Hopf-Frobenius Algebras

PhD Thesis

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October 3, 2024

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

Abstract

Hopf-Frobenius algebras are an algebraic structure present in two distinct areas of applied category theory. They consist of two Frobenius algebras and two Hopf algebras such that their structure maps overlap – i.e. a Frobenius algebra shares its monoid with one Hopf algebra, and its comonoid with the other Hopf algebra.

Hopf-Frobenius algebras are present in ZX-calculus, a model for quantum circuits, and the category of linear relations, which is used to model signal-flow graphs and graphical linear algebra. Both of these are exemplary examples of how string diagrams can be used, and the algebras are both commutative.

This thesis focuses on the noncommutative case of Hopf-Frobenius algebras. We examine the conditions under which a Hopf algebra is Hopf-Frobenius, and show that the conditions are relatively minor - every Hopf algebra in the category of Vector spaces is a Hopf-Frobenius algebra. We have provided several conditions which are all equivalent to when a Hopf algebra is Hopf-Frobenius, which makes checking if a given Hopf algebra is Hopf-Frobenius relatively straightforward. This is beneficial, as when a Hopf algebra is Hopf-Frobenius, we have more morphisms and equations to work with, and the string diagrams of Hopf-Frobenius algebras have a pleasing topology. In addition, we demonstrate in the final section of this thesis that many theorems about Hopf algebras in finite dimensional vector spaces can be lifted to the Hopf-Frobenius case. Hence when a Hopf algebra from a category other than vector spaces is Hopf-Frobenius, it will inherit machinery from the category of finite vector spaces.

We develop the theory of Hopf-Frobenius algebras by proving that Hopf algebra isomorphisms preserve Frobenius algebra structure, and using these to construct the category of Hopf-Frobenius algebras.

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Acknowledgements

I would like to begin these acknowledgements by thanking my supervisor, Ross Duncan. My PhD ended up taking much longer than we hoped, but despite this, he stuck with me all of the way. Without Ross, this thesis would never have come to fruition. He gave me structure, encouragement, and a great deal of advice and edits, and it is only through his help that this thesis has come about. I feel that the kindness that Ross has shown me over this PhD has been exceedingly rare and unique, a kind of kindness that I could never expect from someone. I am extremely grateful to Ross, and lucky that he was my supervisor.

I also would like to thank my family for all of their love. I have been living with my parents for the past few years, and without the encouragement and security that they have provided, I would not have accomplished half of what I did. My parents, Steve and Aine, and my brother Brian have been constant sources of unconditional love, and have pulled me through many, many tough spots. I have no idea what person I would be without them.

I want to thank the Mathematically Structured Programming (MSP) group and I especially want to focus on the other PhD students of MSP – Ezra, Bruno, Alasdair, Georgi, Sean, Malin, James, Matteo, Stuart, Riu, Eigil, Dilsat, Andre and Dylan. You are all such fantastic people, and it is because of all of you that I am not a gibbering wreck. These past 6 years have been so special and life changing, and I wouldn't trade them for anything. There are so many genuinely kind people in MSP, so many people who are willing to give up their time and their energy to help others, it is truly absurd. I have found myself utterly overwhelmed and surrounded by positive role models. The companionship that has been given to me over this PhD has been so precious and valuable, I can never begin to repay it. Thank you for spectacular banter over lunch and at the pub, and for all of the great discussions, and the never ending tide of curiosity. This atmosphere of MSP - characterised by kindness, curiosity and friendship - did not come about by accident. It was built and is maintained by the members of MSP, very intentionally. I believe that my time within the MSP group has made me a better person.

I especially want to thank Ezra, the smartest and kindest person that I know, for being with me in the final stretch of the thesis. You make me so happy, and your passion and curiosity creates so much beauty in the world. You make me excited to live.

I would like to extend my thanks to the Carnegie foundation for all of their funding and support, and to the University of Strathclyde for the use of its services, especially the Health and Wellbeing centre.

I would also like to express whatever the opposite of thanks is to the novel coronavirus, Covid-19. A pox upon your house!

Chapter 1

Introduction

Hopf-Frobenius algebras are algebraic structures that have a pivotal role in two distinct parts of applied category theory which would not be a priori associated with each other – categorical quantum computing [19, 31] and graphical linear algebra [13, 68].

Hopf algebras and Frobenius algebras are the two algebraic structures which appear in Hopf-Frobenius algebras, and they will be the main focus of this thesis. They are similar in a number of ways: they both consist of a monoid and a comonoid, such that their respective monoids preserve the structure of their respective comonoids. Specifically, in a Hopf algebra, the monoid structure maps are comonoid homomorphisms, and in a Frobenius algebra the monoid preserves the comodule structure induced by the comonoid[1]. We say that a Frobenius algebra and a Hopf algebra *overlap* when they share their monoid. In other words, they overlap when the monoid of one of these structures is the same as the monoid of the other. A Frobenius algebra and a Hopf algebra may also overlap on their comonoid in much the same way. With this in mind, a Hopf-Frobenius algebra consists of two Hopf algebras and two Frobenius algebras such that each Frobenius algebra overlaps on its monoid with one Hopf algebra, and on its comonoid with the other Hopf algebra.

The structure *Interacting Frobenius Algebras* [26] is a commutative example of Hopf-Frobenius algebras, and forms the mathematical backbone of ZX-calculus [18, 67], a model for quantum computing. On the other hand, *Interacting Hopf algebras* [13] are

[1]For a more in depth survey on the history of Hopf algebras, see [3]. For a more in-depth survey of Frobenius algebras, see [28]

also a commutative case of Hopf-Frobenius algebras which are used to model signal-flow graphs and graphical linear algebra [57]. We may ask, why does this algebraic structure appear in these two places? Are there other places where this structure appears, but we are not aware of it?

In this thesis, we provide several equivalent conditions for when a Hopf algebra is a Hopf-Frobenius algebra. These mostly revolve around the existence of an *integral* (Definition 4.2.2), and we show that they are equivalent to the Larson-Sweedler theorem [41] which states that every Hopf algebra in the category of finite dimensional vector spaces ($\mathbf{FVect_k}$) may be equipped with a Frobenius algebra. Hence, one of the main results of this thesis is that every finite dimensional Hopf algebra is Hopf-Frobenius.

While none of the conditions that we provide for when a Hopf algebra is Hopf-Frobenius are constructed purely from the structural maps of the Hopf algebra, we still believe that they are relatively straightforward to confirm. As we mentioned previously, the conditions mostly revolve around the construction of an integral. In particular, the construction of an *integral Hopf algebra* (Definition 4.2.3). However, it is not a trivial matter to construct an integral. As such, it is useful to consider the *integral morphism* (Definition 4.4.1). When you can construct a trace on the Hopf algebra object, or it has a dual structure, you may construct the integral morphism. It may be that this is sufficient, as in Lemma 5.2.1. However, in Lemma 4.4.7, we discuss how, given the integral morphism, we may construct an integral Hopf algebra from a given Hopf algebra. If the Hopf algebra is equipped with a dual structure, and the category has equalisers or coequalisers, then Lemma 5.2.5 is a constructive proof that will produce an integral if one exists. On the other hand, if you can prove that the Hopf algebra does not have any dual structure, then this will prove that the Hopf algebra must not be Hopf-Frobenius, as every Frobenius algebra comes equipped with a dual structure (see Remark 2.4.4). Hence, we believe that if you are operating in a string diagram with Hopf algebra structure, it is relatively straightforward to check if the Hopf algebra is Hopf-Frobenius.

When we find that a Hopf algebra is Hopf-Frobenius, then this gives us several benefits. First of all, the theory of Hopf-Frobenius algebras is simply a stronger theory than Hopf algebras. In addition, the equations that we get are well suited to string diagrams. Commutative Frobenius algebras are well known for their topological properties - for example see 2 dimensional TQFTs [39] and the spider theorem [18] - but even without commutativity, Frobenius algebras retain some topologically pleasing properties - for example, see the planar spider theorem [31]. Of course, we cannot prove the statement "Hopf-Frobenius algebras make good string diagrams", but we hope that as you read this thesis and the proofs contained within, you will find that the diagrammatic language of Hopf-Frobenius algebras is pleasing.

In addition, we find that many theorems about finite dimensional Hopf algebras only rely on their Hopf-Frobenius structure. This means that it is possible to lift theorems directly from a finite dimensional context to the Hopf-Frobenius context. For example, in the final section of the thesis, we have proven Radford's theorem on the order of the antipode for general Hopf-Frobenius algebras, and shown a connection between semisimplicity of Hopf algebras and the symmetry of Frobenius algebras.

1.1 Hopf Algebras

We may see the motivation behind a symmetric monoidal category as a way of generalising the Cartesian product - for example, in **Set** we can take any two sets, A and B, or functions, f and g, and we get $A \times B$ (or $f \times g$ respectively). To generalise this, we say that a *monoidal category* is a category $\mathcal{C}$ equipped with a functor $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$, called the *monoidal product* and object I, called the *monoidal unit*, such that $(\mathcal{C}, \otimes, I)$ behaves somewhat like a monoid. It becomes a *symmetric* monoidal category when we equip it with a natural transformation, $\{\sigma_{A,B} : A \otimes B \to B \otimes A\}$, called the *symmetry*, where A, B are objects in $\mathcal{C}$.

The notion that a monoidal product is a straightforward generalisation of the Cartesian product motivated us to generalise concepts that are traditionally defined in **Set** to an arbitrary monoidal category. For example, when we generalise the concept of a *monoid* in Set to monoidal categories, we find concepts like an *algebra*, which is a monoid in the category of vector spaces (denoted $\mathbf{Vect_k}$); a *monad*, which is a monoid in the category of endofunctors of an arbitrary category; or a strict monoidal category, which is a monoid in **Cat**. When we generalise the concept of group to a symmetric[2] monoidal category, the result is a *Hopf algebra* [3].

Let us unpack exactly what we mean by that. Recall that the definition of a *group* is a monoid (G, μ, e) with an inverse operation $\_^{-1} : G \to G$, where $gg^{-1} = e$ and $g^{-1}g = e$ for all $g \in G$. This is a sufficient definition for groups in **Set**, but for our purposes it is helpful to state the definition in terms of commutative diagrams.

In the category of sets and functions, a *group* is a set G with a binary operation $\mu : G \times G \to G$, an element $e : 1 \to G$, and an operation $\_^{-1} : G \to G$ such that

$$
\begin{tikzcd}
1 \times G \arrow[r, "e \times 1_G"] \arrow[dr, "\sim"'] & G \times G \arrow[d, "\mu"'] & G \times 1 \arrow[l, "1_G \times e"'] \arrow[dl, "\sim"] \\
& G &
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \times G \arrow[r, "1_G \times \mu"] \arrow[d, "\mu \times 1_G"'] & G \times G \arrow[d, "\mu"] \\
G \times G \arrow[r, "\mu"] & G
\end{tikzcd}
\qquad
\begin{tikzcd}
G \times G \arrow[d, "\_^{-1} \times 1_G"] & G \arrow[l, "\delta"'] \arrow[r, "\delta"] \arrow[d, "!"'] & G \times G \arrow[d, "1_G \times \_^{-1}"] \\
G \times G \arrow[dr, "\mu"'] & 1 \arrow[d, "e"'] & G \times G \arrow[dl, "\mu"] \\
& G &
\end{tikzcd}
$$

where $\delta : G \to G \times G$ is the *diagonal map* (or the *copy map*) induced by the cartesian product, and $! : G \to 1$ is the terminal map from G. Note also that, within **Set**, a function with type $1 \to G$ is the same as an element of G. We claim that when we generalise this definition to an arbitrary symmetric monoidal category, we will get the definition of a Hopf algebra. However, it is not immediately obvious how to accomplish this – the definition of a group uses the concept of the diagonal map δ, the terminal object 1, and a terminal map !. Given an arbitrary symmetric monoidal category, these maps may not exist. Hence, to define a Hopf algebra, we must first define concepts that are analogous to the terminal object, a monoid, δ and !.

The role that the terminal object plays in the above definition of a group is that of the monoidal unit – that is, 1 is the monoidal unit of the Cartesian product. Hence, given a symmetric monoidal category, $(\mathcal{C}, \otimes, I)$, we use $\otimes$ in place of $\times$ and I in place of 1. This makes it relatively straightforward to define a monoid in $\mathcal{C}$.

Recall that a monoid consists of a set, M, an associative multiplication $M \times M \to M$, and a unit element $e \in M$ such that $me = em = m$ for all elements m in M. We

[2] Throughout this thesis we're going to focus on symmetric monoidal categories, however many of the structures here can also be defined in *braided* monoidal categories.

[3] Hopf algebras origins lie in algebraic topology and algebraic groups. For an in-depth review of the history of Hopf algebras, see [3].

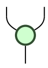
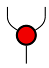
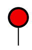
generalise this to the symmetric monoidal setting as follows.

Let $\mathcal{C}$ be an arbitrary symmetric monoidal category. A *monoid* (M, μ, e) in $\mathcal{C}$ is an object M in $\mathcal{C}$, morphisms $\mu : M \otimes M \to M$ and $e : I \to M$ such that

$$\begin{tikzcd}
I \otimes M \arrow[r, "e\otimes 1_M"] \arrow[dr, "\sim"'] & M \otimes M \arrow[d, "\mu"'] & M \otimes I \arrow[l, "1_M\otimes e"'] \arrow[dl, "\sim"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "1_M\otimes\mu"] \arrow[d, "\mu\otimes 1_M"'] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[r, "\mu"] & M
\end{tikzcd}$$

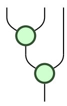
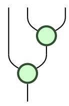
It is clear that when $\mathcal{C}$ is **Set**, then this corresponds to the familiar definition of a monoid.

At this point, we shall also introduce the string diagram notation that we shall be using throughout this thesis. We shall draw the following pictures to denote the multiplication and unit of the monoid

$\mu :=$  or  and $e :=$  or 

Throughout the thesis, we may have to deal with multiple monoids on the same object, and the way that we shall notationally differentiate between the two monoids is by their colours. Note also that we draw our diagrams to be read from top to bottom - so $M \otimes M \to M$ is denoted .

Every commutative diagram may be redrawn in the language of string diagrams. For example, the associativity axiom may be drawn as

 = 

As such, within the body of the thesis we shall tend to draw our equations as string diagrams rather than commutative diagrams.

To capture the essence of $\delta : G \to G \times G$ and $! : G \to 1$, we define the concept of a *comonoid*. This is simply the dual concept of a monoid – that is, a comonoid in $\mathcal{C}$ is a monoid in $\mathcal{C}^{\mathrm{op}}$. Explicitly, a comonoid (C, Δ, ϵ) in $\mathcal{C}$ is an object C in $\mathcal{C}$, morphisms

$\Delta : C \to C \otimes C$ and $\epsilon : C \to I$ such that

$$
\begin{tikzcd}
 & H \arrow[dl, "\sim"'] \arrow[d, "\Delta"] \arrow[dr, "\sim"] & \\
I \otimes H & H \otimes H \arrow[l, "\epsilon \otimes 1_H"] \arrow[r, "1_H \otimes \epsilon"'] & H \otimes I
\end{tikzcd}
\qquad
\begin{tikzcd}
H \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & H \otimes H \arrow[d, "1_H \otimes \Delta"] \\
H \otimes H \arrow[r, "\Delta \otimes 1_H"'] & H \otimes H \otimes H
\end{tikzcd}
$$

Since comonoids are the dual concept of monoids, we will draw them simply as upside down monoids (i.e. , , etc.)

We shall show later, in Example 2.3.3, that for each set A there is a unique comonoid $(A, \delta, !)$ called the *copy comonoid*. Hence, the concept of a comonoid generalises the concept of the diagonal map in **Set**.

So far, in pursuit of our goal to generalise groups to symmetric monoidal categories, we have captured the notions of the group multiplication and the diagonal map. However, an arbitrary monoid and comonoid are not sufficient for our purposes - the multiplication and the diagonal map interact in a specific way.

Given any function $f : A \to B$, we have

$$
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\delta_A"'] & B \arrow[d, "\delta_B"] \\
A \times A \arrow[r, "f \times f"'] & B \times B
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "!_A"'] & B \arrow[d, "!_B"] \\
 & 1
\end{tikzcd}
$$

in other words, every function $f : A \to B$ is a comonoid homomorphism . In particular, given a monoid in set (G, μ, e), both μ and e are comonoid homomorphisms. This only follows when the monoidal product is the Cartesian product.

To generalise the concept of a group to a symmetric monoidal category, we require that we have a monoid (H, μ, e) and a comonoid (H, Δ, ϵ) such that μ and e are both comonoid homomorphisms. We call such a structure $(H, \mu, e, \Delta, \epsilon)$ a *bialgebra* .

Explicitly, the axioms that this structure must fulfil are

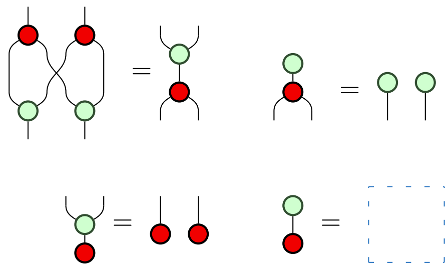
$$
\begin{tikzcd}
H \otimes H \arrow[rr, "\mu"'] \arrow[d, "\Delta\otimes\Delta"'] & & H \arrow[dd, "\Delta"'] \\
H \otimes H \otimes H \otimes H \arrow[d, "1_H\otimes\sigma\otimes 1_H"'] & & \\
H \otimes H \otimes H \otimes H \arrow[rr, "\mu\otimes\mu"] & & H \otimes H
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "e"] \arrow[d, equal, "\wr"'] & H \arrow[d, "\Delta"] \\
I \otimes I \arrow[r, "e\otimes e"'] & H \otimes H
\end{tikzcd}
$$

$$
\begin{tikzcd}
H \otimes H \arrow[r, "\epsilon\otimes\epsilon"] \arrow[d, "\mu"'] & I \otimes I \arrow[d, equal, "\wr"] \\
H \arrow[r, "\epsilon"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
I \arrow[r, "e"] \arrow[dr, equal] & H \arrow[d, "\epsilon"] \\
 & I
\end{tikzcd}
$$

where σ is the symmetry. Note that this implies that every monoid in **Set** forms a bialgebra with the copy comonoid, as every function is a comonoid homomorphism.

It is worth noting that this definition is self-dual. Hence, we may equivalently define a bialgebra as a monoid (H, μ, e) and a comonoid (H, Δ, ϵ) such that Δ and ϵ are both monoid homomorphisms.

We represent these equations graphically as follows



where they are presented in the same order as the commutative diagrams above. We are denoting the multiplication and comultiplication maps as  and  respectively, and the unit and counit as  and . Not that the colours are mere notation - for example, later on we will have green comonoids and red monoids.

Finally, we now need to capture the concept of the group inverse. This is the rightmost commutative diagram in the definition of a group above. Let us simply replace the inverse operation $\_^{-1} : G \to G$ with a morphism $S : H \to H$

$$\begin{tikzcd}
H \otimes H \arrow[dd, "1_H \otimes S"'] & H \arrow[l, "\Delta"'] \arrow[r, "\Delta"] \arrow[d, "\epsilon"] & H \otimes H \arrow[dd, "S \otimes 1_H"] \\
& I \arrow[d, "e"] & \\
H \otimes H \arrow[r, "\mu"'] & H & H \otimes H \arrow[l, "\mu"]
\end{tikzcd}$$

This is denoted graphically as

where we denote the antipode S graphically as . When we have a morphism S that fulfils the above condition, we call this an *antipode*. Given a bialgebra H, there is only ever a unique antipode (see Proposition 2.5.6) that we may equip it with, in which case we call such a bialgebra a *Hopf Algebra*. We define Hopf Algebras and antipodes fully in terms of string diagrams in Definition 2.5.3. To summarise, a Hopf algebra $(H, \mu, e, \Delta, \epsilon, S)$ consists of a monoid (H, μ, e) and a comonoid (H, Δ, ϵ) such that they form a bialgebra $(H, \mu, e, \Delta, \epsilon)$, which is then equipped with an antipode. We find that the Hopf algebras in **Set** are exactly the groups, where the antipode of a group is the group inverse.

We may see this by looking at some familiar properties of the group inverse. For example, it is true that $(gh)^{-1} = h^{-1}g^{-1}$ in groups. As we will see below, this property of group inverses is carried over to Hopf algebras, as $S \circ \mu = \mu \circ \sigma \circ (S \otimes S)$, where σ is the symmetry. We call this *anticommutativity* and is denoted graphically as

However, not every property of the inverse is shared by the antipode: the inverse operation of a group is an involution, but it is not necessarily the case in general. Indeed, there are Hopf algebras that have an antipode that are not even invertible [62], though in vector spaces, this is only possible for Hopf algebras that are infinite dimensional. We shall see in Proposition 2.5.6 that whenever either the monoid or comonoid are commutative, the antipode is involutive, which is why it is always the case that group algebras have an involutive Hopf algebra - the copy map is always cocommutative. More generally, we see in Lemma 5.4.14 that it is also implied by symmetry.

The primary reason that Hopf algebras have garnered interest is due to their role in representation theory. It can be shown that the modules of a Hopf algebra always form a monoidal category, and that any dual structure or closed monoidal structure of the base category is lifted to the category of modules. In addition, if we can equip the Hopf algebra with a *quasitriangular* structure, then this will tell us that the modules are braided [47, 60]. In physics, this fact is used in the field of quantum groups, where there are quasitriangular Hopf algebras that are represented in certain operator algebras, implying that the operators have a braided monoidal structure. See Majid [43] for more information.

1.2 Frobenius Algebras

Given an arbitrary symmetric monoidal category $\mathcal{C}$ and object F, a *Frobenius algebra* [4] $(F, \mu, e, \Delta, \epsilon)$ is similar in some ways to a bialgebra, in that we have a monoid (F, μ, e) and a comonoid (F, Δ, ϵ) [5], but we do not have the bialgebra rules of interaction. Instead, let us see F as a left and right F-module, where the action is simply induced by the multiplication. For example, in vector spaces this would be written as $a \triangleright b := \mu(a, b)$ and $a \triangleleft b := \mu(a, b)$, where $\triangleright$ and $\triangleleft$ are the left and right module actions respectively. In a Frobenius algebra, the comultiplication is *F-linear*, so in vector spaces, this would be written as $a \triangleright \Delta(b) = \Delta(a \triangleright b)$, and similarly for $\triangleleft$. Thus, we write this in the language

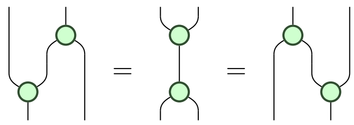
[4] Frobenius algebras were named after Georg Frobenius by Curtis and Reiner [23], and were initially used in representation theory.

[5] This definition of Frobenius algebras is due to Carboni and Walters [16]

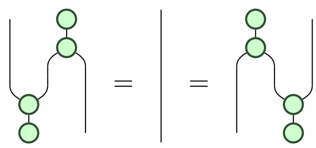
of commutative diagrams as

$$
\begin{tikzcd}
 & F \otimes F \arrow[dl, "\Delta \otimes 1_F"'] \arrow[d, "\mu"] \arrow[dr, "1_F \otimes \Delta"] & \\
F \otimes F \otimes F \arrow[dr, "1_F \otimes \mu"'] & F \arrow[d, "\Delta"] & F \otimes F \otimes F \arrow[dl, "\mu \otimes 1_F"] \\
 & F \otimes F &
\end{tikzcd}
$$

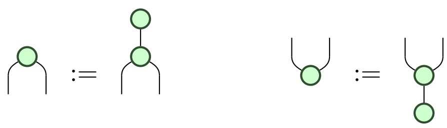
We denote this graphically as



We see from this definition that we may also write the following equation



This motivates us to denote the following morphisms

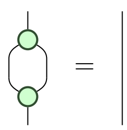


which call the cap and cup respectively.

The monoid is commutative exactly when the comonoid is cocommutative (See Lemma 2.4.3), so in such a case, we shall say that the Frobenius algebra as a whole is commutative. We say that it is special when

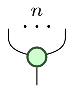
$$
\begin{tikzcd}
F \arrow[r, "\Delta"] \arrow[dr, "1_F"'] & F \otimes F \arrow[d, "\mu"] \\
 & F
\end{tikzcd}
$$

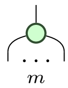
We denote this graphically as



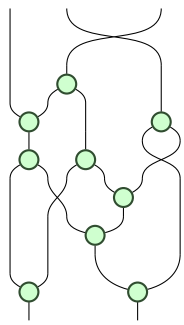
When a Frobenius algebra is both commutative and special, then it has a normal form due the *spider theorem*[6] (see [18]). The easiest way to state this theorem is graphically, in terms of string diagrams[7].

We define *n-ary multiplication* as the morphism $\mu_n : F^{\otimes n} \to F$, where $\mu_0 := \eta$ and $\mu_n := \mu \circ (1 \otimes \mu_{n-1})$. Under this notation, we have $\mu_1 = 1_F$ and $\mu_2 = \mu$. Since μ is commutative and associative, μ_n is the unique composition of $n-1$ μ maps together. We define *n-ary comultiplication* $\Delta_n : F \to F^{\otimes n}$ analogously. We depict these morphisms graphically as

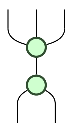
$\mu_n :=$ 

$\Delta_m :=$ 

Consider any morphism term $t : F^{\otimes n} \to F^{\otimes m}$ in the language of Frobenius algebras – i.e. some term t generated by the Frobenius algebra structure maps, $(\mu, e, \Delta, \epsilon)$, and the structure maps of a symmetric monoidal category, $(\otimes, I, \sigma)$, such that the string diagram is a connected graph. For example, consider the following string diagram.



The spider theorem tells us that we may construct a proof that t is equal to $\Delta_n \circ \mu_n : F^{\otimes n} \to F^{\otimes m}$. So the above diagram would be equal to



[6]Commutativity is not necessary for the spider theorem. A term without any instances of the symmetry $\sigma_{F,F}$ also has a normal form - see [31]. Hence, every special Frobenius algebra in a non-symmetric monoidal category has a normal form.

[7]In his work on PROPs, Lack [40] constructed a PROP from cospans on the finite ordinals that is equivalent to commutative, special Frobenius algebras. The spider theorem is equivalent to the distributive law present in this PROP.

Since every term in a symmetric monoidal category can be seen as a sequence of connected graphs in the language of string diagrams, the spider theorem extends to every term in the language of Frobenius algebras.

Frobenius algebras are relatively common algebraic structures – for example, every semisimple algebra in the category of vector spaces may be equipped with a Frobenius structure (See Example 2.2.17 in Kock [39]).

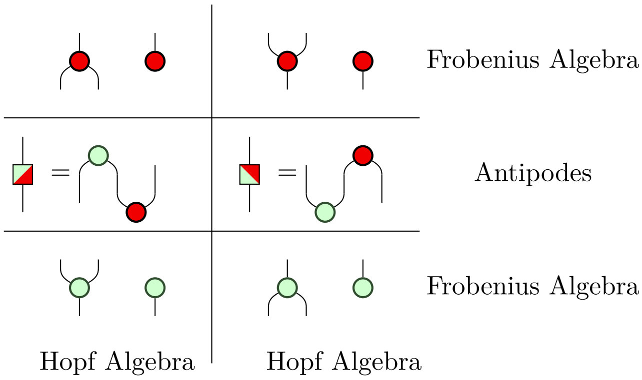
In more recent times Frobenius algebras are studied in the context of topological quantum field theories (TQFTs). Quantum field theories are famously difficult to model in a mathematically rigorous manner. TQFTs (Atiyah [4]) can be seen as a way of capturing some of the dynamics of a quantum field theory, without these issues of rigour. We construct a category of n-dimensional cobordisms, $\mathbf{Cord}_n$, where the objects are $(n-1)$-dimensional manifolds, and a morphism $X \to Y$ is an isotopy class of n dimensional manifold such that its boundary is the disjoint union of X and Y. A *topological quantum field theory* is a strong monoidal functor $\mathbf{Cord}_n \to \mathbf{Vect}_\mathbf{k}$. We find that when the cobordisms are 2-dimensional, then a TQFT is exactly the same as identifying a commutative Frobenius algebra in $\mathbf{Vect}_\mathbf{k}$ (See Kock [39]).

1.3 Hopf-Frobenius Algebras

In this section, we will introduce *Hopf-Frobenius algebras*, which are the main focus of this thesis. They are an algebraic structure that combines both a Hopf algebra and a Frobenius algebra, and therefore they have a complex definition. Despite this, they naturally appear in a variety of contexts, as we will see.

A Hopf-Frobenius algebra consists of two monoids, (H, ,) and (H, ,), and two comonoids, (H, ,) and (H, ,) such that when we pair these monoids and comonoids together, we require that they form a Frobenius algebra or a Hopf algebra. To distinguish between the different algebraic structures, we use a naming convention where the structure is named after the colour of its monoid – for example, (H, , , , ,) is called the *green Hopf algebra*, and (H, , , ,) is called the *red Frobenius algebra*. The comonoids (H, ,) and (H, ,) are named after their colours. Note that the colours are merely notation, used to distinguish different algebraic structures.

The following diagram captures all of the above information



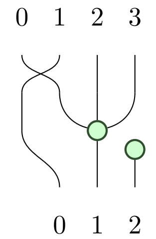
where we have a 2×3 grid with two monoids, two comonoids, and two antipodes. Each column is a distinct Hopf algebra, the top and bottom rows are Frobenius algebras, and the centre row is how the antipodes are constructed. Each monoid and comonoid is part of one Hopf algebra and one Frobenius algebra, and each of the Hopf algebras and Frobenius algebras overlap.

1.4 PROPs and Interacting Hopf Algebras

In the above sections, we have defined algebraic structures by stating an object, structural morphisms, and axioms that those morphisms follow. For example, a monoid on M consists of a multiplication map $\mu : M \otimes M \to M$ and a unit $\eta : I \to M$ such that

$$\begin{tikzcd}
I \otimes M \arrow[r, "\eta \otimes 1_M"] \arrow[dr, "\sim"'] & M \otimes M \arrow[d, "\mu"'] & M \otimes I \arrow[l, "1_M \otimes \eta"'] \arrow[dl, "\sim"] \\
& M &
\end{tikzcd}
\qquad
\begin{tikzcd}
M \otimes M \otimes M \arrow[r, "1_M \otimes \mu"] \arrow[d, "\mu \otimes 1_M"'] & M \otimes M \arrow[d, "\mu"] \\
M \otimes M \arrow[r, "\mu"] & M
\end{tikzcd}$$

However, we may define algebraic structures in alternative manner using *PROPs*. We define a PROP as a strict symmetric monoidal category, where the objects are natural numbers, and the monoidal product on objects is addition. This tells us that to define a PROP, we only need to state the morphisms. For example, consider the PROP where a morphism $n \to m$ is a function from the finite ordinal with n elements to the one with m elements. Let us call this category **FinOrd**. A string diagram for such a morphism might look like



where this morphism is the function where $0, 2, 3 \mapsto 1$ and $1 \mapsto 0$. Note how similar this diagram looks to a morphism in the language of monoids. Indeed, we find that this PROP is able to capture the theory of commutative monoids perfectly. By this, we mean that, given a symmetric monoidal category $\mathcal{C}$, every commutative monoid in $\mathcal{C}$ is equivalent to a strong monoidal functor from **FinOrd** to $\mathcal{C}$. More generally, we say that, given a PROP P, an *algebra of P in $\mathcal{C}$* is a strong monoidal functor $P \to \mathcal{C}$. In this manner, we may use PROPs to define many algebraic theories. This is typically done by defining a PROP by the structure maps and equations present in the algebraic theory. For example, we may define a PROP for noncommutative monoids by having a PROP generated by the maps $\mu : 2 \to 1$ and $\eta : 0 \to 1$, which is then quotiented by the equations

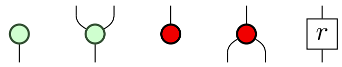
$$
\begin{tikzcd}
1 \arrow[r, "\eta\otimes 1"] \arrow[dr, "\sim"'] & 2 \arrow[d, "\mu"'] & 1 \arrow[l, "1\otimes\eta"'] \arrow[dl, "\sim"] \\
& 1 &
\end{tikzcd}
\qquad
\begin{tikzcd}
3 \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & 2 \arrow[d, "\mu"] \\
2 \arrow[r, "\mu"] & 1
\end{tikzcd}
$$

Let us call this PROP **Mon**. It is clear from how this PROP is defined that every algebra of this PROP will be a monoid, and indeed, we find that **Mon** is equivalent to **FinOrd**. We may do something similar with every algebraic theory that we have defined previously (Comonoids, Hopf algebras, Frobenius algebras, Hopf-Frobenius algebras).

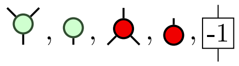
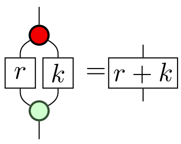
We may define a PROP, $\mathbf{Mat}_R$, where a morphism $n \to m$ is a matrix from $R^n \to R^m$ where R is a unital ring. For example, the following matrix

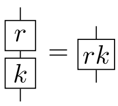
$$A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix}$$

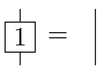
is a morphism $A : 2 \to 3$. Bonchi, Sobociński and Zanasi [13] proved the classic folklore that this PROP is equivalent to the PROP of *commutative and cocommutative Hopf algebras over* R, denoted $\mathbb{HA}_R$. We construct this PROP by taking the PROP of commutative Hopf algebras and equipping it with extra morphisms $r : 1 \to 1$ for each $r \in R$, called *ring elements*, which each function as homomorphisms. Explicitly, we generate a PROP via the morphisms

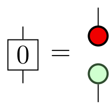


quotiented such that

- (), where -1 is the additive inverse of R, forms a commutative and cocommutative Hopf algebra
- Each ring element is a Hopf algebra homomorphism
- 

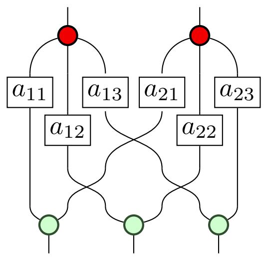
- 

- 

and 

We see that this PROP is equivalent to $\mathbf{Mat}_R$ via the functor $\mathbb{HA}_R \to \mathbf{Mat}_R$ defined in such a manner that

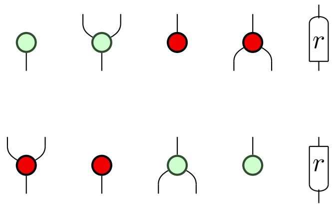
$$\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{pmatrix} \mapsto$$



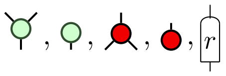
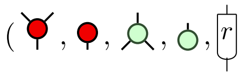
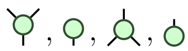
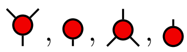
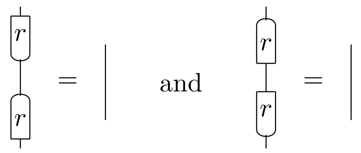
where each row is a comultiplication, and each column is a multiplication.

This equivalence is extended by the main result of Bonchi et al. [13]. They construct *the PROP of linear relations*, denoted $\mathbf{LinRel_R}$, in a similar manner to the way that **Rel** is constructed from **Set**. In **Rel**, a morphism $R : A \to B$ is a subset $R \subseteq A \times B$, while in $\mathbf{LinRel_R}$ a morphism $L : n \to m$ is a subspace $L \subseteq R^n \oplus R^m$.

Bonchi et al. [13] construct an algebraic theory whose PROP is equivalent to $\mathbf{LinRel_R}$, called *interacting Hopf algebras*, denoted $\mathbb{IH}_R$. Essentially, it is a commutative and cocommutative Hopf-Frobenius algebra equipped with ring elements in a similar manner to $\mathbb{HA}_R$. Explicitly, the PROP is generated by



and quotiented such that

- both () and () are $\mathbb{HA}_R$,
- both () and () are commutative special Frobenius algebras
- 

Interacting Hopf algebras are used to model signal-flow graphs, as in [11, 12, 29, 68]. They function well as denotational semantics, which introduces compositionality into signal flow diagrams. It also functions as operational semantics, with the caveat that $\mathbb{IH}_R$ does not capture deadlocks. This allows it to also model control theory [6].

Since the morphisms of $\mathbf{LinRel_R}$ are linear subspaces, we see that $\mathbb{IH}_R$ also captures the linear subspaces. This allows us to present an entirely graphical presentation of linear algebra, including concepts such as image, kernel and matrix multiplication [57].

1.5 Categorical quantum mechanics and Interacting Frobenius Algebras

Categorical quantum mechanics [1, 19, 31] is a framework for modelling quantum information and computing processes that focuses on the abstract structures underlying quantum mechanics. A foundational structure in this framework is a type of Hopf-Frobenius algebra called an *interacting Frobenius algebra*. In this section, we will briefly discuss quantum computing and categorical quantum mechanics, followed by an explanation of interacting Frobenius algebras and their use. The next section, on *ZX-calculus* [18, 67], will explore the most important application of interacting Frobenius algebras to date.

To understand quantum computing, it is important to first discuss quantum information. In a *classical* (non-quantum) computer, a bit can take a value from the set $\mathbf{2} := \{0, 1\}$, so that the state space with n bits is $\mathbf{2}^n$. One important fact to keep in mind is that each bit in a classical computer corresponds to the state of a physical system, such as the on/off state of a transistor. In a quantum computer, the basic unit of information is the *quantum bit*, or *qubit*. A qubit is similar to a classical bit, but instead of being an element of a set with 2 elements, it is a vector in a 2 dimensional vector space over $\mathbb{C}$, and we denote such a quantum state as a *ket*, such as $|x\rangle$. This means that the state of the qubit can be modelled as a linear combination of multiple elements of an orthonormal basis, which we can view as classical states. This is the phenomena which within physics is referred to as superposition. In this way, qubits can be thought of as a more general version of classical bits that are based on the principles of quantum mechanics.

In quantum mechanics, we may model observable quantities with an orthonormal basis, where each value of the basis represents a possible classical value that we could measure. We distinguish between the outcomes of a measurement, x, and the states that correspond to those outcomes, $|x\rangle$. Measurement is nondeterministic – that is, given a quantum state $|\phi\rangle$, there is some probability of measuring $|x\rangle$. This probability is $|\langle x \mid \phi\rangle|^2$, where $\langle x \mid \phi\rangle$ is the inner product of $|x\rangle$ and $|\phi\rangle$. If we do happen to measure x, then the state of the quantum system becomes $|x\rangle$. In other words, the act of measuring a quantum state will change the system. This is very different from the classical setting where, for example, to know the state of a bit I only need to look it up, and this will not effect the system at all.

We may also contrast the quantum scenario with a classical nondeterministic process. For example, if I flip a coin and see that it is ‘heads’, my measuring of the coin did not cause it to change – the coin’s state was heads, independent of my measurement of it. However, in quantum mechanics, Bell’s theorem [7] tells us that this is not the case. Prior to measurement, the outcome of the measurement was not determined – the state of the system was a superposition of all possible outcomes. Mathematically, it is a linear combination of the elements of the orthonormal basis.

Every quantum transformation must be *unitary* – i.e. its inverse is equal to its conjugate transpose – and therefore, may be reversed. Hence, in every quantum transformation, no information may ever be lost. Note how this is contrasted with measurement, where a state cannot be measured without losing information. In quantum computing, a quantum transformation is usually presented as a *quantum circuit*, which is the quantum equivalent to a logic circuit, and is the main focus of the next section.

Categorical quantum mechanics (CQM) is an approach to modelling quantum mechanics that focusses more on abstract structures, such as algebra and compositionality. For example, a fundamental structure in CQM is the *dagger* functor [1, 54]. This is a involutive strict monoidal functor on our category, with type $\dagger : \mathcal{C} \to \mathcal{C}^{\mathrm{op}}$. This functor is able to capture adjoint operators on Hilbert spaces, where every operator $U : H \to K$ has a pair, $U^\dagger : K \to H$ that reverses the type. Since the functor is involutive strict monoidal, we know $U^{\dagger\dagger} = U$, and $(U \otimes V)^\dagger = U^\dagger \otimes V^\dagger$. Hence, the dagger functor is used to represent unitarity in an abstract setting.

One of the key structures in categorical quantum mechanics is the way in which classical information is modelled using *dagger special commutative Frobenius algebras* (†-SC Frobenius algebra). We covered special commutative Frobenius algebras in Section 1.2. Given such a Frobenius algebra, $F = (F, \mu, e, \Delta, \epsilon)$ we say that it is a *dagger* Frobenius algebra when $\mu^\dagger = \Delta$ and $e^\dagger = \epsilon$. We find that every †-SC Frobenius algebra on a finite dimensional Hilbert space corresponds exactly with an orthonormal basis [17].

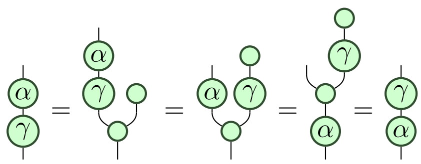
Given an orthonormal basis $A := \{|j\rangle \,|0 \leq j < n\}$, we construct the corresponding †-SC Frobenius algebra as

 $:= |j\rangle \otimes |k\rangle \mapsto \begin{cases} |j\rangle & \text{if } j = k \\ 0 & \text{otherwise} \end{cases}$  $:= \sum_{j=0}^{n-1} |j\rangle$

 $:= |j\rangle \mapsto |j\rangle \otimes |j\rangle$  $:= |j\rangle \mapsto 1$

In addition, each Frobenius algebra comes with a set of morphisms called *phases*. These are unitary morphisms $\alpha : F \to F$ with the following property

These morphisms are automorphisms, and therefore form a group. We see that this is in fact an abelian group as



In the category of Hilbert spaces, these are rotation matrices. Concretely, given basis $A := \{|j\rangle \,|0 \leq j < n\}$, then the phases are maps of the form

$$|j\rangle \mapsto \begin{cases} |0\rangle & \text{if } j = 0 \\ e^{i\alpha_j}\,|j\rangle & \text{otherwise} \end{cases}$$

where α_j is an angle, and i is the complex number i.

Recall that when we measure a quantum state, $|\phi\rangle$, we measure with respect to a basis and the outcome is nondeterministic. The different members of that basis are the possible outcomes of the measurement, and $|\phi\rangle$'s representation in that basis tells us the probability of measuring any particular outcome. Suppose that we have two bases, A and B. For $a \in A$, if $|\phi\rangle = |a\rangle$ there is a 100% probability of measuring a from $|\phi\rangle$ when measured with respect to the A basis. Suppose that, for all a in A, when we measure $|a\rangle$ with respect to the B basis, there is an equal probability of measuring any given b in B. So if B has n elements, there is a $\frac{1}{n}$ chance of measuring any b in B. In this case, we call A and B *mutually unbiased* [53] or *strongly complementary*. Mutually unbiased bases are used in quantum key distribution [8] and quantum error correction [30], and are important in quantum information.

We find that two bases are mutually unbiased if their respective Frobenius algebras interact to form a Hopf-Frobenius algebra. More specifically, it is a *scaled* Hopf-Frobenius algebra. A scaled Hopf-Frobenius algebra is an algebraic structure with almost exactly the same equations as a Hopf-Frobenius algebra, except each equation is modified by some invertible scalar factor. This allows us to define the *classical maps* of the Frobenius algebra. The classical maps of the green Frobenius algebra are the subgroup of the phases of the red Frobenius algebra that are also homomorphisms of the green Frobenius algebra. In other words, they are phases of the red Frobenius algebra such that

In the Hilbert space setting, given a basis X, the classical maps of the X Frobenius algebra correspond exactly with the endofunctions on the basis elements of X. We now move on to a specific instance of categorical quantum mechanics and interacting Frobenius algebras – ZX-Calculus.

1.6 Quantum Circuits and ZX-calculus

In this section, we will focus on a specific way of modelling quantum transformations on the state space of a quantum computer, called a *quantum circuit*. We will also look at the CQM approach to quantum circuits, a formal language based on interacting Frobenius algebras called *ZX-calculus*.

A quantum circuit is analogous to a logic circuit in the classical setting. For a classical computer, we may see a computer process as a function from one state space to another. A logic circuit models this function via logic gate. In the same way a quantum circuit models the quantum transformation on a state space with n qubits, $(\mathbb{C}^2)^{\otimes n}$, as the composition of a series of *quantum gates*. A quantum gate may be seen as any quantum transformation on a small number of qubits (typically 1 or 2). In the classical setting, the AND, OR and NOT gates are sufficient to model every logic gate, and we call this a *universal gate set*. We get something similar in the quantum setting, where the following gates are a universal set

$$R_Z(\alpha) := \begin{pmatrix} 1 & 0 \\ 0 & e^{i\alpha} \end{pmatrix} \qquad R_X(\alpha) := \begin{pmatrix} \cos(\alpha/2) & -i\sin(\alpha/2) \\ -i\sin(\alpha/2) & \cos(\alpha/2) \end{pmatrix}$$

$$CNOT := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad H := \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

We call the R_Z and R_X matrices the *rotation* matrices, where α takes the value of an angle. $CNOT$ stands for *controlled-not*, and H stands for *Hadamard*. Note that this universal gate set is uncountable. In fact, we only need one family of rotation matrices – the R_Z gates and the Hadamard gate is sufficient to generate the R_X gates. However, this presentation shall make our discussion of ZX-calculus easier.

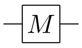
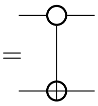
In quantum computing, we are primarily concerned with two bases, the Z basis and the X basis. The Z basis is denoted $|0\rangle$ and $|1\rangle$, while we define the X basis as

$$|+\rangle := \frac{|0\rangle + |1\rangle}{\sqrt{2}} \qquad |-\rangle := \frac{|0\rangle - |1\rangle}{\sqrt{2}}$$

These two bases are mutually unbiased, and indeed we may define the Z basis in terms of the X basis as follows

$$|0\rangle := \frac{|+\rangle + |-\rangle}{\sqrt{2}} \qquad |1\rangle := \frac{|+\rangle - |-\rangle}{\sqrt{2}}$$

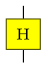
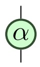
We represent quantum circuits graphically using string diagrams in a PROP, whch we shall call **QCIRCUIT**. The notation is fairly straightforward, as follows

$M :=$  $\qquad CNOT :=$ 

where M is either $R_Z(\alpha), R_X(\alpha)$ or H. This notation has no equations, and is therefore purely syntactical[8]. Regardless, it is a PROP and it does have an interpretation functor, $\mathbf{QCIRCUIT} \to \mathbf{FHilb}_{\mathbf{C}}$, defined in the obvious way.

ZX-calculus is a major approach that categorical quantum mechanics uses to model quantum circuits. It is the interacting Frobenius algebra based off of the Z and X bases, as they are mutually unbiased. In this scenario, the phases are the rotation matrices that we defined previously. We also equip the structure with a *Hadamard morphism*, which functions as a Frobenius algebra isomorphism between the red and green Frobenius structures, is unitary and self-inverse. This structure, equipped with some extra axioms (see Duncan and Coecke [18]) gives us a sound and complete [64, 65] model of finite dimensional Hilbert spaces – in other words, there is an interpretation functor, $\mathbf{ZX} \to \mathbf{FHilb}_{\mathbf{C}}$ and this functor is an equivalence. To define this functor, we define what the Frobenius algebras map to, then the phase maps and the Hadamard. The red and green Frobenius algebras map to the †-SC Frobenius algebras in $\mathbf{FHilb}_{\mathbf{C}}$that correspond with the Z and X bases respectively. The antipodes of the Hopf algebra map to the identity morphism.

The Hadamard map and the phases map to the unitaries that we have defined previously

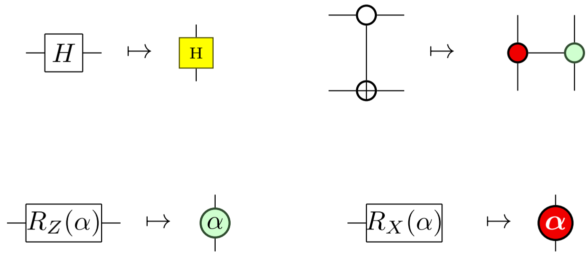
$\Big[$  $\Big] := H \qquad \Big[$  $\Big] := R_Z(\alpha) \qquad \Big[$  $\Big] := R_X(\alpha)$

We find that the green Hopf algebra, ( ,  ,  , ), maps to a scaled version of the $\mathbb{Z}_2$ group algebra[9]. Specifically, we find that the comonoid ( , ) is the comultiplication of $\mathbb{Z}_2$, while the monoid ( , ) is the monoid of $\mathbb{Z}_2$ multiplied by $\sqrt{2}$.

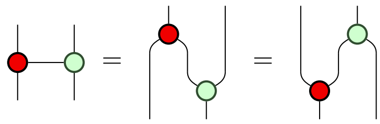
[8]There are ways that we may apply equations to this formalism - see [21] for example. However, traditionally, these diagrams are used purely for notational purposes

[9]A *group algebra* is a group that has been lifted to vector spaces from **Set**. See Example 2.3.4 for more information.

There is a functor from **QCIRCUIT** to **ZX**, defined as



where for the $CNOT$ gate, we have written



This interacts with the interpretation functors that we have previously defined in the following way

$$\begin{tikzcd} \mathbf{QCIRCUIT} \arrow[r] \arrow[dr] & \mathbf{ZX} \arrow[d] \\ & \mathbf{FHilb}_{\mathbf{C}} \end{tikzcd}$$

The presence of this functor informs us of how we use ZX calculus. Given a quantum circuit, we may convert it into the language of matrices in $\mathbf{FHilb}_{\mathbf{C}}$, or the language of ZX-calculus. By converting it into ZX-calculus, we maintain the graphical aspect of circuit notation, while also being able to use the algebraic rules of ZX-calculus to reason about and perform calculations on quantum circuits. This means that it is often easier to reason using ZX diagrams than matrices. Since the above diagram commutes, we know that, given a quantum circuit, the unitary matrix that we get shall be the same as if we convert the circuit into ZX-calculus, perform algebraic rewrites, and then convert it into a matrix. This means that we may convert a circuit into ZX and then prove properties about it. For example, see [22]. We may also convert a circuit into ZX, optimise it according to some criteria, then convert it back into a circuit. An excellent example of this are the results by Kissinger and van der Wetering [38]. The phase gate $R_Z(\frac{\pi}{4})$ is often called the T-*gate*, and it is both necessary for quantum effects[10]

[10]Together, the $CNOT$ gate, the Hadamard gate, $R_Z(\frac{\pi}{2})$ and the T-gate form an approximately

and it is difficult to implement in a noise resistant fashion [15]. Kissinger and Van der Wetering [38] provide an algorithm that takes quantum circuits and, using the ZX rules, returns an equivalent circuit with fewer T-gates.

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1.7 Outline of Thesis and Original Contributions

In the Background (Chapter 2), we define a few monoidal categories that we will be working with - such as modules and vector spaces. Most of the literature on Hopf algebras is within these categories, and most of our examples of Hopf algebras are in the category of finite dimensional vector spaces. We are working almost entirely in the language of string diagrams, so we briefly cover this topic - however, we leave a full discussion of string diagrams for the appendix (See Section A.1). We shall cover dual structures – i.e. cups and caps – and define the algebraic structures that we will be mostly concerning ourselves with in this thesis – namely monoids, comonoids, Frobenius algebras and Hopf algebras – in the general context of a symmetric monoidal category. Note in particular Lemma 2.5.10, which was originally mentioned in the Appendix of Collins and Duncan [20]. It is also worth noting that in the appendix of this thesis, in Section A.1, we construct a definition of string diagrams without any reference to topology. Compare this with the traditional definition given by Joyal and Street [33].

We follow up the background chapter with a chapter on the trace. In this thesis, we do not assume that we are working in a traced monoidal category [34], so we introduce the novel concept of a traced family. The definition of a traced family comes from a requirement to have a subset of objects in a category on which a trace may be defined, as opposed to the definition of a traced monoidal category which has a traced for every object. We introduce the concept of a traced family, and prove several lemmas that will help us work with them. Lemma 3.0.10 is of particular note. We introduce an axiom for traced families, and by extension, traced monoidal categories, that replaces two axioms - *dinaturality* and *the second vanishing axiom* (See Joyal, Street and Verity [34]), which we call Axioms $T4$ and $T5$.

universal gate set (See Nielsen and Chuang [48])

In Chapter 4, we introduce the most central concepts in the paper – integrals and cointegrals. This section refers heavily to *Hopf-Frobenius Algebras and a Simpler Drinfeld Double* [20] by the author of this thesis, and develops its ideas. Every result connecting Hopf algebras to Frobenius algebras is related to integrals. We introduce the integral Hopf algebra, Definition 4.2.3, which is a concept in between a Hopf algebra and a Frobenius algebra: integral Hopf algebras are weaker than a Hopf-Frobenius algebra[11], but they do still have many properties that are similar. We define the concept of a half-dual (Definition 4.3.3), which is a weakening of the standard notion of a dual (Definition 2.3.7). It does not obey the full snake equation, but it is strong enough to define a trace. This allows us to define the integral morphism (Definition 4.4.1), a morphism that is able to capture the essence of integrals in a Hopf algebra. We show that we can use it to construct integrals (Lemma 4.4.7 and Remark 4.4.8) and that we can use it to capture the idea of a space of integrals (Remark 4.4.5).

We expand upon previous results by strengthening the link between integrals and Frobenius algebras by proving new results and talking more about traced families. We see that we may equip the comonoid with a multiplication that behaves much like a Frobenius algebra if and only if the Hopf algebra can be equipped with an integral (Lemma 4.2.12). We develop the idea of half duals and traced families, by showing that they are sufficient to define a unique trace (Lemma 4.3.7 and 4.3.8), and while the integral morphism is defined in [20], we are able to define it in terms of the trace, rather than half duals, thus weakening the definition (Definition 4.4.1).

The penultimate chapter of this thesis, Chapter 5, is primarily about the several equivalent theorems that state when a Hopf algebra is Hopf-Frobenius. These are all defined in terms of the concepts that we introduced in the previous chapter, and they are summarised in Theorem 5.2.8. Again, much of these results come from Collins and Duncan [20]. However, we develop these results by examining the integral morphism, and showing how is behaves in a comparable manner to it's counterpart in the category of finite vector spaces, and how these results compare with Hopf algebra folklore, Definition 5.1.1, Lemma 5.1.2 and Lemma 5.2.5, which were communicated by Gabriella Böhm.

[11]We have not been able to prove that integral Hopf algebras are strictly weaker than Hopf-Frobenius algebras, however. This is a conjecture

Section 5.3 is original work, where we develop the theory of Hopf-Frobenius algebras by defining the concept of a morphism between Hopf-Frobenius algebras, and hence, we introduce the category of Hopf-Frobenius algebras. We find that morphisms that preserve Hopf algebra structure and integrals are a class of Hopf algebra isomorphisms. We also provide a condition that allows us to check with relative ease whether a Hopf algebra morphism preserves Hopf-Frobenius structure (Corollary 5.3.10). We link this definition of morphisms to Hadamard gates, which are a staple of ZX-calculus. Hadamard morphisms exist whenever we have a self-dual Hopf algebra, and we provide conditions under which we may construct a Hadamard morphism from a quasitriangular Hopf algebra - thereby providing conditions under which a quasitriangular Hopf algebra is self-dual.

In the final section of the chapter, we prove several results for finite dimensional Hopf algebras in the context of Hopf-Frobenius algebras. This shows how much of the structure of Hopf-Frobenius algebras is preserved between finite dimensional Hopf algebras and Hopf-Frobenius algebras, and demonstrates how, when we have a Hopf-Frobenius algebra, we can transfer theorems from finite dimensional Hopf algebras to Hopf-Frobenius algebras.

We provide a new proof of Radford's theorem [51]. This is a well established theorem in Hopf algebra theory, but the proof that we provide is original - Lemma 5.4.2 until Corollary 5.4.7 are all independent. The proof that we provide is shorter and easier to follow than the original proof, and demonstrates why I believe that string diagrams are the "right" language for Hopf-Frobenius algebras.

The section ends with a demonstration of the link between the symmetry of the Frobenius algebra, and semisimplicity of the Hopf algebra - i.e. the diagrams are very similar. These are two old concepts in algebra (from 1939 [**?**] and 1969 [41] respectively). Without string diagrams, this is not an obvious perspective, but string diagrams make the connection between the two concepts clear. However, while the work was done independently, the work does appear in later work by Radford [52]. The definitions and proofs differ significantly, where Radford uses module actions instead of string diagrams, and our results are simpler and take less space, despite being graphical.

Finally, we conclude and consider further work based on the results of this thesis.

Chapter 2

Mathematical Background

We begin this thesis by introducing concepts that may be familiar to the reader. We will assume that the reader is familiar with the basic concepts of monoidal category theory — the definition of a category, functors, natural transformations, adjunctions, limits, colimits and their monoidal counterparts. We refer to Mac Lane [42] for an explanation of these concepts if the reader is not familiar. We begin in Section 2.1 by defining two categories that we shall reference throughout this thesis, the category of modules over a ring, $\mathbf{Mod}_{\mathbf{R}}$, and the category of vector spaces, $\mathbf{Vect}_{\mathbf{k}}$. We will use these categories for the majority of our examples of Hopf algebras. For more on these topics, see Street [60]. We shall be primarily working in an arbitrary symmetric monoidal category, which we shall discuss in Section 2.2. We shall also develop the graphical language that we shall be primarily working in — that of string diagrams. Given a category $\mathcal{C}$, we construct the category of string diagrams, and we state how this connects to the free symmetric monoidal category $\mathcal{C}$. We do this briefly in this chapter, but we cover it in much more depth in the appendix, in Section A.1. For more on this topic, see Selinger [55]. We shall define the various equational theories that we will be working with, such as monoids and dual structures (Section 2.3), Frobenius algebras (Section 2.4), bialgebras and Hopf algebras (Section 2.5). For more on these topics, see Kock [39] and Street [60].

We begin the background chapter by introducing categories that we will refer to later.

Example 2.0.1. We denote the category of sets and functions by **Set**.

Example 2.0.2. The category of sets and relations, denoted **Rel**, has sets as objects and a subset $R \subseteq A \times B$ is a morphism $R : A \to B$. When $(a, b) \in R$, we say that aRb.

Given $R : A \to B$ and $R' : B \to C$, the composition $R' \circ R$ is the relation where $a(R' \circ R)c$ if there exists some $b \in B$ such that aRb and $bR'c$.

Let $f : A \to B$ be a function in **Set**. We set $G(f)$ to be the corresponding relation in **Rel**,

$$G(f) := \{(a, f(a)) : a \in A\} \subseteq A \times B.$$

Note how $G(f) \circ G(g) = G(f \circ g)$. Hence, we may define a functor $G : \mathbf{Set} \to \mathbf{Rel}$ where $G(A) := A$.

2.1 Modules

Most of the theory of Hopf algebras comes from the category of vector spaces and modules over a ring, and so most of the examples of Hopf algebras are in these categories. In this section, we define these categories and introduce some terminology that we will be using.

Definition 2.1.1. Let R be a commutative ring, and let $(M, +, 0)$ be an abelian group. We say that an operation $\cdot : R \times M \to M$ is an *R-action on M* when for all $r, r' \in R$ and $m, m' \in M$

1. $1 \cdot m = m$
2. $r \cdot (m + m') = r \cdot m + r \cdot m'$
3. $(r + r') \cdot m = r \cdot m + r' \cdot m$
4. $(rr') \cdot m = r \cdot (r' \cdot m)$.

We say that an *R-module* $M = (M, +, 0, \cdot)$, is an abelian group $(M, +, 0)$ equipped with $\cdot$, an R-action on M. In this case, we say that the elements of R are the *scalars* of M. We may also refer to the R-action $\cdot$ as *scalar multiplication*.

A linear transformation f between R-modules M and N is a function $f : M \to N$ such that

$$f(x + y) = f(x) + f(y) \qquad \text{and} \qquad r \cdot f(x) = f(r \cdot x).$$

Note how the composition of two linear transformations is a linear transformation. The category of R-modules, denoted $\mathbf{Mod_R}$, is defined as the category where each R-module is an object, and the linear transformations are the morphisms. We will often suppress the notation $\cdot$, and instead write rx for $r \cdot x$ when it is clear.

Remark 2.1.2. In the above example, we limited ourselves to the case when R is a *commutative* ring. When R is not commutative, we must instead specify if R is acting on M from the left or the right. However, in this thesis, we shall only refer to the case when R is commutative.

Definition 2.1.3. Given a set A, the *free R-module generated by A* is the module, denoted $F(A)$, where the elements are every term of the form

$$r_0 a_0 + r_1 a_1 + \ldots + r_n a_n$$

with $r_i \in R$, $a_i \in A$ and n is a natural number. This includes the empty term, which we will denote 0. This is then quotiented such that

$$ra + r'b = r'b + ra \qquad (r + r')a = ra + r'a \qquad 0a = 0$$

We shall show that $F(A)$ is a module. We begin by showing that $F(A)$ is an abelian group. Clearly $+$ and 0 give us a commutative monoid, so we merely need to show that each element has an inverse. The inverse of $r_0 a_0 + \ldots + r_n a_n$ is simply $(-r_0)a_0 + \ldots + (-r_n)a_n$. Hence we have an abelian group. We define the R-action as

$$r \cdot (r_0 a_0 + \ldots + r_n a_n) = r r_0 a_0 + \ldots + r r_n a_n.$$

Hence, $F(A)$ is a module.

There is straightforward way of defining linear transformations with type $f : F(A) \to M$. We define what $f(a)$ is for each element $a \in A$. This then tells us how f is defined over all of $F(A)$, as

$$f(r_0 a_0 + \ldots + r_n a_n) = r_0 f(a_0) + \ldots + r_n f(a_n)$$

where $r_i \in R$ and $a_i \in A$.

Given $f : A \to B$, we define $F(f)$ to be the linear transformation $F(f) : F(A) \to F(B)$ where $F(f)(a) = f(a)$ for each $a \in A$. Since the set A generates the module $F(A)$, and f is defined on each $a \in A$, we see that this sufficient to define $F(f)$.

This assignment of $A \mapsto F(A)$ and $f \mapsto F(f)$ defines a functor $F : \mathbf{Set} \to \mathbf{Mod_R}$. We may define another functor, $U : \mathbf{Set} \to \mathbf{Mod_R}$ that maps R-module $(M, +, 0, \cdot)$ to its generating set M, and linear transformation $f : M \to N$ to the underlying function f. We remark without proof that F is the left adjoint functor to U.

Definition 2.1.4. Let M be an R-module. We say that subset $B \subseteq M$ is *a generating set* of M when, for every $x \in M$, there exists a set of scalars $\{r_b \in R | b \in B\}$ such that

$$x = \sum_{b \in B} r_b b$$

We say that the set $\{r_b\}$ is a set of *coefficients* of x. We shall see that $\{r_b\}$ is not necessarily unique in Example 2.1.7. Consider the free module $F(B)$, and the linear map $\phi : F(B) \to M$ defined as $\phi(b) = b$. Note the abuse of notation, where the b on the left hand side is a free term, while the b on the right hand side is an element of M. Since B generates M, we see that ϕ is always surjective. When ϕ is an isomorphism, we say that B is a *basis* of M. In this case, we say that it is *freely generated by* B. It should be noted that a basis is not necessarily unique, as we shall see in Example 2.1.6.

Example 2.1.5. The integers, $\mathbb{Z}$, form a commutative ring. We may therefore see $\mathbb{Z}$ itself as a $\mathbb{Z}$-module, where the $\mathbb{Z}$-action is multiplication. This module is generated by the set $\{1\}$, but it is also generated by the set $\{2, 3\}$. This is because $3 - 2 = 1$ so the set $\{2, 3\}$ is able to generate everything that $\{1\}$ is able to generate. Note also how since $2 \cdot 1 = 2$ and $3 \cdot 1 = 3$, the set $\{1\}$ is able to generate everything that $\{2, 3\}$. Hence, they are equivalent as generating sets. However, only $F(\{1\})$ is isomorphic to $\mathbb{Z}$, and therefore $\{1\}$ is a basis of $\mathbb{Z}$.

Example 2.1.6. The rational numbers, $\mathbb{Q}$, form a field, so we may see $\mathbb{Q}$ itself as a $\mathbb{Q}$-module. We see that, as in Example 2.1.5 above, $\{1\}$ is a basis of $\mathbb{Q}$. However, so is $\{2\}$, and every other element of $\mathbb{Q}$. Hence, we see that the basis of $\mathbb{Q}$ is not unique.

Example 2.1.7. Not every module with a generating set is freely generated. For example, we may see the cyclic group of order 2, $\mathbb{Z}_2$ as a $\mathbb{Z}$-module, where we set the $\mathbb{Z}$-action $\cdot : \mathbb{Z} \times \mathbb{Z}_2 \to \mathbb{Z}_2$ as $n \cdot k = nk$ modulo 2. Then $\{1\}$ is a generating set of $\mathbb{Z}_2$. However, clearly $F(\{1\})$ is not isomorphic to $\mathbb{Z}_2$, as the free module generated by $\{1\}$ is $\mathbb{Z}$. In other words, $\mathbb{Z}_2$ is generated by $\{1\}$, but it is not *freely* generated by $\{1\}$. So does $\mathbb{Z}_2$ have a basis? The answer is no - $\mathbb{Z}_2$ has only 2 elements, while every freely generated $\mathbb{Z}$-module has an infinite number of elements (apart from $F(\emptyset)$). Hence, $\mathbb{Z}_2$ is not a free $\mathbb{Z}$-module.

Lemma 2.1.8. *Let M be an R-module with generating set B, where each $x \in M$ has a unique set of coefficients. Then given any other R-module, N, we may define a morphism $f : M \to N$ by stating what $f(b)$ is equal to, for each $b \in B$. Then, for any $x \in M$ with coefficients $\{r_b\}$, we have*

$$f(x) = f(\sum_{b \in B} r_b b) = \sum_{b \in B} r_b f(b).$$

Proof. To prove that this is a well-defined function, we point out how it is clear that for all $x \in M$, there is a unique value of $f(x)$ due to the uniqueness of $\{r_b\}$ for each x. That this function is a linear map follows by definition. $\square$

For R-modules without this property, there is no guarantee that this morphism will be well defined - consider $\mathbb{Z}_2$ from Example 2.1.7. Recall that $\{1\}$ is a generating set of $\mathbb{Z}_2$. If we were to try to define a linear map $f : \mathbb{Z}_2 \to \mathbb{Z}$, where $f(1) = 1$, then since

$0 = 2 \cdot 1 = 4 \cdot 1$, we would have

$$\begin{aligned} f(0) &= f(2 \cdot 1) = 2 \cdot f(1) = 1 \\ &= f(4 \cdot 1) = 4 \cdot f(1) = 4 \end{aligned}$$

Hence, f would not be well defined.

Lemma 2.1.9. *Let R-module M have a generating set B. Then B is a basis of M if and only if, for each $x \in M$, its set of coefficients $\{r_b\}$ is unique.*

Proof. First off, note that in $F(B)$, the coefficients of any $v \in F(B)$ are unique by construction.

Suppose that M has B as a basis. By definition, the morphism $\phi : F(B) \to M$ is an isomorphism. Then there exists an inverse $\phi^{-1} : M \to F(B)$. Suppose that there exists $x \in M$ which has two distinct sets of coefficients, $\{r_b\}$ and $\{r'_b\}$. So we have

$$x = \sum_{b \in B} r_b b = \sum_{b \in B} r'_b b$$

such that $\{r_b\} \neq \{r'_b\}$. But this implies that

$$\phi^{-1}(x) = \sum_{b \in B} r_b \phi^{-1}(b) \neq \sum_{b \in B} r'_b \phi^{-1}(b).$$

This is a contradiction. Hence, this implies that x must have a unique set of coefficients.

On the other hand, suppose that for all $x \in M$, each x has a unique set of coefficients. We now need to show that ϕ is an isomorphism.

From Lemma 2.1.8, we see that we may define a linear map $\psi : M \to F(B)$ as $\psi(b) = b$. Clearly, ψ is the inverse of ϕ. Hence, $F(B)$ is isomorphic to M and B forms a basis. □

Definition 2.1.10. A *vector space* is a k-module, where k is a field. We remark without proof that, when we assume the axiom of choice, every vector space has a basis. This property makes vector spaces a critical subset of modules. In addition, vector spaces have the property that, while the basis of vector space V is not unique, we cannot have two bases of V with different cardinalities. We say that the *dimension of* V is the cardinality of its basis set.

Let V and W be vector spaces with bases B_V and B_W. If V and W have the same dimension, then there is a bijection between the sets B_V and B_W. Hence $F(B_V)$ will be isomorphic to $F(B_W)$. Therefore, since $V \cong F(B_V)$ and $W \cong F(B_W)$, this implies that when vector spaces have the same dimension, they will be isomorphic.

We denote the *category of vector spaces* as $\mathbf{Vect_k}$, where the objects are vector spaces and the morphisms are linear maps. The *category of finite dimensional vector spaces*, denoted $\mathbf{FVect_k}$ is the full subcategory of $\mathbf{Vect_k}$ where the objects are only the vector spaces with finite dimensions.

Definition 2.1.11. Let M and N be R-modules. The *tensor product* of M and N, denoted $M \otimes N$ is the module freely generated by ordered pairs

$$m \otimes n \qquad\qquad \text{where } m \in M, n \in N.$$

which is then quotiented such that the following identities hold

$$(m \otimes n) + (m \otimes n') = m \otimes (n + n')$$
$$(m \otimes n) + (m' \otimes n) = (m + m') \otimes n$$
$$r(m \otimes n) = rm \otimes n = m \otimes rn.$$

Given morphisms, $f : M_1 \to M_2$ and $g : N_1 \to N_2$, the tensor product of f and g is defined as a morphism $f \otimes g : M_1 \otimes N_1 \to M_2 \otimes N_2$ where

$$(f \otimes g)(m \otimes n) = f(m) \otimes g(n).$$

Definition 2.1.12. Recall the definition of the free functor $F : \mathbf{Set} \to \mathbf{Mod_R}$ from Definition 2.1.1. Recall that the monoidal unit with respect to the Cartesian product in

Set is the singleton set, $1 = \{e\}$. This functor is strong monoidal, as follows.

$$\text{if } x \in F(A \times B), \text{ then } x \text{ has the form } \sum_i r_i(a_i, b_i) \tag{2.1}$$

$$\text{if } x \in F(A) \otimes F(B), \text{ then } x \text{ has the form } \sum_i r_i a_i \otimes \sum_j k_j b_j = \sum_{i,j} r_i k_j a_i \otimes b_j \tag{2.2}$$

$$\text{if } x \in F(1), \text{ then } x \text{ has the form } re \tag{2.3}$$

where $r_i, k_j, r \in R$, $a_i \in A$ and $b_j \in B$.

Note that the basis elements of $F(A \times B)$ are $(a, b) \in A \times B$, and the basis elements of $F(A) \otimes F(B)$ are $a \otimes b$, where $a \in A$ and $b \in B$. Hence, we define the natural transformation $\gamma_{A,B} : F(A) \otimes F(B) \to F(A \times B)$ as $\gamma_{A,B}(a \otimes b) = (a, b)$, and the morphism $\tau : R \to F(1)$ as $\tau(r) = re$. We sketch the proof that the equations are respected. For the associativity axiom, this is merely the statement that

$$\begin{aligned}
\gamma_{A\otimes B,C}(\gamma_{A,B}(a \otimes b) \otimes c) &= \gamma_{A\otimes B,C}((a, b) \otimes c) = (a, b, c) \\
&= \gamma_{A,B\otimes C}(a \otimes (b, c)) \\
&= \gamma_{A,B\otimes C}(a \otimes \gamma_{B,C}(b \otimes c))
\end{aligned}$$

for $a \in A, b \in B$ and $c \in C$. For the unitality axioms, we will only look at the left hand axiom.

$$\lambda_{FA}(re \otimes a) = ra \qquad \text{and} \qquad \begin{aligned}
F\lambda_A(\gamma_{1,A}(\tau(r) \otimes a)) &= F\lambda_A(\gamma_{1,A}(re \otimes a)) \\
&= F\lambda_A(r(e, a)) \\
&= ra
\end{aligned}$$

where $r \in R$ and $a \in A$ It is clear that both γ and τ are isomorphisms, and that they fulfil the appropriate axioms. Hence, F is a strong monoidal functor.

2.2 Monoidal Categories and String Diagrams

The categories of modules over a ring and vector spaces are both examples of symmetric monoidal categories. This thesis is not limiting itself to these categories however – we will be proving our results for an arbitrary symmetric monoidal category.

This section of the thesis will cover some results and definitions for symmetric monoidal categories, and we will discuss our method for reasoning within symmetric monoidal categories - string diagrams.

In this thesis, we are going to be denoting the natural isomorphisms as follows: the associator is denoted $\alpha_{A,B,C} : (A \otimes B) \otimes C \to A \otimes (B \otimes C)$, the left and right unitors are respectively $\lambda_A : I \otimes A \to A$ and $\rho_A : A \otimes I \to A$, and the symmetry is $\sigma_{A,B} : A \otimes B \to B \otimes A$.

However, we will note that the following classic result tells that, even though our results are true for arbitrary symmetric monoidal categories, we only need to prove our results for strict symmetric monoidal categories.

Theorem 2.2.1. *Given any (symmetric) monoidal category $\mathcal{C}$, we can construct a strict (symmetric) monoidal category $\overline{\mathcal{C}}$ such that there is a strong monoidal equivalence between $\mathcal{C}$ and $\overline{\mathcal{C}}$.*

We cite MacLane [42] for the proof of this theorem.

Lemma 2.2.2. *Let $\mathcal{C}$ be a monoidal category. Then $\rho_I : I \otimes I \to I$ is equal to $\lambda_I : I \otimes I \to I$.*

Proof. We refer to Kelly [37] for the proof of the above Lemma. □

Definition 2.2.3. Let $\mathcal{C}$ be a monoidal category (note: not necessarily symmetric), and let A be an object in $\mathcal{C}$. We say that a morphism with type $I \to A$ is a *point*, a morphism with type $A \to I$ is a *copoint*, and a morphism with type $I \to I$ is a *scalar*. We may define a multiplication on scalars as follows.

Recall from Lemma 2.2.2 that $\rho_I = \lambda_I$. Given scalars $a, b : I \to I$, the multiplication is defined as $a \bullet b := \rho_I^{-1} \circ (a \otimes b) \circ \rho_I$. In addition, by the natural transformation property of ρ, we see that the following diagram commutes

$$\begin{tikzcd} I \otimes I \arrow[r, "a\otimes 1_I"] & I \otimes I \arrow[d, "\rho_I"] \\ I \arrow[u, "\rho_I^{-1}"] \arrow[r, "a"] & I \end{tikzcd}$$

The same is true for $1_I \otimes a$. Therefore, 1_I is the unit of the multiplication. Finally, note that

$$\begin{aligned} a \circ b &= (a \bullet 1_I) \circ (1_I \bullet b) = a \bullet b \\ &= (1_I \bullet a) \circ (b \bullet 1_I) = b \bullet a \end{aligned}$$

so the monoid via $\circ$ is the same as the multiplication via $\bullet$, and this monoid is commutative. This is known as the Eckmann-Hilton argument [27]. We see that this also implies that $\bullet$ is associative, since $\circ$ is associative. As such, we say that a scalar a is *invertible* when there exists some a^{-1} such that $a \bullet a^{-1} = 1_I$.

Remark 2.2.4. In **Set**, the unit is the terminal object, a singleton set, denoted 1. A point in **Set** $a : 1 \to A$ is therefore an element of A. In **Rel**, a point $I \to A$ is a subset of A. A similar fact holds true in $\mathbf{Mod_R}$; R is the unit object in $\mathbf{Mod_R}$, and to define a morphism $m : R \to M$, we state what element in M the multiplicative identity[1] $1 \in R$ is being mapped to. This then tells us that $m(r) = rm(1)$. Hence, in $\mathbf{Mod_R}$, each point $m : R \to M$ corresponds to an element in M.

This implies that a scalar in the module sense (i.e. an element of R) corresponds with a scalar in the categorical sense, a morphism $r : R \to R$. Hence we refer to morphisms with type $I \to I$ as scalars.

Let V be a finite dimensional vector space with basis B. The dual basis, denoted B^*, is the set of copoints where, for each $b \in B$, we define $b^* : V \to k$ as

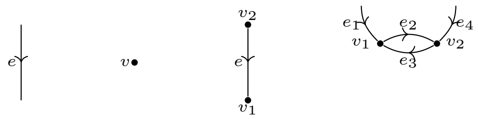
$$v \mapsto \begin{cases} 1 & \text{if } v = b \\ 0 & \text{otherwise.} \end{cases}$$

[1]For modules non-unital rings, there is not a bijective correspondence between elements of the module and maps from the ring. For non-unital ring R and module A, with element $x \in A$, we may define a morphism $f_x : R \to A$ such that $f_x(r) = rx$. However, there may be an element $y \in A$ such that $rx = ry$, and therefore $f_x = f_y$. Hence, there is no bijective correspondence.

In this case, we say that b^* is the *dual* of b. In Lemma 2.3.17 we will show how B^* forms a basis of the *dual space* of V.

The vast majority of proofs in this paper are done in the language of string diagrams. As such, we shall devote the remainder of this section to how we shall be using string diagrams. This section is primarily a sketch of the full definition of string diagrams. We have included the full discussion in the appendix, in Section A.1, including a more rigourous definition of open graphs (Definition A.1.1), string diagrams (Definition A.1.5), composition (Definitions A.1.13 and A.1.10), the category of string diagrams (Lemma A.1.17) and the free symmetric monoidal category (Definition A.1.19). We also provide concrete examples of each of the string diagram concepts. In this section, however, we shall merely sketch these concepts.

A graph G has a set of *vertices*, V and a set of *edges*, E such that each edge connects one vertex to another vertex – i.e. we have source and target functions, $s, t : E \to V$. In an *open graph*, we do not require that each edge is connected to a vertex. In other words, s and t are partial functions. When an edge's source is undefined, then we say that it is in the *domain* of G. Likewise, when an edge's target is undefined, then we say that it is in the *codomain* of G. This essentially allows us to define a notion of composition on open graphs. For example, the following are all examples of open graphs

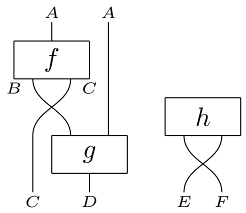


We explain how these are open graphs in Example A.1.2.

A *string diagram* is a certain type of open graph that represents a morphism term in a given symmetric monoidal category. So, for example, given a symmetric monoidal category $\mathcal{C}$, with morphisms $f : A \to B \otimes C$, $g : B \otimes A \to D$ and $h : I \to E \otimes F$, we may represent the morphism term

$$((1_C \otimes g) \circ ((\sigma_{B,C} \circ f) \otimes 1_A)) \otimes (\sigma_{E,F} \circ h)$$

as the string diagram

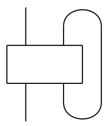


We do this by labelling edges as objects of $\mathcal{C}$, and vertices as morphisms in $\mathcal{C}$. For each vertex v, we say that the domain of v is the set of edges e such that $t(e) = v$. Likewise, we may define the codomain of v in a similar manner. When labelling a vertex v with a morphism f, we require that the object labels of the domain and codomain of v match with the domain and codomain of f.

However, we find that open graphs do not have enough structure to accurately represent morphism terms in particular, we do not want the following string diagrams to be equal

We accomplish this by requiring that the domain and codomain of the string diagram have a total order. We also require that the domain and codomain of each vertex have a total order.

Finally, we require that no string diagrams exist of the following form



we do this by saying that there are no *cycles* – paths from a vertex v back to v (see Definition A.1.1).

We use this definition of string diagram to construct a category of string diagrams, SD($\mathcal{C}$), of a given symmetric monoidal category $\mathcal{C}$. We use this category to state the following theorem. We only paraphrase here, but we will state the full theorem in the appendix.

Theorem 2.2.5. *Let $\mathcal{C}$ be a strict symmetric monoidal category. Then given two free morphism terms, t_1 and t_2, there are corresponding string diagrams $\hat{t}_1$ and $\hat{t}_2$ in SD($\mathcal{C}$)*

such that there is a proof following from the strict symmetric monoidal category axioms that $t_1 = t_2$ *if and only if* $\hat{t}_1 = \hat{t}_2$*.*

We do not prove this theorem in this thesis – instead, we refer to Joyal and Street [33] for the proof of the theorem, and for a full discussion of the theorem, we refer to Selinger [55].

Practically, this means that instead of using free morphism terms in definitions and proofs, we may instead use string diagrams. This carries the following advantage; with the term language we may have two terms that are not equal as free morphism terms, but are equal in the free symmetric monoidal category. Then these terms will be mapped to the same string diagram. This is advantageous, as it can often be difficult to prove that two terms are equal in the free symmetric monoidal category, but it is trivial in the category of string diagrams. Hence, from now on the majority of our reasoning will be graphical.

However, the equations of the free symmetric monoidal category are rarely enough. How might we reason about, for example, monoids? We will approach this problem by quotienting $\mathrm{SD}(\mathcal{C})$, as follows

Definition 2.2.6. Let $E = \{s_i = t_i\}$ be a finite family of equations, where s_i and t_i are morphism terms in $\mathcal{C}$ and their types match. Let the category of *graphical reasoning with respect to* E be the category $\mathrm{GR}_E(\mathcal{C})$ that is $\mathrm{SD}(\mathcal{C})$ quotiented such that $\hat{s}_i = \hat{t}_i$. By Theorem 2.2.5, we know that each symmetric monoidal term t has a corresponding string diagram $\hat{t}$. Since $\mathrm{GR}_E(\mathcal{C})$ is a quotient of $\mathrm{SD}(\mathcal{C})$, each string diagram in $\mathrm{SD}(\mathcal{C})$ has a corresponding string diagram in $\mathrm{GR}_E(\mathcal{C})$. Hence, we set $\{t\}$ as the term t's corresponding string diagram in $\mathrm{GR}_E(\mathcal{C})$. We shall state without proof that we may define composition and monoidal product in $\mathrm{GR}_E(\mathcal{C})$ as

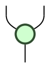
$$\{t_1\} \circ \{t_2\} := \{t_1 \circ t_2\}$$
$$\{t_1\} \otimes \{t_2\} := \{t_1 \otimes t_2\}$$

Corollary 2.2.7. *Let* t_1 *and* t_2 *be morphism terms in* $\mathcal{C}$*. Then there is a proof that* $t_1 = t_2$ *in* $\mathcal{C}$ *following from the axioms of strict symmetric monoidal categories and the* *equations E if and only if* $\{t_1\} = \{t_2\}$.

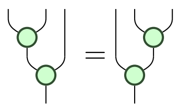
2.3 Monoids, Comonoids and Duals

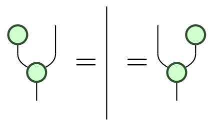
We begin this section by providing an example of how we use graphical reasoning by defining a *monoid* in an arbitrary monoidal category. Recall from Definition 2.2.3 and the subsequent Remark 2.2.4 that a point is a morphism with type $I \to A$ for some object A, and it corresponds to an element of A when the category is **Set** or $\mathbf{Mod_R}$.

Definition 2.3.1. A *monoid* in a monoidal category $\mathcal{C}$ consists of an object M, a binary multiplication $\mu : M \otimes M \to M$ and a unit point $\eta : I \to M$, denoted graphically as

$\mu :=$  and $\eta :=$ 

respectively. These obey the familiar associativity and unit laws, shown diagrammatically below.



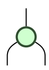
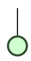
 .

The set of equations that define the above graphical reasoning for monoids is expressed below as a family of commutative diagrams

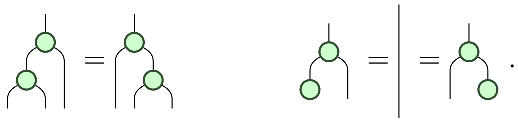
$$\begin{tikzcd} M \otimes M \otimes M \arrow[r, "1\otimes\mu"] \arrow[d, "\mu\otimes 1"'] & M \otimes M \arrow[d, "\mu"] \\ M \otimes M \arrow[r, "\mu"] & M \end{tikzcd} \qquad \begin{tikzcd} M \arrow[r, "\eta\otimes 1"] \arrow[dr, "1"'] & M \otimes M \arrow[d, "\mu"] & M \arrow[l, "1\otimes\eta"'] \arrow[dl, "1"] \\ & M & \end{tikzcd}$$

Note how these equations can be inferred from the string diagrams. Therefore, since it is not necessary to state the equations as terms in the language of $\mathcal{C}$, we will refrain from doing so from now on.

A *comonoid* in $\mathcal{C}$ is the dual concept of a monoid. Concretely, a comonoid C consists of a *comultiplication* $\Delta : C \to C \otimes C$ and *counit* $\varepsilon : C \to I$ depicted below.

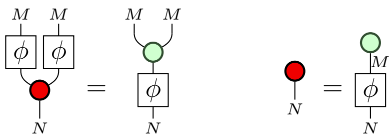
$\delta =$  and $\epsilon =$  .

These obey the dual of the monoid axioms, known as the *coassociativity* and *counit* laws, as follows.



A (co)monoid is called (co)commutative if its (co)multiplication is invariant under symmetry, as depicted below.

Let $(M,$  ,  $)$ and $(N,$  ,  $)$ be monoids in $\mathcal{C}$. A *monoid homomorphism* is a morphism in $\mathcal{C}$, $\phi : M \to N$, such that



A *comonoid homomorphism* is the dual notion of this.

Example 2.3.2. This more general definition of a monoid aligns with the familiar definition of a monoid in **Set**. In the category $\mathbf{Mod_R}$, a monoid is typically referred to as an *algebra*. Likewise, a comonoid in $\mathbf{Mod_R}$ is typically referred to as a coalgebra.

When we are referring to a monoid $(M, \cdot : M \otimes M \to M, 1)$ in **Set** or $\mathbf{Mod_R}$, we will often suppress the notation $\cdot$ and instead write ab for $a \cdot b$.

We may see a homomorphism between monoids $(M, \cdot, 1_M)$ and $(N, \cdot, 1_N)$ as a function on **Set** (resp. a linear map in $\mathbf{Mod_R}$) $\phi : M \to N$ where $\phi(ab) = \phi(a)\phi(b)$ and $\phi(1_M) = 1_N$.

Example 2.3.3. Suppose that $\mathcal{C}$ has products. Then there is exactly one comonoid for each object – the *copy comonoid*. The comultiplication is commonly referred to as the *diagonal map*, and it comes from the universal property of the product

$$\begin{tikzcd} & C \arrow[dl, "1_C"'] \arrow[d, dashed, "\Delta_C"] \arrow[dr, "1_C"] & \\ C & C \times C \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & C \end{tikzcd} .$$

By definition, the counit must have type $C \to 1$, where 1 is the terminal object. Hence, by the universal property of the terminal object, there is only one morphism that the counit could be.

To see that this comonoid is unique for each object, suppose that for some object A, we have a comonoid, $(A, \delta : A \to A \times A, \epsilon : A \to 1)$. Since δ maps to $A \times A$, by the properties of the product there must be maps $\delta_1, \delta_2 : A \to A$ such that

$$\begin{tikzcd} & A \arrow[dl, "\delta_1"'] \arrow[d, "\delta"] \arrow[dr, "\delta_2"] & \\ A & A \times A \arrow[l, "\pi_1"] \arrow[r, "\pi_2"'] & A \end{tikzcd} \tag{$\star$}$$

However, note that

$$\pi_1 = A \times A \xrightarrow{1_A \times \epsilon} A \times 1 \cong A$$

since there is only a single morphism from A to 1. This means that the following diagram commutes

$$\begin{tikzcd} A & A \arrow[l, "\delta_1"'] \arrow[d, "\delta"] \\ A \times 1 \arrow[u, "\wr" description, equal] & A \times A \arrow[l, "1_A \times \epsilon"] \end{tikzcd}$$

This is simply the counit axiom for comonoids. Hence, δ_1 must be equal to the identity 1_A. The same follows for δ_2. If we now refer back to the commutative diagram $\star$, we see that by the uniqueness property of products, δ must equal Δ_A, the copy map. Hence, there is only one comonoid defined on A.

Recall that in **Set**, the terminal object is the singleton set, $1 = \{\star\}$. The diagonal map is $\Delta_C : C \to C \times C$ is defined as $\Delta_C(c) = (c, c)$, and the counit $\epsilon_C : C \to 1$ maps each $c \in C$ to $\star$.

Example 2.3.4. Consider a monoid $M = (M, \cdot, e)$ in **Set**. Recall that the free module functor is strong monoidal from Definition 2.1.12. It follows from the definition of a monoidal functor that $F(M)$ is a monoid in $\mathbf{Mod_R}$. The multiplication is

$$F(M) \otimes F(M) \xrightarrow{\gamma_{M,M}} F(M \times M) \xrightarrow{F(\cdot)} F(M),$$

with unit $R \xrightarrow{\tau} F(1) \xrightarrow{F(e)} F(M)$. In $\mathbf{Mod_R}$, monoids are typically called *algebras*, and the algebra $F(M)$ is called the *free algebra generated by M*. An important case of this is when M is a group. In this case, it is called a *group algebra*.

For a concrete definition of the free algebra, recall that M is a basis of $F(M)$. Hence, we only need to define the multiplication and unit maps on the elements of M. Given $m, n \in M$, the free algebra would map $m \otimes n$ to mn. The unit is simply the unit element of M.

Example 2.3.5. Let C be a set. We may apply the free functor $F : \mathbf{Set} \to \mathbf{Mod_R}$ to the copy comonoid to get the *copy coalgebra*. Let M be a free module with basis B. Since the copy comonoid exists for every set, we may define a copy coalgebra on M as (M, Δ, ϵ), where $\Delta : M \to M \otimes M$ and $\epsilon : M \to R$ are defined as $\Delta := b \mapsto b \otimes b$ and $\epsilon := b \mapsto 1$ for all $b \in B$, where 1 is the unit in the ring R.

When B is finite, we may also define an algebra on M that we will refer to as the *copy algebra*. The multiplication is defined for $b, b' \in B$ as

$$b \otimes b' \mapsto \begin{cases} b & \text{if } b = b' \\ 0 & \text{otherwise} \end{cases}$$

and the unit is the element

$$\sum_{b \in B} b.$$

In certain circumstances, this is actually a monoid algebra. Suppose that $B = \{b_0, b_1, b_2\}$, and that we are working over the complex numbers[2]. Then we set $\omega = e^{\frac{2}{3}\pi i}$, and define the basis $\mathbb{Z}_3$, where

$$\begin{aligned} \mathbf{0} &:= \frac{1}{3}b_0 + \frac{1}{3}b_1 + \frac{1}{3}b_2 \\ \mathbf{1} &:= \frac{1}{3}b_0 + \frac{1}{3}\omega b_1 + \frac{1}{3}\omega^2 b_2 \\ \mathbf{2} &:= \frac{1}{3}b_0 + \frac{1}{3}\omega^2 b_1 + \frac{1}{3}\omega b_2 \end{aligned}$$

[2]This construction works with the free module of a set of p elements, over a commutative ring with a primitive pth root of unity, where p is a prime number, and invertible within the ring. This was communicated to me via private correspondence with Ezra Schoen.

In this case, this copy algebra is the free algebra of the cyclic group of order 3, $\mathbb{Z}_3$.

Example 2.3.6. Let $M = (M, \cdot : M \times M \to M, e)$ be a finite monoid. We may define a coalgebra that bears similarities with the free algebra. The comultiplication $F(M) \to F(M) \otimes F(M)$ is defined

$$m \mapsto \sum_{m=ab} a \otimes b.$$

with counit $F(M) \to k$, defined

$$m \mapsto \begin{cases} 1 & \text{if } m = e \\ 0 & \text{otherwise.} \end{cases}$$

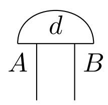
We will refer to this coalgebra as the *monoid coalgebra*.

When we have a finite group G, the group coalgebra has the following equivalent definition.

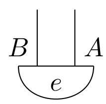
$$g \mapsto \sum_{h \in G} gh \otimes h^{-1}.$$

because $g = ghh^{-1}$

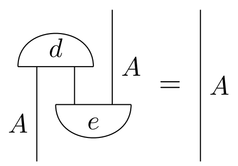
Definition 2.3.7. Let A, B be objects in monoidal category $\mathcal{C}$. We say that B is *left dual* to A if there exist morphisms $d : I \to A \otimes B$ and $e : B \otimes A \to I$ with graphical representations



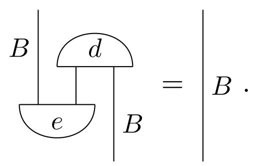
and



such that

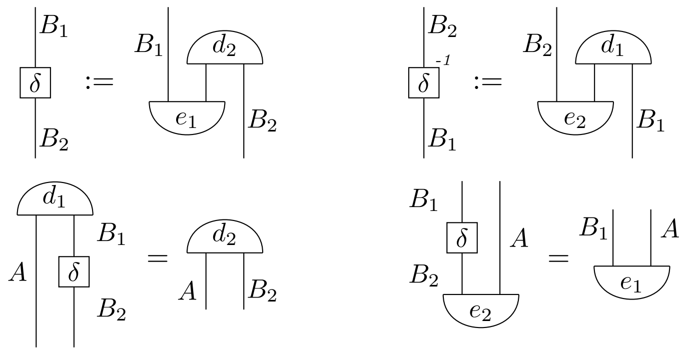


and



In this circumstance A is *right dual* to B. Note that if $\mathcal{C}$ is symmetric then each left dual is also a right dual.

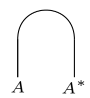
Proposition 2.3.8. *In monoidal category $\mathcal{C}$, suppose that A has two right duals (B_1, d_1, e_1) and (B_2, d_2, e_2); then there exists an isomorphism $\delta : B_1 \cong B_2$, satisfying the equations shown below.*

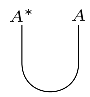


Proof. Define δ as shown above; the required equations follow immediately. □

The morphisms d and e are often referred to as the *unit* and *counit* respectively. In this thesis, there are already morphisms that we refer to as the unit and counit, so we avoid that terminology. Due to their graphical representations, we refer to d as the *cap* and e as the *cup*.

Definition 2.3.9. A *compact closed category* [36] is a symmetric monoidal category where every object A has an assigned dual (A^*, d_A, e_A). In the graphical notation we depict the cap and cup as

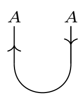
$d_A :=$ 

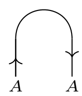
$e_A :=$ 

.

Note how we are abusing the notation above, where we have drawn the above morphisms as though they were a single edge. The cap and the cup are morphisms of type $I \to A \otimes A^*$ and $A^* \otimes A \to I$ respectively. Therefore, their string diagrammatic representations consist of a vertex and two edges.

Remark 2.3.10. When drawing compact closed structure, some authors will use *arrow notation*,

$d_A :=$ 

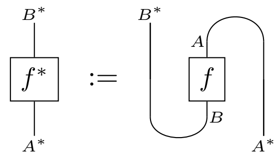
$e_A :=$ 

where when a wire has type A^*, instead of being labelled as such, it will be labelled with A and annotated with an arrow to indicate whether or not it is the dual of A.

Corollary 2.3.11. *Let $\mathcal{C}$ be a compact closed category. Then $A \cong A^{**}$ for all objects A.*

Proof. This follows from Proposition 2.3.8 and the fact that both A and A^{**} are both dual to A^*. ☐

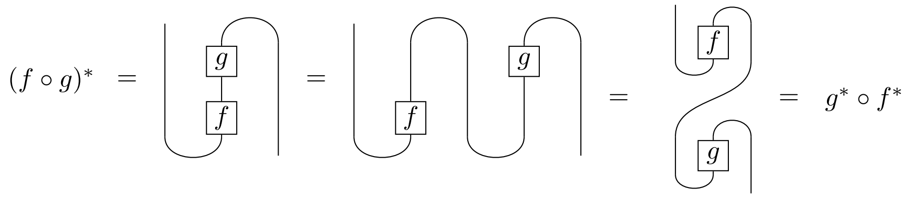
Proposition 2.3.12. *Let A and B have duals, A^*, B^*. We set the* dual of $f : A \to B$ *as the morphism $f^* : B^* \to A^*$, defined as*



Assigning objects and morphisms to their duals has the following properties

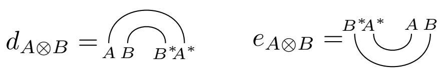
$$(f \circ g)^* = g^* \circ f^* \qquad 1_A^* = 1_{A^*} \qquad (A \otimes B)^* \cong B^* \otimes A^* \qquad I^* \cong I$$

Proof. We begin by showing that $(\cdot)^*$ preserves composition. This follows graphically



It is obvious from the definition that $(1_A)^* = 1_{A^*}$.

The proof that $(A \otimes B)^* \cong B^* \otimes A^*$ follows graphically, as we see below that the cup and cap of $A \otimes B$ must be defined as follows



The isomorphism $(A \otimes B)^* \cong B^* \otimes A^*$ then follows from Proposition 2.3.8. Note that in a symmetric monoidal category, since $A \otimes B \cong B \otimes A$, it is also the case that $A^* \otimes B^*$ is dual to $A \otimes B$.

Finally, since we are working in a strict monoidal category, $I \otimes I = I$. Then the identity map 1_I functions as a cap $I \to I \otimes I$ and cup $I \otimes I \to I$. This clearly fulfils the dual equations. Hence, Proposition 2.3.8 gives us the isomorphism $I^* \cong I$.

$\square$

Corollary 2.3.13. *When $\mathcal{C}$ is a compact closed category, the assignment of $A \mapsto A^*$ and $f \mapsto f^*$ defines a strong monoidal functor, $(\cdot)^* : \mathcal{C} \to \mathcal{C}^{op}$. We call this the* dual functor.

Lemma 2.3.14. *In monoidal category $\mathcal{C}$, suppose that A and B have two right duals each: (A^*, d_{A^*}, e_{A^*}) and $(A^\diamond, d_{A^\diamond}, e_{A^\diamond})$ shall be the duals of A; (B^*, d_{B^*}, e_{B^*}) and $(B^\diamond, d_{B^\diamond}, e_{B^\diamond})$ shall be the duals of B. Let us denote the dual actions on morphism $f : A \to B$ as f^* and $f^\diamond$ respectively.*

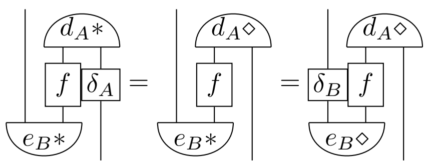
In Proposition 2.3.8, we construct an isomorphism between mutual dual structures on the same object. Let us denote these isomorphisms as $\delta_A : A^* \cong A^\diamond$ and $\delta_B : B^* \cong B^\diamond$.

Then the following square commutes

$$\begin{tikzcd} B^* \arrow[r, "f^*"] \arrow[d, "\delta_B"'] & A^* \arrow[d, "\delta_A"] \\ B^\diamond \arrow[r, "f^\diamond"] & A^\diamond \end{tikzcd}$$

for all morphisms $f : A \to B$.

Proof. We see this from the string diagram



$\square$

Corollary 2.3.15. *Let $\mathcal{C}$ be a symmetric monoidal category with two compact closed structures, denoted $*$ and $\diamond$. Then their respective functors, $(\cdot)^* : \mathcal{C} \to \mathcal{C}^{op}$ and $(\cdot)^\diamond : \mathcal{C} \to \mathcal{C}^{op}$ are naturally isomorphic via $\delta : (\cdot)^* \cong (\cdot)^\diamond$, as defined in Lemma 2.3.14.*

Example 2.3.16. The category of finite vector spaces, $\mathbf{FVect_k}$, is compact closed. Given an n-dimensional vector space V, the *dual-space* of V, denoted V^* is the space populated by copoints $V \to k$. To see that this is a vector space, simply observe that for $f, g \in V^*$, both $f + g$ and rf are in V^* for all $r \in k$.

We defined a *dual basis* in Remark 2.2.4, but we repeat the definition here. Suppose that V has finite basis B. The dual basis of B is the subset $\{b^* \in V^* | \text{ for each } b \in B\}$ where each $b^* : V \to k$ is defined as

$$v \mapsto \begin{cases} 1 & \text{if } v = b \\ 0 & \text{otherwise.} \end{cases}$$

In this case, we say that b^* is the *dual* of b.

Lemma 2.3.17. *Suppose a finite dimensional vector space V has basis B. The set B^* forms a basis for V^*.*

Proof. We start by proving that the basis is linearly independent. Suppose that there is some n such that for $b_n^* \in B^*$, we have

$$b_n^* = \sum_{i \neq n} a_i b_i^*$$

Then we would have

$$b_n^*(b_n) = \sum_{i \neq n} a_i b_i^*(b_n)$$

However, since $b_n^*(b_n) = 1$, and $b_i^*(b_n) = 0$ for all $i \neq n$, we have a contradiction. Hence, B^* must be linearly independent.

Let $f \in V^*$. Then f has type $f : V \to k$. Consider the linear map

$$g := \sum_{b \in B} f(b) b^*$$

Since $g(b) = f(b)$ for each basis element $b \in B$, this implies by linearity that $g = f$. Hence, each $f \in V^*$ is equal to a linear combination of the elements of B^*, so B^* forms a basis of V^*. □

We will now show that V^* is the dual of V. To prove this, we will define the cup and cap of V. Recall from Remark 2.2.4 that for any vector space A, a morphism of type $\mathbf{k} \to A$ is equivalent to an element of A. The cap $\mathbf{k} \to V \otimes V^*$ is $\sum_{b \in B} b \otimes b^*$, and the cup $V^* \otimes V \to \mathbf{k}$ is the map that maps $f \otimes v$ to $f(v)$, where $f \in V^*$ and $v \in V$. It is straightforward to show that this cup and cap fulfils the appropriate axioms, making V^* the dual of V.

Example 2.3.18. Note that the above proof does not hold for $\mathbf{Mod_R}$ in general, as there is no guarantee that an R-module will have a finite basis. For an R-module M to have a dual, we require that M is

- *Projective*: There exist sets $\{m_i \in M\}$ and $\{f_i \in \mathbf{Mod_R}(M, R)\}$, both indexed over some indexing set I, where for all $x \in M$, $f_i(x)$ is nonzero for finitely many f_i, and

$$x = \sum_{i \in I} f_i(x) m_i$$

- *Finitely generated*: There exists some set $\{n_j \in M\}$, indexed over finite set J, such that for any $x \in M$ there exists $\{r_j \in R\}$ such that

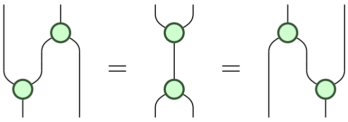
$$x = \sum_{j \in J} r_j n_j$$

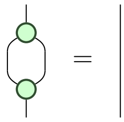
This forms the category of *finitely generated projective modules*, $\mathbf{FPMod_R}$, which is the largest full subcategory of $\mathbf{Mod_R}$ that is compact closed. Recall from the definition of a generating set (Definition 2.1.4) that even though a finitely generated module has a generating set, this set may not be a basis – i.e. $\{r_j \in R\}$ may not be unique.

2.4 Frobenius Algebras

In this thesis, we will focus primarily on the algebraic structures *Frobenius algebras* and *Hopf algebras*. Both of these structures consist of a monoid, a comonoid, and some laws of interaction between the two. We begin with the definition of a Frobenius algebra.

Definition 2.4.1. A *Frobenius algebra* $F = (F,$  ,  ,  ,  $)$ in a symmetric monoidal category $\mathcal{C}$ consists of a monoid $(F,$  ,  $)$ and a comonoid $(F,$  ,  $)$ on the same object F, obeying the *Frobenius law*, shown below on the left:

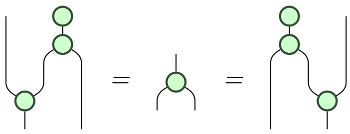


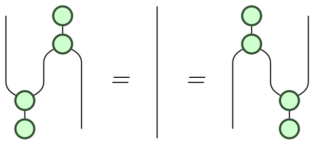


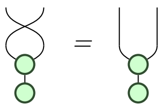
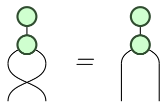
A Frobenius algebra is called *special* or *separable* when it obeys the equation above on the right, and *quasispecial* when it obeys the special equation up to an invertible scalar. A Frobenius algebra is *commutative* when its monoid $(F,$  ,  $)$ is commutative.

Remark 2.4.2. Note how the Frobenius law does not use the unit or counit of the Frobenius algebra. In Lemma 4.2.12, we will refer to the Frobenius law when we only have a multiplication and a comultiplication map, as opposed to a monoid and comonoid.

Lemma 2.4.3. *Let* $F = (F,$  ,  ,  ,  $)$ *be a Frobenius algebra. Then the following identities hold*

1. 

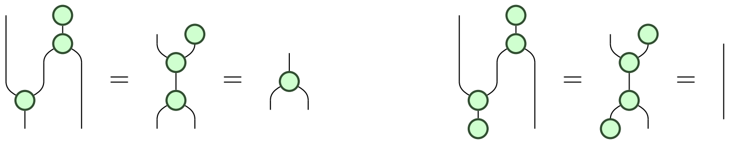
2. 

3.  *if and only if* 

4. *The monoid* $(F,$  ,  $)$ *is commutative if and only if the comonoid* $(F,$  ,  $)$ *is cocommutative*

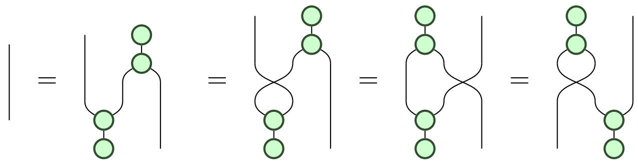
When the Frobenius algebra fulfils the equations in 3, we say that it is symmetric*. When either the monoid* $(F,$  ,  $)$ *or the comonoid* $(F,$  ,  $)$ *are commutative, we say that the Frobenius algebra is commutative.*

Proof. 1 and 2 both follow immediately from the Frobenius law, and the properties of the unit and counit, as follows

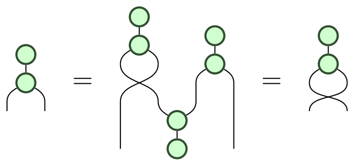


.

For 3, suppose that the left hand of the equivalence holds. We use 2 to get the following identity

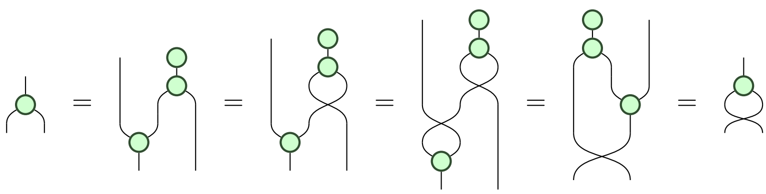


This implies therefore that



The implication in the other direction follows similarly.

Finally, suppose that the monoid (F, , ) is commutative. Following from 3, this implies that the Frobenius algebra will also be symmetric. Therefore,



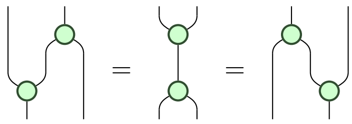
It is clear that the converse holds as well. □

Remark 2.4.4. Note how part 2 of the above Lemma tells us that for every Frobenius algebra, (F, , , , ), F is dual with itself, as in Definition 2.3.7, where we set the cup and cap as

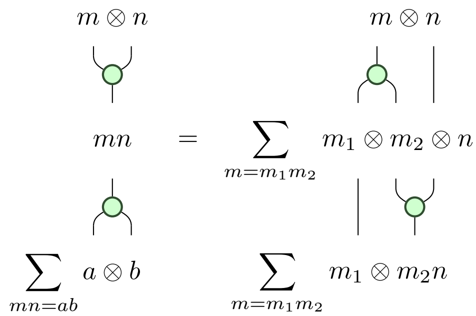
When using this dual structure, we denote the dual of object A and morphism f as A° and f° respectively.

Example 2.4.5. In Examples 2.3.4 and 2.3.6, we defined the free algebra and monoid coalgebra. This forms a Frobenius algebra $(M, \ , \ , \ , \ )$ in $\mathbf{Vect_k}$ if and only if M is a group algebra, as we shall see. Let us denote $\overline{M}$ as the generating monoid of M in **Set** (i.e. $F(\overline{M}) \cong M$).

For $(M, \ , \ , \ , \ )$ to be a Frobenius algebra, we need the following equations to hold



Since $\overline{M}$ forms a basis for this vector space, we shall denote the right hand equation with basis elements. Let $m, n \in \overline{M}$. Then from the definition of free algebra and coalgebra, we get that



In other words, a free algebra is a Frobenius algebra if and only if the sets

$$A(m,n) = \{(a,b) | a, b \in \overline{M}, ab = mn\}$$

and

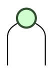
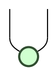
$$B(m,n) = \{(m_1, m_2 n) | m_1, m_2 \in \overline{M}, m_1 m_2 = m\}$$

are equal. Consider the sets $A(1,m)$ and $B(1,m)$ for some $m \in \overline{M}$. Clearly, $(m,1)$ is an element of $A(1,m)$, so it must therefore be an element of $B(1,m)$. Hence, there exists some $k \in \overline{M}$ such that $(m, km) = (m, 1) \in B(1, m)$ – i.e. $mk = 1$ and $km = 1$. Therefore, every $m \in \overline{M}$ has a left and right inverse, so $\overline{M}$ must be a group.

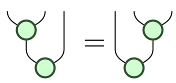
Example 2.4.6. Recall that in Example 2.3.5, we defined the copy coalgebra and copy algebra. Together, these form a commutative Frobenius algebra, that we call the *copy Frobenius algebra.*

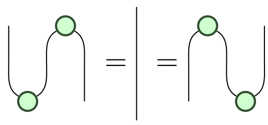
Definition 2.4.1, due to Carboni and Walters [16], has a pleasing symmetry between the monoid and comonoid parts. However, the following equivalent definition will be useful in later sections[3].

Definition 2.4.7. A *Frobenius algebra* in a symmetric monoidal category $\mathcal{C}$ consists of a monoid (F, , ) and a *Frobenius form* $\beta : F \otimes F \to I$, which admits an inverse, $\bar{\beta} : I \to F \otimes F$ (i.e. F is self-dual). These are denoted graphically as

$\beta :=$  $\bar{\beta} :=$ 

and they satisfy the equations



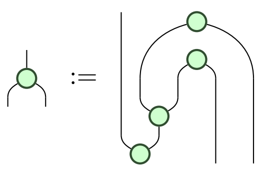


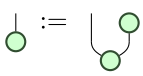
The left hand equation tells us that β is *associative.*

Lemma 2.4.8. *Definition 2.4.1 and Definition 2.4.7 are equivalent definitions of Frobenius algebras.*

Proof. To see that Definition 2.4.1 implies Definition 2.4.7, it suffices to take the cup and cap defined in Remark 2.4.4 as β and $\bar{\beta}$.

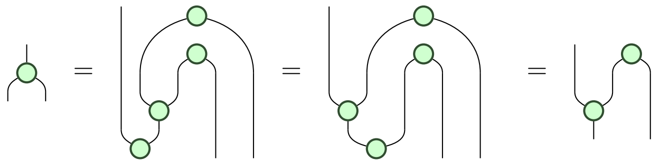
For the converse, we dualise  and  using β as in Proposition 2.3.12



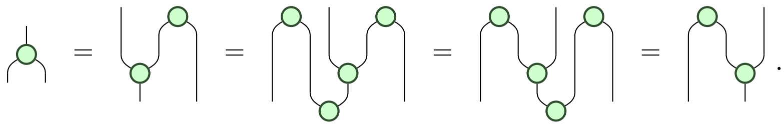


[3]See Fauser's survey [28] for several equivalent definitions.

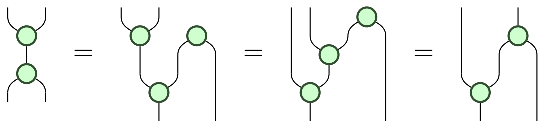
It follows from Proposition 2.3.12 that (F, ,) is therefore a comonoid. To prove that the Frobenius law is satisfied, we first prove the following identities for the comultiplication



and

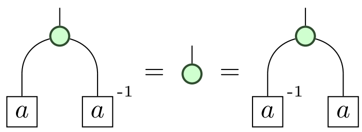


We only prove one side of the Frobenius law, as the proof of the other side is similar.



Hence, Definition 2.4.7 implies Definition 2.4.1. $\square$

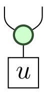
Given a comonoid (C, ,), a copoint $a : C \to I$ is *coinvertible* if there exists copoint $a^{-1} : C \to I$ such that

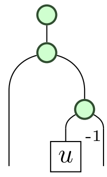


Lemma 2.4.9. *There is a bijective correspondence between invertible points for the monoid and coinvertible copoints for the comonoid of a Frobenius algebra.*

Proof. Recall the duality $(\cdot)^\circ$ induced by the cup and cap of the Frobenius algebra, as in Remark 2.4.4; then $u : I \to F$ is invertible iff and only if the dual $u^\circ : F \to I$ is coinvertible. $\square$

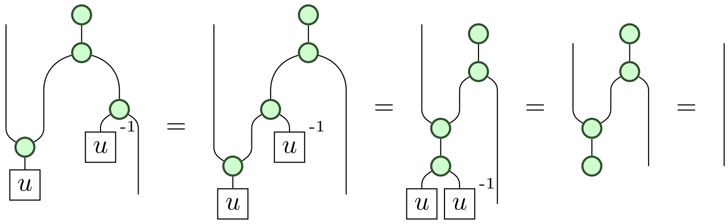
Lemma 2.4.10. *Let u be a coinvertible copoint of the comonoid. Define*

$\beta_u :=$ 

$\bar{\beta}_u :=$ 

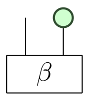
Then β_u is a Frobenius form for the monoid $(F,$  *,*  *$)$.*

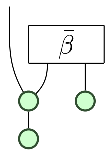
Proof. We must show that the equations of Definition 2.4.7 hold. The first follows from associativity of the monoid. For the second we have:



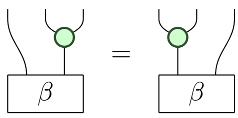
and similarly for the other side. Note that $\beta_u = \beta_v$ implies $u = v$ by the uniqueness of inverses. □

Lemma 2.4.11. *Suppose that β is a Frobenius form on Frobenius algebra F; then we obtain a coinvertible copoint $u : F \to I$ as follows:*

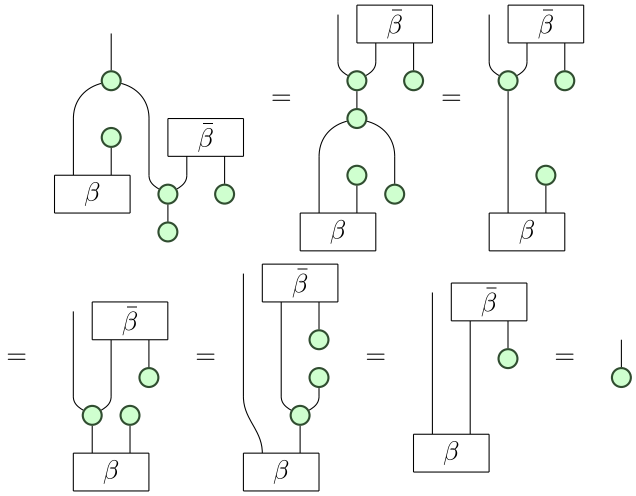
$u :=$ 

$u^{-1} :=$ 

Proof. Recall that for any Frobenius form



We need only to show that u^{-1} is the coinverse of u. The calculation is as follows



□

Combining the three preceding lemmas we obtain:

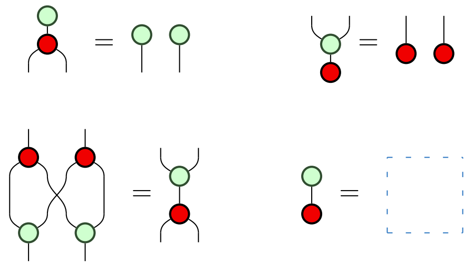
Proposition 2.4.12. *There is a bijective correspondence between the invertible elements of a monoid and the Frobenius forms definable on it.*

Frobenius algebras are merely one structure that is composed of a monoid and a comonoid. We now move onto the main algebraic structure of this thesis.

2.5 Bialgebras and Hopf algebras

Note. Unlike the preceding section, in our discussion of bialgebras and Hopf algebras, we will use different colours for the monoid and comonoid parts of the structure. We will refer to *green* and *red* morphisms, but this is mere notation.

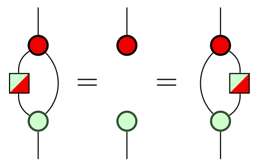
Definition 2.5.1. A *bialgebra* $B = (B,$ , , , $)$ in a symmetric monoidal category $\mathcal{C}$ consists of a monoid $(B,$ , $)$ and a comonoid $(B,$ , $)$ on the same object, which jointly obey the *copy*, *cocopy*, *bialgebra*, and *scalar* laws depicted below.



We may equivalently define a bialgebra as a monoid and a comonoid such that the comonoid is a monoid homomorphism. Given bialgebras $(B, \;, \;, \;, \;)$ and $(B', \;, \;, \;, \;)$, a *bialgebra morphism* is an morphism $\phi : B \to B'$ in $\mathcal{C}$ which is both a monoid homomorphism from $(B, \;, \;)$ to $(B', \;, \;)$ and a comonoid homomorphism from $(B, \;, \;)$ to $(B', \;, \;)$. Note that $B \otimes B'$ is also a bialgebra, with structure maps

Remark 2.5.2. Some works, notably on the ZX-calculus [5, 18, 32] and related theories [26], the scalar law is dropped and the other equations modified by a scalar factor, to give a *scaled bialgebra*.

Definition 2.5.3. A *Hopf algebra* consists of a bialgebra $(H, \;, \;, \;, \;)$ and a morphism $\;: H \to H$ called the *antipode* which satisfies the *Hopf law*:

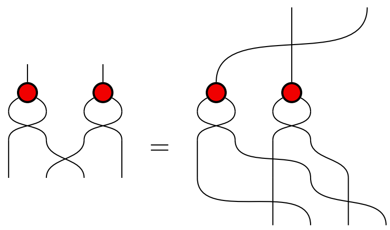


Given two Hopf algebras, H, K, with antipodes $\;_H, \;_K$, we may make the bialgebra $H \otimes K$, as defined in Definition 2.5.1, a Hopf algebra by equipping it with antipode $\;_H \otimes \;_K$.

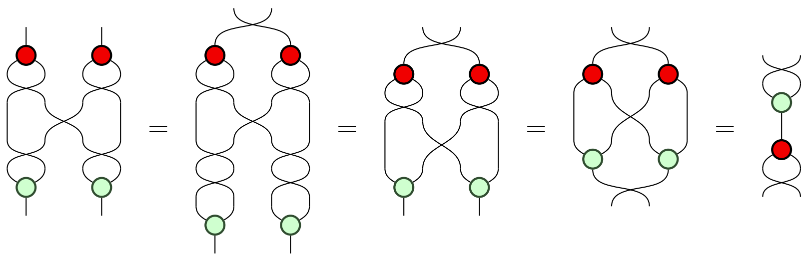
Proposition 2.5.4. *We can define another Hopf algebra H^{op} on the same object, having the same unit and counit, but the multiplication and comultiplication are composed with* *the symmetry:*

When we only replace the multiplication as above, we get a bialgebra, denoted H^σ which is not necessarily Hopf.

Proof. To prove that H^{op} is a Hopf algebra, we simply show that each of the axioms are fulfilled. We shall begin by proving that both H^{op} and H^σ are bialgebras. It is clear that H^{op} fulfils the copy, cocopy and scalar laws, so we shall only prove the bialgebra law. First off, note that

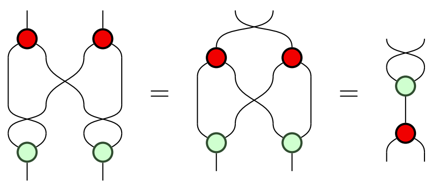


where we have simply swapped the two  morphisms. It then follows that

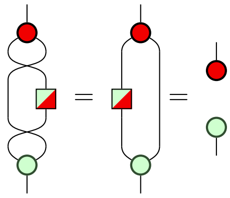


where the third equality is from swapping the two  morphisms. Hence, H^{op} fulfils the bialgebra law, and is therefore a bialgebra.

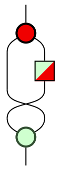
Showing that H^σ is a bialgebra is similar, as the copy, cocopy and scalar laws follow immediately. As such, we shall only prove the bialgebra law here.



To see that H^{op} is a Hopf algebra, simply see that



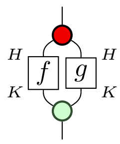
However, H^{σ} fails here, as we only get



which we cannot □

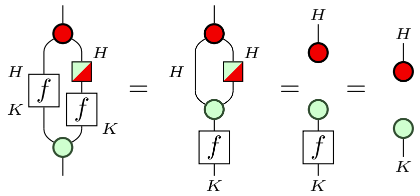
Definition 2.5.5. The *category of Hopf algebras on $\mathcal{C}$* is the category **Hopf** where the objects are Hopf algebras (H, , , , , ) and the morphisms are bialgebra homomorphisms. We show below that bialgebra morphisms are sufficient, i.e. when $f : H \to K$ is a bialgebra morphism, $f \circ$ $_H =$ $_K \circ f$.

Given bialgebras H, K, we may define a monoid structure, called the *convolution algebra*, on the homset $\mathcal{C}(H, K)$, where the multiplication and unit are respectively

$f \star g :=$ 

$1_\star :=$ 

for morphisms $f, g : H \to K$. Suppose that H and K are Hopf algebras. We find that when $f : H \to K$ is a monoid homomorphism or comonoid homomorphism, then f has an inverse when composed with the antipode. For example, when $f : H \to K$ is a monoid homomorphism, then $f \circ {}_H$ is the convolution inverse of f

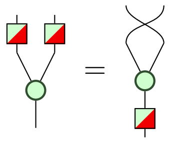


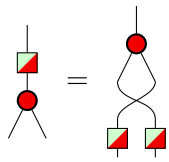
This implies that, since inverses are unique, when $f : H \to K$ is a bialgebra homomorphism, then $f \circ {}_H = {}_K \circ f$. This also tells us that $\mathbf{Hopf}(H, K)$ is a group[4].

Proposition 2.5.6. *For a Hopf algebra $H = (H, \ , \ , \ , \ , \ )$:*

1. *The antipode is unique.*

2. *$: H^{\mathrm{op}} \to H$ is a bialgebra homomorphism, i.e.*





3. *The bialgebra H^σ, as defined in Proposition 2.5.4, is a Hopf algebra if and only if is invertible, in which case the antipode of H^σ is the inverse of the antipode of H, which is denoted $^{-1}$.*

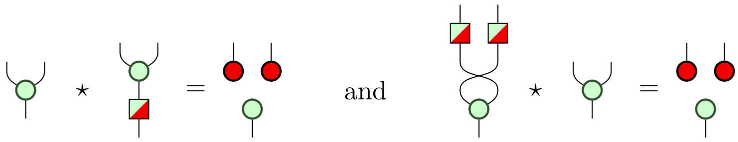
4. *If H is commutative or cocommutative then the antipode is an involution, i.e. $\circ = \mathrm{id}_H$.*

Proof. 1. This follows from the fact that the antipode is the inverse of 1_H in the convolution algebra of $\mathbf{Hopf}(H, H)$, and that inverses are unique.

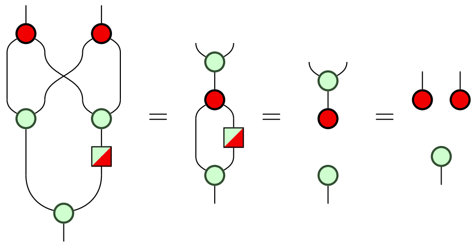
2. We will show that the antipode is a monoid homomorphism. The fact that it is a comonoid homomorphism will follow dually. To do this, we will show that both sides of the equation are inverses of the multiplication map . That is, we will

[4]This does not mean that the category is enriched in the category of groups (**Group**), and indeed, **Hopf** fails to be enriched over **Group** when the monoidal product is the Cartesian product.

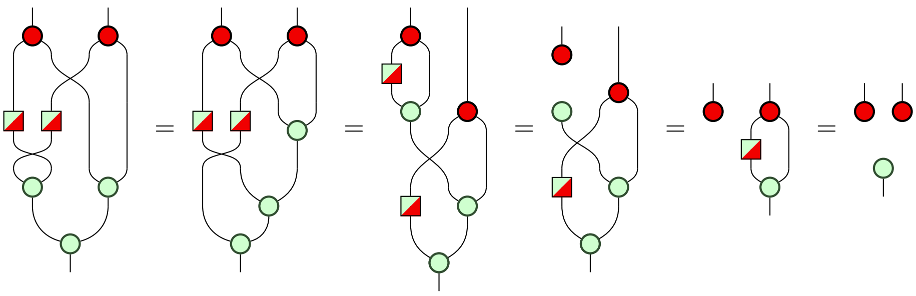
show that



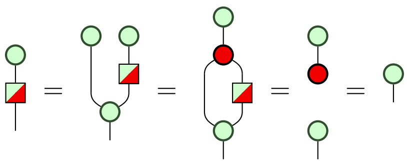
where the right hand side of each of the above equations is the convolution unit of **Hopf**$(H \otimes H, H)$. We prove that equation on the left is true as follows



To prove the equation on the right, we use associativity of the monoid, in addition to the Hopf law



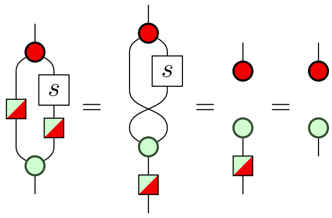
We now need to show that  $\circ$  $=$ .



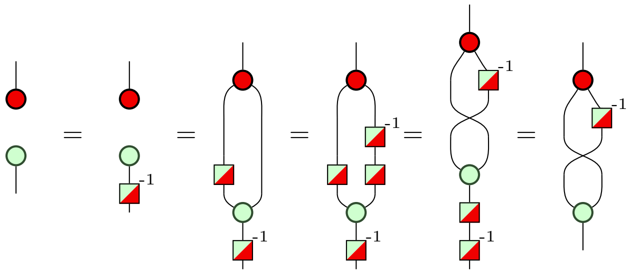
The proof for  and  are similar.

3. The bialgebra H^σ is the underlying bialgebra of H, except  is replaced with  $\circ\, \sigma_{H,H}$, where $\sigma_{H,H}$ is the symmetry natural transformation on H. Since we know the H^σ is a bialgebra, we only need to prove that the Hopf law follows for

H^σ if and only if is invertible. Suppose that H^σ has antipode s. Then



where we used the fact that the antipode is a bialgebra homomorphism. The above proof implies that $\circ\, s$ is the convolution inverse of . Since the antipode is the convolution inverse of 1_H, this implies that $\circ\, s = 1_H$. By a similar argument, we see that $s \circ = 1_H$. On the other hand, suppose that the antipode has an inverse. Then note that this implies that the antipode inverse is a bialgebra homomorphism. Then we have



Hence, we have proven our result.

4. This follows from the the previous result, as when the Hopf algebra is commutative then $H = H^\sigma$ as Hopf algebras. Since the antipode is unique, we get that the antipode is it's own inverse.

□

Example 2.5.7. In **Set**, any monoid forms a bialgebra when paired with the copy comonoid. This structure is a Hopf algebra if and only if the monoid is a group – the antipode is the operation where $g \mapsto g^{-1}$. This also clearly implies that, the free algebra defined in Example 2.3.4 is a bialgebra when paired with the copy coalgebra, defined in Example 2.3.5, and becomes a Hopf algebra if and only if it is a group algebra.

Example 2.5.8. Unlike **Set**, there are Hopf algebras in $\mathbf{FVect_k}$ which are not group algebras. An example of this is the *Taft Hopf algebra.* Let k be a field with a *primitive n^{th} root of unity*, which we denote as z. This means that there exists some element $z \in k$ such that $z^n = 1$ and $z^m \neq 1$ for $m < n$. For example, the complex numbers have a primitive 2^{nd} root of unity, i. To define the Taft Hopf algebra, $T = (T, \mu, e, \Delta, \epsilon, S)$, we must define the monoid (T, μ, e), the comonoid (T, Δ, ϵ), and the antipode $S : T \to T$.

Let T be the vector space constructed by taking the free algebra generated by $\{x, g\}$ over the field k, and quotienting it such that $x^n = 0$, $g^n = e$ and $gx = zxg$, where e is the unit of the monoid and z is the n^{th} root of unity of k. We denote this multiplication as μ. This tells us that T has a basis of elements of the form $g^a x^b$ for $a, b < n$. Hence, T is finite dimensional. The monoid (T, μ, e) is unital and associative by construction.

For the comonoid, we must define structure maps $\Delta : T \to T \otimes T$ and $\epsilon : T \to k$. We shall require Δ and ϵ to be monoid homomorphisms, as $(T, \mu, e, \Delta, \epsilon)$ will form a bialgebra. Hence, while we would typically only need to define linear maps on basis elements, it is actually sufficient to define Δ and ϵ on the generators of (T, μ, e). We define Δ and ϵ as

$$\begin{aligned} \Delta(x) &= e \otimes x + x \otimes g && \text{and} && \Delta(g) = g \otimes g \\ \epsilon(x) &= 0 && \text{and} && \epsilon(g) = 1 \end{aligned}$$

For the proof that (T, Δ, ϵ) is coassociative and counital, it is sufficient to prove it on the generators – i.e. proving, for example, that $(1 \otimes \Delta) \circ \Delta(x) = (\Delta \otimes 1) \circ \Delta(x)$. This follows from straightforward calculation.

Finally, we now need to define an antipode $S : T \to T$ such that $(T, \mu, e, \Delta, \epsilon, S)$ forms a Hopf algebra. Such an antipode is a bialgebra homomorphism $T^{op} \to T$. Hence, we only need to state the action of S on the generators of (T, μ, e). Set

$$S(x) = -xg^{-1} \qquad \text{and} \qquad S(g) = g^{-1}$$

where we note that $g^{-1} = g^{n-1}$. To prove that S satisfies the Hopf law, it is sufficient to prove that it holds for the generators. This follows as

$$
\begin{aligned}
\mu \circ (1 \otimes S) \circ \Delta(g) = gg^{-1} = 1 = g^{-1}g = \mu \circ (S \otimes 1) \circ \Delta(g) \\
\mu \circ (1 \otimes S) \circ \Delta(x) = e \otimes S(x) + x \otimes S(g) = -xg^{-1} + xg^{-1} = 0 \\
\mu \circ (S \otimes 1) \circ \Delta(x) = S(e) \otimes x + S(x) \otimes g = x - -xg^{-1}g = 0
\end{aligned}
$$

Hence, S fulfils the Hopf law, and therefore $(T, \mu, e, \Delta, \epsilon, S)$ forms a Hopf algebra.

Example 2.5.9 (Sweedler [61], pg. 89)**.** Consider a vector space A over a field k, generated by elements $1, g, g^{-1}$ and x such that 1 is the unit, and the following relations hold

$$
gg^{-1} = g^{-1}g = 1 \qquad x^2 = 0 \qquad xg = -gx
$$

This gives us an algebra on A. Note that this space has basis elements g^n and $g^n x$ for all integers n. Hence, A is infinite dimensional.

We may define a comonoid (Δ, ϵ) on A as follows:

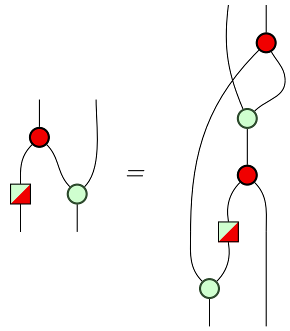
$$
\Delta(g) = g \otimes g \qquad \Delta(x) = x \otimes g + 1 \otimes x \qquad \epsilon(g) = 1 \qquad \epsilon(x) = 0
$$

Together with the monoid, this forms a bialgebra. Finally, we may define an antipode S on A, where

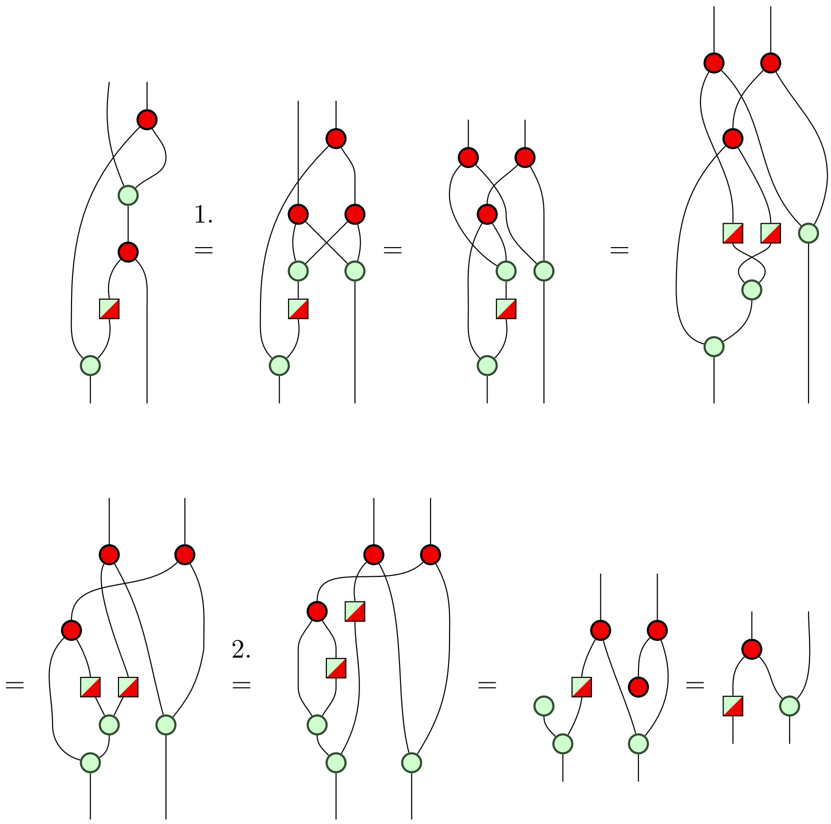
$$
S(g) = g^{-1} \qquad S(x) = g^{-1}x
$$

This then gives us a Hopf algebra.

Lemma 2.5.10. *Let $H = (H, \ , \ , \ , \ , \ )$ be a Hopf algebra. Then*



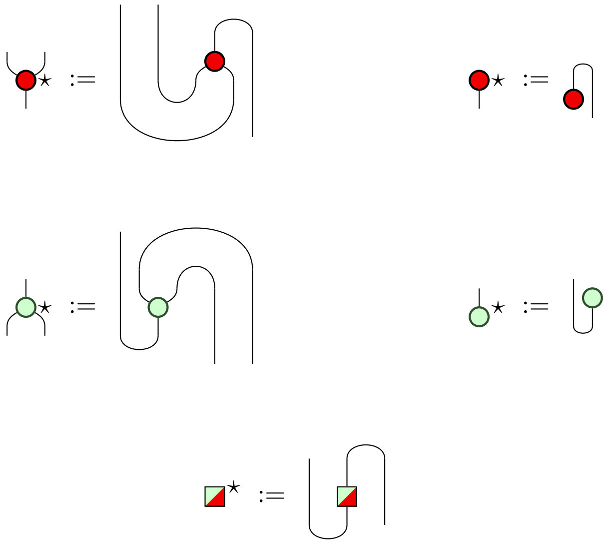
Proof. Observe that



where 1. comes from the bialgebra rule and 2. comes from the Hopf law. $\square$

Note that the above lemma does not require any additional structure upon the Hopf algebra.

Definition 2.5.11. Let $(H, \ , \ , \ , \ , \ )$ be a Hopf algebra, and suppose that the object H has a left dual H^*. We define the *dual Hopf algebra* (H^*, *, *, *, *, *) as in Proposition 2.3.12 as :



Note the above abuse of notation where, for example, * is the dual of – i.e. $^* = ( )^*$. It is clear that since the dual action $(\cdot)^*$ preserves composition that the dual Hopf algebra is still a Hopf algebra.

We shall be using the dual structure throughout this thesis, in particular in Theorem 5.2.9, where we shall use the dual Hopf algebra to construct out Hopf-Frobenius algebra. The dual structure may also be used to construct a trace, as we shall see in Lemma 4.3.7. We make the concept of a trace more concrete in the following chapter, where we shall generalise traced monoidal categories to define *traced families*.

Chapter 3

Trace

In this short chapter, we introduce the notion of a *trace*. This is usually defined in the context of a *traced monoidal category*, but in this thesis we are concerned with the case when we may not necessarily be able to define a trace on every object in the category. Hence, we introduce the notion of a *traced family*.

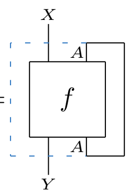
Definition 3.0.1. Given a symmetric monoidal category $\mathcal{C}$, a *pre-traced family of* $\mathcal{C}$ is a family $\mathcal{T}$ of objects of $\mathcal{C}$ such that

1. If A and B are elements of $\mathcal{T}$, then $A \otimes B$ is in $\mathcal{T}$
2. I is in $\mathcal{T}$
3. For each object A in $\mathcal{T}$, and each X, Y in $\mathcal{C}$, we may define a function

$$\mathrm{Tr}^A_{X,Y} : \hom(X \otimes A, Y \otimes A) \to \hom(X, Y)$$

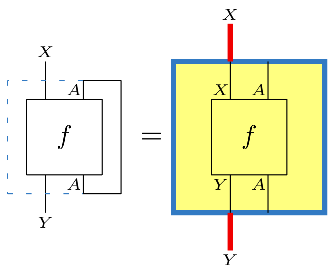
We will say that $\mathrm{Tr}^A_{X,Y}(f)$ is the *trace of f with respect to $\mathcal{T}$*, or simply the *trace of f*.

Notation 3.0.2. Given morphism $f : X \otimes A \to Y \otimes A$, we denote the trace of f, $\mathrm{Tr}^A_{X,Y}(f)$ as the string diagram

$\mathrm{Tr}^A_{X,Y}(f) :=$ 

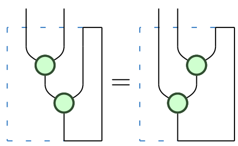
Compare this notation to the notation used by Joyal, Street and Verity [34] when defining traced monoidal categories, and the notation used by Malherbe, Scott and Sellinger [46] when defining partially traced categories.

Recall from Definition A.1.5 that string diagrams are defined not to have any cycles. If we did not have the dashed box, then our string diagram would include a cycle. Instead, we read the above diagram in the following manner: it is a string diagram with 1 vertex and 2 edges. The edges are labelled by X and Y, and the vertex is labelled *by the string diagram* of the morphism $f : X \otimes A \to Y \otimes A$. We have highlighted the different parts of the string diagram below



The edges of the string diagram are red, the vertex is the blue box, and the label of the vertex is the string diagram which is highlighted yellow. Note how the dashed lines and the border of the trace "loop" are all part of the same blue box.

Keep in mind that f in the above picture is a string diagram in $\mathrm{GR}_E(\mathcal{C})$. If the term f is equal to another term f' in $\mathcal{C}$ with respect to the equations E, then it follows that $\mathrm{Tr}^A_{X,Y}(f) = \mathrm{Tr}^A_{X,Y}(f')$, and we will rewrite the diagram to reflect this. For example, given a monoid (M,  , ), associativity of the monoid implies that we may write

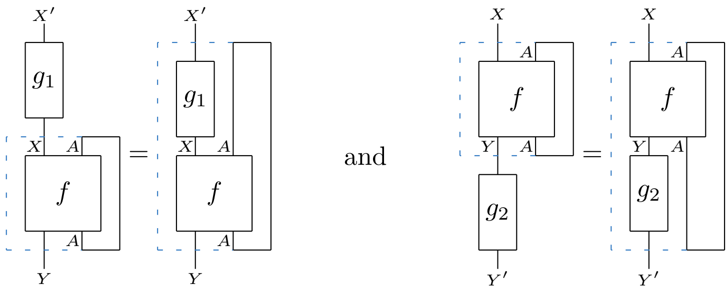


Definition 3.0.3. We say that $\mathcal{T}$ is a *traced family* when the following axioms are fulfilled. Let $A, B \in \mathcal{T}$, and $X, X', Y, Y' \in \mathcal{C}$.

($T1$) $\mathrm{Tr}^A_{X,Y}$ is natural in both X and Y. This means that, for all $g_1 : X' \to X$ and $g_2 : Y \to Y'$, we have

- $\mathrm{Tr}^A_{X,Y}(f) \circ g_1 = \mathrm{Tr}^A_{X',Y}(f \circ (g_1 \otimes 1_A))$
- $g_2 \circ \mathrm{Tr}^A_{X,Y}(f) = \mathrm{Tr}^A_{X,Y'}((g_2 \otimes 1_A) \circ f)$

This is denoted graphically as

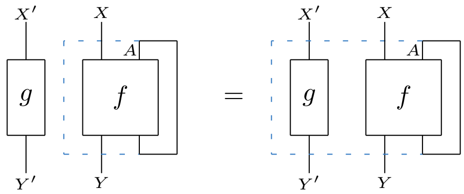


(T1)

$(T2)$ Let $g : X' \to Y'$. Then

$$g \otimes \mathrm{Tr}^A_{X,Y}(f) = \mathrm{Tr}^A_{X' \otimes X, Y' \otimes Y}(g \otimes f).$$

This is represented graphically as

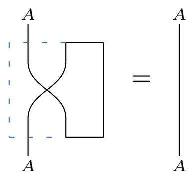


(T2)

$(T3)$ The trace respects the symmetry of $\mathcal{C}$ as

$$\mathrm{Tr}^A_{A,A}(\sigma_{A,A}) = 1_A,$$

where $\sigma_{A,A}$ is the symmetry on A. This is depicted graphically as



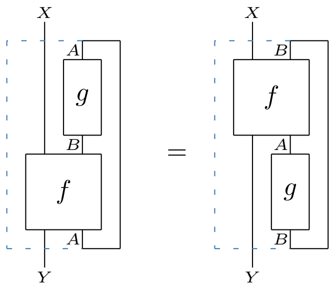
(T3)

$(T4)$ For any B in $\mathcal{T}$, let $g : A \to B$ be a morphism, and $f : X \otimes B \to Y \otimes A$ in $\mathcal{C}$.

Then

$$\mathrm{Tr}^{A}_{X,Y}(f \circ (1_X \otimes g)) = \mathrm{Tr}^{B}_{X,Y}((1_Y \otimes g) \circ f).$$

Graphically, this is depicted as

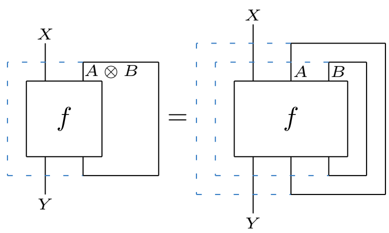


$$(T4)$$

($T5$) Suppose that B is in $\mathcal{T}$. Then $A \otimes B$ is in $\mathcal{T}$, with trace

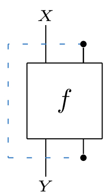
$$\mathrm{Tr}^{A\otimes B}_{X,Y} = \mathrm{Tr}^{A}_{X,Y} \circ \mathrm{Tr}^{B}_{X\otimes A.Y\otimes A}.$$

The graphical depiction of this is



$$(T5)$$

($T6$) Recall that we denote the left unitor of a monoidal category as ρ. The monoidal unit I is in $\mathcal{T}$, where the trace is defined as $f : X \otimes I \to Y \otimes I$ maps to $\rho_Y \circ f \circ \rho_X^{-1}$. We denote this as



$$(T6)$$

Remark 3.0.4. We may define a *traced monoidal category* as a symmetric monoidal category $\mathcal{C}$ equipped with a traced family that includes every object in $\mathcal{C}$. This implies that, given a symmetric monoidal category with traced family $\mathcal{T}$, the full subcategory consisting of the objects in $\mathcal{T}$ is a traced monoidal category.

Selinger in Theorem 5.22 of [55] provides a seperate graphical language for traced symmetric monoidal categories. This notation is similar to our string diagrams, except it does allow cycles. This notation effectively means that when we write string diagrams of this form, we would be able to ignore the trace axioms in a similar manner to how string diagrams allow us to ignore the symmetric monoidal category axioms. However, we will not use this notation, as it requires that we are working in a traced monoidal category. In this thesis, we will not work in such a category, as to do so would require us to assume that every object has a trace. Instead, we will stick with our definition of string diagrams given in Definition A.1.5. However, it would be likely be possible to develop a graphical language for traced families where loops are permitted, but only if they are labelled by a certain set of objects.

Remark 3.0.5. Note the similarity of the concept of a traced family and *partially traced categories* (which are themselves a categorification of traced ideals). Partially traced categories are effectively traced monoidal categories where the trace function

$$\mathrm{Tr}^A_{X,Y} : \hom(X \otimes A, Y \otimes A) \to \hom(X, Y)$$

is a partial function. The two are comparable, as a traced family defines a trace on a subset of the objects in the category.

Notation 3.0.6. Let $\mathcal{T}$ be a pre-traced family in $\mathcal{C}$. The following notation will aid us when we are proving when $\mathcal{T}$ is a traced family.

Let $A \in \mathcal{T}$. We say that $T1(A)$ holds in $\mathcal{T}$ if Axiom $T1$ holds for A. That is, $T1(A)$ holds if for all $X, X', Y, Y' \in \mathcal{C}$, all $g_1 : X' \to X$ and $g_2 : Y \to Y'$, and $f : X \otimes A \to Y \otimes A$, we have

- $\mathrm{Tr}^A_{X,Y}(f) \circ g_1 = \mathrm{Tr}^A_{X',Y}(f \circ (g_1 \otimes 1_A))$
- $g_2 \circ \mathrm{Tr}^A_{X,Y}(f) = \mathrm{Tr}^A_{X,Y'}((g_2 \otimes 1_A) \circ f)$

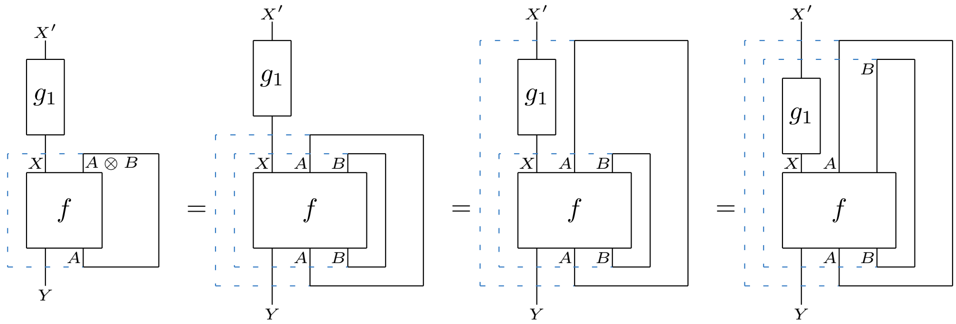
We will use similar notation for the other axioms; when Axioms $T2$ or $T3$ hold for $A \in \mathcal{T}$, we say that $T2(A)$ or $T3(A)$ respectively holds in $\mathcal{T}$. When Axioms $T4$ or $T5$

hold for $A, B \in \mathcal{T}$, then we say that $T4(A, B)$ or $T5(A, B)$ respectively holds in $\mathcal{T}$. We do not need such notation for $T6$.

Lemma 3.0.7. *Suppose that, for pre-traced family $\mathcal{T}$, and for some $A, B \in \mathcal{T}$, $T5(A, B)$ holds. Then*

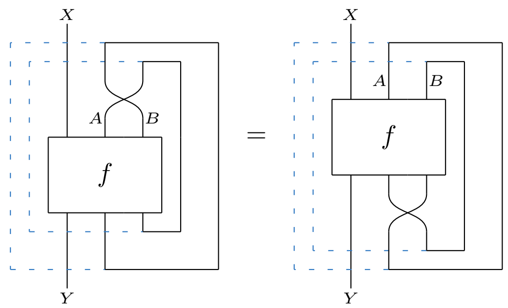
- *If $T1(A)$ and $T1(B)$ hold, then $T1(A \otimes B)$ holds*
- *If $T2(A)$ and $T2(B)$ hold, then $T2(A \otimes B)$ holds*
- *If $T3(A)$ and $T3(B)$ hold, then $T3(A \otimes B)$ holds*

Proof. We shall only prove this Lemma for $T1$, as the proof for $T2$ and $T3$ are similar.



$\square$

Definition 3.0.8. Suppose that $\mathcal{T}$ is a pre-traced family of $\mathcal{C}$. We denote the following equation as $P(A, B)$

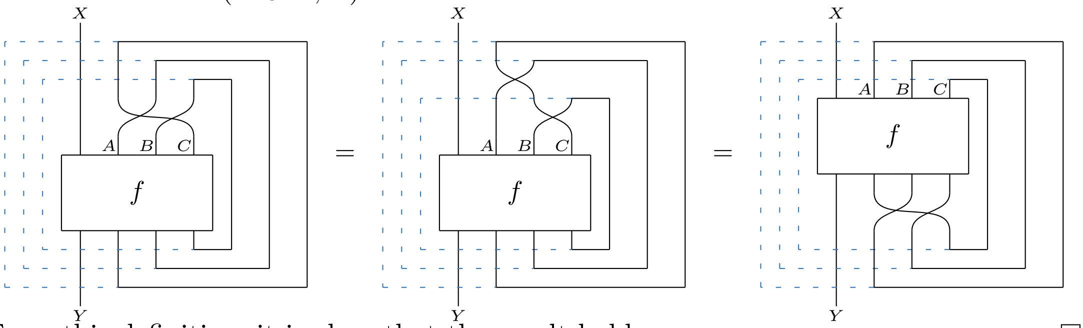


$(P(A, B))$

where $A, B \in \mathcal{T}$.

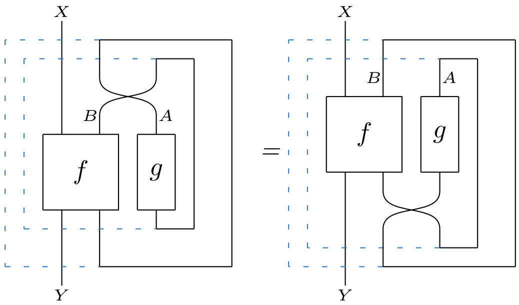
Lemma 3.0.9. *Suppose that $\mathcal{T}$ is a pre-traced family of $\mathcal{C}$ that fulfils $T1$, $T2$, $T3$ and $T5$. Let $A, B, C \in \mathcal{T}$. If $P(A, C)$ and $P(B, C)$ are true, then $P(A \otimes B, C)$ is true. Likewise, if $P(A, B)$ and $P(A, C)$ are true, then $P(A, B \otimes C)$ is true.*

Proof. Suppose that $P(A, C)$ and $P(B, C)$ are true. Since $T5$ holds, we simply consider the definition of $P(A \otimes B, C)$.



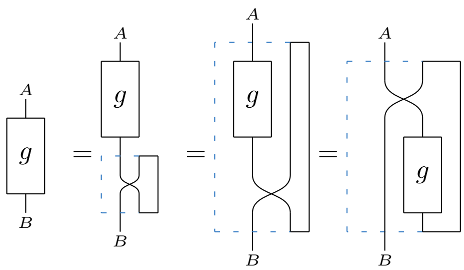
From this definition, it is clear that the result holds. $\square$

Lemma 3.0.10. *Suppose that we have a pre-traced family $\mathcal{T}$ that fulfils $T1$, $T2$ and $T3$. Let $A, B \in \mathcal{T}$. Then $T4(A, B)$ is equivalent to the following: for all $X, Y \in \mathcal{C}$, $g : X \to Y$ and $f : X \otimes B \to Y \otimes A$,*

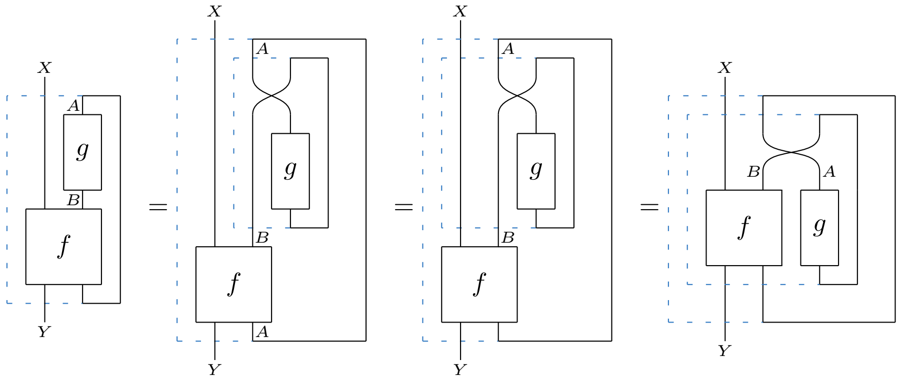


In particular, $P(A, B)$ implies $T4(A, B)$.

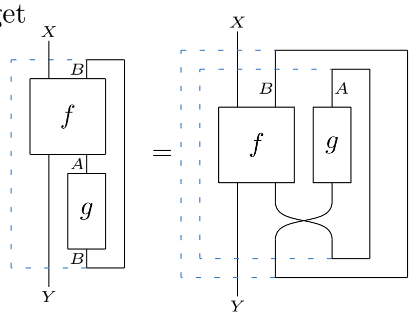
Proof. First off, note that the following identity is a result of $T1$ and $T3$



$T1$ and $T2$ then then leads us to the result that



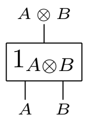
By similar reasoning, we get



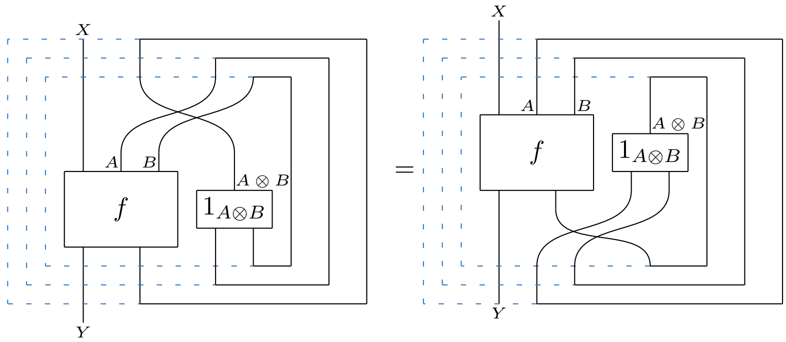
The result follows. ☐

Lemma 3.0.11. *Suppose that we have a pre-traced family $\mathcal{T}$ that fulfils $T1$, $T2$ and $T3$. Then $P(A \otimes B, A)$ and $P(A \otimes B, B)$ imply $T5(A, B)$.*

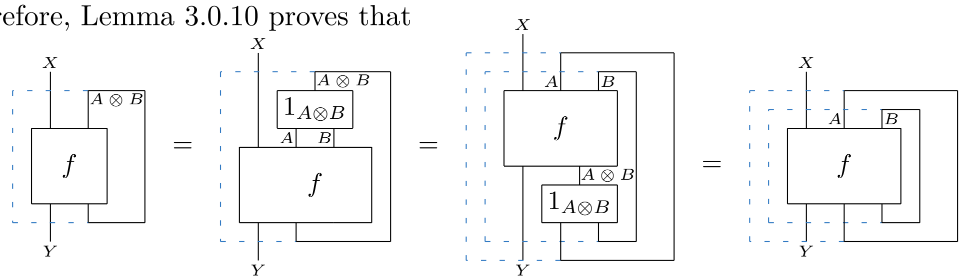
Proof. Suppose that $P(A \otimes B, A)$ and $P(A \otimes B, B)$ hold in $\mathcal{T}$ and consider the identity map, $1_{A \otimes B} : A \otimes B \to A \otimes B$. Graphically, we represent this morphism as



The reason why we are using this notation instead of the standard notation (i.e. two disconnected wires labelled with A and B) is so that we can graphically distinguish between $\mathrm{Tr}^{A \otimes B}_{X,Y}$ and $\mathrm{Tr}^{A}_{X,Y} \circ \mathrm{Tr}^{B}_{X \otimes A, Y \otimes A}$, as we shall see below. Lemma 3.0.9 tells us that



Therefore, Lemma 3.0.10 proves that



Hence, we have our result. □

Remark 3.0.12. Sprunger and Katsumata [59] define a *delayed trace*, which is a trace that drops axioms $T3$ and $T4$ to model feedback. In our case, we find that we are able to replace $T4$ and $T5$ with $\mathcal{P}$, but this requires that $T3$ holds.

Definition 3.0.13. Let $\mathcal{T}$ and $\mathcal{R}$ be traced families of category $\mathcal{C}$ with respective trace functions Tr and Tr′ such that $\mathcal{T}$ is a subset of $\mathcal{R}$, and for all $A \in \mathcal{T}$, $\mathrm{Tr}^A = \mathrm{Tr}'^A$. Then we say that the traced family $\mathcal{T}$ is a *sub-family* of $\mathcal{R}$, and we denote this as $(\mathcal{T}, \mathrm{Tr}) \subseteq (\mathcal{R}, \mathrm{Tr}')$

Lemma 3.0.14. *Let W be an object in $\mathcal{C}$ such that we may define a function*

$$\underline{\mathrm{Tr}}^W_{X,Y} : \hom(X \otimes W, Y \otimes W) \to \hom(X, Y)$$

for all $X, Y \in \mathcal{C}$ that fulfils $T1(W), T2(W), T3(W)$ and $P(W, W)$. Then

1. *We may define a traced family, $\mathcal{T}_W$ with trace* Tr *such that $\underline{\mathrm{Tr}}^W = \mathrm{Tr}^W$*
2. *If there exists some other traced family $\mathcal{R}$ with trace* Tr′ *such that $A \in \mathcal{R}$ and $\underline{\mathrm{Tr}}^W = \mathrm{Tr}'^W$, then $(\mathcal{T}_W, \mathrm{Tr}) \subseteq (\mathcal{R}, \mathrm{Tr}')$*

We say that Tr_W *is the* minimal traced family *of W.*

Proof. Let us inductively define a family of objects of $\mathcal{C}$, denoted as $\mathcal{T}_W$, as

- $W \in \mathcal{T}_W$
- $I \in \mathcal{T}_W$
- If $A, B \in \mathcal{T}_W$, then $A \otimes B \in \mathcal{T}_W$

We shall begin by showing that $\mathcal{T}_W$ is a pre-traced family, then proving that it fulfils the trace axioms (Definition 3.0.3). We define $\mathrm{Tr}^W_{X,Y}$ as $\underline{\mathrm{Tr}}^W = \mathrm{Tr}^W$, and $\mathrm{Tr}^I_{X,Y}$ as in Axiom $T6$ of the definition of a traced family. For $A, B \in \mathcal{T}_W$, $\mathrm{Tr}^{A \otimes B}_{X,Y}$ is defined as in Axiom $T5$. In other words,

- Given $f : X \otimes W \to Y \otimes W$, $\mathrm{Tr}^W_{X,Y}(f) := \underline{\mathrm{Tr}}^W_{X,Y}(f)$
- Given $f : X \otimes I \to Y \otimes I$, $Tr^I_{X,Y}(f) = \rho_Y \circ f \circ \rho_X^{-1}$
- Given $f : X \otimes A \otimes B \to Y \otimes A \otimes B$, such that $A, B \in \mathcal{T}_W$ and both Tr^A and Tr^B are defined, then $\mathrm{Tr}^{A \otimes B}_{X,Y} = \mathrm{Tr}^A_{X,Y} \circ \mathrm{Tr}^B_{X \otimes A . Y \otimes A}$

Therefore, $\mathcal{T}_W$ is a pre-traced family.

Now we need to show that $\mathcal{T}_W$ fulfils the trace axioms. It is clear that it fulfils Axioms $T5$ and $T6$ by construction. Note that $T1(I)$, $T2(I)$ and $T3(I)$ all hold. The axioms $T1(W)$, $T2(W)$ and $T3(W)$ all hold by assumption. Lemma 3.0.7 tells us that, since Axiom $T5$ holds, and $T1$, $T2$ and $T3$ hold for both I and W, then $T1$, $T2$ and $T3$ all hold for every object in $\mathcal{T}_W$.

Hence, we now only need to prove that $T4$ holds. Recall the definition of $P(A, B)$ from Lemma 3.0.9. Lemma 3.0.10 tells us that $P(A, B)$ implies $T4(A, B)$, so we shall focus on proving $P(A, B)$ for all $A, B \in \mathcal{T}_W$.

To prove that $P(A, B)$ hold for all A and B, we note that since each $A \in \mathcal{T}_W$ is generated by W, I and $\otimes$, we may proceed by induction on the generators. We know that $P(W, W)$ by assumption, and it is clear from $T6$ that $P(A, I)$ and $P(I, A)$ hold for all $A \in \mathcal{T}_W$. This tells us that, given some $A \in \mathcal{T}_W$ such that $P(A, W)$ holds, it follows from Lemma 3.0.9 that $P(W \otimes A, W)$ and $P(I \otimes A, W)$ hold. Hence, by induction, and the definition of $\mathcal{T}_W$, $P(A, W)$ holds for all $A \in \mathcal{T}_W$. Likewise, if $P(A, B)$ holds for some $B \in \mathcal{T}_W$, then $P(A, W \otimes B)$ and $P(A, I \otimes B)$ hold. Hence, $P(A, B)$ holds for all $A, B \in \mathcal{T}_W$. Hence, by Lemma 3.0.10, $\mathcal{T}_W$ must fulfil $T4$. Hence, $\mathcal{T}_W$ fulfils all of the trace axioms, and is therefore a traced family.

For part 2, we see that since $W \in \mathcal{R}$, each of the the elements of $\mathcal{T}_W$ are contained within $\mathcal{R}$ by construction. Hence, $\mathcal{T}_W$ is a subset of $\mathcal{R}$.

We also see that since

- $\mathrm{Tr}^W = \underline{\mathrm{Tr}}^W = \mathrm{Tr}'^W$
- Tr^I and Tr'^I are both determined by Axiom $T6$

we only need to prove that $\mathrm{Tr}^{A \otimes B} = \mathrm{Tr}'^{A \otimes B}$ for $A, B \in \mathcal{T}_W$. However, this follows immediately from Axiom $T5$. Hence, $\mathrm{Tr}^A = \mathrm{Tr}'^A$ for all $A \in \mathcal{T}_W$, and therefore $(\mathcal{T}_W, \mathrm{Tr}) \subseteq (\mathcal{R}, \mathrm{Tr}')$ □

Developing a language that allows us to talk about the trace of a morphism in a category that is not traced monoidal will give us a straightforward language to talk about traces. We will now move on to the main topic of this thesis.

Chapter 4

Integrals

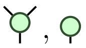
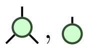
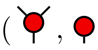
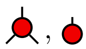
We begin this section by formally defining Hopf-Frobenius algebras and providing familiar examples. In the following section, we will focus on the conditions for when a Hopf algebra is Hopf-Frobenius, so this section will be focussed on introducing concepts that we shall need later on. Primary among these concepts are integrals and cointegrals, which are a type of copoint and point respectively. When a Hopf algebra is equipped with an integral and cointegral, then it resembles a Frobenius algebra. Specifically, every Frobenius algebra is equipped with a comonoid that fulfils the Frobenius law (Definition 2.4.1). We see that the presence of an integral implies the presence of a comultiplication that obeys the Frobenius law, but not necessarily a counit (Lemma 4.2.12). In addition, a Frobenius algebra is equipped with a cup and a cap, making it self-dual (Remark 2.4.4). The presence of an integral gives us a *half-dual* – a cup and cap that are only one sided (Corollary 4.3.5). While strictly weaker than a standard dual, it is sufficient to define a trace on H. Our final concept is the *Integral morphism*. This is a morphism with type $\mathcal{I} : H \to H$ on a Hopf algebra H, that behaves like an integral and a cointegral. This is despite the fact that it is neither a point, nor a copoint. We may use the integral morphism to map any point or copoint to a cointegral or integral respectively. It essentially acts as a “projection” to the “space” of integrals and cointegrals. Because of this, the integral morphism is instrumental in informing us about the Frobenius structure of a Hopf algebra.

Throughout this chapter, we will be using concepts and results that the author of this thesis originally stated in [20]. In particular, the definitions of Hopf-Frobenius algebras (Definition 4.1.1), integral Hopf algebras (Definition 4.2.3), half-duals (Definition 4.3.3), nondegeneracy (Definition 5.1.1) and the integral morphism (Definition 4.4.1) were all defined previously. The results of Lemma 4.3.1, Lemma 5.2.1, Lemma 5.2.5, Theorem 5.2.6, Theorem 5.2.8, Theorem 5.2.9 and Lemma 5.2.15 were all proven previously.

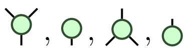
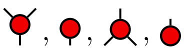
4.1 Hopf-Frobenius Algebras

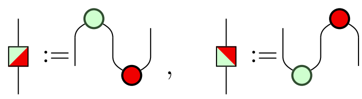
The main topic of this thesis is the subject of *Hopf-Frobenius algebras*. We covered this in the introduction, but now we introduce the concept concretely.

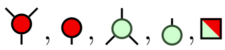
Definition 4.1.1. Let $\mathcal{C}$ be a strict symmetric monoidal category. A *Hopf-Frobenius Algebra*, or *HF-Algebra* on $\mathcal{C}$ is an algebraic structure consisting of

- An object H in $\mathcal{C}$
- a green monoid (), a green comonoid (), a red monoid (), and a red comonoid ()

We require the above structure to have the following properties

- () and () are Frobenius algebras
- When we set



then () and () are Hopf algebras

The algebraic structures that we referenced in the introduction, interacting Frobenius algebras [26] and interacting Hopf algebras [13], are both instances of this structure, wherein each monoid and comonoid is commutative and both Frobenius algebras are special.

Example 4.1.2. Consider a finite group algebra G over field k, with generating group $\overline{G} = (\overline{G}, \mu, e)$. Recall from Example 2.4.5 that we may define a Frobenius algebra on G with the group coalgebra, and from Example 2.5.7 that we may define a Hopf algebra on

G with the copy coalgebra. In addition, recall from Example 2.4.6 that the copy algebra and coalgebra form a Frobenius algebra. We now summarise each of these algebraic structures.

The *group algebra* $(G, \ , \ )$ is defined on the basis $\overline{G}$. It is simply the underlying group multiplication where $g \otimes h \mapsto gh$, and the unit of $\overline{G}$ is the unit e of G.

This is paired with the *group coalgebra* $(G, \ , \ )$, where

$$:= g \mapsto \sum_{ab=g} a \otimes b = \sum_{h \in \overline{G}} h \otimes h^{-1} g$$

with counit

$$:= g \mapsto \begin{cases} 1 & \text{if } g = e \\ 0 & \text{otherwise.} \end{cases}$$

where 1 is the unit of the field k, to give us our *green Frobenius algebra* $(G, \ , \ , \ , \ )$.

The group algebra may be paired with the *copy coalgebra*, defined in Example 2.3.5, which we denote as $(G, \ , \ )$. This is defined as $:= g \mapsto g \otimes g$ and $:= g \mapsto 1$ for all $g \in \overline{G}$, where 1 is the unit of the field k. When we pair the group algebra and the copy coalgebra together, we get our *green Hopf algebra*, $(G, \ , \ , \ , \ , \ )$, where $: G \to G$ is the inverse operation, $:= g \mapsto g^{-1}$, on the elements of $\overline{G}$.

The copy coalgebra may be paired with the *copy algebra*, $(G, \ , \ )$, as defined in Example 2.3.5. Recall that the multiplication is defined

$$:= g \otimes h \mapsto \begin{cases} g & \text{if } g = h \\ 0 & \text{otherwise.} \end{cases}$$

for all $g \in \overline{G}$. The unit is the following element of G

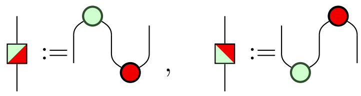
$$:= \sum_{g \in \overline{G}} g.$$

When we pair this monoid with the copy coalgebra, we get our *red Frobenius algebra*,

$(G,$, , , $)$.

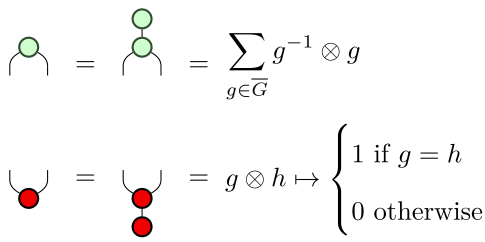
Finally, pairing the copy algebra with the group coalgebra, we get our red Hopf algebra, $(G,$, , , , $)$, where the antipode, , is the inverse operation $g \mapsto g^{-1}$.

Recall that, in the definition of Hopf-Frobenius algebras, we require that the antipodes have the form

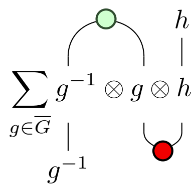


To prove that this group algebra structure is Hopf-Frobenius, we must show that the antipode has this form. We shall show this for , as the proof for is similar.

Recall that the green cap and the red cup are defined as



Hence, the composition of the two is

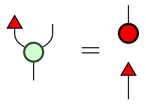


so this is simply the group's inverse operation.

4.2 Integrals and Cointegrals

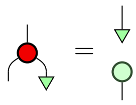
As we saw above, every group algebra is a Hopf-Frobenius algebra. This motivates us to ask the following question – given a Hopf algebra, H, on an arbitrary symmetric monoidal category, when is it a Hopf-Frobenius algebra? To answer this we introduce the dual concepts of *integrals* and *cointegrals*.

Definition 4.2.1. A *left cointegral* on H is a point  $: I \to H$, satisfying the equation:



A *right cointegral*  is defined similarly.

Definition 4.2.2. A *right integral* on H is a copoint  $: H \to I$, satisfying the equation:



A *left integral*  is defined similarly.

We provide special attention to the above definitions. Note that in $\mathbf{Mod_R}$ and $\mathbf{Vect_k}$, the zero map, 0, is trivially a cointegral. As we shall see in Lemma 4.2.12 the presence of a cointegral is the same as a comultiplication that fulfils the Frobenius law. In fact, we will see later in Lemma 5.1.2 the conditions under which these two coincide. We will also see that the cointegral functions as a unit for the red multiplication.

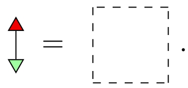
Larson and Sweedler [41] have previously showed that every finite dimensional Hopf algebra has an associated Frobenius algebra. The space of integrals of H formed a central part of their proof. This was later expanded upon by Paregis [49], who provided the conditions under which a Hopf algebra in $\mathbf{FPMod_R}$ (Example 2.3.18) is Frobenius.

Both of their proofs follow a similar structure. Recall that $\mathbf{FVect_k}$ and $\mathbf{FPMod_R}$ are compact closed, so each object has a dual. In addition, recall from Definition 2.4.7 that one condition for a Frobenius algebra is that it is self-dual, such that the dual structure is associative. To show that H is self-dual, we must merely provide an isomorphism $H \cong H^*$.

Larson and Sweedler show that every Hopf algebra H may be equipped with an isomorphism $H \cong H^* \otimes P(H)$, where H^* is the dual space of H, and $P(H)$ is the space of integrals on H. Hence, when $P(H)$ is isomorphic to the monoidal unit, we have our self dual structure. Larson and Sweedler showed that for every Hopf algebra in $\mathbf{FVect_k}$, $P(H)$ is isomorphic to the underlying field, and that the self dual structure that this provides is associative.

Clearly, this condition is not sufficient for us, as we cannot define the space of integrals in an arbitrary symmetric monoidal category. Though we do define a set analogous to this in Remark 4.4.5. We shall be focussing on the integrals themselves and their properties. With that in mind, consider the following definition.

Definition 4.2.3. A *left-right integral Hopf algebra* or *LR integral Hopf algebra* $(H, \uparrow, \downarrow)$ is a Hopf algebra H equipped with a choice of left cointegral $\uparrow$, and right integral $\downarrow$, such that



An *RL integral Hopf algebra* or *right-left integral Hopf algebra* is defined in a similar manner, except we have a right cointegral and left integral.

Remark 4.2.4. The above requirement that $\downarrow \circ \uparrow = 1_I$ rules out the possibility that either the integral or cointegral are 0 in $\mathbf{Mod_R}$ or $\mathbf{Vect_k}$ (except for the 0 ring).

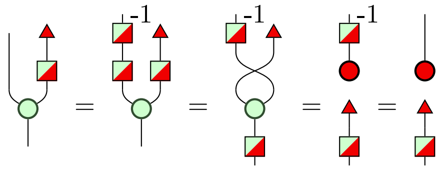
Definition 4.2.5. Unless we state otherwise, we shall only talk about LR integral Hopf algebras, and refer to them simply as *integral Hopf algebras.* This is because the distinction between LR and RL integral Hopf algebras is relatively minor. We understand intuitively that there is a sense in which every theorem about LR integral Hopf algebras may be restated in terms of RL integral Hopf algebras: we may take any statement in terms of LR integral Hopf algebras, flip it along the vertical axis and replace left cointegrals (resp. right integrals) with right cointegrals (resp. left integrals). We state this more concretely in the appendix, in Section A.2. As such, every theorem that we prove about LR integral Hopf algebras is also a theorem about RL integral Hopf algebras. As follows, there is another sense in which every LR integral Hopf algebra gives us an RL integral Hopf algebra

Lemma 4.2.6. *Let H be an LR integral Hopf algebra, and suppose that the antipode has an inverse. Consider the following morphisms*

These morphisms are a right cointegral and a left integral respectively. The same holds for the inverse of the antipode.

Proof. We shall only prove this for the cointegral and the antipode, as the rest of the proof follows similarly.

This follows from the fact that the antipode is a homomorphism.



□

Corollary 4.2.7. *Given an LR integral Hopf algebra, we may construct an RL integral Hopf algebra, with integral/cointegral pair*

Consider the following examples. We shall denote the left cointegral as Λ, and the right integral as $\int : H \to I$.

Example 4.2.8. Consider a finite group algebra G, with generating group $\overline{G}$. A left cointegral on G would be an element $\Lambda \in G$ such that $\Lambda g = \Lambda$ for all g. A candidate for this is

$$\Lambda := \sum_{g \in \overline{G}} g$$

Likewise, an integral on G is a linear map $\int : G \to k$ such that

$$\int (g) \otimes g = \int (g) e$$

for all g[1]. Hence we set $\int (e) = 1$ and $\int (g) = 0$ otherwise.

We note that since $\int (\Lambda) = 1$, this choice of integral and cointegral gives us an integral Hopf algebra.

[1]Note that since $\int (g)$ is an element of k, we are abusing the notation here - since $\int (g) \otimes g = 1 \otimes \int (g) g$

Note also that the integral and cointegral are the counit and unit respectively that we defined in Example 4.1.2.

Example 4.2.9. Recall that we defined the Taft Hopf algebra T in Example 2.5.8. For the Taft Hopf algebra, we have left cointegral $\Lambda = \sum_{i=1}^{n} z^{-i} g^i x^{n-1}$ and the right integral is the map where $\int(x^{n-1}) = 1$ and every other basis element maps to 0. Again, these compose together to give us $\int(\Lambda) = 1$.

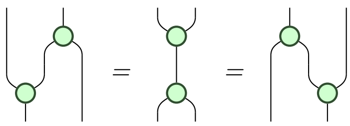
Example 4.2.10. Recall from Example 2.5.9 that we defined a Hopf algebra A that was infinite dimensional, and that it has basis g^n and $g^n x$ for all integers n. Consider the linear map $\int : A \to k$ that maps x to 1 and every other basis element to 0. Since we get

$$(1 \otimes \int) \circ \Delta(g) = 0 \qquad \textit{and} \qquad (1 \otimes \int) \circ \Delta(x) = 1$$

it follows that this is therefore a right integral.

Remark 4.2.11. In the literature, the term *co-Frobenius* is used to describe a Hopf algebra with an integral, but not necessarily a cointegral (we refer to Theorem 3 of Lin [9]).

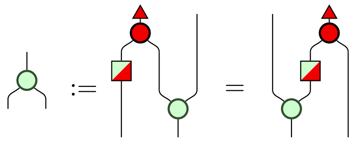
Lemma 4.2.12. *A Hopf algebra* $(H,$, , , , $)$ *has a left cointegral* *if and only if there exists a comultiplication map* *such that it fulfils the Frobenius law:*



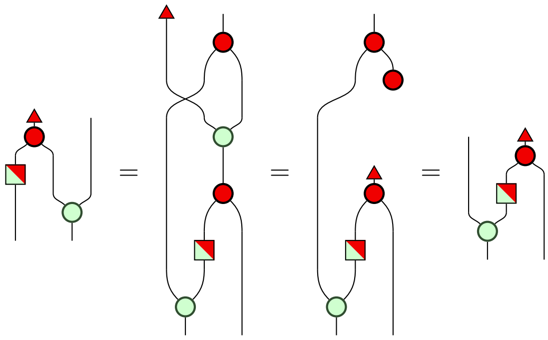
and *is coassociative.*

Remark 4.2.13. Note that the above lemma is not saying that a Hopf algebra with a left cointegral is a Frobenius algebra. The comultiplication given by the cointegral does not necessarily have a counit , and therefore it does not give us a comonoid.

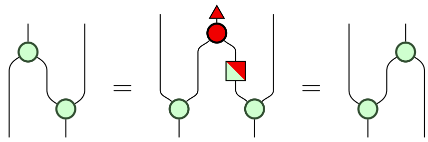
Proof. Suppose that H has a left cointegral,  , and we set



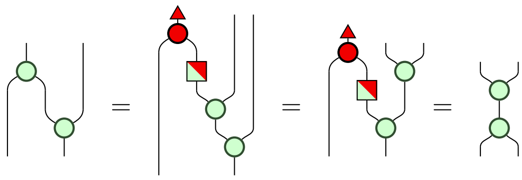
We use the Lemma 2.5.10, and we get



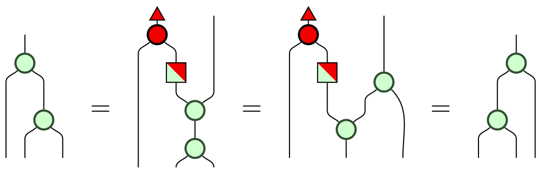
where we have also used the definition of the cointegral. This relation then implies the Frobenius law



and

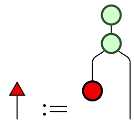


All that remains to be proven is that  is coassociative. This follows from the Frobenius law

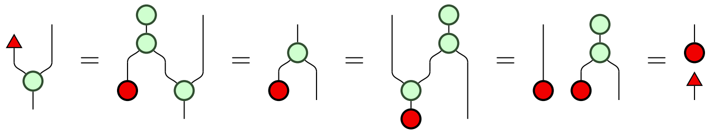


Hence we see that the presence of a cointegral allows us to construct the comultiplication.

For the other direction, suppose that we have a comultiplication map  that fulfils the Frobenius law and is coassociative. We set

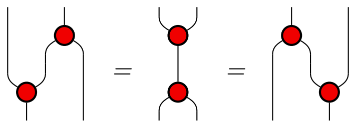


It follows that this is a left cointegral



and we have proven our result. $\square$

Corollary 4.2.14. *A Hopf algebra* $(H,$ , , , , $)$ *has a right integral*  *if and only if there exists a multiplication map*  *such that it fulfils the Frobenius law:*



and  *is associative.*

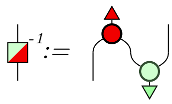
Example 4.2.15. Consider the infinite dimensional Hopf algebra that we have defined previously, in Example 2.5.9 and Example 4.2.10. Recall that the basis elements are of the form g^n and xg^n, where n is any positive or negative integer. This has an integral, so according to the above Corollary, we must be able to define another multiplication on H that is associative and fulfils the Frobenius law. When we follow the above construction, we get the following map $\bullet : A \otimes A \to A$.

$$
\begin{aligned}
&g^n \bullet g^m = 0 && xg^n \bullet g^m = g^n \text{ if } n = m, \text{ otherwise } 0 \\
&g^n \bullet xg^m = g^n \text{ if } n = m+1, \text{ otherwise } 0 && xg^n \bullet xg^m = xg^n \text{ if } n = m, \text{ otherwise } 0
\end{aligned}
$$

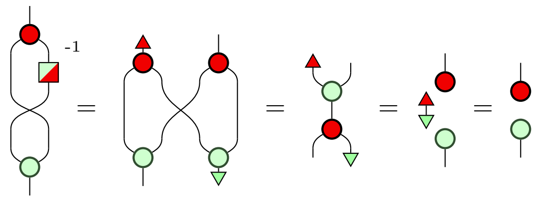
4.3 Integrals and the Trace

Our goal is to produce several equivalent conditions for a Hopf algebra to be Hopf-Frobenius. In the following chapter, we will be providing a condition that implies that we may equip our Hopf algebra with a Frobenius algebra, and it requires that we may define a trace on H, as in Definition 3.0.3. Below, we show that when we have an integral Hopf algebra we may construct a unique trace on H.

Lemma 4.3.1. *Let* $(H,$ , $)$ *be an integral Hopf algebra. Then the following map is the inverse of the antipode.*



Proof. From the definition of $^{-1}$, we see that

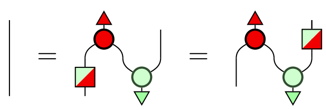


Recall from Proposition 2.5.4 the definition of H^σ. The above diagram implies that H^σ is a Hopf algebra with $^{-1}$ as the antipode. Proposition 2.5.6 tells us that the antipode of H^σ is the inverse of the antipode of H.

□

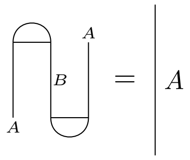
Remark 4.3.2. It was known from the inception of the concept of the integral [41] that when a Hopf algebra has an integral and cointegral pair, the Hopf algebra's antipode is an isomorphism. The above lemma tells us that the antipode inverse has a form  . We see something similar in ZX calculus [66] and in interacting Frobenius algebras [26]. The fact that we can construct the antipode inverse in this manner implies that when you have a bialgebra with an integral and cointegral pair, then you have a Hopf algebra.

Note that the above lemma implies that



Note how similar this is to duals (see Definition 2.3.7). This motivates us to weaken the definition of a dual as follows.

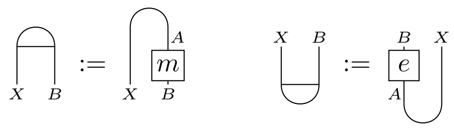
Definition 4.3.3. Let A and B be objects in a symmetric monoidal category $\mathcal{C}$. A is a *right half dual of* B if there exists morphisms $\cap : I \to A \otimes B$ and $\cup : B \otimes A \to I$ which satisfy the following equation



In this circumstance, B is a *left half dual of* A.

Compare the above definition with Definition 2.3.7 of duals. To make the distinction between duals and half duals more clear, we will sometimes refer to duals as *full duals*. Every full dual is trivially an example of a half dual, but not necessarily the other way around.

Example 4.3.4. Let X be a right full dual to A such that there exists morphisms $m : A \to B$ and $e : B \to A$ such that $e \circ m = 1_A$. Then we show that X is a right half dual to B, but not necessarily a full dual, by defining the cup and the cap in the following manner.

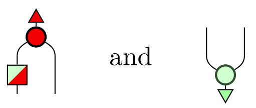


Clearly, X is only a full dual to B if m and e are mutually inverse.[2]

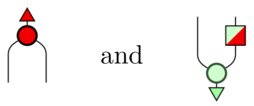
This definition motivates the following Lemma

Lemma 4.3.5. *Let* $(H, \uparrow, \downarrow)$ *be an integral Hopf algebra. Then* H *is half dual to itself.*

Proof. There are two ways that we may assign a half dual structure to H. We may define the cup and cap as



or we could set them as

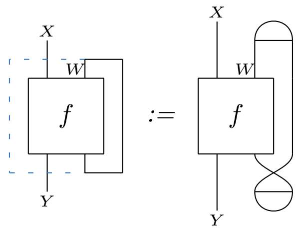


[2]Note that our primary example of a half-dual throughout this thesis will be the half dual constructed from an integral Hopf algebra, which, unlike this example, is not constructed via a retract of a full dual.

These follow from Lemma 4.3.1. □

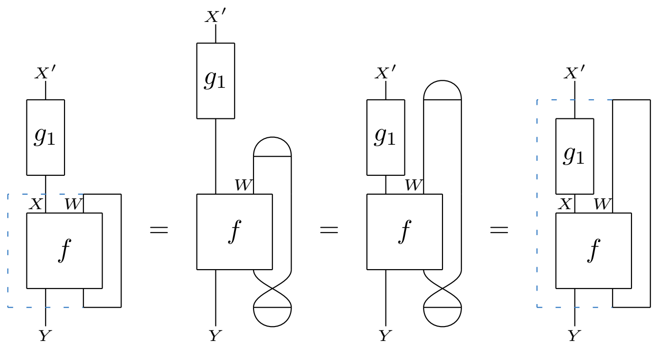
Definition 4.3.6. We say that an object A *has a trace* when there exists a traced family $\mathcal{T}$ where A is a member of $\mathcal{T}$. We say that the trace of A is unique when we can show that, for all X, Y in $\mathcal{C}$, the function $\mathrm{Tr}^A_{X,Y}$ is the same for every traced family that A belongs to.

Lemma 4.3.7. *Suppose that W has a left half dual W^*, with cup $d : W \otimes W^* \to I$ and cap $e : I \to W^* \otimes W$. Then W has a trace, defined as*

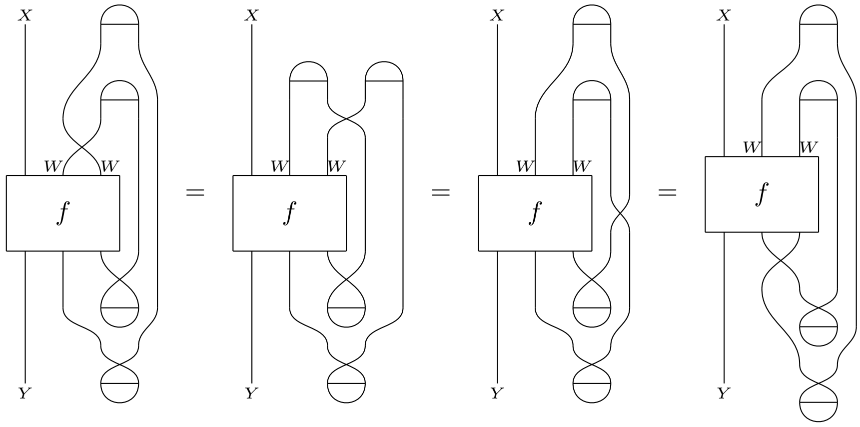


Proof. As per Definition 4.3.6, we say that W has a trace when there exists some traced family that contains W. We claim that W fulfils the conditions for a minimal traced family, as in Lemma 3.0.14. We define $\mathrm{Tr}^W_{X,Y}$ as above, and we now only need to show that it fulfils $T1(W), T2(W), T3(W)$ and $P(W, W)$.

These all follow fairly straightforwardly. We shall prove $T1(W)$, as $T2(W)$ and $T3(W)$ follow similarly



We can then show that $P(w, W)$ holds

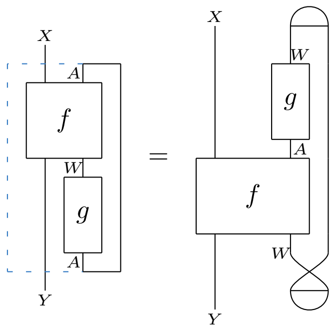


Hence, W has a minimal traced family.

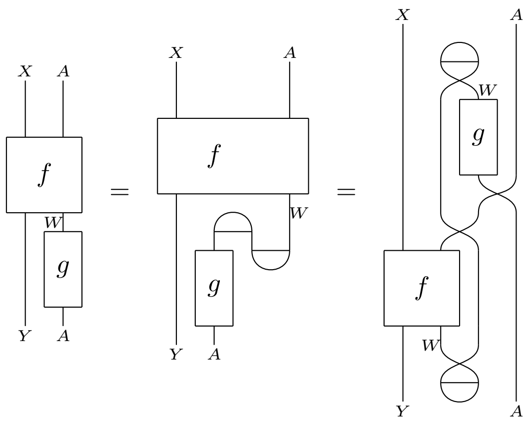
$\square$

Lemma 4.3.8. *Suppose that W has a left half dual W^*, with cup $d : W \otimes W^* \to I$ and cap $e : I \to W^* \otimes W$. The trace defined in Lemma 4.3.7 is unique.*

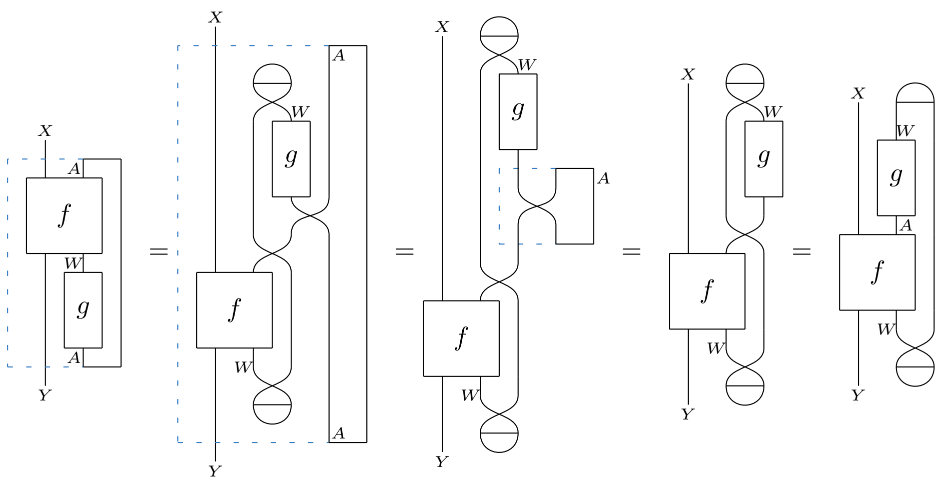
Proof. Suppose that there is some traced family $\mathcal{T}$ that W is in, with trace Tr^W. To prove that the trace of W is unique, we shall show that the half-dual trace defined as above is equal to Tr^W. We shall actually prove a stronger result, as follows: for all $A \in \mathcal{T}$, $X, Y \in \mathcal{C}$, and $g : W \to A$ and $f : X \otimes A \to Y \otimes W$, we have



Let $A \in \mathcal{T}$ and $g : W \to A$. Let us first state the following



This follows from graphical reasoning and the definition of the half dual. We then use the above identity and $T1$, $T2$ and $T3$.



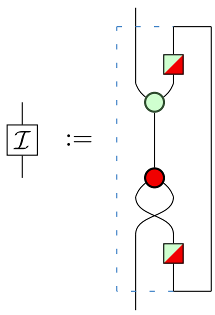
Then, to get our result, we shall only need to set $A = W$ and $g = 1_W$. Hence, we have proven our result.

□

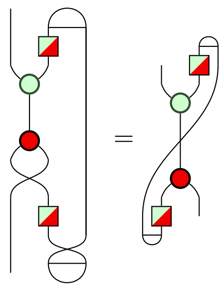
Remark 4.3.9. Compare the above results to the result in the original paper on traced monoidal categories by [34], where it is proven that every compact closed category is traced monoidal.

4.4 Integral Morphism

Definition 4.4.1. Let the object H have a trace. The *integral morphism*, denoted $\mathcal{I} : H \to H$, is defined as shown below.



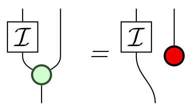
Note that when H has a half dual, then we may draw $\mathcal{I}$ as



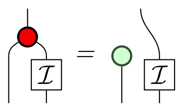
Remark 4.4.2. The integral morphism was constructed from the isomorphism in the fundamental theorem of Hopf Modules (see Larson and Sweedler [41]). However, the two are not equivalent, and their definitions differ considerably.

We defined in our original paper [20] in Definition 3.10, and afterwards we found that this exact morphism was used previously (See Fig. 3 of [10]). The following Lemmas were all discovered independently.

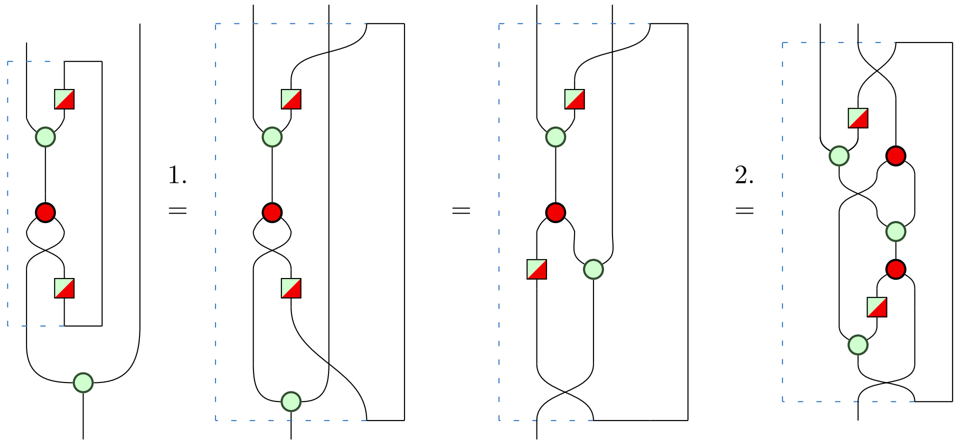
Lemma 4.4.3. *Let H be a Hopf algebra such that H is traced. Then*



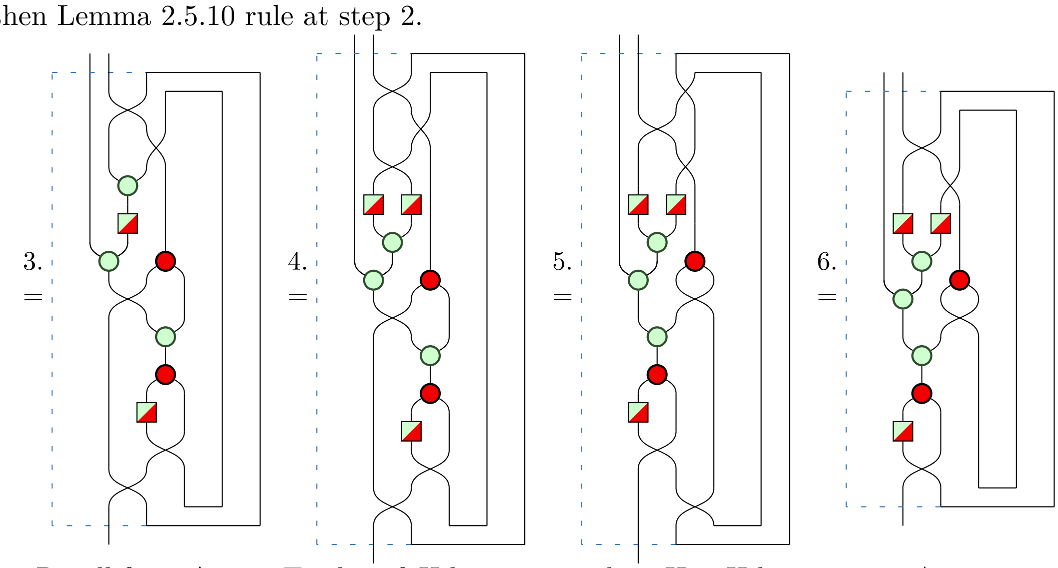
and



Proof. we compose $\mathcal{I}$ with  and use Lemma 2.5.10.

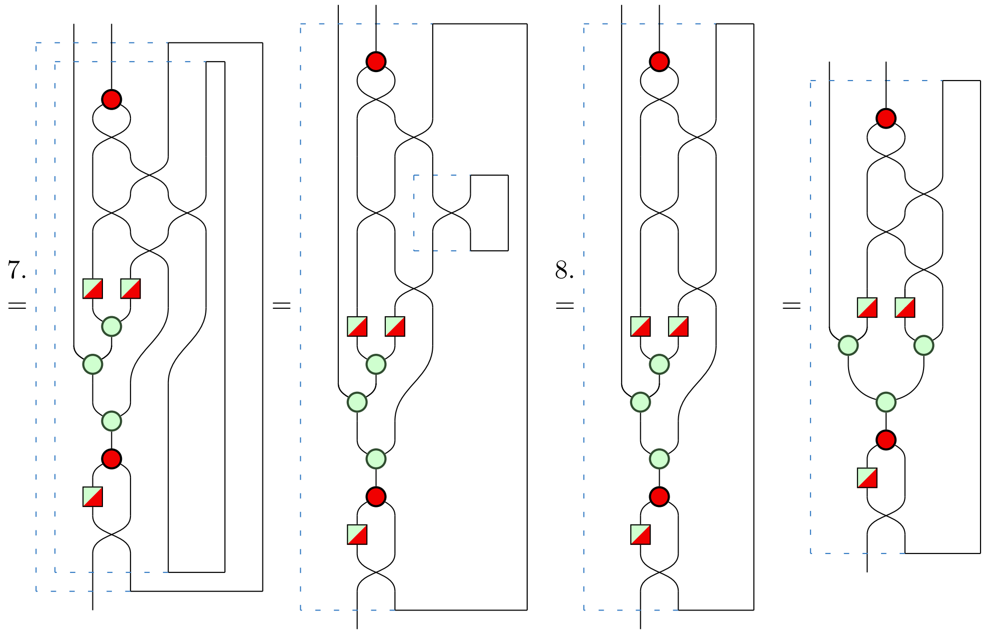


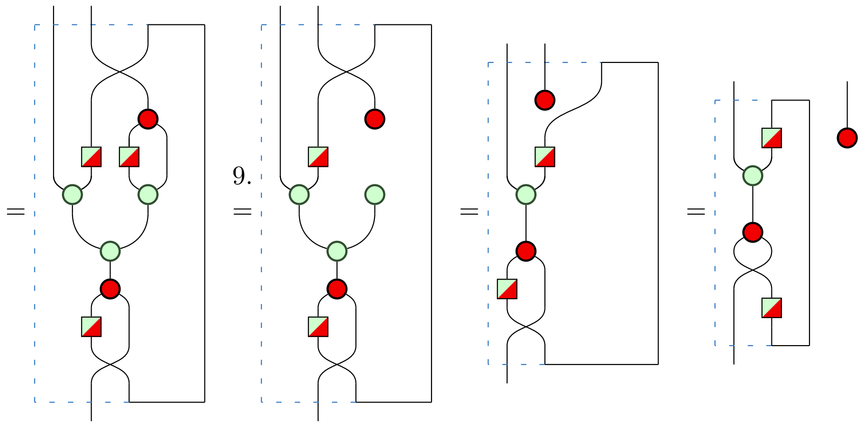
After composing $\mathcal{I}$ with , we use the trace axioms at step 1. (Definition 3.0.3), and then Lemma 2.5.10 rule at step 2.



Recall from Axiom $T5$ that if H has a trace then $H \otimes H$ has a trace. At step 3., we use this and Axiom $T4$ of the trace (Definition 3.0.3) to move through the trace. At step 4., we use the fact that the antipode is a Hopf algebra homomorphism $H^{\mathrm{op}} \to H$, while step 5. is simply string diagrammatic reasoning. Finally, at step 6. we use Axiom

$T4$ of the trace to move the symmetry $\sigma_{H,H}$ through the trace.



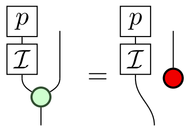


At step 7., we move . We also use the definition of $\mathrm{Tr}^{H\otimes H}$ as defined in Axiom $T5$ of the trace. At step 8., we use the trace axioms to get rid of one of the traces. This allows us to reposition , which then allows us to use the Hopf law at step 9.. All that remains is using the axioms of the trace and string diagramatic reasoning, and we have our result.

The proof for the comultiplication is similar. □

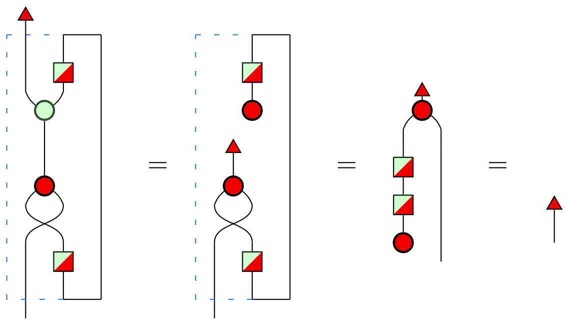
Lemma 4.4.4. *Given a point $p : I \to H$, and copoint $q : H \to I$, the morphism $\mathcal{I} \circ p$ is* *a left cointegral, and* $q \circ \mathcal{I}$ *is a right integral. In addition,* p *is a left cointegral if and only if* $\mathcal{I} \circ p = p$*, and* q *is a right integral if and only if* $q \circ \mathcal{I} = q$*.*

Proof. We begin by proving that $\mathcal{I} \circ p$ is a left cointegral. In other words, for all points p,



This follows immediately from Lemma 4.4.3.

Suppose that we have a point p such that $\mathcal{I} \circ p = p$. Since we know that $\mathcal{I} \circ p$ is a cointegral, p must therefore be a cointegral. For the converse, let be a left cointegral. We then get

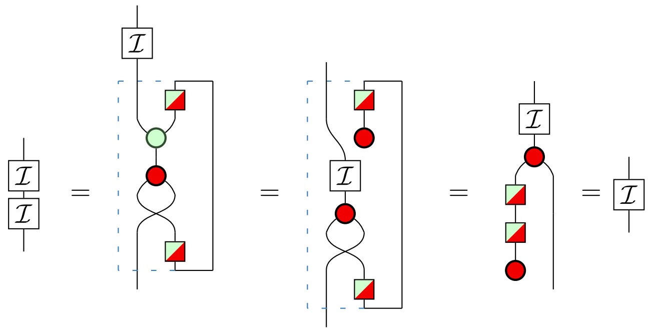


where we use the fact that $\circ$ $=$. The proof for right integrals is similar. $\square$

Remark 4.4.5. Let us set $P(H)$ as the set of left cointegrals $p : I \to H$. The above lemma tells us that that since $\mathcal{I} \circ p$ is always an integral, we can see $\mathcal{I}$ as a function $\mathcal{I} \circ \_ : \mathcal{C}(I, H) \to P(H)$. We see that when we restrict the domain of $\mathcal{I} \circ \_$ to $P(H)$, then we get the identity function, as $\mathcal{I} \circ p = p$ for all $p \in P(H)$.

Lemma 4.4.6. *The integral morphism* $\mathcal{I}$ *is idempotent.*

Proof. Recall that idempotent means $\mathcal{I} \circ \mathcal{I} = \mathcal{I}$. The result follows in a straightforward manner



∎

Recall that the definition of an integral Hopf algebra (Definition 4.2.3) requires that our Hopf algebra is equipped with an integral and a cointegral pair. This raises the question – given an arbitrary Hopf algebra H, is there a reliable way to find an integral and cointegral pair for H? If H has a trace, then we may use the integral morphism. Consider the following Lemma.

Lemma 4.4.7. *Let H be Hopf Algebra that has a trace. Suppose that we have a point $p : I \to H$ and a copoint $q : H \to I$ such that $q \circ \mathcal{I} \circ p$ is an invertible scalar, with inverse $k : I \to I$. Then $(H, \mathcal{I} \circ p \circ k, q \circ \mathcal{I})$ is an integral Hopf algebra.*

Proof. Recall from Lemma 4.4.4 that given any point $p : I \to H$, the composition $\mathcal{I} \circ p$ is a cointegral. Likewise, given copoint q, the composition $q \circ \mathcal{I}$ is an integral. We require that the composition of the integral and cointegral is the identity, but this follows directly from our assumption, and the fact that the integral morphism is idempotent (Lemma 4.4.6)

$$q \circ \mathcal{I} \circ \mathcal{I} \circ p \circ k = q \circ \mathcal{I} \circ p \circ k = 1_I$$

Hence, we have our result. ∎

Note that in a strict symmetric monoidal category, $p \circ k = p \otimes k$. As such, we use the two interchangeably from now on.

Remark 4.4.8. Let us use the category of Vector spaces to examine the above Lemma. In the category of vector spaces, every nonzero point $m : I \to H$ is a monomorphism, and for every monomorphism there exists an epimorphism $e : H \to I$ such that $e \circ m = 1_I$. Also, note that every point $p : I \to H$ is an element of H. As such, if we have an element p such that $\mathcal{I}(p)$ is not zero, it is fairly simple to find a morphism $q : H \to I$ such that $q(\mathcal{I}(p)) = 1$. Note, however, that since infinite dimensional vector spaces do not have traces, this approach will not work for infinite dimensional Hopf algebras. The only category of modules that have traces are finitely generated projective modules [50].

Chapter 5

Hopf-Frobenius Algebras

We say that a Hopf algebra (H, , , , ,) *is Frobenius* when we may equip the Hopf algebra with a comonoid (,) such that (H, , , ,) is a Frobenius algebra. In this chapter, we will be providing several conditions for when a Hopf algebra is Frobenius.

Recall from Lemma 4.2.12 that when a Hopf algebra has a cointegral, we may construct a comultiplication that fulfils the Frobenius law with the multiplication. This provides some glimpse into the connection between Frobenius algebras and integrals.

Given a Hopf algebra with a cointegral, we could ask when the constructed comultiplication has a counit. To answer this, we introduce *nondegeneracy* (Definition 5.1.1) and show in Lemma 5.1.2 that the constructed comultiplication has a counit exactly when the Hopf algebra is a *nondegenerate* integral Hopf algebra.

The condition defined in Definition 5.1.5 uses the integral morphism from Definition 4.4.1. This condition may have greater utility than nondegeneracy, as it only requires that the Hopf algebra H has a trace and, depending on the category, it may provide to us a way that may construct the integral and cointegral pair.

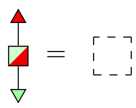
Finally, we provide a condition that shows that the Hopf algebra is nondegenerate when a certain limit exists, and we use this to show that when the Hopf algebra is Frobenius, we may construct an integral and cointegral pair making the Hopf algebra a nondegenerate integral Hopf algebra. This shows that all of our separate conditions are equivalent to nondegeneracy.

Theorem 5.2.9 is the main result of the thesis, where we show that when a Hopf algebra is a nondegenerate integral Hopf algebra, then it must also be Hopf-Frobenius. Since nondegeneracy is equivalent to the Hopf algebra being Frobenius, and since every Hopf algebra in $\mathbf{FVect_k}$ is Frobenius, we see that every finite dimensional Hopf algebra must therefore be Hopf-Frobenius. We illustrate this by showing how some standard concepts in the field of finite dimensional Hopf algebras relate to the Hopf-Frobenius structure, and then we finish the chapter by proving Radford's form for the antipode to the power of 4, in Lemma 5.4.4.

Throughout this chapter, we will be using concepts and results that the author of this thesis originally stated in [20]. In particular, nondegeneracy (Definition 5.1.1) was defined previously, as well as the results of Lemma 5.2.1, Lemma 5.2.5, Theorem 5.2.6, Theorem 5.2.8, Theorem 5.2.9 and Lemma 5.2.15, which were all proven previously.

5.1 Non-degeneracy

Definition 5.1.1. An integral Hopf algebra $(H, ,)$ is *nondegenerate*[1] when

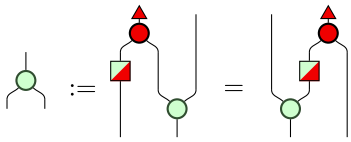


Lemma 5.1.2. *Suppose that the Hopf algebra* $(H, , , , ,)$ *has left cointegral . By Lemma 4.2.12, we know that we may equip* H *with a green comultiplication that fulfils the Frobenius law with . Suppose that there exists a copoint . Then the following two statements are equivalent:*

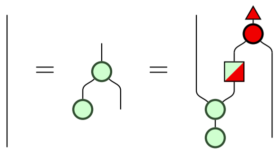
- *is the counit of the comonoid* $(H, ,)$*.*
- *is a right integral, and forms a nondegenerate integral Hopf algebra* $(H, ,)$*.*

[1] The above concept was brought to our attention by Gabriella Böhm in a private correspondence where she explained several pieces of Hopf algebra folklore. Since this is folklore, we are unsure of it's origin.

Proof. Recall from Lemma 4.2.12 that we define  as



Suppose that we have  such that (H, , ) is a comonoid. Then



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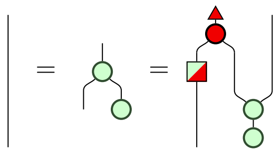
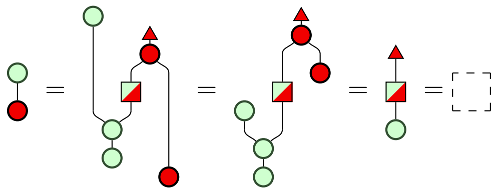
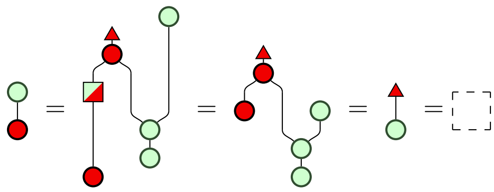


We use these two equations to get

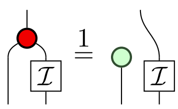


and

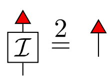


respectively.

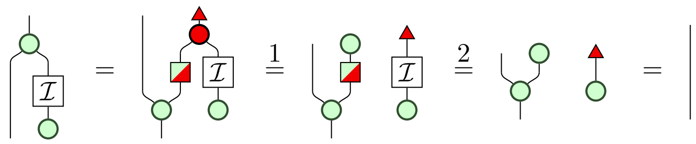
Recall the Definition of $\mathcal{I}$ from Definition 4.4.1. Integral Hopf algebras have a half-dual (Lemma 4.3.5), which means that we may define a trace on H (Lemma 4.3.8). Lemma 4.4.3 tells us that



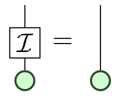
and



We use these facts below.

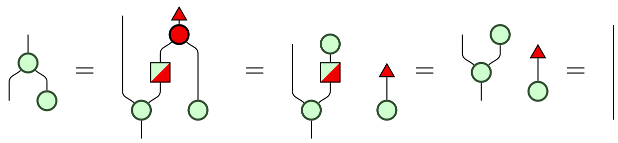


Hence, $\circ\, \mathcal{I}$ is a right counit. The counit of a comonoid is unique, so we see that

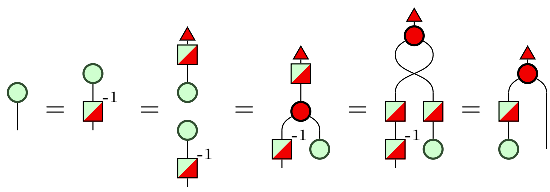


Lemma 4.4.4 tells us that copoint q is an integral if and only if $q \circ \mathcal{I} = q$. Hence, is an integral. Since both $\circ$ $= 1_I$ and $\circ$ $\circ$ $= 1_I$, we see that $(H,$ $,$ $)$ is a nondegenerate integral Hopf algebra.

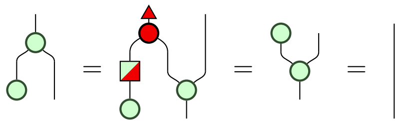
On the other hand, suppose that we have a right integral such that $(H,$ $,$ $)$ is nondegenerate. We now show that is the counit of . The right side follows immediately from the fact that is a right integral, and that the composition of and is 1_I.



For the left hand side, recall that since we have an integral Hopf algebra, the antipode has an inverse $^{-1}$ by Lemma 4.3.1. Since is a homomorphism, $^{-1}$ must also be a homomorphism. Hence, we see that



where we have used the nondegeneracy of and . From this, we use the definition of to see that

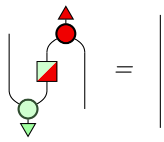


Hence, is a left and right counit to . □

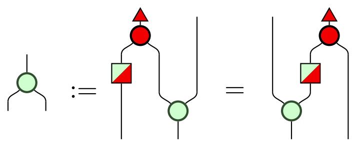
Corollary 5.1.3. *Suppose that the Hopf algebra H has right integral . By Corollary 4.2.14, we know that we may equip H with a red multiplication that fulfils the Frobenius law with . Suppose that there exists a point . Then the following two statements are equivalent:*

- *is the unit of the monoid* $(H, \ , \ )$.
- *is a left cointegral, and forms a nondegenerate integral Hopf algebra* $(H, \ , \ )$.

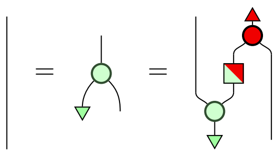
Lemma 5.1.4. *An integral Hopf algebra* $(H, \ , \ )$ *is nondegenerate if and only if*



Proof. Suppose that we have nondegenerate $(H, \ , \ )$. Then by Lemma 5.1.2, we have a comonoid $(H, \ , \ )$, defined as

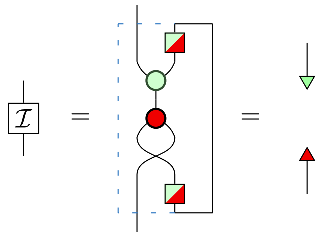


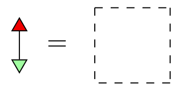
This follows from Lemma 5.1.2, as we know that we must have a comonoid $(H, \ , \ )$. Hence



□

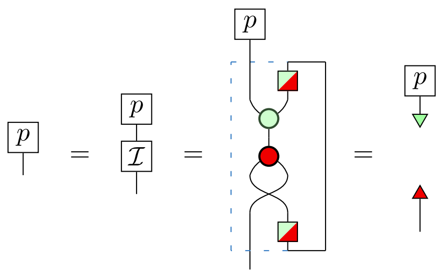
Definition 5.1.5. Recall the definition of $\mathcal{I}$ from Definition 4.4.1. Suppose that we may define a trace on H. We say that $\mathcal{I}$ *factors through the unit* when there exists copoint $\downarrow : H \to I$ and point $\uparrow : I \to H$ such that





Note how, since $\downarrow \circ \uparrow = 1_I$, $\downarrow$ is a section of $\uparrow$, hence $\downarrow$ is a monomorphism and $\uparrow$ is an epimorphism. When we want to specify $\uparrow$ and $\downarrow$, we say that $\mathcal{I}$ *factors through the unit with* $\uparrow$ *and* $\downarrow$. Note that we are not assuming a factorisation system in our category - we are still working within an arbitrary symmetric monoidal category.

Remark 5.1.6. Recall Lemma 4.4.4 and Remark 4.4.5. The above definition tells us that, when $\mathcal{I}$ factors through the unit, then for every cointegral $p \in P(H)$, we have the identity



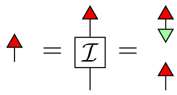
Hence, every cointegral is simply a scalar multiple of $\uparrow$. Since the scalar multiple of a cointegral is always a cointegral, there is a bijection from $\mathcal{C}(I, I)$ to $P(H)$, the set of cointegrals.

In $\mathbf{FPMod_R}$ and $\mathbf{Vect_k}$, every morphism $f : A \to B$ may be factorised into an epimorphism $e : A \to X$ and a monomorphism $m : X \to B$ such that $m \circ e = f$. In this case, we would say that X is the *image* of f. In both of these cases, the image of $\mathcal{I}$ is equal to $P(H)$. Hence, the above condition is analgous to the condition presented by Paregis [49], where $P(H)$ is isomorphic to the monoidal unit.

Lemma 5.1.7. *Suppose that $\mathcal{I}$ factors through the unit with $\uparrow$ and $\downarrow$. Then* $(H, \uparrow, \downarrow)$ *is an integral Hopf algebra*

Proof. We only need to show that $\uparrow$ and $\downarrow$ are a cointegral and integral respectively.

Recall from Lemma 4.4.4 that a point p is an integral if and only if $\mathcal{I} \circ p = p$. We see that this is true as follows



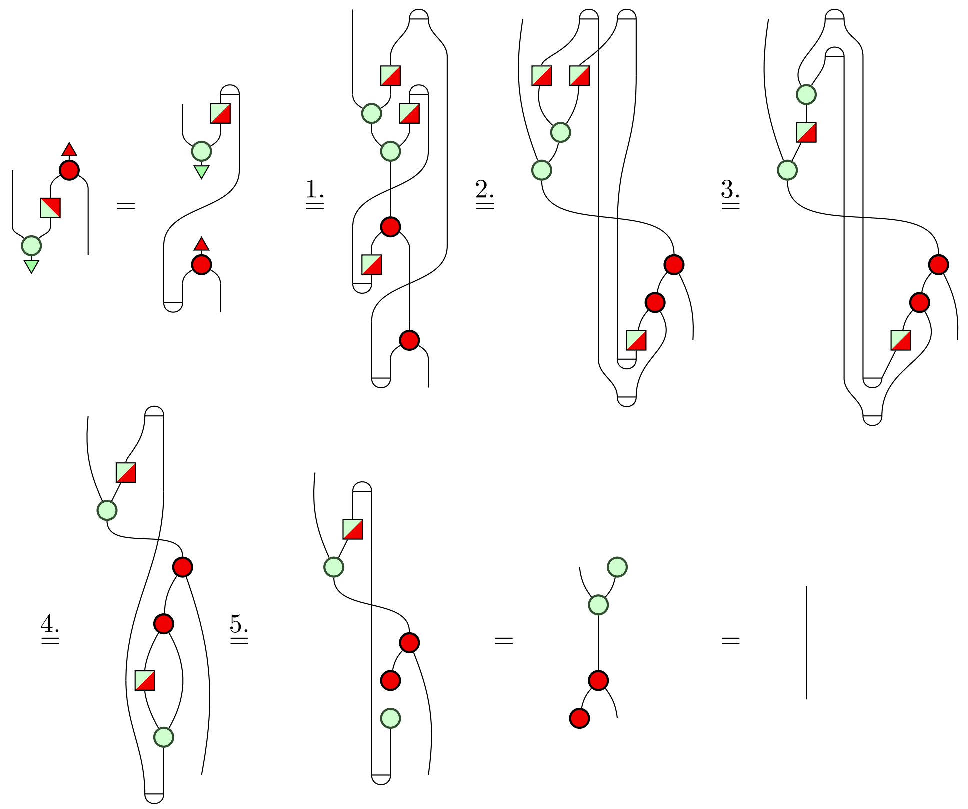
By a similar argument, is an integral. HEnce, $(H, ,)$ is an integral Hopf algebra

□

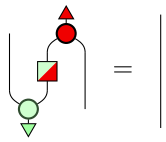
5.2 Equivalences with Frobenius structure

Lemma 5.2.1. *If $\mathcal{I}$ factors though the unit with and , then $(H, ,)$ is a nondegenerate integral Hopf algebra*

Proof. Lemma 5.1.7 tells us that H is an integral Hopf algebra. Therefore, by Corollary 4.3.5, H is half dual to itself. By Lemma 4.3.7 and Lemma 4.3.8, the trace on H is the same as the trace given by the half dual. Hence, in the following proof, we use the half dual trace. Observe that

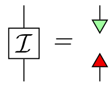


where step 1 is due to the Frobenius condition, 2 comes from associativity and at step 3 we use the fact that the antipode is a homomorphism $H^{\mathrm{op}} \to H$. The presence of half duals gives us 4, and 5 is due to the Hopf law. We then get the following identity

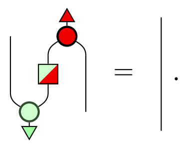


Lemma 5.1.4 tells us that this is the same as H being a nondegenerate integral Hopf algebra. $\square$

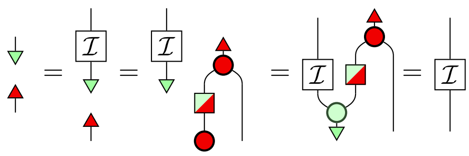
Lemma 5.2.2. *Let* $(H, \ , \ )$ *be a nondegenerate integral Hopf algebra. Then* $\mathcal{I}$ *factors through the unit with* *and* *.*



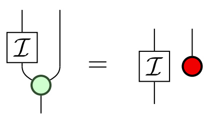
Proof. Recall that by Lemma 4.4.4, $\circ\, \mathcal{I} =$, and by Lemma 5.1.4, it is true that



We combine these to see that



where we have used the result from Lemma 4.4.3



All we need to show now is that $\circ$ $= 1_I$. However, this follows from the definition of an integral Hopf algebra $\square$

We have now shown that the integral morphism factorises through the unit exactly when we have a nondegenerate integral Hopf algebra. In what follows, we will provide another equivalent condition to when a Hopf algebra is nondegenerate, and we shall show that all of this is equivalent to our Hopf algebra being Frobenius.

Definition 5.2.3. Suppose that we have $e : E \to A$ and morphisms $f, g : A \to B$ such that $f \circ e = g \circ e$. In addition, suppose that there exists morphisms $s : A \to E$ and $t : B \to A$ such that $s \circ e = 1_E$, $t \circ f = 1_A$ or $t \circ g = 1_A$, and the following diagram commutes

$$\begin{tikzcd}
A \arrow[r, "g"] \arrow[d, "s"'] & B \arrow[d, "t"] \\
E \arrow[r, "e"] & A
\end{tikzcd}$$

In this case, we say that e is a *split equaliser of f over g*, with structure maps s and t.

Note how the above definition does not assume that a split equaliser is a limit. Hence, we need the following lemma.

Lemma 5.2.4. *Suppose that $e : E \to A$ is a split equaliser of f over g, with $f, g : A \to B$, with structure maps $s : A \to E$ and $t : B \to A$. Then e is an equaliser of f and g.*

Proof. As part of the definition of a split equaliser, we are assuming that $f \circ e = g \circ e$. Hence, we only need to prove that e is universal in equalising f and g. Suppose that $x : X \to A$ equalises f and g. To prove that e is an equaliser, we need to show that there exists some unique morphism $i : X \to E$ such that $x = e \circ i$. In other words, the following diagram commutes

$$\begin{tikzcd}
E \arrow[r, "e"] & A \arrow[r, shift left, "f"] \arrow[r, shift right, "g"'] & B \\
X \arrow[u, dashed, "i"] \arrow[ur, "x"'] & &
\end{tikzcd}$$

Consider the morphism $s \circ x : X \to E$. It follows from the definition of the split equaliser, and the fact that x equalises f and g, that the following diagram commutes.

$$\begin{tikzcd}
E \arrow[r, "e"] & A \\
A \arrow[u, "s"] \arrow[r, "g"] & B \arrow[u, "t"'] \\
X \arrow[u, "x"] \arrow[r, "x"] & A \arrow[u, "f"'] \arrow[uu, bend right, "1_A"']
\end{tikzcd}$$

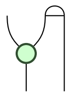
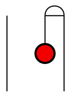
Hence, $x = e \circ s \circ x$. Now all we need to show that this morphism is unique. Suppose that there exists some other morphism $j : X \to E$ such that $x = e \circ j$. Then the following diagram commutes

$$\begin{tikzcd} & E \\ E \arrow[ur, "1_E"] \arrow[r, "e"] & A \arrow[u, "s"'] \\ X \arrow[u, "j"] \arrow[ur, "x"'] & \end{tikzcd}$$

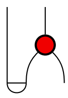
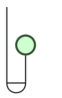
Hence, $j = s \circ x$, and we have our result. □

Lemma 5.2.5. *Let H have a right half dual H^*. Then the following statements are equivalent*

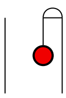
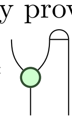
1. *$\mathcal{I}$ factors through the unit with*  *and* 
2. *There is a point*  $: I \to H$ *that is the equaliser*[2] *of*

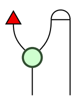
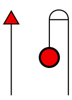
 *and* 

3. *There is a copoint*  $: H \to I$ *which is a coequaliser of*

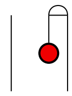
 *and* 

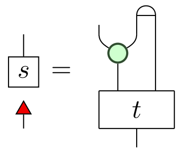
Proof. We will only prove that point 1. and 2. are equivalent, as the proof that 1. and 3. are equivalent follows from duality.

Suppose that $\mathcal{I}$ factors though the unit. We shall actually prove a stronger result, that  is a split equaliser of f over g, where $f =$  and $g =$ . Lemma 5.2.1 tells us that we have  and  such that $(H,$ , $)$ is a nondegenerate integral Hopf algebra. Then by definition

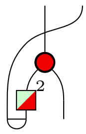
 $=$ 

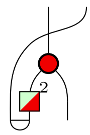
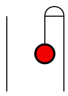
[2]In the same correspondence that brought Definition 5.1.1 to our attention, Gabriella Böhm brought this lemma to our attention. This is also Hopf algebra folklore, and we are unsure of it's origin.

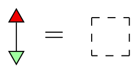
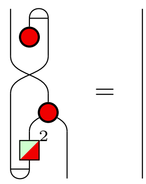
Hence,  equalises the appropriate maps. To prove that  is a split equaliser, we need to find the appropriate structure maps. So we need $s : H \to I$ and $t : H \otimes H^* \to H$ such that s is a retract of , t is a retract of , and the following identity holds



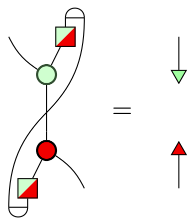
We shall accomplish this by setting

$s :=$  $\qquad t :=$ 

Therefore, we need to prove that  is the retract of , and  is a retract of . In other words

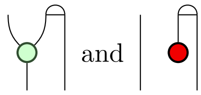
 

Both of these are clearly true. Hence, the final condition required to show that  is a split equaliser is that

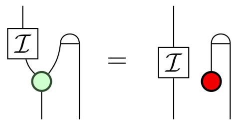


but this follows immediately from the fact that $\mathcal{I}$ factors through the unit. The result that that  is a split coequaliser follows similarly.

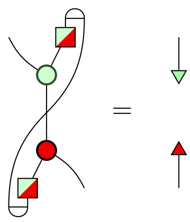
For the other direction, suppose that we have a point $\uparrow$ which acts as an equaliser of



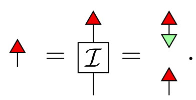
To achieve our result, we shall be proving that $\mathcal{I}$ factors through the unit (Lemma 5.2.1). By Lemma 4.4.4 we have



Thus, $\mathcal{I}$ is a cone of the appropriate diagram. We are assuming that $\uparrow : I \to H$ is an equaliser, so by the universal property of the equaliser there is a unique morphism $\downarrow : H \to I$ such that



Now we merely need to show that $\uparrow$ and $\downarrow$ are a monomorphism and epimorphism respectively. Since $\uparrow$ is an equaliser, it must be an monomorphism. Hence, all we need to show is that $\downarrow$ is an epimorphism. Note that, by assumption, $\uparrow$ is a cointegral, so by Lemma 4.4.4 we get that $\mathcal{I} \circ \uparrow = \uparrow$. Hence

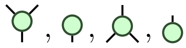


Since $\uparrow$ is a monomorphism, and $\uparrow = \uparrow \circ \downarrow \circ \uparrow$, it follows that $\downarrow \circ \uparrow = 1_I$. This implies that $\downarrow$ must therefore be an epimorphism, as it is a left handed inverse. Hence, $\mathcal{I}$ factorises though the unit, and we have our result. □

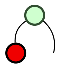
Theorem 5.2.6. *Let H be a Hopf algebra. $\mathcal{I}$ factors through the unit (Definition 5.1.5) if and only if H is Frobenius*[3].

[3] As we shall see later on, a Hopf algebra is Frobenius if and only if it is Hopf-Frobenius (Corollary 5.2.10)

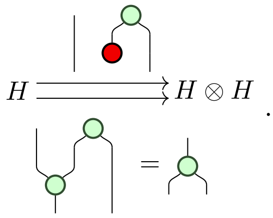
Proof. Suppose that $\mathcal{I}$ factors through the unit. By Lemma 5.2.1, we know that H must be a nondegenerate integral Hopf algebra, and by Lemma 5.1.2, we may construct a Frobenius algebra on H.

For the other direction, suppose that H has a green Frobenius algebra, (H, ). This implies that H is self dual by Definition 2.4.7.

Let us set

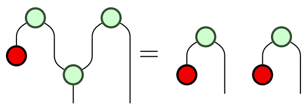
$\alpha :=$  .

We will show that $\alpha : I \to H$ is a split equaliser of f over g, for $f = 1_H \otimes \alpha$ and $g =$ . In other words, it is a split equaliser of the diagram

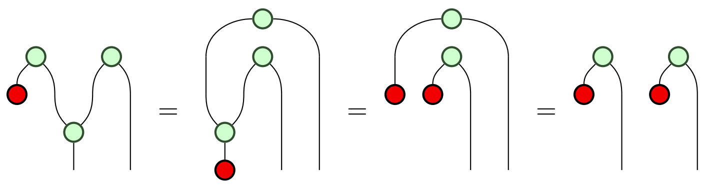


which, by Lemma 5.2.5, this will prove our assertion.

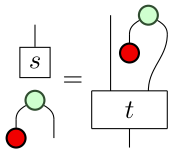
To show that α is a split equaliser, we must first show that it is a cone of the appropriate diagram. That is, we will show that



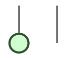
This follows from the properties of the Frobenius algebra and the Hopf algebra.



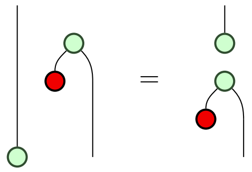
We now need to find structure maps $s : H \to I$ and $t : H \otimes H \to H$ such that s is a retract of α, t is a retract of , and the following identity holds



Let us set

$s :=$  $t :=$ .

We see that is a retract of α, and is a retract of . The final condition for α to be a split equaliser is

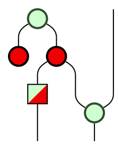


Thus, α is a split equaliser, and we have our result.

□

Remark 5.2.7. Recall that when we have a nondegenerate integral Hopf algebra, we may construct a Frobenius algebra (Lemma 5.1.2) from the integral and cointegral pair. In the above theorem, we show that I factors through the unit when a Hopf algebra is Frobenius, which is equivalent to constructing a nondegenerate Frobenius algebra (Lemma 5.2.2). It is therefore worth asking the following: given Hopf algebra (H, , , , ,) with Frobenius structure (H, , , ,), we construct a nondegenerate integral Hopf algebra (H, ,) via Theorem 5.2.6. Is the Frobenius algebra that we may construct from (H, ,) via Lemma 5.1.2 equal to our original Frobenius algebra, (H, , , ,)?

The answer to this is no in general. The above theorem gives us left cointegral . This implies that the corresponding green comultiplication given by Lemma 4.2.12 is defined as



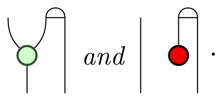
Likewise, Lemma 5.2.5 tells us that the integral is $\circ\,\mathcal{I}$, which is not necessarily equal to . We see that this is equal to and only if they are a cointegral and integral respectively.

However, the converse is true – given nondegenerate integral Hopf algebra $(H, ,)$, we construct a Frobenius algebra $(H, , , ,)$ via Lemma 5.1.2. We construct a nondegenerate integral Hopf algebra as above, and we find that is our cointegral, and is our integral.

We summarize the results of section by stating the main theorem of the paper.

Theorem 5.2.8. *Let H be a Hopf algebra. The following conditions are equivalent*

1. *The Hopf algebra H is Frobenius.*
2. *We may define a trace on H, and the integral morphism, $\mathcal{I}$, factors through the unit with and .*
3. *There exists left cointegral and right integral such that $(H, ,)$ is a nondegenerate integral Hopf algebra.*
4. *We may define a half dual on H, and H admits an equaliser of*



Proof. Theorem 5.2.6 tells us that 1 is equivalent to 2. Lemma 5.2.1 and Lemma 5.2.2 tells us that 2 is equivalent to 3 and finally Lemma 5.2.5 tells us that 2 is equivalent to 4. □

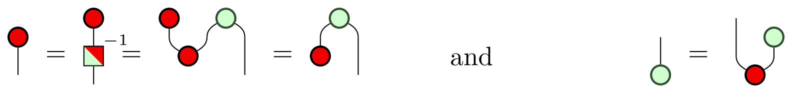
The most straightforward of these conditions to check is likely to be 2, with the manner that we describe in Lemma 4.4.7. We have provided a set of conditions that are equivalent to when a Hopf algebra is Frobenius. In this final section, we show that these conditions are all equivalent to a Hopf algebra being Hopf-Frobenius (Definition 4.1.1).

Theorem 5.2.9. *Let $H = (H,$, , , , $)$ be a Hopf algebra. Then H is Hopf-Frobenius with red Hopf algebra $(H,$, , , , $)$ if and only if H is a nondegenerate integral Hopf algebra $(H,$, $)$.*

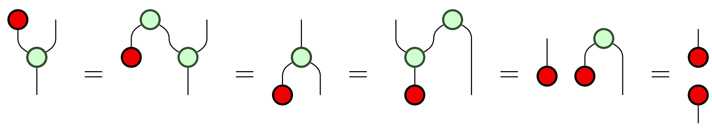
Proof. Suppose that H is Hopf-Frobenius, with red Hopf algebra $(H,$, , , , $)$. Recall from the definition of a Hopf-Frobenius algebra (Definition 4.1.1 that

Our goal is to show that and are a left integral and right cointegral respectively, and that they form a nondegenerate integral pair.

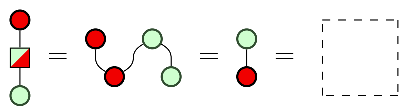
Since is the unit of a Hopf algebra, we know that $^{-1} \circ$ $=$. Similarly, we know that $\circ$ $^{-1} =$. This therefore implies that



We use this to show that is a left cointegral as follows

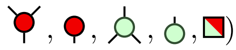


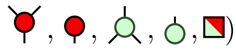
The proof that is a right integral is similar. Note that since these are the unit and counit of a Hopf algebra, it is immediate that $\circ$ $= 1$. Hence, we only need to show nondegeneracy (Definition 5.1.1). This follows from the fact that the antipode has the form $=$, as



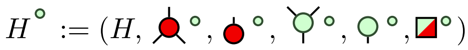
Hence, $(H,$, $)$ is a nondegenerate integral Hopf algebra.

We now assume the converse, that H is a nondegenerate integral Hopf algebra $(H,$, $)$. Using the nondegenerate integral structure $(H,$, $)$, we set $(H,$, , , $)$ as the Frobenius algebra that we construct via Lemma 5.1.2, and $(H,$, , , $)$ as the Frobenius structure that we get from Corollary 5.1.3. To show that H is Hopf-

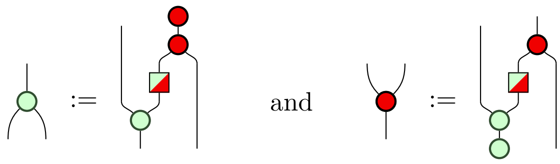
Frobenius, we only need to prove that $(H,$  $)$ forms a Hopf algebra, where

recall that we call the Hopf algebra $(H,$  $)$ the *red Hopf algebra*.

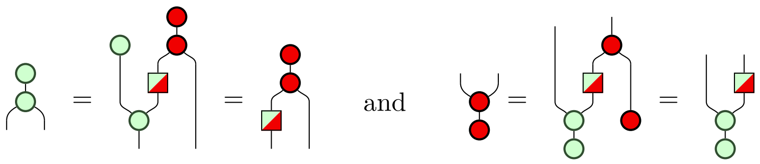
Recall from Definition 2.5.11 the definition of the dual Hopf algebra. Since H is Frobenius, then H is self-dual as we stated in Remark 2.4.4. We may apply the dual structure of the green Frobenius algebra to get a dual Hopf algebra. In other words, we know immediately that



is a Hopf algebra. Note that  and  are constructed as follows

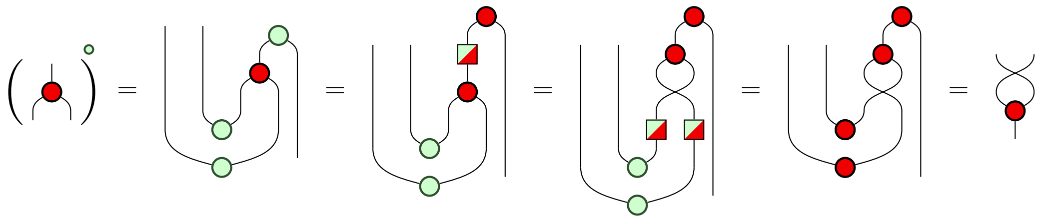


Therefore, we may use the antipode to transfer between the green dual structure and the red dual structure as follows



We now apply the green dual to our Hopf algebra. It is clear from the definition that

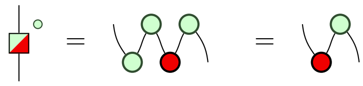
On the other hand, when we apply the green dual to the red comonoid, we find that



In a similar manner, we see that $(\;)^{\circ} = $.

Recall from Proposition 2.5.4 that, given an arbitrary Hopf algebra $G = (G, \;, \;, \;, \;, S)$, we get bialgebra $G^{\sigma} = (G, \;^{\sigma}, \;, \;, \;)$ where $\;^{\sigma} := \; \circ \sigma_{G,G}$. Recall from Proposition 2.5.6 that G^{σ} is a Hopf algebra if and only if S, the antipode of G, is invertible.

We see, therefore, that the red Hopf algebra $(H, \;, \;, \;, \;)$ is simply $(H^{\circ})^{\sigma}$ when viewed as a bialgebra. Therefore, all we need to show now is that $(H^{\circ})^{\sigma}$ has an antipode. We therefore only need to show that $\;^{\circ}$ is invertible. We see that $\;^{\circ}$ is equal to



The fact that $\;^{\circ}$ has an inverse follows from the fact that has an inverse (Lemma 4.3.1), and that the dual action preserves composition (Proposition 2.3.12), so $(\;^{\circ})^{-1} = (\;^{-1})^{\circ}$. We see that the antipode of the red Hopf algebra is therefore

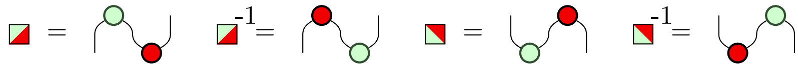
We have therefore proved the result that H is a Hopf-Frobenius algebra. $\square$

Corollary 5.2.10. *Let H be a Hopf algebra. Then H is Frobenius if and only if H is Hopf-Frobenius.*

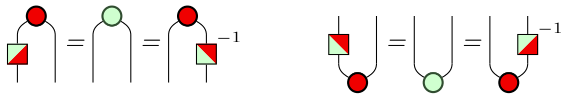
Corollary 5.2.11. *Every Hopf algebra in* $\mathbf{FVect_k}$ *is Hopf-Frobenius.*

Proof. This follows from Larson and Sweedler's theorem [41] that every Hopf algebra in $\mathbf{FVect_k}$ is Frobenius. $\square$

Remark 5.2.12. Let us recall the various antipode forms and antipode inverses

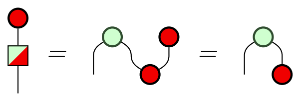


Corollary 5.2.13. *Let H be a Hopf-Frobenius algebra. Then*

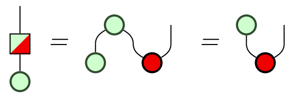


Proof. The above corollary follows directly from the antipode forms in Remark 5.2.12. □

Corollary 5.2.14. *Let H be a Hopf-Frobenius algebra. Then*



and

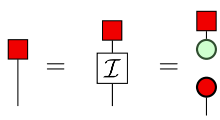


Recall from Proposition 2.4.12 that each invertible element of a Frobenius algebra gives us a new Frobenius algebra. This means that there is no canonical choice for a Frobenius structure in general. However, with Hopf-Frobenius algebras, this is not the case, as the following Lemma shows.

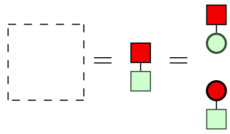
Lemma 5.2.15. *Let H admit a Hopf-Frobenius algebra structure. Then this structure is unique up to invertible scalar.*

Proof. Suppose that Hopf algebra $H = (H, \ , \ , \ , \ , \ )$ admits two Hopf-Frobenius structures, with red Hopf algebras $H_\bullet = (H, \ , \ , \ , \ , \ )$ and $\widehat{H}_\bullet = (H, \ , \ , \ , \ , S)$. Our goal is to show that the structure maps of $H_\bullet$ and $\widehat{H}_\bullet$ only differ by an invertible scalar factor. Recall that we refer to $(H, \ , \ , \ , \ , \ )$ as the green Hopf algebra.

By Theorem 5.2.9 we know that $(H, \ , \ )$ must be a nondegenerate integral Hopf algebra so therefore Theorem 5.2.8 tells us that $\mathcal{I}$ must factor through the unit with and . Hence, by Lemma 5.2.2, we get that

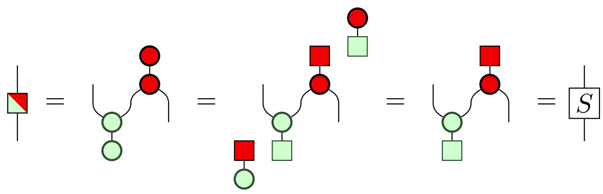


In a similar manner, $= \circ$. These two scalar factors are inverses of each other, as follows

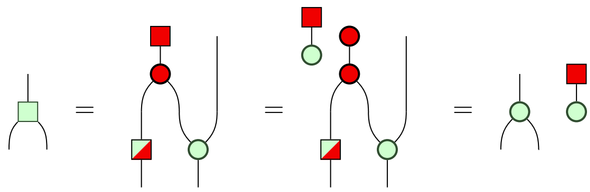


So and are inverses. Hence, the respective units and counits of $H_\bullet$ and $\widehat{H}_\bullet$ are an invertible scalar multiple of each other.

This implies that the antipodes of the two red Hopf algebras are equal, as follows



which then allows us to show that the green comultiplication maps are scalar multiples of each other. Recall from Lemma 5.1.2 how the comultiplication is constructed



The proof for the red multiplication maps is similar. Thus, we have proven our result. $\square$

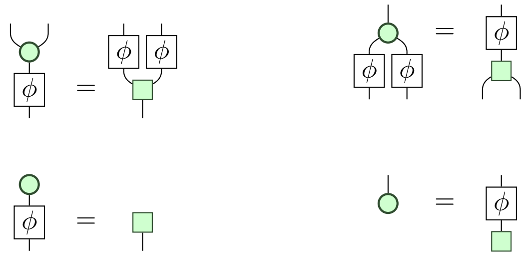
5.3 Morphisms Between Hopf-Frobenius Algebras

Now that we have established Hopf-Frobenius algebras, it is natural to ask what the category of Hopf-Frobenius algebras is? What is the appropriate notion of morphism between two Hopf-Frobenius algebras?

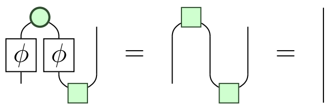
In Definition 2.5.5, we showed that bialgebra morphisms preserve Hopf algebra structure. However, maps between Frobenius algebras are much more limiting:

Lemma 5.3.1. *Let* $(F, \ , \ , \ , \ )$ *and* $(G, \ , \ , \ , \ )$ *be two Frobenius algebras, and let* $\phi : F \to G$ *be a Frobenius algebra homomorphism. The* ϕ *must be an isomorphism.*

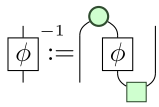
Proof. A Frobenius algebra homomorphism is a homomorphism of both the monoid and the comonoid. In other words, we have



This implies the following equation



We may therefore define an inverse to ϕ as follows

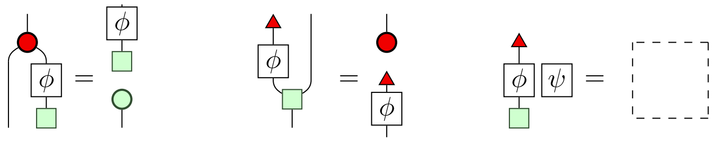


$\square$

There is no requirement for bialgebra morphisms to be an isomorphism, however. This suggests that there are morphisms between Hopf algebras that do not preserve Frobenius structure. For example, in $\mathbf{Vect_k}$, finite dimensional Hopf algebras must be Hopf-Frobenius by Corollary 5.2.11, while infinite dimensional Hopf algebras cannot be Frobenius, as Frobenius structure implies the existence of a dual structure (Definition 2.3.7). However, there are bialgebra morphisms from finite dimensional Hopf algebras to infinite dimensional Hopf algebras – from the group algebras $\mathbb{Z}_n$ to $\mathbb{Z}$, for example.

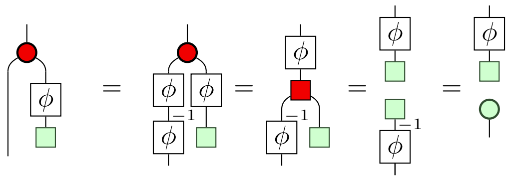
Below, when we refer to H and K, we will mean integral Hopf algebras (H, , , , ,) and (K, , , , , S) with integrals (,) and (,) respectively.

Definition 5.3.2. We say that a bialgebra homomorphism $\phi : H \to K$ *preserves integrals* if $\circ\, \phi$ is a right integral for H, $\phi \circ$ is a left cointegral for K, and $\circ\, \phi \circ$ is an invertible scalar with inverse ψ. In other words, there exists scalar $\psi : I \to I$ such that



Lemma 5.3.3. *Let $\phi : H \to K$ be a bialgebra isomorphism for integral Hopf algebras H and K. Then $\circ\, \phi$ is a right integral for H, and $\phi \,\circ$ is a left cointegral for K.*

Proof. We shall only prove this for the integral, as the proof for the cointegral follows similarly. This follows from

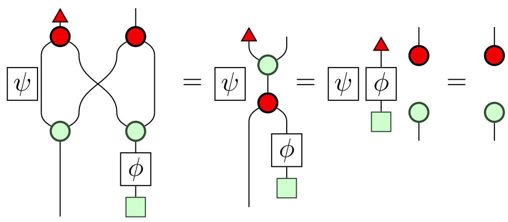


□

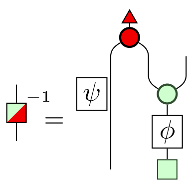
Lemma 5.3.4. *Let $\phi : H \to K$ be a bialgebra homomorphism such that $\circ\, \phi \,\circ$ is an invertible scalar. Then ϕ is a bialgebra isomorphism if and only if it preserves integrals.*

Proof. Suppose that ϕ is an isomorphism. We see that it preserves integrals immediately as a result of Lemma 5.3.3.

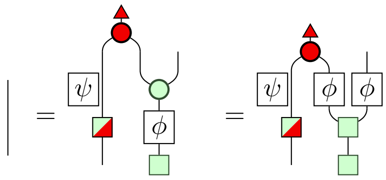
For the converse, we recall how we proved that the presence of integrals gives us the inverse of the antipode, from Lemma 4.3.1. Consider the following equation



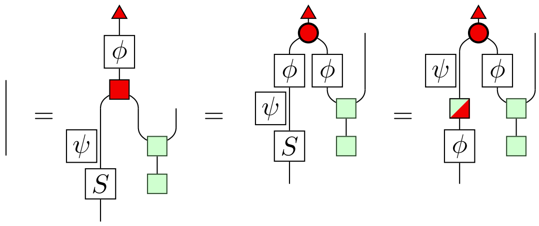
This implies that the antipode has the following morphism as its inverse



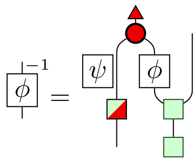
Hence,



By a similar argument, we have



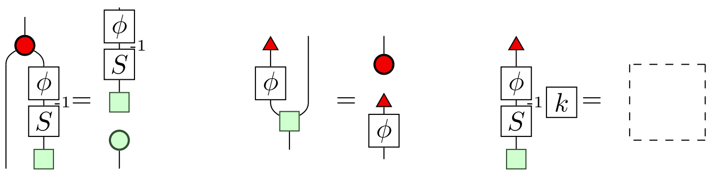
Therefore, ϕ must have the following morphism as its inverse.



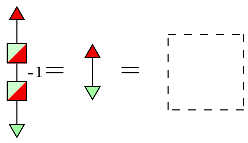
□

Recall from Definition 4.2.3 the distinction between LR integral Hopf algebras and RL integral Hopf algebras. Via Corollary 4.2.7, we see that we may extend the definition of an integral preserving morphism to one from an LR integral Hopf algebra to an RL integral Hopf algebra as follows:

Definition 5.3.5. Let H be an LR integral Hopf algebra and K be an RL integral Hopf algebra. Via Corollary 4.2.7, we may construct an LR integral Hopf algebra on K. An *LR to RL morphism* is an integral preserving morphism between these two LR integral Hopf algebras. In other words, we have the following equations



Example 5.3.6. Let H be an LR integral Hopf algebra. The antipode $: H \to H^{\text{op}}$ is an LR to RL morphism. As established in Proposition 2.5.6, the antipode is a Hopf algebra homomorphism, and from Lemma 4.3.1 we have that the antipode is an isomorphism. Finally, we see that



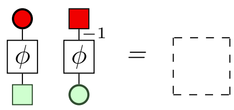
Hence, by Lemma 5.3.4, we have our result.

Below, H and K will refer to Hopf-Frobenius algebras, with green Hopf algebras $(H, , , , ,)$ and $(K, , , , , S)$, and red Hopf algebras $(H, , , , ,)$ and $(K, , , , , S^\dagger)$.

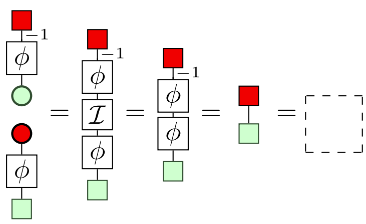
Corollary 5.3.7. *Let $\phi : H \to K$ a bialgebra homomorphism. Then ϕ preserves integrals if and only if ϕ is an isomorphism.*

Proof. This follows immediately from Lemma 5.3.4 and Lemma 5.3.8. $\square$

Lemma 5.3.8. *Let H and K be Hopf-Frobenius algebras, and let $\phi : H \to K$ be a bialgebra isomorphism between their respective green Hopf algebras. Then*

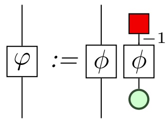


Proof. We recall from Lemma 5.2.1 that H and K are Hopf-Frobenius if and only if $\mathcal{I}$ factor through the unit. Therefore, we have



where we use the fact that when ϕ is an isomorphism, $\circ\, \phi$ is an integral from Lemma 5.3.3, and Lemma 4.4.4 which says that, for any integral $q : H \to I$, $q \circ \mathcal{I} = q$. $\square$

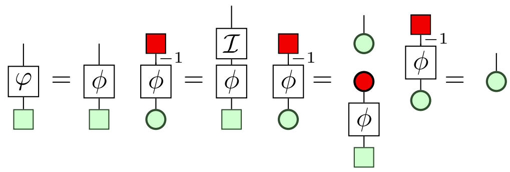
Lemma 5.3.9. *Let H and K be Hopf-Frobenius algebras, and let $\phi : H \to K$ be a bialgebra isomorphism between their respective green Hopf algebras. Then the following morphism*



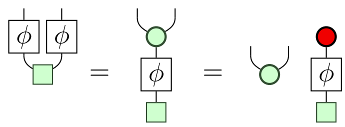
is a bialgebra isomorphism between $(H, \ , \ , \ , \ , \ )$ and $(K, \ , \ , \ , \ , S^{\dagger})$.

Proof. We shall prove this for the counit and the comultiplication, and the proof for the unit and multiplication follow similarly.

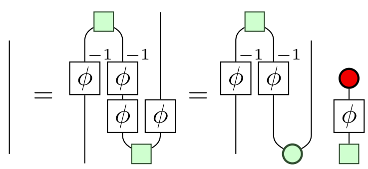
Consider the following



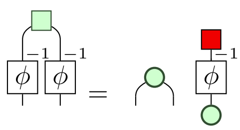
where we used the fact that $\circ\, \phi$ is an integral from Lemma 5.3.3, and Lemma 4.4.4 which says that, for any integral $q : H \to I$, $q \circ \mathcal{I} = q$. Since H is Hopf-Frobenius, $\mathcal{I}$ factors through the unit, and then Lemma 5.3.8 gives us the final equality. This informs us that ϕ will preserve the cup between the Frobenius algebras up to a scalar factor, as follows.



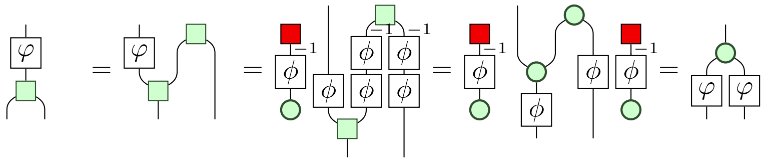
This implies that the cap will also be preserved. We see that



Therefore



Hence, we find that when we compose φ with the green comultiplication, , we get



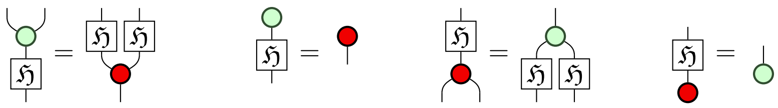
□

Corollary 5.3.10. *Let* $(H,$, , , , $)$ *and* $(K,$, , , , $S)$ *be Hopf-Frobenius algebras, and let* $\phi : H \to K$ *be a bialgebra isomorphism. Then* ϕ *preserves Frobenius structure if and only if* $\circ \phi \circ$ $= 1$.

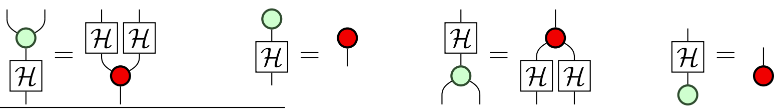
Definition 5.3.11. The category of Hopf-Frobenius algebras is a category where the objects are Hopf-Frobenius algebras, and the morphisms are bialgebra isomorphisms such that $\circ \phi \circ$ $= 1$.

A *Hadamard morphism* [4] is a type of morphism that acts on the Hopf-Frobenius algebra itself. Majid [44], defines two types of Hadamard morphisms:

Definition 5.3.12. A *type 1 Hadamard*, $\mathfrak{H} : H \to H$ is a Frobenius homomorphism from the green Frobenius algebra to the red Frobenius algebra. In other words, we have the following equations



A *type 2 Hadamard*[5] $\mathcal{H} : H \to H$ is a bialgebra isomorphism from the green Hopf algebra to the red Hopf algebra. Note that the green Hopf algebra is an LR integral Hopf algebra with (,), while the red Hopf algebra is an RL integral Hopf algebra with (,). So we have the equations



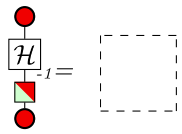
[4]This terminology originates from ZX-calculus (See Duncan and Coecke [18]) where the Hadamard gate is a unitary map that maps between the Z basis and X basis. In the context of ZX-calculus, this means that it maps between the green Frobenius algebra and red Frobenius algebra.

[5]The terminology, type 1 and 2 Hadamard, comes from Majid [44]

Lemma 5.3.13. *Hopf-Frobenius algebra H has a type 2 Hadamard if and only if the green Hopf algebra of H is isomorphic to the dual Hopf algebra, $H^{*\sigma}$, as defined in Definition 2.5.11 and Definition 2.5.4.*

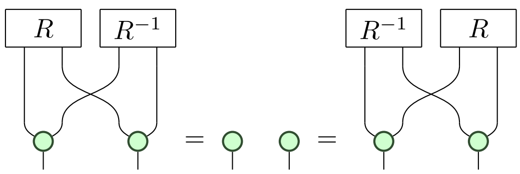
Proof. The red Hopf algebra is equal to $H^{\bullet\sigma}$, so this follows from Lemma 2.3.14. □

Corollary 5.3.14. *A type 2 Hadamard $\mathcal{H} : H \to H$ will be a type 1 Hadamard if*

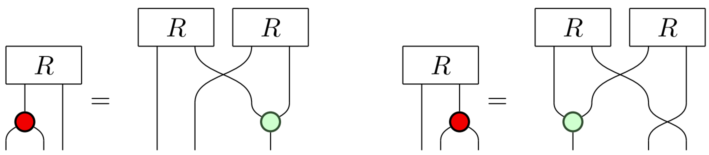


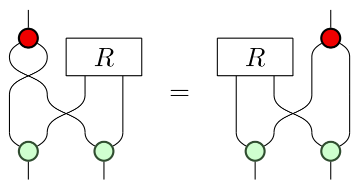
Proof. A type 2 Hadamard is an LR to RL morphism, so this follows from Definition 5.3.5 and Corollary 5.3.10. Note that we are not stating that a type 1 Hadamard is a type 2 Hadamard when the above condition is fulfilled. □

A *quasitriangular Hopf algebra* is a Hopf algebra equipped with points, R, R^{-1} : $I \to H \otimes H$ such that R^{-1} is the multiplicative inverse of R – i.e.



and the following equations are met

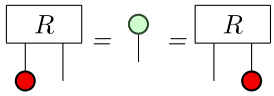


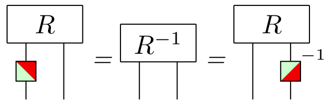


This point R is called the *R-matrix* of H. We shall show later how an R-matrix is similar to a type 2 Hadamard.

Quasitriangular Hopf algebras were introduced by Drinfeld [25], where he showed that a Hopf algebra H has an R-matrix if and only if the category of modules of H is braided monoidal.

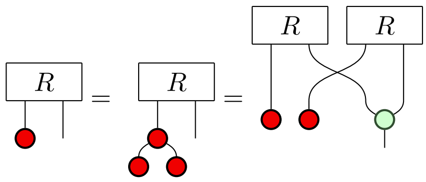
Lemma 5.3.15. *For quasitriangular Hopf algebra H, with R-matrix $R : I \to H \otimes H$, we have*



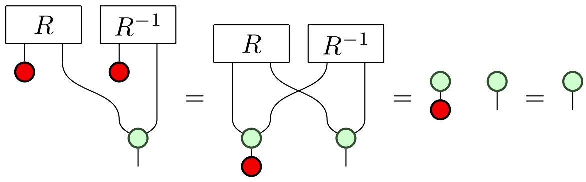


Proof. We begin by proving the first equations. We shall only prove the left equality, as the right equality follows similarly.

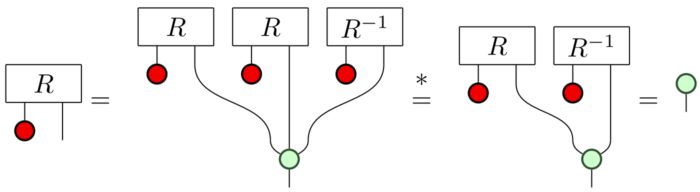
Note first that



We then see that

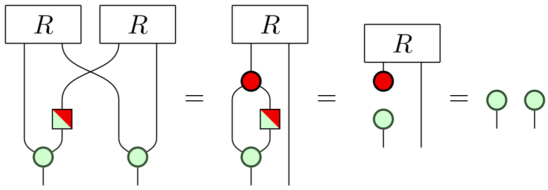


These two facts tell us that ( $\otimes\, 1) \circ R$ is idempotent and it has an inverse with respect to the multiplication . Together, this informs us that it must be equal to the multiplicative unit.



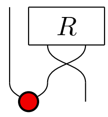
where $*$ follows from the fact that ( $\otimes\, 1) \circ R$ with respect to the multiplication.

For the second equations, we again will prove the left equality, as the right equality follows similarly. We first note that, since R^{-1} is the multiplicative inverse of R on $H \otimes H$, we only need to show that our proposed morphism acts as the inverse of R, since inverses are unique. This follows straightforwardly.



Hence, we have our results. $\square$

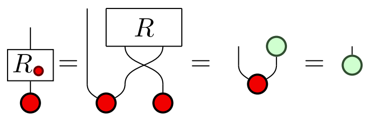
Definition 5.3.16. We define $R_{\bullet}$ as

$R_{\bullet} :=$ 

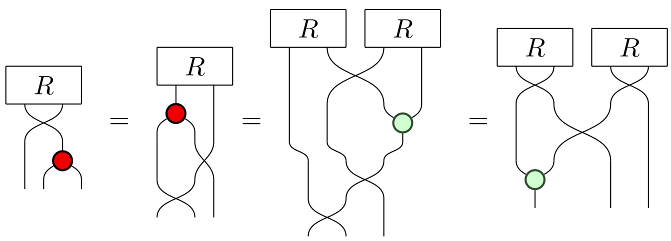
Lemma 5.3.17. *The morphism $R_{\bullet} : H \to H$ is a Hopf algebra homomorphism from the red Hopf algebra to the green Hopf algebra.*

Proof. To prove this, we must show that $R_{\bullet}$ is a bialgebra homomorphism. We shall show that $R_{\bullet}$ preserves the structure of the green comonoid, as the proof for the monoid is similar.

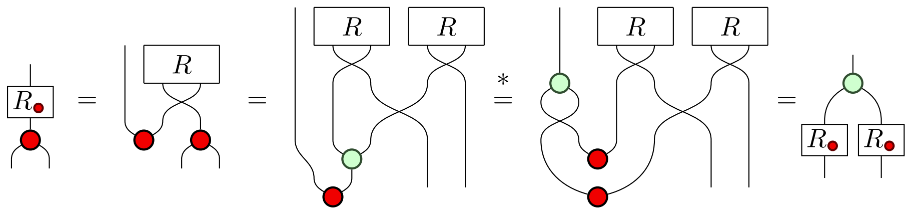
The proof for the counit is simple



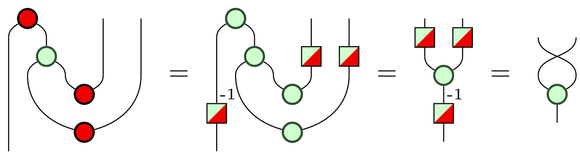
For the comonoid, first note that



Therefore, we see that



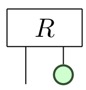
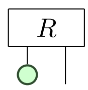
where at *, we use the fact that



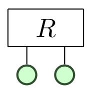
where we have used Corollary 5.2.13. Hence, the result follows. ☐

In other words, the R-matrix of a quasitriangular Hopf algebra gives us a morphism that is close to a type 2 Hadamard morphism, except it is not invertible. This motivates the following corollary.

Lemma 5.3.18. *For quasitriangular Hopf algebra H, $R_\bullet$ is a type 2 Hadamard if and only if $R_\bullet$ preserves integrals. This is equivalent to when the following morphisms*

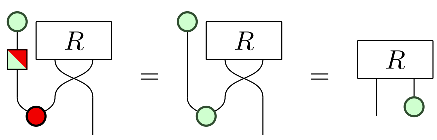
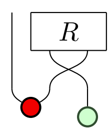
 and 

are left and right cointegrals of the green Hopf algebra respectively, and the following scalar is invertible

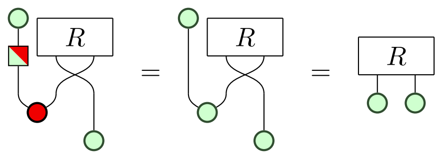


Proof. This follows on from Lemma 5.3.17 and Corollary 5.3.7.

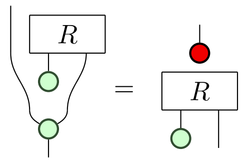
Note that since the red Hopf algebra and green Hopf algebra are RL and LR integral Hopf algebras respectively, which means that $R_\bullet$ is an RL to LR morphism. Hence we require that

 and 

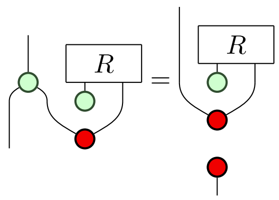
are a left cointegral of the green Hopf algebra of H, and a right integral of the red Hopf algebra of H respectively, and



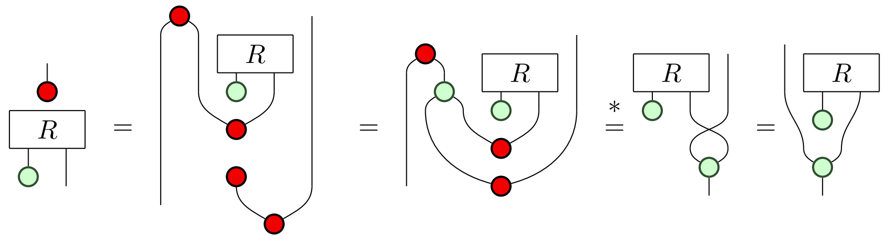
is invertible. Hence, all we have left to prove is that



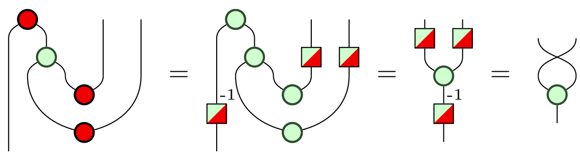
We already have



Hence, it follows that



where at $*$, we use the fact that



where we have used Corollary 5.2.13. □

Note that R is always invertible with respect to the multiplication on $H \otimes H$. However, this does not necessarily imply that $R_\bullet$ will be invertible with respect to composition.

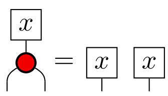
5.4 Relation to Finite Dimensional Hopf algebras

We now introduce one aspect of the significance of Hopf-Frobenius algebras. In this section, we will show that a Hopf-Frobenius algebra recovers much of the classic theorems of finite dimensional Hopf algebras.

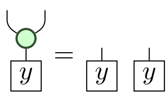
In this section, we will be including the definitions of group-like points [41] (Definition 5.4.1), unimodularity [41] (Definition 5.4.8), the Nakayama automorphism [?] (Definition 5.4.12) and semisimplicity [41](Definition 5.4.19). These are all well known concepts with well known properties in Hopf algebra theory. The proofs that we used in this section were all done independently, with the exception of Lemma 5.4.23, and they differ significantly from their counterparts. From Definition 5.4.15 until Corollary 5.4.18 we define and use the morphism ν, and we use it to demonstrate similarities between symmetric Frobenius algebras (Definition 2.4.1) and semisimple Hopf algebras (Definition 5.4.19). We later discovered that this morphism was originally defined by Radford [52].

Let us recall the definition of a right cointegral, as in Definition 4.2.1. Note that we have kept to left cointegrals and right integrals in this thesis. Hence, all of the previous results of this thesis are dependant on this choice. However, there are clearly equivalent results if we were to instead use right cointegrals or left integrals.

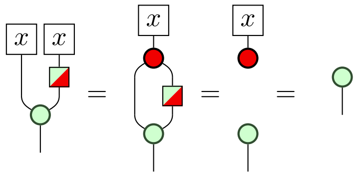
Definition 5.4.1. Let $x : I \to H$. We say that x is *group-like* when



In the same way, we would refer to a copoint $y : H \to I$ as group-like.



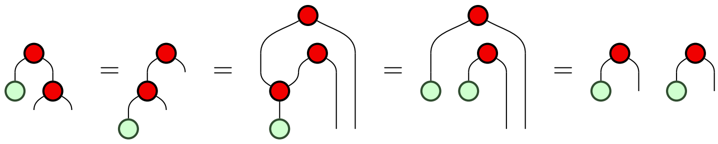
When $x : I \to H$ is group-like, if  $\circ\, x = 1_I$ then  $\circ\, x =$  $^{-1} \circ x$ is the multiplicative inverse of x, as follows. We provide the proof for , but the proof for $^{-1}$ is similar.



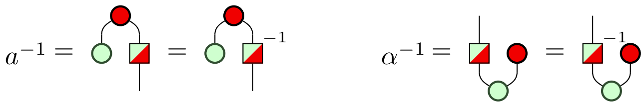
Likewise, when $y : H \to I$ is group-like, if $y \circ$ $= 1_I$ then $y \circ$ $= y \circ$ $^{-1}$ is the comultiplicative inverse of y.

Given a Hopf-Frobenius algebra H, we define the following point and copoint as the *distinguished group-like point* a and *copoint* α respectively

We will show that a is group-like, and a similar proof follows for α



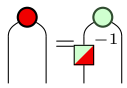
Hence, the above explanation tells us that the inverses of the distinguished group-like point and copoint are



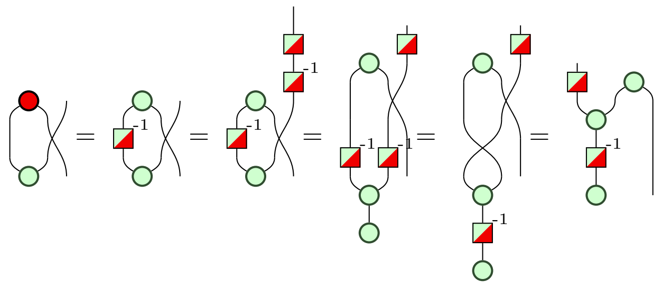
Note also that in literature related to ZX-calculus, the term *set-like* or *classical* may be used instead of group-like [18].

Lemma 5.4.2. *Let H be Hopf-Frobenius. Then*

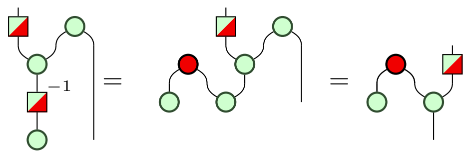
Proof. Recall from Corollary 5.2.13 that



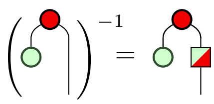
The proof follows pictorially



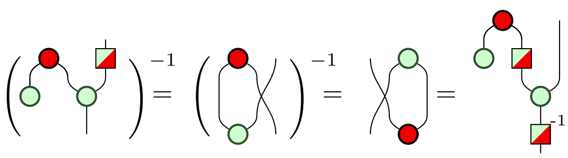
We then use the form of the antipode inverse (Remark 5.2.12) to get



Note the presence of the distinguished group-like point (Definition 5.4.1). We note that the distinguished group-like point is invertible, with the following inverse



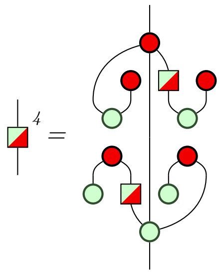
Since both the distinguished group-like point and the antipode are invertible, the inverse of our morphism is



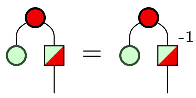
Hence, we have our result. □

Corollary 5.4.3. *Let H be Hopf-Frobenius. Then*

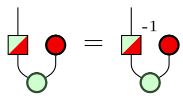
Lemma 5.4.4. *We may write the antipode as follows*



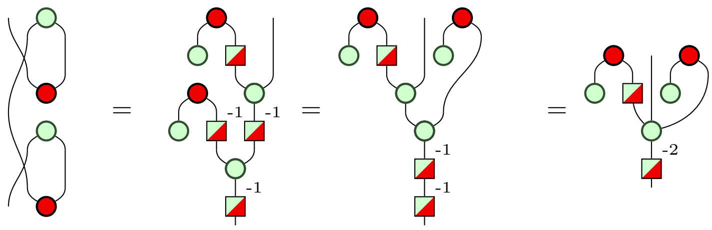
Proof. Recall from the definition of the distinguished group-like point and copoint (Definition 5.4.1) that the following morphisms



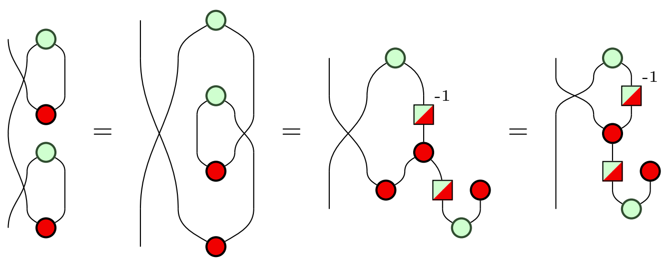
and



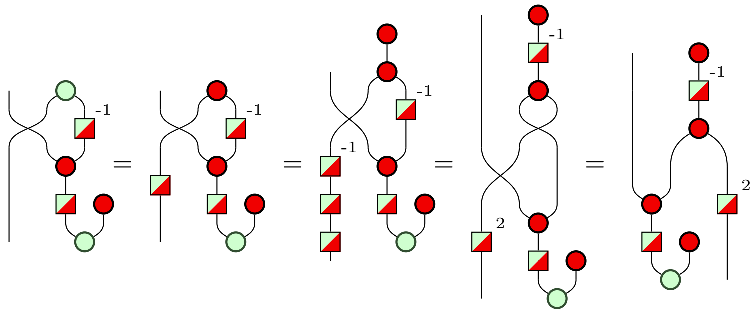
are equal to the inverse of the distinguished group-like point and copoint respectively. We use this with Lemma 5.4.2 to see that



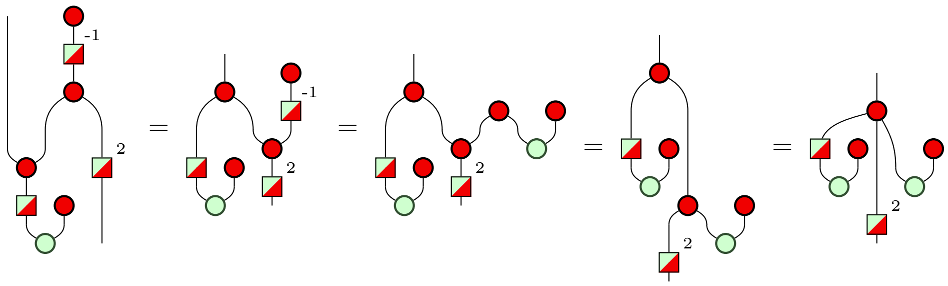
On the other hand, we use Corollary 5.4.3 to show that



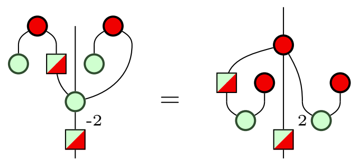
We use Corollary 5.2.13 to transfer between the green cup and the red cup, and then introduce another $^{-1}$ by using the fact that $\circ$ $^{-1} = 1$ to get



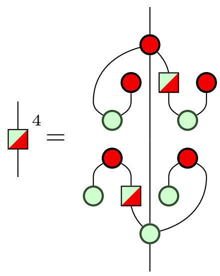
Finally, we use the form of $^{-1}$ from Remark 5.2.12 and see that



Hence, we have shown that

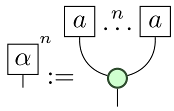


Since both the antipode and the group-like points and copoints are invertible, we may therefore derive our result

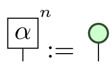


□

Definition 5.4.5. We denote the powers of a point as follows. Given a point $a : I \to H$, we write

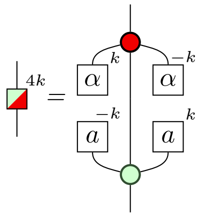


We say that the *order of a point* is the smallest finite n such that

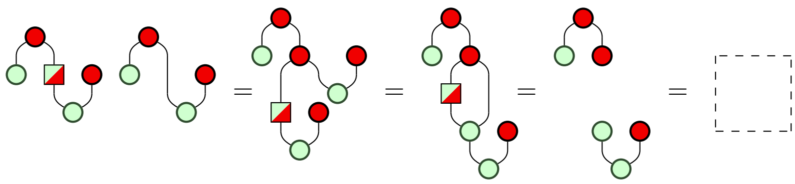


Note that not every point has an order.

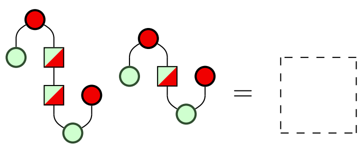
Lemma 5.4.6. *Let us denote the group-like point and copoint as a and α respectively. Then*



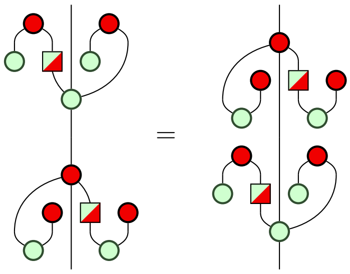
Proof. We begin by proving the following property of the distinguished group-like point and copoint.



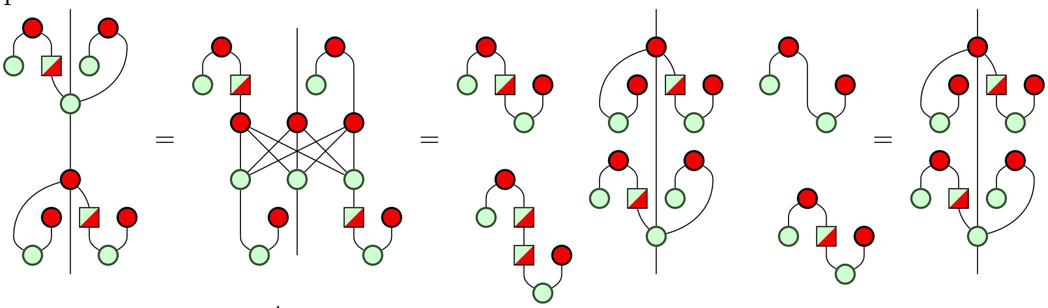
Where we have used the group-like property and the Hopf law. It follows similarly that



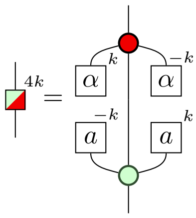
We will use this to show that



This follows from the bialgebra law and the properties of the group-like point and copoint.



Hence, when we take $\square^4$ to the power of k, and set the group-like point and copoint as a and α respectively, we get



□

Corollary 5.4.7 (Radford)**.** *Suppose that the distinguished group-like point and copoint have finite orders n and m respectively, and set l as the lowest common multiple of n and m. Then the order of the antipode will be a divisor of $4l$.*

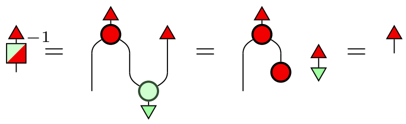
Radford's theorem [51] is a staple of finite dimensional Hopf algebra theory, so it is interesting that this theorem may be generalised to the Hopf-Frobenius case. We will now show how concepts such as unimodularity and semisimplicity may be generalised to the Hopf-Frobenius case, and how symmetry of the Frobenius algebra links them.

Definition 5.4.8. Let Hopf algebra H have left cointegral ↑ . We say that H is *unimodular* when is also a right cointegral. Likewise, we say that H is counimodular when right integral is a left integral.

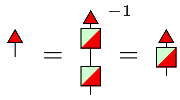
Lemma 5.4.9. *Suppose that H is an integral Hopf algebra. Then H is unimodular*[6] *if and only if* $\circ$ $=$.

Proof. It is clear from the definition of unimodular (Definition 5.4.8) and Lemma 4.2.6 that when $\circ$ $=$, H must be unimodular.

On the other hand, suppose that H is unimodular. Consider the following identity



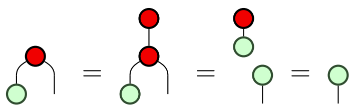
The result follows from this, as



□

Corollary 5.4.10. *When Hopf-Frobenius algebra H is unimodular, the distinguished group-like point and copoint are equal to the unit and counit respectively.*

Proof. This follows immediately from the definition



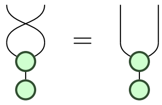
□

Corollary 5.4.11. *When Hopf-Frobenius algebra H is unimodular and counimodular, the antipode has an order of 4*

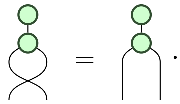
Proof. This follows from Corollaries 5.4.7 and 5.4.10. □

[6] The terminology *unimodular* comes from the way that unimodular Hopf algebras are analogous to unimodular Lie groups [41]

Recall that the definition of a symmetric Frobenius algebra (Lemma 2.4.3) is that

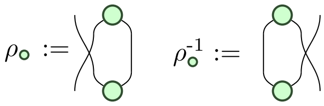


or, equivalently

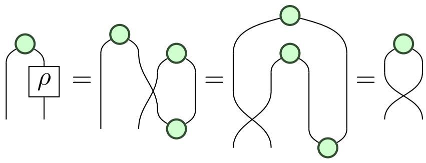


Since a Hopf-Frobenius algebra H has two Frobenius structures, a green Frobenius algebra (H, , , ,) and a red Frobenius algebra (H, , , ,), we shall say that H is -*symmetric* when the green Frobenius algebra is symmetric. Likewise, we shall say that H is -*symmetric* when the red Frobenius algebra is symmetric. When it is unambiguous which structure we are talking about, we shall simply say that H is symmetric.

Definition 5.4.12. Consider the green Frobenius algebra of H, (H, , , ,). Let us define the -*Nakayama automorphism*, $\rho_{\circ} : H \to H$ and its inverse as



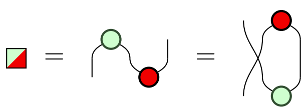
We define the -*Nakayama automorphism*, $\rho_{\bullet}$, in a similar manner. It is clear from the definition that the Nakayama automorphism has the following property



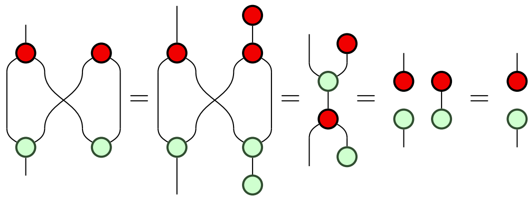
Therefore, it is equal to the identity if and only if H is symmetric

Lemma 5.4.13. *Let H be Hopf-Frobenius. Then if H is unimodular, the square of the antipode is equal to the -Nakayama automorphism*

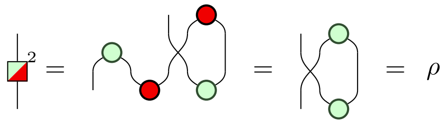
Proof. Suppose that H is unimodular. Then we have another form for the antipode



as follows



When we compose this with the standard form for the antipode (Remark 5.2.12), we get

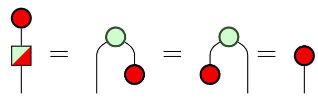


□

Lemma 5.4.14. *Let H be Hopf-Frobenius. Then H is ○-symmetric if and only if H is unimodular and* ◪$^2 = 1$.

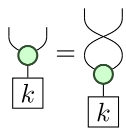
Proof. Lemma 5.4.13 tells us that when H is unimodular, ◪$^2 = \rho_{\circ}$. Since $\rho_{\circ} = 1$ if and only if H is ○-symmetric (See Definition 5.4.12), we see that unimodularity implies that ◪$^2 = 1$ if and only H is ○-symmetric. This tells us that unimodularity and ◪$^2 = 1$ imply ○-symmetry. Note that by duality, if H is counimodular and ◪$^2 = 1$, then H is ●-symmetric.

To show the converse, suppose that H has ○-symmetry. Then if we show that ○ -symmetry implies unimodularity, we also get that ◪$^2 = 1$. Simply note from Corollary 5.2.14 that

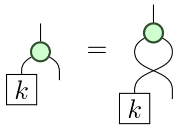


Hence, it follows from Lemma 4.2.6 that ● is a right cointegral, so H is unimodular. Hence, symmetry implies unimodularity, which in turn implies that ◪$^2 = 1$. Therefore, we have our result. □

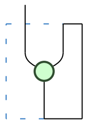
Definition 5.4.15. Let H be Hopf-Frobenius. We say that a copoint $k : H \to I$ is ○-*symmetrising* if



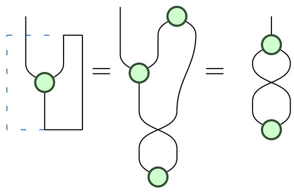
We say that k is -*cocentral* if



We denote the trace of  as $\nu : H \to I$, as follows

$\nu :=$ 

Note that to define ν, we only need a monoid (M, , ) with a trace Tr_M. Since F is a Frobenius algebra, we know it has a trace via Lemma 4.3.7. In this case, we may write

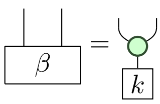


Lemma 5.4.16. *Let* (F, , , , ) *be a Frobenius algebra.*

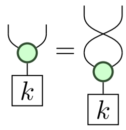
1. *There exists an additional Frobenius structure on F that is symmetric if and only if there exists some symmetrising, coinvertible copoint $k : H \to I$*

2. *If there exists a symmetrising copoint $k : H \to I$ that is both coinvertible and cocentral, then F is symmetric.*

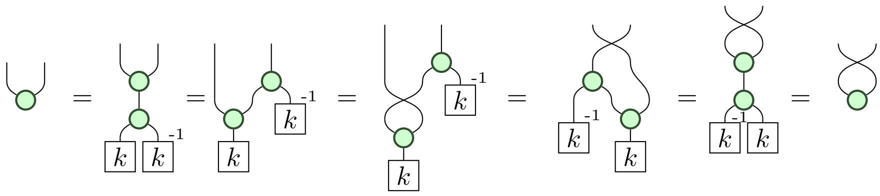
Proof. 1. Suppose that F has an additional Frobenius structure, $(F, \beta, \bar{\beta})$, where $\beta : F \otimes F \to I$ and $\bar{\beta} : I \to F \otimes F$ are symmetric. Proposition 2.4.12 tells us that there is a bijective correspondence between additional Frobenius structures on F and coinvertible copoints. Via Lemma 2.4.10 and Lemma 2.4.11, we see that we may construct a coinvertible copoint $k : F \to I$ such that



We are assuming that β is symmetric, so



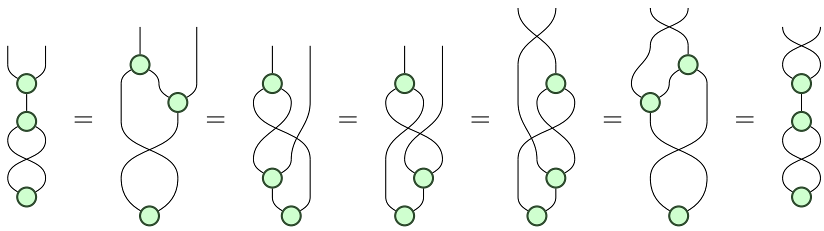
2. Note that if k is cocentral, this implies that k^{-1} is also cocentral. Hence



□

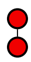
Lemma 5.4.17. *Let H be a Hopf-Frobenius algebra. Then ν is symmetrising.*

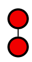
Proof. When we compose the cup with ν, we get the following equation



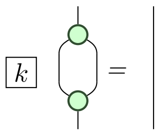
□

Corollary 5.4.18. *If the morphism ν is coinvertible and cocentral then H is* *-symmetric.*

Definition 5.4.19. Let H be a Hopf-Frobenius algebra. We say that H is *semisimple* when the scalar  has an inverse.

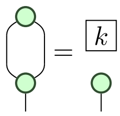
Semisimplicity is an important algebraic property for finite dimensional Hopf algebras. In $\mathbf{FVect_k}$, a semisimple algebra is equivalent to the direct sum of matrix algebras – See 2.2.17 in Kock [39]. For finite dimensional Hopf algebras, we find that the traditional definition of semisimplicity is equivalent  is non-zero, due to Maschke's theorem for Hopf algebras (See Larson and Sweedler [41]). On the other hand, special Frobenius algebras allow us to perform the *spider theorem*, and achieve a normal form for Frobenius terms – see Majid [45].

Lemma 5.4.20. *Recall the definition of quasispecial from Definition 2.4.1 that there exists an invertible scalar $k : I \to I$ such that*

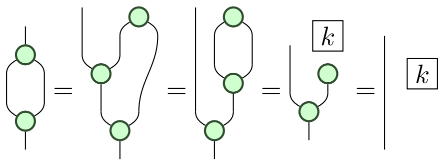


Let H be Hopf-Frobenius. Then H is semisimple if and only if the green Frobenius algebra is quasispecial.

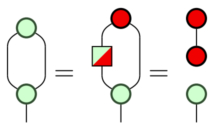
Proof. Suppose that there exists invertible scalar k such that

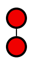
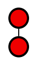


Then we see that



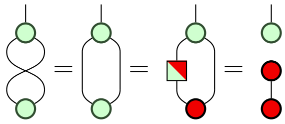
Hence, the above relation holds if and only if the green Frobenius algebra is quasispecial. We see that this equation relates to semisimplicity as



where we have used Corollary 5.2.13, and the Hopf law. We see that fulfils the definition of semisimplicity by setting $k =$  . Hence, H is semisimple if and only if  has an inverse, which is true if and only if the green Frobenius algebra of H is quasispecial. $\square$

Lemma 5.4.21. *If H is* *-symmetric, then ν is cocentral.*

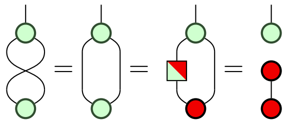
Proof. This follows in a similar manner to the proof that quasispecialness is equivalent to semisimplicity, as follows

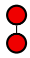


Where we have used Remark 5.2.12, and we see that  is the antipode of the red Hopf algebra in H. This morphism is clearly cocentral. □

Lemma 5.4.22. *If the morphism ν is coinvertible and cocentral then H is semisimple.*

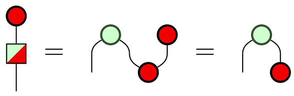
Proof. Via Lemma 5.4.18, we see that H is -symmetric. Hence we get that

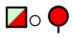


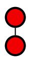
as in Lemma 5.4.21. ν is coinvertible, which implies that  has an inverse. Hence, H is semisimple. □

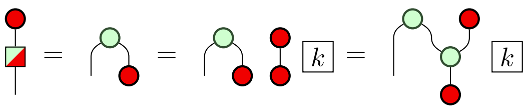
Lemma 5.4.23. *Let H be semisimple. Then H is unimodular.*

Proof. Recall from Corollary 5.2.14 and Lemma 4.2.6 that when we compose  with the antipode

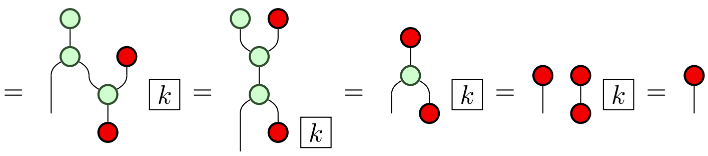


we get a right cointegral. We may show that H is unimodular by showing that  = .

Suppose that H is semisimple. Then there exists a scalar $k : I \to I$ that is the inverse of . Therefore



where we used the copy rule. We then use the Frobenius law to get



$\square$

As we have seen, there are strong links between $\bigcirc$-symmetry, semisimplicity, and ν. We see from Corollary 5.4.18 and Lemma 5.4.22 then when ν is cocentral and coinvertible, H is both $\bigcirc$-symmetric and semisimple. When H is $\bigcirc$-symmetric, Lemma 5.4.21 tells us that ν is cocentral.

To summarise

- H is $\bigcirc$-symmetric if and only if H is unimodular and $\boxslash^2 = 1$ (Lemma 5.4.14)
- If H is semisimple, then H is unimodular (Lemma 5.4.23)
- If ν is cocentral and coinvertible, then H is $\bigcirc$-symmetric and semisimple (Corollary 5.4.18 and Lemma 5.4.22)
- If H is $\bigcirc$-symmetric, then ν is cocentral (Lemma 5.4.21)

We see that both semisimplicity and $\bigcirc$-symmetry are linked together by ν. The proposal that semisimplicity implies that $\boxslash^2 = 1$ in $\mathbf{FVect_k}$ is known as the *Kaplinsky's fourth conjecture*, and is an open problem in the theory of Hopf algebras. We see that this conjecture is equivalent to the conjecture that semisimplicity implies $\bigcirc$-symmetry.

Here we see a related result: Suppose that it is possible to prove that when H is $\bigcirc$-symmetric, ν is coinvertible. Then $\bigcirc$-symmetry implies semisimplicity.

Chapter 6

Conclusions and Further Work

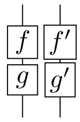
6.1 Conclusion

We will conclude this thesis with a summary of the results of this thesis, how they might be connected to other results in the literature and further work that may be done. We will begin by stating that, even though all of the examples of Hopf-Frobenius algebras that we have referenced in this thesis are in either $\mathbf{FPMod_R}$ or $\mathbf{FVect_k}$, the results of this thesis do apply outside of these two categories. Recall that, for any Hopf algebra for which we may define a trace, we can define the integral morphism (Definition 4.4.1). In Lemma 4.4.7 we discussed how we may construct an integral Hopf algebra using the integral morphism. Hence, the results of this thesis are relevant beyond the categories $\mathbf{FPMod_R}$ and $\mathbf{FVect_k}$.

Throughout this thesis, we have been using string diagrams to reason about Hopf-Frobenius algebras. One of the arguments of this thesis is that the string diagrams of Hopf-Frobenius algebras have advantages over the standard term language of Hopf-Frobenius algebras. For general string diagrams, the primary advantage can be seen in the following equation

$$(f \circ g) \otimes (f' \circ g') = (f \otimes f') \circ (g \otimes g')$$

This equation may be effectively ignored in string diagrams, as both sides of the equation are represented by the diagram



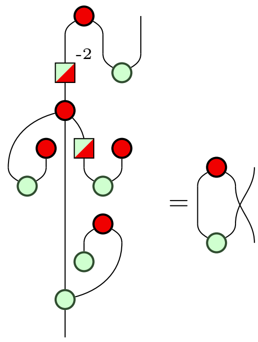
The way that we have used string diagrams in this paper may be contrasted with the way that Hopf algebras are typically reasoned with. Compare the proof of Radford's proof of the order of the antipode in the original paper with the one in this thesis. Our proof of Radford's theorem, (Corollary 5.4.7) uses only a handful of results (Corollary 5.2.13, Definition 5.4.1, Lemma 5.4.2, Corollary 5.4.3, Lemma 5.4.4 and Lemma 5.4.6), and the definition of a Hopf-Frobenius algebra. This is to be contrasted with Radford's proof of the order of the antipode [51] which is very involved. Part of the simplicity of our proofs comes from the fact that the only morphisms that we use in our proofs are the structural morphisms of Hopf-Frobenius algebras. For example, in Lemma 3 of [51], the proof of b) has the following equality for all $p \in H^*$:

$$\beta_l \beta_r^*(\gamma^{-2}(\alpha^{-1}(p \cdot a^{-1})\alpha)) = \beta_l \beta_l^*(p)$$

Let us have Hopf algebra $(H, \mu, e, \Delta, \epsilon, S)$ with integral $\int$ and cointegral Λ. In this statement we have

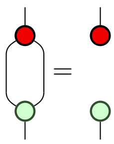
$$\beta_l(q) = \int \circ \mu(q, \_) \qquad \beta_r^*(p) = (p \otimes 1) \circ \Delta \circ \Lambda$$
$$\gamma = s^* \qquad p \cdot q = p \circ \mu(p, \_)$$

for all $p \in H^*$ and $q \in H$. The group like point and copoint are represented as a and α respectively. The equivalent statement in our presentation would be the following equation



So we see that string diagrams are often able to avoid this particular type of obfuscation.

As an example of the way that one may approach Hopf-Frobenius algebras differently as a result of their string diagrams, consider ZX calculus. The Hopf-Frobenius algebras of ZX calculus are commutative, cocommutative and special. The spider theorem tells us that connected Frobenius terms have a normal form, as we talked about in the introduction (See Section 1.2). Due to the spider theorem, we know that every connected Frobenius term may be represented as a graph with a single node. Because of this and because ZX-calculus has a trivial antipode – i.e.  $= 1_H$

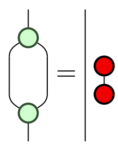


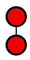
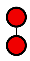
there will only be a single edge between any two vertices. To see this, suppose that we have a term that has $k > 1$ edges between two nodes. The nodes will be either red or green. If the nodes are the same colour, then they are part of the same Frobenius algebra, so by the spider rule, they may be joined. If they are different colours, then since the identity wire is the antipode, we may rewrite the graph term so that there are k mod 2 edges connecting the two nodes - i.e. either 0 or 1 edges.

This results in ZX-calculus terms being seen as graphs rather than algebraic terms, and instead of term rewriting, ZX-calculus uses *graph rewriting*. For example Kissinger and van der Wetering [38] optimise a quantum circuit with respect to T gates using a concept called *Gflow*, which is a graph theoretic property that is not easily stated algebraically. Using graph theoretic ideas makes it easier for ZX-calculus to state theorems about terms that involve a large number of structural morphisms. A quantum circuit is constructed from an arbitrarily large number of gates which, when they are from certain universal gate sets (such as Clifford+T), are each constructed from at least one structural morphism when represented in ZX-calculus. Hence, standard term rewriting is not suited for these kinds of equations. Even when computer scientists are not using ZX-calculus, they still use circuit diagrams [48].

This Hopf-Frobenius algebra that is used in ZX-calculus is particularly suited to being represented as a graph. As such, for the remander of this chapter, we will call Hopf-Frobenius algebras where the Frobenius algebras are both commutative and special *graph-like*[1].

In general, Hopf-Frobenius algebras are not represented as simple graphs, but it is still worth asking the question – can we use graph theory to develop tools in our proofs about Hopf-Frobenius algebras? In order to do this, we will need to state an appropriate graph representation for Hopf-Frobenius algebras. Recall from Lemma 5.4.20 that the Frobenius algebra of a Hopf-Frobenius algebra is quasispecial if and only if the Hopf algebra is semisimple. However, the proof tells us that it is always the case that



It is just the case that the scalar  will not always be invertible - i.e. in a non-semisimple Hopf algebra in $\mathbf{FVect_k}$,  is equal to 0. As such, in every Hopf-Frobenius algebra, the Frobenius terms may be described by the *planar spider theorem* (See Majid [45]) when there is no use of the symmetry morphism.

However, in a general symmetric monoidal category, planarity is not an assumption that we can make – indeed, the bialgebra rule will always causes wires to cross, making the graph non-planar. We also must contend with the fact that since the antipode is not equal to the identity morphism in general, the graphical representation of an arbitrary

[1]Note how we are not assuming that the antipode is equal to 1. While this is critical to the graph structure of ZX-calculus, it is not very common. Indeed, it does not hold for the ZX-calculus that describes qutrits [66]

Hopf-Frobenius morphism will not be a simple graph.

Since the graphs involved are not, in general, planar, we may describe the graphs by ordering the edges that are incedent to a given vertex – see Definition A.1.5 in the appendix. Compare this approach to work on rotation systems – see Altenmüller et al. [2] where a similar approach is taken to embed string diagrams into different topological surfaces. We conjecture that the graphs of Hopf-Frobenius algebras will have the following properties:

1. They are open graphs, where the domain and codomain of the graph, and of each of the vertices, are ordered.
2. We colour the vertices of the graph with two colours to indicate which Frobenius algebra the node belongs to.
3. We may label the edges with an integer n to indicate the presence of n antipodes.

The graphical representation becomes more simple for certain flavours of Hopf-Frobenius algebras. For example, if the algebra is either ○-symmetric or ●-symmetric, then ◩$^2 = 1$, so an edge will at most only ever have a single antipode on it.

While most graph theoretic concepts are not made with the idea that edges are ordered in mind, we believe that many standard graph theory definitions should be able to be generalised to this setting.

In this thesis, we introduced the notion of a traced family of objects in a symmetric monoidal category. This is a generalisation of the notion of a traced monoidal category. We introduced it so that we would not have to require that an entire category was traced, and model how finite dimensional vector spaces operate within $\mathbf{Vect_k}$. In the same way that in a monoidal category, a single object may have a dual without the entire category being compact closed, using this definition we may say that a single object has a trace without it belonging to a traced monoidal category. This concept leads directly onto the concept of half-duals. Half-duals emerged naturally when defining an integral Hopf algebra, and it allows for a concept in-between a trace and a dual. It is known that compact closed categories and traced monoidal categories are intimately connected, as every compact closed category is traced, and the INT construction on a traced monoidal category gives us a free compact closed category (See Joyal, Street and Verity [34]). An object with a half dual implies that there is a trace on that object, but it is strictly weaker than a full dual. This raises the question, what concepts does a half dual inherit from a full dual? For example, half duals are not sufficient to define an internal Hom, but we do get something similar - where when for all objects A, we have a dual, A^*, we have a natural isomorphism

$$\mathcal{C}(A \otimes B, C) \cong \mathcal{C}(A, C \otimes B^*).$$

When every object A has a half dual A^*, we have natural monomorphisms and epimorphisms

$$\begin{tikzcd}
\mathcal{C}(A \otimes B, C) \arrow[r, two heads] \arrow[dr, equal] & \mathcal{C}(A, C \otimes B^\times) \arrow[d, hook] \\
& \mathcal{C}(A \otimes B, C)
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}(A, B \otimes C) \arrow[r, two heads] \arrow[dr, equal] & \mathcal{C}(B^\times \otimes A, C) \arrow[d, hook] \\
& \mathcal{C}(A, B \otimes C)
\end{tikzcd}$$

Curiously, this implies that in a symmetric monoidal category where every object has half duals, then there is both a monomorphism and an epimorphism $A \to A^{\times\times}$. We have shown in Lemma 4.3.7 and Lemma 4.3.8 that the mere presence of a half dual implies that the trace is equal to the half dual trace. Indeed, if we have a pair of morphisms, $I \to A \otimes B$, $B \otimes A \to I$, and we construct a trace in the same way that we did with the half dual in Lemma 4.3.7, then the pair of morphisms must be half-duals. This might suggest that there is a construction similar to the Int construction (cf. Joyal, Street and Verity [34]) for half duals. In general, it remains to be seen exactly how half duals fit into the gap between traced monoidal categories and closed monoidal categories.

In Section 4.2, we introduced the concept of integrals, and showed that the presence of a cointegral is equivalent to the presence of a comultiplication that fulfils the Frobenius law in Lemma 4.2.12. This demonstrates the deep connection between integrals and Frobenius algebras. Integrals are a concept that has been well studied in the theory of Hopf algebras, and the connection between Frobenius algebras and integrals is well understood. We introduce the concept of an integral Hopf algebra as an intermediate concept between a standard Hopf algebra and a Hopf-Frobenius algebra. An integral

Hopf algebra inherits some concepts that a Hopf-Frobenius algebra has – an integral Hopf algebra has an invertible antipode, and half-duals. Due to Lemma 4.2.12 and Corollary 4.2.14, it has an extra multiplication and comultiplication. As of yet, we have not found any examples of an integral Hopf algebra that are not Hopf-Frobenius. There are no integral Hopf algebras in $\mathbf{Vect_k}$ that are infinite dimensional, as Lemma 4.3.7 tells us that the object must have a trace, and only finite dimensional vector spaces have a trace (to see this, note that the trace of the identity map does not converge for infinite dimensions). However, it would be interesting to find an example of such a Hopf algebra.

Finally, we provide several equivalent conditions for when a Hopf algebra is Hopf-Frobenius, summarised in Theorem 5.2.8. We believe that, if you have a Hopf algebra in a symmetric monoidal category, it is worth checking if the Hopf algebra is Hopf-Frobenius. Depending on the category, this may be relatively easy. First off, if you can show that it is impossible to define a trace on your Hopf algebra, then the Hopf algebra cannot be Hopf-Frobenius. If you can define a trace on the Hopf algebra, then you may construct the integral morphism (Definition 4.4.1). As we state in Lemma 4.4.7, if we can find point p and copoint q such that $q \circ \mathcal{I} \circ p$ is invertible, then we may construct an integral Hopf algebra, and from there, it is trivial to check if the Hopf algebra is Hopf-Frobenius (It is Hopf-Frobenius if and only if it is nondegenerate – see Definition 5.1.1). If the category has equalisers or coequalisers, and H has a dual, then Lemma 5.2.5 tells us that we only need to check if a particular equaliser is a map $I \to H$.

The Larson-Sweedler theorem [41] tells us that every Hopf algebra in the category of finite dimensional vector spaces is a Hopf algebra. More generally, when we show that a particular class of Hopf algebras has a Frobenius algebra, we say that we are proving the Larson-Sweedler theorem for this class (For example, see [14, 35, 63]). As such, every class of Hopf algebra in a symmetric monoidal category for which the Larson-Sweedler theorem has been proven is also Hopf-Frobenius.

As we showed in section 5.4, we may transfer several classic theorems of Hopf algebras into the language of Hopf-Frobenius algebras. This means that the theory of Hopf-Frobenius algebras is already fairly well developed. When you prove that a

Hopf algebra is Hopf-Frobenius, then this Hopf algebra will inherit many results from the theory of finite dimensional Hopf algebras. A potential avenue for further work would be to develop this statement and make it more explicit. In general, whenever a proof of Hopf algebras only uses morphisms constructed from the structural maps of Hopf algebras, integrals and cointegrals, then this may be straightforwardly translated into the language of Hopf-Frobenius algebras. There is less of a guarantee that this is possible for proofs that use, for example, properties of abelian categories.

6.2 Further Work

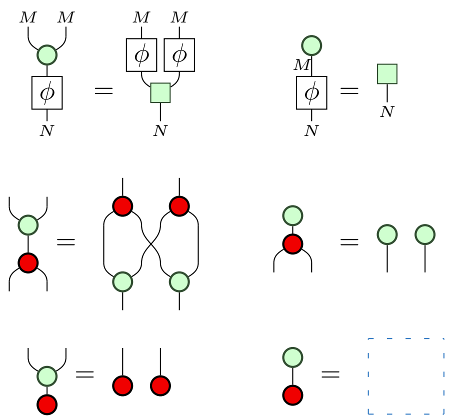
This work has defined and explored the construction of a Hopf-Frobenius algebra, and shown its connection to finite dimensional Hopf algebras in $\mathbf{Vect_k}$. We now explore several future directions for Hopf-Frobenius algebras.

In their work on interacting Hopf algebras, Zanasi et al. [13] proved an equivalence between the PROP of graph-like Hopf-Frobenius algebras and the PROP of linear relations, $\mathbf{LinRel}_{\mathbb{Z}}$ where we are setting the ring to be $\mathbb{Z}$, as we established in Section 1.4 of the Introduction. This implies that, given a symmetric monoidal category $\mathcal{C}$, every graph-like Hopf-Frobenius algebra in $\mathcal{C}$ has an associated functor from $\mathbf{LinRel}_{\mathbb{Z}}$ to $\mathcal{C}$ that preserves the structure of the linear relations. For example, given the cyclic group of order n, $\mathbb{Z}_n$, we may construct the group algebra in $\mathbf{FVect_k}$. This group algebra will be a graph-like Hopf-Frobenius algebra, via Example 4.1.2. There is a functor $\mathbf{LinRel}_{\mathbb{Z}} \to \mathbf{FVect_k}$ that maps the matrix $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ to the group multiplication .

This tells us that every graph-like Hopf-Frobenius algebra has an underlying structure of $\mathbf{LinRel}_{\mathbb{Z}}$. It also implies that every term in the language of graph-like Hopf-Frobenius algebras has a corresponding linear relation, and two terms are equal if and only if their respective linear relations are equal. This is fascinating, as it implies that to work out if two morphisms are equal in the PROP of graph-like Hopf-Frobenius algebras, instead of using rewrite rules one can prove equality of subspaces of $\mathbb{Z}$. This begs the question, can this be done with ZX-calculus? ZX-calculus is a graph-like scaled Hopf-Frobenius algebra, so for terms that only use the structure maps, the PROP of

ZX-calculus is equivalent to $\mathbf{LinRel}_{\mathbf{R}}$ where $R = \mathbb{Z}_2$ such that we set the scalars of ZX-calculus equal to 1. However, the inclusion of the phases and the Hadamard gate, as defined in Section 1.6, means that this equivalence no longer holds (This follows from Duncan and Dunne [26]). Still, is it possible that this equivalence may be utilised to make proving equality of two circuits in ZX more efficient?

We find that the construction done by Zanasi et al. [13] is more generic than merely applying to the PROP of matrices and $\mathbf{LinRel}_{\mathbb{Z}}$. Given a category with biproducts, every object has a unique bialgebra associated with it. To see this, consider Example 2.3.3, where we show how in a category with products every object has a unique comonoid associated with it. By duality, we know that when a category has coproducts, every object must have a unique monoid with respect to the coproduct. Recall that we may define a bialgebra $(B, \mu, e, \delta, \epsilon)$ as a monoid (B, μ, e) and comonoid (B, δ, ϵ) such that $\delta : B \to B \otimes B$ and $\epsilon : B \to I$ are monoid homomorphisms. To see this, note the similarities between the homomorphism equations and the bialgebra equations



When a category has products then the comonoid must be *natural*. This means that every morphism in the category must act as a comonoid homomorphism. The same is true for categories with coproducts with respect to the canonical monoid. In particular, this means that in a category with biproducts, the comonoid must be a monoid homomorphism. Hence, in a category with biproducts, every object has a unique bialgebra. Since each bialgebra B is unique for each object, we see that both B^{op} and B^{σ} must be equal to B. Hence, the bialgebra must be commutative and cocommutative.

This ends up making the category enriched in commutative monoids. To see this, recall how we defined the convolution algebra in Definition 2.5.5. Every morphism in this category is a bialgebra morphism, so the entire category is enriched in commutative monoids.

The bialgebra becomes a Hopf algebra if and only if this category is enriched in abelian groups. To see this, note that when we have a category with biproducts such that each object is a Hopf algebra, the antipode acts as an inverse operator we see in Definition 2.5.5. For the other direction, suppose that we have a category $\mathcal{C}$ with biproducts that is enriched in abelian groups. Then for each object B, the homset $\mathcal{C}(B, B)$ is a group. Set the antipode of B to be -1_B. It follows that B is therefore a Hopf algebra.

Finally, if this category is regular, then we may take the category of internal relations (via Carboni and Walters [16]), then the comonoid will become a Frobenius algebra. Since every Hopf algebra that is Frobenius is a Hopf-Frobenius algebra, via Corollary 5.2.10, we find that in such a category, every object has a unique graph-like Hopf-Frobenius algebra. Every abelian category has biproducts, is enriched in abelian groups, and is regular. Hence, the category of relations of any abelian category has this structure. This work was explored in unpublished work by Spivak and Fong [58].

It was shown by Zanasi et. al. [13] that since the morphisms of $\mathbf{LinRel}_{\mathbb{Z}}$ are subspaces, we may describe the kernel and image of morphisms in the language of graph-like Hopf-Frobenius algebras. This was expanded by Spivak and Fong, where they showed that it was possible to prove theorems about abelian categories using this language. It remains an open question on how much further can we take this work? It also suggests that it may be possible to prove theorems about Hopf-Frobenius algebra using abelian categories.

Appendix A

Appendix

A.1 Graphical Language

In the following section, we shall develop the graphical language that we will use in most of our proofs. String diagrams were formally defined by Joyal and Street [33], and we shall use them in this thesis because we find that the algebraic equations of Hopf algebras and Frobenius algebras become much easier to follow when they are presented graphically than in their typical symbolic language. We define an open graph (Definition A.1.1), and from this definition, we may define a string diagram (Definition A.1.5). We show how to compose string diagrams, and therefore we may construct a category of string diagrams. This allows us to state Theorem 2.2.5, which says that every term in the language of monoidal categories has a corresponding string diagram, meaning that string diagram notation is equivalent to the conventional symbolic language.

We will begin by defining an open graph. We may think of an open graph as a graph that can be composed with other open graphs. This composition is done by joining edges together. This is done by allowing the edges to not necessarily be connected to a vertex - i.e. the source and target functions are partially defined. We may then compose two open graphs, G_1 and G_2 by identifying the edges in G_1 that don't have a target with the edges in G_2 that don't have a source. However, even though this is the motivation behind how we define open graphs, we shall only formally define composition after we have formally defined string diagrams.

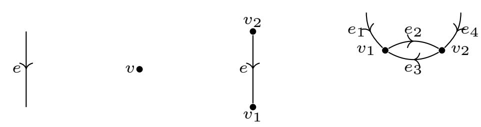
Definition A.1.1. An *open graph* G consists of a set of *vertices*, V and a set of *edges* E, and two partially defined functions, $s, t : E \to V$, the *source* and *target* functions. When we are dealing with multiple graphs, these components will be denoted with the appropriate subscript – V_G, s_G for example. Let $e \in E$. When we write a formula using the term $s(e)$ or $t(e)$, it is to be understood that there is an implicit assumption that $s(e)$ or $t(e)$ is defined.

In G, there will be a subset of E which are the edges that do not have a source – i.e. $s(e)$ is not defined. We call this subset the *domain of* G, and denote it as $\operatorname{dom} G$. We similarly define the *codomain of* G as the subset of E such that $t(e)$ is not defined, and denote it as $\operatorname{cod} G$. The intuition behind this is that, when we have graphs G_1 and G_2, such that $\operatorname{cod} G_1$ is the same as $\operatorname{dom} G_2$, then we may compose G_1 and G_2 by identifying $\operatorname{cod} G_1$ with $\operatorname{dom} G_2$.

Let v be a vertex. We set the *domain* (resp. *codomain*) *of* v, denoted $\operatorname{dom} v$ (resp. $\operatorname{cod} v$), as the set of edges e for which $s(e) = v$ (resp. $t(e) = v$). We do this because when we later define string diagrams, vertices will be labelled with morphisms, and the domain and codomain of v will be identified with the domain and codomain of those morphisms.

A *path* in an open graph G is a finite sequence $p = e_1, \ldots, e_n$ of edges $e_i \in E$ such that $t(e_i) = s(e_{i+1})$. We say that $s(p) = s(e_1)$, and $t(p) = t(e_n)$. A *cycle* is a path where $s(p) = t(p)$. Given paths p_1 and p_2 such that $t(p_1) = s(p_2)$, we may *append* p_1 and p_2, denoted $p_1@p_2$, by appending their lists. We say that q is a *subpath* of p if there exists paths p_l and p_r such that $p_l@q@p_r = p$.

Example A.1.2. We draw open graphs as follows, where edges are drawn as wires, and vertices are denoted as dots.



Above, we have four open graphs. From left to right, the open graphs above are

1. A graph consisting of a single edge, e, and no vertices. The domain and codomain of the graph will therefore both contain e.

2. A graph with no edges and a single vertex, v. The set of edges is empty, so the domain and codomain of the graph is also empty.

3. A graph with two vertices, $\{v_1, v_2\}$ and a single edge e such that $s(e) = v_1$ and $t(e) = v_2$. Since $s(e)$ and $t(e)$ are both defined, the domain and codomain of the graph will be empty.

4. A graph with two vertices, $\{v_1, v_2\}$ and four edges, $\{e_1, e_2, e_3, e_4\}$. The source and target functions are defined

$$t(e_1) = t(e_3) = v_1 \qquad t(e_2) = t(e_4) = v_2$$
$$s(e_3) = v_2 \qquad s(e_2) = v_1$$

Since $t(e)$ is defined for all edges e, the codomain of the graph will be empty. However, both $s(e_1)$ and $s(e_4)$ are undefined, so $\mathrm{dom} = \{e_1, e_4\}$.

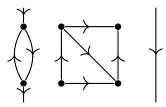
Note how in the rightmost graph we have a cycle – the path e_2, e_3 is a cycle, since $s(e_2) = t(e_3) = v_1$.

Definition A.1.3. Let G be an open graph. We define a relation $\sim$ on the set $V + E$, where

$$v \sim e \qquad \text{if } s(e) = v \text{ or } t(e) = v$$

We then take the transitive, symmetric and reflexive closure of the above relation. We say that if $x \sim y$, then x is *connected* to y. The set of *connected components of* G is the set of equivalence classes generated by $\sim$.

For example, if we consider the following diagram as a single graph



then the above graph contains 3 connected components.

Lemma A.1.4. *The connected components of an open graph either contain a vertex, or are a single edge e with both $s(e)$ and $t(e)$ undefined.*

Proof. Let G be an open graph. Suppose that we have connected component c, and c does not contain a vertex. A connected component is an equivalence class of $V + E$, and c is non empty, c must only contain edges. Let e be an edge in c. If $s(e)$ or $t(e)$ were defined, then c would contain a vertex. Hence, there is no d in c such that $e \sim d$, other than $d = e$. Hence we have proven our result. □

Definition A.1.5. Let $\mathcal{C}$ be a symmetric monoidal category. A *string diagram* over $\mathcal{C}$ is an open graph G such that

1. The sets $\operatorname{dom} G$ and $\operatorname{cod} G$ have a total order,
2. The sets $\operatorname{dom} v$ and $\operatorname{cod} v$ have a total order for each $v \in V$,
3. There are no cycles in G,
4. We equip G with a function that assigns each edge $e \in E$ to some object $O(e)$ in $\mathcal{C}$. We say that e is *labelled* by $O(e)$.
5. Let $v \in V$, and set

$$D(v) := \bigotimes_{e \in \operatorname{dom} v} O(e), \text{ and } C(v) := \bigotimes_{e \in \operatorname{cod} v} O(e).$$

   such that if $\operatorname{dom} v$ is empty, then $D(v) = I$. Likewise, if $\operatorname{cod} v$ is empty then $C(v) = I$. We equip G with a function that maps each $v \in V$ to some $M(v) \in \mathcal{C}(D(v), C(v))$. We say that v is *labelled* by $M(v)$.

We denote the set of string diagrams over $\mathcal{C}$ as $\mathrm{SD}(\mathcal{C})$.

Let us define functions $D, C : \mathrm{SD}(\mathcal{C}) \to obj\mathcal{C}^\star$, where $obj\mathcal{C}^\star$ is the set of finite lists of objects of $\mathcal{C}$. We denote elements of $obj\mathcal{C}^\star$ as bold letters (i.e. $\mathbf{A}$) or as a list of objects (i.e. $[A, \ldots, C]$). Recall that $\operatorname{dom} G$ is an ordered set of edges, $e_1 < \ldots < e_n$. The function $D(G)$ is defined as the list $[O(e_1), \ldots, O(e_n)]$. Likewise, $C(G)$ is the list $[O(e'_1), \ldots, O(e'_m)]$ where $e'_1 < \ldots < e'_m$ are the edges in $\operatorname{cod} G$. We denote that a graph has $D(G) = \mathbf{A}$ and $C(G) = \mathbf{B}$ by writing $G : \mathbf{A} \to \mathbf{B}$. We note that by using morphism notation, we could conceivably call both $\operatorname{dom} G$ and $D(G)$ the domain of G. However, in this thesis, we shall stick to calling $\operatorname{dom} G$ the domain of G.

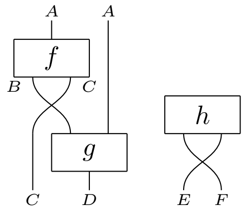
Remark A.1.6. In this thesis, we will be using string diagrams to reason about symmetric monoidal categories. It is shown in Selinger [55] that it is possible to use an alternate definition of a string diagram to talk about *compact closed categories* (Definition 2.3.9) and *traced monoidal categories* (Definition 3.0.3). In this thesis, we will refer to compact closed categories and traced monoidal categories, but we will not change our definition of string diagram. Instead, we will treat them in a similar manner as other equational theories, as in Definition 2.2.6.

Definition A.1.7. In general we will depict string diagrams in the following manner

- Edges are depicted as wires, and vertices are depicted as shapes (i.e. boxes or circles).
- If edge e has $s(e) = v$, then the wire depicting e will be drawn going into the top of the shape depicting v. Likewise, if $t(e) = v$, then e is drawn coming from the bottom of v.
- When $e \in \operatorname{dom} G$, then e will be drawn emerging from the top of the picture. Likewise, for all $e \in \operatorname{cod} G$, e is drawn going into the bottom of the picture.
- The order of $\operatorname{dom} G$ is depicted as the order in which the edges are drawn from left to right. So if $e_i < e_j$, then the place that e_i emerges from will be to the left of e_j. Similar conventions hold for $\operatorname{cod} G, \operatorname{dom} v$ and $\operatorname{cod} v$, where if $e_i < e_j$, then the place that e_i emerges from or descends to will be to the left of e_j.
- If $O(e) = A$, then in general we will write A next to the side of e. However, we will avoid doing this when $O(e) = A$ for all edges e.
- If $M(v) = f$, then we will typically write f in the centre of the shape depicting v. An exception to this is that we will occasionally fill the shape depicting v with a colour. In this case, we will make it clear that $M(v) = f$ in another manner. For

example, in Definition 2.3.1, we have a multiplication map $\mu : H \otimes H \to H$, and we explicitly state that we are setting $\mu =$  .

Example A.1.8. Let $f : A \to B \otimes C$, $g : B \otimes A \to D$ and $h : I \to E \otimes F$ be morphisms in $\mathcal{C}$. Consider the following diagram



This is the string diagram that represents the morphism term

$$((1_C \otimes g) \circ ((\sigma_{B,C} \circ f) \otimes 1_A)) \otimes (\sigma_{E,F} \circ h)$$

We may infer from the above picture how the string diagram is defined. It is clear that this is a string diagram $G : [A, A] \to [C, D, E, F]$. Edges are represented by wires, so we have 7 edges, $e_1, \ldots, e_7$. Boxes represent vertices, so we have three vertices, v_1, v_2 and v_3. The source and target functions are defined

$$t(e_1) = v_1 \qquad t(e_2) = t(e_4) = v_2$$
$$s(e_2) = s(e_3) = v_1 \qquad s(e_5) = v_2 \qquad s(e_6) = s(e_7) = v_3$$

The dom and cod sets have orders

$$\operatorname{dom} G : e_1 < e_4 \qquad \operatorname{dom} v_1 : e_1 \qquad \operatorname{dom} v_2 : e_2 < e_4$$
$$\operatorname{cod} G : e_3 < e_5 < e_6 < e_7 \qquad \operatorname{cod} v_1 : e_2 < e_3 \qquad \operatorname{cod} v_2 : e_5 \qquad \operatorname{cod} v_3 : e_7 < e_6$$

Finally, the labelling functions are defined as

$$O(e_1) = O(e_4) = A \qquad O(e_2) = B \qquad O(e_3) = C \qquad O(e_5) = D \qquad O(e_6) = E$$
$$O(e_7) = F$$

and

$$M(v_1) = f \qquad\qquad M(v_2) = g \qquad\qquad M(v_3) = h$$

Remark A.1.9. Notice how in the previous example, how well the graphical presentation captures the string diagram. Instead of defining our string diagrams explicitly, as we did in Example A.1.8, we will instead simply draw the picture of the string diagram, and infer from the picture what the string diagram is.

Definition A.1.10. Let $G : \mathbf{A} \to \mathbf{B}$ and $H : \mathbf{C} \to \mathbf{D}$ be string diagrams over $\mathcal{C}$. The *parallel composition* of G and H, is denoted $G \otimes H : \mathbf{A}@\mathbf{C} \to \mathbf{B}@\mathbf{D}$ where @ denotes the concatenation of lists. The set of edges of $G \otimes H$ is $E_G + E_H$, the set of vertices is $V_G + V_H$, and the labelling functions are $O_G + O_H$ and $M_G + M_H$. The order on $\operatorname{dom} G \otimes H$ and $\operatorname{cod} G \otimes H$ is defined such that

$$e \leq e' \text{ if and only if } \begin{cases} e, e' \in E_G \text{ and } e \leq e' \text{ in } \operatorname{cod} G \text{ or } \operatorname{dom} G \\ e, e' \in E_H \text{ and } e \leq e' \text{ in } \operatorname{cod} H \text{ or } \operatorname{dom} H \\ e \in E_G \text{ and } e' \in E_G \end{cases}$$

The source and target functions are defined as expected, where $s_{G\otimes H} := s_G + s_H$ and $t_{G\otimes H} := t_G + t_H$. Following from Definition A.1.7, we depict $G \otimes H$ as G parallel to the left of H.

Lemma A.1.11. *Let $G : \mathbf{A_1} \to \mathbf{B_1}$, $H : \mathbf{A_2} \to \mathbf{B_2}$ and $S : \mathbf{A_3} \to \mathbf{B_3}$ be string diagrams. Then*

1. *$(S \otimes G) \otimes H = S \otimes (G \otimes H)$ up to isomorphism of coproducts*
2. *$G \otimes H : \mathbf{A_1}@\mathbf{A_2} \to \mathbf{B_1}@\mathbf{B_2}$ is a string diagram.*

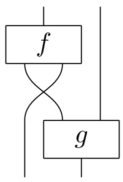
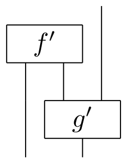
Proof. 1. This follows immediately from the definition of parallel composition. For example, the set of edges on the left hand side of the equation is $(E_S + E_G) + E_H$, while the set of edges on the right hand side is $E_S + (E_G + E_H)$. These sets are equal up to isomorphism of coproducts. The same holds for the set of vertices, the source and target functions, and the labelling functions.

2. As we have shown what the vertices and edges are, how the source and targets are defined, and what the orders are. It is clear how the type of $G \otimes H$ is $G \otimes H : \mathbf{A}@\mathbf{C} \to \mathbf{B}@\mathbf{D}$. Hence, all we need to show is that there are no cycles in $G \otimes H$.

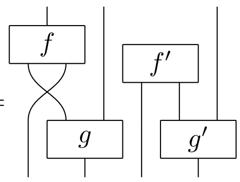
   Suppose that there exists a cycle $p = e_1, \ldots, e_n$ in $G \otimes H$. Without loss of generality, suppose that e_1 is G. Since there are no cycles in G, there must be some e_i in p that is in H. This implies that there is some e_j with $s(e_j) \in V_G$ and $t(e_j) \in V_H$. However, this is impossible. Hence, there are no cycles in $G \otimes H$.

□

Example A.1.12. Consider the following string diagrams

$G :=$  $H :=$ 

The parallel composition of these two string diagrams is depicted

$G \otimes H =$ 

Definition A.1.13. Let $G : \mathbf{A} \to \mathbf{B}$ and $H : \mathbf{B} \to \mathbf{C}$ be string diagrams over $\mathcal{C}$. Each edge in $\operatorname{cod} G$ has a corresponding member in the list $\mathbf{B}$, as does each edge in $\operatorname{dom} H$. Hence, there exists an order preserving isomorphism $\psi : \operatorname{cod} G \cong \operatorname{dom} H$ such that the following diagram commutes

$$\begin{tikzcd} \operatorname{cod} G \arrow[r, "O_G"] \arrow[d, "\psi"'] & \mathcal{C} \\ \operatorname{dom} H \arrow[ur, "O_H"'] & \end{tikzcd}$$

The *sequential composition* of G and H is denoted $H \circ G$ where

- The set of vertices is $V_G + V_H$

- The set of edges is the union of E_G and E_H where we identify $\operatorname{cod} G$ and $\operatorname{dom} H$ via the isomorphism ψ. We may equivalently define it as the pushout $E_G \cup E_H$ in the diagram

$$\begin{tikzcd} \operatorname{cod} G \arrow[r, hook] \arrow[d, "\psi"'] & E_H \arrow[d, hook] \\ E_G \arrow[r, hook] \arrow[ur, phantom, "\ulcorner", very near end] & E_G \cup E_H \end{tikzcd}$$

- The source and target functions are defined as expected, where the source function is defined as

$$s_{H \circ G}(e) = \begin{cases} s_G(e) \text{ if } e \in E_G \\ s_H(e) \text{ if } e \in E_H \end{cases}$$

  and the target function is defined in a similar manner. In particular, we are identifying the edges in $\operatorname{cod} G$ and $\operatorname{dom} H$, and for any edge e in these sets, if $s_G(e)$ is defined, then $s_H(e)$ is not defined, and if $s_H(e)$ is defined, then $s_G(e)$ is not defined. The same holds true for the target function.

- The labelling function for the vertices is $M_G + M_H$.

- For the labelling function of the edges, recall that for $e \in \operatorname{cod} G$, $O_G(e) = O_H(\psi(e))$. We are identifying the edges in $\operatorname{cod} G$ and $\operatorname{dom} H$, so it makes sense to define the labelling function as

$$O_{H \circ G}(e) := \begin{cases} O_G(e) \text{ if } e \in E_G \\ O_H(e) \text{ if } e \in E_H \end{cases}$$

- The orders of domains and codomains are as expected, where $\operatorname{dom} H \circ G = \operatorname{dom} G$, $\operatorname{cod} H \circ G = \operatorname{cod} H$, and

$$\operatorname{dom}_{H \circ G}(v) := \begin{cases} \operatorname{dom}_G(v) \text{ if } v \in V_G \\ \operatorname{dom}_H(v) \text{ if } v \in V_H \end{cases}$$

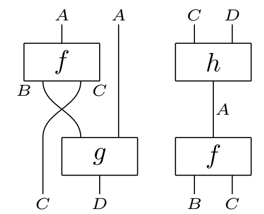
  and similarly for $\operatorname{cod}_{H \circ G}$.

Lemma A.1.14. *For* $\mathbf{X} = [X_1, \dots, X_n]$*, set* $1_\mathbf{X}$ *as the string diagram with* n *edges,* $e_1 < \dots < e_n$*, and no vertices, where* $O(e_i) = X_i$*. Then for all string diagrams* $G : \mathbf{A} \to \mathbf{B}$*, we have*

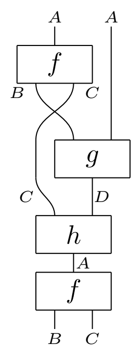
$$1_{\boldsymbol{B}} \circ G = G = G \circ 1_{\boldsymbol{A}}$$

Proof. In $G \circ 1_\mathbf{A}$, since $\operatorname{cod} 1_\mathbf{A}$ is the same as the edges of $1_\mathbf{A}$, we are simply identifying all of the edges of $1_\mathbf{A}$ with $\operatorname{dom} G$. Hence the composition $G \circ 1_\mathbf{A}$ will be equal to G. □

Example A.1.15. Consider the following string diagrams



The sequential composition of the above diagrams would be



Lemma A.1.16. *Consider string diagrams* G*,* H*, and* S *in* $\mathcal{C}$*. Then*

1. $H \circ G$ *is a string diagram*

2. $S \circ (H \circ G) = (S \circ H) \circ G$ *up to equality of colimits*

where the above terms are defined.

Proof. 1. The definition tells us how the structure of the string diagram is defined, we only need to prove that there are no cycles in $H \circ G$. Let $p = e_1, \dots, e_n$ be a path in $H \circ G$, and suppose that p is a cycle. Without loss of generality, we suppose that that $s(e_1) = t(e_n)$ is in V_G. If p is contained entirely within G or H, then p cannot be a cycle as G is a string diagram. Hence, since $v \in V_G$, there is exists some e in p such that $e \in E_H$.

If it were the case that $e_n \in \operatorname{cod} G = \operatorname{dom} H$ then $t(e_n) \in V_H$. Hence, $e_n \notin \operatorname{cod} G = \operatorname{dom} H$. Let p' be the path from e to e_n that is a subpath of p. Then there is some edge f in p' such that $s(f) \in V_H$ and $t(f) \in V_G$. This implies that $f \in E_G$ and E_H. Given the definition of sequential composition, $f \in \operatorname{dom} G = \operatorname{cod} H$. However, this would imply that $s(f) \in V_G$ and $t(f) \in V_H$. This is a contradiction, so p must not be a cycle.

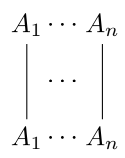
2. Both sides of the equation are equal as follows. The set of vertices is $V_G + V_H + V_S$. The set of edges of $S \circ (H \circ G)$ is $(E_G \cup E_H) \cup E_S$, which is a pushout. Therefore, this is equal to $E_G \cup (E_H \cup E_S)$ up to equality of colimits. The rest of the proof follows from similar reasoning.

□

Lemma A.1.17. *We may define a strict monoidal category of* string diagrams over $\mathcal{C}$, *denoted SD*$(\mathcal{C})$. *The objects of this category are finite lists* $[A_1, A_2, \ldots, A_n]$ *of the objects of* $\mathcal{C}$. *A morphism* $G : [A_1, \ldots, A_n] \to [B_1, \ldots, B_m]$ *is a string diagram* G *(Definition A.1.5) in* $\mathcal{C}$ *where*

- *There are* n *edges,* $e_1 < \ldots < e_n$ *in* $\operatorname{dom} G$, *with* $O(e_i) = A_i$.
- *Likewise, there are* m *edges,* $e'_1 < \ldots < e'_m$ *in* $\operatorname{cod} G$, *with* $O(e_i) = B_i$.

The identity morphism $1 : [A_1, \ldots, A_n] \to [A_1, \ldots, A_n]$ *is the string diagram with no vertices and* n *edges such that* $e_1 < \ldots < e_n$ *in both* $\operatorname{dom} G$ *and* $\operatorname{cod} G$ *and* $O(e_i) = A_i$.



Composition of morphisms is sequential composition of string diagrams (Definition A.1.13). The monoidal product of two objects is the concatenation of lists, and the monoidal product on morphisms is parallel composition (Definition A.1.10), where the monoidal unit object I *is the empty list.*

Proof. As we showed in Lemma A.1.14, for each object in the category there is an identity morphism. In Lemma A.1.16, we have shown that the composition of two string diagrams gives us another string diagram, and that this composition is associative. Hence, this forms a category.

Now all we need to do is show that the category is strictly monoidal. As we mentioned above, the monoidal product is defined where the action on objects is concatenation of lists, and the action on morphisms is parallel composition. We need to show that this action is well defined, is functorial, and that it is unital and associative.

We showed that parallel composition results in another string diagram and is associative in Lemma A.1.11. The fact that it is unital is trivial, as the unit is the empty list is clearly unital with respect to concatenation.

We now need to show that, for string diagrams G, H, G', H', then

$$(G \circ G') \otimes (H \circ H') = (G \otimes H) \circ (G' \otimes H')$$

when the above equation is defined.

It is clear that the set of vertices for both sides of the equation is $V_G + V_H + V_{G'} + V_{H'}$. For the set of edges, the left side of the equation is $(E_G \cup E_{G'}) + (E_H \cup E_{H'})$, and the right side is $(E_G + E_H) \cup (E_{G'} + E_{H'})$.

First off, note that for $(G \otimes H) \circ (G' \otimes H')$ to be defined, it must be true that the codomain of $G \otimes H$ is $\operatorname{cod} G + \operatorname{cod} H$, which will be equal to $\operatorname{dom} G' + \operatorname{dom} H'$. Recall that we may define $E_G \cup E_{G'}$ and $E_H \cup E_{H'}$ as pushouts

$$\begin{tikzcd}
 & \operatorname{cod} G \arrow[dl] \arrow[dr] & \\
E_G \arrow[dr] & & E_{G'} \arrow[dl] \\
 & E_G \cup E_{G'} &
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \operatorname{cod} H \arrow[dl] \arrow[dr] & \\
E_H \arrow[dr] & & E_{H'} \arrow[dl] \\
 & E_H \cup E_{H'} &
\end{tikzcd}$$

We may take the coproduct of these two pushouts, and we find the set of edges on the left side of the equation.

$$\begin{tikzcd}
& \operatorname{cod} G + \operatorname{cod} H \arrow[dl] \arrow[dr] & \\
E_G + E_H \arrow[dr] & & E_{G'} + E_{H'} \arrow[dl] \\
& (E_G \cup E_{G'}) + (E_H \cup E_{H'}) &
\end{tikzcd}$$

However, note that the codomain of $G \otimes H$ is $\operatorname{cod} G + \operatorname{cod} H$. Hence, it is also true that $(E_G + E_H) \cup (E_{G'} + E_{H'})$ is the pushout of the above diagram. Hence,

$$(E_G \cup E_{G'}) + (E_H \cup E_{H'}) = (E_G + E_H) \cup (E_{G'} + E_{H'})$$

A similar proof tells us that the labelling functions M and O, and the source and target functions are equal. Hence, we see that $\otimes$ is functorial. $\square$

Remark A.1.18. We note that since sequential and parallel composition are defined by colimits, they are unique only up to isomorphism p to coproduct. It is possible to define composition and the monoidal product up to equality on the nose by defining a choice function for the coproduct. However, we will ignore this for the sake of notational simplicity.

Definition A.1.19. A *monoidal signature D* (referred as a *tensor scheme* in Joyal and Street [33]) is a pair of sets, $objD$ and $morD$ with functions $\operatorname{dom}, \operatorname{cod} : morD \to objD^*$, where $objD^*$ is the set of finite lists over $objD$. We denote the empty list as $[]$.

Let $\mathcal{C}$ be a small symmetric monoidal category. An *interpretation of D to $\mathcal{C}$*, denoted $K : D \to \mathcal{C}$ is a pair of functions, $K_0 : objD \to obj\mathcal{C}$ and $K_1 : morD \to mor\mathcal{C}$ such that given $f : [A_1 \ldots A_n] \to [B_1 \ldots B_m] \in morD$, the action of K_1 on f has type

$$K_1 f : K_0 A_1 \otimes \ldots \otimes K_0 A_n \to K_0 B_1 \otimes \ldots \otimes K_0 B_m$$

We denote $x@y$ as the concatenation of lists. We say that the set of *free morphism terms on D* is the set $morD^{\otimes}$ where the elements are generated by

- The set $morD$.

- The set $\{1_x : x \to x\}$, indexed over $x \in objD^*$
- The set $\{\sigma_{x,y}\}$, indexed over $x, y \in objD^*$, where $\sigma_{x,y}$ has type $x@y \to y@x$.
- The operation $\otimes$, where $f \otimes g$ has type $\mathrm{dom}(f)@\,\mathrm{dom}(g) \to \mathrm{cod}(f)@\,\mathrm{cod}(g)$
- The operation $\circ$, where $f \circ g$ is defined only if $\mathrm{cod}(g) = \mathrm{dom}(f)$, in which case it has type $f \circ g : \mathrm{dom}(g) \to \mathrm{cod}(f)$.

The *free symmetric monoidal category on* D, $M(D)$, is the category where the set of objects is $objD^*$, and morphisms are the free morphism terms on D, quotiented such that

$$
\begin{aligned}
&(f \circ g) \circ h = f \circ (g \circ h) && 1_{\mathrm{cod}(f)} \circ f = f = f \circ 1_{\mathrm{dom}(f)} \\
&(f \otimes g) \otimes h = f \otimes (g \otimes h) && 1_x \otimes 1_y = 1_{x@y} \\
&(f \circ g) \otimes (f' \circ g') = (f \otimes f') \circ (g \otimes g') && 1_{[]} \otimes f = f = f \otimes 1_{[]} \\
&(1_y \otimes \sigma_{x,z}) \circ (\sigma_{x,y} \otimes 1_z) = \sigma_{x,y@z} && (\sigma_{x,z} \otimes 1_y) \circ (1_x \otimes \sigma_{y,z}) = \sigma_{x@y,z} \\
&\sigma_{y,x} \circ \sigma_{x,y} = 1_{x@y}
\end{aligned}
$$

where [] is the empty list. This gives us a strict symmetric monoidal category, with $\circ$ as composition and 1_x as the identity morphism. The monoidal product of objects x and y is $x@y$, the monoidal product of morphisms is $\otimes$, and the monoidal unit is the empty list, [].

Lemma A.1.20. *Given monoidal signature D, there exists an interpretation $K : D \to M(D)$ such that if there is an interpretation $L : D \to \mathcal{C}$, then there exists a unique strict monoidal functor $U : M(D) \to \mathcal{C}$ such that the below diagram commutes.*

$$
\begin{tikzcd}
D \arrow[r, "K"] \arrow[dr, "L"'] & M(D) \arrow[d, dashed, "U"] \\
& \mathcal{C}
\end{tikzcd}
$$

We call K the canonical interpretation of D

Proof. There are two parts to this proof – we must first define K, then for any given L, we must define U. To define K, we define the object map K_0 and the morphism map K_1.

Since $objM(D) = objD^*$, K_0 has type $K_0 : objD \to objD^*$, and is defined as $K_0(A) = [A]$, the single element list. We have $morM(D) = morD^{\otimes}$, and $morD$ is contained within $morD^{\otimes}$. Hence, $K_1 : morD \to morD^{\otimes}$ is defined as $K_1(f) = f$. We see that this fulfils the definition of an interpretation, as if we suppose that $f \in D$ has type $f : [A_1, A_2, A_3] \to [B_1, B_2]$, then

$$K(f) : K_0(A_1)@K_0(A_2)@K_0(A_3) \to K_0(B_1)@K_0(B_2) = f : [A_1, A_2, A_3] \to [B_1, B_2]$$

in D.

Suppose that we have an interpretation $L : D \to \mathcal{C}$. Our goal is to define a functor $U : M(D) \to \mathcal{C}$ that fulfils the above property and show that it is unique. We shall show that U must defined as $U([A_1, \ldots, A_n]) := LA_1 \otimes \ldots \otimes LA_n$ for all objects $A \in morD$, and

$$U(f) := L(f) \text{ for all } f \in morD \tag{A.1}$$

$$U(1_x) := 1_{Ux} \text{ for all } x \in objD^*D \tag{A.2}$$

$$U(\sigma_{x,y}) := \sigma_{Ux,Uy} \text{ for all } x, y \in objD^*D \tag{A.3}$$

$$U(t \circ t') = Ut \circ Ut' \text{ for all } t, t' \in morD^{\otimes}D \tag{A.4}$$

$$U(t \otimes t') = Ut \otimes Ut' \text{ for all } t, t' \in morD^{\otimes}D \tag{A.5}$$

For $U : M(D) \to \mathcal{C}$ to be defined with the above property, we see that the action on objects must be $UK(A) = L(A)$ for each $A \in objD$. Since $K(A) = [A]$, we see that $U([A]) = L(A)$ for all single element lists.

Since U is strict monoidal, then $U(x@y) = Ux \otimes Uy$. Hence, we see that

$$U([A_1, \ldots, A_n]) := LA_1 \otimes \ldots \otimes LA_n.$$

Therefore, if a strict monoidal functor exists that fulfils the above property, then there is only one possible way that U's action on objects may be defined.

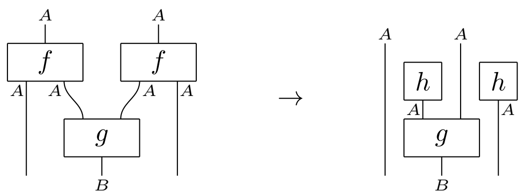
For a similar reason, we see that $U(f) = UK(f) = L(f)$ for all $f \in morD$. Since U is a strict functor, we also require that Equations A.2 through A.5 hold. Hence, we have shown that U exists and is unique. $\square$

Theorem 2.2.5. *Let $\mathcal{C}$ be a strict symmetric monoidal category. We may see $\mathcal{C}$ as a monoidal signature in the obvious way. Then the category of string diagrams over $\mathcal{C}$, denoted SD($\mathcal{C}$), is isomorphic to the free symmetric monoidal category on $\mathcal{C}$, denoted $M(C)$. We denote the canonical interpretation from $\mathcal{C}$ to SD($\mathcal{C}$) as $\hat{\_}$.*

A.2 LR and RL Integral Hopf Algebras

In this section, we cover how any statement about LR integral Hopf algebras implies an equivalent one about RL integral Hopf algebras.

Definition A.2.1. Let G be a string diagram of $\mathcal{C}$, f a morphism in $\mathcal{C}$ and let $V_G(f) \subseteq V_G$ be the subset of vertices v such that $M(v) = f$. Let g be a morphism with the same type signature as f. We define the string diagram $G[g/f]$ as the same string diagram as G, except for each vertex v in $V_G(f)$, we have $M(v) = g$. We say that in $G[g/f]$, we have *substituted g for f*. For example, let $f : A \to A \otimes A, g : A \otimes A \to B$ and $h : I \to A$ be morphisms in $\mathcal{C}$. In the following figure



we have substituted $1_A \otimes h$ for f.

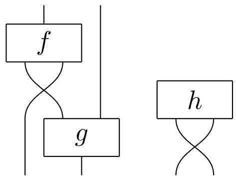
Definition A.2.2. Let S be a string diagram in SD($\mathcal{C}$) such that for every edge e in S, we have $O(e) = A$ for some $A \in \mathcal{C}$. We define the *horizontal dual* of S as the string diagram $H(S)$ where

- The set of vertices and edges of $H(S)$ are the same as those in S. Likewise, the source and target functions are the same for both $H(S)$ and S, and the labelling functions O and M are the same.

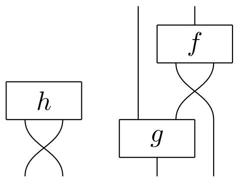
- The edges in $\operatorname{dom} H(S)$ are the same as the edges in $\operatorname{dom} S$, except that the order is reversed. The same is true for $\operatorname{cod} H(S)$.

- For vertex v in $H(S)$, the edges in $\operatorname{dom} v$ in are the same as the edges $\operatorname{dom} v$ in S, except that the order is reversed.

For example, consider the following string diagram

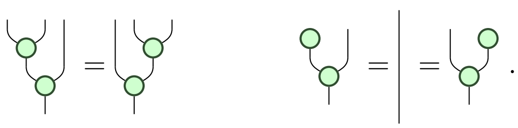


The horizontal dual of this string diagram is



Let $S = T$ be an equation in $\mathrm{GR}_E(\mathcal{C})$. The *horizontal dual of* $S = T$ is $H(S) = H(T)$. We say that an equation is *horizontally self-dual* when it is equal to its horizontal dual. Likewise, a set of equations may be horizontally self-dual if when we take the horizontal dual of each of the equations, we get back equations that are equivalent to the original set of equations. When we use a horizontally self-dual set of equations to form a proof, then the proof will be horizontally self dual.

For example, consider the monoid axioms



Note how, for the associativity axiom, neither string diagram is self-dual, but the horizontal dual of the equation is still the associativity axiom. Likewise, the unit axiom contains two equalities, neither of which are horizontally self-dual, but the set of equalities are.

As such, the monoid axioms as horizontally self-dual. This implies that any proof for an arbitrary monoid will be horizontally self-dual. We see that the same holds for comonoids.

When we look at the defining equations of Frobenius algebras, bialgebras and Hopf algebras, we find that they are all similarly horizontally self-dual.

Lemma A.2.3. *Let t be a true statement about an arbitrary LR integral Hopf algebra. If we take the horizontal dual of t, then substitute left cointegrals for right cointegrals, and right integrals for left integrals, then we shall have a true statement about arbitrary RL integral Hopf algebras.*

Proof. Using the notation for substitution, we would write the above construction on t as $H(t)[\ /\ ][\ /\ ]$, and we shall refer to this as the *mirror* of t.

Let the axiom that defines left cointegrals and right integrals be denoted LC and RI respectively. We see that $H(LC)[\ /\ ][\ /\ ]$ and $H(RI)[\ /\ ][\ /\ ]$ are equal to

In other words, they are equal to the axioms of right cointegrals and left integrals respectively.

Recall that the axioms of Hopf algebras are vertically self dual. Hence, if we mirror the axioms of an LR integral Hopf algebra, then we get the axioms of an RL integral Hopf algebra. Thus, we have proven our result. $\square$

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