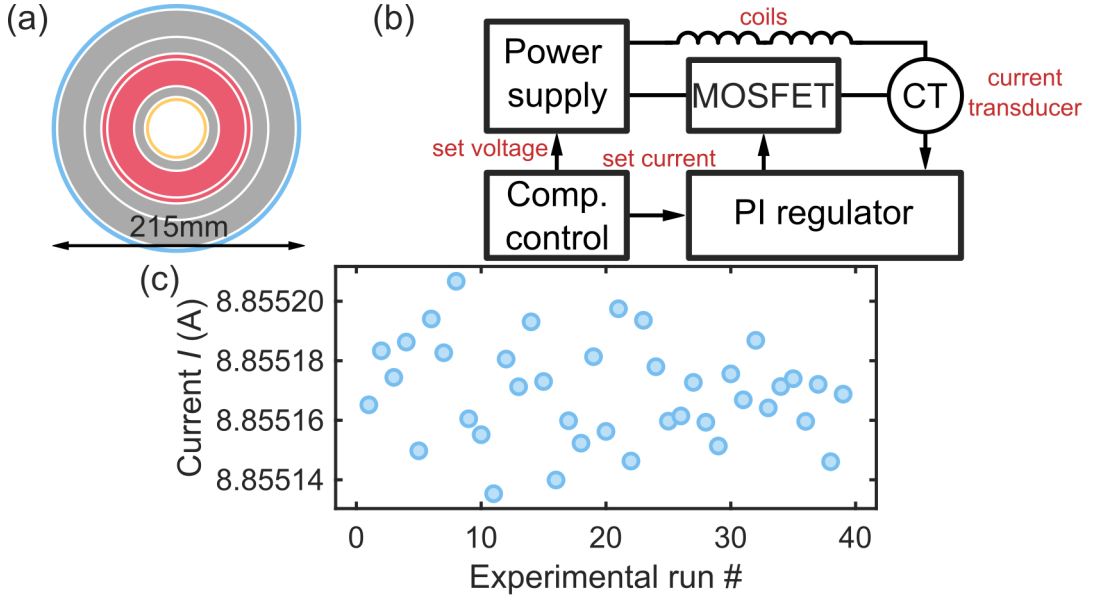


Figure 3.2: Overview of the coil configuration. (a) Coil assembly with seven independently controllable sections per coil, enabling flexible magnetic field configurations; three are used here: levitation gradient (red), field offset (blue), and fast field creation (yellow). Figure adapted from Ref. [83]. (b) Current regulation. The computer control system sets the target current, which is regulated by a custom-built PI controller. A MOSFET controls the current, which is continuously monitored using a current transducer. The power supply operates in voltage-controlled mode, with the output voltage also set by the control system. (c) Measured current stability over experimental runs (~ 15 s duration, 100 ms measurement time).

cal traps. To compensate it, a magnetic-field gradient is applied that levitates the atoms against gravity. Since different internal states have different Landé g -factors and m_F values, the levitation condition is state dependent. For caesium-133 in the $|F = 3, m_F = 3\rangle$ state, the magnetic field gradient required is approximately [64]

$$\frac{\partial B}{\partial z} \approx 31.137 \frac{\text{G}}{\text{cm}} - \frac{0.0119}{\text{cm}} B. \quad (3.1)$$

The levitating coils are driven by a switch-mode power supply (Delta Elektronika SM7020-D) operated in an anti-Helmholtz configuration, providing up to 20 A in current, with a maximum gradient of 56.9 G/cm [83].

For caesium, the presence of Feshbach resonances (Sec. 4.3.1) at low external magnetic field tunes the interaction strength between the atoms. This requires an additional magnetic field that shifts (“offsets”) the centre of the levitation potential, which is provided by the “offset” coils [Fig. 3.2(a)]. The coils are driven by a second power supply (Delta Elektronika SM120-50) with a maximum output current of 50 A. In practice, adding an offset field shifts the centre of the quadrupole field produced by the levitation coils along the vertical direction.

The internal current regulation of the power supplies did not provide sufficient long-term stability, likely due to heating of components within the devices. To achieve precise current control, we implemented a feedback loop that actively monitors and

3.3. Laser setup

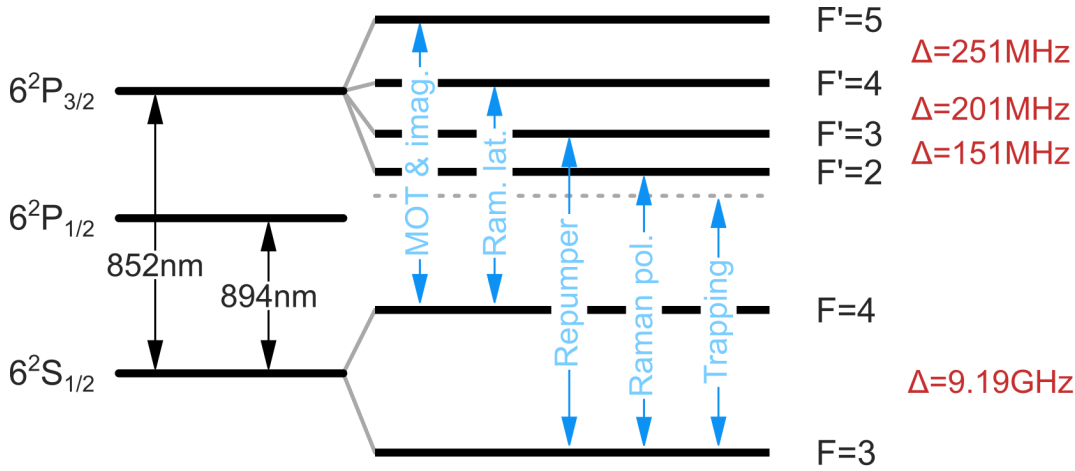


Figure 3.3: Energy level diagram of ^{133}Cs showing the hyperfine splitting of states $6S_{1/2}$ and $6P_{3/2}$. The transitions used in the experimental setup are shown in blue. The term “trapping” refers to off-resonant lasers with $\lambda = 1064\text{nm}$. The frequency differences between hyperfine states are given in red.

stabilises the current in both coil pairs [Fig. 3.2(b)].

The current is measured using a current transducer (LEM IT 60-S Ultrastab) with bandwidth 600kHz. Stabilisation is achieved via a MOSFET controlled by a proportional–integral (PI) feedback loop. In this configuration, the power supplies operate in constant-voltage mode rather than constant-current mode, with the output voltage set sufficiently high to enable effective regulation. To minimise excess heating in the MOSFET, this voltage is also adjusted through the experimental cycle using our computer control system. This configuration allows us to regulate currents with a long time precision of 10^{-4}A [Fig. 3.2(c)].

In addition to the main coils in the 2D+MOT and ultra-low pressure sections, smaller coils, referred to as shim coils, are used to fine-tune the magnetic fields and compensate for stray or time-varying background fields. In the 2D+MOT section there are two pairs located in the x and y direction, while in the experimental section there is a pair for each direction x , y and z . When in the co-propagating current configuration each pair can generate magnetic fields of up to 5 G.

3.3 Laser setup

The main fine-structure transition used for laser cooling of caesium is the D_2 line, ($6S_{1/2} \rightarrow 6P_{3/2}$), with a wavelength of 852nm (352 THz) [85]. This transition provides a nearly closed cycling transition between $|F = 4\rangle$ and the excited state $|F' = 5\rangle$, which is used for both absorption imaging and operation of the magneto-optical trap (Fig. 3.3). Atoms that decay into the $|F = 3\rangle$ hyperfine state are returned to the cooling cycle using a repump laser resonant with the $|F = 3\rangle$ to $|F' = 3\rangle$ transition.

The light for laser cooling is generated by diode lasers, which can be divided into two groups that drive transitions on the D_2 line: one addressing transitions originating from $|F = 4\rangle$ and the other from $|F = 3\rangle$. Lasers addressing the $|F = 4\rangle$ manifold are

3.4. Cooling cycle

used for laser cooling processes, such as operation of a MOT, Raman cooling, and absorption imaging. Lasers addressing transitions from $|F = 3\rangle$ are used for repumping and for polarising the atomic sample.

To generate sufficient optical power, each group consists of a master laser stabilised using a polarisation gradient lock, injection-locked slave lasers, and two tapered amplifiers. Far-detuned light for optical trapping is generated is produced by a Coherent Mephisto Laser operating at 1064 nm and subsequently amplified to 10 W using a fibre amplifier.

3.4 Cooling cycle

Caesium is well suited for Bose–Einstein condensation (BEC) experiments due to its broad Feshbach resonances at low magnetic fields, and its large atomic mass, which results in low recoil energy favourable for laser cooling. However, early attempts proved that creating caesium BECs was challenging, with schemes using the hyperfine states $|F = 4, m_F = 4\rangle$ [86, 87] and $|F = 3, m_F = -3\rangle$ [88, 89] failing due to spin relaxation rates and low phase-space densities respectively. Work then began to instead use the $|F = 3, m_F = 3\rangle$ state, however this added more complexity to the system as this state is a high-field seeking state, meaning atoms are attracted to regions of high magnetic field, eliminating the use of magnetic traps. The following sections describe the cooling stages used in our experiment to produce a BEC of ^{133}Cs atoms, following the scheme developed in Refs. [60, 61].

3.4.1 Trapping atoms in a 3D magneto-optical trap

Magneto-optical traps (MOTs) are a standard tool in cold-atom experiments. In our setup, atoms at room temperature are first cooled in a separate glass cell (see Sec. 3.1) in a two-dimensional MOT (2D MOT). In this configuration, atoms are cooled in two transverse dimensions using laser beams in a $\sigma^+ + \sigma^-$ configuration and confined by two pairs of anti-Helmholtz coils. To increase the atomic flux, an additional pair of laser beams creates an optical molasses [90] along the longitudinal direction. This configuration, known as a 2D+MOT, significantly enhances the atom number delivered to the 3D MOT [91].

To transfer the cooled atoms from the 2D+MOT section to the 3D-MOT, a resonant laser beam is used to “push” the atoms. Once transferred, the atoms are captured by a 3D-MOT, which is formed of three pairs of beams and uses the main experimental coils to generate a quadrupole field in the three dimensions. The pairs of beams are generated from a single laser beam 25 mm in diameter and a power of 500 mW and are directed towards the centre of the cell with periscopes.

To increase the phase-space density of the atomic gas, the MOT is compressed by increasing the detuning of the MOT beams. Although this compression increases the atomic density, it typically results in heating (typical temperatures are in the range 50 μK –100 μK). In our experimental sequence, this heating is mitigated by the subsequent application of degenerate Raman sideband cooling, which efficiently captures the atoms and provides substantial additional cooling.

3.4. Cooling cycle

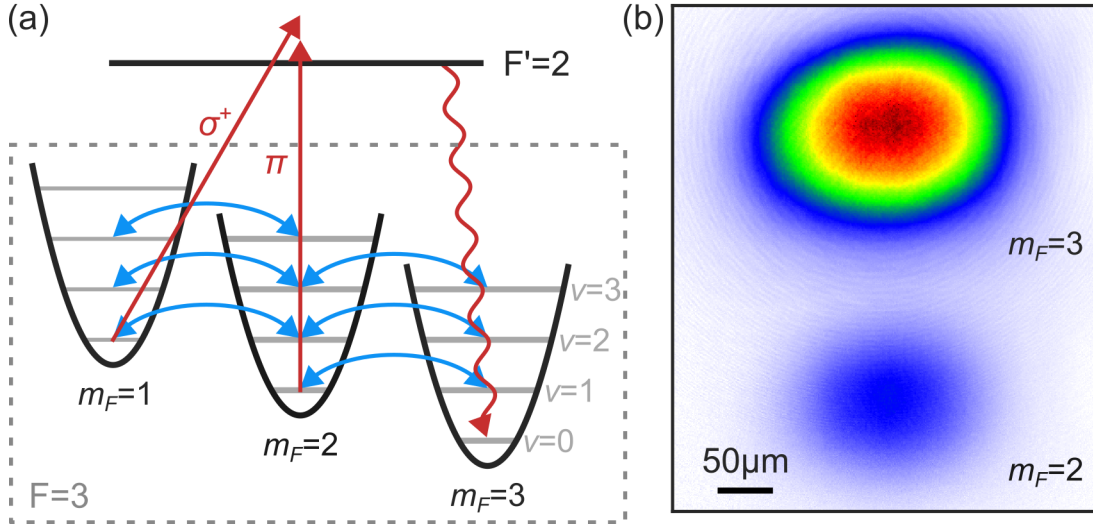


Figure 3.4: Degenerate Raman sideband cooling. (a) Schematic of the cooling process with harmonic oscillator states, v , for different hyperfine states, m_F . Blue arrows indicate Raman coupling while red arrows show real transitions to an excited state. (b) Absorption image of the cloud after Raman cooling and an expansion time of 50 ms. Atoms remaining in different m_F states are no longer effectively levitated so fall at different rates during expansion.

3.4.2 Degenerate Raman sideband cooling

Immediately after the compression of the MOT, degenerate Raman sideband cooling is used to capture and cool the atoms [64, 92–94]. The scheme works as follows: after release from the 3D-MOT the atoms are immediately captured in a 3D optical lattice created by two pairs of lasers resonant to the $|F=4\rangle \rightarrow |F'=4\rangle$ transition. This acts as a repumper system and allows the atoms to quickly decay into the $|F=3\rangle$ state. In the lattice potential, thermal excitations manifest as population of higher harmonic-oscillator states (labelled by the quantum number v). Figure 3.4(a) illustrates these oscillator states for a single lattice site in a weak magnetic field, showing the corresponding Zeeman sublevels m_F . At this stage the atoms occupy an excited vibrational state. The magnetic field is tuned to bring the vibrational level v of an atom in the state m_F into degeneracy with $v+1$ of state m_F+1 . These degenerate vibrational states are coupled via Raman transitions [red arrows in Fig. 3.4(a)], which are driven by the lattice light.

Through the Raman coupling, an atom initially in the state $|m_F=k, v=l\rangle$ is coherently transferred to the state $|m_F=k-1, v=l-1\rangle$, effectively removing one quantum of harmonic-oscillator excitation. The cooling cycle is completed by a real excitation using a σ^+ polarised beam, followed by spontaneous decay, which on average increases m_F by one. If the lattice potential is sufficiently deep, such that the system operates in the Dicke regime [95], this excitation-decay process does not change the harmonic-oscillator quantum number, leaving the atom in the state $|m_F=k, v=l-1\rangle$. This entire process reduces the vibrational state by one quantum when compared to its original state. To reach the final state $|m_F=3\rangle$, an additional π -polarised beam is necessary which resonantly excites the $|F'=2, m_F=2\rangle$ state that can decay to

3.4. Cooling cycle

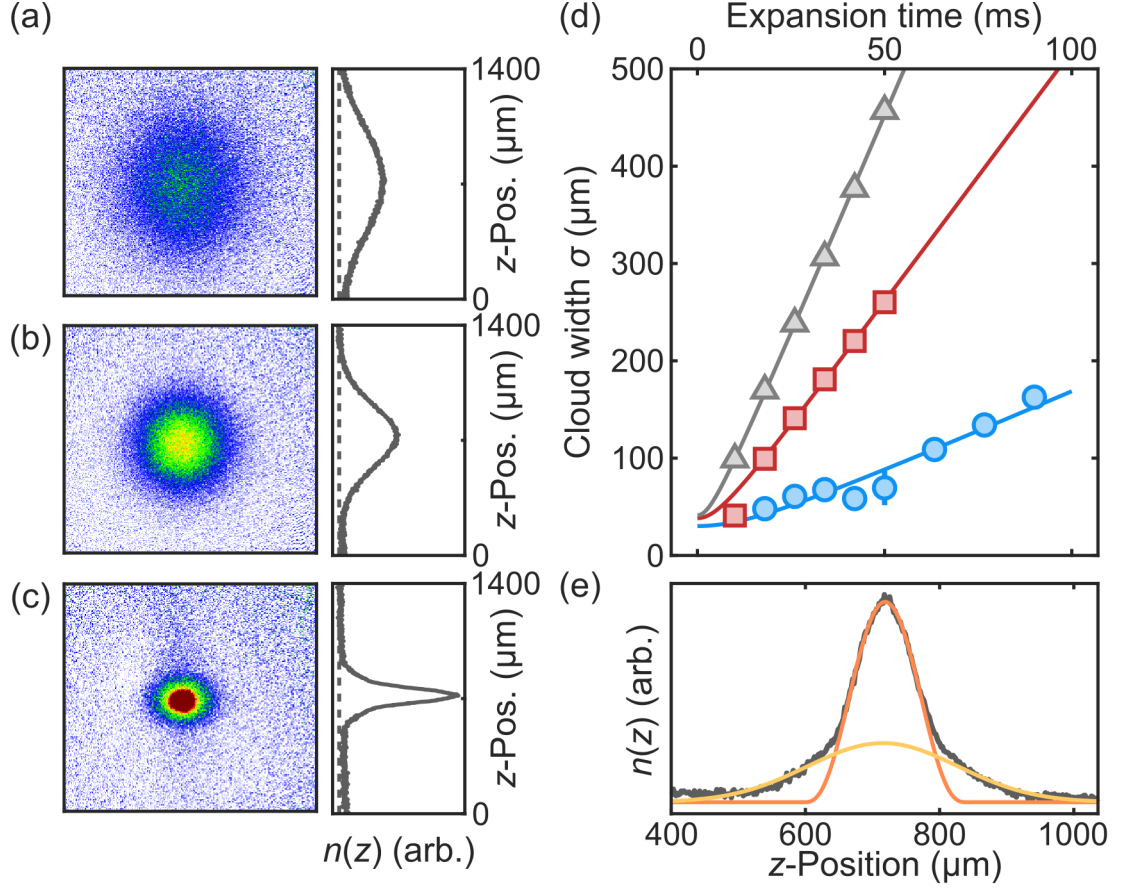


Figure 3.5: Stages of evaporative cooling. (a)-(c) Absorption images after 60ms expansion time of cloud after the three stages of evaporative cooling (left) and the extracted vertical density profiles (right). (d) Measured width of the cloud after varying expansion times. The three data sets account for cooling stages one (grey), two (red) and three (blue). The straight lines are fits to the data using Eq. (3.2), giving temperatures of 0.65, 0.21, and 0.02 μK respectively. (e) Density profile of a cloud showing the TF ground state (orange) and thermal fraction (yellow).

$$|F = 3, m_F = 3\rangle.$$

After 3 ms of Raman cooling, the lattice is removed in 600 μs , and a magnetic field gradient is rapidly switched on to levitate the $|F = 3, m_F = 3\rangle$ state. We measure typically 3×10^8 atoms in this state with a temperature close to 1 μK . An order of magnitude of atoms is lost during the process, primarily due to atoms not being efficiently captured by the lattice potential at the onset of the Raman cooling. In addition, there are atoms in the $|F = 3, m_F = 2\rangle$ and $|F = 3, m_F = 1\rangle$ states. These states are not magnetically levitated and the atoms fall due to gravity [Fig. 3.4(b)].

3.4.3 Dipole trapping

After the Raman cooling stage, the atoms are transferred into a far-detuned crossed optical dipole trap at a wavelength of 1070 nm. To maximise the number of captured atoms, the trapping beams have a waist of 800 μm and a maximum power of 100 W

3.4. Cooling cycle

(Ytterbium fibre laser, IPG Photonics YLR-LP-200). The two beams intersect at the centre of the glass cell at an angle of 12° , resulting in a trap depth of $\sim 25 \mu\text{K}$. This trap is referred to as the “reservoir” trap due to its ability to hold a large number of atoms. After a hold time of 2 s, we typically obtain 3×10^7 at a temperature of $\sim 9 \mu\text{K}$.

The low trapping frequencies of the reservoir traps make further cooling of the atomic cloud inefficient. To overcome this limitation, we employ an additional trap formed by superimposing two tightly focused laser beams, horizontal and vertical beams D1 and D2, onto the reservoir trap. This configuration creates a second, strongly confining potential located at the bottom of the reservoir trap, visible as a dimple in the overall potential landscape [60, 96, 97]. The beams are generated by a 50 W fibre amplifier (Nufern NUA-1064-PD-0050-DO), seeded by an Nd:YAG laser (Coherent Mephisto). During loading, the powers of beams D1 and D2 are 50 mW and 300 mW, respectively, reaching an atom number of $\sim 1 \times 10^6$ at $1 \mu\text{K}$.

3.4.4 Evaporative cooling

The final cooling stage used to generate BECs in the experiment is forced evaporative cooling. This is the process of ejecting the hottest atoms out of the experiment through the lowering of the trap depth. We make use of three stages of evaporation, each of which involves reducing the power of the dimple beams at different rates.

In the first stage, the powers in beams D1 and D2 are reduced to 30 and 70 mW respectively, over 500 ms, resulting in 8×10^5 atoms at 230 nK [Fig. 3.5(a)]. To suppress three-body losses we quickly adjust the scattering length to $210 a_0$.

After the initial ramp, the cloud is close to the phase transition of a BEC. To account for the thermalisation associated with fast ramps, in this stage the dimples are ramped from their previous values to 10 and 26 mW in 1 s. After this stage there is 5×10^5 atoms at 60 nK [Fig. 3.5(b)].

The final ramp generates a BEC of 1.5×10^5 atoms at 14 nK [Fig. 3.5(c)] by ramping the dimple dipole traps to 1 and 15 mW in 3 s. In total, the evaporation stage takes 4.5 s to complete. If required, the atom number can be further reduced by adjusting the magnetic levitation over 3 s to tilt the trapping potential, thereby removing a controlled fraction of atoms proportional to the change in the levitation gradient.

3.4.5 Temperature measurement

We use expansion measurements to determine the temperature of the gas during the cooling stages. After release from the trap, the rate of expansion of the cloud is proportional to its temperature, assuming weak interactions. For a thermal gas, the time-dependent width is given by [64]

$$\sigma_{x,y}(t) = \sqrt{\sigma_0^2 + \frac{2k_B T_{x,y}}{m} t^2}, \quad (3.2)$$

where σ_0 is the initial width of the cloud, k_B is Boltzmann’s constant, m is the mass of a caesium atom and $T_{x,y}$ is the temperature in each direction.

3.5. Imaging setup

Thermal fraction of the BEC

For an interacting quantum gas, time-of-flight expansion measurements reveal a bimodal density distribution [Fig. 3.5(e)], consisting of a broad Gaussian component associated with the thermal fraction and a Thomas–Fermi profile corresponding to the condensed fraction. We use measure the width of the Gaussian background together with Eq. (3.2) to determine the temperature of the gas.

3.5 Imaging setup

To detect the density distribution of the atoms, we use absorption imaging. In this technique, a resonant imaging beam is passed through the atomic cloud before being recorded on a camera (MatrixVision BlueFox3-1012bG). As the beam is much larger than the cloud, the resulting intensity pattern has a shadow at the position of the atoms. For light propagating in y -direction, the intensity profile is given by

$$I(x, z) = I_0(x, z)e^{-D(x, z)}, \quad (3.3)$$

where the optical density of the atoms, $D(x, z)$ is given by

$$D(x, z) = D_0(x, z) \frac{1}{1 + I/I_{\text{sat}}}, \quad (3.4)$$

$$\text{with } D_0(x, z) = \sigma_0 \int n(x, y, z) dy. \quad (3.5)$$

Here I is the light intensity, I_{sat} is the saturation intensity for caesium, $\sigma_0 = 3\lambda^2/(2\pi)$ is the absorption cross section and $n(x, y, z)$ is the density distribution of the atoms. The intensity of the laser beam must be smaller than the saturation intensity ($I/I_{\text{sat}} \ll 1$) to avoid saturation corrections.

The atomic density $n(x, z)$ can be determined by taking two images: one with the atoms present $I(x, z)$ and one without $I_0(x, z)$. To account for background noise in the images, a dark frame is always recorded without light and subtracted from the other two images.

Our imaging sequence proceeds as follows. First, all trapping potentials are removed, allowing the levitated atoms to expand for typically 20-100ms. To minimise the effects of interatomic interactions, the scattering length can be tuned to $0a_0$ during this time via the magnetic offset field. Subsequently, all magnetic fields are switched off for resonant absorption imaging. This introduces an additional delay of 2ms before the first image is acquired. This delay does not significantly affect the atomic density distribution.

3.6 Conclusion

This chapter has provided an overview of the experimental setup used to conduct the experiments shown in the following chapters. The vacuum system consists of two chambers where different parts of the experiment take place (Sec. 3.1). A first higher-pressure glass cell is used to create a 2D+MOT, while a low-pressure cell is used for

3.6. Conclusion

3D-MOT creation and subsequent experimental sequences. After the creation of a 3D-MOT (Sec. 3.4.1), we make use of degenerate Raman sideband cooling (Sec. 3.4.2) and evaporative cooling (Sec. 3.4.4) to reach the phase-space densities required for Bose-Einstein condensation. The state of the atoms is measured by using absorption imaging (Sec. 3.5) to cast a shadow of the cloud onto a CCD camera.

Chapter 4

Accordion lattice setup and experimental techniques

The aims of this chapter are twofold, first, it provides the setup and calibration of an optical lattice with variable lattice spacing, an optical accordion lattice [98–101], which is used in experiments in the following chapter (Chap. 5). This is largely based on the publication: R. Cruickshank, A. La Rooij, E. F. Kerr, T. Hilker, S. Kuhr, and E. Haller, “Tunable optical lattices for the creation of matter-wave lattice solitons”, *Opt. Express* **34**, 623-637 (2026), the article is cited as Ref. [102] in the rest of the thesis.

Second, it aims to connect how some of the theoretical concepts discussed in Chap. 2 are implemented into physical systems and the experimental nuances associated with this. This extends to a discussion about the methods used to calibrate the system, allowing for the conversion of units, e.g., the beam power in Watts to trap depth in recoil energies.

It begins with an overview of the accordion lattice, with methods used to calibrate the lattice spacing (Sec. 4.1.1) and one of its main use cases- to magnify the lattice by increasing the spacing, providing better imaging detection (Sec. 4.1.2). Section 4.2 discusses the methods used to calibrate the depth of both a static lattice and the accordion lattice. The tuning of interactions between atoms is a vital part of the experiments in the following chapters. Section 4.3 discusses the method to implement this, Feshbach resonances, and the associated need to calibrate the magnetic field and microwave generation. The chapter closes with methods used to prepare atoms in a flexible number of sites using microwave pulses (Sec. 4.4).

4.1 Accordion lattice setup

Accordion lattices provide a versatile tool for cold-atom experiments in preparation and imaging schemes [98–101]. They operate by varying the angle θ_L between two interfering laser beams, which generate the lattice potential through a periodic intensity pattern. Several methods for controlling θ_L have been demonstrated, including adjusting the relative angle between two mirrors [98, 103], changing the beam separation with a translating stage [100, 104], and employing an acousto-optic deflector (AOD) [99, 105]. In our setup, we combine the mechanically stable beam-splitting setup of Ref. [104] with the fast angle control provided by an AOD [Fig. 4.1(b)]. We

4.1. Accordion lattice setup

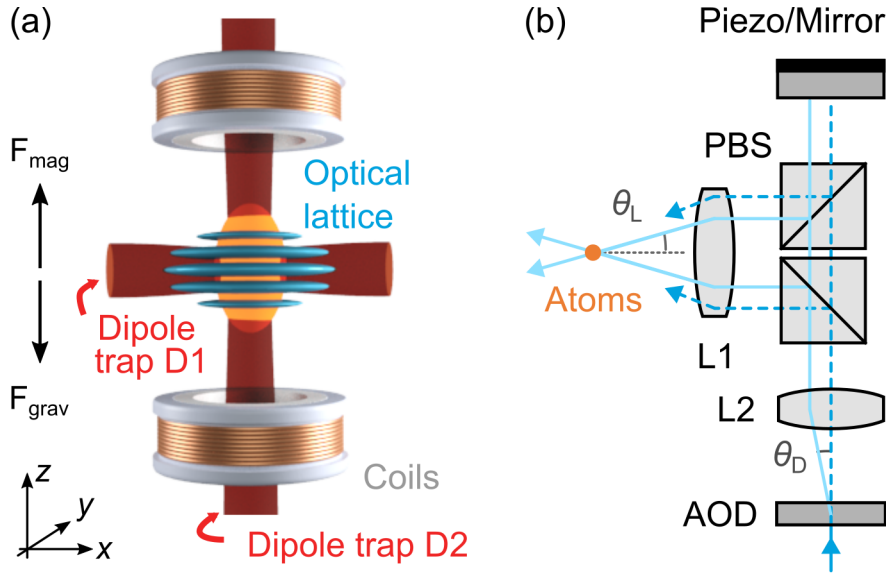


Figure 4.1: Experimental setup of the accordion lattice. (a) Configuration of laser beams and coils, and (b) of the accordion lattice. A beam is diffracted by an acousto-optical deflector (AOD) and split into two parallel beams by polarising beam splitters (PBS). These beams are focused by a lens (L1) onto the atomic cloud and interfere creating the accordion lattice potential.

achieve a tunable lattice spacing $d_L = 1.4(2) \mu\text{m}$ to $43.1(4) \mu\text{m}$, allowing access both to Bose–Hubbard physics with short tunnelling times and to individual potential wells of atoms with long tunnelling times, suitable for resolving individual lattice sites.

The working principle of the accordion lattice is illustrated in Fig. 4.1(b). An incident laser beam is diffracted by an AOD and split into two parallel beams by polarising beam splitters (PBS) [104]. These beams are focused by a lens (L1) onto the atomic cloud and interfere creating the accordion lattice potential. The lattice spacing at the position of the atoms is given by

$$d_L = \frac{\lambda}{2 \sin \theta_L}, \quad \text{with} \quad \tan \theta_L = \tan \theta_D \frac{f_{L2}}{f_{L1}}, \quad (4.1)$$

where $\lambda = 780\text{nm}$ is the blue detuned laser wavelength and θ_L is the angle between the interfering beams. This angle θ_L is determined by the focal lengths of lenses L1 ($f_{L1} = 100 \text{ mm}$) and L2 ($f_{L2} = 500 \text{ mm}$), and the lateral displacement of their input beams, which again depends on the deflection angle θ_D and f_{L2} . The AOD provides a bandwidth of 20 kHz, enabling precise and dynamic control of d_L and allowing for the generation of time-averaged lattice potentials. The lattice position is actively monitored via a separate interference pattern imaged on a CCD camera. Feedback stabilisation is achieved with a piezo-mechanical actuator that adjusts the position of a mirror, thereby controlling the lattice phase ϕ_L .

4.1.1 Calibration of the accordion lattice spacing

Our lattice setup enables dynamic control of the lattice spacing d_L over a wide range, from $1.4(2) \mu\text{m}$ to $43.1(4) \mu\text{m}$, allowing us to explore the system in three regimes. In

4.1. Accordion lattice setup

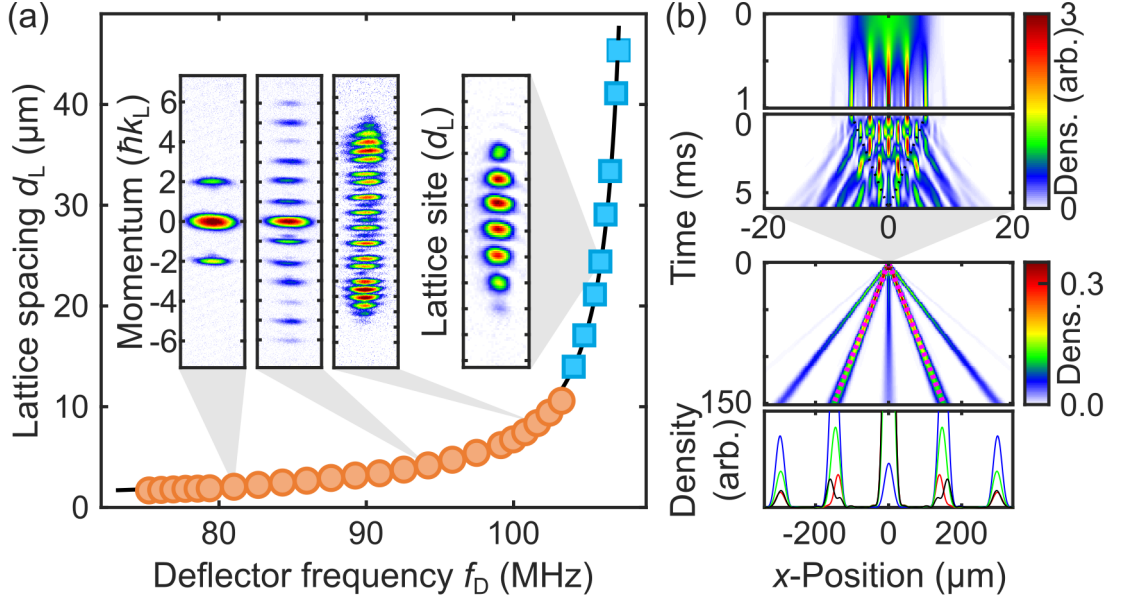


Figure 4.2: Calibration of accordion lattice spacing d_L . (a) Measurement of lattice spacing by determining the atom position (blue squares) and the momentum after expansion from the lattice (orange circles). Insets show absorption images after an expansion time 122 ms for deflector frequency $f_D = 82.7$ MHz, 94.2 MHz, 101.6 MHz, and in position space for 104.9 MHz (left to right). (b) Numerical simulation of the time evolution shows the density distribution for parameters $V_0 = 20E_r$, $d_L = 3\mu\text{m}$, $a_s = 10a_0$, $N = 10,000$ atoms. Panels, from top to bottom, show the evolution during 1 ms lattice pulse, the initial evolution with Talbot revivals, and the full evolution over 150 ms. Black dotted lines indicate the $\pm 2\hbar k_L$ momentum. Bottom: The final momentum peaks of the density distribution show a small dependence on pulse duration: 1.00 ms (blue line), 1.45 ms (green line), 1.70 ms (red line), and 2.00 ms (black line).

the tight-binding limit, characterised by deep lattice potentials and small lattice spacing, the system exhibits a well-defined band structure with its characteristic resonances. In contrast, in the shallow lattice regime, the periodic potential acts as a perturbation to the underlying continuous medium. For the experiments discussed in Chap. 5, we typically used atom numbers N between 200 and 3000 and lattice depth $V_0 \leq 6E_r$. The parameters were chosen to study the system with low three-body loss rates and on experimentally convenient times scales below one second. The third regime arises during the magnification process, where we suppress tunnelling in a deep lattice with large spacing, and the system can be effectively described as an array of isolated harmonic traps.

Due to its variability, accurately calibrating the lattice spacing requires two experimental methods, each suited to different regimes of d_L [98]. For $d_L > 10\mu\text{m}$, we directly resolved individual lattice sites by imaging the atomic density distribution after a time-of-flight expansion of 2 ms, during which we switch off the magnetic field [rightmost inset in Fig. 4.2(a)]. We extracted the centre positions of the density peaks using Gaussian fits and determined d_L by averaging the distance between neighbouring peaks [blue squares in Fig. 4.2(a)]. The image pixel size was calibrated independently

4.1. Accordion lattice setup

through comparison with the gravitational acceleration of the BEC.

For smaller spacings, we applied a short pulse of the lattice potential and determined the lattice momentum $\hbar k_L$ from the atoms' momentum distribution after 122 ms of free expansion [three leftmost insets in Fig. 4.2(a)]. This method, introduced in Refs. [106, 107], relies on the fact that the momentum profile after release from the lattice is dominated by discrete peaks at multiples of $2\hbar k_L$, which are visible in the absorption images. We extracted the centre momentum of each peak and calculated k_L from the spacing between the peaks. The lattice spacing was obtained using the relation $k_L = \pi/d_L$ [orange circles in Fig. 4.2(a)]. A variation of this method, where the atoms were adiabatically transferred into the lattice rather than pulsing the lattice potential, proved less suitable due to the loss of coherence at large lattice spacings.

The two methods diverge by approximately 10% around $d_L = 10.5 \mu\text{m}$ [Fig. 4.2(a)] (see below). We expect the direct spatial imaging to be accurate within the limits of our optical resolution and our ability to determine the centre position at each site using a Gaussian fit. In contrast, the momentum-based measurement increasingly deviates and becomes unreliable for larger lattice spacings, as discussed below. To perform a complete calibration of d_L , we excluded data points in the range $d_L = 6 - 10.5 \mu\text{m}$, and fit the remaining data using

$$d_L = A \frac{\sqrt{1 + B^2(f_D - C)^2}}{B(f_D - C)}, \quad (4.2)$$

which is a reformulation of Eq. (4.1) for small angles θ_D and deflector frequency f_D with fitting parameters A , B , and C [black line in Fig. 4.2(a)]. We found the lower limit of d_L to be set by the maximum angle θ_L permitted by the optical setup, while the upper limit is determined by the minimum achievable distance between the parallel beams produced by the beam splitters, typically constrained by the beam size [Fig. 4.1(b)].

To understand the small mismatch between the two methods we simulated the momentum profile of the atoms when pulsing the lattice potential with variable duration t_p [Fig. 4.2(b)]. First, the wave packet's ground state in the crossed dipole trap was determined with imaginary time propagation [108]. Second, a split-step Fourier method was used to propagate the wave function in time with the Gross-Pitaevskii equation [109]. Top to bottom panels in Fig. 4.2(b) show the atoms' time evolution during the different steps of the simulation. During the pulse duration, the wave packet oscillates between different lattice bands [106, 107] with the density distribution reflecting the states in the bands. After the pulse follows a brief time interval during which a Talbot-like interference pattern develops [110]. This pattern results from the interference of matter waves expanding from different lattice sites. It starts to fade once the wavefronts with the dominant momentum components $\pm 2\hbar k_L$ from the outermost sites reach the central site (black dashed lines). Finally, after 150 ms of free expansion a clear pattern of momentum components is visible. We used the peak positions of this pattern to determine d_L , assuming a linear expansion with peaks at multiples of $h/(md_L)t$.

Determining d_L by pulsing the lattice potential has three potential limitations at large values of d_L . First, resolving the $2\hbar k_L$ momentum components after expansion can become increasingly difficult, as the distances between the peaks scale with $1/d_L$. However, this is not a limiting factor in our case due to long expansion times at negligible interaction strength. Second, coherence between lattice sites can decrease at large

4.1. Accordion lattice setup

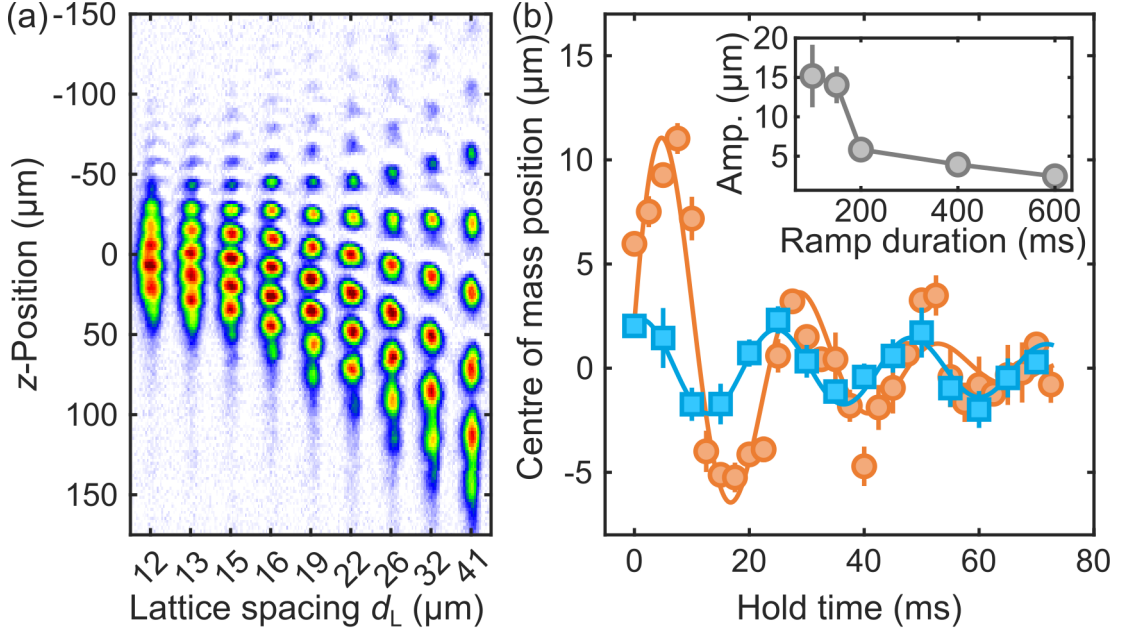


Figure 4.3: Magnification of density profile. (a) Absorption images of density profile in the optical lattice after a short expansion time of 2 ms. Parameters $V_0 > 200 E_r$, ramp $\Delta t = 400$ ms. Shift of centre position of $0.87 d_L$ for the image where $d_L = 22 \mu\text{m}$. (b) Oscillation of wave packet in central lattice site after magnification to $d_L = 30 \mu\text{m}$ in 150 ms (orange circles) and 600 ms (blue squares). Solid lines denote damped sine fits to the data. Inset: measured oscillation amplitude for varying ramp duration Δt , error bars denote the 2σ error, line is a guide to the eye.

lattice spacings, leading to a diminished interference pattern. For our measurements, this effect is negligible due to the short pulse durations. However, the loss of coherence becomes a significant limitation when measuring the free expansion after adiabatically loading into a lattice. Finally, the simulation reveals that the peak position can shift depending on the pulse duration [bottom panel in Fig. 4.2(b)]. The momentum of the peaks varies slightly as the atoms oscillate between bands during the pulse duration. We believe this last effect contributes to the discrepancy observed between the two calibration methods in Fig. 4.2(a).

4.1.2 Detection with magnification

One of the primary roles of the accordion lattice in our sequence is to increase the lattice spacing at the end of the experimental cycle, 'magnifying' the sites to resolve them individually [105, 111]. In our setup, individual sites are distinguishable for $d_L \geq 6 \mu\text{m}$. However, we magnified the lattice to approximately $d_L = 20 \mu\text{m}$ to provide a clear separation of sites for measurements of atom number. To magnify the atomic density distribution we suppressed tunnelling by increasing the lattice depth to $V_0 = 100 E_r$ in 5 ms, followed by a slower increase of d_L to $20 \mu\text{m}$ in 400 ms. The final lattice depth was greater than $300 E_r$ with a tunnelling time of $\hbar/J > 10$ s, where J is the tunnelling matrix element for atoms in the lowest band. In addition to an improved resolution, this magnification sequence reduces the atomic density in each lattice site,

4.1. Accordion lattice setup

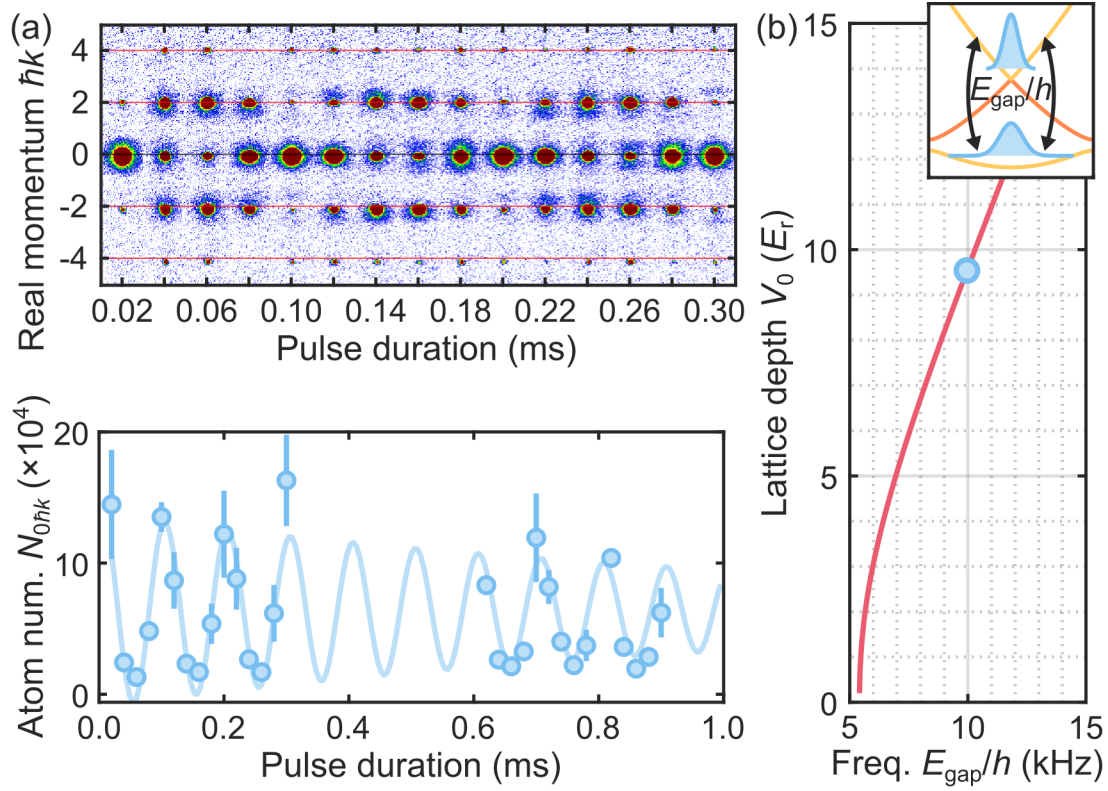


Figure 4.4: Optical lattice depth calibration by measurement of inter-band oscillation frequencies for lattice with constant spacing, $d_L = 1064\text{nm}$. (a) Upper: real momentum distribution of atoms after exposed to the lattice of varying duration, the different atom peaks are the various momenta. Images taken after 70 ms of expansion duration. Lower: extracted atom number $N_{0\hbar k}$ in the $0\hbar k_L$ momentum mode (central peak in images). (b) Calculated oscillation frequencies between the first and third bands for varying lattice depths. The blue point indicates a depth of $9.54 E_r$, found using the data in (a). Inset: visualisation of the inter-band oscillations observed.

avoiding the re-absorption of imaging light which could produce an apparent increase in atom number [112].

The magnification inherently shifts off-centre lattice sites due to an increase in the spacing d_L . Additionally, we observed a small shift of the cloud position [Fig. 4.3(a)] caused by variations in the intersection point of the lattice beams as the angle θ_L was adjusted. Both displacements can induce centre-of-mass oscillations of the atoms within each site that persist even after the magnification process. However, the measurements of the final occupation numbers per site were unaffected by those oscillations, and we ignored them except for the calibration of d_L . The amplitude of those oscillations was reduced by increasing the duration of the magnification ramp (see below). A second challenge arises from the finite size of the lattice beams with a waist of $200\mu\text{m}$. Depending on the magnification, this finite size limited the number of observable lattice sites, e.g., to approximately ten sites for a final spacing of $d_L = 20\mu\text{m}$. Due to the Gaussian intensity profile of the lattice beams, the lattice depth decreased at larger distances from the center, allowing atoms in outer sites to escape.

Absorption images in Fig. 4.3(a) illustrate the density distribution for increasing

4.2. Calibration of the optical lattice depth

d_L after magnification. The images reveal the shift of the lattice centre position (here $0.87 d_L$ for $d_L = 22 \mu\text{m}$) as well as the loss of atoms at the edge of the lattice beams. For the study of solitons, we adjusted the magnification to the specific problem, using a larger magnification for single-site solitons compared to multi-site solitons. The amplitude of the centre-of-mass oscillations depends strongly on the duration of the magnification ramp Δt [Fig. 4.3(b)]. For $\Delta t = 150 \text{ ms}$, the oscillations show an initial amplitude of $14(3) \mu\text{m}$ ($0.46 d_L$), which was significantly reduced to $2.4(7) \mu\text{m}$ for a slower ramp of $\Delta t = 600 \text{ ms}$. To balance experimental run time and minimise excitations, we chose an intermediate value of $\Delta t = 400 \text{ ms}$ [see inset of Fig. 4.3(b)].

4.2 Calibration of the optical lattice depth

As discussed in Sec. 2.2, the depth of an optical lattice potential is directly proportional to the power of the laser beams. In practice, however, it can be challenging to accurately determine parameters of an existing optical setup, such as beam waist. For this reason, it is beneficial to calibrate the lattice depth experimentally using the atoms themselves. Over the years many different techniques have been developed to do this such as [7]: parametric heating, Landau-Zener tunnelling or simply measuring the separation of the momentum peaks after a time-of-flight. For lattices with a wavelength of $d_L = 532 \text{ nm}$, the common method is to measure the frequency of band oscillations when pulsing on the lattice potential for short time intervals [107]. As the separation between different energy bands scales with the depth V_0 , measuring this energy scale provides a direct way to retrieve V_0 .

The time evolution of the atomic wavefunction in the lattice is most naturally described in the Bloch basis (Sec. 2.2.1), which consists of the eigenstates of the system Hamiltonian. The initial state of the BEC $|\phi\rangle$ can be projected onto this basis of Bloch states $|n, q\rangle$ with complex amplitudes

$$a_{n,q}(0) = \langle n, q | \phi \rangle, \quad (4.3)$$

where $n = 0, 1, 2, 3, \dots$ is the band number and q is the quasimomentum. In the lattice the wavepacket evolves according to

$$|\psi(t)\rangle = \sum_{n,q=0}^{\infty} a_{n,q}(0) e^{\frac{-iE_n(q)t}{\hbar}} |n, q\rangle. \quad (4.4)$$

As a result, wave packets with a narrow momentum distribution in the centre of the band i.e., $q \approx 0$, show oscillation frequencies caused by the energy differences between the bands, $\omega_{\text{gap}} = (E_{n,0} - E_{n',0})/\hbar$ [inset in Fig. 4.4(b)].

The calibration began with a BEC of 1×10^5 atoms. To initiate the atoms in the zeroth and second bands the lattice was non-adiabatically pulsed on for fractions of a second. After a variable hold time t the lattice potential was removed, projecting the time evolved state onto plane waves and revealing the momentum distribution. The resulting distribution [upper plot in Fig. 4.4(a)] shows the oscillating atom number in each mode. The extracted frequency in the atom number $N_{0\hbar k_L}$ in the $0\hbar k_L$ [lower plot in Fig. 4.4(a)] can be compared directly to the calculated values [Fig. 4.4(b)] to find the lattice depth.

4.2. Calibration of the optical lattice depth

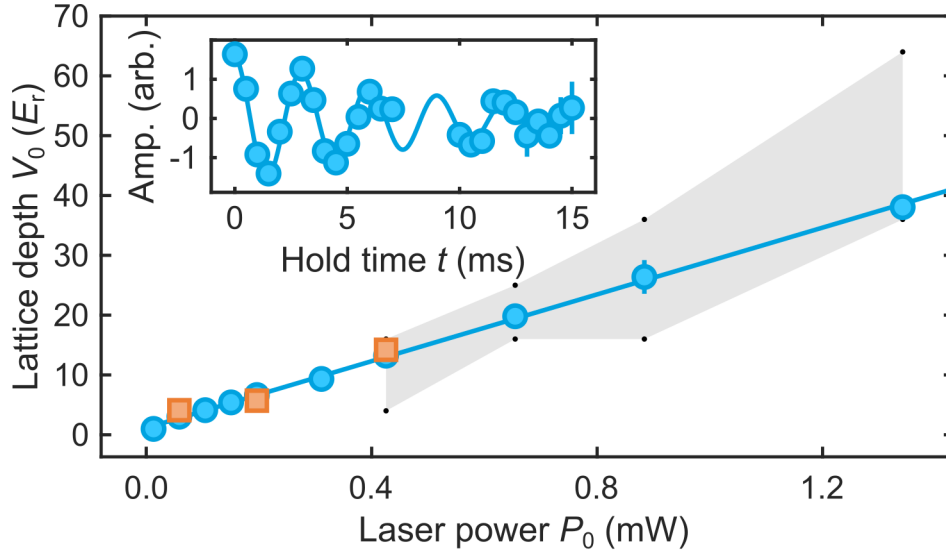


Figure 4.5: Calibration of lattice depth V_0 of the accordion lattice. Determination of V_0 by measuring the trap frequency ω_L at each lattice site via centre-of-mass oscillations (blue circles), and by measuring oscillations between energy bands (orange squares). The gray shaded area indicates the range of depths achievable using the method described in Ref. [103], with black dots showing the data points ($d_L = 3 \mu\text{m}$ throughout). Inset shows measured oscillation for $P_0 = 0.75 \text{ mW}$, solid line is a damped sinusoidal fit.

As the relationship between V_0 and laser power P is expected to be linear the measurement shown in Fig. 4.4(a) was repeated with different values of P to gain the gradient $\Delta V_0 / \Delta P$. In our setup, typical values of laser power per recoil energy were $7.6 \text{ mW} / E_r$.

4.2.1 Accordion lattice depth calibration

Compared to the previous section, the calibration of the lattice depth of the accordion lattice is more complicated due to its variability in spacing. Similar to the calibration of d_L in Sec. 4.1.1, it requires multiple methods to cover the full range of lattice spacings. We again determined V_0 in units of the recoil energy, however this now depends on the laser power and lattice spacing. We calibrated it across the full parameter space spanned by P_0 and d_L . For clarity, however, we present here only a representative slice of this space at fixed lattice spacing $d_L = 3.0 \mu\text{m}$.

We first measured V_0 by measuring the trap frequency ω_L at individual sites by directly monitoring the oscillation of atoms within a single well. To prepare the system, we loaded approximately 2,000 atoms into the central lattice site using a tightly confining dipole trap (beam D1) with an axial trapping frequency of $\omega_z = 2\pi \times 100 \text{ Hz}$. We then slowly shifted the phase ϕ_L of the optical lattice by adjusting a piezo-driven mirror [Fig. 4.1(b)] over 100 ms, and simultaneously removed the potential of D1. Oscillations were initiated by rapidly returning ϕ_L to its original value within 2 ms. The resulting oscillations of the atoms were measured after an expansion time of 22 ms [inset in Fig. 4.5]. To ensure that the atoms remained within a single site, we adjusted

4.3. Calibration of the magnetic field

the magnitude of the phase shift for different lattice depths. The lattice depth in E_r was calculated using

$$\frac{V_0}{E_r} = \frac{\omega_L^2 m^2}{\hbar^2 k_L^4} \text{ and } V_0 = CP_0, \quad (4.5)$$

where C is a constant that contains beam parameters, such as waist, overlap, and frequency detuning from atomic transitions. The resulting measurements show the expected linear dependence of lattice depth on beam power P_0 [blue circles in Fig. 4.5]. This method is limited to sufficiently large values of P_0 and d_L , such that the lattice trap frequency remains larger than the frequency of inter-site tunnelling.

The second method for calibrating V_0 was based on interband oscillations of the atoms [7, 106, 107], used in the previous section. This method was better suited for lattices with $d_L \leq 3 \mu\text{m}$ as there is a well defined band structure. In this approach, the lattice was rapidly pulsed on for a short duration t_p , placing the atoms in a superposition of states in the odd energy bands. Band populations oscillate during t_p with frequencies that depend on the energy gaps between the bands [106]. We determined the band population in quasi-momentum space using a band-mapping technique and a free expansion time of 120ms. Our measurements [orange squares in Fig. 4.5] are in good agreement with the values obtained from the position-space oscillation method. The method works usually best for shallow lattices when only one energy band is excited.

Finally, the last method to calibrate V_0 employed a variation of the previous technique [103]. The lattice is again imposed onto the atoms for only a short period of time ($t_p \sim 1 \text{ ms}$) until the momentum distribution shows a maximum number of modes N_{max} . Assuming that the momentum modes are created by atoms in odd bands, the lattice depth can then be approximated using $V_0/E_r = 4N_{\text{max}}^2$. This method works best for deep lattices when many bands are excited. However, the discreteness of N_{max} presents a challenge for shallow lattices due to the a large relative uncertainty of N_{max} . We used the shaded area in Fig. 4.5 to indicate this uncertainty, assuming $\Delta N_{\text{max}} = \pm 1$. Though less precise, the results agree well with the previous measurements.

4.3 Calibration of the magnetic field

Precise knowledge of the magnetic field is essential in ultracold atomic physics experiments. In an ideal scenario, the background field would remain constant, and a calibration would only be needed on once. However, in practice, environmental fluctuations lead to gradual drifts in the magnetic field over timescales of days to weeks. As a result, regular recalibration is necessary. Fortunately, atoms themselves provide a highly sensitive probe for measuring magnetic fields.

For the experiments described in this thesis, one of the main use cases of the magnetic field was to tune the interatomic interactions. This is achieved through the use of Feshbach resonances, and as such to motivate the calibration, these are described in Sec. 4.3.1.

4.3. Calibration of the magnetic field

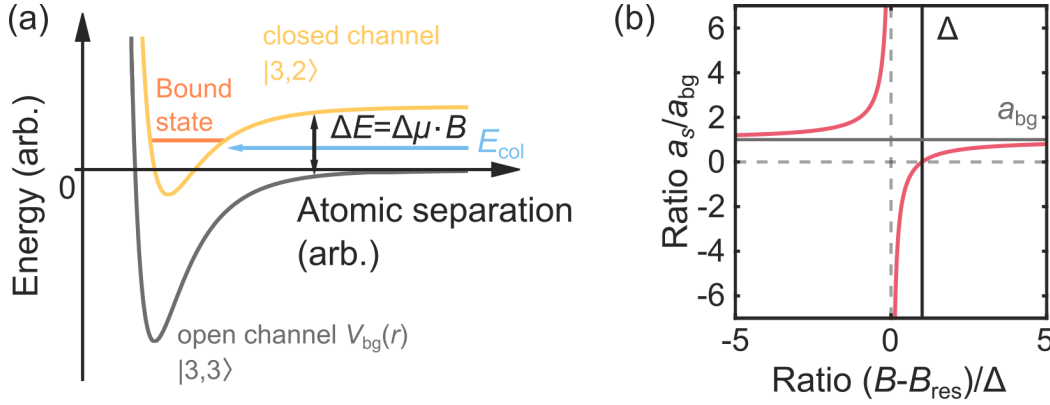


Figure 4.6: Feshbach resonance in ultracold atoms. (a) Visualisation of the mechanism which causes Feshbach resonances. Two colliding atoms have an associated energy E_{col} and experience a potential known as the open channel. A separate channel (the closed channel) is inaccessible and supports a molecular bound state. Due to the difference in atomic energy states between the two channels the energy of the closed channel can be shifted to bring the bound state into degeneracy with the collisional energy. (b) Feshbach resonance of width ΔB showing divergence of scattering length around the magnetic field strength B_{res} . A constant background scattering length a_{bg} is present from the potential V_{bg} and shifts the resonance vertically upwards.

4.3.1 Feshbach resonances

To control the scattering length between the atoms, we make use of Feshbach resonances. These were first theorised independently by Feshbach [113, 114] in the field of nuclear physics and Fano [115] in the field of atomic physics to explain resonance phenomena. Magnetically induced Feshbach resonances were first discussed in the 1970s [116], before being used experimentally in the late 1990s to tune the scattering length in a quantum gas of sodium atoms [117].

The general model to describe a Feshbach resonance involves two collision channels [Fig. 4.6(a)]. When two atoms collide with relative energy E_{col} [denoted by an arrow in Fig. 4.6(a)], their interaction is described by the so-called open channel [8]. In addition, there exists a second channel, called the closed channel, which is not energetically accessible. This channel supports a molecular bound state, in which the two colliding atoms can temporarily form a molecule. Atoms in different channels are in different atomic states meaning the energy level of the closed channel can be shifted through an external magnetic field. The two channels can couple if the energy of the bound state matches to the collision energy E_{col} , this is the mechanism of the Feshbach resonance. For magnetic fields close to this degeneracy, the s -wave scattering length can be approximated by [8]

$$a_s(B) = a_{\text{bg}} \left(1 - \frac{\Delta}{B - B_{\text{res}}} \right). \quad (4.6)$$

Here, a_{bg} is the background scattering length, Δ is the resonance width, B is the applied magnetic field strength and B_0 is resonance position. The scattering length diverges as the magnetic field approaches B_{res} , while it tends towards the background value a_{bg} far

4.3. Calibration of the magnetic field

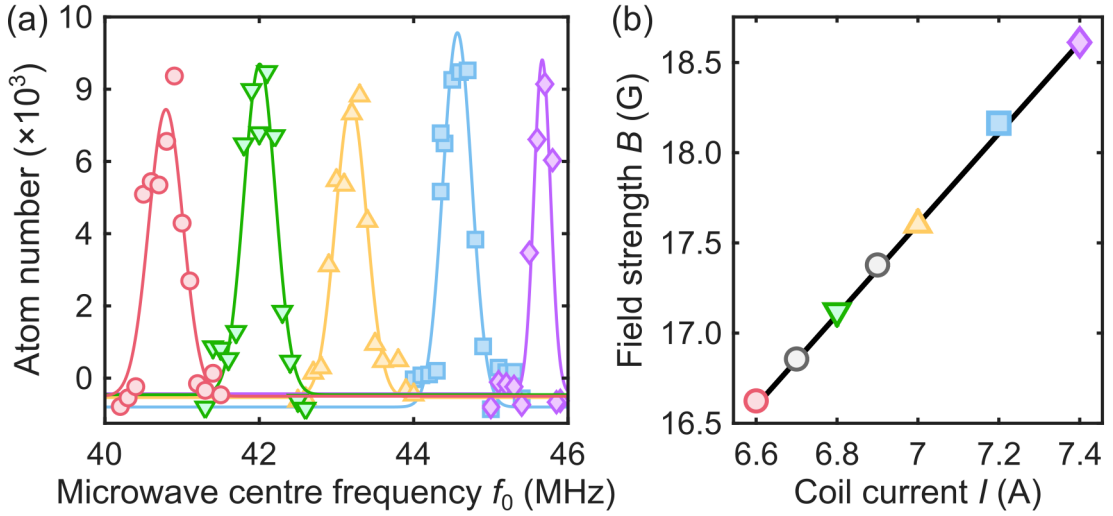


Figure 4.7: Calibration of the magnetic field strength using microwave spectroscopy. (a) Measured atom number after Gaussian microwave pulses of varying frequency for different currents supplied to the offset coils (red 6.6 A, green 6.8 A, yellow 7.0 A, blue 7.2 A and purple 7.4 A). The centre frequency f_0 is relative to the $|F = 3\rangle \rightarrow |F = 4\rangle$ transition of 9.19 GHz. (b) Calculated B-field strength from the resulting Zeeman splitting for the measurements in (a). Straight line is a fit to the data with equation $B = 2.5162I + 0.0098$. Colours of data points indicate field strength found from corresponding data sets in (a), grey points indicate data sets not shown in (a).

from resonance [Fig. 4.6(b)].

For collisions of atoms in the $|F = 3, m_F = 3\rangle$ state caesium possesses more than sixty Feshbach resonances at magnetic fields $B < 250$ G [118], with a zero crossing of a_s at 17.1 G. This zero crossing allows us to tune interactions from repulsive to attractive.

4.3.2 Microwave spectroscopy

Microwave spectroscopy provides a powerful technique to calibrate the magnetic field strength at the position of the atoms. The approach relies on the Zeeman shift of the internal hyperfine atomic states in a magnetic field, meaning the transition frequencies are dependent on the field strength.

To measure the magnetic field, we prepare the Bose–Einstein condensate (BEC) in the $|F = 3, m_F = 3\rangle$ state and drive a microwave transition that transfers the atoms to the $|F = 4, m_F = 4\rangle$ state. Absorption imaging is then performed using the cycling transition from $|F = 4\rangle$ to $|F' = 5\rangle$ [see Fig. 3.3]. In contrast to our standard absorption imaging procedure, the microwave transition replaces the usual optical repumping step. The microwave frequency depends on the shift of the Zeeman levels with magnetic field. For low magnetic field strength in the Zeemann regime, the energy separation between the states is directly proportional to the applied magnetic field strength.

Microwave pulses were applied for 1 ms with a linear frequency sweep and with a varying centre frequency f_0 . This frequency f_0 was scanned around the bare transition frequency of 9.19 GHz. To ensure overlapping scans, the width of the frequency

4.4. Microwave-assisted atom removal

sweep was twice as large as the measurement steps. Typical widths were 20 kHz with measurement steps spaced by 10 kHz.

The number of transferred and detected atoms shows a peak when level spacing matches to the microwave frequency, $dE = \hbar\omega$ [Fig. 4.7(a)]. We determined the transition frequency at the centre of the peak using a Gaussian fit. To calibrate the relation between current in the offset coils and the resulting magnetic field, we repeated the procedure for different offset currents.

For improved precision, we use the Breit Rabi formula to determine the magnetic field [85]

$$E = -\frac{\Delta E_{\text{hfs}}}{2(2I+1)} + g_I \mu_B m_F B \pm \frac{E_{\text{hfs}}}{2} \left(1 + \frac{4m_F x}{2I+1} + x^2 \right)^{1/2}. \quad (4.7)$$

Here, E is the measured transition energy, E_{hfs} is the bare hyperfine splitting $h \cdot 9.19$ GHz, the quantum number $I = 7/2$ and

$$x = \frac{(g_J - g_I) \mu_B B}{\Delta E_{\text{hfs}}}. \quad (4.8)$$

The relationship between coil current and the magnetic field is linear [Fig. 4.7(b)], which can be used to convert the current in the coils to the field generated. This allows for the scattering length to be known from the following equation [64, 119]

$$a_s(B) = a_{\text{bg}} \prod_j^4 \left(\frac{B - B_{s,j}}{B - B_{\text{res},j}} \right), \quad (4.9)$$

where the background scattering length for caesium is $a_{\text{bg}} = 1875 a_0$, $B_{s,j} = 17.119, 47.944, 53.457, 19.805$ G and $B_{\text{res},j} = -11.1, 47.78, 53.449, 19.8$ G.

4.4 Microwave-assisted atom removal

We employed a combination of microwave transitions and short pulses of resonant light to prepare a single occupied lattice site, which served as the starting point for soliton creation described in the next chapter. Microwave-assisted atom removal is a common technique for creating single occupied layers in quantum gas microscopes [73, 120–122]. Addressing individual lattice sites is challenging at small lattice spacings of only a few hundred nanometres, as it requires precise control of magnetic field gradients and lattice angles [122]. Accordion lattices simplify this preparation process due to their variable lattice spacing. We studied the preparation process in two steps: we first optimised the removal of atoms in a single site of the accordion lattice before implementing the removal of atoms in all sites but one.

We use the levitating magnetic field gradient, $\partial_z B$ (see Sec. 3.2), to enable selective spatial addressing of atoms along the z -direction. The gradient shifts the energy levels of the Zeeman substates $|F=3, m_F=3\rangle$ and $|F=4, m_F=4\rangle$ that were used for the microwave transition. The gradient makes the microwave transitions spatially dependent with a frequency shift of 7.5 kHz/ μm . To achieve site-selective transfer, we employ hyperbolic secant (HS1) pulses [123], which maximise population transfer at

4.4. Microwave-assisted atom removal

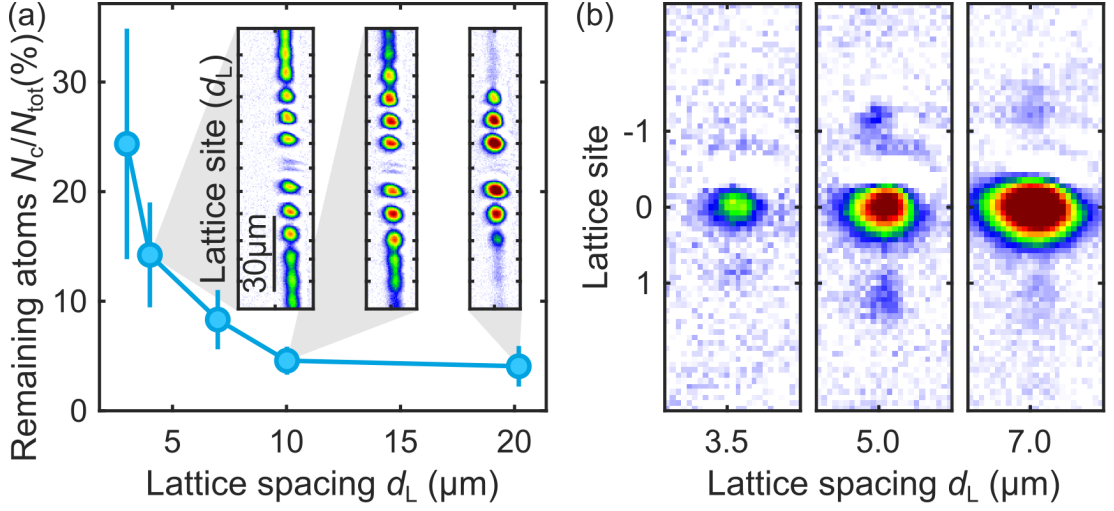


Figure 4.8: Microwave-assisted atom removal. (a) Fraction of atoms remaining in the central site after the removal process for varying d_L . Sweep parameters of the HS1 pulse were optimised for each lattice spacing with widths of $\delta_0 = 15 \text{ kHz}$ and 90 kHz for $d_L = 3 \mu\text{m}$ and $20 \mu\text{m}$, respectively (lattice depth was $V_0 = 216 E_r$ defined at $d_L = 3 \mu\text{m}$ throughout). The insets show averaged absorption images for different d_L . (b) Averaged images of single-site preparation in lattices with spacing $d_L = 3.5 \mu\text{m}$, $5 \mu\text{m}$, and $7 \mu\text{m}$. Images show density profile after magnification.

resonant lattice sites while suppressing transfer in neighbouring sites. These pulses produce a flat-top transfer profile with sharp spectral edges [122, 124] by sweeping the microwave detuning $\delta(t)$ and coupling amplitude $\Omega(t)$ according to hyperbolic tangent and secant functions, respectively

$$\delta(t) = (\delta_0/2) \tanh(2t/\tau), \Omega(t) = \Omega_0 \text{sech}(2t/\tau), \quad (4.10)$$

where $\Omega_0 = 9.9(1.0) \text{ kHz}$ is the Rabi frequency of the transition, δ_0 is the frequency width of the pulse and τ is a characteristic timescale. The smoothness of the switch on was controlled using the ratio $\tau/T = 1/3$, where T is the pulse duration. After transfer, atoms in the $|F = 4\rangle$ state are no longer levitated and accelerate downwards. To prevent collisions with other atoms, we remove them using a 1 ms pulse of a horizontal laser beam resonant with the $|F = 4 \rightarrow F' = 5\rangle$ transition [125].

We first characterised the removal efficiency of our scheme for atoms in the central lattice sites [Fig. 4.8(a)]. For each lattice spacing d_L , we optimised the HS1 pulse parameters to minimise the number of atoms remaining in the target site. Depending on d_L , the frequency sweep widths ranged from $\delta_0 = 15$ to 90 kHz . After all sweeps, the neighbouring sites remained unaffected, allowing us to apply the pulses more than once to achieve larger efficiencies. However, for a single pulse in lattices with spacings larger than $10 \mu\text{m}$, approximately 5% of the atoms remained, whereas for $d_L = 3 \mu\text{m}$ this fraction increased to 25%. The remaining atoms show as two weak horizontal stripes in the *in situ* images [insets Fig. 4.8(a)], which is an artifact of our imaging system caused by diffraction from small objects.

In a second measurement, we studied the preparation of only one occupied site. Our scheme used two broad HS1 pulses with frequency sweeps that addressed atoms

4.4. Microwave-assisted atom removal

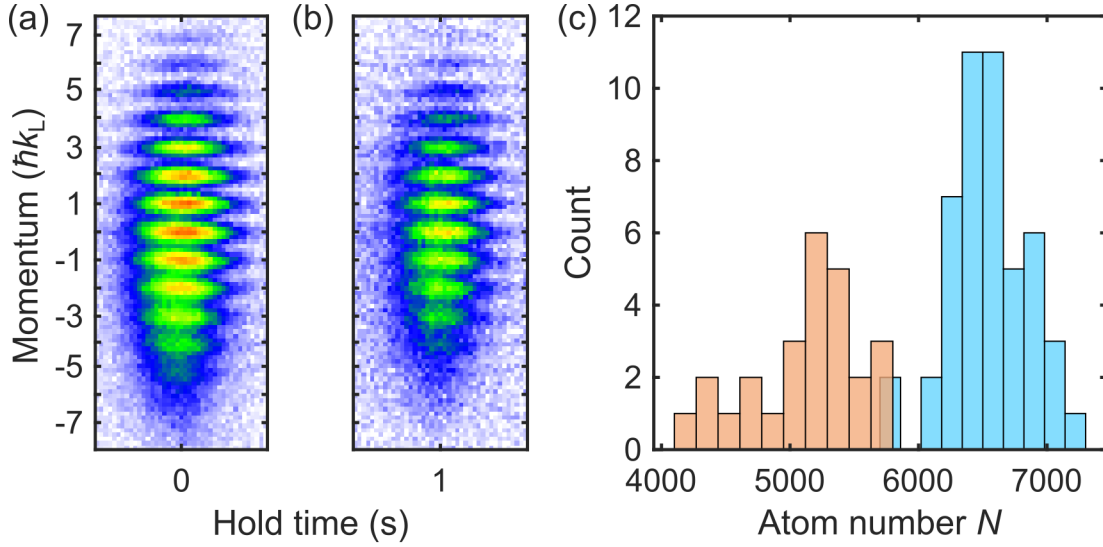


Figure 4.9: Momentum distribution of two interfering sites after a hold duration in the lattice potential. (a) Averaged image of momentum distribution of two occupied lattice sites after 0s hold duration and 40ms expansion duration. Lattice spacing of $d_L = 10.3 \mu\text{m}$, and depth of $V_0 = 200 E_r$. (b) Momentum distribution of the atoms after a hold duration of 1 s. (c) Histograms of total atom number after 0s (blue) and 1 s (orange). Mean of $6.5(3) \times 10^3$ (blue) and $5.1(4) \times 10^3$ (orange).

above and below the site. Figure 4.8(b) shows images of the preparation of a single lattice site for varying lattice spacings, where a single site could be reliably prepared in lattices with spacing $d_L = 3 - 3.5 \mu\text{m}$ with minimal atoms in neighbouring sites [left-most image in Fig. 4.8(b)]. The variation of the atom number between the images is a result of the loading process and was not from the removal of atoms by the microwave scheme. All images in Fig. 4.8(b) show the preparation after the two frequency sweeps, each of $T = 20 \text{ ms}$ duration, however we increased the sweep width δ_0 by $\sim 20 \text{ kHz}$ when $d_L = 7 \mu\text{m}$ to account for the increased lattice spacing.

For the single-site preparation, we achieved a removal efficiency of 90% for atoms in the outer sites, while less than 6% of the atoms in the central site were removed. To verify that the preparation sequence did not induce significant heating of the remaining atoms, we repeated the sequence while leaving two sites occupied and performed expansion-time measurements to observe interference fringes [Fig. 4.9(a)]. The fringes persisted even after a 1 s hold time in the accordion lattice [Fig. 4.9(b)], confirming the absence of any significant excitations. There was however, a decrease in atom number observed [Fig. 4.9(c)], with a mean atom number of $6.5(3) \times 10^3$ immediately after loading [blue data in Fig. 4.9(c)] and $5.1(4) \times 10^3$ after the hold duration [orange data in Fig. 4.9(c)].

Chapter 5

Experimental observation of single- and multisite matter-wave solitons in an optical accordion lattice

This chapter presents the experimental observation of single- and multisite matter-wave solitons in an optical accordion lattice. While solitons in continuous systems have been extensively studied, the presence of a periodic potential introduces new length and energy scales, leading to the emergence of lattice solitons with distinct properties. In particular, the interplay between atomic interactions and the lattice potential allows for the formation of localised states that can extend over a single lattice site or multiple sites, with stability determined by the underlying energy landscape. This chapter is an extension of the article R. Cruickshank, F. Lorenzi, A. La Rooij, E. F. Kerr, T. Hilker, S. Kuhr, L. Salasnich, and E. Haller, “Experimental Observation of Single- and Multisite Matter-Wave Solitons in an Optical Accordion Lattice”, *Phys. Rev. Lett.* **135**, 263404 (2025), the article is cited as Ref. [111] in the rest of the thesis.

This chapter is structured as follows. Section 5.2 introduces the theoretical framework for solitons in continuous media, followed by an extension to lattice systems in Sect. 5.3. Several theoretical approaches can be used to analyse solitons and their stability, including the one-dimensional Gross–Pitaevskii equation, a general variational method, and the nonpolynomial Schrödinger equation. However, accurately reproducing the experimental observations requires a full three-dimensional Gross–Pitaevskii description, which is therefore also introduced in Sec. 5.3, for more details on the theoretical methods the reader should refer to the thesis of F. Lorenzi [126].

Section 5.4 describes the experimental preparation of the initial atomic states in the accordion lattice and the techniques used to control the occupation of individual sites. Results for single and multisite solitons are presented in Sec. 5.6 and Sec. 5.7. The chapter closes with a study of the dynamical collapse of multisite solitons (Sec. 5.7.1).

5.1 Introduction

Solitons are localised, non-dispersive wave packets that preserve their shape during propagation due to a balance between dispersion and nonlinearity [127]. In Bose–Einstein condensates (BECs), solitons arise as nonlinear matter-waves governed by the

5.2. Solitons in continuous media

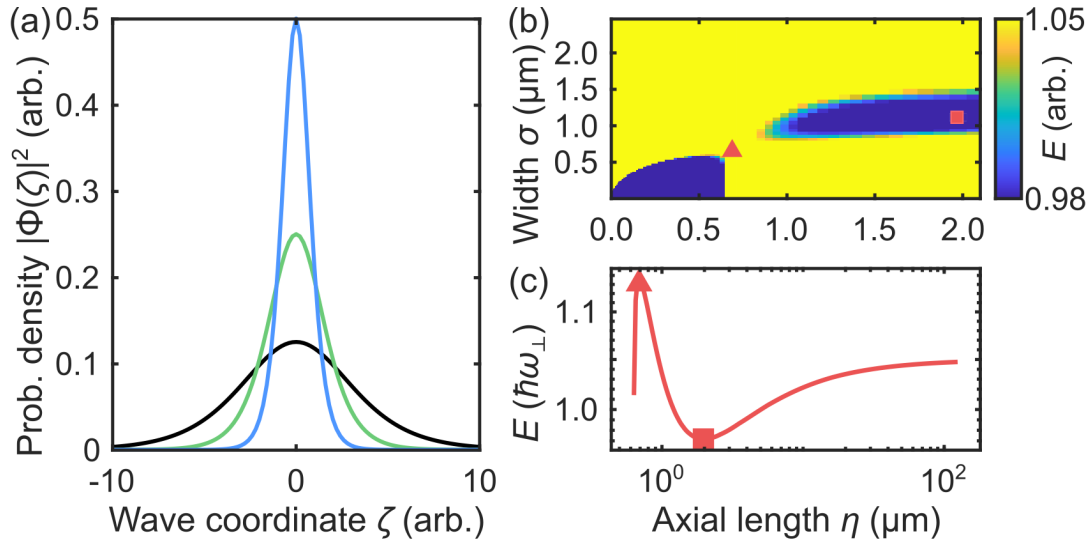


Figure 5.1: Soliton in a continuous potential. (a) Axial density profiles of soliton solutions calculated using the 1D-GPE [Eq. (5.3)] for nonlinear parameters $\gamma = -1$ (black), -2 (green), -4 (blue). (b) Energy landscape calculated using the approximate variational approach [Eq. (5.10)] for different axial lengths η and radial widths σ of a Gaussian wavepacket. The soliton forms at the minima with non-zero length scales, marked with the square. Parameters: $N = 3000$, $a_s = -5a_0$, $\omega_{\perp} = 2\pi \times 50\text{Hz}$. (c) Minimum energy line of data in (b), the soliton minimum is prevented from collapse by the left barrier (triangle) and dispersion from the right barrier.

Gross–Pitaevskii equation with two prominent examples [31]. Firstly, bright matter-wave solitons, which are localised density peaks on a low-density background, have been subject of extensive experimental and theoretical investigations [128], studying their formation [20, 21], collapse dynamics [129–131], collisions [132], and interactions with external potentials, such as reflection and transmission at barriers [23, 133], among other phenomena [134, 135]. Secondly, dark matter-wave solitons are characterised by density dips accompanied by a phase slip in a higher-density background [25] and have been created [26, 27] and observed interacting with other solitons [136, 137]. Various techniques have been used to realise matter-wave solitons, including rapid or adiabatic changes in the atomic interaction strength [21, 138, 139], phase imprinting [26, 27], and the exploitation of modulational instabilities in the carrier medium [140].

5.2 Solitons in continuous media

Bright matter-wave solitons form when the kinetic and nonlinear terms in the governing Gross-Pitaevskii equation [Eq. (2.32)] are balanced. For BECs in continuous systems stable bright solitons are only expected to form in approximately one dimensional systems. Solving the 3D-GPE can be difficult, therefore it can be useful to derive an approximate equation in only 1D [141]. In this context, the system is effec-

5.2. Solitons in continuous media

tively one-dimensional due to strong radial confinement of the form

$$V_{\text{ext}}(\mathbf{r}) = \frac{1}{2}m\omega_{\perp}^2(x^2 + y^2), \quad (5.1)$$

where $\omega_{\perp}$ is the harmonic oscillator frequency in the radial dimension. For the wave function of the BEC, we use an Ansatz with a Gaussian distribution in the radial direction and the wave function $\phi(z, t)$ along the longitudinal direction

$$\psi(\mathbf{r}, t) = \phi(z, t) \frac{1}{\sqrt{\pi}a_{\perp}} \exp\left(-\frac{x^2 + y^2}{2a_{\perp}^2}\right). \quad (5.2)$$

Minimising the energy functional of the GPE, one can derive the 1D-GPE [141]

$$i\hbar \frac{\partial}{\partial t} \phi(z, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + \gamma |\phi(z, t)|^2 \right] \phi(z, t), \quad (5.3)$$

where

$$\gamma = \frac{(N-1)g}{2\pi a_{\perp}^2}. \quad (5.4)$$

is the nonlinear strength from the interactions. Equation (5.3) admits solutions of the form [142, 143]

$$\Phi(\zeta) = \sqrt{\frac{\gamma}{2}} \text{sech}(\gamma\zeta), \quad (5.5)$$

where $\zeta = z - vt$. These solutions propagate with velocity v while maintaining their shape, characteristic of solitonic behaviour, with the radius reducing as γ gets larger [Fig. 5.1(a)].

Nonpolynomial Schrödinger equation

The 1D-GPE derived above assumes a constant width in the radial direction. A more realistic assumption uses an Ansatz with a width $\sigma(z, t)$ that varies with z -position and time [141],

$$\psi(\mathbf{r}, t) = \phi(z, t) \frac{1}{\sqrt{\pi}\sigma(z, t)} \exp\left(-\frac{x^2 + y^2}{2\sigma(z, t)^2}\right). \quad (5.6)$$

Using this ansatz allows for the 1D nonpolynomial Schrödinger equation (1D NPSE) to be derived

$$i\hbar \frac{\partial}{\partial t} \phi(z, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_{\text{ext}}(z) + \frac{\hbar^2}{2m\sigma^2} + \frac{m\omega_{\perp}^2\sigma^2}{2} + \frac{2\hbar^2 a_s(N-1)|\phi(z, t)|^2}{m\sigma^2} \right] \phi(z, t), \quad (5.7)$$

where

$$\sigma^2 = a_{\perp}^2 \sqrt{1 + 2a_s |\phi(z, t)|^2}. \quad (5.8)$$

For weak interactions the 1D NPSE and the 1D GPE converge with $\sigma \simeq a_{\perp}$. Equation (5.7) also permits solitonic solutions in the form of Eq. (5.5). However, in this

5.3. Solitons in lattice potentials

case a critical interaction strength emerges, beyond which the balance between kinetic and nonlinear terms is lost, leading to collapse of the soliton. This occurs when

$$\frac{Na_s}{a_\perp} > -\frac{2}{3}. \quad (5.9)$$

The value of this critical interaction strength g_c depends on geometry and confinement, and various methods have been used for its predictions, e.g., numerically solving the full 3D Gross-Pitaevskii equation [144, 145] with a variational approach [144, 146], or using the nonpolynomial Gross-Pitaevskii equation [141].

Variational approach

Another method which can be used to investigate soliton stability without doing full GPE calculations is to use a variational method [143, 144, 146, 147]. Here, it is assumed that the soliton is Gaussian in the axial and radial directions allowing for the decoupling of the components. In this case the energy per atom is [143]

$$E = \frac{1}{2} \left(\frac{1}{\sigma^2} + \frac{1}{2\eta^2} + \sigma^2 + \frac{g}{\sigma^2\eta^2} \right), \quad (5.10)$$

where $g = \gamma\sqrt{2/\pi}$, the energy is in units of $\hbar\omega_\perp$ and lengths in units of $a_\perp$. The two variational parameters are the axial length η and the radial width σ . A stable soliton is formed when the energy Eq. (5.10) has a local minimum [Fig 5.1(b)]. For a simplified illustration [Fig. 5.1(c)], we determined the value $\sigma_{\min}$ that minimises $E(\eta, \sigma)$ for each value of η [144, 147]. The resulting energy $E(\eta) = E(\eta, \sigma_{\min}(\eta))$ shows a local minima is bounded by two barriers. The right barrier prevents dispersion ($\eta \rightarrow \infty$), while the left barrier prevents collapse ($\eta \rightarrow 0$), which disappears above the critical interaction strength g_c [147].

5.3 Solitons in lattice potentials

Based on a seminal theoretical insight by Davydov [32], solitons have also been studied in systems with periodic potentials. Such “lattice solitons” [148] arise in wide range of systems [148], including molecular chains [32, 149], nonlinear optical waveguide arrays [40, 150–152], and quantum gases in optical lattices [33, 153, 154]. They exist in both one-dimensional and two-dimensional geometries [34, 150, 151, 155] and are predicted to exhibit intricate transport properties [33–35, 156]. However, despite considerable theoretical interest [31, 33, 157–163], the experimental realisation of lattice solitons with attractive matter waves has remained an open challenge.

Variational approach

The variational approach from the previous section can also be applied to the case of a 1D optical lattice [160]. The energy per atom is then

$$E = \frac{1}{2} \left(\frac{1}{2\eta^2} + \frac{1}{\sigma^2} + \sigma^2 \right) + \frac{g}{2\sqrt{2\pi}} \frac{1}{\sigma^2\eta} - V_0 \exp(-k_L^2 \eta^2), \quad (5.11)$$

5.3. Solitons in lattice potentials

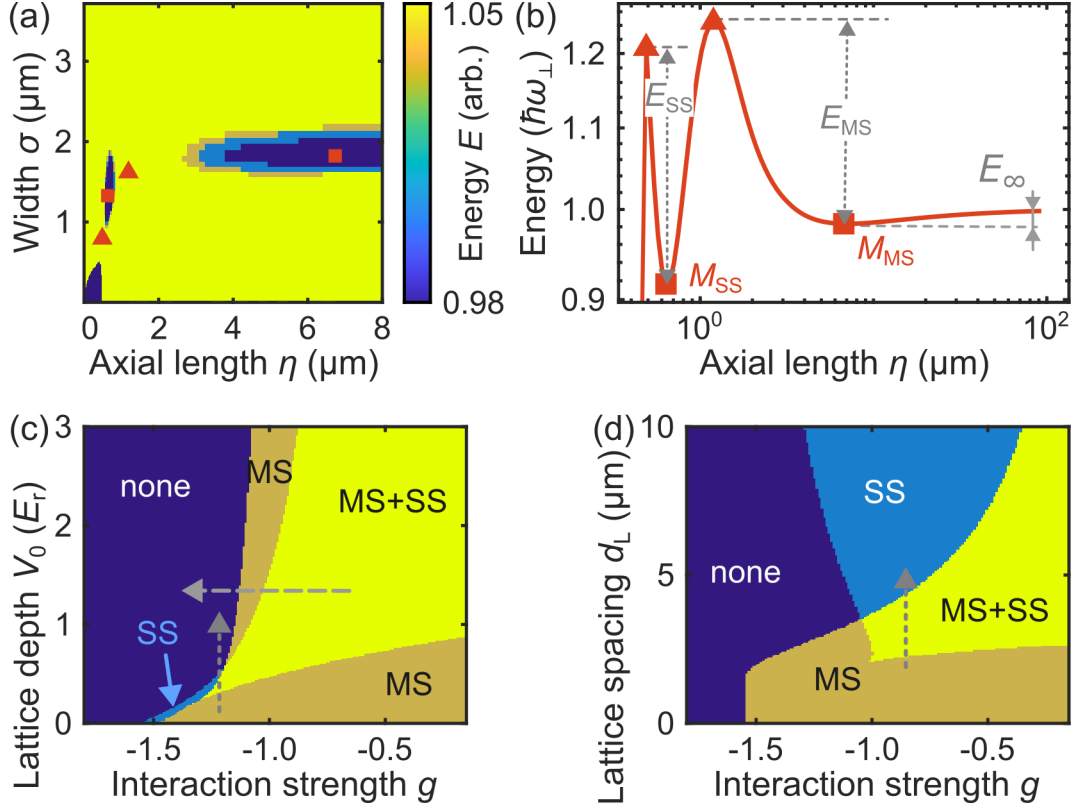


Figure 5.2: Solitons in periodic potentials. (a) Energy landscape $E(\eta)$ for a Gaussian wave packet with $V_0 = 1.1 E_r$, $a_s = -6.2 a_0$, $d_L = 2 \mu\text{m}$. Single-site (SS) and multi-site (MS) solitons are stable at minima M_{SS} and M_{MS} with barriers B_{SS} and B_{MS} . (b) Extracted minimum energy line from (a), The barrier heights E_{SS} and E_{MS} are illustrated. (c) Stable regions of SS and MS solitons for varying parameters g and V_0 , with $N = 1800$, $\omega_\perp = 2\pi \times 30 \text{ Hz}$, $d_L = 3.2 \mu\text{m}$. No solitons exist in dark blue regions. (d) Stable regions for varying d_L , same parameters as (c) with constant $V_0 = 1.3 E_r$ set at $d_L = 3.2 \mu\text{m}$.

The first term again provides the kinetic energy of the soliton, while the second term describes the interaction energy using the interaction strength $g = 2a_s N/a_\perp$, where a_s is the s-wave scattering length and N the total atom number. The third (new) term contains the lattice contribution, with lattice depth V_0 in units of $\hbar\omega_\perp$ and wave number $k_L = \pi/d_L$.

The presence of the new length scale from the lattice spacing introduces additional structure to the energy landscape [Fig. 5.2(a)] when compared to its continuous counterpart. For some parameter regimes there are two local minima instead of one. Depending on the ratio of η/d_L these minima classify lattice solitons into either single-site (SS) or multi-site (MS) solitons, which extend over different numbers of lattice sites. These solitons form near the centre of the Brillouin zone with energies below the lowest lattice band [158, 164]. This is in contrast to gap solitons with repulsive interactions [154, 165] that appear in the energy gap near the band edge.

Again, we determined the value $\sigma_{\min}$ that minimises $E(\eta, \sigma)$ for each value of η [Fig. 5.2(b)] [144, 147]. The minimum energy line shows the two minima where stable

5.3. Solitons in lattice potentials

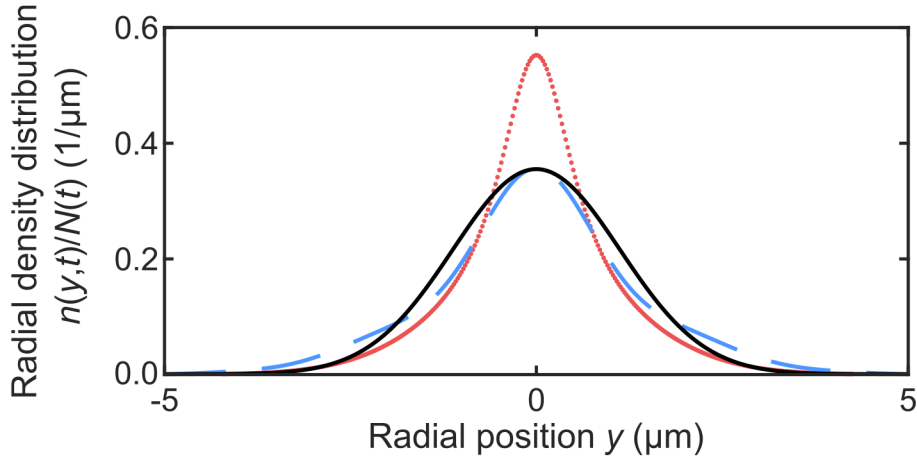


Figure 5.3: Radial density profile $n(y) = \int dz dx |\psi(x, y, z)|^2$, calculated using the 3D-GPE for a scattering length of $a_s = -18.17 a_0$. The simulation follows the protocol described in the main text. The profile is shown at two different times: $t = 5.13$ ms (red dotted line) and $t = 10.26$ ms (blue dashed line). The solid black line is a Gaussian profile for comparison. Figure from Ref. [111].

single-site and multi-site solitons form [M_{SS} and M_{MS}]. Collapse towards smaller axial length η is prevented by two barriers B_{SS} and B_{MS} . With a lattice potential, the barrier heights depend also on lattice depth and spacing, and g_c is replaced by surfaces in the (g, V_0, d_L) -parameter space that indicate the disappearance of the barriers.

By examining the existence and position of minima in the energy landscape for varying parameters stability diagrams can be constructed [Figs. 5.2(c) and 5.2(d)]. Here, different colours represent stable regions with non-zero energy barriers for parameters (g, V_0) and (g, d_L) . The critical interaction strength is given by the boundary that separates parameter regions without solitons (dark blue patches) and with solitons. The interplay between V_0 , d_L , and g , and the barrier heights E_{SS} and E_{MS} is not straightforward. For instance, decreasing g at a fixed lattice depth [dashed horizontal arrow in Fig. 5.2(c)], lowers the barriers due to strong attractive interactions and leads to the eventual collapse, first of the single-site soliton followed by the multi-site soliton. Conversely, when the interaction strength is held constant [dotted vertical arrow in Fig. 5.2(c)], the multi-site soliton can already exist at shallow lattice depths, whereas a larger value of V_0 is required to form the energy minimum M_{SS} that supports the single-site soliton. A further increase of V_0 eventually eliminates both barriers. Both types of solitons connect to bright 1D solitons without a lattice, either in the limit of vanishing lattice depth or in the limit of large lattice spacing for single-site solitons [111].

3D-GPE

While the aforementioned methods are useful to gain an insight into the general stability of lattice solitons, to accurately explain the collapse dynamics observed experimentally in the proceeding sections it was found that a full numerical simulation of the 3D-GPE was required. In some circumstances, lattice solitons are expected to show a dynamical collapse whereby the density profile both axially and radially change drastically over a short period, increasing to the point where three-body loss occurs. The

5.3. Solitons in lattice potentials

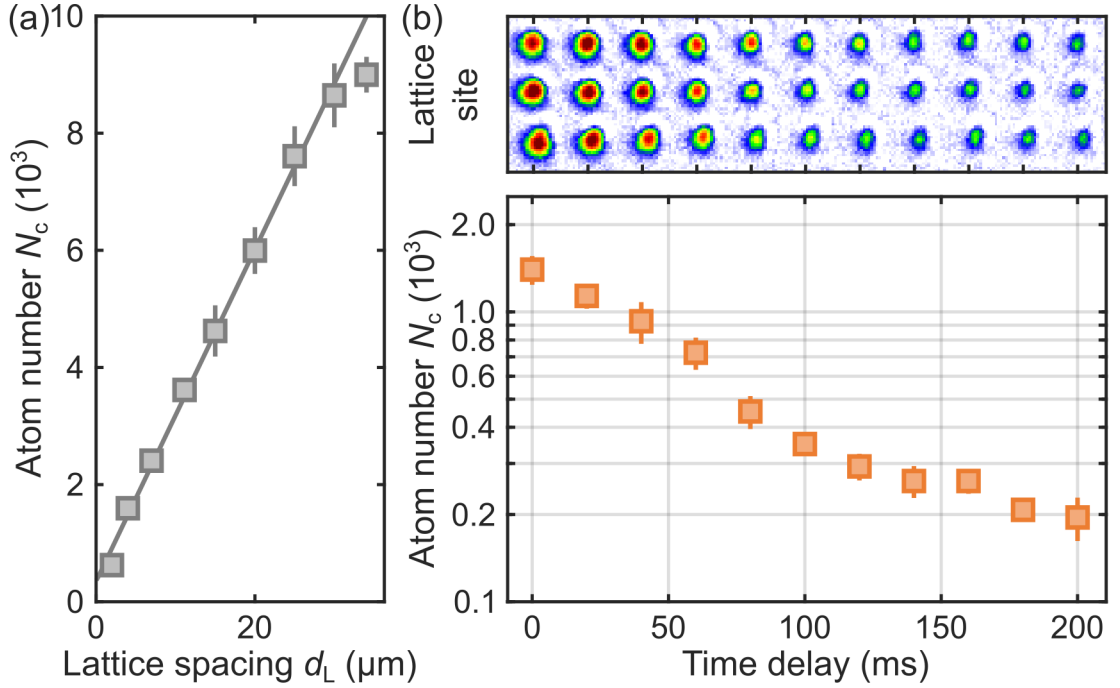


Figure 5.4: Atom number preparation in central lattice site. (a) Atom number reduction by loading into a lattice with spacing d_L and depth $V_0 \approx 30 E_r$ (gray squares), where E_r is defined at $d_L = 3 \mu\text{m}$. $N_{\text{tot}} = 13.0(4) \times 10^4$ atoms, error bars denote the standard deviation over five repetitions, solid line is a linear fit to the data. (b) Atom number in central site after a time delay between switching off the dipole trap D1 and adding the accordion lattice. Time $t = 0$ is the moment immediately after the dipole trap is removed. (Top panel) Absorption images of corresponding density profile of central three lattice sites, dashed red lines delineate centre site. (Lower panel) Atom number in the central site, error bars denote the standard deviation over seven repetitions. Figure from Ref. [102].

NPSE is insensitive to this kind of collapse due to the vanishing of the width of the wavepacket radially [111, 126]. To further match the experiment an additional quintic loss term was added to account for three-body losses

$$i\hbar \frac{\partial}{\partial t} \phi(z, t) = \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial z^2} + V_{\text{ext}}(z) + g|\phi(z, t)|^2 - ig_5|\phi(z, t)|^4 \right] \phi(z, t), \quad (5.12)$$

where $g_5 = \hbar L_3/2$, with L_3 being the three-body loss coefficient [166–168]. There have been various studies aimed at measuring the approximate value of coefficient L_3 [60], here, to match the experimentally observed loss rates L_3 was varied between $5 \times 10^{-39} \text{m}^6 \text{s}^{-1}$ and $5 \times 10^{-38} \text{m}^6 \text{s}^{-1}$. The resulting simulations show the variation of the radial wavefunction during a collapse event at different times [Fig. 5.3, $a_s = -18.17 a_0$, $N = 1700$]. When compared with after the collapse (blue, $t = 10.26 \text{ms}$), the density profile before the collapse (red, $t = 5.13 \text{ms}$) is more 'peaked' with a smaller waist. For comparison a Gaussian profile is also plotted (black), the difference in profiles between the 3D-GPE simulations and the Gaussian show the need to use the full 3D-GPE rather than the NPSE.

5.4 Initial state preparation

Lattice solitons show a strong dependence on atom number (a change of hundreds of atoms can induce collapse), making precise control over the initial occupation of individual lattice sites a central challenge in preparing the initial state. An advantage of using an accordion lattice is the ability to dynamically control this occupation during loading, not only by altering the lattice depth but also the lattice spacing. We employed a two-step process for the preparation of the density distribution, first by preparing a small sample of $N \lesssim 30,000$ in the lattice potential, and second, by removing atoms in unwanted sites with a combination of a microwave transfer and a resonant laser beam. We adjusted the trap frequency of dipole trap D1 during the process to keep the density distribution close to the system's ground state, and provided a final waiting period to allow the system to settle.

We found it essential to prepare the desired atom number in the central lattice site, N_c , already during the lattice loading procedure, thereby reducing excitations during the subsequent atom removal process and quench. To prepare the sample in the lattice, we first reduced the atom number of the Bose-Einstein condensate in the crossed dipole trap from 130,000 to 30,000 by reducing our magnetic levitation gradient for 3 s. We then increased the lattice depth V_0 over 150 ms to a final value while simultaneously adjusting the trapping frequency ω_z of the dipole trap to create the intended density distribution. The final atom number per site is controlled by ω_z , a_s , V_0 , and d_L . Especially the reduction d_L allows us to vary N_c by a factor of ten [Fig. 5.4(a)]. The approach works well down to approximately 1,000 atoms per site.

To further reduce N_c to a few hundred atoms, we lowered the depth of the dipole potential to zero in 250 ms and added a time delay, allowing the atoms to slowly expand in the vertical dipole potential of beam D2 before increasing V_0 in 150 ms. Fine control of the atom number was possible by increasing the time delay between these two ramps, allowing the atoms to expand further before their motion was frozen by the lattice potential. The absorption images in Fig. 5.4(b) demonstrate this reduction for the central three sites, while the lower plot shows the extracted atom number in the central site.

5.5 Creation of lattice solitons

Lattice solitons preserve their shape through a balance of kinetic, interaction, and potential energies. Their shape remains stable because density variations increase the system's energy, with large deviations suppressed by energy barriers. Remarkably, in the presence of a lattice potential, this stabilisation mechanism applies not only to attractive but also to repulsive interactions. These wave packets, sometimes called “repulsive solitons” [33], are related to repulsively bound pairs in the two-atom limit [169] and to self-trapping in two-site Josephson junctions [72, 170]. We demonstrate in this section that repulsive solitons provide a convenient intermediate stage to create attractive single-site solitons with an interaction quench.

A common approach to generate solitons in quantum gases involves a quench of both the interaction strength and the confining potential [111, 131, 135, 138]. Typically, a trapped wave packet is first prepared with repulsive interactions. Then, the

5.5. Creation of lattice solitons

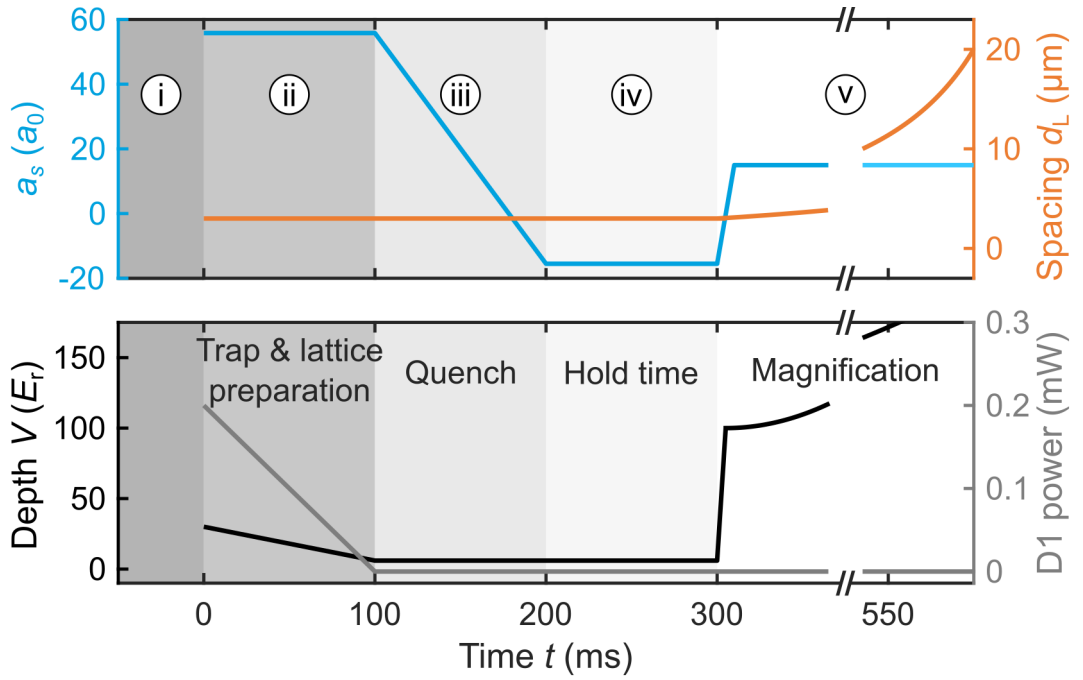


Figure 5.5: Sketch of the experimental sequence for the generation of lattice solitons. Values as a function of time of system variables such as scattering length (blue), lattice spacing (orange), lattice depth (black) and power of horizontal trapping potential D1 (gray). The different experimental phases are marked (i) preparation of lattice site occupations, (ii) slow reduction of trapping potentials, (iii) quench of scattering length, (iv) soliton generation and subsequent hold time, (v) magnification sequence and imaging. The start point $t = 0$ occurs immediately after the microwave preparation discussed in Sec. 4.4. Figure from Ref. [102].

confining potential is switched off, and the scattering length is tuned from positive to negative. The quench is designed so that trapping and interaction parameters closely match the initial density distribution, usually a Thomas-Fermi profile, to the density of a stable soliton, often approximated by a sech-squared shape. Strong mismatch in the density profiles can lead to excitations, resulting in atom loss by shedding or to oscillations [138, 139].

Using repulsive solitons in a lattice potential as an intermediate step is convenient, because they can be created by directly reshaping the trapping potential without the possibility of collapse. Our quench protocol to create an attractive single-site soliton is sketched in Fig. 5.5, with different periods in the experiment designated by Roman numerals. We first prepared a wave packet of 500 atoms in a single lattice site following the procedure outlined in Sec. 5.4, [region (i) in Fig. 5.5]. During this stage, the scattering length was fixed at $a_s = +55 a_0$, with lattice parameters $d_L = 3 \mu\text{m}$ and $V_0 = 30 E_r$ [upper and lower plot in Fig. 5.5]. We then used the stability of the repulsive soliton to slowly approach the target configuration by reducing the lattice depth to $V_0 = 6 E_r$ and switching off the external trap over 100 ms [region (ii)]. Finally, we quenched the system by reducing the scattering length to its final value within a time t_q [region (iii)] and allowed the soliton to evolve for 100 ms [region (iv)]. The density distribution was subsequently measured using absorption imaging after a magnification sequence

5.5. Creation of lattice solitons

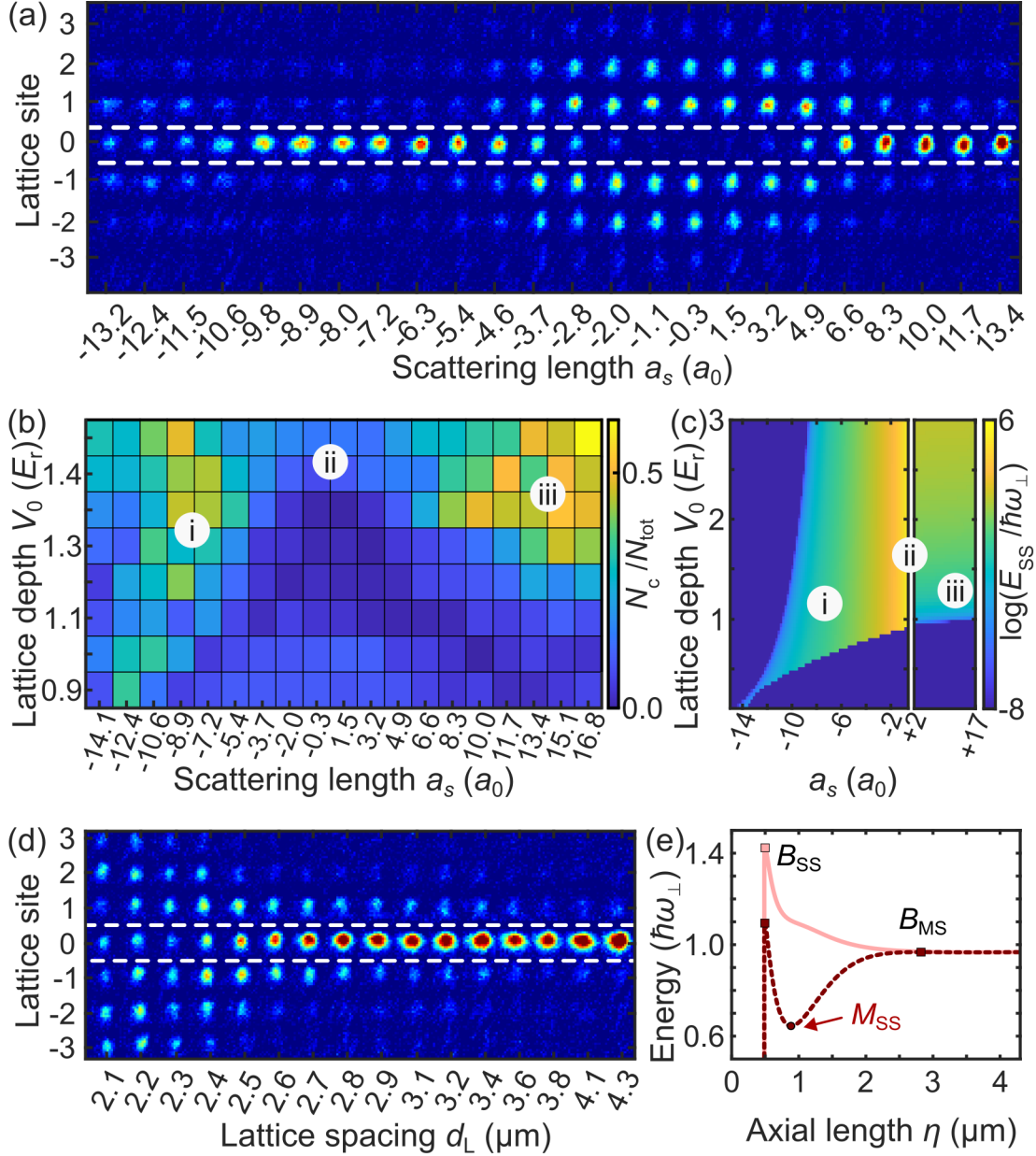


Figure 5.6: Stability of single-site solitons. (a) Measured density distribution after a quench of a_s and $t = 100$ ms hold time with $d_L = 3.2(2) \mu\text{m}$, $V_0 = 1.3(1) E_r$, $\omega_{\perp} = 2\pi \times 40(1) \text{ Hz}$, $N \approx 1800$. White lines mark atoms in the central site. (b) Measured relative central-site atom number N_c/N_{tot} vs. a_s and V_0 with same parameters as (a). (c) Energy E_{SS} of the barrier B_{SS} ; (i-iii) indicate regions of varying stability in (b,c). (d) Density distribution for varying d_L after 100 ms with $a = -6.4 a_0$, $N \approx 1800$, constant $V_0 = 1.3(1) E_r$ set at $d_L = 3.2(2) \mu\text{m}$. (e) Calculated $E(\eta)$ for (d) with $d_L = 3.5 \mu\text{m}$ (dotted line) and $d_L = 2.0 \mu\text{m}$ (solid line). Measured data is averaged over typically seven repetitions. Figure from Ref. [111].

5.6. Observation of single-site solitons

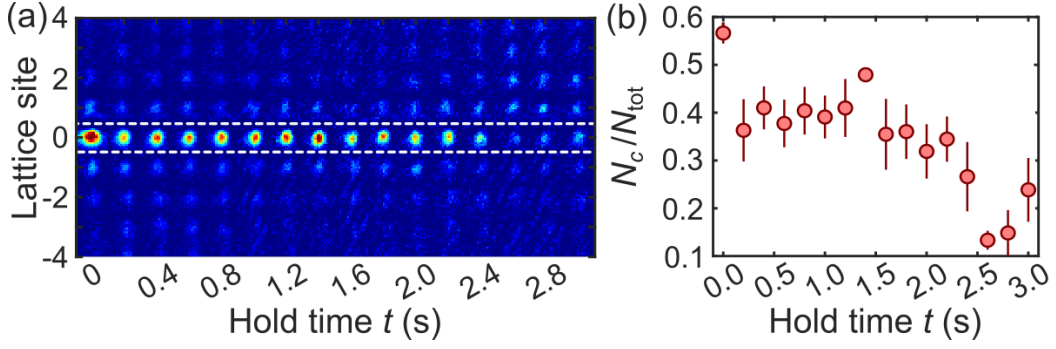


Figure 5.7: Time evolution of single-site soliton. (a) Average measured density distribution of the wave packet after a quench of the scattering length a_s and for a varying hold time with $d_L = 3.2(2) \mu\text{m}$, $V_0 = 1.3(1) E_r$, $a_s = -6 a_0$, $\omega_\perp = 2\pi \times 40(1) \text{ Hz}$, $N \approx 1800$. White lines mark atoms at the central lattice site. (b) Relative atom number N_c/N in the central site for the data in (a). Error bars denote standard errors. Figure from Ref. [111].

[region (v)], quickly freezing out the occupation number with an increase in lattice depth to $V_0 = 100 E_r$ in 5 ms and a slow increase of the lattice spacing as described in Sec. 4.1.2. To reduce three-body loss and excitations caused by attractive interactions we tuned the scattering length to $+15 a_0$ during magnification.

5.6 Observation of single-site solitons

In a first measurement, we demonstrate the existence and properties of single-site solitons [111]. After preparing approximately 1800 atoms at a single site, we quenched a_s and measured the density profile and the atom number per lattice site after a hold time of 100 ms. Absorption images of the density profile show a strong dependence on a_s [Fig. 5.6(a)]. For $a_s \approx -8 a_0$, the wave packet remained localised at the central lattice site, which indicates the formation of a single-site soliton. Except for some initial shedding of atoms, we found this soliton to be stable for a hold time up to 2 s (Fig. 5.7, see below). For stronger attractive interaction, $a_s < -10 a_0$, the soliton collapsed, and the remaining atoms spread along the lattice direction. Weak attractive and repulsive interactions, $-5 a_0 < a_s < +5 a_0$, resulted in the dispersion of the wave packet, with a minimum at the central lattice site after the given hold time, while larger scattering lengths $a_s > +7 a_0$ lead again to the localisation of the wave packet. In the two-particle limit, this localised state corresponds to repulsively bound pairs [169], whereas in the context of two lattice sites and Josephson oscillations, it is associated with macroscopic quantum self-trapping [72, 170].

We extended the study to different lattice depths and determined the relative atom number in the central site, N_c/N , as a measure of the system stability [Fig. 5.6(b)]. The data reveals the three regimes: (i) a stable single-site soliton, (ii) a free dispersion of the wave packet close to $0 a_0$, and (iii) the self-trapping for repulsive interaction and sufficient lattice depth. The regimes can be explained by the height of the barriers B_{SS} and B_{MS} . For comparison, Fig. 5.6(c) shows the energy E_{SS} , which is the height of barrier B_{SS} [see Fig. 5.2(b)]. Large values of E_{SS} align well with the experimental

5.6. Observation of single-site solitons

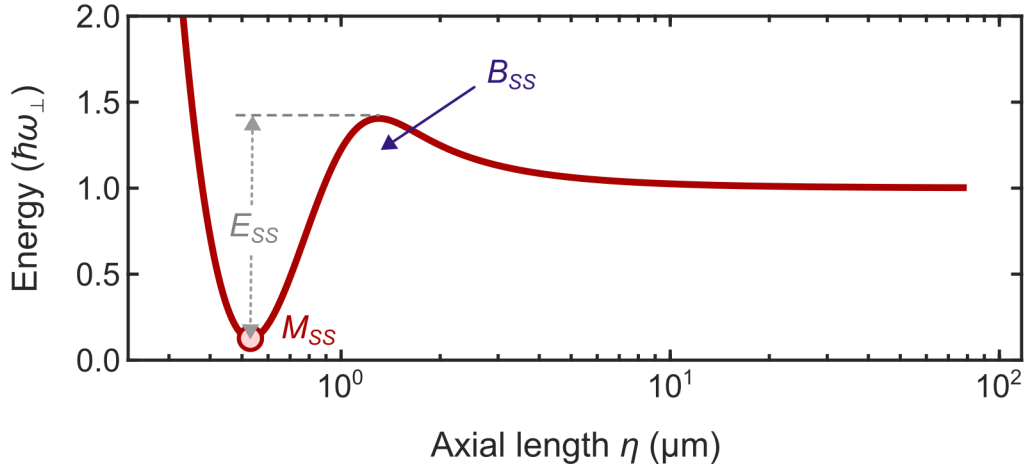


Figure 5.8: Energy minimum for repulsive interaction. Energy $E(\eta)$ in Eq. (5.11) for a Gaussian wave packet with axial length η and lattice depth $V_0 = 2.2 E_r$, scattering length $a_s = +5.0 a_0$, and lattice spacing $d_L = 2.0 \mu\text{m}$. A minimum M_{SS} forms with an energy barrier B_{SS} that prevents spreading of the wave packet. Figure from Ref. [111].

data in Fig. 5.6(c), accurately predicting the stable regions (i) and (iii). However, E_{SS} does not capture the evolution of the wave packet close to zero scattering length in region (ii), where the wave packet spreads. While collapse is prevented by barrier B_{SS} , spreading is inhibited by the barriers at larger values of η [Fig. 5.2(b)].

In addition to the measurements in Fig. 5.6, we determined the lifetime of the single-site soliton. Averaged absorption images show the density profile of the wave packet after a variable hold time t following the quench [Fig. 5.7(a)].

The wave packet remains stable for approximately 2 s before drifting away from the central site and beginning to spread. This behaviour is also reflected in the extracted atom number at the central site N_c , normalised to the total atom number within the central 9 sites [Fig. 5.7(b)]. We also observe a rapid drop in atom number within the first 200 ms, which we attribute to shedding of atoms and three-body loss after quenching to attractive interaction [60].

To investigate the effect of the lattice spacing on the stability of the solitons, we varied d_L while keeping V_0 and a_s constant [Fig. 5.6(d)]. Absorption images taken after a hold time of 100 ms show a spreading of the wave packet for $d_L \lesssim 2.5 \mu\text{m}$ [Fig. 5.6(d)]. Our calculations of $E(\eta)$ indicate that as d_L decreases, the minimum M_{SS} disappears, while the barrier B_{SS} persists [Fig. 5.6(e)]. Consequently, the observed spreading after the interaction quench is not due to a collapse, as observed in Fig. 5.6(a), but rather due to the absence of an energy minimum. The calculated minimum M_{SS} vanishes for $d_L \approx 2.2 \mu\text{m}$, which agrees with our experimental data ($d_L \approx 2.5 \mu\text{m}$). This measurement also aligns well with the stability diagram in Fig. 5.2(d) (dotted arrow).

To further illustrate the localisation of a single-site wave packet with repulsive interaction, as observed in Fig. 5.6(a), we calculated the curve $E(\eta) = E(\eta, \sigma_{\min}(\eta))$ for a wave packet with scattering length $a_s = +5 a_0$ [Fig. 5.8]. The resulting energy curve [144, 147] shows a single minimum, M_{SS} , that allows for the formation of stable wave packets that are localised on a single lattice site. Although repulsive interactions typically lead to spreading, this behaviour is suppressed by the energy barrier B_{SS} of

5.7. Observation of multi-site solitons

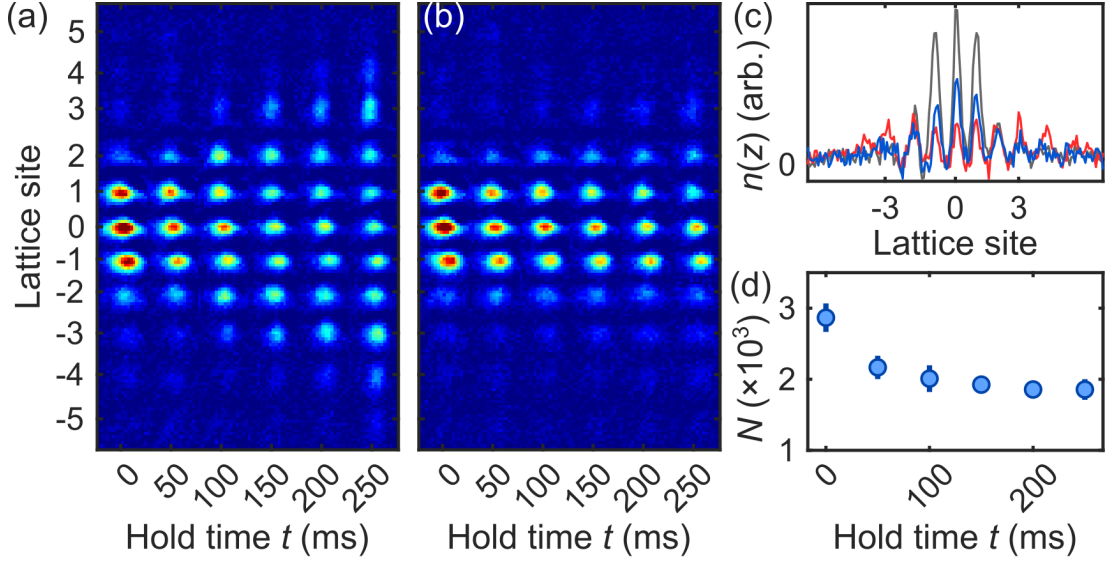


Figure 5.9: Stability of a multi-site wave packet. (a,b) Time evolution of a wave packet after a quench of a_s , averaged over ten repetitions with $V_0 = 1.3 E_r$, $d_L = 2.6 \mu\text{m}$, $\omega_\perp = 2\pi \times 25 \text{ Hz}$, $\omega_z = 2\pi \times 25 \text{ Hz}$, $N \approx 2900$. (a) The wave packet disperses for a quench to $a_s = +2.0a_0$ and (b) mostly retains its overall shape for $-5.7a_0$. Site occupation numbers for both data sets are provided in Ref. [111]. (c) Density profiles of the wave packet immediately after the quench (grey), and after 250 ms for $+2.0a_0$ (red) and $-5.7a_0$ (blue). (d) Atom number for data in (b), error bars denote the standard deviation. Figure from Ref. [111].

height E_{SS} , which stabilises the localised state.

This behaviour closely resembles that of a single-site soliton with attractive interaction. However, instead of collapsing, the wave packet here tends to spread due to repulsion. For consistency, we applied the same labels B_{SS} and E_{SS} as used for the single-site soliton. Figure 5.6(c) shows the barrier height E_{SS} for attractive and repulsive interactions.

5.7 Observation of multi-site solitons

In a second measurement, we investigated the stability of multi-site wave packets. To prepare the initial state, we adjusted the microwave transfer to remove all but the atoms in three adjacent lattice sites. During the subsequent waiting period, this density profile evolved toward a Gaussian envelope spanning 3–5 lattice sites, determined by the trapping frequency ω_z . After the quench, we observed stronger density fluctuations compared to single-site solitons, showing in some cases a splitting of the soliton with moving fractions. Quantum fluctuations have been suggested as a possible cause of this fragmentation [171–173]. However, here, we attributed it to technical noise and the low binding energy of multi-site solitons, E_∞ [Fig. 5.2(a)].

We studied the time evolution of the wave packet over 250 ms following the quench. For scattering lengths near zero [Fig. 5.9(a)], the wave packet shows dispersion, while for $a_s = -5.7a_0$ [Fig. 5.9(b)], it remains mostly localised. We attribute this localisa-

5.7. Observation of multi-site solitons

tion to the formation of a multi-site soliton-like state. The density profiles show little change in site occupations between $t = 100$ ms and 250 ms, indicating a stable configuration. However, during the initial tens of milliseconds, we observe a loss of atoms in the three central sites. We attribute this initial depletion to three-body loss, which predominantly occurs in high-density regions where attractive interactions enhance the local density [174]. The subsequent atom loss is strongly reduced, and we speculate that the wave packet gradually adjusts its radial and axial sizes in response to the slowly decreasing atom number.

The 1D density profiles [Fig. 5.9(c)] and the measured total atom number support this interpretation [Fig. 5.9(d)]. Comparing the initial 1D density profile [gray line in Fig. 5.9(c)] with the profile after 250 ms reveals a reduction in atom number at the three central sites for the soliton-like wave packet (blue line), while the overall profile remains localised. In contrast, for near-zero interactions (red line), the wave packet undergoes significant spreading, with increased occupation of the outermost sites. To quantify the spreading, we calculated the relative site occupations N_j/N , and extracted the wave packet width. The non-interacting and soliton-like wave packets exhibit linear dispersion velocities of 12 and 7 sites/s, respectively. For the soliton-like wave packet, this apparent spreading mainly reflects the flattening of the density profile rather than significant mass transport. We also found good agreement of the density profiles and atom numbers with simulations of the 3D-GPE [111].

5.7.1 Collapse dynamics of multi-site solitons

Finally, we quantitatively analysed the collapse by measuring the wave packets' density profiles at $t = 150$ ms over a broad range of scattering lengths [Fig. 5.10(a)]. To account for varying density distributions, we calculated the second-moment width w_m of the site occupations, defined as

$$w_m^2 = \frac{1}{N} \sum_j N_j (z_j - \bar{z})^2, \quad \text{with } j = -4, \dots, 4.$$

Here, z_j is the position of the j th lattice site and $\bar{z}$ is the centre-of-mass position. The value of w_m indicates the varying stability of the wave packet depending on a_s . It spreads for $a_s \approx 0 a_0$, shrinks towards the central site for $a_s \approx -10 a_0$, and spreads after collapse for strong attractive interactions $a_s < -13 a_0$. Single absorption images illustrate the spreading and shrinking of the wave packet in the different regions [inset in Fig. 5.10(a)].

The variational approach used in Eq. (5.11) provides a simple model for predicting the evolution of the wave packet after quenching to scattering length a_s . Within the model, stability is achieved when the initial parameters of the wave packet, $N(0)$ and η_0 , closely match those of a multi-site soliton with length η_{MS} . In our experimental protocol, $N(0)$ and η_0 are set during preparation, while only a_s can be varied. A soliton is created by quenching the scattering length to a_s^* with $\eta_0 = \eta_{MS}(N(0), a_s^*)$. For other quench values close to a_s^* , the wave packet is expected to exhibit small breathing oscillations [160], unless its initial energy $E(\eta_0)$ exceeds one of the barrier energies, E_∞ or E_{MS} , leading to dispersion or collapse. Calculating the barrier energy barriers E_∞ and E_{MS} using Eq. (5.11) allows us to predict the stability regions. The brown patch in

5.7. Observation of multi-site solitons

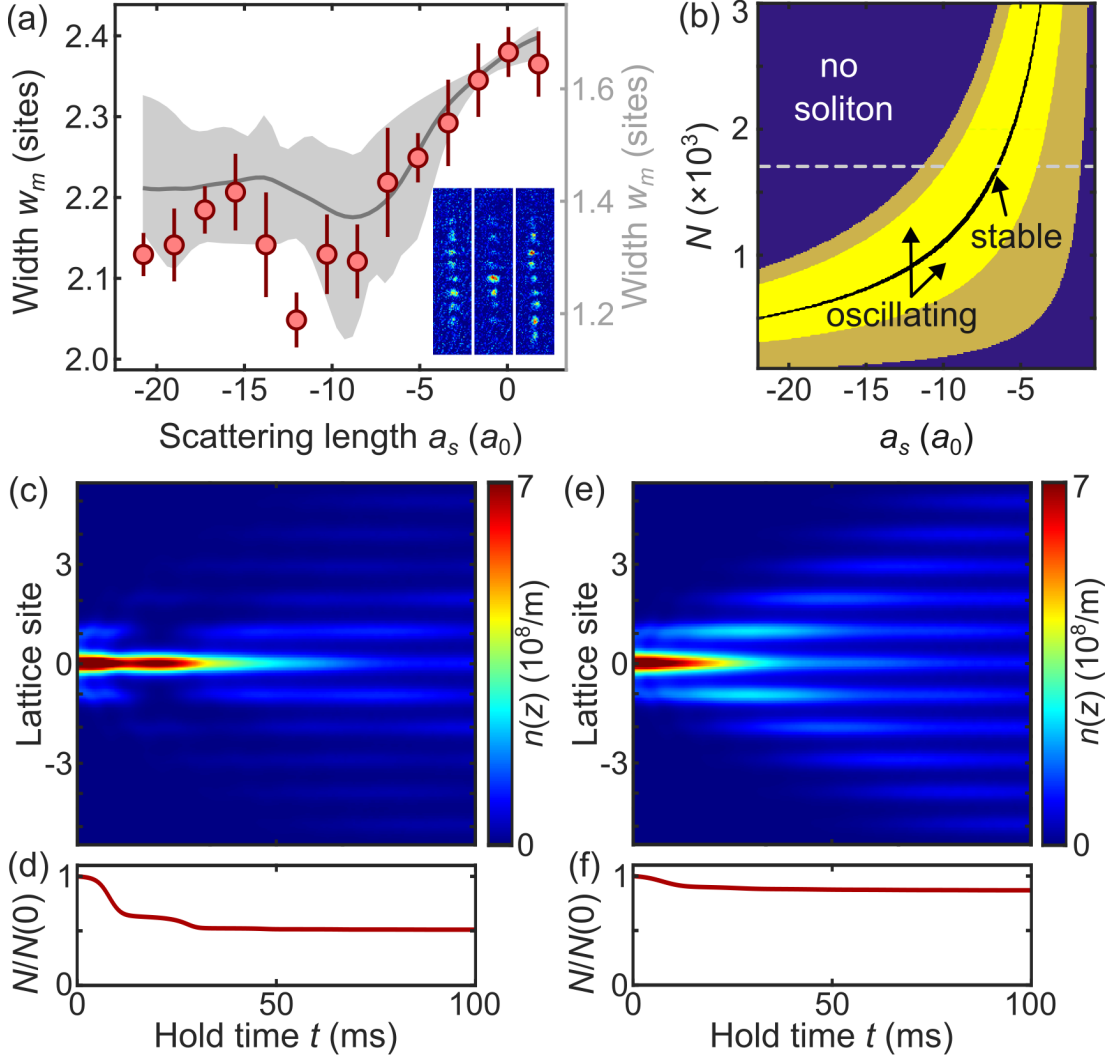


Figure 5.10: Collapse of a multi-site wave packet. (a) Width w_m of the wave packet at $t = 150$ ms after quenching to different values of a_s , with $V_0 = 1.4E_r$, $d_L = 2.6 \mu\text{m}$, $\omega_\perp = 2\pi \times 30$ Hz, $N \approx 1700$. The gray patch shows the variation in w_m , calculated using the 3D-GPE, resulting from uncertainties in the three-body loss coefficient L_3 and N . The line is an average of the calculations. (Inset, left to right) Typical images of the density profiles after collapse ($a_s = -17a_0$), shrinking towards the central site ($a_s = -10a_0$) and expanding wave packet ($a_s = 0a_0$). (b) Stability regions calculated using Eq. (5.11) with an existing minimum M_{MS} (brown) and without (blue), with breathing oscillations for $E(\eta) < E_\infty$ and E_{MS} (yellow), and with stable multi-site solitons for $E(\eta_0) \approx E_{MS}$ (black). (c),(d) The calculated time evolution of the density distribution and relative atom number show collapse followed by expansion for $a_s = -9.5a_0$, and (e),(f) dispersion for $a_s = -1.7a_0$. Calculations use $L_3 = 5 \times 10^{-39} \text{m}^6 \text{s}^{-1}$ and other parameters as in (a). Figure from Ref. [111].

5.8. Conclusion

Fig. 5.10(b) marks where a minimum M_{MS} exists. Stable solitons form only along the black line, while breathing oscillations occur within the yellow regions. Assuming a fixed atom number further constrains the choice of a_s to lie on the dashed line, though in practice N decreases due to three-body loss.

To capture the full evolution of the wave packet beyond this simple model, we numerically simulated the dynamics of the multi-site soliton using a modified 3D Gross-Pitaevskii equation (GPE) with a quintic term that accounts for three-body loss [Eq. 5.12] [166, 167]. The simulations show two distinct dynamical regimes. In the first regime, corresponding to large negative values of a_s , the wave packet begins to collapse, leading to an increase in local density at the central site [Fig. 5.10(c)]. However, a further shrinking of the wave packet is suppressed by the enhanced loss and a rapid shedding of atoms [Fig. 5.10(d)]. The second regime, which occurs for less negative values of a_s , is marked by a slow dispersion of the matter wave, and has lower and more gradual atom loss [Fig. 5.10(e) and (f)].

The simulation agrees well with the observed shrinking in w_m during the collapse process. However, it is sensitive to the precise values of atom number and the three-body loss coefficient [174, 175], resulting in an uncertainty of the predicted dynamics [gray patch in Fig. 5.10(a)]. In addition, imaging noise in the experimental data increases w_m , leading to an offset and reduced contrast compared to the simulation. While the observed atom loss was sufficiently low to permit the formation and investigation of lattice solitons, its inclusion in our simulation was still essential to reproduce our observations. Interestingly, at strong attractive interactions, the loss helped to suppress collapse and enhanced the stability of the system.

5.8 Conclusion

In conclusion, we have demonstrated the existence and stability of both single-site and multi-site solitons that extend over varying numbers of lattice sites. Using an accordion lattice with adjustable lattice spacing, we examined their stability across various lattice depths, spacings, and interaction strengths and compared our findings with theoretical predictions. A variational model based on a Gaussian approximation for the solitons was used to identify stable parameter regions, while numerical simulations of the 3D-GPE with a three-body loss term captured the solitons' time evolution. We found both types of solitons to be stable for hundreds of milliseconds, allowing ample time for further studies.

Chapter 6

Theoretical background of instabilities in Floquet driven systems

This chapter provides the motivation and theoretical insights into understanding Floquet induced heating in driven lattice experiments. This includes a brief introduction into Floquet theory (Sec. 6.2), the basics of implementing periodic driving in our experiment (Sec. 6.2.1), and general theoretical insights into analysis of the resulting instabilities (Sec. 6.3). We split the heating effect into two different types of instabilities depending on their mechanisms. Negative-mass instabilities (discussed in Sec. 6.3.3) occur whenever a wavepacket is in parts of the Brillouin zone associated with negative-mass, parametric instabilities (discussed in Sec. 6.3.4) occur whenever the driving frequency resonantly matches the energy of a phonon mode. The chapter closes with a discussion on how these instabilities change in a tilted system (Sec. 6.3.5), that is a lattice subjected to a constant force.

Parts of this chapter are taken from the publications A. Di Carli, R. Cruickshank, M. Mitchell, A. La Rooij, S. Kuhr, C. Creffield, and E. Haller, “Instabilities of interacting matter waves in optical lattices with Floquet driving”, *Phys. Rev. Res.* **5**, 033024 (2023) and R. Cruickshank, A. Di Carli, M. Mitchell, A. La Rooij, S. Kuhr, C. Creffield, and E. Haller, “Phonon excitations of Floquet-driven superfluids in a tilted optical lattice”, *Phys. Rev. Res.* **6**, 023323 (2024), which are cited as Ref. [176] and Ref. [177] respectively.

6.1 Introduction

Periodically driven quantum gases in optical lattices provide versatile settings to investigate condensed matter phenomena [178]. The periodic drive allows for coherent control over variables like atom tunnelling rates, band-structures and transport [41]. Recent experiments include a dynamically driven quantum phase transition between a bosonic Mott insulator and a superfluid [43], generation of solitons [165] and the generation of kinetic frustration on a triangular lattice [179]. Perhaps the area which has attracted the most attention recently has been that of artificial gauge fields [180–182]. Owing to the neutral charge of the atoms, synthetic magnetic fields are generated by the atoms acquiring a Peierls phase whenever they tunnel around a plaquette in a two-dimensional lattice [183]. Such phases are analogous to the Aharonov-Bohm phase

6.2. Floquet Theory

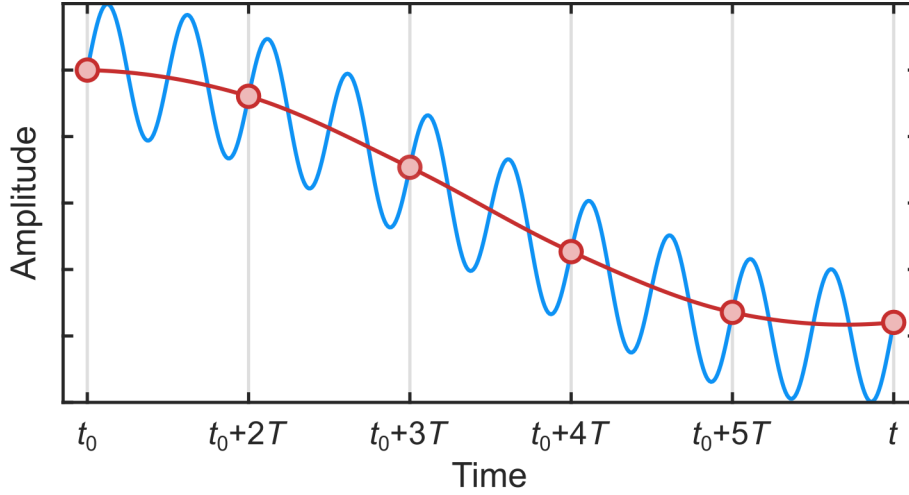


Figure 6.1: Example of fast and slow dynamics. Fast oscillations, the micromotion (blue line) is captured by the kick operators $\hat{K}(t)$, however when stroboscopically measured at times of $t_0 + nT$ the Floquet Hamiltonian only captures certain points, resulting in long time dynamics of the red line.

acquired by a charged particle moving along a closed path in the presence of a magnetic field. One advantage of synthetic magnetic fields is they can have a flux as large as π producing field strengths on the order of $B \approx 10^4$ T [41]. At this strength, which is several orders of magnitude greater than physical magnets, the fields can provide access to the Harper-Hofstadter model [184, 185], which in turn can be used to investigate the quantum Hall effect [186] and superconductivity.

One significant challenge posed by periodically driven experiments is that the driving can introduce heating and instabilities into the system. This destroys the coherence of the quantum state and limits experimental observation times [187–189]. The goal of the following chapters is to find the sources of this driving induced heating and search for parameter regimes whereby the growth of instabilities in the system is suppressed.

6.2 Floquet Theory

A driven system can be described using the so-called Floquet formalism, which seeks to map the time-dependent system to a time-*independent* Hamiltonian [190]. For a time-periodic Hamiltonian operator with period T , $\hat{H}(t) = \hat{H}(t + T)$, Floquet theory uses an effective Hamiltonian $\hat{H}_F$ to describe the system's evolution over one driving period. If one starts with the unitary time-evolution operator, $\hat{U}(t, t_0)$, which evolves a state from some initial time t_0 to a time t so that [191]

$$|\psi(t)\rangle = \hat{U}(t, t_0) |\psi(t_0)\rangle, \quad (6.1)$$

and is also a solution of the Schrödinger equation

$$i\hbar \frac{\partial}{\partial t} \hat{U}(t, t_0) = \hat{H}(t) \hat{U}(t, t_0), \quad (6.2)$$

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it can be shown that the time-evolution operator obeys the identities

$$\begin{aligned}\hat{U}(t+T, 0) &= \hat{U}(t, 0)\hat{U}(T, 0), \\ \hat{U}(t+T, T) &= \hat{U}(t, 0).\end{aligned}\tag{6.3}$$

A consequence of this is that the information contained within a single period is sufficient to determine its dynamics at long time scales [191]. Specifically, if we measure the system stroboscopically at time multiples of the driving period $t = nT$, where $n \in \mathbb{N}$, the operator $\hat{U}(t, t_0)$ can be known exactly, that is

$$\hat{U}(nT, t_0) = [\hat{U}(T, t_0)]^n.\tag{6.4}$$

The operator for one period of driving can be expressed in the form of an exponential [190, 191]

$$\hat{U}(T, t_0) = e^{-i\hat{H}_F[t_0]T/\hbar},\tag{6.5}$$

where $\hat{H}_F$ is the time-independent Floquet Hamiltonian [41].

The Hamiltonian $\hat{H}_F[t_0]$ possesses a set of complex eigenvalues known as *quasienergies*, ε_n , which are unique in the interval $[0, T]$ [192]. We retain the square brackets notation in $\hat{H}_F[t_0]$ to emphasise that, although the Hamiltonian is time-independent, it still depends on the initial time t_0 , making this time a gauge choice [193]. From Eq. (6.5) one can see that the eigenvalues of $\hat{U}(T, t_0)$ are $e^{-i\varepsilon_n T/\hbar}$. Its eigenstates are known as Floquet states

$$|\Psi_n(t)\rangle = e^{-i\varepsilon_n T/\hbar} |\Phi_n(t)\rangle,\tag{6.6}$$

where $|\Phi_n(t)\rangle = |\Phi_n(t+T)\rangle$ are T periodic Floquet modes [190, 191]. While Eq. (6.5) defines the dynamics at stroboscopic time intervals, any dynamics that occur on a faster timescale than $\omega = 2\pi/T$ are lost in the description. This is illustrated in Fig. 6.1 when the fast dynamics, referred to as the micromotion, is sampled stroboscopically at times $t_0 + nT$, the resulting slow-time evolution is only exact at those discrete times. In order to correctly capture the micromotion, one can encapsulate all time-dependence into an operator $\hat{K}(t)$, which is known as a *kick operator* [194, 195]. The evolution operator can then be written as

$$U(t_f, t_i) = e^{-i\hat{K}(t_f)} e^{-i\hat{H}_F[t_0](t_f-t_i)/\hbar} e^{-i\hat{K}(t_i)},\tag{6.7}$$

where $\hat{K}(t_0 + nT) = 0$, t_i and t_f are the initial and final points of evolution respectively [193]. Equation (6.7) is composed of three sections: an initial kick at t_i , evolution according to $\hat{H}_F$ and a final kick at t_f [195]. This stroboscopic Floquet description works well for single particles, but interacting many-body systems present challenges. Interactions introduce additional time and energy scales with non-linear properties that break the periodic time evolution.

6.2.1 Driven lattices

Floquet driving in lattice systems can be realised experimentally through several methods, including periodic modulation of the lattice depth (amplitude modulation) or by periodically shifting the centre of the lattice sites (phase modulation). All experiments in this thesis used the latter. An acousto-optic modulator was used to impart a periodic

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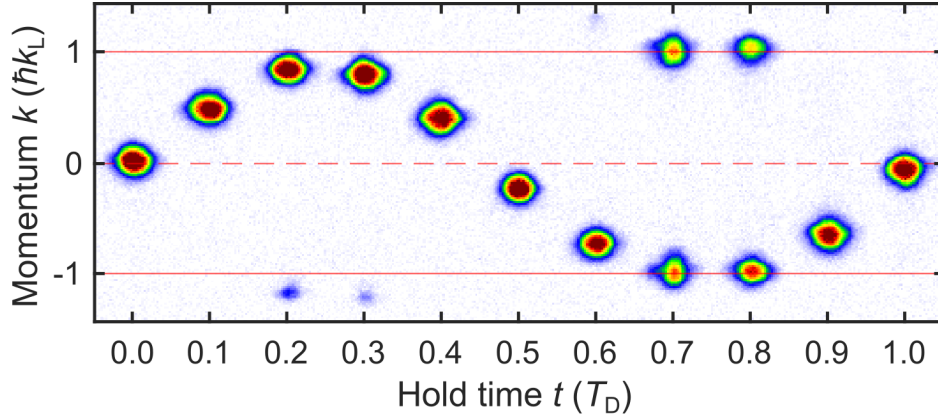


Figure 6.2: Condensate's response to an external driving force. The condensate traces a sinusoidal path in the Brillouin zone per driving period T_D . The extra peaks at the peaks and troughs for the micromotion are due to Bragg scattering.

frequency difference $\Delta v(t)$ between the two beams which create the lattice potential. As a result, the lattice moves with velocity

$$v(t) = \Delta v_{\max} d_L \sin(\omega_D t), \quad (6.8)$$

where $d_L = \lambda/2$ is the lattice constant, ω_D the angular frequency of the driving, and $\Delta v_{\max}$ is the maximum frequency difference between the two laser beams.

In the reference frame of the lattice, this frequency modulation creates an inertia force that drives the atoms with

$$F(t) = -F_0 \cos(\omega_D t + \phi) \Theta(t) - \frac{F_0}{\omega_D} \sin(\phi) \delta(t), \quad (6.9)$$

where $F_0 = m d_L \Delta v_{\max} \omega_D$ is the amplitude of the shaking, and $\Theta(t)$ the Heaviside function [80]. The first term in Eq. (6.9) represents the sinusoidal force of the drive for $t > 0$, while the second is a sudden 'kick' at $t = 0$ when the driving is turned on, which exists for driving forces with a phase shift ϕ . This initial kick is related to the kick operators $\hat{K}(t)$ discussed in Sec. 6.2 [80, 195]. For our setup, one can characterise the degree of shaking with the dimensionless driving strength [196]

$$K = \frac{F_0 d_L}{\hbar \omega_D} = \frac{m v_{\max} d_L^2}{\hbar}. \quad (6.10)$$

The driving force accelerates the wave packet through the first Brillouin zone according to a micromotion

$$k(t) = k_0 + (K/d_L) \sin(\omega_D t), \quad (6.11)$$

where $\hbar k(t)$ is the quasimomentum of the carrier wave and $\hbar k_0$ is the initial (gauge) momentum. Note that the shape of the micromotion depends only on the initial quasimomentum $\hbar k_0$, ϕ and on K , which is independent of ω_D when we provide the driving force by lattice shaking [80]. This micromotion is observable experimentally through TOF measurements (Fig. 6.2).

6.3. Instabilities (“Floquet Heating”)

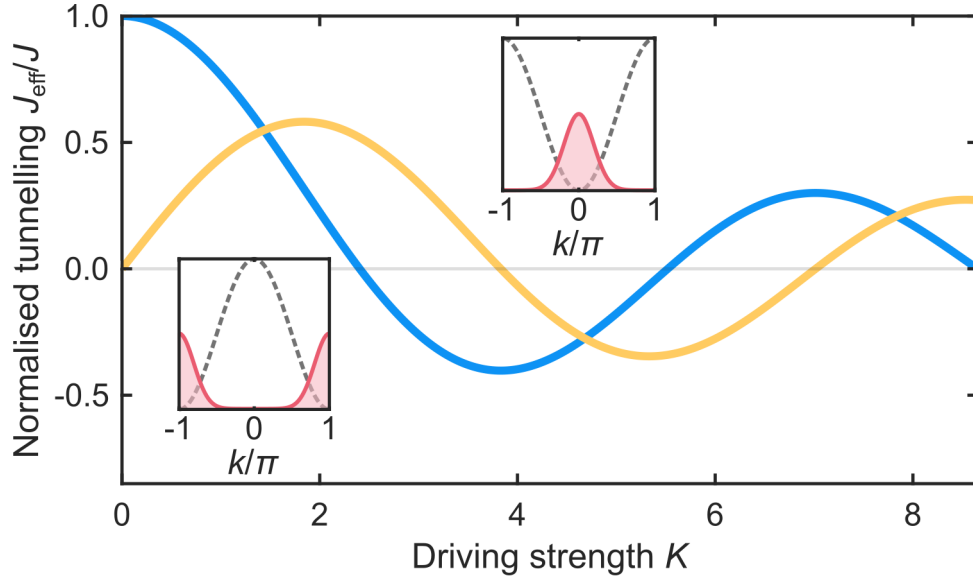


Figure 6.3: Renormalisation of tunnelling parameter. Zeroth Bessel function (blue) and first Bessel function (yellow) show how the tunnelling parameter is normalised for driven lattices and driven tilted lattices (see Sec. 6.3.5 and Chap. 8) respectively. The ground states (red) in the Brillouin zone (dashed grey line) for $J_{\text{eff}}/J > 0$ and $J_{\text{eff}}/J < 0$ are shown by the insets above and below the horizontal line.

For increasing values of K , the wave packet samples larger regions of the Brillouin zone, crossing into regions with a negative effective mass for $K > \pi/2$, until it reaches the edge of the Brillouin zone at $K = \pi$. For even larger values of K , Bragg-scattering occurs at the band edge and folds the micromotion back into the first Brillouin zone. To simplify the description, we refer to weak and strong driving strengths for K below and above $\pi/2$.

Another consequence of periodic driving is the renormalisation of the inter-site tunnelling parameter J . This occurs due to the bandwidth of the effective dispersion ϵ_{eff} , which is found by averaging the single-particle micromotion over one period

$$\epsilon_{\text{eff}} = \frac{1}{T} \int_0^T E(k(t)d_L) dt = -2J_{\text{eff}} \cos(k_0 d_L), \quad (6.12)$$

where J_{eff} follows the zeroth order Bessel function $J_0(K)$ as a function of the driving strength [41, 197]

$$J_{\text{eff}}(K) = J J_0(K). \quad (6.13)$$

The zeros of the Bessel function indicate driving strengths where tunnelling is inhibited, resulting in dynamical localisation [197]. For cases with $J_{\text{eff}} < 0$, the band structure flips (insets in Fig. 6.3) and the ground state occurs for momenta $k = \pi/d_L$, with a phase factor $\exp(i\pi)$ for the tunnelling matrix element.

6.3 Instabilities (“Floquet Heating”)

Simulating quantum many-body physics in Floquet-driven lattice potentials is challenging for interacting particles. Interactions can create instabilities and heating on

6.3. Instabilities (“Floquet Heating”)

Table 6.1: Comparison of instability mechanisms for phonon excitation in a superfluid in a lattice potential. The parameter E_p is the energy of a phonon, while α is introduced in Sec. 6.3.6.

Instability	Origin	Onset (1st resonance)	Growth	Energy / Momentum
Landau (LI)	Energetic instability	$E_p(k, q) < 0$	Exponential	Not conserved
Negative-mass (NMI)				
- static	Negative effective mass	$k > k_L/2$	Exponential, $\Gamma_q = \text{Im}(E_p)$	Both conserved
- driven	Pulsed negative effective mass	$k(t) > k_L/2,$ $\hbar\omega_D = 2\bar{E}_p/\alpha$	Exponential, $\Gamma_q = \text{Im}(\bar{E}_p)$	Both conserved
Parametric (PI)	Periodic drive	$\hbar\omega_D = E_p^{av}$	Exponential, $\Gamma_q = \text{Im}(E_p^{av})$	Momentum conserved

short timescales, which are comparable to the modulation period and which quickly destroy the coherence of the system [198]. To prevent this, it is crucial to develop an understanding of excitation mechanisms and to find an optimal window for the driving frequency [199]. Instabilities observed in the experiment can largely be categorised into three types, each with different underlying mechanisms.

Some instabilities may result from forces other than driving such as constant gradients, however when the lattice is subject to periodic driving many instabilities occur in the same parameter regimes simultaneously, making separating them difficult. Furthermore, signatures of the instabilities are near identical despite different underlying mechanisms causing them, in the experiment this is the growth of phonon modes out-with the ground state.

A general overview of different types of instabilities and their properties are listed in Table 6.1. Landau instabilities occur when phonon modes can lower the energy of the system. These will not be discussed here and are only included in Table 6.1 for comparison. The term “negative-mass instability” (NMI) is adopted in place of the common “dynamical instability” or “modulational instability” [200, 201] as these refer to exponential growth and density modulation respectively, which can be observed in other types of instabilities. The following chapters will focus on negative-mass instabilities and parametric instabilities.

6.3.1 Theoretical model for instabilities in lattices

We provide a brief summary of the description of excitation modes within Bogoliubov theories before introducing periodic driving into the system. An excited BEC can be described by a carrier wave with quasimomentum $\hbar k$ and a weak perturbation, $\delta\phi_k(z, t)$, [202]

$$\psi(z, t) = e^{-i\mu_k t/\hbar} e^{ikz} [\phi_k(z) + \delta\phi_k(z, t)]. \quad (6.14)$$

Here, μ_k is the chemical potential and ϕ_k is the solution of the stationary Gross-Pitaevskii equation. We write the perturbation as a superposition of Bogoliubov ex-

6.3. Instabilities (“Floquet Heating”)

citations each with quasimomentum $\hbar q$, amplitudes u_{kq}, v_{kq} , and energy $\hbar\omega_q(k)$

$$\delta\phi_k(z, t) = \sum_q u_{kq}(z) e^{i(qz - \omega_q(k)t)} + v_{kq}^*(z) e^{-i(qz - \omega_q(k)t)}. \quad (6.15)$$

This perturbation can exist in the form of phonon modes in the system. The time evolution of an excitation mode can be described by the Bogoliubov-de Gennes equations [203]. In static systems, and without interactions, the components u and v evolve with energy $\varepsilon_{\pm}(q)$, while the interaction energy U couples the evolution with

$$i\partial_t \begin{pmatrix} u_q \\ v_q \end{pmatrix} = \begin{pmatrix} \varepsilon_+(q) + U & U \\ -U & -\varepsilon_-(q) - U \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix} \quad (6.16)$$

where

$$\varepsilon_{\pm}(q) = 4J \sin\left(\frac{q}{2}d_L\right) \sin\left(\frac{q}{2}d_L \pm k_0 d_L\right). \quad (6.17)$$

The index k is omitted for (u_q, v_q) by assuming that the momentum of the carrier wave is k without driving.

An analytical approximation for the energy of an excitation can be derived using Eq. (6.16) [200, 204]

$$\hbar\omega_q(k) = 2J \sin(kd_L) \sin(qd_L) \pm 2\sqrt{4J^2 \cos^2(kd_L) \sin^4\left(\frac{qd_L}{2}\right) + 2JU \cos(kd_L) \sin^2\left(\frac{qd_L}{2}\right)}. \quad (6.18)$$

Here, J is the tunnelling matrix element, and U is an interaction coefficient that depends on the number of atoms, N_j , and the two-particle interaction energy at a lattice site j . Dipole traps in our experimental setup create a harmonic potential with a position-dependent N_j along the lattice direction. For simplicity, we describe the system by its properties near the trap centre where N_j is largest. The lattice potential is defined by the lattice spacing d_L , lattice momentum $k_L = \pi/d_L$, and a lattice depth V measured in recoil energies, E_r . Figure 6.4 illustrates the energies of phonons and anti-phonons given by the plus and minus signs in Eq. (6.18) [202]. For a stationary carrier medium with $k = 0$, the energies are symmetric with respect to q [dark blue lines in Fig. 6.4(a)] and the system is energetically stable for $q = 0$. Landau instabilities can occur when the carrier wave moves above a critical velocity [light blue lines in Fig. 6.4(a)] and excitations, which move in the same direction, have an increased energy, while the energy is reduced for excitations with a negative q . The time scale for the creation of those Landau instabilities is often large and usually does not limit the stability of periodically driven BECs [200].

6.3.2 Calculation of instability rate

The growth rate Γ_q of phonon modes with momentum q can be calculated using Eq. (6.16) and the diagonalisation procedure in Refs. [203, 205]. This was done using an ordinary differential equation solver ode45 in MATLAB. In the limit of fast

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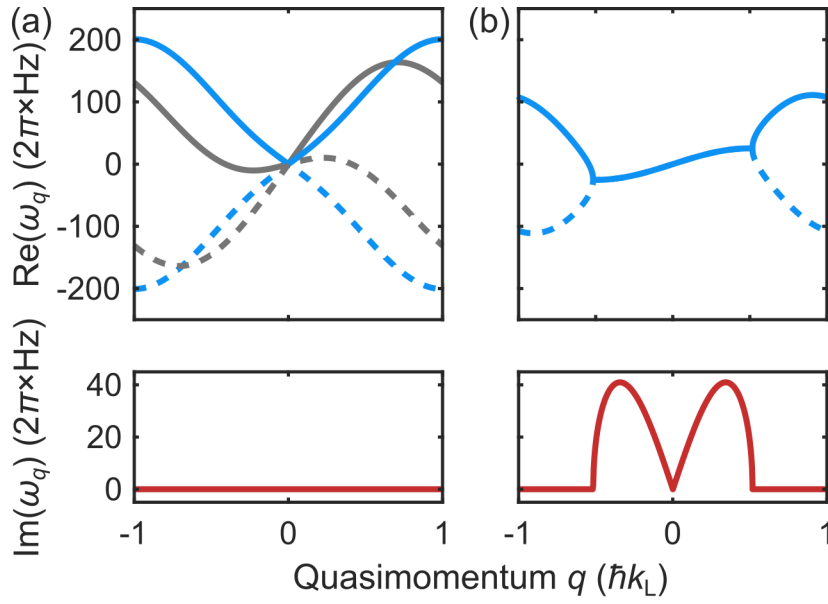


Figure 6.4: Excitation energy of phonon modes with no driving. (a) Real part of the energy for the phonon (solid lines) and anti-phonon (dashed lines) modes for $k = 0$ (blue) and $k = 0.3k_L$ (grey) in Eq. (6.18). All imaginary components are zero (red line) for parameters $V_0 = 9E_r$, $U/J = 1$. (b) Phonon and anti-phonon modes for $k = 0.9k_L$. The energy’s imaginary component is non-zero when real energies of phonon and anti-phonon modes match. Figure from Ref. [176].

driving Eq. (6.16) is solvable analytically [189], however for slower driving frequencies this is solvable only numerically. The excitation energy of the modes were then extracted from the eigenvalues λ_q where $\varepsilon_q = \log(\lambda_q)/T$. The resulting growth of the phonon mode was found using $\Gamma_q = |\text{Re}(\varepsilon_q)|$. To generate stability diagrams, such as in Fig. 6.5 and Fig. 6.7, we calculated the growth rate for various parameters, e.g. driving strength K , driving period T_D , or interaction strength U .

For strong interaction $U > 4J$, the mode that dominates the stability diagrams is usually $q = k_L$. However, to include the effect that all modes can occur, we calculated the growth for many evenly spaced modes between $q = 0$ and π and averaged the growth.

6.3.3 Negative-mass instability (modulational instability)

Static systems

NMIs appear when the argument of the root in Eq. (6.18) is negative and the energy $\hbar\omega_q$ becomes complex. As a result, excitations grow exponentially in strength with $\exp(\Gamma t)$, where the growth rate Γ is given by $\text{Im}(\omega_q)$. Such modulational instabilities have been studied in detail for quantum gases in lattices without periodic driving [202, 204, 206]. They occur when the root’s second term is negative and dominates the first, which happens either for attractive interactions or for a carrier wave with negative effective

6.3. Instabilities (“Floquet Heating”)

mass,

$$m_{\text{eff}}(k) = \hbar^2 \frac{\partial^2 E(k)}{\partial^2 k} < 0, \quad (6.19)$$

meaning they can already occur in systems without periodic driving by accelerating the superfluid by e.g. a magnetic field gradient. As a result, NMIs appear for repulsive interactions and negative effective mass when [204]

$$2J \cos^2(kd_L) \sin^2\left(\frac{q}{2}d_L\right) < -U \cos(kd_L). \quad (6.20)$$

For weak interactions with $U \ll 2J$, this condition is only fulfilled for small values of $|q|$ [red line in Fig. 6.4(b)]. For increasing interaction strength, this region of q -values with complex energy grows until it covers the entire Brillouin zone ($U = 2J$) and instabilities reach maximal growth rates at $q = \pm k_L$ for $U \geq 4J$.

Driven systems

In driven systems the energy ε in the Bogoliubov-de Gennes equations is time depending, meaning the equations take the form

$$i\partial_t \begin{pmatrix} u_q \\ v_q \end{pmatrix} = \begin{pmatrix} \varepsilon_+(q,t) + U & U \\ -U & -\varepsilon_-(q,t) - U \end{pmatrix} \begin{pmatrix} u_q \\ v_q \end{pmatrix} \quad (6.21)$$

where

$$\varepsilon_{\pm}(q,t) = 4J \sin\left(\frac{q}{2}d_L\right) \sin\left(\frac{q}{2}d_L \pm k_0 d_L \mp K \sin(\omega_D t)\right). \quad (6.22)$$

These equations can be simplified for slow and fast driving frequencies. As a reference, we use the largest energy that an excitation mode can acquire in the band due to the micromotion of the carrier wave, $\hbar\omega_{\text{max}} = 2\sqrt{4J^2 + 2JU}$. This energy provides an upper limit for the oscillation frequencies of the u, v components during one driving period. For faster frequencies, $\omega_D \gg \omega_{\text{max}}$, the response of the system is too slow, and $\varepsilon_{\pm}(q,t)$ can be replaced by time-averaged values [203]. Diagonalising the resulting matrix in Eq. (6.21) provides the energy of phonon and anti-phonon modes in the fast driving limit

$$\hbar\omega_q^f(k_0, K) = 2J_{\text{eff}} \sin(k_0 d_L) \sin(q d_L) \pm \sqrt{4J_{\text{eff}}^2 \sin^2\left(\frac{q d_L}{2}\right) \left(4J_{\text{eff}}^2 \cos^2(k_0 d_L) \sin^2\left(\frac{q d_L}{2}\right) + 2U \cos(k_0 d_L)\right)} \quad (6.23)$$

where $J_{\text{eff}} = JJ_0(K)$ is the renormalised tunnelling matrix element and J_0 is the zeroth-order Bessel function.

Equation (6.23) is similar to the effective Bogoliubov dispersion relation in [189, 203, 207], but extended by the parameter k_0 . This extension allows us to use the same representation for the effective dispersion relation in a driven system as in a non-driven system, i.e., Eq. (6.18) maps directly to Eq. (6.23) for fast driving and a stroboscopic description, by replacing J with J_{eff} , and k with the initial momentum k_0 . This result implies that the concept of NMIs and their properties can be well extended to systems

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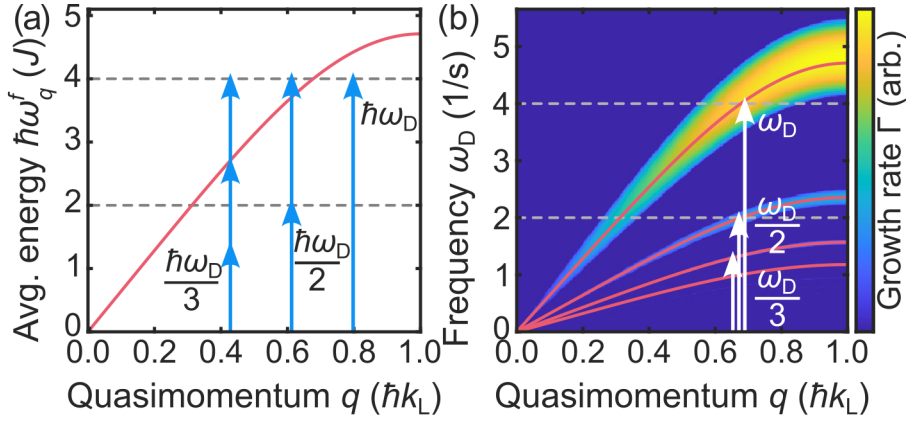


Figure 6.5: Parametric resonances. (a) Calculated excitation energy $\hbar\omega_q^f$ of modes with momentum q ($V_0 = 8E_r$, $U = 4J$, $K = 1.4$). Resonances occur when integer multiples of the driving frequency ω_D match ω_q^f . (b) Calculated growth rate Γ for excitations modes with momentum q and driving frequency ω_D . Red lines indicate the frequency matching condition $\omega_q^f = n\omega_D$. Figure from Ref. [176].

with fast driving. As in the static case, NMIs appear in the time-averaged system when $\omega_q^f(k_0, K)$ has complex components, i.e., the system is unstable for

$$2J_{\text{eff}}^2 \cos^2(k_0 d_L) \sin^2\left(\frac{q d_L}{2}\right) < -J_{\text{eff}} U \cos(k_0 d_L), \quad (6.24)$$

which can be fulfilled if one of the parameters J_{eff} , $\cos(k_0 d_L)$, or U , is negative. For example, the initial evolution of a BEC after quenching the sign of J_{eff} [197] can be interpreted as the growth of NMIs.

Slow driving with $\omega_D \ll \omega_{\text{max}}$ requires the opposite approach. The oscillations of the u, v -components and the system’s response are fast compared to T_D and the excitation energies are given by Eq. (6.18) for each moment in time. Large growth rates with $\Gamma \gg \omega_D$ limit the application of the model as excitation modes can grow quickly and break the perturbative ansatz in Eq. (6.16) before even a single driving cycle is completed. However, we find it helpful to define the average energy of an existing excitation as the system cycles through one micromotion

$$\hbar\omega_q^s(k_0, K) = \frac{1}{T_D} \int_0^{T_D} \hbar\omega_q(k(t)) dt, \quad (6.25)$$

where $k(t)$ depends on k_0 and K . The values of ω_q^s and ω_q^f are closely matched for weak driving strengths but deviate when the micromotion crosses into regions with negative effective mass and $\omega_q^s(k_0, K)$ acquires imaginary components. Both parameters, ω_q^s and ω_q^f , have limitations when describing the interesting regime of intermediate driving frequencies with comparable values for ω_D and ω_{max} .

6.3.4 Parametric instabilities

Parametric instabilities occur from the time-periodic modulation of the phonon energy $E_p[k(t), q]$, caused by the micromotion $k(t)$ of the medium. Phonon modes un-

6.3. Instabilities (“Floquet Heating”)

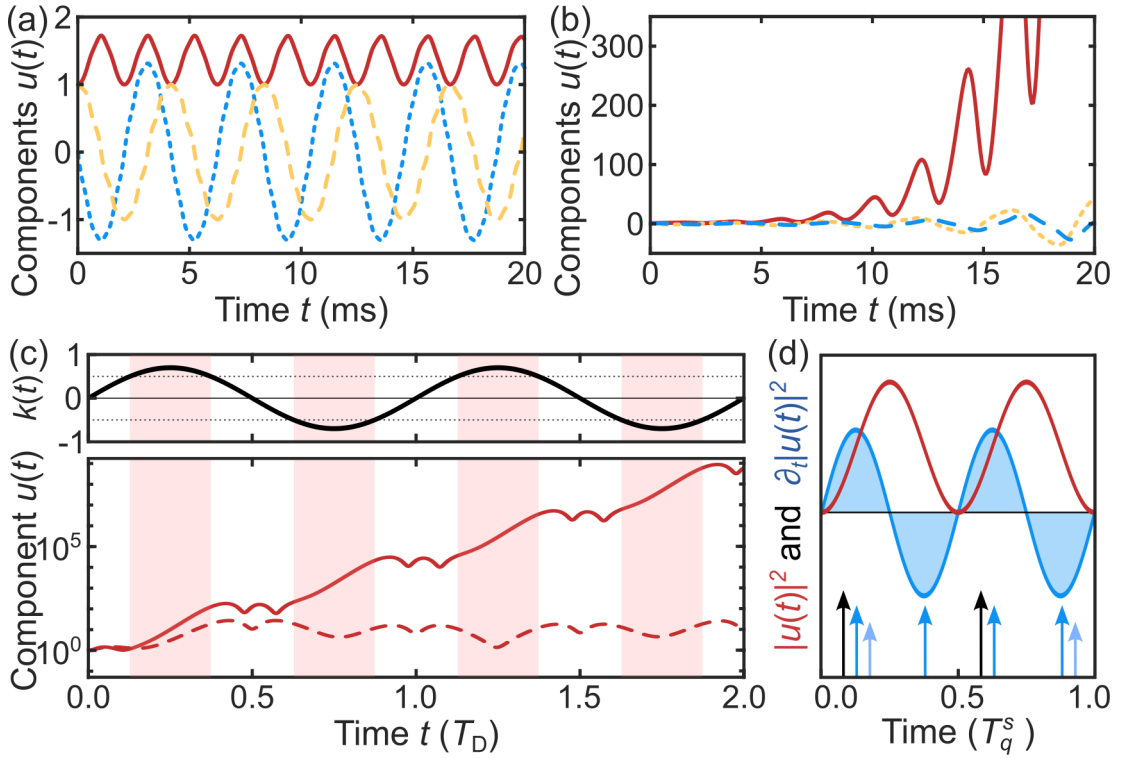


Figure 6.6: Calculated time evolution of an excitation mode ($u = 1, v = 0$). (a) Components of $u(t)$ oscillate with the mode’s intrinsic frequency ω_q^f in the fast driving limit. Yellow, blue, and red lines show $\text{Im}[u(t)]$, $\text{Re}[u(t)]$, $|u(t)|^2$, for parameters $\omega_D = 4\omega_q^f$, $K = 1.4$, $q = \hbar k_L$, $U = 10J$. (b) A parametric instability appears when ω_D matches this intrinsic frequency and the excitation mode grows exponentially. (c) Modulational instabilities require the micromotion (black line top panel) to cross into regions with negative effective mass (red patches). The system is modulationally unstable for $T_D = 11.3$ ms (solid red line), but stable for $T_D = 8.3$ ms (dashed red line) with parameters ($K = 2.2$, $k_0 = 0$, $U = 10J$). (d) Sketch of the oscillation of $|u|^2$ (red line) and its derivative (blue line). The system is stable when the effective mass turns negative (arrows) in intervals with alternating slope of $|u|^2$. Figure from Ref. [176].

dergo dynamical growth when the driving frequency is within the Bogoliubov band, $\hbar\omega_D < \Delta W$, and resonantly matches the period-averaged phonon energy $\bar{E}_p(k_0, K, q)$ [189, 207], causing an exponential growth in phonon mode population. We illustrate this condition in Fig. 6.5(a), which shows the calculated phonon energy $\hbar\omega_q^f$ for varying phonon momentum q . Parametric resonances occur at quasimomenta for which ω_q^f is equal to integer multiples of ω_D [blue arrows in Fig. 6.5(a)], exciting the fundamental or higher-order excitations. This concept works well to explain the numerically calculated growth rate for weak driving.

We tested this resonance condition by numerically calculating the phonon growth rate. We propagate Eq. (6.21) for u and v over one period, diagonalise the resulting matrix, and use its eigenvalues to determine $\Gamma_q(K, T_D)$ [203]. The regions with large values of Γ_q [green-to-yellow colours in Fig. 6.5(b)] show good agreement with the predicted resonance condition, marked by red lines $\omega_q^f = n\omega_D$ with integer n . Higher-

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order resonances are also visible and indicated by arrows.

We further investigated the cause of the instabilities by studying the time evolution of the excitation mode ($u = 1, v = 0$) over a few oscillation periods using Eq. (6.21). In the case of fast driving and weak driving strength, the real and imaginary components of $u(t)$ oscillate with frequency ω_q^f [blue lines in Fig. 6.6(a)] and $|u|^2$ oscillates with frequency $2\omega_q^f$ [red line in Fig. 6.6(a)]. Parametric instabilities arise when the driving frequency approaches ω_q^f , causing $|u|^2$ to diverge exponentially [Fig. 6.6(b)]. In contrast, NMIs only occur for strong driving strength with $K > \pi/2$ when the micromotion crosses into regions of the Brillouin zone with negative effective mass. For instance, Fig. 6.6(c) illustrates the micromotion and critical regions with negative effective mass with a black line and red patches, respectively. The excitation mode grows whenever the micromotion crosses through such critical regions (solid red line). It would seem reasonable to expect that interacting systems are always unstable for strong driving. However, we also observe narrow intervals of ω_D with large yet stable oscillations (dashed red line).

Those stable intervals in the driving frequency are caused by an interplay between the periodic growth of NMIs and the oscillation of $|u|^2$ with period $T_q^s/2$. Crossings into critical regions with negative effective mass cause the system to be periodically unstable. Surprisingly, this can reduce the oscillation amplitude of the mode, when the crossing occurs on a downward slope of $|u|^2$. The stability of the system depends on the number of crossings into critical regions of the Brillouin zone per interval T_q^s . It is unstable for two crossings [black arrows in Fig. 6.6(d)], but stable for at least four crossings (dark blue arrows) when the growth during upward and downward slopes of $|u|^2$ cancels out. We developed a general conditions to predicts driving resonances in the context of systems with a potential tilt [177]. This condition applies to both in tilted and untilted systems and will be discussed in Sec. 6.3.6 and Chap. 8.

6.3.5 Tilted systems

Tilted lattice systems arise when a lattice is subjected to a constant force F_B , resulting in an energy offset of $F_B d_L$ between neighbouring sites, also known as a washboard potential [71]. These systems have attracted interest in part due to their use in the implementation of artificial gauge fields [44, 45]. However, the addition of the tilt introduces another energy and timescale to the system, which can cause additional phonon resonances. These resonances further complicate the system and can induce different types of instabilities in the same parameter regime.

A wavepacket subjected to a constant static force in addition to periodic driving acquires the following micromotion

$$k(t) = k_0 + \frac{F_B}{\hbar} t - \frac{K}{d_L} \sin(\omega_D t + \varphi). \quad (6.26)$$

The second term describes the contribution of the force F_B , while the third term comes from the periodic driving. The phase φ is set by the switch-on procedure of the forces and their initial directions. We used opposite directions for F_B and $F(0)$ in calculations and experiments to minimise the growth of excitations for $k_0 = 0$.

6.3. Instabilities (“Floquet Heating”)

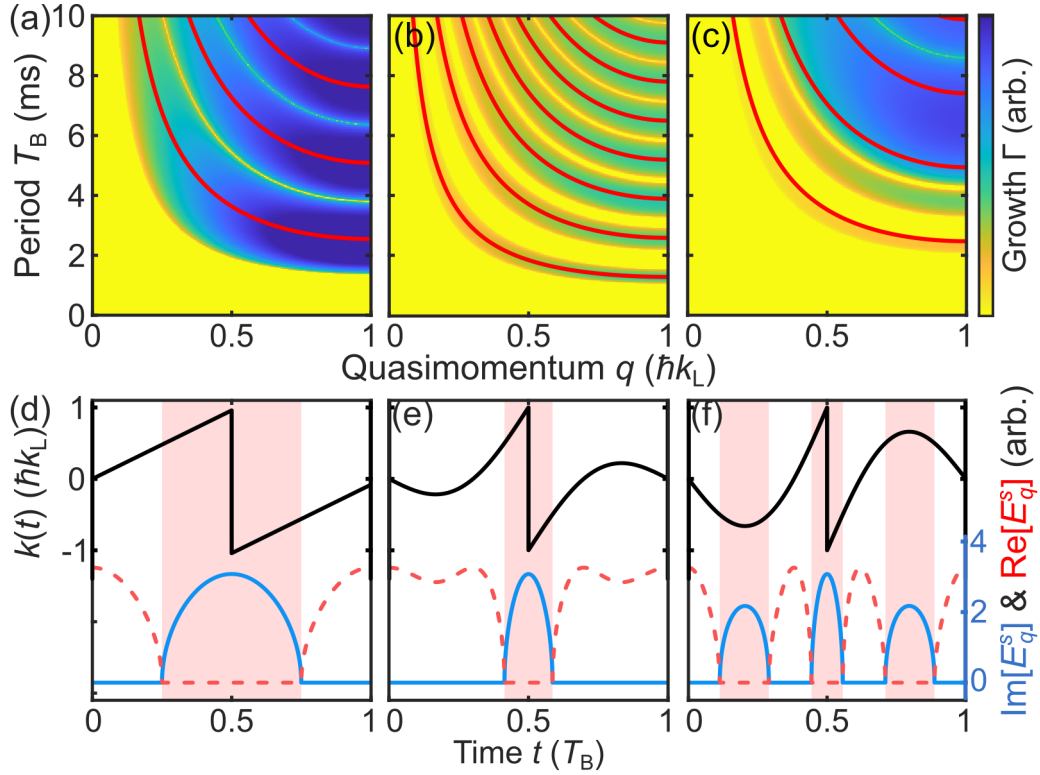


Figure 6.7: Calculated growth rate for phonon modes with momentum q . (a)-(c) Growth Γ_q for resonant driving $h\nu_D = F_B d_L = h \times 1 \text{ kHz}$, $V_0 = 10 E_r$, $U = 35 J$, and driving strength with (a) $K = 0$, (b) $K = 2$, (c) $K = 3.5$. Red lines indicate Bloch periods for phonon resonances with energy $(2E_q^s)/m_p$ and integer values of m_p . (d)-(f) Micromotion $k(t)$ (black lines), phonon energy $\text{Re}[E_{k_L}^s]$ (red lines), and growth rate $\text{Im}[E_{k_L}^s]/h$ (blue lines) for (d) $K = 0$, (e) $K = 2$, (f) $K = 3.5$. Figure from Ref. [177].

The micromotion in Eq. (6.26) alters the energy ε [Eq. 6.22]

$$\varepsilon_{\pm}(q, t) = 4J \sin\left(\frac{q}{2}d_L\right) \sin\left(\frac{q}{2}d_L \pm k_0 d_L \pm \omega_D t \mp K \sin(\omega_D t)\right), \quad (6.27)$$

with the inclusion of the additional term $\omega_D t$ to account for the force. This term includes the driving frequency ω_D as it holds for resonant driving, i.e. the driving frequency is resonant with the energy offset between neighbouring sites.

An effective dispersion for the driven lattice system, $-2J_{\text{eff}}(K) \cos(d_L k_0)$, can be derived by time-averaging the single-particle energy over one period of the micromotion $k(t)$ [208]. The phonon dispersion in the fast-driving limit is identical to that of Eq. (6.23), however now with the effective tunnelling renormalised according to the *first* Bessel function $J_{\text{eff}}(K) = J J_1(K)$. The sign of $J_1(K)$ is negative in the interval $3.8 < K < 7.0$.

The calculated growth rate Γ_q for the undriven tilted system [$K = 0$, Fig. 6.7(a)] is similar to that of the driven system [Fig. 6.5(b)]. the micromotion passes only once per cycle through a critical region with non-zero growth [Fig. 6.7(d)]. The corresponding time-averaged growth rate shows unstable regions that are boomerang-shaped and align with the predicted resonances.

6.3. Instabilities (“Floquet Heating”)

With the introduction of a driving force the superfluid’s micromotion is distorted [black line in Fig. 6.7(e)] and reduces the time it spends in critical regions of the Brillouin zone (red patch). This distortion of the micromotion changes the real and imaginary components of the time-averaged energy E_q^s . Spending less time in the critical region increases $\text{Re}[E_q^s]$ as the real component of $E_q[k(t)]$ is zero in the critical regions, while it decreases $\text{Im}[E_q^s]$ as the imaginary component is zero in the non-critical regions of the Brillouin zone. As a result, unstable regions shown in Fig. 6.7(b) move towards smaller values of T_B and shrink in size compared to Fig. 6.7(a).

6.3.6 Resonance condition

For our analysis we assume that phonon modes with momentum $\hbar q$ are resonantly excited when the frequency ν_c of the carrier wave crossing into critical regions of the Brillouin zone matches twice the time-averaged Bogoliubov energy of the mode, $\langle E_q \rangle$. This resonance condition for negative-mass instabilities is similar to the condition for parametric instabilities $\hbar \nu_D = 2\langle E_q \rangle$ in Sec. 6.3.4 and Refs. [182, 189, 207, 209], but it relies on the frequency ν_c . For strong driving forces, several crossings of critical regions can occur per Bloch period [e.g. Fig. 6.7(f)], and we introduce the parameter α to relate the resulting frequencies ν_c to ν_D . The parameter α provides the fraction of time between two consecutive crossings per Bloch period with $\nu_c = \nu_D/\alpha$. As a result, our resonance condition predicts a series of driving periods ν_D with strong growth of excitation modes for

$$\hbar \nu_D = \frac{h}{T_D} \approx \frac{\alpha}{m_p} 2\langle E_q \rangle, \quad (6.28)$$

where m_p is an integer that indicates higher-order phonon resonances (see Sec. 8.2). For the micromotions in Fig. 6.7(d)-(f), in a tilted system with resonant driving, the crossing parameter $\alpha = 1, 1, 1/3$ respectively. With this knowledge, Eq. (6.28) can be used to predict the unstable resonances for varying Bloch periods [red lines in Fig. 6.7(a)-(c)]. This prediction agrees well with the numerical results, however for strong driving strengths, e.g. $K = 3.5$ in Figs. 6.7(c) and (f), the resonance prediction begins to fail. This is suspected to be caused by the fact that the time between crossings is no longer uniform and changes throughout a single Bloch period [Fig. 6.7(f)].

Chapter 7

Measurement of instabilities of interacting matter waves in optical lattices with Floquet driving

This chapter presents a series of experimental measurements that demonstrate the existence of both negative-mass instabilities (NMIs) and parametric instabilities (PIs) and studies their properties. We found it helpful to probe the properties of excitation modes separately for non-driven and driven systems, and for NMIs and PIs, before combining our results for the analysis of the main stability measurement in Sec. 7.4. For non-driven systems, we demonstrated the growth of excitation modes when the carrier wave has a negative effective mass (Sec. 7.2.2), and we observed the decay of modes in stable regions of the Brillouin zone (Sec. 7.2.3). Combining both concepts allowed us to analyse the observed evolution of excitation modes in driven systems when the micromotion cycles through the Brillouin zone (Sec. 7.3.1). Demonstrating the existence of PIs was challenging, as we never observed the formation and growth of excitations in parameter regimes which allow for PIs but not for NMIs. Instead, we created excitation modes directly with the help of Bragg scattering pulses and measured the mode's reduced decay due to PIs (Sec. 7.3.2).

The content of the chapter is based on the article A. Di Carli, R. Cruickshank, M. Mitchell, A. La Rooij, S. Kuhr, C. Creffield, and E. Haller, “Instabilities of interacting matter waves in optical lattices with Floquet driving”, *Phys. Rev. Res.* **5**, 033024 (2023), the article is cited as Ref. [176] in the rest of the thesis.

7.1 Extracting growth rate from images

Due to the stochastic nature of the seeding of Bogoliubov modes, measuring instabilities can be challenging. For detection, we used absorption imaging to measure the momentum distribution of the wave packets after rapid switch-off of the lattice potential and after a levitated expansion time of approximately 70 ms (Fig. 7.1). In the case of periodic driving, the time evolution was analysed stroboscopically at times t that were multiples of $T_D = 2\pi/\omega_D$, while in the case of static systems the time evolution was measured at any given time t . The resulting distribution of the gas in real momentum space shows the carrier wave at k_0 and peaks of the excitation modes at $k_0 \pm q$,

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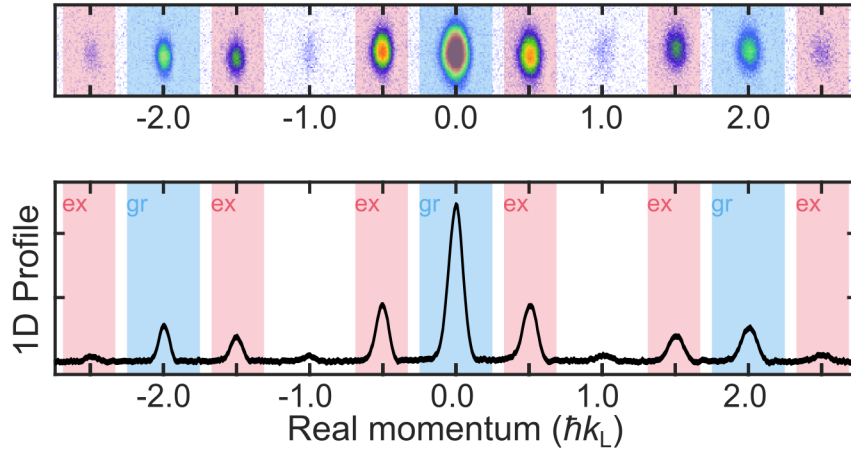


Figure 7.1: Extracting stability of system through expansion measurements. (upper) Absorption image showing the real momentum distribution after release from the lattice (average of eight repetitions). Red and blue patches indicate atoms in excited modes (N_{ex}) and in the carrier (N_{gr}) of the wave packet, respectively. (lower) Integrated density profile to count the atoms in the excited “ex” and ground-state “gr” regions.

both with repetitions at $\pm 2k_L$ (Fig. 7.1). The blue and red regions indicate atoms in the main carrier wave and in excitation modes, respectively. We calculated the atom number N_{ex} in regions “ex” that enclose the excitation modes in the momentum distribution and scaled N_{ex} to the total atom number, N_{tot} , for each image.

The initial wave packets were prepared in the ground states of the time-averaged potentials, which are centred at $k_0 = 0$ for driving strengths $K = 0 - 2.4$ ($J_{\text{eff}} > 0$) and at $k_0 = k_L$ for $K = 2.4 - 5.5$ ($J_{\text{eff}} < 0$, see Fig. 6.3). The ratio of excited atoms and total atom number, $N_{\text{ex}}/N_{\text{tot}}$, was determined after driving duration t . The exact duration was always adjusted to a multiple of T_D , and it was sufficiently short to reduce the effects of transversal [182, 210] and longitudinal excitations [165]. The system’s stability is indicated by $N_{\text{ex}}/N_{\text{tot}}$. Instabilities typically spread the wave packet in momentum space, and, as a result, cause a rapid reduction of $N_{\text{gr}}/N_{\text{tot}}$.

7.2 Negative-mass instabilities without periodic driving

Without periodic driving, NMIs were experimentally studied in detail during the 2000s (known as modulational or dynamical instabilities) [201, 211–213]. For example, modulational instabilities can grow in homogeneous superfluids with attractive interactions, leading to the formation of soliton trains [140], and they can occur for repulsive interactions in lattice potentials with quasimomenta that have a negative effective mass [201, 211]. In this section, we experimentally measure properties of NMIs with relevance to the driven system, such as the momentum distribution, growth rate, and time evolution.

7.2.1 Momentum distribution

We first studied the momentum distribution of excitation modes that develop without driving due to NMIs for carrier momentum $\hbar k_L$. We accelerated the wave packet with a magnetic field gradient to the carrier momentum, waited for approximately 10 ms and measured its quasimomentum distribution. The resulting momentum profiles [Fig. 7.2(a)] show the carrier wave at k_L and two peaks, which are caused by pairs of excitation modes with opposite quasimomenta ($+\hbar q, -\hbar q$) [214].

The position of the two peaks is related to the momentum of the most unstable mode, q_{mum} . Using Eq. (6.18), q_{mum} can be calculated as

$$\cos\left(\frac{q_{\text{mum}}d_L}{2}\right) = \begin{cases} -\sqrt{1 - \frac{U}{4J|\cos(kd_L)|}}, & \text{if } U \leq 4J|\cos(kd_L)| \\ 0, & \text{if } U > 4J|\cos(kd_L)| \end{cases} \quad (7.1)$$

Depending on the ratio U/J , the value of q_{mum} shifts from the centre of the Brillouin zone ($U \ll J$) to the edge of the BZ when U is equal to the band width $4J$. Note that the quasimomentum $\hbar q$ of an excitation mode is defined in the reference frame of the carrier wave. As a result, excitation modes have a momentum $\hbar k \pm \hbar q$ in the lattice frame, and they appear in the images at the edge of the Brillouin zone for $q = 0$ and at the centre for $q = k_L$.

In addition to the expected peaks at momentum $\pm q_{\text{mum}}$, we observed images with more than two modes and shot-to-shot fluctuations of their momenta [see 2nd example image for $V_0 = 12.5 E_r$ in Fig. 7.2(a)]. To account for these fluctuations, we used a statistical analysis of the NMIs. We took approximately 150 images, applied a peak-finding algorithm to detect excitation modes and calculated histograms of their positions, Fig. 7.2(b). The shapes of the histograms match well with the predicted growth rates $\Gamma_q = \text{Im}(\omega_q)$ for increasing interaction strengths U/J [solid lines in Fig. 7.2(b)]. The absolute values of the growth rates will be discussed in Sec. 7.2.2.

For the measurement of q_{mum} , we determined the two peaks with the most likely quasimomentum for positive and negative values of q [Fig. 7.2(c)]. Connected pairs of data points indicate the momenta of the two peaks per histogram. Histograms with $U/J > 4$ have a single peak at $q = 0$. Our data matches well to the predicted values for q_{mum} in Eq. (7.1), and we expect that the observed U/J -dependence of q_{mum} can be extended to driven systems. However, there is a discrepancy with the models used in previous studies [182, 189, 215]. In our measurements, the most unstable mode does not dominate the system's time evolution, but it is merely the most likely mode in a distribution. While, in a scenario of infinite carrier medium and infinite observation time, the most unstable mode would outgrow all others, such conditions do not exist in our system.

7.2.2 Growth rates

The growth rate Γ_q of excitations is difficult to measure directly due to their random occurrence. To illustrate this challenge, we used a magnetic field gradient that accelerated the wave packet to momentum $\hbar k$, waited for a variable hold time, and evaluated the momentum profile using absorption imaging. We observed that excitation modes grow predominantly near the centre of the Brillouin zone [region “ex” in Fig. 7.3(a)],

7.2. Negative-mass instabilities without periodic driving

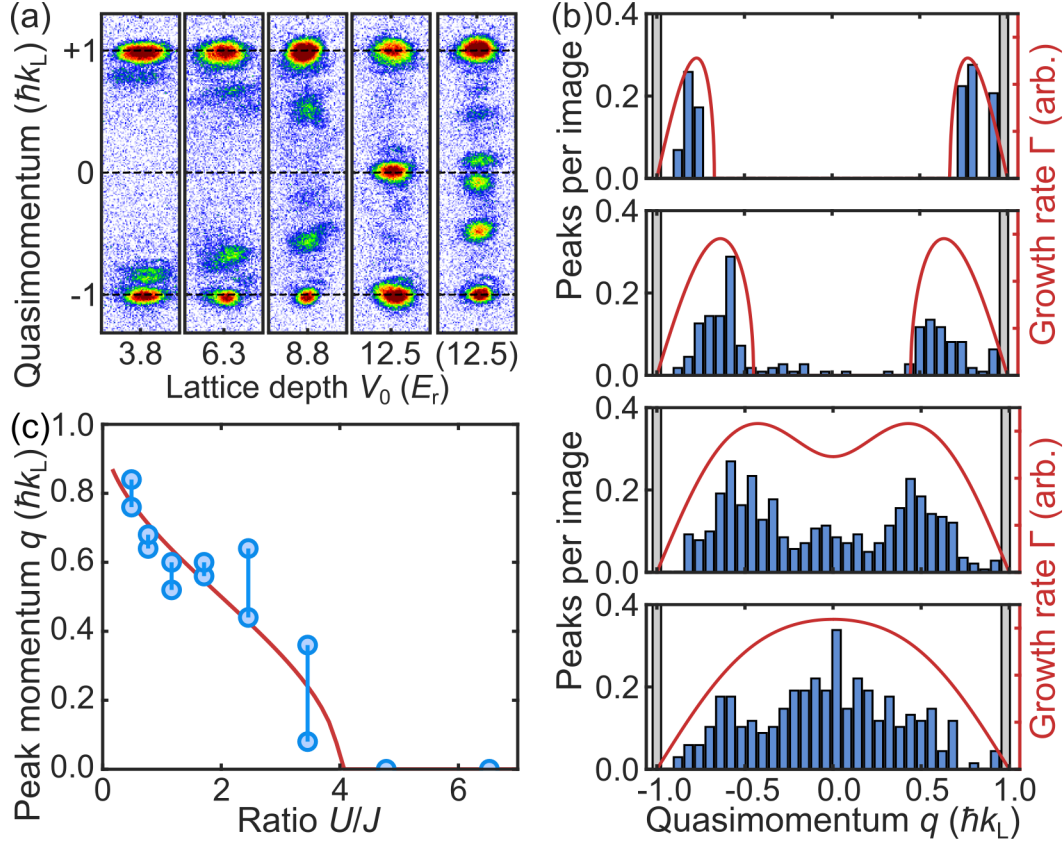


Figure 7.2: Momentum distribution of NMIs without driving. (a) Example images of NMIs for different lattice depths V and $k = k_L$ after approximately 10 ms ($a_s = 6a_0$, $N = 3 \times 10^4$). We observe images with matching pairs of excitation modes ($V_0 = 3.8 - 12.5 E_r$), but also images with multiple modes ($V_0 = 12.5 E_r$ right). (b) Histogram of peak positions for increasing lattice depths and $U/J = 0.5, 1.2, 2.5, 6.5$ (top-to-bottom). Red lines indicate $\Gamma = \text{Im}(\omega_q)$. (c) The measured peak positions of the histograms match well to the predicted positions in Eq. (7.1) (red line). Figure from Ref. [176].

7.2. Negative-mass instabilities without periodic driving

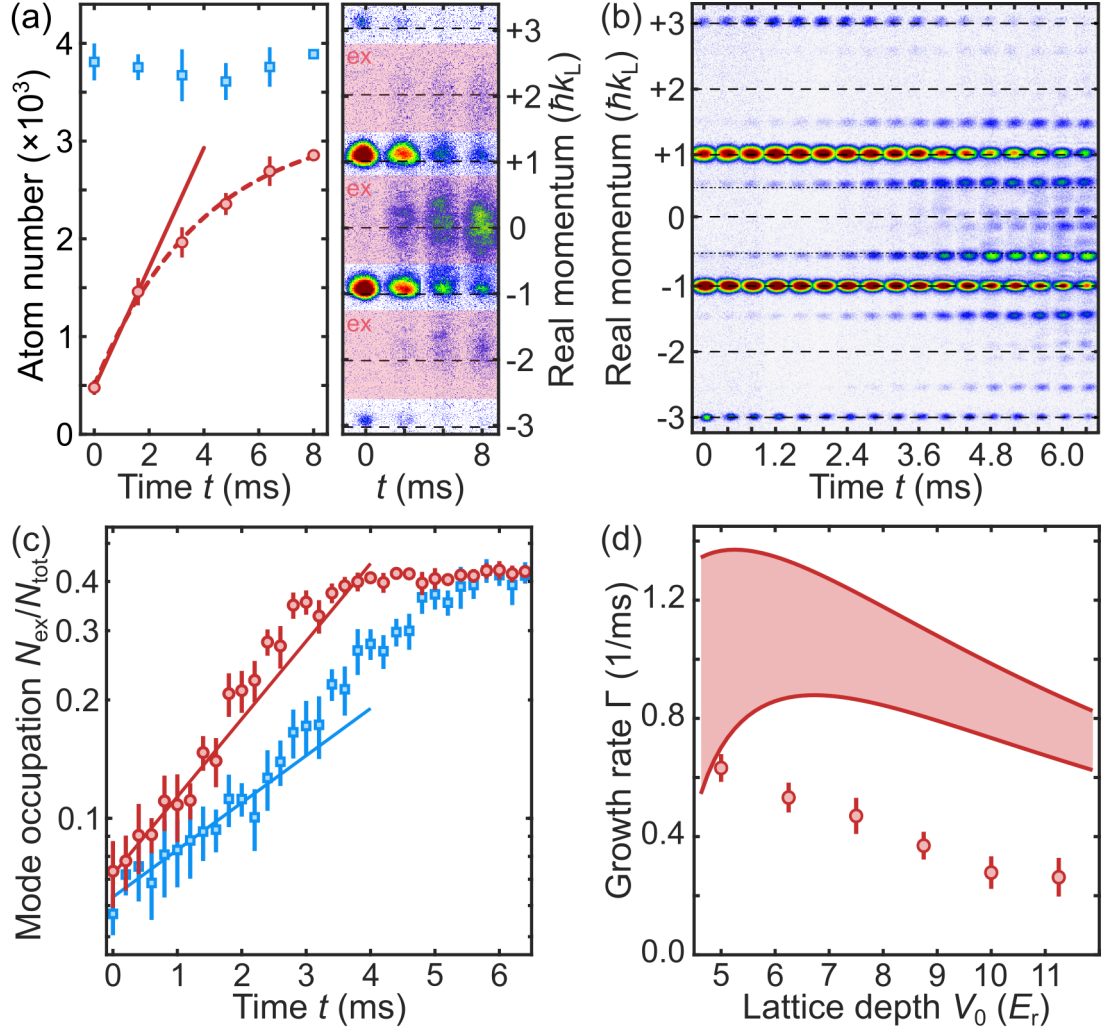


Figure 7.3: Growth rate of NMIs without driving. (a) Time evolution of excitations with N_{ex} atoms at the centre of the BZ, (red circles, red patches) and total atom number, N_{tot} (blue squares), $k = 0.95k_L$, $V_0 = 10E_r$, $a_s = 90a_0$. Lines are added to guide the eye. (b) Absorption images of the time evolution after seeding excitations at $q = \pm 0.46$ ($k = k_L$, $V_0 = 10E_r$, $a_s = 14a_0$, $N_{\text{tot}} \approx 41,000$, $U/J = 8.9$, 5 repetitions). (c) Relative atom number in seeded excitation modes for $V_0 = 10E_r$ (blue squares) and $V_0 = 6.3E_r$ (red circles). The lines are exponential fits to the data points with $t < 1.5$ ms. (d) Measured growth rate Γ_q (red circles) and range of expected rates $\text{Im}(\omega_q)$ assuming lattice site occupations between 60% and 100% of the peak value (red patch). Figure from Ref. [176].

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resulting in an increase in atom number N_{ex} in that region, note that the region occurs at $\hbar k = 0$ instead of the usual $\hbar k = 1$ due to a flip in the dispersion. Instead of exhibiting exponential growth as expected, the increase in N_{ex} shows a logarithmic-like curvature. This curvature results from the simultaneous or sequential growth of multiple excitation modes. Similar logarithmic-like growth of excitation modes was also reported for a driven system in [215], and for non-driven systems when using atom loss as an indicator for mode growth [201].

To enhance the predictability of NMIs, we seeded the modes directly using Bragg pulses generated by a second lattice. The pulse intensity and duration were adjusted to create a seed with 5-8% of the atoms in the carrier wave. The absorption images of the time evolution show an initial increase in the excited modes for 4 ms, followed by coupling to higher order modes at $\pm 2q$ [Fig. 7.3(b)]. To evaluate the growth, we measured the relative atom numbers $N_{\text{ex}}/N_{\text{tot}}$ in the excitation modes [Fig. 7.3(c)]. Initially, the seeded modes experience exponential growth that is followed by saturation or even a decrease, which is due to the limited number of atoms and the growth of higher-order excitations. When determining the exponential growth rate within the first 1.5 ms, the measured values of Γ_q are significantly smaller than $\text{Im}(\omega_q)$, which are the values suggested by Eq. (6.18) [Fig. 7.3(d)]. This difference between the measured and expected growth rates increases further when we consider that the measurement probes the density of the excitation modes, which should show twice the growth rate [215].

We speculate that this discrepancy between predicted and measured growth rates is caused by different initial states of the excitation modes. $\text{Im}(\omega_q)$ is calculated for the eigenstates (u_q, v_q) of the matrix in Eq. (6.18). However, these states may not match the modes created by Bragg scattering or by random seeding, leading to initial oscillations [red line Fig. 6.6(a)] and an inaccurate measurement of exponential growth. In addition, the calculated growth rates also become unreliable for longer hold times, as the perturbative ansatz in Eq. (6.14) is no longer applicable for large excitation modes.

7.2.3 Decay rates

Before discussing driven systems in the next section, it is instructive to study the evolution of existing excitation modes in a stable region of the Brillouin zone. Compared to Sec. 7.2.2, we used stronger Bragg pulses that excited approximately 40% of the carrier wave to momentum $\pm q$ and prepared the carrier wave in the stable region at $k = 0$. The resulting time evolution [Fig. 7.4(a,b)] shows a fast oscillation and a slow decay of the number of atoms in the excitation modes, N_{ex} . The first minimum of N_{ex} is at $T_h \approx 3$ ms, the maximum at 6 ms, and exponential decay times are approximately 15 ms. We find that T_h depends on lattice depth and interaction strength, but not on the initial occupation strength of the excitation mode. Measured values of T_h show a similar trend for varying lattice depths as $2\pi/\omega_q$ [red circles and blue squares in Fig. 7.4(c)].

We speculate that the time evolution of N_{ex} in Fig. 7.4 is caused by an oscillation between two configurations of a density wave. After the Bragg pulse, the initial state consists of a carrier wave and two excitation modes of similar strength and quasimomentum $\pm q$. Interfering these modes creates a density modulation of the carrier wave with wavelength $\lambda_q = 2\pi/q$ [sketch in Fig. 7.4(c)]. This density wave grows due to

7.2. Negative-mass instabilities without periodic driving

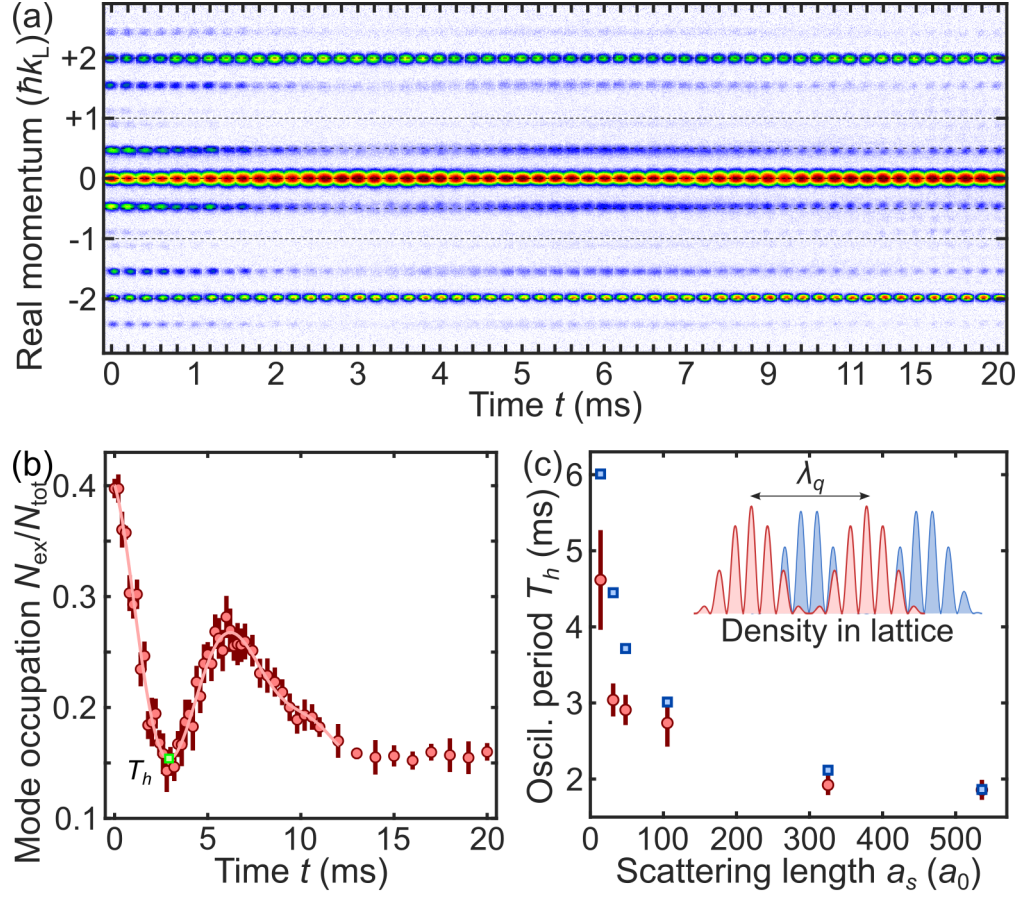


Figure 7.4: Decay of excitations in stable regions without driving. (a) Images of the time evolution of excitation modes $k = 0$ ($a_s = 31 a_0$, $V_0 = 10 E_r$, $q = 0.47 k_L$, eight repetitions). The excitation modes are created by a Bragg pulse which excites approximately 40 % of the atoms. Images are centred to the $q = 0$ peak. (b) Fraction of atoms $N_{\text{ex}}/N_{\text{tot}}$ in excited mode for data in (a). We determine the time at the minimum T_h by locally fitting a 2nd order polynomial. (c) T_h for various scattering lengths (red circles) and period $2\pi/\omega_q$ in Eq. (6.18) (blue squares). The inset shows a sketch of two density profiles (red and blue) for a pair of excitation modes with $\pm q$. Figure from Ref. [176].

7.3. NMIs and PIs with periodic driving

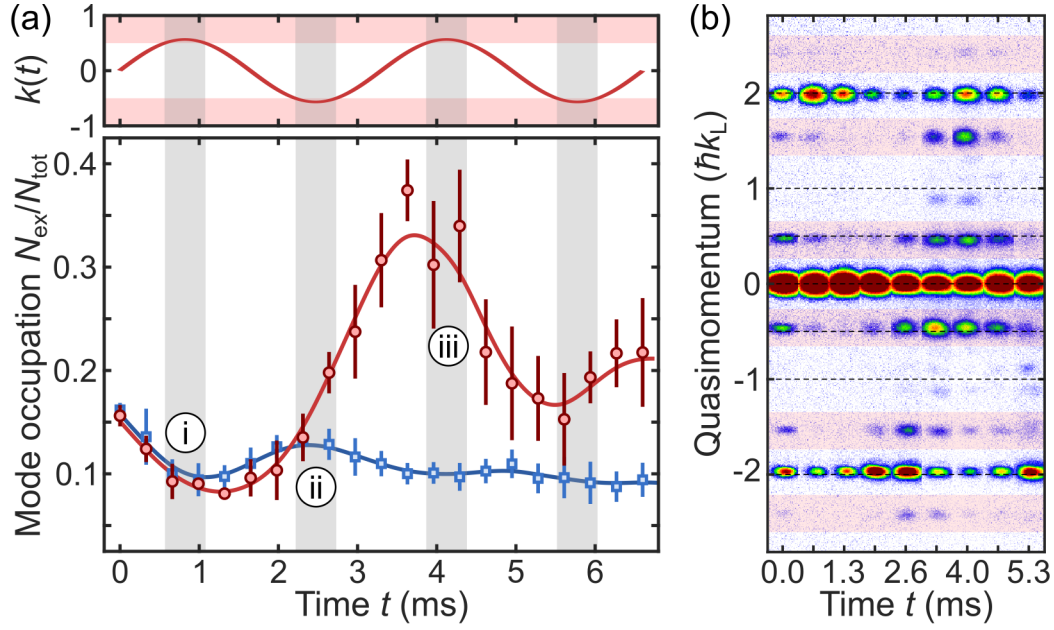


Figure 7.5: Excitation growth due to NMIs with driving. (a) Time evolution of a seeded excitation mode without driving (blue squares) and with driving (red circles), $q = 0.48 k_L$, $V_0 = 6.3 E_r$, $a_s = 105 a_0$, $T_D = 3.3$ ms. Top panel: Micromotion for $K = 1.8$, $k_0 = 0$. Critical regions of the BZ are indicated by red patches and time intervals of $k(t)$ in those critical regions by grey patches. The lines in the lower panel are smoothing splines to guide the eye. (b) Averaged absorption images for the driven system. We observe no significant heating during the initial evolution time. Figure from Ref. [176].

NMIs for $|k| > k_L/2$ (Sec. 7.2.2), but it decays for a stable system with $|k| < k_L/2$. For those stable carrier momenta, repulsive interactions cause the wave to spread and to regenerate at a shifted position $\lambda_q/2$. As a result, the time evolution of N_{ex} shows an oscillation with period $2\pi/\omega_q$ between these two configurations of the density wave.

The damping of the oscillations can be attributed to our trapping potential and the finite system size. In our experimental setup, the trapping potential breaks the translational symmetry and lifts the energy degeneracy for the two configurations of the density wave, which results in the dephasing and decay of the oscillations. However, the measured timescale of 10 to 20 ms for damping is shorter than expected for our trap periods of approximately 100 ms, and we did not observe dephasing or heating in the momentum profiles in Fig. 7.4(a).

7.3 NMIs and PIs with periodic driving

We separately examine in this section both types of instabilities at intermediate driving frequencies. For NMIs, we demonstrate the impact of micromotion on excitation modes by observing a complex pattern of growth and decay as the carrier wave cycles through the Brillouin zone (Sec. 7.3.1). For PIs, we provide evidence of their existence in a parameter regime where NMIs are not possible (Sec. 7.3.2).

7.3.1 Negative-mass instabilities

In driven systems, NMIs occur when the carrier wave periodically passes through regions of the Brillouin zone with positive and negative-effective mass. As we demonstrated in Sec. 7.2.2 and 7.2.3, NMIs grow in unstable regions but oscillate and decay in stable regions of the Brillouin zone. In this section, we show that, for periodic driving, both effects combine to create complex oscillations of the excitation modes.

In a first step, we applied a Bragg pulse that generated a pair of excitation modes with 15% of the carrier wave. Without driving, the resulting density wave was unstable for $k = 0$ and exhibited damped oscillations of the mode occupation with a first minimum at $T_h \approx 1.1$ ms [blue squares in Fig. 7.5(a)]. In a second step, we added a driving force with period T_D and observed its impact on the mode occupation [red circles in Fig. 7.5(a)]. For a driving strength $K = 1.8$ and $T_D \approx 3T_h$, the micromotion crossed briefly into regions with negative effective mass whenever the non-driven oscillation of the excitation mode was close to a maximum or minimum. Red and grey patches in Fig. 7.5(a) indicate intervals in quasimomentum space and time with negative effective mass.

The first crossing of the micromotion into an unstable region of the Brillouin zone [marked by (i) in Fig. 7.5(a)] has little effect on the excitation mode, as the mode's occupation is small and decreasing. The driving force even reduces the mode occupation between 1 ms and 2 ms compared to the non-driven system. However, when the occupation of the non-driven mode is at a maximum at the second crossing (ii), the driven excitation mode grows strongly by almost a factor three. At the third crossing (iii), as the mode occupation begins to decrease, the oscillation amplitude is reduced again. We found that the mode's oscillation amplitude continued to fluctuate with a complex interplay between micromotion and mode occupation.

We did not observe a significant increase of the number of thermal atoms during the first 5 ms [Fig. 7.5(b)] which suggests that the process was mostly coherent. For NMIs, excitation modes must occur in pairs with opposite momenta ($+\hbar q, -\hbar q$) to conserve the total momentum. Also the energy is conserved due to the symmetry of Eq. (6.18) for which the real parts of ω_{+q} and ω_{-q} have opposite signs [interval with overlapping blue lines in Fig. 6.4(b)]. As a result, pairs of excitation modes can grow for $|k| > 0.5k_L$ without changing the system's energy [e.g. Fig. 7.3(b)]. However, we expect that the observed oscillations result in a slow heating of the system as the micromotion drags existing modes through the Brillouin zone. Further studies are needed to evaluate this mechanism on longer time scales. The mechanism is different for PIs that can be created at $|k| < 0.5k_L$ when the total energy of the modes is positive.

7.3.2 Parametric instabilities

Weak driving strengths seem to be ideally suited for the search of PIs. For $K < \pi/2$, the micromotion does not cross into regions of the Brillouin zone with negative effective mass and NMIs are not possible. However, we did not detect the growth of excitation modes in this region in our measurements for the stability diagram (see Sec. 7.4). Instead, we employed a different approach by demonstrating the reduced decay of existing modes due to PIs. Using a Bragg pulse, we created excitations at $q = 0.46k_L$ and monitored their decay.

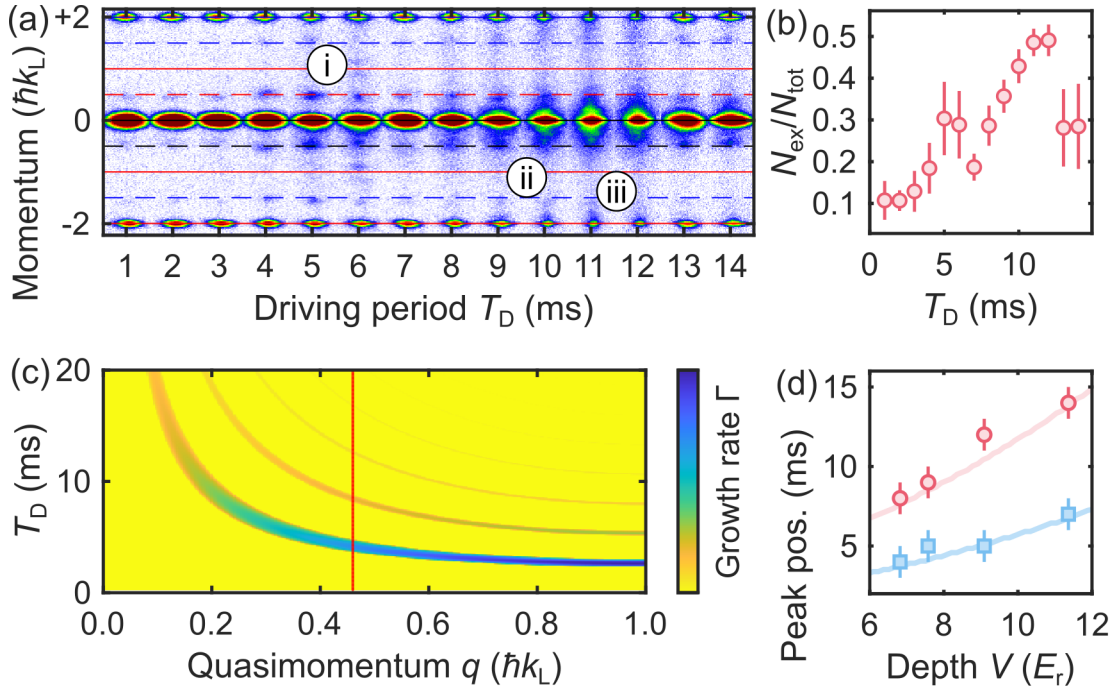


Figure 7.6: Measurement of parametric instabilities. (a) Absorption images with momentum distribution after driving for approximately 30 ms ($a_s = 73 a_0$, $K = 1.3$, $V_0 = 9 E_r$, $N_{\text{tot}} = 3.4 \times 10^4$). An initial Bragg pulse seeds excitation modes at $q = 0.46 \hbar k_L$. (b) Number of atoms in the excited state, N_{ex} , for the data in (a). (c) Numerically calculated growth rates using Eq. (6.16) and parameters in (a). The line indicates the quasimomentum of the excited excitation mode. (d) Measured driving periods of the first (blue squares) and second (red circles) peak of N_{ex} for varying lattice depths. Solid lines show calculated peak positions.

7.4. Stability diagram

For most driving periods, those excitation modes were no longer visible in the momentum profiles after 30 ms [Fig. 7.6(a)]. However, some weak excitations persisted for $T_D \approx 5$ ms (i) and $T_D \approx 10$ ms (ii). We determined the number of atoms in the ground state, N_{gr} , from Gaussian fits to the peaks at $0, \pm 2\hbar k_L$, and calculated the relative number of excited atoms as $N_{\text{ex}} = 1 - N_{\text{gr}}/N_{\text{tot}}$. The two peaks of N_{ex} in Fig. 7.6(b) match well to the observed excitation modes in the absorption images. Figure 7.6(c) compares the measured resonance positions with the calculated growth rates. The initially excited mode at $q = 0.46k_L$ is indicated by the red line. We measured the two resonance positions for varying lattice depth and compared them to the calculated growth rates [Fig. 7.6(d)]. We observed good agreement between the measurements and the predictions, except for the large strength of the second resonance at $T_D \approx 10$ ms. The width and height of its peak in Fig. 7.6(b) are likely due to the coupling with other excitation modes at smaller q , which can also be seen in the absorption images at (iii).

This measurement demonstrates the existence of PIs, but their growth rates seem too small to excite the system within 30 ms by random seeding. We find for the parameters in Fig. 7.6(a) that the maximum calculated growth rate for weak driving (only PIs) is a factor of 3 smaller than the growth rate for stronger driving with $\pi/2 < K < 2.4$ (PIs and NMIs). This variation in growth rates may be due to the different coupling mechanisms between the carrier and excitation modes for PIs and NMIs. PIs require an oscillating driving force to excite the carrier wave and add energy, whereas NMIs are achieved by periodically switching the instability on and off without adding energy.

7.4 Stability diagram

To probe the stability of the system for a range of driving parameters initial wave packets were prepared in the ground states of the time-averaged potentials, which are centred at $k_0 = 0$ for driving strengths $K = 0 - 2.4$ and at $k_0 = k_L$ for $K = 2.4 - 5.5$ [Fig. 7.7(a)]. For this measurement, we found it easier to determine the ratio of atoms in the carrier wave and the total atom number, $N_{\text{gr}}/N_{\text{tot}}$. The ratio was determined after a driving duration close to 30 ms. The exact duration was always adjusted to a multiple of T_D , and it was sufficiently short to reduce the effects of transversal [182, 210] and longitudinal excitations [165].

With this, a stability diagram was created [Fig. 7.7(a)], with the system's stability indicated by $N_{\text{gr}}/N_{\text{tot}}$. Instabilities typically spread the wave packet in momentum space, and, as a result, cause a rapid reduction of $N_{\text{gr}}/N_{\text{tot}}$. The measurement shows areas of high and low stability (yellow and blue colours) with clear boundaries in between. For example, the system is stable for weak driving strength K and for fast driving frequencies [regions (i) and (ii)], but it is unstable for K values close to 2.5 [region (iii)]. Strong driving strengths with $K \approx 4.4$ increase the system's stability again [region (iv)]. For very fast driving, $T_D < 0.6$ ms [region (v)], instabilities are caused by the coupling to higher bands [216] and will not be considered here.

The observed regions (i-iv) do not match well to predictions based on parametric instabilities [189, 193]. Instead, we compare our measurement result to our own numerical simulations, which calculated the growth rates $\Gamma_q(K, T_D)$ using the Bogoliubov-de Gennes equations [Eq. (6.16)] [203] [Fig. 7.7(b)]. We account for the variation of quasimomenta (Sec. 7.2.2) by averaging the growth rate over a collection of q values

7.4. Stability diagram

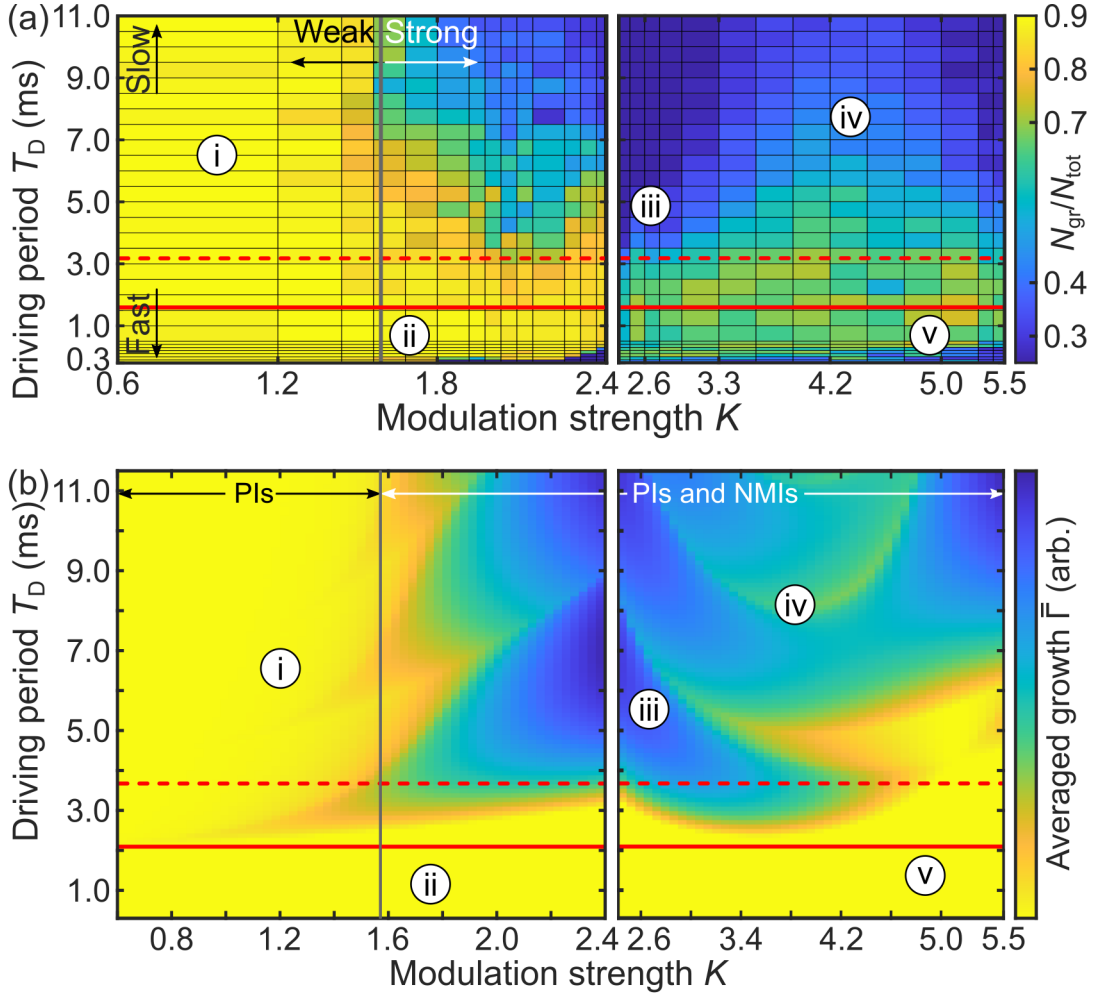


Figure 7.7: Stability diagrams. (a) Measured stability diagram for a driven BEC in a 1D lattice potential ($V_0 = 8.8 E_r$, $U/J = 22$). The number of excited atoms, N_{ex} is measured after driving for approximately 30ms with strength K and driving period T_D . Left and right panels indicate different initial states with $k_0 = 0$ and $k_0 = k_L$ for positive and negative values of J_{eff} . (b) Numerically calculated stability using $\bar{\Gamma}(K, T_D)$ for parameters as in (a). The black vertical line at $K = \pi/2$ separates weak from strong driving regimes. Solid and dashed horizontal red lines provide $2\pi/\omega_{\text{max}}$ and $h/8J$, respectively. Regions indicated by Roman numerals are discussed in the text.

7.5. Conclusion

in the Brillouin zone, $\bar{\Gamma} = \langle \Gamma_q(K, T_D) \rangle_q$. The parameter $\bar{\Gamma}$ provides an indicator for the system's stability. The structure of $\bar{\Gamma}$ along the T_D axis results from higher-order resonances (see Fig. 6.5).

The calculated parameter $\bar{\Gamma}$ provides a good qualitative estimate for the observed regions of stability. The system is least stable for $K \approx 2.5$ in region (iii), and the transition line increases to larger values of T_D at point (iv). Even the small cusp in the transition line at $K \approx 2.4$ matches between prediction and experiment. However, the agreement is only qualitative and values of T_D for the transition line are smaller in the calculation than in our measurement. For example, the transition from stable to unstable occurs at $K = 2.4$ for $T_D \approx 6$ ms in the experiment and for 3 ms in the calculation. In general, the system is more stable in our experimental measurement than predicted by the calculation, which might be due to the short observation time. Using longer driving durations would improve our measurement sensitivity, but requires the consideration of additional time scales, such as trapping periods.

Roman numerals in Fig. 7.7 indicate the same characteristic regions between sub-figures. We did not observe any excitation modes in the weak driving regime (i) as discussed in Sec. 7.3.2, although numerical calculations predict the creation of PIs. We speculate that the PIs' growth rates are insufficient for our observation times and for random seeding. The system starts to become unstable when we cross into the region of strong driving strength at $K = \pi/2$ (vertical grey lines in Fig. 7.7). In addition, the measurement shows that the system is stable for small values of T_D at all driving strengths [region (ii)], which corresponds to the fast driving regime. The energy threshold $\hbar\omega_{\max}$ (Sec. 6.3.1) predicts the fast driving limit correctly (solid red line), however, the condition is strict and not useful to describe the transition line between stable and unstable driving periods. As a reference, we also provide the time scale $\hbar/8J$ of twice the width of the lattice band (red dashed lines).

7.5 Conclusion

In conclusion, we studied the stability of driven matter waves with repulsive interactions in a 1D lattice. By measuring the momentum distribution of the matter waves after a fixed hold time, we identified stable and unstable regions based on the driving period T_D and strength K . Our results emphasise the importance of making a clear distinction between modulation and parametric instabilities in driven systems. Parametric instabilities arise when the drive resonantly couples to excitation modes, while negative-mass instabilities stem from inherent properties of the medium, such as attractive interactions or a negative effective mass. Our key findings are summarised as follows. Negative-mass instabilities are well suited for describing intra-band excitations in the limits of fast and slow driving frequencies. We demonstrated for the fast driving limit that phonon energies and stability criteria map directly to the non-driven system. This enables the use of established concepts for the stability analysis of interacting many-body states in Floquet-driven systems. The description is applicable for the initial growth of excitation modes while perturbation theories remain valid.

Of particular interest is the regime of intermediate driving frequencies, where both negative-mass and parametric instabilities are present. The micromotion of the driven

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wave packet periodically crosses into regions of the Brillouin zone that are unstable due to negative-mass instabilities, resulting in a complex growth and decay of excitation modes. Directly demonstrating parametric instabilities was challenging due to their low growth rates. As an alternative, we demonstrated a reduced decay of existing excitation modes at the driving parameters where only parametric instabilities are expected.

We compared the measured stability diagram with predictions for the growth of excitation modes based on the Bogoliubov-de Gennes equations. There is a good qualitative agreement of stable and unstable zones, however, our experimental system was more stable than predicted and the stable fast-driving regime extended to larger driving periods. We found that the time-averaged energies $\hbar\omega_q^f$ and $\hbar\omega_q^s$ provide good estimates for the position of resonances and boundaries at weak driving strength and for $K \approx 2.4$.

Our results have important implications for experiments using Floquet-engineered potentials, particularly for applications that require minimal excitations and heating. By identifying stable and unstable parameter regions, we can predict which experimental conditions will lead to unwanted excitations. Energy transfer and excitation mechanisms are different for negative-mass and parametric instabilities, and a clear distinction between these two types of instabilities will be instrumental to study the system over longer time scales as required, e.g., for the creation of discrete time crystals [217] or dynamical gauge fields [218].

Chapter 8

Phonon excitations of Floquet-driven superfluids in a tilted optical lattice

This chapter provides an investigation of instabilities of a superfluid trapped in a tilted optical lattice potential. The additional energy scale due to the tilt creates new regimes depending on the ratio of the driving frequency and energy offset between adjacent lattice sites. As such, the measurements are categorised depending on the value of this ratio. In Sec. 8.1 the system is without driving, meaning any instabilities are solely due to the tilt. Section 8.2 deals with the case where the driving frequency is resonant with the offset, while Sec. 8.3 deals with non-resonant driving. All sections provide a measured stability diagram to probe the system’s stability across a range of parameters.

The results shown in the chapter are based on the article R. Cruickshank, A. Di Carli, M. Mitchell, A. La Rooij, S. Kuhr, C. Creffield, and E. Haller, “Phonon excitations of Floquet-driven superfluids in a tilted optical lattice”, Phys. Rev. Res. **6**, 023323 (2024), the article is cited as Ref. [177] in the rest of the thesis.

8.1 Excitations without periodic driving

To gain an understanding of the system, we first studied excitation modes for only a constant force, F_B . Without driving, the superfluid Bloch oscillates in the first Brillouin zone with its micromotion, $k(t)$, following straight lines in quasimomentum space [219]. Absorption images, showing the momentum distribution in quasimomentum space [Fig. 8.1(a)], were used to determine the fraction of atoms in excitation modes $N_{\text{ex}}/N_{\text{tot}}$ [blue circles in Fig. 8.1(b)]. Whenever the superfluid crosses into critical regions of the Brillouin zone with negative effective mass, $|k(t)| > k_L/2$, small perturbations of the stationary state grow exponentially in time [201, 220], while excitations are steady or decay in stable regions [176].

We quantified the system’s stability for a varying constant force and lattice depth by measuring $N_{\text{ex}}/N_{\text{tot}}$ after approximately $t = 20$ ms. Yellow and blue colours in the (V, T_B) -stability diagram [Fig. 8.1(c)] indicate stable and unstable regions with weak and strong growth of phonon modes, respectively. We expect the superfluid to become stable in deep lattices and for large values of F_B due to the suppression of tunnelling when the energy gap between neighbouring sites, $F_B d_L$, exceeds the width of the lattice band, $4J$, with tunnelling matrix element J . However, we find that the decoupling of

8.1. Excitations without periodic driving

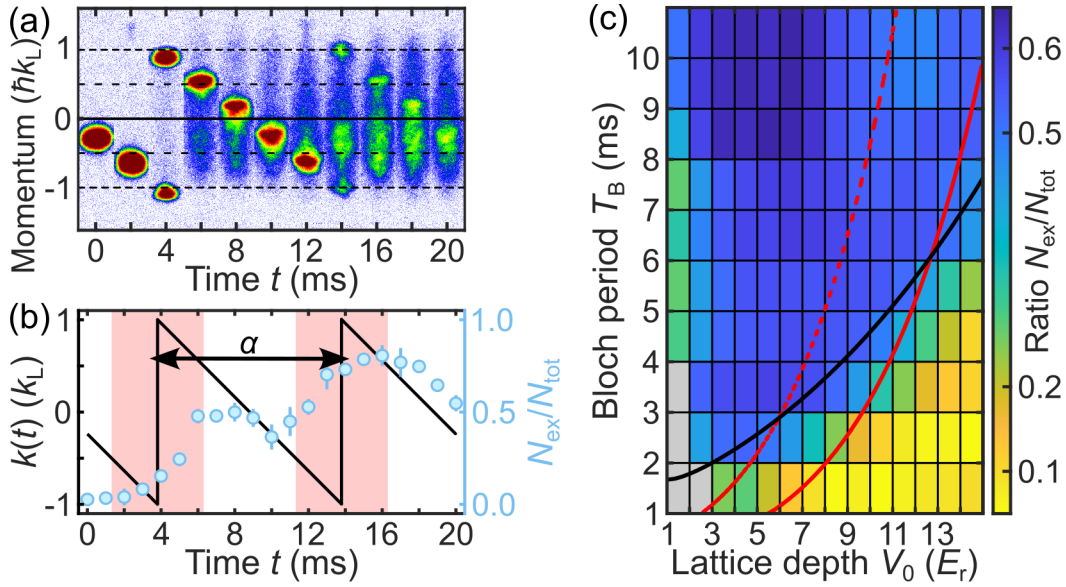


Figure 8.1: Negative-mass instabilities without periodic driving. (a) Absorption images of Bloch-oscillating atoms and excitation modes in quasimomentum space for $T_B = 10$ ms, $V_0 = 6 E_r$, $U/J = 3$. (b) Atoms in excited modes $N_{\text{ex}}/N_{\text{tot}}$ (blue circles) and calculated micromotion $k(t)$ (black line) for parameters in (a). Red patches indicate time intervals when micromotion crosses into unstable regions of the Brillouin zone. (c) Measured ratio $N_{\text{ex}}/N_{\text{tot}}$ after a hold time of $t \approx 20$ ms. Lines indicate Bloch periods that match to the single particle bandwidth $4J/h$ (dashed red line), the approximated phonon energy $2E_{k_L}^s/h$ (solid red line), and the predicted transition line between unstable and stable parameters in [220] (black line). Grey patches indicate strong atom loss for small V and large values of F_B . Figure from Ref. [177].

lattice sites is far less abrupt for a 1D-superfluid with large atom numbers per site than for single atoms [221], and the superfluid remains phase coherent beyond $F_B d_L = 4J$ [dashed red line in Fig. 8.1(c)]. This allows us to observe the reduction of negative-mass instabilities when the system becomes stable in the fast cycling regime. This onset of stability shifts for increasing interaction strength towards smaller values of T_B .

The time-averaged energy of a phonon mode $\langle E_q[k(t)] \rangle$ with micromotion $k(t)$ can be approximated by two methods, both of which are based on the Bogoliubov-de-Gennes equations (BdG) [189, 205, 209, 222]. Time-averaging the single-particle energy in the BdG equations provides an approximation, $E_q^f(K, k_0)$, that depends on the driving strength K and on the initial momentum $k_0 = k(0)$ [203, 205, 209, 220] (see Sec. 6.3.5). For fast driving frequencies, this analytic expression maps directly to the energy of a phonon $E_q(k)$ in a non-driven system when we identify k with k_0 and the tunnelling matrix element J with $J_{\text{eff}}(K)$ (see Sec. 6.3.5). This mapping indicates that negative-mass instabilities also exist in periodically driven systems for $|k_0| > 0.5k_L$. However, the approximation assumes fast driving frequencies and neglects the short time intervals of phonon growth when the micromotion cycles through the Brillouin zone.

For a low-frequency approximation, we numerically time-average the energy $E_q[k(t)]$

8.2. Excitations with resonant driving

of an existing phonon over $k(t)$ [176], as in Eq. (6.25) but now over the Bloch period T_B

$$E_q^s = \frac{1}{T_B} \int_0^{T_B} E_q[k(t)] dt. \quad (8.1)$$

Due to negative-mass instabilities, the phonon energy $E_q(k)$ is a complex number with an imaginary component that indicates the mode's growth rate Γ_q and a real component that provides the frequency of the mode's phase oscillations [200, 202]. The same complex description applies to E_q^s , but not for the approximation E_q^f . Only the real component of E_q^s is used in Eq. (6.28) to predict resonant cycling frequencies.

Without driving, the micromotion crosses once per Bloch period into critical regions with $\alpha = 1$ [Fig. 8.1(a)] resulting in a periodic increase of the phonon mode occupation [Fig. 8.1(b)]. Resonances occur when the frequency ν_c of these crossings matches to the natural oscillation frequency of the phonon density $2\langle E_q \rangle$ (with $\alpha = m_p = 1$). For faster frequencies, the system becomes stable when $h\nu_c$ exceeds the energy of all phonon resonances, i.e., when the cycling due to Bloch oscillations becomes faster than possible response times of the phonons. We find that the transition to this fast cycling regime (or fast driving regime in later sections) is well approximated by Eq. (6.28) [solid red line in Fig. 8.1(c)]. An even better prediction for the transition from an unstable to a stable system is provided by $T_B = h/(2.96\sqrt{2JU})$ [black line in Fig. 8.1(c)], which is an analytical approximation to a numerical solution of the Bogoliubov-de Gennes equations in [220].

We observed a second stable region at shallow lattice depths $V_0 < 2E_r$ [Fig. 8.1(c)]. For repulsive interactions, negative-mass instabilities are induced by the lattice potential and we expect the superfluid to become stable in the limit of vanishing lattice depths. We omit the discussion of instabilities in shallow lattice potentials as identifying unstable intervals of the micromotion is more challenging beyond the tight-binding regime [200].

8.2 Excitations with resonant driving

8.2.1 Experimental measurements

In a second series of experiments, we added a resonant driving force $F(t)$ with a driving period that matches T_B . Despite a strongly tilted lattice potential, resonant driving re-introduces coupling between the lattice sites with an effective tunnelling matrix element, $J_{\text{eff}}(K) = JJ_1(K)$ [208]. Here, $J_1(K)$ is the first-order Bessel function and $K = F_B d_L / (h\nu_D)$ is the dimensionless driving strength. For increasing driving strength, the straight lines of the micromotion [$K = 0$, dotted black line in Fig. 8.2(a)] start bending upwards or downwards [red and blue lines in Fig. 8.2(a)], depending on the initial directions of F_B and $F(0)$. Coloured patches in Fig. 8.2(a) indicate time intervals during which the superfluid's micromotion crosses into critical regions of the Brillouin zone. We used opposite initial directions of the forces to reduce these time intervals [red and orange lines in Fig. 8.2(a)], as identical initial directions extend them (blue dashed line).

8.2. Excitations with resonant driving

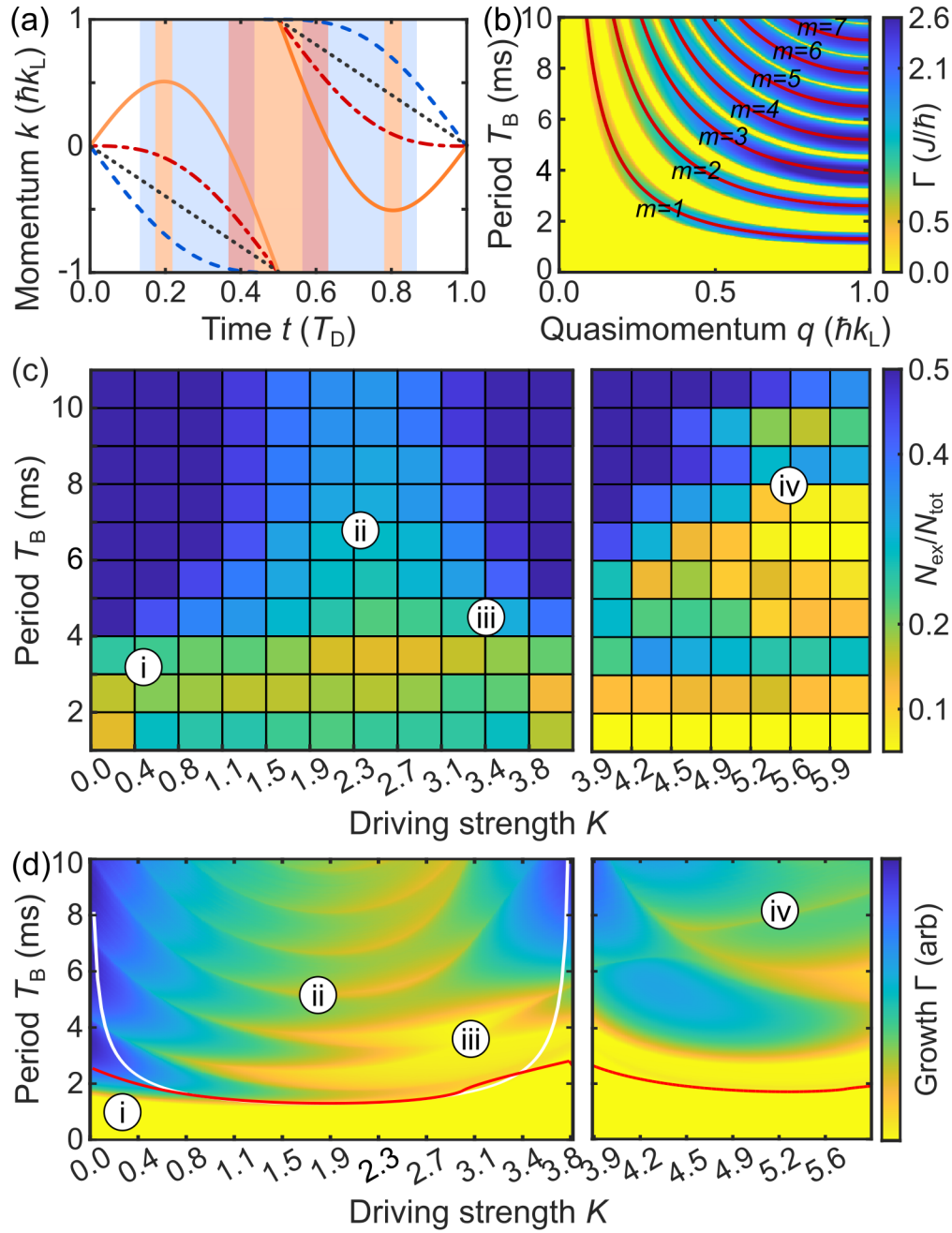


Figure 8.2: Stability of a superfluid in a tilted potential with a resonant driving force. (a) Calculated micromotion for $K = 0, 1$ (black and blue lines) and for $K = 1, 3$ with opposite initial direction of $F(0)$ and F_B (red and orange lines). Patches with corresponding colours indicate time intervals in critical regions of the Brillouin zone. (b) Numerical calculation of the growth rate Γ for $K = 2$, $V_0 = 10 E_r$, $U/J = 35$ together with predicted resonance (red line) with energy $2\text{Re}[E_{k_L}^s]$ in Eq. (6.28). (c) Experimental measurement of atoms in excitation modes after driving for approximately 30 ms with strength K and parameters $V_0 = 10 E_r$, $N_{\text{tot}} \approx 60,000$, $a_s = 107 a_0$. Left and right panels use initial momentum $k_0 = 0$ and $k_0 = k_L$. Average over typically six repetitions. (d) Numerical calculation of Γ with same parameters as in (c). Red and white lines indicate the predicted position of the fundamental resonance in Eq. (6.28) when approximating the phonon energy by E_q^s and E_q^f . Figure from Ref. [177].

8.2. Excitations with resonant driving

To develop a better understanding of the system's stability, we calculated the growth rate, $\Gamma_q(T_B, K)$, of phonons with momentum q by integrating and diagonalising the Bogoliubov-de-Gennes equations [203, 205, 209] (see Sec. 6.3.2). The resulting (q, T_B) -stability diagram for $K = 2$ [Fig. 8.2(b)] shows a sequence of unstable regions with strong growth (blue colour) which belong to fundamental and higher-order phonon resonances $m_p = 1 - 7$. The resonance condition Eq. (6.28) provides a good approximation of these resonances (red lines with $\alpha = 1$). Successive higher-order resonances are separated by a stable region (yellow colour). Position, width, and growth rate for these stable and unstable regions can be related to the time that the micromotion spends in critical regions of the Brillouin zone (see Sec. 6.3.5). However, this q -dependence of Γ is usually not directly visible in experiments due to the stochastic nature of negative-mass instabilities. Phonon modes require small initial modulations of the medium to start growing [220]. Those seeds are of random nature, caused by thermal, quantum or technical fluctuations, which result in the occurrence of modes with varying values of q [223].

We experimentally determined the growth of phonon modes by measuring $N_{\text{ex}}/N_{\text{tot}}$ after a driving period of approximately 30 ms. Two different initial quasimomenta, $\hbar k_0 = 0$ and $\hbar k_L$, were used to prepare the superfluid in the ground state of the time-averaged lattice band [panels in Fig. 8.2(c)]. The ground state's micromotion starts with $k_0 = 0$ for $K < 3.83$ when $J_{\text{eff}}(K)$ is positive and with $k_0 = k_L$ for $3.83 < K < 7.02$ for negative values of $J_{\text{eff}}(K)$. To push the superfluid to the edge of the Brillouin zone, $k_0 = k_L$, we applied a weak magnetic field gradient before starting the drive [177].

We compare the measured (T_B, K) -stability diagram in Fig. 8.2(c) to the calculated growth rate $\Gamma(T_B, K)$ averaged over 11 modes with equally spaced values of $q = 0 - k_L$ [Fig. 8.2(d)]. Measurements and numerical calculations show good qualitative agreement for characteristic parameter regions that are indicated by Roman numerals in the diagrams. Particularly important for future experiments is the stable region at high driving frequencies [region i in Fig. 8.2(d)] which, for weak driving strengths, starts approximately at the first resonance with $m_p = 1$. Both E_q^s and $E_{k_L}^f$ provide a good approximation for $\langle E_q \rangle$ in Eq. (6.28), except for K values close to the zeros of $J_1(K)$ [red and white lines in Fig. 8.2(d)].

Stronger driving strengths, e.g. $1.5 < K < 3.0$, shorten the time that the superfluid spends in critical regions and reduce growth rates and widths of the resonances. As a result, even large driving periods show an increased stability, both in our data and in calculations (region ii). Even stronger driving strengths, e.g. with $K \approx 3$, cause three crossings of the micromotion into critical regions of the Brillouin zone. This reduces the α parameter and shifts the stable fast-driving regime towards larger values of T_B (region iii). Even stronger driving strengths, $3.9 < K < 5.9$, for which the ground state has an initial momentum $k_0 = k_L$, cause multiple crossings into critical regions with varying values of α . Despite the resulting complex structure of stable and unstable regions in our calculation (region iv), we find good agreement between calculations and experiment within our measurement resolution.

8.2.2 Interpretation of the stability diagram

The structure of stable and unstable regions in our calculation of Γ can be explained using Eq. (6.28). For a direct comparison with Eq. (6.28), we omitted the averaging and calculated $\Gamma_q(K, T_B)$ for a single mode, $q = k_L$, and increasing values of K [Fig. 8.3(a)]. The local maxima of Γ_{k_L} were determined using a peak finding algorithm (dashed black lines) and we compared them to the predicted resonances with parameters (m_p, α) (solid red and blue lines in Fig. 8.3).

We find that unstable regions on the left side of each panel show a mostly regular pattern of higher-order resonances [regions (L) in Fig. 8.3(a)]. The vertical distance between two resonances increases for each panel with $1/\alpha$ (blue lines) as predicted by Eq. (6.28). α was determined using the micromotion for each value of K [Figs. 8.3(b)-(d) for each panel in (a)]. The largest values of α [arrows in Figs. 8.3(b)-(d)] provide a good prediction for the resonance positions in regions (L). However, we find a more complex pattern of intersecting resonances on the right side of each panel in regions (R), and our prediction of the resonance position works less well [red lines in Fig. 8.3(a)]. The difference between regions (L) and (R) is caused by different number of crossings of the micromotion (coloured patches), which have similar, but slightly different, time intervals between them in regions (R). We speculate that the structure of resonances is caused by this multitude of slightly different α values and their corresponding cycling frequencies ν_c .

Equation (6.28) provides a general condition for resonances in lattice systems with negative-mass instabilities and a cycling micromotion, irrespective of whether the cycling is caused by driving, a tilted lattice potential, or both. For systems without tilt, ν_B must be replaced by the driving frequency ν_D , and α increases from 0 for weak driving strength, when the micromotion never crosses into critical regions of the Brillouin zone, to $1/2$ for moderate driving strength. The resonance condition for this special case with $\alpha = 1/2$ and $m_p = 1$ is $h\nu_D = \langle E_q \rangle$ which matches the condition suggested in Ref. [189].

8.3 Excitations with non-resonant driving

Finally, we examine the stability for off-resonant driving frequencies with detuning $\Delta\nu = \nu_B - \nu_D$. Off-resonant driving in tilted lattice potentials can induce super-Bloch oscillations with large amplitudes in position space [224–226] that make the study of phonon modes challenging. To reduce the amplitude of those oscillations, we chose large detunings with $\Delta\nu > 20$ Hz and a strong force with $\nu_B = 1$ kHz. This Bloch frequency also ensures that the system is well in the fast-driving regime and any growth of phonon modes can be attributed to $\Delta\nu$. In momentum space, super-Bloch oscillations show the characteristic straight lines of Bloch oscillations in the Brillouin zone (as in Fig. 8.1) when evaluating $k(t)$ stroboscopically at integer multiples of the driving frequency. The period of those oscillations is $\Delta T = 1/\Delta\nu$.

To understand the growth of phonon modes for non-resonant driving, it is helpful to study the complete micromotion $k(t)$. The shape of the micromotion gradually shifts, resulting in predominantly stable and predominantly unstable time intervals [red patches in Figs. 8.4(a) and (b)], and it repeats itself after a period $T_{\text{tot}} = s/\nu_B = r/\nu_D$ for

8.3. Excitations with non-resonant driving

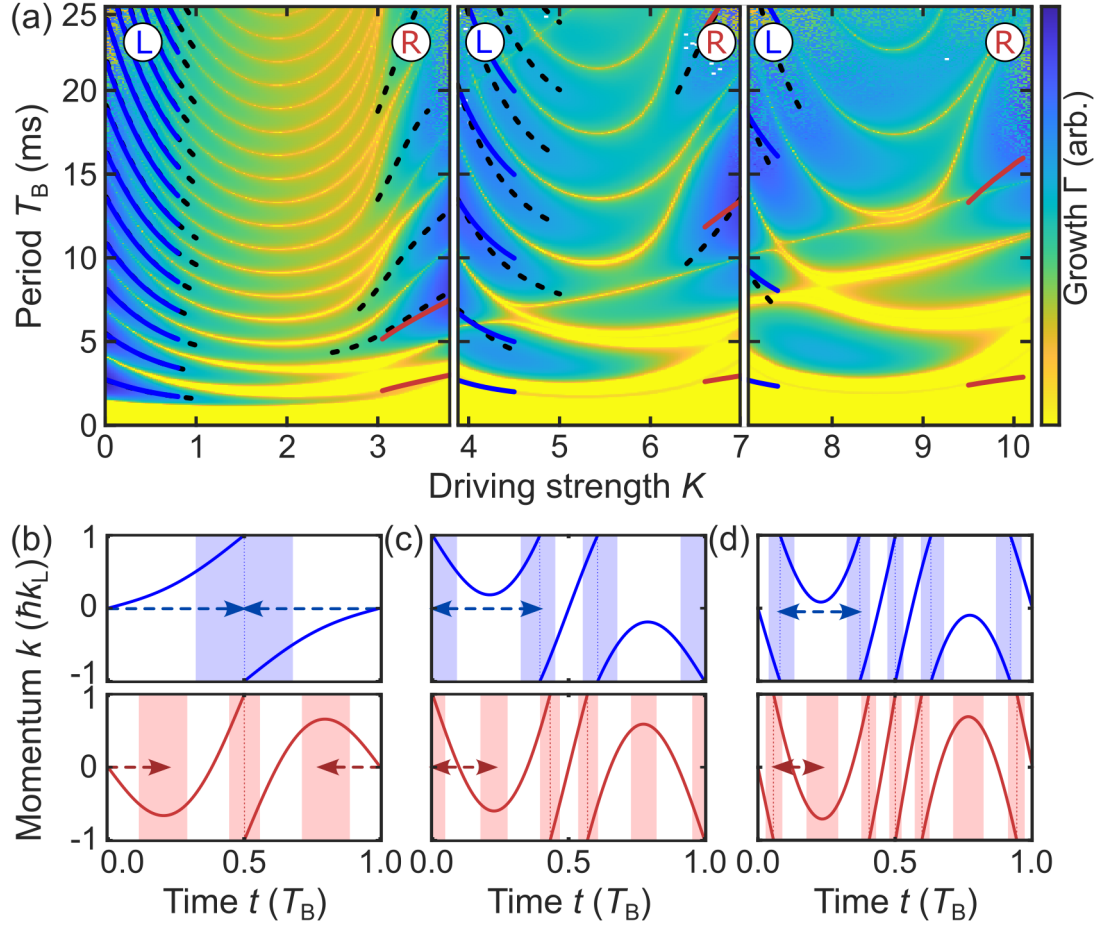


Figure 8.3: Growth rate and driving resonances. (a) Calculated growth rate Γ_{k_L} for $V_0 = 10E_r$ and $U = 30J$. Dashed black lines indicate local maxima of Γ_{k_L} that were determined with a peak-finding algorithm. Solid blue and red lines show the predicted resonance positions in Eq. (6.28). Panels indicate different signs of $J_{\text{eff}}(K)$. (b)-(d) Examples of the micromotion for the three panels in (a), top and bottom panels refer to regions (L) and (R) with blue and red line colours, respectively. (b) $K = 0.5$ (top) $K = 3.5$ (bottom), (c) $K = 4.0$ (top) 6.5 (bottom), (d) $K = 7.5$ (top) 10.0 (bottom). Coloured patches indicate time intervals of the micromotion in regions of the Brillouin zone with NMIs. Arrows show α for the longest time intervals between consecutive crossings. Figure from Ref. [177].

8.3. Excitations with non-resonant driving

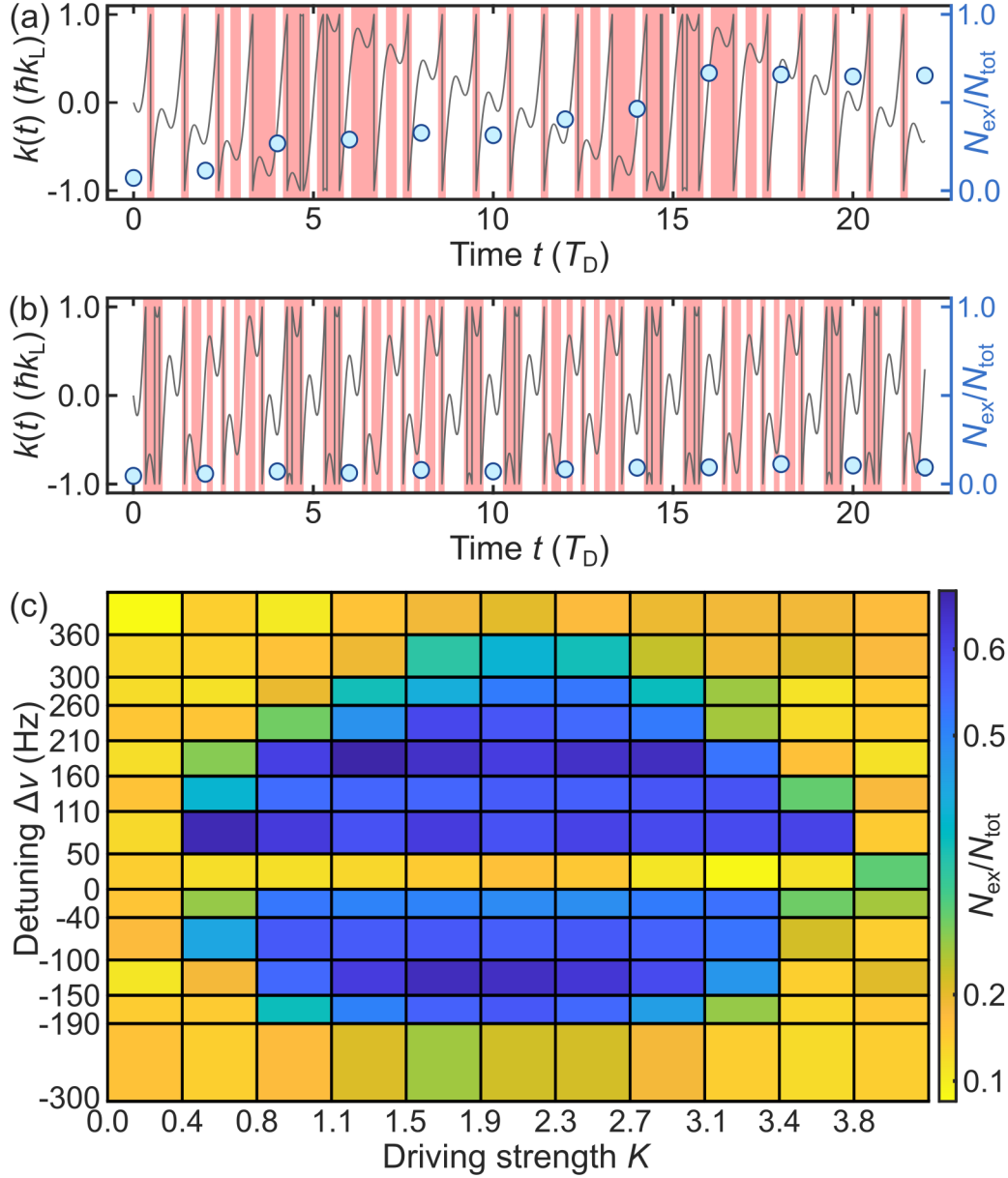


Figure 8.4: Stability of the superfluid in a tilted potential with off-resonant driving. (a) Micromotion (grey line) and measured number of atoms in phonon modes N_{ex} (blue circles) for small detuning $\Delta\nu = 100$ Hz with parameters $K = 1.5$, $V_0 = 8 E_r$, $k_0 = 0$, $T_B = 0.998$ ms, $U/J = 15$, and (b) for large detuning $\Delta\nu = 800$ Hz. Coloured patches indicate time intervals in the critical region of the Brillouin zone. (c) Measured stability diagram showing $N_{\text{ex}}/N_{\text{tot}}$ after approx. 30 ms of driving and parameters $V_0 = 8 E_r$, $N_{\text{tot}} \approx 60,000$, $a_s = 107 a_0$. Figure from Ref. [177].

8.3. Excitations with non-resonant driving

integer values s, r and rational ratios v_D/v_B . The period T_{tot} depends on the choice of s, r and is less relevant for the phonon growth than the period between predominantly stable and unstable intervals, ΔT . We found that the growth of phonon modes shows a similar time dependence for super-Bloch oscillations as for Bloch oscillations in Sec. 8.1. For instance, we observe for $\Delta T = 10$ ms that excitation modes grow in the predominately unstable time intervals at 5 ms and 15 ms [blue circles in Fig. 8.4(a)], and the system once again attains stability for larger detunings, e.g., for $\Delta v = 800$ Hz [blue circles in Fig. 8.4(b)].

As in Sec. 8.2, we quantified the growth of phonon modes by measuring $N_{\text{ex}}/N_{\text{tot}}$ after approximately 30 ms of driving. The resulting $(K, \Delta v)$ -stability diagram shows a stable region for zero detuning [yellow colour in Fig. 8.4(c)], as expected in the fast-driving regime. Small values of Δv cause instabilities as the superfluid spends longer time intervals in predominantly unstable regions (blue colours), while we find the system to be again stable for large detunings with $|\Delta v| > 360$ Hz. The overall shape of the unstable region resembles an ellipse with a stable region in the form of a horizontal line at the centre.

We compared our measurement results to a numerical calculation of the phonon growth rate Γ_q that was again based on the Bogoliubov-de Gennes equations. Instead of integrating the Bogoliubov-de Gennes equations over T_B , as in Sec. 8.2, we used the complete driving cycle T_{tot} . To match our experimental parameters and to provide rational values of v_B/v_D , we chose $v_B = 1$ kHz, $T_{\text{tot}} = sT_B$, and $T_D = (s/r)T_B$ with $s = 500$ and integer values of r . We again used the phonon momentum $q = k_L$ that provides the strongest growth [Fig. 8.5(a)]. The shape of the unstable region and the stable line at resonance match well to our experimental results. As for normal Bloch-oscillations in Fig. 8.1(c), the transition lines between stable and unstable parameters regions are again well predicted by the approximation $\hbar|\Delta v| = 2.96\sqrt{2J_{\text{eff}}(K)U}$ [220], where we replaced J by the renormalised $J_{\text{eff}}(K)$ [black line in Fig. 8.5(a)].

To explain the shape of the unstable regions in Fig. 8.5(a), we use a resonance condition similar to the one in Eq. (6.28). We replace the Bloch frequency with the super-Bloch frequency Δv and use $\alpha = 1$ for Bloch oscillations,

$$\hbar\Delta v \approx \frac{1}{m_p}2\langle E_q \rangle. \quad (8.2)$$

Due to the detuning, the averaged phonon energy continues to oscillate when we time-average over the Bloch period, and we instead approximate $\langle E_q \rangle$ with a long time average E_q^{tot} calculated over T_{tot} . For a direct comparison, we use the momentum $q = k_L$ instead of averaging over different values of q . Equation (8.2) provides a good approximation for the resonances [blue regions and dashed red lines in Fig. 8.5(a)], however, E_q^{tot} is independent of K and our approach only predicts resonance positions close the maximum of $J_{\text{eff}}(K)$, i.e. at $K \approx 1.85$. There we recovered the full K -dependence of the resonance position by scaling the black transition line to $E_{k_L}^{\text{tot}}$, with

$$E_{k_L}^{\text{tot}}(K) = 1.64\sqrt{2J_{\text{eff}}(K)U}. \quad (8.3)$$

Gray lines in Fig. 8.5(a) indicate the predicted positions of the phonon resonances with $m_p = 1, 2, 3$. For completeness, we also compared the interaction dependence of the

8.3. Excitations with non-resonant driving

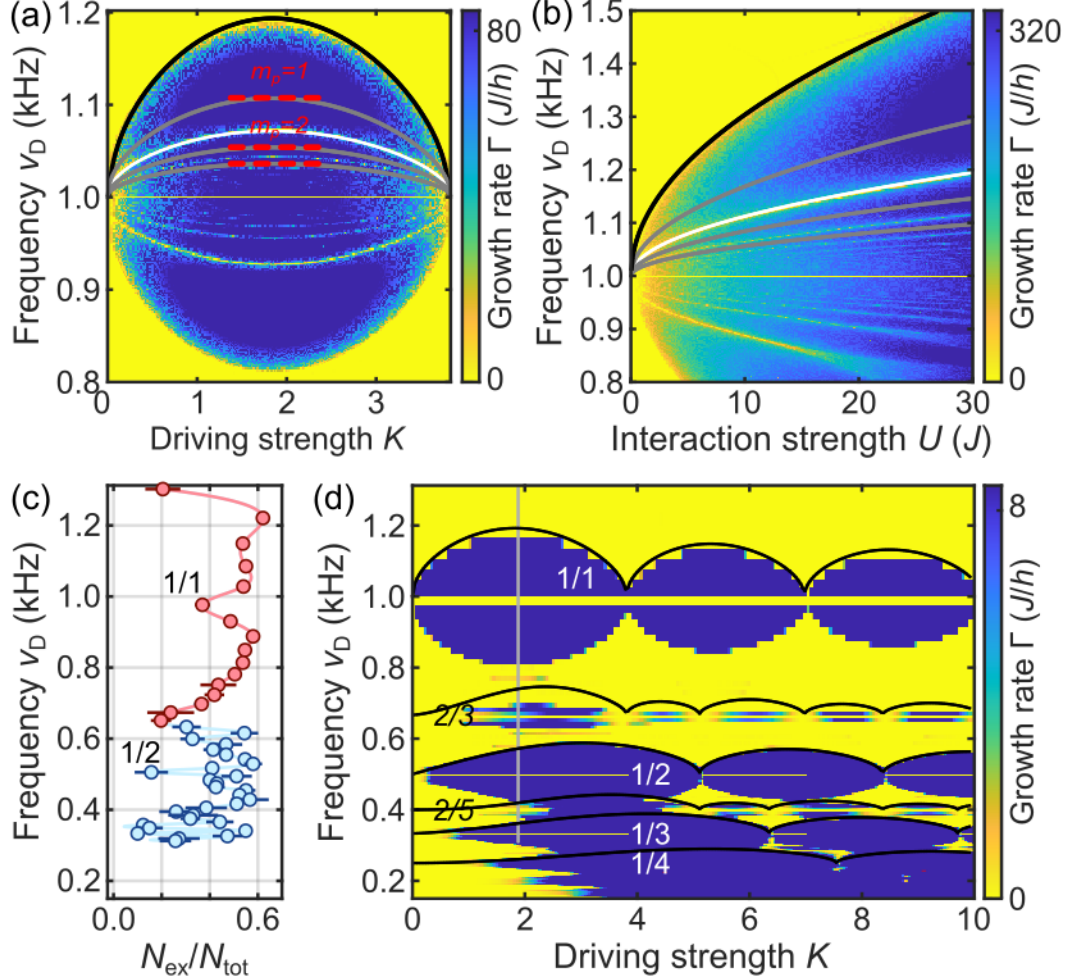


Figure 8.5: Phonon growth rate for off-resonant driving. (a) Calculated growth rate Γ_{k_L} for $V_0 = 10E_r$, $U = 4J$, $k_0 = 0$, $s = 500$, $\nu_B = 1$ kHz. Dashed red lines show the resonances $m_p = 1, 2, 3$ for $K = 1.85$ calculated using Eq.(8.2). Other lines are based on the approximation $E_{k_L}^{\text{tot}}(K)$ in Eq.(8.3) predicting the resonances $m_p = 1, 2, 3$ (grey lines), the stable region between $m_p = 1$ and 2 (white line), and the transition between unstable and stable (black line). (b) Interaction dependence of Γ_{k_L} for $K = 1.85$. Other parameters and line colours match (a). (c) Measurement of $N_{\text{ex}}/N_{\text{tot}}$ for the parameters in Fig. 8.4(c) and $K = 1.90$ [dashed grey line in (d)]. Data colours indicate the tunnelling resonances 1/1 (red colour) and 1/2 (blue colour). (d) Calculated growth rate for a large range of K and ν_D values. Phonon resonances occur at tunnelling resonances with frequency $\nu_D = (n_t/m_t)\nu_B$ and parameters $s = 42$, $V_0 = 10E_r$, $U = 4J$, $q = k_L$. Figure from Ref. [177].

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growth rate for $K = 1.85$ to Eq. (8.2) [Fig. 8.5(b)] and found excellent agreement. The lines in Fig. 8.5(b) refer to the same resonances as in Fig. 8.5(a).

We extended our calculation of Γ_{k_L} to larger driving strengths and detuning to demonstrate the connection between phonon resonances and tunnelling resonances [Fig. 8.5(d)]. As in Figs. 8.2(c) and (d), we used the momentum of the superfluid's ground state in the fast driving limit as the initial momentum k_0 . The driving frequency ν_D is provided on the vertical axis instead of $\Delta\nu$ to allow for an easy recognition of tunnelling resonances. For reference, the unstable region $m_p = 1$ in Fig. 8.5(a) matches to the first region at $\nu_D = 1$ kHz in the top left corner of Fig. 8.5(d). The unstable region is repeated for increasing K values with a shape that is given by the Bessel function in $E_q^{\text{tot}}(K)$. Solid black lines indicate the transition lines between stable and unstable regions.

We found that phonon resonances also occur at other driving frequencies due to additional tunnelling resonances [Fig. 8.5(d)]. In strongly tilted lattice potentials, phonon resonances can only exist on top of a tunnelling resonance, as phonons typically spread over several lattice sites which requires tunnelling and coherence between the sites. Periodic driving restores the tunnelling in tilted potentials when the driving frequency is resonant to the energy shift $h\nu_B$. For driving frequencies $\nu_D = (n_t/m_t)\nu_B$, the tunnelling resonances are of order m_t and couple lattice sites at a distance $n_t d_L$ [225, 227]. Combining all contributions, the complete resonance condition for phonon resonances of order m_p and for tunnelling resonance (n_t/m_t) is

$$h\nu_D = \frac{n_t}{m_t} \left(h\nu_B + \frac{2\langle E_q \rangle}{m_p} \right). \quad (8.4)$$

Higher-order tunnelling resonances require the regularisation of J with the m_t -order Bessel function $J_{m_t}(K)$ [226] which must be included when approximating the average phonon energy with $E_q^{\text{tot}}(K)$. We indicate the predicted driving frequencies for transition lines in Fig. 8.5(d) with black lines and find good agreement with the unstable regions in our calculation. For the tunnelling resonances with $n_t = 2$, we used products of Bessel functions to provide a guide to the eye.

To experimentally demonstrate the existence of phonon resonances at the next tunnelling resonance we measured $N_{\text{ex}}/N_{\text{tot}}$ after 30 ms of driving with a constant value $K = 1.9$ [dashed grey line in Fig. 8.5(d)]. We observe two regions with strong growth of phonon modes and stable centre points at $\nu_D = 1.0$ kHz and 0.5 kHz [data points with red and blue colours Fig. 8.5(d)]. The first region matches to our previous measurement in Fig. 8.4(c) for the tunnelling resonance $m_t = n_t = 1$, while the second region occurs at tunnelling resonance $m_t = 2, n_t = 1$. For our measurement parameters in Fig. 8.4(c) and 8.5(d), the unstable regions of phonon resonances start merging at higher-order tunnelling resonances and we used a smaller interaction strength of $U = 4J$ and a larger lattice depth $V_0 = 10 E_r$ for the calculations to clearly show disjunct regions with phonon growth.

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In conclusion, we have studied the growth of phonons in a superfluid with a tilted 1D lattice in three scenarios: non-driven, resonantly driven and non-resonantly driven. To

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determine the phonon growth, we measured the momentum distribution of the system after a fixed hold time and obtained the fraction of atoms that were not in the ground state. For all settings, we found stable and unstable parameter regimes which can be explained by analysing the micromotion of the superfluid through the first Brillouin zone. Due to the tilted potential, the micromotion always crosses into critical regions of the Brillouin zone and negative-mass instabilities make the superfluid unstable for short time intervals. We found that the duration and multitude of those crossings per cycle determine the growth rate of phonon modes. Time-averaging over the micromotion, e.g. within a Floquet-description, loses this information and makes it challenging to predict the stability of the system.

To determine the resonance condition, we matched the phonon energy with the frequency at which the micromotion crosses critical regions of the Brillouin zone. This cycling frequency allowed us to predict the fundamental and higher-order phonon modes of negative-mass instabilities in all three scenarios. For off-resonant driving, we replaced the driving frequency with the detuning Δv between Bloch frequency and driving frequency. In all cases, a stable, fast-driving limit is reached when the cycling frequency exceeds the energy of the fundamental phonon mode. In addition to phonon resonances, band excitations and tunnelling excitations, a complete stability analysis must include the role of the trapping potential [165]. Understanding the joint effects of these excitation mechanisms will be instrumental for quantum simulation experiments that study interacting many-body states with periodic driving over long time scales [198].

Chapter 9

Outlook

This thesis has presented a collection of experiments centred around the topics of matter-wave solitons and Floquet driven lattice systems. Chapter 5 presented the observation of new types of matter-wave solitons which exist within a periodic potential, these lattice solitons exist across different number of sites and in the centre of the Brillouin zone. The experimental methods were discussed, where an optical accordion lattice was used to allow for the variation of both the lattice spacing and the lattice depth. The accordion lattice also allowed for the magnification of the soliton state *in situ*, giving the final density profile of the soliton at the end of the experiment. The collapse properties of both single-site and multisite solitons were probed, showing that a dynamical collapse occurs with some associated three-body losses due to the increased population in the central site.

These types of solitons are expected to show phenomena that have been developed in solid state systems such as the Peierls-Nabarro barrier [34, 228–230], where a pushed soliton is expected to not move until the force overcomes a potential. Also predicted are interesting properties of multiple interacting solitons such as co-accelerating solitons which are possible with the lattice potential [36]. Investigation of the motion of lattice solitons requires greater control over many experimental parameters such as magnetic field strengths as small fluctuations can cause a change in the forces applied to induce motion. By generating the accordion lattice with multiple deflector frequencies, more complex lattice geometries, such as superlattices with several lattice constants, can be engineered, enabling studies of topological insulators [37]. In particular, this approach could realise a Su–Schrieffer–Heeger chain, where topological solitons emerge from the system’s dimerised structure [38, 39]. Further extensions also include generating higher-dimensional lattices [155] and synthetic gauge fields [231], introducing controlled disorder [232, 233], and creating bound states of lattice solitons [234].

The creation of stable lattice solitons can also act as a precursor to the generation of solitons with time-periodic stable excitations, known as discrete breathers [35]. Also known as intrinsic-localised modes, these excitation modes are well documented theoretically (see Refs. [235, 236] for reviews), in comparison, however, experimental studies are scarcer due to the complexity of the lattice structures [237]. Nevertheless, they have been observed in 1D polymers using spectroscopy techniques [238], waveguide arrays [239], and spin lattices [240], among others [241–244]. Due to its

flexibility, the setup discussed in Chap. 4 and Chap. 5 could provide a useful test-bed to investigate discrete breathers in Bose-Einstein condensates [33, 35, 159, 245]. Due to their fundamental nature, discrete breathers are expected to be relevant across many systems [246]. They have been proposed to explain heat transport and conductivity in materials [247], and other phenomena such as melting [236] and fracture [248, 249].

Chapters 6-8 dealt with the heating and instabilities associated with Floquet-driven lattice systems. According to their underlying mechanisms two main types of instabilities were identified: parametric instabilities, which occur from the driving frequency matching some internal energy of the system, such as the energy of a phonon mode. This results in an exponential growth of the population of the phonon mode. Second, we identified negative-mass instabilities, which instead result from the periodic entering of the wavepacket into unstable sections of the Brillouin zone associated with negative-mass. These also grow exponentially but can dominate the system depending on the duration within the unstable zones. The instabilities were investigated in a range of physical systems such as with untilted and tilted lattices, tilted lattices are the starting point of many different effects such as artificial gauge fields [44, 45]. The energy scale associated with the offset of neighbouring lattice sites further complicates the occurrence and growth of instabilities, producing a range of new resonances which can be excited.

A natural extension of the work involving instabilities in Floquet driven systems is to develop methods to mitigate heating. Traditional methods of limiting the driving strength and frequency restricts access to some regimes experimentally. Another method to limit instability growth is the engineering of bespoke driving protocols. Currently, we are investigating the effectiveness of higher harmonic driving schemes such as pulsed driving, where the micromotion follows a square waveform rather than a sinusoidal one, and two-tone driving where two different sinusoidal frequencies are supplied simultaneously. Both of which are expected to limit or remove the presence of parametric instabilities, allowing for increased experimental observation times and with that access to strongly interacting models, such as Laughlin [250, 251] and Pfaffian [252–255] states, chiral spin liquids [256, 257] and strongly correlated electrons [258].

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