Stochastic Modelling in Finance

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Abstract

The trading of financial derivatives and products in financial markets has influenced the development of the world economy. Over the last few decades, a rapid growth in complex financial systems, which can generate unstable conditions in financial markets, has been observed. Therefore models are being developed to study and examine the uncertainty surrounding these financial systems in different circumstances.

The important milestone of this work can be traced to the Black-Scholes formula for option pricing which was published in 1973 and revolutionized the financial industry by introducing the no-arbitrage principle [8]. This model assumed that the average rates of return and volatility are constant, however, this is not realistic. Therefore, several models have been developed, based on pragmatic studies, which generalize the Black-Scholes formula to acquire more knowledge for these financial systems.

In this project, we will focus on Stochastic Differential Equations (SDEs) models in finance which do not have explicit solutions so far. In particular, Lewis [47] developed the mean-reverting-theta processes which can not only model the volatility but also the asset price. Therefore, we will establish the Euler-Maruyama (EM) numerical schemes to approximate the solution to this model and show that the EM approximate solution will converge in probability to the true solution

under certain conditions. The convergence property of the corresponding step process will be examined under the same conditions to determine its application in finance. In addition, the Markov-switching format of this model can be used to explain some erratic situations observed in financial data. Under the same conditions on parameters of mean-reverting-theta model, the Markov-switching model will be examined to show that the EM approximate solution to this model will converge in probability to the true solution.

Although previous models fit to a certain type of financial data, they can not be used to explain behaviour of the unpredictable abrupt structural changes in financial markets. However, the mean-reverting-theta stochastic volatility model driven by a Poisson jump process explains some of this phenomenon. Therefore, we will examine the analytical properties of EM approximate solutions to this model for two conditions of the parameters theta and beta.

Since it is possible to obtain a more generalized formula for this stochastic volatility jump process, by incorporating a hybrid concept into this SDE model, we will consider the mean-reverting-theta volatility model with Poisson jumps driven by two independent Markov processes. Existing financial instruments are not strong enough to examine the convergence property of the approximate solution to this model. Therefore, we will establish EM approximate solutions to this model and examine their convergence property, when we assume similar parameter conditions to the mean-reverting-theta model. Finally, we will show that these approximate solutions of the SDE models can be used to evaluate financial quantities, options and bonds for example.

Contents

List of Tables

List of Figures

General Notation

positive	:	> 0.
negative	:	< 0.
non-positive	:	≤ 0.
non-negative	:	≥ 0.
$\emptyset$	:	the empty set.
$a.s.$	:	almost surely or with probability 1.
I_A	:	the indicator function of a set A
		i.e $I_A(x) = 1$ if $x \in A$ or otherwise zero.
A^C	:	the complement of A in Ω i.e $A^C = \Omega - A$.
$A \subseteq B$	:	$A \cap B^c = \emptyset$.
$A \subseteq B$ a. s.	:	$\mathbb{P}(A \cap B^c = \emptyset) = 1$.
$\sigma(C)$	:	the σ-algebra generated by C.
$a \wedge b$	:	$\min\{a,\ b\}$.
$a \vee b$	:	$\max\{a,\ b\}$.
$f : A \to B$	:	the mapping f from A to B.
$\mathbb{R} = \mathbb{R}^1$	:	the real line.
$\mathbb{R}_+$	:	the set of all nonnegative real numbers, i.e $\mathbb{R}_+ = [0, \infty)$.
$\mathcal{B}$	:	the Borel σ-algebra on $\mathbb{R}$.
$\|x\|$	:	the Euclidean norm of a vector x.

$C(D;\mathbb{R})$	:	the family of continuous $\mathbb{R}-$ valued functions defined on D.
$C^m(D;\mathbb{R})$	:	the family of continuously m-times differentiable $\mathbb{R}-$ valued functions defined on D.
$C^{2,1}(D\times\mathbb{R}_+;\mathbb{R})$	:	the family of all real-valued functions $V(x,t)$ defined on $D\times\mathbb{R}_+$ which are continuously twice differentiable in $x\in D$ and once differentiable in $t\in\mathbb{R}_+$.
$\|x\|_{L^p}$	:	$\Big(\|x\|^p\Big)^{\frac{1}{p}}$.
$L^p(\Omega;\mathbb{R})$	:	the family of $\mathbb{R}$-valued random variables X with $\mathbb{E}\|X\|^p<\infty$.
$L^p_{\mathcal{F}_t}(\Omega;\mathbb{R})$	:	the family of $\mathbb{R}$-valued $\mathcal{F}_t$-measurable random variables X with $\mathbb{E}\|X\|^p<\infty$.
$L^p([a,b];\mathbb{R})$	:	the family of Borel measurable functions $h:[a,b]\to\mathbb{R}$ such that $\int_a^b \|h(t)\|^p dt<\infty$.
$\mathcal{L}^p([a,b];\mathbb{R})$	:	the family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ such that $\int_a^b \|h(t)\|^p dt<\infty$ a. s..
$\mathcal{M}^p([a,b];\mathbb{R})$	:	the family of $\mathbb{R}$-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a\le t\le b}$ in $\mathcal{L}^p([a,b];\mathbb{R})$ such that $\mathbb{E}\int_a^b \|h(t)\|^p dt<\infty$.

Other notation will be explained where it first appears.

Chapter 1

Introduction

1.1 Background

In modern society the modelling of financial systems has gained significant attention due to rapid and complex behaviour in financial markets. More precisely, it has been observed that random changes in these systems are dependent on an unmeasurable distribution or an unknown system of parameters that are subject to some uncertainty and several deterministic factors. However, without having knowledge of these system distributions and their parameters, the prediction of future events and the examination of their current status is impossible. Therefore, the accurate modelling of financial systems can be very useful in understanding the governing forces and their parameters. However, this has become a challenging task because of briskly changing conditions. Consequently, relaxing some of the conditions in these interesting systems, applying developing mathematical instruments with knowledge of stochastic processes could achieve better predictions and understanding of the driving forces behind the financial markets.

The knowledge of the Itô stochastic system and the stochastic process, that are widely used in the modelling of a vast number of dynamical systems in many disciplines, plays a major role in mathematical finance. The main idea behind this interesting subject was first examined by Scottish botanist, Robert Brown in 1827 [65]. While studying pollen grains of a plant suspended in water under a microscope, Brown observed that tiny particles which were ejected by the pollen grains performed jittery motions. However, T. N. Thiele was the first person who explained the mathematical concept behind this Brownian motion. Thiele published a paper on "The method of least squares" in 1880 [28]. In 1905, 70 years after Brownian motion was discovered, Albert Einstein succeeded in explaining the irregular movements of small particles suspended in a liquid as visible evidence for the molecular motion [67]. It is therefore not surprising that Einstein's work on Brownian motion also became one of the pillars of modern statistical thermodynamics, more generally, the physics of stochastic processes [66]. In 1923, the existence of Brownian motion and construction of a continuous-time stochastic process, which is often called standard Brownian motion, was clearly established by Norbert Wiener. The Wiener process has become a synonym of Brownian motion and the measure is called the Wiener measure[1] in his honour [10].

The next generation of stochastic processes was laid out by Russian mathematician, Andrey Kolmogorov. Kolmogorov contributed to the development of the fundamental theory of Markov processes which motivated the beginning of the theory of stochastic integration [44]. Most notably in this paper, Kolmogorov proved that continuous-time Markov processes (diffusions) fundamentally depend on only two parameters: the speed of the drift and the size of the purely random part (the diffusive component). From studying the work of Kolmogorov, Kiyosi Itô, the father of stochastic integration made an outstanding contribution to the

[1] Probability law on the space of continuous functions. See Chapter 2, Probability theory

development of stochastic analysis. In his seminal work, he attempted to establish a true stochastic differential equation to be used in the study of Markov processes, which was one of Itô's primary motivations for studying stochastic integrals [40, 37, 38, 39]. Although Wolfgang Doeblin was the first person who contributed to the development of the theory of Markov processes, his exceptional research work was secretly hidden away in the safe of the French Academy of Science until 2000 [19]. Meanwhile, in the same year as Itô's first paper was published, S. Kakutani published his brief explanation on two-dimensional Brownian motion and harmonic functions [41]. However, for the first time in stochastic research, Joseph Leo Doob clearly explained and established the strong Markov property in 1941 [20]. A few years later Deny, E. Hille and K. Yosida contributed to the development of stochastic processes, stochastic differential equations, stochastic integrals and martingale properties [79]. All of this work helped to establish stochastic analysis as a complete and interesting subject.

Both the history of stochastic integration and the modelling of risky asset prices began with Brownian motion. L. Bachelier developed a model of the Paris stock market while deriving the dynamic behaviour of Brownian motion in 1900. This work can be treated as the earliest attempt of using Brownian motion in finance [6, 11]. Preceding the work of Einstein, Bachelier attempted to model the market noise of the Paris Bourse while exploiting the ideas of the Central Limit Theorem. He also argued that increments of stock prices should be independent and normally distributed but realized that market noise should be without memory. He is now seen as the founder of modern mathematical finance by many people. However, his research concept met with disfavour in the Paris mathematical community, mostly because Bachelier's work was way ahead of that time, thus it was ignored until it was re-discovered by the mathematical statistician L. J. Savage in

1955. Paul Samuelson further developed Bachelier's model to include stock prices that evolved according to a geometric Brownian motion, following work by A. Cowles, M. Kendall and M. F. M. Osborne and others [12, 43, 62, 63]. Moreover, Samuelson focussed on the concept of a martingale[2] rather than a random walk, which was neatly summarized by the title of his article: "Proof that properly anticipated prices fluctuate randomly" [68]. E. Fama worked on the same topic and formed the basis of the efficient market hypothesis [22, 23]. This efficient market concept caused a revolution in empirical finance though the debate and empirical investigation of this hypothesis is still continuing [24].

The concept of a good model for stock price movements, which is today known as geometric Brownian motion[3], is explained in Samuelson's companion paper [69] together with H. P. McKean Jr.. Samuelson also showed that Bachelier's model failed to guarantee that stock prices will always be positive as his model leads to inconsistencies with economic principles, where geometric Brownian motion avoids these difficulties. Furthermore, this was the paper that first coined the terms of European and American options and derived valuation formulas for both European and American options. Thanks to his efforts, based on the profound insights of the governing body of the asset price or continuous-time stock market, economists postulated that the general stochastic differential equation has the form

$$dX(t) = \alpha X(t)dt + \sigma X(t)dW(t), \quad \text{for all} \quad t \in [0, T]. \tag{1.1}$$

Here $W(t)$ represents a Brownian motion which can ensure a positive value of the asset price $X(t)$, α is the estimated average rate or return of the asset price, σ is the standard deviation of the asset price, which is often called volatility. In

[2]The expectation of the next value in the sequence is equal to the present observed value. See Definition 2.3

[3]is a continuous-time stochastic process in which the logarithm of the randomly varying quantity follows a Brownian motion. See (1.1).

this process he again postulated the condition that the discounted options payoffs follow a martingale, although the derivation of options was almost identical to the Black-Scholes formula which was derived nearly a decade later [69].

The development of financial asset pricing theory over the 45 years since Samuelson's seminal work had been intertwined with the development of stochastic analysis. However in the early 1970's, Myron Scholes and Fisher Black made an important breakthrough in the pricing of complex financial tools by developing what has today become known as the "Black-Scholes formula". In this process, geometric Brownian motion was applied as the key factor by invoking the no-arbitrage[4] principle to derive their famous formula for pricing option via an explicit solution. The seminal work of the paper [8] on the pricing and hedging of European call and put options ushered in the modern era of derivative securities.

Robert C. Merton, who was influenced by Samuelson, clarified and improved the option price problem which was derived by Myron Scholes and Fisher Black [55, 56, 57, 58]. An option pricing formula is derived in his paper for the more general case when the underlying stock returns are generated by a mixture of both continuous and jump processes. However, this derived formula has most of the attractive features of the original Black and Scholes formula [8] which does not depend on investor preferences or knowledge of the expected return on the underlying stock. In 1974, Merton made another major contribution to assess the credit risk model while assuming that a company has a certain amount of zero-coupon debt (no interest, and often issued at a price lower than the principal amount) that will be due at a future time T [60]. Furthermore, the model can be used to estimate either the risk-neutral-probability[5] that the company will

[4] is a trade which does not require any initial funds, never loses money and produces strictly positive cash flows with strictly positive probability.

[5] is probability of future outcome adjusted for risk which can be used to compute expected asset values.

default or the credit spread on the debt. However, Merton's model requires the current value of the company's assets, the volatility of the company's assets, the outstanding debt and the debt maturity as inputs. In 1979, J. M. Harrison and D. M. Kreps constructed a classical securities market model for the asset price which is described by a real-valued process [29]. Under certain assumptions, their model is arbitrage-free if and only if there exists an equivalent probability measure under which the price process is a martingale.

By rapid development of the stochastic calculus and option pricing concept, their contribution to the improvement of field of modern mathematical finance is clear. In addition, these financial instruments can also be utilized to examine more and more important applications in complex financial quantities. However, some of these financial instruments are specific to solving certain types of problems and their generalizations are capable of giving some extended results. Theoretically, alternative methods for these findings have been investigated by several authors but in applications most of these techniques can only be used under certain conditions. Another milestone of this process has been examined by John C. Cox and Stephen A. Ross [14]. In this paper, they developed several alternative jump and diffusion processes and established solutions for the limiting diffusion cases. However, explicit solutions presented in this paper have potential empirical applications, while a comparative study of them should give additional insight into the structure of security valuation. A special limiting case of the famous Black-Scholes model which has previously been derived only by much more difficult methods, was discussed by C. Cox, A. Ross and M. Rubinstein [15]. Because of this generalization it gives rise to a simple and efficient numerical procedure for valuing options whose premature exercise price is optimal. On the other hand J. M. Harrison and S. R. Pliska showed that the continuous limit of the model having a deterministic bond and two independent stocks following geometric Brownian motion is completed. This also demonstrates that the finite market is nearly complete, or each contingent-claim (option pricing) is nearly computable [30].

With the rapid development of the theory of stochastic analysis and stochastic modelling, a significant number of problems related to financial economics have been unveiled in the last few decades. However, the extension of these existing financial instruments is still continuing as a result of jittery and complex situations. More precisely, continuous-time stochastic analysis still dominates the stochastic modelling of asset price and novel ideas are employed to construct new financial instruments while generalizing the existing formulas. One of the renowned concepts behind this development process is the behaviour of the asset price volatility. According to empirical studies it was revealed that volatility of the asset price does not have the property to be a constant in different situations, which deviates from the previous assumption of geometric Brownian motion. However, several alternative techniques have been utilized to overcome modelling difficulties. The stochastic volatility model is one such alternative method which explains in a self consistent way as options with different strikes and expirations have different volatility. Furthermore, this attractive feature was pointed out by John C. Hull and Alan White in 1987 [34] that volatility follows a Itô process which is driven by another Brownian motion. A few years later, in 1991, the structure of the stochastic volatility model was also discussed by E. M. Stein and J. C. Stein under the topic of "Stock price distributions with stochastic volatility: an analytic approach" in 1993 [72]. Steven Heston extended the Hull-White volatility model which is known as the mean-reverting stochastic volatility process [31].

However, due to the computational complexity of stochastic volatility models and the extreme difficulty of fitting parameters to the current option pricing,

B. Dupire [21], E. Derman and I. Kani [17] proposed another concept which is known as the unique state-dependent diffusion coefficient or clearly local volatility function. Moreover, they observed that there was a unique diffusion process consistent with these distributions under risk-neutrality. In the meantime people such as R.C. Morton [61], C. Aha and H. Thompson [1] discussed and gave an outstanding contribution to the development of mathematical instruments when unpredictable upsurges occurred in financial markets.

It is therefore clear that knowledge of stochastic modelling and stochastic analysis form a platform from which to study and understand the present financial quantities and some of its related issues, while establishing a strong foundation in mathematical finance that also gears up curiosity about the use of results in this field.

1.2 Overview of the study

This thesis mainly concentrates on extended Black-Scholes type stochastic models, which can be seen in many situations of modern financial markets in practice. However, these extended stochastic models which describe movements of the asset price or portfolio data in financial markets, have no explicit solutions as expected by the Black-Scholes formula. Numerical approximation techniques have become a popular and powerful tool to study and understand behaviour of these complicated systems, especially the consequences of their application in finance. In this process, the Euler-Maruyama (EM) scheme will therefore be used to approximate the solutions of these SDE models and effective techniques will be developed to compute their analytical properties.

Chapter 2 introduces basic concepts in mathematical finance, especially knowledge of stochastic analysis. It begins with probability theory and evolves into general structure of stochastic models while explaining the important features of Brownian motion, stochastic analysis and stochastic differential equations. Moreover, uniform notation and concepts are also established while providing related references in order to help readers gain required knowledge to understand the following chapters conveniently. In addition, some important mathematical tools like the generalized Gronwall's inequality are stated in the latter part of this chapter for the reader's convenience. The book written by X. Mao [52] contributes to the development of this thesis as the main source of reference but readers can find most of these mathematical concepts in many fine books that can also be used to improve their knowledge in the fields of mathematical finance and stochastic analysis [5, 25, 36, 42, 49, 46, 70].

In Chapter 3, we will concentrate on the mean-reverting-beta processes developed by Lewis [47], which can not only model the volatility but also the asset price. Although this widely used stochastic volatility process fits to certain types of financial data there is no explicit solution like the Black-Scholes formula. However, when the parameters θ and β [6] are greater than 1, the diffusion coefficients only satisfy the local Lipschitz condition and existing mathematical techniques can not be used to determine convergence in second moment[7] property of its approximate solution. In this chapter, we will therefore establish the EM numerical approximate solution and show convergence in probability of this approximation while deriving related analytical properties.

According to the empirical studies, there is more and more evidence to suggest that average rate of return has no significant reason to be a constant as assumed by the Black-Scholes formula. Moreover, evidence suggests that the rate of return

[6]parameters of diffusion coefficients of the SDE model (3.6).

[7]see page (16), section (c)

and volatility follows a jump-process known as a Markov chain. In Chapter 4, we will therefore consider the mean-reverting-theta stochastic volatility model driven by a Markov chain in which the parameters θ and β are greater than 1. Although this highly sensitive volatility model plays a very important role in modern financial market, like the previous model, there is no information on its the solution so far. Therefore, an approximate solution to this SDE model will be established by applying the EM numerical scheme in order to examine and study this highly sensitive volatility model of asset price. A strong error bound of this approximate solution can not be obtained since its diffusion coefficients only satisfy the local Lipschitz condition. We will therefore compute an explicitly computable error bound over finite time and derive the convergence in probability of this EM approximate solution by removing its stopping time.

However existing stochastic volatility models, which capture some dynamic behaviour of the asset price in financial markets, do not explain unpredictable abrupt structural changes that are independent of average rate return and volatility. Meanwhile, pragmatic studies show that some of these phenomena can be examined using properties of the Poisson-jump process, so study of the mean-reverting-theta stochastic volatility mode driven by the Poisson-jump process unveils some important features in financial markets. Furthermore, if θ and β vary between $\frac{1}{2}$ and 1, the mean-reverting-theta stochastic volatility model satisfies the global Lipschitz condition as well as the linear growth condition. In Chapter 5, a strong error bound of the EM approximate solution to this model will be obtained over finite time intervals while deriving the other supportive mathematical instruments. Finally, we will derive the convergence in second moment property of the EM approximate solution by the strong error bound when the time step is sufficiently small. However, we can not appeal to the strong error bound of the

EM approximate solution to this Poisson jump model over finite time, when the parameters θ and β are greater than 1. In Chapter 6, we will therefore use the stochastic convergence technique to show convergence in probability of this EM approximate solution to the true solution by removing its stopping time.

In Chapters 7 and 8, we will discuss one of the generalized formulas of the Poisson-jump stochastic volatility models already discussed in Chapter 5 and 6, which can usually be seen in financial markets in practice. More precisely, the mean-reverting-theta stochastic volatility Poisson-jump model driven by a Markov process is more appropriate to describe the higher dimension of the asset price, interest rate and stochastic volatility though there is so far no explicit solution.

In Chapter 7, we will therefore consider this stochastic jump model in the case of $\frac{1}{2} \leq \theta, \beta \leq 1$, where diffusion coefficients satisfy the global Lipschitz condition and the linear growth condition. Hence, we will establish an EM approximate solution to this model and examine a strong error bound for this approximate solution over finite time intervals which gives the convergence in second moment property of this EM approximate solution when the time step is sufficiently small.

However, this hybrid stochastic volatility model using a Poisson-jump process, when the parameters θ and β are greater than 1, satisfies the local Lipschitz condition though it does not obey the linear growth condition. So we can not derive the convergence in second moment property of the EM numerical approximate solution to this highly sensitive volatility model. In Chapter 8, we will establish an error bound for this EM approximate solution using the stopping time and the property of the Markov chain. Convergence in probability of the EM approximate solution will then be obtained by removing the stopping time.

Finally in Chapter 9, we will show that EM approximate solutions to generalized asset price models can be used to evaluate financial quantities in practice.

Chapter 2

Basic Stochastic Analysis

2.1 Introduction

In the modern financial world, knowledge of stochastic processes and stochastic analysis play a major role where effects of random changes generate complex situations. It is therefore necessary to understand basic concepts which lie behind these interesting systems in order to study and examine their behaviour.

In this chapter, definitions and theories of stochastic processes, stochastic analysis and stochastic differential equations are explained along with probability theory, the global and the local Lipschitz conditions, and the linear growth condition. In addition, the generalized Itô integral and its related properties are also mentioned here to lay out strong foundations. Furthermore, some important tools like Gronwall's inequality which contribute significantly throughout this thesis are stated in the latter part of this chapter. Although this chapter introduces very important theorems, required proofs are omitted here. However, these related proofs can be found from the textbooks written by X. Mao [50, 53], which contributed to the development of this thesis as the main source of reference, but readers can find most of these basic mathematical concepts and their proofs in many text books [42, 50, 46, 53, 70].

The mathematical theory of probability was first coined by two French mathematicians in 1654, Blaise Pascal and Pierre de Fermat, from the gambler's argument occurred. This interesting phenomenon got great attention from researchers, since more and more applications related to this concept can be found in the real world when a situation is complicated. Since then, these concepts have formed a complete new subject, in mathematical random-fields[1].

2.2 Probability theory

Probability theory is a branch of mathematics which has relates to uncertainty. The outcome of an experiment can not be precisely predicted though it can be identified as an element in a set of possibilities. We call this set the sample space and denote it by Ω. In addition, each element of this set, $\omega \in \Omega$, denotes only one possible outcome of the experiment. However, generally not every subset of the sample space, Ω, is an observable or interesting event. We therefore denote the family of these observable or interesting events by $\mathcal{F}$, which satisfies the following properties:

(1) $\emptyset \in \mathcal{F}$, where $\emptyset$ denotes the empty set,

(2) if $A \in \mathcal{F}$, then its complement $A^c = \Omega \setminus A \in \mathcal{F}$,

(3) If $\forall \quad i : a \leq i < \infty, A_i \in \mathcal{F}$, then $\cup_{i=1}^{\infty} A_i \in \mathcal{F}$.

The family $\mathcal{F}$ with these three properties is called a $\sigma-$algebra. The pair $(\Omega, \mathcal{F})$ is called a measurable space, the elements of $\mathcal{F}$ are hereafter called $\mathcal{F}-$measurable sets. Let us state another useful function as follows.

[1] is simply a stochastic process which takes values in a Euclidean space

A *probability measure* $\mathbb{P}$ on a measurable space $(\Omega, \mathcal{F})$ is a function $\mathbb{P} : \mathcal{F} \to [0, 1]$ such that

(1) $\mathbb{P}(\Omega) = 1$;

(2) for any disjoint sequence $\{A_i\}_{1\leq i} \subset \mathcal{F}$ (i.e. $A_i \cap A_j = \emptyset$ if $i \neq j$),

$$\mathbb{P}\Big(\bigcup_{i=1}^{\infty} A_i\Big) = \sum_{i=1}^{\infty} \mathbb{P}(A_i).$$

If $\mathcal{C}$ is a collection of subsets of Ω, then there exists a smallest $\sigma-$algebra $\sigma(\mathcal{C})$ on Ω which contains $\mathcal{C}$. Hence, this $\sigma(\mathcal{C})$ is called the $\sigma-$algebra generated by $\mathcal{C}$. If $\Omega = \mathbb{R}^d$ and $\mathcal{C}$ is a collection of all open sets in $\mathbb{R}^d$, then $\mathcal{B}^d = \sigma(\mathcal{C})$ is called the Borel $\sigma-$algebra and the elements of $\mathcal{B}^d$ are called the Borel sets.

A real-valued function $X : \Omega \to \mathbb{R}$ is said to be $\mathcal{F}-$measurable if

$$\{\omega : X(\omega) \leq a\} \in \mathcal{F} \quad \text{for all} \quad a \in \mathbb{R}.$$

The function X is also called a real-valued ($\mathcal{F}$-measurable) random variable. An $\mathbb{R}^d$-valued function $X(\omega) = (X_1(\omega), ..., X_d(\omega))^T$ is said to be $\mathcal{F}$-measurable if all the elements X_i are $\mathcal{F}$-measurable. The *indicator function* I_A of a set $A \in \Omega$ is defined by

$$I_A(\omega) = \begin{cases} 1 \;\; ; \;\; if \;\; \omega \in A, \\ 0 \;\; ; \;\; if \;\; \omega \notin A. \end{cases} \tag{2.1}$$

If I_A is $\mathcal{F}$-measurable if and only if A is an $\mathcal{F}$-measurable set (i.e. $A \in \mathcal{F}$).

The triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a *probability space*. If $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we set $\bar{\mathcal{F}} = \{A \subset \Omega : \exists\, B, C \in \mathcal{F}$ such that $B \subset A \subset C, \mathbb{P}(B) = \mathbb{P}(C)\}$.

Then $\bar{\mathcal{F}}$ is a $\sigma-$algebra and is called the *completion* of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, then the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete.

Then, for a real-valued random variable X on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we define the moment of random variable X that gives the integration of X by measure theory. A real-valued random variable X on $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be $\mathbb{P}-$integrable, if the integral $\int_\Omega X(\omega)d\mathbb{P}(\omega)$ is finite. Therefore, the expectation of this real valued random variable can be given by

$$\mathbb{E}(X) = \int_\Omega X(\omega)d\mathbb{P}(\omega).$$

Then, the variance of this real-valued variable X can be defined by $V(X) = \mathbb{E}(X - \mathbb{E}(X))^2$. Moreover, the number $\mathbb{E}|X|^p$ is called the p^{th} moment of X where $(p > 0)$ and $\mathbb{E}|X|^p = \int_\Omega |X(\omega)|^p d\mathbb{P}(\omega)$. Now we define L^p space:

$$L^p = L^p(\Omega, \mathcal{F}, \mathbb{P}) = \{X : X \text{ is an } \mathbb{R} - \text{valued random variable}, \mathbb{E}|X|^p < \infty\}.$$

For every X in L^1, we have $|\mathbb{E}(X)| \leq \mathbb{E}|(X)|$. If Y is another random variable, the covariance of these two variables X and Y can be given by

$$Cov(X, Y) = \mathbb{E}((X - \mathbb{E}(X))(Y - \mathbb{E}(Y))).$$

If $Cov(X, Y) = 0$, then variables X and Y said to be uncorrelated, in other words the two variables are independent of each other. The following sections establish very useful inequalities which are related to the integration or the moments.

(1) **Hölder's inequality** (for p=2, this is known as **Schwarz's inequality**)

$$|\mathbb{E}(X^TY)| \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} (\mathbb{E}|X|^q)^{\frac{1}{q}} \quad \text{where} \quad p > 1,\ \frac{1}{p}+\frac{1}{q} = 1,\ X \in L^p, Y \in L^q.$$

(2) **Minkowski's inequality**

$$(\mathbb{E}|X + Y|^p)^{\frac{1}{p}} \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} + (\mathbb{E}|Y|^p)^{\frac{1}{p}} \quad \text{where} \quad p > 1,\ X, Y \in L^p.$$

(3) **Chebyshev's inequality**

$$\mathbb{P}\{\omega : |X(\omega)| \geq c\} \leq c^{-p}\mathbb{E}|X|^p \quad \text{where} \quad c, p > 0,\ X \in L^p.$$

An application of Hölder's inequality can be given by

$$(\mathbb{E}|X|^r)^{\frac{1}{r}} \leq (\mathbb{E}|X|^p)^{\frac{1}{p}} \quad \text{where} \quad 0 < r < p < \infty,\ X \in L^p.$$

Let X and X_n $(n \geq 1)$, be $\mathbb{R}$-valued random variables defined on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. The following four convergence concepts of the sequence play a major role in mathematical modelling.

(a) *Almost sure* (or with probability 1) convergence:

If there exists a $\mathbb{P}-$null set[2] $\Omega_0 \in \mathcal{F}$ such that for every $\omega \notin \Omega_0$, the sequence $\{X_n(\omega)\}$ converges to $X(\omega)$ in the usual in $\mathbb{R}^d$, then $\{X_n\}$ is said to converge to X *almost surely*, and can be written as

$$\lim_{n\to\infty} X_n = X\ a.s.$$

(b) *Stochastic* (or in probability) convergence:

if for any given $\varepsilon > 0$,

$$\mathbb{P}(\omega : |X_n(\omega) - X(\omega)| > \varepsilon) \to 0\ as\ n \to \infty.$$

(c) *Convergence in the P^{th} moment* (or in L^p):

if X and X_n belong to L^p and $\mathbb{E}|X_n - X|^p \to 0$. Then $\{X_n\}$ is said to converge to X in P^{th} moment or L^p. This process is called convergence in *mean square* or in *quadratic mean*, when $p = 2$.

(d) *Convergence in distribution* :

If for every real-valued continuous and bounded function g defined on $\mathbb{R}^d$,

$$\lim_{n\to\infty} \mathbb{E}g(X_k) = \mathbb{E}g(X),$$

then $\{X_k\}$ is said to converge to X in *distribution.*

[2] A set which has zero measure.

These convergence concepts have the following relationships:

$$\text{convergence in the } q^{th} \text{ moment}$$
$$\Downarrow$$
$$\text{convergence in the } p^{th} \text{ moment}(p \leq q)$$
$$\Downarrow$$
$$\text{almost surely} \Rightarrow \text{convergence in probability} \Rightarrow \text{convergence in distribution.}$$

Furthermore, a sequence converges in probability if and only if every subsequence of it contains an almost surely convergent subsequence.

Now, we establish the following theorems to explain two very important convergence properties.

Theorem 2.1. (Monotonic convergence theorem)
If $\{X_n\}$ is an increasing sequence[3] *of non-negative random variables, then*

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}\left(\lim_{n\to\infty} X_n\right).$$

Theorem 2.2. (Dominated convergence theorem)
Let $p \geq 1$, $\{X_n\} \subset L^p(\Omega, \mathbb{R})$ and $Y \in L^p(\Omega, \mathbb{R})$. Assume that $|X_n| \leq Y$ almost surely and $\{X_n\}$ converges to X in probability. Then $X \in L^p(\Omega, \mathbb{R})$, $\{X_n\}$ converges to X in L^p, and

$$\lim_{n\to\infty} \mathbb{E}X_n = \mathbb{E}(X).$$

Let I be an index set and $(\Omega, \mathcal{F}, \mathbb{P})$ denote the probability space. A collection of sets $\{A_i : i \in I\} \subset \mathcal{F}$ is said to be independent if

$$\mathbb{P}\left(A_{i_1} \bigcap ... \bigcap A_{i_k}\right) = \mathbb{P}(A_{i_1}) ... \mathbb{P}(A_{i_k}),$$

[3] $X_n \leq X_{n+1}$ for all n.

for all possible choice of indices $i_1, ...i_k \in I$. Then, a collection of sub-σ-algebras $\{\mathcal{F}_i : i \in I\}$ is said to be independent if every possible choice of indices $i_1, ..., i_k \in I$,

$$\mathbb{P}\left(A_{i_1} \bigcap ... \bigcap A_{i_k}\right) = \mathbb{P}\left(A_{i_1}\right) ... \mathbb{P}\left(A_{i_k}\right)$$

holds for all $A_{i1} \in \mathcal{F}_{i_1}, ..., A_{i_k} \in \mathcal{F}_{i_k}$. Then a family of random variables $\{X_i : i \in I\}$ is said to be independent if the $\sigma-$algebra $\sigma(X_i), i \in I$ generated by them are independent.

Let $\{A_k\}$ be a sequence of sets in $\mathcal{F}$. The *inferior limit* of A_k is denoted by

$$\liminf_{k\to\infty} A_k = \bigcup_{i=1}^{\infty}\bigcap_{k=i}^{\infty} A_k,$$

which contains all finite points belonging to the almost all A_k (all but any finite number). The set of all those points which belong to infinitely many A_k is called the *superior limit* of A_k and is denoted by

$$\limsup_{k\to\infty} A_k = \bigcap_{i=1}^{\infty}\bigcup_{k=i}^{\infty} A_k.$$

On the other hand,

$$\liminf_{k\to\infty} A_k \subset \limsup_{k\to\infty} A_k.$$

The following lemma describes another useful property in probability.

Theorem 2.3. (Borel-Cantelli's lemma)

(1) If $\{A_n\} \subset \mathcal{F}$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then

$$\mathbb{P}\left(\limsup_{k\to\infty} A_k\right) = 0,$$

(2) If the sequence $\{A_n\} \subset \mathcal{F}$ is independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then

$$\mathbb{P}\left(\limsup_{k\to\infty} A_k\right) = 1.$$

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a probability space and $A, B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$. The conditional probability of A under condition B is

$$\mathbb{P}(A \backslash B) = \frac{\mathbb{P}(A \bigcap B)}{\mathbb{P}(B)}.$$

2.3 Basic concepts of stochastic processes

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A filtration is a family $\{\mathcal{F}_t\}_{t\geq 1}$ of increasing sub-σ-algebra of $\mathcal{F}$(i.e. $\mathcal{F}_t \subset \mathcal{F}_s \subset \mathcal{F}$ for all $0 \leq t < s < \infty$). The filtration is said to be *right continuous* if $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$. This means that the family of events in t time point contains all the information of past up to t^{th} time point. Accordingly, this creates a platform to keep observed information and compare all the knowledge of the present with past or even future events. When the probability space is complete, the filtration is said to satisfy the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all null sets. From now on, we shall always work on a given complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets).

A stochastic process is a family of real-valued random variables $\{X_t\}_{t\in I}$ with parameter set I (or index set) and state space $\mathbb{R}$. The parameter set I is usually the half-line, $\mathbb{R}_+ = [0, \infty)$. For each fixed $\omega \in \Omega$ we have a function $I \ni t \to X_t(\omega) \in \mathbb{R}$ which is called a *sample path* of the process, and we shall write $X_.(\omega)$ for the path. These sample paths provide information on random effects of continuous random experiments with time (e.g. random changes of trajectory in financial markets driven by Brownian motions).

On the other hand, note that we have a random variable for each fixed time $t \in I$, $\Omega \ni \omega \to X_t(\omega) \in \mathbb{R}$ which provides possible outcomes of the random process at time t. When stochastic process $X_t(\omega)$ is considered as a function of two variables (t, ω), then it can be written as $X(t, \omega)$ which is $X : I \times \Omega \to \mathbb{R}$.

Definition 2.1. *An $\mathbb{R}$-valued stochastic process X is said to be measurable if the stochastic process regarded as a function of two variables (t, ω) from $\mathbb{R}_+ \times \Omega \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R}_+) \times \mathcal{F}$-measurable, i.e. for every $A \in \mathcal{B}$, the set $\{(t, \omega) : X_t(\omega) \in A\}$ belongs to the product $\sigma-$field $\mathbb{R}_+ \times \Omega$.*

Definition 2.2. *A stochastic process X is said to be $\mathcal{F}_t-$adapted (or simply, adapted) if for every t, X_t is $\mathcal{F}_t-$measurable random variable.*

The concept of stopping time plays a very important role in the following chapters. We will therefore introduce that key result in the following way:

A random variable $\tau : \Omega \to [0, \infty]$ is called a $\mathcal{F}_t-$stopping time (or simply, stopping time) if $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for any $t \geq 0$. The following two theorems explain a few properties of this stopping time.

Theorem 2.4. *If $\{X_t\}_{t \geq 0}$ is a progressively measurable process and τ is a stopping time, then $X_\tau I_{\{\tau < \infty\}}$ is $\mathcal{F}_\tau-$measurable. In particular, if τ is finite, then X_τ is $\mathcal{F}_\tau-$measurable.*

Theorem 2.5. *If $\{X_t\}_{t \geq 0}$ be an $\mathbb{R}$-valued càdlàg (right continuous with left limit) $\mathcal{F}_t-$adapted process, and $\mathcal{D}$ an open subset of $\mathbb{R}$. Define $\tau = \inf\{t \geq 0 : X_t \notin \mathcal{D}\}$, where we use the convention $\inf \emptyset = \infty$. Then τ is a $\mathcal{F}_t-$stopping time and is called the first exit time from $\mathcal{D}$. Moreover, if ρ is a stopping time, then $\theta = \inf\{t \geq \rho : X_t \notin \mathcal{D}\}$ is also an $\mathcal{F}_t-$stopping time, and is called the first exit time from $\mathcal{D}$ after ρ.*

Martingales

The martingale property explains a special case of stochastic processes that can be defined based on knowledge of conditional expectation. Let $\{M_t\}_{t\geq 0}$ denote an $\mathbb{R}$-valued integrable stochastic process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, adapted to a given filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

Definition 2.3. *A integrable process $\{M_t\}_{t\geq 0}$ is said to be martingale if, for every $0 \leq s < t < \infty$, we have,*

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \ a.\ s..$$

If we replace the equality sign in the above formula with $\leq$ or $\geq$, the process is called a *supermartingale* or a *submartingale* respectively. Moreover, a right continuous adapted process $\{M_t\}_{t\geq 0}$ is called a local martingale if there exists a nondecreasing sequence $\{\tau_k\}_{k\geq 1}$ of stopping time with $\lim_{k\to\infty} \tau_k \to \infty$ a.s. such that every $\{M_{\tau_k \wedge t} - M_0\}_{t\geq 0}$ is a martingales.

The martingale property is one of the most important tools in the theory of stochastic processes and has been widely employed. Also the concept of this martingale property can be explained as an abstract of a fair game where no knowledge of past events can help to predict future winnings. It is therefore more useful to discuss important properties of martingales.

A stochastic process $X = \{X_t\}_{t\geq 0}$ is called square-integrable if $\mathbb{E}|X_t|^2 < \infty$ for every $t \geq 0$. So if $M = \{M_t\}_{t\geq 0}$ is a real valued square-integrable continuous martingale, then there exists a unique continuous integrable adapted increasing process denoted by $\{< M, M >_t\}$ such that $\{M_t^2 - < M, M >_t\}$ is a continuous martingale vanishing at $t = 0$. The process $\{< M, M >_t\}$ is called the *quadratic variation* of M. The property of this process can be given by

Theorem 2.6. (Strong law of large numbers)

Let $M = \{M_t\}_{t\geq 0}$ be a real-valued martingale vanishing at $t = 0$. Then

$$\limsup_{t\to\infty} \frac{< M_t, M_t >}{t} < \infty \ a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{t} = 0 \ a.s.,$$

and also

$$\lim_{t\to\infty} < M_t, M_t >= \infty \ a.s. \quad \Rightarrow \quad \lim_{t\to\infty} \frac{M_t}{< M_t, M_t >} = 0 \ a.s..$$

Very useful inequalities of a martingale can be expressed as the following theorem:

Theorem 2.7. (Doob's Martingale inequality)

Let $M = \{M_t\}_{t\geq 0}$ be a $\mathbb{R}$-valued martingale. Let $[a, b]$ be a bounded interval in $\mathbb{R}_+$.

(1) If $p \geq 1$ and $M_t \in L^p(\Omega; \mathbb{R})$, then

$$\mathbb{P}\{\omega : \sup_{a\leq t\leq b} |M_t(\omega)| \geq c\} \leq \frac{\mathbb{E}|M_b|^p}{c^p} \quad \text{holds for all} \quad c > 0.$$

(2) If $p > 1$ and $M_t \in L^p(\Omega; \mathbb{R})$, then

$$\mathbb{E}\left(\sup_{a\leq t\leq b} |M_t|^p\right) \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}|M_b|^p.$$

2.4 Brownian motion

Brownian motion is one of the standard examples of the continuous-time martingale. However, due to its very interesting properties, this process gains quite significant attention. The Scottish botanist Robert Brown observed this important property and described irregular movement of pollen suspended in water, in 1828. However, T. N. Thiele was the first person to describe the mathematics behind the Brownian motion, in a paper published in 1880. Meanwhile, this was examined independently by Louis Bachelier in 1900. A. Einstein derived another property of Brownian motion, i.e. the transition density in 1905. Then, the mathematical foundation for Brownian motion as a stochastic process was strongly laid out by N. Wiener in 1931, and this process is also called the Wiener process. Now, Brownian motion has become the basic theory of stochastic analysis. In this section, let us discuss its basic mathematical concepts.

Definition 2.4. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$. A (standard) one-dimensional Brownian motion is a real valued continuous function $\{\mathcal{F}_t\}-$adapted process $\{W_t\}_{t\geq 0}$ with the following properties:*

(i) $W_0 = 0$ a.s.;

(ii) for $0 \leq s < t < \infty$, the increment $W_t - W_s$ is normally distributed with mean zero and variance $t - s$;

(iii) for $0 \leq s < t < \infty$, the increment $W_t - W_s$ is independent of $\mathcal{F}_s$.

Moreover, $0 \leq t_0 < t_1 < t_2 < ... < t_k < \infty$, then the increments $W_{t_i} - W_{t_{i-1}}, 1 \leq i \leq k$ are independent, and we say that Brownian motion has independent increments and we also say it has stationary increments since the distribution of $W_{t_i} - W_{t_{i-1}}$ depends only on the difference $t_i - t_{i-1}$.

Brownian motion has many important properties. Some of them are summarized below:

(a) $\{-W_t\}$ is a Brownian motion with respect to the same filtration $\{\mathcal{F}_t\}_{t\geq 0}$.

(b) Let $c > 0$. Define

$$X_t = \frac{W_{ct}}{\sqrt{c}} \text{ for } t \geq 0.$$

Then $\{X_t\}$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_{ct}\}$.

(c) $\{W_t\}$ is a continuous square-integrable martingale and its quadratic variation $< W, W >_t = t$ for all $t \geq 1$.

(d) The strong law of large numbers states that

$$\lim_{t\to\infty} \frac{W_t}{t} = 0 \text{ a.s.}.$$

(e) For almost every[4] $\omega \in \Omega$, the Brownian sample path $W.(\omega)$ is nowhere-differentiable[5].

(f) For almost every $\omega \in \Omega$, the Brownian sample path $W.(\omega)$ is locally Hölder continuous with exponent δ if $\delta \in (0, \frac{1}{2})$. However, for almost every $\omega \in \Omega$, the Brownian sample path $W.(\omega)$ is nowhere Hölder continuous with exponent $\delta > \frac{1}{2}$.

2.5 Stochastic integrals

Since for almost all $\omega \in \Omega$, the Brownian sample path $W_t(\omega)$ is nowhere differentiable, the integral can not be defined in the ordinary way. K. Itô was the first person to define this stochastic integral for a large class of stochastic processes by making use of the stochastic nature of Brownian motion and now it is known as the *Itô stochastic integral.*

Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions. Let $\{W_t\}_{t\geq 0}$ be a one-dimensional Brownian motion defined on the probability space adapted to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$. Then the space of all $\mathbb{R}$-valued, $\{\mathcal{F}_t\}-$adapted stochastic process $\{f(t,\omega)\}_{a\leq t\leq b}$ such that:

$$\int_a^b |f(t)|^p dt < \infty \qquad \text{a.s. for all } 0 \leq a < b < \infty$$

is denoted by $\mathcal{L}^p\left([a,b];\mathbb{R}\right)$. The $\mathcal{M}^p\left([a,b];\mathbb{R}\right)$ denotes the space of all processes

[4] except $\{\omega : \mathbb{P}(\omega) = 0\}$

[5] A function $f : \Omega \to \mathbb{R}$ is said to be nowhere differentiable if it is not differentiable at any point in the domain Ω of f

$\{f(t,\omega)\}_{a\leq t\leq b} \in \mathcal{L}^p([a,b];\mathbb{R})$ such that:

$$\mathbb{E}\int_a^b |f(t)|^p dt < \infty \qquad \text{for all } 0 \leq a < b < \infty.$$

Accordingly, a real-valued stochastic process $g \in \mathcal{L}^p([a,b];\mathbb{R})$ is called a step process if there exists a partition $a = t_0 < t_1 < ... < t_k = b$ of $[a,b]$ and bounded random variable ξ_i, $0 \leq i \leq k-1$ such that ξ_i is $\{\mathcal{F}_{t_i}\}-$ measurable and

$$g(t) = \xi_0 I_{[t_0,t_1]}(t) + \sum_{i=1}^{k-1} \xi_i I_{(t_i,t_{1+i}]}(t),$$

where I is an indicator function (see (2.1)). The Itô stochastic integral driven by such a step process of g with respect to W_t is defined as

$$\int_a^b g(t)dW_t = \sum_{i=1}^{k-1} \xi_i \left(W_{t_{i+1}} - W_{t_i}\right).$$

Definition 2.5. *Let $f \in \mathcal{L}^2([a,b];\mathbb{R})$. The Itô stochastic integral of f with respect W_t is defined by*

$$\int_a^b f(t)dW_t = \lim_{n\to\infty}\int_a^b g_n(t)dW_t \qquad \text{in } \mathcal{L}^2(\Omega,\mathbb{R})$$

where $\{g_n\}$ is a sequence of step process such that

$$\lim_{n\to\infty}\mathbb{E}\int_a^b |f(t) - g_n(t)|^2 dt = 0.$$

The stochastic integral has nice properties. We first observe the following:

Theorem 2.8. *Let $f, g \in \mathcal{M}^2([a,b];\mathbb{R})$, and let α, β be two real numbers, Then*

(1) $\int_a^b f(t)dW_t$ *is* $\{\mathcal{F}_b\}-$ *measurable;*

(2) $\mathbb{E}\left(\int_a^b f(t)dW_t|\mathcal{F}_b\right) = 0;$

(3) $\mathbb{E}\left(|\int_a^b f(t)dW_t|^2|\mathcal{F}_b\right) = \mathbb{E}\left(\int_a^b |f(t)|^2 dt|\mathcal{F}_b\right);$

(4) $\int_a^b[\alpha f(t) + \beta g(t)]dW_t = \alpha\int_a^b f(t)dW_t + \beta\int_a^b g(t)dW_t.$

Definition 2.6. *Let* $f \in \mathcal{L}^2([a,b];\mathbb{R})$*. Define*

$$I(t) = \int_0^t f(s)dW_s \quad for \quad 0 \le t \le T$$

where, by definition $I(0) = 0$*. We call* $I(t)$ *the indefinite Itô integral of* f*.*

Clearly, $\{I(t)\}$ is $\{\mathcal{F}_t\}$-adapted. The following theorem shows the very important martingale property of Itô integral.

Theorem 2.9. *If* $f \in \mathcal{M}^2([a,b];\mathbb{R})$*, then the indefinite integral* $\{I(t)\}_{0\le t\le T}$ *is a square-integrable martingale with respect to the filtration* $\{\mathcal{F}_t\}$*. Then,*

$$\mathbb{E}\left(\sup_{0\le t\le T}\left|\int_0^t f(s)dW_s\right|^2\right) \le 4\mathbb{E}\left(\int_0^T |f(s)|^2\right) ds.$$

2.6 Itô's formula

The basic definition of the Itô's integral which was defined in the previous section is not useful when we attempt to evaluate a given integrals. This is similar to the situation for classical Lebesgue integrals. We do not use the basic definition but rather the fundamental theorem of calculus plus the chain rule in the explicit calculation. On the other hand, we do not have differential theories though we have its integration concept. In this section, we therefore establish a stochastic version of chain rule for the Itô's integral which is known as Itô's formula. This concept has become the most fundamental theorem in the area of stochastic analysis and this chain rule can be explained in the following way.

Let $X(t)$ be a continuous $\{\mathcal{F}_t\}-$adapted process on $t \ge 0$ and have the form

$$X_t = X_0 + \int_0^t f(u)du + \int_0^t g(u)dW_u \tag{2.2}$$

where $f \in \mathcal{L}^1(\mathbb{R}_+;\mathbb{R})$ and $g \in \mathcal{L}^2(\mathbb{R}_+;\mathbb{R})$. Then, the differential form of this process can be given by

$$dX_t = f(t)dt + g(t)dW_t. \tag{2.3}$$

Then, a one-dimensional Itô's formula can be established by the following theorem.

Theorem 2.10. (1-dimensional Itô's formula)

Let X_t be an Itô process given by (2.3). Let $V(t, X_t) \in C^{2,1}(\mathbb{R}_+ \times \mathbb{R}; \mathbb{R})$. Then $Y_t = V(t, X_t)$ has the form of (2.2) and

$$dY_t = \left[\frac{\partial V(t,X_t)}{\partial t} + \frac{\partial V(t,X_t)}{\partial x} f(t) + \frac{1}{2}\frac{\partial^2 V(t,X_t)}{\partial^2 x} g(t)\right] dt + \frac{\partial V(t,X_t)}{\partial x} g(t) dW_t \text{ a.s..}$$

Moreover, in this section, we concentrate on properties of Itô stochastic integral which are very useful for examining financial models, established in the following chapters. These properties are called *moment inequalities* and we have already discussed the 1^{st} and 2^{nd} moment in Theorem 2.8. Now, the problem is what would be the p^{th} moment inequality when $p \geq 2$.

Theorem 2.11. *Let $p \geq 2$, let $g \in \mathcal{M}^2([0,T]; \mathbb{R})$ such that*

$$\mathbb{E}\int_0^T |g(s)|^p ds < \infty.$$

Then

$$\mathbb{E}\left|\int_0^T g(s)dW(s)\right|^p \leq \left[\frac{p(p-1)}{2}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \mathbb{E}\int_0^T |g(s)|^p ds.$$

In particular, for $p = 2$, there is equality.

Theorem 2.12. (Burkholder-Davis-Gundy Inequality)

Let $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R})$. For $t \geq 0$, define,

$$x(t) = \int_0^t g(s)dW(s) \quad \text{and} \quad A(t) = \int_0^t |g(s)|^2 ds,$$

then for every $p > 0$, there exists a universal positive constant c_p, C_p (depending *only on p), such that*

$$c_p \mathbb{E}|A(t)|^{\frac{p}{2}} \le \mathbb{E}\left(\sup_{0\le s\le t}|x(s)|^p\right) \le C_p \mathbb{E}|A(t)|^{\frac{p}{2}}$$

for all $t \ge 0$. *In particular one may take*

$$\begin{aligned} &c_p = (\tfrac{p}{2})^p, && C_p = (\tfrac{32}{p})^{\frac{p}{2}} && \textit{if} \quad 0 < p < 2; \\ &c_p = 1 && C_p = 4, && \textit{if} \quad p = 2; \\ &c_p = (2p)^{-2p}, && C_p = (\tfrac{p^{p+1}}{2(p-1)^{p-1}})^{\frac{p}{2}} && \textit{if} \quad p > 2. \end{aligned}$$

2.7 Stochastic differential equations

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge 0}$ satisfying the usual conditions and let $W(t)$, $t \ge 0$ be a 1-dimensional Brownian motion defined on the same probability space adapted to the filtration. Let $f : \mathbb{R} \times [t_0, T] \to \mathbb{R}$ and $g : \mathbb{R} \times [t_0, T] \to \mathbb{R}$ both be Borel measurable. Also let $0 \le t < T < \infty$ and x_0 be an $\{\mathcal{F}_{t_0}\}-$measurable, $\mathbb{R}$-valued random variable such that $\mathbb{E}|X_0|^2 < \infty$. Then, consider the 1-dimensional stochastic differential equation of Itô type for the $\mathbb{R}$-valued stochastic process $\{X_t\}_{t\in[t_0,T]}$ and with initial value $X_{t_0} = X_0$. It has the following form:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t, \quad \text{on} \quad t_0 \le t \le T. \tag{2.4}$$

A definition of the solution to this SDE can be given by:

Definition 2.7. *An* $\mathbb{R}$*-valued stochastic process* $\{X_t\}_{t_0\le t\le T}$ *is called a solution of the SDE (2.4) if it has the following properties with initial value* $X_{t_0} = X_0$*:*

(1) $\{X(t)\}$ *is continuous and* $\{\mathcal{F}_t\}-$*adapted;*

(2) $\mathbb{P}\{X_{t_0} = X_0\} = 1$*;*

(3) $\{f(X(t), t)\} \in \mathcal{L}^1([t_0, T]; \mathbb{R})$ *and* $\{g(X(t), t)\} \in \mathcal{L}^2([t_0, T]; \mathbb{R})$*;*

(4) the integral version of (2.4) holds for every $t \in [t_0, T]$ with probability 1.

A solution $\{X_t\}$ is said to be unique if any other solution $\{\bar{X}_t\}$ indistinguishable from $\{X_t\}$, that is

$$\mathbb{P}\{X(t) = \bar{X}_t \quad \textit{for all} \quad t_0 \le t \le T\} = 1.$$

Now, we turn to discuss the condition that guarantees existence and uniqueness of the solution to the SDE model (2.4).

Theorem 2.13. *Assume that there exist two positive constants K and $\bar{K}$ such that*

(1) (Lipschitz condition) for all $x, y \in \mathbb{R}$ and $t \in [t_0, T]$,

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \le \bar{K} \, |x - y|^2 \, ; \tag{2.5}$$

(2) (Linear growth condition) for all $(x, t) \in \mathbb{R} \times [t_0, T]$,

$$|f(x,t)|^2 \vee |g(x,t)|^2 \le K(1 + |x|^2); \tag{2.6}$$

then there exists a unique solution $X(t)$ to equation (2.4) with initial value $X_{t_0} = X_0$ and solution belongs to $\mathcal{M}^2([t_0, T]; \mathbb{R})$.

Theorem 2.14. *Assume that the linear growth condition (2.6) holds. But the Lipschitz condition (2.5) is replaced by the following condition:*

(1)(local Lipschitz condition) for every integer $R \ge 1$, there exists a positive constant K_R such that, for all $t \in [t_0, T]$ and all $x, y \in \mathbb{R}$ with $|x| \vee |y| \le R$,

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \le K_R \, |x - y|^2 \, , \tag{2.7}$$

then there exists a unique local solution $X(t)$ to equation (2.4) with initial value $X_{t_0} = X_0$ and the solution belongs to $\mathcal{M}^2([t_0, T]; \mathbb{R})$.

Let us now focus on another useful and important property of the models in the following chapters.

2.8 Poisson process

The Poisson process $N(t)$ counts the number of jumps that occur at or before time t. Namely

$$N(t) = \begin{cases} 0 \quad ; \quad if \quad 0 \le t < S_1, \\ 1 \quad ; \quad if \quad S_1 \le t < S_2, \\ \vdots \qquad\qquad\qquad \vdots \\ n \quad ; \quad if \quad S_n \le t < S_{n+1}, \\ \vdots \qquad\qquad\qquad \vdots \end{cases} \tag{2.8}$$

where $S_n = \sum_{k=1}^{n} \tau_k$ is called the *arrival time* (i.e S_n is the time of the n^{th} jump) and the exponential random variables τ_k are called the *interarrival times*. The jump time $N(t)$ is a *right continuous* function which gives $N(t) = {}_{s\downarrow t}N(s)$ and $\mathcal{F}$ the $\sigma-$algebra containing the observed information on $N(t)$ for $0 \le s \le t$. Moreover, since the expected time between jumps is $\frac{1}{\lambda}$, the Poisson process has *intensity* λ.

The following section gives more useful information about this Poisson process.

Theorem 2.15. *For $n \ge 1$, the random variable $S_n = \sum_{k=1}^{n} \tau_k$ has the gamma density*

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!}\lambda e^{-\lambda s}, \quad s \ge 0. \tag{2.9}$$

Theorem 2.16. *The Poisson process $N(t)$ with intensity λ has the distribution*

$$\mathbb{P}\{N(t) = k\} = \frac{(\lambda t)^k}{k!}e^{-\lambda t}, \quad k = 0, 2, 3, ... \tag{2.10}$$

Theorem 2.17. *Let $N(t)$ be the Poisson process with intensity λ. Then we define* *the compensated Poisson process,*

$$\bar{N}(t) = N(t) - \lambda t. \tag{2.11}$$

Then $\bar{N}(t)$ is a martingale.

2.9 Stochastic differential equations with a Poisson jump process

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $W(t)$ be a Brownian motion relative to this filtration. We say that a Poisson process $N(.)$ is a Poisson process relative to filtration, if $N(t)$ is $\{\mathcal{F}_t\}-$measurable for every t and for every $u > t$ the increment $N(u) - N(t)$ is independent of $\{\mathcal{F}_t\}$.

Then, let $f : \mathbb{R} \times [t_0, T] \to \mathbb{R}$, $g : \mathbb{R} \times [t_0, T] \to \mathbb{R}$ and $h : \mathbb{R} \times [t_0, T] \to \mathbb{R}$ are all Borel measurable. Consider a Poisson jump stochastic differential equation of Itô type for the $\mathbb{R}$-valued stochastic process $\{X_t\}_{t\in[t_0,T]}$ and with initial value $X_{t_0} = X_0$, which the following form:

$$dX_t = f(t, X_t)dt + g(t, X_t)dW_t + h(t, X_t)d\bar{N}_t \quad \text{on} \quad t_0 \leq t \leq T, \tag{2.12}$$

where $\bar{N}(t) = N(t) - \lambda t$, $\lambda \in R^+$ and N(t) is a stationary $\mathcal{F}_t-$Poisson point process with intensity λ.

2.10 Continuous-time Markov process

In this section, we will recall some basic facts about a continuous-time Markov process which is known as the continuous-time version of a Markov chain. Let $X = \{X(t)\}_{t\geq 0}$ be a d-dimensional stochastic process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ while taking values in a countable space Ξ, which is called the state space of the process.

Definition 2.8. *The d-dimensional $\{\mathcal{F}_t\}$-adapted process $X = \{X(t)\}_{t\geq 1}$ is called a continuous-time Markov process if it satisfies $\mathbb{P}\{X(t) = j | X_{t_1} = i_1, ..., X_{t_k} = i_k\} = \mathbb{P}\{X(t) = j | X_{t_k} = i_k\}$, for all $0 < t_1, ..., < t_k < t < \infty$ and all $j, i_x \in \Xi$ such that $1 \leq x \leq k$.*

Definition 2.9. *A function $p(s, i : t, j) = p_{ij}(s, t)$ defined on $0 \leq s \leq t < \infty$ and $i, j \in \Xi$ is said to be the transition probability of the continuous-time Markov process $X = \{X(t)\}_{t\geq 0}$ and $P(s, t) = (p_{ij}(s, t))_{i,j\in\Xi}$ is said to be the transition probability matrix of X if the following properties are satisfied:*

(i) $p_{ij}(s, t) = \mathbb{P}\{X_t = j | X_s = i\}$ for all $0 \leq s \leq t$ and $i, j \in \Xi$;

(ii) $p_{ij}(s, s) = \kappa_{ij}$ for all $0 \leq s$ and $i, j \in \Xi$;

(iii) $\sum_{j\in\Xi} p_{ij}(s, t) = 1$ for all $0 \leq s \leq t$ and $i, j \in \Xi$;

(iv) the Chapman-Kolmogorov equation

$$p_{ij}(s, t) = \sum_{k\in\Xi} p_{ik}(s, u) p_{kj}(u, t)$$

or in matrix form

$$P(s, t) = P(s, u)P(u, t)$$

holds for all $0 \leq s \leq u \leq t$.

The continuous-time Markov process $X = \{X(t)\}_{t\geq 0}$ is said to be homogenous if its transition probability $p_{ij}(s, t)$ is stationary, which depends only on the difference $t - s$ for all $0 \leq s \leq t < \infty$ and $i, j \in \Xi$, namely

$$P(s, s + u) = P(u),$$

for all $s \geq 0$ and $u \geq 0$. Furthermore, the corresponding transition probability and transition probability matrix can be given by $p_{ij}(t)$ and $P(t)$ for all $t \geq 0$ respectively. If $\lim_{t\to 0} p_{ii} = 1$ for all $i \in \Xi$ then the transition probability matrix $P(t) = (p_{ij}(t))_{ij\in\Xi}$ is called standard.

Theorem 2.18. *[3] Let $p_{ij}(t)$ be a standard transition function. Then*

$$\kappa_i = \lim_{t\to 0} \frac{1 - p_{ii}(t)}{t}$$

exists (but may be ∞) for all $i \in \Xi$.

A state $i \in \Xi$ is said to be stable if $\kappa_i < \infty$.

Theorem 2.19. *[3] Let $p_{ij}(t)$ be a stranded transition function and j be a stable state. Then*

$$\kappa_{ij} = p'_{ij}(0) = \lim_{h\to 0} \frac{p_{ij}(t+h) - p_{ij}(t)}{h}$$

exists and is finite for every $i \in \Xi$.

On the other hand, let $\kappa_{ij} = -\kappa_i$ and $\Gamma = (\kappa_{ij})_{ij\in\Xi}$ is called the generator of the Markov chain. If this process is said to be a continuous time Markov process then it has a finite state space and we can take to $\mathbb{S} = \{1, 2, ..., N\}$. Hereafter we assume that all Markov chains are finite and states are stable. Moreover, almost every sample path of these Markov chains is a right continuous step function.

Theorem 2.20. *[3] Let $P(t) = (p_{ij}(t))_{N\times N}$ be the stranded transition matrix and $\Gamma = (\kappa_{ij})$ be the generator of the finite continuous-time Markov process. Then*

$$P(t) = e^{t\Gamma} \quad \textit{for all} \quad t \geq 0.$$

It is useful to see that a continuous-time Markov process $X = \{X(t)\}_{t\geq 0}$ with the generater $\Gamma = (\kappa_{ij})_{ij\in\Xi}$ can be represented as a stochastic integral with respect to a Poisson random measure [4, 71]. Let Δ_{ij} be consecutive, left closed, right open intervals of the real line each having length κ_{ij} such that

$$\begin{aligned}
\Delta_{12} &= \Big[0, \kappa_{12}\Big), \\
\Delta_{13} &= \Big[\kappa_{12}, \kappa_{12} + \kappa_{13}\Big), \\
&\vdots \\
\Delta_{1N} &= \Big[\sum_{j=2}^{N-1} \kappa_{1j}, \sum_{j=2}^{N} \kappa_{1j}\Big), \\
\Delta_{21} &= \Big[\sum_{j=2}^{N} \kappa_{1j}, \sum_{j=2}^{N} \kappa_{1j} + \kappa_{21}\Big), \\
\Delta_{23} &= \Big[\sum_{j=2}^{N} \kappa_{1j} + \kappa_{21}, \sum_{j=2}^{N} \kappa_{1j} + \kappa_{21} + \kappa_{23}\Big), \\
&\vdots \\
\Delta_{2N} &= \Big[\sum_{j=2}^{N} \kappa_{1j}, \sum_{j=1, j\neq 2}^{N-1} \kappa_{2j}, \sum_{j=2}^{N} \kappa_{1j} + \sum_{j=1, j\neq 2}^{N} \kappa_{2j}\Big)
\end{aligned} \tag{2.13}$$

and so on. Define a function $h : \mathbb{S} \times \mathbb{R} \to \mathbb{R}$ by

$$h(i, y) = \begin{cases} j - i & ; \quad if \quad y \in \Delta_{ij}, \\ 0 & ; \quad otherwise. \end{cases} \tag{2.14}$$

Then

$$dX_t = \int_{\mathbb{R}} h(X_{t-}, y) v(dt, dy),$$

with initial condition $X_0 = i_0 \in \mathbb{S}$, where $v(dt, dy)$ is a Poisson measure with intensity $dt \times m(dy)$, in which m is the Lebesgue measure on $\mathbb{R}$.

Now, we introduce the following important SDE which can be seen in practice.

2.11 Stochastic differential equation with Markov-swiching

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the complete probability space $(\Omega, \mathcal{F}, \mathcal{F}_{t\geq 0}, \mathbb{P})$ taking values in a finite state space $\mathbb{S} = \{1, 2, ..., N\}$ with generator $\Gamma = (\kappa_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j | r(t) = i\} = \begin{cases} \kappa_{ij}\Delta + 0(\Delta) & ; \quad if \quad i \neq j, \\ 1 + \kappa_{ij}\Delta + 0(\Delta) & ; \quad if \quad i = j, \end{cases}$$

where $\Delta > 0$ and $\kappa_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$, while $\kappa_{ii} = -\sum_{i\neq j} \kappa_{ij}$. It is also known that almost all sample paths of $r(t)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+ = [0, \infty)$. Then, we assume that $W(t)$ is the m-dimensional Brownian motion defined on the same probability space which is independent of the Markov chain $r(t)$.

Now, consider the 1-dimensional stochastic differential equation with Markovian switching

$$dX(t) = f(X(t), t, r(t))dt + g(X(t), t, r(t))dW(t) \tag{2.15}$$

on $t \geq 0$ with initial values $X(t_0) = X_0 \in \mathcal{L}^2_{\mathcal{F}_{t_0}}(\Omega; \mathbb{R})$ and $r(t_0) = r_0$, where r_0 is an $\mathbb{S}-$valued $\mathcal{F}_{t_0}$-measurable random variable and

$$f : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R} \quad \text{and} \quad g : \mathbb{R} \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}.$$

On the other hand, let $\mathbb{C}^{2,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$ denote the family of all real-valued functions $V(X, t, i)$ on $\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}$ which are continuously twice differentiable in X and once in t. If $V(X, t, i) \in \mathbb{C}^{2,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$, define an operator $\mathcal{L}V$ from

$\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(X,t,i) =& V_t(X,t,i) + V_X(X,t,i)f(X(t),t,r(t)) \\ &+ \frac{1}{2}\left[g^2(X(t),t,r(t))V_{XX}(X,t,i)\right] + \sum_{j=1}^{N} \kappa_{ij}V(X,t,j),\end{aligned} \tag{2.16}$$

where $V_t(X,t,i) = \frac{\partial V(X,t,i)}{\partial t}$, $V_X(X,t,i) = \frac{\partial V(X,t,i)}{\partial X}$ and $V_{XX}(X,t,i) = \left(\frac{\partial^2 V(X,t,i)}{\partial X^2}\right)$. The following Theorem shows the transformation of the paired process $(X(t), r(t))$ into a new process $V(X(t), t, r(t))$ which is known as the generalized Itô formula.

Theorem 2.21. *If $V(X,t,i) \in \mathbb{C}^{2,1}(\mathbb{R} \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$, then*

$$\begin{aligned}& V(X(t),t,r(t)) \\ =& V(X(0),0,r(0)) + \int_0^t \mathcal{L}V(X(s),s,r(s))ds \\ &+ \int_0^t V_X(X(s),s,r(s))f(X(t),t,r(t))dW(s) \\ &+ \int_0^t \int_{\mathbb{R}} \left(V(X(s),s,r(0)+h(r(s),l)) - V(X(s),s,r(s))\right)\mu(ds,dl)\end{aligned} \tag{2.17}$$

for any $t \geq 0$, where function $h(.,.)$ is defined by (2.14) and $\mu(ds,dl) = v(ds,dl) - \mu(dl)ds$ is a martingale measure (see also [53], [4]).

2.12 Approximate solutions

The criteria which were established in the previous section show the property of uniqueness and existence of the solution of SDEs. However, most of the SDEs do not have explicit solutions. Study of the numerical method is therefore more useful to find the approximate solution of these SDEs. In this section, we will therefore concentrate on the Euler-Maruyama method, one of the most powerful numerical schemes.

To define the Euler-Maruyama approximate solution, first recall the discrete-time Markov chain: Given time step $\Delta > 0$, let $r_k^\Delta = r(k\Delta)$ for $k > 0$. Then

$\{r_k^\Delta, k = 0, 1, 2, 3, ...\}$ is a discrete time Markov-chain with one step transition probability matrix:

$$P(\Delta) = (p_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}.$$

Given a step size $\Delta > 0$, the discrete-time Markov chain $\{r_k^\Delta, k = 0, 1, 2, 3, ...\}$ can be simulated as follows: Compute the one step transition probability matrix

$$P(\Delta) = (p_{ij}(\Delta))_{N\times N} = e^{\Delta\Gamma}.$$

Let $r_0^\Delta = r_0$ and generate a random number ξ_1 which is uniformly distributed in [0,1]. Define

$$r_1^\Delta = \begin{cases} i_1 \quad ; \quad if \quad i_1 \in \mathbb{S} - \{N\} \\ \qquad\qquad \text{is such that} \quad \sum_{j=1}^{i_1-1} P(\Delta)_{i_0 j}(\Delta) \le \xi_1 < \sum_{j=1}^{i_1} P(\Delta)_{i_0 j}(\Delta), \\ N \quad ; \quad if \quad \sum_{j=1}^{N-1} P(\Delta)_{i_0 j}(\Delta) \le \xi_1, \end{cases}$$

and let $r_1^\Delta = r_1$, where we set $\sum_{i=1}^{0} P(\Delta)_{i_0 j}(\Delta) = 0$ as usual. Generate a independently new random number ξ_2 which is again uniformly distributed in $[0, 1]$ and define

$$r_2^\Delta = \begin{cases} i_2 \quad ; \quad if \quad i_2 \in \mathbb{S} - \{N\} \\ \qquad\qquad \text{is such that} \quad \sum_{j=1}^{i_2-1} P(\Delta)_{i_1 j}(\Delta) \le \xi_2 < \sum_{j=1}^{i_2} P(\Delta)_{i_1 j}(\Delta), \\ N \quad ; \quad if \quad \sum_{j=1}^{N-1} P(\Delta)_{i_1 j}(\Delta) \le \xi_2, \end{cases}$$

Repeating this procedure, a trajectory of $\{r_k^\Delta, k = 0, 1, 2, 3, ...\}$ can be generated. This procedure can be carried out independently to obtain more trajectories. Now based on this discrete-time Markov-chain r_k^Δ, we can define the Euler-Maruyama approximate solution to equation (2.15). Given time step $\Delta > 0$, let $t_k = k\Delta$ for

$k \geq 0$, compute the discrete approximation $x_k \approx X(t_k)$ by setting $x_0 = X_0$ and r_0^Δ and forming

$$x_{k+1} = x_k + f(x_k, t_k, r_k^\Delta)\Delta + g(x_k, t_k, r_k^\Delta)\Delta W_k, \tag{2.18}$$

where $\Delta W_k = W(t_{k+1}) - W(t_k)$. Let $\bar{x}(t) = x_k$, $\bar{r}(t) = r_k^\Delta$ for $t \in [t_k, t_{k+1})$ and define the continuous EM approximate solution

$$x(t) = x_0 + \int_0^t f(\bar{x}(s), s, \bar{r}(s))ds + \int_0^t g(\bar{x}(s), s, \bar{r}(s))dW(s). \tag{2.19}$$

Note that $x(t_k) = \bar{x}(t_k) = x_k$, that is $x(t)$ and $\bar{x}$ coincide with the discrete solution at the grid points. Moreover, applying similar techniques, the EM approximate solution to the SDE (2.4) and (2.12) can easily be obtained.

Let us now present useful techniques for the following chapters.

Theorem 2.22. *Assume that f and g satisfy the linear growth condition (2.6). Then for any $p \geq 2$, there is a constant H, which is dependent on only p, T, K, X_0 but independent of Δ, such that the true solution and the continuous EM approximate solution to equation (2.15) have the property that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T} |x(t)|^p\right) \leq H. \tag{2.20}$$

Together with this lemma, we can discuss the convergence in second moment of the EM approximate solution to the true solution of equation (2.15) under the global Lipschitz condition (2.5).

Theorem 2.23. *Under the global Lipschitz condition (2.5),*

$$\lim_{\Delta\to 0} \mathbb{E}\left(\sup_{0\leq t\leq T} [X(t) - x(t)]^2\right) = 0. \tag{2.21}$$

Now, we concentrate on the local Lipschitz condition (2.7), without linear growth condition (2.6). The following theorem describes the convergence in probability of the EM approximate solution under some additional conditions.

Theorem 2.24. *Let the local Lipschitz condition (2.7) hold. Assume that there exists a C^2 function $V : \mathbb{R} \times \mathbb{S} \to \mathbb{R}_+$ satisfying the following three conditions:*

(i) $\lim_{|X| \to \infty} V(X, i) = \infty$ *for any* $i \in \mathbb{S}$*;*

(ii) for some $h > 0$, $LV(X, i) \leq h(1 + V(X, i))$ *for all* $(X, i) \in \mathbb{R} \times \mathbb{S}$*, where*

$$LV(X, i) = V_X f(X, i) + \frac{1}{2} g^2(X, i) V_{XX} + \sum_{j=1}^{N} \kappa_{ij} V(X, j);$$

(iii) for each $R > 0$ *there exists a positive constant* K_R *such that for all* $i \in \mathbb{S}$ *and those* $X, Y \in \mathbb{R}$ *with* $|X| \vee |Y| \leq R$,

$$|V(X,i) - V(Y,i)| \vee |V_X(X,i) - V_Y(Y,i)| \vee |V_{XX}(X,i) - V_{XX}(Y,i)| \leq K_R |X - Y|.$$

Then

$$\lim_{\Delta \to 0} \left(\sup_{0 \leq t \leq T} |X(t) - x(t)|^2 \right) = 0 \quad \text{in probability.} \tag{2.22}$$

2.13 Gronwall-type integral inequalities

The Gronwall-type integral inequalities have been widely used in many branches of ordinary differential equations (ODEs) and stochastic differential equations to prove the required theorems and results on existence, uniqueness, boundlessness, etc. Also this concept is useful to prove theorems in the following chapters. Therefore, this useful tool with other inequalities can be expressed in the following way.

Theorem 2.25. (Gronwall's inequality)

Let $T > 0$, *and* $c > 0$. *Let* $u(.)$ *be a Borel measurable bounded nonnegative function on [0,T], and let* $v(.)$ *be a nonnegative integrable function on [0,T]. If*

$$u(t) \leq c + \int_0^t v(s) u(s) ds \qquad \text{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq c \exp\left(\int_0^t v(s)ds\right) \qquad \text{for all} \quad 0 \leq t \leq T.$$

Theorem 2.26. (Bihari's inequality)

Let $T > 0$, and $c > 0$. Let $K : \mathbb{R}_+ \to \mathbb{R}_+$ be a continuous nondecreasing function such that $K(t) > 0$ for all $t > 0$. Let $u(.)$ be a Borel measurable bounded nonnegative function on [0,T], and let $v(.)$ be a nonnegative integrable function on [0,T]. If

$$u(t) \leq c + \int_0^t v(s)K(u(s))ds \qquad \text{for all} \quad 0 \leq t \leq T,$$

then

$$u(t) \leq G^{-1}\left(G(c) + \int_0^t v(s)ds\right)$$

holds for all $0 \leq t \leq T$ such that

$$G(c) + \int_0^t v(s)ds \in Dom(G^{-1}),$$

where

$$G(r) = \int_r^1 \frac{ds}{K(s)} \quad \text{on } r > 0,$$

and G^{-1} is the inverse function of G.

Chapter 3

Euler-Maruyama Approximation for Mean-Reverting-Theta Stochastic Volatility Model

3.1 Introduction

In general, the rate of the change of an asset price $X(t)$ consists of random changes and deterministic changes. The well-known Black-Scholes [8] model of the asset price is described by the linear SDE

$$dX(t) = \alpha_1 X(t)dt + \sigma_1 X(t)dW_1(t), \tag{3.1}$$

where W_1 is a scalar Brownian motion and the rate of return α_1 and the volatility σ_1 are assumed to be constants. Later, Vasicek [74] developed the mean-reverting model and Cox, Ingersoll and Ross (CIR) [13] modified it into the mean-reverting square root process which has the SDE form

$$dX(t) = \alpha_1(\mu_1 - X(t))dt + \sigma_1\sqrt{X(t)}dW_1(t). \tag{3.2}$$

According to empirical studies, many authors have shown that the volatility is a stochastic process and it can be modelled by an SDE in many situations [31, 34, 13, 59, 7]. In particular, Hull and White [34] observed that the instantaneous variance $V = \sigma_1^2$ is governed by another Brownian motion W_2 and can be described by the SDE

$$dV(t) = \alpha_2 V(t)dt + \sigma_2 V(t)dW_2(t), \tag{3.3}$$

where α_2, σ_2 are constants. Heston [31] proposed to model the variance by the mean reverting square root process

$$dV(t) = \alpha_2(\mu_2 - V(t))dt + \sigma_2\sqrt{V(t)}dW_2(t). \tag{3.4}$$

Lewis [47] developed this into the more general mean-reverting-theta process

$$dV(t) = \alpha_2(\mu_2 - V(t))dt + \sigma_2 V(t)^\theta dW_2(t), \tag{3.5}$$

which can not only model the volatility but also the asset price, where $\theta \geq 1/2$. Accordingly, we will, in this chapter, consider the following mean-reverting-theta stochastic volatility model

$$\begin{aligned} dX(t) &= \alpha_1(\mu_1 - X(t))dt + \sigma_1\sqrt{V(t)}X(t)^\theta dW_1(t), \\ dV(t) &= \alpha_2(\mu_2 - V(t))dt + \sigma_2 V(t)^\beta dW_2(t), \end{aligned} \tag{3.6}$$

where W_1 and W_2 are scalar Brownian motions with correlation coefficient ρ, defined on the same probability space. This SDE model has no explicit solutions. Hence numerical techniques have become one of the most popular and powerful tools to find the approximate solution [51, 54, 2, 32, 33, 64]. In the case when $1/2 \leq \beta, \theta \leq 1$, the convergence (in L^2) of the Euler-Maruyama (EM) approximate solution has been established by Mao et al [52]. In this paper, the expected upper bound for the EM approximate solution and the true solution to this SDE model (3.6) have been obtained under the linear growth condition. Moreover, the diffusion coefficients of this model satisfy the global Lipschitz condition. Therefore, the convergence in second moment property of the EM approximate solution to the volatility has been obtained by applying Itô formula. Finally, they proved the convergence in second moment property of the EM approximate solution to the asset price when the time step is sufficiently small. However, there is so far no result on the numerical solutions for the SDE model (3.6) when $\theta, \beta > 1$. The aim of this chapter is to close this gap.

It is essential for the SDE model (3.6) to have its non-negative solution. Given that the SDEs do not obey the linear growth condition though it satisfies the local Lipschitz condition, there is so far no result on the non-negative solution. We will therefore in the following section develop a technique to prove the non-negativity of the solution to the model. We will then define the EM approximate solutions to the volatility process $V(t)$ and the underlying asset price process $X(t)$. To guarantee the non-negativity of the EM solutions, we will use the technique of stopping times. We will finally show that the EM numerical solutions will converge in probability to the true solution.

3.2 Non-negative solution

The SDE model (3.6) describes the asset price and its volatility in the financial market. It is therefore essential to prove that the solution of (3.6) is non-negative with probability 1. The following lemmas in fact show that the solution is positive with probability 1.

Lemma 3.1. *Let $\beta > 1$. Then, for any given initial value $V(0) = V_0 > 0$, the solution $V(t)$ of the SDE model (3.6) will be positive for all $t \in [0, T]$ almost surely.*

Proof. Treat the second SDE in (3.6) as an SDE in the whole real space $\mathbb{R} =$

$(-\infty,\infty)$ by setting its coefficients be 0 when $V(t) < 0$. Clearly, the coefficients obey the local Lipschitz condition. Hence, there exists a unique maximal local solution $V(t)$ on $t \in [0,\rho_e)$, where ρ_e is the explosion time[1] . We also set $\inf\emptyset = \infty$ (as usual, $\emptyset$ denotes the empty set). For any sufficiently large positive number M, namely $\frac{1}{M} < V(0) < M$, define a stopping time $\rho_M{=}\rho_e \wedge \inf\big\{t \in [0,\rho_e) : |V(t)| \notin [\frac{1}{M}, M]\big\}$ and set $\rho_\infty = \lim_{M\to\infty} \rho_M$.

Now, define a C^2-function $H : (0,\infty) \to (0,\infty)$ by

$$H(V) = V^{\frac{1}{2}} - 1 - \frac{1}{2}\ln V, \quad V > 0.$$

Applying the Itô formula 2.10 yields

$$\begin{aligned}
\mathbb{E}[H(V(T\wedge\rho_M))] =& H(V_0) + \mathbb{E}\int_0^{T\wedge\rho_M} H^{'}(V(u))\alpha_2[\mu_2 - V(u)]du \\
&+ \frac{1}{2}\mathbb{E}\int_0^{T\wedge\rho_M} H^{''}(V(u))\sigma_2^2|V(u)|^{2\beta}du \\
\le& H(V_0) + \mathbb{E}\int_0^{T\wedge\rho_M} \frac{1}{2}\Big[(V(u))^{-\frac{1}{2}} - (V(u))^{-1}\Big]\alpha_2\Big[\mu_2 - V(u)\Big]du \\
&+ \frac{1}{4}\mathbb{E}\int_0^{T\wedge\rho_M} \sigma_2^2\Big[(V(u))^{-2} - \frac{(V(u))^{-\frac{3}{2}}}{2}\Big]|V(u)|^{2\beta}du \\
\le& H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} \\
&+ \frac{\sigma_2^2}{4}\mathbb{E}\int_0^{T\wedge\rho_M} \Big[1 - \frac{(V(u))^{\frac{1}{2}}}{2}\Big]|V(u)|^{2\beta-2}du.
\end{aligned} \tag{3.7}$$

Since

$$\Big[1 - \frac{y^{\frac{1}{2}}}{2}\Big] = \begin{cases} < 0 \quad ; \quad if \quad 4 < y; \\ < 1 \quad ; \quad if \quad 0 \le y \le 4, \end{cases} \tag{3.8}$$

we then have

$$\mathbb{E}[H(V(T\wedge\rho_M))] \le H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} + \sigma_2^2 4^{2\beta-3}T. \tag{3.9}$$

[1] $\rho_e = \inf\{t \ge 0 : V(t) \notin \mathbb{R}\}$

Noting

$$\begin{aligned}\mathbb{E}[H(V(T\wedge\rho_M))] &\geq \mathbb{E}[H(V(T\wedge\rho_M))1_{[\rho_M\leq T]}]\\ &\geq [H(M^{-1})\wedge H(M)]\mathbb{P}(\rho_M\leq T),\end{aligned} \tag{3.10}$$

we see from (3.9) that

$$[H(M^{-1})\wedge H(M)]\mathbb{P}(\rho_M\leq T)\leq H(V_0)+\frac{\alpha_2\mu_2 T}{2}+\frac{\alpha_2 T}{2}+\sigma_2^2 4^{2\beta-3}T, \tag{3.11}$$

namely

$$\mathbb{P}(\rho_M\leq T)\leq \frac{H(V_0)+\frac{\alpha_2\mu_2 T}{2}+\frac{\alpha_2 T}{2}+\sigma_2^2 4^{2\beta-3}T}{H(M^{-1})\wedge H(M)}. \tag{3.12}$$

Now letting $M\to\infty$ we have $\mathbb{P}(\rho_\infty\leq T)=0$. This implies that $\mathbb{P}(\rho_\infty>T)=1$, which means $\mathbb{P}\left(0<V(t)<\infty \text{ for all } t\in[0,T]\right)=1$ as required. □

Lemma 3.2. *Let $\theta>1$ and $\beta>1$. Then, for any given initial values $V(0)=V_0>0$ and $X(0)=X_0>0$, the solution $X(t)$ of (3.6) will be positive for all $t\in[0,T]$ almost surely.*

Proof. Once again, treat the SDE model (3.6) as an SDE in $\mathbb{R}^2$ by setting its coefficients to 0 whenever $V(t)<0$ or $X(t)<0$. Clearly, the coefficients obey the local Lipschitz condition. Hence, there exists a unique maximal local solution $(X(t),V(t))$ on $t\in[0,\rho_e)$, where ρ_e is defined as before.

For any sufficiently large positive values M and N, namely $\frac{1}{M}<V(0)<M$ and $\frac{1}{N}<X(0)<N$, define stopping times ρ_M=$\rho_e\wedge\inf\{t\in[0,\rho_e]:V(t)\notin[\frac{1}{M},M]\}$ and $\tau_N=\rho_e\wedge\inf\{t\in[0,\rho_e]:|X(t)|\notin[\frac{1}{N},N]\}$ and let $\eta=\rho_M\wedge\tau_N$. Then set $\rho_\infty=\lim_{M\to\infty}\rho_M$ (as before) and $\tau_\infty=\lim_{N\to\infty}\tau_N$. Let the C^2-function H be the same as before. Applying the Itô formula yields

$$\begin{aligned}\mathbb{E}[H(X(T\wedge\eta))]=&H(X_0)+\mathbb{E}\int_0^{T\wedge\eta}H'(X(u))\alpha_1[\mu_1-X(u)]du\\ &+\frac{1}{2}\mathbb{E}\int_0^{T\wedge\eta}H''(X(u))\sigma_1^2V(u)|X(u)|^{2\theta}du\end{aligned} \tag{3.13}$$

$$
\begin{aligned}
&\mathbb{E}[H(X(T\wedge\eta))]\\
\le& H(X_0)+\mathbb{E}\int_0^{T\wedge\eta}\frac{1}{2}[(X(u))^{-\frac{1}{2}}-(X(u))^{-1}]\alpha_1[\mu_1-X(u)]du\\
&+\frac{1}{4}\mathbb{E}\int_0^{T\wedge\eta}\sigma_1^2[(X(u))^{-2}-\frac{(X(u))^{-\frac{3}{2}}}{2}]V(u)|X(u)|^{2\theta}du\\
\le& H(X_0)+\frac{\alpha_1\mu_1T}{2}+\frac{\alpha_1T}{2}+\frac{\sigma_1^2}{4}\mathbb{E}\int_0^{T\wedge\eta}[1-\frac{(X(u))^{\frac{1}{2}}}{2}]|X(u)|^{2\theta-2}V(u)du.
\end{aligned}
\tag{3.14}
$$

By (3.8), we have

$$
\mathbb{E}[H(X(T\wedge\eta))]\le H(X_0)+\frac{\alpha_1\mu_1T}{2}+\frac{\alpha_1T}{2}+\sigma_1^2 4^{2\beta-3}MT. \tag{3.15}
$$

Since

$$
\begin{aligned}
\mathbb{E}[H(X(T\wedge\eta))]&=\mathbb{E}[H(X(T\wedge\rho_M\wedge\tau_N))]\ge\mathbb{E}[H(X(\tau_N))1_{[\tau_N\le T\wedge\rho_M]}]\\
&\ge[H(N^{-1})\wedge H(N)]\mathbb{P}(\tau_N\le T\wedge\rho_M),
\end{aligned}
\tag{3.16}
$$

we have

$$
[H(N^{-1})\wedge H(N)]\mathbb{P}(\tau_N\le T\wedge\rho_M)\le H(X_0)+\frac{\alpha_1\mu_1T}{2}+\frac{\alpha_1T}{2}+\sigma_1^2 4^{2\beta-3}MT, \tag{3.17}
$$

that is

$$
\mathbb{P}(\tau_N\le T\wedge\rho_M)\le\frac{H(X_0)+\frac{\alpha_2\mu_2T}{2}+\frac{\alpha_2T}{2}+\sigma_2^2 4^{2\beta-3}MT}{H(N^{-1})\wedge H(N)}. \tag{3.18}
$$

Now letting $N\to\infty$ we have $\mathbb{P}(\tau_\infty\le T\wedge\rho_M)=0$. Then letting $M\to\infty$ and using Lemma 3.1, we can get $\mathbb{P}(\tau_\infty\le T)=0$. This gives that $\mathbb{P}(\tau_\infty>T)=1$ which implies our assertion easily. □

3.3 Convergence in probability

The SDE model (3.6) has no explicit solution, hence the study of its numerical approximate solutions has become more and more useful. In this section we will investigate the EM numerical approximate solutions to the SDE model (3.6).

Euler-Maruyama approximation

Given the time step $\Delta \in (0,1)$, we let $t_k = k\Delta$ for $k = 0, 1, 2, 3, ..., \left[\frac{T}{\Delta}\right]$, where $\left[\frac{T}{\Delta}\right]$ denotes the integer part of $\frac{T}{\Delta}$. The discrete time EM approximate solution to the SDE model (3.6) can be defined by setting $x_0 = X(0)$, $v_0 = V(0)$ and forming

$$\begin{aligned} x_{k+1} &= x_k + \alpha_1(\mu_1 - x_k)\Delta + \sigma_1\sqrt{|v_k|}|x_k|^{\theta}\Delta W_{1k}, \\ v_{k+1} &= v_k + \alpha_2(\mu_2 - v_k)\Delta + \sigma_2|v_k|^{\beta}\Delta W_{2k}, \end{aligned} \tag{3.19}$$

where $\Delta W_{1k} = [W_1(t_{k+1}) - W_1(t_k)]$ and $\Delta W_{2k} = [W_2(t_{k+1}) - W_2(t_k)]$. The corresponding continuous EM approximate solution to this model is defined by

$$\begin{aligned} x(t) &= x_0 + \int_0^t \alpha_1(\mu_1 - \bar{x}(u))du + \int_0^t \sigma_1\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^{\theta}dW_1(u), \\ v(t) &= v_0 + \int_0^t \alpha_2(\mu_2 - \bar{v}(u))du + \int_0^t \sigma_2|\bar{v}(u)|^{\beta}dW_2(u), \end{aligned} \tag{3.20}$$

where $\bar{x}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} x_k 1_{[t_k,t_{k+1})}(t)$ and $\bar{v}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} v_k 1_{[t_k,t_{k+1})}(t)$ are step processes. That is, $\bar{x}(t) = x_k$ and $\bar{v} = v_k$ for $t \in [t_k, t_{k+1})$ when $k = 0, 1, 2, 3, ..., \left[\frac{T}{\Delta}\right]$.

Convergence of $v(t)$ in probability

In this chapter we are concerned with the case when both parameters θ and β are greater than 1, as the case $\frac{1}{2} \leq \theta, \beta \leq 1$ is proven by Mao et al [52]. So the diffusion coefficients of the SDE model (3.6) do not follow the linear growth condition although they obey the local Lipschitz condition. The existing results on the finite-time convergence of the EM approximate solutions can not be applied. It is therefore necessary to establish a new theory on the convergence property of the EM approximate solution to the SDE model (3.6). For this purpose, let us first discuss the convergence property for the volatility process.

Theorem 3.1. *Let $V(t)$ be the solution and $v(t)$ be continuous EM approximate solution to the second SDE of (3.6). For any positive number M, define the stopping time $q = \rho_M \wedge \gamma_M \wedge T$, where $\rho_M = \inf\{t \in [0,T]; V(t) \notin [\frac{1}{M}, M]\}$ and $\gamma_M = \inf\{t \in [0,T]; |v(t)| \notin [\frac{1}{M}, M]\}$. Then, for any integer $p \geq 2$,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[V(t\wedge q) - v(t\wedge q)]^2\right) \leq C_{1,2}(M,p)\Delta^{1-\frac{1}{p}}, \tag{3.21}$$

where $C_{1,2} = C_{1,2}(M,p)$ is a constant independent of Δ.

To prove this theorem, we need to establish a useful lemma which shows that the continuous EM approximate solution $v(t)$ and its step process $\bar{v}(t)$ are close to each other.

Lemma 3.3. *There exists a constant $C_{1,1}(M,p)$ dependent on M and p but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[v(t\wedge q) - \bar{v}(t\wedge q)]^2\right) \leq C_{1,1}(M,p)\Delta^{1-\frac{1}{p}}. \tag{3.22}$$

Proof. For $t \in [0,T]$, let $\left[\frac{t}{\Delta}\right]$ be the integer part of $\frac{t}{\Delta}$. Then we have

$$v(t\wedge q) - \bar{v}(t\wedge q) = \int_{[\frac{t\wedge q}{\Delta}]\Delta}^{t\wedge q} [\alpha_2(\mu_2 - \bar{v}(u))]\, du + \int_{[\frac{t\wedge q}{\Delta}]\Delta}^{t\wedge q} \sigma_2|\bar{v}(u)|^\beta dW_2(u), \tag{3.23}$$

which gives

$$[v(t\wedge q) - \bar{v}(t\wedge q)]^2 \leq 4\alpha_2^2(\mu_2^2 + M^2)\Delta^2 + 2\sigma_2^2 M^{2\beta}\left[W_2(t\wedge q) - W_2([\frac{t\wedge q}{\Delta}]\Delta)\right]^2.$$

We hence have

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leq t\leq T}[v(t\wedge q) - \bar{v}(t\wedge q)]^2\right) \\ &\leq 4\alpha_2^2(\mu_2^2 + M^2)\Delta^2 + 2\sigma_2^2 M^{2\beta}\mathbb{E}\left(\sup_{0\leq t\leq T\wedge q}\left[W_2(t) - W_2([\frac{t}{\Delta}]\Delta)\right]^2\right) \\ &\leq 4\alpha_2^2(\mu_2^2 + M^2)\Delta^2 + 2\sigma_2^2 M^{2\beta}\mathbb{E}\left(\sup_{0\leq t\leq T}\left[W_2(t) - W_2([\frac{t}{\Delta}]\Delta)\right]^2\right). \end{aligned} \tag{3.24}$$

By the Hölder inequality,

$$\mathbb{E}\left(\sup_{0\le t\le T}\left[W_2(t)-W_2(\Delta[\frac{t}{\Delta}])\right]^2\right)\le\left(\mathbb{E}\left(\sup_{0\le t\le T}\left[W_2(t)-W_2(\Delta[\frac{t}{\Delta}])\right]^{2p}\right)\right)^{\frac{1}{p}}. \tag{3.25}$$

Using the Doob martingale inequality, we get

$$\begin{aligned}\mathbb{E}\left(\sup_{0\le t\le T}\left[W_2(t)-W_2(\Delta[\frac{t}{\Delta}])\right]^{2p}\right)=&\mathbb{E}\left(\sup_{0\le k\le[\frac{T}{\Delta}]}\sup_{k\Delta\le r\le\Delta(k+1)}[W_2(r)-W_2(\Delta k)]^{2p}\right)\\ \le&\sum_{k=0}^{[\frac{T}{\Delta}]}\mathbb{E}\left[\sup_{k\Delta\le r\le\Delta(k+1)}|W_2(r)-W_2(\Delta k)|^{2p}\right]\\ \le&\left(\frac{2p}{2p-1}\right)^{2p}\sum_{k=0}^{[\frac{T}{\Delta}]}\mathbb{E}\left|W_2(\Delta(k+1))-W_2(\Delta k)\right|^{2p}\\ \le&\left(\frac{2p}{2p-1}\right)^{2p}\sum_{k=0}^{[\frac{T}{\Delta}]}(2p-1)!!\Delta^p\\ \le&\left(\frac{2p}{2p-1}\right)^{2p}(2p-1)!!\Delta^{p-1}(T+1),\end{aligned} \tag{3.26}$$

where $(2p-1)!!=(2p-1)\times(2p-3)\times\cdots 3\times 1$. Substituting (3.26) with (3.25) into (3.24) yields

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le T}[v(t\wedge q)-\bar{v}(t\wedge q)]^2\right)\\ \le&4\alpha_2(\mu_2^2+M^2)\Delta^2+2\sigma_2^2M^{2\beta}\left[\left(\frac{2p}{2p-1}\right)^{2p}(2p-1)!!\Delta^{p-1}(T+1)\right]^{\frac{1}{p}}\\ \le&C_{1,1}(M,p)\Delta^{1-\frac{1}{p}},\end{aligned} \tag{3.27}$$

as required. The proof of Lemma 3.3 is complete. □

Proof. (of Theorem 3.1)

For any $0\le t\le T$, we clearly have that

$$[V(t\wedge q)-v(t\wedge q)]^2\le 2\alpha_2^2\left[\int_0^{t\wedge q}(V(u)-\bar{v}(u))du\right]^2$$

$$+ 2\sigma_2^2 \left[\int_0^{t\wedge q} (|V(u)|^\beta - |\bar{v}(u)|^\beta) dW_2(u) \right]^2 .$$

For any $t_1 \in [0,T]$, by the Doob martingale inequality and the Hölder inequality, we then compute

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1} [V(t\wedge q) - v(t\wedge q)]^2 \right) \\
\le& 2\alpha_2^2 T\mathbb{E} \int_0^{t_1\wedge q} [V(u) - \bar{v}(u)]^2 \, du \\
&+ 2\sigma_2^2 \mathbb{E}\left(\sup_{0\le t\le t_1} \left[\int_0^{t\wedge q} (|V(u)|^\beta - |\bar{v}(u)|^\beta) dW_2(u) \right]^2 \right) \\
\le& 2\alpha_2^2 T\mathbb{E} \int_0^{t_1\wedge q} [V(u) - \bar{v}(u)]^2 \, du + 8\sigma_2^2 \mathbb{E} \int_0^{t_1\wedge q} \left[|V(u)|^\beta - |\bar{v}(u)|^\beta \right]^2 du \\
\le& 4\alpha_2^2 T\mathbb{E} \int_0^{t_1\wedge q} [V(u) - v(u)]^2 + [v(u) - \bar{v}(u)]^2 \, du \\
&+ 16\sigma_2^2 \mathbb{E} \int_0^{t_1\wedge q} \left[|V(u)|^\beta - |v(u)|^\beta \right]^2 du + \left[|v(u)|^\beta - |\bar{v}(u)|^\beta \right]^2 du.
\end{aligned} \tag{3.28}$$

Applying the well-known mean value theorem gives

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1} [V(t\wedge q) - v(t\wedge q)]^2 \right) \\
\le& 4\alpha_2^2 T\mathbb{E} \int_0^{t_1} [V(u\wedge q) - v(u\wedge q)]^2 \, du \\
&+ 4\alpha_2^2 T\mathbb{E} \int_0^{t_1} [v(u\wedge q) - \bar{v}(u\wedge q)]^2 \, du \\
&+ 16\sigma_2^2 \mathbb{E} \int_0^{t_1} \beta^2 M^{2\beta-2} [V(u\wedge q) - v(u\wedge q)]^2 \, du \\
&+ 16\sigma_2^2 \mathbb{E} \int_0^{t_1} \beta^2 M^{2\beta-2} [v(u\wedge q) - \bar{v}(u\wedge q)]^2 \, du.
\end{aligned} \tag{3.29}$$

By Lemma 3.3, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1} [V(t\wedge q) - v(t\wedge q)]^2 \right) \\
\le& \left(16\sigma_2^2 \beta^2 M^{2\beta-2} + 4\alpha_2^2 T \right) T C_{1,1}(M,p) \Delta^{1-\frac{1}{p}} \\
&+ \left(4\alpha_2^2 T + 16\sigma_2^2 \beta^2 M^{2\beta-2} \right) \int_0^{t_1} \mathbb{E} [V(u\wedge q) - v(u\wedge q)]^2 \, du.
\end{aligned} \tag{3.30}$$

An application of Gronwall's inequality (see Theorem 2.25) will complete the proof.

□

Now we remove the stopping time of volatility and establish the following theorem to show that the EM approximate solution will converge in probability to the true solution.

Theorem 3.2. *Let $V(t)$ be the true solution to the second SDE of (3.6) and $v(t)$ be its continuous EM approximate solution. Then*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}[V(t)-v(t)]^2\right)=0 \quad \textit{in probability.} \tag{3.31}$$

Proof. The proof is rather technical so we divide the whole proof into three steps.
Step 1. Using the technique to prove Lemma 3.1, but with the stopping time ρ_M, we derive that, for $t_1\in[0,T]$,

$$\mathbb{P}(\rho_M\le T)\le \frac{H(V_0)+\frac{\alpha_2\mu_2}{2}T+\frac{\alpha_2}{2}T+\sigma_2^2 4^{2\beta-3}T}{H(M^{-1})\wedge H(M)}, \tag{3.32}$$

where function $H(.)$ has been defined in Lemma 3.1.
Step 2. Applying the Itô formula for continuous EM approximate solution $v(t)$ with stopping time γ_M, we derive that, for $t_1\in[0,T]$,

$$\begin{aligned}
&\mathbb{E}\left[H(v(t_1\wedge\gamma_M))\right]\\
=&H(V_0)+\mathbb{E}\int_0^{t_1\wedge\gamma_M}H'(v(u))\alpha_2[\mu_2-\bar{v}(u)]du\\
&+\frac{\sigma_2^2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}H''(v(u))|\bar{v}(u)|^{2\beta}du\\
\le&H(V_0)+\mathbb{E}\int_0^{t_1\wedge\gamma_M}\frac{1}{2}[v(u)^{-\frac{1}{2}}-v(u)^{-1}]\alpha_2[\mu_2-\bar{v}(u)]du\\
&+\frac{\sigma_2^2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\frac{1}{2}[v(u)^{-2}-\frac{1}{2}v(u)^{-\frac{3}{2}}]|\bar{v}(u)|^{2\beta}du.
\end{aligned} \tag{3.33}$$

Rearranging the terms on the right hand side,

$$\begin{aligned}
\le&H(V_0)+\frac{\alpha_2\mu_2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}[v(u)^{-\frac{1}{2}}-v(u)^{-1}]du+\frac{\alpha_2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}[1-v(u)^{\frac{1}{2}}]du\\
&+\frac{\sigma_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M}[1-\frac{1}{2}v(u)^{\frac{1}{2}}]|v(u)|^{2\beta-2}du
\end{aligned}$$

$$+ \frac{\alpha_2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}[v(u)^{-\frac{1}{2}} - v(u)^{-1}][v(u) - \bar{v}(u)]du$$
$$+ \frac{\sigma_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M}[v(u)^{-2} - \frac{1}{2}v(u)^{-\frac{3}{2}}][|\bar{v}(u)|^{2\beta} - |v(u)|^{2\beta}]du.$$

By (3.8) and the well-known mean value theorem, we further get

$$\begin{aligned}
&\mathbb{E}\left[H(v(t_1\wedge\gamma_M))\right] \\
\le& H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} + \frac{\sigma_2^2 4^{2\beta-2}T}{4} \\
&+ \frac{\alpha_2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}[v(u)^{-\frac{1}{2}} - v(u)^{-1}][v(u) - \bar{v}(u)]du \\
&+ \frac{\sigma_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M}2\beta\sup[u]^{2\beta-1}[v(u)^{-2} - \frac{1}{2}v(u)^{-\frac{3}{2}}]\Big|\bar{v}(u) - v(u)\Big|du.
\end{aligned} \tag{3.34}$$

Note that $\bar{v}(u)\in[M^{-1},M]$ whenever $v(u)\in[M^{-1},M]$. We can then compute

$$\begin{aligned}
&\mathbb{E}\left[H(v(t_1\wedge\gamma_M))\right] \\
\le& H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} + \frac{\sigma_2^2 4^{2\beta-2}T}{4} \\
&+ \frac{\alpha_2}{2}\mathbb{E}\int_0^{t_1}[v(u\wedge\gamma_M)^{-\frac{1}{2}} - v(u\wedge\gamma_M)^{-1}][v(u\wedge\gamma_M) - \bar{v}(u\wedge\gamma_M)]du \\
&+ \frac{2\beta M^{2\beta-1}\sigma_2^2}{4}\mathbb{E}\int_0^{t_1}[v(u\wedge\gamma_M)^{-2} - \frac{1}{2}v(u\wedge\gamma_M)^{-\frac{3}{2}}] \\
&\qquad\times\Big|\bar{v}(u\wedge\gamma_M) - v(u\wedge\gamma_M)\Big|du,
\end{aligned} \tag{3.35}$$

which has the form

$$\begin{aligned}
&\mathbb{E}\left[H(v(t_1\wedge\gamma_M))\right] \\
\le& H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} + \frac{\sigma_2^2 4^{2\beta-2}T}{4} \\
&+ \left[\frac{[M^{\frac{1}{2}} + M]\alpha_2}{2}\right]\int_0^{t_1}\mathbb{E}|v(u\wedge\gamma_M) - \bar{v}(u\wedge\gamma_M)|du \\
&+ \left[\frac{[M^2 + \frac{1}{2}M^{\frac{3}{2}}]\beta M^{2\beta-1}\sigma_2^2}{2}\right]\int_0^{t_1}\mathbb{E}|v(u\wedge\gamma_M) - \bar{v}(u\wedge\gamma_M)|du.
\end{aligned} \tag{3.36}$$

In the meantime, in the same way as Lemma 3.3 was proved, we can compute

$$\mathbb{E}\left(\sup_{0\le t\le T}[v(u\wedge\gamma_M) - \bar{v}(u\wedge\gamma_M)]^2\right) \le C_{1,1}^*(M,p)\Delta^{1-\frac{1}{p}}, \tag{3.37}$$

where $C^*_{1,1}(M,p)$ is dependent on M and p but independent of Δ.

Substituting this (3.37) into (3.36) yields

$$
\begin{aligned}
&\mathbb{E}\left[H(v(t_1 \wedge \gamma_M))\right] \\
\le &H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} + \frac{\sigma_2^2 4^{2\beta-2} T}{4} \\
&+ \left[\frac{[M^{\frac{1}{2}} + M]\alpha_2}{2}\right] \left[C^*_{1,1}(M,p)\Delta^{1-\frac{1}{p}}\right]^{\frac{1}{2}} T \\
&+ \left[\frac{[M^2 + \frac{1}{2}M^{\frac{3}{2}}]\beta M^{2\beta-1}\sigma_2^2}{2}\right] \left[C^*_{1,1}(M,p)\Delta^{1-\frac{1}{p}}\right]^{\frac{1}{2}} T \\
\le &H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} + \frac{\sigma_2^2 4^{2\beta-2} T}{4} + \bar{C}_{1,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}.
\end{aligned}
\tag{3.38}
$$

If $(\gamma_M \le T)$, by a similar technique used to compute (3.12), we can derive

$$
\mathbb{P}(\gamma_M \le T) \le \frac{H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{\alpha_2 T}{2} + \frac{\sigma_2^2 4^{2\beta-2} T}{4} + \bar{C}_{1,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)}. \tag{3.39}
$$

Step 3. For arbitrarily small constants $\varepsilon > 0$ and $\delta \in (0,1)$, set

$$
\bar{\Omega}_1 = \left[\omega; \sup_{0\le t\le T} [V(t) - v(t)]^2 \ge \delta\right]. \tag{3.40}
$$

Then, we have

$$
\begin{aligned}
\delta\mathbb{P}(\bar{\Omega}_1 \cap (q \ge T)) =&\delta\mathbb{E}\left[I_{(\bar{\Omega}_1 \cap (q \ge T))}\right] \\
\le&\mathbb{E}\left[I_{(q\ge T)} \sup_{0\le t\le T} [V(t) - v(t)]^2\right] \\
\le&\mathbb{E}\left[\sup_{0\le t\le T\wedge q} [V(t) - v(t)]^2\right].
\end{aligned}
\tag{3.41}
$$

By Theorem 3.1,

$$
\mathbb{P}(\bar{\Omega}_1 \cap (q \ge T)) \le \frac{C_{1,2}(M,p)\Delta^{[1-\frac{1}{p}]}}{\delta}. \tag{3.42}
$$

On the other hand, we can compute

$$
\begin{aligned}
\mathbb{P}(\bar{\Omega}_1) &\le \mathbb{P}(\bar{\Omega}_1 \cap (q \ge T)) + \mathbb{P}(q \le T) \\
&\le \mathbb{P}(\bar{\Omega}_1 \cap (q \ge T)) + \mathbb{P}(\rho_M \le T) + \mathbb{P}(\gamma_M \le T).
\end{aligned}
\tag{3.43}
$$

Substituting results of (3.32), (3.39) and (3.42) into (3.43) yields

$$\begin{aligned}\mathbb{P}(\bar{\Omega}_1) \le & \frac{C_{1,2}(M,p)\Delta^{[1-\frac{1}{p}]}}{\delta} \\ & + \frac{2[H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{\alpha_2}{2}T + \sigma_2^2 4^{2\beta-3}T] + \bar{C}_{1,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)}.\end{aligned} \tag{3.44}$$

Now, choose M sufficiently large for

$$\frac{2[H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{\alpha_2}{2}T + \sigma_2^2 4^{2\beta-3}T]}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{2} \tag{3.45}$$

and then choose Δ sufficiently small for

$$\frac{C_{1,2}(M,p)\Delta^{[1-\frac{1}{p}]}}{\delta} + \frac{\bar{C}_{1,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{2}. \tag{3.46}$$

Hence, we have

$$\mathbb{P}\left(\sup_{0\le t\le T}\Big[V(t) - v(t)\Big]^2 \ge \delta\right) < \varepsilon, \tag{3.47}$$

which completes the proof of our theorem. □

Convergence of $x(t)$ in probability

We can now proceed to establish our key results which show the finite-time convergence property of the EM approximate solution to the true solution of the underlying asset price.

Theorem 3.3. *Let $X(t)$ be the solution and $x(t)$ be continuous EM approximate solution to the asset price. For any positive numbers N and M, define stopping time $s = q \wedge \tau_N \wedge \zeta_N \wedge T$, where q is the same as before while $\tau_N = \inf\{t \in [0,T] : X(t) \notin [\frac{1}{N}, N]\}$, $\zeta_N = \inf\{t \in [0,T] : |x(t)| \notin [\frac{1}{N}, N]\}$. Then, for any integer $p \ge 2$,*

$$\mathbb{E}\left(\sup_{0\le t\le T}[X(t\wedge s) - x(t\wedge s)]^2\right) \le C_{1,3}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{3.48}$$

where $C_{1,3}(M, N, p)$ *is a constant independent of* Δ*.*

The proof needs the following Lemma which can be proved in the same way as Lemma 3.3 was proved.

Lemma 3.4. *There exists a constant* $C_{1,4}(M, N, p)$ *dependent on* M*,* N *and* p *but independent of* Δ*, such that*

$$\mathbb{E}\left(\sup_{0\le t\le T}\left[x(t\wedge s)-\bar{x}(t\wedge s)\right]^2\right)\le C_{1,4}(M,N,p)\Delta^{1-\frac{1}{p}}. \tag{3.49}$$

Proof. (of Theorem 3.3)

For any $t \in [0, T]$, we have

$$\begin{aligned}
&\Big[X(t\wedge s)-x(t\wedge s)\Big]^2\\
\le&2\alpha_1^2\left[\int_0^{t\wedge s}(X(u)-\bar{x}(u))du\right]^2\\
&+2\sigma_1^2\left[\int_0^{t\wedge s}(|X(u)|^\theta\sqrt{V(u)}-|\bar{x}(u)|^\theta\sqrt{|\bar{v}(u)|})dW_1(u)\right]^2\\
\le&2\alpha_1^2T\int_0^{t\wedge s}\left[X(u)-\bar{x}(u)\right]^2du\\
&+2\sigma_1^2\left[\int_0^{t\wedge s}(|X(u)|^\theta\sqrt{V(u)}-|\bar{x}(u)|^\theta\sqrt{|\bar{v}(u)|})dW_1(u)\right]^2.
\end{aligned} \tag{3.50}$$

Hence, for any $t_1 \in [0, T]$, we further have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}\left[X(t\wedge s)-x(t\wedge s)\right]^2\right)\\
\le&2\alpha_1^2T\mathbb{E}\int_0^{t_1\wedge s}\left[X(u)-\bar{x}(u)\right]^2du\\
&+2\sigma_1^2\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge s}\left(|X(u)|^\theta\sqrt{V(u)}-|\bar{x}(u)|^\theta\sqrt{|\bar{v}(u)|}\right)dW_1(u)\right]^2\right).
\end{aligned} \tag{3.51}$$

By the Doob martingale inequality,

$$\begin{aligned}
\le&2\alpha_1^2T\mathbb{E}\int_0^{t_1\wedge s}\left[X(u)-\bar{x}(u)\right]^2du\\
&+8\sigma_1^2\mathbb{E}\int_0^{t_1\wedge s}\left[|X(u)|^\theta\sqrt{V(u)}-|\bar{x}(u)|^\theta\sqrt{|\bar{v}(u)|}\right]^2du,
\end{aligned} \tag{3.52}$$

which gives

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}[X(t\wedge s)-x(t\wedge s)]^2\right)\\
\le&2\alpha_1^2T\mathbb{E}\int_0^{t_1\wedge s}[X(u)-\bar{x}(u)]^2\,du\\
&+16\sigma_1^2M\mathbb{E}\int_0^{t_1\wedge s}\left[|X(u)|^\theta-|\bar{x}(u)|^\theta\right]^2du\\
&+16\sigma_1^2N^{2\theta}\mathbb{E}\int_0^{t_1\wedge s}\left[\sqrt{V(u)}-\sqrt{|\bar{v}(u)|}\right]^2du.
\end{aligned}
\tag{3.53}
$$

Using the mean value theorem, we then compute

$$
\begin{aligned}
\le&4\alpha_1^2T\int_0^{t_1}\mathbb{E}\,[X(u\wedge s)-x(u\wedge s)]^2\,du\\
&+4\alpha_1^2T\int_0^{t_1}\mathbb{E}\,[x(u\wedge s)-\bar{x}(u\wedge s)]^2\,du\\
&+32\sigma_1^2\theta^2N^{2\theta-2}M\int_0^{t_1}\mathbb{E}\,[X(u\wedge s)-x(u\wedge s)]^2\,du\\
&+32\sigma_1^2\theta^2N^{2\theta-2}M\int_0^{t_1}\mathbb{E}\,[x(u\wedge s)-\bar{x}(u\wedge s)]^2\,du\\
&+16\sigma_1^2N^{2\theta}\int_0^{t_1}\mathbb{E}\,|V(u\wedge s)-v(u\wedge s)|\,du\\
&+16\sigma_1^2N^{2\theta}\int_0^{t_1}\mathbb{E}\,|v(u\wedge s)-\bar{v}(u\wedge s)|\,du.
\end{aligned}
\tag{3.54}
$$

Substituting Lemma 3.3, Lemma 3.4 and Theorem 3.1 into (3.54) yields

$$
\begin{aligned}
\le&[4\alpha_1^2T+32\sigma_1^2\theta^2N^{2\theta-2}M]C_{1,4}(M,N,p)\Delta^{1-\frac{1}{p}}T\\
&+16\sigma_1^2N^{2\theta}T\left[C_{1,2}(M,p)^{\frac{1}{2}}+C_{1,1}(M,p)^{\frac{1}{2}}\right]\Delta^{\frac{1}{2}[1-\frac{1}{p}]}\\
&+(4\alpha_1^2T+32\sigma_1^2\theta^2N^{2\theta-2}M)\int_0^{t_1}\mathbb{E}\,[X(u\wedge s)-x(u\wedge s)]^2\,du.
\end{aligned}
\tag{3.55}
$$

By the Gronwall inequality, we have

$$
\begin{aligned}
\le&\Big([4\alpha_1^2T+32\sigma_1^2\theta^2N^{2\theta-2}M]C_{1,4}(M,N,p)\Delta^{1-\frac{1}{p}}T\\
&+16\sigma_1^2N^{2\theta}T[C_{1,2}(M,p)^{\frac{1}{2}}+C_{1,1}(M,p)^{\frac{1}{2}}]\Delta^{\frac{1}{2}[1-\frac{1}{p}]}\Big)\,e^{[(4\alpha_1^2T+32\sigma_1^2\theta^2N^{2\theta-2}M)T]}\\
\le&C_{1,3}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]},
\end{aligned}
\tag{3.56}
$$

as desired. The proof of Theorem 3.3 is therefore complete. □

In the following theorem we will remove the stopping time and show that the continuous EM approximate solution will converge in probability to the true solution.

Theorem 3.4. *Let $X(t)$ be the true solution of the SDE model (3.6) and $x(t)$ be the continuous EM approximate solution. Then*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|X(t)-x(t)|^2\right)=0 \quad \textit{in probability.} \tag{3.57}$$

Proof. Here we will also apply a similar technique to how Theorem 3.2 was proved. Thus, we divide the whole proof into three steps.

Step 1. By the same way as computation of (3.15), but with stopping time $s_1 = \tau_N \wedge \rho_M$, we compute that, for $t_1 \in [0,T]$,

$$\mathbb{E}\left[H(X(t_1\wedge s_1))\right] \le H(X_0) + \frac{\alpha_1\mu_1}{2}T + \frac{\alpha_1}{2}T + M\sigma_1^2 4^{2\theta-3}T, \tag{3.58}$$

where function $H(.)$ is same as before defined in Lemma 3.1.

If $(s_1 \le T)$, we further get

$$\begin{aligned} H(X_0) + \frac{\alpha_1\mu_1 T}{2} + \frac{\alpha_1 T}{2} + M\sigma_1^2 4^{2\theta-3}T &\ge \mathbb{E}\left[H(X(T\wedge s_1))\right] \\ &\ge \mathbb{E}\left[H(X(\tau_N))I_{(\tau_N<\rho_M)}I_{(\tau_N\wedge\rho_M\le T)}\right] \\ &\ge \left[H(N^{-1})\wedge H(N)\right]\mathbb{P}(\tau_N\le T), \end{aligned}$$

which gives

$$\mathbb{P}(\tau_N\le T) \le \frac{H(X_0)+\frac{\alpha_1\mu_1 T}{2}+\frac{\alpha_1 T}{2}+M\sigma_1^2 4^{2\theta-3}T}{H(N^{-1})\wedge H(N)}. \tag{3.59}$$

Step 2. Repeating the same procedure which is used in *(Step 2.)* of Theorem 3.2 but with the stopping time $s_2 = \zeta_N \wedge \gamma_M$, we further get that

$$\begin{aligned} &\mathbb{E}\left[H(x(t_1\wedge s_2))\right] \\ &\le H(X_0) + \frac{\alpha_1\mu_1 T}{2} + \frac{\alpha_1 T}{2} + \frac{\sigma_1^2 M 4^{2\theta-2}T}{4} + \bar{C}_{1,4}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}. \end{aligned} \tag{3.60}$$

where $\bar{C}_{1,4}(M, N, p)$ is dependent on M, N and p but independent of Δ.

If $(s_2 \leq T)$, the same way as (3.59) was obtained, we can easily compute

$$\mathbb{P}(\zeta_N \leq T) \leq \frac{H(X_0) + \frac{\alpha_1 \mu_1 T}{2} + \frac{\alpha_1 T}{2} + \frac{\sigma_1^2 M 4^{2\theta-2} T}{4} + \bar{C}_{1,4}(M, N, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)}. \quad (3.61)$$

Step 3. For arbitrarily small constants $\varepsilon > 0$ and $\delta \in (0, 1)$, set

$$\Omega_1 = \left[\omega; \sup_{0 \leq t \leq T} [X(t) - x(t)]^2 \geq \delta\right]. \quad (3.62)$$

By Theorem 3.3, we then easily get

$$\begin{aligned} \mathbb{P}(\Omega_1 \cap (s \geq T)) \leq & \frac{1}{\delta}\mathbb{E}\left(\sup_{0 \leq t \leq T \wedge s} [X(t) - x(t)]^2\right) \\ \leq & \frac{C_{1,3}(M, N, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}. \end{aligned} \quad (3.63)$$

Now, we compute

$$\begin{aligned} \mathbb{P}(\Omega_1) \leq & \mathbb{P}(\Omega_1 \cap (s \geq T)) + \mathbb{P}(s \leq T) \\ \leq & \mathbb{P}(\Omega_1 \cap (s \geq T)) + \mathbb{P}(\rho_M \leq T) + \mathbb{P}(\tau_N \leq T) \\ & + \mathbb{P}(\gamma_M \leq T) + \mathbb{P}(\zeta_N \leq T). \end{aligned} \quad (3.64)$$

Substituting (3.63),(3.32),(3.39),(3.59) and (3.61) into (3.64), we further get

$$\begin{aligned} \mathbb{P}(\Omega_1) \leq & \frac{C_{1,3}(M, N, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta} + \frac{H(V_0) + \frac{\alpha_2 \mu_2}{2} T + \frac{\alpha_2}{2} T + \sigma_2^2 4^{2\beta-3} T}{H(M^{-1}) \wedge H(M)} \\ & + \frac{H(V_0) + \frac{\alpha_2 \mu_2 T}{2} + \frac{\alpha_2 T}{2} + \frac{\sigma_2^2 4^{2\beta-2} T}{4} + \bar{C}_{1,1}(M, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} \\ & + \frac{H(X_0) + \frac{\alpha_1 \mu_1}{2} T + \frac{\alpha_1}{2} T + M\sigma_1^2 4^{2\theta-3} T}{H(N^{-1}) \wedge H(N)} \\ & + \frac{H(X_0) + \frac{\alpha_1 \mu_1 T}{2} + \frac{\alpha_1 T}{2} + \frac{\sigma_1^2 M 4^{2\theta-2} T}{4} + \bar{C}_{1,4}(M, N, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)}. \end{aligned} \quad (3.65)$$

Now, choose M sufficiently large for

$$2\left[\frac{H(V_0) + \frac{\alpha_2 \mu_2}{2} T + \frac{\alpha_2}{2} T + \sigma_2^2 4^{2\beta-3} T}{H(M^{-1}) \wedge H(M)}\right] < \frac{\varepsilon}{3}, \quad (3.66)$$

then choose N sufficiently large for

$$2\left[\frac{H(X_0)+\frac{\alpha_1\mu_1}{2}T+\frac{\alpha_1}{2}T+M\sigma_1^2 4^{2\theta-3}T}{H(N^{-1})\wedge H(N)}\right]<\frac{\varepsilon}{3} \tag{3.67}$$

and further choose Δ sufficiently small for

$$\left[\frac{C_{1,3}(M,N,p)}{\delta}+\frac{\bar{C}_{1,1}(M,p)}{H(M^{-1})\wedge H(M)}+\frac{\bar{C}_{1,4}(M,N,p)}{H(N^{-1})\wedge H(N)}\right]\Delta^{\frac{1}{2}[1-\frac{1}{p}]}<\frac{\varepsilon}{3}. \tag{3.68}$$

We then have

$$\mathbb{P}\left(\sup_{0\le t\le T}[X(t)-x(t)]^2\ge\delta\right)<\varepsilon, \tag{3.69}$$

which completes the proof of Theorem 3.4. □

Theorem 3.4 shows that the continuous EM approximate solution $x(t)$ will converge in probability to the true solution $X(t)$. However, the continuous EM approximate solution is in general not computable in practice but the EM step process $\bar{x}(t)$ is computable. It is therefore more useful to show that the EM step process $\bar{x}(t)$ will converge in probability to the true solution $X(t)$.

Theorem 3.5. *Let $X(t)$ be the true solution of the SDE model (3.6) and $\bar{x}(t)$ be the EM step process. Then*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|X(t)-\bar{x}(t)|\right)=0 \quad \textit{in probability.} \tag{3.70}$$

The proof of this theorem is based on the following lemma which shows that the continuous EM approximate solution $x(t)$ and the EM step process $\bar{x}(t)$ will converge in probability to each other.

Lemma 3.5. *Let $x(t)$ be the continuous EM approximate solution and $\bar{x}(t)$ be the EM step process. Then*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|x(t)-\bar{x}(t)|\right)=0 \quad \textit{in probability.} \tag{3.71}$$

Proof. In the same way as Lemma 3.3 was proved, we can show that

$$\mathbb{E}\left(\sup_{0\le t\le T}[x(t\wedge s_2)-\bar{x}(t\wedge s_2)]^2\right)\le C_{1,5}(M,N,p)\Delta^{1-\frac{1}{p}}. \tag{3.72}$$

For any arbitrarily small $\varepsilon>0$ and $\delta\in(0,1)$, set

$$\Omega_1^*=\left[\omega;\ \sup_{0\le t\le T}[x(t)-\bar{x}(t)]^2\ge\delta\right]. \tag{3.73}$$

It is easy to show that

$$\mathbb{P}(\Omega_1^*\cap(s_2\ge T))\le\frac{C_{1,5}(M,N,p)\Delta^{1-\frac{1}{p}}}{\delta}. \tag{3.74}$$

But

$$\begin{aligned}\mathbb{P}(\Omega_1^*)&\le\mathbb{P}(\Omega_1^*\cap(s_2\ge T))+\mathbb{P}(s_2\le T)\\&\le\mathbb{P}(\Omega_1^*\cap(s_2\ge T))+\mathbb{P}(\zeta_N\le T)+\mathbb{P}(\gamma_M\le T).\end{aligned} \tag{3.75}$$

Now, substituting (3.39), (3.61) and (3.74) into (3.75) yields that

$$\begin{aligned}\mathbb{P}(\Omega_1^*)\le&\frac{C_{1,5}(M,N,p)\Delta^{[1-\frac{1}{p}]}}{\delta}\\&+\frac{H(X_0)+\frac{\alpha_1\mu_1T}{2}+\frac{\alpha_1T}{2}+\frac{\sigma_1^2M4^{2\theta-2}T}{4}+\bar{C}_{1,4}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1})\wedge H(N)}\\&+\frac{H(V_0)+\frac{\alpha_2\mu_2T}{2}+\frac{\alpha_2T}{2}+\frac{\sigma_2^24^{2\beta-2}T}{4}+\bar{C}_{1,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1})\wedge H(M)}.\end{aligned} \tag{3.76}$$

Choose M sufficiently large such that

$$\frac{H(V_0)+\frac{\alpha_2\mu_2T}{2}+\frac{\alpha_2T}{2}+\frac{\sigma_2^24^{2\beta-2}T}{4}}{H(M^{-1})\wedge H(M)}<\frac{\varepsilon}{3}, \tag{3.77}$$

then choose N sufficiently large such that

$$\frac{H(X_0)+\frac{\alpha_1\mu_1T}{2}+\frac{\alpha_1T}{2}+\frac{\sigma_1^2M4^{2\theta-2}T}{4}}{H(N^{-1})\wedge H(N)}<\frac{\varepsilon}{3} \tag{3.78}$$

and further choose Δ sufficiently small such that

$$\left[\frac{C_{1,5}(M,N,p)\Delta^{[1-\frac{1}{p}]}}{\delta}+\frac{\bar{C}_{1,4}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1})\wedge H(N)}+\frac{\bar{C}_{1,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1})\wedge H(M)}\right]<\frac{\varepsilon}{3}. \tag{3.79}$$

We then have

$$\mathbb{P}\left(\sup_{0\le t\le T}[x(t)-\bar{x}(t)]^2 \ge \delta\right) < \varepsilon, \tag{3.80}$$

which gives the desired assertion of Lemma 3.5. □

Proof. (of Theorem 3.5)

Let $\varepsilon > 0$ and $\delta \in (0,1)$ be arbitrarily small. By Lemma 3.5 and Theorem 3.4, we see that for any sufficiently small step size Δ, we have

$$\mathbb{P}\left(\sup_{0\le t\le T}|x(t)-\bar{x}(t)| \ge \frac{\delta}{2}\right) < \frac{\varepsilon}{2} \quad \text{and} \quad \mathbb{P}\left(\sup_{0\le t\le T}|X(t)-x(t)| \ge \frac{\delta}{2}\right) < \frac{\varepsilon}{2}.$$

We then compute

$$\begin{aligned}
&\mathbb{P}\left(\sup_{0\le t\le T}|X(t)-\bar{x}(t)| \ge \delta\right)\\
\le&\mathbb{P}\left(\sup_{0\le t\le T}|X(t)-x(t)| + \sup_{0\le t\le T}|x(t)-\bar{x}(t)| \ge \delta\right)\\
\le&\mathbb{P}\left(\sup_{0\le t\le T}|X(t)-x(t)| + \sup_{0\le t\le T}|x(t)-\bar{x}(t)| \ge \delta, \sup_{0\le t\le T}|x(t)-\bar{x}(t)| \ge \frac{\delta}{2}\right)\\
&\quad + \mathbb{P}\left(\sup_{0\le t\le T}|X(t)-x(t)| + \sup_{0\le t\le T}|x(t)-\bar{x}(t)| \ge \delta, \sup_{0\le t\le T}|x(t)-\bar{x}(t)| \le \frac{\delta}{2}\right)\\
\le&\mathbb{P}\left(\sup_{0\le t\le T}|x(t)-\bar{x}(t)| \ge \frac{\delta}{2}\right) + \mathbb{P}\left(\sup_{0\le t\le T}|X(t)-x(t)| \ge \frac{\delta}{2}\right) < \varepsilon.
\end{aligned} \tag{3.81}$$

The proof is therefore complete. □

According to Theorem 3.5, it is now clear that the step process of the EM approximate solution to SDE model (3.6) will converge to the true solution when the time step is sufficiently small. Thus, let us choose initial condition ($X(0) = 10.5, V(0) = 3.25$), ($\theta = 1.2, \beta = 1.1$), coefficients of the SDE model (3.6) (see Table 3.1) and $\rho = 0.5$, and apply MATLAB® software (see Appendix A for code) to illustrate the behaviour of the EM approximate solution in practice (see Figure 3.1).

Table 3.1: Coefficients of the SDE model (3.6)

Case	Parameters				
SDE 1	$\theta = 1.2$	$X(0) = 10.5$	$\alpha_1 = 1.21$	$\mu_1 = 10.4$	$\sigma_1 = 0.05$
SDE 2	$\beta = 1.1$	$V(0) = 3.25$	$\alpha_2 = 2.3$	$\mu_2 = 2.13$	$\sigma_2 = 0.054$

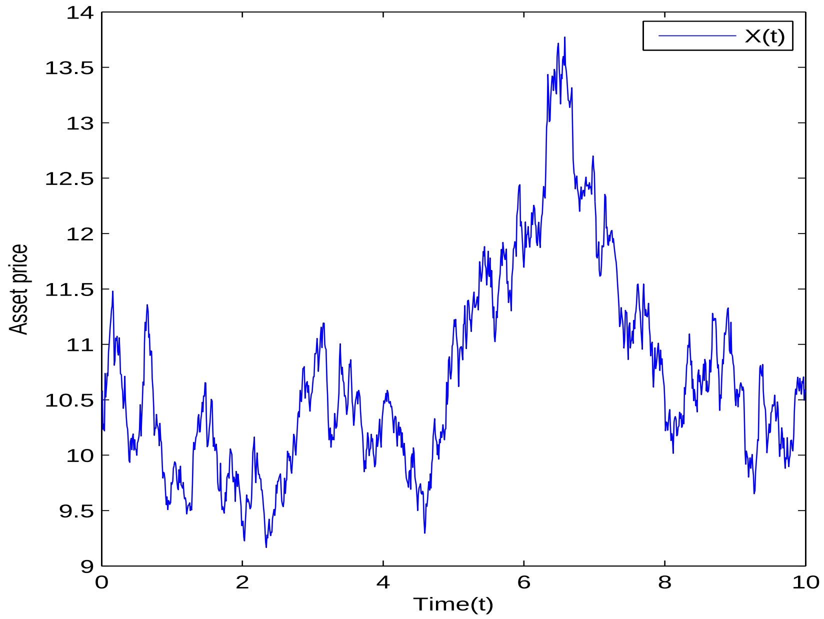

Figure 3.1: A sample path of the asset price $X(t)$ which is generated by the EM approximate solution to the mean-reverting-theta stochastic volatility model over finite time, where $\theta = 1.2$ and $\beta = 1.1$.

3.4 Summary

In this chapter, we have focussed on the EM approximate solution to the mean-reverting-theta stochastic volatility model for the asset price, which has no explicit solution so far. Thus, we have first proved that the unique local solution to SDE model (3.6) is positive with probability 1. However, we can not appeal to the convergence in second moment property of the EM approximate solution to this model under the local Lipschitz condition. Therefore, the convergence property of the EM approximate solution to this model has been examined in probability.

Finally, we have obtained the convergence in probability of the corresponding step process to show that it can be used to evaluate applications in finance.

Chapter 4

Hybrid Mean-Reverting-Theta Stochastic Volatility and Its Euler-Maruyama Approximation

4.1 Introduction

The mean-reverting-theta stochastic volatility model which was examined in the previous chapter gives a significant contribution to the evaluation of financial securities. However, many people have seen several deviations from this concept when upsurges have occurred. The empirical studies show that some of these fluctuations are dependent on the average rate of return and volatility of asset price. In the meantime, many authors have revealed that rate of return can not be a constant as is assumed by the Black-Scholes formula [8]. There is strong evidence to show that the rate of return obeys the property of a Markov-jump process and volatility follows this as well [16, 18, 27, 78, 9, 73, 75]. Therefore, the mean-reverting-theta stochastic volatility model driven by a Markov-jump process can be used to explain some of these phenomena in financial markets which have the SDE form:

$$\begin{aligned} dX(t) &= \alpha_1(r(t))(\mu_1(r(t)) - X(t))dt + \sigma_1(r(t))\sqrt{V(t)}X(t)^{\theta}dW_1(t), \\ dV(t) &= \alpha_2(r(t))(\mu_2(r(t)) - V(t))dt + \sigma_2(r(t))V(t)^{\beta}dW_2(t), \end{aligned} \tag{4.1}$$

where W_1 and W_2 are defined as before in the previous chapter, and $r(t)$ is a right-continuous Markov chain on the same probability space taking values in a finite state space $\mathbb{S} = \{1, 2, ..., N\}$. The diffusion coefficients of the SDE model (4.1), when the parameters θ and β are between $\frac{1}{2}$ and 1, obey the global Lipschitz condition (2.5) as well as satisfying the linear growth condition (2.6). Therefore, the convergence in second moment (in L^2) of Euler-Maruyama (EM) approximate solution to the SDE model (4.1) has been examined by X. Mao et al [52]. However, there is no information so far on the convergence property of the EM approximate solution to the SDE model (4.1), when the parameters θ and β are greater than 1. So we will fill this gap in this chapter.

The highly sensitive SDE model (4.1) describes the asset price, volatility and interest rate in financial markets. It is therefore necessary to have a non-negative solution in practice. Accordingly, we will first prove that the solution to the SDE model (4.1) will be positive with probability 1. Provided that the coefficients of SDE model (4.1) satisfy the local Lipschitz condition though do not obey the linear growth condition, an error bound for the EM approximate solution can be obtained under stopping time technique. Therefore, we will define the EM approximate solution to this SDE model and show convergence in probability of the EM approximate solution to the true solution when the time step is sufficiently small. However, the continuous EM approximate solution to this SDE model is not computable but its corresponding step process is computable in practice. Thus, the corresponding step process can be used to evaluate its applications in finance.

Therefore, we will finally show that the step process will converge in probability to the true solution of SDE model (4.1).

4.2 Non-negative solution

As the SDE model (4.1) mainly describes behaviour of the asset price and its volatility in financial markets, a natural requirement is to have non-negative solution $(V(t), X(t))$. The following lemmas in fact show that the solution will be positive with probability 1.

Lemma 4.1. *Assume $\beta > 1$, Then, for any given initial values $V(0) = V_0 > 0$ and $r(0) = i_0 \in \mathbb{S}$, the solution $V(t)$ to the second SDE of (4.1) will be positive for all $t \in [0, T]$ almost surely.*

Proof. Assume that the solution to the second SDE of (4.1) is in the real space $\mathbb{R}^2$ while setting coefficients in the second SDE of (4.1) to be 0 when $V(t) < 0$. In addition, the coefficients obey the local Lipschitz condition. Hence, there exists a unique maximal local solution $V(t)$ on $t \in [0, \rho_e]$, where ρ_e defined as before. For a sufficiently large positive value M, namely $\frac{1}{M} < V(0) < M$, define a stopping time $\rho_M = \rho_e \wedge \inf\{t \in [0, \rho_e] : |V(t)| \notin [\frac{1}{M}, M]\}$ and set $\rho_\infty = \lim_{M\to\infty} \rho_M$. Applying the Itô formula with C^2 function H which has been defined in Lemma 3.1 yields

$$
\begin{aligned}
&\mathbb{E}[H(V(T \wedge \rho_M))] \\
=&H(V_0) + \mathbb{E}\int_0^{T\wedge\rho_M} H^{'}(V(u))\alpha_2(r(u))[\mu_2(r(u)) - V(u)]du \\
&+ \frac{1}{2}\mathbb{E}\int_0^{T\wedge\rho_M} H^{''}(V(u))\sigma_2^2(r(u))|V(u)|^{2\beta}du \\
\le&H(V_0) + \mathbb{E}\int_0^{T\wedge\rho_M} \frac{1}{2}\left[(V(u))^{-\frac{1}{2}} - (V(u))^{-1}\right]\alpha_2(r(u))[\mu_2(r(u)) - V(u)]du \\
&+ \frac{1}{4}\mathbb{E}\int_0^{T\wedge\rho_M} \sigma_2^2(r(u))\left[(V(u))^{-2} - \frac{(V(u))^{-\frac{3}{2}}}{2}\right]|V(u)|^{2\beta}du
\end{aligned}
$$

$$\leq H(V_0) + \frac{\bar{\alpha}_2 \bar{\mu}_2 T}{2} + \frac{\bar{\alpha}_2 T}{2} + \frac{\bar{\sigma}_2^2}{4} \mathbb{E} \int_0^{T \wedge \rho_M} \left[1 - \frac{(V(u))^{\frac{1}{2}}}{2} \right] |V(u)|^{2\beta - 2} du,$$

where $\bar{\alpha}_j = \max_{i \in \mathbb{S}} \alpha_i$, $\bar{\mu}_j = \max_{i \in \mathbb{S}} \mu_i$ and $\bar{\sigma}_j = \max_{i \in \mathbb{S}} \sigma_i$.
By (3.8), we further get

$$\mathbb{E}[H(V(T \wedge \rho_M))] \leq H(V_0) + \frac{\bar{\alpha}_2 \bar{\mu}_2 T}{2} + \frac{\bar{\alpha}_2 T}{2} + \bar{\sigma}_2^2 4^{2\beta - 3} T. \tag{4.2}$$

Finally, applying the technique used to compute (3.12), we compute that

$$\mathbb{P}(\rho_M \leq T) \leq \frac{H(V_0) + \frac{\bar{\alpha}_2 \bar{\mu}_2 T}{2} + \frac{\bar{\alpha}_2 T}{2} + \bar{\sigma}_2^2 4^{2\beta - 3} T}{H(M^{-1}) \wedge H(M)}. \tag{4.3}$$

Now letting $M \to \infty$ we have $\mathbb{P}(\rho_\infty \leq T) = 0$. This implies that $\mathbb{P}(\rho_\infty > T) = 1$, which means $\mathbb{P}(0 < V(t) < \infty \ \forall \ t \in [0, T]) = 1$ as required. □

Lemma 4.2. *Assume $\theta > 1$ and $\beta > 1$. Then, for any given initial values $V(0) = V_0 > 0$, $X(0) = X_0 > 0$ and $r(0) = i_0 \in \mathbb{S}$, the solution $X(t)$ to the SDE of (4.1) will be positive for all $t \in [0, T]$ almost surely.*

Proof. In the same way, the SDE model of (4.1) is treated as an SDE in the real space $\mathbb{R}^2$ by setting its coefficients to be 0, when $V(t) < 0$ and $X(t) < 0$. Since the coefficients obey the local Lipschitz condition, there exists a unique maximal local solution $(X(t), V(t))$ on $t \in [0, \rho_e]$, where ρ_e defined as before.

For sufficiently large positive values M and N, namely $\frac{1}{M} < V(0) < M$ and $\frac{1}{N} < X(0) < N$, define stopping times $\rho_M = \rho_e \wedge \inf\{t \in [0, \rho_e] : V(t) \notin [\frac{1}{M}, M]\}$ and $\tau_N = \rho_e \wedge \inf\{t \in [0, \rho_e] : |X(t)| \notin [\frac{1}{N}, N]\}$ and let $\eta = \rho_M \wedge \tau_N$. Then set $\rho_\infty = \lim_{M \to \infty} \rho_M$ (as before) and $\tau_\infty = \lim_{N \to \infty} \tau_N$.
Applying the Itô formula with C^2 function H (same as before), we get that

$$\mathbb{E}[H(X(T \wedge \eta))]$$
$$= H(X_0) + \mathbb{E} \int_0^{T \wedge \eta} H'(X(u)) \alpha_1(r(u)) [\mu_1(r(u)) - X(u)] du$$

$$
\begin{aligned}
&+\frac{1}{2}\mathbb{E}\int_0^{T\wedge\eta} H''(X(u))\sigma_1^2(r(u))V(u)|X(u)|^{2\theta}du \\
\leq &H(X_0)+\mathbb{E}\int_0^{T\wedge\eta}\frac{1}{2}\left[(X(u))^{-\frac{1}{2}}-(X(u))^{-1}\right]\alpha_1(r(u))[\mu_1(r(u))-X(u)]du \\
&+\frac{1}{4}\mathbb{E}\int_0^{T\wedge\eta}\sigma_1^2(r(u))\left[(X(u))^{-2}-\frac{(X(u))^{-\frac{3}{2}}}{2}\right]V(u)|X(u)|^{2\theta}du,
\end{aligned}
$$

which gives

$$
\begin{aligned}
&\mathbb{E}[H(X(T\wedge\eta))] \\
\leq &H(X_0)+\frac{\bar{\alpha}_1\bar{\mu}_1T}{2}+\frac{\bar{\alpha}_1T}{2}+\frac{\bar{\sigma}_1^2}{4}\mathbb{E}\int_0^{T\wedge\eta}\left[1-\frac{(X(u))^{\frac{1}{2}}}{2}\right]|X(u)|^{2\theta-2}V(u)du.
\end{aligned}
\tag{4.4}
$$

By (3.8), we then have

$$
\mathbb{E}[H(X(T\wedge\eta))] \leq H(X_0)+\frac{\bar{\alpha}_1\bar{\mu}_1T}{2}+\frac{\bar{\alpha}_1T}{2}+\bar{\sigma}_1^2 4^{2\beta-3}MT. \tag{4.5}
$$

As the same way (3.18) was obtained, we further get that

$$
\mathbb{P}(\tau_N\leq T\wedge\rho_M)\leq\frac{H(X_0)+\frac{\bar{\alpha}_2\bar{\mu}_2T}{2}+\frac{\bar{\alpha}_2T}{2}+\bar{\sigma}_2^2 4^{2\beta-3}MT}{H(N^{-1})\wedge H(N)}. \tag{4.6}
$$

Now letting $N\to\infty$ we have $\mathbb{P}(\tau_\infty\leq T\wedge\rho_M)=0$. Then letting $M\to\infty$ and Lemma 4.1, we can get $\mathbb{P}(\tau_\infty\leq T)=0$. This gives that $\mathbb{P}(\tau_\infty>T)=1$ which follows our desired assertion. □

4.3 Convergence in probability

As we demonstrated, the unique solution $(V(t),X(t))$ to the SDE model is positive with probability 1. However, an explicit solution to the SDE model (4.1) can not be obtained within the existing theory. Therefore, we will establish an Euler-Maruyama numerical approximation to the SDE model (4.1) and examine its convergence property in the following section.

Euler-Maruyama approximation

Given time step $\Delta \in (0,1)$, we let $t_k = k\Delta$ and $r_k^\Delta = r(k\Delta)$ for $k = 0,1,2,...[\frac{T}{\Delta}]$, where $[\frac{T}{\Delta}]$ denotes the integer part of $\frac{T}{\Delta}$. Then set $x_0 = X(0)$, $v_0 = V(0)$ and $r_0^\Delta = \bar{r}(0) = i_0 \in \mathbb{S}$. The discrete time EM approximation to the SDE model (4.1) can be defined by

$$\begin{aligned} x_{k+1} &= x_k + \alpha_1(r_k^\Delta)(\mu_1(r_k^\Delta) - x_k)\Delta + \sigma_1(r_k^\Delta)\sqrt{|v_k|}|x_k|^\theta \Delta W_{1,k}, \\ v_{k+1} &= v_k + \alpha_2(r_k^\Delta)(\mu_2(r_k^\Delta) - v_k)\Delta + \sigma_2(r_k^\Delta)|v_k|^\beta \Delta W_{2,k}, \end{aligned} \tag{4.7}$$

where $\Delta = (t_{k+1} - t_k)$ and $\Delta W_{i,k} = (W_i(t_{k+1}) - W_i(t_k))$ for $i = 1,2$. Then, the corresponding continuous EM approximate solution can be defined by

$$\begin{aligned} x(t) &= x_0 + \int_0^t \alpha_1(\bar{r}(u))(\mu_1(\bar{r}(u)) - \bar{x}(u))du + \int_0^t \sigma_1(\bar{r}(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta dW_1(u), \\ v(t) &= v_0 + \int_0^t \alpha_2(\bar{r}(u))(\mu_2(\bar{r}(u)) - \bar{v}(u))du + \int_0^t \sigma_2(\bar{r}(u))|\bar{v}(u)|^\beta dW_2(u), \end{aligned} \tag{4.8}$$

where $\bar{x}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} x_k 1_{[t_k,t_{k+1})}(t)$ and $\bar{v}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} v_k 1_{[t_k,t_{k+1})}(t)$ are step functions. That is $\bar{x}(t) = x_k$, $\bar{v}(t) = v_k$ and $\bar{r}(t) = r_k^\Delta$ for $t \in [t_k, t_{k+1})$, $k = 0,1,2,3,...,[\frac{T}{\Delta}]$.

Convergence of $v(t)$ in probability

In this work, we concentrate on the SDE model (4.1) for underlying asset price where θ and β are greater than 1. The coefficients of this SDE model obey the local Lipschitz condition though do not satisfy the linear growth condition, and we can not appeal to finite time convergence within the existing results. Thus, we need new techniques to examine the convergence property of the EM approximate solution. Accordingly, the following theorem will establish a strong error bound of the EM approximate solution to the volatility under stopping time.

Theorem 4.1. *Let $V(t)$ be the solution and $v(t)$ be continuous EM approximate solution to the second SDE of (4.1). For any positive number M, define the stopping time $q = \rho_M \wedge \gamma_M \wedge T$, where $\rho_M = \inf\{t \in [0,T]; V(t) \notin [\frac{1}{M}, M]\}$ and $\gamma_M = \inf\{t \in [0,T]; |v(t)| \notin [\frac{1}{M}, M]\}$. Then, for any integer $p \geq 2$,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[V(t\wedge q) - v(t\wedge q)]^2\right) \leq C_{2,6}(M,p)\Delta^{1-\frac{1}{p}}, \tag{4.9}$$

where $C_{2,6} = C_{2,6}(M,p)$ is a constant independent of Δ.

To prove Theorem 4.1, we need the following lemma that shows the property for closeness of $v(t)$ and its step process $\bar{v}(t)$ when the time step is sufficiently small.

Lemma 4.3. *There is a constant $C_1(M,p)$ dependent on M and p but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[v(t\wedge q) - \bar{v}(t\wedge q)]^2\right) \leq C_{2,1}(M,p)\Delta^{1-\frac{1}{p}}. \tag{4.10}$$

Proof. For any $t \in [0,T]$, let $\left[\frac{t}{\Delta}\right]$ be the integer part of $\frac{t}{\Delta}$. We then have

$$\begin{aligned} &v(t\wedge q) - \bar{v}(t\wedge q) \\ =&\int_{[\frac{t\wedge q}{\Delta}]\Delta}^{t\wedge q} \alpha_2(\bar{r}(u))[\mu_2(\bar{r}(u)) - \bar{v}(u)]du + \int_{[\frac{t\wedge q}{\Delta}]\Delta}^{t\wedge q} \sigma_2(\bar{r}(u))|\bar{v}(u)|^{\beta}dW_2(u), \end{aligned} \tag{4.11}$$

which gives

$$\begin{aligned} &\Big[v(t\wedge q) - \bar{v}(t\wedge q)\Big]^2 \\ \leq& 4\bar{\alpha}_2^2(\bar{\mu}_2^2 + M^2)\Delta^2 + 2\bar{\sigma}_2^2 M^{2\beta}\left[W_2(t\wedge q) - W_2([\frac{t\wedge q}{\Delta}]\Delta)\right]^2. \end{aligned} \tag{4.12}$$

Taking expectation, we get

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leq t\leq T}[v(t\wedge q) - \bar{v}(t\wedge q)]^2\right) \\ \leq& 4\bar{\alpha}_2^2(\bar{\mu}_2^2 + M^2)\Delta^2 + 2\bar{\sigma}_2^2 M^{2\beta}\mathbb{E}\left(\sup_{0\leq t\leq T}\left[W_2(t\wedge q) - W_2([\frac{t\wedge q}{\Delta}]\Delta)\right]^2\right), \end{aligned} \tag{4.13}$$

Applying the technique used to compute (3.25) and (3.26), we further get that

$$\mathbb{E}\left(\sup_{0\le t\le T}\left[v(t\wedge q)-\bar{v}(t\wedge q)\right]^2\right)\le C_{2,1}(M,p)\Delta^{1-\frac{1}{p}}, \tag{4.14}$$

as expected. The proof of Lemma 4.3 is therefore complete. □

Proof. (of Theorem 4.1) For any $0\le t\le T$, we compute

$$\begin{aligned}
&\Big[V(t\wedge q)-v(t\wedge q)\Big]^2\\
\le&3\left[\int_0^{t\wedge q}\alpha_2(r(u))\mu_2(r(u))-\alpha_2(\bar{r}(u))\mu_2(\bar{r}(u))du\right]^2\\
&+3\left[\int_0^{t\wedge q}\alpha_2(r(u))V(r)-\alpha_2(\bar{r}(u))\bar{v}(r)du\right]^2\\
&+3\left[\int_0^{t\wedge q}\sigma_2(r(u))|V(r)|^\beta-\sigma_2(\bar{r}(u))|\bar{v}(r)|^\beta dW_2(u)\right]^2.
\end{aligned}\tag{4.15}$$

Taking expectation for $t_1\in[0,T]$, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}\left[V(t\wedge q)-v(t\wedge q)\right]^2\right)\\
\le&3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\alpha_2(r(u))\mu_2(r(u))-\alpha_2(\bar{r}(u))\mu_2(\bar{r}(u))du\right]^2\right)\\
&+3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\alpha_2(r(u))V(r)-\alpha_2(\bar{r}(u))\bar{v}(r)du\right]^2\right)\\
&+3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\sigma_2(r(u))|V(r)|^\beta-\sigma_2(\bar{r}(u))|\bar{v}(r)|^\beta dW_2(u)\right]^2\right).
\end{aligned}\tag{4.16}$$

By Hölder's inequality, we further get that

$$\begin{aligned}
\le&3T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r(u))\mu_2(r(u))-\alpha_2(\bar{r}(u))\mu_2(\bar{r}(u))\right]^2du\\
&+3T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r(u))V(r)-\alpha_2(\bar{r}(u))\bar{v}(r)\right]^2du\\
&+3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\sigma_2(r(u))|V(r)|^\beta-\sigma_2(\bar{r}(u))|\bar{v}(r)|^\beta dW_2(u)\right]^2\right).
\end{aligned}\tag{4.17}$$

Applying properties of the Markov-chain defined in Chapter 2 yields

$$A(t)=3T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r(u))\mu_2(r(u))-\alpha_2(\bar{r}(u))\mu_2(\bar{r}(u))\right]^2du$$

$$\leq 3T\mathbb{E}\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}} 4\bar{\alpha}_2^2\bar{\mu}_2^2(I_{r(u)\neq r(t_d)})du \leq 12\bar{\alpha}_2^2\bar{\mu}_2^2 T\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}} \mathbb{E}(I_{r(u)\neq r(t_d)})du,$$

which has the form

$$\begin{aligned}
A(t) \leq & 12\bar{\alpha}_2^2\bar{\mu}_2^2 T\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}} \mathbb{P}(r(u)\neq r(t_d))du \\
\leq & 12\bar{\alpha}_2^2\bar{\mu}_2^2 T\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}} \sum_{i\in\mathbb{S}}\mathbb{P}(r(t_d)=i)\mathbb{P}(r(u)\neq i/r(t_d)=i)du \\
\leq & 12\bar{\alpha}_2^2\bar{\mu}_2^2 T\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}} \sum_{i\in\mathbb{S}}\mathbb{P}(r(t_d)=i)\sum_{i\neq j}(\kappa_{ij}(u-t_d)+0(u-t_d))du \\
\leq & 12\bar{\alpha}_2^2\bar{\mu}_2^2 T\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}} [\max_{0\leq i\leq N}(-\kappa_{ii})\Delta+0(\Delta)]du \leq [C_{2,2}\Delta+0(\Delta)].
\end{aligned} \tag{4.18}$$

Similarly, we compute

$$\begin{aligned}
B(t) = & 3T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r(u))V(r)-\alpha_2(\bar{r}(u))\bar{v}(r)\right]^2du \\
\leq & 6T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r(u))-\alpha_2(\bar{r}(u))\right]^2V^2(u)du \\
& + 6T\mathbb{E}\int_0^{t_1\wedge q}\alpha_2^2(\bar{r}(u))\left[V(r)-\bar{v}(r)\right]^2du \\
\leq & 6TM^2\mathbb{E}\int_0^{t_1}[\alpha_2(r(u\wedge q))-\alpha_2(\bar{r}(u\wedge q))]^2du \\
& + 12T\bar{\alpha}_2^2\mathbb{E}\int_0^{t_1}[V(u\wedge q)-v(u\wedge q)]^2+[v(u\wedge q)-\bar{v}(u\wedge q)]^2du \\
\leq & 24T\bar{\alpha}_2^2M^2[\max_{0\leq i\leq N}(-\kappa_{ii})\Delta+0(\Delta)](T+1) \\
& + 12T\bar{\alpha}_2^2\int_0^{t_1}\mathbb{E}[V(u\wedge q)-v(u\wedge q)]^2+\mathbb{E}[v(u\wedge q)-\bar{v}(u\wedge q)]^2du.
\end{aligned} \tag{4.19}$$

By Lemma 4.3, we further get that

$$\begin{aligned}
B(t) \leq & [C_{2,3}\Delta+0(\Delta)]+12T^2\bar{\alpha}_2^2C_{2,1}(M,p)\Delta^{1-\frac{1}{p}} \\
& + 12T\bar{\alpha}_2^2\int_0^{t_1}\mathbb{E}[V(u\wedge q)-v(u\wedge q)]^2du \\
\leq & [C_{2,4}(M,p)\Delta^{1-\frac{1}{p}}+0(\Delta)]+12T\bar{\alpha}_2^2\int_0^{t_1}\mathbb{E}[V(u\wedge q)-v(u\wedge q)]^2du.
\end{aligned} \tag{4.20}$$

Applying the Doob martingale inequality and the well-known mean value theorem, we then have

$$
\begin{aligned}
D(t) =& 3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\sigma_2(r(u))|V(u)|^\beta - \sigma_2(\bar{r}(u))|\bar{v}(u)|^\beta dW_2(u)\right]^2\right)\\
\le& 12\mathbb{E}\int_0^{t_1\wedge q}\left[\sigma_2(r(u))|V(u)|^\beta - \sigma_2(\bar{r}(u))|\bar{v}(u)|^\beta\right]^2 du\\
\le& 24\mathbb{E}\int_0^{t_1\wedge q}\sigma_2^2(\bar{r}(u))[|V(u)|^\beta - |\bar{v}(u)|^\beta]^2 du\\
&+ 24\mathbb{E}\int_0^{t_1\wedge q}V^{2\beta}(u)[\sigma_2(r(u)) - \sigma_2(\bar{r}(u))]^2 du\\
\le& 48\mathbb{E}\int_0^{t_1\wedge q}\sigma_2^2(\bar{r}(u))\Big([|V(u)|^\beta - |v(u)|^\beta]^2 + [|v(u)|^\beta - |\bar{v}(u)|^\beta]^2\Big)du\\
&+ 24\mathbb{E}\int_0^{t_1\wedge q}V^{2\beta}(u)[\sigma_2(r(u)) - \sigma_2(\bar{r}(u))]^2 du\\
\le& 48\beta^2\bar{\sigma_2}^2M^{(2\beta-2)}\int_0^{t_1\wedge q}\mathbb{E}[V(u)-v(u)]^2 + \mathbb{E}[v(u)-\bar{v}(u)]^2 du\\
&+ 24\mathbb{E}\int_0^{t_1\wedge q}V^{2\beta}(u)[\sigma_2(r(u) - \sigma_2(\bar{r}(u))]^2 du.
\end{aligned}
\tag{4.21}
$$

By Lemma 4.3, applying the technique used to compute (4.18), we further get that

$$
\begin{aligned}
D(t) \le& 48\beta^2\bar{\sigma_2}^2M^{(2\beta-2)}\int_0^{t_1}\mathbb{E}[V(u\wedge q) - v(u\wedge q)]^2 du\\
&+ 48\beta^2\bar{\sigma_2}^2M^{(2\beta-2)}TC_{2,1}(M,p)\Delta^{1-\frac{1}{p}}\\
&+ 96\bar{\sigma_2}^2M^{2\beta}\Big[\max_{0\le i\le N}(-\kappa_{ii})\Delta + 0(\Delta)\Big](T+1)\\
=& [C_{2,5}(M,p)\Delta^{1-\frac{1}{p}} + 0(\Delta)]\\
&+ 48\beta^2\bar{\sigma_2}^2M^{(2\beta-2)}\int_0^{t_1}\mathbb{E}[V(u\wedge q) - v(u\wedge q)]^2 du.
\end{aligned}
\tag{4.22}
$$

Now, substituting (4.18),(4.20) and (4.22) into (4.17), we have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}[V(t\wedge q) - v(t\wedge q)]^2\right)\\
&\le \Big[C_{2,2}\Delta + C_{2,4}(M,p)\Delta^{1-\frac{1}{p}} + C_{2,5}(M,p)\Delta^{1-\frac{1}{p}} + 0(\Delta)\Big]\\
&\quad + \Big[12T\bar{\alpha}_2^2 + 48\beta^2\bar{\sigma_2}^2M^{(2\beta-2)}\Big]\int_0^{t_1}\mathbb{E}[V(u\wedge q) - v(u\wedge q)]^2 du.
\end{aligned}
\tag{4.23}
$$

as required. An application of Gronwall's inequality will therefore complete the proof of Theorem 4.1. $\square$

Now, we will remove the stopping time and establish the following theorem to show that the EM approximate solution $v(t)$ will converge in probability to the true solution .

Theorem 4.2. *Let $V(t)$ be the true solution of the second SDE model (4.1) and $v(t)$ be the continuous EM approximate solution. Then*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}[V(t)-v(t)]^2\right)=0 \quad \textit{in probability.} \tag{4.24}$$

Proof. The proof can be obtained the same way as Theorem 3.2 was proved, but with the conditions of $r(t)$. Thus, we will divide the whole proof into three steps.
Step 1. The same way as in computation of (4.3), we obtain

$$\mathbb{P}(\rho_M\le T)\le \frac{H(V_0)+\frac{\bar{\alpha}_2\bar{\mu}_2}{2}T+\frac{\bar{\alpha}_2}{2}T+\bar{\sigma}_2^2 4^{2\beta-3}T}{H(M^{-1})\wedge H(M)}, \tag{4.25}$$

where the function $H(.)$ is same as before.
Step 2. For any $0\le t_1\le T$, applying the Itô formula for continuous EM approximate solution of volatility process with stopping time γ_M, we compute that

$$\begin{aligned}
&\mathbb{E}\left[H(v(t_1\wedge\gamma_M))\right]\\
=&H(V_0)+\mathbb{E}\int_0^{t_1\wedge\gamma_M} H^{'}(v(u))\alpha_2(r(u))[\mu_2(r(u))-\bar{v}(u)]du\\
&+\frac{1}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M} H^{''}(v(u))\sigma_2^2(r(u))|\bar{v}(u)|^{2\beta}du\\
\le&H(V_0)+\mathbb{E}\int_0^{t_1\wedge\gamma_M}\frac{1}{2}\left[v(u)^{-\frac{1}{2}}-v(u)^{-1}\right]\alpha_2(r(u))\left[\mu_2(r(u))-\bar{v}(u)\right]du\\
&+\frac{1}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\frac{1}{2}\left[v(u)^{-2}-\frac{1}{2}v(u)^{-\frac{3}{2}}\right]\sigma_2^2(r(u))|\bar{v}(u)|^{2\beta}du.
\end{aligned} \tag{4.26}$$

Rearranging the terms in right hand side, we then have

$$\le H(V_0)+\frac{\bar{\alpha}_2\bar{\mu}_2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[v(u)^{-\frac{1}{2}}-v(u)^{-1}\right]du+\frac{\bar{\alpha}_2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[1-v(u)^{\frac{1}{2}}\right]du$$

$$
\begin{aligned}
&+ \frac{\bar{\sigma}_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M} \left[1 - \frac{1}{2}v(u)^{\frac{1}{2}}\right] |v(u)|^{2\beta-2}du \\
&+ \frac{\bar{\alpha}_2}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M} \left[v(u)^{-\frac{1}{2}} - v(u)^{-1}\right] (v(u) - \bar{v}(u))du \\
&+ \frac{\bar{\sigma}_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M} \left[v(u)^{-2} - \frac{1}{2}v(u)^{-\frac{3}{2}}\right] [|\bar{v}(u)|^{2\beta} - |v(u)|^{2\beta}]du.
\end{aligned}
$$

By similar techniques which were used to obtain (3.34) to (3.39), we get that

$$
\mathbb{P}(\gamma_M \le T) \le \frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{\bar{\alpha}_2 T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4} + \bar{C}_{2,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} \tag{4.27}
$$

where constant $\bar{C}_{2,1}(M,p)$ is dependent on M and p but independent of Δ.

Step 3. For arbitrarily small constants $\varepsilon > 0$ and $\delta \in (0,1)$, set

$$
\bar{\Omega}_2 = \left[\omega; \sup_{0\le t\le T}[V(t) - v(t)]^2 \ge \delta\right]. \tag{4.28}
$$

The same way as in computation of (3.42), but with Theorem 4.1, we obtain

$$
\mathbb{P}(\bar{\Omega}_2 \cap (q \ge T)) \le \frac{C_{2,6}(M,p)\Delta^{1-\frac{1}{p}}}{\delta}. \tag{4.29}
$$

On the other hand, we can derive

$$
\mathbb{P}(\bar{\Omega}_2) \le \mathbb{P}(\bar{\Omega}_2 \cap (q \ge T)) + \mathbb{P}(\rho_M \le T) + \mathbb{P}(\gamma_M \le T). \tag{4.30}
$$

Now, substituting (4.25), (4.29) and (4.27) into (4.30) yields

$$
\begin{aligned}
\mathbb{P}(\bar{\Omega}_2) \le &\frac{C_{2,6}(M,p)\Delta^{1-\frac{1}{p}}}{\delta} + \frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{\bar{\alpha}_2 T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4} + \bar{C}_{2,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} \\
&+ \frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2}{2}T + \frac{\bar{\alpha}_2}{2}T + \bar{\sigma}_2^2 4^{2\beta-3}T}{H(M^{-1}) \wedge H(M)},
\end{aligned}
$$

namely

$$
\mathbb{P}(\bar{\Omega}_2) \le \frac{C_{2,6}(M,p)\Delta^{1-\frac{1}{p}}}{\delta} + \frac{\bar{C}_{2,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} + \frac{2[H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{\bar{\alpha}_2 T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4}]}{H(M^{-1}) \wedge H(M)}.
$$

Now, choose M sufficiently large for

$$
2\left[\frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2}{2}T + \frac{\bar{\alpha}_2}{2}T + \bar{\sigma}_2^2 4^{2\beta-3}T}{H(M^{-1}) \wedge H(M)}\right] < \frac{\varepsilon}{2}, \tag{4.31}
$$

then further choose Δ sufficiently small for

$$\frac{C_{2,6}(M,p)\Delta^{1-\frac{1}{p}}}{\delta} + \frac{\bar{C}_{2,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{2}. \tag{4.32}$$

We then have

$$\mathbb{P}\left(\sup_{0\le t\le T}[V(t)-v(t)]^2 \ge \delta\right) < \varepsilon \tag{4.33}$$

as required. The proof is therefore complete. □

Convergence of $x(t)$ in probability

In this section, we proceed with these obtained results to compute the key result of this chapter, which gives convergence of the approximate solution to the underlying asset price. The following theorem first shows the strong error bound of the EM approximate solution of asset price with stopping time.

Theorem 4.3. *Let $X(t)$ be the true solution and $x(t)$ be continuous EM approximate solution of asset price. For any positive numbers N and M, define stopping time $s = q \wedge \tau_N \wedge \zeta_N \wedge T$, where q is same as before, $\tau_N = \inf\{t \in [0,T] : X(t) \notin [\frac{1}{N}, N]\}$ and $\zeta_N = \inf\{t \in [0,T] : |x(t)| \notin [\frac{1}{N}, N]\}$. Then, for any integer $p \ge 2$,*

$$\mathbb{E}\left(\sup_{0\le t\le T}[X(t\wedge s)-x(t\wedge s)]^2\right) \le C_{2,11}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{4.34}$$

where $C_{2,11}(M,N,p)$ is a constant independent of Δ.

To prove Theorem 4.3, we establish the following lemma that can be obtained in the same way as Lemma 4.3 was proved.

Lemma 4.4. *There is a constant $C_{2,7}(M,N,p)$ dependent on M, N and p but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\le t\le T}[x(t\wedge s)-\bar{x}(t\wedge s)]^2\right) \le C_{2,7}(M,N,p)\Delta^{1-\frac{1}{p}}. \tag{4.35}$$

Proof. (of Theorem 4.3) For any $0 \le t \le T$, we compute

$$\begin{aligned}
&[X(t\wedge s) - x(t\wedge s)]^2 \\
&\le 3\left[\int_0^{t\wedge s} \alpha_1(r(u))\mu_1(r(u)) - \alpha_1(\bar{r}(u))\mu_1(\bar{r}(u))du\right]^2 \\
&\quad + 3\left[\int_0^{t\wedge s} \alpha_1(r(u))X(u) - \alpha_1(\bar{r}(u))\bar{x}(u)du\right]^2 \\
&\quad + 3\left[\int_0^{t\wedge s} \sigma_1(r(u))\sqrt{V(u)}|X(u)|^\theta - \sigma_1(\bar{r}(u))\sqrt{|\bar{v}(u)|}\bar{x}(u)|^\theta dW_1(u)\right]^2.
\end{aligned} \tag{4.36}$$

Taking the expectation for any $t_1 \in [0, T]$, by the Hölder inequality and the Doob martingale inequality, we further compute

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1} [X(t\wedge s) - x(t\wedge s)]^2\right) \\
&\le 3T\mathbb{E}\int_0^{t_1\wedge s} [\alpha_1(r(u))\mu_1(r(u)) - \alpha_1(\bar{r}(u))\mu_1(\bar{r}(u))]^2 du \\
&\quad + 3T\mathbb{E}\int_0^{t_1\wedge s} [\alpha_1(r(u))X(u) - \alpha_1(\bar{r}(u))\bar{x}(u)]^2 du \\
&\quad + 12\mathbb{E}\int_0^{t_1\wedge s} \left[\sigma_1(r(u))\sqrt{V(u)}|X(u)|^\theta - \sigma_1(\bar{r}(u))\sqrt{|\bar{v}(u)|}\bar{x}(u)|^\theta\right]^2 du \\
&= K(t) + F(t) + G(t).
\end{aligned} \tag{4.37}$$

The same way as in computation of (4.18), we have

$$\begin{aligned}
K(t) =& 3T\mathbb{E}\int_0^{t_1\wedge s} [\alpha_1(r(u))\mu_1(r(u)) - \alpha_1(\bar{r}(u))\mu_1(\bar{r}(u))]^2 du \\
\le& [C_{2,8}\Delta + 0(\Delta)].
\end{aligned} \tag{4.38}$$

Similarly, we compute

$$\begin{aligned}
F(t) =& 3T\mathbb{E}\int_0^{t_1\wedge s} [\alpha_2(r(u))X(u) - \alpha_1(\bar{r}(u))\bar{x}(u)]^2 du \\
\le& 6T\mathbb{E}\int_0^{t_1\wedge s} [\alpha_1(r(u)) - \alpha_1(\bar{r}(u))]^2 X^2(u) + \alpha_1^2(\bar{r}(u))[X(u) - \bar{x}(u)]^2 du \\
\le& 24T\bar{\alpha}_1^2 M^2[\max_{0\le i\le N}(-\kappa_{ii})\Delta + 0(\Delta)](T+1) \\
&+ 12T\bar{\alpha}_1^2\int_0^{t_1} \mathbb{E}[X(u\wedge s) - x(u\wedge s)]^2 + \mathbb{E}[x(u\wedge s) - \bar{x}(u\wedge s)]^2 du.
\end{aligned}$$

Applying Lemma 4.4 yields

$$\begin{aligned}F(t) \leq &24T\bar{\alpha}_1^2 M^2[\max_{0\leq i\leq N}(-\kappa_{ii})\Delta + 0(\Delta)](T+1)\\&+ 12\bar{\alpha}_1^2\Big[T^2 C_{2,7}(M,N,p)\Delta^{1-\frac{1}{p}} + T\int_0^{t_1}\mathbb{E}[X(u\wedge s) - x(u\wedge s)]^2 du\Big]\\ \leq&[C_{2,9}(M,N,p)\Delta^{1-\frac{1}{p}} + 0(\Delta)]\\&+ 12T\bar{\alpha}_1^2\int_0^{t_1}\mathbb{E}[X(u\wedge s) - x(u\wedge s)]^2 du.\end{aligned} \tag{4.39}$$

Now, consider

$$\begin{aligned}G(t) =&12\mathbb{E}\int_0^{t_1\wedge s}\left[\sigma_1(r(u))\sqrt{V(u)}|X(u)|^\theta - \sigma_1(\bar{r}(u))\sqrt{|\bar{v}(u)|}\bar{x}(u)|^\theta\right]^2 du\\ \leq&36\mathbb{E}\int_0^{t_1\wedge s} V(u)X(u)^{2\theta}[\sigma_1(r(u)) - \sigma_1(\bar{r}(u))]^2 du\\&+ 36\mathbb{E}\int_0^{t_1\wedge s}\sigma_1^2(\bar{r}(u))X(u)^{2\theta}\left[\sqrt{V(u)} - \sqrt{|\bar{v}(u)|}\right]^2 du\\&+ 36\mathbb{E}\int_0^{t_1\wedge s}\sigma_1^2(\bar{r}(u))|\bar{v}(u)|\left[|X(u)|^\theta - |\bar{x}(u)|^\theta\right]^2 du.\end{aligned} \tag{4.40}$$

By the mean value theorem, applying the technique used to compute (4.18), we then have

$$\begin{aligned}G(t) \leq &36MN^{2\theta}[\max_{0\leq i\leq N}(-\kappa_{ii})\Delta + 0(\Delta)]T\\&+ 36\bar{\sigma}_1^2 N^{2\theta}\mathbb{E}\int_0^{t_1}|V(u\wedge s) - v(u\wedge s)| + |v(u\wedge s) - \bar{v}(u\wedge s)|du\\&+ 36\bar{\sigma}_1^2 M\mathbb{E}\int_0^{t_1}\left[|X(u\wedge s)|^\theta - |\bar{x}(u\wedge s)|^\theta\right]^2 du\\ \leq&36MN^{2\theta}[\max_{0\leq i\leq N}(-\kappa_{ii})\Delta + 0(\Delta)]T\\&+ 36\bar{\sigma}_1^2 N^{2\theta}\int_0^{t_1}\left(\mathbb{E}[v(u\wedge s) - \bar{v}(u\wedge s)]^2\right)^{\frac{1}{2}} du\\&+ 36\bar{\sigma}_1^2 N^{2\theta}\int_0^{t_1}\left(\mathbb{E}[V(u\wedge s) - v(u\wedge s)]^2\right)^{\frac{1}{2}} du\\&+ 72\bar{\sigma}_1^2 M\theta^2 N^{2\theta-2}\int_0^{t_1}\mathbb{E}[X(u\wedge s) - x(u\wedge s)]^2 du\\&+ 72\bar{\sigma}_1^2 M\theta^2 N^{2\theta-2}\int_0^{t_1}\mathbb{E}[x(u\wedge s) - \bar{x}(u\wedge s)]^2 du.\end{aligned} \tag{4.41}$$

Substituting Lemma 4.3, Lemma 4.4 and Theorem 4.1 into (4.41) yields

$$\begin{aligned}G(t) \leq & 36MN^{2\theta}[\max_{0\leq i\leq N}(-\kappa_{ii})\Delta + 0(\Delta)]T + 36\bar{\sigma}_1^2 N^{2\theta}[C_{2,1}(M,p)\Delta^{1-\frac{1}{p}}]^{\frac{1}{2}}T \\ & + 72\bar{\sigma}_1^2 M\theta^2 N^{2\theta-2}C_{2,7}(M,N,p)\Delta^{1-\frac{1}{p}}T + 36\bar{\sigma}_1^2 N^{2\theta}[C_{2,6}(M,p)\Delta^{1-\frac{1}{p}}]^{\frac{1}{2}}T \\ & + 72\bar{\sigma}_1^2 M\theta^2 N^{2\theta-2}\mathbb{E}\int_0^{t_1\wedge s}\mathbb{E}[X(u)-x(u)]^2 du \\ \leq & [C_{2,10}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]} + 0(\Delta)] \\ & + 72\bar{\sigma}_1^2 M\theta^2 N^{2\theta-2}\mathbb{E}\int_0^{t_1}\mathbb{E}[X(u\wedge s)-x(u\wedge s)]^2 du.\end{aligned} \tag{4.42}$$

Now, substituting $F(t)$, $G(t)$ and $K(t)$ into (4.37) we get

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\leq t\leq t_1}[X(t\wedge s)-x(t\wedge s)]^2\right) \\ &\leq [C_{2,8}\Delta + 0(\Delta)] + \left[C_{2,9}(M,N,p)\Delta^{1-\frac{1}{p}} + C_{2,10}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]} + 0(\Delta)\right] \\ &\quad + [12T\bar{\alpha}_1^2 + 72\bar{\sigma}_1^2 M\theta^2 N^{2\theta-2}]\int_0^T \mathbb{E}[X(u\wedge s)-x(u\wedge s)]^2 du.\end{aligned} \tag{4.43}$$

as desired. The proof of our theorem follows finally from Gronwall's inequality. □

The following theorem will be established to show that the continuous EM approximation will converge in probability to the true solution of asset price. In this process, we will remove the stopping time condition.

Theorem 4.4. *Let $X(t)$ be the true solution of asset price and $x(t)$ be the its continuous EM approximate solution. Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\leq t\leq T}[X(t)-x(t)]^2\right) = 0 \quad \textit{in probability.} \tag{4.44}$$

Proof. In this process, we also divide the whole proof into three steps.

Step 1. In the same way as in computation of (4.5), by a similar technique as was used to compute (3.59), we have

$$\mathbb{P}(\tau_N \leq T) \leq \frac{H(X_0) + \frac{\bar{\alpha}_1\bar{\mu}_1}{2}T + \frac{\bar{\alpha}_1}{2}T + M\bar{\sigma}_1^2 4^{2\theta-3}T}{H(N^{-1})\wedge H(N)}. \tag{4.45}$$

Step 2. By a similar technique used to compute (4.27) but with the EM approximate solution to the asset price, we obtain

$$\mathbb{P}(\zeta_N \le T) \le \frac{H(X_0) + \frac{\bar{\alpha}_1\bar{\mu}_1 T}{2} + \frac{\bar{\alpha}_1 T}{2} + \frac{\bar{\sigma}_1^2 M 4^{2\theta-2} T}{4} + \bar{C}_{2,7}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)}. \quad (4.46)$$

where $\bar{C}_{2,7}(M,N,p)$ is a constant dependent on M, N and p but independent of Δ.

Step 3. For arbitrarily small $\varepsilon > 0$ and $\delta \in (0,1)$, set

$$\Omega_2 = \left(\omega;\ \sup_{0\le t\le T}[X(t) - x(t)]^2 \ge \delta\right). \quad (4.47)$$

By Theorem 4.3, in the same way as in computation of (3.42), we further get that

$$\mathbb{P}(\Omega_2 \cap (s \ge T)) \le \frac{C_{2,11}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}. \quad (4.48)$$

In the meantime, we have

$$\begin{aligned}\mathbb{P}(\Omega_2) \le &\mathbb{P}(\Omega_2 \cap (s \ge T)) \\ &+ \mathbb{P}(\rho_M \le T) + \mathbb{P}(\gamma_M \le T) + \mathbb{P}(\tau_N \le T) + \mathbb{P}(\zeta_N \le T).\end{aligned} \quad (4.49)$$

Substituting (4.25), (4.27), (4.45), (4.46) and (4.48) into (4.49), yields

$$\begin{aligned}\mathbb{P}(\Omega_2) \le &\frac{C_{2,11}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta} + \frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2}{2}T + \frac{\bar{\alpha}_2}{2}T + \bar{\sigma}_2^2 4^{2\beta-3}T}{H(M^{-1}) \wedge H(M)} \\ &+ \frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{\bar{\alpha}_2 T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4} + \bar{C}_{2,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} \\ &+ \frac{H(X_0) + \frac{\bar{\alpha}_1\bar{\mu}_1}{2}T + \frac{\bar{\alpha}_1}{2}T + M\bar{\sigma}_1^2 4^{2\theta-3}T}{H(N^{-1}) \wedge H(N)} \\ &+ \frac{H(X_0) + \frac{\bar{\alpha}_1\bar{\mu}_1 T}{2} + \frac{\bar{\alpha}_1 T}{2} + \frac{\bar{\sigma}_1^2 M 4^{2\theta-2}T}{4} + \bar{C}_{2,7}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)}.\end{aligned} \quad (4.50)$$

Now, choose M sufficiently large for

$$2\left[\frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2}{2}T + \frac{\bar{\alpha}_2}{2}T + \bar{\sigma}_2^2 4^{2\beta-3}T}{H(M^{-1}) \wedge H(M)}\right] < \frac{\varepsilon}{3}, \quad (4.51)$$

then choose N sufficiently large for

$$2\left[\frac{H(X_0)+\frac{\bar{\alpha}_1\bar{\mu}_1}{2}T+\frac{\bar{\alpha}_1}{2}T+M\bar{\sigma}_1^2 4^{2\theta-3}T}{H(N^{-1})\wedge H(N)}\right]<\frac{\varepsilon}{3} \tag{4.52}$$

and further choose Δ sufficiently small for

$$\left[\frac{C_{2,11}(M,N,p)}{\delta}+\frac{\bar{C}_{2,1}(M,p)}{H(M^{-1})\wedge H(M)}+\frac{\bar{C}_{2,7}(M,N,p)}{H(N^{-1})\wedge H(N)}\right]\Delta^{\frac{1}{2}[1-\frac{1}{p}]}<\frac{\varepsilon}{3}. \tag{4.53}$$

We then have

$$\mathbb{P}\left(\sup_{0\le t\le t_1}[X(t)-x(t)]^2\ge\delta\right)<\varepsilon. \tag{4.54}$$

This completes the required proof of our Theorem. □

In practice, the EM approximate solution to the asset price is not computable though its corresponding step process is computable. Therefore we will establish the following theorem to show that the step process will converge to the real solution when the time step is sufficiently small.

Theorem 4.5. *Let $X(t)$ be the true solution of the SDE model (4.1) and $\bar{x}(t)$ be the EM step process. Then*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|X(t)-\bar{x}(t)|\right)=0 \quad \textit{in probability.} \tag{4.55}$$

Repeating the same technique which was used to prove Theorem 3.5, a proof of Theorem 4.5 can be obtained.

Theorem 4.4 shows that the EM approximate solution will converge to the true solution of SDE model (4.1) when the time step is sufficiently small. Therefore, we will choose initial condition ($X(0)=1.4, V(0)=0.24$), ($\theta=1.2, \beta=1.1$), $\rho=0.8$, generator of Markov chain $\Gamma_{3,3}$ and coefficients of the SDE model (4.1) (see Table 4.1) to demonstrate its behaviour in practice. Accordingly, we apply MATLAB® software (see Appendix A for code) to obtain the following graph (see Figure 4.1).

$$\Gamma_{3,3} = \begin{pmatrix} -4 & 3 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix}$$

.

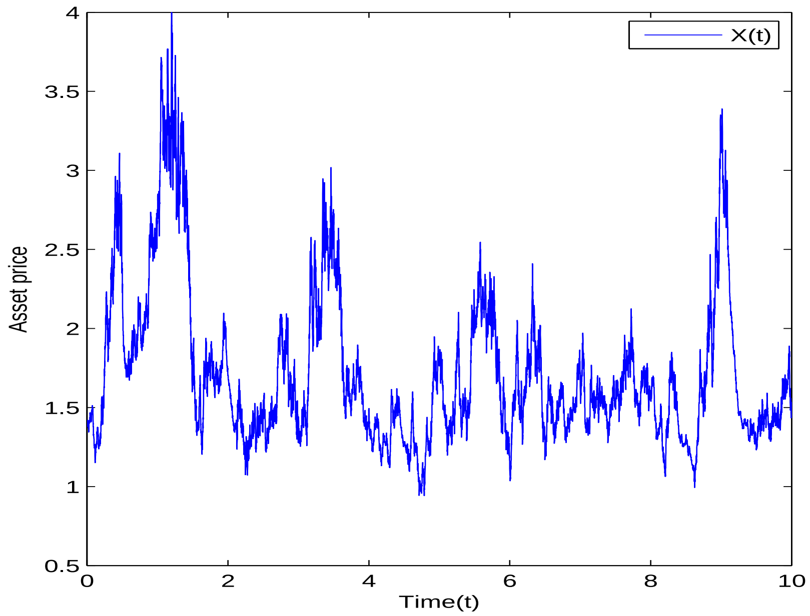

Figure 4.1: A sample path of the asset price $X(t)$ which is generated by the EM approximate solution to the hybrid mean-reverting-theta stochastic volatility model over finite time, where $\theta = 1.2$ and $\beta = 1.1$.

Table 4.1: Coefficients of the SDE model (4.1)

State	(α_1, α_2)	(μ_1, μ_2)	(σ_1, σ_2)
1	$(1.2, 1)$	$(1, 0.2)$	$(0.6, 0.84)$
2	$(4, 4)$	$(2, 0.7)$	$(0.9, 0.6)$
3	$(7, 1.6)$	$(1, 0.3)$	$(0.24, 0.56)$

4.4 Summary

The mean-reverting-theta stochastic volatility model driven by a Markov chain for asset price gives a more generalized formula of the SDE model (3.6). Even though it can be seen in financial markets, an explicit solution can not be obtained within the existing theories. Therefore, we have first proved that the SDE model has a local unique positive solution with probability 1. Then, an EM approximate solution to this mode has been established and we obtained a strong error bound on this approximate solution under the local Lipschitz condition. Finally, we have shown that the EM approximate solution will converge in probability to the true solution and obtained the convergence in probability of its corresponding step process to examine its applications in finance.

Chapter 5

Mean-Reverting-Theta Stochastic Volatility Model with Poisson Jump

5.1 Introduction

The values of financial securities affected by some unpredictable random disturbances gain quite a lot of attention due to their jittery movements in financial markets. Empirical studies show that these financial quantities which give abrupt structural changes do not perform as assumed by the Black-Scholes type formulas. However, mean-reverting-theta stochastic volatility models which were discussed in previous chapters can not be used to examine these jumpy situations. On the other hand, pragmatic studies of asset price modelling reveal that jump processes can be used to explain some of these phenomena [1, 45, 48]. Thus, generalization of existing financial models along with jump processes helps to make appropriate stochastic models which describe the behaviour of the underlying asset price when unpredictable disturbances are present. In other words, a mean-reverting-theta stochastic volatility model driven by a Poisson jump process can be considered as one such model that has the SDE form:

$$\begin{aligned} dX(t) &= \alpha_1(\mu_1 - X(t^-))dt + \sigma_1\sqrt{V(t^-)}X(t^-)^\theta dW_1(t) + \delta_1 X(t^-)d\bar{N}_1(t), \\ dV(t) &= \alpha_2(\mu_2 - V(t^-))dt + \sigma_2 V(t^-)^\beta dW_2(t) + \delta_2 V(t^-)d\bar{N}_2(t), \end{aligned} \tag{5.1}$$

where $V(t^-) = \lim_{u\uparrow t} V(t)$ and $X(t^-) = \lim_{u\uparrow t} X(t)$, W_1 and W_2 are two scalar Brownian motions (as before). Moreover $\alpha_i, \mu_i, \sigma_i$ and $(\delta_i > -1)$ are assumed to be constants, while $(\alpha_i + \lambda_i\delta_i) > 0$ for $i = 1, 2$. Further, we let $\bar{N}_i(t)$ to be a compensated Poisson process which has the form $\bar{N}_i(t) = N_i(t) - \lambda_i t$, where $N_i(t)$ is a Poisson process with intensity λ_i and $\bar{N}_i$ and W_i are independent for $i = 1, 2$.

Since this SDE model has no explicit solution like the Black-Scholes formula [8], numerical techniques have become a more popular and powerful tool to find its approximate solution. More precisely, the Euler-Maruyama (EM) numerical scheme can be used to approximate a solution to the SDE model (5.1).

In the case of $\theta = 1$, convergence of the EM approximate solution to the SDE model (5.1) has been established by F. Wu et al [76] for constant volatility. However, there is no information so far for numerical approximate solution to the SDE model (5.1) when $\frac{1}{2} \leq \theta, \beta \leq 1$. In this chapter we will fill this gap.

As the SDE model (5.1) describes asset price, interest rate and volatility in the financial markets, its solution should be non-negative in practice. Therefore, we will first prove that the solution to the SDE model (5.1) will be non-negative with probability 1. Given that the diffusion coefficients of the SDE model (5.1) not only satisfy the global Lipschitz condition but also the linear growth condition, strong error bounds of the approximate solution to this model can be obtained. Thus, we will define the EM approximate solution to this SDE model and establish upper bounds for the expected value of volatility and asset price by applying the generalized Itô-Doeblin formula. We will then establish a strong error bound (in L^2) for this EM approximate solution and show that the EM approximate solution will converge to the true solution as the time step goes to zero. In general, the continuous EM approximate solution is not computable though its corresponding step process is computable. Therefore, we will finally establish the convergence property of the corresponding step process to show that it can be used to examine applications of the EM approximate solution in finance.

5.2 Non-negative solution

The SDE model (5.1) describes underlying asset price, volatility and interest rate in financial markets, so a natural requirement is that the solution $(X(t), V(t))$ to this SDE model should be non-negative. Hence, the following lemmas show that the solution to this SDE model will be non-negative with probability 1.

Lemma 5.1. *Assume $\frac{1}{2} \leq \beta \leq 1$. Then, given any initial value $V(0) = V_0 > 0$, the second SDE of (5.1) has unique global solution $V(t)$ which will be non-negative for all $t \in [0, T]$ almost surely.*

Proof. It is sufficient to show that SDE, $dV(t) = \alpha_2(\mu_2 - V(t^-))dt + \sigma_2|V(t^-)|^\beta dW_2(t) + \delta_2 V(t^-)d\bar{N}_2(t)$, which has unique global solution $\forall\ t \geq 0$, is non-negative with probability 1.

To show this, let $a_0 = 1$ and for each integer $k = 1, 2, 3, ...,$

$$a_k = \begin{cases} e^{-\frac{k(k+1)}{2}} & ; \quad if \quad \beta = \frac{1}{2}, \\ \left[\frac{(2\beta-1)k(k+1)}{2}\right]^{\frac{1}{(1-2\beta)}} & ; \quad if \quad \frac{1}{2} < \beta \leq 1, \end{cases}$$

so that

$$\int_{a_k}^{a_{k-1}} \frac{1}{u^{2\beta}} du = k.$$

For each $k = 0, 1, 2, 3, ...$, there exists a continuous function $\psi_k(u)$ with support in (a_k, a_{k-1}) such that $0 \leq \psi_k(u) \leq \frac{2}{ku^{2\beta}}$ for $a_k < u < a_{k-1}$ and $\int_{a_k}^{a_{k-1}} \frac{1}{ku^{2\beta}} du = 1$. Define $\varphi_k(d) = 0$ for $d \geq 0$ and

$$\varphi_k(d) = \int_0^{-d} dy \int_0^y \psi_k(u) du \ \text{ for } \ d < 0.$$

Then it is very easy to observe that $\varphi_k(d) \in C^2(\mathbb{R}, \mathbb{R})$ has the following properties:

(i) $-1 \leq \varphi_k^{'}(d) \leq 0$ for $a_k < |d| < a_{k-1}$ or otherwise $\varphi_k^{'}(d) = 0$;

(ii) $|\varphi_k^{''}(d)| \leq \frac{2}{k|d|^{2\beta}}$ for $a_k < |d| < a_{k-1}$ or otherwise $\varphi_k^{'}(d) = 0$;

(iii) $d^- - a_{k-1} \leq \varphi_k(d) \leq d^-$ for all $d \in \mathbb{R}$, where $d^- = -d$ if $d < 0$ or otherwise $d^- = 0$.

Now for any $t \in [0, T]$, by the Itô-Doeblin formula for one jump process [70], we have

$$\begin{aligned}
\mathbb{E}[\varphi_k(V(t))] =& \varphi_k(V_0) + \mathbb{E}\int_0^t \varphi_k^{'}(V(u^-))[\alpha_2\mu_2 - (\alpha_2 + \lambda_2\delta_2)V(u^-)]du \\
&+ \frac{1}{2}\mathbb{E}\int_0^t \varphi_k^{''}(V(u^-))\sigma_2^2|V(u^-)|^{2\beta}du \\
&+ \mathbb{E}\int_0^t \varphi_k[(1+\delta_2)V(u^-)] - \varphi_k[V(u^-)]dN_2(u).
\end{aligned} \tag{5.2}$$

Here, we have replaced $V(u^-)$ by $V(u)$, because this will not have any effect on the Lebesgue integrals involved. This property will be called throughout this thesis where necessary. Applying the mean value theorem and properties of the function $\varphi_k(.)$ yields

$$\begin{aligned}
\mathbb{E}[\varphi_k(V(t))] \leq& \frac{\sigma_2^2 T}{k} + \lambda_2\mathbb{E}\int_0^t \sup_{s\in\mathbb{R}} |\varphi_k^{'}(s)||\delta_2 V(u)|du \\
\leq& \frac{\sigma_2^2 T}{k} + \lambda_2|\delta_2|\mathbb{E}\int_0^t \varphi_k(V(u)) + a_{k-1}du \\
\leq& \frac{\sigma_2^2 T}{k} + \lambda_2|\delta_2|\mathbb{E}\int_0^t \varphi_k(V(u))du + \lambda_2|\delta_2|a_{k-1}T.
\end{aligned} \tag{5.3}$$

By Gronwall's inequality, we further get that

$$\mathbb{E}[\varphi_k(V(t))] \leq \left[\frac{\sigma_2^2 T}{k} + \lambda_2|\delta_2|a_{k-1}T\right] \exp\left(\lambda_2|\delta_2|T\right). \tag{5.4}$$

Using the (iii) property of $\varphi_k(.)$, it is very easy to observe that

$$\mathbb{E}[V^-(t)] - a_{k-1} \leq \left[\frac{\sigma_2^2 T}{k} + \lambda_2|\delta_2|a_{k-1}T\right] \exp\left(\lambda_2|\delta_2|T\right), \tag{5.5}$$

namely

$$\mathbb{E}[V^-(t)] \leq \left[\frac{\sigma_2^2 T}{k} + \lambda_2|\delta_2|a_{k-1}T\right] \exp\left(\lambda_2|\delta_2|T\right) + a_{k-1}. \tag{5.6}$$

Now letting $k \to \infty$ we have $\mathbb{E}[V^-(t)] = 0$ for all $t \geq 0$, that implies $\mathbb{P}(V(t) < 0) = 0$ for all $t \in [0,T]$, which means $\mathbb{P}\left(V(t) \geq 0 \ \forall \ t \in [0,T]\right) = 1$ as required. □

Lemma 5.2. *Assume $\frac{1}{2} \leq \theta \leq 1$ and $\frac{1}{2} \leq \beta \leq 1$. Then, given any initial values $V(0) = V_0 > 0$ and $X(0) = X_0 > 0$, the SDE model (5.1) has unique global solution $X(t)$ which will be non-negative for all $t \in [0,T]$ almost surely.*

Proof. For any $0 \leq t \leq T$, we can easily compute

$$\mathbb{E}[V(t)] = V_0 + \mathbb{E}\int_0^t \alpha_2[\mu_2 - V(u^-)]du \leq V_0 + \mathbb{E}\int_0^t \alpha_2[\mu_2 + V(u)]du. \tag{5.7}$$

By Gronwall's inequality,

$$\mathbb{E}[V(t)] \leq R, \tag{5.8}$$

where $R = [V_0 + \mu_2\alpha_2 T]e^{\alpha_2 T}$. Applying similar conditions and techniques used in Lemma 5.1 but with parameter θ, we can define a new function $\phi_k(b) \in C^2(\mathbb{R},\mathbb{R})$ for $b < 0$ having the same properties as before. For any $0 \leq t \leq T$, by the Itô-Doeblin formula for one jump process, we have

$$\begin{aligned}\mathbb{E}[\phi_k(X(t))] =& \phi_k(X_0) + \mathbb{E}\int_0^t \phi_k^{'}(X(r))[\alpha_1\mu_1 - (\alpha_1 + \lambda_1\delta_1)X(u^-)]du \\ &+ \frac{1}{2}\mathbb{E}\int_0^t \phi_k^{''}(X(u))\sigma_1^2|V(u^-)||X(u^-)|^{2\theta}du\end{aligned}$$

$$+\mathbb{E}\int_0^t \phi_k[(1+\delta_1)X(u^-)] - \phi_k[X(u^-)]dN_2(u).$$

Applying the mean value theorem, (5.8), properties of the function $\phi_k(.)$ and the technique used to compute (5.6), we can easily obtain

$$\mathbb{E}[X^-(t)] \leq \left[\frac{\sigma_1^2 RT}{k} + \lambda_1|\delta_1|a_{k-1}T\right]\exp\{\lambda_1|\delta_1|T\} + a_{k-1}. \tag{5.9}$$

Then letting $k \to \infty$ we have $\mathbb{E}[X^-(t)] = 0$ for all $t \geq 0$. That implies $\mathbb{P}(X(t) < 0) = 0$ for all $t \in [0,T]$, which gives our required assertion, $\mathbb{P}(X(t) \geq 0 \;\forall\; t \in [0,T]) = 1$. □

5.3 Convergence in second moment

As the jump-diffusion SDE model (5.1) has no explicit solution so far, let us establish the Euler-Maruyama numerical approximate solution and examine its convergence in second moment.

Euler-Maruyama approximation

Given time step $\Delta \in (0,1)$, let $t_k = k\Delta$ for $k = 0,1,2,...[\frac{T}{\Delta}]$ while set $v_0 = V(0)$ and $x_0 = X(0)$, where $[\frac{T}{\Delta}]$ is integer part of $\frac{T}{\Delta}$. The discrete time EM approximation of the model is defined by

$$x_{k+1} = x_k + \alpha_1(\mu_1 - x_k)\Delta + \sigma_1\sqrt{|v_k|}|x_k|^{\theta}\Delta W_1 + \delta_1 x_k \Delta\bar{N}_1,$$
$$v_{k+1} = v_k + \alpha_2(\mu_2 - v_k)\Delta + \sigma_2|v_k|^{\beta}\Delta W_2 + \delta_2 x_k \Delta\bar{N}_2,$$

where $\Delta W_i = [W_i(t_{k+1}) - W_i(t_k)]$ and $\Delta\bar{N}_i = [\bar{N}_i(t_{k+1}) - \bar{N}_i(t_k)]$ for $i = 1,2$. The corresponding continuous EM approximate solution is defined by

$$x(t) = x_0 + \int_0^t \alpha_1(\mu_1 - |\bar{x}(u)|)du + \int_0^t \sigma_1\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^{\theta}dW_1(u) + \int_0^t \delta_1|\bar{x}(u)|d\bar{N}_1(u),$$

$$v(t) = v_0 + \int_0^t \alpha_2(\mu_2 - |\bar{v}(u)|)du + \int_0^t \sigma_2|\bar{v}(u)|^\beta dW_2(u) + \int_0^t \delta_2|\bar{v}(u)|d\bar{N}_2(u),$$

where $\bar{x}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} x_k 1_{[t_k,t_{k+1})}(t)$ and $\bar{v}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} v_k 1_{[t_k,t_{k+1})}(t)$ are step processes. That is $\bar{x}(t) = x_k$ and $\bar{v} = v_k$ for $t \in [t_k, t_{k+1})$, $k = 0, 1, 2, ...[\frac{T}{\Delta}]$.

Upper bound

The coefficients of the SDE model (5.1), when $\frac{1}{2} \leq \theta, \beta \leq 1$, satisfy not only the global Lipschitz condition but also the linear growth condition. Therefore, we will establish an important property by the following theorems, i.e. the convergence in second moment (in L^2) of the EM approximate solution.

Theorem 5.1. *Let $V(t)$ be the true solution and $v(t)$ be the EM approximate solution to the second SDE of model (5.1). Then, for any $p \geq 2$, there is a constant $R_1(p)$ dependent on p, T, V_0 but independent of Δ, such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |V(t)|^p\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T} |v(t)|^p\right) \leq R_1(p). \tag{5.10}$$

Proof. For any $0 \leq t \leq T$, compute

$$\begin{aligned} |V(t)|^p \leq & 4^{p-1}|V_0|^p + 4^{p-1}\left|\int_0^t \alpha_2(\mu_2 - V(u^-))du\right|^p \\ & + 4^{p-1}\left|\int_0^t \sigma_2|V(u^-)|^\beta dW_2(u)\right|^p + 4^{p-1}\left|\int_0^t \delta_2|V(u^-)|d\bar{N}_2(u)\right|^p. \end{aligned} \tag{5.11}$$

Taking the expectation for any $t_1 \in [0, T]$, we then have

$$\begin{aligned} \mathbb{E}\left(\sup_{0\leq t\leq t_1} |V(t)|^p\right) \leq & 4^{p-1}|V_0|^p + 4^{p-1}\mathbb{E}\left(\sup_{0\leq t\leq t_1}\left|\int_0^t \alpha_2(\mu_2 - V(u^-))du\right|^p\right) \\ & + 4^{p-1}\mathbb{E}\left(\sup_{0\leq t\leq t_1}\left|\int_0^t \sigma_2|V(u^-)|^\beta dW_2(u)\right|^p\right) \\ & + 4^{p-1}\mathbb{E}\left(\sup_{0\leq t\leq t_1}\left|\int_0^t \delta_2|V(u^-)|d\bar{N}_2(u)\right|^p\right). \end{aligned} \tag{5.12}$$

By Burkholder-Davis-Gundy's inequality and Höder's inequality. we further get that

$$
\begin{aligned}
\leq & 4^{p-1}|V_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_2^p \mu_2^p + 4^{p-1}T^{p-1}2^{p-1}\alpha_2^p \mathbb{E}\int_0^{t_1} |V(u)|^p du \\
& + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \mathbb{E}\int_0^{t_1} \sigma_2^p |V(u)|^{p\beta} du \\
& + \lambda_2 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \mathbb{E}\int_0^{t_1} |\delta_2|^p |V(u)|^p du,
\end{aligned} \tag{5.13}
$$

which takes the form

$$
\begin{aligned}
\leq & 4^{p-1}|V_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_2^p \mu_2^p + 4^{p-1}T^{p-1}2^{p-1}\alpha_2^p \mathbb{E}\int_0^{t_1} |V(u)|^p du \\
& + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \sigma_2^p \mathbb{E}\int_0^{t_1} |V(u)|^{\frac{p}{2}} + |V(u)|^p du \\
& + \lambda_2 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \mathbb{E}\int_0^{t_1} |\delta_2|^p |V(u)|^p du \\
\leq & 4^{p-1}|V_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_2^p \mu_2^p + 4^{p-1}T^{p-1}2^{p-1}\alpha_2^p \mathbb{E}\int_0^{t_1} |V(u)|^p du \\
& + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \sigma_2^p T \\
& + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \left(2\sigma_2^p + \lambda_2|\delta_2|^p\right) \mathbb{E}\int_0^{t_1} |V(u)|^p du.
\end{aligned} \tag{5.14}
$$

By Gronwall's inequality, we then compute

$$
\mathbb{E}\left(\sup_{0\leq t\leq T} |V(t)|^p\right) \leq A_1 e^{A_2 T}, \tag{5.15}
$$

where $A_1 = \left[4^{p-1}|V_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_2^p \mu_2^p + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \sigma_2^p T\right]$ and $A_2 = \left[4^{p-1}T^{p-1}2^{p-1}\alpha_2^p + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}} \left(2\sigma_2^p + \lambda_2|\delta_2|^p\right)\right]$.

Analogously, we can derive

$$
\mathbb{E}\left(\sup_{0\leq t\leq T} |v(t)|^p\right) \leq A_3 e^{A_4 T}. \tag{5.16}
$$

By (5.15) and (5.16), we have

$$
\mathbb{E}\left(\sup_{0\leq t\leq T} |V(t)|^p\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T} |v(t)|^p\right) \leq A_1 e^{A_2 T} \vee A_3 e^{A_4 T} = R_1(p), \tag{5.17}
$$

as required. The proof of Theorem 5.1 is therefore complete. □

Theorem 5.2. *Let $X(t)$ be the true solution and $x(t)$ be the EM approximate solution to the asset price. Then, for any $p \geq 2$, there is a constant $R_2(p)$, dependent on p, T, X_0, $R_1(p)$ but independent of Δ, such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T}|x(t)|^p\right) \leq R_2(p). \tag{5.18}$$

Proof. In the same way as in computation of (5.13), we can easily compute

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq t\leq t_1}|X(t)|^p\right) \\
\leq & 4^{p-1}|X_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_1^p\mu_1^p + 4^{p-1}T^{p-1}2^{p-1}\alpha_1^p\mathbb{E}\int_0^{t_1}|X(u^-)|^p du \\
&+ 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\mathbb{E}\int_0^{t_1}\sigma_1^p|V(u^-)|^{\frac{p}{2}}|X(u^-)|^{p\theta}du \\
&+ \lambda_1 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\mathbb{E}\int_0^{t_1}|\delta_1|^p|X(u^-)|^p du.
\end{aligned} \tag{5.19}$$

By Höder's inequality, we then have

$$\begin{aligned}
\leq & 4^{p-1}|X_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_1^p\mu_1^p + 4^{p-1}T^{p-1}2^{p-1}\alpha_1^p\mathbb{E}\int_0^{t_1}|X(u^-)|^p du \\
&+ 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\mathbb{E}\int_0^{t_1}\sigma_1^p|V(u^-)|^{p} + \sigma_1^p|X(u^-)|^{2p\theta}du \\
&+ \lambda_1 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\mathbb{E}\int_0^{t_1}|\delta_1|^p|X(u^-)|^p du.
\end{aligned} \tag{5.20}$$

Applying Theorem 5.1, we further get that

$$\begin{aligned}
\leq & 4^{p-1}|X_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_1^p\mu_1^p + 4^{p-1}T^{p-1}2^{p-1}\alpha_1^p\mathbb{E}\int_0^{t_1}|X(u)|^p du \\
&+ 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\left[R_1(p)T + \mathbb{E}\int_0^{t_1}\sigma_1^p|X(u)|^{2p}du\right] \\
&+ 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\left(\lambda_1|\delta_1|^p + \sigma_1^p\right)\mathbb{E}\int_0^{t_1}|X(u)|^p du \\
\leq & B_1 + B_2\int_0^{t_1}\mathbb{E}|X(u)|^p du + B_3\int_0^{t_1}\mathbb{E}|X(u)|^{2p}du,
\end{aligned} \tag{5.21}$$

where $\quad B_1 = \left[4^{p-1}|X_0|^p + 4^{p-1}T^p 2^{p-1}\alpha_1^p\mu_1^p + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}R_1(p)T\right],$

$B_2 = \left[4^{p-1}T^{p-1}2^{p-1}\alpha_1^p + 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\left(\lambda_1|\delta_1|^p + \sigma_1^p\right)\right]$ and
$B_3 = 4^{p-1}\left[\frac{p^{p+1}}{2(p-1)^{p-1}}\right]^{\frac{p}{2}} T^{\frac{p-2}{2}}\sigma_1^p.$

On the other hand for any $0 \leq t \leq T$, applying the Itô-Doeblin formula of one jump process to the asset price yields

$$
\begin{aligned}
\mathbb{E}[\Gamma(X)] =& \Gamma(X_0) + \mathbb{E}\int_0^t p|X(u)|^{(p-1)}\left[\alpha_1\mu_1 - (\alpha_1+\delta_1\lambda_1)X(u^-)\right]du \\
&+ \frac{1}{2}\mathbb{E}\int_0^t \sigma_1^2 p(p-1)|X(u)|^{p-2}|V(u^-)||X(u^-)|^{2\theta}du \\
&+ \mathbb{E}\int_0^t \lambda_1\left[[(1+\delta_1)X(u^-)]^p - [X(u^-)]^p\right]du \\
\leq& \Gamma(X_0) + \mathbb{E}\int_0^t p|X(u)|^{(p-1)}[\alpha_1\mu_1 - (\alpha_1+\delta_1\lambda_1)X(u)]du \\
&+ \frac{1}{2}\mathbb{E}\int_0^t \sigma_1^2 p(p-1)|X(u)|^{p-2}|V(u)||X(u)|^{2\theta}du \\
&+ \mathbb{E}\int_0^t \lambda_1(1+\delta_1)^p|X(u)|^p du \\
\leq& \Gamma(X_0) + \mathbb{E}\int_0^t p|X(u)|^{(p-1)}\alpha_1\mu_1\left[1 - \frac{(\alpha_1+\delta_1\lambda_1)}{2\alpha_1\mu_1}X(u)\right]du \\
&+ \frac{\sigma_1^2 p(p-1)}{2}\mathbb{E}\int_0^t |X(u)|^{p+2\theta-2}V(u)\left[1 - \frac{(\alpha_1+\delta_1\lambda_1)}{V(u)\sigma_1^2(p-1)}|X(u)|^{2-2\theta}\right]du \\
&+ \mathbb{E}\int_0^t \lambda_1(1+\delta_1)^p X(u)^p du.
\end{aligned}
$$

By

$$
\left[1 - \frac{X(t)^{2-2\theta}}{M}\right]\begin{cases} < 0 \quad ; \quad if \quad M < X(t); \\ < 1 \quad ; \quad if \quad 0 \leq X(t) \leq M, \end{cases} \tag{5.22}
$$

we further get that

$$
\begin{aligned}
\mathbb{E}[\Gamma(X)] \leq& \Gamma(X_0) + p\alpha_1\mu_1\int_0^t\left[\frac{2\alpha_1\mu_1}{(\alpha_1+\delta_1\lambda_1)}\right]^{(p-1)}du + \mathbb{E}\int_0^t \lambda_1(1+\delta_1)^p|X(u)|^p du \\
&+ \frac{\sigma_1^2 p(p-1)}{2}\mathbb{E}\int_0^t\left[\frac{V(u)\sigma_1^2(p-1)}{(\alpha_1+\delta_1\lambda_1)}\right]^{\frac{p+2\theta-2}{2-2\theta}} V(u)du
\end{aligned}
$$

$$\leq \Gamma(X_0) + p\alpha_1\mu_1 \left[\frac{2\alpha_1\mu_1}{(\alpha_1+\delta_1\lambda_1)}\right]^{(p-1)} T + \lambda_1(1+\delta_1)^p \int_0^t \mathbb{E}|X(u)|^p du$$
$$+ \frac{\sigma_1^2 p(p-1)}{2} \left[\frac{\sigma_1^2(p-1)}{(\alpha_1+\delta_1\lambda_1)}\right]^{\frac{p+2\theta-2}{2-2\theta}} \mathbb{E}\int_0^t |V(u)|^{\frac{p}{2-2\theta}} du$$
$$\leq \Gamma(X_0) + p\alpha_1\mu_1 \left[\frac{2\alpha_1\mu_1}{(\alpha_1+\delta_1\lambda_1)}\right]^{(p-1)} T + \lambda_1(1+\delta_1)^p \mathbb{E}\int_0^t |X(u)|^p du$$
$$+ \frac{\sigma_1^2 p(p-1)}{2} \left[\frac{\sigma_1^2(p-1)}{(\alpha_1+\delta_1\lambda_1)}\right]^{\frac{p+2\theta-2}{2-2\theta}} T\left[R_1(p)\right]^{\frac{p}{(2-2\theta)p_1}},$$

where $p_1 \geq \frac{p}{2-2\theta}$. By Gronwall's inequality, we then have

$$\mathbb{E}|X(t)|^p \leq D(p)_1 e^{D(p)_2 T}, \tag{5.23}$$

where $D(p)_2 = \Big[\lambda_1(1+\delta_1)^p\Big]$ and
$D(p)_1 = \Big[\Gamma(X_0) + p\alpha_1\mu_1\Big[\frac{2\alpha_1\mu_1}{(\alpha_1+\delta_1\lambda_1)}\Big]^{(p-1)}T + \frac{\sigma_1^2 p(p-1)}{2}\Big[\frac{\sigma_1^2(p-1)}{(\alpha_1+\delta_1\lambda_1)}\Big]^{\frac{p+2\theta-2}{2-2\theta}} T\Big[R_1(p)\Big]^{\frac{p}{(2-2\theta)p_1}}\Big]$.
Now, substituting (5.23) into (5.21) yields

$$\mathbb{E}\left(\sup_{0\leq t\leq t_1}|X(t)|^p\right) \leq B_1 + B_3 D(2p)_1 e^{D(2p)_2 T} T + B_2\int_0^{t_1}\mathbb{E}\left(\sup_{0\leq u_1\leq u}|X(u_1)|^p\right) du.$$

Applying Gronwall's inequality, we further get that

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X(t)|^p\right) \leq \left(B_1 + B_3 D(2p)_1 e^{D(2p)_2 T} T\right) e^{B_2 T} = B_4. \tag{5.24}$$

Analogously, we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|x(t)|^p\right) \leq B_5. \tag{5.25}$$

By (5.24) and (5.25), we then have

$$\mathbb{E}\left(\sup_{0\leq t\leq T}|X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T}|x(t)|^p\right) \leq B_4 \vee B_5 = R_2(p), \tag{5.26}$$

as required. The proof is therefore complete. □

Convergence in second moment of $v(t)$

The solution to SDE model (5.1) is non-negative with probability 1, and an upper bound for the expected value of the solution to this SDE model can be obtained under the linear growth condition. Let us establish the following theorem to show that the EM approximate solution to the volatility will converge to the true solution in L^1.

Theorem 5.3. *Let $V(t)$ be the true solution and $v(t)$ be the EM approximate solution to the second SDE of (5.1). Then,*

$$\lim_{\Delta \to 0} \left(\sup_{0 \le t \le T} \mathbb{E} \left| V(t) - v(t) \right| \right) = 0. \tag{5.27}$$

To prove Theorem 5.3, we need the following Lemma that shows the closenesses of $v(t)$ and $\bar{v}(t)$ when Δ is sufficiently small.

Lemma 5.3. *There exists a constant $C_{3,1}$ independent of Δ such that*

$$\mathbb{E} \left[v(t) - \bar{v}(t) \right]^2 \le C_{3,1} \Delta. \tag{5.28}$$

Proof. For any $0 \le t \le T$, let $\left[\frac{t}{\Delta}\right]$ be the integer part of $\frac{t}{\Delta}$. We have

$$\begin{aligned} & v(t) - \bar{v}(t) \\ &= \int_{[\frac{t}{\Delta}]\Delta}^{t} \alpha_2 \left[\mu_2 - \bar{v}(u) \right] du + \int_{[\frac{t}{\Delta}]\Delta}^{t} \sigma_2 |\bar{v}(u)|^{\beta} dW_2(u) + \int_{[\frac{t}{\Delta}]\Delta}^{t} \delta_2 \bar{v}(u) d\bar{N}_2(u), \end{aligned}$$

which gives

$$\begin{aligned} & \mathbb{E} \left[v(t) - \bar{v}(t) \right]^2 \\ \le & 3\mathbb{E} \left[\alpha_2 (\mu_2 - |v_{[\frac{t}{\Delta}]}|) \right]^2 (t - [\frac{t}{\Delta}]\Delta)^2 + 3\sigma_2^2 \mathbb{E} |v_{[\frac{t}{\Delta}]}|^{2\beta} \left[W_2(t) - W_2([\frac{t}{\Delta}]\Delta) \right]^2 \\ & + 3\mathbb{E} \delta_2^2 |v_{[\frac{t}{\Delta}]}|^2 \left[\bar{N}_2(t) - \bar{N}_2([\frac{t}{\Delta}]\Delta) \right]^2 . \end{aligned} \tag{5.29}$$

Applying the Hölder inequality and Theorem 5.1 yields

$$\begin{aligned}
&\mathbb{E}\left[v(t)-\bar{v}(t)\right]^2\\
\leq &6\alpha_2^2(\mu_2^2+R_1(2))\Delta^2+3\sigma_2^2[\mathbb{E}|v_{[\frac{t}{\Delta}]}|^{4\beta}]^{\frac{1}{2}}\left[\mathbb{E}\left[W_2(t)-W_2([\frac{t}{\Delta}]\Delta)\right]^4\right]^{\frac{1}{2}}\\
&+3\delta_2^2[\mathbb{E}|v_{[\frac{t}{\Delta}]}|^{4}]^{\frac{1}{2}}\left[\mathbb{E}\left[\bar{N}_2(t)-\bar{N}_2([\frac{t}{\Delta}]\Delta)\right]^4\right]^{\frac{1}{2}}\\
\leq &6\alpha_2^2(\mu_2^2+R_1(2))\Delta^2+3\sigma_2^2[R_1(4)]^{\frac{\beta}{2}}\left[3\Delta^2\right]^{\frac{1}{2}}+3\delta_2^2[R_1(4)]^{\frac{1}{2}}\left[3\lambda_2^2\Delta^2\right]^{\frac{1}{2}}\leq C_{3,1}\Delta,
\end{aligned} \tag{5.30}$$

as required. The proof of Lemma 5.3 is therefore complete. □

Proof. (of Theorem 5.3) For any $0\leq t\leq T$, compute

$$\begin{aligned}
&\Big(V(t)-v(t)\Big)\\
&=-(\alpha_2+\lambda_2\delta_2)\int_0^t\big(V(u^-)-\bar{v}(u)\big)\,du+\int_0^t\sigma_2\left(V(u^-)^\beta-|\bar{v}(u)|^\beta\right)dW_2(u)\\
&\quad+\delta_2\int_0^t\big(V(u^-)-\bar{v}(u)\big)\,dN_2(u).
\end{aligned}$$

Now, set $e(u)=(V(u)-v(u))$ and $e(u^-)=(V(u^-)-v(u))$. Then, applying Itô-Doeblin's formula for one jump process yields

$$\begin{aligned}
\mathbb{E}(\varphi_k(e(t)))\leq&\mathbb{E}\int_0^t|\varphi_k^{'}(e(u^-))|(\alpha_2+\lambda_2\delta_2)\left|V(u^-)-\bar{v}(u)\right|du\\
&+\frac{\sigma_2^2}{2}\mathbb{E}\int_0^t|\varphi_k^{''}(e(u^-))|\left|V(u^-)^\beta-|\bar{v}(u)|^\beta\right|^2du\\
&+\lambda_2\mathbb{E}\int_0^t\left|\varphi_k((1+\delta_2)e(u^-))-\varphi_k(e(u^-))\right|du.
\end{aligned} \tag{5.31}$$

By the well-known mean value theorem, we have

$$\begin{aligned}
\mathbb{E}(\varphi_k(e(t)))\leq&\mathbb{E}\int_0^t(\alpha_2+\lambda_2\delta_2)\left|V(u)-\bar{v}(u)\right|du\\
&+\frac{\sigma_2^2}{2}\mathbb{E}\int_0^t|\varphi_k^{''}(e(u))|\left|V(u)^\beta-|\bar{v}(u)|^\beta\right|^2du\\
&+\lambda_2|\delta_2|\mathbb{E}\int_0^t|\sup_{s\in\mathbb{R}}\varphi_k^{'}(s)|\left|V(u)-v(u)\right|du.
\end{aligned} \tag{5.32}$$

Substituting the properties of $\varphi_k(.)$ and Lemma 5.3, we obtain

$$\begin{aligned}\mathbb{E}(\varphi_k(e(t))) &\leq \int_0^t \mathbb{E}(\alpha_2 + 2\lambda_2|\delta_2|)\,|V(u) - v(u)|\,du \\ &\quad + \int_0^t \mathbb{E}(\alpha_2 + \lambda_2\delta_2)\,|v(u) - \bar{v}(u)|\,du \\ &\quad + \sigma_2^2\mathbb{E}\int_0^t |\varphi_k^{''}(e(t))|\left[V(u)^\beta - |v(u)|^\beta\right]^2 du \\ &\quad + \sigma_2^2\mathbb{E}\int_0^t |\varphi_k^{''}(e(t))|\left[|v(u)|^\beta - |\bar{v}(u)|^\beta\right]^2 du \\ &\leq \int_0^t \mathbb{E}(\alpha_2 + 2\lambda_2|\delta_2|)\,|V(u) - v(u)|\,du + (\alpha_2 + \lambda_2\delta_2)C_{3,1}^{\frac{1}{2}}\Delta^{\frac{1}{2}}T \\ &\quad + \frac{2\sigma_2^2T}{k} + \frac{\sigma_2^2TC_{3,1}^\beta\Delta^\beta}{ka_k^{2\beta}}.\end{aligned} \tag{5.33}$$

This, together with (iii) property of the function $\varphi_k(.)$, gives

$$\begin{aligned}\mathbb{E}(e^-(t)) \leq \int_0^t \mathbb{E}(\alpha_2 + 2\lambda_2|\delta_2|)\,|V(u) - v(u)|\,du + (\alpha_2 + \lambda_2\delta_2)C_{3,1}^{\frac{1}{2}}\Delta^{\frac{1}{2}}T \\ + \frac{2\sigma_2^2T}{k} + \frac{\sigma_2^2TC_{3,1}^\beta\Delta^\beta}{ka_k^{2\beta}} + a_{k-1}.\end{aligned} \tag{5.34}$$

By Gronwall's inequality, we have

$$\begin{aligned}&\sup_{0\leq t\leq T} \mathbb{E}\,|V(t) - v(t)| \\ &\leq \left[(\alpha_2 + \lambda_2\delta_2)C_{3,1}^{\frac{1}{2}}\Delta^{\frac{1}{2}}T + \frac{\sigma_2^2}{ka_k^{2\beta}}TC_{3,1}^\beta\Delta^\beta + \frac{2\sigma_2^2T}{k} + a_{k-1}\right]e^{(\alpha_2+2\lambda_2|\delta_2|)T}.\end{aligned} \tag{5.35}$$

Now, choose k sufficiently large for

$$\left[\frac{2\sigma_2^2T}{k} + a_{k-1}\right]e^{(\alpha_2+2\lambda_2|\delta_2|)T} < \frac{\varepsilon}{2} \tag{5.36}$$

and further choose Δ sufficiently small for

$$\left[(\alpha_2 + \lambda_2\delta_2)C_{3,1}^{\frac{1}{2}}\Delta^{\frac{1}{2}}T + \frac{\sigma_2^2}{ka_k^{2\beta}}TC_{3,1}^\beta\Delta^\beta\right]e^{(\alpha_2+2\lambda_2|\delta_2|)T} < \frac{\varepsilon}{2}. \tag{5.37}$$

We then have

$$\sup_{0\leq t\leq T} \mathbb{E}\,|V(t) - v(t)| < \varepsilon, \tag{5.38}$$

which complete the proof of Theorem 5.3. $\square$

The following theorem shows that the EM approximate solution to the volatility will converge to the true solution in L^2, with the support of Theorem 5.3.

Theorem 5.4. *Let $V(t)$ be the true solution and $v(t)$ be the EM approximate solution of the second SDE of (5.1). Then,*

$$\lim_{\Delta\to 0}\mathbb{E}\left(\sup_{0\le t\le T}[V(t)-v(t)]^2\right)=0. \tag{5.39}$$

Proof. For any $0\le t\le T$, we compute

$$\begin{aligned}&[V(t)-v(t)]^2\\ \le&3\alpha_2^2\left[\int_0^t\big(V(u^-)-\bar{v}(u)\big)\,du\right]^2+3\sigma_2^2\left[\int_0^t\big(V(u^-)^\beta-|\bar{v}(u)|^\beta\big)\,dW_2(u)\right]^2\\ &+3\delta_2^2\left[\int_0^t\big(V(u^-)-\bar{v}(u)\big)\,d\bar{N}_2(u)\right]^2.\end{aligned} \tag{5.40}$$

Taking the expectation for any $t_1\in[0,T]$, and applying the Burkholder-Davis-Gundy inequality and the Hölder inequality, we then have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le t_1}[V(t)-v(t)]^2\right)\\ \le&\Big[3\alpha_2^2T+12\lambda_2\delta_2^2\Big]\mathbb{E}\int_0^{t_1}\big[V(u^-)-\bar{v}(u)\big]^2du+12\sigma_2^2\mathbb{E}\int_0^{t_1}\big[V(u^-)^\beta-|\bar{v}(u)|^\beta\big]^2du.\end{aligned} \tag{5.41}$$

Rearranging the terms on the left hand side, we further get that

$$\begin{aligned}&\le\left[3\alpha_2^2T+12\lambda_2\delta_2^2\right]\mathbb{E}\int_0^{t_1}\big[V(u^-)-\bar{v}(u)\big]^2du+12\sigma_2^2\mathbb{E}\int_0^{t_1}\big[V(u^-)-\bar{v}(u)\big]^{2\beta}du\\ &\le\left[3\alpha_2^2T+12\lambda_2\delta_2^2+12\sigma_2^2\right]\mathbb{E}\int_0^{t_1}[V(u)-\bar{v}(u)]^2du+12\sigma_2^2\mathbb{E}\int_0^{t_1}|V(u)-\bar{v}(u)|\,du.\end{aligned}$$

By Lemma 5.3, we then compute

$$\begin{aligned}\le&2[3\alpha_2^2T+12\lambda_2\delta_2^2+12\sigma_2^2]\int_0^{t_1}\mathbb{E}\,[V(u)-v(u)]^2du+12\sigma_2^2\int_0^{t_1}\mathbb{E}\,|V(u)-v(u)|\,du\\ &+C_{3,2}\Delta^{\frac{1}{2}}.\end{aligned}$$

Now, applying Gronwall's inequality, we have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}[V(t)-v(t)]^2\right)\\
&\quad\le\left[12\sigma_2^2T\left(\sup_{0\le t\le T}\mathbb{E}\left|V(t)-v(t)\right|\right)+C_{3,2}\Delta^{\frac{1}{2}}\right]e^{[6\alpha_2^2T+24\lambda_2\delta_2^2+24\sigma_2^2]T}.
\end{aligned} \tag{5.42}$$

The proof of Theorem 5.4 finally is completed by Theorem 5.3 and letting $\Delta \to 0$. □

Convergence in second moment of $x(t)$

Now, we will establish a necessary condition for convergence in second moment of the asset price by the following theorem.

Theorem 5.5. *Let $X(t)$ be the true solution and $x(t)$ be the EM approximate solution to the asset price: Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}\mathbb{E}\left|X(t)-x(t)\right|\right)=0. \tag{5.43}$$

To prove Theorem 5.5, we need the following lemma which gives the close form of $x(t)$ and $\bar{x}(t)$ to each other when the time step Δ is small enough.

Lemma 5.4. *There exists a constant $C_{3,3}$ independent of Δ such that*

$$\mathbb{E}\left[x(t)-\bar{x}(t)\right]^4\le C_{3,3}\Delta^2. \tag{5.44}$$

The proof can be obtained by the same way as Lemma 5.3 was proved.

Proof. (of Theorem 5.5) For any $0\le t\le T$, compute

$$\begin{aligned}
(X(t)-x(t)) = &-(\alpha_1+\lambda_1\delta_1)\int_0^t\left(X(u^-)-\bar{x}(u)\right)du\\
&+\int_0^t\sigma_1\left(X(u^-)^\theta\sqrt{|V(u^-)|}-|\bar{x}(u)|^\theta\sqrt{|\bar{v}(u)|}\right)dW_1(u)\\
&+\delta_1\int_0^t\left(X(u^-)-\bar{x}(u)\right)d\bar{N}_1(u).
\end{aligned} \tag{5.45}$$

Now, set $e(u) = (X(u) - x(u))$ and $e(u^-) = (X(u^-) - x(u))$.

Applying the Itô-Doeblin formula for one jump process yields

$$\begin{aligned}\mathbb{E}(\phi_k(e(t))) \leq & \mathbb{E}\int_0^t |\phi_k^{'}(e(u^-))|[\alpha_1 + \lambda_1\delta_1]\left|X(u^-) - \bar{x}(u)\right| du \\ & + \frac{\sigma_1^2}{2}\mathbb{E}\int_0^t |\phi_k^{''}(e(u^-))|\left|X(u^-)^\theta\sqrt{|V(u^-)|} - |\bar{x}(u)|^\theta\sqrt{|\bar{v}(u)|}\right|^2 du \\ & + \lambda_1\mathbb{E}\int_0^t \left|\phi_k((1+\delta_1)e(u^-)) - \phi_k(e(t^-))\right| du.\end{aligned}$$

By the mean value theorem, we then have

$$\begin{aligned}& \mathbb{E}(\phi_k(e(t))) \\ \leq & \mathbb{E}\int_0^t [\alpha_1 + \lambda_1\delta_1]\left|X(u^-) - \bar{x}(u)\right| du \\ & + \sigma_1^2\mathbb{E}\int_0^t |\phi_k^{''}(e(u^-))|X(u^-)^{2\theta}\left[\sqrt{|V(u^-)|} - \sqrt{|\bar{v}(u)|}\right]^2 du \\ & + \sigma_1^2\mathbb{E}\int_0^t |\phi_k^{''}(e(u^-))||\bar{v}(u)|\left[X(u^-)^\theta - |\bar{x}(u)|^\theta\right]^2 du \\ & + \lambda_1|\delta_1|\mathbb{E}\int_0^t \sup_{b\in\mathbb{R}}|\phi_k^{'}(b)|\left|X(u^-) - x(u)\right| du, \\ \leq & \mathbb{E}\int_0^t [\alpha_1 + \lambda_1\delta_1]\left|X(u) - \bar{x}(u)\right| du + \lambda_1|\delta_1|\mathbb{E}\int_0^t |X(u) - x(u)| du \\ & + \sigma_1^2\mathbb{E}\int_0^t |\phi_k^{''}(e(u))|X(u)^{2\theta}\left|V(u) - \bar{v}(u)\right| du \\ & + \sigma_1^2\mathbb{E}\int_0^t |\phi_k^{''}(e(u))||\bar{v}(u)|\left[X(u) - \bar{x}(u)\right]^{2\theta} du.\end{aligned} \tag{5.46}$$

Applying the property of $\varphi_k(.)$ we further compute

$$\begin{aligned}\leq & \mathbb{E}\int_0^t [\alpha_1 + 2\lambda_1|\delta_1|]\left|X(u) - x(u)\right| du + \mathbb{E}\int_0^t [\alpha_1 + \lambda_1\delta_1]\left|x(u) - \bar{x}(u)\right| du \\ & + \frac{2\sigma_1^2}{ka_k^{2\theta}}\mathbb{E}\int_0^t X(u)^{2\theta}\left|V(u) - v(u)\right| du + \frac{2\sigma_1^2}{ka_k^{2\theta}}\mathbb{E}\int_0^t X(u)^{2\theta}\left|v(u) - \bar{v}(u)\right| du \\ & + \frac{4\sigma_1^2}{k}\mathbb{E}\int_0^t |\bar{v}(u)|du + \frac{4\sigma_1^2}{ka_k^{2\theta}}\mathbb{E}\int_0^t |\bar{v}(u)|\left[|x(u) - \bar{x}(u)| + |x(u) - \bar{x}(u)|^2\right] du.\end{aligned}$$

Using Hölder's inequality, we further get that

$$\leq [\alpha_1 + 2\lambda_1|\delta_1|]\int_0^t \mathbb{E}\left|X(u) - x(u)\right| du + [\alpha_1 + \lambda_1\delta_1]\int_0^t \left(\mathbb{E}\left|x(u) - \bar{x}(u)\right|^2\right)^{\frac{1}{2}} du$$

$$
\begin{aligned}
&+ \frac{2\sigma_1^2}{ka_k^{2\theta}} \int_0^t \left(\mathbb{E}X(u)^{4\theta}\right)^{\frac{1}{2}} \left(\mathbb{E}\left|V(u)-v(u)\right|^2\right)^{\frac{1}{2}} du \\
&+ \frac{2\sigma_1^2}{ka_k^{2\theta}} \int_0^t \left(\mathbb{E}X(u)^{4\theta}\right)^{\frac{1}{2}} \left(\mathbb{E}\left|v(u)-\bar{v}(u)\right|^2\right)^{\frac{1}{2}} du \\
&+ \frac{4\sigma_1^2}{k} \int_0^t \left(\mathbb{E}|\bar{v}(u)|^2\right)^{\frac{1}{2}} du + \frac{4\sigma_1^2}{ka_k^{2\theta}} \int_0^t \left(\mathbb{E}|\bar{v}(u)|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left|x(u)-\bar{x}(u)\right|^2\right)^{\frac{1}{2}} du \\
&+ \frac{4\sigma_1^2}{ka_k^{2\theta}} \int_0^t \left(\mathbb{E}|\bar{v}(u)|^2\right)^{\frac{1}{2}} \left(\mathbb{E}\left|x(u)-\bar{x}(u)\right|^4\right)^{\frac{1}{2}} du.
\end{aligned}
$$

Applying the natural relationship

$$
\sup_{0\le t\le T} |\bar{v}(u)|^2 \le \sup_{0\le t\le T} |v(u)|^2, \tag{5.47}
$$

by Theorem 5.1, Theorem 5.2, Lemma 5.3 and Lemma 5.4, we then compute

$$
\begin{aligned}
&\le [\alpha_1 + 2\lambda_1|\delta_1|] \int_0^t \mathbb{E}\left|X(u)-x(u)\right| du + [\alpha_1+\lambda_1\delta_1] \left(C_{3,3}\Delta^2\right)^{\frac{1}{4}} T \\
&\quad + \frac{2\sigma_1^2 \left(R_2(4)\right)^{\frac{\theta}{2}}}{ka_k^{2\theta}} \int_0^t \left(\mathbb{E}\left|V(u)-v(u)\right|^2\right)^{\frac{1}{2}} du \; + \frac{2\sigma_1^2 \left(R_2(4)\right)^{\frac{\theta}{2}}}{ka_k^{2\theta}} \left(C_{3,1}\Delta\right)^{\frac{1}{2}} T \\
&\quad + \frac{4\sigma_1^2 \left(R_1(2)\right)^{\frac{1}{2}} T}{k} + \frac{4\sigma_1^2 \left(R_1(2)\right)^{\frac{1}{2}} \left(C_{3,3}\Delta^2\right)^{\frac{1}{4}} T}{ka_k^{2\theta}} + \frac{4\sigma_1^2 \left(R_1(2)\right)^{\frac{1}{2}} \left(C_{3,3}\Delta^2\right)^{\frac{1}{2}} T}{ka_k^{2\theta}}.
\end{aligned}
$$

Now, substituting Theorem 5.4,

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T} [V(t)-v(t)]^2\right) \\
&\le \left[12\sigma_2^2 T \left(\sup_{0\le t\le T} \mathbb{E}\left|V(t)-v(t)\right|\right) + C_{3,2}\Delta^{\frac{1}{2}}\right] e^{[6\alpha_2^2 T + 24\lambda_2\delta_2^2 + 24\sigma_2^2]T} \\
&= \Lambda(\Delta) + \Upsilon(k\Delta) + \Theta(k),
\end{aligned} \tag{5.48}
$$

yields

$$
\begin{aligned}
&\mathbb{E}(\phi_k(e(t))) \\
&\le [\alpha_1 + 2\lambda_1|\delta_1|] \int_0^t \mathbb{E}\left|X(u)-x(u)\right| du + [\alpha_1+\lambda_1\delta_1] \left(C_{3,3}\Delta^2\right)^{\frac{1}{4}} T \\
&\quad + \frac{2\sigma_1^2 \left(R_2(4)\right)^{\frac{\theta}{2}} \left(\Lambda(\Delta)+\Upsilon(k\Delta)+\Theta(k)\right)^{\frac{1}{2}} T}{ka_k^{2\theta}} + \frac{2\sigma_1^2 \left(R_2(4)\right)^{\frac{\theta}{2}}}{ka_k^{2\theta}} \left(C_{3,1}\Delta\right)^{\frac{1}{2}} T \\
&\quad + \frac{4\sigma_1^2 \left(R_1(2)\right)^{\frac{1}{2}} T}{k} + \frac{4\sigma_1^2 \left(R_1(2)\right)^{\frac{1}{2}} T \Big[\left(C_{3,3}\Delta^2\right)^{\frac{1}{4}} + \left(C_{3,3}\Delta^2\right)^{\frac{1}{2}} \Big]}{ka_k^{2\theta}}.
\end{aligned} \tag{5.49}
$$

Noting $\quad e^{-}(t)-a_{k-1}\leq\phi_k(e(t))\leq e^{-}(t)$, we further get that

$$
\begin{aligned}
&\mathbb{E}(e^{-}(t))\\
\leq & a_{k-1}+[\alpha_1+2\lambda_1|\delta_1|]\int_0^t\mathbb{E}\left|X(u)-x(u)\right|du+[\alpha_1+\lambda_1\delta_1]\left(C_{3,3}\Delta^2\right)^{\frac{1}{4}}T\\
&+\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}\left(\Lambda(\Delta)+\Upsilon(k\Delta)+\Theta(k)\right)^{\frac{1}{2}}T}{ka_k^{2\theta}}+\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}}{ka_k^{2\theta}}\left(C_{3,1}\Delta\right)^{\frac{1}{2}}T\\
&+\frac{4\sigma_1^2\left(R_1(2)\right)^{\frac{1}{2}}T}{k}+\frac{4\sigma_1^2\left(R_1(2)\right)^{\frac{1}{2}}T\left[\left(C_{3,3}\Delta^2\right)^{\frac{1}{4}}+\left(C_{3,3}\Delta^2\right)^{\frac{1}{2}}\right]}{ka_k^{2\theta}}.
\end{aligned}
\tag{5.50}
$$

By Gronwall's inequality, we then have

$$
\begin{aligned}
&\sup_{0\leq t\leq T}\mathbb{E}\left|X(u)-x(u)\right|\\
\leq&\left[a_{k-1}+\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}\Theta(k)^{\frac{1}{2}}T}{ka_k^{2\theta}}+\frac{4\sigma_1^2\left(R_1(2)\right)^{\frac{1}{2}}T}{k}\right.\\
&+\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}\left(\Lambda(\Delta)+\Upsilon(k\Delta)\right)^{\frac{1}{2}}T}{ka_k^{2\theta}}+[\alpha_1+\lambda_1\delta_1]\left(C_{3,3}\Delta^2\right)^{\frac{1}{4}}T\\
&+\frac{4\sigma_1^2\left(R_1(2)\right)^{\frac{1}{2}}\left[\left(C_{3,3}\Delta^2\right)^{\frac{1}{4}}+\left(C_{3,3}\Delta^2\right)^{\frac{1}{2}}\right]T}{ka_k^{2\theta}}\\
&\left.+\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}}{ka_k^{2\theta}}\left(C_{3,1}\Delta\right)^{\frac{1}{2}}T\right]e^{[\alpha_1+2\lambda_1|\delta_1|]T}.
\end{aligned}
\tag{5.51}
$$

Now choose k sufficiently large such that

$$
\left[\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}\Theta(k)^{\frac{1}{2}}T}{ka_k^{2\theta}}+\frac{4\sigma_1^2\left(R_1(2)\right)^{\frac{1}{2}}T}{k}+a_{k-1}\right]e^{[\alpha_1+2\lambda_1|\delta_1|]T}<\frac{\varepsilon}{2}
\tag{5.52}
$$

and further choose Δ sufficiently small such that

$$
\begin{aligned}
&\left[\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}\left(\Lambda(\Delta)+\Upsilon(k\Delta)\right)^{\frac{1}{2}}T}{ka_k^{2\theta}}+[\alpha_1+\lambda_1\delta_1]\left(C_{3,3}\Delta^2\right)^{\frac{1}{4}}T\right.\\
&+\frac{2\sigma_1^2\left(R_2(4)\right)^{\frac{\theta}{2}}}{ka_k^{2\theta}}\left(C_{3,1}\Delta\right)^{\frac{1}{2}}T\\
&\left.+\frac{4\sigma_1^2\left(R_1(2)\right)^{\frac{1}{2}}\left[\left(C_{3,3}\Delta^2\right)^{\frac{1}{4}}+\left(C_{3,3}\Delta^2\right)^{\frac{1}{2}}\right]T}{ka_k^{2\theta}}\right]e^{[\alpha_1+2\lambda_1|\delta_1|]T}<\frac{\varepsilon}{2}.
\end{aligned}
\tag{5.53}
$$

Hence, we have

$$\sup_{0 \le t \le T} \mathbb{E}\,|X(u) - x(u)| < \varepsilon \tag{5.54}$$

as required. The proof of Theorem 5.5 is therefore complete. □

Now, we will establish the following theorem to show convergence in second moment of the EM approximate solution to the asset price.

Theorem 5.6. *Let $X(t)$ be the true solution and $x(t)$ be the EM approximate solution to the asset price. Then,*

$$\lim_{\Delta \to 0} \mathbb{E}\left(\sup_{0 \le t \le T} [X(t) - x(t)]^2 \right) = 0. \tag{5.55}$$

Proof. For any $0 \le t_1 \le T$, we compute

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0 \le t \le t_1} [X(t) - x(t)]^2 \right) \\
\le& 3\alpha_1^2 T \mathbb{E} \int_0^{t_1} \left[X(u^-) - \bar{x}(u)\right]^2 du \\
&+ 12\sigma_1^2 \mathbb{E} \int_0^{t_1} \left[\sqrt{V(u^-)} X(u^-)^\theta - \sqrt{|\bar{v}(u)|} \bar{x}(u)^\theta\right]^2 du \\
&+ 12\lambda_1 \delta_1^2 \mathbb{E} \int_0^{t_1} \left[X(u^-) - \bar{x}(u)\right]^2 du \\
\le& [3\alpha_1^2 T + 12\lambda_1 \delta_1^2] \mathbb{E} \int_0^{t_1} \left[X(u^-) - \bar{x}(u)\right]^2 du \\
&+ 12\sigma_1^2 \mathbb{E} \int_0^{t_1} \left[\sqrt{V(u^-)} X(u^-)^\theta - \sqrt{|\bar{v}(u)|} |\bar{x}(u)|^\theta\right]^2 du \qquad (5.56) \\
\le& [3\alpha_1^2 T + 12\lambda_1 \delta_1^2] \mathbb{E} \int_0^{t_1} [X(u) - \bar{x}(u)]^2 du \\
&+ 48\sigma_1^2 \mathbb{E} \int_0^{t_1} |X(u)|^{2\theta} \left[\sqrt{V(u)} - \sqrt{|v(u)|}\right]^2 du \\
&+ 48\sigma_1^2 \mathbb{E} \int_0^{t_1} |x(u)|^{2\theta} \left[\sqrt{|v(u)|} - \sqrt{|\bar{v}(u)|}\right]^2 du \\
&+ 48\sigma_1^2 \mathbb{E} \int_0^{t_1} |v(u)| \left[X(u)^\theta - |x(u)|^\theta\right]^2 du \\
&+ 48\sigma_1^2 \mathbb{E} \int_0^{t_1} |\bar{v}(u)| \left[|x(u)|^\theta - |\bar{x}(u)|^\theta\right]^2 du.
\end{aligned}$$

Rearranging the terms on the right hand side, we then have

$$\begin{aligned}
&\leq 2[3\alpha_1^2 T + 12\lambda_1\delta_1^2]\mathbb{E}\int_0^{t_1} [X(u) - x(u)]^2 + [x(u) - \bar{x}(u)]^2 \, du \\
&\quad + 48\sigma_1^2\mathbb{E}\int_0^{t_1} |X(u)|^{2\theta} |V(u) - v(u)| + |x(u)|^{2\theta} |v(u) - \bar{v}(u)| \, du \\
&\quad + 48\sigma_1^2\mathbb{E}\int_0^{t_1} |v(u)| \, |X(u) - x(u)|^2 + |v(u)| \, |X(u) - x(u)| \, du \\
&\quad + 48\sigma_1^2\mathbb{E}\int_0^{t_1} |\bar{v}(u)| \, |x(u) - \bar{x}(u)|^2 + |\bar{v}(u)| \, |x(u) - \bar{x}(u)| \, du.
\end{aligned} \tag{5.57}$$

By Theorem 5.1, Theorem 5.2, Lemma 5.3 and Lemma 5.4, this yields

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq t\leq t_1} [X(t) - x(t)]^2\right) \\
\leq& 2[3\alpha_1^2 T + 12\lambda_1\delta_1^2]\left(\int_0^{t_1} \mathbb{E}\,[X(u) - x(u)]^2 \, du + [C_{3,3}\Delta^2]^{\frac{1}{2}} T\right) \\
&+ 48\sigma_1^2 [R_2(4)]^{\frac{\theta}{2}} \int_0^{t_1} \left[\mathbb{E}\left(\sup_{0\leq u_1\leq u} [V(u_1) - v(u_1)]^2\right)\right]^{\frac{1}{2}} du \\
&+ 48\sigma_1^2 [R_2(4)]^{\frac{\theta}{2}} [C_{3,1}\Delta]^{\frac{1}{2}} T + 48\sigma_1^2 [R_1(2)]^{\frac{1}{2}} T\left([C_{3,3}\Delta^2]^{\frac{1}{2}} + [C_{3,3}\Delta^2]^{\frac{1}{4}}\right) \\
&+ 48\sigma_1^2 [R_1(2)]^{\frac{1}{2}} \int_0^{t_1} (\mathbb{E}\,|X(u) - x(u)|)^{\frac{1}{4}} \left(\mathbb{E}\,|X(u) - x(u)|^3\right)^{\frac{1}{4}} du \\
&+ 48\sigma_1^2 [R_1(2)]^{\frac{1}{2}} \int_0^{t_1} (\mathbb{E}\,|X(u) - x(u)|)^{\frac{1}{4}} \left(\mathbb{E}\,|X(u) - x(u)|^7\right)^{\frac{1}{4}} du \\
\leq& 2[3\alpha_1^2 T + 12\lambda_1\delta_1^2] \int_0^{t_1} \mathbb{E}\,|X(u) - x(u)|^2 \, du + C_{3,4}\Delta^{\frac{1}{2}} \\
&+ 48\sigma_1^2 [R_2(4)]^{\frac{\theta}{2}} \int_0^{t_1} \left[\mathbb{E}\left(\sup_{0\leq u_1\leq u} [V(u_1) - v(u_1)]^2\right)\right]^{\frac{1}{2}} du \\
&+ 48\sigma_1^2 [R_1(2)]^{\frac{1}{2}} [8R_2(3)]^{\frac{1}{4}} \int_0^{t_1} \left(\sup_{0\leq u_1\leq u} \mathbb{E}\,|X(u_1) - x(u_1)|\right)^{\frac{1}{4}} du \\
&+ 48\sigma_1^2 [R_1(2)]^{\frac{1}{2}} [2^7 R_2(7)]^{\frac{1}{4}} \int_0^{t_1} \left(\sup_{0\leq u_1\leq u} \mathbb{E}\,|X(u_1) - x(u_1)|\right)^{\frac{1}{4}} du \\
\leq& C_{3,4}\Delta^{\frac{1}{2}} + 2[3\alpha_1^2 T + 12\lambda_1\delta_1^2] \int_0^{t_1} \left(\mathbb{E}\sup_{0\leq u_1\leq u} [X(u_1) - x(u_1)]^2\right) du \\
&+ 48\sigma_1^2 [R_2(4)]^{\frac{\theta}{2}} \left(\mathbb{E}\sup_{0\leq u\leq T} [V(u) - v(u)]^2\right)^{\frac{1}{2}} T + C_{3,5}\left(\sup_{0\leq u\leq T} \mathbb{E}\,|X(u) - x(u)|\right)^{\frac{1}{4}} T.
\end{aligned} \tag{5.58}$$

Applying Gronwall's inequality, we have

$$\mathbb{E}\left(\sup_{0\le t\le T}[X(t)-x(t)]^2\right) \le \Bigg[C_{3,4}\Delta^{\frac{1}{2}} + 48\sigma_1^2 T[R_2(4)]^{\frac{\theta}{2}}\left(\mathbb{E}\sup_{0\le u\le T}[V(u)-v(u)]^2\right)^{\frac{1}{2}}$$
$$+ C_{3,5}\left(\sup_{0\le u\le T}\mathbb{E}\left|X(u)-x(u)\right|\right)^{\frac{1}{4}} T\Bigg] e^{2\left(3\alpha_1^2 T + 12\lambda_1\delta_1^2\right)T}. \tag{5.59}$$

The proof is therefore completed by Theorem 5.4, Theorem 5.5 and letting $\Delta \to 0$. □

Theorem 5.6 shows that the continuous EM approximate solution will converge to the true solution though it is not computable in practice. Therefore, it is required to show that the corresponding step process, which is computable, will converge to the true solution of the asset price when the time step is sufficiently small. Thus, the following theorem will establish the convergence property of this step process which can also be used to examine applications of the EM approximate solution in finance.

Theorem 5.7. *Let $X(t)$ be the true solution and $\bar{x}(t)$ be the step process of the EM approximate solution to the asset price. Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}\mathbb{E}\Big|X(t)-\bar{x}(t)\Big|\right) = 0. \tag{5.60}$$

Proof. We can easily compute

$$\left(\sup_{0\le t\le T}\mathbb{E}\left|X(t)-\bar{x}(t)\right|\right) \le \left(\sup_{0\le t\le T}\mathbb{E}\left|X(t)-x(t)\right|\right) + \left(\sup_{0\le t\le T}\mathbb{E}\left|x(t)-\bar{x}(t)\right|\right). \tag{5.61}$$

Then, the required proof of Theorem 5.7 will be completed by Lemma 5.4, Theorem 5.6 and letting $\Delta \to 0$. □

Theorem 5.7 shows that the step process of the EM approximate solution will converge to the true solution. We will apply MATLAB® software (see Appendix A for code) with initial condition $(X(0) = 0.3, V(0) = 0.5)$, $\rho = 0.1$, $(\theta = 1, \beta = 0.5)$, $\lambda_1 = 1, \lambda_2 = 2$ and coefficients of the SDE model (5.1)(see Table 5.1) to illustrate its behaviour in practice (see Figure 5.1).

Table 5.1: Coefficients of the SDE model (5.1)

Case	Parameters					
SDE 1	$\theta = 1.0$	$X(0) = 0.3$	$\alpha_1 = 2$	$\mu_1 = 2$	$\sigma_1 = 2$	$\delta_1 = 2$
SDE 2	$\beta = 0.5$	$V(0) = 0.5$	$\alpha_1 = 1$	$\mu_2 = 0.5$	$\sigma_2 = 1$	$\delta_2 = 1$

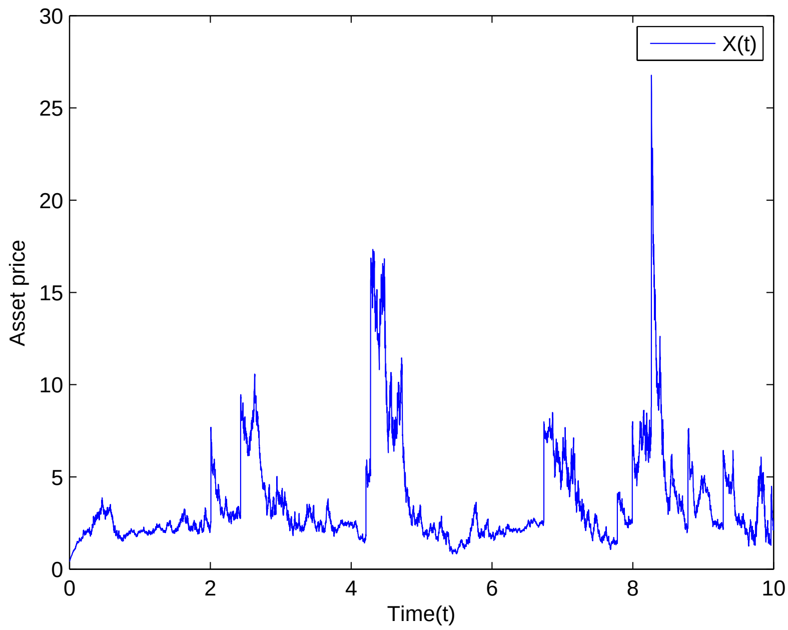

Figure 5.1: A sample path of the asset price $X(t)$ which is generated by the EM approximate solution to the mean-reverting-theta stochastic volatility model with Poisson jump over finite time, where $\theta = 1$ and $\beta = 0.5$.

5.4 Summary

The convergence property of the EM approximate solution to the mean-reverting-theta stochastic volatility model driven by a Poisson jump process has been examined in this chapter, for parameters $\frac{1}{2} \leq \theta, \beta \leq 1$. In this process, we have first proved that the unique global solution to SDE model (5.1) is non-negative with probability 1. Then, we have obtained convergence in second moment of the EM approximate solution to this SDE model when the time step is sufficiently small. Finally, the convergence property of the corresponding step process has been obtained to show that it can be used to evaluate applications in finance.

Chapter 6

Jump-Diffusion Stochastic Volatility Model for Asset Price

6.1 Introduction

In contrast to the SDE model (5.1) examined in the previous chapter for parameters $\frac{1}{2} < \theta, \beta < 1$, we will examine the same SDE model in this chapter but with parameters θ and β greater than 1. Even though many applications of this highly sensitive SDE model can be seen in financial markets where random disturbances occurred, explicit solution to this model can not be obtained within the existing theory. Thus, the Euler-Maruyama (EM) numerical scheme has become a more appropriate way to study and examine its behaviour in practice. In this process, the convergence property of the EM approximate solution to the true solution plays a very important role though existing techniques are not strong enough to show this important task. Thus, we will develop the necessary financial instruments to fill this gap in this chapter,.

The SDE model (5.1) was mainly developed to examine behaviour of the underlying asset price and volatility, even though there is no information on the non-negative solution when $1 < \theta, \beta < \infty$. Therefore, we will first prove that solution to this SDE model is non-negative with probability 1. We will then consider the EM approximate solution to this model which has been defined in Chapter 5 but with parameters θ and β are greater than 1. Since this model satisfies the local Lipschitz condition, the strong error bound of this EM approximate solution can be obtained with stopping time. Therefore, we will show that the continuous EM approximate solution to the SDE model (5.1) when $1 < \theta, \beta < \infty$ will converge in probability to the true solution. However, this continuous EM approximate solution is not computable in practice, but its corresponding step process is computable. Hence, we will finally show that the step process of the EM approximate solution will converge in probability to the true solution when the parameters θ and β are greater than 1 and this property can also be used to examine applications of this EM approximate solution in finance.

6.2 Non-negative solution

As the SDE model (5.1) mainly describes behaviour of the underlying asset price and its volatility in financial markets, a natural requirement is to have a non-negative solution $(X(t), V(t))$ in practice. Thus, we will establish the following lemmas to show that the solution to the SDE model (5.1) will be non-negative with probability 1 when parameters θ and β are greater than 1. However, we will omit the proofs of these lemmas since they can be obtained in the same way as Lemma 5.1 and Lemma 5.2 were proved but with

$$a_k = \left[\frac{(2A-1)k(k+1)}{2}\right]^{\frac{1}{(1-2A)}}$$

for $k = 1, 2, 3, ...$ and $1 < A$ where $A \in \{\beta, \theta\}$.

Non-negative $V(t)$

Lemma 6.1. *Assume* $1 < \beta < \infty$. *Then, given any initial value* $V(0) = V_0 > 0$, *the second SDE of the model (5.1) has unique local solution* $V(t)$ *which will be non-negative for all* $t \in [0, T]$ *almost surely.*

Non-negative $X(t)$

Lemma 6.2. *Assume* $1 < \theta < \infty$ *and* $1 < \beta < \infty$. *Then, given any initial values* $V(0) = V_0 > 0$ *and* $X(0) = X_0 > 0$, *the SDE model (5.1) has unique local solution* $X(t)$ *which will be non-negative for all* $t \in [0, T]$ *almost surely.*

6.3 Convergence in probability

In general, SDE models have no explicit solutions, hence study of numerical approximate solutions is useful to understand the behaviour of systems. In this section, we will examine the EM numerical approximate solution to the SDE model (5.1) defined in Chapter 5 but with the parameters θ and β greater than 1.

Convergence of $v(t)$ in probability

The diffusion coefficients of SDE model obey the local Lipschitz condition though do not satisfy the linear growth conditions. We can not appeal to convergence in second moment of its approximate solution within the existing theory. Thus, the following theorem will establish a strong error bound of the EM continuous approximate solution to volatility with stopping time.

Theorem 6.1. *Let* $V(t)$ *be the true solution and* $v(t)$ *be continuous EM approximate solution to the second SDE of (5.1) when* $1 < \beta < \infty$. *For any positive* *number M, define the stopping time $q = \rho_M \wedge \gamma_M \wedge T$, where $\rho_M = \inf\{t \in [0,T] : V(t) \notin [\frac{1}{M}, M]\}$ and $\gamma_M = \inf\{t \in [0,T] : |v(t)| \notin [\frac{1}{M}, M]\}$. Then, for any integer $p \geq 2$,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} [V(t\wedge q) - v(t\wedge q)]^2\right) \leq C_{4,2}(M,p)\Delta^{1-\frac{1}{p}}, \tag{6.1}$$

where $C_{4,2} = C_{4,2}(M,p)$ is a constant independent of Δ.

To prove Theorem 6.1, we need the following lemma that shows the closeness of the $x(t)$ and $\bar{x}(t)$ when time step is so small.

Lemma 6.3. *There exists a constant $C_{4,1}(M,p)$ dependent on M and p but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} [v(t\wedge q) - \bar{v}(t\wedge q)]^2\right) \leq C_{4,1}(M,p)\Delta^{1-\frac{1}{p}}. \tag{6.2}$$

Proof. For $0 \leq t \leq T$, let $\left[\frac{t}{\Delta}\right]$ be the integer part of $\frac{t}{\Delta}$. We then have

$$\begin{aligned} &v(t\wedge q) - \bar{v}(t\wedge q) \\ =&\int_{[\frac{t\wedge q}{\Delta}]\Delta}^{t\wedge q} [\alpha_2(\mu_2 - \bar{v}(u))]\, du + \int_{[\frac{t\wedge q}{\Delta}]\Delta}^{t\wedge q} \sigma_2|\bar{v}(u)|^\beta dW_2(u) + \int_{[\frac{t\wedge q}{\Delta}]\Delta}^{t\wedge q} \delta_2\bar{v}(u)d\bar{N}_2(u), \end{aligned} \tag{6.3}$$

which gives

$$\begin{aligned} \Big[v(t\wedge q) - \bar{v}(t\wedge q)\Big]^2 \leq& 6\alpha_2^2(\mu_2^2 + M)\Delta^2 + 3\sigma_2^2 M^{2\beta}\left[W_2(t\wedge q) - W_2([\frac{t\wedge q}{\Delta}]\Delta)\right]^2 \\ &+ 3\delta_2^2 M\left[\bar{N}_2(t\wedge q) - \bar{N}_2([\frac{t\wedge q}{\Delta}]\Delta)\right]^2. \end{aligned} \tag{6.4}$$

Taking the expectation, we further get that

$$\begin{aligned} &\mathbb{E}\left(\sup_{0\leq t\leq T} [v(t\wedge q) - \bar{v}(t\wedge q)]^2\right) \\ \leq& 6\alpha_2^2(\mu_2^2 + M)\Delta^2 + 3\sigma_2^2 M^{2\beta}\mathbb{E}\left(\sup_{0\leq t\leq T\wedge q}\left[W_2(t) - W_2([\frac{t}{\Delta}]\Delta)\right]^2\right) \end{aligned}$$

$$
\begin{aligned}
&+ 3\delta_2^2 M\mathbb{E}\left(\sup_{0\le t\le T\wedge q}\left[\bar{N}_2(t) - \bar{N}_2([\frac{t}{\Delta}]\Delta)\right]^2\right)\\
\le & 6\alpha_2^2(\mu_2^2 + M)\Delta^2 + 3\sigma_2^2 M^{2\beta}\mathbb{E}\left(\sup_{0\le t\le T}\left[W_2(t) - W_2([\frac{t}{\Delta}]\Delta)\right]^2\right)\\
&+ 3\delta_2^2 M\mathbb{E}\left(\sup_{0\le t\le T}\left[\bar{N}_2(t) - \bar{N}_2([\frac{t}{\Delta}]\Delta)\right]^2\right).
\end{aligned}
$$

Applying the technique used to compute (3.27) yields

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le T}[v(t\wedge q) - \bar{v}(t\wedge q)]^2\right)\\
\le & 6\alpha_2(\mu_2^2 + M^2)\Delta^2 + 3\sigma_2^2 M^{2\beta}\left[\left(\frac{2p}{2p-1}\right)^{2p}(2p-1)!!\Delta^{p-1}(T+1)\right]^{\frac{1}{p}}\\
&+ 3\delta_2^2 M\left[\lambda_2^p\left(\frac{2p}{2p-1}\right)^{2p}(2p-1)!!\Delta^{p-1}(T+1)\right]^{\frac{1}{p}}\\
\le & C_{4,1}(M,p)\Delta^{1-\frac{1}{p}}.
\end{aligned}
\tag{6.5}
$$

as required. The proof of Lemma 6.3 is therefore complete. □

Proof. (of Theorem 6.1) For any $0 \le t \le T$, compute

$$
\begin{aligned}
\left[V(t\wedge q) - v(t\wedge q)\right]^2 \le & 3\left[\int_0^{t\wedge q}\alpha_2[V(u^-) - \bar{v}(u)]du\right]^2\\
&+ 3\left[\int_0^{t\wedge q}\sigma_2[|V(u^-)|^\beta - |\bar{v}(u)|^\beta]dW_2(u)\right]^2\\
&+ 3\left[\int_0^{t\wedge q}\delta_2[V(u^-) - \bar{v}(u)]d\bar{N}_2(u)\right]^2.
\end{aligned}
\tag{6.6}
$$

Taking the expectation for $t_1 \in [0, T]$, we then have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}[V(t\wedge q) - v(t\wedge q)]^2\right)\\
\le & 3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\alpha_2[V(u^-) - \bar{v}(u)]d(u)\right]^2\right)\\
&+ 3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\sigma_2\left[|V(u^-)|^\beta - |\bar{v}(u)|^\beta\right]dW_2(u)\right]^2\right)
\end{aligned}
$$

$$+ 3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge q}\delta_2[V(u^-)-\bar{v}(u)]d\bar{N}_2(u)\right]^2\right).$$

By the Burkholder-Davis-Gundy inequality and the Hölder inequality, we further get that

$$\begin{aligned}\le & 3\alpha_2^2 T\mathbb{E}\int_0^{t_1\wedge q}\left[V(u^-)-\bar{v}(u)\right]^2 du + 12\sigma_2^2\mathbb{E}\int_0^{t_1\wedge q}\left[|V(u^-)|^\beta - |\bar{v}(u)|^\beta\right]^2 du \\ & + 12\delta_2^2\lambda_2\mathbb{E}\int_0^{t_1\wedge q}\left[V(u^-)-\bar{v}(u)\right]^2 du.\end{aligned} \tag{6.7}$$

Applying the mean value theorem yields

$$\begin{aligned}&\le \left[3\alpha_2^2 T + 12\delta_2^2\lambda_2\right]\mathbb{E}\int_0^{t_1}\left[V(u\wedge q)-\bar{v}(u\wedge q)\right]^2 du \\ &\quad + 12\sigma_2^2\beta^2 M^{2\beta-2}\mathbb{E}\int_0^{t_1}[V(u\wedge q)-\bar{v}(u\wedge q)]^2 du \\ &\le \left[6\alpha_2^2 T + 24\delta_2^2\lambda_2 + 24\sigma_2^2\beta^2 M^{2\beta-2}\right]\mathbb{E}\int_0^{t_1}\left[V(u\wedge q)-v(u\wedge q)\right]^2 du \\ &\quad + \left[6\alpha_2^2 T + 24\delta_2^2\lambda_2 + 24\sigma_2^2\beta^2 M^{2\beta-2}\right]\mathbb{E}\int_0^{t_1}\left[v(u\wedge q)-\bar{v}(u\wedge q)\right]^2 du.\end{aligned} \tag{6.8}$$

By Lemma 6.3, we then have

$$\begin{aligned}&\mathbb{E}\left(\sup_{0\le t\le t_1}\left[V(t\wedge q)-v(t\wedge q)\right]^2\right) \\ &\le \left[6\alpha_2^2 T + 24\delta_2^2\lambda_2 + 24\sigma_2^2\beta^2 M^{2\beta-2}\right]\int_0^{t_1}\mathbb{E}\left[V(u\wedge q)-v(u\wedge q)\right]^2 du \\ &\quad + \left[6\alpha_2^2 T + 24\delta_2^2\lambda_2 + 24\sigma_2^2\beta^2 M^{2\beta-2}\right] C_{4,1}(M,p)\Delta^{1-\frac{1}{p}}T,\end{aligned} \tag{6.9}$$

An application of Gronwall's inequality will therefore complete the proof. □

Now, we will remove the stopping time of volatility by the following theorem and show that the EM continuous approximate solution of the volatility will converge in probability to the true solution.

Theorem 6.2. *Let $V(t)$ be the true solution and $v(t)$ be the approximate solution to second SDE of (5.1) when $1<\beta<\infty$. Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}\left[V(t)-v(t)\right]^2\right) = 0 \quad \textit{in probability.} \tag{6.10}$$

To prove Theorem 6.2, we need the following lemma which gives the upper bound for expected value of the continuous EM approximate solution to the volatility.

Lemma 6.4. *There exists a constant $C_{4,4}(M,p)$ which dependent on M and p, but independent of Δ such that*

$$\mathbb{E}(v(t \wedge \gamma_M)) \leq Z + C_{4,4}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{6.11}$$

where Z is a constant independent of Δ.

Proof. For any $0 \leq t \leq T$, we compute

$$\mathbb{E}(v(t \wedge \gamma_M)) \leq V_0 + \alpha_2\mu_2 T + \alpha_2 \int_0^{t\wedge\gamma_M} \mathbb{E}|\bar{v}(u)|du. \tag{6.12}$$

Rearranging the terms in right hand side, we further get that

$$\begin{aligned}
&\mathbb{E}(v(t \wedge \gamma_M)) \\
\leq& V_0 + \alpha_2\mu_2 T + \alpha_2 \int_0^{t\wedge\gamma_M} \mathbb{E}[|\bar{v}(u)| - v(u)]du + \alpha_2 \int_0^{t\wedge\gamma_M} \mathbb{E}v(u)du \\
\leq& V_0 + \alpha_2\mu_2 T + \alpha_2 \int_0^{t\wedge\gamma_M} \mathbb{E}|v(u) - \bar{v}(u)|du + \alpha_2 \int_0^{t\wedge\gamma_M} \mathbb{E}(v(u))du.
\end{aligned} \tag{6.13}$$

On the other hand, in the same way as in computation of Lemma 6.3, we obtain

$$\mathbb{E}\left(\sup_{0\leq t\leq T} [v(t \wedge \gamma_M) - \bar{v}(t \wedge \gamma_M)]^2\right) \leq C_{4,3}(M,p)\Delta^{1-\frac{1}{p}}, \tag{6.14}$$

where $C_{4,3}(M,p)$ is a constant independent of Δ. Now substituting (6.14) in (6.13), we further get that

$$\begin{aligned}
&\mathbb{E}(v(u \wedge \gamma_M)) \\
\leq& V_0 + \alpha_2\mu_2 T + \alpha_2 \left(C_{4,3}(M,p)\Delta^{1-\frac{1}{p}}\right)^{\frac{1}{2}} T + \alpha_2 \int_0^{t\wedge\gamma_M} \mathbb{E}(v(u))du.
\end{aligned} \tag{6.15}$$

By Gronwall's inequality, we have

$$\begin{aligned}
\mathbb{E}(v(u \wedge \gamma_M)) \leq& \left[V_0 + \alpha_2\mu_2 T + \alpha_2 \left(C_{4,3}(M,p)\Delta^{1-\frac{1}{p}}\right)^{\frac{1}{2}} T\right] e^{\alpha_2 T} \\
=& Z + C_{4,4}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]},
\end{aligned} \tag{6.16}$$

as required. The proof of Lemma 6.4 is therefore complete. □

Proof. (of Theorem 6.2)

The proof is rather complex, so we will therefore divide the whole proof into 3 steps.

Step 1: Applying the Itö-Doeblin formula for one jump process for $t_1 \in [0,T]$ yields

$$\begin{aligned}
&\mathbb{E}\left[H(V(t_1 \wedge \rho_M))\right] \\
=&H(V_0) + \mathbb{E}\int_0^{t_1\wedge\rho_M} H^{'}(V(u^-))\left[\alpha_2\mu_2 - (\alpha_2+\lambda_2\delta_2)V(u^-)\right]du \\
&+\frac{1}{2}\mathbb{E}\int_0^{t_1\wedge\rho_M} H^{''}(V(u^-))\sigma_2^2 V(u^-)^{2\beta}du \\
&+\mathbb{E}\int_0^{t_1\wedge\rho_M} H([1+\delta_2]V(u^-)) - H(V(u^-))dN_2(u),
\end{aligned} \tag{6.17}$$

where function $H(.)$ has been defined in Lemma 3.1. By the mean value theorem, we then have

$$\begin{aligned}
&\mathbb{E}\left[H(V(t_1 \wedge \rho_M))\right] \\
\leq&H(V_0) + \mathbb{E}\int_0^{t_1\wedge\rho_M} \frac{1}{2}[V(u)^{-\frac{1}{2}} - V(u)^{-1}][\alpha_2\mu_2 - (\alpha_2+\lambda_2\delta_2)V(u)]du \\
&+\frac{1}{4}\mathbb{E}\int_0^{t_1\wedge\rho_M}[V(u)^{-2} - \frac{1}{2}V(u)^{-\frac{3}{2}}]\sigma_2^2|V(u)|^{2\beta}du \\
&+\lambda_2\mathbb{E}\int_0^{t_1\wedge\rho_M}\sup_{s\in\mathbb{R}}[s^{-\frac{1}{2}} - s^{-1}]|\delta_2|V(u)du \\
\leq&H(V_0) + \frac{\alpha_2\mu_2}{2}\mathbb{E}\int_0^{t_1\wedge\rho_M}[V(u)^{-\frac{1}{2}} - V(u)^{-1}]du \\
&+\frac{(\alpha_2+\lambda_2\delta_2)}{2}\mathbb{E}\int_0^{t_1\wedge\rho_M}[1 - V(u)^{\frac{1}{2}}]du \\
&+\frac{\sigma_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\rho_M}[1-\frac{1}{2}V(u)^{\frac{1}{2}}]|V(u)|^{2\beta-2}du + \lambda_2|\delta_2|\int_0^T\mathbb{E}V(u)du.
\end{aligned} \tag{6.18}$$

Applying (3.8), we further get that

$$\begin{aligned}
&\mathbb{E}\left[H(V(t_1 \wedge \rho_M))\right] \\
\leq&H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{(\alpha_2+\lambda_2\delta_2)}{2}T + \frac{\sigma_2^2 4^{2\beta-2}T}{4} + \lambda_2|\delta_2|\int_0^T\mathbb{E}V(u)du.
\end{aligned} \tag{6.19}$$

On the other hand, in the same way as in the computation of (5.8), we derive

$$\mathbb{E}[V(t)] \leq R. \tag{6.20}$$

This together with (6.19) yields

$$\begin{aligned}
&\mathbb{E}\left[H(V(t_1 \wedge \rho_M))\right] \\
\leq &H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{(\alpha_2 + \lambda_2\delta_2)}{2}T + \frac{\sigma_2^2 4^{2\beta-2}T}{4} + \lambda_2|\delta_2|RT.
\end{aligned} \tag{6.21}$$

Now, repeating the technique which was used to compute (3.12), we have

$$\mathbb{P}(\rho_M \leq T) \leq \frac{H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{(\alpha_2+\lambda_2\delta_2)}{2}T + \frac{\sigma_2^2 4^{2\beta-2}T}{4} + \lambda_2|\delta_2|RT}{H(M^{-1}) \wedge H(M)}. \tag{6.22}$$

Step 2: Then, applying a similar technique as used in *(Step 1)* for the EM continuous approximate solution to the volatility, we get

$$\begin{aligned}
&\mathbb{E}\left[H(v(t_1 \wedge \gamma_M))\right] \\
=&H(V_0) + \mathbb{E}\int_0^{t_1\wedge\gamma_M} H^{'}(v(u))\left[\alpha_2\mu_2 - (\alpha_2 + \lambda_2\delta_2)\bar{v}(u)\right]du \\
&+ \frac{1}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M} H^{''}(v(u))\sigma_2^2|\bar{v}(u)|^{2\beta}du \\
&+ \mathbb{E}\int_0^{t_1\wedge\rho_M} H([1+\delta_2]\bar{v}(u)) - H(\bar{v}(u))dN_2(u).
\end{aligned} \tag{6.23}$$

By the mean value theorem, we then obtain

$$\begin{aligned}
\leq &H(V_0) + \mathbb{E}\int_0^{t_1\wedge\gamma_M} \frac{1}{2}\left[v(u)^{-\frac{1}{2}} - v(u)^{-1}\right]\left[\alpha_2\mu_2 - (\alpha_2 + \lambda_2\delta_2)\bar{v}(u)\right]du \\
&+ \frac{1}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M} \frac{1}{2}\left[v(u)^{-2} - \frac{1}{2}v(u)^{-\frac{3}{2}}\right]\sigma_2^2|\bar{v}(u)|^{2\beta}du \\
&+ |\delta_2|\lambda_2\mathbb{E}\int_0^{t_1\wedge\gamma_M} \sup_{s\in\mathbb{R}}\left[s^{-\frac{1}{2}} - s^{-1}\right]\bar{v}(u)du.
\end{aligned} \tag{6.24}$$

Rearranging the terms on the righthand side, we further get that

$$\begin{aligned}
\leq &H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{(\alpha_2 + \lambda_2\delta_2)}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[1 - v(u)^{\frac{1}{2}}\right]du \\
&+ \frac{\sigma_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[1 - \frac{1}{2}v(u)^{\frac{1}{2}}\right]|v(u)|^{2\beta-2}du + |\delta_2|\lambda_2\mathbb{E}\int_0^{t_1\wedge\gamma_M} v(u)du
\end{aligned}$$

$$+\frac{(\alpha_2+\lambda_2\delta_2)}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[v(u)^{-\frac{1}{2}}-v(u)^{-1}\right](v(u)-\bar{v}(u))du$$
$$+\frac{\sigma_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[v(u)^{-2}-\frac{1}{2}v(u)^{-\frac{3}{2}}\right]\left[|\bar{v}(u)|^{2\beta}-|v(u)|^{2\beta}\right]du$$
$$+|\delta_2|\lambda_2\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[\bar{v}(u)-v(u)\right]du.$$

By (3.8), Lemma 6.4 and the well-known mean value theorem, we compute

$$\begin{aligned}\leq & H(V_0)+\frac{\alpha_2\mu_2 T}{2}+\frac{(\alpha_2+\delta_2\lambda_2)T}{2}+\frac{\sigma_2^2 4^{2\beta-2}T}{4}+|\delta_2|\lambda_2 TZ\\ &+\frac{(\alpha_2+\lambda_2\delta_2)}{2}\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[v(u)^{-\frac{1}{2}}-v(u)^{-1}\right](v(u)-\bar{v}(u))du\\ &+|\delta_2|\lambda_2\mathbb{E}\int_0^{t_1\wedge\gamma_M}\left[\bar{v}(u)-v(u)\right]du++|\delta_2|\lambda_2 C_{4,4}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}T\\ &+\frac{\bar{\sigma}_2^2}{4}\mathbb{E}\int_0^{t_1\wedge\gamma_M}2\beta\sup[u]^{2\beta-1}\left[v(u)^{-2}-\frac{1}{2}v(u)^{-\frac{3}{2}}\right]|\bar{v}(u)-v(u)|du.\end{aligned}\tag{6.25}$$

Note that $\bar{v}(u)\in[M^{-1},M]$ whereas $v(u)\in[M^{-1},M]$. we then have

$$\begin{aligned}\leq & H(V_0)+\frac{\alpha_2\mu_2 T}{2}+\frac{(\alpha_2+\delta_2\lambda_2)T}{2}+\frac{\sigma_2^2 4^{2\beta-2}T}{4}+|\delta_2|\lambda_2 TZ\\ &+|\delta_2|\lambda_2 C_{4,4}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}T+|\delta_2|\lambda_2\mathbb{E}\int_0^{t_1}|\bar{v}(u\wedge\gamma_M)-v(u\wedge\gamma_M)|du\\ &+\left[\frac{|M^{\frac{1}{2}}+M|(\alpha_2+\lambda_2\delta_2)}{2}\right]\int_0^{t_1}\mathbb{E}|v(u\wedge\gamma_M)-\bar{v}(u\wedge\gamma_M)|du\\ &+\left[\frac{|M^2+\frac{1}{2}M^{\frac{3}{2}}|\beta M^{2\beta-1}\sigma_2^2}{2}\right]\int_0^{t_1}\mathbb{E}|v(u\wedge\gamma_M)-\bar{v}(u\wedge\gamma_M)|du.\end{aligned}\tag{6.26}$$

By Lemma (6.14), we get

$$\begin{aligned}\leq & H(V_0)+\frac{\alpha_2\mu_2 T}{2}+\frac{(\alpha_2+\delta_2\lambda_2)T}{2}+\frac{\sigma_2^2 4^{2\beta-2}T}{4}+|\delta_2|\lambda_2 TZ\\ &+|\delta_2|\lambda_2 C_{4,4}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}T+|\delta_2|\lambda_2(C_{4,3}(M,p)\Delta^{1-\frac{1}{p}})^{\frac{1}{2}}T\\ &+\left[\frac{|M^{\frac{1}{2}}+M|(\alpha_2+\lambda_2\delta_2)}{2}\right](C_{4,3}(M,p)\Delta^{1-\frac{1}{p}})^{\frac{1}{2}}T\\ &+\left[\frac{|M^2+\frac{1}{2}M^{\frac{3}{2}}|\beta M^{2\beta-1}\sigma_2^2}{2}\right](C_{4,3}(M,p)\Delta^{1-\frac{1}{p}})^{\frac{1}{2}}T,\end{aligned}\tag{6.27}$$

which is

$$\begin{aligned}\mathbb{E}\left[H(v(t_1 \wedge \gamma_M))\right] \leq & H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{(\alpha_2 + \delta_2\lambda_2)T}{2} + \frac{\sigma_2^2 4^{2\beta-2}T}{4} \\ & + |\delta_2|\lambda_2 TZ + \bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}.\end{aligned} \tag{6.28}$$

Recalling the technique which was used to obtain (3.12), we further get that

$$\begin{aligned}&\mathbb{P}(\gamma_M \leq T) \\ &\leq \frac{H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{(\alpha_2+\delta_2\lambda_2)T}{2} + \frac{\sigma_2^2 4^{2\beta-2}T}{4} + |\delta_2|\lambda_2 TZ + \bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)}.\end{aligned} \tag{6.29}$$

Step 3: Let $\varepsilon > 0$ and $\delta \in (0,1)$ be arbitrarily small, then define

$$\bar{\Omega}_4 = \left[\omega; \sup_{0\leq t\leq T} [V(t) - v(t)]^2 \geq \delta\right]. \tag{6.30}$$

In the same way as in the computation of (3.42) but with Theorem 6.1, we obtain

$$\mathbb{P}(\bar{\Omega}_4 \cap (q \geq T)) \leq \frac{C_{4,2}(M,p)\Delta^{1-\frac{1}{p}}}{\delta}. \tag{6.31}$$

On the other hand, we compute

$$\begin{aligned}\mathbb{P}(\bar{\Omega}_4) &\leq P(\bar{\Omega}_4 \cap (q \geq T)) + \mathbb{P}(q \leq T) \\ &\leq P(\bar{\Omega}_4 \cap (q \geq T)) + \mathbb{P}(\gamma_M \leq T) + \mathbb{P}(\rho_M \leq T).\end{aligned} \tag{6.32}$$

Substituting (6.22), (6.29) and (6.31) into (6.32) yields

$$\begin{aligned}\mathbb{P}(\bar{\Omega}_4) \leq & \frac{C_{4,2}(M,p)\Delta^{1-\frac{1}{p}}}{\delta} \\ & + \frac{H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{[\alpha_2+\delta_2\lambda_2]T}{2} + \frac{\sigma_2^2 4^{2\beta-2}T}{4} + |\delta_2|\lambda_2 TZ + \bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} \\ & + \frac{H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{(\alpha_2+\lambda_2\delta_2)}{2}T + \frac{\sigma_2^2 4^{2\beta-2}T}{4} + \lambda_2|\delta_2|RT}{H(M^{-1}) \wedge H(M)}.\end{aligned}$$

Now, choose M sufficiently large for

$$\frac{2\left[H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{(\alpha_2+\lambda_2\delta_2)}{2}T + \frac{\sigma_2^2 4^{2\beta-2}T}{4}\right] + \lambda_2|\delta_2|RT + \lambda_2|\delta_2|ZT}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{2} \tag{6.33}$$

and further choose Δ sufficiently small for

$$\frac{C_{4,2}(M,p)\Delta^{1-\frac{1}{p}}}{\delta} + \frac{\bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{2}. \tag{6.34}$$

Hence, we have

$$\mathbb{P}\left(\sup_{0\le t\le T} [V(t) - v(t)]^2 \ge \delta\right) < \varepsilon, \tag{6.35}$$

as required. The proof of Theorem 6.2 is therefore complete. □

Convergence of $x(t)$ in probability

The main result of this chapter, which gives convergence of the continuous EM approximate solution to the asset price, will be examined in this section. Therefore, we will first establish the following theorem which shows the strong error bound for this EM approximate solution with stopping time.

Theorem 6.3. *Let $X(t)$ be the true solution and $x(t)$ be the continuous EM approximate solution to the SDE of (5.1) when $1 < \theta, \beta < \infty$. For any positive numbers N and M, define stopping time $s = q \wedge \tau_N \wedge \zeta_N \wedge T$, where q is the same as before while $\tau_N = \inf\{t \in [0,T] : X(t) \notin [\frac{1}{N}, N]\}$, $\zeta_N = \inf\{t \in [0,T] : |x(t)| \notin [\frac{1}{N}, N]\}$. Then, for any $p \ge 2$,*

$$\mathbb{E}\left(\sup_{0\le t\le T} \Big[X(t\wedge s) - x(t\wedge s)\Big]^2\right) \le C_{4,7}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{6.36}$$

where $C_{4,7}(M,N,p)$ is a constant independent of Δ.

The proof of Theorem 6.3 needs the following lemma which can be obtained in the same way as Lemma 6.3 was proved.

Lemma 6.5. *There exists a constant $C_{4,5}(M,N,p)$ dependent on M, N and p but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\le t\le T}\Big[x(t\wedge s)-\bar{x}(t\wedge s)\Big]^2\right)\le C_{4,5}(M,N,p)\Delta^{1-\frac{1}{p}}. \tag{6.37}$$

Proof. (of Theorem 6.3) For any $0\le t\le T$, we compute

$$\begin{aligned}
&\Big[X(t\wedge s)-x(t\wedge s)\Big]^2\\
\le&3\left[\int_0^{t\wedge s}\alpha_1\Big[X(u)-\bar{x}(u)\Big]du\right]^2+3\left[\int_0^{t\wedge s}\delta_1\left[X(u)-\bar{x}(u)\right]d\bar{N}_1(u)\right]^2\\
&+3\left[\int_0^{t\wedge s}\sigma_1\left[\sqrt{V(u)}|X(u)|^\theta-\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\right]dW_1(u)\right]^2.
\end{aligned} \tag{6.38}$$

Taking the expectation for any $t_1\in[0,T]$, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}\left[X(t\wedge s)-x(t\wedge s)\right]^2\right)\\
\le&3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge s}\alpha_1\left[X(u^-)-\bar{x}(u)\right]du\right]^2\right)\\
&+3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge s}\sigma_1\left[\sqrt{V(u^-)}|X(u^-)|^\theta-\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\right]dW_1(u)\right]^2\right)\\
&+3\mathbb{E}\left(\sup_{0\le t\le t_1}\left[\int_0^{t\wedge s}\delta_1[X(u^-)-\bar{x}(u)]d\bar{N}_1(u)\right]^2\right).
\end{aligned} \tag{6.39}$$

By the Burkholder-Davis-Gundy inequality and Hölder's inequality, we get that

$$\begin{aligned}
\le&3\alpha_1^2T\mathbb{E}\int_0^{t_1\wedge s}\left[X(u^-)-\bar{x}(u)\right]^2du\\
&+12\sigma_1^2\mathbb{E}\int_0^{t_1\wedge s}\left[\sqrt{V(u^-)}|X(u^-)|^\theta-\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\right]^2du\\
&+12\delta_1^2\lambda_1\mathbb{E}\int_0^{t_1\wedge s}\left[X(u^-)-\bar{x}(u)\right]^2du.
\end{aligned} \tag{6.40}$$

Rearranging the terms on the right hand side, we obtain

$$\begin{aligned}
&\le\left[3\alpha_1^2T+12\delta_1^2\lambda_1\right]\mathbb{E}\int_0^{t_1}\left[X(u\wedge s)-\bar{x}(u\wedge s)\right]^2du\\
&\quad+24\sigma_1^2M\mathbb{E}\int_0^{t_1}\left[|X(u\wedge s)|^\theta-|\bar{x}(u\wedge s)|^\theta\right]^2du\\
&\quad+24\sigma_1^2N^{2\theta}\mathbb{E}\int_0^{t_1}|V(u\wedge s)-\bar{v}(u\wedge s)|\,du.
\end{aligned} \tag{6.41}$$

By Theorem 6.1, Lemma 6.3, Lemma 6.5 and the mean value theorem, we get

$$
\begin{aligned}
\leq & 2\left[3\alpha_1^2 T + 12\delta_1^2\lambda_1\right]\mathbb{E}\int_0^{t_1}\left[X(u\wedge s) - x(u\wedge s)\right]^2 du \\
& + 2\left[3\alpha_1^2 T + 12\delta_1^2\lambda_1\right]\mathbb{E}\int_0^{t_1}\left[x(u\wedge s) - \bar{x}(u\wedge s)\right]^2 du \\
& + 48\sigma_1^2 M\mathbb{E}\int_0^{t_1}\left[|X(u\wedge s)|^\theta - |x(u\wedge s)|^\theta\right]^2 + \left[|x(u\wedge s)|^\theta - |\bar{x}(u\wedge s)|^\theta\right]^2 du \\
& + 24\sigma_1^2 N^{2\theta}\mathbb{E}\int_0^{t_1}|V(u\wedge s) - v(u\wedge s)| + |v(u\wedge s) - \bar{v}(u\wedge s)|\, du,
\end{aligned}
$$

which gives

$$
\begin{aligned}
& \mathbb{E}\left(\sup_{0\leq t\leq t_1}\left[X(t\wedge s) - x(t\wedge s)\right]^2\right) \\
& \leq \left[6\alpha_1^2 T + 24\delta_1^2\lambda_1\right]\mathbb{E}\int_0^{t_1}\left[X(u\wedge s) - x(u\wedge s)\right]^2 du \\
& \quad + 48\sigma_1^2\theta^2 M N^{2\theta-2}\mathbb{E}\int_0^{t_1}\left[X(u\wedge s) - x(u\wedge s)\right]^2 du \\
& \quad + [6\alpha_1^2 T + 24\delta_1^2\lambda_1] C_{4,5}(M,N,p)\Delta^{1-\frac{1}{p}} T + 24\sigma_1^2 N^{2\theta}[C_{4,2}(M,p)\Delta^{1-\frac{1}{p}}]^{\frac{1}{2}} T \\
& \quad + 24\sigma_1^2 N^{2\theta}[C_{4,1}(M,p)\Delta^{1-\frac{1}{p}}]^{\frac{1}{2}} T + 48\sigma_1^2 M N^{2\theta-2}\theta^2 C_{4,5}(M,N,p)\Delta^{1-\frac{1}{p}} T.
\end{aligned}
\tag{6.42}
$$

The proof of Theorem 6.3 is finally completed by application of Gronwall's inequality. □

Now, we remove the condition of stopping time and establish the following theorem to show that the continuous EM approximate solution will converge in probability to the true solution.

Theorem 6.4. *Let $X(t)$ be the true solution and $x(t)$ be the approximate solution to the SDE of (5.1) when $1 < \theta, \beta < \infty$. Then,*

$$
\lim_{\Delta\to 0}\left(\sup_{0\leq t\leq T}\left[X(t) - x(t)\right]^2\right) = 0 \quad \textit{in probability.} \tag{6.43}
$$

To prove Theorem 6.4, we will establish the following lemma that gives an upper bound for the expected value of the EM approximate solution to the asset price.

Lemma 6.6. *There exists a constant $C_{4,9}(M,p)$ which is dependent on M, N and p, but independent of Δ such that*

$$\mathbb{E}(x(t \wedge h)) \leq L + C_{4,9}(N, M, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{6.44}$$

where $h = \gamma_M \wedge \zeta_N$ and constant L is independent of Δ.

The required proof of Lemma 6.6 can be obtained in the same way as Lemma 6.4 was proved.

Proof. (Theorem 6.4)

In this process, we also divide the whole proof into 3 steps.

Step 1: In the same way as in computation of (6.21) but stopping time $g = \tau_N \wedge \rho_M$, we have

$$\mathbb{E}\left[H(X(t_1 \wedge g))\right] \leq H(X_0) + \frac{\alpha_1 \mu_1 T}{2} + \frac{(\alpha_1 + \lambda_1 \delta_1)T}{2} + \frac{\sigma_1^2 R 4^{2\theta-2} T}{4} + \lambda_1 |\delta_1| \bar{R} T, \tag{6.45}$$

where $\bar{R}$ is the upper bound for the expected value of the asset price that can be obtained by applying a similar technique as used in (5.8). Now, in the same way as in computation of (3.59), we further get that

$$\mathbb{P}(\tau_N \leq T) \leq \frac{H(X_0) + \frac{\alpha_1 \mu_1 T}{2} + \frac{(\alpha_1 + \lambda_1 \delta_1)T}{2} + \frac{\sigma_1^2 R 4^{2\theta-2} T}{4} + \lambda_1 |\delta_1| \bar{R} T}{H(N^{-1}) \wedge H(N)}. \tag{6.46}$$

Step 2: Repeating the similar technique used to compute (6.28) but with Lemma 6.6 and

$$\mathbb{E}\left(\sup_{0 \leq t \leq T} \left[x(t \wedge h) - \bar{x}(t \wedge h)\right]^2\right) \leq C_{4,8}(M, N, p)\Delta^{1-\frac{1}{p}}, \tag{6.47}$$

which can be obtained in the same way as Lemma 6.3 was proved, we get

$$\begin{aligned}\mathbb{E}\left[H(x(t_1 \wedge h))\right] \leq & H(X_0) + \frac{\alpha_1 \mu_1 T}{2} + \frac{(\alpha_1 + \lambda_1 \delta_1)T}{2} \\ & + \frac{\sigma_1^2}{4} M 4^{2\theta-2} T + |\delta_1| \lambda_1 L T + \bar{C}_{4,2}(M, N, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}.\end{aligned} \tag{6.48}$$

By the technique which was used to compute (3.59), we then have

$$\mathbb{P}(\zeta_N \leq T) \leq \frac{H(X_0) + \frac{\alpha_1\mu_1 T}{2} + \frac{(\alpha_1+\lambda_1\delta_1)T}{2} + \frac{\sigma_1^2 M 4^{2\theta-2} T}{4} + |\delta_1|\lambda_1 LT + \bar{C}_{4,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)}. \tag{6.49}$$

Step 3: Let $\varepsilon > 0$ and $\delta \in (0,1)$ be arbitrarily small, then define

$$\Omega_4 = \left[\omega; \sup_{0\leq t\leq T} [X(t) - x(t)]^2 \geq \delta\right]. \tag{6.50}$$

Therefore, repeating the technique used in (3.42), but with Theorem 6.3, we get that

$$\mathbb{P}(\Omega_4 \cap (s \geq T)) \leq \frac{C_{4,7}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}. \tag{6.51}$$

On the other hand, we compute

$$\begin{aligned}\mathbb{P}(\Omega_4) &\leq \mathbb{P}(\Omega_4 \cap (s \geq T)) + \mathbb{P}(s \leq T)\\ &\leq \mathbb{P}(\Omega_4 \cap (s \geq T)) + \mathbb{P}(\gamma_M \leq T) + \mathbb{P}(\rho_M \leq T) + \mathbb{P}(\zeta_N \leq T) + \mathbb{P}(\tau_N \leq T).\end{aligned} \tag{6.52}$$

Now, substituting (6.22), (6.29), (6.46), (6.49) and (6.51) into (6.52) yields

$$\begin{aligned}\mathbb{P}(\Omega_4) \leq& \frac{C_{4,7}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}\\
&+ \frac{H(V_0) + \frac{\alpha_2\mu_2}{2}T + \frac{(\alpha_2+\lambda_2\delta_2)}{2}T + \frac{\sigma_2^2 4^{2\beta-2} T}{4} + \lambda_2|\delta_2|RT}{H(M^{-1}) \wedge H(M)}\\
&+ \frac{H(V_0) + \frac{\alpha_2\mu_2 T}{2} + \frac{(\alpha_2+\delta_2\lambda_2)T}{2} + \frac{\sigma_2^2 4^{2\beta-2} T}{4} + |\delta_2|\lambda_2 TZ + \bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)}\\
&+ \frac{H(X_0) + \frac{\alpha_1\mu_1 T}{2} + \frac{(\alpha_1+\lambda_1\delta_1)T}{2} + \frac{\sigma_1^2 R 4^{2\theta-2} T}{4} + \lambda_1|\delta_1|\bar{R}T}{H(N^{-1}) \wedge H(N)}\\
&+ \frac{H(X_0) + \frac{\alpha_1\mu_1 T}{2} + \frac{(\alpha_1+\lambda_1\delta_1)T}{2} + \frac{\sigma_1^2 M 4^{2\theta-2} T}{4} + |\delta_1|\lambda_1 LT + \bar{C}_{4,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)}.\end{aligned}$$

Choose M sufficiently large for

$$\frac{2\left[H(V_0)+\frac{\alpha_2\mu_2 T}{2}+\frac{[\alpha_2+\delta_2\lambda_2]T}{2}+\frac{\sigma_2^2 4^{2\beta-2}T}{4}\right]+|\delta_2|\lambda_2 TZ+\lambda_2|\delta_2|RT}{H(M^{-1})\wedge H(M)}<\frac{\varepsilon}{3},$$

then choose N sufficiently large for

$$\frac{2\left[H(X_0)+\frac{\alpha_1\mu_1 T}{2}+\frac{(\alpha_1+\lambda_1\delta_1)T}{2}\right]+\frac{\sigma_1^2 R4^{2\theta-2}T}{4}+\frac{\sigma_1^2 M4^{2\theta-2}T}{4}+\lambda_1|\delta_1|\bar{R}T+|\delta_1|\lambda_1 LT}{H(N^{-1})\wedge H(N)}<\frac{\varepsilon}{3}$$

and further choose Δ sufficiently small for

$$\frac{C_{4,7}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}+\frac{\bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1})\wedge H(M)}+\frac{\bar{C}_{4,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1})\wedge H(N)}<\frac{\varepsilon}{3}. \quad (6.53)$$

We then have

$$\mathbb{P}\left(\sup_{0\le t\le T}[X(t)-x(t)]^2\ge\delta\right)<\varepsilon, \quad (6.54)$$

as required. The proof is therefore complete. □

Theorem 6.4 shows that the continuous EM approximate solution of the asset price will converge in probability to the true solution, though this continuous EM approximation is not computable in practice. It is therefore necessary to show that this step process, which is computable, will converge to the true solution when time step is sufficiently small. The following theorem will show the convergence in probability of this step process .

Theorem 6.5. *Let $X(t)$ be the true solution and $\bar{x}(t)$ be the step process of EM approximate solution $x(t)$. Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|X(t)-\bar{x}(t)|\right)=0 \quad \textit{in probability.} \quad (6.55)$$

To prove Theorem 6.5, we need the following Lemma.

Lemma 6.7. *Let $x(t)$ be the continuous EM approximate solution and $\bar{x}(t)$ be the* *corresponding step process of $x(t)$. Then*

$$\lim_{\Delta \to 0} \left(\sup_{0 \le t \le T} [x(t) - \bar{x}(t)]^2 \right) = 0 \quad \textit{in probability.} \tag{6.56}$$

Proof. Let $\varepsilon > 0$ and $\delta \in (0,1)$ be arbitrarily small, then define

$$\Omega_4^* = \left[\omega; \sup_{0 \le t \le T} [x(t) - \bar{x}(t)]^2 \ge \delta \right]. \tag{6.57}$$

In the same way as in computation of (3.42), together with (6.57) and (6.47), we have

$$\mathbb{P}(\Omega_4^* \cap (h \ge T)) \le \frac{C_{4,8}(M,N,p)\Delta^{1-\frac{1}{p}}}{\delta}. \tag{6.58}$$

Note that

$$\mathbb{P}(\Omega_4^*) \le \mathbb{P}(\Omega_4^* \cap (h \ge T)) + \mathbb{P}(\gamma_M \le T) + \mathbb{P}(\zeta_N \le T). \tag{6.59}$$

Substituting (6.29), (6.46) and (6.58) yields

$$\begin{aligned} &\mathbb{P}(\Omega_4^*) \\ \le& \frac{C_{4,8}(M,N,p)\Delta^{1-\frac{1}{p}}}{\delta} \\ &+ \frac{H(V_0) + \frac{\alpha_2 \mu_2 T}{2} + \frac{(\alpha_2 + \delta_2 \lambda_2)T}{2} + \frac{\sigma_2^2 4^{2\beta-2} T}{4} + |\delta_2|\lambda_2 T Z + \bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} \\ &+ \frac{H(X_0) + \frac{\alpha_1 \mu_1 T}{2} + \frac{(\alpha_1 + \lambda_1 \delta_1)T}{2} + \frac{\sigma_1^2 M 4^{2\theta-2} T}{4} + |\delta_1|\lambda_1 L T + \bar{C}_{4,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)}. \end{aligned}$$

Choose M sufficiently large such that

$$\frac{H(V_0) + \frac{\alpha_2 \mu_2 T}{2} + \frac{(\alpha_2 + \delta_2 \lambda_2)T}{2} + \frac{\sigma_2^2 4^{2\beta-2} T}{4} + |\delta_2|\lambda_2 T Z}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{3}, \tag{6.60}$$

then choose N sufficiently large such that

$$\frac{H(X_0) + \frac{\alpha_1 \mu_1 T}{2} + \frac{(\alpha_1 + \lambda_1 \delta_1)T}{2} + \frac{\sigma_1^2}{4} M 4^{2\theta-2} T + |\delta_1|\lambda_1 L T}{H(N^{-1}) \wedge H(N)} < \frac{\varepsilon}{3} \tag{6.61}$$

and choose Δ sufficiently small such that

$$\frac{C_{4,8}(M,N,p)\Delta^{1-\frac{1}{p}}}{\delta} + \frac{\bar{C}_{4,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} + \frac{\bar{C}_{4,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1}) \wedge H(N)} < \frac{\varepsilon}{3}. \quad (6.62)$$

Hence, we have

$$\mathbb{P}\left(\sup_{0\le t\le T} [x(t) - \bar{x}(t)]^2 \ge \delta\right) < \varepsilon, \quad (6.63)$$

as required. The proof is therefore complete. □

The proof of Theorem 6.5 therefore can be obtained in the same way as Theorem 3.5 was proved but with Lemma 6.7 and Theorem 6.4.

Theorem 6.5 shows that the step process will converge to the true solution when the time step is sufficiently small. Therefore, let us choose initial condition ($X_0 = 0.5, V_0 = 0.04$), $\rho = 0.1$, parameters ($\theta = 1.2, \beta = 1.5$), $\lambda_1 = 1, \lambda_2 = 2$ with coefficients of the SDE model (5.1) (see Table 6.1) to illustrate the behaviour of the approximate solution to the SDE model (5.1) when $1 < \beta, \theta < \infty$. In this process, we use MATLAB® software (see Appendix A for code) to obtain the following graph (see Figure 3.1).

Table 6.1: Coefficients of the SDE model (5.1) when $\beta, \theta > 1$

Case	Parameters					
SDE 1	$\theta = 1.2$	$X(0) = 0.5$	$\alpha_1 = 4.21$	$\mu_1 = 3.4$	$\sigma_1 = 1.05$	$\delta_1 = 2$
SDE 2	$\beta = 1.1$	$V(0) = 0.04$	$\alpha_1 = 1.3$	$\mu_2 = 1.03$	$\sigma_2 = 1.054$	$\delta_2 = 1$

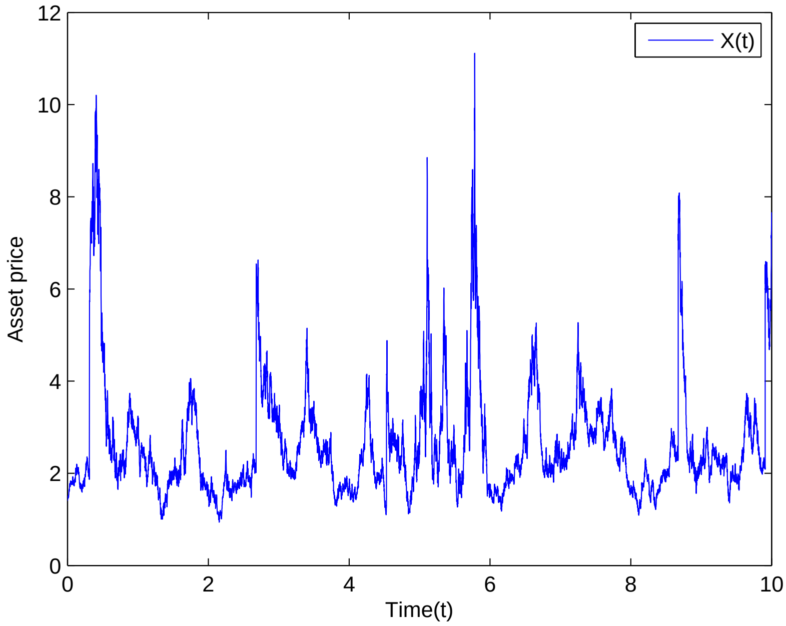

Figure 6.1: A sample path of the asset price $X(t)$ which is generated by the EM approximate solution to the mean-reverting-theta stochastic volatility model with Poisson jump over finite time, where $\theta = 1.2$ and $\beta = 1.5$.

6.4 Summary

In this chapter, we have focussed on the SDE model (5.1) in Chapter 5 but with parameters θ and β greater than 1. Since this model satisfies the local Lipschitz condition but does not obey the linear growth condition, we have examined the convergence in probability of the EM approximate solution to this model. In addition, the convergence property of the step process has been examined to show that it can be used to evaluate applications of this EM approximate solution in finance.

Chapter 7

Hybrid Poisson-Jump Stochastic Volatility Model for Asset Price

7.1 Introduction

In the financial world, Black-Scholes type market models which are driven by diffusion processes are widely used to evaluate various financial quantities. However, real market data show some deviations from this concept where unpredictable abrupt structural changes are present. Meanwhile, more and more empirical studies reveal that some of these extra properties can be modelled by a Poisson process. Thus, a mean-reverting-theta stochastic volatility model driven by a Poisson-jump process can be treated as one such processes that explains some of these phenomena. In the meantime, there is strong evidence to show that rate of return obeys the property of a Markov jump process and volatility jumps accordingly [16, 18, 27, 78, 9, 73, 75]. Therefore, a hybrid mean-reverting-theta stochastic volatility model with jumps can be transformed into a generalized financial market model by Markov jump processes. Thus, this hybrid stochastic volatility model has the SDE form:

$$\begin{aligned} dX(t) =& \alpha_1(r_1(u))(\mu_1(r_1(u)) - X(t^-))dt + \sigma_1(r_1(u))\sqrt{V(t^-)}X(t^-)^{\theta}dW_1(t) \\ &+ \delta_1(r_2(u))X(t^-)d\bar{N}_1(t), \\ dV(t) =& \alpha_2(r_1(u))(\mu_2(r_1(u)) - V(t^-))dt + \sigma_2(r_1(u))V(t^-)^{\beta}dW_2(t) \\ &+ \delta_2(r_2(u))V(t^-)d\bar{N}_2(t), \end{aligned} \tag{7.1}$$

where $V(t)$, $X(t)$, W_1, W_2, $\bar{N}_1(t)$ and $\bar{N}_1(t)$ are the same as defined in Chapter 5 with parameters θ and β greater than $\frac{1}{2}$. It is well known that almost every sample path of $r_1(.)$ and $r_2(.)$ is a right-continuous step function with finite number of sample jumps in any finite subinterval of $\mathbb{R}_+ := [0, \infty)$ and $\mathbb{R}_+ := (-1, \infty)$ respectively. We further assume that $r_1(.)$ and $r_2(.)$ are independent Markov chains that are also independent from the Brownian motions $W_1(.)$ and $W_2(.)$. In addition, the SDE model (7.1) has the parameters $\alpha_i(r_1(.)), \mu_i(r_1(.)), \sigma_i(r_1(.))$ and $\delta_i(r_1(.))(> -1)$ with the condition of $(\alpha_i(r_1(t)) + \lambda_i\delta_i(r_2(t))) > 0$ for $i = 1, 2$.

Unlike deterministic models such as ordinary differential equation models, which have an explicit unique solution for each initial condition, SDEs have no explicit solutions. Therefore, the method of finding computational solutions to SDEs models have become a more and more popular and powerful topic in mathematical finance. Thus, analytical properties of Euler-Maruyama (EM) numerical approximate solution to the SDE model (7.1) when $\frac{1}{2} \leq \theta, \beta \leq 1$ will be examined in this chapter.

As the SDE model (7.1) describes the asset price, interest rate and volatility, a natural requirement is to have a non-negative solution in practice. Provided that the diffusion coefficients of the SDE model (7.1) satisfy the global Lipschitz condition and the linear growth condition, we will first prove that the solution to the SDE model (7.1) will be non-negative with probability 1. We will then define the EM approximate solution to this SDE model and establish an upper bound for the expected value of the solution under the linear growth condition. On the other hand, a strong error bound for the EM approximate solution can be obtained over a finite time interval. Hence, we will show that the continuous EM approximate solution will converge to the true solution under the convergence in second moment (in L^2) property. However, this continuous EM approximate solution is not computable in practice. Therefore, the corresponding step process, which is computable, can be used to examine financial quantities. Thus, we will finally show that this step process will converge to the true solution of SDE model (7.1) when $\frac{1}{2} \leq \theta, \beta \leq 1$.

7.2 Non-negative solution

The natural requirement is that the solution to the asset price model should be non-negative in practice. The following lemmas state that the solution to the SDE model (7.1) will be non-negative with probability 1 when $\frac{1}{2} \leq \theta, \beta \leq 1$. In this process, we set $\bar{\alpha}_j = \max_{i \in \mathbb{S}_1} \alpha_i$, $\bar{\mu}_j = \max_{i \in \mathbb{S}_1} \mu_i$, $\bar{\sigma}_j = \max_{i \in \mathbb{S}_1} \sigma_i$ and $\bar{\delta}_j = \max_{i \in \mathbb{S}_2} |\delta_i|$ for $j = 1, 2$.

Non-negative $V(t)$

Lemma 7.1. *Assume $\frac{1}{2} \leq \beta \leq 1$. Then, given any initial values $V(0) = V_0 > 0$, $r_1(0) = i_0 \in \mathbb{S}_1$ and $r_2(0) = j_0 \in \mathbb{S}_2$, the second SDE of the model (7.1) has unique global solution $V(t)$ which will be non-negative for all $t \in [0, T]$ almost surely.*

Non-negative $X(t)$

Lemma 7.2. *Assume $\frac{1}{2} \leq \beta \leq 1$ and $\frac{1}{2} \leq \theta \leq 1$. Then, for given any initial values $V(0) = V_0 > 0$, $X(0) = X_0 > 0$, $r_1(0) = i_0 \in \mathbb{S}_1$ and $r_2(0) = j_0 \in \mathbb{S}_2$,* *the SDE of (7.1) has unique global solution $X(t)$ which will be non-negative for all $t \in [0,T]$ almost surely.*

The proofs of the above lemmas can be obtained in the same way as Lemma 5.1 and Lemma 5.2 were proved.

7.3 Convergence in second moment

The SDE model (7.1) has no explicit solution, so examination of its numerical approximate solution is helpful to understand its behaviour in financial systems. Accordingly, we will establish the EM approximate solution to this SDE model to examine its analytical properties.

Euler-Maruyama approximation

Given time step $\Delta \in (0,1)$, let $t_k = k\Delta$ and $r_{ak}^{\Delta} = r_a(k\Delta)$ for $k = 0, 1, ...[\frac{T}{\Delta}]$ and $a = 1, 2$, where $[\frac{T}{\Delta}]$ is the same as before, while setting $x_0 = X(0)$ and $v_0 = V(0)$, and $r_{a0}^{\Delta} = r_a(0) \in \mathbb{S}_a$ for $a = 1, 2$. The discrete time EM approximate solution is defined by

$$
\begin{aligned}
x_{k+1}(t) =& x_k + \alpha_1(r_{1k}^{\Delta})(\mu_1(r_{1k}^{\Delta}) - x_k)\Delta + \sigma_1(r_{1k}^{\Delta})\sqrt{|v_k|}|x_k|^{\theta}\Delta W_{1,k} + \delta_1(r_{2k}^{\Delta})|x_k|\Delta\bar{N}_{1,k},\\
v_{k+1}(t) =& v_k + \alpha_2(r_{1k}^{\Delta})(\mu_2(r_{1k}^{\Delta}) - v_k)\Delta + \sigma_2(r_{1k}^{\Delta})|v_k|^{\beta}\Delta W_{2,k} + \delta_2(r_{2k}^{\Delta})|v_k|\Delta\bar{N}_{2,k},
\end{aligned}
$$

where $\Delta = (t_{k+1} - t_k)$, $\Delta W_{i,k} = (W_i(t_{k+1}) - W_i(t_k))$ and $\Delta\bar{N}_{i,k} = (\bar{N}_i(t_{k+1}) - \bar{N}_i(t_k))$ for $i = 1, 2$. The corresponding continuous EM approximate solution is defined by

$$
\begin{aligned}
x(t) =& x_0 + \int_0^t \alpha_1(\bar{r}_1(u))(\mu_1(\bar{r}_1(u)) - \bar{x}(u))du + \int_0^t \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^{\theta}dW_1(u)\\
&+ \int_0^t \delta_1(\bar{r}_2(u))|\bar{x}(u)|d\bar{N}_1(u),
\end{aligned}
$$

$$v(t) = v_0 + \int_0^t \alpha_2(\bar{r}_1(u))(\mu_2(\bar{r}_1(u)) - \bar{v}(u))du + \int_0^t \sigma_2(\bar{r}_1(u))|\bar{v}(u)|^\beta dW_2(u)$$
$$+ \int_0^t \delta_2(\bar{r}_2(u))|\bar{v}(u)|d\bar{N}_2(u),$$

where $\bar{x}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} x_k 1_{[t_k,t_{k+1})}(t)$ and $\bar{v}(t) = \sum_{k=0}^{[\frac{T}{\Delta}]} v_k 1_{[t_k,t_{k+1})}(t)$ are step processes. That is $\bar{x}(t) = x_k$, $\bar{v}(t) = v_k$ and $\bar{r}(t) = r_k^\Delta$ for $t \in [t_k, t_{k+1})$, $k = 0, 1, 2, 3, ...[\frac{T}{\Delta}]$.

Upper bound

In the case of $\frac{1}{2} \leq \beta, \theta, \leq 1$, the diffusion coefficients of the SDE model (7.1) satisfy the linear growth condition. Therefore, the following theorems will establish an upper bound for the expected value of asset price and volatility which also help to compute the strong error bound of the continuous EM approximate solution to the SDE model (7.1) when $\frac{1}{2} \leq \theta, \beta \leq 1$.

Theorem 7.1. *Let $V(t)$ be the true solution and $v(t)$ be the continuous EM approximate solution to the second SDE in model (7.1). Then, for any $p \geq 2$, there is a constant $R_1(p)$ dependent on p, T, V_0 but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |V(t)|^p\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T} |v(t)|^p\right) \leq R_1(p). \tag{7.2}$$

The proof of Theorem 7.1 can be obtained in the same way as Theorem 5.1 was proved.

Theorem 7.2. *Let $X(t)$ be the true solution and $x(t)$ be the continuous EM approximate solution to the asset price. Then, for any $p \geq 2$, there is a constant $R_2(p)$ dependent on p, T, X_0, $R_1(p)$ but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0\leq t\leq T} |x(t)|^p\right) \leq R_2(p). \tag{7.3}$$

In the same way as in computation of Lemma 5.2, we will get the required proof.

Convergence in second moment of $v(t)$

The solution to SDE model (7.1) when $\frac{1}{2} \leq \theta, \beta \leq 1$ is non-negative with probability 1. The diffusion coefficients of this SDE model satisfy the linear growth condition, and the upper bound for the expected value of asset price and volatility have been established by Theorem 7.1 and Theorem 7.2. Thus, the following theorem will establish an error bound for the EM approximate solution to the volatility which gives one of the necessary conditions for convergence in second moment (in L^2) of the EM approximate solution to asset price.

Theorem 7.3. *Let $V(t)$ be the true solution and $v(t)$ be the EM approximate solution to the second SDE of (7.1). Then,*

$$\lim_{\Delta \to 0} \left(\sup_{0 \leq t \leq T} \mathbb{E}\Big| V(t) - v(t) \Big| \right) = 0. \tag{7.4}$$

To prove Theorem 7.3, we need the following lemma that shows the closeness of $v(t)$ and $\bar{v}(t)$ when the time step Δ is sufficiently small.

Lemma 7.3. *There is a constant $C_{5,1}$ independent of Δ such that*

$$\mathbb{E}\Big| v(t) - \bar{v}(t) \Big|^2 \leq C_{5,1}\Delta. \tag{7.5}$$

The proof can be obtained in the same way as Lemma 5.3 was proved.

Proof. For any $0 \leq t \leq T$, we easily compute

$$\begin{aligned} V(t) - v(t) = & \int_0^t \Big[\alpha_2(r_1(u))\mu_2(r_1(u)) - (\alpha_2(r_1(u)) + \lambda_2\delta_2(r_2(u)))V(u^-) \Big] \\ & - \Big[\alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u)) - (\alpha_2(\bar{r}_1(u)) + \lambda_2\delta_2(\bar{r}_2(u)))\bar{v}(u) \Big] du \\ & + \int_0^t \Big[\sigma_2(r_1(u))V(u^-)^\beta - \sigma_2(\bar{r}_1(u))\bar{v}(u)^\beta \Big] dW_2(u) \\ & + \int_0^t \Big[\delta_2(r_2(u))V(u^-) - \delta_2(\bar{r}_2(u))\bar{v}(u) \Big] dN_2(u). \end{aligned} \tag{7.6}$$

Now, set $e(u) = (V(u) - v(u))$ and $e(u^-) = (V(u^-) - v(u))$. Applying the itö-Doeblin formula for one jump process, we have

$$
\begin{aligned}
&\mathbb{E}\left(\varphi_k(e(t))\right) \\
\leq&\mathbb{E}\int_0^t |\varphi_k^{'}(e(u^-))|\, |\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|\, du \\
&+ \mathbb{E}\int_0^t |\varphi_k^{'}(e(u^-))| \Big| (\alpha_2(\bar{r}_1(u)) + \lambda_2\delta_2(\bar{r}_2(u)))\bar{v}(u) \\
&\qquad\qquad - (\alpha_2(r_1(u)) + \lambda_2\delta_2(r_2(u)))V(u^-)\Big| du \\
&+ \frac{1}{2}\mathbb{E}\int_0^t |\varphi_k^{''}(e(u^-)|\left(\sigma_2(r_1(u))V(u^-)^\beta - \sigma_2(\bar{r}_1(u))|\bar{v}(u)|^\beta\right)^2 du \\
&+ \lambda_2\mathbb{E}\int_0^t \left|\varphi_k(e(u)) - \varphi_k(e(u^-))\right| du,
\end{aligned}
$$

where $\varphi_k(.)$ has been defined in Lemma 5.1,

$$
\begin{aligned}
\leq&\mathbb{E}\int_0^t |\varphi_k^{'}(e(u^-))|\, |\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|\, du \\
&+ \mathbb{E}\int_0^t |\varphi_k^{'}(e(u^-))| \left|(\alpha_2(\bar{r}_1(u)))\bar{v}(u) - (\alpha_2(r_1(u)))V(u^-)\right| du \\
&+ \lambda_2\mathbb{E}\int_0^t |\varphi_k^{'}(e(u^-))| \left|\delta_2(\bar{r}_2(u))\bar{v}(u) - \delta_2(r_2(u))V(u^-)\right| du \\
&+ \mathbb{E}\int_0^t |\varphi_k^{''}(e(u^-))||\bar{v}(u)|^{2\beta}\left(\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right)^2 du \\
&+ \mathbb{E}\int_0^t |\varphi_k^{''}(e(u^-))|\sigma_2(r_1(u))^2\left(V(u^-)^\beta - |\bar{v}(u)|^\beta\right)^2 du \\
&+ \lambda_2\mathbb{E}\int_0^t \left|\varphi_k([1+\delta_2(r_2(u))]e(u^-)) - \varphi_k(e(u^-))\right| du.
\end{aligned}
\tag{7.7}
$$

By the property of $\varphi_k(u)$ and the mean value theorem, we then have

$$
\begin{aligned}
\leq&\mathbb{E}\int_0^t |\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|\, du \\
&+ \mathbb{E}\int_0^t |\bar{v}(u)|\, |\alpha_2(\bar{r}_1(u)) - \alpha_2(r_1(u))|\, du + \mathbb{E}\int_0^t |\alpha_2(r_1(u))|\, |\bar{v}(u) - V(u)|\, du \\
&+ \lambda_2\mathbb{E}\int_0^t |\bar{v}(u)|\, |\delta_2(\bar{r}_2(u)) - \delta_2(r_2(u))|\, du + \lambda_2\mathbb{E}\int_0^t |\delta_2(r_2(u))|\, |\bar{v}(u) - V(u)|\, du \\
&+ \mathbb{E}\int_0^t \frac{2}{ka_k^{2\beta}}|\bar{v}(u)|^{2\beta}\left(\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right)^2 du
\end{aligned}
$$

$$+\,\mathbb{E}\int_0^t \frac{2}{ke^{2\beta}(u)}\sigma_2(r_1(u))^2\Big|V(u)^\beta - |\bar{v}(u)|^\beta\Big|^2 du$$
$$+\,\lambda_2\bar{\delta}_2\mathbb{E}\int_0^t |\max_{q\in\Re}\varphi_k^{'}(q)||V(u)-v(u)|du.$$

Rearranging the terms on the right hand side, we further get that

$$\begin{aligned}
\leq & \mathbb{E}\int_0^t |\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|\,du \\
&+ [\bar{\alpha}_2 + \lambda_2\bar{\delta}_2]\mathbb{E}\int_0^t |\bar{v}(u) - V(u)|\,du + \lambda_2\bar{\delta}_2\int_0^t \mathbb{E}|V(u)-v(u)|du \\
&+ \mathbb{E}\int_0^t |\bar{v}(u)|\Big[\,|\alpha_2(r_1(u)) - \alpha_2(\bar{r}_1(u))| + \lambda_2\,|\delta_2(r_2(u)) - \delta_2(\bar{r}_2(u))|\,\Big]du \\
&+ \frac{2}{ka_k^{2\beta}}\mathbb{E}\int_0^t |\bar{v}(u)|^{2\beta}\left(\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right)^2 du \\
&+ \mathbb{E}\int_0^t \frac{4\bar{\sigma}_2^2}{ke^{2\beta}(u)}\,|V(u)-v(u)|^{2\beta}\,du + \mathbb{E}\int_0^t \frac{4\bar{\sigma}_2^2}{ka_k^{2\beta}}\,|v(u)-\bar{v}(u)|^{2\beta}\,du \\
\leq & \mathbb{E}\int_0^t |\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|\,du \\
&+ [\bar{\alpha}_2 + 2\lambda_2\bar{\delta}_2]\mathbb{E}\int_0^t |V(u)-v(u)|\,du + [\bar{\alpha}_2 + \lambda_2\bar{\delta}_2]\int_0^t \left[\mathbb{E}\,|v(u)-\bar{v}(u)|^2\right]^{\frac{1}{2}}du \\
&+ \mathbb{E}\int_0^t |\bar{v}(u)|\Big[\,|\alpha_2(r_1(u)) - \alpha_2(\bar{r}_1(u))| + \lambda_2\,|\delta_2(r_2(u)) - \delta_2(\bar{r}_2(u))|\,\Big]du \\
&+ \frac{2}{ka_k^{2\beta}}\mathbb{E}\int_0^t |\bar{v}(u)|^{2\beta}\left(\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right)^2 du \\
&+ \frac{4\bar{\sigma}_2^2 T}{k} + \int_0^t \frac{4\bar{\sigma}_2^2}{ka_k^{2\beta}}\left(\mathbb{E}\,|v(u)-\bar{v}(u)|^2\right)^\beta du.
\end{aligned}$$

By Lemma (7.3), we obtain

$$\begin{aligned}
\leq & \mathbb{E}\int_0^t |\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|\,du + \frac{4\bar{\sigma}_2^2 T}{k} \\
&+ [\bar{\alpha}_2 + 2\lambda_2\bar{\delta}_2]\int_0^t \mathbb{E}\,|V(u)-v(u)|\,du + [\bar{\alpha}_2 + \lambda_2\bar{\delta}_2]\,[C_{5,1}\Delta]^{\frac{1}{2}}\,T \\
&+ \mathbb{E}\int_0^t |\bar{v}(u)|\Big[\,|\alpha_2(r_1(u)) - \alpha_2(\bar{r}_1(u))| + \lambda_2\,|\delta_2(r_2(u)) - \delta_2(\bar{r}_2(u))|\,\Big]du \\
&+ \frac{4\bar{\sigma}_2^2}{ka_k^{2\beta}}[C_{5,1}\Delta]^\beta T + \frac{2}{ka_k^{2\beta}}\mathbb{E}\int_0^t |\bar{v}(u)|^{2\beta}\left(\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right)^2 du,
\end{aligned} \tag{7.8}$$

which gives

$$
\begin{aligned}
&\mathbb{E}\left(\varphi_k(e(t))\right)\\
=&\frac{4\bar{\sigma}_2^2 T}{k}+\frac{4\bar{\sigma}_2^2}{k a_k^{2\beta}}[C_{5,1}\Delta]^{\beta}T+[\bar{\alpha}_2+2\lambda_2\bar{\delta}_2]\int_0^t \mathbb{E}\left|V(u)-v(u)\right|du\\
&+[\bar{\alpha}_2+\lambda_2\bar{\delta}_2]\left[C_{5,1}\Delta\right]^{\frac{1}{2}}T+A(t)+B(t)+\lambda_2 D(t)+\frac{2}{k a_k^{2\beta}}E(t).
\end{aligned}
\tag{7.9}
$$

By a similar technique as used to compute (4.18), we get

$$
\begin{aligned}
A(t)=&\mathbb{E}\int_0^t |\alpha_2(r_1(u))\mu_2(r_1(u))-\alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|du\\
\leq&\mathbb{E}\int_0^T |\alpha_2(r_1(u))\mu_2(r_1(u))-\alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))|du\leq(C_{5,2}\Delta+0\Delta).
\end{aligned}
\tag{7.10}
$$

Let

$$
\begin{aligned}
B(t)=&\mathbb{E}\int_0^t |\bar{v}(u)|\left|\alpha_2(r_1(u))-\alpha_2(\bar{r}_1(u))\right|du\\
\leq&\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}}\mathbb{E}\left[\mathbb{E}\left[|\alpha_2(r_1(u))-\alpha_2(\bar{r}_1(u))||v_d|\backslash(I_{r_1(u)\neq r_1(t_d)})\right]\right]du\\
=&\sum_{d=0}^{[\frac{T}{\Delta}]}\int_{t_d}^{t_{d+1}}\mathbb{E}\left[\mathbb{E}\left[|\alpha_2(r_1(u))-\alpha_2(\bar{r}_1(u))|\backslash(I_{r_1(u)\neq r_1(t_d)})\right]\mathbb{E}\left[|v_d|\backslash(I_{r_1(u)\neq r_1(t_d)})\right]\right]du.
\end{aligned}
$$

where v_k and $(I_{r_1(u)\neq r_1(t_d)})$ are conditionally independent with respect to the σ-algebra generated by $r_1(t_d)$. Applying as similar technique as was used to obtain (4.18) and (5.47), together with Theorem 7.1 yields

$$
\begin{aligned}
B(t)\leq& 2\bar{\alpha}_2[\max_{0\leq i\leq N_1}(-\kappa_{1ii})\Delta+0\Delta]\int_0^T \mathbb{E}|\bar{v}(u)|du\\
\leq& 2\bar{\alpha}_2[\max_{0\leq i\leq N_1}(-\kappa_{1ii})\Delta+0\Delta](R_1(2))^{\frac{1}{2}}T\leq(C_{5,3}\Delta+0\Delta).
\end{aligned}
\tag{7.11}
$$

Similarly, we compute

$$
\begin{aligned}
D(t)=&\mathbb{E}\int_0^t |\bar{v}(u)|\left|\delta_2(r_2(u))-\delta_2(\bar{r}_2(u))\right|du\\
\leq&2\bar{\delta}_2[\max_{0\leq i\leq N_2}(-\kappa_{2ii})\Delta+0\Delta]\int_0^T \mathbb{E}|\bar{v}(u)|du\\
\leq&2\bar{\delta}_2[\max_{0\leq i\leq N_2}(-\kappa_{2ii})\Delta+0\Delta]\sqrt{R_1(2)}T\leq(C_{5,4}\Delta+0\Delta)
\end{aligned}
\tag{7.12}
$$

and

$$
\begin{aligned}
E(t) =&\mathbb{E}\int_0^t |\bar{v}(u)|^{2\beta}\left(\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right)^2 \\
\leq& 4\sigma_2^2[\max_{0\leq i\leq N_1}(-\kappa_{1ii})\Delta + 0\Delta]R(2)^{\beta}T = (C_{5,5}\Delta + 0\Delta).
\end{aligned} \tag{7.13}
$$

Substituting $A(t)$, $B(t)$, $D(t)$ and $F(t)$ into (7.9), we then have

$$
\begin{aligned}
\mathbb{E}\left(\varphi_k(e(t))\right) \leq& \frac{4\bar{\sigma}_2^2 T}{k} + \frac{4\bar{\sigma}_2^2[C_{5,1}\Delta]^{\beta}T}{ka_k^{2\beta}} + [\bar{\alpha}_2 + \lambda_2\bar{\delta}_2]\,[C_{5,1}\Delta]^{\frac{1}{2}}\,T \\
&+ (C_{5,2}\Delta + 0\Delta) + (C_{5,3}\Delta + 0\Delta) + \lambda_2(C_{5,4}\Delta + 0\Delta) \\
&+ \frac{2}{ka_k^{2\beta}}(C_{5,5}\Delta + 0\Delta) + [\bar{\alpha}_2 + 2\lambda_2\bar{\delta}_2]\int_0^t \mathbb{E}\,|V(u) - v(u)|\,du.
\end{aligned} \tag{7.14}
$$

By (iii) properties of the function φ_k defined in Lemma 5.1 and Gronwall's inequality, we further get that

$$
\begin{aligned}
\sup_{0\leq t\leq T}\mathbb{E}\,|V(u) - v(u)| \leq& \Big[(C_{5,6}\Delta + C_{5,7}\Delta^{\frac{1}{2}} + 0\Delta) + \frac{4[C_{5,1}\Delta]^{\beta}T\bar{\sigma}_2^2}{ka_k^{2\beta}} \\
&+ \frac{2(C_{5,5}\Delta + 0\Delta)}{ka_k^{2\beta}} + \frac{4\bar{\sigma}_2^2 T}{k} + a_{k-1}\Big]e^{\left(\bar{\alpha}_2 + 2\lambda_2\bar{\delta}_2\right)T}.
\end{aligned} \tag{7.15}
$$

Now, choose k sufficiently large for

$$
\left[\frac{4\bar{\sigma}_2^2 T}{k} + a_{k-1}\right]e^{\left(\bar{\alpha}_2 + 2\lambda_2\bar{\delta}_2\right)T} < \frac{\varepsilon}{2} \tag{7.16}
$$

and then choose Δ sufficiently small for

$$
\left[(C_{5,6}\Delta + C_{5,7}\Delta^{\frac{1}{2}} + 0\Delta) + \frac{4[C_{5,1}\Delta]^{\beta}T\bar{\sigma}_2^2}{ka_k^{2\beta}} + \frac{2(C_{5,5}\Delta + 0\Delta)}{ka_k^{2\beta}}\right]e^{\left(\bar{\alpha}_2 + 2\lambda_2\bar{\delta}_2\right)T} < \frac{\varepsilon}{2}. \tag{7.17}
$$

Hence, we have

$$
\sup_{0\leq t\leq T}\mathbb{E}\,|V(u) - v(u)| < \varepsilon, \tag{7.18}
$$

as required. The proof of our theorem is therefore complete. □

Let us establish the following theorem which shows convergence in second moment (in L^2) of the EM approximate solution to the volatility when the time step is small enough.

Theorem 7.4. *Let $V(t)$ be the true solution and $v(t)$ be the continuous EM approximate solution to the second SDE of (7.1). Then,*

$$\lim_{\Delta \to 0} \mathbb{E}\left(\sup_{0 \le t \le T} [V(t) - v(t)]^2 \right) = 0. \tag{7.19}$$

Proof. For any $t \in [0, T]$, we compute

$$\begin{aligned} [V(t) - v(t)]^2 \le & 4 \left[\int_0^t \alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u)) du \right]^2 \\ & + 4 \left[\int_0^t \left(\alpha_2(r_1(u))V(u^-) - \alpha_2(\bar{r}_1(u))\bar{v}(u) \right) du \right]^2 \\ & + 4 \left[\int_0^t \left(\sigma_2(r_1(u))V(u^-)^\beta - \sigma_2(\bar{r}_1(u))|\bar{v}(u)|^\beta \right) dW_2(u) \right]^2 \\ & + 4 \left[\int_0^t \left(\delta_2(r_2(u))V(u^-) - \delta_2(\bar{r}_2(u))\bar{v}(u) \right) d\bar{N}_2(u) \right]^2. \end{aligned} \tag{7.20}$$

Taking the expectation for any $t_1 \in [0, T]$, by the Burkholder-Davis-Gundy inequality and the Hölder inequality, we then have

$$\begin{aligned} & \mathbb{E}\left(\sup_{0 \le t \le t_1} [V(t) - v(t)]^2 \right) \\ \le & 4T \int_0^{t_1} \mathbb{E}\left[\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u)) \right]^2 du \\ & + 4T \int_0^{t_1} \mathbb{E}\left[\alpha_2(r_1(u))V(u^-) - \alpha_2(\bar{r}_1(u))\bar{v}(u) \right]^2 du \\ & + 16 \int_0^{t_1} \mathbb{E}\left[\sigma_2(r_1(u))V(u^-)^\beta - \sigma_2(\bar{r}_1(u))|\bar{v}(u)|^\beta \right]^2 du \\ & + 16\lambda_2 \int_0^{t_1} \mathbb{E}\left[\delta_2(r_2(u))V(u^-) - \delta_2(\bar{r}_2(u))\bar{v}(u) \right]^2 du. \end{aligned} \tag{7.21}$$

Rearranging the terms on the right hand side, we further get that

$$\begin{aligned} \le & 4T \int_0^{t_1} \mathbb{E}\left[\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u)) \right]^2 du \\ & + 8T \int_0^{t_1} \mathbb{E}|\bar{v}(u)|^2 \left[\alpha_2(r_1(u)) - \alpha_2(\bar{r}_1(u)) \right]^2 + \bar{\alpha}_2^2 \mathbb{E}\left[V(u) - \bar{v}(u) \right]^2 du \end{aligned}$$

$$+\,32\lambda_2\int_0^{t_1}\mathbb{E}|\bar{v}(u)|^2\left[\delta_2(r_2(u))-\delta_2(\bar{r}_2(u))\right]^2+\bar{\delta}_2^2\mathbb{E}\left[V(u)-\bar{v}(u)\right]^2du$$

$$+\,32\int_0^{t_1}\mathbb{E}|\bar{v}(u)|^{2\beta}\left[\sigma_2(r_1(u))-\sigma_2(\bar{r}_1(u))\right]^2+\mathbb{E}\bar{\sigma}_2^2\left[V(u)-\bar{v}(u)\right]^{2\beta}du$$

$$\le 4T\int_0^{t_1}\mathbb{E}\left[\alpha_2(r_1(u))\mu_2(r_1(u))-\alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))\right]^2du$$

$$+\,8T\int_0^{t_1}\mathbb{E}|\bar{v}(u)|^2\left[\alpha_2(r_1(u))-\alpha_2(\bar{r}_1(u))\right]^2du$$

$$+\,32\lambda_2\int_0^{t_1}\mathbb{E}|\bar{v}(u)|^2\left[\delta_2(r_2(u))-\delta_2(\bar{r}_2(u))\right]^2du$$

$$+\,32\int_0^{t_1}\mathbb{E}|\bar{v}(u)|^{2\beta}\left[\sigma_2(r_1(u))-\sigma_2(\bar{r}_1(u))\right]^2du$$

$$+\,[8T\bar{\alpha}_2^2+32\lambda_2\bar{\delta}_2^2+32\bar{\sigma}_2]\int_0^{t_1}\mathbb{E}\left[V(u)-v(u)\right]^2+\mathbb{E}\left[v(u)-\bar{v}(u)\right]^2du$$

$$+\,32\bar{\sigma}_2^2\int_0^{t_1}\mathbb{E}\left|V(u)-v(u)\right|+\mathbb{E}\left|v(u)-\bar{v}(u)\right|du.$$

Applying the techniques used to compute (4.18) and (7.11), by Lemma 7.3, we then obtain

$$\le 16T\bar{\alpha}_2^2\bar{\mu}_2^2(C_{5,8}\Delta+0(\Delta))+32TR_1(2)\bar{\alpha}_2^2(C_{5,9}\Delta+0(\Delta))$$

$$+\,128\lambda_2\bar{\delta}_2^2TR_1(2)(C_{5,10}\Delta+0(\Delta))+128\bar{\sigma}_2^2T[R_1(2)]^\beta(C_{5,11}\Delta+0(\Delta))$$

$$+\,[8T\bar{\alpha}_2^2+32\lambda_2\bar{\delta}_2^2+32\bar{\sigma}_2]\int_0^{t_1}\mathbb{E}\left[V(u)-v(u)\right]^2du$$

$$+\,32\bar{\sigma}_2^2\int_0^{t_1}\mathbb{E}\left|V(u)-v(u)\right|du+[8T\bar{\alpha}_2^2+32\lambda_2\bar{\delta}_2^2+32\bar{\sigma}_2]C_{5,1}\Delta T+32\bar{\sigma}_2^2[C_{5,1}\Delta]^{\frac{1}{2}}T$$

$$\le\left[C_{5,12}\Delta+32\bar{\sigma}_2^2[C_{5,1}\Delta]^{\frac{1}{2}}T+0(\Delta)\right]+32\bar{\sigma}_2^2\int_0^{t_1}\left(\sup_{0\le u_1\le u}\mathbb{E}\left|V(u_1)-v(u_1)\right|\right)du$$

$$+\,[8T\bar{\alpha}_2^2+32\lambda_2\bar{\delta}_2^2+32\bar{\sigma}_2]\int_0^{t_1}\mathbb{E}\left(\sup_{0\le u_1\le u}\left[V(u)-v(u)\right]^2\right)du.$$

By Gronwall's inequality, we have

$$\mathbb{E}\left(\sup_{0\le t\le t_1}\left[V(t)-v(t)\right]^2\right)\le\Big[\left(\sup_{0\le t\le T}\mathbb{E}\left|V(t)-v(t)\right|\right)32\bar{\sigma}_2^2T+C_{5,12}\Delta$$
$$+\,32\bar{\sigma}_2^2[C_{5,1}\Delta]^{\frac{1}{2}}T+0(\Delta)\Big]e^{[8T\bar{\alpha}_2^2+32\lambda_2\bar{\delta}_2^2+32\bar{\sigma}_2]T},\qquad(7.22)$$

as required. The proof of our theorem will therefore be completed by Theorem 7.3

and letting $\Delta \to 0$. $\square$

Convergence in second moment of $x(t)$

Let us focus on our main result which gives convergence in second moment (in L^2) of the continuous EM approximate solution to the true solution of SDE model (7.1). Thus, we will first establish convergence (in L^1) of the EM approximate value to the asset price by the following theorem.

Theorem 7.5. *Let $X(t)$ be the true solution and $x(t)$ be the continuous EM approximate solution to the SDE model (7.1). Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}\mathbb{E}\left|X(t)-x(t)\right|\right)=0. \tag{7.23}$$

To prove Theorem 7.5, we need the following Lemma that can be obtained in the same way as Lemma 5.3 was proved.

Lemma 7.4. *There exists a constant $C_{5,13}$ independent of Δ such that*

$$\mathbb{E}\left[x(t)-\bar{x}(t)\right]^4 \le C_{5,13}\Delta^2. \tag{7.24}$$

Proof. (of Theorem 7.5) For any $0\le t\le T$, we have

$$\begin{aligned}
&(X(t)-x(t))\\
&=\int_0^t \Big[\alpha_1(r_1(u))\mu_1(r_1(u))-(\alpha_1(r_1(u))+\lambda_1\delta_1(r_1(u)))X(u^-)\Big]\\
&\qquad -\Big[\alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))-(\alpha_1(\bar{r}_1(u))+\lambda_1\delta_1(\bar{r}_1(u)))\bar{x}(u)\Big]du \qquad (7.25)\\
&\quad+\int_0^t \Big[\sigma_1(r_1(u))\sqrt{V(u^-)}X(u^-)^{\theta}-\sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^{\theta}\Big]dW_1(u)\\
&\quad+\int_0^t \Big[\delta_1(r_2(u))X(u^-)-\delta_1(\bar{r}_2(u))\bar{x}(u)\Big]dN_1(u).
\end{aligned}$$

Then, set $e(u)=(X(u)-x(u))$ and $e(u^-)=(X(u^-)-x(u))$. Now, applying Itô-Doeblin's formula for one jump process with function $\phi_k(.)$ which has been defined in Lemma 5.2, we obtain

$$
\begin{aligned}
&\mathbb{E}\left(\phi_k(e(t))\right.\\
=&\mathbb{E}\int_0^t \phi_k^{'}(e(u^-))\Big[\alpha_1(r_1(u))\mu_1(r_1(u))-\alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))\Big]du\\
&+\mathbb{E}\int_0^t \phi_k^{'}(e(u^-))\Big[(\alpha_1(\bar{r}_1(u))+\lambda_1\delta_1(\bar{r}_2(u)))\bar{x}(u)\\
&\qquad\qquad -(\alpha_1(r_1(u))+\lambda_1\delta_1(r_2(u)))X(u^-)\Big]du\\
&+\frac{1}{2}\mathbb{E}\int_0^t \phi_k^{''}(e(u^-))\Big[\sigma_1(r_1(u))\sqrt{V(u^-)}X(u^-)^\theta-\sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2du\\
&+\lambda_1\mathbb{E}\int_0^t\Big[\phi_k(e(u))-\phi_k(e(u^-))\Big]du.
\end{aligned}
$$

Rearranging the terms on right hand side, we further compute that

$$
\begin{aligned}
\leq&\mathbb{E}\int_0^t |\phi_k^{'}(e(u^-))|\,|\alpha_1(r_1(u))\mu_1(r_1(u))-\alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))|\,du\\
&+\mathbb{E}\int_0^t |\phi_k^{'}(e(u^-))|\,\Big|\alpha_1(\bar{r}_1(u)))|\bar{x}(u)|-\alpha_1(r_1(u))X(u^-)\Big|\,du\\
&+\lambda_1\mathbb{E}\int_0^t |\phi_k^{'}(e(u^-))|\,\Big|\delta_1(\bar{r}_2(u))|\bar{x}(u)|-\delta_1(r_2(u))X(u^-)\Big|\,du\\
&+\frac{1}{2}\mathbb{E}\int_0^t |\phi_k^{''}(e(u^-))|\Big[\sigma_1(r_1(u))\sqrt{V(u^-)}X(u^-)^\theta-\sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2du\\
&+\lambda_1\mathbb{E}\int_0^t\Big[\phi_k([1+\delta_1(r_2(u))]e(u^-))-\phi_k(e(u^-))\Big]du.
\end{aligned}
$$

By the mean value theorem and properties of $\phi_k(u)$, we then have

$$
\begin{aligned}
\leq&\mathbb{E}\int_0^t \Big|\alpha_1(r_1(u))\mu_1(r_1(u))-\alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))\Big|du\\
&+\mathbb{E}\int_0^t \Big|\alpha_1(\bar{r}_1(u)))\bar{x}(u)-\alpha_1(r_1(u))X(u)\Big|du\\
&+\lambda_1\mathbb{E}\int_0^t \Big|\delta_1(\bar{r}_2(u))\bar{x}(u)-\delta_1(r_2(u))X(u)\Big|du\\
&+\frac{1}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}\Big[\sigma_1(r_1(u))\sqrt{V(u)}X(u)^\theta-\sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2du\\
&+\lambda_1\mathbb{E}\int_0^t \Big|\sup_{q\in\mathbb{R}}\phi_k^{'}(q)\Big|\Big|\delta_1(r_2(u))\Big|\Big|X(u^-)-x(u)\Big|du
\end{aligned}
$$

$$
\begin{aligned}
\leq & \mathbb{E}\int_0^t \Big|\alpha_1(r_1(u))\mu_1(r_1(u)) - \alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))\Big| du \\
& + [\bar{\alpha}_1 + 2\lambda_1\bar{\delta}_1]\mathbb{E}\int_0^t \Big|X(u) - x(u)\Big| du + \mathbb{E}\int_0^t |\bar{x}(u)|\Big|\alpha_1(\bar{r}_1(u))) - \alpha_1(r_1(u))\Big| du \\
& + \lambda_1\mathbb{E}\int_0^t |\bar{x}(u)|\Big|\delta_1(\bar{r}_2(u)) - \delta_1(r_2(u))\Big| du + [\bar{\alpha}_1 + \lambda_1\bar{\delta}_1]\mathbb{E}\int_0^t \Big|x(u) - \bar{x}(u)\Big| du \\
& + \frac{1}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}\Big[\sigma_1(r_1(u))\sqrt{V(u)}X(u)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2 du.
\end{aligned}
$$

In the same way as in computation of (7.10) and (7.11) but with Lemma 7.4, we further get that

$$
\begin{aligned}
& \mathbb{E}\left(\phi_k(e(t))\right) \\
\leq & 2\bar{\alpha}\bar{\mu}_1[C_{5,14}\Delta + \Delta(0)]T + 2\bar{\alpha}_1[R_2(2)]^{\frac{1}{2}}[C_{5,15}\Delta + \Delta(0)]T + [\bar{\alpha}_1 + \lambda_1\bar{\delta}_1][C_{5,13}\Delta^2]^{\frac{1}{4}}T \\
& + \lambda_1\bar{\delta}_1[R_2(2)]^{\frac{1}{2}}[C_{5,16}\Delta + \Delta(0)]T + [\bar{\alpha}_1 + 2\lambda_1\bar{\delta}_1]\mathbb{E}\int_0^t \Big|X(u) - x(u)\Big| du \\
& + \mathbb{E}\int_0^t \frac{1}{ke(u)^{2\theta}}\Big[\sigma_1(r_1(u))\sqrt{V(u)}X(u)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2 du.
\end{aligned}
\tag{7.26}
$$

Let us consider

$$
\begin{aligned}
Q(t) = & \frac{1}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}\left[\sigma_1(r_1(u))\sqrt{V(u)}X(u)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\right]^2 du \\
\leq & \frac{5}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}|\bar{v}(u)||\bar{x}(u)|^{2\theta}\left|\sigma_1(\bar{r}_1(u)) - \sigma_1(r_1(u))\right|^2 du \\
& + \frac{5}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}|\sigma_1(r_1(u))|^2|\bar{x}(u)|^{2\theta}\left|\sqrt{V(u)} - \sqrt{|v(u)|}\right|^2 du \\
& + \frac{5}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}|\sigma_1(r_1(u))|^2|\bar{x}(u)|^{2\theta}\left|\sqrt{|v(u)|} - \sqrt{|\bar{v}(u)|}\right|^2 du \\
& + \frac{5}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}|\sigma_1(r_1(u))|^2|V(u)|\left|X(u)^\theta - |x(u)|^\theta\right|^2 du \\
& + \frac{5}{2}\mathbb{E}\int_0^t \frac{2}{ke(u)^{2\theta}}|\sigma_1(r_1(u))|^2|V(u)|\left||x(u)|^\theta - |\bar{x}(u)|^\theta\right|^2 du.
\end{aligned}
$$

In the same way as in computation of (7.11) and Lemma 7.4, Lemma 7.3, we have

$$
\leq \frac{20\bar{\sigma}_1^2 R_1^{\frac{1}{2}}(2)R_2^{\frac{\theta}{2}}(4)}{ka_k^{2\theta}}[C_{5,17}\Delta + \Delta(0)]T + \frac{5\bar{\sigma}_1^2 R_2^\theta(4)T}{ka_k^{2\theta}}
$$

$$
\begin{aligned}
&+\frac{5\bar{\sigma}_1^2R_2^{\theta}(4)T}{ka_k^{2\theta}}+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}[C_{5,1}\Delta]T+\int_0^t\mathbb{E}\frac{5\bar{\sigma}_1^2}{ke(u)^{2\theta}}|\bar{v}(u)|\left[X(u)-x(u)\right]^{2\theta}du\\
&+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\int_0^t\mathbb{E}|\bar{v}(u)|\left[x(u)-\bar{x}(u)\right]^{2\theta}du+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\int_0^t\mathbb{E}\left[V(u)-v(u)\right]^2du\\
\leq&\frac{20\bar{\sigma}_1^2R_1^{\frac{1}{2}}(2)R_2^{\frac{\theta}{2}}(4)}{ka_k^{2\theta}}[C_{5,17}\Delta+\Delta(0)]T+\frac{10\bar{\sigma}_1^2R_2^{\theta}(4)T}{ka_k^{2\theta}}+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}[C_{5,1}\Delta]T\\
&+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}T}{k}+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}}{ka_k^{2\theta}}\int_0^t\left(\mathbb{E}\left[x(u)-\bar{x}(u)\right]^2\right)^{\frac{1}{2}}du\\
&+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}}{ka_k^{2\theta}}\int_0^t\left(\mathbb{E}\left[x(u)-\bar{x}(u)\right]^4\right)^{\frac{1}{2}}du+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\int_0^t\mathbb{E}\left[V(u)-v(u)\right]^2du,
\end{aligned}
$$

which gives

$$
\begin{aligned}
\leq&\frac{20\bar{\sigma}_1^2R_1^{\frac{1}{2}}(2)R_2^{\frac{\theta}{2}}(4)}{ka_k^{2\theta}}[C_{5,17}\Delta+\Delta(0)]T+\frac{10\bar{\sigma}_1^2R_2^{\theta}(4)T}{ka_k^{2\theta}}+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}[C_{5,1}\Delta]T\\
&+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}T}{k}+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}}{ka_k^{2\theta}}\left(C_{5,13}\Delta^2\right)^{\frac{1}{4}}T+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}}{ka_k^{2\theta}}\left(C_{5,13}\Delta^2\right)^{\frac{1}{2}}T\\
&+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\int_0^t\mathbb{E}\left[V(u)-v(u)\right]^2du\\
=&\frac{10\bar{\sigma}_1^2R_2^{\theta}(4)T}{ka_k^{2\theta}}+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}T}{k}+\frac{[C_{5,18}\Delta+C_{5,19}\Delta^{\frac{1}{2}}+\Delta(0)]}{ka_k^{2\theta}}\\
&+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\int_0^t\mathbb{E}\left[V(u)-v(u)\right]^2du.
\end{aligned}
\tag{7.27}
$$

Substituting (7.27) into (7.26) yields

$$
\begin{aligned}
\mathbb{E}\left(\phi_k(e(t))\right)\leq&2\bar{\alpha}\bar{\mu}_1[C_{5,14}\Delta+\Delta(0)]T+2\bar{\alpha}_1[R_2(2)]^{\frac{1}{2}}[C_{5,15}\Delta+\Delta(0)]T\\
&+\lambda_1\bar{\delta}_1[R_2(2)]^{\frac{1}{2}}[C_{5,16}\Delta+\Delta(0)]T+[\bar{\alpha}_1+\lambda_1\bar{\delta}_1][C_{5,13}\Delta^2]^{\frac{1}{4}}T\\
&+\frac{10\bar{\sigma}_1^2R_2^{\theta}(4)T}{ka_k^{2\theta}}+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}T}{k}+\frac{[C_{5,18}\Delta+C_{5,19}\Delta^{\frac{1}{2}}+\Delta(0)]}{ka_k^{2\theta}}\\
&+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\int_0^t\mathbb{E}\left[V(u)-v(u)\right]^2du+[\bar{\alpha}_1+\lambda_1\bar{\delta}_1]\mathbb{E}\int_0^t|X(u)-x(u)|\,du\\
\leq&[C_{5,20}\Delta+C_{5,21}\Delta^{\frac{1}{2}}+\Delta(0)]T+\frac{10\bar{\sigma}_1^2R_2^{\theta}(4)T}{ka_k^{2\theta}}+\frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}T}{k}\\
&+\frac{[C_{5,18}\Delta+C_{5,19}\Delta^{\frac{1}{2}}+\Delta(0)]}{ka_k^{2\theta}}+\frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\mathbb{E}\left(\sup_{0\leq t\leq T}\left[V(u)-v(u)\right]^2\right)T
\end{aligned}
$$

$$+ [\bar{\alpha}_1 + \lambda_1 \bar{\delta}_1]\mathbb{E}\int_0^t |X(u) - x(u)|\, du.$$

Applying (iii) property of function ϕ_k, by Gronwall's inequality, we then have

$$\sup_{0\le t\le T} \mathbb{E}\,|X(u) - x(u)| \le \Bigg[[C_{5,20}\Delta + C_{5,21}\Delta^{\frac{1}{2}} + \Delta(0)]T + \frac{[C_{5,18}\Delta + C_{5,19}\Delta^{\frac{1}{2}} + \Delta(0)]}{ka_k^{2\theta}}$$
$$+ \frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\mathbb{E}\left(\sup_{0\le t\le T}[V(u) - v(u)]^2\right)T\Bigg] e^{[\bar{\alpha}_1 + 2\lambda_1\bar{\delta}_1]T}$$
$$+ \left[a_{k-1} + \frac{10\bar{\sigma}_1^2 R_2^\theta(4)T}{ka_k^{2\theta}} + \frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}T}{k}\right] e^{[\bar{\alpha}_1 + 2\lambda_1\bar{\delta}_1]T}.$$

Now, choose k sufficiently large such that

$$\left[a_{k-1} + \frac{10\bar{\sigma}_1^2 R_2^\theta(4)T}{ka_k^{2\theta}} + \frac{5\bar{\sigma}_1^2[R_1(2)]^{\frac{1}{2}}T}{k}\right] e^{[\bar{\alpha}_1 + 2\lambda_1\bar{\delta}_1]T} < \frac{\varepsilon}{2} \tag{7.28}$$

and then choose Δ sufficiently small such that

$$\Bigg[[C_{5,20}\Delta + C_{5,21}\Delta^{\frac{1}{2}} + \Delta(0)]T + \frac{[C_{5,18}\Delta + C_{5,19}\Delta^{\frac{1}{2}} + \Delta(0)]}{ka_k^{2\theta}}$$
$$+ \frac{5\bar{\sigma}_1^2}{ka_k^{2\theta}}\mathbb{E}\left(\sup_{0\le t\le T}[V(u) - v(u)]^2\right)T\Bigg] e^{[\bar{\alpha}_1 + 2\lambda_1\bar{\delta}_1]T} < \frac{\varepsilon}{2}. \tag{7.29}$$

We then have

$$\sup_{0\le t\le T} \mathbb{E}\,|X(u) - x(u)| < \varepsilon, \tag{7.30}$$

as required. The proof of Theorem 7.5 is therefore complete. □

The following theorem will establish convergence in second moment of the continuous EM approximate solution to the model (7.1) which gives the main result of this chapter.

Theorem 7.6. *Let $X(t)$ be the true solution and $x(t)$ be the continuous EM approximate solution to the asset price. Then*

$$\lim_{\Delta\to 0}\mathbb{E}\left(\sup_{0\le t\le T}[X(t) - x(t)]^2\right) = 0. \tag{7.31}$$

Proof. For any $0 \leq t \leq T$, we compute

$$\begin{aligned}
&\Big[X(t) - x(t)\Big]^2 \\
\leq& 4\left[\int_0^t \alpha_1(r_1(u))\mu_1(r_1(u)) - \alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))du\right]^2 \\
&+ 4\left[\int_0^t \big(\alpha_1(r_1(u))X(u^-) - \alpha_1(\bar{r}_1(u))\bar{x}(u)\big)\, du\right]^2 \\
&+ 4\Big[\int_0^t \Big(\sigma_1(r_1(u))\sqrt{V(u^-)}X(u^-)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big)dW_1(u)\Big]^2 \\
&+ 4\Big[\int_0^t \Big(\delta_1(r_2(u))X(u^-) - \delta_1(\bar{r}_2(u))\bar{x}(u)\Big)d\bar{N}_1(u)\Big]^2.
\end{aligned}$$

Taking expectation for any $0 \leq t_1 \leq T$, by the Burkholder-Davis-Gundy inequality and the Hölder inequality, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq t\leq t_1} [X(t) - x(t)]^2\right) \\
\leq& 4T\int_0^{t_1} \mathbb{E}\Big[\alpha_1(r_1(u))\mu_1(r_1(u)) - \alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))\Big]^2 du \\
&+ 4T\int_0^{t_1} \mathbb{E}\Big[\alpha_1(r_1(u))X(u^-) - \alpha_1(\bar{r}_1(u))\bar{x}(u)\Big]^2 du \\
&+ 16\int_0^{t_1} \mathbb{E}\Big[\sigma_1(r_1(u))\sqrt{V(u^-)}X(u^-)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2 du \\
&+ 16\lambda_1\int_0^{t_1} \mathbb{E}\Big[\delta_1(r_2(u))X(u^-) - \delta_1(\bar{r}_2(u))\bar{x}(u)\Big]^2 du \qquad (7.32) \\
\leq& 4T\int_0^{t_1} \mathbb{E}\left[\alpha_1(r_1(u))\mu_1(r_1(u)) - \alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))\right]^2 du \\
&+ 8T\int_0^{t_1} \mathbb{E}|\bar{x}(u)|^2\left[\alpha_1(r_1(u)) - \alpha_1(\bar{r}_1(u))\right]^2 + \bar{\alpha}_1^2\mathbb{E}\left[X(u) - \bar{x}(u)\right]^2 du \\
&+ 32\lambda_1\int_0^{t_1} \bar{\delta}_1^2\mathbb{E}\left[X(u) - \bar{x}(u)\right]^2 + \mathbb{E}|\bar{x}(u)|^2\Big[\delta_1(r_2(u)) - \delta_1(\bar{r}_2(u))\Big]^2 du \\
&+ 16\int_0^{t_1} \mathbb{E}\Big|\sigma_1(r_2(u))\sqrt{V(u)}X(u^-)^\theta - \sigma_1(\bar{r}_2(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big|^2 du.
\end{aligned}$$

Applying the techniques used to compute (7.10) and (7.11), we further get that

$$\leq 16T^2\bar{\alpha}_1^2\bar{\mu}_1^2(C_{5,22}\Delta + 0(\Delta)) + 32T^2R_2(2)\bar{\alpha}_1^2[C_{5,23}\Delta + 0(\Delta)]$$

$$
\begin{aligned}
&+ 16T\bar{\alpha}_1^2 \int_0^{t_1} \mathbb{E}\left[X(u) - x(u)\right]^2 + \mathbb{E}\left[x(u) - \bar{x}(u)\right]^2 du \\
&+ 128\lambda_1[R_2(2)]\bar{\delta}_1^2[C_{5,24}\Delta + 0(\Delta)] \\
&+ 64\lambda_1\bar{\delta}_1^2 \int_0^{t_1} \mathbb{E}\left[X(u) - x(u)\right]^2 + \mathbb{E}\left[x(u) - \bar{x}(u)\right]^2 du \\
&+ 16 \int_0^{t_1} \mathbb{E}\Big[\sigma_1(r_1(u))\sqrt{V(u)}X(u^-)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2 du.
\end{aligned}
$$

Applying Lemma 7.4 yields

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1} [X(t) - x(t)]^2\right) \\
\le&[C_{5,25}\Delta + 0(\Delta)] + [16T\bar{\alpha}_1^2 + 64\lambda_1\bar{\delta}_1^2]\left(C_{5,13}\Delta^2\right)^{\frac{1}{2}} T \\
&+ [16T\bar{\alpha}_1^2 + 64\lambda_1\bar{\delta}_1^2] \int_0^{t_1} \mathbb{E}\left[X(u) - x(u)\right]^2 du \\
&+ 16 \int_0^{t_1} \mathbb{E}\Big[\sigma_1(r_1(u))\sqrt{V(u)}X(u^-)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]^2 du.
\end{aligned} \tag{7.33}
$$

Then, consider

$$
\begin{aligned}
Z(t) =&16 \int_0^{t_1} \mathbb{E}\left|\sigma_1(r_1(u))\sqrt{V(u)}X(u)^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\right|^2 du \\
\le&80 \int_0^{t_1} \mathbb{E}|\bar{v}(u)|\bar{x}(u)^{2\theta}\left[\sigma_1(r_1(u)) - \sigma_1(\bar{r}_1(u))\right]^2 du \\
&+ 80 \int_0^{t_1} \mathbb{E}|\sigma_1(r_1(u))|^2|x(u)|^{2\theta}\left[\sqrt{V(u)} - \sqrt{|v(u)|}\right]^2 du \\
&+ 80 \int_0^{t_1} \mathbb{E}|\sigma_1(r_1(u))|^2|x(u)|^{2\theta}\left[\sqrt{|v(u)|} - \sqrt{|\bar{v}(u)|}\right]^2 du \\
&+ 80 \int_0^{t_1} \mathbb{E}|\sigma_1(r_1(u))|^2 V(u)\left[X(u)^\theta - |x(u)|^\theta\right]^2 du \\
&+ 80 \int_0^{t_1} \mathbb{E}|\sigma_1(r_1(u))|^2 V(u)\left[|x(u)|^\theta - |\bar{x}(u)|^\theta\right]^2 du.
\end{aligned} \tag{7.34}
$$

Repeating similar technique used in (7.10), we have

$$
\begin{aligned}
Z(t) \le&320[R_1(2)]^{\frac{1}{2}}[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2[C_{5,26}\Delta + 0(\Delta)] \\
&+ 80[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2 \int_0^{t_1} \left(\mathbb{E}\left[V(u) - v(u)\right]^2\right)^{\frac{1}{2}} + \left(\mathbb{E}\left[v(u) - \bar{v}(u)\right]^2\right)^{\frac{1}{2}} du \\
&+ 80\bar{\sigma}_1^2 \int_0^{t_1} \mathbb{E}|\bar{v}(u)||X(u) - x(u)| + \mathbb{E}|\bar{v}(u)|\left[X(u) - x(u)\right]^2 du
\end{aligned}
$$

$$+ 80\bar{\sigma}_1^2 \int_0^{t_1} \mathbb{E}|\bar{v}(u)| \left[x(u) - \bar{x}(u)\right] + \mathbb{E}|\bar{v}(u)| \left[x(u) - \bar{x}(u)\right]^2 du.$$

By Theorem 7.1, Theorem 7.2, Lemma 7.3 and Lemma 7.4 this yields

$$\begin{aligned}
Z(t) \leq & 320[R_1(2)]^{\frac{1}{2}}[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2[C_{5,26}\Delta + 0(\Delta)] + 80[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2 \left(C_{5,1}\Delta\right)^{\frac{1}{2}} T \\
& + 80[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2 \int_0^{t_1} \left(\mathbb{E}\left[V(u) - v(u)\right]^2\right)^{\frac{1}{2}} du \\
& + 80\bar{\sigma}_1^2 \int_0^{t_1} \left(\mathbb{E}|\bar{v}(u)|^2|X(u) + x(u)|\right)^{\frac{1}{2}} \left(\mathbb{E}|X(u) - x(u)|\right)^{\frac{1}{2}} du \\
& + 80\bar{\sigma}_1^2 \int_0^{t_1} \left(\mathbb{E}|\bar{v}(u)|^2|X(u) + x(u)|^3\right)^{\frac{1}{2}} \left(\mathbb{E}|X(u) - x(u)|\right)^{\frac{1}{2}} du \\
& + 80[R_1(2)]^{\frac{1}{2}}\bar{\sigma}_1^2 \left[C_{5,13}\Delta^2\right]^{\frac{1}{4}} T + 80[R_1(2)]^{\frac{1}{2}}\bar{\sigma}_1^2 \left[C_{5,13}\Delta^2\right]^{\frac{1}{2}} T \\
\leq & [C_{5,27}\Delta + C_{5,28}\Delta^{\frac{1}{2}} + 0(\Delta)] + 80[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2 \left(\mathbb{E} \sup_{0\leq t\leq T} [V(t) - v(t)]^2\right)^{\frac{1}{2}} T \\
& + 80\bar{\sigma}_1^2(R_1(4))^{\frac{1}{4}} \left[(2R_2(6))^{\frac{1}{4}} + (2R_2(2))^{\frac{1}{4}}\right] \left(\sup_{0\leq t\leq T} \mathbb{E}|X(t) - x(t)|\right)^{\frac{1}{2}} T.
\end{aligned} \tag{7.35}$$

Substituting (7.35) into (7.33), we further get that

$$\begin{aligned}
& \mathbb{E}\left(\sup_{0\leq t\leq t_1} [X(t) - x(t)]^2\right) \\
\leq & [C_{5,25}\Delta + 0(\Delta)] + [16T\bar{\alpha}_1^2 + 64\lambda_1\bar{\delta}_1^2]\left(\left(C_{5,13}\Delta^2\right)^{\frac{1}{2}} T + \int_0^{t_1} \mathbb{E}\left[X(u) - x(u)\right]^2 du\right) \\
& + [C_{5,27}\Delta + C_{5,28}\Delta^{\frac{1}{2}} + 0(\Delta)] + 80[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2 \left(\mathbb{E} \sup_{0\leq t\leq T} [V(t) - v(t)]^2\right)^{\frac{1}{2}} T \\
& + 80\bar{\sigma}_1^2 \left[(R_1(4))^{\frac{1}{4}}(2R_2(2))^{\frac{1}{4}} + (R_1(4))^{\frac{1}{4}}(2R_2(6))^{\frac{1}{4}}\right] \left(\sup_{0\leq t\leq T} \mathbb{E}|X(t) - x(t)|\right)^{\frac{1}{2}} T \\
= & [16T\bar{\alpha}_1^2 + 64\lambda_1\bar{\delta}_1^2] \int_0^{t_1} \mathbb{E}\left[X(u) - x(u)\right]^2 du \\
& + [C_{5,29}\Delta + C_{5,28}\Delta^{\frac{1}{2}} + 0(\Delta)] + 80[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2 \left(\mathbb{E} \sup_{0\leq t\leq T} [V(t) - v(t)]^2\right)^{\frac{1}{2}} T \\
& + 80\bar{\sigma}_1^2(R_1(4))^{\frac{1}{4}} \left[(2R_2(2))^{\frac{1}{4}} + (2R_2(6))^{\frac{1}{4}}\right] \left(\sup_{0\leq t\leq T} \mathbb{E}|X(t) - x(t)|\right)^{\frac{1}{2}} T.
\end{aligned}$$

By Gronwall's inequality, we then have

$$\mathbb{E}\left(\sup_{0\le t\le t_1}[X(t)-x(t)]^2\right)$$

$$\le\Bigg[80\bar{\sigma}_1^2(R_1(4))^{\frac{1}{4}}\left[(2R_2(2))^{\frac{1}{4}}+(2R_2(6))^{\frac{1}{4}}\right]\left(\sup_{0\le t\le T}\mathbb{E}|X(t)-x(t)|\right)^{\frac{1}{2}}T$$

$$+80[R_2(4)]^{\frac{\theta}{2}}\bar{\sigma}_1^2\left(\mathbb{E}\sup_{0\le t\le T}\Big[V(t)-v(t)\Big]^2\right)^{\frac{1}{2}}T+[C_{5,29}\Delta+C_{5,28}\Delta^{\frac{1}{2}}+0(\Delta)]\Bigg]e^{(16T\bar{\alpha}_1^2+64\lambda_1\bar{\delta}_1^2)T}.$$

The proof of our theorem is finally completed by Theorem 7.4, Theorem 7.5 and letting $\Delta\to 0$. □

In practice, the continuous EM approximate solution $x(t)$ to the asset price, which has the convergence in second moment described by Theorem 7.6, is not computable. However, its corresponding step process $\bar{x}(t)$ is computable. Therefore, we will establish the following theorem to show that the step process will converge to the true solution when the time step is sufficiently small.

Theorem 7.7. *Let $X(t)$ be the true solution and $\bar{x}(t)$ be the step process of the continuous EM approximate solution to the SDE model (7.1). Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}\mathbb{E}\Big|X(t)-\bar{x}(t)\Big|\right)=0. \tag{7.36}$$

The proof can be obtained in the same way as Theorem 5.7 was proved.

Theorem 7.7 shows that the step process will converge to the true solution of the SDE model (7.1). Let us choose initial condition $(X(0)=0.5, V(0)=0.2)$, $\rho=0.2$, parameters $(\theta=1,\beta=0.5)$, $\lambda_1=1,\lambda_2=2$ and coefficients of SDE model (7.1) (see Table 7.1) to illustrate its behaviour in practice. In addition, we use MATLAB® software (see Appendix A for code) with generators of Markov chains Γ_{r_1} and Γ_{r_2} to obtain the following graph (see Figure 7.1).

$$\Gamma_{r_1} = \begin{pmatrix} -4 & 3 & 1 \\ 1 & -2 & 1 \\ 1 & 3 & -4 \end{pmatrix} \text{ and } \quad \Gamma_{r_2} = \begin{pmatrix} -2 & 1 & 1 \\ 3 & -5 & 2 \\ 1 & 1 & -2 \end{pmatrix}.$$

Table 7.1: Coefficients of SDE model (7.1)

State	(α_1, α_2)	(μ_1, μ_2)	(σ_1, σ_2)	(δ_1, δ_2)
1	$(1.2, 1)$	$(1, 0.2)$	$(0.6, 0.84)$	$(0.5, 2)$
2	$(4, 4)$	$(2, 0.7)$	$(0.9, 0.6)$	$(2, 2)$
3	$(7, 1.6)$	$(1, 0.3)$	$(0.24, 0.56)$	$(1, -0.9)$

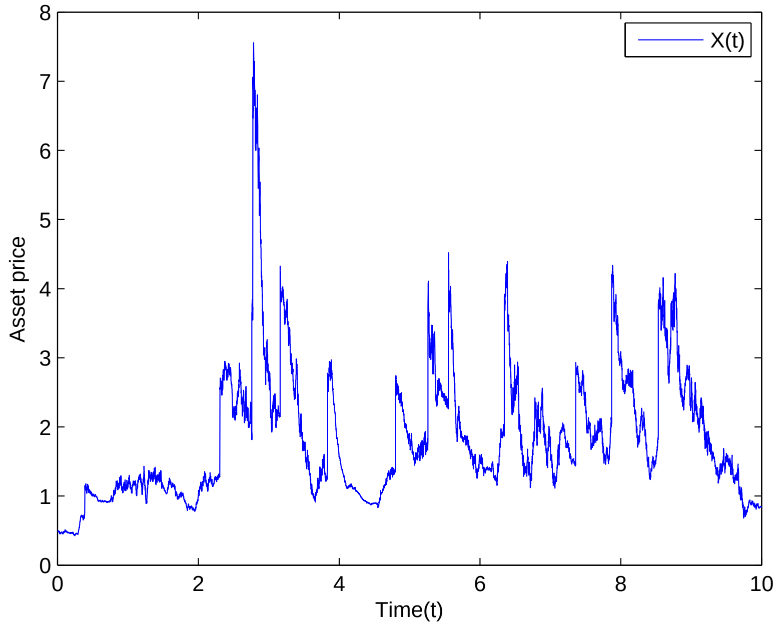

Figure 7.1: A sample path of the asset price $X(t)$ which is generated by the EM approximate solution to the hybrid mean-reverting-theta stochastic volatility model with Poisson jump over finite time, where $\theta = 1$ and $\beta = 0.5$.

7.4 Summary

In this chapter, we have focussed on the Markov switching form of the SDE model examined in Chapter 5. Thus, we have first proved that the solution to this model is non-negative with probability 1. Since this generalized model has no explicit solution, convergence in second moment of the EM approximate solution to this SDE model has been examined when the time step is sufficiently small. In addition, we have proved that the convergence property of the corresponding step process which can also be used to evaluate application to financial quantities.

Chapter 8

A Highly Sensitive Hybrid, Poisson Jump Stochastic Volatility Model

8.1 Introduction

In contrast to the SDE model examined in Chapter 7 which not only obeys the global Lipschitz condition but also the linear growth condition, the SDE model (7.1), when parameters θ and β greater than 1, will be examined in this chapter. Although many applications of this highly sensitive SDE model can be seen in financial markets, explicit solution to this model can not be obtained within the existing theory. Therefore, the Euler-Maruyama (EM) approximation method is more appropriate to study and examine its behaviour in practice. However, some techniques which have been developed in previous chapters are not strong enough to obtain analytical properties of the EM approximate solution to this SDE model. Thus, we will proceed to develop the required tools in this chapter.

In the SDE model (7.1) when the parameters $\theta, \beta > 1$ describe the asset price and interest rate in financial markets, the diffusion coefficients of the model obey the local Lipschitz condition though do not satisfy the linear growth condition. So there is no information about the non-negative solution so far. Therefore, we will prove that the solution to this SDE model will be non-negative with probability 1. We will then show that the continuous EM approximate solution to this model will converge in probability to the true solution when the time step is sufficiently small. On the other hand, the continuous EM approximate solution is not computable but its corresponding step process is computable in practice. Thus, we will show that this corresponding step process will converge in probability to the true solution, which can be used to evaluate applications of this SDE model in finance.

8.2 Non-negative solution

The SDE model (7.1) mainly describes the asset price and volatility in financial markets, so a natural requirement is that the solution $(V(t), X(t))$ to this model is non-negative. The following lemma in fact shows that the solution to this SDE model will be non-negative with probability 1.

Non-negative $V(t)$

Lemma 8.1. *Assume $\beta > 1$. Then, given any initial values $V(0) = V_0 > 0$, $r_1(0) = i_0 \in \mathbb{S}_1$ and $r_2(0) = j_0 \in \mathbb{S}_2$, the second SDE of the model (7.1) has unique local solution $V(t)$ which will be non-negative for all $t \in [0, T]$ a.s..*

Non-negative $X(t)$

Lemma 8.2. *Assume $\theta > 1$ and $\beta > 1$. Then, given any initial values $V(0) = V_0 > 0$, $X(0) = X_0 > 0$, $r_1(0) = i_0 \in \mathbb{S}_1$ and $r_2(0) = j_0 \in \mathbb{S}_2$, the SDE of (7.1) has unique local solution $X(t)$ which will be non-negative for all $t \in [0, T]$ a.s..*

The proofs of Lemma 8.1 and Lemma 8.2 can easily be obtained in the same way as Lemma 6.1 and Lemma 6.2 were proved but with $\bar{\alpha}_j = \max_{i \in \mathbb{S}_1} \alpha_i$, $\bar{\mu}_j = \max_{i \in \mathbb{S}_1} \mu_i$, $\bar{\sigma}_j = \max_{i \in \mathbb{S}_1} \sigma_i$ and $\bar{\delta}_j = \max_{i \in \mathbb{S}_2} |\delta_i|$ for $j = 1, 2$.

8.3 Convergence in probability

The SDE model (7.1) has no explicit solution, so study of an numerical approximate solution is appropriate to understand its behaviour in financial markets. Therefore, we will consider the EM numerical approximate solution to this SDE model defined in Chapter 7, but with parameters θ and β greater than 1.

Convergence of $v(t)$ in probability

The unique solution (V(t), X(t)) to the SDE model (7.1) is non-negative with probability 1. We will examine convergence of the continuous EM approximate solution to this SDE model in the following section. Thus, we will first establish the following theorem which gives a strong error bound to the EM approximate solution of volatility. In addition, this can also be used to examine convergence in probability of this EM approximate solution.

Theorem 8.1. *Let $V(t)$ be the true solution and $v(t)$ be the continuous EM approximate solution to the second SDE of (7.1) when $\beta > 1$. For any positive number M, define the stopping time $q = \rho_M \wedge \gamma_M \wedge T$, where $\rho_M = \inf\{t \in [0, T]; V(t) \notin$*

$[\frac{1}{M}, M]\}$ *and* $\gamma_M = \inf\{t \in [0,T]; |v(t)| \notin [\frac{1}{M}, M]\}$. *Then, for any integer* $p \geq 2$,

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[V(t\wedge q) - v(t\wedge q)]^2\right) \leq C_{6,7}(M,p)\Delta^{1-\frac{1}{p}}, \tag{8.1}$$

where $C_{6,7}(M,p)$ *is a constant independent of* Δ.

To prove Theorem 8.1, we need the following lemma which can be obtained in the same way as Lemma 6.3 was proved.

Lemma 8.3. *There exists a constant* $C_{6,1}(M,p)$ *dependent on* M *and* p *but independent of* Δ *such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[v(t\wedge q) - \bar{v}(t\wedge q)]^2\right) \leq C_{6,1}(M,p)\Delta^{1-\frac{1}{p}}. \tag{8.2}$$

Proof. *(of Theorem 8.1)* For any $0 \leq t \leq T$, compute

$$\begin{aligned}
&\Big[V(t\wedge q) - v(t\wedge q)\Big]^2\\
\leq&4\left[\int_0^{t\wedge q}\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))du\right]^2\\
&+4\left[\int_0^{t\wedge q}\alpha_2(r_1(u))V(u^-) - \alpha_2(\bar{r}_1(u))\bar{v}(u)du\right]^2\\
&+4\left[\int_0^{t\wedge q}\sigma_2(r_1(u))|V(u^-)|^\beta - \sigma_2(\bar{r}_1(u))|\bar{v}(u)|^\beta dW_2(u)\right]^2\\
&+4\left[\int_0^{t\wedge q}\delta_2(r_2(u))V(u^-) - \delta_2(\bar{r}_2(u))\bar{v}(u)d\bar{N}_2(u)\right]^2.
\end{aligned} \tag{8.3}$$

Taking the expectation for $t_1 \in [0,T]$, by the Burkholder-Davis-Gundy inequality and the Hölder inequality, we then have

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\leq t\leq t_1}[V(t\wedge q) - v(t\wedge q)]^2\right)\\
\leq&4T\mathbb{E}\int_0^{t_1\wedge q}\Big[\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))\Big]^2du\\
&+4T\mathbb{E}\int_0^{t_1\wedge q}\Big[\alpha_2(r_1(u))V(u) - \alpha_2(\bar{r}_1(u))\bar{v}(u)\Big]^2du\\
&+16\mathbb{E}\int_0^{t_1\wedge q}\Big[\sigma_2(r_1(u))|V(u)|^\beta - \sigma_2(\bar{r}_1(u))|\bar{v}(u)|^\beta\Big]^2du
\end{aligned}$$

$$+ 16\lambda_2\mathbb{E}\int_0^{t_1\wedge q}\Big[\delta_2(r_2(u))V(u) - \delta_2(\bar{r}_2(u))\bar{v}(u)\Big]^2 du.$$

Rearranging the terms on the right hand side, we further get that

$$\begin{aligned}
\leq & 4T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))\right]^2 du \\
& + 12T\mathbb{E}\int_0^{t_1\wedge q}|\bar{v}(u)|^2\left[\alpha_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\right]^2 du \\
& + [12T\bar{\alpha}_2^2 + 48\lambda_2\bar{\delta}_2^2]\mathbb{E}\int_0^{t_1\wedge q}\left[V(u) - v(u)\right]^2 + \left[v(u) - \bar{v}(u)\right]^2 du \\
& + 48\mathbb{E}\int_0^{t_1\wedge q}|\bar{v}(u)|^{2\beta}\left[\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right]^2 du \\
& + 48\bar{\sigma}_2^2\mathbb{E}\int_0^{t_1\wedge q}\left[|V(u)|^\beta - |v(u)|^\beta\right]^2 + \left[|v(u)|^\beta - |\bar{v}(u)|^\beta\right]^2 du \\
& + 48\lambda_2\mathbb{E}\int_0^{t_1\wedge q}|\bar{v}(u)|^2\left[\delta_2(r_2(u)) - \delta_2(\bar{r}_2(u))\right]^2 du.
\end{aligned} \tag{8.4}$$

Applying Lemma 8.3 and the mean value theorem yields

$$\begin{aligned}
\leq & 4T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))\right]^2 du \\
& + 12M^2T\mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\right]^2 du \\
& + 48M^{2\beta}\mathbb{E}\int_0^{t_1\wedge q}\left[\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))\right]^2 du \\
& + 48M^2\lambda_2\mathbb{E}\int_0^{t_1\wedge q}\left[\delta_2(r_2(u)) - \delta_2(\bar{r}_2(u))\right]^2 du \\
& + \left[12\bar{\alpha}_2^2T + 48\bar{\sigma}_2^2\beta^2M^{2\beta-2} + 48\bar{\delta}_2^2\lambda_2\right]\mathbb{E}\int_0^{t_1\wedge q}\left[V(u) - v(u)\right]^2 du \\
& + \left[12\bar{\alpha}_2^2T + 48\bar{\sigma}_2^2\beta^2M^{2\beta-2} + 48\bar{\delta}_2^2\lambda_2\right]TC_{6,1}(M,p)\Delta^{1-\frac{1}{p}}.
\end{aligned} \tag{8.5}$$

On the other hand, in the same way as in computation of (4.18), we get

$$\begin{aligned}
A(T) =& \mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r_1(u))\mu_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\mu_2(\bar{r}_1(u))\right]^2 du \\
\leq & (C_{6,2}\Delta + 0\Delta),
\end{aligned} \tag{8.6}$$

$$B(T) = \mathbb{E}\int_0^{t_1\wedge q}\left[\alpha_2(r_1(u)) - \alpha_2(\bar{r}_1(u))\right]^2 du \leq (C_{6,3}\Delta + 0(\Delta)) \tag{8.7}$$

and

$$D(T) = \mathbb{E}\int_0^{t_1\wedge q} [\sigma_2(r_1(u)) - \sigma_2(\bar{r}_1(u))]^2\, du \le (C_{6,4}\Delta + 0(\Delta)). \tag{8.8}$$

Similarly, but by the properties of the Markov chain $r_2(.)$, we further get that

$$E(T) = \mathbb{E}\int_0^{t_1\wedge q} [\delta_2(r_2(u)) - \delta_2(\bar{r}_2(u))]^2\, du \le (C_{6,5}\Delta + 0(\Delta)). \tag{8.9}$$

Substituting (8.6), (8.7), (8.8), (8.9) into (8.5) gives

$$\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1} [V(t\wedge q) - v(t\wedge q)]^2\right)\\
&\le 4T(C_{6,2}\Delta + 0(\Delta)) + 12M^2T(C_{6,3}\Delta + 0(\Delta))\\
&+ 48M^{2\beta}(C_{6,4}\Delta + 0(\Delta)) + 48M^2\lambda_2(C_{6,5}\Delta + 0(\Delta))\\
&+ \left[12\bar{\alpha}_2^2T + 48\bar{\sigma}_2^2\beta^2M^{2\beta-2} + 48\bar{\delta}_2^2\lambda_2\right]\mathbb{E}\int_0^{t_1\wedge q} [V(u) - v(u)]^2\, du \qquad (8.10)\\
&+ \left[12\bar{\alpha}_2^2T + 48\bar{\sigma}_2^2\beta^2M^{2\beta-2} + 48\bar{\delta}_2^2\lambda_2\right] TC_{6,1}(M,p)\Delta^{1-\frac{1}{p}}\\
&\le [C_{6,6}(M,p)\Delta^{1-\frac{1}{p}} + 0(\Delta)]\\
&+ \bar{C}_{6,6}(M)\int_0^{t_1}\mathbb{E}\left(\sup_{0\le u_1\le u} [V(u_1\wedge q) - v(u_1\wedge q)]^2\right) du,
\end{aligned}$$

as required. Therefore, an application of the Gronwall's inequality will complete the proof of our theorem. □

Now, we will remove the stopping time of volatility and establish the following theorem to show that the continuous EM approximate solution of volatility will converge to the true solution in probability.

Theorem 8.2. *Let $V(t)$ be the true solution and $v(t)$ be the continuous EM approximate solution to the second SDE of (7.1) when $\beta > 1$. Then,*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T} [V(t) - v(t)]^2\right) = 0 \quad \textit{in probability.} \tag{8.11}$$

To prove Theorem 8.2, we establish the following lemma which gives an upper bound for the expected value of the EM approximate solution to volatility.

Lemma 8.4. *There exists a constant $C_{6,1}(M,p)^{**}$ which is dependent on M and p, but independent of Δ such that*

$$\mathbb{E}(v(t \wedge \gamma_M)) \leq Z + C_{6,1}^{**}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{8.12}$$

where Z is a constant independent of Δ.

The proof can be obtained by applying the technique with which Lemma 6.4 was proved.

Proof. (of Theorem 8.2)

The proof of the theorem is rather difficult. We will therefore divide the whole proof into 3 steps.

Step 1: Applying a similar technique as used in *(Step 1 :)* Theorem 6.2, we have

$$\mathbb{P}(\rho_M \leq T) \leq \frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2}{2}T + \frac{(\bar{\alpha}_2+\lambda_2\bar{\delta}_2)}{2}T + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4} + \lambda_2\bar{\delta}_2 RT}{H(M^{-1}) \wedge H(M)}. \tag{8.13}$$

Step 2: In the same way as in computation of *(Step 2 :)* of Theorem 6.2, we further get that

$$\mathbb{P}(\gamma_M \leq T) \leq \frac{H(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{(\bar{\alpha}_2+\bar{\delta}_2\lambda_2)T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4} + \bar{\delta}_2\lambda_2 TZ + \bar{C}_{6,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)}. \tag{8.14}$$

Step 3: Let $\varepsilon > 0$ and $\delta \in (0,1)$ be arbitrarily small, then define

$$\bar{\Omega}_6 = \left[\omega; \sup_{0\leq t\leq T} [V(t) - v(t)]^2 \geq \delta\right]. \tag{8.15}$$

In the same way as computation of (3.42) but with Theorem 8.1, we then compute

$$\mathbb{P}(\bar{\Omega}_6 \cap (q \geq T)) \leq \frac{C_{6,7}(M,p)\Delta^{1-\frac{1}{p}}}{\delta}. \tag{8.16}$$

On the other hand, we easily obtain

$$\mathbb{P}(\bar{\Omega}_6) \leq \mathbb{P}(\bar{\Omega}_6 \cap (q \geq T)) + \mathbb{P}(\gamma_M \leq T) + \mathbb{P}(\rho_M \leq T). \tag{8.17}$$

Substituting (8.13), (8.14) and (8.16) into (8.17) yields

$$\begin{aligned}
&\mathbb{P}(\bar{\Omega}_6) \\
\le& \frac{C_{6,7}(M,p)\Delta^{1-\frac{1}{p}}}{\delta} \\
&+ \frac{\phi(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{(\bar{\alpha}_2+\bar{\delta}_2\lambda_2)T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4} + \bar{\delta}_2\lambda_2 TZ + \bar{C}_{6,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} \\
&+ \frac{\phi(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{(\bar{\alpha}_2+\lambda_2\bar{\delta}_2)T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4} + \lambda_2\bar{\delta}_2 RT}{H(M^{-1}) \wedge H(M)}.
\end{aligned} \tag{8.18}$$

Now, choose M sufficiently large for

$$\frac{2\left[\phi(V_0) + \frac{\bar{\alpha}_2\bar{\mu}_2 T}{2} + \frac{(\bar{\alpha}_2+\lambda_2\bar{\delta}_2)T}{2} + \frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4}\right] + \lambda_2\bar{\delta}_2 RT + \lambda_2\bar{\delta}_2 ZT}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{2} \tag{8.19}$$

and then choose Δ sufficiently small for

$$\frac{C_{6,7}(M,p)\Delta^{1-\frac{1}{p}}}{\delta} + \frac{\bar{C}_{6,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1}) \wedge H(M)} < \frac{\varepsilon}{2}. \tag{8.20}$$

Hence we have

$$\mathbb{P}\left(\sup_{0\le t\le T} [V(t) - v(t)]^2 \ge \delta\right) < \varepsilon, \tag{8.21}$$

as required. The proof is therefore complete.

□

Convergence of $x(t)$ in probability

Theorem 8.2 shows that the EM approximate solution of volatility will converge in probability to the true solution. In this section, we will therefore examine the main result of this chapter, which gives the convergence in probability of the continuous EM approximate solution to the asset price . Accordingly, we will first establish the following theorem which shows the strong error bound on the continuous EM approximate solution with stopping time.

Theorem 8.3. *Let $X(t)$ be the true solution and $x(t)$ be the continuous EM approximate solution to the SDE model (7.1) when $\theta > 1$ and $\beta > 1$. For any positive numbers N and M, define stopping time $s = q \wedge \tau_N \wedge \zeta_N \wedge T$, where q is the same as before, while $\tau_N = \inf\{t \in [0,T] : X(t) \notin [\frac{1}{N}, N]\}$ and $\zeta_N = \inf\{t \in [0,T] : |x(t)| \notin [\frac{1}{N}, N]\}$. Then, for any $p \geq 2$,*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[X(t\wedge s) - x(t\wedge s)]^2\right) \leq C_{6,13}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{8.22}$$

where $C_{6,13}(M,N,p)$ is a constant independent of Δ.

To prove Theorem 8.3, we need the following lemma, which can be obtained the same way as Lemma 6.1 was proved.

Lemma 8.5. *There exists a constant $C_{6,8}(M,N,p)$ dependent on M, N and p but independent of Δ such that*

$$\mathbb{E}\left(\sup_{0\leq t\leq T}[x(t\wedge s) - \bar{x}(t\wedge s)]^2\right) \leq C_{6,8}(M,N,p)\Delta^{1-\frac{1}{p}}. \tag{8.23}$$

Proof. (of Theorem 8.3)

For any $0 \leq t \leq T$, compute

$$\begin{aligned}
&\Big[X(t\wedge s) - x(t\wedge s)\Big]^2 \\
\leq& 4\Big[\int_0^{t\wedge s} \alpha_1(r_1(u))\mu_1(r_1(u)) - \alpha_1(\bar{r}_1(u))\mu_1(\bar{r}_1(u))du\Big]^2 \\
&+ 4\Big[\int_0^{t\wedge s} [\alpha_1(r_1(u))X(u^-) - \alpha_1(\bar{r}_1(u))\bar{x}(u)]du\Big]^2 \\
&+ 4\Big[\int_0^{t\wedge s} \Big[\sigma_1(r_1(u))\sqrt{V(u^-)}|X(u^-)|^\theta - \sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\Big]dW_1(u)\Big]^2 \\
&+ 4\Big[\int_0^{t\wedge s} [\delta_1(r_2(u))X(u^-) - \delta_1(\bar{r}_2(u))\bar{x}(u)]d\bar{N}_1(u)\Big]^2.
\end{aligned} \tag{8.24}$$

Taking the expectation for any $t_1 \in [0,T]$, by the Burkholder-Davis-Gundy inequality and the Hölder inequality, we then have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}[X(t\wedge s)-x(t\wedge s)]^2\right)\\
\le&4T\mathbb{E}\int_0^{t_1\wedge s}\Big[\alpha_1(r_1(u))\mu_1(r_1(u))-\alpha_1(\bar r_1(u))\mu_1(\bar r_1(u))\Big]^2du\\
&+4T\mathbb{E}\int_0^{t_1\wedge s}\Big[\alpha_1(r_1(u))X(u^-)-\alpha_1(\bar r_1(u))\bar x(u)\Big]^2du\\
&+16\mathbb{E}\int_0^{t_1\wedge s}\Big[\sigma_1(r_1(u))\sqrt{V(u^-)}|X(u^-)|^\theta\\
&\qquad\qquad-\sigma_1(\bar r_1(u))\sqrt{|\bar v(u)|}|\bar x(u)|^\theta\Big]^2du\\
&+4\lambda_1\mathbb{E}\int_0^{t_1\wedge s}\Big[\delta_1(r_2(u))X(u^-)-\delta_1(\bar r_2(u))\bar x(u)\Big]^2du.
\end{aligned}
\tag{8.25}
$$

Now, in the same way as in computation of (4.18), we compute

$$
\begin{aligned}
G=&4T\mathbb{E}\int_0^{t_1\wedge s}\Big[\alpha_1(r_1(u))\mu_1(r_1(u))-\alpha_1(\bar r_1(u))\mu_1(\bar r_1(u))\Big]^2du\\
\le&4T(C_{6,9}+0(\Delta)).
\end{aligned}
\tag{8.26}
$$

Similarly, but applying Lemma 8.5, we get that

$$
\begin{aligned}
H=&4T\mathbb{E}\int_0^{t_1\wedge s}[\alpha_1(r_1(u))X(u)-\alpha_1(\bar r_1(u))\bar x(u)]^2\,du\\
\le&12T\mathbb{E}\int_0^{t_1\wedge s}X(u)^2[\alpha_1(r_1(u))-\alpha_1(\bar r_1(u))]^2\,du\\
&+12T\bar\alpha_1^2\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2+[x(u)-\bar x(u)]^2\,du\\
\le&12TN^2(C_{6,10}+0(\Delta))+12T\bar\alpha_1^2C_{6,8}(M,N,p)\Delta^{1-\frac{1}{p}}T\\
&+12T\bar\alpha_1^2\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2\,du.
\end{aligned}
\tag{8.27}
$$

Analogously, but with the property of $r_2(u)$, we have

$$
\begin{aligned}
I=&4\lambda_1\mathbb{E}\int_0^{t_1\wedge s}[\delta_1(r_2(u))X(u)-\delta_1(\bar r_2(u))\bar x(u)]^2\,du\\
\le&12\lambda_1N^2(C_{6,11}\Delta+0(\Delta))+12\bar\delta_1^2\lambda_1C_{6,8}(M,N,p)\Delta^{1-\frac{1}{p}}T\\
&+12\bar\delta_1^2\lambda_1\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2\,du.
\end{aligned}
\tag{8.28}
$$

Now, compute

$$
\begin{aligned}
J =&16\mathbb{E}\int_0^{t_1\wedge s}\left[\sigma_1(r_1(u))\sqrt{V(u^-)}|X(u^-)|^\theta-\sigma_1(\bar{r}_1(u))\sqrt{|\bar{v}(u)|}|\bar{x}(u)|^\theta\right]^2du\\
\le&80\mathbb{E}\int_0^{t_1\wedge s}|V(u)|X(u)^{2\theta}\left[\sigma_1(r_1(u))-\sigma_1(\bar{r}_1(u))\right]^2du\\
&+80\bar{\sigma}_1^2\mathbb{E}\int_0^{t_1\wedge s}X(u)^{2\theta}\left[\sqrt{V(u)}-\sqrt{|v(u)|}\right]^2+X(u)^{2\theta}\left[\sqrt{|v(u)|}-\sqrt{|\bar{v}(u)|}\right]^2du\\
&+80\bar{\sigma}_1^2\mathbb{E}\int_0^{t_1\wedge s}|\bar{v}(u)|\left[|X(u)|^\theta-|x(u)|^\theta\right]^2+|\bar{v}(u)|\left[|x(u)|^\theta-|\bar{x}(u)|^\theta\right]^2du.
\end{aligned}
$$

Applying the technique used to compute (4.18), the mean value theorem yields

$$
\begin{aligned}
J \le&80NM^{2\theta}(C_{6,12}+0(\Delta))+80\bar{\sigma}_1^2N^{2\theta}\mathbb{E}\int_0^{t_1\wedge s}|V(u)-v(u)|+|v(u)-\bar{v}(u)|\,du\\
&+80\bar{\sigma}_1^2M\theta^2N^{2\theta-2}\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2+[x(u)-\bar{x}(u)]^2du.
\end{aligned}
$$

Substituting Theorem 8.1, Lemma 8.3 and Lemma 8.5 gives

$$
\begin{aligned}
J \le&80NM^{2\theta}(C_{6,12}+0(\Delta))+80\bar{\sigma}_1^2N^{2\theta}\left[C_{6,7}(M,p)\Delta^{1-\frac{1}{p}}\right]^{\frac{1}{2}}T\\
&+80\bar{\sigma}_1^2N^{2\theta}\left[C_{6,1}(M,p)\Delta^{1-\frac{1}{p}}\right]^{\frac{1}{2}}T\\
&+80\bar{\sigma}_1^2M\theta^2N^{2\theta-2}\Big[\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2du+C_{6,8}(M,N,p)\Delta^{1-\frac{1}{p}}T\Big].
\end{aligned}
\tag{8.29}
$$

Applying (8.26), (8.27), (8.28) and (8.29) into (8.25), we then have

$$
\begin{aligned}
&\mathbb{E}\left(\sup_{0\le t\le t_1}[X(t\wedge s)-x(t\wedge s)]^2\right)\\
\le&4T(C_{6,9}+0(\Delta))+12TN^2(C_{6,10}+0(\Delta))+12T\bar{\alpha}_1^2C_{6,8}(M,N,p)\Delta^{1-\frac{1}{p}}T\\
&+12T\bar{\alpha}_1^2\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2du+12\bar{\delta}_1^2\lambda_1\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2du\\
&+12\lambda_1N^2(C_{6,11}\Delta+0(\Delta))+12\bar{\delta}_1^2\lambda_1C_{6,8}(M,N,p)\Delta^{1-\frac{1}{p}}T\\
&+80NM^{2\theta}(C_{6,12}+0(\Delta))+80\bar{\sigma}_1^2N^{2\theta}\left[C_{6,7}(M,p)\Delta^{1-\frac{1}{p}}\right]^{\frac{1}{2}}T\\
&+80\bar{\sigma}_1^2N^{2\theta}\left[C_{6,1}(M,p)\Delta^{1-\frac{1}{p}}\right]^{\frac{1}{2}}T+80\bar{\sigma}_1^2M\theta^2N^{2\theta-2}C_{6,8}(M,N,p)\Delta^{1-\frac{1}{p}}T\\
&+80\bar{\sigma}_1^2M\theta^2N^{2\theta-2}\mathbb{E}\int_0^{t_1\wedge s}[X(u)-x(u)]^2du,
\end{aligned}
\tag{8.30}
$$

as desired. The proof of our theorem finally will follow by the application of Gronwall's inequality. $\square$

Next, we will establish the following theorem to show convergence in probability of the continuous EM approximate solution to the true solution.

Theorem 8.4. *Let $X(t)$ be the true solution and $x(t)$ be the continuous approximate solution to the SDE model (7.1) when $\theta > 1$ and $\beta > 1$. Then,*

$$\lim_{\Delta \to 0} \left(\sup_{0 \le t \le T} [X(t) - x(t)]^2 \right) = 0 \quad \text{in probability.} \tag{8.31}$$

To prove Theorem 8.4, we need the following lemma which can be obtained in the same way as computation of Lemma 6.4.

Lemma 8.6. *There exists a constant $C_{6,2}(M,p)^{**}$ which is dependent on M, N and p, but independent of Δ such that*

$$\mathbb{E}(x(t \wedge h)) \le L + C^{**}_{6,2}(N, M, p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}, \tag{8.32}$$

where constant L is independent of Δ, and $h = \gamma_M \wedge \zeta_N$.

Proof. (of Theorem 8.4)
In this process, we also divide the whole proof into 3 steps.
Step 1: In the same way as in computation of (6.21) but with stopping time $g = \tau_N \wedge \rho_M$, we have

$$\mathbb{E}\left[H(X(T \wedge g))\right] \le H(X_0) + \frac{\bar{\alpha}_1 \bar{\mu}_1 T}{2} + \frac{(\bar{\alpha}_1 + \lambda_1 \bar{\delta}_1)T}{2} + \frac{\bar{\sigma}_1^2 R 4^{2\theta-2} T}{4} + \lambda_1 \bar{\delta}_1 \bar{R} T,$$

where $\bar{R}$ is the upper bound for the expected value of the asset price that can be obtained by applying a similar technique as used in (5.8). Now, in the same way as in computation of (3.59), we further get that

$$\mathbb{P}(\tau_N \le T) \le \frac{H(X_0) + \frac{\bar{\alpha}_1 \bar{\mu}_1 T}{2} + \frac{(\bar{\alpha}_1 + \lambda_1 \bar{\delta}_1)T}{2} + \frac{\bar{\sigma}_1^2 R 4^{2\theta-2} T}{4} + \lambda_1 \bar{\delta}_1 \bar{R} T}{H(N^{-1}) \wedge H(N)}. \tag{8.33}$$

Step 2: Repeating the technique used in *(Step 2:) of* Theorem 8.2, Lemma 8.6 and

$$\mathbb{E}\left(\sup_{0\le t\le T}[x(t\wedge h)-\bar{x}(t\wedge h)]^2\right)\le C^*_{6,2}(M,N,p)\Delta^{1-\frac{1}{p}}, \tag{8.34}$$

which can be proved in the same way as Lemma 8.3 was proved, we obtain

$$\begin{aligned}\mathbb{P}(\zeta_N\le T)\le&\frac{H(X_0)+\frac{\bar{\alpha}_1\bar{\mu}_1 T}{2}+\frac{(\bar{\alpha}_1+\lambda_1\bar{\delta}_1)T}{2}}{H(N^{-1})\wedge H(N)}\\&+\frac{\frac{\bar{\sigma}_1^2}{4}M4^{2\theta-2}T+\bar{\delta}_1\lambda_1 LT+\bar{C}_{6,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1})\wedge H(N)}.\end{aligned} \tag{8.35}$$

Step 3: For any arbitrarily small constants $\varepsilon>0$ and $\delta\in(0,1)$, then define

$$\Omega_6=\left[t\in[0,T];\sup_{0\le t\le T}[X(t)-x(t)]^2\ge\delta\right]. \tag{8.36}$$

Repeating the technique used in (3.42), but with Theorem 8.3, we further get that

$$\mathbb{P}(\Omega_6\cap(s\ge t_1))\le\frac{C_{6,13}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}. \tag{8.37}$$

On the other hand, we can compute

$$\begin{aligned}\mathbb{P}(\Omega_6)\le&\mathbb{P}(\Omega_6\cap(s\ge T))+\mathbb{P}(\gamma_M\le T)\\&+\mathbb{P}(\rho_M\le T)+\mathbb{P}(\zeta_N\le T)+\mathbb{P}(\tau_N\le T).\end{aligned} \tag{8.38}$$

Substituting (8.13), (8.14), (8.33) and (8.35) into (8.38) yields

$$\begin{aligned}\mathbb{P}(\Omega_6)\le&\frac{C_{6,13}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}\\&\frac{H(V_0)+\frac{\bar{\alpha}_2\bar{\mu}_2}{2}T+\frac{(\bar{\alpha}_2+\lambda_2\bar{\delta}_2)}{2}T+\frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4}+\lambda_2\bar{\delta}_2 RT}{H(M^{-1})\wedge H(M)}\\&\frac{H(V_0)+\frac{\bar{\alpha}_2\bar{\mu}_2 T}{2}+\frac{(\bar{\alpha}_2+\bar{\delta}_2\lambda_2)T}{2}+\frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4}+\bar{\delta}_2\lambda_2 TZ+\bar{C}_{6,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1})\wedge H(M)}\\&\frac{H(X_0)+\frac{\bar{\alpha}_1\bar{\mu}_1 T}{2}+\frac{(\bar{\alpha}_1+\lambda_1\bar{\delta}_1)T}{2}+\frac{\bar{\sigma}_1^2 R4^{2\theta-2}T}{4}+\lambda_1\bar{\delta}_1\bar{R}T}{H(N^{-1})\wedge H(N)}\\&\frac{H(X_0)+\frac{\bar{\alpha}_1\bar{\mu}_1 T}{2}+\frac{(\bar{\alpha}_1+\lambda_1\bar{\delta}_1)T}{2}+\frac{\bar{\sigma}_1^2 M4^{2\theta-2}T}{4}+\bar{\delta}_1\lambda_1 LT+\bar{C}_{6,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1})\wedge H(N)}.\end{aligned}$$

Choose M sufficiently large for

$$\frac{2\left[H(V_0)+\frac{\bar{\alpha}_2\bar{\mu}_2 T}{2}+\frac{(\bar{\alpha}_2+\bar{\delta}_2\lambda_2)T}{2}+\frac{\bar{\sigma}_2^2 4^{2\beta-2}T}{4}\right]+\bar{\delta}_2\lambda_2 TZ+\lambda_2\bar{\delta}_2 RT}{H(M^{-1})\wedge H(M)}<\frac{\varepsilon}{3}, \tag{8.39}$$

then choose N sufficiently large for

$$\frac{2\left[H(X_0)+\frac{\bar{\alpha}_1\bar{\mu}_1 T}{2}+\frac{(\bar{\alpha}_1+\lambda_1\bar{\delta}_1)T}{2}\right]+\frac{\bar{\sigma}_1^2 R 4^{2\theta-2}T}{4}+\frac{\bar{\sigma}_1^2 M 4^{2\theta-2}T}{4}+\lambda_1\bar{\delta}_1\bar{R}T+\bar{\delta}_1\lambda_1 LT}{H(N^{-1})\wedge H(N)}<\frac{\varepsilon}{3} \tag{8.40}$$

and further choose Δ sufficiently small for

$$\frac{C_{6,13}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{\delta}+\frac{\bar{C}_{6,1}(M,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(M^{-1})\wedge H(M)}+\frac{\bar{C}_{6,2}(M,N,p)\Delta^{\frac{1}{2}[1-\frac{1}{p}]}}{H(N^{-1})\wedge H(N)}<\frac{\varepsilon}{3}. \tag{8.41}$$

Hence, we have

$$\mathbb{P}\left(\sup_{0\le t\le T}[X(t)-x(t)]^2\ge\delta\right)<\varepsilon, \tag{8.42}$$

as required. The proof is therefore complete now. □

Even though the continuous EM approximate solution will converge to the true solution, it is not computable in practice. Therefore, we will establish the following theorem to show that the corresponding step process will converge in probability to the true solution $X(t)$.

Theorem 8.5. *For any $t\in[0,T]$, there exists a step function $\bar{x}(t)$ of the EM approximate solution to the $X(t)$ such that*

$$\lim_{\Delta\to 0}\left(\sup_{0\le t\le T}|X(t)-\bar{x}(t)|\right)=0 \quad \textit{in probability.} \tag{8.43}$$

The proof of Theorem 8.4 can be obtained in the same way as in computation of Theorem 6.5.

Clearly, the computable step process will converge to the true solution of the SDE model (7.1) when parameters θ and β are greater than 1. Therefore, let us choose initial condition $(X(0)=0.5, V(0)=0.04)$, correlation coefficient of two

Brownian motions $\rho = 0.6$, $(\theta = 1.5, \beta = 1.6)$ and $\lambda_1 = 1, \lambda_2 = 2$ with coefficients of the SDE model (see Table 7.1) to illustrate its behaviour in practice. Thus, we apply MATLAB® software (see Appendix A for code) and generators of two Markov chains Γ_{r_1} and Γ_{r_2} to obtain the following graph (see Figure 8.1).

$$\Gamma_{r_1} = \begin{pmatrix} -4 & 1 & 3 \\ 5 & -7 & 2 \\ 2 & 1 & -3 \end{pmatrix} \qquad \Gamma_{r_2} = \begin{pmatrix} -3 & 1 & 2 \\ 3 & -9 & 6 \\ 2 & 1 & -3 \end{pmatrix}$$

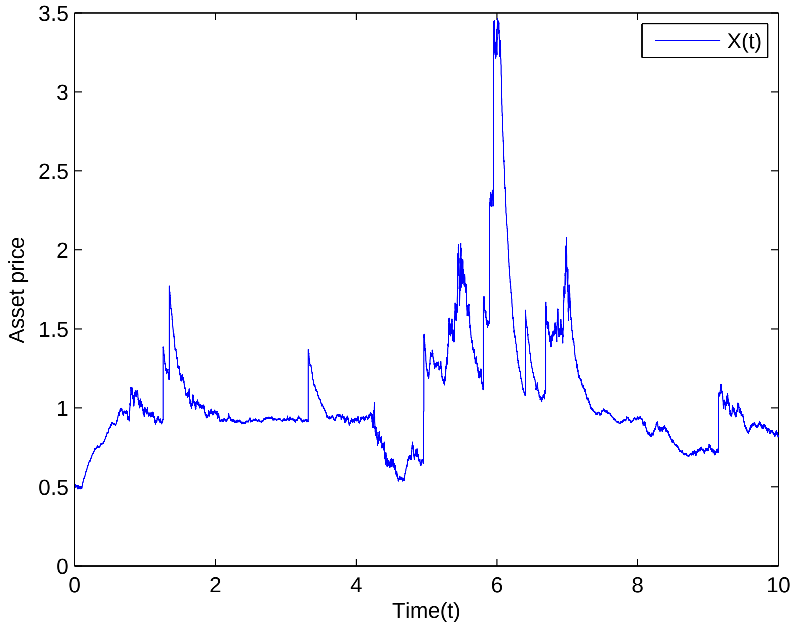

Figure 8.1: A sample path of the asset price $X(t)$ which is generated by the EM approximate solution to the hybrid mean-reverting-theta stochastic volatility model with Poisson jump over finite time, where $\theta = 1.5$ and $\beta = 1.6$.

8.4 Summary

The convergence property of the EM approximate solution to the SDE model (7.1) has been examined in this chapter when $\theta, \beta > 1$. In this process, we have first proved that the unique local solution to this SDE model is non-negative with probability 1. Then, we have obtained the convergence property of the EM approximate solution to this SDE model in probability when the time step is sufficiently small. However, this EM approximate solution is not computable in practice. Therefore, the convergence property of the corresponding step process has been obtained to examine its application in finance.

Chapter 9

Applications in Finance

The generalized Black-Scholes type formulas which were examined in previous chapters give a significant contribution to understanding behaviour of the underlying asset prices in financial markets even though they have no explicit solutions. However, the final result of the each previous chapters shows that the EM approximate solution to the each SDE model will converge to the corresponding true solution when the time step is sufficiently small.

In this chapter, we will therefore show that these EM solutions can be used to evaluate financial quantities. However, we will omit some information that can be found in several research papers under the same conditions on parameters θ and β. Furthermore, details of these financial quantities are omitted here, as these can also be found in textbooks [35, 50].

9.1 Bonds

Assume that the SDE model (5.1) when $\frac{1}{2} \leq \theta, \beta \leq 1$ describes the short-term interest rate dynamics. Then the price of a bond at the end of period is given by

$$B(T) = \mathbb{E}\left[\exp(-\int_0^T X(t)dt)\right].$$

Given that the step process $\bar{x}(t)$ is computable and it converges to the true solution $X(t)$ in probability, we would naturally compute $B(T)$ approximately by

$$\bar{B}_\Delta(T) = \mathbb{E}\left[\exp(-\int_0^T |\bar{x}(t)|dt)\right].$$

The question is: does $\bar{B}_\Delta(T)$ approximate $B(T)$ well whenever the step size Δ is sufficiently small? The following theorem confirms this.

Theorem 9.1. *In the notation above, we have*

$$\lim_{\Delta \to 0} |B(T) - \bar{B}_\Delta(T)| = 0. \tag{9.1}$$

The proof of our theorem can be obtained in the same way as in computation of Theorem 4.1 in [32] but with Theorem 5.7.

On the other hand, when short-term interest rate dynamics follows the SDE model (3.6) which was examined in chapter 3, we will get the required proof for Theorem 9.1 in the same way as in computation of Theorem 5.1 in [77], but applying our new Theorem 3.5.

9.2 Path dependent options

Let us now consider a barrier option under the SDE model (4.1) when parameters θ and β are greater than 1. That is, consider a down-and-out European put option, which, at expiry time T, pays the European put value $(E - X(T))^+$ if $X(t)$ never decreases below the fixed barrier B, and pays zero otherwise, where

E is the exercise price. We suppose that the expected payoff is computed from a Monte Carlo simulation (see [26]) based on the EM step process $\bar{x}(t)$. The following theorem uses our new convergence theorem to show that the expected payoff from the numerical method converges to the correct expected payoff as $\Delta \to 0$.

Theorem 9.2. *Let $X(t)$ be the solution of the SDE model (4.1) when parameters θ and β are greater than 1 and $\bar{x}(t)$ be the EM step process. Consider a down-and-out European put option with the exercise price E, the fixed barrier B and the expiry date T. The expected payoff of the down-and-out call option is*

$$O = \mathbb{E}\left[(E - X(T))^+ \, 1_{\left(B \le \inf\limits_{0\le t\le T} X(t)\right)} \right],$$

while the estimated expected payoff based on the EM step process $\bar{x}(t)$ is

$$\hat{O}_\Delta = \mathbb{E}\left[(E - |\bar{x}(T)|)^+ \, 1_{\left(B \le \inf\limits_{0\le t\le T} |\bar{x}(t)|\right)} \right].$$

Then

$$\lim_{\Delta\to 0} \left| O - \hat{O}_\Delta \right| = 0. \tag{9.2}$$

Proof. Let

$$\mathfrak{A} = \left(B \le \inf_{0\le t\le T} X(t) \right) \quad \text{and} \quad \mathfrak{B} = \left(B \le \inf_{0\le t\le T} |\bar{x}(t)| \right).$$

We will complete the proof, if we can prove that

$$\lim_{\Delta\to 0} \left| (E - X(T))^+ 1_{\mathfrak{A}} - (E - |\bar{x}(T)|)^+ 1_{\mathfrak{B}} \right| = 0 \quad \text{in probability.}$$

In other words, the theorem holds as long as we can show that for any small constants $\varepsilon > 0$ and $\delta \in (0,1)$, the following

$$\mathbb{P}\left(\left| (E - X(T))^+ 1_{\mathfrak{A}} - (E - |\bar{x}(T)|)^+ 1_{\mathfrak{B}} \right| \ge \delta \right) < \varepsilon \tag{9.3}$$

holds for all sufficiently small Δ. To prove this, we set $\mathfrak{A}' = \Omega - \mathfrak{A}$ and $\mathfrak{B}' = \Omega - \mathfrak{B}$.

It is easy to verify that

$$|(E - X(T))^+ - (E - |\bar{x}(T)|)^+| \le |X(T) - |\bar{x}(T)|| \le |X(T) - \bar{x}(T)|.$$

We then compute

$$\begin{aligned}
&\mathbb{P}\left(|(E - X(T))^+ 1_{\mathfrak{A}} - (E - |\bar{x}(T)|)^+ 1_{\mathfrak{B}}| \ge \delta\right) \\
\le &\mathbb{P}\left[(|(E - X(T))^+ 1_{\mathfrak{A}} - (E - |\bar{x}(T)|)^+ 1_{\mathfrak{B}}| \ge \delta) \cap (\mathfrak{A} \cap \mathfrak{B})\right] \\
&+ \mathbb{P}\left[(|(E - X(T))^+ 1_{\mathfrak{A}} - (E - |\bar{x}(T)|)^+ 1_{\mathfrak{B}}| \ge \delta) \cap (\mathfrak{A}' \cap \mathfrak{B})\right] \\
&+ \mathbb{P}\left[(|(E - X(T))^+ 1_{\mathfrak{A}} - (E - |\bar{x}(T)|)^+ 1_{\mathfrak{B}}| \ge \delta) \cap (\mathfrak{A} \cap \mathfrak{B}')\right] \\
\le &\mathbb{P}\left(|X(T) - \bar{x}(T)| \ge \delta\right) + \mathbb{P}\left(\mathfrak{A}' \cap \mathfrak{B}\right) + \mathbb{P}\left(\mathfrak{A} \cap \mathfrak{B}'\right).
\end{aligned} \tag{9.4}$$

By Theorem 4.5, for all sufficiently small Δ, we have

$$\mathbb{P}\left(|X(T) - \bar{x}(T)| \ge \delta\right) < \frac{\varepsilon}{3}. \tag{9.5}$$

Now, let $z \in (0, B)$ be any sufficiently small number. Write

$$\begin{aligned}
\mathfrak{A}' &= \left[\inf_{0 \le t \le T} X(t) < B\right] \\
&= \left[\inf_{0 \le t \le T} X(t) < B - z\right] \cup \left[B - z \le \inf_{0 \le t \le T} X(t) < B\right] \\
&:= \mathfrak{A}_1' \cup \mathfrak{A}_2'.
\end{aligned} \tag{9.6}$$

We hence compute

$$\begin{aligned}
\mathbb{P}\left(\mathfrak{A}' \cap \mathfrak{B}\right) &= \mathbb{P}\left(\mathfrak{A}_1' \cap \mathfrak{B}\right) + \mathbb{P}\left(\mathfrak{A}_2' \cap \mathfrak{B}\right) \\
&\le \mathbb{P}\left(|\inf_{0 \le t \le T} X(t) - \inf_{0 \le t \le T} |\bar{x}(t)|| \ge z\right) + \mathbb{P}\left(\mathfrak{A}_2'\right) \\
&\le \mathbb{P}\left(\sup_{0 \le t \le T} |X(t) - |\bar{x}(t)|| \ge z\right) + \mathbb{P}\left(\mathfrak{A}_2'\right) \\
&\le \mathbb{P}\left(\sup_{0 \le t \le T} |X(t) - \bar{x}(t)| \ge z\right) + \mathbb{P}\left(\mathfrak{A}_2'\right).
\end{aligned} \tag{9.7}$$

Since $\inf_{0 \le t \le T} X(t)$ is a continuously distributed random variable, we can choose z so small that

$$\mathbb{P}\left(\mathfrak{A}_2'\right) < \frac{\varepsilon}{6},$$

while by Theorem 4.5, we can choose Δ so small for

$$\mathbb{P}\left(\sup_{0\le t\le T} |X(t) - \bar{x}(t)| \ge z\right) < \frac{\varepsilon}{6}.$$

We hence see that for all sufficiently small Δ,

$$\mathbb{P}\left(\mathfrak{A}' \cap \mathfrak{B}\right) < \frac{\varepsilon}{3}. \tag{9.8}$$

Similarly, we can show that for all sufficiently small Δ,

$$\mathbb{P}\left(\mathfrak{A} \cap \mathfrak{B}'\right) < \frac{\varepsilon}{3}. \tag{9.9}$$

Substituting (9.5), (9.8) and (9.9) into (9.4) yields

$$\mathbb{P}\left(\left|(E - X(T))^+ 1_{\mathfrak{A}} - (E - |\bar{x}(T)|)^+ 1_{\mathfrak{B}}\right| \ge \delta\right) < \varepsilon, \tag{9.10}$$

as required. The proof is therefore complete. □

9.3 Lookback put options

The fixed strike lookback put option differs from the standard European put option in that when we compute the payoff, the price at the expiry date is replaced by the smallest asset price observed. So the expected payoff of the fixed strike lookback put is given by

$$L = \mathbb{E}\Big[\Big(E - \inf_{0\le t\le T} X(t)\Big)^+\Big],$$

where E is the exercise prise. Analogously, our numerical approximation to this payoff is

$$\hat{L}_\Delta = \mathbb{E}\Big[\Big(E - \inf_{0\le t\le T} |\bar{x}(t)|\Big)^+\Big].$$

Theorem 9.3. *In the notation above, we have*

$$\lim_{\Delta\to 0} |L - \hat{L}_\Delta| = 0.$$

Here, we proceed with the asset price model examined in Chapter 8. Thus, we will assume that the lookback put option follows the SDE model (7.1) when parameters θ and β are greater than 1.

Proof. Clearly, it is sufficient to prove

$$\lim_{\Delta\to 0} \left| \Big(E - \inf_{0\le t\le T} X(t)\Big)^+ - \Big(E - \inf_{0\le t\le T} |\bar{x}(t)|\Big)^+ \right| = 0 \quad \text{in probability.}$$

In other words, the theorem holds as long as we can show that for any small constants $\varepsilon > 0$ and $\delta \in (0,1)$, the following

$$\mathbb{P}\left(\left| \Big(E - \inf_{0\le t\le T} X(t)\Big)^+ - \Big(E - \inf_{0\le t\le T} |\bar{x}(t)|\Big)^+ \right| \ge \delta \right) < \varepsilon \tag{9.11}$$

holds for all sufficiently small Δ. On the other hand, it is easy to show that

$$\begin{aligned} &\left| \Big(E - \inf_{0\le t\le T} X(t)\Big)^+ - \Big(E - \inf_{0\le t\le T} |\bar{x}(t)|\Big)^+ \right| \\ &\le \left| \inf_{0\le t\le T} X(t) - \inf_{0\le t\le T} |\bar{x}(t)| \right| \\ &\le \sup_{0\le t\le T} \big| X(t) - |\bar{x}(t)| \big| \\ &\le \sup_{0\le t\le T} \big| X(t) - \bar{x}(t) \big|. \end{aligned} \tag{9.12}$$

We therefore have that

$$\mathbb{P}\left(\left| \Big(E - \inf_{0\le t\le T} X(t)\Big)^+ - \Big(E - \inf_{0\le t\le T} |\bar{x}(t)|\Big)^+ \right| \ge \delta \right) \le \mathbb{P}\left(\sup_{0\le t\le T} \big| X(t) - \bar{x}(t) \big| \ge \delta \right). \tag{9.13}$$

But, by Theorem 8.5, we have

$$\mathbb{P}\left(\sup_{0\le t\le T} \big| X(t) - \bar{x}(t) \big| \ge \delta \right) < \varepsilon \tag{9.14}$$

for all sufficiently small Δ. Combining (9.13) and (9.14) we obtain the required

(9.11). The proof is therefore complete. □

9.4 Summary

In this chapter, we have proved that the step processes of the corresponding EM approximate solutions to the SDE models which were discussed in previous chapters can be used to evaluate financial quantities. In this process, SDE models have been divided into two categories, with $1/2 \leq \beta, \theta \leq 1$ and with $1 < \theta, \beta$. However, each SDE model in one category needs similar techniques to investigate its corresponding application in finance. Therefore, we have examined a few models to complete this process.

Chapter 10

Conclusions

This research program focussed on several SDE models which are widely used to examine asset price or portfolio data but so far have no explicit solutions. In this process, we have established Euler-Maruyama (EM) numerical approximations to these highly sensitive volatility models which help to study and understand effects and movements of financial markets. However, existing financial instruments are not strong enough to derive analytical properties of more generalized Black-Scholes formulas. Therefore, necessary effective theories have been developed under certain assumptions. Clearly, these newly developed financial tools can also be used to scrutinize some other financial quantities.

The first model can be treated as having constant coefficients with parameters θ and β are greater than 1. Since this model satisfies the local Lipschitz condition, the expected error bound of the continuous EM approximation has been used to show that convergence in probability of the EM approximate solution. In practice, the continuous EM approximate solution to this model is not computable. Thus, the convergence in probability of the step process has also been established.

The Markov switching concept changes the direction of the first model to explain the higher dimension of some financial quantities. The coefficients of this model also satisfy the local Lipschitz condition. Therefore, we have obtained convergence in probability of the EM approximate solution to the true value. In addition, convergence of the corresponding step process has also been established to examine application of this approximate solution in finance.

The mean-reverting-theta stochastic volatility model driven by a Poisson-jump process with constant coefficients and its Markov switching form created more generalized Black-Scholes formulas which can also be seen in financial markets. In the case of $\frac{1}{2} \leq \theta, \beta \leq 1$, the convergence in second moment of the EM approximate values to these SDE models have been examined under the global Lipschitz condition and the linear growth conditions. In order to show applications of these EM approximate solutions in finance, convergence properties of corresponding step processes have been obtained.

In contrast to the stochastic volatility model discussed in Chapter 5 and 7, when the parameters θ and β are greater than 1, these hybrid SDE models obey the local Lipschitz condition. Therefore, we can not appeal to convergence in second moment of the EM approximate solutions to these SDE models. Thus, new techniques have been developed and we have examined analytical properties of the EM approximate solutions in probability. Then, we extended this process to show convergence in probability of the corresponding step processes which give the necessary condition to evaluate their applications in finance.

These developed techniques have proved that the approximate values generate the expected result of the true solutions to the SDE models when the time step is sufficiently small. In Chapter 9, several applications of these financial models have been discussed to show that their solutions can be used to evaluate financial quantities in practice.

In addition, these EM approximate solutions can be used to evaluate investment risk of financial quantities in financial markets. If the short-term interest rate of a bond is high then the bond price becomes low, and if short-term interest rate is low then bond price becomes high. Therefore, the risk of bond price can be examined by taking the variance of the short-term interest rate which gives the dispersion of the short-term interest rate throughout the mean.

On the other hand, implied volatility gives the significant effect for the decision of the option price. If the implied volatility of the asset price is high then it gives the higher option price, and if it is low then gives the lower option price. Therefore, investors can evaluate the risk of the option by examining the variance of the implied volatility.

Appendix A

MATLAB® Codes

Model 1

```
function [x,v]=svm_ccoefficient_stime(x0,v0,T,n,M,N,rho,NS)

%x0:Initial value of asset price
%v0:Initial value of volatility
%T:Time period
%n:Number of steps
%NS:Number of simulations
%M:Upper bound of x
%N:Upper bound of v
%rho: correlation coefficient of w1 and w2

theta=1.2;   beta=1.1;
alpha1=0.21; alpha2=0.3;
miyu1=10.4;   miyu2=0.13;
sigma1=0.05; sigma2=0.054;

delta=T/n;
x=zeros(NS,n+1); v=zeros(NS,n+1);
v(:,1)=v0*ones(NS,1); x(:,1)=x0*ones(NS,1);

for i=2:1:n+1

normrand1=randn(NS,1); normrand2=randn(NS,1);

 x(:,i)=x(:,i-1)+alpha1*(miyu1-x(:,i-1))*delta+...
 sigma1*(x(:,i-1).^theta).*sqrt(v(:,i-1)).*...
 normrand1*sqrt(delta);

          if (abs(x(:,i))< N)
               x(:,i)=abs(x(:,i));
           else
               x(:,i)=N;
          end
```

```
v(:,i)=v(:,i-1)+alpha2*(miyu2-v(:,i-1))*delta+...
sigma2*(v(:,i-1).^beta).*...
(rho*normrand1+sqrt(1-rho^2)*normrand2)*sqrt(delta);

        if (abs(v(:,i))<M)
              v(:,i)=abs(v(:,i));
        else
            v(:,i)=M;
        end

end
```

Model 2

```
function [x,v]=svm_makove_stime(x0,v0,T,n,C1,Q1,s,M,N,rho,NS)

%x0:Initial value of asset price,
%v0:Initial value of volatility
%T:Time period
%n:Number of steps
%NS:Number of simulations
%M:Upper bound of x
%N:Upper bound of v
%rho: correlation coefficient w1 and w2
%C1: states value matrix;
%Q1: generator of Markov-chain;
%s : Initial state;
theta=1.2;   beta=1.1;

delta=T/n;
x=zeros(NS,n+1); v=zeros(NS,n+1);
v(:,1)=v0*ones(NS,1); x(:,1)=x0*ones(NS,1);

for i=2:1:n+1

s=markovs(Q1,T, n, s);

alpha1=C1(s,1); alpha2=C1(s,4);
miyu1 =C1(s,2); miyu2 =C1(s,5);
sigma1=C1(s,3); sigma2=C1(s,6);

normrand1=randn(NS,1); normrand2=randn(NS,1);

  x(:,i)=x(:,i-1)+alpha1*(miyu1-x(:,i-1))*delta+...
  sigma1*(x(:,i-1).^theta).*sqrt(v(:,i-1)).*...
  normrand1*sqrt(delta);

       if (abs(x(:,i))< N)
          x(:,i)=abs(x(:,i));
       else
          x(:,i)=N;
       end
```

```
        v(:,i)=v(:,i-1)+alpha2*(miyu2-v(:,i-1))*delta+...
            sigma2*(v(:,i-1).^beta).*...
            (rho*normrand1+sqrt(1-rho^2)*normrand2)*sqrt(delta);

          if (abs(v(:,i))<M)
            v(:,i)=abs(v(:,i));
          else
               v(:,i)=M;
          end

end
```

Model 3

```
function [x,v]=svm_ccofficient_jump(x0,v0,T,n,rho,NS)

%x0:Initial value of asset price
%v0:Initial value of volatility
%T:Time period
%n:Number of steps
%NS: Number of simulations
%rho: correlation coefficient w1 and w2

theta=0.5;     beta=0.6;
alpha1=0.21;  alpha2=0.3;
miyu1=0.4;    miyu2=0.03;
sigma1=0.05;  sigma2=0.054;
delta1=0.09;  delta2=0.07;
lambda1=1;    lambda2=2;

delta=T/n;
x=zeros(NS,n+1);
v=zeros(NS,n+1);
v(:,1)=v0*ones(NS,1);  x(:,1)=x0*ones(NS,1);

for i=2:1:n+1

normrand1=randn(NS,1);  normrand2=randn(NS,1);

 x(:,i)=x(:,i-1)+alpha1*(miyu1-x(:,i-1))*delta+...
 sigma1*(x(:,i-1).^theta).*sqrt(v(:,i-1)).*...
 normrand1*sqrt(delta)...
 +delta1*x(:,i-1)*(poissrnd(lambda1*delta)-lambda1*delta);
 x(:,i)=abs(x(:,i));

 v(:,i)=v(:,i-1)+alpha2*(miyu2-v(:,i-1))*delta+...
 sigma2*(v(:,i-1).^beta).*...
 (rho*normrand1+sqrt(1-rho^2)*normrand2)*sqrt(delta)...
 +delta2*v(:,i-1)*(poissrnd(lambda2*delta)-lambda1*delta);
 v(:,i)=abs(v(:,i));
```

```
end
```

Model 4

```
function [x,v]=svm_ccoefficient_jump_stime(x0,v0,T,n,M,N,rho,NS)

%x0:Initial value of asset price,
%v0:Initial value of volatility
%T:Time period
%n:Number of steps
%NS:Number of simulations
%M:Upper bound of x
%N:Upper bound of v
%rho: correlation coefficient w1 and w2

theta=1.2;      beta=1.1;
alpha1=0.21;    alpha2=0.3;
miyu1=0.4;      miyu2=0.13;
sigma1=0.05;    sigma2=0.054;
delta1=0.09;    delta2=0.07;
lambda1=1;      lambda2=2;

delta=T/n;
x=zeros(NS,n+1); v=zeros(NS,n+1);
v(:,1)=v0*ones(NS,1); x(:,1)=x0*ones(NS,1);

for i=2:1:n+1

normrand1=randn(NS,1); normrand2=randn(NS,1);

 x(:,i)=x(:,i-1)+alpha1*(miyu1-x(:,i-1))*delta+...
 sigma1*(x(:,i-1).^theta).*sqrt(v(:,i-1)).*...
 normrand1*sqrt(delta)...
 +delta1*x(:,i-1)*(poissrnd(lambda1*delta)-lambda1*delta);

        if (abs(x(:,i))< N)
            x(:,i)=abs(x(:,i));
        else
            x(:,i)=N;
        end

 v(:,i)=v(:,i-1)+alpha2*(miyu2-v(:,i-1))*delta+...
 sigma2*(v(:,i-1).^beta).*...
 (rho*normrand1+sqrt(1-rho^2)*normrand2)*sqrt(delta)...
 +delta2*v(:,i-1)*(poissrnd(lambda2*delta)-lambda1*delta);

       if (abs(v(:,i))<M)
           v(:,i)=abs(v(:,i));
       else
           v(:,i)=M;
       end
```

```
end
```

Model 5

```
function
[x,v]=svm_makove_jump(x0,v0,T,n,C1,C2,Q1,Q2,s1,s2,l1,l2,rho,NS)

%x0:Initial value of asset price,
%v0:Initial value of volatility
%T:Time period
%n:Number of steps
%NS:Number of simulations
%rho: correlation coefficient w1 and w2
%C1,C2: states value matrix;
%Q1,Q2: generator of Markov-chain;
%s : Initial state;
%l1 : lamda1;
%l2 : lamda2;

theta=1;  beta=0.5;

delta=T/n;
x=zeros(NS,n+1); v=zeros(NS,n+1);
v(:,1)=v0*ones(NS,1); x(:,1)=x0*ones(NS,1);

for i=2:1:n+1

s1=markovs(Q1,T, n, s1); s2=markovs(Q2,T, n, s2);

alpha1=C1(s1,1); alpha2=C1(s1,4);
miyu1 =C1(s1,2); miyu2 =C1(s1,5);
sigma1=C1(s1,3); sigma2=C1(s1,6);
delta1=C2(s2,1); delta2=C2(s2,2);

normrand1=randn(NS,1); normrand2=randn(NS,1);

  x(:,i)=x(:,i-1)+alpha1*(miyu1-x(:,i-1))*delta+...
  sigma1*(x(:,i-1).^theta).*sqrt(v(:,i-1)).*...
  normrand1*sqrt(delta)...
  +delta1*x(:,i-1)*(poissrnd(l1*delta)-l1*delta);

        x(:,i)=abs(x(:,i));

  v(:,i)=v(:,i-1)+alpha2*(miyu2-v(:,i-1))*delta+...
  sigma2*(v(:,i-1).^beta).*...
  (rho*normrand1+sqrt(1-rho^2)*normrand2)*sqrt(delta)...
  +delta2*v(:,i-1)*(poissrnd(l2*delta)-l2*delta);

        v(:,i)=abs(v(:,i));

end
```

Model 6

```
function [x,v]=svm_makove_jump_stime
              (x0,v0,T,n,C1,C2,Q1,Q2,s1,s2,l1,l2,M,N,rho,NS)

%x0:Initial value of asset price,
%v0:Initial value of volatility
%T:Time period
%n:Number of steps
%NS:Number of simulations
%M:Upper bound of x
%N:Upper bound of v
%rho: correlation coefficient w1 and w2
%C1,C2: states value matrixs;
%Q1,Q2: generator of Markov-chains;
%s : Initial state;
%l1 : lamda1;
%l2 : lamda2;

theta=1.5;   beta=1.2;

delta=T/n;
x=zeros(NS,n+1); v=zeros(NS,n+1);
v(:,1)=v0*ones(NS,1); x(:,1)=x0*ones(NS,1);

for i=2:1:n+1

s1=markovs(Q1,T, n, s1); s2=markovs(Q2,T, n, s2);

alpha1=C1(s1,1); alpha2=C1(s1,4);
miyu1 =C1(s1,2); miyu2 =C1(s1,5);
sigma1=C1(s1,3); sigma2=C1(s1,6);
delta1=C2(s2,1); delta2=C2(s2,2);

normrand1=randn(NS,1); normrand2=randn(NS,1);

 x(:,i)=x(:,i-1)+alpha1*(miyu1-x(:,i-1))*delta+...
 sigma1*(x(:,i-1).^theta).*sqrt(v(:,i-1)).*...
 normrand1*sqrt(delta)...
 +delta1*x(:,i-1)*(poissrnd(l1*delta)-l1*delta);

   if (abs(x(:,i))< N)
     x(:,i)=abs(x(:,i));
   else
     x(:,i)=N;
   end

 v(:,i)=v(:,i-1)+alpha2*(miyu2-v(:,i-1))*delta+...
 sigma2*(v(:,i-1).^beta).*...
 (rho*normrand1+sqrt(1-rho^2)*normrand2)*sqrt(delta)...
 +delta2*v(:,i-1)*(poissrnd(l2*delta)-l2*delta);
```

```
    if (abs(v(:,i))<M)
        v(:,i)=abs(v(:,i));
    else
        v(:,i)=M;
    end

end
```

Markov-chain

```
function [s]=markovs(Q,T, n, s)

%Q = Transition rate matrix;
%u = values of states matrix;
%T = Time
%n = No of steps;
%s = Present state;
%R = Randam number;
%L = Numbrer of states;

i=1;
R=rand(1);
probability=10;
p=expm((T/n)*Q);
L=length(p);

while (R<probability)
          if (R <sum(p(s,1:i)))
              s=i;
            probability =R;
          elseif (i<L-1)
               i=i+1;
          else
              s=L;
            probability =R;
          end
end
```

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