# Essays on the Economics of Information

***Matthew Robertson***

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Department of Economics
University of Strathclyde
Glasgow

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# Declaration

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## Abstract

In three distinct, yet interrelated, essays I examine the effects of asymmetric information and imperfect information on economic decision makers' incentives and behaviour. To do so I employ, and modify, the methodology of Bayesian games.

In chapter one, I analyse an unconventional contest inspired by the real world. In this contest, players are ranked by a scoring rule based on both their realised performance and how close this performance is to a target set before the contest, which is private information. I elucidate and analyse the incentive properties of these rules then characterise the equilibrium behaviour of the players.

In chapter two, I integrate aspects from adverse selection and moral hazard models to provide a unified theory of securitisation under asymmetric information. I show that introducing skin in the game increases signalling costs for originators who performed sufficient due-diligence yet still improves incentives by making high effort relatively more likely. I relax the conventional assumption of risk neutrality and show that risk-sharing concerns are sufficient for the aforementioned qualitative properties of equilibrium to hold. Finally, I demonstrate that, depending on the severity of the originator's preference for liquidity or need to share risk, each setting may be more conducive for signalling.

In chapter three, I propose a simple and intuitive way to transform canonical signalling games with exogenous types into games in which the informed agent endogenously generates her private information through an unobservable costly effort decision. I provide portable results on the differentiability of action functions and existence of equilibrium. I then apply these results to classic models of security design and the job market to demonstrate the practical usefulness of endogenous effort. In particular, my approach in these applications lends theoretical support to stylised facts that cannot be derived from the standard signalling framework.

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# Introduction

A fundamental tenet of economic theory is that individuals respond to incentives. Whilst incentives can be determined by myriad factors, one especially salient aspect is the information available to economic decision makers. The importance of information is ubiquitous in economic situations: many sellers have additional information about the products they sell relative to potential buyers; bidders in a large number of auction formats are unaware of their opponents' valuations for the items; individuals taking part in a contest cannot precisely estimate the abilities of the other contestants; and, when setting wages, firms may be unable to observe how hard their employees have worked. Consequently, over the latter half of the twentieth century economic theorists have incorporated information in game theoretic and economic models. This has been done in a variety of different forms including imperfect information, incomplete information, and asymmetric information. Imperfect information arises when economic agents are unsure of the actions chosen by the other players of a game, whilst incomplete information occurs when certain characteristics of the other players are unknown. Finally, asymmetric information characterises situations when one party involved in a transaction has more information than others. Economic theorists then devised ingenious methods for solving such problems, both analytically and intuitively. The significance of these developments to the Economics profession are such that that five Nobel prizes have been awarded to economists for their work on incentives, asymmetric information and uncertainty.

One of the most important contributions in this field was put forth by the 1994 Nobel laureate John C. Harsanyi. In a series of papers (Harsanyi, 1967, 1968a,b) he introduced the concept of Bayesian games, which have been widely studied and applied since this inception. In this class of games, players with different information are modelled as having different 'types', which are a theoretical construction that encodes the player's private information as a real number. To incorporate this idea into traditional game theory, Harsanyi proposed adding an initial stage in which a fictitious player known as Nature randomly determines the players' private information. This method converts a game with incomplete information to one of complete, but imperfect, information as long as the probability distribution from which the players' information is drawn is common knowledge. Given that each player has a common prior over the information possessed by the other players, it is possible for each player to use Bayes rule to update her beliefs as the game is played and actions are taken. This methodology has enabled theorists to conceptually treat an economic agent who can possess two mutually exclusive pieces of information as two distinct players, which means that Nash's (1951) seminal result can be applied to conclude that every finite Bayesian game has a potentially degenerate mixed-strategy Nash equilibrium.

Asymmetric information can lead to market failure by giving rise to moral hazard and adverse selection; that is, problems of hidden action and private information. This has been demonstrated in a variety of settings by the 1996 Nobel laureates James Mirrlees and William Vickrey and the 2001 Nobel laureates George Akerlof, Michael Spence and Joseph Stiglitz, respectively. Akerlof (1970) demonstrates that, in markets where the seller is privately informed about the quality of the good being sold, as in used car markets, this informational asymmetry will reduce the number of high quality products being traded, and may even lead to no trade whatsoever. This celebrated result inspired other economists who aimed to show that even a small amount of informational asymmetry could generate market failure (Stiglitz, 2002).

These novel findings, which stood in contrast to conventional wisdom, inspired researchers to pursue methods by which the problems of asymmetric information could be alleviated. Two such mechanisms are screening and signalling, proposed by Stiglitz (1975) and Spence (1973), respectively. In a screening model the uninformed party moves first and offers contracts to the informed individual that are designed to elicit the informed individual's private information. Thus, in a screening model, the burden of information elicitation lies with the uninformed party. Conversely, in a signalling model, the informed agent moves first and takes a costly action to convey her private information to the uninformed party. Rothschild and Stiglitz (1976) and Spence (1973) applied these concepts to models of insurance and the job market and provided the precise conditions under which individuals with different information will be able to distinguish themselves from one another in equilibrium. Such an equilibrium is characterised as separating, whilst an equilibrium in which individuals are unable to distinguish themselves is known as pooling.

Since these initial contributions, Bayesian games have been employed by economists in a diverse range of economic settings in which information plays a key role. These applications include: auctions (Amann and Leininger, 1996); bargaining (Chatterjee and Samuelson, 1983); contests (Fey, 2008); industrial organisation (Milgrom and Roberts, 1982); communication (Crawford and Sobel, 1982); and, finance (Leland and Pyle, 1977).

In three distinct, yet interrelated, essays I examine the effects of asymmetric information and imperfect information on economic decision maker's incentives and behaviour. To do so I make use of, and modify, the methodology of Bayesian games for two purposes.

The first is to analyse an unconventional contest hitherto unconsidered by the Economics literature. The motivation for this idea arose when I discovered a real world contest that did not seem designed to always award the prize to either the contestant with the highest ability or the contestant that exerted the most effort. This is in stark contrast to classical contest and tournament models: in the canonical rent-seeking contest, for example, used to model political lobbying, the player who exerts the most effort relative to her peers obtains either the highest probability of winning the prize or the largest share of the prize (Tullock, 1980). In tournament models of the labour market, the players' equilibrium efforts are symmetric, leading to the player with the highest ability receiving the largest prize and the second highest ability player obtaining the second largest prize, and so on (Lazear and Rosen, 1981).

The contest I analyse in the first chapter of this thesis is based on the Apnea World Free-Diving Competition. The rules state that each player must submit a private target of her performance to a referee before the contest takes place. This referee then collates the targets of all competitors and makes them common knowledge on the day the contest takes place. The contestants then compete by producing a level of output that is determined by a combination of natural ability and the realisation of the aggregate uncertainty, which represents such pervasive factors as the weather and water conditions. The crucial twist is that the contestant with the objectively highest performance may not always win the prize, as players are ranked by a scoring role that penalises any negative difference between ex-ante targets and realised performances. Moreover, the target effectively caps a player's score, so that there is no benefit from a performance exceeding it. The rules therefore provide incentives for both a high performance and an accurate prediction of this performance. I model this contest as a Bayesian game due to the two forms of uncertainty present. First, there is uncertainty about the abilities of the other contestants, as each player only has private information about her own ability. Second, there is aggregate uncertainty, captured by states of nature, that acts to symmetrically suppress or enhance the players' abilities on the day of the contest, and which is not known when targets are set. By implementing the common prior assumption over both the distribution of the players' abilities and the possible states of nature I am able to solve the model using standard techniques.

To make a contribution to the literature I first formalise this unconventional contest using game theoretic tools. I then provide novel results by analysing this model with the aim of characterising Bayesian Nash equilibria under a variety of assumptions to develop an understanding of the incentives that players face under different sets of conditions. These characterisations include demonstrating that, when the state of nature is common knowledge, players truthfully reveal their abilities, the intuition for which is akin to the classic auction of Vickrey (1961). I show that, when the players' abilities are common knowledge, low ability players obtain a strictly positive probability of winning the contest by using a mixed-strategy that puts more probability weight on the target that takes the less likely state of nature into account. Finally, when the possible states of nature are equally likely, the players use a mixed-strategy target setting rule that approximates the expected effect of the possible states of nature on their realised performances.

The setting I analyse in most depth is when each player's ability is private information and when there is uncertainty over the weather and water conditions; that is, when the state of nature is not known. I focus attention on pure strategy equilibria in this setting for two reasons: the first is that there is evidence to suggest that even professionals often do not employ the mixed strategies prescribed by economic theory (Brown and Rosenthal, 1990; Mookherjee and Sopher, 1994); the second is due to analytical difficulties as mixed strategies would need to be defined over a four-by-four game. By focusing on pure strategy equilibria, I am able to derive intuitive sufficient conditions on both the distribution of the players' abilities and the common prior over the possible states, which can be represented geometrically, such that pure strategies are optimal. If these conditions are satisfied, the players are sufficiently confident in one state of nature arising, and so use a simple target setting rule that accounts for the effect of this state of nature.

These results contribute to several strands of the Economics literature. The first is the literature on rank-order tournaments, as my analysis of equilibrium without aggregate uncertainty demonstrates that under these conditions the complex rules of this contest are superfluous, as the same outcome can be obtained by a standard rank-order tournament. This outcome involves the player with the highest ability winning the prize. The formalisation of the function that ranks the competitors contributes to the literature on scoring rules, often used to study the elicitation of an agent's belief about a probabilistic event, by providing a novel application for such rules and defining one that possesses three important properties in an unusual economic setting. Additionally, I provide the conditions under which the scoring rule I employ is proper (Savage, 1971). I contribute to a subset of the sports economics literature that aims to understand when it is optimal for players to use mixed strategies, with seminal papers such as Chiappori et al. (2002) and Walker and Wooders (2001). These papers focus on complete and perfect information games, with the exception of Coloma (2012) who extends Chiappori et al. (2002) to include uncertainty. My contribution is to analyse a richer theoretical model that includes two distinct forms of uncertainty and characterise conditions under which the players will employ both pure and mixed strategies in equilibrium. A central contribution to the literature on contest design is to show that this unusual contest generates a large degree of outcome uncertainty. This is an important finding as outcome uncertainty is of crucial significance to contest designers and has been shown by Szymanski (2003b) to be a significant determinant in the consumer demand for contests. My analysis demonstrates that, even when the players' abilities are common knowledge, a lower ability player has a strictly positive probability of winning the contest. The only setting without outcome uncertainty is when there is no aggregate uncertainty, or only one state of nature. In this case, once the targets have been collated and made public, the player who has set the highest target has the highest ability, and will therefore win the contest with certainty. Whilst I believe my modelling approach captures many of the interesting aspects of this contest the most apparent limitations are not incorporating an effort choice on the part of the contestants and focusing on systematic uncertainty rather than idiosyncratic. I have chosen to model the problem in this way to narrow the focus on the problem of strategic target setting, which is the most novel element and main contribution of this chapter.

What one learns from this chapter is that this contest generates interesting incentives for the players, which leads to diverse behaviour. This is because players employ both pure and mixed strategies in equilibrium, depending on their beliefs about both the abilities of their competitors and the possible weather and water conditions. However, these two beliefs are not treated equally. One learns that more precise information about the weather and water conditions is more valuable than that about competitor abilities. Finally, one learns that, with only one exception, these incentives and resulting behaviour give rise to a large degree of outcome uncertainty. Even when players are ranked by a public ranking, lower ranked players have a strictly positive probability of winning. This implies that these rules could be implemented in a wide variety of sporting circumstances where consumer demand could be increased.

The second purpose of employing the methodology of Bayesian games in this thesis is "to try and create models with assumptions – and conclusions – closer to those that accorded with the world I saw, with all of its imperfections" (Stiglitz, 2002). One strategy to achieve this goal involves combining the two problems of informational asymmetry, moral hazard and adverse selection, in a single model. Such a combination model could be utilised in an applied setting, and then developed and studied in a more general framework, which are the respective aims of my second and third chapters. My second chapter attempts to reconcile these two aspects in an applied model that considers the market for securitised assets. I choose to study this market because the two phenomena of informational asymmetry are prevalent. Gorton and Pennacchi (1988) and Pennacchi (1988) find evidence of both ex-ante and ex-post moral hazard, respectively, and Berndt and Gupta (2009) provide evidence of adverse selection.

Ex-ante moral hazard arises in this market as the underwriter of the loans makes an unobservable decision about the degree of due-diligence to carry out when granting loans. A greater degree of due-diligence leads to higher quality assets, however, as the firm that issues the securities knows that, once the securities have been sold to investors, any default risk will be transferred from the firm to the investors, the firm has the incentive to shirk. Moreover, once the securities have been sold, the firm has the responsibility of following up on the monitoring and servicing of the loans, but again, as this cannot impact the firm's payoff once the securities have been sold, the firm has a further ex-post incentive to shirk. Finally, adverse selection becomes apparent as the issuer of asset-backed securities has private information with respect to the quality of the assets that back the security.

The existing literature has employed either principal-agent models to study moral hazard or signalling models to analyse adverse selection. The central results from these two disjoint strands of the literature demonstrate that, through analysis of principal-agent models, retaining a fraction of the security issue on their books improves effort incentives for issuers (Fender and Mitchell, 2009; Malekan and Dionne, 2014). Whilst signalling models have shown that the same mechanism allows the issuer to transmit its private information about the quality of the underlying assets to uninformed investors (Leland and Pyle, 1977; DeMarzo and Duffie, 1999). Contemporary literature has attempted to synthesise aspects of both strands of research and it is this literature that I extend in my second chapter.

In this chapter, I model a monopolist issuer of asset-backed securities who also has responsibility for originating and underwriting the loans that back the securities. In the model, the issuer first makes an unobservable costly effort decision about the degree of due-diligence to carry out when underwriting the loans. This effort choice endogenously determines the payoff of the assets that back the securities, rather than the quality of the assets being determined exogenously, as would be standard in the literature. This is the first innovation of the chapter. The issuer then uses the quantity of the security that is offered for sale as a signal of this previous choice of effort, as this comprises the firm's private information. This is an important research topic as numerous economists have suggested that the prolific rise in securitisation from 1980 to 2005 was a salient precipitating factor in the 2007-09 Financial Crisis (Keys et al., 2010; Mian and Sufi, 2009; Ashcraft and Schuermann, 2008). The theoretical results have inspired a policy intervention known as 'skin in the game', which requires issuers of asset-backed securities to retain a five percent vertical tranche of any issued security. However, whilst this policy intervention possesses intuitive similarities with the theoretical results, it remains unclear how the introduction of an exogenous lower bound on an issuer's retention decision would affect both its ability to signal and its effort incentives. This ambiguity has arisen because in economists' models the firms endogenously decide to retain a fraction of their assets, rather it being a mandatory condition. It is also unclear whether this policy would successfully improve issuer's effort incentives when risk-sharing is the dominant driving force behind securitisation. Given that securitisation is greatly motivated by risk-sharing it is important to know how robust this policy intervention is to different risk attitudes.

I utilise my model to address these questions and contribute to the literature on informational asymmetries in the market for securitised assets. I first show that introducing a rule akin to 'skin in the game' increases signalling costs for the firm; that is, a firm that has carried out sufficient due-diligence and wishes to signal this choice of effort to investors must retain a strictly greater quantity of the security than when the regulation is not in place. Intuitively, introducing skin in the game increases the payoff wedge between high and low underwriting effort. This increases the incentive for a firm that has chosen low effort to mimic the retention strategy of a high effort firm. To reduce this incentive, a firm that has chosen high effort must increase its equilibrium retention strategy. I then show that effort incentives are improved by skin in the game; consequently, generating an increased probability of high effort in equilibrium. The originator's effort incentives are improved as skin in the game generates increased incentives for the issuer to dedicate more time to due-diligence, as a firm with poor quality assets will be exposed to potential default risk that reduces its expected payoff. The effect is to increase the wedge between the payoffs from high and low effort, but, as ex-ante incentives are affected, this increases the likelihood that the issuer will choose high effort given the equilibrium retention strategies. The result is an interesting contribution as it echoes findings from the principal-agent literature yet arises in a model that shares more similarities with classic signalling models.

The final contribution is to analyse the model under risk aversion and risk seeking behaviour on the part of the issuer. Whilst some more recent principal-agent models consider securitisation under different risk attitudes, such as Malamud et al. (2013), the adverse selection literature has remained wedded to the assumption of risk neutrality coupled with a discount factor since DeMarzo and Duffie (1999). Numerous studies, however, including Mohanty (2005) provide evidence that risk-sharing is a principal incentive for firms to securitise assets. The analysis of the incentive effects and signalling costs of skin in the game under different risk preferences is, therefore, important to develop a complete understanding of the impact of this regulation in different settings.

I find that both the increased signalling costs and improved effort incentives carry over when the originating firm securitises assets to transfer risk rather than to improve liquidity. Specifically, the key qualitative properties of equilibrium continue to hold when the originator is risk-averse, but not when it is risk-seeking. This is a novel result, as the analysis of securitisation motivated by risk-sharing has fallen out of favour with the literature. Moreover, I compare the required levels of retention necessary to signal high effort under both a preference for liquidity and a need to share risk to characterise which setting is more conducive for signalling. When the originator is characterised by a relatively high level of either risk aversion or liquidity preference, the minimum level of retention required to signal high effort is higher under a need for liquidity. Conversely, when it is characterised by relatively mild levels of risk aversion or liquidity preference, the minimum level of retention is higher under risk aversion. This analysis provides an answer to an interesting research question that had not yet been analysed by the literature and suggests that the enforced level of skin in the game could be allowed to change depending on the salient driver behind securitisation.

Other central differences that I model in this chapter to extend and contribute to the literature include that, in my model, the originator must pay a cost to learn the quality of an asset. In the only other paper to consider a setting such as this (Chemla and Hennessy, 2014) the originator learns this payoff without cost. Moreover, in contrast to Chemla and Hennessy (2014), my model does not admit pooling. These authors focus on designing optimal retention clauses with respect to welfare whilst I focus on the signalling and incentive effects of a fixed exogenous retention rule more akin to that imposed in practice. Finally, they focus on only the risk-neutral case.

What one learns from this chapter is that the intuition that skin in the game improves incentives in a pure moral hazard setting is a more general result that can also arise in adverse selection environments. Despite leading to originators retaining more of a security issue than otherwise, increasing signalling costs, this regulation leads to a higher probability of sufficient due-diligence. Moreover, one learns that originators of asset backed securities do not need to be solely motivated by a need for liquidity for this effect to hold, as incentives for risk-sharing are sufficient for this regulation to have a positive impact on effort incentives. Finally, one learns that, depending on the relative magnitude of the need for liquidity or the need to share risk, it may be easier for some firms to communicate their private information than others.

In my third chapter, I build on the central idea from the second chapter of endogenising the process by which the informed agent's private information in generated in a signalling game and develop this in a more general framework. The motivation for this work is that the assumption of exogenous private information is not the most intuitive or realistic in many situations one can think of involving adverse selection, not just in the market for securitised assets. Consider a simple used car market, for example, where the quality of any specific car can most naturally be thought of as the combination of some random aspect and the effort put into the maintenance and upkeep of the vehicle by the current owner. Alternatively, consider the job market where the productivity of a candidate is determined largely by the effort put into studying and learning. As such, portable results for setting up a signalling game where the informed agent's private information is determined endogenously would be useful for applied researchers.

Therefore, the objective of my third chapter is to introduce endogenous private information into the canonical signalling framework developed by George Mailath

(Mailath, 1987; Mailath and von Thadden, 2013). I have chosen this framework as it provides a set of sufficient conditions for a separating equilibrium to exist, which have been applied in a diverse range of fields including job markets with matching (Hopkins, 2012); cheap talk with lying costs (Kartik, 2008); and altruism (Glazer and Konrad, 1996; Andreoni and Bernheim, 2009). I modify this framework so that the informed agent's private information is determined endogenously via an unobservable costly effort decision, rather than being exogenously determined as the realisation of a random variable drawn from some probability distribution. This modification is simple yet novel and enables the model to more closely approximate reality. I then work to answer three distinct research questions. The first is the characterisation of the function the informed agent uses to separate and signal her choice of effort; that is, what are the additional conditions on the informed agent's production function, responsible for mapping effort choices into outcomes, which ensure separation. The second, and more interesting, research question is, given behaviour analogous to that which arises in separating equilibria, how does the informed agent decide upon her choice of effort. Is it possible for the informed agent to choose her effort in a manner that is consistent with conventional notions of optimality, and if so will there be any key parameters that govern this choice. The final question relates to what additional insight and intuition is gained from this approach relative to the standard model.

To characterise the function the informed agent uses to signal her choice of effort I directly extend a result of Mailath and von Thadden (2013) to cover this modified framework. The additional necessary conditions on the informed agent's production function are intuitive: it must be monotonic and concave, echoing classic producer theory production functions. The characterisation demonstrates that, in contrast to the standard setting, the informed agent's signalling function takes account of the marginal effect of her effort production function. This result is portable and can be used in applied settings.

By analysing the informed agent's effort optimisation problem, I provide one set of sufficient conditions that ensure that, given the method by which the informed agent communicates her private information, she is able to optimally choose her effort. These conditions provide restrictions on the informed agent's payoff function such that it will be concave in equilibrium, and hence the optimisation problem over effort will have a unique solution. Separability of the informed agent's payoff in the signal and the response of the uninformed is one of the key requirements. This is the section of the chapter that could be most improved in the future as the conditions that I provide are relatively restrictive.

Finally, I elucidate the novel insights gained from this approach by considering two applications based on seminal signalling papers. In these applications, I employ my modified framework to provide theoretical support for commonly reported stylised facts that cannot be derived under the standard assumptions. In the first application, I extend the classic security design model of DeMarzo and Duffie (1999) so that the quality of the assets are determined endogenously. I then show that firms that are in greater need of liquidity will find it optimal to exert less effort when underwriting the assets and will subsequently securitise and sell a larger proportion of these assets, which aligns with commonly reported stylised facts (Cardone-Riportella et al., 2010; Bannier and Hansel, 2008; Martin-Oliver and Saurina, 2007; Agostino and Mazzuca, 2009; Affinito and Tagliaferri, 2010). In the second application, I endogenise a worker's productivity in Spence's (1973) model of the job market and, without further extensions to the model, I show that all workers will exert the same effort and acquire the same education in equilibrium. As I make an assumption which implies that I only consider graduate job markets, this result is akin to the widespread standard of an undergraduate degree. Upon introducing a parameter that differentiates workers through their cost of effort functions, however, I show that optimal effort is increasing in this parameter. One possible interpretation of this parameter, supported to some degree by the empirical literature, is each worker's socioeconomic background (James, 2002; Cameron and Heckman, 1998; Mare, 1980). In principle, comparative statics results similar to what I obtain in these applications should be attainable from endogenising private information in the manner I suggest in any signalling model that makes use of the framework of Mailath and von Thadden (2013).

This chapter makes contributions to various strands of literature. The first is to the blossoming literature on signalling games with endogenous private information, such as In and Wright (2017), to which I contribute a portable framework based on Mailath (1987) and Mailath and von Thadden (2013) that could be applied and extended by other researchers. This portable framework contributes to this literature by answering the question of how the informed agent will signal her endogenous effort choice in such a way that other researchers can make use of the result. My results on the characterisation of the informed agent's signalling strategy and existence of a solution to her effort optimisation problem provide additional contributions above laying out a method for setting up the model with endogenous private information. The individual applications considered in this chapter contribute to their respective literatures on signalling in securitisation and education. I provide theoretical support for stylised facts that cannot be derived under the standard assumption of exogenous private information by extending the baseline models considered in a manner that has hitherto not been analysed.

What one learns from this chapter is that a simple change in the set-up of signalling games allows them to capture the more realistic feature of informed agents choosing their private information through an unobservable effort decision. One learns what additional intuitive assumptions are necessary for classic results, which characterise the function informed agents use to communicate their private information, to continue to hold. Finally, one learns why endogenising private information in signalling games is useful. It provides comparative static results that explain how informed agents would choose their private information, which are often supported empirically, that are unavailable in the standard framework.

In this thesis, I utilise and modify the methodology of Bayesian games to study economic situations that feature uncertainty and asymmetric information. In the first chapter, I use a Bayesian game to formalise an unconventional contest that has hitherto not been subjected to economic analysis. This analysis provides new insights on contest design and outcome uncertainty, which enables the chapter to make contributions to several literatures. In the second and third chapters, I modify signalling games, which are a class of dynamic Bayesian game, such that the informed agent's private information is generated endogenously. This means that the informed agent has control over her private information, which more accurately captures reality. In the second chapter, I employ this modification to study how the securitisation regulation skin in the game' affects signalling costs and effort incentives for issuing firms under different risk preferences. My contribution involves showing how signalling and incentive effects of this regulation are affected by risk preferences. This contributes to the literatures on both adverse selection and moral hazard in the market for asset-backed securities. In the third chapter, I develop the core idea of endogenous private information in signalling games in a general framework, and provide portable results that could be applied by researchers studying problems involving an informed agent who makes an unobservable costly choice to determine her private information. I then demonstrate the practical usefulness of this modification by studying several applications in which I provide theoretical support for stylised facts that are unobtainable under standard assumptions.

# Chapter 1

# Contests with Ex-Ante Target Setting

## 1.1 Introduction

Uncertainty surrounding outcomes is a salient determinant of the consumer demand for contests and is, consequently, of crucial importance for contest designers (Neale, 1964; Knowles et al., 1992; Forrest and Simmons, 2002; Szymanski, 2003b,a; Borland and MacDonald, 2003). The implications of this are such that, when spectators are unable to determine ex-ante who will win a contest, there will be a higher willingness to pay. Therefore, contests that generate such uncertainty, often by creating competitive balance, are important objects of study. This motivation has lead to the genesis of unconventional contest rules that have hitherto not been subjected to economic analysis. A notable example is the Apnea World Free-Diving Championship that involves competitors attempting to dive to the greatest depth in the ocean, without the use of breathing equipment, before successfully returning to the surface. This contest does not use the methodology of the traditional winner-take-all format, under which the contestant with the deepest dive would win, and the analysis of which is captured by existing theoretical models. Instead, each contestant submits a private target of her performance to a referee before the contest takes place, which functions in a manner similar to a bid in an auction, albeit with a twist[1]. Each contestant is then evaluated according to a common knowledge scoring rule based on both her realised dive depth and any difference between her ex-ante target and this dive depth. In particular, if a

[1] "The contest officially starts the night before a dive, when divers secretly submit the proposed depths of the next day's dive attempts to a panel of judges. It's basically a bid, and there's gamesmanship involved as each diver tries to guess what the other divers will do" (Nestor, 2014).

contestant's dive depth is less than the target she set, she faces a negative penalty on her score such that her penalised score is less than her realised performance. The target also acts as an upper bound on contestants' scores, implying that there is no benefit from diving deeper than the target. The winner of the contest is the player with the highest score. These rules therefore reward, and provide incentives for, both a high performance *and* an accurate prediction of this performance. The problem facing competitors is that their performance is subject to aggregate uncertainty, which impacts everyone symmetrically, and cannot be perfectly estimated in advance. Prominent free-diver Guillaume Nery[2] suggests that, if contestants faced no uncertainty and told the truth with respect to their abilities, the contest outcome would be predetermined. However, as this uncertainty could represent such pervasive factors as the weather and water conditions, he states that "it will never happen like that". Divers with relatively lower ex-ante ability can, consequently, gain a competitive edge through strategic target setting with respect to this uncertainty. Moreover, it is a priori unclear whether these rules would indeed incentivise contestants to truthfully reveal their abilities, even if there was no uncertainty. These unconventional rules, coupled with the contestants' private information, suggest a game theoretic analysis should be employed to understand the degree of outcome uncertainty the contest generates. To do so would involve elucidating and understanding the incentives the contestants face, and their subsequent equilibrium behaviour. If these rules can be shown to demonstrate considerable outcome uncertainty then they could be utilised by contest designers whose aim is to capture significant consumer demand.

Consequently, this chapter formally models these contest rules as a game of incomplete information. In this game, each player's realised output is determined by a combination of their natural ability, which is each player's private information, and the realisation of the state of nature, which captures the aggregate uncertainty. Each player's decision problem is to optimally choose a target that takes into account her private information about her own ability and her beliefs over the abilities of the other players in conjunction with the possible states of nature. Players are then ranked using a common knowledge scoring rule that possesses three important properties: first, if a player's ex-post performance is less than her ex-ante target, her score is discounted to less than her ex-post performance; second, the ex-ante target acts as an upper bound on each player's score; and third, if the ex-post performance is equal to the ex-ante target, then the player's score is also equal to this value. The motivation for this game theoretic

[2] Of viral internet video 'Freefall' fame.

analysis is to understand the incentive properties of the unusual rules featured in this contest, and the conditions responsible for generating these incentives. Given the incentives facing the players, one would then like to understand the conditions on the primitives of the model such that Bayes-Nash equilibria exist, and the associated equilibrium behaviour of the players. Finally, once the incentive properties and equilibria are understood, how uncertain is the outcome and to what economic settings could these rules, and my findings, be applied.

The results are closely related to the above motivation. To facilitate a tractable analysis of this contest, I study a setting involving two players, each of whom can have either low or high ability, throughout most of the chapter. I assume that there are two states of nature, one that is beneficial for the players' ex-post performances and one that is neutral, which does not affect performance. Finally, I assume each player has two actions available, one that sets a target equal to the ex-post performance in the neutral state and one equal to the ex-post performance in the enhancing state, which gives each player four possible strategies. In this setting, the players' incentives are generated by three main assumptions. These assumptions identify and determine the various settings in which a low ability player can obtain a greater score than a high ability player through strategic target setting.

I first analyse the case in which the state of nature is known to the competitors and formally confirm the intuition of Guillaume Nery that the contest outcome is determined solely by the distribution of the players' abilities. This demonstrates that, when the state of nature is known, the contest rules posses incentive compatibility properties, as players truthfully reveal their abilities. This result is related to findings in the rank-order tournament literature such as Lazear and Rosen (1981) and shares intuition with equilibrium in the classic Vickrey (1961) second-price auction. I show in an extension that this result can be generalised to a setting with many players and a continuous distribution over their abilities.

I then turn to the analysis of the most general setting when there is uncertainty over both the abilities of the other players and the state of nature. When the two states are equally likely there are no pure strategy Bayes-Nash equilibria, even when low ability players have the most opportunity to gain from strategic target setting. The intuition is similar to that found in the game of matching pennies and arises because players cannot account for the expected effect of the possible states using pure strategies. I exclude such strategies as the target must take an integer value in many applications. Therefore, incorporating a pure strategy that was based on the expected effect of the possible states of nature would violate this requirement in my model. The mixed strategy equilibrium, however, confirms that this is optimal. Upon reducing the scenarios in which a lower ability player can gain a competitive edge through strategic target setting, I find that there is a unique pure strategy Bayes-Nash equilibrium. In this equilibrium, the players set their targets as if they knew that the performance enhancing state would be realised with certainty. This is because this reduction in the gains from from strategic target setting effectively increases the skill gap between low and high ability players; consequently, decreasing the potential risk of overestimation.

My two main findings provide sufficient conditions on the primitives of the model, the distribution of the players' ex-ante abilities and the common prior over the possible states, such that unique pure strategy Bayes-Nash equilibria exist. These sufficient conditions provide bounds on these probabilities that incentivise the players to use simple target setting rules and have a useful geometric representation. Specifically, the probability of one state of nature needs to be at least three times greater than the other for players to set their targets in line with this state when they have the least information about the abilities of their competitors. For example, if a player is close to certain that her opponent has high ability she will not choose a simple pure strategy in equilibrium. As the probability of a state approaches one, the players' behaviour converges to that when there is no uncertainty, and the distribution of the players' abilities becomes less important. The intuition is that players need to be sufficiently confident in one particular state arising to be incentivised to take this state into account when setting their targets when they are equally likely to face high or low ability opponents. By increasing the skill gap between high and low ability players, I find that there is now a larger set of probabilities that will incentivise players to set their targets as if the performance enhancing state will arise. The intuition is similar to when I reduce the set of scenarios in which a low ability player can obtain a better score than a high ability player, through strategic target setting, when the two states are equally likely. Finally, I consider some extensions of this simplified model. These extensions characterise equilibria when the players' abilities are common knowledge to capture the effect of a published ranking on the players' behaviour; and, when there are many players and a continuous distribution over their abilities. Additionally, I describe conditions under which a player maximising her expected score is equivalent to maximising her expected payoff.

Economics has, in general, taken two approaches to modelling situations where individuals compete with each other for a prize. The first is the rent-seeking approach, in which players compete for a rent through expenditure of effort (Tullock, 1980). The central conceptual apparatus of this approach is the contest success function, which determines either the probability of winning, or share of the prize, and is determined by a given player's effort and the sum of the efforts of her competitors. Whilst an extensive literature has arisen that analyses specifications of contest success functions, the contest considered in this chapter is closer to the contests studied by Dixit (1987) and Che and Gale (1997). These authors analyse the incentive effects of pre-committing to effort, and constraints and caps on the level of effort that can be chosen, respectively. Since the scoring rule in this chapter effectively caps a player's potential score, and players need to make an ex-ante commitment, there are similarities; however, as I do cannot employ a contest success function, the link is conceptual rather than methodological. This is a consequence of my decision to model the player's abilities as random variables rather than as a result of an optimal effort choice.

Tournament models take a different approach, compensating workers not according to individual output but by ordinal rank. These models do not employ contest success functions in the sense of Tullock (1980), instead using probability tools commonly seen in auction theory. This literature began by extending the canonical models of Lazear and Rosen (1981), Green and Stokey (1983) and Nalebuff and Stiglitz (1983), who compare individual contracts with rank-order tournaments in terms of the alleviation of moral hazard in a variety of settings. The contemporary literature on tournaments has examined more unconventional rules including the construction of optimal seeding (Groh et al., 2012) and Round-Robin brackets (Arad and Rubinstein, 2013). Very recently, Vong (2017) considers when teams will strategically lose matches to manipulate a bracket in their favour when effort is costless.

I contribute to both strands of literature by analysing hitherto unconsidered rules and making use of conceptual aspects from the literature on contests and methodology drawn from the literature on tournaments. Moreover, I contribute to the contemporary tournament literature that focuses on more intricate rules by providing the conditions under which the unconventional rules considered in this chapter collapse to those akin to standard tournaments. In particular, I obtain the standard tournament result when there is only one, common knowledge, state of nature[3]. It should be noted, however, that the game studied in this chapter is winner-take-all and so this chapter formally models contests not tournaments[4].

[3] The result is that the player with the highest ability wins.

[4] Although Konrad (2009) defines both contests and tournaments as "games that are defined by a prize $b$, a set $N$ of players, pure strategy spaces defined by the sets $X_i$ of feasible pure strategies and by the set of payoff functions".

A central element of my model is the scoring rule used to rank competitors. Such scoring rules have been analysed in a variety of economic circumstances including auctions (Asker and Cantillon, 2008) and voting (Lepelley, 1996; Baharad and Nitzan, 2007). My use of a scoring rule is more closely related to papers that study the elicitation of an agent's belief about a probabilistic event (Hossain and Okui, 2013; Nelson and Bessler, 1989; Karni, 2009). This literature originated with Brier (1950) and the seminal work of Savage (1971), who introduced the concept of a proper scoring rule which creates the incentive for forecasters to report their true subjective probabilities. I use the scoring rule differently, as players have a common objective prior over relevant uncertainty, yet, conceptually, the scoring rule still functions in an analogous manner. I contribute to this literature by employing the concept of a scoring rule in a contest and providing a functional form for a scoring rule that possesses three important properties. I then show under what conditions this scoring rule could be described as proper.

Lastly, I contribute to the literature on the economics of sport that seeks to understand when players will employ mixed strategies. Seminal papers are Chiappori et al. (2002) and Walker and Wooders (2001). These authors analyse theoretical models and characterise the resulting mixed strategy Nash equilibria, before taking the results to the data. Both papers study complete information games as Walker and Wooders (2001) do not take into account different player abilities or the effect of weather conditions on the players' abilities and, whilst Chiappori et al. (2002) have distributions of goal scoring probabilities, these are assumed to be common knowledge. Coloma (2012) extend the model of Chiappori et al. (2002) to a setting with incomplete information, which incorporates two different types of strikers. Whilst I do not utilise empirical analysis, I contribute to this literature by analysing a more complex theoretical model, which features uncertainty with respect to the players' abilities and the effect of weather conditions, and derive the conditions under which players would find it optimal to use both pure and mixed strategies.

This rest of the chapter is structured as follows. Section 1.2 lays out the general framework within which I work and establishes existence of equilibrium. Section 1.3 employs specific assumptions to make this framework more tractable to characterise the resulting incentives facing the players. Section 1.4 analyses equilibria in a number of settings. In Section 1.5 I consider extensions of the model to provide additional results and generalisations, and in Section 1.6, I discuss some implications of these results in the context of sales-forecasting, auctions with budget constraints and oligopolistic competition with asymmetric costs and demand shocks. Section 1.7 concludes and discusses what one learns in terms of outcome uncertainty and contest design.

## 1.2 General Framework

Consider a winner-take-all contest between a finite set of players for a fixed, indivisible prize. Each player competes in the contest by producing output determined by a combination of her natural ability and the aggregate uncertainty, which affects all players symmetrically. Unlike a standard winner-take-all contest, each player must first set a private target of her ex-post output before the contest takes place and the aggregate uncertainty is realised. This target not only acts as an upper bound on each player's performance in the contest, but is crucial in determining the player's rank. Specifically, there is a common knowledge scoring rule that penalises players for any negative difference between their ex-ante target and their realised ex-post output. The winner of the contest is the player with the highest score after taking any penalties into account; therefore, the player with the objectively highest output may not always win. The rules provide incentives for both an accurate ex-ante target and a high level of realised output. The problem faced by the players is that they are unable to perfectly estimate their ex-post performance due to the aggregate uncertainty, which may act to enhance their natural ability. Moreover, as each player observes only her own natural ability, and not the abilities of the others or their targets, her beliefs over the abilities of the other players will be of central importance when setting her target. Each player's strategic decision problem involves setting her choice of an ex-ante target given how she believes the other players will set their targets, taking into account the possible states of nature.

I model this contest as a Bayesian game of incomplete information as each player faces uncertainty with respect to both the abilities of her competitors *and* the effect that external factors, such as the weather, will have on performance. Formally, the contest involves a set $\mathcal{I} = \{1, \ldots, i, \ldots, N\}$ of players who compete for a fixed, indivisible prize $\psi \in \mathbb{R}_{++}$. Each player $i$'s ex-ante ability in the contest is captured by her type $\theta_i \in \Theta_i \subset \mathbb{R}_{++}$, which is player $i$'s private information. I assume that $\theta_i > \theta_j$ implies player $i$ has higher ability than player $j$. In the diving context, this would imply that, without taking the weather considerations into account, player $i$ can dive deeper than player $j$. Each type $\theta_i$ is associated with a belief about the state of nature $k \in \mathcal{K} \subset \mathbb{R}$ and the other contestants' abilities $\theta_{-i} \in \Theta_{-i}$. The set $\mathcal{K}$ of states of nature represents the aggregate uncertainty, which in the context of the diving competition could capture all realisations of the weather and water conditions. The state of nature has a symmetric impact on the performance of all players. Player $i$'s beliefs are $\rho(\theta_i) \in \Delta(\mathcal{K} \times \Theta_{-i})$, an element of the set of joint probability distributions over the product set $\mathcal{K} \times \Theta_{-i}$. The states of the world are each possible combination of the aggregate uncertainty and the abilities of all players $\Omega = \mathcal{K} \times \prod_{i \in I} \Theta_i$. As in tournament models[5], player $i$'s ex-post output in the contest $Q : \Theta_i \times \mathcal{K} \to \mathbb{R}_{++}$ is a function of her ex-ante ability and the realisation of the aggregate uncertainty. The lack of subscript implies that $Q$ is symmetric for all players: $Q_i = Q_j = Q$ for each $i, j \in \mathcal{I}$. I assume that for $\theta_i, \theta'_i \in \Theta_i$, where $\theta_i > \theta'_i$, and for $k, k' \in \mathcal{K}$, such that $k > k'$, I have $Q(\theta_i, \cdot) > Q(\theta'_i, \cdot)$ and $Q(\cdot, k) > Q(\cdot, k')$, which implies $Q(\theta_i, k) > Q(\theta'_i, k')$. This modelling choice means that, once the uncertainty has been resolved, each player's ex-post output is given. I purposefully model the contest with exogenous output to focus attention on each player's target setting problem, which is one of the unique aspects of this competitive environment[6]. Shifting focus away from the optimal choice of effort is not without precedent, as a recent tournament study has considered strategic settings with costless effort (Vong, 2017).

Consequently, unlike conventional contests, each player's decision variable is not her output, but is instead her choice of an ex-ante target. Specifically, player $i$, who knows her own ability $\theta_i$ and has beliefs $\rho(\theta_i)$ over the abilities of the other players and the aggregate uncertainty $\mathcal{K} \times \Theta_{-i}$, must strategically set a target output to achieve in the contest. In particular, given any realisation of ex-ante ability and associated beliefs, each player $i$ privately chooses a target $t_i \in \mathcal{T} \subset \mathbb{R}_{++}$ that are then collated, ordered and made publicly available, for example as the poset $(T, >) = (t_i > t_j > \cdots > t_N)$. The pure strategy of player $i$ is therefore a mapping $\sigma_i : \Theta_i \to \mathcal{T}$ that yields a target for each type, with $\sigma_i(\theta_i) \in \mathcal{T}$ for each $\theta_i \in \Theta_i$, and a mixed strategy would be $\sigma_i(\theta_i) \in \Delta(\mathcal{T})$. A Bayes-Nash equilibrium is a vector of strategies $\sigma^* = (\sigma_i^*)_{i \in I}$ such that $\sigma_i^*$ is a best response to $\sigma_{-i}^*$ for each $i \in \mathcal{I}$ and $\theta_i \in \Theta_i$.

Given the announced targets $T$, and the ex-post output $Q(\theta_i, k)$ for each player $i$ is determined once the state of nature $k$ has been realised. The players are then ranked according to both their realised output in the contest and any negative or positive deviations from their ex-ante targets using the following common

[5] Such as Lazear and Rosen (1981) and He and Gerchak (2003).

[6] Note that my framework does not exclude the possibility of including an effort choice. This could be accommodated by assuming that player $i$'s production function is $Q : \mathcal{E} \times \Theta_i \times \mathcal{K} \to \mathbb{R}_{++}$ defined by $(e_i, \theta_i, k) \mapsto Q(e_i, \theta_i, k)$, or more simply $Q : \mathcal{E} \times \mathcal{K} \to \mathbb{R}_{++}$ defined by $(e_i, k) \mapsto Q(e_i, k)$. Associated with this choice would be a convex cost of effort $\psi : \mathcal{E} \to \mathbb{R}_+$. Each player's problem would then be to choose a pair $(e_i, t_i)$ to maximise her expected utility.

knowledge scoring rule.

**Definition 1.** The *scoring rule* is a function $S : \mathbb{R}_{++} \times \mathcal{T} \times \mathcal{D} \to \mathbb{R}_{++}$, where $\delta \in \mathcal{D}$ scales the penalty applied to negative deviations from target and $Q(\theta_i, k) \in \mathbb{R}_{++}$ is the player's ex-post output. $S$ has the following properties:

i Given $t_i$, such that $Q(\theta_i, k) < t_i$, $S(Q(\theta_i, k), t_i) < Q(\theta_i, k)$;

ii Given $t_i$, such that $Q(\theta_i, k) \geq t_i$, $S(Q(\theta_i, k), t_i) = t_i$;

iii Given $Q(\theta_i, k)$ and $t_i$ such that $Q(\theta_i, k) < t_i$, $S(Q(\theta_i, k), t_i)$ is decreasing in $\delta$.

Definition 1 formalises the three salient properties of the scoring rule discussed in the introduction. Specifically, conditions i and iii capture the property that a player's score is decreasing in the difference between her ex-ante target and her ex-post performance. That is, negative deviations from target are costly for the players. Property ii implies that the ex-ante target acts as an upper bound on each player's score, so that each player accrues no benefit from an ex-post performance greater than the ex-ante target, and that, if a player's realised performance is equal to her target, then her score is also equal to this value. Finally, property iii states that whenever a player overestimates her ability, her score is decreasing in the penalty applied to negative deviations. With the ordering of targets $(T, >)$ and scoring rule $S$, in state $k$ player $i$ has an ex-post payoff of $u_i^k(\mathcal{T}) = \psi \mathbb{1}_{S^*}(\mathcal{S})$, where $S^* = \max\{S(Q(\theta_1, k), t_1), \ldots, S(Q(\theta_N, k), t_N)\} = \max \mathcal{S}$, that is equal to $\psi$ if $S(Q(\theta_i, k), t_i) = S^*$ and zero otherwise. If $S^* \neq \{S(\theta_i, k), t_i)\}$ for some $i \in \mathcal{I}$, then I adopt a tie-breaker rule that assigns each player in $S^* = \{S(Q(\theta_i, k), t_i), \ldots, S(Q(\theta_j, k), t_j)\}$ a $\frac{1}{|S^*|}$ probability of winning the prize. The expected payoff of player $i$, when the set of states and the set of types are finite, is given by

$$U_i(\sigma, \theta_i) = \sum_{(k, \theta_{-i}) \in \mathcal{K} \times \Theta_{-i}} \mathrm{Pr.}\Big( S(Q(\theta_i, k), t_i) > S(Q(\theta_j, k), t_j) \, \forall \, j \neq i \Big) \psi \times \rho(\theta_i)(k, \theta_{-i}). \tag{1.1}$$

Given a strategy profile of the other players, $\sigma_{-i}$ the strategy $\sigma_i$ is a best response for player $i$ if $U_i((\sigma_i, \sigma_{-i}), \theta_i) \geq U_i((\sigma_i', \sigma_{-i}), \theta_i)$ for each $\theta_i \in \Theta_i$ and all other strategies $\sigma_i'$. The following proposition clarifies the assumptions on this general environment required for the existence of Bayes-Nash equilibrium.

**Proposition 1.** *Suppose that $\mathcal{I}$, $\Theta_i$, $\mathcal{K}$ and $\mathcal{T}$ are finite. Then, a (potentially degenerate) mixed strategy Bayes-Nash equilibrium exists in the contest with ex-ante target setting.*

*Proof of Proposition 1.* All proofs for this chapter are given in Section B.1. □

## 1.3 Simplified Model

Given that existence of equilibrium has been established by Proposition 1, I now invoke several assumptions to characterise the incentives of the players and the resulting equilibrium behaviour. Assume that there are two players $\mathcal{I} = \{1, 2\}$, each of whom can be one of two types $\Theta_i = \{\theta_L, \theta_H\}$ where $\theta_H > \theta_L$. The probability of player $i$ being of high ex-ante ability is $\Pr.(\theta_i = \theta_H) = \mu \in (0, 1)$, which is common knowledge, and is identical and independent for each $i \in \mathcal{I}$. Moreover, suppose there are two possible states of nature $\mathcal{K} = \{0, 1\} = \{k_0, k_1\}$ with generic element $k_\ell$ for $\ell \in \{0, 1\}$. The state of nature affects all players' ex-post performances symmetrically. When $k = k_0$ the state does not affect the players' realised performances. Whilst, when $k = k_1$, each player's realised performance increases relative to her ex-ante ability. This is due to the assumption that each player's production function, $Q(\theta_i, k_\ell) = \theta_i + k_\ell$, is additively separable in ability and the realised state[7]. The probability of the neutral state is given by the common prior $\Pr.(k = k_0) = \lambda \in [0, 1]$. Additionally, I set the value of the prize $\psi$ to one, as this implies that player $i$'s expected payoff is equal to her expected probability of winning the contest.

I assume that each player has the following action set

$$\mathcal{T} = \{t_{i0}, t_{i1}\} = \{Q(\theta_i, k_0), Q(\theta_i, k_1)\} = \{\theta_m, \theta_m + k_1\} \text{ for } m \in \{L, H\} \text{ and } i \in \mathcal{I}$$

with generic element $t_{i\ell}$. My justification for this action set is threefold. First, given that players can randomise over these actions, each player's strategy set is in essence the union of intervals with a non-empty intersection[8]. This implies that player $i$'s strategy set is itself the interval $[Q(\theta_L, k_0), Q(\theta_H, k_1)] = [\theta_L, \theta_H + k_1]$. To justify the bounds on this interval, note that the maximum ex-post performance players can obtain is $\max Q(\theta_i, k_\ell) = Q(\theta_H, k_1)$. Hence, there is no feasible benefit in using a strategy such as $Q(\theta_H, \gamma k_1)$ for $\gamma > 1$ as they will never be able to achieve this target with their realised performance. This argument also applies to the lower bound as $\min Q(\theta_i, k_\ell) = Q(\theta_L, k_0)$, and so players will never benefit from setting a target equal to $Q(\beta\theta_L, k_0)$ for $\beta < 1$ as they could obtain a greater score by using $Q(\theta_L, k_0)$ in every state of nature. Second, there is precedent set by seminal papers in the sports economics literature for this discretisation. One

[7] An interesting extension could consider multiplicative production functions of the form $Q(\theta_i, k_\ell) = k_\ell \cdot \theta_i$ with $k_\ell > 1$ for all $\ell$, where supermodularity would imply that greater states of nature provider greater marginal production returns to those with greater ex-ante abilities. See He and Gerchak (2003) for such a production function in a tournament setting

[8] One subset available to each player would be $[\theta_L + k_1, \theta_H] \cup [\theta_L, \theta_H]$ when she mixes over $Q(\theta_L, k_1)$ with $Q(\theta_H, k_0)$ and $Q(\theta_L, k_0)$ with $Q(\theta_H, k_0)$, for example.

example is Chiappori et al. (2002), who study penalty-kicks in soccer. In reality, the striker effectively has access to a continuum of actions as they can aim the ball anywhere between the two goal posts. This would most accurately be modelled as a symmetric interval $[L, R] = [-R, R]$ centred at zero, however, they assume that each striker's action set is $\{L, C, R\}$. This modelling choice is echoed in Walker and Wooders (2001), who study the serving behaviour of tennis players. Again, the action set that most accurately captures reality would be $[-L, L] \cup [-R, R]$, but they assume that each server's action set is simply $\{L, R\}$. Third, as noted above, the assumption of a finite action set implies the existence of equilibrium.

Finally, I assume that the scoring rule takes the form

$$S(Q(\theta_i, k_\ell), t_{i\ell}) = t_{i\ell} - \delta \cdot \max\{0, t_{i\ell} - Q(\theta_i, k_\ell)\}, \tag{1.2}$$

where if $S(Q(\theta_i, k_\ell), t_{i\ell}) = S(Q(\theta_j, k_\ell), t_{j\ell})$ I adopt a tie-breaker rule under which players $i$ and $j$ each win with equal probability. My first assumption concerns the construction of the scoring rule and a necessary condition on the penalty imposed on negative deviations from target such that (1.2) satisfies Definition 1.

### 1.3.1 Assumptions

**Assumption 1.**

$$S(Q(\theta_i, k_0), t_{i0}) > S(Q(\theta_i, k_0), t_{i1}) \Leftrightarrow 2 > \delta > 1 \Leftrightarrow \theta_m > \theta_m + k_1 - \delta(\theta_m + k_1 - \theta_i)$$

*for* $m \in \{L, H\}$.

Assumption 1 says that the penalty applied to negative deviations is such that, if a player's realised performance falls short of her target, the punishment from deviation is such that a player with the same ex-ante ability, who set her target in accordance with the neutral state, will receives a higher score[9]. If a player's realised performance is less than her target, her score must be discounted to less than her performance. This condition creates a setting in which another player of equivalent ability can win the contest through strategic target setting. Technically, I assume that $\delta > 1$ so that property i of Definition 1 is satisfied. Intuitively, if $\delta \leq 1$, falling short of one's target is not punished. To see this, let $t' > t$ and $Q(\theta_i, k) < t$. Then, if $\delta \leq 1$, I would have $S(Q(\theta_i, k), t') = (1 - \delta)t' + \delta Q(\theta_i, k) > (1 - \delta)t + \delta Q(\theta_i, k) = S(Q(\theta_i, k), t)$ and player $i$ would

[9]That is, if a player sets a target of $\hat{t}_{i\ell}$ and has a realised performance of $Q(\theta_i, k_{\ell'}) < \hat{t}_{i\ell}$, her score will be less than $Q(\theta_i, k_{\ell'})$ for each $\ell \neq \ell'$.

always choose $t_i = \max \mathcal{T}$. Assumption 1 will hold throughout this chapter.

Combined with Assumption 1, the following two conditions are responsible for generating the players' incentives in this finite environment. They determine the settings under which a player with relatively lower ability ex-ante is able to obtain a greater score than a high ability player through strategic target setting.

**Assumption 2.**

$$S(Q(\theta_L, k_1), t_{i1}) > S(Q(\theta_H, k_1), t_{i0}) \;\Leftrightarrow\; 1 > \theta_H - \theta_L.$$

Assumption 2 states that, if a low ability player sets her target in accordance with the performance enhancing state, and this state is subsequently realised, whilst a high ability player set his target in line with the neutral state, the low ability player will receive the greater score. This assumption is necessary to generate interesting behaviour. Without it, a high ability player would always win the contest by simply setting a target equal to her natural ability, removing any possibility of strategic target setting by a lower ability player; that is, any possibility of outcome uncertainty. Intuitively, it states that, when the weather is good, a less skilled diver can dive deeper than a highly skilled diver who dives in bad weather.

**Assumption 3.**

$$S(Q(\theta_L, k_0), t_{i0}) > S(Q(\theta_H, k_0), t_{i1}) \;\Leftrightarrow\; 1 > \delta - 1 > \theta_H - \theta_L.$$

Assumption 3 states that, if a low ability player correctly anticipates the neutral state, and sets a target that takes this state into account, whilst a high ability player incorrectly predicts the performance enhancing state, the punishment from deviation applied to the high ability player's score is sufficient for the low ability player's score to be greater. This condition helps generate an incentive for players with lower ex-ante ability to risk overestimation, given a positive probability of being able to obtain a greater score than a player with higher ex-ante ability who chooses more prudent target setting behaviour. If Assumption 3 does not hold, the set of scenarios in which a low ability player could obtain a greater score than a high ability player through strategic target setting would be smaller, which would generate different incentives for the players[10]. Intuitively, Assumption 3

[10]This is because Assumption 3 is more restrictive than Assumption 2, which would be satisfied by $\Theta_i = \{(\theta_L, \theta_H) \in \mathbb{R}^2_{++} : 1 > \theta_H > \theta_L > 0\}$ for example, in terms of the convex feasible parameter space within $(\theta_L, \theta_H) \in \Theta_i^2$. In particular, $\mathcal{A}_3 = \{(\theta_L, \theta_H) \in \mathbb{R}^2_{++} : \delta - 1 > \theta_H - \theta_L)\} \subset \mathcal{A}_2 = \{(\theta_L, \theta_H) \in \mathbb{R}^2_{++} : 1 > \theta_H - \theta_L)\}$ for all $\delta \in (1, 2)$.

will hold when the skill gap between high and low ability players is sufficiently small, so that the variance in the players' abilities is not too large.

I assume that $\delta \neq 2$ so that the conditions of Assumption 2 and Assumption 3 are not equivalent. This implies that, for fixed $\delta$, the smaller the difference in the players' abilities, $\theta_H - \theta_L$, the more likely Assumption 3 is to be satisfied. Moreover, I assume that $\delta < 2$ as Assumption 3 is designed to capture a subset of the parameter space allowed by Assumption 2, as shown in Figure 1.1. If $\delta > 2$,

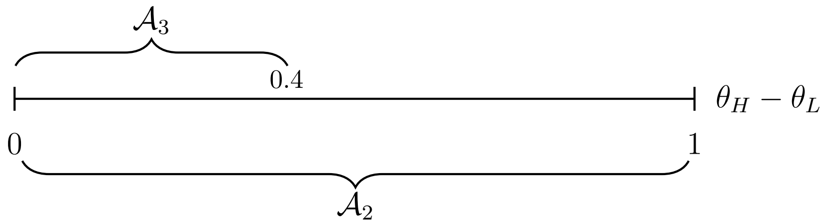


Figure 1.1: Assumption 2 and 3 for $\delta = 1.4$.

then Assumption 3 subsumes Assumption 2, which is contrary to its design.

## 1.4 Equilibrium Analysis

### 1.4.1 Equilibrium without Aggregate Uncertainty

Before studying equilibria under aggregate uncertainty, I first solve for Bayes-Nash equilibria when there is only one state of nature $\mathcal{K} = \{k\}$ so that $\Pr.(k = k_0) \in \{0, 1\}$, which is common knowledge between the players. The purpose is to answer whether Guillaume Nery's intuition that the contest outcome will be predetermined by the distribution of the players' abilities holds in this formal game theoretic framework. Moreover, this will enable one to understand the incentive properties that this contest possesses absent aggregate uncertainty.

Supposing that Assumption 2 and Assumption 3 hold, the four strategic form games the player would compete in when the only state is neutral $\mathcal{K} = \{k_0\}$ are given in Figure 1.2. The ex-post payoffs in Figure 1.2 capture the probability of a player having a greater score than her opponent for a given action profile. Given that the prize is equal to one, these payoffs also represent that player's probability of winning the contest, implying each payoff is constant sum to one. The actions represent the different targets that the players can set, recalling that $t_{i0} = Q(\theta_m, k_0) = \theta_m$ and $t_{i1} = Q(\theta_m, k_1) = \theta_m + k_1$ for $m \in \{L, H\}$. In the top

| 1 \ 2 | $Q(\theta_H, k_0)$ | $Q(\theta_H, k_1)$ |
|---|---|---|
| $Q(\theta_H, k_0)$ | $\frac{1}{2}, \frac{1}{2}$ | 1, 0 |
| $Q(\theta_H, k_1)$ | 0, 1 | $\frac{1}{2}, \frac{1}{2}$ |

| 1 \ 2 | $Q(\theta_L, k_0)$ | $Q(\theta_L, k_1)$ |
|---|---|---|
| $Q(\theta_H, k_0)$ | 1, 0 | 1, 0 |
| $Q(\theta_H, k_1)$ | 0, 1 | 1, 0 |

| 1 \ 2 | $Q(\theta_H, k_0)$ | $Q(\theta_H, k_1)$ |
|---|---|---|
| $Q(\theta_L, k_0)$ | 0, 1 | 1, 0 |
| $Q(\theta_L, k_1)$ | 0, 1 | 0, 1 |

| 1 \ 2 | $Q(\theta_L, k_0)$ | $Q(\theta_L, k_1)$ |
|---|---|---|
| $Q(\theta_L, k_0)$ | $\frac{1}{2}, \frac{1}{2}$ | 1, 0 |
| $Q(\theta_L, k_1)$ | 0, 1 | $\frac{1}{2}, \frac{1}{2}$ |

Figure 1.2: Strategic Form when $\mathcal{K} = \{k\} = \{k_0\}$.

left game, both players are of high ability, which occurs with probability

$$\Pr.(\theta_1 = \theta_H \cap \theta_2 = \theta_H) = \Pr.(\theta_1 = \theta_H) \cdot \Pr.(\theta_2 = \theta_H) = \Pr.(\theta_i = \theta_H) \cdot \Pr.(\theta_i = \theta_H).$$

In the top right and bottom left games, one of the players is of high ability, whilst the other player is of low ability. Each of these games occurs with probability

$$\Pr.(\theta_1 = \theta_H \cap \theta_2 = \theta_L) = \Pr.(\theta_1 = \theta_L \cap \theta_2 = \theta_H) = \Pr.(\theta_i = \theta_H) \cdot \Pr.(\theta_j = \theta_L,\ j \neq i).$$

Finally, the bottom right game, in which both players are of low ability, occurs with probability

$$\Pr.(\theta_1 = \theta_L \cap \theta_2 = \theta_L) = \Pr.(\theta_1 = \theta_L) \cdot \Pr.(\theta_2 = \theta_L) = \Pr.(\theta_i = \theta_L) \cdot \Pr.(\theta_i = \theta_L).$$

Using these probabilities, one can derive the expected payoffs of the players and use these to look for mutually consistent best responses in the Bayesian strategic form. I derive the players' expected payoffs by fixing the strategy of one player and then varying the strategy of the other for all possible strategy profiles. For example, fixing player 2's strategy as using the target $t_{i0} = Q(\theta_i, k_0)$ when she is of either high or low ability, that is $\sigma_2(\theta_2) = (Q(\theta_H, k_0)Q(\theta_L, k_0)) = (\theta_H\theta_L)$. I calculate player 1's expected payoff from following strategy $t_{i0} = Q(\theta_i, k_0)$ when she is of high ability and $t_{i1} = Q(\theta_i, k_1)$ when she is of low ability, that is $\sigma_1(\theta_1) = (Q(\theta_H, k_0)Q(\theta_L, k_1)) = (\theta_H\theta_L + k_1)$, by weighting each ex-post payoff by the probability that the specific payoff occurs

$$U_1(\sigma, \theta_1) = U_1((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_1) = \underbrace{\mu^2 \cdot \frac{1}{2}}_{\text{Pr.}(S_1(\cdot) \geq S_2(\cdot) | \theta_1 = \theta_2 = \theta_H)} + \underbrace{\mu(1-\mu) \cdot 1}_{\text{Pr.}(S_1(\cdot) \geq S_2(\cdot) | \theta_1 = \theta_H, \theta_2 = \theta_L)}$$
$$+ \underbrace{\mu(1-\mu) \cdot 0}_{\text{Pr.}(S_1(\cdot) \geq S_2(\cdot) | \theta_1 = \theta_L, \theta_2 = \theta_H)} + \underbrace{(1-\mu)^2 \cdot 0}_{\text{Pr.}(S_1(\cdot) \geq S_2(\cdot) | \theta_1 = \theta_2 = \theta_L)},$$

and $U_2((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_2) = 1 - U_1((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_1)$. After performing this calculation for each possible strategy profile $\sigma = (\sigma_1(\theta_1), \sigma_2(\theta_2))$ one can formulate the Bayesian strategic form of the game to find the following mutually consistent best responses.

**Proposition 2.** *Suppose that either Assumption 2 and Assumption 3 hold or Assumption 2 holds but Assumption 3 does not. Then, for all* $Pr.(\theta_i = \theta_H)$*, the unique pure strategy Bayes-Nash equilibrium when* $Pr.(k = k_0) \in \{0, 1\}$ *is the target setting rule* $t_i : \Theta_i \times \mathcal{K} \to \Theta_i \times \mathcal{K}$ *where* $t_i(\theta_m) = Q(\theta_m, k_\ell)$ *for each* $i \in \mathcal{I}$*,* $\ell \in \{0, 1\}$ *and* $m \in \{L, H\}$*.*

Proposition 2 shows that, when there is no aggregate uncertainty, it is optimal for each player to truthfully set her targets in line with her natural ability. This confirms the statement and intuition of Guillaume Nery and demonstrates that these contest rules possess incentive compatible properties, as players find it optimal to tell the truth. This equilibrium, which identifies a fixed point in target-output space, implies that the player with the highest ability wins the contest and holds when the state is both neutral or performance enhancing. Moreover, the equilibrium behaviour is independent of the distribution of the players' abilities. This means that, irrespective of the specified common prior over the players' abilities, the strategy of truthful revelation given the known state remains optimal. This result continues to hold when I relax Assumption 3, which only impacts the players' incentives when the state is unknown.

The intuition is akin to that behind the equilibrium of second-price auctions, in which bidders find it optimal to bid their true value (Vickrey, 1961). When the state is neutral, for example, each player finds no benefit in using a strategy that either over-or underestimates her target relative to her ability. If a player sets a target above or below her ability, she will receive a score less than what she would have received if she set her target in line with her ability. Thus, as in a second-price auction, the dominant strategy involves a target setting rule that equates each player's target with her ability in the contest. The contest without aggregate uncertainty is, therefore, ex-post dominant strategy incentive compatible and ex-post efficient. However, this implies that the amount of outcome uncertainty is reduced. If the players' ex-ante abilities were endogenously determined through an effort choice, which generated a corresponding disutility, this result may change to reflect the optimal strategy in a first-price auction, in which players shade their bids with respect to the number of bidders (Krishna, 2009).

One conclusion is that, if players in a contest formulated under the current rules are able to deduce the state of nature, these unconventional rules may be superfluous. Such a situation could arise through expertise, repetition, or if the effect of aggregate uncertainty was very small[11]. This result is linked to the seminal work of Lazear and Rosen (1981) and Green and Stokey (1983), as a simple rank-order tournament will lead to the same outcomes, namely the player with the highest ability receiving the prize. The key difference is that here there is no prize spread and each player's performance is determined without an endogenous effort component; however, upon introducing an effort choice it can be shown that under the conditions of Proposition 2 each player's expected payoff collapses to that studied by Lazear and Rosen (1981). Moreover, given that the demand for sporting contests is often increasing in outcome uncertainty (Neale, 1964; Szymanski, 2003b,a; Borland and MacDonald, 2003), if contestants are able to deduce the state of nature a contest designer may choose to employ different rules to induce more outcome uncertainty, and hence consumer demand.

### 1.4.2 Equilibria with Aggregate Uncertainty

I now analyse the game with two states of nature $\mathcal{K} = \{k_0, k_1\}$ with $\Pr.(k = k_0) = \lambda \in (0, 1)$, and initially assume that Assumption 2 and Assumption 3 hold. To construct this game, one takes the payoffs the players would obtain in each of the Bayesian strategic form games derived from holding the state of nature fixed and weights these payoffs by their probability of occurring. As an illustration, fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H \theta_L)$ and taking player 1's as $\sigma_1(\theta_1) = (\theta_H \theta_L + k_1)$, one can calculate that when the state is neutral $\{k\} = \{k_0\}$ player 1 has an expected payoff of $U_1((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_1 | k = k_0) = \mu(1 - \mu/2)$. Conversely, when the state is performance enhancing $\{k\} = \{k_1\}$, player 1 has an expected payoff of $U_1((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_1 | k = k_1) = 1 - \mu^2/2$. Therefore, in the complete Bayesian game, one can express player 1's expected payoff from following strategy $\sigma_1(\theta_1) = (\theta_H \theta_L + k_1)$ when player 2's strategy is held fixed at

[11] As shall be shown in Theorem 1.

$\sigma_2(\theta_2) = (\theta_H \theta_L)$ as the weighted sum of these two payoffs

$$U_1((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_1) = \underbrace{\lambda \cdot \mu(1 - \frac{1}{2}\mu)}_{\text{Pr.}(S_1(\cdot) \geq S_2(\cdot)|\ k=k_0)} + \underbrace{(1-\lambda)\cdot(1 - \frac{1}{2}\mu^2)}_{\text{Pr.}(S_1(\cdot) \geq S_2(\cdot)|\ k=k_1)},$$

and player 2's as $U_2((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_2) = 1 - U_1((\sigma_1(\theta_1), \sigma_2(\theta_2)), \theta_1)$. Repeating these calculations for each possible strategy profile of the players leads to the Bayesian strategic form with aggregate uncertainty. It is here that the assumption that the prize is equal to one helps with tractability, as it allows me to express the players' expected payoffs as a probability defined on the two salient primitives of the model: the common prior over the possible states and the distribution of the players' abilities.

With this Bayesian strategic form representation, I once again look for mutually consistent best responses that will form unique, pure strategy, Bayes-Nash equilibria. I mostly focus on equilibria in pure strategies in this section for two reasons. The first is that many papers suggest that, in practice, amateur competitors do not employ the mixed strategies prescribed by theory (Brown and Rosenthal, 1990; Mookherjee and Sopher, 1994)[12]. Moreover, there is inconsistent evidence as to whether even professional competitors employ mixed strategies. Chiappori et al. (2002) and Walker and Wooders (2001) find evidence supporting the hypothesis that they do, whilst Kovash and Levitt (2009) and Levitt et al. (2009) find contrary evidence. Secondly, because each player has four possible strategies, mixed strategies would be defined over a $4 \times 4$ game, which would lead to analytical difficulties. The strategic form representation suggests three distinct cases to be studied, all of which restrict the common prior over the neutral state $\text{Pr.}(k = k_0)$ in different ways. The first case leads to a nonexistence result.

**Proposition 3.** *Suppose that $Pr.(k = k_0) = Pr.(k = k_1)$. Then, under Assumption 2 and Assumption 3, for all $Pr.(\theta_i = \theta_H)$, there are no pure strategy Bayes-Nash equilibria. Conversely, if Assumption 2 holds but Assumption 3 does not, then, for all $Pr.(\theta_i = \theta_H)$, the unique pure strategy Bayes-Nash equilibrium, by iterated elimination of weakly dominated strategies, is $\sigma^* = (\theta_H + k_1\theta_L + k_1, \theta_H + k_1\theta_L + k_1) = Q(\theta_i, k_1)$ for $\theta_i \in \Theta_i$.*

Proposition 3 first shows that, when the neutral and performance enhancing states are equally likely, the players are unable to find mutually consistent best responses in pure strategies. This arises as, excluding strategy profiles where the players have an equal probability of winning, the structure of the game is such

[12] Chiappori et al. (2002) provide a comprehensive summary.

that players have opposing preferences. This creates an incentive to unilaterally deviate from any candidate equilibrium for one player, which in turn generates an incentive for the other player to deviate and change strategy, ad infinitum. This structure links this contest to the game of matching pennies, whilst being constant sum rather than zero-sum, that has no equilibria in pure strategies. When the two states are equally likely, the expected gain from these states to each player's realised performance is $\mathbb{E}[k] = \Pr.(k = k_\ell)$. Due to my assumption on the players' action set, they cannot choose a pure strategy that takes this expectation into account and so are left without mutually consistent best responses in pure strategies. If the target $t_{i2} = \mathbb{E}_{\mathcal{K}}[Q(\theta_i, k)] = \theta_m + \mathbb{E}[k]$ for $m \in \{L, H\}$ was included in each player's action set, then it seems likely that a pure strategy Bayes-Nash equilibrium would exist in which the players' strategies would consist of this target for each ex-ante ability. Indeed, I will show that, intuitively, as players cannot use such a pure strategy, it is optimal for them to mix over their available strategies to approximate this form of target setting rule.

Unlike Proposition 2, once Assumption 3 is relaxed the equilibrium behaviour of players changes significantly. The widened gap between the players' abilities reduces the possible penalty that results from incorrectly setting a target in line with the performance enhancing state, and so, it becomes a dominant strategy for each player to risk overstating her target. That is, players are willing to engage in more optimistic target setting behaviour given the lower risk of such a strategy. Iterated elimination of weakly dominated strategies removes three equilibria, which involve the players only setting the more optimistic target when they are one particular type and not the other. For example, players' strategies could use the optimistic target when they are of higher ex-ante ability and the more prudent target when they are of relatively lower ability.

Whilst a full characterisation of mixed-strategy Nash equilibria will not be provided, I do examine some particular cases. If $\Pr.(\theta_i = \theta_H) = \Pr.(\theta_i = \theta_L)$ there are four explicit mixed-strategy Bayes-Nash equilibria[13], and a continuum of convex combinations of these equilibria. In these, each player employs the same two strategies $\varepsilon_1 \equiv (0, 1/2, 1/2, 0)$ and $\varepsilon_2 \equiv (1/2, 0, 0, 1/2)$ and the equilibria are characterised by the set $\Gamma = \{(\alpha\varepsilon_1 + (1-\alpha)\varepsilon_2), (\beta\varepsilon_1 + (1-\beta)\varepsilon_2)\}$ for $\alpha, \beta \in [0, 1]$. These equilibria correspond to each player choosing a strategy equal to $\mathbb{E}[Q(\theta_i, k_\ell)]$, as discussed above. When both players use strategies $\sigma^* \in \Gamma$, each player has an equal probability of winning the contest as $U(\sigma^*, \theta_i) = U(\sigma^*, \theta_j)$,

[13] These mixed strategy equilibria are calculated using Game Theory Explorer (Savani and von Stengel, 2015), the output of which is contained within Section A.1.

which generates maximum outcome uncertainty. If $\Pr.(\theta_i = \theta_H) \neq \Pr.(\theta_i = \theta_L)$, then the payoffs have a convenient structure that implies the analysis of each fixed value simultaneously determines two cases and, hence, I can show that the equilibrium set $\Gamma$ remains the same whether $\Pr.(\theta_i = \theta_H)$ is close to one or zero[14].

I now consider two more general cases by assuming that the common prior puts more probability mass on one state than another, and then assuming the converse. I will first assume that the performance enhancing state is more likely, and then, that the neutral state is more likely. In each case, the following two main results will characterise the equilibrium strategy profiles and subsequent behaviour of the players under two different sets of assumptions. These cases can be summarised respectively as $\Pr.(k = k_0) < \Pr.(k = k_1)$ and $\Pr.(k = k_0) > \Pr.(k = k_1)$. My first theorem characterises the equilibrium behaviour of players in each of these two cases, respectively, under the first set of assumptions.

**Theorem 1.** *Suppose that* $Pr.(\theta_i = \theta_H) \in \Omega_+ \equiv (2\lambda, 1 - 2\lambda)$. *Then, under Assumption 2 and Assumption 3,* $Pr.(k = k_0) \in (0, 1/4)$ *and the strategy profile*

$$\sigma^* = (\sigma_1^*(\theta_1), \sigma_2^*(\theta_2)) = (\theta_H + k_1\theta_L + k_1, \theta_H + k_1\theta_L + k_1) = Q(\theta_i, k_1) \text{ for } i \in \{1, 2\} \tag{1.4}$$

*is the unique pure strategy Bayes-Nash equilibrium. Conversely, suppose that* $Pr.(\theta_i = \theta_H) \in \Omega_= \equiv (2(1-\lambda), 2\lambda - 1)$. *Then, under Assumption 2 and Assumption 3,* $Pr.(k = k_0) \in (3/4, 1)$ *and the strategy profile*

$$\sigma^* = (\sigma_1^*(\theta_1), \sigma_2^*(\theta_2)) = (\theta_H\theta_L, \theta_H\theta_L) = Q(\theta_i, k_0) \text{ for } i \in \{1, 2\} \tag{1.5}$$

*is the unique pure strategy Bayes-Nash equilibrium.*

Theorem 1 derives endogenous sufficient conditions on the two key parameters of the model, the probabilities of a given player being high ability and the state of nature being neutral, such that unique pure strategy Bayes-Nash equilibria exist. I look for these pure strategy Bayes-Nash equilibria to characterise the conditions under which players find it optimal to follow relatively simple target setting rules, as these strategies may be more likely to arise or be applied in real world settings. The support for this decision comes from the literature previously discussed that demonstrates that often players, whether amateur or professional,

[14]The relevant payoffs can be summarised by the equation:

$$\mu(1 - \mu) = (1 - \mu)(1 - (1 - \mu)). \tag{1.3}$$

The other case analysed is $\Pr.(\theta_i = \theta_H) = \mu = 0.9$, where (1.3) implies that I simultaneously consider $\Pr.(\theta_i = \theta_H) = 0.1$.

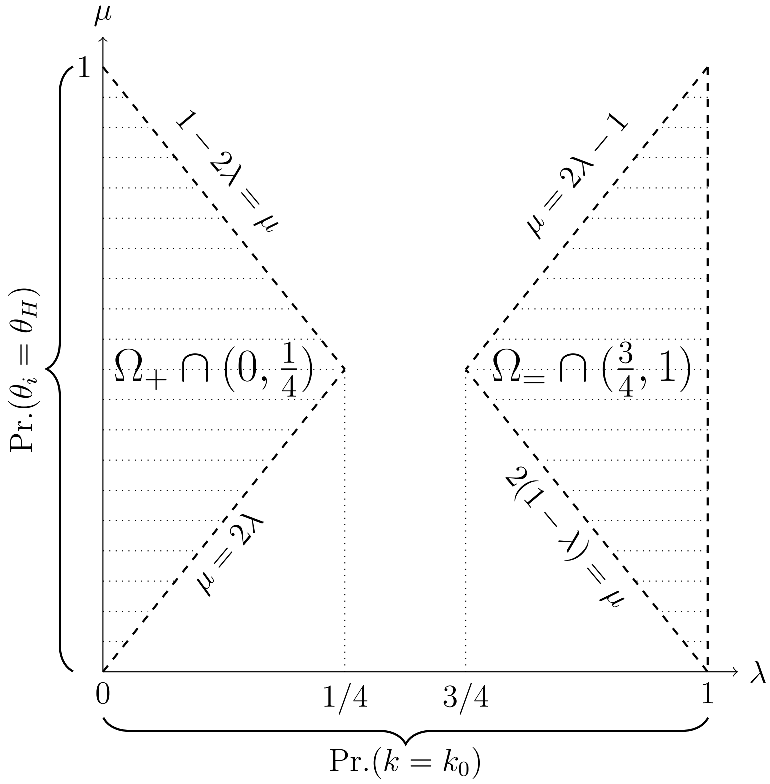


Figure 1.3: Parameter Space for Theorem 1.

may not use mixed strategies. In these equilibria, the players' strategies prescribe following the same target setting rule irrespective of their ability. These strategies guarantee each player the same expected payoff, and hence each player is ex-ante equally likely to win the contest before the players' abilities and the state of nature are realised. Once again, these equilibria demonstrate maximum outcome uncertainty. The sufficient conditions imply that the joint parameter space in Figure 1.3 is convex and each individual parameter is contained within an open interval defined by linear inequalities. In the equilibrium characterised by (1.4), as the probability of the neutral state increases, the abilities of the other players becomes more important; conversely, in the equilibrium characterised by (1.5), as the probability of the neutral state decreases, the abilities of the other players becomes more important. When the probability of a state is close to one the players' incentives to use the strategy that incorporates the effect of this state is so dominant that the distribution of the players' abilities does not matter. The intuition is that, when the players know that one state will arise almost certainly, it is optimal for them to take this state into account when setting their targets, irrespective of their ex-ante abilities. This is because, as the probability of a particular state approaches one, the players' equilibrium behaviour approaches that detailed in Proposition 2. However, as this probability approaches the upper or lower bounds determined by Theorem 1, players will only follow this strategy when they have the least information about their competitors' abilities. If more information about competitors' abilities is available, player $i$'s joint common prior $(\lambda, \mu) \in (0,1)^2$ will lie outside the convex parameter space detailed in Theorem 1 and a more complex target setting rule will be optimal. This form of target setting rule may involve choosing different targets conditional upon realising different ex-ante abilities, unlike these equilibria, and may feature mixed strategies.

Theorem 1 shows that, for each player to find it optimal to risk overestimating her target relative to her ability, the probability of the neutral state must be sufficiently low; specifically, it must be below one quarter. Conversely, for each player to set her target in line with her abilities, which risks receiving a lower score than a player who gambles successfully on the performance enhancing state, the probability of the neutral state must be sufficiently high. The threshold value for this to occur is when $\Pr.(k = k_0)$ is greater than three quarters. Intuitively, as there are no pure strategy Bayes-Nash equilibria when the two states are equally likely, the probability of one state needs to be sufficiently greater than the probability of the other for pure strategy Bayes-Nash equilibria to exist. The requirement is that one state is three times more likely to occur than the other for each player to find it optimal to set her target in accordance with that state. Moreover, at this minimum probability, the distribution of the players' ex-ante abilities must be such that each player must be equally likely to be either low or high ability. As the probability of a particular state approaches one, the distribution of the players' ex-ante abilities can be given more variance. That is, as information about the state of nature becomes more precise, information about the players' abilities becomes less valuable.

I now assume that Assumption 3 no longer holds and look for the conditions under which equivalent pure strategy Bayes-Nash equilibria exist to understand the incentives generated by each set of assumptions.

**Theorem 2.** *Suppose that $Pr.(\theta_i = \theta_H) \in \Omega'_+ \equiv (0,1)$. Then, under Assumption 2 but not Assumption 3, if $Pr.(k = k_0) < Pr.(k = k_1)$ the strategy profile*

$$\sigma^* = (\sigma_1^*(\theta_1), \sigma_2^*(\theta_2)) = (\theta_H + k_1\theta_L + k_1, \theta_H + k_1\theta_L + k_1) = Q(\theta_i, k_1) \text{ for } i \in \{1,2\} \tag{1.6}$$

*is the unique pure strategy Bayes-Nash equilibrium. Conversely, suppose that $Pr.(\theta_i = \theta_H) \in \Omega_= \equiv (2(1-\lambda), 2\lambda - 1)$. Then, under Assumption 2 but not*

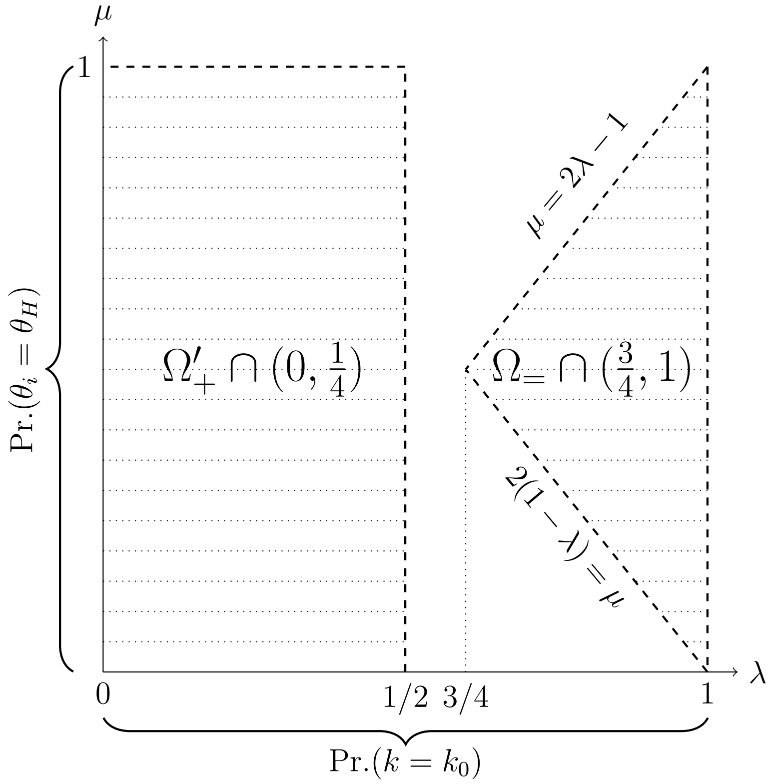


Figure 1.4: Parameter Space for Theorem 2.

*Assumption 3, Pr.$(k = k_0) \in (3/4, 1)$ and the strategy profile*

$$\sigma^* = (\sigma_1^*(\theta_1), \sigma_2^*(\theta_2)) = (\theta_H\theta_L, \theta_H\theta_L) = Q(\theta_i, k_0) \text{ for } i \in \{1, 2\} \tag{1.7}$$

*is the unique pure strategy Bayes-Nash equilibrium.*

By relaxing Assumption 3, each player has a greater incentive to risk overestimating her targets relative to her ex-ante ability. This is because the parameter space that determines when players find it optimal to set the relatively higher target has increased in magnitude. Specifically, the restrictions detailed in Theorem 1 that relate the primitives of the model, Pr.$(\theta_i = \theta_H)$ and Pr.$(k = k_0)$, are removed when the performance enhancing state has a strictly higher probability of occurring. That is, when Pr.$(k = k_1) >$ Pr.$(k = k_0)$, the players' equilibrium strategy, characterised by (1.6), involves using the target that takes account of the performance enhancing state for any distribution of ex-ante abilities. In contrast to the previous result, players now require a lower probability of the performance enhancing state to set the target that takes this state into account. This change in behaviour is intuitive as, by relaxing Assumption 3, there is less risk in overestimation. There is now a smaller expected penalty that will be imposed when a realised performance is less than the target, as players with higher ex-ante ability who end up overestimating will still obtain a greater score than lower ability players irrespective of their target setting behaviour. This arises as Assumption 3 will only hold when the players are relatively close in ability. By relaxing this, the players can have a wider skill gap. Then, given that the distribution of types is common knowledge, each player is more willing to overestimate her target since the other potential competitors may be sufficiently far from her own ability to challenge her. This change in equilibrium behaviour is similar to that which occurs in Proposition 3 when Assumption 3 is relaxed. The equilibrium behaviour of the players characterised by (1.7) does not change when the neutral state has a strictly greater probability of occurring, $\Pr.(k = k_0) > \Pr.(k = k_1)$, as in this case, overestimation still carries the same penalty as in Theorem 1.

## 1.5 Extensions

### 1.5.1 Equilibria when Abilities are Common Knowledge

I now briefly consider the setting under which the players' abilities are common knowledge. This extension is motivated by developing an understanding of the effect of a published ranking on the players' incentives, and whether private abilities are necessary for outcome uncertainty. Generally, in this chapter, I focus on the case when abilities are private information for two reasons. The first is that a published ranking will have less impact under these rules, given that ranking player $i$ above player $j$ does not provide any information about how big of a gap in ability there is between player $i$ and $j$. As my assumptions have shown, this skill gap is important in determining the players' incentives. The second reason is that private abilities allows one to study a more general setting that obtains common knowledge abilities as a special case.

Once again, I analyse the case with two players $i \in \{1, 2\} \equiv \mathcal{I}$ and assume that $\theta_i \in (0, 1) \equiv \Theta$. I first assume, without loss of generality, that player 1 is ranked higher than player 2 so that $\theta_1 > \theta_2$. Since $\sup \Theta = 1$, this implies that $\theta_2 + k_1 > \theta_1$. I also assume that $\theta_2 > \theta_1 - (\delta - 1)$[15]. Given that $\Pr.(k = 0) = \lambda$, the players' expected payoffs are summarised by the following strategic form:

[15]These assumptions are the analogues of Assumption 2 and Assumption 3.

**2**

| | | $Q(\theta_2, k_0)$ | $Q(\theta_2, k_1)$ |
|---|---|---|---|
| **1** | $Q(\theta_1, k_0)$ | $\underline{1}, 0$ | $\lambda, \underline{1-\lambda}$ |
| | $Q(\theta_1, k_1)$ | $1-\lambda, \underline{\lambda}$ | $\underline{1}, 0$ |

Figure 1.5: Strategic Form when Types are Common Knowledge.

As can be seen in Figure 1.5, there are no equilibria in pure strategies as the game has the same structure as matching pennies, except being constant sum rather than zero sum. This implies that there is no pair of pure strategies such that player 1 can best respond to player 2, whilst player 2 best responds to player 1. As there is always an incentive to deviate, I look for mixed strategy equilibria. Suppose that $\Pr.(\sigma_1 = Q(\theta_1, k_0)) = p$ and $\Pr.(\sigma_2 = Q(\theta_2, k_0)) = q$. To make player 2 indifferent between her two pure strategies player 1 chooses $p$ such that

$$p \cdot 0 + (1-p) \cdot \Pr.(k = k_0) = p \cdot (1 - \Pr.(k = k_0)) + (1-p) \cdot 0 \;\Rightarrow\; p^* = \Pr.(k = k_0).$$

Conversely, to make player 1 indifferent between his pure strategies player 2 chooses $q$ such that

$$q \cdot 1 + (1-q) \cdot \Pr.(k = k_0) = q \cdot (1 - \Pr.(k = k_0)) + (1-q) \cdot 1$$
$$\Rightarrow\; q^* = 1 - \Pr.(k = k_0) = \Pr.(k = k_1).$$

The mixed strategy equilibria of the game is, therefore,

$$\sigma^* = (p^* Q(\theta_1, k_0), q^* Q(\theta_2, k_0)) = (\Pr.(k = k_0) Q(\theta_1, k_0), \Pr.(k = k_1) Q(\theta_2, k_0)).$$

In this equilibrium, the higher ability player uses the action that sets a target in line with the neutral state with the same probability that this state occurs. So that, if the neutral state has a forty percent chance of occurring, the higher ability player sets her target equal to her performance in the neutral state forty percent of the time. The lower ability player's behaviour is more interesting. For $\Pr.(k = k_0) > \Pr.(k = k_1)$ the lower ability player puts more probability weight on using the target that takes the performance enhancing state into account. Using the previous example, the lower ability player would set her target equal to her performance in the performance enhancing state sixty percent of the time.

This is because the lower ability player has less to lose, and subsequently more to gain from this strategic behaviour relative to the higher ability player. This strategy gives the lower ability player a strictly positive probability of winning the contest, generating significant outcome uncertainty even when abilities are common knowledge. For $\Pr.(k = k_0) < \Pr.(k = k_1)$, the lower ability player puts more probability weight on the more conservative target in the hope that the high ability player 'busts' by failing to reach her target[16].

Now supposing that the players are ranked equally, so that $\theta_1 = \theta_2$, the strategic form is

**2**

| | | $Q(\theta_2, k_0)$ | $Q(\theta_2, k_1)$ |
|---|---|---|---|
| **1** | $Q(\theta_1, k_0)$ | $\frac{1}{2}, \frac{1}{2}$ | $\lambda,\ 1-\lambda$ |
| | $Q(\theta_1, k_1)$ | $1-\lambda,\ \lambda$ | $\frac{1}{2}, \frac{1}{2}$ |

Figure 1.6: Strategic Form when $\theta_1 = \theta_2$.

Figure 1.6 suggests that two distinct cases must be studied to determine the equilibria. Assuming first that $\Pr.(k = k_0) > \Pr.(k = k_1)$ then each player has a dominant strategy in $Q(\theta_i, k_0)$, and hence the pure strategy dominance solvable Nash equilibrium is

$$\sigma^* = (Q(\theta_1, k_0), Q(\theta_2, k_0)).$$

Conversely, if $\Pr.(k = k_0) < \Pr.(k = k_1)$, then each player has a dominant strategy in $Q(\theta_i, k_1)$ and hence the pure strategy dominance solvable Nash equilibrium is

$$\sigma^* = (Q(\theta_1, k_1), Q(\theta_2, k_1)).$$

Hence, players will only find it optimal to mix in a common knowledge setting when they have different ex-ante abilities. This makes intuitive sense as following these pure strategies in equilibrium gives each player an equal probability of winning the contest. When the players have the same ability, the equilibrium simply involves each player setting her target equal to her performance in the most likely state.

[16] "It's like playing poker", Trubridge told me."You are playing the other divers as much as you are playing yourself". The hope is that your foes will choose a shallower dive than you can do, or that they'll choose a deeper dive than they can do and end up "busting" (Nestor, 2014).

### 1.5.2 Equilibria when Abilities are Continuously Distributed

I now consider an extension to the model of Section 1.3 that features many players and a continuous, common knowledge, distribution over the space of ex-ante abilities. I study this extension as Proposition 2 can be generalised to this setting. To see this, suppose that player $i \in \mathcal{I} = \{1, \ldots, i, \ldots, N\}$'s ex-ante ability is drawn from the closed interval $\theta_i \in \Theta \equiv [\underline{\theta}, \overline{\theta}]$ with associated distribution $F : \Theta \to [0, 1]$. Then, to fix the state of nature, let $\Pr.(k = k_0) = 1$, which is without loss of generality as analogous results hold if $\Pr.(k = k_0) = 0$.

Now consider the following three strategies: first, a target equal to player $i$'s ex-ante ability $\sigma_1(\theta_i) = \theta_i$; second, a target equal to player $i$'s ability minus some constant $c \in \mathbb{R}_{++}$ such that $\sigma_2(\theta_i) = \theta_i - c > 0$; and lastly, a target equal to player $i$'s ability plus $c$, that is $\sigma_3(\theta_i) = \theta_i + c$. Since $c$ is arbitrary, I effectively consider strategies $\sigma_i(\theta_i) \in [\theta_i - c, \theta_i + c]$. I focus on symmetric equilibria, in which each player uses the same strategy, and show that $\sigma_1(\theta_i)$ is a dominant strategy for each $i \in \mathcal{I}$. Suppose first that this is true, so that the strategy profile $\sigma_1^* = (\sigma_1(\theta_1), \ldots, \sigma_1(\theta_N))$ is the unique pure strategy Bayes-Nash equilibrium. In this equilibrium, each player has an expected payoff equal to the second highest order statistic of the distribution of ex-ante abilities

$$U(\sigma_1^*, \theta_i) = \int \Pr.\Big(S(Q(\theta_i, k_0), \sigma_1(\theta_i)) > S(Q(\theta_j, k_0), \sigma_1(\theta_j)), \forall\ j \neq i\Big) dF(\theta_{i-1})$$
$$= \Pr.(\theta_i > \theta_j, \forall\ j \neq i) = (F(\theta_i))^{N-1}.$$

Hence, player $i$'s expected probability of winning the contest is the probability that each of the other $N - 1$ players have a lower ex-ante ability than player $i$. I now show that player $i$ has no incentive to deviate from this candidate equilibrium. If player $i$ deviates to $\sigma_2(\theta_i)$ her expected payoff becomes

$$U((\sigma_2(\theta_i), \sigma_{1_{-i}}^*), \theta_i) = \Pr.(\theta_i - c > \theta_j, \forall\ j \neq i) = (F(\theta_i - c))^{N-1}$$

where $\sigma_{1_{-i}}^* = (\sigma^*(\theta_1), \ldots, \sigma_1(\theta_{i-1}), \sigma_1(\theta_{i+1}), \ldots, \sigma_1(\theta_N))$, whilst by deviating to $\sigma_3(\theta_i)$ one has, using the properties of the scoring rule (1.2)

$$U((\sigma_3(\theta_i), \sigma_{1_{-i}}^*), \theta_i) = \Pr.(\theta_i + c - \delta((\theta_i + c) - \theta_i) > \theta_j, \forall\ j \neq i)$$
$$= \Pr.(\theta_i - c(\delta - 1) > \theta_j, \forall\ j \neq i) = (F(\theta_i - c(\delta - 1)))^{N-1}.$$

There is no profitable deviation as

$$(F(\theta_i))^{N-1} > (F(\theta_i - c))^{N-1} > (F(\theta_i - c(\delta - 1)))^{N-1}$$

because $F$ is nondecreasing by definition, $c > 0$ by construction and $\delta > 1$ by Assumption 1. As an example, this inequality chain is equivalent to $\theta_i > \theta_i - c > \theta_i - c(\delta - 1)$ if $[\underline{\theta}, \overline{\theta}] \sim U([0,1])$.

I now show that there exists a profitable deviation for player $i$ when other players use the strategies $\sigma_2(\theta_j) = \theta_j - c$ or $\sigma_3(\theta_j) = \theta_j + c$ for all $j \neq i$. Suppose now that the strategy profile $\sigma_2^*$ is the Bayes-Nash equilibrium. By similar methods as above, I find that each player $i \in \mathcal{I}$ has an expected payoff of $(F(\theta_i))^{N-1}$ from following this strategy profile. At this strategy profile, however, player $i$ has an incentive to deviate to $\sigma_1(\theta_i)$, as this yields her an expected payoff of $(F(\theta_i + c))^{N-1} > (F(\theta_i))^{N-1}$ since $F$ is nondecreasing. Therefore, player $i$ can profitably deviate. Finally, suppose now that the strategy profile $\sigma_3^*$ is the Bayes-Nash equilibrium. In this case, each player has an expected payoff equal to $(F(\theta_i))^{N-1}$ as before, and, once again, player $i$ can profitably deviate to $\sigma_1(\theta_i)$ obtaining an expected payoff of $(F(\theta_i + c(\delta - 1)))^{N-1} > (F(\theta_i))^{N-1}$.

The last case to check is when $\sigma_i(\theta_i) \in [\theta_i - c_1, \theta_i + c_2]$ for $c_1 \neq c_2$ and $c_1, c_2 > 0$. Under this assumption $\sigma_1(\theta_i)$ remains the dominant strategy for each player as $(F(\theta_i))^{N-1} > (F(\theta_i - c_1))^{N-1}$ and $(F(\theta_i))^{N-1} > (F(\theta_i - c_2(\delta - 1)))^{N-1}$. This case implies that the result also holds when $\sigma_2(\theta_i) = \theta_i - c_i$ and $\sigma_3(\theta_i) = \theta_i + c_i$ for $c_i \neq c_j$ for all $i \neq j$ as one has

$$(F(\theta_i))^{N-1} > F(\theta_1 - c_1) \cdots F(\theta_{i-1} - c_{i-1}) F(\theta_{i+1} - c_{i+1}) \cdots F(\theta_N - c_N)$$

and

$$(F(\theta_i))^{N-1} >$$
$$F(\theta_1 - c_1(\delta - 1)) \cdots F(\theta_{i-1} - c_{i-1}(\delta - 1)) F(\theta_{i+1} - c_{i+1}(\delta - 1)) \cdots F(\theta_N - c_N(\delta - 1)),$$

respectively.

Therefore, player $i$ has a dominant strategy in $\sigma_1(\theta_i)$ irrespective of the strategies of the rest of the players $\mathcal{I} \setminus \{i\}$. Since player $i$ is chosen arbitrarily from the set of players, this implies that all players have the same dominant strategy, which constitutes an equilibrium in dominant strategies. Consequently, the unique symmetric pure strategy Bayes-Nash equilibrium is the strategy profile $\sigma_1^* = (\sigma_1(\theta_1), \dots, \sigma_1(\theta_N)) = (\theta_1, \dots, \theta_N)$. This result continues to hold in a setting with multiplicative strategies, such as $\sigma(\theta_i) = c \cdot \theta_i$, and provides a further generalisation of the result given in Proposition 2. The extended result states that, when the players' ex-ante abilities are continuously distributed, each player truthfully reveals her ex-ante abilities in equilibrium without shading or exaggerating.

I now turn to examine the equilibria when the players' abilities are continuously distributed and there are two states of nature captured by $\mathcal{K} = \{k_0, k_1\}$, where $k_0 = 0$ and $k_1 \in (0, 1)$. In this setting, I consider the following two strategies $\sigma_1(\theta_i) = Q(\theta_i, k_0)$ and $\sigma_2(\theta_i) = Q(\theta_i, k_1)$. Each player $i \in \mathcal{I}$ now has an expected payoff of

$$U_i(\sigma, \theta_i) = \sum_{k \in \mathcal{K}} \int \text{Pr.}\Big(S(Q(\theta_i, k_\ell), \sigma_m(\theta_i)) > S(Q(\theta_j, k_\ell), \sigma_n(\theta_j)) \,\forall\, j \neq i\Big) \text{Pr.}(k) dF(\theta_{-i}),$$

for $m, n \in \{1, 2\}$. Suppose all players are characterised by the strategy profile $\sigma^*_{1_{-i}}$, then player $i$'s expected payoff from following strategy $\sigma_1(\theta_i)$ is

$$\begin{aligned} U_i(\sigma^*_1, \theta_i) &= \text{Pr.}(k = k_0) \cdot \text{Pr.}(\theta_i > \theta_j \;\forall\; j \neq i) + \text{Pr.}(k = k_1) \cdot \text{Pr.}(\theta_i > \theta_j \;\forall\; j \neq i), \\ &= (F(\theta_i))^{N-1}, \end{aligned} \tag{1.8}$$

since $\text{Pr.}(k = k_1) = 1 - \text{Pr.}(k = k_0)$. If instead, player $i$ uses the strategy $\sigma_2(\theta_i)$, then her expected payoff is

$$\begin{aligned} &U_i((\sigma_2(\theta_i), \sigma^*_{1_{-i}}), \theta_i) \\ &= \text{Pr.}(k = k_0) \cdot \text{Pr.}(\theta_i - k_1(\delta - 1) > \theta_j \,\forall\, j \neq i) + \text{Pr.}(k = k_1) \cdot \text{Pr.}(\theta_i + k_1 > \theta_j \,\forall\, j \neq i), \\ &\;= \text{Pr.}(k = k_0) \cdot (F(\theta_i - k_1(\delta - 1)))^{N-1} + \text{Pr.}(k = k_1) \cdot (F(\theta_i + k_1))^{N-1}. \end{aligned} \tag{1.9}$$

Suppose instead all players $\mathcal{I} \setminus \{i\}$ are characterised by the strategy profile

$$\sigma^*_{2_{-i}} = (\sigma_2(\theta_1), \ldots, \sigma_2(\theta_{i-1}), \sigma_2(\theta_{i+1}), \ldots, \sigma_2(\theta_N)).$$

Then player $i$'s expected payoff from following strategy $\sigma_2(\theta_i)$ is

$$\begin{aligned} U_i(\sigma^*_2, \theta_i) &= \text{Pr.}(k = k_0) \cdot \text{Pr.}(\theta_i > \theta_j \;\forall\; j \neq i) + \text{Pr.}(k = k_1) \cdot \text{Pr.}(\theta_i > \theta_j \;\forall\; j \neq i), \\ &= (F(\theta_i))^{N-1}, \end{aligned} \tag{1.10}$$

and from following strategy $\sigma_1(\theta_i)$

$$
\begin{aligned}
&U_i((\sigma_1(\theta_i), \sigma^*_{2_{-i}}), \theta_i)\\
&= \Pr.(k = k_0)\cdot\Pr.(\theta_i + k_1(\delta-1) > \theta_j \,\forall\, j \neq i) + \Pr.(k = k_1)\cdot\Pr.(\theta_i - k_1 > \theta_j \,\forall\, j \neq i).\\
&= \Pr.(k = k_0)\cdot(F(\theta_i + k_1(\delta - 1)))^{N-1} + \Pr.(k = k_1)\cdot(F(\theta_i - k_1))^{N-1}. \quad (1.11)
\end{aligned}
$$

As (1.8)-(1.11) are more complex than in the case with one state of nature, I invoke a simplifying assumption to solve for the Bayes-Nash equilibria.

**Assumption 4.** *Let $\mathcal{I} = \{1, 2\}$, implying $N = 2$, and let $\theta_i \sim U([0, 1])$ for all $i \in \mathcal{I}$.*

Assumption 4 then implies that (1.8) and (1.10) become $U_i(\sigma^*_1, \theta_i) = U_i(\sigma^*_2, \theta_i) = \theta_i$, whilst (1.9) becomes

$$
\begin{aligned}
U_i((\sigma_2(\theta_i), \sigma_1(\theta_j)), \theta_i) &= \lambda \cdot F(\theta_i - k_1(\delta - 1)) + (1 - \lambda)\cdot F(\theta_i + k_1),\\
&= \lambda \cdot \int_0^{\theta_i - k_1(\delta-1)} \mathrm{d}\theta_j + (1-\lambda)\cdot \int_0^{\theta_i + k_1} \mathrm{d}\theta_j,\\
&= \theta_i + k_1(1 - \lambda\delta).
\end{aligned}
$$

Analogously, (1.11) becomes $U_i((\sigma_1(\theta_i), \sigma_2(\theta_j)), \theta_i) = \theta_i - k_1(1 - \lambda\delta)$. Under Assumption 4 the salient condition responsible for determining the Bayes-Nash equilibria is

$$
\frac{1}{\Pr.(k = k_0)} \gtreqless \delta. \quad (1.12)
$$

Condition (1.12) determines two possible equilibria, if one rules out the case of $\frac{1}{\lambda} = \delta$. The first is displayed in Figure 1.7.

| | | **2** | |
|---|---|---|---|
| | | $Q(\theta_2, k_0)$ | $Q(\theta_2, k_1)$ |
| **1** | $Q(\theta_1, k_0)$ | $\theta_1, \theta_2$ | $\theta_1 - k_1(1-\lambda\delta)$, $\underline{\theta_2 + k_1(1-\lambda\delta)}$ |
| | $Q(\theta_1, k_1)$ | $\underline{\theta_1 + k_1(1-\lambda\delta)}$, $\theta_2 - k_1(1-\lambda\delta)$ | $\underline{\theta_1}, \underline{\theta_2}$ |

Figure 1.7: Equilibrium when $\frac{1}{\lambda} > \delta$.

The second equilibrium is given in Figure 1.8.

| | | **2** | |
|---|---|---|---|
| | | $Q(\theta_2, k_0)$ | $Q(\theta_2, k_1)$ |
| **1** | $Q(\theta_1, k_0)$ | $\underline{\theta_1}, \underline{\theta_2}$ | $\underline{\theta_1 - k_1(1-\lambda\delta)}, \theta_2 + k_1(1-\lambda\delta)$ |
| | $Q(\theta_1, k_1)$ | $\theta_1 + k_1(1-\lambda\delta), \underline{\theta_2 - k_1(1-\lambda\delta)}$ | $\theta_1, \theta_2$ |

Figure 1.8: Equilibrium when $\frac{1}{\lambda} < \delta$.

The greater $\Pr.(k = k_0)$, the smaller is $[\Pr.(k = k_0)]^{-1}$ and hence, for fixed $\delta$, the lower is the probability that the equilibrium is the one characterised in Figure 1.7. Intuitively, as the probability of the neutral state increases, the players are more likely to use the strategy that is in accordance with this state. Moreover, by Wilson (1971), since each case displays an odd number of equilibria, I can conclude that there are no non-degenerate mixed strategy equilibria.

Assumption 4 gives the game an interesting structure in that it is equivalent for player $i$ to maximise her expected payoff as it is for her to maximise her expected score, as both problems lead to the same decision rule. To see this, note that player $i$'s expected score is

$$\mathbb{E}[S(Q(\theta_i, k), t_i)] = t_i - \delta \cdot \mathbb{E}[\max\{0, t_i - Q(\theta_i, k)\}].$$

As each player has only two pure strategies I can calculate the expected score from following each strategy as

$$\mathbb{E}[S(Q(\theta_i, k), \sigma_1(\theta_i))] = \theta_i + \lambda \cdot 0 + (1-\lambda) \cdot 0 = \theta_i$$

and

$$\mathbb{E}[S(Q(\theta_i, k), \sigma_2(\theta_i))] = \theta_i - \lambda \cdot [\delta(\theta_i + k_1 - \theta_i)] + (1-\lambda) \cdot 0 = \theta_i + k_1(1-\lambda\delta),$$

which are equivalent to (1.8) and (1.9), respectively. Hence, player $i$ chooses the first strategy if $\delta > \frac{1}{\lambda}$ as per (1.12). However, whilst being simpler to compute, this process does not identify the outcome that arises if player 1 uses the first strategy and player 2 uses the second. In the above setting, it does not characterise the payoff (1.10). In this particular case, whilst the payoffs change, player 1's strategy of choosing the first strategy is still optimal, but this might not hold in more complex scenarios. This gives less intuition than maximising expected payoffs, as it characterises only a subset of the payoffs that can occur in the full game.

## 1.6 Applications

To further motivate the relevance of this form of strategic environment to economic situations I will now informally discuss how these results could be applied to other settings.

### 1.6.1 Sales Forecasting

Consider the problem of a manager of a firm that sells a physical product, which must be purchased in advance of any sales, who is trying to forecast the future sales made by her salespeople. Herbig et al. (1993) describes this activity as the one managers dislike more than any other as if the manager orders too many units there will be superfluous costs, whilst too few units will lead to excess demand. Many firms employ a sales-force composite approach to forecast sales, which involves simply asking salespeople for their opinion on their expected sales; however, Mentzer and Cox Jr. (1984) show that a significant proportion of these firms find this method inaccurate. A salient reason for this inaccuracy is that this method provides either no incentives or perverse incentives for the salespeople. For example, if salespeople receive bonuses for simply achieving a target set by themselves then underreporting expected sales becomes a dominant strategy. Cox Jr. (1989) suggests one mechanism that may improve the accuracy of the sales force composite is to provide incentives for accurate sales forecasting. As the scoring rule implemented in the diving contest provides incentives for both objective dive depth, or total sales, and the accuracy of the target, this form of competitive structure may be able to provide the correct incentives for the salespeople with respect to reporting their expected sales. Proposition 2 shows that if the salespeople are able to infer any demand shocks that may arise, the analogue of aggregate uncertainty in this case, then these contest rules will incentivise them to truthfully set their targets. Theorem 1 shows that, when salespeople are sufficiently confident in their estimates of the possible demand shocks, they will set their targets in line with one specific state of nature, and will otherwise randomise. Theorem 2 shows that, if there is a wide discrepancy between the skills of the salespeople, then these contest rules benefit the relatively higher skilled salespeople significantly more than those who are relatively lower skilled.

### 1.6.2 Auctions with Budget Constraints and Deposits

Auction theorists have studied situations with similarities to the setting considered in this chapter. Considering auctions with budget constraints Che and Gale (1998) remark that "before proceeding I must deal with the possibility that a buyer bids more than his budget and then reneges on his bid. I assume that the seller does not award the object to a buyer who reneges, and that she also imposes a small penalty on him. In a first-price auction the winner pays his bid so it would never be optimal to bid more than ones budget, given such a response from the seller". The idea of reengaging on a bid is conceptually similar to overestimating ones target except not as terminal as, unlike in Che and Gale (1998), players in this contest may still win if they overestimate their target. Thinking about the target contest as an auction, Assumption 2 and Assumption 3 characterise the conditions on the penalty imposed by the seller such that bidders are willing to risk overestimating their bid relative to their budget whilst Theorem 1 provides the conditions under which they will find this optimal. The fact that bidders are willing to bid more than their ex-ante budget characterises the key difference in behaviour between the setting considered in this chapter and that considered by Che and Gale (1998).

### 1.6.3 Oligopoly with Cost Uncertainty and Demand Shocks

In Cournot (Bertrand) competition under uncertainty each firm must choose the quantity (price) it produces (sets), knowing its own cost of production and its beliefs over its competitors' costs of production. This is analogous to the target contest in which each player knows her own ex-ante ability and holds beliefs over the distribution of the other players' abilities. This setting has been studied by Gal-Or (1986) and Shapiro (1986). Moreover, the firms may be subject to exogenous symmetric demand shocks, which can either positively or negatively impact the demand faced by the firms, that is similar to the aggregate uncertainty featured in this contest. This environment has been analysed by Novshek and Sonnenschein (1982), Vives (1984) and Gal-Or (1985). The key difference between Cournot competition and this setting is that the prize, the equivalent of the price the firms receive for their output, is fixed and does not depend on the output of the players. The salient similarity between Bertrand competition and the contest formulation is that only one firm will service the market in equilibrium, depending on their marginal cost. In such a setting Theorem 1 provides conditions under which the firms will set prices in line with the possible demand shocks, irrespective of being high or low cost and when they will mix over their available prices conditional upon being either high or low cost. Firms will produce the high quantity to satisfy the positive demand shock when this demand shock is at least three times more likely than the neutral shock that does not impact demand.

## 1.7 Conclusion

This chapter has analysed an unconventional contest drawn from the real world Apnea Freediving Championships using game theoretic tools. The central innovation is that the players must submit a private target of their performance before the contest takes place, which they are unable to perfectly estimate due to a random component. Given each player's ex-ante target and ex-post performance, they are ranked using a common knowledge scoring rule that penalises players for negative deviations from the target. This form of contest has hitherto been not considered by the literature and so I provided a first step in elucidating the incentives facing the players and characterise their equilibrium behaviour.

The contest was modelled as a game of incomplete information. In the game each player's ex-post output was determined by a combination of her ex-ante ability, which is private information, and a random symmetric component. Each player optimally choose a target, which is an estimate of her ex-post performance, knowing her own ability and having beliefs about the abilities of the other players and the possible states of nature. Players were then ranked by both their output and the accuracy of their target.

I focused mainly on a setting with two players, types, states and actions as this tractable setting provided a means to derive the conditions responsible for generating the players' incentives under these rules. In particular, Assumption 1, Assumption 2 and Assumption 3 defined the set of scenarios in which relatively lower ability players can win the contest through strategic target setting. I then turned to analyse the equilibria of the game and characterise the resulting equilibrium behaviour of the players. Proposition 2 showed that when state of nature is common knowledge the players act as in a second-price auction. That is, the players find it optimal not to shade or exaggerate their targets given their ex-ante abilities. This result was extended to a setting with many players and a continuous distribution of ex-ante abilities in Section 1.5. Proposition 3 demonstrated that, when the two states of nature are equally likely, the only equilibria are in mixed-strategies as a consequence of a structure akin to the classic matching pennies game. Upon relaxing Assumption 3, which increases the skill gap between players with low and high ex-ante ability, the pure-strategy equilibrium involved players setting an optimistic target that takes the state of nature that benefits their performance into account. This equilibrium behaviour arose because the penalty to overestimation has been reduced.

The two central results Theorem 1 and Theorem 2 provided sufficient conditions on the distribution of the players' ex-ante abilities and the common belief over the possible realisations of the aggregate uncertainty such that pure strategy Bayes-Nash equilibria exist. This approach lends itself to a useful geometric representation and bounds these probabilities in such a way that players use relatively simple target setting rules. Theorem 1 showed that the players will set their targets in line with one particular state of nature if the probability of this state is three times more likely than the other and their competitors are equally likely to possess high or low ability. The intuition is that players need to be sufficiently confident in one particular state of nature arising to be incentivised to take this state of nature into account when setting their targets when they are the most unsure of the abilities of their opponents. As players become more confident in one state of nature, that is at least three times more likely to occur, the distribution of the players' abilities becomes less important. Upon relaxing Assumption 3, Theorem 2 showed that there is now a larger set of pairs of probabilities that will incentivise players to set their targets as if the performance enhancing state of nature will arise. The intuition behind this result echoes that of the second part of Proposition 3.

One learned from these results that this contest generates significant outcome uncertainty, when the players' abilities are both common knowledge and private information. Relatively lower ability players have a strictly positive probability of winning the contest in all settings that have more than one state of nature. Given that outcome uncertainty has been shown to be a significant determinant in the demand for contests, and in particular contests between individuals rather than terms (Szymanski, 2003b), these rules could be implemented in individual contests in which consumer interest has stagnated due to low outcome uncertainty. Moreover, these rules could also be applied to economic scenarios such as sales-forecasting, where less competent sales-people may be inspired to set higher targets compared to a situation than when the same strong sales-person continues to win the bonus.

One limitation of the methodology employed is that the analysis is restricted to a simplified setting with two players, two states of nature and two actions available to each player. Whilst this method did prove useful in determining the conditions responsible for generating the players' incentives and characterising their equilibrium behaviour, it is at the cost of sacrificing some generality. A second limitation is that players' ex-ante abilities, and ex-post performances, are determined exogenously. Introducing an optimal effort choice that, combined with a random component, endogenously generating each player's ability would facilitate easier comparison with existing contest and tournament models and would lead to additional insights. Future work could, therefore, consist of extending the framework to provide a characterisation of both the player's optimal target setting rule and effort strategies.

# Chapter 2

# Effort-Signalling under Different Preferences for Risk

## 2.1 Introduction

In classical models of signalling, the sender's type is drawn from some cumulative probability distribution. While this is an apt description of myriad scenarios in which the type represents an intrinsic ability, economic environments exist where the sender may be able to manipulate the value of the variable that she will consequently signal. One such environment is that of financial securitisation, a complex process that converts manifold heterogenous illiquid loans into tradable liquid securities that can be sold to third-party investors on the secondary, or over the counter, markets. Whilst securitisation provides multitudinous benefits for issuing firms, including increased ability to raise capital and liquidity(Agostino and Mazzuca, 2009; Martin-Oliver and Saurina, 2007), numerous economists have suggested that the prolific rise of securitisation from 1980 to 2005 was a salient precipitating factor in the 2007-09 Financial Crisis (Keys et al., 2010; Mian and Sufi, 2009; Ashcraft and Schuermann, 2008). This is due to the dichotomy of ownership and control engendered by securitisation, as lending institutions moved from the classic model of 'originating-and-holding' mortgages to one of 'originating-and-distributing' mortgages as repackaged securities (Brunnermeier, 2009; Stiglitz, 2009; Loutskina and Strahan, 2006). This separation gave rise to agency problems in the form of adverse selection *à* la Akerlof (1970) and both ex-ante and ex-post moral hazard. Adverse selection refers to the private knowledge that an originator has over the composition of borrowers in the market whilst moral hazard, which takes two forms, refers to the incentive an originator has to shirk on underwriting effort, both at the due-diligence and monitoring and servicing stage (Gorton and Pennacchi, 1988; Pennacchi, 1988). An originator may have this incentive to shirk as she alone knows the true value of the asset pool, as investors cannot observe each individual asset included, and, moreover, she knows that, once the security has been sold, any default risk will be transferred from her onto the investors.

I investigate the problem faced by a lone originator who raises capital by underwriting loans that are subsequently securitised. This originator faces opportunistic borrowers whose probability of repayment is random, and so she knows that, to acquire high quality assets, she will need to implement costly underwriting effort. However, she also knows that third party investors, to whom she sells the resulting securities, are unable to observe her choice of effort and so may believe that she is presenting poor quality asset pools as high quality securities. Thus, as the originator and investors are both aware that the act of retention is costly for the originator, and more so under relatively low effort, the originator uses retention as a signal of her underwriting effort. The resulting equilibrium involves the originator retaining distinct levels of the security for each choice of effort, allowing investors to accurately update their beliefs upon observing the retained fraction and correctly price the securities. After looking forward and computing the retention strategies for each choice of effort, the originator then chooses the level of effort that gives her the greatest payoff. Once equilibrium strategies and prices have been fixed, this effort decision is driven by the relative magnitude of the originator's liquidity preference and her effort disutility. The conceptual difference between the aforementioned equilibrium and the canonical separating equilibrium is that, in this setting, the originator signals her choice of effort, which is determined endogenously, whilst in the classical case the signal reveals the exogenously endowed type of the sender. A model of this structure can, therefore, be interpreted as a modified signalling game in which an agent must choose both her type and her investment in the signal.

After the benchmark equilibrium has been characterised the structure of the model is adjusted to incorporate a notion akin to the 'skin in the game' rule[1]. This perturbation generates two novel qualitative properties of the equilibrium, which state that skin in the game increases the amount of retention required to signal high effort, and simultaneously makes it relatively more likely that the originator will choose high effort in equilibrium. This suggests that, while skin in the game is harmful to the originator, it is relatively more harmful under low

[1]Skin in the game is a minimum 5% retention share imposed on any issuer of asset-backed securities introduced in practice in the United States in 2009.

effort, and so improves her incentives for due-diligence. The analysis up until this point rests on the conventional assumption that the originator is risk-neutral and discounts retained earnings via a liquidity preference. As such, these assumptions are then relaxed, and the two comparative statics effects of skin in the game are shown to continue to hold when the originator is risk-averse, but not when she is risk-seeking. This is a robustness result, and implies that, even when the originator is motivated to securitise assets solely by risk-sharing concerns, skin in the game continues to be able to correct some of the problems arising from adverse selection and moral hazard.

In a seminal contribution, DeMarzo and Duffie (1999) conceptually motivate securitisation via liquidity concerns in their analysis of a firm looking to raise capital via an asset backed security issue. They show, at a high level of generality, that the quantity of the security offered for sale is decreasing in the originator's private information, generating a downward sloping demand curve. My model shares similarities with the model of DeMarzo and Duffie as, initially, securitisation is motivated by a need for liquidity, and something akin to their payoff function has been adapted for use in my binary model. However, the key difference with my model is that the asset quality is engendered endogenously, rather than being drawn from an exogenous probability distribution, and I consider risk-sharing as the salient motivating factor for securitisation.

Studying the problem from a pure moral hazard standpoint Malekan and Dionne (2014) explore the properties of the optimal securitisation contract between an originator and investors. My work is related to Malekan and Dionne's model as both show that a positive retention clause can improve the originator's incentives, and, moreover, both papers consider a risk-averse originator; therefore, providing a similar result. The major distinction, however, is that Malekan and Dionne's model is built upon a principal-agent framework, whilst I employ a signalling framework more akin to Spence (1973). Fender and Mitchell (2009) also study a principal-agent model in which an originator can exert costly screening effort that increases the expected return of the asset pool. Their model explores the optimal effort under different forms of exogenously given retention including vertical, mezzanine and equity with a risk-neutral originator. Whilst my model examines solely vertical retention, but with a risk-averse originator, and characterises the conditions under which each choice of effort arises. Kiff and Kisser (2010) extend the model of Fender and Mitchell and perform numerical simulations. These two papers focus mainly on the originator's choice of effort, and only briefly consider the dual problem of choosing both effort and retention;

conversely, this is the central problem I consider.

The most closely related model to that which I study is proposed by Chemla and Hennessy (2014), who are the first to implement an effort choice within a securitisation signalling framework based on Tirole (2006). They show that there exists a threshold cost that the originator is willing to pay to increase to high effort in equilibrium, and then design, using welfare maximising measures, retention clauses that enlarge this threshold. My model differs from that of Chemla and Hennessy and extends their results in several directions. Structurally, in Chemla and Hennessy's model, the originator is able to observe the payoff of the asset, regardless of her choice of effort, whereas in this setting, the originator is only able to perfectly observe asset payoffs under high effort. This adds an additional layer of realism. Moreover, Chemla and Hennessy focus on both separating and pooling equilibria, whilst it is shown that, in my framework, an inability to pool is an endogenous consequence. Secondly, I extend their results by introducing a simpler fixed exogenous retention and show that the incentive improving properties continue to hold. Most importantly, I relax the central assumption of risk neutrality and show that the comparative statics results, obtained when the originator is risk-neutral, continue to hold when the originator is risk-averse.

Seeking to incorporate moral hazard elements in a screening model of securitisation, Vanasco (2013) deviates from previous literature by assuming that there cannot be a separating equilibrium. She employs a first-order condition to characterise equilibrium information acquisition in a framework with continuous actions. I study solely separating equilibria and characterise the equilibrium effort choice via an inequality centred around the value of certain key parameters, as here the originator is restricted to a binary effort choice. Attacking the problem from a different angle Coulter (2012) considers a model of securitisation in which the originator may exert costly monitoring effort and obtain ratings from a credit rating agency. He shows that the originator retains the minimum quantity possible regardless of her monitoring choice and that credit rating agencies may lead to ratings inflation. In contrast with Coulter, I show that, to signal high asset quality, the originator will choose to retain a fraction strictly greater than that she is forced to retain by regulation. Secondly, the originator in Coulter's (2012) model is risk-neutral and I do not study the implications of credit rating agencies. Ozerturk (2015) considers a similar model to Coulter (2012). Hartman-Glaser (2013) studies an innovative model in which the asset originator is able to build a reputation through asset reports, which may not be accurate. Hartman-Glaser shows that, in this case, retention is no longer a sufficient signal for investors, as the originator may build up a good reputation and then present a false asset report to cash in on her reputation.

Focusing on the dynamic properties of securitisation under moral hazard, Hartman-Glaser et al. (2012) study a continuous time principal-agent model. They show that the optimal contract involves an endogenously specified waiting period after purchase whereby, if no asset has defaulted, investors make their payments. This form of contract shares similarities with Innes (1990)'s 'live-or-die' contract. The framework I analyse is similar to that of Hartman-Glaser et al.'s only in that the originator discounts retained earnings, creating gains from trade. However, Malamud et al. (2013) extend Hartman-Glaser et al.'s model to study the implications of risk aversion and variations in market power, two issues I discuss. In particular, in this signalling setting, I find support for Malamud et al.'s rejection of their 'conjecture two', that effort is unaffected by changes in market power when the originator is risk-averse. Whilst not explicitly studying securitisation, Leland and Pyle (1977) inspired much of the subsequent financial signalling literature by showing that entrepreneurs, or firms, can signal project quality to investors by retaining a fraction of the equity issue, which is increasing in the project's underlying quality. Whilst later papers specifically modelling security design assumed risk neutrality, Leland and Pyle's (1977) model stands out by focusing on a risk-averse entrepreneur. I reintroduce risk aversion into a financial signalling model but depart from Leland and Pyle by explicitly studying asset backed securities and incorporating a costly effort choice.

Looking more broadly at the literature that integrates adverse selection and moral hazard in a unified model, Laffont and Tirole (1986), in a seminal contribution, propose a model that examines the procurement contracting relationship when the supplier has both hidden information about the cost of the project and is able to engage in hidden cost reduction activities. They focus on the optimal linear contract scheme under risk neutrality and consider the problem of choosing effort and cost reduction simultaneously, whilst in this setting the decision problems are treated sequentially. Picard (1987) shows that these incentive schemes can be approximated by quadratic contracts and Theilen (2003) extends these results to a risk-averse setting. Lewis and Sappington (2000), in a model of contracting, sequentially mitigate the problems arising from asymmetric information by first treating moral hazard and then, secondly, alleviating adverse selection. In contrast, I first consider the problem of private information, before moving on to tackle that of hidden action.

Section 2.2 elucidates the formal details of the model and Section 2.3 defines the solution concept before characterising the benchmark equilibrium. Section 2.4 adapts the model to take into account the imposition of the skin in the game regulation and then contrasts these results with those of Section 2.3 in terms of signalling costs and originator incentives. Finally, Section 2.5 considers the main extension to the model with a risk-averse originator and Section 2.6 concludes.

## 2.2 Model

I now set up the formal details of the agents that interact within the securitisation setting, modelled as a modified, sequential move, Bayesian game. The narrative that pins down these interactions involves a single originator who underwrites loans and securitises them to sell on secondary markets; opportunistic borrowers who apply for loans regardless of creditworthiness; and, a set of investors who compete in prices for the securities offered for sale. Initially, nature determines the probability of repayment of each borrower by endowing them with a type, and then the originator, who at this stage is looking to lend to the borrowers, must choose a level of costly underwriting effort to undertake. High underwriting effort allows the originator to perfectly infer the type of each borrower, enabling her to exclude borrowers who may not repay, but creates disutility. When the originator securitises the assets she must choose the fraction that she will retain on her own balance sheet, or equivalently, how much of the security issue she will sell. Finally, investors bid for the offered quantity of the security on the secondary market via prices. Investors cannot observe the choice of effort, and so they form beliefs about the chosen effort based on the observed retention, which informs their pricing decision. Investors have no outside investment opportunities, and, therefore, their payoffs are determined by the wedge between the return on investment and the price they pay. Once this set of interactions has taken place, each agent realises their payoff.

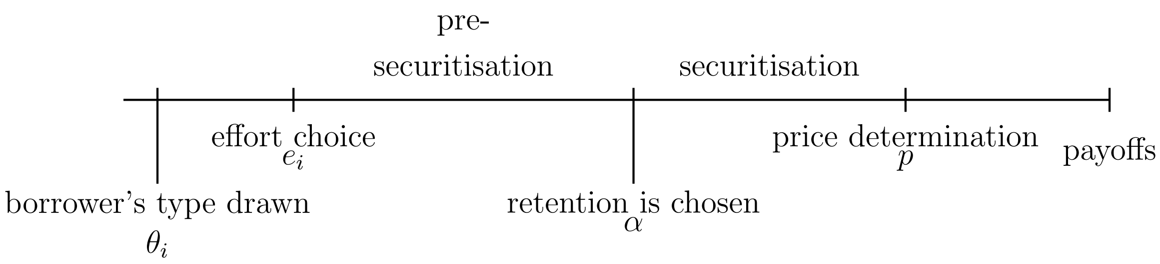


Figure 2.1: Securitisation Game Time-line.

### 2.2.1 Borrowers

Rather than determine the expected payoff of the assets held by the originator as the outcome of some exogenous stochastic process, as is the norm in adverse selection models, asset quality will instead be engendered by the interaction of an originator and automata borrowers. Suppose there is a finite set of borrowers, indexed by $i = \{1, 2, ..., n\}$, each of whom apply for a loan of heterogenous value $l_i \in \mathbb{R}_{++}$. If granted the loan, each borrower spends this loan endowment on an asset of equal value and owes a repayment of $r_i \in \mathbb{R}_{++}$. Each borrower $i$ is characterised by a probability of repayment $\theta_i \in \{\theta_1, \theta_2\}$, which is private information known only by $i$. However, the originator will be able to pay a cost to learn $i$'s type, generating security quality endogenously through selection of high quality assets. Each borrower $i$ has no income at the lending stage of the model, however, she is endowed with an exogenous money income of $\omega_i \in \mathbb{R}_{++}$ after the loans have been granted. I assume $\theta_1 = 1$ and $\theta_2 \in [0, 1)$, and that $\theta_1\omega_i \geq r_i$ and $\theta_2\omega_i < r_i$. These conditions capture the simple idea that there is a set of borrowers who are creditworthy and will repay the loan with certainty, and a disjoint set of borrowers who may not repay, introducing the underlying adverse selection 'lemons problem'. Additionally, to model the idea that borrowers can behave opportunistically, they are protected by limited liability, implying that uncreditworthy borrowers will apply for loans they know they cannot repay. Formulating the problem in this manner, rather than introducing default disutility as in Dubey et al. (2005), helps create the information gain accruing to the originator as a result of costly effort, as both types will appear ex-ante identical. This closely resembles the economic environment that led to the 2007-09 Financial Crisis. Thus, each borrower $i$'s payoff is given by

$$\max\{0, \theta_i\omega_i - r_i\}. \tag{2.1}$$

### 2.2.2 Originator

I study a single originator who combines the actions of two potentially disjoint agents: the lender who underwrites loans and the issuer who securitises these loans to subsequently sell on the secondary market. Assume for the moment that the originator is risk-neutral. The originator has a two-stage decision process: she first must choose a level of costly underwriting effort and then, given this effort, retain a fraction of the security issue. Underwriting effort is a binary choice $e \in \{e_L, e_H\}$ representing ex-ante due-diligence, and can be interpreted as the amount of time that the originator dedicates to determine each borrower's type. Assume that $e_H > e_L$ and, without loss of generality, that $e_L = 0$. If the originator chooses high effort, she becomes perfectly informed of each borrower's type, but pays a cost determined by the disutility that arises from the amount of time she has dedicated to this activity. Let $c : \{e_L, e_H\} \to \mathbb{R}_+$ represent the originator's increasing and convex disutility, where $c(e_L) = 0$ and $c(e_H) > 0$. A natural supposition is that, after paying the cost and becoming informed of the borrower's type, the originator will only grant loans to those borrowers who are creditworthy. This implies that, if the originator chooses high effort, she will only grant loans to type $\theta_1$ borrowers.

The originator's prior belief, and how it is transformed to a posterior as a result of the choice of effort, can be formalised as follows: with low underwriting effort the originator has a posterior belief about the aggregate distribution of borrower types from which she can form an expected probability of repayment. However, with a choice of high underwriting effort, the originator can perfectly differentiate between borrower types. Thus, this posterior belief acts as a summary statistic as to the likelihood of the $i$th borrower's repayment, given the choice of effort. The originator's posterior belief is $\rho : \{e_L, e_H\} \to [0,1]$ defined by $\rho(\theta|e) = \beta(e) \cdot \theta_1 + (1 - \beta(e)) \cdot \theta_2$. The function $\beta : \{e_L, e_H\} \to [0,1]$ represents the probability weight the originator places on the borrower being of type $\theta_1$ and creates the formal connection between eventual asset payoff and the originator's choice of effort.

**Assumption 5.** *Only high effort gives the originator a perfectly accurate posterior belief:*

$$\beta(e_L) \in (0,1) \Rightarrow 0 < \rho(\theta|e_L) < 1;$$

$$\beta(e_H) \in \{0,1\} \Rightarrow \rho(\theta|e_H) \in \{0,1\}.$$

Assumption 5 formalises the notion of an information gain resulting to the originator from a choice of high effort. This implies that, while the originator cannot affect the type of borrower that applies for loans, she is able to affect the make-up of the asset pool that forms the security through her choice of effort. Therefore, Assumption 5 helps to give rise to moral hazard. To prevent a situation in which the originator can increase her profits by exploiting the finiteness of the set of borrowers, it will be assumed that the originator can grant $k < n$ loans irrespective of her effort. The originator will then be unable to choose low effort and grant loans to all $n$ borrowers, whilst a choice of high effort restricts the pool of borrowers to those who are type $\theta_1$, This may arise due to some budgetary constraint and will be relaxed in Section A.2.

The originator is motivated to securitise the asset pool as she has a liquidity preference, denoted by $\delta$, that creates gains from trade and represents the discount that the originator places on any retained earnings. This preference arises as, when the originator retains long term assets, her payoff is effectively a stream of cash payments discounted over time. Whereas, if she sells the assets as securities, she receives an upfront cash price, the proceeds of which can be used to underwrite more loans. These loans can then be securitised and sold on to the secondary market, ad infinitum. In this static setting, this is the essence of what this modelling tool aims to capture.

**Assumption 6.** *The originator has a strict preference for liquidity:* $1 > \delta > 0$.

Once the originator has chosen a level of underwriting effort, and the expected loan repayments have been securitised, her second decision problem is to choose a retention strategy. The retention strategy is a fraction of the security issue that the originator will keep on her own balance sheet and is denoted $\alpha \in [0, 1]$. The originator may try to use her retention as a signal of her effort choice to investors as it is costly, both in terms of forgone revenue and exposure to default risk. In such a case, the retention strategy would be a function $\alpha : \{e_L, e_H\} \to [0, 1]$. I assume that the originator's payoff function displays increasing differences in the level of retention and the choice of effort. This assumption captures the complementarity that arises between the originator's choice of retention and her effort $(\alpha, e) \in [0, 1] \times \{e_L, e_H\}$.

**Definition 2.** The function $f(x, y)$ has *increasing differences* in $(x, y)$ if, for $x_L, x_H \in X$ such that for $x_H > x_L$ and for $y_L, y_H \in Y$ such that $y_H > y_L$, one has

$$f(x_H, y_H) - f(x_L, y_L) \geq f(x_H, y_L) - f(x_L, y_L). \tag{ID}$$

When applied to the originator's payoff, increasing differences implies that, for any given level of retention, the incremental cost associated with this retention is decreasing in the originator's choice of effort. This condition arises naturally as, under high effort, the originator is not exposed to default risk when she retains a positive fraction of the issue. She faces only the implicit cost of foregone revenue, the opportunity cost, engendered by her liquidity preference. After deciding on a level of retention, the originator sells the remaining fraction of the security issue and receives a price-per-unit denoted $p \in [0, 1]$. The originator's payoff is, therefore, defined by the mapping $u : \{e_L, e_H\} \times [0, 1]^2 \to \mathbb{R}_+$, and her expected payoff is

$$\mathbb{E}[u(e,\alpha;p)] = \rho(\theta|e)\left(\delta\alpha\sum\nolimits_{i=1}^{k} r_i + p(1-\alpha) - \sum\nolimits_{i=1}^{k} l_i\right) \\ +(1-\rho(\theta|e))\left(p(1-\alpha) - \sum\nolimits_{i=1}^{k} l_i\right) - c(e). \tag{2.2}$$

### 2.2.3 Investors

There is a set of risk-neutral investors of measure one who bid for the securities on the secondary market. The interplay of this bidding behaviour and market conditions determine the prices that investors pay for the offered securities. Investors' information is such that they are aware of the implications of the originator's effort choice, as the effort technology is common knowledge, but are unable to observe the chosen effort. Therefore, investors form a common prior, represented by a belief function, that provides an ex-ante probability that the originator has chosen high effort. Once the originator presents a quantity of the security issue for sale, which is observed by investors who know retention is costly, this function is updated to form a posterior belief. Denote this posterior belief by

$$\gamma(\alpha) = \text{Pr.}(e = e_H|\alpha), \tag{2.3}$$

and the payoff of an arbitrary investor as $\pi : [0,1]^3 \to \mathbb{R}_+$. Prices are pinned down via the standard convention of competition, implying prices are pushed up by bids until they arrive at the expected payoff of the security.

**Assumption 7.** *Investors are competitive, hence, in any equilibrium, they make zero expected profits.*

Suppose that there are pairs of scalars $(\alpha_H, p_H)$ and $(\alpha_L, p_L)$ that arise when the originator chooses high and low effort, respectively. Note that, at this point, the only property these four scalars possess is that they are all drawn from the unit interval[2]. An investor's ex-post payoffs, in the high and low effort cases, are given by the return of the securitised asset minus the price-per-unit paid, multiplied by the quantity that has been put up for sale by the originator

$$\pi(p_H, \alpha_H, 1) = (1-\alpha_H)\left(\sum_{i=1}^{k} r_i - p_H\right),$$

[2]To be more explicit: at this stage there is nothing to prevent, for example, $\alpha_H = \alpha_L$.

and

$$\pi(p_L, \alpha_L, 0) = (1 - \alpha_L)\left(\rho(\theta|e_L)\sum_{i=1}^{k} r_i - p_L\right).$$

Therefore, taking (2.3) into account means that an investor has an expected payoff given by

$$\mathbb{E}[\pi(p, \alpha, \gamma(\alpha))] = \gamma(\alpha) \cdot \pi(p_H, \alpha_H, 1) + (1 - \gamma(\alpha)) \cdot \pi(p_L, \alpha_L, 0). \tag{2.4}$$

The assumption of competition between investors enables one of the equilibrium conditions to be derived endogenously.

**Lemma 1.** *If investors make zero expected profits then $p = \mathbb{E}[\rho(\theta|e)]$.*

*Proof of Lemma 1.* All proofs for this chapter are given in Section B.2. □

## 2.3 Equilibrium

The solution concept used to solve the securitisation game is a modified perfect Bayesian equilibrium. Equilibrium occurs when the originator, for each choice of underwriting effort, chooses a level of retention that maximises her payoff, given how this choice of retention level will impact investors' posterior beliefs and, hence, the price they pay for the security. It must also be the case that investors, knowing the optimising behaviour of the originator, form accurate posterior beliefs and offer a price that is consistent with the chosen equilibrium action of the originator. Only when both of these events occur simultaneously can an equilibrium arise.

There is a conceptual difference between the equilibrium to follow, in which the retention strategy that results from a choice of high underwriting effort is distinct from that which results from low effort, and the usual separating equilibrium of signalling models. Here, the action of the originator, the sender, does not signal an exogenously endowed type, nor the expected payoff of the asset. Instead, the fraction retained by the originator signals an endogenous choice previously made by that same originator viz-a-viz high or low underwriting effort. The crucial component is the originator's underwriting technology, which maps effort into expected asset payoffs, and is assumed to be common knowledge. Investors, therefore, learn implicitly the payoff of the underlying assets, which is determined by the originator's effort choice, signalled through the level of retention. As such the equilibrium is coined 'effort-signalling'.

**Definition 3.** A retention function $\alpha^* : \{e_L, e_H\} \to \mathbb{R}_+$ and a corresponding price $p$ is an *equilibrium* of the signalling game if the following conditions are satisfied:

1. $\alpha(e_i) \in \arg\max_{\alpha \in [0,1]} u(e_i, \alpha; p_i)$ for $i \in \{L, H\}$.

2. $p = \mathbb{E}[\rho(\theta|e)]$.

The equilibrium is *effort-signalling* if price is a function of retention such that

3. $p^*(\alpha(e_i)) = \rho(\theta|e_i)$ for $\alpha(e_i) \neq \alpha(e_{-i})$ and $i \in \{L, H\}$.

Condition 1 states that the retention strategy chosen by the originator maximises her payoff for each choice of underwriting effort[3], and condition 2 reflects investor competition. Finally, condition 3 states that for separation, or effort-signalling, to occur the signal must fully reveal the originator's choice of underwriting effort via distinct fractions of retention for each level of effort. Moreover, condition 3 implies that the beliefs of investors must become accurate, and hence, consistent with the optimal actions of the originator, reflecting the key restriction imposed in a perfect Bayesian equilibrium.

The method by which the equilibrium is arrived at can be best inferred as a hypothetical thought experiment of the lone originator. The originator knows that whether or not a borrower is creditworthy is random, and that borrowers behave opportunistically. Therefore, she realises that, if she want to underwrite high quality loans, she will have to put in costly high effort. However, at the same time, the originator knows that investors are unable to observe her choice of effort and so she may try and present low quality assets as high quality. As a result, the originator can deduce that, if she does choose high effort, she will have to find a means by which she can credibly convey her choice of effort to investors. Her chosen mechanism is retention, as she knows that investors know that this action is costly. Thus, the originator has to compute the optimal retention strategies for each choice of effort, such that under high effort she can try and signal her effort choice. If she can successfully signal this information, then competitive investors will be able to accurately update their beliefs and offer the correct prices. Then, given these two optimal strategies, she computes the equilibrium payoff that would result by following the prescribed strategy for each choice of effort. The originator then compares these two equilibrium payoffs and chooses the effort level that satisfies the following definition:

[3]Note this is for *given* prices, here represented by arbitrary scalars. It will become clear that the originator will realise that effort-signalling prices are the correct ones to anticipate and optimise against.

**Definition 4.** $e^* \in \{e_L, e_H\}$ is the *equilibrium effort choice* of the effort-signalling equilibrium given in Definition 3 if $e^* = \arg\max_{e \in \{e_L, e_H\}} u(e, \alpha^*(e); p^*(\alpha))$.

Once retention strategies are fixed, the choice of effort will centre around the value of certain key parameters. The intuition behind the effort signalling equilibrium is that, if investors observe a relatively high level of retention, then their beliefs are updated in such a way that they now place probability one on the originator having performed adequate due-diligence via high effort. The game is solved by characterising the range of levels of retention that are 'large enough' to signal high effort to investors. Once this range is found, if the originator chooses high effort, she selects the level of retention from this range that maximises her payoff.

In order for effort-signalling to take place, the originator must be willing to enter the secondary market for asset-backed securities, regardless of her choice of effort. She must have, in expectation, a greater payoff from selling some positive fraction of the security issue compared to retaining the full issue on her balance sheet. The constraints that ensure this are the participation constraints, one for each choice of effort:

$$\mathbb{E}[u(e_L, \alpha(e_L); p(\alpha(e_L)))] \geq \mathbb{E}[u(e_L, 1; 0)], \tag{PC1}$$

$$u(e_H, \alpha(e_H); p(\alpha(e_H))) \geq u(e_H, 1; 0). \tag{PC2}$$

**Lemma 2.** *The originator's participation constraints are strictly satisfied in an effort-signalling equilibrium.*

Lemma 2 assumes that the equilibrium is effort-signalling without considering pooling. The following proposition demonstrates that this assumption is not necessary, as pooling equilibria are ruled out by the structure of the model. The intuition lies in the subtle difference between my model and the canonical signalling framework.

**Proposition 4.** *If the originator can deviate across effort there can be no pooling in equilibrium. If the originator's effort is fixed, then, for $i \in \{L, H\}$, there can be pooling at $\alpha^*(e_i) = 0$ in equilibrium if $\delta < \mathbb{E}[\rho(\theta|e)]$.*

Given that, in my model, the originator can choose her effort, which can be interpreted as being able to deviate between types, Proposition 4 implies she can focus her attention on computing optimal retention strategies that will comprise an effort-signalling equilibrium. If the originator's effort is fixed, assuming that her discounted payoff for the high quality asset is greater than the expected payoff of any given asset rules out pooling. This condition arises, as unlike many signalling settings, the reservation payoffs are nonzero and asymmetric.

If the originator chooses low effort, her optimal strategy involves issuing the entire security, as at effort-signalling prices her payoff is decreasing in the level of retention. Therefore, the originator can fix this strategy and compute the optimal retention under high effort, given that low effort leads to a full issue. This is a more delicate decision than her choice of retention under low effort. In this case, in order for effort-signalling behaviour to arise, the originator's retention must be able to credibly convey her choice of effort to the investors. To successfully signal a choice of high effort, the originator's retention must be large enough so that investors can deduce that, if the originator had chosen low effort, she would find mimicking this retention strategy and receiving the higher price unprofitable, relative to the strategy of issuing the entire security. The constraints that ensure that the optimal retention strategy under high effort has this property are the incentive compatibility constraints:

$$u(e_L, 0; p(0)) \geq \mathbb{E}[u(e_L, \alpha(e_H); p(\alpha(e_H)))], \tag{IC1}$$

$$u(e_H, \alpha(e_H); p(\alpha(e_H))) \geq u(e_H, 0; p(0)). \tag{IC2}$$

Note that these conditions are local in the sense that they are conditional on the originator issuing everything if she chooses low effort, and on prices being determined by equilibrium condition 3. The intuition is that investors know that retention is costly for the originator, and in particular, that it is relatively more costly for the originator if she chooses low effort. Thus, with a high enough choice of retention, the retention cost that would be imposed on the originator if she were to choose low effort and mimic this strategy, is large enough to prevent this imitation.

**Definition 5.** The *incentive compatible set* is $\Lambda = \left\{\alpha(e_H) \in \mathbb{R}_+ : \underline{\Phi} \leq \alpha(e_H) \leq \overline{\Phi}\right\}$, where

$$\underline{\Phi} \equiv \min\{\alpha(e_H) \in \mathbb{R}_+ : u(e_L, 0; p(0)) \geq \mathbb{E}[u(e_L, \alpha(e_H); p(\alpha(e_H)))]\},$$

and

$$\overline{\Phi} \equiv \max\{\alpha(e_H) \in \mathbb{R}_+ : u(e_H, \alpha(e_H); p(\alpha(e_H))) \geq u(e_H, 0; p(0))\}.$$

The lower bound $\underline{\Phi}$ is the minimum level of retention that satisfies the local incentive compatibility conditions, whilst the upper bound $\overline{\Phi}$ is the maximum.

Note that $\underline{\Phi} > 0$ for all $p_H > p_L$, which is implied by the definition of an effort-signalling equilibrium, and $\Lambda$ is multi-valued for all $\delta < 1$ and $\rho(\theta|e_L) < 1$, which is ensured by Assumption 5 and Assumption 6. In order for the originator to adopt a strategy dictated by incentive compatibility it must also be feasible.

**Definition 6.** The *feasible set* is $\Lambda^F = \{\alpha(e_H) \in \mathbb{R}_+ : 0 \leq \alpha \leq 1\}$.

The intersection of the incentive compatible and feasible sets defines the domain of the optimisation problem of the originator if she wants to signal high underwriting effort.

$$\sup_{\alpha(e_H)\in\ \Lambda^F\cap\ \Lambda} u\left(e_H, \alpha; p(\alpha(e_H))\right). \tag{2.5}$$

The originator's decision problem (2.5) is to choose the level of retention that will both signal her high effort and maximise her payoff.

### 2.3.1 Benchmark Equilibrium

**Proposition 5.** *Suppose that $\Lambda^F \cap\ \Lambda \neq \emptyset$. Let $\underline{\Phi}^* \equiv \underline{\Phi}(\delta, p^*(\alpha))$ denote the solution to (2.5). Then, the least-cost effort-signalling retention is defined by $\alpha^* : \{e_L, e_H\} \to [0,1]$, where*

$$\alpha^*(e) = \begin{cases} \underline{\Phi}^* & \text{if } e = e_H, \\ 0 & \text{if } e = e_L. \end{cases} \tag{2.6}$$

*Let $p^*(\alpha)$ denote the equilibrium price offered by investors and $\gamma^*(\alpha)$ investors' equilibrium beliefs. Then*

$$\begin{aligned} p^*(\alpha) &= \gamma^*(\alpha)p(\alpha(e_H)) + (1-\gamma^*(\alpha))p(\alpha(e_L)), \\ &= \gamma^*(\alpha)\rho(\theta|e_H) + (1-\gamma^*(\alpha))\rho(\theta|e_L). \end{aligned}$$

*Beliefs are defined by the linear step function*

$$\gamma^*(\alpha) = \begin{cases} 1 & \text{if } \tilde{\alpha} \geq\ \underline{\Phi}^*, \\ 0 & \text{if } \tilde{\alpha} <\ \underline{\Phi}^*, \end{cases}$$

*and $\tilde{\alpha}$ is the observed retention. Finally, $e^* = e_H$ if*

$$\delta > \vartheta^{-1}c(e_H)$$

*where* $\vartheta^{-1} > 1$ *if* $2 - \rho(\theta|e_L) > c(e_H)$.

The characterisation given in Proposition 5 is the canonical separating equilibrium of signalling games with a conceptual twist. Rather than signal an exogenously endowed type, in this setting, the originator signals a prior endogenous action, viz-a-viz the choice of effort. Thus, as I study a single originator, in equilibrium only one action will arise. Intuitively, as the originator's retention cost is increasing in the quantity of retention, she retains the minimum amount of the security necessary to signal her choice of effort. Moreover, it can be shown[4] that the following comparative statics properties hold:

$$\underline{\Phi}^*{}_{\rho(\theta|e_L)} < 0$$

and

$$\underline{\Phi}^*{}_{\delta} > 0.$$

The first property states that, given effort-signalling prices, as the probability of repayment that arises if the originator chooses low effort increases, the minimum level of retention required to signal high effort decreases. Intuitively, as the asset pool becomes less risky under low effort, the originator's payoff under low effort increases, and so her incentive to mimic her high effort strategy decreases; therefore, as the wedge between the prices received for the security in each effort case decreases, the incentive to imitate also decreases.

The second property affirms that, as the originator's liquidity preference becomes closer to one, the minimum level of retention required to signal high effort increases. Intuitively, as the part of the retention cost that arises from discounting retained earnings decreases, the originator is more willing to hold some fraction of the asset under low effort to try and mimic the high effort retention strategy. To counteract this incentive, the high effort retention strategy needs to increase by just enough to again make truth telling a dominant strategy in the post effort game.

Figure 2.2 graphs the originator's indifference curves that result from each choice of underwriting effort in retention-price space, where the familiar single-crossing point, which arises due to the assumption of increasing differences, identifies the minimum level of retention required for effort-signalling. Investors' equilibrium beliefs are defined to put probability one on the originator having chosen low underwriting effort for any realised level of retention less than the minimum effort-signalling level, and vice versa for levels of retention greater than

[4] Comparative statics derivations are included at the end of the proof of Proposition 5.

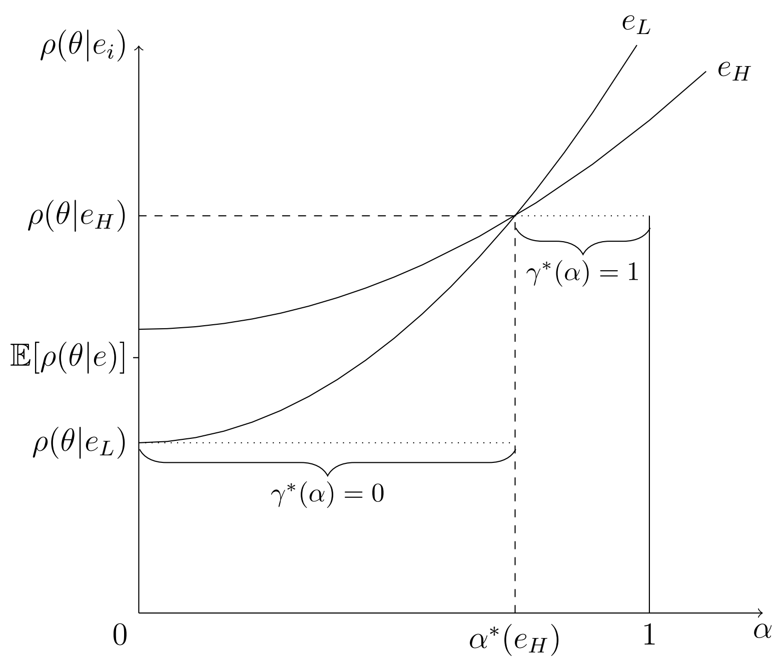


Figure 2.2: Illustration of Equilibrium.

or equal to this minimum. This can be seen in Figure 2.2 as the dotted line represents the equilibrium price as a step function that jumps from the lower price to the higher price for any level of retention greater than that which solves (2.5).

Figure 2.3 illustrates the final aspect of Proposition 5, which concerns a sufficient condition on the liquidity preference, such that, if this condition is satisfied, the originator will choose high effort in the effort-signalling equilibrium, taking $c : \{e_L, e_H\} \rightarrow \mathbb{R}_+$ to be fixed[5]. The intuition is simple: the originator's liquidity preference needs to be sufficiently low that, taking into account her retention cost and disutility, her payoff is still greater than that which obtains under low effort.

Hitherto, there has been the implicit assumption that an effort-signalling equilibrium can exist; however, there may be combinations of parameter values and assumptions which will not engender such an equilibrium. Recall that the incentive compatible set is always nonempty, but that its intersection with the feasible set may be empty under certain conditions. Assumption 5 and Assumption 6 will be taken as given and, hence, the parameter that determines the existence of the effort-signalling equilibrium is the price that investors pay for the security. This implies that the crucial assumption is on the market forces that help to determine equilibrium prices.

[5]In Section 2.4 I allow for variations in $e_H$, and hence in $c(e_H)$

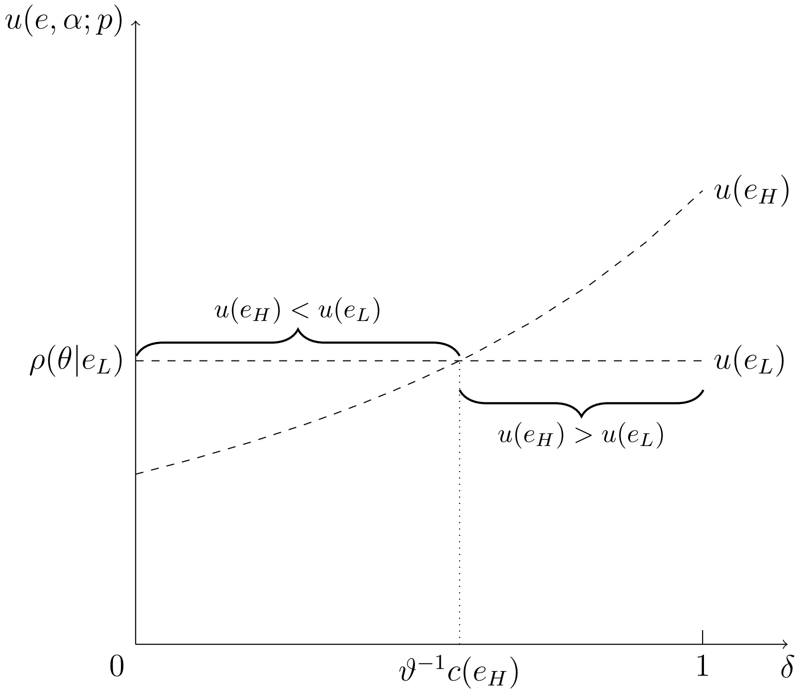


Figure 2.3: Equilibrium Payoffs Conditional on $\delta$.

**Proposition 6.** *If the originator's payoff function is continuous and Assumption 5 and Assumption 6 hold, then, a sufficient condition for the existence of an effort signalling equilibrium is that* $u_\alpha(e_L, \alpha; p_L) < 0$*, since* $u_\alpha(e_L, \alpha; p_L) < 0 \Leftrightarrow p_L > \delta\rho(\theta|e_L) \Leftrightarrow \Lambda^F \cap \Lambda \neq \emptyset$.

Therefore, the key assumption required for existence of equilibrium is that investors are competitive. Proposition 6 suggests that, while this assumption can be weakened, as it is weakened and prices drift towards the originator's discounted value, it becomes more difficult for effort-signalling behaviour to arise. This suggests an inherent trade-off between the arising consumer surplus when investors yield market power and the inefficiency generated by asymmetric information. In fact, when this assumption is completely relaxed, the effort-signalling equilibrium ceases to exist; the originator must be able to extract a positive surplus.

**Corollary 1.** *An effort-signalling equilibrium cannot exist when prices are such that the originator's participation constraints hold with equality for each level of effort.*

## 2.4 Skin in the Game

Treating Proposition 5 as a benchmark, I will now analyse how the level of retention necessary to signal high effort changes, relative to this benchmark, for any retention strategy that may be implemented by the originator if she chooses low effort. Assume now that the originator faces a strictly positive lower bound on her retention choice set and can no longer issue the entire security, irrespective of her choice of effort. The motivation for studying this case, which would not arise endogenously, is that the setting resembles the skin in the game regulation. Skin in the game requires any asset-backed security issuer to retain a minimum five percent vertical[6] slice of that security. The regulation, therefore, moves the originator, if she chooses low effort, away from her boundary optimum of retaining no part of the security. A question of interest is how the level of retention needed to signal high effort changes when this positive lower bound, or skin in the game, is imposed.

Intuitively, when skin in the game is introduced, it reduces the originator's payoff if she chooses low effort as she now holds this minimum positive quantity. This increases her incentive to mimic the retention strategy that previously signalled high effort. Let $\mu \in (0, 1)$ denote the lower bound, or skin in the game, then $\alpha^*(e_L | \, \mu \in (0, 1)) = \mu \; \forall \; \mu \in (0, 1)$. In this setting, if the originator instead chooses high effort and wants to signal this to investors, she now solves the following problem taking equilibrium prices and the low effort retention strategy as given:

$$\max_{\alpha(e_H) \in \; \Lambda^F \cap \; \Lambda(\mu)} u(e_H, \alpha; p(\alpha(e_H))), \tag{2.7}$$

where

$$\Lambda^F \cap \Lambda(\mu) = \{\alpha(e_H) \in \mathbb{R}_+ : \mu \leq \alpha(e_H) \leq 1 \text{ and } \underline{\Phi}\,(\mu) \leq \alpha(e_H) \leq \overline{\Phi}\,(\mu)\}.$$

The boundaries of this set are defined by

$$\underline{\Phi}\,(\mu) \equiv \min\{\alpha(e_H) \in \mathbb{R}_+ : \mathbb{E}[u(e_L, \mu; p(\mu))] \geq \mathbb{E}[u(e_L, \alpha(e_H); p(\alpha(e_H)))] | \, \mu \in (0, 1)\},$$

and

$$\overline{\Phi}\,(\mu) \;\equiv \max\{\alpha(e_H) \in \mathbb{R}_+ : u(e_H, \alpha(e_H); p(\alpha(e_H))) \geq u(e_H, \mu; p(\mu)) | \, \mu \in (0, 1)\}.$$

[6] A fraction of the entire asset pool rather than a fraction of some subset of the total pool, such as tranche retention.

Problem (2.7) differs from (2.5) as the boundaries of the incentive compatible set are now defined for when $\alpha^*(e_L|\ \mu) = \mu$. If the originator chooses high effort, and then purposefully retains a fraction of the security to signal her effort to investors, the retention cost she faces, ignoring the effort disutility, is the opportunity cost generated by her liquidity preference. This cost represents more profitable investments, a need to satisfy capital adequacy requirements etc. and is increasing in the fraction that she retains, as this reduces the potential sale revenue that can be raised.

### 2.4.1 Signalling Effect of Skin in the Game

As shown in the following proposition, imposing a positive lower bound on the level of retention that can be chosen by the originator increases these costs of signalling. That is, the originator must hold a strictly greater fraction of the security issue to signal high effort when there is a positive lower bound on her choice set.

**Proposition 7.** *Let* $\underline{\Phi}^*(\mu) \equiv \underline{\Phi}(\delta, p^*(\alpha), \mu)$ *denote the solution to* (2.7). *Then*

$$\underline{\Phi}^*(\mu) > \underline{\Phi}^* \ \forall \ \mu \in (0, 1).$$

*Moreover,* $\underline{\Phi}^*(\mu)$ *is a linear function of* $\mu$ *with*

$$\underline{\Phi}^*{}_{\mu}(\mu) > 0 \ \forall \ \mu \in (0, 1),$$

*where rate of increase in retention from* $\mu > 0$ *is strictly less than* $\mu$.

Proposition 7 formalises the intuition that, when the originator faces a positive minimum level of retention irrespective of her effort choice, a strictly greater level of retention is required to signal high effort than when no such lower bound existed. However, this also reduces the equilibrium payoff of the originator if she chooses high effort; therefore, the imposed lower bound increases the costs of effort-signalling relative to the benchmark case. This reduction in payoff $\frac{\partial}{\partial \mu} u(e_H, \alpha; p) < 0$ is a direct result of the originator's strict preference for liquidity. The comparative statics results of $\underline{\Phi}^*$, detailed in Section 2.3, continue to hold and, in addition, there is now the property that as the lower bound increases, the least-cost effort signalling level of retention is also increasing, but at a rate strictly less than $\mu$. The equilibrium price, and investor's beliefs, can be specified in a manner commensurate to that of Proposition 5, except now the cut-off value of retention is the solution to (2.7). Figure 2.4 demonstrates the increase in retention necessary to signal high effort with the skin in the game rule, and the associated reduction in payoff. Note that, when investors are competitive, there is no value of $\mu \in (0,1)$ such that the signalling equilibrium breaks down. This occurs as a result of two factors: the first is that, if the originator chooses high effort, she faces no default risk by assumption; and second is that, when investors are competitive, the prices that arise in the separating equilibrium transfer all surplus to the originator. If these two assumptions are relaxed, there may be a level of skin in the game such that the originator faces a better payoff by retaining the entire asset.

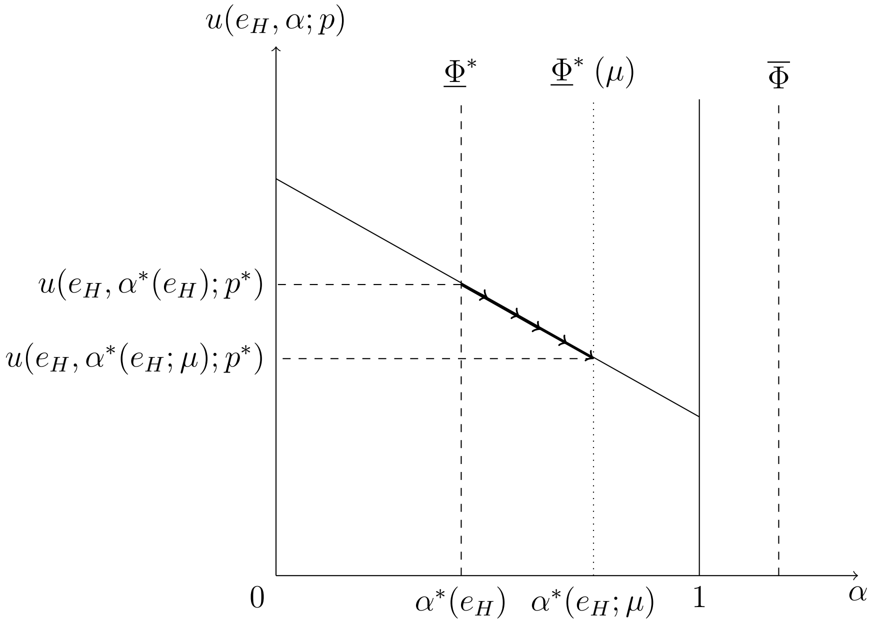


Figure 2.4: Increased Retention Costs.

### 2.4.2 Incentive Effect of Skin in the Game

As I have shown, imposing skin in the game increases the originator's retention cost of signalling high underwriting effort. However, this is not the exclusive effect that skin in the game has on the originator, as her incentives with respect to her choice of effort are also affected. As such, I now turn to examine the incentive effects of the lower bound. This refers to the change in the originator's incentive to choose high effort in the effort-signalling equilibrium, after skin in the game has been introduced. Suppose now that the value of high effort fluctuates, but that when the originator makes her choice of effort she faces a fixed realised value, so that her choice remains binary. Under the proposed interpretation, the amount of time the originator needs to dedicate in order to determine whether a borrower is creditworthy now varies. This variation could arise due to exogenous macroeconomic factors, as in times of general prosperity the originator may need to dedicate less time to determine whether a borrower is creditworthy.

Let $e_H \in [\underline{e}_H, \overline{e}_H] \subset \mathbb{R}_{++}$ represent the continuum of potential values that, when mapped by $c : [\underline{e}_H, \overline{e}_H] \to \mathbb{R}_{++}$, gives rise to a hypothetical smooth convex curve. When $\mu = 0$, there is a range of values of high effort[7], or equivalently a range of disutility, for which the originator will strictly prefer high effort in equilibrium. Intuition suggests that when the lower bound becomes positive this range of values will increase in size. As, when skin in the game is introduced, if the originator chooses low effort she will be exposed to some default risk that reduces her payoff relative to that which obtains with high effort. The originator's incentive to choose high effort in equilibrium then increases due to this effect. This suggests that, while skin in the game makes the originator worse off irrespective of her effort choice, the decrease in payoff is greater under low effort for a larger set of values of $e_H$. Several papers that study the problem of securitisation from a pure moral hazard viewpoint, employing the principal-agent methodology, have shown that a positive level of retention improves originators' incentives for high effort. However, this form of result has not yet been shown to hold when the securitisation problem is imbedded in an adverse selection signalling model. I now define some objects that enable one to demonstrate that this intuition continues to hold despite the change in modelling framework.

To formalise this notion, and analyse whether introducing skin in the game increases the range of high effort realisations that would incentivise a choice of high effort in equilibrium, I define the value of $e_H$ for which the equilibrium payoffs under high and low effort are equalised when $\mu = 0$

$$\{\hat{e}_H\} = \{e_H \in [\underline{e}_H, \overline{e}_H] : u(e_H, \alpha^*(e_H); p^*(\alpha)) = u(e_L, 0; p^*(\alpha))\}.$$

Therefore, one can state that, for any realisation of the high effort variable less that $\hat{e}_H$, the originator will choose high effort in equilibrium, and vice-versa for any realisation strictly greater than $\hat{e}_H$. The intuition is best explained via a concrete example: suppose that $\hat{e}_H = 5$, then, with the current interpretation, if it will take the originator between 0 and 4 hours to determine the borrower's creditworthiness she will choose high effort in equilibrium. However, if it will take 6 or more hours to verify the borrower's type, then, the originator will choose

[7] That is, there exists a subset $B \subset [\underline{e}_H, \overline{e}_H]$ such that the originator prefers high effort for any realisation $e_H \in B$.

low effort. Alternatively, an isomorphic interpretation is that this is the level of disutility arising from high effort that the originator is willing to tolerate before being incentivised to choose low effort. Using this knowledge allows me to define the set

$$E^H = \{e_H \in [\underline{e}_H, \overline{e}_H] : e_H < \hat{e}_H\},$$

which is semi-open, being bounded above by the value of effort defined by $\hat{e}_H$. As such, the set has no maximum and, hence, $\sup E^H = \hat{e}_H$. Now, when skin in the game is introduced, I define the value of high effort that equalises the effort-signalling equilibrium payoffs under high and low effort with $\mu > 0$ as

$$\{\hat{e}_H(\mu)\} = \{e_H \in [\underline{e}_H, \overline{e}_H] : u(e_H, \alpha^*(e_H); p^*(\alpha)) = \mathbb{E}[u(e_L, \mu; p^*(\alpha))] | \, \mu \in (0,1)\}.$$

Then, defining in an analogous manner to $E^H$ I have

$$E^{H(\mu)} = \{e_H \in [\underline{e}_H, \overline{e}_H] : e_H < \hat{e}_H(\mu)\}.$$

This is the set of all realisations of high effort such that, for values within this set, when there is a positive lower bound on the originator's choice of retention, the originator strictly prefers to choose high effort in equilibrium. As this set is also semi-open, I have $\sup E^{H(\mu)} = \hat{e_H}(\mu)$. The following proposition uses these sets to describe the qualitative effect that skin in the game has on the originator's effort incentives. This is done by characterising the range of values for which the originator will be incentivised to choose high effort in the effort-signalling equilibrium, relative to the benchmark case when there is no such rule in place. Proposition 8 states that restricting the choice of the level of retention by imposing a lower bound improves incentives for originators. That is, it creates a larger set of realisations of high effort for which the disutility of these realisations implies the payoff to the originator is strictly greater under high effort.

**Proposition 8.**

$$E^H \subset E^{H(\mu)} \quad \textit{and} \quad \sup E^{H(\mu)} > \sup E^H.$$

The intuition behind Proposition 8 is that, when skin in the game is introduced, the originator is willing to dedicate a strictly greater amount of time to determining borrowers' types. This is because she knows that, if she chooses low effort, she will be exposed to potential default loss that will reduce her expected payoff. The gap between the equilibrium payoffs that arise from each choice of effort then increases and, therefore, improves incentives for high effort relative to those without skin in the game. Figure 2.5 demonstrates this incentive effect of the skin in the game. The distance $\triangle = \hat{e_H}(\mu) - \hat{e_H}$ represents the increase in incentives for high effort when the originator must hold a positive minimum level of retention for either effort choice, which arises in part due to the assumption of increasing differences. Moreover, as I assume that the value of high effort is drawn from a compact interval, introducing skin in the game increases the relative probability that the originator will choose high effort in equilibrium if I augment this with the assumption that each value is equally likely. For example, if $e_H \sim U([\underline{e}_H, \overline{e}_H])$.

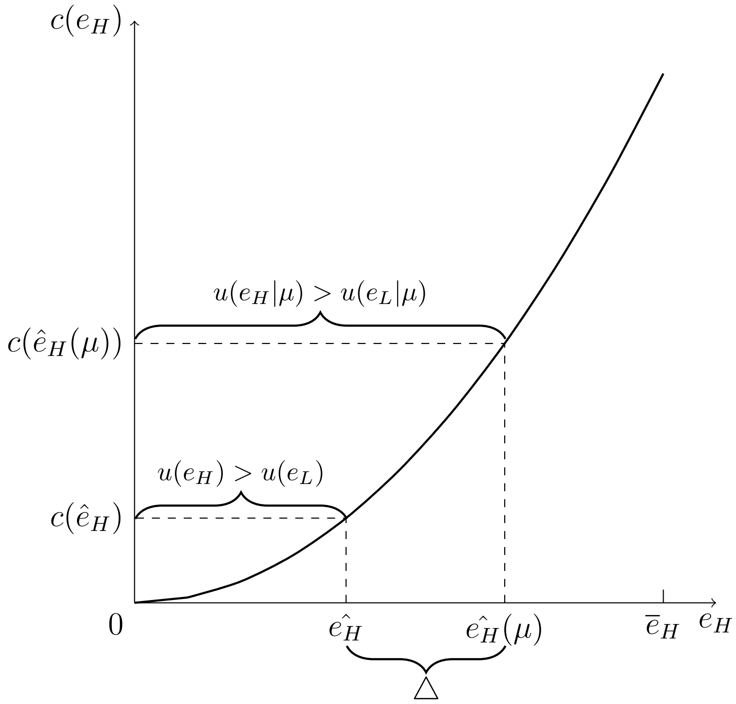


Figure 2.5: Incentive Effect of Skin in the Game.

## 2.5 Risk-Averse Originator

Initial models of financial signalling, featuring entrepreneurs looking to raise capital via issuing equity, assumed risk aversion on the part of agents. However, after DeMarzo and Duffie's seminal contribution introduced a risk-neutral model of a firm or bank looking to raise capital via asset-backed securities, motivated by liquidity concerns, risk aversion fell out of favour within the literature. This may have been for three reasons: tractability; a desire to direct focus away from standard risk-sharing notions; or conceptual beliefs, as banks with their diversified portfolio of investments may not be risk-averse when compared to a lone entrepreneur. As a result, whilst the literature that studies securitisation from a pure moral hazard standpoint has begun to re-introduce risk aversion (Malamud et al., 2013; Malekan and Dionne, 2014) it has not yet been re-introduced and studied in an adverse selection signalling model or combination framework. This is an important point of study as risk-sharing is a salient incentive behind securitisation. Therefore, understanding how skin in the game affects signalling costs and effort incentives when the originator is motivated by risk-sharing, rather than a need for liquidity, is necessary to determine whether this regulation will continue to perform as desired.

In this section, I will depart from this convention and characterise the effort signalling equilibrium of the game when the originator is risk-averse, coupled with no preference for liquidity: $\delta = 1$. The motivation for this modification is that, whilst the originator may be driven to securitise assets to invest in more profitable ventures, there is also an inherent risk-sharing aspect to securitisation, as the originator can transfer risky assets from her balance sheet as securities. Risk aversion helps to accentuate this aspect of the transaction. As such, the preference for liquidity is removed to isolate the effect that risk aversion has on the level of retention required to signal high effort. Moreover, this formulation enables a second analysis that develops an understanding of the conditions under which risk aversion, or risk neutrality alongside a liquidity preference, is a stronger driver of effort-signalling. This is a question hitherto unconsidered by the literature.

To this end, I now assume that the originator is risk-averse with parameter $0 < \sigma < 1$ such that $u(x) = x^{\sigma}$. The degree of risk aversion is captured by the Arrow-Pratt measure of absolute risk aversion, defined by $A(x) = -u''(x)/u'(x) = (1-\sigma)/x$. This implies that, the lower is the risk aversion parameter, the greater is the the originator's risk aversion. When she is risk-averse, the originator's payoff is decreasing-concave in the level of retention. As such, her optimal retention strategy, if she chooses low underwriting effort, continues to occur on the lower boundary of her choice domain and specifies selling the entire asset. This will be the case as long as equilibrium prices are equal to the expected payoff of the assets[8]. If the originator chooses high effort, her least-cost effort-signalling level of retention is found, as in the risk-neutral case, at the intersection of the now decreasing-concave payoff function and the minimum level of retention that satisfies incentive compatibility.

[8]The proof of this statement is included in the proof of Proposition 9

### 2.5.1 Comparison of Solutions under Risk Neutrality and Risk Aversion

Knowing that the required level of retention is calculated analogously in both the risk-neutral and risk-averse cases helps facilitate a characterisation of when each setting will be more conducive for effort-signalling. The more conducive setting is the environment that features the relatively smaller effort-signalling retention, as each key parameter is varied whilst the other is held fixed. This process creates two curves, each born from a specific case: in the risk-averse case, the parameter representing risk aversion is varied between $(0,1)$, whilst the preference for liquidity is held fixed at one; in the risk-neutral case, the liquidity preference is varied over $(0,1)$ and risk aversion is held fixed and equal to one.

**Proposition 9.** *Let* $\rho(\theta|e_L) = 1/2$ *then*

$$\min\{\alpha_\delta, \alpha_\sigma\} = \alpha_\sigma \text{ for } \delta, \sigma \in (0, \rho(\theta|e_L)),$$

$$\min\{\alpha_\delta, \alpha_\sigma\} = \alpha_\delta \text{ for } \delta, \sigma \in (\rho(\theta|e_L), 1),$$

*where* $\alpha_\delta \equiv \underline{\Phi}(\delta, p^*(\alpha))$ *and* $\alpha_\sigma \equiv \underline{\Phi}(\sigma, p^*(\alpha))$.

Proposition 9 elucidates an interesting property of the level of retention required to signal high effort when $\rho(\theta|e_L) = 1/2$. When the originator is characterised by a relatively high level of either risk aversion or liquidity preference, the minimum level of retention required to signal high effort is higher under a need for liquidity. Conversely, when she is characterised by relatively mild levels of risk aversion or liquidity preference, the minimum level of retention required to separate is higher under risk aversion. That is, with relatively low parameter values risk aversion is more conducive for effort-signalling, whilst with relatively high parameter values, a preference for liquidity is more conducive. Moreover, at the point $\delta = \sigma = \frac{1}{2}$, the fraction required to signal effort is equal in either case and for $\sigma \in [0, \tilde{\sigma}]$ there is a range of rigidity where the effort signalling level of retention is constant. These details are illustrated in Figure 2.6.

The choice of $\rho(\theta|e_L) = 1/2$ is mostly for illustration, as the crossing point arises at numerous other values. In particular, without fixing the level of $\rho(\theta|e_L)$ one can state that for realisations of the risk aversion parameter that satisfy

$$\sigma > \frac{\ln(\rho(\theta|e_L))}{\ln(\rho(\theta|e_L) \cdot \delta)} \tag{2.8}$$

the level of retention required to signal effort is strictly greater under risk aversion

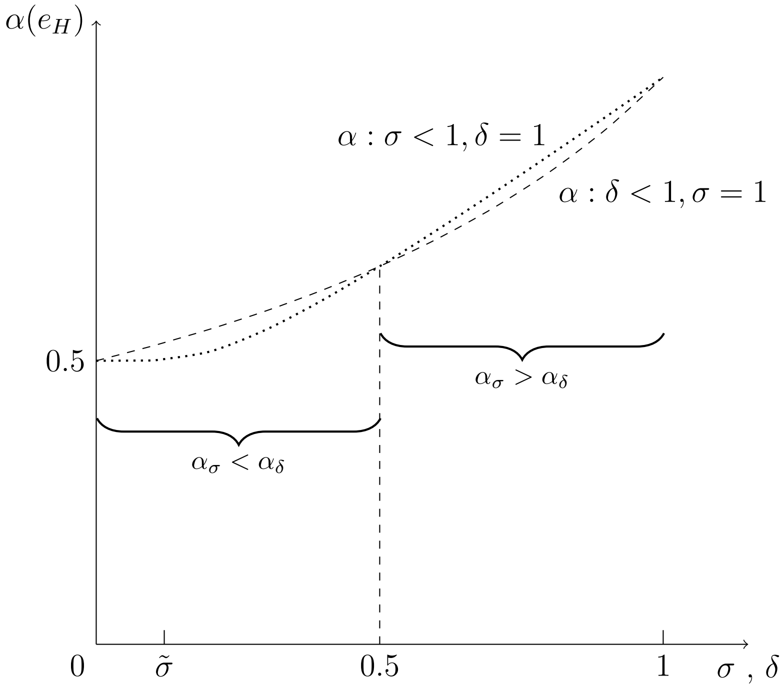


Figure 2.6: Liquidity Preference vs. Risk Aversion.

that under a preference for liquidity, and the converse holds when (2.8) is not satisfied.

### 2.5.2 Signalling and Incentive Effect under Risk Aversion

Finally, it remains to be seen whether the qualitative properties obtained as a result of introducing skin in the game in the risk-neutral setting continue to hold when the originator is risk-averse. That is, does $\mu > 0$ continue to simultaneously increase the costs of signalling and improve incentives for high underwriting effort when one moves to an environment of risk aversion.

**Proposition 10.** *When investors are competitive the qualitative properties of the effort-signalling equilibrium detailed in Proposition 7 and Proposition 8 continue to hold when the originator is risk-averse.*

These qualitative properties of the effort-signalling equilibrium arise in the risk-neutral case as, when skin in the game in introduced, if the originator chooses low effort, she both loses out on potential revenue and is exposed to default risk. It is this combination that drives the increase in effort incentives. In the risk-averse setting, these properties continue to hold, except they are now driven solely by the originator's exposure to default risk. Thus, the intuition that suggests that imposing a positive lower bound on the originator's retention domain both requires a greater level of retention to signal high effort and leads to an increase in incentives is robust to this change in risk preference. Proposition 10 implies that exposure to default risk is sufficient for the regulation to perform as intended. Therefore, irrespective of the reason behind securitisation, be it based on a need for liquidity or a need to share risk, the skin in the game regulation leads to improved effort incentives for originators. Note, however, that this intuition does not carry over to the case when the originator is risk-seeking.

**Corollary 2.** *When investors are competitive and the originator is risk-seeking, there cannot exist an effort-signalling equilibrium.*

## 2.6 Conclusion

In this chapter aspects from both the adverse selection and moral hazard literatures have been combined to provide a simple unified model of securitisation under asymmetric information. I studied an originator who first decides whether to exert costly underwriting effort before granting loans and securitizing the resulting assets to sell to investors on the secondary market. The originator realises that, depending on which effort she selects, she may have to try and credibly convey this information to the investors, who cannot observe the originator's effort, to receive the correct price for the security. As such, an equilibrium was characterised in which the originator signals high effort, which implicitly suggests asset quality, by retaining a fraction of the security issue that is large enough to convince investors that, if she had chosen low effort, she would find it unprofitable to hold a retention of this size. Then, referring to the aforementioned case as a benchmark, a rule akin to the skin in the game regulation was introduced. It was demonstrated that this rule leads to a larger required retention in equilibrium, yet at the same time, it improved the originator's incentives for high effort relative to the benchmark. The intuition behind this result is as follows: when skin in the game is introduced, the originator's equilibrium payoff is reduced irrespective of her choice of effort, yet this reduction in payoff is relatively larger if the originator chooses low effort, and so this increases her incentive to select high effort in equilibrium.

The second set of results arose as a consequence of relaxing the assumption that the originator is risk-neutral, a conventional assumption within the securitisation and signalling literature. It was shown that the qualitative properties that were engendered as a result of introducing skin in the game continue to hold when the originator is risk-averse. The intuition is that exposure to default loss is sufficient for the improvement of originator's incentives when the originator is risk-averse, whilst under risk neutrality the originator needs to be exposed to both default loss and have a preference for liquidity in order for her incentives to be improved by skin in the game. Secondly, conditions under which risk aversion is more conducive for signalling were derived, and vice-versa for risk neutrality with a preference for liquidity.

There are several assumptions that helped to make the model more tractable that, if relaxed, may provide some more insights into the problem of effort-signalling. In particular, in this model the probability of default of an asset when the originator has chosen high effort is zero. Whilst relaxing this assumption should not change any qualitative properties, it may provide more realism and more stringent conditions for existence. Secondly, the binary choice of underwriting effort is restrictive and precludes the possibility of a continuous equilibrium strategy. Lastly, as has been discussed in the text, as prices move away from the true value of the asset, it becomes more difficult for the originator to signal effort. Eventually, when prices are equal to the originator's discounted value for the assets, the ability to signal effort breaks down. This suggests a trade off between inefficiency due to informational asymmetry and inefficiency due to market power. Thus, as a key restrictive assumption is that investors make zero profits in equilibrium, a possible direction for future work is to study, in a more formal manner, the variation of market power in a combination securitisation model.

# Chapter 3

# Signalling with Endogenous Private Information

## 3.1 Introduction

Signalling games, a class of dynamic Bayesian game, are employed to model situations in which an informed agent tries to communicate some private information to one, or potentially many, uninformed agents. In much of the existing literature, this private information is exogenously endowed, implying the informed agent has no control over her information. This exogenous private information is typically modelled as the realisation of a random variable drawn from some probability distribution. The uninformed agent(s) have knowledge about this probability distribution, but do not observe the realisation, and so, there is asymmetric information between the two parties. In an attempt to credibly convey her information, the informed agent sends an observable signal to the uninformed agents, which is commonly known to be costly for the informed agent. Specifically, this cost varies depending on the informed agent's private information. Uninformed agents then update their beliefs about the informed agent's private information, given the observed signal, before responding in a way that maximises their payoffs. In many applications of signalling games, the salient object of study is a separating equilibrium. This is an equilibrium in which the informed agent's private information is completely revealed via the signal.

In terms of application, formal signalling models have been introduced into a wide range of settings including: biology (Grafen, 1990; Smith, 1991); philosophy (Lewis, 1969); and, linguistics (Rooy, 1982)[1]. The most widespread utilisation, however, has, of course, been economics. Since the inception of economic applica-

[1]See Sobel (2009) and Riley (2001) for in-depth reviews.

tions in Spence (1973) signalling games have been used by economists in a diverse assortment of fields: industrial organization (Milgrom and Roberts, 1982); bargaining (Fudenberg and Tirole, 1983); finance (Leland and Pyle, 1977; DeMarzo and Duffie, 1999); and, most notably, to models of market interactions that have a flavour of adverse selection *à* la Akerlof (1970)[2]. However, in many situations involving adverse selection, the assumption of exogenous private information is not the most intuitive. Consider a simple used car market, for example. The quality of any given car, which represents the owner's private information, can naturally be thought of as comprising two aspects. The first is the endogenous effort that the current owner has dedicated to the maintenance of the car, whilst the second is an exogenous component including the manufacturing conditions of the car and the servicing carried out by previous owners. Alternatively, consider a firm that underwrites assets before transforming them into securities to sell on secondary markets. The payoff of any particular asset will be determined to a large extent by the effort of the underwriter during ex-ante due-diligence and ex-post monitoring and servicing. Finally, consider workers pondering higher education. Each worker's productivity is at least partially engendered through the effort that is dedicated to studying. To capture this more realistic framework, I introduce endogenous private information, modelled as an unobservable costly effort decision, into the canonical signalling framework developed by George Mailath. Moreover, to fully capture the intuition expressed in the above examples, I augment this endogenous effort decision with an exogenous component that determines the returns to effort.

In a seminal contribution Mailath (1987) provided widely applicable conditions under which the informed agent's action function in a signalling game is differentiable when the type space is a continuum. This function is responsible for mapping exogenously given types into observable actions. The crucial requirements are that the informed agent's private information is fully revealed by a unique action and the usual incentive compatibility condition is satisfied. In this case, the informed agent finds it optimal to truthfully reveal her private information, given the response each signal generates, and the equilibrium is separating. If these conditions are satisfied, Mailath identifies structural assumptions such that the informed agent's separating action function can be obtained as the solution of the ordinary differential equation implied by incentive compatibility, without resorting to ad-hoc assumptions. These results, both regarding differ-

[2] Relaxing the assumption that signalling is costly has also spawned the large cheap-talk literature based upon Crawford and Sobel (1982).

entiability of action functions and existence of separating equilibria, have been applied in a wide range of settings including: job markets with matching (Hopkins, 2012); cheap talk with lying costs (Kartik, 2008); and, altruism (Glazer and Konrad, 1996; Andreoni and Bernheim, 2009). However, two of the structural assumptions, unbounded action spaces and concavity of the informed agent's payoff in her signal, are not satisfied in the classic models of corporate finance and security design proposed by Leland and Pyle (1977) and DeMarzo and Duffie (1999). These models feature compact action spaces and linearity of the informed agent's payoff in the signal, respectively. Consequently, Mailath and von Thadden (2013) generalised the results of Mailath (1987) to cover these useful settings, amongst other extensions. Crucially, however, these existing results do not apply to the case when private information is endogenous.

I introduce a model in which the informed agent's private information, or type, arises endogenously as the result of an effort decision, which generates an associated disutility, rather than being exogenously drawn from a probability distribution. This effort choice creates an additional stage in the game in the form of a second optimisation problem faced by the informed agent. I also consider a natural extension to this framework that involves a combination of both an exogenous component and an endogenous effort choice, as in the previously discussed examples. I endogenise private information in the signalling framework of Mailath and von Thadden (2013) to answer two main research questions. The first question I answer is the characterisation of the action function responsible for mapping each choice of effort into a unique signal. This function is of central importance in applied models of signalling where it characterises the signalling strategy of the informed agent in the usual separating equilibrium. My first portable result provides an answer in a general setting by directly extending a result of Mailath and von Thadden (2013) to an environment in which the informed agent is endowed with an effort production function. This production function determines a unique output for each effort input[3] and I state the extra assumptions required on this for the result to hold. Compared to the classic differential equation implied by incentive compatibility in the standard game, the differential equation in this setting takes account of the marginal effect of the informed agent's effort production function in an intuitive way. The second question, and the more important conceptually, can be phrased loosely as: given the behaviour that arises in a separating equilibrium in which the informed agent truthfully signals her private

[3]For example this output could be a probability of default, monetary payment, or productivity.

information, allowing uninformed agents to correctly respond given their updated beliefs, how does the informed agent choose her effort. More specifically, can the informed agent optimally choose her effort, given the separating behaviour governed by incentive compatibility and the uninformed agent's equilibrium strategy, in a manner that is congruous to that seen in producer theory and principal-agent models[4]. My second portable result provides a partial answer to this latter question by employing the recent insights of In and Wright (2017), who provide an equilibrium refinement for this class of endogenous signalling games known as Reordering Invariance. My result identifies one set of sufficient conditions that characterise when the informed agent's effort optimisation problem will have a solution, taking into account the signal derived in my first result. These two results formalise when a Reordering Invariance equilibrium will exist. In doing so, I provide a natural way to transform the usual exogenous signalling game into one with an endogenous costly effort decision using the framework developed in Mailath (1987) and Mailath and von Thadden (2013).

I then apply these results to the signalling games of two seminal papers: the security design model of DeMarzo and Duffie (1999) and the job market model of Spence (1973). By doing so, I demonstrate the practical usefulness of these results on the characterisation of the equilibrium signal in these transformed games and the existence of a solution to the informed agent's effort optimisation problem. I then obtain intuitive results as to how the informed agent optimally chooses her effort in the Reordering Invariance equilibria of these two games, and the conditions under which these equilibria exist. In the extension of DeMarzo and Duffie, the firm's optimal effort is decreasing in the rate at which it discounts any retained earnings. I show that, when the firm is in greater need of liquidity, it underwrites relatively poorer quality assets, and subsequently sells a relatively larger quantity of these assets to investors on the secondary market. Moreover, I also show that, under economically appealing assumptions on the informed agent's effort production function and associated disutility, in Spence's model of education the informed agent optimally chooses her effort by equating marginal benefit with marginal cost. Upon introducing a parameter capturing an agent's socioeconomic background, optimal effort is increasing in this measure. These results provide theoretical support for two stylised facts often documented in the respective empirical literatures that cannot be derived in the standard exogenous model. The first, is that firms with relatively low liquidity will have relatively

[4]Formally, can the problem be set up such that the informed agent's payoff has a form similar to $\max_{K,L}\ \Pi(K,L) = AK^{\alpha}L^{1-\alpha} - rL - wL$ for $\alpha \in (0,1)$ or $\max_e\ U(w,e) = u(w) - c(e)$ where $u'' \leq 0$ and $c'' > 0$.

lower quality assets, and in turn, securitise and sell a greater proportion of these assets (Cardone-Riportella et al., 2010; Bannier and Hansel, 2008; Martin-Oliver and Saurina, 2007; Agostino and Mazzuca, 2009; Affinito and Tagliaferri, 2010). The second, is that socioeconomic background explains large proportion of students' acquisition of higher education (James, 2002; Cameron and Heckman, 1998; Mare, 1980).

This chapter is related to three entwined strands of the literature on the economics of information. First of all, to the small but growing literature on signalling games with endogenous choices. The closest to this chapter is the recent work of In and Wright (2017), who provide an equilibrium refinement for this class of games. This often enables one to pin down a unique equilibrium outcome by appealing to a notion of Reordering Invariance (RI). In particular, they show that, when the informed agent does not gain any new payoff relevant information between the two decision nodes, and the uninformed agent's strategies and beliefs do not change, the game can be reordered and solved by first considering the choice of signal before the choice of effort. This technique also pins down the equilibrium beliefs of the uninformed agents in a way that aligns closely with ex-ante intuition. In and Wright (2016, 2017) study several examples, but without a consistent framework, and assuming differentiability of the informed agent's action function and existence of equilibrium in the reordered game. In contrast, the contribution of this chapter is to propose a consistent framework, based on that of Mailath (1987) and Mailath and von Thadden (2013), with which to set up applied endogenous signalling games. I then state the precise conditions required for the action function to be differentiable and for the informed agent's effort maximisation problem to have a solution. If these conditions are satisfied equilibrium exists in the reordered game. Moreover, I develop intuitive results as to how the informed agent optimally chooses her effort in the RI equilibrium of several seminal signalling games by applying the aforementioned results.

This chapter also relates to the literature grown out of the two classic signalling games I consider. The model of DeMarzo and Duffie (1999) has been extended to analyse and explain the following corporate finance phenomena: pooling and tranching of securities (DeMarzo, 2005); screening (Vanasco, 2013); and, the impact of market power by taking a mechanism design approach (Biais and Mariotti, 2005). I extend this literature by interpolating a costly effort decision into the model and employing this extension to provide a rationale for the determinants of security payoffs. The contribution is to produce theoretical support for a stylised fact often reported in the literature, which states that firms with a greater need for liquidity will dedicate less time to underwriting assets and will subsequently securitise a greater proportion of these relatively riskier assets, that cannot be derived in the standard model. The original model of Spence (1973) has been greatly extended and merged with other theoretical frameworks by several authors. Hopkins (2012) considers a matching model of the labour market where workers signal productivity via education, whilst Spiewanowski (2010) considers a version of the Spencian universe in which firms observe a noisy signal of a worker's education. Feltovich et al. (2002) introduce 'medium' productivity types. Finally, Eichberger and Kelsey (1999) consider a version of the model where firms are uncertain over the equilibrium actions of workers, in the sense of Choquet expected utility, and they show that the only equilibrium involves pooling. I endogenise productivity in a continuous version of the Spence model, and use this adaption to offer an explanation of how productivity is engendered when workers use education as a signal, and compare this with the situation under symmetric information. The contribution of this section is again to produce theoretical support for the stylised fact that socioeconomic background impacts higher education attainment.

The rest of the chapter is structured as follows. Section 3.2 introduces the general model, defines the equilibrium concept, and provides the two portable results. Section 3.3 and Section 3.4 applies these results to the signalling models of two seminal papers. Finally, Section 3.5 concludes.

## 3.2 Model and Equilibrium

The framework I extend is the canonical approach of Mailath (1987), specifically employing the conditions proposed in Mailath and von Thadden (2013). Rather than the informed agent's private information, or type, being exogenously endowed I model the informed agent's private information as being endogenously determined via an unobservable costly effort decision. This adds an additional stage to the standard signalling model, transforming it into a three stage game. In the first stage, the informed agent chooses her unobservable effort that generates a unique outcome along with an associated disutility. This outcome could represent a monetary payoff, a probability of default or the productivity of an agent. The process by which effort maps into outcomes is common knowledge between all agents, but the outcome is known only by the informed agent. In the second stage, there is the usual signalling subgame in which the informed agent chooses an observable costly action in an attempt to signal her choice of effort in the first stage to the uninformed agent. The cost of the chosen signal will depend upon her choice of effort. The uninformed agent cares about the informed agent's choice of effort as the particular outcome arising from this choice will impact both his action and payoff. Therefore, he wants to know the choice of effort to respond optimally. In the third stage, the uninformed agent observes the signal, and forms beliefs about the chosen effort conditional upon this signal. These beliefs, and any relevant equilibrium conditions arising from the uninformed agent's optimisation problem, will determine the uninformed agent's best response.

This additional stage means that the game may have many sequential equilibria, and the usual method for solving standard exogenous signalling games may not be sufficient for pinning down a unique equilibrium. This is due to ambiguity over the uninformed agent's beliefs in relation to the informed agent's choice of effort upon observing signals that are off the equilibrium path. Consequently, In and Wright (2017) propose an equilibrium refinement, Reordering Invariance, which can lead to a unique equilibrium outcome in this class of games. They show that, when the informed agent's payoff-relevant information, and the uninformed agent's strategies and beliefs, do not change between the stages of the game, the payoffs in the reordered game will be identical to those in the original game if the reordering does not change the possible outcomes. When this requirement is satisfied, In and Wright show that, in an RI equilibrium, the informed agent will rationally choose the same effort and signal pair if the game is reordered, so that the second stage is played first. Consequently, in this chapter, the informed agent will choose her signalling strategy before her effort. This reordering enables one to use the uninformed agent's strategies and beliefs to shrink the set of equilibria; specifically, In and Wright suggest a form of forward induction for the equilibrium beliefs of the uninformed. This belief formation centres around the uninformed agent making inferences about the costly effort decision. The uninformed agents assume that the chosen effort was an optimal choice made by the informed agent with both the observed signal, and equilibrium response of the uninformed in mind. It is the concept of RI equilibrium that I employ in this chapter.

Formally, the informed agent begins by choosing an action, most naturally interpreted as effort, which is observable only by herself, and, therefore, represents her private information. This effort is denoted $\omega \in [\omega_1, \omega_2] \equiv \Omega \subset \mathbb{R}_+$, where $\Omega$ is assumed to be compact. Each effort $\omega \in \Omega$ generates a unique outcome via the $\mathbb{C}^2$ mapping $\varphi : \Omega \to \mathbb{R}_+$, which can be intuitively thought of as a production function[5]. I assume that $\varphi$ is strictly increasing over the domain of effort except

[5] In Section A.3 I discuss this assumption and how it is equivalent to assuming that $\varphi$ is a

at the left end-point, so that $\varphi'(\omega) > 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$. Equally important, I assume that the mapping $\varphi$ is common knowledge, but the outcome $\varphi(\omega)$ is known only by the informed agent. Furthermore, each effort $\omega \in \Omega$ generates a corresponding disutility, captured by $\psi : \Omega \to \mathbb{R}_+$. This is also assumed to be $\mathbb{C}^2$ and strictly increasing, where $\psi(\omega_1) = 0$ and $\psi'(\omega) > 0$ for all $\omega \in \Omega \setminus \{\omega_1\}$.

After the informed agent selects an unobservable effort, and computes the associated outcome via $\varphi$, she takes a second observable action as a means to signal her prior choice of effort to the uninformed agent(s). Specifically, given any effort $\omega \in \Omega$, the informed agent chooses a costly signal $x \in \mathcal{X} \subset \mathbb{R}_+$. The uninformed agent uses this signal to form beliefs about the unobservable effort $\omega$, and its associated payoff $\varphi(\omega)$, via his knowledge of $\varphi$. The uninformed agent cares about the chosen effort as, given any signal, he responds with an action $r \in \mathcal{R} \subset \mathbb{R}_+$ through which he aims to maximise his payoff. If the uninformed agent is unable to infer $\omega$ he may calculate the incorrect outcome, and his subsequent response may not be optimal. Therefore, given any effort $\omega$ and subsequent signal $x$ chosen by the informed agent, and response $r$ by the uninformed, I write the informed agent's payoff as $v(x, r, \omega)$. I denote the uninformed agent's best response after observing signal $x$ and forming subsequent beliefs $\hat{\omega}$ about the chosen effort, which is used to calculate $\varphi(\hat{\omega})$, as $\rho(x, \varphi(\hat{\omega}))$. This best response determines the optimal action for the uninformed agent, for any signal and belief. In applications, this action is often derived from an optimisation problem that takes account of market conditions. Incorporating the uninformed agent's best response implies the informed agent's payoff is now defined by the function $V : \Omega^2 \times \mathcal{X} \to \mathbb{R}_+$, which will be $\mathbb{C}^2$ in each of its arguments if $\varphi$ is $\mathbb{C}^2$, where

$$V(\omega, \varphi(\hat{\omega}), x) \equiv v(x, \rho(x, \varphi(\hat{\omega})), \omega). \tag{3.1}$$

I assume that $V_3(\omega, \varphi(\hat{\omega}), x) \neq 0$ for all $x \in \mathcal{X}$ and that $V_2(\omega, \varphi(\hat{\omega}), x) \neq 0$ for all $\hat{\omega} \in \Omega$, where subscripts denote partial derivatives[6]. Note that the assumption on $V_3$ implies that, with symmetric information, the problem $\max_{x \in \mathcal{X}} V(\omega, \varphi(\omega), x)$ has a unique solution for each $\omega \in \Omega$. Under symmetric information, the solution, $X^{FB}(\omega)$, lies on the boundary of $\Omega$. This arises as $V_3(\omega, \varphi(\hat{\omega}), x) \neq 0$ implies the informed agent's payoff is strictly decreasing (increasing) in the signal, and as such, when the uninformed can observe the choice of effort, the informed agent's payoff is maximised by not investing (fully investing) in the signal.

conditional expectation under first-order stochastic dominance.

[6] This last assumption can also be relaxed at one boundary in applications.

### 3.2.1 Signalling Stage

After reordering the game, however, the informed agent first chooses the observable action. This stage is much like the construction of a separating equilibrium in the standard game, if one exists. As such, the informed agent must find it optimal to use a different signal for each choice of effort, given how the uninformed agent will respond to any credible signal. Formally, the informed agent's signal in this stage of the game, denoted $X : \Omega \to \mathcal{X}$, must be one-to-one so each $\omega \in \Omega$ maps into a unique signal $X(\omega) \in \mathcal{X}$. It must also be the case that, for any $\omega$, the signal must be incentive compatible so that

$$X(\omega) \in \underset{x \in X(\Omega)}{\arg\max}\ V(\omega, \varphi(X^{-1}(x)), x) \tag{IC}$$

holds and the informed agent finds it optimal to truthfully signal her choice of effort. I show in the proof of Theorem 4 that, if $\varphi$ is differentiable and increasing, then (IC) will continue to hold in this setting with an endogenous effort choice. As noted, one crucial requirement for a separating equilibrium to exist is that the signal, or separating action $X$, is one-to-one so that $\omega \neq \omega'$ implies $X(\omega) \neq X(\omega')$. Consequently, if such a function exists, any additional properties of $X$ one may be able to elucidate will be of salient importance in applied models of endogenous signalling. One property of particular relevance is differentiability as, if this holds a priori, one can obtain $X$ as the solution to the ordinary differential equation implied by incentive compatibility. Mailath (1987) first provided a set of widely applicable assumptions that, if employed, generate a differentiable observable action function. Whilst these conditions have been successfully applied in many fields they do not apply in this setting with endogenous effort. Moreover, the results of Mailath and von Thadden (2013), whilst greatly expanding the possible assumptions that generate a differentiable $X$ to cover linearity of the informed agent's payoff in the signal, amongst others, do not apply here. This is due to the introduction of the informed agent's production technology $\varphi$. I extend Mailath and von Thadden's Theorem 3 to include models in which the private information, or type, is endogenously generated through an unobservable effort choice.

**Theorem 3.** *Let $X : \Omega \to \mathcal{X}$ be one-to-one, where $\mathcal{X}$ is compact, and satisfy (IC). Let $\varphi : \Omega \to \mathbb{R}_+$ be differentiable and one-to-one with $\varphi'(\omega) \neq 0\ \forall\ \omega \in \Omega$, except at either $\omega_1$ or $\omega_2$. Then, for any $\omega \in \Omega$, if $V_3(\omega, \varphi(\hat{\omega}), x) \neq 0\ \forall\ x \in \mathcal{X}$,*

*$X$ is differentiable at $\omega$. At all points of differentiability $X$ satisfies*

$$X'(\omega) = -\frac{V_2(\omega, \varphi(\omega), X(\omega)) \cdot \varphi'(\omega)}{V_3(\omega, \varphi(\omega), X(\omega))}\bigg|_{\hat{\omega}=\omega, x=X(\omega)}. \tag{DE}$$

*Proof of Theorem 3.* All proofs for this chapter are given in Section B.3. □

Theorem 3 enables one to derive the function the informed agent will use to signal her choice of endogenous effort, which will play a salient role in applications. Adapting Mailath and von Thadden's theorem to a setting with endogenous private information changes the classic differential equation of separating equilibria. The signal now takes into account the marginal effect of the informed agent's effort technology. As a result, several conceptual and technical assumptions are required on $\varphi$. Conceptually, the function $\varphi$ must be monotonic, or one-to-one, so that each $\omega \in \Omega$ maps into a unique $\varphi(\omega) \in \varphi(\Omega)$. This condition ensures that, if the informed agent is able to fully reveal her private information, there is a unique best response for the uninformed agent. For example, if the production technology is a correspondence, $\varphi : \Omega \rightrightarrows \mathbb{R}_{++}$, this ceases to be the case, as each effort no longer generates a unique outcome that the uninformed agent can respond with. Technically, I require that $\varphi$ is differentiable for all $\omega \in \Omega$ so that (DE) has a solution, and that $\varphi'(\omega) \neq 0$ for all $\omega$ so that $X(\omega) \neq 0$ for all $\omega \in \Omega$. As noted, however, this can often be relaxed at one boundary of $\Omega$ in applications. Note also, that, at this stage, Theorem 3 does not require $\varphi$ to be increasing or decreasing, nor do I make assumptions on its second derivative.

### 3.2.2 Effort Stage

After computing the signalling strategy for any effort in the reordered game, the informed agent turns to her choice of unobservable costly effort. In many applications, the informed agent's choice of effort, after determining the signal that will be used for any effort, will be dependant upon a parameter $\theta \in \Theta \cup \{1\}$. This parameter will play a key role in both applications considered in this chapter. The reason for this is that the parameter $\theta$ will generate ex-ante asymmetric informed agents, when a setting with multiple informed agents is considered. I write the informed agent's payoff in this stage as a function defined by $\mathcal{V} : \Omega \times \Theta \cup \{1\} \to \mathbb{R}_+$, where

$$\mathcal{V}(\omega, \theta) \equiv V(\omega, \varphi(\omega), X(\omega)).$$

This payoff function takes account of the signalling strategy and the uninformed agent's equilibrium response. Given this set-up, the optimisation problem that determines the informed agent's optimal effort is

$$\omega(\theta) \in \arg\max_{\omega \in \Omega} \ \mathcal{V}(\omega, \theta). \tag{E0}$$

Any solution to (EO) will provide the informed agent with a means by which to optimally choose her effort $\omega$, given how she will subsequently signal this effort through $X(\omega)$, and receive response $\varphi(\omega)$ from the uninformed agent, whose updated beliefs of $\hat{\omega} = \omega$ will be confirmed.

**Definition 7.** The triple of functions $(\varphi(\omega(\theta)), X(\omega(\theta)), \omega(\theta))$ and associated beliefs constitute an *Reordering Invariance Equilibrium* if they satisfy the equilibrium condition on the uninformed, (IC), and (EO), respectively.

The following theorem provides one set of sufficient conditions that will lead to the existence of an RI equilibrium in endogenous signalling games based on the framework and assumptions of Mailath and von Thadden (2013). Note that these are not the only set of conditions that will give rise to such an equilibrium.

**Theorem 4.** *Suppose the informed agent's payoff* (3.1) *is linear in the signal and the response of the uninformed, additively separable in effort and the response, concave in effort, and increasing in the response. Finally, suppose that the effort technology is concave. Then* (EO) *has a solution if* (3.1) *is decreasing (increasing) in the signal and $X$ is convex (concave),* (3.1) *is additively separable in the signal and the response, and*

$$-\left\{V_{11}(\omega, \varphi(\omega), X(\omega)) + V_2(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}\varphi(\omega) + V_3(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2} X(\omega)\right\}$$
$$> 2 \cdot V_{13}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega} X(\omega). \tag{3.2}$$

The first set of conditions I have employed are relatively uncontroversial. Linearity in the signal and concavity of the informed agent's payoff in effort, before optimising over the signal, holds in in the two applications considered in Section 3.3 and 3.4. Moreover, additive separability of the informed agent's payoff in effort and the response of the uninformed agent will be satisfied in general. Indeed, it is in my applications. For example, any model where the response of the uninformed is a wage or price will lead to (3.1) being increasing in this argument, and additively separable. Combined with a concave effort technology, the other statements lead to the first three terms of (3.2) being negative.

The second set of conditions required for the theorem are somewhat more restrictive. To make the conclusion of existence one possible 'recipe' is to assume additive separability of (3.1) in the signal and the response of the uninformed agent, which is satisfied in Section 3.4 but not in Section 3.3. Unfortunately, the following expression holds

$$V_{13}(\omega, \varphi(\omega), X(\omega) \cdot \frac{d}{d\omega} X(\omega) > 0, \tag{3.3}$$

for all signalling models. The reason is that $V_{13}$ captures increasing (decreasing) differences in the informed agent's objective function in effort and signal, which in turn plays a role in determining whether the equilibrium signal is increasing (decreasing) in effort. Hence, if $V_{13} > 0$ then $X'(\omega) > 0$ and (3.3) holds, whilst if $V_{13} < 0$ then $X'(\omega) < 0$ and one is again left with (3.3). Once (3.2) has been derived in specific applications, it can be critiqued for its applicability, realism and intuitive properties as I do in Section 3.3.

## 3.3 Application 1 : DeMarzo and Duffie (1999)

In DeMarzo and Duffie's model of security design, the informed agent, a monopolist issuer of asset-backed securities, has private information related to the payoff of the assets that back the securities. The firm uses this private information to compute the conditional expected value of the security. Once it has performed this computation, it uses the quantity it puts up for sale to signal this expected value to uninformed market investors. DeMarzo and Duffie assume that the firm's private information is exogenously endowed through the realisation of a random variable. However, there are many cases in which a firm that issues securities is also responsible for underwriting the assets that back each security. In such a setting, the firm's private information, and hence, the payoff of the security, are engendered endogenously via due-diligence and adequate monitoring and servicing. As such, I extend DeMarzo and Duffie's signalling model to a setting in which the firm chooses an unobservable effort $\omega \in [\omega_1, \omega_2] \equiv \Omega$ that generates a unique monetary payoff $\varphi(\omega)$ at cost $\psi(\omega)$, where it will be assumed that $\varphi(\omega_1) > 0$. This effort then informs the firm's choice of signal $x \in [0, 1] \equiv \mathcal{X}$, where $x$ is the quantity of the security offered for sale by the firm. The firm discounts the fraction $(1 - x)$ of retained earnings at rate $\delta \in (0, 1)$, which is strictly less than the discount rate of market investors. By adapting the model to include an endogenous effort choice, I can offer theoretical support for a stylised fact often reported by the empirical literature, which cannot be derived in the standard model. This result states that firms more in need of liquidity securitise a relatively larger amount of their assets, and that these assets are of relatively poorer quality. My intuitive explanation relates to the means by which firms determine the payoff of assets that they securitise and sell, given that they will signal effort through the quantity offered for sale, and centres on the firm's discount factor, or preference for liquidity.

Rather than solve the game in the natural order of the moves, I reorder the game so that the firm first computes the quantity that it will sell as a signal, for any effort, before turning to the decision problem of choosing effort. The payoff of the firm is set up before I impose a zero profit condition on market investors to solve their optimisation problem. From this I obtain their best response to any signal and belief about the firm's chosen effort. This best response enables one to apply Theorem 3 to the firm's payoff to obtain the quantity that allows the firm to truthfully reveal its effort. Subsequently, taking this derived quantity strategy and investors' optimal response as given, I propose a means by which the firm optimally chooses its effort. In particular, when certain conditions are satisfied, optimal effort is born out of the interplay between the firm's discount factor and behaviour defined by equating marginal effort benefit with marginal cost. Under key conditions on the firm's production function, responsible for mapping effort into monetary outcomes, the firm's optimal effort in the RI equilibrium of the security design signalling game is monotonically decreasing in the discount rate it applies to retained earnings. So that, the more the firm discounts retained earnings, relative to the market, the lower is the effort chosen; therefore, echoing the stylised fact.

Formally, given any effort $\omega$, quantity for sale $x$ and price received $r$, the firm's payoff is written

$$v(x, r, \omega) = \delta(1 - x)\varphi(\omega) + r - \psi(\omega), \tag{3.4}$$

and market investors' $m(r, x, \omega) = x\varphi(\omega) - r$. Imposing the standard zero profit condition on market investors, $m(r, x, \omega) = 0$ for all $\omega \in \Omega$, I solve for their best response, given signal $x$ and belief that the chosen effort is $\hat{\omega}$,

$$\rho(x, \varphi(\hat{\omega})) \equiv x\varphi(\hat{\omega}) - r^* = 0. \tag{3.5}$$

I now employ (3.5) to transform (3.4) into a form analogous to (3.1) so that I can apply Theorem 3

$$\begin{aligned} V(\omega, \varphi(\hat{\omega}), x) &= \delta(1-x)\varphi(\omega) + x\varphi(\hat{\omega}) - \psi(\omega), \\ &= \delta\varphi(\omega) + (\varphi(\hat{\omega}) - \delta\varphi(\omega))x - \psi(\omega). \end{aligned} \tag{3.6}$$

In the re-ordered game I first look for a separating action $X : \Omega \to \mathcal{X}$, which is a unique fraction offered for sale for each choice of effort, that satisfies incentive compatibility

$$X(\omega) \in \operatorname*{arg\,max}_{x \in X(\Omega)} \; V(\omega, \varphi(X^{-1}(x)), x). \tag{IC}$$

If (IC) is satisfied then the fraction offered for sale will truthfully reveal the firm's choice of effort, allowing market investors to calculate the correct price to pay using their knowledge of $\varphi$, and the firm will find this optimal. To derive $X$ in this setting with endogenous private information, I apply Theorem 3 to (3.6), which provides both a means by which to obtain $X$ but also to conclude that $X$ is differentiable. The application of Theorem 3 yields the following differential equation

$$X'(\omega) = -\frac{x \cdot \varphi'(\hat{\omega})}{\varphi(\hat{\omega}) - \delta\varphi(\omega)} = \frac{1}{\delta - 1} \frac{X(\omega) \cdot \varphi'(\omega)}{\varphi(\omega)} \Bigg|_{\hat{\omega}=\omega, x=X(\omega)}.$$

I obtain $X(\omega)$ by noting that, as a result of Theorem 3, $X$ is differentiable and by employing the boundary condition $X(\omega_1) = X^{FB}(\omega_1) = 1$. This boundary condition arises naturally as when the firm chooses the lowest possible effort $\omega = \omega_1$, where $\omega_1$ is the left endpoint of $\Omega$, it obtains its first best outcome of selling the entire security. The key difference between $X(\omega)$ and the separating action obtained in the standard model is the presence of the firm's effort technology. The assumptions on this technology greatly impacts the magnitude of the quantity offered for sale for any non boundary effort. Example 1 illustrates this for noncontroversial effort technologies and a suitably defined domain.

**Example 1.** *Suppose that $\Omega = [1, 2]$ and consider the following two cases: $\varphi_1(\omega) = \omega^{1/2}$ and $\varphi_2(\omega) = \omega$. The firm's payoff in case 1 is*

$$V(\omega, \hat{\omega}^{\frac{1}{2}}, x) = \delta\omega^{1/2} + (\hat{\omega}^{1/2} - \delta\omega^{1/2})x - \psi(\omega). \tag{E1}$$

*Applying Theorem 3 to (E1) yields the differential equation*

$$X_1'(\omega) = \frac{1}{\delta - 1} \frac{X(\omega) \cdot \frac{1}{2\omega^{1/2}}}{\omega^{1/2}} \Bigg|_{\hat{\omega}=\omega, x=X(\omega)} = \frac{1}{2(\delta - 1)} \frac{X(\omega)}{\omega} \Bigg|_{\hat{\omega}=\omega, x=X(\omega)},$$

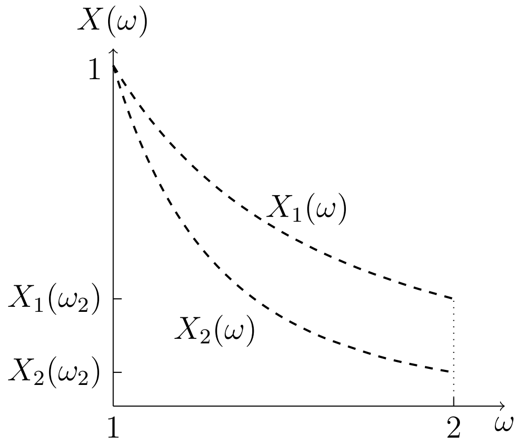


Figure 3.1: Decreasing vs Constant Returns to Scale.

*with solution* $X_1(\omega) = [\frac{1}{\omega}]^{\frac{1}{2(1-\delta)}} = \omega^{\frac{1}{2(\delta-1)}}$*. Repeating the process for the second case obtains* $X_2(\omega) = \omega^{\frac{1}{\delta-1}}$*. Note that* $X_2(\omega)$ *is the analogue of the solution found in DeMarzo and Duffie (1999). Figure 3.1 highlights the shift in magnitude of the fraction offered for sale under a effort production function with decreasing returns to scale.*

Given (3.5) and $X(\omega)$, I can write the firm's payoff when it uses the signal defined by $X$ for each effort $\omega$, and subsequently receives price $\varphi(\omega)$ from market investors, as

$$\mathcal{V}(\omega,\delta) \equiv V(\omega,\varphi(\omega),X(\omega)) = \delta\varphi(\omega) + (1-\delta)\varphi(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} - \psi(\omega), \quad (3.7)$$
$$\text{where} \quad X(\omega) = \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} \ \forall\, \omega \in \Omega.$$

In the re-ordered game, the firm now has to choose its effort to maximise (3.7). This decision problem takes account of how effort will impact the retained earnings, the fraction that will be offered for sale, which itself feeds through to the price received, and the associated disutility. The firm's effort optimisation problem is written

$$\omega(\delta) \in \arg\max_{\omega\in\Omega} \ \mathcal{V}(\omega,\delta), \quad \text{(EO)}$$

where any solution to (EO) is said to be 'effort optimal'.

I first consider the problem under complete information, which implies symmetric information between the firm and market investors, before analysing the true problem (EO). In this first-best outcome, the firm sells all of the security, $X(\omega)^{FB} = 1$, for every effort, as investors observe the choice of effort and can, therefore, calculate the relevant monetary outcome. The firm's problem is then $\max_{\omega\in\Omega} V(\omega, \varphi(\omega), 1) = \max_{\omega\in\Omega} \varphi(\omega) - \psi(\omega)$, which provides the unique optimal effort $\omega^*$ defined by the solution of $\varphi'(\omega) = \psi'(\omega)$ if $\varphi''(\omega) \leq 0$ and $\psi''(\omega) > 0$. Note that this solution is independent of $\delta$, whilst this parameter, the firm's discount rate, will play a central role under asymmetric information. Specifically, under asymmetric information, and when certain conditions are satisfied, a strictly increasing mapping $\delta \mapsto \omega^*(\delta) \mapsto X(\omega^*(\delta))$ will exist. This monotonicity enables the RI equilibrium to exist with endogenous effort. To make this statement, one can take two approaches, the first is to make suitable assumptions on the firm's effort production function that ensure that the second-order condition of the problem (EO) is satisfied.

**Assumption 8.** *The function $\varphi : \Omega \to \mathbb{R}_+$ satisfies*

$$\delta\left[\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} - 1\right]\varphi''(\omega) > \varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) > 0,$$

*for each $\omega \in \Omega \setminus \{\omega_1\}$ and all $\delta \in (0, 1)$.*

Assumption 8 identifies a class of functions $\varphi \in \mathbb{C}^2$ for which, after taking account of the magnitude of the fraction offered for sale and the discount factor, the second derivative is greater than the derivative of the log of the function. Note that Assumption 8 rules out effort production functions that display constant or increasing returns to scale. This intuitively leaves only those functions with decreasing returns to scale. In particular, concavity of the effort production function is a necessary condition for concavity of the function $\mathcal{V}$. This implies that, without strong assumptions on $\psi$, the firm's payoff is convex in effort under a linear or convex production technology.

**Proposition 11.** *Suppose the effort production function satisfies Assumption 8 and effort disutility is increasing and strictly convex. Then, in the RI equilibrium of the security design signalling game characterised by (3.5) and (IC), for $\delta \in (0, 1)$, there exists a unique effort optimal $\omega^* : (0, 1) \to \Omega$ satisfying (EO), which is continuous, implicitly defined by*

$$\delta\varphi'(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right] = \psi'(\omega), \tag{3.8}$$

*and strictly decreasing in a firm's preference for liquidity*

$$\frac{d}{d\delta}\omega^*(\delta) > 0.$$

Unlike under complete information, optimal effort implicitly defined by (3.8) takes account of the firm's discount factor, and as $\delta \in (0, 1)$ and $X(\omega) \in [0, 1]$, any solution implicitly defined by (3.8) is strictly less than that defined under complete information, which will also be the case in Section 3.4. The mechanics of the comparative statics result can be explicated as follows. By (3.8) the firm's discount rate affects only the firm's marginal effort benefit. Hence, holding effort fixed, as the firm's discount rate decreases, the firm's marginal benefit increases whilst marginal cost remains constant; therefore, (3.8) no longer holds. Consequently, the firm will increase its effort until (3.8) once again holds with equality. (3.8) will return to equality if the effort disutility rises at a rate greater than that of the effort production function. This increase in effort then improves the monetary value of the asset pool, which causes the firm to sell less of the security to investors; however, this smaller fraction will be sold at a relatively greater price. Knowing this, investors also learn implicitly of the firm's discount rate when they observe a specific retention. The intuition is clear: when the firm's payoff from retaining the assets is very low, perhaps due to a need for liquidity, it has a much larger incentive to exert less effort and securitise a greater fraction of these poorer quality assets to sell to investors, as the relatively low price will be preferred to retention. Conversely, when the firm does not discount retained earnings to such a degree, it is more willing to spend time and care on the due-diligence process knowing that this will feed through to a relatively smaller amount sold to investors, but at a relatively greater price.

To provide conceptual support for the intuition of Proposition 11 I briefly draw on empirical evidence to informally motivate how a market comprised of $N$ firms may behave in a setting with endogenous private information. Suppose that $N$ is finite, that each $i \in \{1, \ldots, N\}$ has discount factor $\delta_i$, where $\delta_i \neq \delta_j$ for $i \neq j$ and $\Delta = \{\delta_1, \ldots, \delta_N\}$, and that Assumption 8 holds. Moreover, suppose that the effort production function and disutility, $\varphi_i$ and $\psi_i$, are symmetric so that $\varphi_i = \varphi_j = \varphi \ \forall \ i$ and similarly for $\psi$. Proposition 11 then implies that the partial ordering $(\Delta, <)$ uniquely determines both the partial ordering of effort by each firm and the subsequent partial ordering of quantities offered for sale. Specifically, $\delta_i > \delta_j$ implies that $\omega^*(\delta_i) > \omega^*(\delta_j)$, which in turn implies that $X(\omega^*(\delta_i)) < X(\omega^*(\delta_j))$. Therefore, firms with relatively high discount rates will underwrite assets of relatively lower value than those firms that have relatively low discount rates. The firms with assets with relatively lower payoffs will then securitise a relatively larger proportion of these assets to sell to market investors. If one interprets $\delta_i$ as firm $i$'s preference for liquidity, which can arise due to more profitable investment opportunities, or a need to satisfy capital adequacy requirements, this informal discussion of the implications of Proposition 11 is supported empirically. Cardone-Riportella et al. (2010) show that, generally, firms with relatively low liquidity will have relatively lower performance and, subsequently, securitise more. Bannier and Hansel (2008) support this finding, as they provide evidence that banks with lower liquidity, greater credit risk exposure and worse performance measures are more likely to securitise and sell a larger proportion of their assets. Martin-Oliver and Saurina (2007) and Agostino and Mazzuca (2009) find that the key motivating factor behind bank securitisation in Spanish and Italian banks, respectively, over the period 1999-2006 was a need for liquidity. Finally, Affinito and Tagliaferri (2010) find that banks that are less profitable, less liquid and with more troubled loans are more likely to securitise assets and at a larger quantity than otherwise.

Unfortunately, Assumption 8 is restrictive and difficult to check in practice without first fixing a value of $\delta$. To circumvent this assumption one can turn to the tools of monotone comparative statics and, instead, make use of the property of increasing differences. The usual assumption when applying Topkis's Theorem (Amir, 2005) is that (3.7) satisfies increasing differences in the choice variable and parameter of interest, here the firm's effort and discount factor. I begin with a preliminary result that demonstrates that this assumption is unnecessary as, intuitively, this property arises endogenously in this setting.

**Lemma 3.** $\mathcal{V}(\omega, \delta)$ *is strictly supermodular and differentiable on* $\Omega \setminus \{\omega_1\} \times (0, 1)$ *so that* $\mathcal{V}_{12}(\omega, \delta) > 0$.

Hence, for any fixed effort, the marginal effect of increasing effort on $\mathcal{V}$ is increasing in the firm's discount factor. This key condition, and $\mathcal{X}$ being a compact real interval, allows me to apply Edlin and Shannon (1998)'s Strict Monotonicity Theorem and Corollary 1 to (3.7). As a result, one can conclude that, if the problem (EO) has an interior solution, by Lemma 3 $\omega^*(\delta) = \arg\max_{\omega \in \text{int}(\Omega)} \mathcal{V}(\omega, \delta)$ is strictly increasing in $\delta$. Despite supermodularity arising endogenously, the conditions present in this setting are stronger than those sufficient for applying Topkis's Theorem. Namely, I do not argue that the argmax correspondence is increasing in the strong set order. In this differentiable setting, inherited from assumptions on the firm's payoff, effort production function, and the signal, given Theorem 3, one can apply Edlin and Shannon's stronger result for a unique solution strictly increasing in $\delta$. The assumption that $\mathcal{V}$ attains an interior maximum is less restrictive than concavity but still enforces some structure that may not be inherently present.

**Example 2.** *Suppose that* $\Omega = [0,2]$*,* $\psi(\omega) = 0$ *and take* $\delta = 1/2$*. Then the firm's payoff with costless effort is written*

$$\mathcal{V}(\omega,\delta) = \delta\varphi(\omega)[1 + X(\omega)]. \tag{E2.}$$

*Supposing further that* $\varphi(\omega) = 1+\omega^{1/2}$ *and applying Theorem 3 to* $V(\omega, 1+\hat{\omega}^{1/2}, x)$ *yields* $X(\omega) = \varphi(\omega)^{\frac{1}{\delta-1}}$*, which I substitute into (E2.) to give*

$$\mathcal{V}(\omega,1/2) = \left[1 + \left(\frac{1}{\varphi(\omega)}\right)^{\frac{1}{1-\frac{1}{2}}}\right]\frac{\varphi(\omega)}{2} = \left[1 + \left(\frac{1}{\varphi(\omega)}\right)^{2}\right]\frac{\varphi(\omega)}{2}$$
$$= \frac{1+\varphi(\omega)^2}{2\varphi(\omega)} = \frac{1+(1+w^{1/2})^2}{2(1+\omega^{1/2})}.$$

*Partially differentiating* $\mathcal{V}(\omega,1/2)$ *twice with respect to* $\omega$ *obtains*

$$\mathcal{V}_{11}(\omega,1/2) = -\frac{1}{8}\frac{w^2+3w^{3/2}}{w^2(1+w^{1/2})^3} < 0 \;\forall\; \omega \in \Omega \setminus \{\omega_1\}.$$

$\mathcal{V}(\omega,1/2)$ *with* $\psi(\omega) = 0$ *is illustrated in Figure 3.2.*

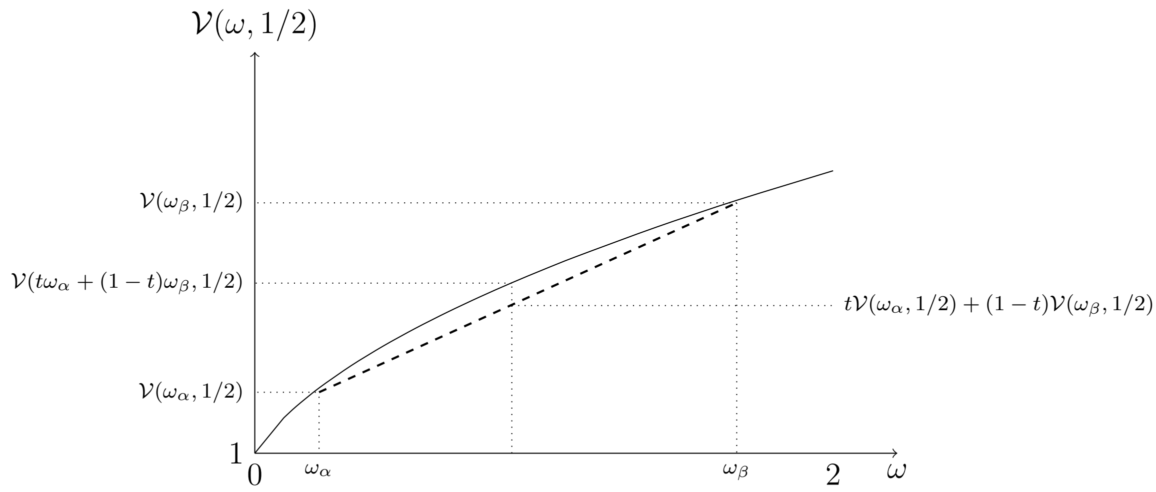


Figure 3.2: Firm's Payoff with Costless Effort.

*Thus, when I include convex costs of effort* $(-\psi''(\omega) < 0)$ *I will have a strengthening of the concavity of* $\mathcal{V}$ *as the sum of a concave function and a strictly concave function is strictly concave. In Figure 3.3 I illustrate* $\mathcal{V}(\omega,\delta)$ *with* $\psi(\omega) = w^2/2$*, which implies* $\psi''(\omega) > 0$*, for* $\delta \in \Delta \equiv \{\delta_1, \delta_2\}$*, where* $\delta_2 = 0.8 > \delta_1 = 0.5$*, and the associated quantities* $X(\omega^*(\delta_i))$*,* $i \in \{1,2\}$*.*

Whilst the firm's value function, $\mathfrak{V}(\delta) \equiv \max_{\omega\in\Omega}\ \mathcal{V}(\omega,\delta) = \mathcal{V}(\omega^*(\delta),\delta)$, is clearly increasing in Figure 3.3 this is not true in general. Specifically, the firm's

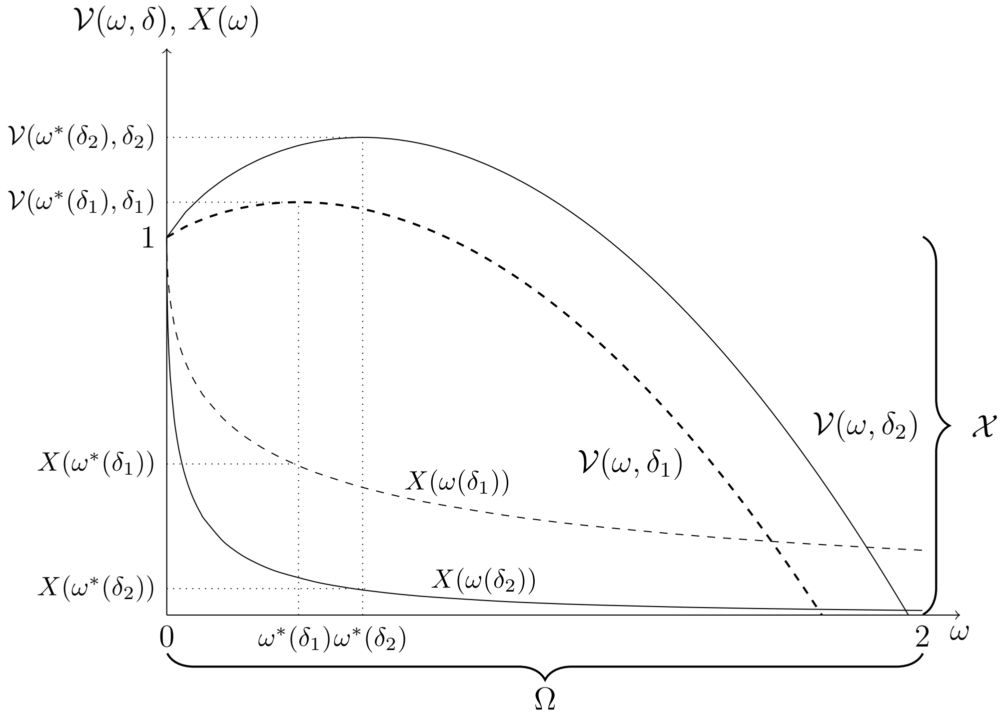


Figure 3.3: RI Equilibrium in DeMarzo and Duffie's (1999) Model.

value function is increasing, $\frac{\partial \mathfrak{V}(\delta)}{\partial \delta} = \frac{\partial}{\partial \delta}\mathcal{V}(\omega,\delta)|_{\omega=\omega^*(\delta)} > 0$, when

$$(1-\delta)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right] > -\ln\left(\frac{\varphi(\omega_1)}{\varphi(\omega)}\right)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}, \tag{3.9}$$

and decreasing otherwise. Condition (3.9), whilst admittedly abstract, holds for effort production functions such as $\varphi(\omega) = \ln(\omega) + 1$ and $\varphi(\omega) = 1 - \frac{1}{2}e^{-x}$ for $\Omega \subset \mathbb{R}_+$, or $\varphi(\omega) = \ln(\omega)$ for $\omega_1 > 1$. These functions satisfy the intuitive requirement of being increasing and concave, as well as $\varphi(\omega) = 1+\omega^{1/2}$ as in Example 2. Note that the right hand side of (3.9) can be written $-X(\omega)[\ln(\varphi(\omega_1))-\ln(\varphi(\omega))]$. This more stringent condition arises because the signal depends also on the parameter $\delta$, unlike Section 3.4 where the signal will be independent of the relevant parameter.

Finally, I consider the case when the firm does not know the value of its discount factor, and instead knows only its distribution $G(\delta|\tau)$ parameterised by $\tau$. This modification enables the model to more closely approximate reality, as the firm may be unable to precisely estimate its liquidity preference in advance of committing to a level of underwriting effort, and subsequent quantity of the security offered on the market. The firm's expected profit prior to choosing effort is then

$$\mathbb{E}[\mathcal{V}(\omega,\tau)] = \varphi(\omega)\int \delta dG(\delta|\tau) + \varphi(\omega)\int (1-\delta)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} dG(\delta|\tau) - \psi(\omega).$$

By following similar methods to the proof of Proposition 13 one can show that if $\tau' > \tau$ implies $G(\delta|\tau') \leq G(\delta|\tau)$, then both the firm's optimal choice of effort, $\omega^u(\delta) = \arg\max_{\omega\in\Omega} \mathbb{E}[\mathcal{V}(\omega,\tau)]$, and the firm's expected profit, are increasing in $\tau$. The key difference in the proof compared to Proposition 13 consists of using the fact that the left-hand side of the first-order condition is positive, implying that the terms involving $\tau$ form an increasing function of $\omega$, to show that the first-order stochastic dominance inequality holds in this case. If instead, I assume that $G$ satisfies the monotone likelihood ratio property, which is a more restrictive assumption that first-order stochastic dominance, then Lemma 3 implies the firm's optimal effort is increasing in $\tau$ by Athey (2002). Proposition 11 is, therefore, robust to the addition of uncertainty, increasing its generality.

## 3.4 Application 2: Spence (1973, 1974)

In the classic model of job market signalling proposed by Spence (1973), uninformed firms are unable to distinguish between high and low productivity workers. Each worker's productivity is exogenously endowed private information, whilst the distribution of the workers' productivities across the population is common knowledge. In the unique separating equilibrium of the game with two levels of productivity, which survives Cho and Kreps's (1987) Intuitive Criterion, the subset of workers endowed by nature with high productivity acquire the minimum education sufficient to incentivise low productivity workers not to invest at all. This enables firms to correctly deduce the realised productivity of workers and pay them the correct wage, maximising their payoffs. In Spence (1974), these ideas are extended to setting with a continuum of types, and the separating equilibrium involves workers with different productivities acquiring divergent levels of education. This provides a natural explanation for the various degrees on offer at higher education institutions, and the different skill set required for each. It is in this setting that I endogenise the process by which each worker's productivity is determined. I endogenise productivity as, in reality, the productivity of a worker, whilst still potentially being impacted by some random aspect, is largely a function of a worker's effort. Specifically, each worker's productivity will be engendered through an unobservable costly effort decision, where each choice of effort generates a unique level of productivity. I seek to answer how, knowing that education may be used as a signal of productivity, workers will choose this unobservable effort that will determine their productivity, which may affect their education decision and the wages paid.

I will show that, by reordering the game and first computing the education that fully reveals the worker's choice of effort, when certain natural assumptions on each worker's production function[7] and effort disutility are satisfied, workers will be able to optimally choose their effort. This method takes account of how additional effort will impact productivity, which feeds through to the choice of education, and the wages received. Without further extensions to the model, I show that all workers will exert the same effort and acquire the same education in equilibrium. As I make an assumption that implies I only consider graduate job markets, this result is akin to the widespread standard of an undergraduate degree. I then analyse how introducing a parameter that differentiates workers through their cost of effort functions changes the workers' effort responses. Intuitively, each worker's effort optimisation will depend on this parameter, which can be interpreted as the worker's socioeconomic background. In the RI equilibrium of the game, the optimal choice of effort will be increasing in this measure. The intuition is that, while workers with lower socioeconomic backgrounds are able to put in more effort than other workers with higher socioeconomic backgrounds to acquire the same productivity, which is ruled out of the standard exogenous setting, in the RI equilibrium of the job market signalling game, workers will not find this optimal. Instead, each worker's optimal effort will be uniquely determined by her socioeconomic background, so that workers with different backgrounds will exert different amounts of effort, and subsequently acquire different education signals[8]. There are two points that are important to note: the first, is that socioeconomic background does not represent the type of the worker, instead it acts a parameter that governs the returns to a worker's choice of effort; the second, is that this result holds in a setting in which education acts as a pure signal. If the model were modified so that education, as well as effort, increased the productivity of workers then a higher degree of social mobility should exist.

After laying out the details of the job market signalling game, and highlighting the differences that arise in this endogenous setting, I apply Theorem 3 to the transformed worker's payoff function to characterise the separating action function that maps effort into education, given incentive compatibility. I then turn

[7]Responsible for mapping effort into productivity.

[8]A simpler, but less empirically meaningful, interpretation of this parameter is as a measure of each worker's preference for studying.

to analysing optimal effort in this setting. Each worker chooses an unobservable effort $\omega \in [\omega_1, \omega_2] \equiv \Omega$ that determines her productivity $\varphi(\omega)$ via the mapping $\varphi : \Omega \to \mathbb{R}_+$. This is assumed to be common knowledge, continuously differentiable and strictly increasing. The associated disutility of $\omega$ is $\Psi(\omega, \lambda) = \lambda \cdot \psi(\omega)$, which can be thought of as the physical and mental strain of effort in higher education, and is also continuously differentiable and strictly increasing in each of its arguments. The parameter $\lambda \in \Lambda \cup \{1\}$ represents a worker's socioeconomic background, which may be taken to include parental education, class and number of siblings (Micklewright, 1989), or family income (Kohn et al., 1976) and it is assumed that a 'smaller' $\lambda$ implies a lower disutility. As a concrete example of why this may be the case consider two individuals who are enrolled in higher education. Suppose the first is fully funded by her family or a scholarship, whilst the second requires part time employment to fund her studies. It makes intuitive sense that the second may feel a greater degree of mental and physical strain given her increased obligations and reduction in the amount of hours available to study. Moreover, the added financial stress faced by the second student, due to the need to manage personal finances, may have a further negative impact. A similar formulation was employed by Lee (2007), who assumes this for high school students, but not in higher education for simplicity.

After choosing $\omega$ each worker chooses a level of observable education $e \in [e_1, e_2] \equiv \mathcal{E}$ to obtain as a means to signal her effort, and hence her productivity, to hiring firms[9]. This education comes at cost $c(e, \varphi(\omega))$, most naturally interpreted as the opportunity cost in terms of time spent studying. I assume this cost is submodular in its arguments so that $c_{12}(e, \varphi(\omega))\varphi'(\omega) < 0$, implying the marginal opportunity cost of education is decreasing in effort, and linear in education $c_{11}(e, \varphi(\omega)) = 0$ so that $c_1(e, \varphi(\omega)) \neq 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$. If a worker is hired by a firm, the firm pays a wage of $r \in \mathcal{R}$, which may depend on the education acquired by the worker.

The game, in its standard form, is solved by first considering the worker's choice of effort and, subsequently, her choice of signal, for any effort. Instead, I re-order the game so that the worker first calculates the education that allows her to both truthfully reveal her choice of effort and maximise her payoff, given the response to this education by firms, for any effort. Each worker then, taking as given the behaviour previously derived through optimising over her choice of signal, chooses her effort, knowing how effort feeds into productivity, education

[9]Note that Spence (1974) does not feature compact action spaces, but I feel this assumption can be justified as one's educational choices are in a sense bounded.

and the wage subsequently received. Formally, this implies that each worker will first consider the problem of incentive compatibility for any $\omega$, and then taking the conditions implied by incentive compatibility as given, each worker optimises with respect to effort to find the RI equilibrium.

Given education $e$, wage $r$ and effort $\omega$ a worker's payoff is written

$$u(e, r, \omega) = r - c(e, \varphi(\omega)) - \lambda\psi(\omega), \tag{3.10}$$

and the firm's $\pi(r, e, \omega) = \varphi(\omega) - r$. Employing the standard assumption of competition between firms implies a natural zero profit condition, so that in any RI equilibrium of the signalling game, the firm's payoff is maximised when it best responds with $\rho(e, \varphi(\hat{\omega}))$ given signal $e$ and belief that the chosen effort is $\hat{\omega}$, where

$$\rho(e, \varphi(\hat{\omega})) \equiv r^* - \varphi(\hat{\omega}) = 0.$$

Substituting the action implied by this best response into (3.10) yields the worker a payoff analogous to (3.1)

$$U(\omega, \varphi(\hat{\omega}), e) = \varphi(\hat{\omega}) - c(e, \varphi(\omega)) - \lambda\psi(\omega).$$

Given that the game has been re-ordered, I first consider the worker's problem of obtaining the mapping that both truthfully reveals the choice of unobservable effort through the observable signal and maximises the worker's payoff with respect to her education decision. The worker, therefore, solves

$$E(\omega) \in \underset{e \in E(\Omega)}{\arg\max}\ U(\omega, \varphi(E^{-1}(e)), e). \tag{IC}$$

When (IC) is satisfied the worker finds it optimal to use signal $E(\omega)$ when she has chosen effort $\omega$. The firm then uses its knowledge and beliefs to translate this signal into an implied choice of effort that is used to calculate its optimal response. Theorem 3 then allows one to characterise the function $E : \Omega \to \mathcal{E}$ as

$$E(\omega) = \int_\Omega \frac{\varphi'(\hat{\omega})}{c_1(e, \varphi(\omega))}\,\mathrm{d}\omega = \int_\Omega \left.\frac{\varphi'(\omega)}{c_1(E(\omega), \varphi(\omega))}\right|_{e=E(\omega), \hat{\omega}=\omega} \mathrm{d}\omega.$$

To obtain a closed form representation of $E(\omega)$, and to make the worker's decision problem of choosing her effort more tractable, I take inspiration from Spence (1973) and assume that $c(e, \varphi(\omega)) = e \cdot [\varphi(\omega)]^{-1}$. This satisfies the assumptions of linearity in $e$ and submodularity in $(e, \omega)$. Under this assumption $E'(\omega) =$

$\varphi(\omega)\varphi'(\omega)$ and $E(\omega) = 1/2 \cdot \varphi(\omega)^2 + k_1$, after integrating by parts, where $k_1$ is the constant of integration. As in any separating equilibrium, the lowest choice of effort must obtain its first best outcome, so that when $\omega = \omega_1$ one must have $E(\omega_1) = 0$, which implies $k_1 = -1/2 \cdot \varphi(\omega_1)^2$. Together these statements imply that each worker's payoff in the first stage, characterised by (IC) and the zero profit condition imposed on firms, is defined by a function $\mathcal{U} : \Omega \times \Lambda \cup \{1\} \to \mathbb{R}_+$ where

$$\mathcal{U}(\omega, \lambda) \equiv U(\omega, \varphi(\omega), E(\omega)) = \varphi(\omega) - \underbrace{\frac{1}{2}\left\{\frac{\varphi(\omega)^2 - \varphi(\omega_1)^2}{\varphi(\omega)}\right\}}_{E(\omega)/\varphi(\omega) = c(E(\omega), \varphi(\omega))} - \lambda\psi(\omega). \tag{3.11}$$

The payoff (3.11) takes account of the mechanism of separation by which each choice of effort feeds through into a unique level of education, engendering a unique response from the firm.

Taking (3.11) as given, each worker's decision problem is to choose her effort to maximise her payoff in the second stage of the reordered game, where workers signal the choice of effort $\omega$ with education $E(\omega)$ and receive wage $\varphi(\omega)$. That is, each worker solves

$$\omega(\lambda) \in \arg\max_{\omega \in \Omega} \mathcal{U}(\omega, \lambda), \tag{EO}$$

as in Section 3.3. Note that if the solution set of (EO) is empty then signalling may break down, despite (IC) holding[10]. However, before turning to analyse whether the problem (EO) admits a solution, I consider the complete information setting where workers and firms have symmetric information. In such a setting, workers acquire no education, as firms are able to observe each worker's choice of effort and compute the relevant productivity. This first-best education is denoted $E^{FB}(\omega) = 0 \ \forall \ \omega \in \Omega$. Each worker's effort optimisation problem is then $\max_{\omega\in\Omega} \ U(\omega, \varphi(\omega), 0) = \max_{\omega\in\Omega} \ \varphi(\omega) - \lambda\psi(\omega)$, which has unique interior solution $\omega^{FB}(\lambda)$ defined by $\varphi'(\omega) = \lambda\psi'(\omega)$ if $\varphi''(\omega) \leq 0$ and $\psi''(\omega) > 0$ for all $\lambda > 0$. Consequently, with complete information, and under standard assumptions on each worker's effort production function and disutility of effort, each worker optimises with respect to effort by following behaviour that equates marginal benefit with marginal cost. Moreover, it is easy to see that the optimal effort under complete information is decreasing in the parameter lambda[11], so that the worker exerts relatively more effort when her socioeconomic back-

[10]Informally, without a consistent means of choosing effort workers may end up putting in to much effort relative to their ability and 'burn out'.

[11]Specifically, by the implicit function theorem, I have that $\frac{d}{d\lambda}\omega^{FB}(\lambda) = \frac{\psi'(\omega)}{\varphi''(\omega) - \lambda\psi''(\omega)}$.

ground is greater. Turning now to consider the true problem (EO), where the worker takes as given both the education signal $E$ derived in the previous step via (IC) and how the firm will respond $\varphi$. I show that, under the same relatively mild conditions employed in the complete information setting, and an additional boundary condition on offered employment contracts, workers will continue to be able to optimise by following an analogous effort setting rule.

**Proposition 12.** *Suppose the effort disutility is strictly convex and the effort production function is concave. Suppose further that workers who choose zero effort do not take part in the job market. Then, in the RI equilibrium of the job market signalling game characterised by* (3.4) *and* (IC)*, for any* $\lambda > 0$*, there exists a unique effort optimal* $\omega^* : \Lambda \cup \{1\} \to \Omega$ *that satisfies* (EO) *and is a continuous function strictly increasing in socioeconomic background,*

$$\frac{d}{d\lambda}\omega^*(\lambda) < 0.$$

Therefore, under economically appealing assumptions, $\omega^*(\lambda)$ defined by the implicit solution to $\varphi'(\omega) = 2\lambda\psi'(\omega)$ is sufficient for effort optimality in the RI equilibrium defined as the tuple $(\varphi(\omega^*(\lambda)), E(\omega^*(\lambda)), \omega^*(\lambda))$. These assumptions are that the effort production function displays weakly decreasing returns to scale, the effort disutility increases at an increasing rate and the productivity that arises from zero effort is also zero, implying no hiring at this effort. Given that zero effort implies zero education, one can think of the market described by Proposition 12 as a graduate market. The contrast with the symmetric information benchmark is immediate: for any $\lambda > 0$ the marginal benefit of any $\omega \in \Omega \setminus \{\omega_1\}$ in the RI equilibrium is one half of that under symmetric information.

**Remark.** *Optimal effort is higher under complete and symmetric information.*

$$\omega^{FB}(\lambda) > \omega^*(\lambda).$$

I have thus established the following sequence that begins with one parameter and determines two optimal actions progressively with $\lambda \mapsto \omega^*(\lambda) \mapsto E(\omega^*(\lambda))$ as I did in Section 3.3. The additional result, gained from endogenising private information, shows that, when workers signal their productivity via education, optimal effort is increasing in socioeconomic background. This is a common finding in the empirical literature on the demand for higher education and human capital theory, which cannot be derived in an exogenous signalling setting. Specifically, James (2002) states that socioeconomic background is largely responsible for a students evaluation of the attainability of higher education in Australia whilst Cameron and Heckman (1998) suggest that, when agents rationally examine the return and costs of higher education, family environment, including level of permanent income, explains a significant amount of the income-schooling relationship.

Whilst Proposition 12 follows naturally from the assumption that $\Psi_2(\omega, \lambda) > 0$, this comparative static will only hold in a setting in which the choice of effort is determined via an optimisation problem under sufficient conditions for a solution to exist. When types are exogenous, the worker will be unable to optimise against her socioeconomic background; consequently, this result will not hold. Introducing an endogenous effort decision provides the worker with a choice and gives rise to a natural concavity under the economically appealing sufficient conditions of Proposition 12. I can formulate how a job market comprised of $N$ workers will behave under Proposition 12 in a manner analogous to Section 3.3. The key difference here is that, if socioeconomic background is removed from the model by letting $\lambda_i = \lambda = 1$ for all $i = \{1, \ldots, n\}$, each worker chooses the same level of effort and subsequent education in equilibrium. This special case has an analogy with a basic bachelors degree, which has now become the standard required for entry in many graduate schemes, given that I am considering a graduate job market.

To build some graphical intuition for Proposition 12 I will now illustrate some of the key properties with a simple closed form example.

**Example 3.** *Suppose that* $\Omega = [0, 2]$, $\varphi(\omega) = \omega$ *and* $\psi(\omega) = \omega^2/2$. *Then by Theorem 3 I have* $E(\omega) = \frac{1}{2}\omega^2 \ \forall \ \omega \in \Omega$. *The worker's payoff is then given by*

$$\mathcal{U}(\omega, \lambda) = \frac{1}{2}\varphi(\omega) - \lambda\psi(\omega) = \frac{1}{2}\Big(\omega - \lambda\omega^2\Big).$$

*The problem* $\max_{\omega \in \Omega} \frac{1}{2}(\omega - \lambda\omega^2)$ *has a unique interior solution* $w^*(\lambda) = \frac{1}{2\lambda} \ \forall \ \lambda > 0$. *It is easy to see in Figure 3.4 that, because optimal effort* $\omega^*(\lambda)$ *is defined as the half of the inverse of a worker's socioeconomic background, as* $\lambda$ *increases,* $\omega^*(\lambda)$ *is decreasing, as per Proposition 12. For sufficiency, the second order condition is satisfied as* $\frac{\partial^2}{\partial \omega^2}\frac{1}{2}(\omega - \lambda\omega^2) = -\lambda < 0$. *I illustrate this in Figure 3.5 with* $\lambda \in \Lambda = \{\lambda_1, \lambda_2\}$ *where* $\lambda_1 = 0.9 > \lambda_2 = 0.6$.

Figure 3.5 graphically demonstrates that, under the conditions postulated in Example 3, the equilibrium payoff function $\mathcal{U}$ is decreasing in $\lambda$. To formulate this without recourse to a specific form of effort technology I make use of the envelope theorem for unconstrained optimisation. Define the value function

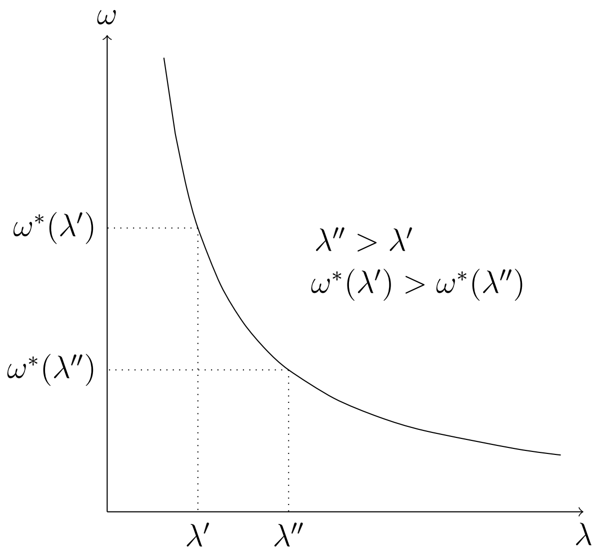


Figure 3.4: Optimal Effort in Spence's (1974) Model: $\omega^*(\lambda) = \frac{1}{2\lambda}$.

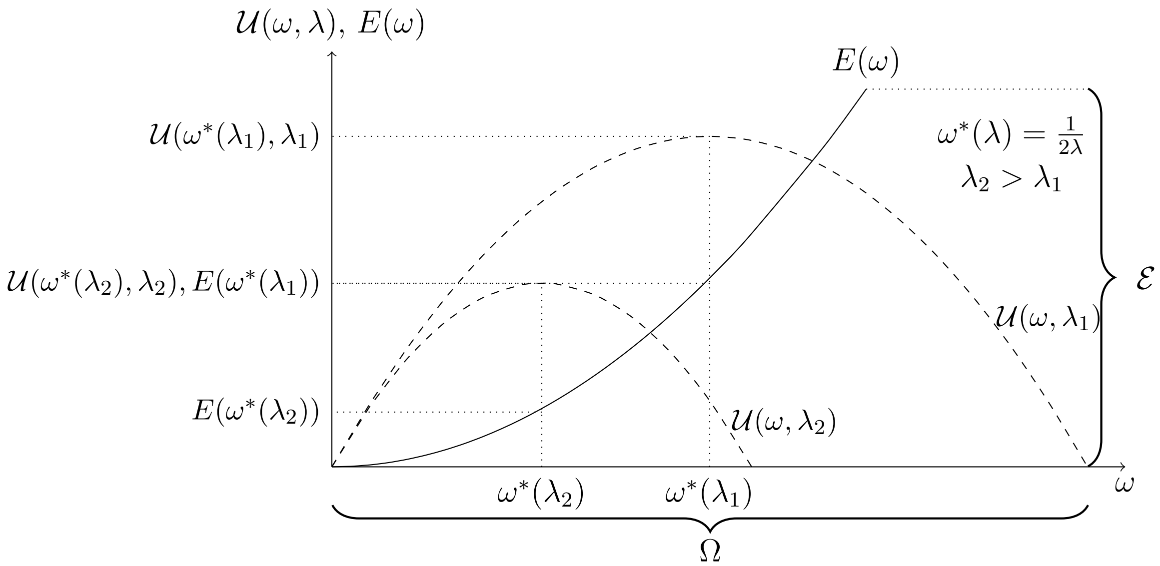


Figure 3.5: RI Equilibrium in Spence's (1974) Model.

$\mathfrak{U}(\lambda) \equiv \max_{\omega\in\Omega} \mathcal{U}(\omega,\lambda) = \mathcal{U}(\omega^*(\lambda),\lambda)$. When the conditions of Proposition 12 hold, I know that $\mathcal{U}$ is concave in effort for all $\omega \in \Omega \setminus \{\omega_1\}$ and that the value of $\omega$ that maximises $\mathcal{U}$, given any $\lambda \in \Lambda$ is given by the solution to (EO) and denoted $\omega^*(\lambda)$. Thus, I can compute that $\frac{\partial \mathfrak{U}(\lambda)}{\partial \lambda} = \frac{\partial}{\partial \lambda}\mathcal{U}(\omega,\lambda)|_{\omega=\omega^*(\lambda)} < 0$, implying the worker's value function $\mathfrak{U}$ is decreasing in $\lambda$ for all $\lambda \in \Lambda \subset \mathbb{R}_{++}$ and each $\omega \in \Omega \setminus \{\omega_1\}$. In simple terms, when education is solely used as a signal of productivity, itself engendered through costly effort, it is beneficial to have a socioeconomic background characterised by a relatively large family income and parental encouragement (Mare, 1980).

Finally, consider the setting in which the worker is unable to estimate her exact socioeconomic background before choosing her effort. As in Section 3.3 this modification provides additional realism and generality by relaxing the assumption that socioeconomic background can be perfectly inferred. I represent the distribution of preferences by the cumulative distribution function $F(\lambda|\tau)$, which is parameterised by $\tau$, and hence a worker's expected payoff, given (IC) and (3.4), is $\mathbb{E}[\mathcal{U}(\omega,\tau)] = \varphi(\omega) - \frac{E(\omega)}{\varphi(\omega)} - \int \Psi(\omega,\lambda)dF(\lambda|\tau)$ where $dF(\lambda|\tau)$ is the density of $F$. I show that, in this case, both the optimal choice of effort and the worker's expected payoff are decreasing in $\tau$, which could represent the mean of the distribution, when $F(\lambda|\tau)$ is decreasing in this parameter.

**Proposition 13.** *Suppose that $\tau' > \tau$ implies $F(\lambda|\tau') < F(\lambda|\tau)$. Then $\omega^u(\tau) = \max_{\omega\in\Omega} \mathbb{E}[\mathcal{U}(\omega,\tau)]$ is decreasing in $\tau$. Moreover, $\mathbb{E}[\mathcal{U}(\omega,\tau)] > \mathbb{E}[\mathcal{U}(\omega,\tau')]$.*

More specifically, Athey (2002) shows that whenever the density $f(\lambda|\tau)d\tau = dF(\lambda|\tau)$ satisfies the monotone likelihood ratio order property, which implies first-order stochastic dominance, and the utility function is supermodular the optimal choice of effort will be weakly increasing in socioeconomic background in the R-I equilibrium of the Spence model with endogenous effort under uncertainty.[12]

## 3.5 Conclusion

In this chapter, I proposed a simple and intuitive way to transform canonical signalling games with exogenous types into three stage games. In these games, the informed agent first endogenously generates the analogue of her type though an unobservable costly effort decision, before attempting to signal her effort in the second stage. My method involved adapting the classic framework of Mailath

[12]Specifically, the monotone likelihood ratio property implies that the density is log-supermodular, as by taking logs of the definition one obtains $\ln(f(\theta^H,s^H)) - \ln(f(\theta^L,s^H)) \geq \ln(f(\theta^H,s^L)) - \ln(f(\theta^L,s^L))$.

(1987), subsequently developed in Mailath and von Thadden (2013), and employing the recent equilibrium refinement propounded by In and Wright (2016, 2017). Their equilibrium refinement, Reordering Invariance, allows one to reorder the game and to first solve the conventional signalling subgame before turning to the choice of optimal unobservable effort.

There are several motivations for extending signalling games in this way. The first is that, in many cases, such as the two applications considered in Section 3.3 and Section 3.4, it is more natural to model the informed agent's private information as arising endogenously via an unobservable effort choice rather than being exogenously endowed; therefore, more closely approximating reality. The second salient motivation is that, by replacing exogenous types with endogenous hidden actions, I increased the explanatory power of signalling games. They now encompass a natural account as to how informed agents would, if given the choice, choose their type in games where this private information is signalled via an observable action. Moreover, I demonstrated that, under certain conditions, this form of choice is consistent with the type of behaviour normally observed in separating equilibria.

The main results were closely related to this motivation. The first set, Theorem 3 and Theorem 4, provided a characterisation of the informed agent's signalling strategy that takes account of her newly endowed effort technology, directly extending a result of Mailath and von Thadden (2013), and a recipe for setting up an endogenous signalling game that will have a unique RI equilibrium, respectively. This recipe is one set of, relatively restrictive, sufficient conditions for the existence of RI equilibria and, therefore, provides a simple first step towards characterising when such an equilibrium exists. Specifically, I focused on the linear setting of Mailath and von Thadden (2013), and so, one further step could be to extend Theorem 4 to the concave setting of Mailath (1987).

The second set of results, Proposition 11, Proposition 12 and Proposition 13, highlighted the additional insight gained by endogenising private information, which I illustrate in two seminal applications: DeMarzo and Duffie's (1999) model of security design; and, Spence's (1973) job market model. In particular, I showed that the informed agent's optimal unobservable effort in the security design setting, in which the firm signals its effort[13] using the quantity of the security offered for sale, is decreasing in the firm's need for liquidity. Conversely, the worker's optimal effort is increasing in her socioeconomic background when investment in

[13]Implying that, since the effort production function is common knowledge, the firm implicitly signals the payoff of the security through the signal of its effort. This holds analogously in the model of the job market.

education is used as a signal of effort, which itself determines the worker's productivity. These intuitive results provided theoretical support for two stylised facts, often documented in the respective empirical literatures, that cannot be derived in standard signalling games. Consequently, demonstrating the practical utility of endogenous private information in such games. In principle, results of this form should be obtainable in any differentiable signalling game, by introducing this form of endogenous effort and any relevant assumptions, which can then be critiqued in terms of applicability and intuition as is done with Assumption 8.

# Conclusion

Information is a salient factor in determining individuals' incentives. Therefore, developing a deeper understanding of how information affects decision makers is a fundamental question in economic theory. In this thesis, I have explored the consequences of information, both imperfect and asymmetric, for economic decision makers in a variety of settings. To do so, I have made use of, and modified, the methodology of Bayesian games. This is a diverse and widely applicable methodology that enables classic game theoretic results to be applied.

In my first chapter, I modelled an unconventional contest, hitherto unconsidered by the literature, as a Bayesian game. In this contest, players submit a private target of their performance before the contest takes place. However, the players are unable to perfectly estimate their performance due to a random component. Each player's ex-ante target, and ex-post performance, determines their relative ranking via a scoring rule. This scoring rule penalises negative deviations from target and acts as an upper bound on the players' scores. I first modelled the contest in full generality and established existence of equilibrium. After establishing existence, I simplified the environment to study the players' incentives and the resulting equilibrium behaviour in a variety of settings. The incentives arose through assumptions on the various settings in which a low ability player can win the contest over a high ability player. I showed that the players' equilibrium behaviour is diverse, in that players employ both pure and mixed strategies. Specifically, when there is no uncertainty affecting the players' performances, they act as in a second price auction and truthfully reveal their abilities. This is a general result, extending beyond the simplified setting considered in much of the chapter. When the uncertainty is equally likely to improve the players' performances as it is to leave performances unchanged, players use mixed strategies to approximate the expected effect that this uncertainty will have on their performances. Finally, I derived sufficient conditions on the distribution of the players' abilities, and their beliefs about the state of nature, such that pure strategy Bayes-Nash equilibria exist. These results demonstrated that information about the state of nature is more valuable than information about the abilities of the other players. Intuitively, players need to be sufficiently sure of one state arising to take that state into account when setting their targets.

This chapter made several contributions to the literature by introducing, and formalising, this unconventional contest. By providing the conditions under which this contest collapses to the standard tournament outcome, I contributed to the literature on contests and tournaments. Through analysis of a richer model than is usually considered, with two forms of uncertainty, I contributed to the literature on the economics of sport that seeks to characterise when players will use either pure or mixed strategies in equilibrium. Finally, by characterising the degree of outcome uncertainty associated with each equilibrium, I contributed to the literature on contest design.

What was learned is that, generally, this contest creates significant outcome uncertainty. This means that, even when the players' abilities are common knowledge, a lower ability player has a strictly positive probability of winning the contest. Moreover, one learned how this contest is related to existing competitive formulations, and when there will be overlap. Finally, one learned how these rules could be adapted and applied to other economic settings, such as sales-forecasting.

One limitation of the chapter is the focus on the simplified setting involving two players, two states of nature and two actions. A second is the lack of a characterisation of mixed strategy equilibria in the most general case. A third is the sole focus on players choosing their targets when performances are exogenous. The final limitation is that the aggregate uncertainty was perfectly correlated between all players. The second limitation could be improved by using a technical computing system, such as Mathematica, which could solve the system of simultaneous equations necessary to characterise mixed-strategy equilibria. The last two limitations could be improved upon by having each player choose both her target and her effort, and by including uncorrelated uncertainty. Each player's performance would then be partially endogenous as in standard tournament models. Future work could consist of extending the framework to move past these limitations.

In my second chapter, I modified the methodology of Bayesian games to study a model of the market for securitised assets. I choose this market as it is characterised by both problems of asymmetric information: adverse selection and moral hazard. In my model, the quality of the assets that are securitised is determined endogenously, rather than exogenously as in the vast majority of the literature. This novel aspect leads to a model that comprises aspects of both adverse selection and moral hazard within a single framework. I studied an originator who chooses whether to exert costly underwriting effort when granting loans that are subsequently securitised. The originator then chooses how much of the security issue to retain in an attempt to credibly convey her choice of effort to uninformed market investors. After characterising a benchmark equilibrium, which defines the levels of retention required to signal each choice of effort and the condition responsible for the choice of effort, I studied the effects of the 'skin in the game' regulation. This is a rule imposed on issuers of asset-backed securities that requires them to retain a fixed exogenous vertical fraction of each security issue on their books. I showed that, whilst this rule increases the signalling costs of an originator who has chosen high underwriting effort, it improves effort incentives for originators. That is, all else being equal, the regulation makes it relatively more likely that an originator will choose sufficient due-diligence. I then relaxed the central assumption of risk neutrality as, while originators are incentivised to securitise assets due to a need for liquidity, they also make use of securitisation to share risk. I showed that the aforementioned qualitative properties of equilibrium continue to hold when the originator is risk-averse. Therefore, exposure to default risk is sufficient for skin in the game to continue to improve the originator's incentives for high effort. Furthermore, I analysed whether a preference for liquidity, or a need to share risk, is more conducive for effort signalling. This is an aspect hitherto unconsidered by the literature. I showed that, depending on the magnitude of the need for liquidity or the need to share risk, each setting can more easily allow the originator to communicate her private effort choice.

This chapter contributed to the literature on both adverse selection and moral hazard in the market for securitised assets. I showed that results previously derived in principal-agent frameworks continue to hold in a modified adverse selection model, thereby increasing the result's generality. In particular, I contributed to the small literature that aims to synthesise both problems of asymmetric information. I did so by showing that pooling cannot be an endogenous outcome of the model and by employing more realistic assumptions on the originator's effort decision. Relaxing the assumption of risk neutrality, coupled with a preference for liquidity, provided a further contribution. This contribution involves demonstrating that the signalling costs and incentive effects of skin in the game are robust to different motivations for securitisation.

What was learned from this chapter is that originators and issuers of asset-backed securities do not need to be solely motivated by a need for liquidity for skin in the game to have a positive effect. Incentives for risk-sharing are sufficient for the originator to be more likely to choose high effort upon skin in the game being introduced. Moreover, one learned that, depending on the relative magnitude of the need for liquidity or the need to share risk, it may be easier for some originators to communicate their private information than others.

Limitations with this chapter involve generalising some of the economic modelling for additional realism. In my model, each asset is independent whilst asset correlation is a key aspect of securitisation in the real world. Future work could, therefore, involve correlation between the assets as well as a less mechanical process for how prices are arrived at. This could lead to further insights as I showed that there is a trade-off between inefficiency due to informational asymmetry and inefficiency due to market power. Another interesting avenue for future research would be to study a dynamic model to fully capture the idea of both ex-ante and ex-post moral hazard.

In my third chapter, I built on the core idea of the second chapter and developed it in a more general framework. This core idea involves endogenising the process by which an informed agent in a signalling game acquires her private information. I did so because the assumption of exogenous private information is not the most intuitive, or realistic, in many situations involving adverse selection. I adapted the canonical signalling framework propounded by George Mailath such that the informed agent's private information is the result of an unobservable, and costly, effort decision. This adds an additional stage to the game and is a simple, yet novel, modification. The informed agent now has to choose both her effort and the means by which she signals this effort to the uninformed agents. I then used this modified general framework to address three distinct research questions. The first research question centres on the method by which the informed agent can communicate her choice of effort. I characterised the additional intuitive conditions required for classic results, which describe the informed agent's means of signalling, to continue to hold in my modified framework. These extra conditions relate to the informed agent's effort technology, the means by which effort choices are converted into outcomes, and are similar to conditions found in producer theory and principal-agent models. The second research question is, given behaviour analogous to that which arises in separating equilibria, how does the informed agent decide upon her effort. To answer this, I analysed the informed agent's novel effort optimisation problem and I provided one set of sufficient conditions required for this problem to have a solution. These conditions place restrictions on the informed agent's payoff function, and separability of her payoff in the signal and the response of the uninformed agents is key. The final research question relates to the additional insight and intuition that is gained by endogenising private information in signalling games. To elucidate the novel insights gained from this approach, I considered two applications based on seminal signalling papers. In these applications, I employed my modified framework to provide theoretical support for commonly reported stylised facts that cannot be derived in the standard models. These results relate to how the informed agent in these models would choose her effort, given her means of signalling, and which parameters affect this choice.

This chapter contributed to the contemporary literature that studies signalling games with endogenous private information. I contributed a portable framework that could be applied and extended by other researchers. My two general results would enable others to set up a signalling game with endogenous private information and to derive the function the informed agent uses to communicate her private effort choice. The individual applications contributed to their respective literatures by providing theoretical support for stylised facts that cannot be derived under the standard assumption of exogenous private information. They provided intuitive explanations of how informed agents would choose their effort when they know they will subsequently signal this effort at a cost.

What was learned from this chapter is that a simple change in the set-up of signalling games allows them to capture more realistic and intuitive features. One learned the additional assumptions required for the informed agent to continue to able to communicate her choice of effort, and the conditions required for her effort optimisation problem to have a solution. Finally, one learned why endogenising private information is useful. It provided novel comparative statics results that explain how informed agents would choose their private information, which are unavailable in the standard setting.

The two main limitations of this chapter are the result on existence of a solution to the informed agent's effort decision problem and a relatively small number of applications. The set of conditions I provide for the informed agent to be able to optimise over her choice of effort is restrictive. Therefore, future work on obtaining more intuitive conditions possessing less restrictions would be useful. Applying this methodology to more applications would also lend additional support to the utility gained by endogenising private information in signalling games. Future research would involve providing a more comprehensive set of applications, which would also give rise to new comparative statics results.

# Appendix A

# Additional Material

# A.1 Chapter 1 Additional Material

## A.1.1 Derivation of the Game with One State of Nature under Assumption 2

Under Assumption 2 the initial strategic form of the game when $\mathcal{K} = \{k\} = \{0\}$ is

| 1 \ 2 | $\theta_H$ | $\theta_H + 1$ |
|---|---|---|
| $\theta_H$ | $\frac{1}{2}, \frac{1}{2}$ | 1, 0 |
| $\theta_H + 1$ | 0, 1 | $\frac{1}{2}, \frac{1}{2}$ |

| 1 \ 2 | $\theta_L$ | $\theta_L + 1$ |
|---|---|---|
| $\theta_H$ | 1, 0 | 1, 0 |
| $\theta_H + 1$ | 1, 0 | 1, 0 |

| 1 \ 2 | $\theta_H$ | $\theta_H + 1$ |
|---|---|---|
| $\theta_L$ | 0, 1 | 0, 1 |
| $\theta_L + 1$ | 0, 1 | 0, 1 |

| 1 \ 2 | $\theta_L$ | $\theta_L + 1$ |
|---|---|---|
| $\theta_L$ | $\frac{1}{2}, \frac{1}{2}$ | 1, 0 |
| $\theta_L + 1$ | 0, 1 | $\frac{1}{2}, \frac{1}{2}$ |

Figure A.1: Strategic form when $\mathcal{K} = \{k\} = 0$.

where the change in payoffs occurs at the strategy profiles $\sigma = (\theta_H + 1, \theta_L)$ and $\sigma = (\theta_L, \theta_H + 1)$. The strategic form when $\mathcal{K} = \{k\} = \{1\}$ remains unchanged from the proof of Proposition 2. Following analogous methods to the derivation of the Bayesian form of the game in Proposition 2 the Bayesian form of the game under Assumption 2 when $\mathcal{K} = \{k\} = \{0\}$ is given in Figure A.2.

## A.1.2 Derivation of Full Probabilistic Game under Assumption 2 and Assumption 3

By first fixing player 2's strategy $\sigma_2(\theta_2) = (\theta_H \theta_L)$ and subsequently varying player 1's strategy $\sigma_1(\theta_1)$ one obtains the following expected payoffs, which are also equal to the players' probabilities of winning:

$$U_1((\theta_H\theta_L, \theta_H\theta_L), \theta_1) = \lambda \cdot \frac{1}{2} + (1 - \lambda) \cdot \frac{1}{2} = \frac{1}{2} = U_2((\theta_H\theta_L, \theta_H\theta_L), \theta_2),$$

| 1 \ 2 | $\theta_H\theta_L$ | $\theta_H\theta_L+1$ | $\theta_H+1\theta_L$ | $\theta_H+1\theta_L+1$ |
|---|---|---|---|---|
| $\theta_H\theta_L$ | $\frac{1}{2},\frac{1}{2}$ | $1-\mu(1-\frac{1}{2}\mu),\mu(1-\frac{1}{2}\mu)$ | $1-\frac{1}{2}(1-\mu^2),\frac{1}{2}(1-\mu^2)$ | $1-\mu(1-\mu),\mu(1-\mu)$ |
| $\theta_H\theta_L+1$ | $\mu(1-\frac{1}{2}\mu),1-\mu(1-\frac{1}{2}\mu)$ | $\frac{1}{2},\frac{1}{2}$ | $\mu,1-\mu$ | $1-\frac{1}{2}(1-\mu^2),\frac{1}{2}(1-\mu^2)$ |
| $\theta_H+1\theta_L$ | $\frac{1}{2}(1-\mu^2),1-\frac{1}{2}(1-\mu^2)$ | $1-\mu,\mu$ | $\frac{1}{2},\frac{1}{2}$ | $1-\mu(1-\frac{1}{2}\mu),\mu(1-\frac{1}{2}\mu)$ |
| $\theta_H+1\theta_L+1$ | $\mu(1-\mu),1-\mu(1-\mu)$ | $\frac{1}{2}(1-\mu^2),1-\frac{1}{2}(1-\mu^2)$ | $\mu(1-\frac{1}{2}\mu),1-\mu(1-\frac{1}{2}\mu)$ | $\frac{1}{2},\frac{1}{2}$ |

Figure A.2: One State of Nature under Assumption 2.

$$U_1((\theta_H\theta_L+1,\theta_H\theta_L),\theta_1) = \lambda\cdot\mu(1-\frac{1}{2}\mu)+(1-\lambda)\cdot(1-\frac{1}{2}\mu^2),$$
$$= 1-\lambda(1-\mu)-\frac{1}{2}\mu^2,$$

and

$$U_2((\theta_H\theta_L+1,\theta_H\theta_L),\theta_2) = \lambda(1-\mu)+\frac{1}{2}\mu^2,$$

$$U_1((\theta_H+1\theta_L,\theta_H\theta_L),\theta_1) = \lambda\cdot\frac{1}{2}(1-\mu)^2+(1-\lambda)\cdot(1-\frac{1}{2}(1-\mu^2),$$
$$= \frac{1}{2}-\mu(\lambda-\frac{1}{2}\mu),$$

and

$$U_2((\theta_H+1\theta_L,\theta_H\theta_L),\theta_2) = 1-\frac{1}{2}+\mu(\lambda-\frac{1}{2}\mu),$$
$$U_1((\theta_H+1\theta_L+1,\theta_H\theta_L),\theta_1) = \lambda\cdot 0+(1-\lambda)\cdot 1 = 1-\lambda,$$

and

$$U_2((\theta_H+1\theta_L+1,\theta_H\theta_L),\theta_2) = \lambda\cdot 1+(1-\lambda)\cdot 0 = \lambda.$$

Now fixing $\sigma_2(\theta_2) = (\theta_H\theta_L+1)$ I calculate the following expected payoffs for each player,

$$U_1((\theta_H\theta_L,\theta_H\theta_L+1),\theta_1) = \lambda\cdot(1-\mu(1-\frac{1}{2}\mu)+(1-\lambda)\cdot\frac{1}{2}\mu^2,$$
$$= \lambda(1-\mu)+\frac{1}{2}\mu^2,$$

and

$$U_2((\theta_H\theta_L,\theta_H\theta_L+1),\theta_2) = 1-\lambda(1-\mu)-\frac{1}{2}\mu^2,$$
$$U_1((\theta_H\theta_L+1,\theta_H\theta_L+1),\theta_1) = \lambda\cdot\frac{1}{2}+(1-\lambda)\cdot\frac{1}{2} = \frac{1}{2} = U_2((\theta_H\theta_L+1,\theta_H\theta_L+1),\theta_2),$$

$$U_1((\theta_H+1\theta_L,\theta_H\theta_L+1),\theta_1) = \lambda\cdot(1-\mu)+(1-\lambda)\cdot\mu,$$
$$= \mu-\lambda(2\mu-1),$$

and

$$U_2((\theta_H+1\theta_L,\theta_H\theta_L+1),\theta_2) = 1-\mu+\lambda(2\mu-1),$$

$$U_1((\theta_H + 1\theta_L + 1, \theta_H\theta_L + 1), \theta_1) = \lambda \cdot \frac{1}{2}(1-\mu^2) + (1-\lambda)\cdot(1-\frac{1}{2}(1-\mu)^2),$$
$$= 1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda),$$

and

$$U_2((\theta_H + 1\theta_L + 1, \theta_H\theta_L + 1), \theta_2) = \frac{1}{2} - \mu(1 - \frac{1}{2}\mu - \lambda).$$

I can use these expected payoffs to construct the first half of the Bayesian strategic form of the game,

| | | **2** | |
|---|---|---|---|
| | | $\theta_H\theta_L$ | $\theta_H\theta_L + 1$ |
| | $\theta_H\theta_L$ | $\frac{1}{2}, \frac{1}{2}$ | $\lambda(1-\mu) + \frac{1}{2}\mu^2, 1 - \lambda(1-\mu) - \frac{1}{2}\mu^2$ |
| | $\theta_H\theta_L + 1$ | $1 - \lambda(1-\mu) - \frac{1}{2}\mu^2, \lambda(1-\mu) + \frac{1}{2}\mu^2$ | $\frac{1}{2}, \frac{1}{2}$ |
| **1** | $\theta_H + 1\theta_L$ | $\frac{1}{2} - \mu(\lambda - \frac{1}{2}\mu), 1 - \frac{1}{2} + \mu(\lambda - \frac{1}{2}\mu)$ | $\mu - \lambda(2\mu - 1), 1 - \mu + \lambda(2\mu - 1)$ |
| | $\theta_H + 1\theta_L + 1$ | $1 - \lambda, \lambda$ | $1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda), \frac{1}{2} - \mu(1 - \frac{1}{2}\mu - \lambda)$ |

Fixing player 2's strategy $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$ the expected payoffs of each player are now given by,

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_1) = \lambda \cdot (1 - \frac{1}{2}(1-\mu)^2) + (1-\lambda)\cdot\frac{1}{2}(1-\mu^2),$$
$$= 1 - \frac{1}{2} + \mu(\lambda - \frac{1}{2}\mu),$$

and

$$U_2((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_2) = \frac{1}{2} - \mu(\lambda - \frac{1}{2}\mu),$$

$$U_1((\theta_H\theta_L + 1, \theta_H + 1\theta_L), \theta_1) = \lambda \cdot \mu + (1-\lambda)\cdot(1-\mu),$$
$$= 1 - \mu + \lambda(2\mu - 1),$$

and

$$U_2((\theta_H\theta_L + 1, \theta_H + 1\theta_L), \theta_2) = \mu - \lambda(2\mu - 1),$$

$$U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L), \theta_1) = \lambda \cdot \frac{1}{2} + (1-\lambda)\cdot\frac{1}{2} = \frac{1}{2} = U_2((\theta_H + 1\theta_L, \theta_H + 1\theta_L), \theta_2),$$

$$U_1((\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L), \theta_1) = \lambda \cdot \frac{1}{2}\mu^2 + (1-\lambda)\cdot(1 - \mu(1 - \frac{1}{2}\mu)),$$
$$= 1 - \mu(1-\frac{1}{2}\mu) - \lambda(1-\mu),$$

and

$$U_2((\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L), \theta_2) = \mu(1-\frac{1}{2}\mu) + \lambda(1-\mu).$$

Finally, fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$, the expected payoffs of each player are calculated as follows,

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_1) = \lambda \cdot 1 + (1-\lambda)\cdot 0 = \lambda,$$

and

$$U_2((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_2) = 1 - \lambda,$$

$$U_1((\theta_H\theta_L + 1, \theta_H + 1\theta_L + 1), \theta_1) = \lambda \cdot (1 - \frac{1}{2}(1-\mu^2)) + (1-\lambda)\cdot\frac{1}{2}(1-\mu)^2,$$
$$= \frac{1}{2} - \mu(1 - \frac{1}{2}\mu - \lambda),$$

and

$$U_2((\theta_H\theta_L + 1, \theta_H + 1\theta_L + 1), \theta_2) = 1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda),$$

$$U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L + 1), \theta_1) = \lambda \cdot (1 - \frac{1}{2}\mu^2) + (1-\lambda)\cdot\mu(1-\frac{1}{2}\mu),$$
$$= \mu(1-\frac{1}{2}\mu) + \lambda(1-\mu),$$

and

$$U_2((\theta_H + 1\theta_L, \theta_H + 1\theta_L + 1), \theta_2) = 1 - \mu(1-\frac{1}{2}\mu) - \lambda(1-\mu),$$

$$U_1((\theta_H{+}1\theta_L{+}1, \theta_H{+}1\theta_L{+}1), \theta_1) = \lambda\cdot\frac{1}{2} + (1-\lambda)\cdot\frac{1}{2} = \frac{1}{2} = U_2((\theta_H{+}1\theta_L{+}1, \theta_H{+}1\theta_L{+}1), \theta_2).$$

Collating the second half of the Bayesian strategic form leads to Figure A.3.

### A.1.3 Derivation of Full Probabilistic Game under Assumption 2

Only certain payoffs are changed in the full probabilistic game when Assumption 3 is relaxed and hence only those payoffs are calculated below, to save space only player 1's payoffs are calculated as player 2's are simply $U_2(\sigma, \theta_2) = 1 - U_1(\sigma, \theta_1)$.

| 1 \ 2 | $\theta_H + 1\theta_L$ | $\theta_H + 1\theta_L + 1$ |
|---|---|---|
| $\theta_H\theta_L$ | $1 - \frac{1}{2} + \mu(\lambda - \frac{1}{2}\mu), \frac{1}{2} - \mu(\lambda - \frac{1}{2}\mu)$ | $\lambda, 1 - \lambda$ |
| $\theta_H\theta_L + 1$ | $1 - \mu + \lambda(2\mu - 1), \mu - \lambda(2\mu - 1)$ | $\frac{1}{2} - \mu(1 - \frac{1}{2}\mu - \lambda), 1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda)$ |
| $\theta_H + 1\theta_L$ | $\frac{1}{2}, \frac{1}{2}$ | $\mu(1 - \frac{1}{2}\mu) + \lambda(1 - \mu), 1 - \mu(1 - \frac{1}{2}\mu) - \lambda(1 - \mu)$ |
| $\theta_H + 1\theta_L + 1$ | $1 - \mu(1 - \frac{1}{2}\mu) - \lambda(1 - \mu), \mu(1 - \frac{1}{2}\mu) + \lambda(1 - \mu)$ | $\frac{1}{2}, \frac{1}{2}$ |

Figure A.3: Second Half of Full Probabilistic Game Under A2 and A3.

Fixing $\sigma_2(\theta_2) = (\theta_H\theta_L)$ I have

$$U_1((\theta_H+1\theta_L, \theta_H\theta_L), \theta_1) = \lambda\cdot\frac{1}{2}(1-\mu^2)+(1-\lambda)\cdot(1-\frac{1}{2}(1-\mu^2)) = 1-\frac{1}{2}+\mu^2(\frac{1}{2}-\lambda),$$

and

$$U_1((\theta_H + 1\theta_L + 1, \theta_H\theta_L), \theta_1) = \lambda \cdot \mu(1-\mu) + (1-\lambda)\cdot 1 = 1 - \lambda(1-\mu(1-\mu)).$$

Changing player 2's strategy to $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$ yields

$$U_1((\theta_H\theta_L, \theta_H+1\theta_L), \theta_1) = \lambda\cdot(1-\frac{1}{2}\mu(1-\mu^2))+(1-\lambda)\cdot\frac{1}{2}\mu(1-\mu^2) = \frac{1}{2}-\mu^2(\frac{1}{2}-\lambda),$$

and

$$U_1((\theta_H+1\theta_L+1, \theta_H+1\theta_L), \theta_1) = \lambda\cdot\mu(1-\frac{1}{2}\mu)+(1-\lambda)\cdot(1-\mu(1-\frac{1}{2}\mu)) = 1-\lambda+\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1).$$

Finally, fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H+1\theta_L+1)$ the changed expected payoffs are

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_1) = \lambda\cdot(1-\mu(1-\mu)) + (1-\lambda)\cdot 0 = \lambda(1-\mu(1-\mu)),$$

and

$$U_1((\theta_H+1\theta_L, \theta_H+1\theta_L+1), \theta_1) = \lambda\cdot(1-\mu(1-\frac{1}{2}\mu))+(1-\lambda)\cdot\mu(1-\frac{1}{2}\mu) = \lambda-\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1).$$

The full game, with only player 1's expected payoffs, under Assumption 2 is represented in Figure A.4.

### A.1.4 Mixed Strategy Equilibria

The mixed strategy Bayes-Nash equilibria discussed in the analysis of the full probabilistic game are reproduced below. The computations were performed using Game Theory Explorer (Savani and von Stengel, 2015). When $\Pr.(\theta_i = \theta_H) = \Pr.(\theta_i = \theta_L)$ the output is:

| | | 2 | | | |
|---|---|---|---|---|---|
| | | $\theta_H\theta_L$ | $\theta_H\theta_L+1$ | $\theta_H+1\theta_L$ | $\theta_H+1\theta_L+1$ |
| **1** | $\theta_H\theta_L$ | $\frac{1}{2}$ | $\lambda(1-\mu)+\frac{1}{2}\mu^2$ | $\frac{1}{2}-\mu^2(\frac{1}{2}-\lambda)$ | $\lambda(1-\mu(1-\mu))$ |
| | $\theta_H\theta_L+1$ | $1-\lambda(1-\mu)-\frac{1}{2}\mu^2$ | $\frac{1}{2}$ | $1-\mu+\lambda(2\mu-1)$ | $\frac{1}{2}-\mu(1-\frac{1}{2}\mu-\lambda)$ |
| | $\theta_H+1\theta_L$ | $1-\frac{1}{2}+\mu^2(\frac{1}{2}-\lambda)$ | $\mu-\lambda(2\mu-1)$ | $\frac{1}{2}$ | $\lambda-\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1)$ |
| | $\theta_H+1\theta_L+1$ | $1-\lambda(1-\mu(1-\mu))$ | $1-\frac{1}{2}+\mu(1-\frac{1}{2}\mu-\lambda)$ | $1-\lambda+\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1)$ | $\frac{1}{2}$ |

Figure A.4: Full Probabilistic Game under Assumption 2.

```
Strategic form:

4 x 4 Payoff player 1

    a   b   c   d
A 1/2 3/8 5/8 1/2
B 5/8 1/2 1/2 3/8
C 3/8 1/2 1/2 5/8
D 1/2 5/8 3/8 1/2

4 x 4 Payoff player 2

    a   b   c   d
A 1/2 5/8 3/8 1/2
B 3/8 1/2 1/2 5/8
C 5/8 1/2 1/2 3/8
D 1/2 3/8 5/8 1/2


EE = Extreme Equilibrium, EP = Expected Payoffs

Rational:

EE 1 P1: (1)   0 1/2 1/2   0 EP= 1/2 P2: (1)   0 1/2 1/2   0 EP= 1/2
EE 2 P1: (1)   0 1/2 1/2   0 EP= 1/2 P2: (2) 1/2   0   0 1/2 EP= 1/2
EE 3 P1: (2) 1/2   0   0 1/2 EP= 1/2 P2: (1)   0 1/2 1/2   0 EP= 1/2
EE 4 P1: (2) 1/2   0   0 1/2 EP= 1/2 P2: (2) 1/2   0   0 1/2 EP= 1/2

Decimal:

EE 1 P1: (1)   0 0.5 0.5   0 EP= 0.5 P2: (1)   0 0.5 0.5   0 EP= 0.5
EE 2 P1: (1)   0 0.5 0.5   0 EP= 0.5 P2: (2) 0.5   0   0 0.5 EP= 0.5
EE 3 P1: (2) 0.5   0   0 0.5 EP= 0.5 P2: (1)   0 0.5 0.5   0 EP= 0.5
EE 4 P1: (2) 0.5   0   0 0.5 EP= 0.5 P2: (2) 0.5   0   0 0.5 EP= 0.5

Connected component 1:
{1, 2}  x  {1, 2}
```

The second case I consider is $\Pr.(\theta_i = \theta_H) = 0.9$, which also covers the case of $\Pr.(\theta_i = \theta_H) = 0.1$ by (1.3). In this case the output is

```
Strategic form:

4 x 4 Payoff player 1

        a       b       c       d
A     1/2  91/200 109/200     1/2
B 109/200     1/2     1/2  91/200
C  91/200     1/2     1/2 109/200
D     1/2 109/200  91/200     1/2

4 x 4 Payoff player 2

        a       b       c       d
A     1/2 109/200  91/200     1/2
B  91/200     1/2     1/2 109/200
C 109/200     1/2     1/2  91/200
D     1/2  91/200 109/200     1/2


EE = Extreme Equilibrium, EP = Expected Payoffs

Rational:

EE 1 P1: (1)   0 1/2 1/2   0 EP= 1/2 P2: (1)   0 1/2 1/2   0 EP= 1/2
EE 2 P1: (1)   0 1/2 1/2   0 EP= 1/2 P2: (2) 1/2   0   0 1/2 EP= 1/2
EE 3 P1: (2) 1/2   0   0 1/2 EP= 1/2 P2: (1)   0 1/2 1/2   0 EP= 1/2
EE 4 P1: (2) 1/2   0   0 1/2 EP= 1/2 P2: (2) 1/2   0   0 1/2 EP= 1/2

Decimal:

EE 1 P1: (1)   0 0.5 0.5   0 EP= 0.5 P2: (1)   0 0.5 0.5   0 EP= 0.5
EE 2 P1: (1)   0 0.5 0.5   0 EP= 0.5 P2: (2) 0.5   0   0 0.5 EP= 0.5
EE 3 P1: (2) 0.5   0   0 0.5 EP= 0.5 P2: (1)   0 0.5 0.5   0 EP= 0.5
EE 4 P1: (2) 0.5   0   0 0.5 EP= 0.5 P2: (2) 0.5   0   0 0.5 EP= 0.5

Connected component 1:
{1, 2}  x  {1, 2}
```

It is easy to see that the equilibrium set remains $\Gamma$ for both specifications.

## A.2 Chapter 2 Additional Material

### A.2.1 Heterogenous Lending Capacity

Hitherto it has been assumed that the originator can grant a homogenous quantity of loans irrespective of her effort, this section relaxes that assumption. Denote the total loan supply under low and high effort, respectively, by $\sum_{i=1}^{k} r_i$ and $\sum_{i=1}^{j} r_i$ where $j < k$. The motivation for this section is due to the fact that, in reality, if the originator chooses low effort she is able to exploit the finiteness of the set of borrowers and grant loans to all borrowers, whilst under high effort only the subset of borrowers who are creditworthy will be granted loans. Assuming that $\delta \sum_{i=1}^{j} r_i > \delta \sum_{i=1}^{k} r_i \rho(\theta|e_L)$, the originator solves the following problem if she wishes to signal high effort

$$\max_{\alpha(e_H)\in[\underline{\Omega},1]} \; -\sum_{i=1}^{j} l_i + \delta\alpha \sum_{i=1}^{j} r_i + p(\alpha(e_H))(1-\alpha) - c(e_H), \tag{A.1}$$

where

$$\underline{\Omega} \equiv \frac{p(\alpha(e_H)) - p(\alpha(e_L)) + \mu(p(\alpha(e_L)) - \sum_{i=1}^{k} r_i \cdot \delta \cdot \rho(\theta|e_L))}{p(\alpha(e_H)) - \sum_{i=1}^{k} r_i \cdot \delta \cdot \rho(\theta|e_L)}.$$

If I assume that $\sum_{i=1}^{j} r_i = 1$ and $\sum_{i=1}^{k} r_i > 1$ then by following arguments essentially identical to those given in the proof of Proposition 7 one can show that with a heterogenous lending capacity the effort-signalling level of retention is lower than the skin in the game case

$$\alpha^*(e_H;\mu) > \alpha^{HLC*}(e_H;\mu),$$

where $\alpha^{HLC*}(e_H;\mu)$ solves (A.1). The intuition is that as a heterogenous lending capacity reduces the wedge between the the prices received for the high and low quality assets, the incentive to mimic decreases. Moreover, with this formulation

$$u(e_H, \alpha^{HLC*}(e_H;\mu); p(\alpha(e_L))) > u(e_H, \alpha^*(e_H;\mu); p(\alpha(e_L))).$$

The originator's payoff is greater under a heterogenous lending capacity, following from the reduction in retention. Thus, this asymmetry turns out to be beneficial for signalling.

# A.3 Chapter 3 Additional Material

## A.3.1 Assumptions on Effort Technology

I now discuss the assumption that the informed agent's effort technology is an injective function. Given her domain of effort $\omega \in \Omega \equiv [\omega_1 \omega_2]$ with $\omega_2 > \omega_1$ the function $\varphi : \Omega \to \mathbb{R}_+$ is assumed to be one-to-one so that $\omega' \neq \omega''$ implies $\varphi(\omega') \neq \varphi(\omega'')$. This is a standard assumption is models of production such as $Y(K, L) = K^{\alpha} L^{1-\alpha}$ for $\alpha \in (0, 1)$ or models of wage setting behaviour, where a worker's output is determined uniquely by her effort. To give $\varphi$ economically meaningful properties it could be assumed that $\varphi'(\omega) > 0$ and $\varphi''(\omega) \leq 0$ for all $\omega \in \Omega \setminus \{\omega_1\}$. Consequently, functional forms that capture these properties include $\varphi(\omega) = \omega^{\alpha}$ for $\alpha \in (0, 1)$ or $\varphi(\omega) = \ln(\omega)$. In a setting where $\omega_1 = 0$ but one requires $\varphi(\omega_1) > 0$ it could be assumed that $\varphi(\omega) = \beta + \omega^{\alpha}$ or $\varphi(\omega) = \beta + \ln(\omega)$ for $\beta \geq 1$.

One criticism of this approach could be that effort does not map monotonically into outcomes. To accommodate such a setting I could instead assume that outcomes $\pi$ are distributed by the conditional cumulative distribution function $F(\pi|\omega)$ with bounded support $\Pi \equiv [\pi_1, \pi_2]$ where $\pi_2 > \pi_1$ and density $f(\pi|\omega)$. As well as assuming that $F$ is common knowledge two structural assumptions are necessary for this distribution to posses economically meaning properties. The first is that $f_{\omega}(\pi|\omega) > 0$ for all $\pi \in \Pi$ and $\omega \in \Omega$ and the second is first-order stochastic dominance $F_{\omega}(\pi|\omega) < 0$ for all $\omega \in \Omega$. First-order stochastic dominance implies that for $\omega'' > \omega'$ one has $F(\pi|\omega'') < F(\pi|\omega')$, which means that when the informed agent increases her effort the probability of an outcome less than or equal to $\pi$ strictly decreases. Additionally, this assumption implies that the expected value of $\pi$ given $\omega''$ is greater than the expected value of $\pi$ given $\omega'$

$$\mathbb{E}[\pi|\omega = \omega''] = \int \pi f(\pi|\omega'') d\pi > \int \pi f(\pi|\omega') d\pi = \mathbb{E}[\pi|\omega = \omega'].$$

As neither $\pi$ nor $\omega$ are observed by the uninformed agents, but instead inferences about $\omega$ are formed given the signal $X(\omega)$, the informed agent signals her choice of effort $\omega$ and the uninformed use their knowledge of the distribution $F$ to calculate the expected value of the outcome. This is because there must be an injective relationship between what is being signalled and the means by which this information is signalled. If the informed agent tried to signal the distribution of outcomes induced by her choice of effort, this injectivity would be lost. Therefore, since the conditional expectation of $\pi$ given some value of $\omega$ is a function of $\omega$ alone I can write

$$\varphi(\omega) = \mathbb{E}[\pi|\omega].$$

Distributions that satisfy the monotone likelihood ratio property, which implies first-order stochastic dominance, include the exponential, binomial, poisson and normal distributions, where the variance must be known for the normal. Two functional form examples (Spaeter, 1998) that also satisfies convexity of the distribution function are

$$F(\pi|\omega) = \left[\frac{1}{(\omega+1)\pi_2}(\pi_2 - \pi) + 1\right]\frac{\pi}{\pi_2}$$

for $\pi \in [0, \pi_2]$ and

$$G(\pi|\omega) = (\omega + k)^{\pi-\pi_2}\left(\frac{\pi - \pi_1}{\pi_2 - \pi_1}\right)$$

for $\pi \in [\pi_1, \pi_2]$.

# Appendix B

# Proofs

## B.1 Chapter 1 Proofs

*Proof of Proposition 1.* Standard arguments imply that the results of Nash (1951) can be applied (Fudenberg and Tirole, 1991). □

*Proof of Proposition 2.* I now derive each player's expected payoff $U_i(\sigma, \theta_i)$ under Assumption 2 and Assumption 3, which is equal to that player's probability of winning the contest given my assumption that the prize is normalised as $R = 1$. Fixing player 2's strategy $\sigma_2(\theta_2) = (\theta_H\theta_L)$ and varying player 1's gives us the following expected payoffs. Specifically, suppose that player 1 also uses the strategy[1] $\sigma_1(\theta_1) = (\theta_H\theta_L)$,

$$\begin{aligned} U_1((\theta_H\theta_L, \theta_H\theta_L), \theta_1) &= \mu^2 \cdot \frac{1}{2} + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot \frac{1}{2} \\ &= \frac{1}{2} = U_2((\theta_H\theta_L, \theta_H\theta_L), \theta_2). \end{aligned}$$

Now changing player 1's strategy to $\sigma_1(\theta_1) = (\theta_H\theta_L + 1)$ one has,

$$U_1((\theta_H\theta_L+1, \theta_H\theta_L), \theta_1) = \mu^2 \cdot \frac{1}{2} + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot 0 = \mu(1-\frac{1}{2}\mu),$$

and

$$U_2((\theta_H\theta_L+1, \theta_H\theta_L), \theta_2) = \mu^2 \cdot \frac{1}{2} + \mu(1-\mu) \cdot 0 + \mu(1-\mu) \cdot 1 + (1-\mu)^2 \cdot 1 = 1-\mu(1-\frac{1}{2}\mu).$$

If instead player 1 plays the strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L)$,

$$U_1((\theta_H\theta_L+1, \theta_H\theta_L), \theta_1) = \mu^2 \cdot 0 + \mu(1-\mu) \cdot 0 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot \frac{1}{2} = \frac{1}{2}(1-\mu)^2,$$

and

$$U_2((\theta_H+1\theta_L, \theta_H\theta_L), \theta_2) = \mu^2 \cdot 1 + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 1 + (1-\mu)^2 \cdot \frac{1}{2} = 1-\frac{1}{2}(1-\mu)^2.$$

Finally, if player 1 employs the strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$,

$$U_1((\theta_H + 1\theta_L + 1, \theta_H\theta_L), \theta_1) = \mu^2 \cdot 0 + \mu(1-\mu) \cdot 0 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot 0 = 0,$$

and

$$U_2((\theta_H + 1\theta_L + 1, \theta_H\theta_L), \theta_2) = \mu^2 \cdot 1 + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 1 + (1-\mu)^2 \cdot 1 = 1.$$

[1] $\sigma = (\sigma_1(\theta_1), \sigma_2(\theta_2))$.

I now fix player 2's strategy $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$ and vary player 1's strategy. When player 1 uses strategy $\sigma_1(\theta_1) = (\theta_H\theta_L)$ the expected payoffs are,

$$U_1((\theta_H\theta_L, \theta_H\theta_L+1), \theta_1) = \mu^2 \cdot \frac{1}{2} + \mu(1-\mu)\cdot 1 + \mu(1-\mu)\cdot 0 + (1-\mu)^2 \cdot 1 = 1 - \mu(1 - \frac{1}{2}\mu),$$

and

$$U_2((\theta_H\theta_L, \theta_H\theta_L+1), \theta_2) = \mu^2 \cdot \frac{1}{2} + \mu(1-\mu)\cdot 0 + \mu(1-\mu)\cdot 1 + (1-\mu)^2 \cdot 0 = \mu(1 - \frac{1}{2}\mu).$$

When player 1 uses the strategy $\sigma_1(\theta_1) = (\theta_H\theta_L + 1)$ one has,

$$\begin{aligned} &U_1((\theta_H\theta_L + 1, \theta_H\theta_L + 1), \theta_1) \\ &= \mu^2 \cdot \frac{1}{2} + \mu(1-\mu)\cdot 1 + \mu(1-\mu)\cdot 0 + (1-\mu)^2 \cdot \frac{1}{2} = \frac{1}{2} = U_2((\theta_H\theta_L+1, \theta_H\theta_L+1), \theta_2), \end{aligned}$$

and under the strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L)$,

$$U_1((\theta_H + 1\theta_L, \theta_H\theta_L + 1), \theta_1) = \mu^2 \cdot 0 + \mu(1-\mu)\cdot 1 + \mu(1-\mu)\cdot 0 + (1-\mu)^2 \cdot 1 = 1 - \alpha,$$

with,

$$U_2((\theta_H + 1\theta_L, \theta_H\theta_L + 1), \theta_2) = \mu^2 \cdot 1 + \mu(1-\mu)\cdot 0 + \mu(1-\mu)\cdot 1 + (1-\mu)^2 \cdot 0 = \alpha.$$

Finally, when player 1 uses the strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ the expected payoffs are,

$$U_1((\theta_H + 1\theta_L + 1, \theta_H\theta_L + 1), \theta_1) = \mu^2 \cdot 0 + \mu(1-\mu)\cdot 1 + \mu(1-\mu)\cdot 0 + (1-\mu)^2 \cdot \frac{1}{2} = \frac{1}{2}(1-\mu^2),$$

and

$$U_2((\theta_H + 1\theta_L + 1, \theta_H\theta_L + 1), \theta_2) = \mu^2 \cdot 1 + \mu(1-\mu)\cdot 0 + \mu(1-\mu)\cdot 1 + (1-\mu)^2 \cdot \frac{1}{2} = 1 - \frac{1}{2}(1-\mu^2).$$

Fixing player 2's strategy now as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$, I first calculate the expected payoffs when player 1 uses the strategy $\sigma_1(\theta_1) = (\theta_H\theta_L)$,

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_1) = \mu^2 \cdot 1 + \mu(1-\mu)\cdot 1 + \mu(1-\mu)\cdot 1 + (1-\mu)^2 \cdot \frac{1}{2} = 1 - \frac{1}{2}(1-\mu)^2,$$

and

$$U_2((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_2) = \mu^2 \cdot 0 + \mu(1-\mu)\cdot 0 + \mu(1-\mu)\cdot 0 + (1-\mu)^2 \cdot \frac{1}{2} = \frac{1}{2}(1-\mu)^2.$$

Now considering $\sigma_1(\theta_1) = (\theta_H\theta_L + 1)$ one has,

$$U_1((\theta_H\theta_L + 1, \theta_H + 1\theta_L), \theta_1) = \mu^2 \cdot 1 + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot 0 = \mu,$$

and

$$U_2((\theta_H\theta_L + 1, \theta_H + 1\theta_L), \theta_2) = \mu^2 \cdot 0 + \mu(1-\mu) \cdot 0 + \mu(1-\mu) \cdot 1 + (1-\mu)^2 \cdot 1 = 1-\mu.$$

If $\sigma_1(\theta_1) = (\theta_H + 1\theta_L)$ then,

$$U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L), \theta_1) = \frac{1}{2} = U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L), \theta_2).$$

Finally, if $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ then the expected payoffs are,

$$U_1((\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L), \theta_1) = \mu^2 \cdot \frac{1}{2} + \mu(1-\mu) \cdot 0 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot 0 = \frac{1}{2} \cdot \mu^2,$$

and

$$U_2((\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L), \theta_2) = \mu^2 \cdot \frac{1}{2} + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 1 + (1-\mu)^2 \cdot 1 = 1 - \frac{1}{2} \cdot \mu^2.$$

For the final set of expected payoffs I fix $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$. Beginning with a player 1 strategy of $\sigma_1(\theta_1) = (\theta_H\theta_L)$, the expected payoffs are,

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_1) = \mu^2 \cdot 1 + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 1 + (1-\mu)^2 \cdot 1 = 1,$$

and

$$U_2((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_2) = \mu^2 \cdot 0 + \mu(1-\mu) \cdot 0 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot 0 = 0.$$

Now letting $\sigma_1(\theta_1) = (\theta_H\theta_L + 1)$ the expected payoffs are,

$$U_1((\theta_H\theta_L + 1, \theta_H + 1\theta_L + 1), \theta_1) = \mu^2 \cdot 1 + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot \frac{1}{2} = 1 - \frac{1}{2}(1-\mu^2),$$

and

$$U_2((\theta_H\theta_L + 1, \theta_H + 1\theta_L + 1), \theta_2) = \mu^2 \cdot 0 + \mu(1-\mu) \cdot 0 + \mu(1-\mu) \cdot 0 + (1-\mu)^2 \cdot \frac{1}{2} = \frac{1}{2}(1-\mu^2).$$

If instead $\sigma_1(\theta_1) = (\theta_H + 1\theta_L)$ the players' expected payoffs are given by,

$$U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L + 1), \theta_1) = \mu^2 \cdot \frac{1}{2} + \mu(1-\mu) \cdot 1 + \mu(1-\mu) \cdot 1 + (1-\mu)^2 \cdot 1 = 1 - \frac{1}{2} \cdot \mu^2,$$

and

$$U_2((\theta_H+1\theta_L,\theta_H+1\theta_L+1),\theta_2)=\mu^2\cdot\frac{1}{2}+\mu(1-\mu)\cdot 0+\mu(1-\mu)\cdot 0+(1-\mu)^2\cdot 0=\frac{1}{2}\cdot\mu^2.$$

The final expected payoffs when $\sigma_1(\theta_1)=(\theta_H+1\theta_L+1)$ are,

$$U_1((\theta_H+1\theta_L+1,\theta_H+1\theta_L+1),\theta_1)=\frac{1}{2}=U_2((\theta_H+1\theta_L+1,\theta_H+1\theta_L+1),\theta_2).$$

Collating these expected payoffs in a Bayesian strategic form representation leads to Figure B.1.

Hence, player 1's strategy $\sigma_1(\theta_1)=(\theta_H\theta_L)$ is a best response to the strategy of player 2 $\sigma_2(\theta_2)=(\theta_H\theta_L)$ if,

$$\begin{aligned}&U_1((\theta_H\theta_L,\theta_H\theta_L),\theta_1)>U_1((\theta_H\theta_L+1,\theta_H\theta_L),\theta_1)\\&\qquad\Leftrightarrow\ \frac{1}{2}>\mu\cdot(1-\frac{1}{2}\cdot\mu)\ \Leftrightarrow\ \frac{1}{2}(1-\mu)^2>0,\end{aligned}\tag{B.1}$$

and

$$\begin{aligned}&U_1((\theta_H\theta_L,\theta_H\theta_L),\theta_1)>U_1((\theta_H+1\theta_L,\theta_H\theta_L),\theta_1)\\&\qquad\Leftrightarrow\ \frac{1}{2}>\frac{1}{2}(1-\mu)^2\ \Leftrightarrow\ 1>\mu.\end{aligned}\tag{B.2}$$

Both condition (B.88) and (B.89) are satisfied by the assumption of $\mu\in(0,1)$. Hence, player 1's strategy $\sigma_1(\theta_1)=(\theta_H\theta_L)$ is a best response to strategy $\sigma_2(\theta_2)=(\theta_H\theta_L)$ of player 2. Moreover, player 1's strategy $\sigma_1(\theta_1)=(\theta_H\theta_L)$ is a best response to player 2's strategy $\sigma_2(\theta_2)=(\theta_H\theta_L+1)$ if,

$$\begin{aligned}&U_1((\theta_H\theta_L,\theta_H\theta_L+1),\theta_1)>U_1((\theta_H\theta_L+1,\theta_H\theta_L+1),\theta_1)\\&\qquad\Leftrightarrow\ 1-\mu(1-\frac{1}{2}\cdot\mu)>\frac{1}{2}\ \Leftrightarrow\ \frac{1}{2}>\mu(1-\frac{1}{2}\cdot\mu),\end{aligned}\tag{B.3}$$

and

$$\begin{aligned}&U_1((\theta_H\theta_L,\theta_H\theta_L+1),\theta_1)>U_1((\theta_H+1\theta_L,\theta_H\theta_L+1),\theta_1)\\&\qquad\Leftrightarrow\ 1-\mu(1-\frac{1}{2}\cdot\mu)>1-\mu\ \Leftrightarrow\ \frac{1}{2}\cdot\mu^2>0,\end{aligned}\tag{B.4}$$

| **1** \ **2** | $\theta_H\theta_L$ | $\theta_H\theta_L+1$ | $\theta_H+1\theta_L$ | $\theta_H+1\theta_L+1$ |
|---|---|---|---|---|
| $\theta_H\theta_L$ | $\frac{1}{2},\frac{1}{2}$ | $1-\mu(1-\frac{1}{2}\mu),\mu(1-\frac{1}{2}\mu)$ | $1-\frac{1}{2}(1-\mu)^2,\frac{1}{2}(1-\mu)^2$ | $1,0$ |
| $\theta_H\theta_L+1$ | $\mu(1-\frac{1}{2}\mu),1-\mu(1-\frac{1}{2}\mu)$ | $\frac{1}{2},\frac{1}{2}$ | $\mu,1-\mu$ | $1-\frac{1}{2}(1-\mu^2),\frac{1}{2}(1-\mu^2)$ |
| $\theta_H+1\theta_L$ | $\frac{1}{2}(1-\mu)^2,1-\frac{1}{2}(1-\mu)^2$ | $1-\mu,\mu$ | $\frac{1}{2},\frac{1}{2}$ | $1-\frac{1}{2}\mu^2,\frac{1}{2}\mu^2$ |
| $\theta_H+1\theta_L+1$ | $0,1$ | $\frac{1}{2}(1-\mu^2),1-\frac{1}{2}(1-\mu^2)$ | $\frac{1}{2}\mu^2,1-\frac{1}{2}\mu^2$ | $\frac{1}{2},\frac{1}{2}$ |

Figure B.1: Bayesian Normal Form when $\{k\}=0$.

and

$$U_1((\theta_H\theta_L, \theta_H\theta_L + 1), \theta_1) > U_1((\theta_H + 1\theta_L + 1, \theta_H\theta_L + 1), \theta_1)$$
$$\Leftrightarrow\ 1 - \mu(1 - \frac{1}{2} \cdot \mu) > \frac{1}{2}(1 - \mu^2)\ \Leftrightarrow\ \frac{1}{2} > 0. \quad \text{(B.5)}$$

Note that condition (B.90) is identical to (B.88) and so is satisfied, whilst (B.91) and (B.92) are trivially satisfied. Therefore, $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is a best response to $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$. I can also show that $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is player 1's best response to player 2's strategy $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$ if

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_1) > U_1((\theta_H\theta_L + 1, \theta_H + 1\theta_L), \theta_1)$$
$$\Leftrightarrow\ 1 - \frac{1}{2}(1 - \mu)^2 > \mu\ \Leftrightarrow\ 1 > \mu^2, \quad \text{(B.6)}$$

and

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_1) > U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L), \theta_1)$$
$$\Leftrightarrow\ 1 - \frac{1}{2}(1 - \mu)^2 > \frac{1}{2}\ \Leftrightarrow\ 1 > \mu, \quad \text{(B.7)}$$

and

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_1) > U_1((\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L), \theta_1)$$
$$\Leftrightarrow\ 1 - \frac{1}{2}(1 - \mu)^2 > \frac{1}{2} \cdot \mu^2\ \Leftrightarrow\ \frac{1}{2} + \mu(1 - \mu) > 0. \quad \text{(B.8)}$$

It is easy to see that conditions (B.93), (B.94) and (B.95) are satisfied for $\mu \in (0, 1)$. Finally, $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is player 1's best response to player 2's strategy of $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$ if,

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_1) > U_1((\theta_H\theta_L + 1, \theta_H + 1\theta_L + 1), \theta_1)$$
$$\Leftrightarrow\ 1 > 1 - \frac{1}{2}(1 - \mu^2) > 0\ \Leftrightarrow\ 0 > -\frac{1}{2}(1 - \mu^2), \quad \text{(B.9)}$$

and

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_1) > U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L + 1), \theta_1)$$
$$\Leftrightarrow\ 1 > 1 - \frac{1}{2} \cdot \mu^2 > 0\ \Leftrightarrow\ 0 > -\frac{1}{2} \cdot \mu^2. \quad \text{(B.10)}$$

Both conditions (A9) and (B.96) are trivially satisfied for $\mu \in (0, 1)$.

Therefore, I have shown that, irrespective of the choice of player 2's strategy, player 1's best response is $\sigma_1(\theta_1) = (\theta_H\theta_L)$. Consequently, as the players' payoffs are symmetric, this implies that, for any strategy of player 1, player 2's best response is defined by the strategy $\sigma_2(\theta_2) = (\theta_H\theta_L)$. Hence the strategy profile

$$\sigma^* = (\sigma_1(\theta_1), \sigma_2(\theta_2)) = (\theta_H\theta_L, \theta_H\theta_L),$$

is the unique pure strategy Nash equilibrium of the game.

To show that this continues to be an equilibrium when $\{k\} = \{1\}$ I perform an analogous analysis to the following games, which occur when the only state of nature is the performance enhancing $\{k\} = 1$:

| **1** \ **2** | $\theta_H$ | $\theta_H + 1$ |
|---|---|---|
| $\theta_H$ | $\frac{1}{2}, \frac{1}{2}$ | 0, 1 |
| $\theta_H + 1$ | 1, 0 | $\frac{1}{2}, \frac{1}{2}$ |

| **1** \ **2** | $\theta_L$ | $\theta_L + 1$ |
|---|---|---|
| $\theta_H$ | 1, 0 | 0, 1 |
| $\theta_H + 1$ | 1, 0 | 1, 0 |

| **1** \ **2** | $\theta_H$ | $\theta_H + 1$ |
|---|---|---|
| $\theta_L$ | 0, 1 | 0, 1 |
| $\theta_L + 1$ | 1, 0 | 0, 1 |

| **1** \ **2** | $\theta_L$ | $\theta_L + 1$ |
|---|---|---|
| $\theta_L$ | $\frac{1}{2}, \frac{1}{2}$ | 0, 1 |
| $\theta_L + 1$ | 1, 0 | $\frac{1}{2}, \frac{1}{2}$ |

Once again fixing $\sigma_2(\theta_2) = (\theta_H\theta_L)$ I calculate the expected payoffs for each player by varying player 1's strategy, the calculations are removed to reduce space.

$$U_1((\theta_H\theta_L, \theta_H\theta_L), \theta_1) = \frac{1}{2} = U_2((\theta_H\theta_L, \theta_H\theta_L), \theta_2),$$

$$U_1((\theta_H\theta_L + 1, \theta_H\theta_L), \theta_1) = 1 - \frac{1}{2} \cdot \mu^2 \text{ and } U_2((\theta_H\theta_L + 1, \theta_H\theta_L), \theta_2) = \frac{1}{2} \cdot \mu^2,$$

$$U_1((\theta_H + 1\theta_L, \theta_H\theta_L), \theta_1) = 1 - \frac{1}{2}(1 - \mu^2) \text{ and } U_2((\theta_H + 1\theta_L, \theta_H\theta_L), \theta_2) = \frac{1}{2}(1 - \mu^2),$$

$$U_1((\theta_H + 1\theta_L + 1, \theta_H\theta_L), \theta_1) = 1 \text{ and } U_2((\theta_H + 1\theta_L + 1, \theta_H\theta_L), \theta_2) = 0.$$

By conditions (A9) and (B.96) one can see that player 1's strategy $\sigma_1(\theta_1) =$

$(\theta_H + 1\theta_L + 1)$ is a best response to strategy $\sigma_2(\theta_2) = (\theta_H\theta_L)$.

Now fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$ and varying player 1's strategy yields the expected payoffs,

$$U_1((\theta_H\theta_L, \theta_H\theta_L + 1), \theta_1) = \frac{1}{2} \cdot \mu^2 \text{ and } U_2((\theta_H\theta_L, \theta_H\theta_L + 1), \theta_2) = 1 - \frac{1}{2} \cdot \mu^2,$$

$$U_1((\theta_H\theta_L + 1, \theta_H\theta_L + 1), \theta_1) = \frac{1}{2} = U_2((\theta_H\theta_L + 1, \theta_H\theta_L + 1), \theta_2),$$

$$U_1((\theta_H + 1\theta_L, \theta_H\theta_L + 1), \theta_1) = \mu \text{ and } U_2((\theta_H + 1\theta_L, \theta_H\theta_L + 1), \theta_2) = 1 - \mu,$$

$$U_1((\theta_H + 1\theta_L + 1, \theta_H\theta_L + 1), \theta_1) = 1 - \frac{1}{2}(1 - \mu)^2$$

and

$$U_2((\theta_H + 1\theta_L + 1, \theta_H\theta_L + 1), \theta_2) = \frac{1}{2}(1 - \mu)^2.$$

By conditions (B.93)-(B.95) one can see that strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response to $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$.

Now considering the expected payoffs of the players when player 2 uses the strategy $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$.

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_1) = \frac{1}{2}(1 - \mu^2) \text{ and } U_2((\theta_H\theta_L, \theta_H + 1\theta_L), \theta_2) = 1 - \frac{1}{2}(1 - \mu^2),$$

$$U_1((\theta_H\theta_L + 1, \theta_H + 1\theta_L), \theta_1) = 1 - \mu \text{ and } U_2((\theta_H\theta_L + 1, \theta_H + 1\theta_L), \theta_2) = \mu,$$

$$U_1((\theta_H + 1\theta_L, \theta_H + 1\theta_L), \theta_1) = \frac{1}{2} = U_2((\theta_H + 1\theta_L, \theta_H + 1\theta_L), \theta_2),$$

$$U_1((\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L), \theta_1) = 1 - \mu(1 - \frac{1}{2} \cdot \mu)$$

and

$$U_2((\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L), \theta_2) = \mu(1 - \frac{1}{2} \cdot \mu).$$

By conditions (B.90)-(B.92) player 1's strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response to player 2's strategy $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$.

Finally, fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$ I calculate the players' expected payoffs.

$$U_1((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_1) = 0 \text{ and } U_2((\theta_H\theta_L, \theta_H + 1\theta_L + 1), \theta_2) = 1,$$

$$U_1((\theta_H\theta_L+1,\theta_H+1\theta_L+1),\theta_1)=\frac{1}{2}(1-\mu)^2$$

and

$$U_2((\theta_H\theta_L+1,\theta_H+1\theta_L+1),\theta_2)=1-\frac{1}{2}(1-\mu)^2,$$

$$U_1((\theta_H+1\theta_L,\theta_H+1\theta_L+1),\theta_1)=\mu(1-\frac{1}{2}\cdot\mu)$$

and

$$U_2((\theta_H+1\theta_L,\theta_H+1\theta_L+1),\theta_2)=1-\mu(1-\frac{1}{2}\cdot\mu)$$

$$U_1((\theta_H+1\theta_L+1,\theta_H+1\theta_L+1),\theta_1)=\frac{1}{2}=U_2((\theta_H+1\theta_L+1,\theta_H+1\theta_L+1),\theta_2).$$

By conditions (B.88) and (B.89) player 1's strategy $\sigma_1(\theta_1)=(\theta_H+1\theta_L+1)$ is a best response to player 2's strategy $\sigma_2(\theta_2)=(\theta_H+1\theta_L+1)$. I can now express the full Bayesian strategic form in Figure B.2.

Therefore I can conclude that as $\sigma_1(\theta_1)=(\theta_H+1\theta_L+1)$ is a best response to any strategy of player 2, and since player's payoffs are symmetric, that $\sigma_2(\theta_2)=(\theta_H+1\theta_L+1)$ is a best response to any strategy of player 1. Hence, the mutually consistent best responses form a pure strategy Nash equilibrium given by,

$$\sigma^*=(\sigma_1(\theta_1),\sigma_2(\theta_2))=(\theta_H+1\theta_L+1,\theta_H+1\theta_L+1).$$

Hence, whenever the players know there is only one state of nature, be this state $\{k\}=\{0\}$ or $\{k\}=\{1\}$, the pure strategy Nash equilibrium of the game is summarised by the target setting rule $t_i(\theta_i)=Q_i=\theta_i+k$.

Under Assumption 2 conditions (B.90)-(B.92) and (B.11)-(B.13) are sufficient for $\sigma=(\theta_H\theta_L,\theta_H\theta_L)$ to continue to be the Nash equilibrium when $\mathcal{K}=\{k\}=\{0\}$. When $\mathcal{K}=\{k\}=\{1\}$, the game is unchanged so the analysis above continues to hold. The Nash equilibrium under only Assumption 2 remains $\sigma=(\theta_H\theta_L,\theta_H\theta_L)$ as $\sigma_1(\theta_1)=(\theta_H\theta_L)$ remains a best response to every strategy of player 2. When $\sigma_2(\theta_2)=(\theta_H\theta_L)$

$$\frac{1}{2}>\mu(1-\frac{1}{2}\mu)\Leftrightarrow\frac{1}{2}(1-\mu)^2>0, \tag{B.11}$$

and

$$\frac{1}{2}>\frac{1}{2}(1-\mu^2)\ \Leftrightarrow\ \mu\in(0,1), \tag{B.12}$$

| | | **2** | | | |
|---|---|---|---|---|---|
| | | $\theta_H\theta_L$ | $\theta_H\theta_L+1$ | $\theta_H+1\theta_L$ | $\theta_H+1\theta_L+1$ |
| **1** | $\theta_H\theta_L$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2}\mu^2, 1-\frac{1}{2}\mu^2$ | $\frac{1}{2}(1-\mu^2), 1-\frac{1}{2}(1-\mu^2)$ | $0,1$ |
| | $\theta_H\theta_L+1$ | $1-\frac{1}{2}\mu^2, \frac{1}{2}\mu^2$ | $\frac{1}{2},\frac{1}{2}$ | $1-\mu,\mu$ | $\frac{1}{2}(1-\mu)^2, 1-\frac{1}{2}(1-\mu)^2$ |
| | $\theta_H+1\theta_L$ | $1-\frac{1}{2}(1-\mu^2), \frac{1}{2}(1-\mu^2)$ | $\mu, 1-\mu$ | $\frac{1}{2},\frac{1}{2}$ | $\mu(1-\frac{1}{2}\mu), 1-\mu(1-\frac{1}{2}\mu)$ |
| | $\theta_H+1\theta_L+1$ | $1,0$ | $1-\frac{1}{2}(1-\mu)^2, \frac{1}{2}(1-\mu)^2$ | $1-\mu(1-\frac{1}{2}\mu), \mu(1-\frac{1}{2}\mu)$ | $\frac{1}{2},\frac{1}{2}$ |

Figure B.2: Bayesian Normal Form when $\{k\}=1$.

and

$$\frac{1}{2} > \mu(1-\mu) \;\Leftrightarrow\; \mu \in (0,1). \tag{B.13}$$

Condition (B.13) holds as $\mu(1-\mu)$ is a concave function, maximised at $\mu^* = 1/2$ that yields a maximum value $V(1/2) = 1/4$. When $\sigma_2(\theta_2) = (\theta_H\theta_L+1)$ conditions (B.90) to (B.92) are sufficient. When $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$

$$1 - \frac{1}{2}(1-\mu^2) > \frac{1}{2} \;\Leftrightarrow\; \frac{1}{2} > \frac{1}{2}(1-\mu^2), \tag{B.14}$$

and

$$1 - \frac{1}{2}(1-\mu^2) > \mu \;\Leftrightarrow\; \frac{1}{2} > \mu(1-\frac{1}{2}\mu), \tag{B.15}$$

and

$$1 - \frac{1}{2}(1-\mu^2) > \mu(1-\frac{1}{2}\mu) \;\Leftrightarrow\; \frac{1}{2} > \mu(1-\mu). \tag{B.16}$$

Conditions (B.14), (B.15) and (B.16) hold by (B.12), (B.11) and (B.13), respectively. Finally, when $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$

$$1 - \mu(1-\mu) > 1 - \frac{1}{2}(1-\mu^2) \;\Leftrightarrow\; \frac{1}{2} > \mu(1-\frac{1}{2}\mu), \tag{B.17}$$

and

$$1 - \mu(1-\mu) > 1 - \mu(1-\frac{1}{2}\mu) \;\Leftrightarrow\; \mu(1-\frac{1}{2}\mu) > \mu(1-\mu), \tag{B.18}$$

and

$$1 - \mu(1-\mu) > \frac{1}{2} \;\Leftrightarrow\; \frac{1}{2} > \mu(1-\mu). \tag{B.19}$$

Condition (B.17) holds by (B.11) whilst (B.19) holds by (B.13), and (B.18) holds for any $\mu > 0$. □

*Proof of Proposition 3.* Under Assumption 2 and Assumption 3 $\sigma_i(\theta_i) = (\theta_H\theta_L+1)$ is a best response to $\sigma_j(\theta_j) = (\theta_H\theta_L)$, $\sigma_i(\theta_i) = (\theta_H+1\theta_L+1)$ is a best response to $\sigma_j(\theta_j) = (\theta_H\theta_L+1)$, $\sigma_i(\theta_i) = (\theta_H\theta_L)$ is a best response to $\sigma_j(\theta_j) = (\theta_H+1\theta_L)$ and $\sigma_i(\theta_i) = (\theta_H+1\theta_L)$ is a best response to $\sigma_j(\theta_j) = (\theta_H+1\theta_L+1)$. Consequently, for any $\mu \in (0,1)$ there are no mutually consistent best response and hence no pure strategy Nash equilibria exist. The strategic form of this game after substituting for $\lambda = 1/2$ is:

| | | **2** | | | |
|---|---|---|---|---|---|
| | | $\theta_H\theta_L$ | $\theta_H\theta_L+1$ | $\theta_H+1\theta_L$ | $\theta_H+1\theta_L+1$ |
| **1** | $\theta_H\theta_L$ | $\frac{1}{2},\frac{1}{2}$ | $B, A$ | $A, B$ | $\frac{1}{2},\frac{1}{2}$ |
| | $\theta_H\theta_L+1$ | $A, B$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{1}{2}$ | $B, A$ |
| | $\theta_H+1\theta_L$ | $B, A$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{1}{2}$ | $A, B$ |
| | $\theta_H+1\theta_L+1$ | $\frac{1}{2},\frac{1}{2}$ | $A, B$ | $B, A$ | $\frac{1}{2},\frac{1}{2}$ |

where

$$A = \frac{1}{2} + \frac{1}{2}\mu(1-\mu) > \frac{1}{2} > \frac{1}{2} + \frac{1}{2}\mu(\mu-1) = B.$$

Under Assumption 2 alone both $\sigma_i(\theta_i) = (\theta_H + 1\theta_L)$ and $\sigma_i(\theta_i) = (\theta_H + 1\theta_L + 1)$ are best responses to $\sigma_j(\theta_j) = (\theta_H\theta_L)$, $\sigma_i(\theta_i) = (\theta_H + 1\theta_L + 1)$ is a best response to $\sigma_j(\theta_j) = (\theta_H\theta_L + 1)$, every strategy of player $i$ constitute best responses to $\sigma_j(\theta_j) = (\theta_H + 1\theta_L)$ and both $\sigma_i(\theta_i) = (\theta_H + 1\theta_L)$ and $\sigma_i(\theta_i) = (\theta_H + 1\theta_L + 1)$ are best responses to $\sigma_j(\theta_j) = (\theta_H + 1\theta_L + 1)$. Therefore, for any $\mu \in (0, 1)$ the strategic form of this game is

| | | **2** | | | |
|---|---|---|---|---|---|
| | | $\theta_H\theta_L$ | $\theta_H\theta_L+1$ | $\theta_H+1\theta_L$ | $\theta_H+1\theta_L+1$ |
| **1** | $\theta_H\theta_L$ | $\frac{1}{2},\frac{1}{2}$ | $B, A$ | $\frac{1}{2},\frac{1}{2}$ | $B, A$ |
| | $\theta_H\theta_L+1$ | $A, B$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{1}{2}$ | $B, A$ |
| | $\theta_H+1\theta_L$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{1}{2}$ |
| | $\theta_H+1\theta_L+1$ | $A, B$ | $A, B$ | $\frac{1}{2},\frac{1}{2}$ | $\frac{1}{2},\frac{1}{2}$ |

and the set of pure strategy Nash Equilibria is given by

$$\{(\theta_H+1\theta_L, \theta_H+1\theta_L), (\theta_H+1\theta_L, \theta_H+1\theta_L+1), (\theta_H+1\theta_L+1, \theta_H+1\theta_L), (\theta_H+1\theta_L+1, \theta_H+1\theta_L)\}.$$

Note that as $\sigma_i(\theta_i) = (\theta_H + 1\theta_L)$ is weakly dominated by $\sigma_i(\theta_i) = (\theta_H + 1\theta_L + 1)$, hence by iterated elimination of weakly dominated strategies I remove three of the equilibria from the above set and are left with the unique pure strategy Nash Equilibrium $\sigma^{*IEWDS} = (\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L + 1)$. □

*Proof of Theorem 1.* I first show that the two inequalities

$$1 - 2\lambda > \mu, \tag{B.20}$$

and

$$\mu > 2\lambda, \tag{B.21}$$

imply that $\lambda < 1/4$. Setting (B.20) and (B.3) as equalities and substituting (B.3) into (B.20) yields,

$$1 - 2\lambda = 2\lambda \;\Leftrightarrow\; \frac{1}{4} = \lambda.$$

Since for the inequalities to hold I must have $1 - 2\lambda > 2\lambda$ I obtain $\lambda < 1/4$.

Now to show that these conditions are sufficient for the existence of a pure strategy Nash equilibrium in which players use the strategy $\sigma_i(\theta_i) = (\theta_H + 1\theta_L + 1)$ I compute player 1's best response for each of the four strategies of player 2. Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H\theta_L)$, the following conditions determine whether $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is player 1's best response:

$$1 - \lambda > \frac{1}{2} \;\Leftrightarrow\; \lambda < \frac{1}{2}, \tag{B.22}$$

$$1 - \lambda > \frac{1}{2} - \mu(\lambda - \frac{1}{2}\mu) \;\Leftrightarrow\; \frac{1}{2}(1 - \mu^2) > \lambda(1 - \mu) \;\Leftrightarrow\; \lambda < \frac{1}{2}, \tag{B.23}$$

$$1 - \lambda > 1 - \lambda(1 - \mu) - \frac{1}{2}\mu^2 \;\Leftrightarrow\; \mu > 2\lambda. \tag{B.24}$$

Conditions (B.22) and (B.23) are satisfied by the assumption of Case 2, whilst (B.24) is satisfied by the postulate of the proposition. Therefore $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response to $\sigma_2(\theta_2) = (\theta_H\theta_L)$.

Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$ the following conditions determine whether $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response:

$$1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda) > \frac{1}{2} \;\Leftrightarrow\; \lambda < \frac{1}{2}, \tag{B.25}$$

$$1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda) > \lambda(1 - \mu) + \frac{1}{2}\mu^2 \;\Leftrightarrow\; \lambda < \frac{1}{2}, \tag{B.26}$$

$$1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda) > \mu - \lambda(2\mu - 1) \;\Leftrightarrow\; \lambda < \frac{1}{2}. \tag{B.27}$$

Therefore, as conditions (B.25)-(B.27) are satisfied by assumption, $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response to $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$.

Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$ the following conditions determine whether $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response:

$$1 - \mu(1 - \frac{1}{2}\mu) - \lambda(1 - \mu) > \frac{1}{2} \;\Leftrightarrow\; \lambda < \frac{1}{2} \text{ and } \mu < 1 - 2\lambda, \tag{B.28}$$

$$1 - \mu(1 - \frac{1}{2}\mu) - \lambda(1 - \mu) > 1 - \mu + \lambda(2\mu - 1) \;\Leftrightarrow\; \lambda < \frac{1}{2} \text{ and } 2\lambda < \mu, \tag{B.29}$$

$$1 - \mu(1 - \frac{1}{2}\mu) - \lambda(1 - \mu) > 1 - \frac{1}{2} + \mu(\lambda - \frac{1}{2}\mu) \;\Leftrightarrow\; \lambda < \frac{1}{4}. \tag{B.30}$$

Note that if conditions (B.28) and (B.29) hold, which they do by the statement of the proposition, they imply that (B.30) holds. Therefore $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response to $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$.

Finally, fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$ the following conditions determine whether $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response:

$$\frac{1}{2} > \lambda, \tag{B.31}$$

$$\frac{1}{2} > \frac{1}{2} - \mu(1 - \frac{1}{2}\mu - \lambda) \;\Leftrightarrow\; \lambda < \frac{1}{2} \tag{B.32}$$

$$\frac{1}{2} > \mu(1 - \frac{1}{2}\mu) + \lambda(1 - \mu) \;\Leftrightarrow\; \lambda < \frac{1}{2} \text{ and } \mu < 1 - 2\lambda. \tag{B.33}$$

Conditions (B.31) and (B.32) are satisfied by the assumption of Case 2, whilst (B.33) is satisfied by the postulate of the proposition. Therefore $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response to $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$.

Hence, when $\mu \in (2\lambda, 1 - 2\lambda)$, $\sigma_i(\theta_i) = (\theta_H + 1\theta_L + 1)$ is a best response to all of player $j$'s strategies. By symmetry of the players' payoffs I can conclude that when player $j$ also uses the strategy $\sigma_j(\theta_j) = (\theta_H + 1\theta_L + 1)$ I have arrived at the unique pure strategy Nash equilibrium.

I first show that the two inequalities

$$2\lambda - 1 > \mu, \tag{B.34}$$

and

$$\mu > 2(1-\lambda), \tag{B.35}$$

imply that $\lambda > 3/4$. Setting (B.34) and (B.35) as equalities and substituting (B.35) into (B.34) yields,

$$2\lambda - 1 = 2(1-\lambda) \;\Leftrightarrow\; \frac{3}{4} = \lambda.$$

Since for the inequalities to hold I must have $2\lambda - 1 > 2(1-\lambda)$ I obtain $\lambda > 3/4$.

Now to show that these conditions are sufficient for the existence of a pure strategy Nash equilibrium in which players use the strategy $\sigma_i(\theta_i) = (\theta_H \theta_L)$ I compute player 1's best response for each of the four strategies of player 2 and look for mutually consistent best responses. Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H \theta_L)$, the following conditions determine whether $\sigma_1(\theta_1) = (\theta_H \theta_L)$ is player 1's best response:

$$\frac{1}{2} > 1 - \lambda \;\Leftrightarrow\; \lambda > \frac{1}{2}, \tag{B.36}$$

$$\frac{1}{2} > \frac{1}{2} - \mu(\lambda - \frac{1}{2}\mu) \;\Leftrightarrow\; \lambda > \frac{1}{2}, \tag{B.37}$$

$$\frac{1}{2} > 1 - \lambda(1-\mu) - \frac{1}{2}\mu^2 \;\Leftrightarrow\; \lambda > \frac{1}{2} \text{ and } 2\lambda - 1 > \mu. \tag{B.38}$$

Conditions (B.36) and (B.37) hold by assumption, whilst (B.38) holds by a combination of my assumption on $\lambda$ and the postulate of the proposition that $\mu \in (2(1-\lambda), 2\lambda - 1)$. Hence, $\sigma_1(\theta_1) = (\theta_H \theta_L)$ is a best response to $\sigma_2(\theta_2) = (\theta_H \theta_L)$.

Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H \theta_L + 1)$, the following conditions determine whether player 1's strategy $\sigma_1(\theta_1) = (\theta_H \theta_L)$ is a best response to this choice of strategy by player 2:

$$\lambda(1-\mu) + \frac{1}{2}\mu^2 > \frac{1}{2} \;\Leftrightarrow\; \lambda > \frac{1}{2} \text{ and } 2\lambda - 1 > \mu, \tag{B.39}$$

$$\lambda(1-\mu) + \frac{1}{2}\mu^2 > \mu - \lambda(2\mu - 1) \;\Leftrightarrow\; \lambda > \frac{1}{2} \text{ and } \mu > 2(1-\lambda), \tag{B.40}$$

$$\lambda(1-\mu) + \frac{1}{2}\mu^2 > 1 - \frac{1}{2} + \mu(1 - \frac{1}{2}\mu - \lambda) \;\Leftrightarrow\; \lambda > \frac{3}{4}. \tag{B.41}$$

Conditions (B.39) and (B.40) are satisfied by the postulate of the proposition, this postulate ensures $\lambda > 3/4$ by bounding $\mu \in (2(1-\lambda), 2\lambda - 1)$.

Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$, the following conditions determine whether player 1's strategy $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is a best response to this choice of strategy by player 2:

$$1 - \frac{1}{2} + \mu(\lambda - \frac{1}{2}\mu) > 1 - \mu + \lambda(2\mu - 1) \;\Leftrightarrow\; \lambda > \frac{1}{2}, \tag{B.42}$$

$$1 - \frac{1}{2} + \mu(\lambda - \frac{1}{2}\mu) > \frac{1}{2} \;\Leftrightarrow\; \lambda > \frac{1}{2}, \tag{B.43}$$

$$1 - \frac{1}{2} + \mu(\lambda - \frac{1}{2}\mu) > 1 - \mu(1 - \frac{1}{2}\mu) - \lambda(1 - \mu) \;\Leftrightarrow\; \lambda > \frac{1}{2}. \tag{B.44}$$

Therefore as conditions (B.42)-(B.44) hold by assumption I can conclude that $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is a best response to $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$.

Finally, fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$, the following conditions determine whether player 1's strategy $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is a best response to this choice of strategy by player 2:

$$\lambda > \frac{1}{2} - \mu(1 - \frac{1}{2}\mu - \lambda) \;\Leftrightarrow\; \lambda > \frac{1}{2}, \tag{B.45}$$

$$\lambda > \mu(1 - \frac{1}{2}\mu) + \lambda(1 - \mu) \;\Leftrightarrow\; \lambda > \frac{1}{2} \text{ and } \mu > 2(1 - \lambda). \tag{B.46}$$

Condition (B.45) is satisfied by assumption and (B.46) holds due to the postulate that $\mu \in (2(1 - \lambda), 2\lambda - 1)$.

Hence, when $\mu \in (2(1 - \lambda), 2\lambda) - 1$, $\sigma_i(\theta_i) = (\theta_H\theta_L)$ is a best response to all of player $j$'s strategies. By symmetry of the players' payoffs I can conclude that when player $j$ also uses the strategy $\sigma_j(\theta_j) = (\theta_H\theta_L)$ I have arrived at the unique pure strategy Nash equilibrium. □

*Proof of Theorem 2.* I first show that in comparison to Theorem 1 $\lambda \in (1/2, 1)$ and $\mu \in (0, 1)$ is sufficient for $\sigma^* = (\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L + 1)$ to be the unique pure strategy Nash equilibrium. I begin by holding player 2's strategy constant at $\sigma_2(\theta_2) = (\theta_H\theta_L)$. In this case player 1's strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ is a best response if

$$1 - \lambda(1 - \mu(1 - \mu)) > \frac{1}{2} \;\Leftrightarrow\; \frac{1}{2} > \lambda(1 - \mu(1 - \mu)) \;\Leftrightarrow\; \lambda \in (0, \frac{1}{2}] \text{ and } \mu \in (0, 1), \tag{B.47}$$

$$1-\lambda(1-\mu(1-\mu)) > 1-\lambda(1-\mu)-\frac{1}{2}\mu^2 \;\Leftrightarrow\; \frac{1}{2}\mu^2 > \lambda\mu^2 \;\Leftrightarrow\; \lambda \in (0, \frac{1}{2}) \text{ and } \mu \in (0,1), \tag{B.48}$$

and

$$1-\lambda(1-\mu(1-\mu)) > 1-\frac{1}{2}+\mu^2(\frac{1}{2}-\lambda) \;\Leftrightarrow\; \frac{1}{2}(1-\mu^2) > \lambda(1-\mu^2)$$
$$\Leftrightarrow\; \lambda \in (0, \frac{1}{2}] \text{ and } \mu \in (0,1). \tag{B.49}$$

Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$ the conditions for $\sigma_1(\theta_H) = (\theta_H + 1\theta_L + 1)$ to be the best response are unchanged from the proof of Theorem 1 and are given by (B.25)-(B.27), which require that $\lambda \in (1/2, 1)$. Changing player 2's fixed strategy to $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$ player 1's strategy $\sigma_1(\theta_1) = (\theta_H + 1\theta_L)$ is the best response if

$$1-\lambda+\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1) > \frac{1}{2} \;\Leftrightarrow\; \frac{1}{2}-\lambda+\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1) > 0$$
$$\Leftrightarrow\; \lambda \in (0, \frac{1}{2}) \text{ and } \mu \in (0,1), \tag{B.50}$$

$$1-\lambda+\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1) > 1-\mu+\lambda(2\mu-1) \;\Leftrightarrow\; \frac{1}{2}\mu^2 > \lambda\mu^2$$
$$\Leftrightarrow \lambda \in (0, \frac{1}{2}) \text{ and } \mu \in (0,1), \tag{B.51}$$

and

$$1-\lambda+\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1) > \frac{1}{2}-\mu^2(\frac{1}{2}-\lambda) \;\Leftrightarrow\; \frac{1}{2}-\lambda+\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1) > -\mu^2(\frac{1}{2}-\lambda)$$
$$\Leftrightarrow\; \lambda \in (0, \frac{1}{2}) \text{ and } \mu \in (0,1). \tag{B.52}$$

Finally, consider the case when player 2's strategy is fixed at $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$. Player 1's best response is $\sigma_1(\theta_1) = (\theta_H + 1\theta_L + 1)$ if

$$\frac{1}{2} > \lambda-\mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1) \;\Leftrightarrow\; \lambda \in (0, \frac{1}{2}) \text{ and } \mu \in (0,1), \tag{B.53}$$

$$\frac{1}{2} > \frac{1}{2}-\mu(1-\frac{1}{2}\mu-\lambda) \;\Leftrightarrow\; 0 > -\mu(1-\frac{1}{2}\mu-\lambda) \;\Rightarrow\; \lambda \in (0, \frac{1}{2}) \text{ and } \mu \in (0,1), \tag{B.54}$$

$$\frac{1}{2} > \lambda(1-\mu(1-\mu))\ \lambda \in (0, \frac{1}{2}] \text{ and } \mu \in (0,1). \tag{B.55}$$

Hence, when $\lambda \in (0, 1/2)$ conditions (B.47)-(B.55) are satisfied. This implies that $\sigma_i(\theta_i) = (\theta_H + 1\theta_L + 1)$ is a best response to all of player $j$'s strategies, and since the player's payoffs are symmetric I have that the unique pure strategy Nash equilibrium is $\sigma^* = (\theta_H + 1\theta_L + 1, \theta_H + 1\theta_L + 1)$.

By Theorem 1 I know that $\mu \in (2(1-\lambda), 2\lambda - 1)$ implies that $\lambda \in (3/4, 1)$. I now show that this remains sufficient for $\sigma^* = (\theta_H\theta_L, \theta_H\theta_L)$ to remain the unique pure strategy Nash equilibrium. Fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H\theta_L)$ the conditions for player 1's strategy of $\sigma_1(\theta_1) = (\theta_H\theta_L)$ to be the best response are

$$\frac{1}{2} > 1 - \lambda(1-\mu) - \frac{1}{2}\mu^2 \ \Leftrightarrow\ \lambda \in (\frac{1}{2}, 1) \text{ and } \mu \in (0, 2\lambda - 1), \tag{B.56}$$

$$\frac{1}{2} > 1 - \frac{1}{2} + \mu^2(\frac{1}{2} - \lambda) \ \Leftrightarrow\ 0 > \mu^2(\frac{1}{2} - \lambda) \ \Leftrightarrow\ \lambda \in (\frac{1}{2}, 1) \text{ and } \mu \in (0,1), \tag{B.57}$$

and

$$\frac{1}{2} > 1 - \lambda(1-\mu(1-\mu)) \ \Leftrightarrow\ \lambda \in (\frac{2}{3}, 1) \text{ and } \mu \in (0,1). \tag{B.58}$$

As in the first part of the proof when player 2's strategy is held fixed at $\sigma_2(\theta_2) = (\theta_H\theta_L + 1)$ conditions (B.39)-(B.41) continue to imply that $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is the best response. These conditions require that $\mu \in (2(1-\lambda), 2\lambda - 1)$, which in turn implies $\lambda \in (3/4, 1)$. Now changing player 2's strategy to $\sigma_2(\theta_2) = (\theta_H + 1\theta_L)$ the conditions for player 1's strategy $\sigma_1(\theta_1) = (\theta_H\theta_L)$ to be the best response are

$$\frac{1}{2} - \mu^2(\frac{1}{2} - \lambda) > 1 - \mu + \lambda(2\mu - 1) \ \Leftrightarrow\ \lambda \in (\frac{1}{2}, 1) \text{ and } \mu \in (0,1), \tag{B.59}$$

$$\frac{1}{2} - \mu^2(\frac{1}{2} - \lambda) > \frac{1}{2}) \ \Leftrightarrow\ \lambda \in (\frac{1}{2}, 1) \text{ and } \mu \in (0,1), \tag{B.60}$$

and

$$\frac{1}{2} - \mu^2(\frac{1}{2} - \lambda) > 1 - \lambda + \mu(\mu(\frac{1}{2} - \lambda) + 2\lambda - 1)) \ \Leftrightarrow\ \lambda \in (\frac{1}{2}, 1) \text{ and } \mu \in (0,1). \tag{B.61}$$

Finally, fixing player 2's strategy as $\sigma_2(\theta_2) = (\theta_H + 1\theta_L + 1)$ player 1's strategy of $\sigma_1(\theta_1) = (\theta_H\theta_L)$ is the best response if the following conditions hold

$$\lambda(1-\mu(1-\mu)) > \frac{1}{2} - \mu(1-\frac{1}{2}-\lambda)) \;\Leftrightarrow\; \lambda \in (\frac{1}{2},1) \text{ and } \mu \in (0,1), \tag{B.62}$$

$$\lambda(1-\mu(1-\mu)) > \lambda - \mu(\mu(\frac{1}{2}-\lambda)+2\lambda-1) \;\Leftrightarrow\; \lambda \in (\frac{1}{2},1) \text{ and } \mu \in (2(1-\lambda),1), \tag{B.63}$$

and

$$\lambda(1-\mu(1-\mu)) > \frac{1}{2} \;\Leftrightarrow\; \lambda \in (\frac{2}{3},1) \text{ and } \mu \in (0,1). \tag{B.64}$$

Hence, when $\mu \in (2(1-\lambda), 2\lambda-1)$ conditions (B.56)-(B.64) are satisfied as this restriction on $\mu$ implies that $\lambda = 3/4 > 2/3 > 1/2$. This implies that $\sigma_i(\theta_i) = (\theta_H\theta_L)$ is a best response to all of player $j$'s strategies, and since the player's payoffs are symmetric I have that the unique pure strategy Nash equilibrium is $\sigma^* = (\theta_H\theta_L, \theta_H\theta_L)$. □

## B.2 Chapter 2 Proofs

*Proof of Lemma 1.* This follows from the competitiveness of investors. Assume not, one investor, $j$, bids with a price $p < \mathbb{E}[\rho(\theta|e)]$ and earns positive profits. But then as the measure of investors is strictly positive, there exists another investor, $k$, who bids $p + \epsilon \leq \mathbb{E}[\rho(\theta|e)]$ who will still make positive profits as the originator will choose to sell the security to investor $k$. Again because of the positive measure of investors this process continues until $\epsilon = 0$ and $p = \mathbb{E}[\rho(\theta|e)]$. Conversely if $p > \mathbb{E}[\rho(\theta|e)]$ then investors make negative profits in expectation, this is a dominated strategy as each investor can earn zero by posting the zero bid. Hence, in equilibrium, $p = \mathbb{E}[\rho(\theta|e)]$.

□

*Proof of Lemma 2.* To show that, if an effort-signalling equilibrium arises, the participation constraints of the originator are satisfied for each choice of effort, conditions under which the participation constraints are satisfied are derived. These conditions turn out to be satisfied by a combination of Assumption 6 and equilibrium condition 3. The originator's participation constraints are given below for low and high effort, respectively

$$u(e_L, \alpha(e_L); p(\alpha(e_L))) \geq u(e_L, 1; 0), \tag{PC1}$$

$$u(e_H, \alpha(e_H); p(\alpha(e_H))) \geq u(e_H, 1; 0). \tag{PC2}$$

Then expanding these two constraints using the functional form specified by (2.2) gives

$$\begin{gathered}
\rho(\theta|e_L) \cdot \left( -\sum_{i=1}^{k} l_i + \alpha(e_L)\delta \sum_{i=1}^{k} r_i + p(\alpha(e_L)) \cdot (1 - \alpha(e_L)) \right) \\
+(1 - \rho(\theta|e_L) \cdot \left( -\sum_{i=1}^{k} l_i + p(\alpha(e_L)) \cdot (1 - \alpha(e_L)) \right) \\
\geq \\
\rho(\theta|e_L) \cdot \left( -\sum_{i=1}^{k} l_i + \delta \sum_{i=1}^{k} r_i \right) \\
+(1 - \rho(\theta|e_L) \cdot \left( -\sum_{i=1}^{k} l_i \right).
\end{gathered} \tag{PC1}$$

The left hand side of (PC1) is the originator's expected payoff from entering the secondary market to sell a positive fraction of the security, if she chooses low effort, whilst the right hand side is her expected payoff from retaining the entire asset pool. (PC2), given below, is analogous to (PC1) except that the inequality is defined for when the originator has chosen high effort, and moreover, due to Assumption 5 with high effort the originator's payoff is no longer an expectation.

$$-\sum_{i=1}^{k} l_i+\alpha(e_H)\delta\sum_{i=1}^{k} r_i+p(\alpha(e_H))\cdot(1-\alpha(e_H))-c(e_H) \geq -\sum_{i=1}^{k} l_i+\delta\sum_{i=1}^{k} r_i-c(e_H). \tag{PC2}$$

Both inequalities can be simplified, normalizing $\sum_{i=1}^{k} r_i = 1$, to

$$p(\alpha(e_i)) \geq \delta \cdot \rho(\theta|e_i) \ i \in \{L, H\}. \tag{PCi}$$

Then using equilibrium condition 3 the effort-signalling equilibrium prices can be substituted into (PC1) and (PC2) to give

$$\rho(\theta|e_i) \geq \delta \cdot \rho(\theta|e_i) \ i \in \{L, H\}. \tag{PCi}$$

Both inequalities can be now be simplified into one inequality that represents the participation constraints under each choice of effort

$$1 \geq \delta. \tag{PCi}$$

Both (PC1) and (PC2) hold with a strict inequality due to Assumption 6. That is, when prices are greater than the discounted retained value, as in an effort-signalling equilibrium, and the the originator has a strict preference for liquidity then her participation constraint holds with a strict inequality. □

*Proof of Proposition 4.* If there is no signalling then the originator retains the entire security issue. In this case, which determines the originator's reservation payoff, the originator has the following simplified payoffs for a choice of high and low effort, respectively,

$$u(e_H, 1; 0) = -\sum_{i=1}^{k} l_i + \delta\sum_{i=1}^{k} r_i - c(e),$$

$$u(e_L, 1; p) = \rho(\theta|e_L) \cdot (-\sum_{i=1}^{k} l_i + \delta\sum_{i=1}^{k} r_i) + (1 - \rho(\theta|e_L)) \cdot (-\sum_{i=1}^{k} l_i).$$

Therefore the choice of effort is determined by the inequality

$$\delta > c(e)/\underbrace{(1-\rho(\theta|e_L))}_{\text{pr. of default}}. \tag{B.65}$$

Assume that (B.65) is satisfied, so that in lieu of securitisation the originator chooses high effort. Note however this assumption is not necessary for what follows.

If we now suppose that there is an effort-signalling equilibrium with $\alpha_H^* > 0$ and $\alpha_L^* = 0$ as the optimal retention strategies for each choice of effort[2], then given equilibrium condition 3 the originators effort-signalling payoffs are, after some simplification,

$$u(e_H, \alpha_H^*; p(\alpha_H^*)) = \delta\alpha_H^* + (1-\alpha_H^*) - c(e)$$

$$u(e_L, 0; p(\alpha_L^*)) = \rho(\theta|e_L)$$

Thus the originator strictly prefers the effort-signalling equilibrium irrespective of the choice of effort for all $\delta < 1$ and $\alpha_H^* < 1$. I will now describe how allowing agents to deviate across 'types', as in this model, prevents pooling from arising. Secondly, I will describe how even if the usual notion of fixed type is adopted, then with an additional justifiable assumption, pooling equilibria cease to exist. Suppose now that there is a pooling equilibrium at

$$\alpha_H^* = \alpha_L^* = 0$$

As[3] retention now communicates no information regarding originator effort, investors offer the price

$$p = \mathbb{E}[\rho(\theta|e)]$$

which obtains for the originator the following payoffs

$$u(e_H, 0; \mathbb{E}[\rho(\theta|e)]) = \mathbb{E}[\rho(\theta|e)] - c(e) \tag{B.66}$$

$$u(e_L, 0; \mathbb{E}[\rho(\theta|e)]) = \mathbb{E}[\rho(\theta|e)] \tag{B.67}$$

There are now two cases to be studied

[2] This supposition turns out to be accurate.

[3] Note that pooling equilibrium with $\alpha_H^* = \alpha_L^* > 0$ are Pareto dominated by the 0 equilibrium.

1. Agents can deviate across type
2. Agents can deviate across action

**Case 1**

This is the scenario of the securitisation game with an effort choice proposed in this chapter. The originator makes a choice between high and low effort, which can be interpreted as choosing her type. Thus in the pooling equilibrium the originator computes that

$$u(e_L, 0; \mathbb{E}[\rho(\theta|e)]) > u(e_H, 0; \mathbb{E}[\rho(\theta|e)])$$

and so she would select low effort in this pooling equilibrium. However, this choice means that the market is composed entirely of low quality assets while investors pay a price greater than the value of the asset, violating the zero profit condition and collapsing the proposed pooling equilibrium. This endogenous no-pooling outcome arises because the originator is able to 'switch' types unlike in a normal signalling game.

**Case 2**

If we suppose now that there are two originators and that their types are fixed, as in the canonical signalling model, pooling can be ruled out by an additional assumption. This assumption, which can be justified conceptually, arises from the fact that for each choice of effort the originator has a heterogenous participation constraint. To see why, in order for pooling to be an equilibrium the originator's payoff must be at least as large under pooling than when there is no signalling. Hence the condition for there to be no pooling is

$$u(e_H, 1; 0) > u(e_H, 0; \mathbb{E}[\rho(\theta|e)])$$

This simplifies to the key condition

$$\delta > \mathbb{E}[\rho(\theta|e)] \tag{B.68}$$

Condition (B.68) states that the discounted value of the high quality asset is greater than the undiscounted expected value of the asset. This condition is, in this authors opinion, conceptually justifiable as it implies that, unless the originator can signal her effort, if she chooses high effort she is better of retaining the assets until she can get a price closer to the true value. This helps motivate the moral hazard aspect of the story as without a suitable mechanism in place the originator will not be incentivised to sell her high quality assets as securities if she chooses high effort. □

*Proof of Proposition 5.* The proof of Proposition 5 will proceed in several steps. First I will show that if the originator's participation constraints are satisfied, as per Lemma 2, then if the originator chooses low effort her optimal strategy is to sell the entire security issue, retaining zero. Second, as a result of this prior step, the incentive compatible set can be reduced to the local incentive compatible set, the boundaries of which are then derived. I will then show, using the Kuhn-Tucker conditions, that the minimum boundary of this set is the level of retention that both maximises the originator's payoff under high effort and allows her to signal this effort. I then derive a condition on the originator's liquidity preference that, when satisfied, will incentivise the originator to choose high effort in equilibrium. Lastly, some comparative statics properties of the minimum boundary of the incentive compatible set are shown.

As the originator's participation constraints are satisfied, if she chooses low effort she can do no better than by selling the entire asset. Suppose not, then the originator could improve her payoff with a deviation to $\hat{\alpha}(e_L) > 0$. However, with effort-signalling prices we have $u_\alpha < 0$ and so the originator, if she has chosen low effort, can do no better than to sell the full security as otherwise she misses out on gains from trade as $\delta < 1$ and hence $p(\alpha(e_L)) > \delta\rho(\theta|e_L)$.

Thus if the originator chooses high effort, she can fix $\alpha(e_L) = 0$ in order to define local incentive compatibility constraints (IC1) and (IC2). Taking (IC1) and low effort first, we can expand using the functional form (2.2) to give

$$\begin{gathered} -\sum_{i=1}^{k} l_i + p(\alpha(e_L)) \\ \geq \\ \rho(\theta|e_L)\cdot\Big(-\sum_{i=1}^{k} l_i + \alpha(e_H)\delta\sum_{i=1}^{k} r_i + p(\alpha(e_H))\cdot(1-\alpha(e_H))\Big) \\ +(1-\rho(\theta|e_L)\cdot\Big(-\sum_{i=1}^{k} l_i + p(\alpha(e_H))\cdot(1-\alpha(e_H))\Big). \end{gathered} \tag{IC1}$$

The left hand side is the payoff to the originator that occurs from following her strategy of issuing the entire security under low effort, whilst the right hand side is the expected payoff that occurs if she retains some positive amount. If this condition is satisfied then the originator prefers a truthful retention strategy in the post effort game if she chooses low effort. (IC1) can be re-arranged to give an implicitly defined lower bound on the levels of retention such that (IC1) holds.

$$\alpha(e_H) \geq \frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - (\delta \cdot \rho(\theta|e_L)}. \tag{IC1}$$

This allows one to define the lower boundary as featured in the text

$$\underline{\Phi} = \frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - (\delta \cdot \rho(\theta|e_L)}$$
$$= \min\left\{\alpha(e_H) \in \mathbb{R}_+ : u(e_L, 0; p(0)) \geq u(e_L, \alpha(e_H); p(\alpha(e_H)))\right\}.$$

Now turning to (IC2) and high effort, the expanded form is

$$-\sum_{i=1}^{k} l_i + \alpha(e_H)\delta \sum_{i=1}^{k} r_i + p(\alpha(e_H)) \cdot (1 - \alpha(e_H)) - c(e_H) \geq -\sum_{i=1}^{k} l_i + p(\alpha(e_L)) - c(e_H).$$

Thus

$$\alpha(e_H) \leq \frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - \delta}, \tag{IC2}$$

and we can define

$$\overline{\Phi} = \frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - \delta}$$
$$= \max\left\{\alpha(e_H) \in \mathbb{R}_+ : u(e_H, \alpha(e_H); p(\alpha(e_H))) \geq u(e_H, 0; p(0))\right\}.$$

The originators problem is to

$$\max_{\alpha(e_H) \in (\Lambda^F \cap \Lambda)} u\left(e_H, \alpha(e_H); p(\alpha(e_H))\right).$$

Now substitute the values for equilibrium prices into the originators payoff using equilibrium condition 3 $p(\alpha(e_i)) = \rho(\theta|e_i)$, which leads to the following Lagrangian, with $\lambda$ and $\mu$ as the Lagrange multipliers for (IC1) and (IC2), respectively,

$$L(\alpha(e_H), \lambda, \mu) = -\sum_{i=1}^{k} l_i + \delta\alpha(e_H) \sum_{i=1}^{k} r_i + 1 \cdot (1 - \alpha(e_H)) - c(e_H)$$
$$+ \lambda(\rho(\theta|e_L) - 1 - \alpha(e_H)(\delta\rho(\theta|e_L) - 1)) + \mu(1 - \rho(\theta|e_L) + \alpha(e_H)(\delta - 1)).$$

With associated first-order condition and complementary-slackness conditions:

$$\frac{\partial L(\alpha(e_H), \lambda, \mu)}{\partial \alpha} = \delta - 1 - \lambda\delta\rho(\theta|e_L) + \lambda + \mu(\delta - 1) = 0, \tag{B.69}$$

$$\lambda(\rho(\theta|e_L) - 1 - \alpha(e_H)(\delta\rho(\theta|e_L) - 1)) = 0, \tag{B.70}$$

$$\mu(1 - \rho(\theta|e_L) + \alpha(e_H)(\delta - 1)) = 0. \tag{B.71}$$

Rearranging (B.69) gives

$$\lambda \underbrace{(1 - \delta\rho(\theta|e_L))}_{>0} + \mu \underbrace{(\delta - 1)}_{<0} = \underbrace{1 - \delta}_{>0}. \tag{B.72}$$

Equation (B.81) implies that $\lambda > 0$. In turn this implies that (IC1) binds as otherwise the complementary slackness condition does not hold. Rearranging the binding (IC1) for the optimal level of effort signalling retention,

$$\alpha^*(e_H) = \frac{1 - \rho(\theta|e_L)}{1 - (\delta \cdot \rho(\theta|e_L))} \equiv \underline{\Phi}^*(\delta, p^*(\alpha)). \tag{B.73}$$

Substituting (B.73) into the complementary slackness condition (B.71),

$$\mu(1 - \rho(\theta|e_L) + (\frac{1 - \rho(\theta|e_L)}{1 - (\delta \cdot \rho(\theta|e_L)})(\delta - 1) = 0, \tag{B.74}$$

as

$$1 > \rho(\theta|e_L) - (\frac{1 - \rho(\theta|e_L)}{1 - (\delta \cdot \rho(\theta|e_L)})(\delta - 1) \quad \forall \quad \rho(\theta|e_L) < 1, \; \delta < 1.$$

(B.74) implies that $\mu = 0$ so that condition (B.71) holds. Then as $\mu = 0$ this in turn implies that (IC2) holds with a strict inequality: $u(e_H, \alpha(e_H); p(\alpha(e_H))) > u(e_H, 0; p(0))$. Substituting $\mu = 0$ into condition (B.69) implies that $\lambda = \frac{1-\delta}{1-(\delta \cdot \rho(\theta|e_L))}$.

To derive the equilibrium effort choice condition I compare the originator's effort-signalling equilibrium payoffs in the case of high and low underwriting effort. Then using this condition I re-arrange to a form with the liquidity preference solely on the right hand side.

$$u(e_H, \alpha^*(e_H); p^*(\alpha)) \geq u(e_L, 0; p^*(\alpha)). \tag{B.75}$$

Expanding using the functional form

$$-\sum_{i=1}^{k} l_i + \delta\alpha^*(e_H)\sum_{i=1}^{k} r_i + 1\cdot(1-\alpha^*(e_H)) - c(e_H) \geq -\sum_{i=1}^{k} l_i + \rho(\theta|e_L).$$

Substitute for $\alpha^*(e_H)$,

$$-\sum_{i=1}^{k} l_i+\delta\frac{(1-\rho(\theta|e_L))}{(1-(\delta\cdot\rho(\theta|e_L)))}\sum_{i=1}^{k} r_i+(1-\frac{(1-\rho(\theta|e_L))}{(1-(\delta\cdot\rho(\theta|e_L)))})-c(e_H) \geq -\sum_{i=1}^{k} l_i+\rho(\theta|e_L).$$

Simplifying terms leads to the following

$$\delta(1-\rho(\theta|e_L))-(1-\rho(\theta|e_L))+(1-\rho(\theta|e_L))(1-\delta\cdot\rho(\theta|e_L)) \geq c(e_H)(1-\delta\cdot\rho(\theta|e_L)).$$

Re-arranging and further simplification gives the result

$$\delta > \frac{1}{(1+(c(e_H)+\rho(\theta|e_L)-2)\cdot\rho(\theta|e_L)}\cdot c(e_H).$$

Denoting $(1+(c(e_H)+\rho(\theta|e_L)-2)\cdot\rho(\theta|e_L) = \vartheta$,

$$\delta > \frac{1}{\vartheta}\cdot c(e_H) \Leftrightarrow \delta > \vartheta^{-1}\cdot c(e_H).$$

To establish the comparative statics properties of $\underline{\Phi}(\delta, p^*(\alpha)$ first note that (B.73) is continuous in both $\delta$ and $\rho(\theta|e_L)$ over the interval $(0,1)$,

$$\begin{aligned}\frac{\partial\underline{\Phi}(\delta,p^*(\alpha)}{\partial\rho(\theta|e_L)} &= \frac{(-1)(1-\delta\rho(\theta|e_L))-[(-\delta)(1-\rho(\theta|e_L))]}{(1-\delta\rho(\theta|e_L))^2}\\ &= \frac{\delta-1}{(\delta\rho(\theta|e_L)-1)^2} < 0 \;\; \forall \;\; \delta<1,\end{aligned}$$

and

$$\begin{aligned}\frac{\partial\underline{\Phi}(\delta,p^*(\alpha)}{\partial\delta} &= (1-\rho(\theta|e_L)(1-\delta\rho(\theta|e_L))^{-2}(-1)(-\rho(\theta|e_L))\\ &= \frac{\rho(\theta|e_L)-\rho(\theta|e_L)^2}{(\delta\rho(\theta|e_L)-1)^2} > 0 \;\; \forall \;\; \rho(\theta|e_L)<1.\end{aligned}$$

□

*Proof of Proposition 6.* To derive a condition which will ensure the existence of an effort signalling equilibrium recall that the solution to both (2.5) and (2.7) involves the originator choosing a level of retention equal to the lower boundary of the incentive compatible set. Hence, if the participation constraints are satisfied, a condition which will lead to an effort signalling equilibrium is that the lower boundary of the incentive compatible set is within the feasible set. That is, it must be nonnegative and strictly less than one. Thus our condition is

$$\min\left\{\alpha(e_H) \in \mathbb{R}_+ : u(e_L, \mu; p(\mu)) \geq u(e_L, \alpha(e_H); p(\alpha(e_H)))\middle|\ \mu \in (0,1)\right\} = \underline{\Phi}\,(\mu) < 1.$$

Substituting for the functional form of $\underline{\Phi}\,(\mu)$ that is given in the proof of Proposition 7

$$\frac{p(\alpha(e_H)) - p(\alpha(e_L)) + \mu \cdot (p(\alpha(e_L)) - \delta \cdot \rho(\theta|e_L))}{p(\alpha(e_H)) - \delta \cdot \rho(\theta|e_L)} < 1. \tag{B.76}$$

Condition (B.76) simplifies to

$$\mu p(\alpha(e_L)) - \mu\delta\rho(\theta|e_L) < p(\alpha(e_L)) - \delta\rho(\theta|e_L). \tag{B.77}$$

One can simplify (B.77) to give

$$\delta \cdot \rho(\theta|e_L) < p(\alpha(e_L)).$$

This condition states that the price paid by an investor for the low effort security must be strictly greater than the payoff the originator would receive by retaining the asset. This strengthens the Participation Constraint of a low effort originator, replacing the weak relation with strict. In the model this condition is satisfied by the assumption of competition between investors, or equivalently, of prices equal to the true value. Moreover, this condition is equivalent to $u_{\alpha(e_L)}(e_L, \alpha(e_L); p(\alpha(e_L))) < 0$. □

*Proof of Corollary 1.* If the participation constraints are satisfied with equality then

$$u(e_L, \alpha(e_L); p(\alpha(e_L))) = u(e_L, 1; 0), \tag{PC1}$$

$$u(e_H, \alpha(e_H); p(\alpha(e_H))) = u(e_H, 1; 0). \tag{PC2}$$

The prices that satisfies these conditions are

$$p(\alpha(e_i)) = \delta \cdot \rho(\theta|e_i).$$

Hence, the condition for existence $p(\alpha(e_L)) > \delta\rho(\theta|e_L)$ is not satisfied. □

*Proof of Proposition 7.* The first part of the proof of Proposition 7 proceeds in the same manner as the proof of Proposition 5. I derive incentive compatible boundaries, which now depend on $\mu$, and show that, the originator, by continuing to retain the amount dictated by the intersection of her payoff function and the newly defined lower boundary maximises her payoff. Secondly I will show that with a positive lower bound on the level of retention, the minimum least-cost effort signalling level of retention is strictly greater than that derived in Proposition 5. In order to show this I will derive a sufficient condition for this to occur and then demonstrate that this condition always holds under the assumptions of the model.

The local incentive compatibility constraints, when $\mu > 0$, are given by

$$u(e_L, \mu; p(\mu)) \geq u(e_L, \alpha(e_H); p(\alpha(e_H))), \tag{IC1.2}$$

$$u(e_H, \alpha(e_H); p(\alpha(e_H))) \geq u(e_H, \mu; p(\mu)). \tag{IC2.2}$$

Substituting for the functional forms and simplifying obtains

$$\alpha(e_H) \geq \frac{p(\alpha(e_H)) - p(\alpha(e_L)) + \mu \cdot (p(\alpha(e_L)) - \delta \cdot \rho(\theta|e_L))}{p(\alpha(e_H)) - \delta \cdot \rho(\theta|e_L)} \equiv \underline{\Phi}\,(\mu),$$

$$\alpha(e_H) \leq \frac{p(\alpha(e_H)) - p(\alpha(e_L)) + \mu \cdot (p(\alpha(e_L)) - \delta)}{p(\alpha(e_H)) - \delta} \equiv \overline{\Phi}\,(\mu).$$

Using the Lagrangian and Kuhn-Tucker conditions in a manner analogous to the proof of Proposition 5 shows that

$$\alpha^*(e_H; \mu) = \underline{\Phi}\,(\delta, p^*(\alpha), \mu).$$

Now to show that originators have to hold a strictly greater level of retention when $\mu > 0$ we need to have

$$\underline{\Phi}\,(\delta, p^*(\alpha), \mu) > \underline{\Phi}(\delta, p^*(\alpha)).$$

This inequality can be equivalently expressed as

$$\frac{p(\alpha(e_H)) - p(\alpha(e_L)) + \mu \cdot (p(\alpha(e_L)) - \delta \cdot \rho(\theta|e_L))}{p(\alpha(e_H)) - \delta \cdot \rho(\theta|e_L)} > \frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - \delta \cdot \rho(\theta|e_L)},$$

which simplifies to

$$\mu p(\alpha(e_L)) - \mu\delta\rho(\theta|e_L) > 0.$$

In an effort-signalling equilibrium one can substitute for price giving

$$\mu\rho(\theta|e_L) - \mu\delta\rho(\theta|e_L) > 0,$$
$$\mu > \delta\mu.$$

This condition holds for all $\delta < 1$, which is automatically satisfied by Assumption 6. Finally to establish the comparative statics properties of Proposition 7

$$\frac{\partial \underline{\Phi}\left(\delta, p^*(\alpha), \mu\right)}{\partial \rho(\theta|e_L)}$$
$$= \frac{(-1+\alpha-\alpha^*(e_L)\delta)(1-\delta\rho(\theta|e_L)) - [(-\delta)(1-\rho(\theta|e_L)+\alpha\rho(\theta|e_L)-\alpha^*(e_L)\delta\rho(\theta|e_L))]}{(\delta\rho(\theta|e_L)-1)^2}$$
$$= -\frac{[(\delta-1)(\alpha^*(e_L)-1)]}{(\delta\rho(\theta|e_L)-1)^2} < 0 \quad \forall \quad \delta < 1, \alpha^*(e_L) < 1,$$

and

$$\frac{\partial \underline{\Phi}\left(\delta, p^*(\alpha), \mu\right)}{\partial \delta}$$
$$= \frac{(-\alpha(e_L)\rho(\theta|e_L))(1-\delta\rho(\theta|e_L)) - [(-\rho(\theta|e_L))(1-\rho(\theta|e_L)+\alpha(e_L)\rho(\theta|e_L)-\alpha(e_L)\delta\rho(\theta|e_L))]}{(1-\delta\rho(\theta|e_L))^2]}$$
$$= \frac{\rho(\theta|e_L)(1-\alpha(e_L)-\rho(\theta|e_L)+\alpha\rho(\theta|e_L))}{(\delta\rho(\theta|e_L)-1)^2} > 0 \quad \forall \quad \rho(\theta|e_L) < 1,$$

and

$$\frac{\partial \underline{\Phi}\left(\delta, p^*(\alpha), \mu\right)}{\partial \alpha(e_L)} = (\rho(\theta|e_L) - \delta\rho(\theta|e_L))(1-\delta\rho(\theta|e_L))^{-1}$$
$$= \frac{\rho(\theta|e_L) - \delta\rho(\theta|e_L)}{1-\delta\rho(\theta|e_L)} > 0 \quad \forall \quad \delta, \rho(\theta|e_L) < 1.$$

□

*Proof of Proposition 8.* To show that a positive minimum level of retention improves incentives for originators I first derive the disutility of effort required to equalize the payoffs to high and low effort when there is no lower bound on the choice of retention, and then again when the lower bound is positive. Using these two expressions I find a sufficient condition for the result, which turns out to be automatically satisfied by the primitives of the model. Initially we look to find the cost of effort such that

$$u(e_H, \alpha^*(e_H); p^*(\alpha)) = (e_L, 0; p^*(\alpha)).$$

Using the functional form (2.2)

$$-\sum_{i=1}^{k} l_i + \delta\alpha^*(e_H)\sum_{i=1}^{k} r_i + (1-\alpha^*(e_H)) - c(e) = -\sum_{i=1}^{k} l_i + \rho(\theta|e_L).$$

Substituting for $\alpha^*(e_H)$,

$$-\sum_{i=1}^{k} l_i + \delta\frac{(1-\rho(\theta|e_L))}{(1-(\delta\cdot\rho(\theta|e_L)))}\sum_{i=1}^{k} r_i + (1-\frac{(1-\rho(\theta|e_L))}{(1-(\delta\cdot\rho(\theta|e_L)))}) - c(e) = -\sum_{i=1}^{k} l_i + \rho(\theta|e_L).$$

Simplification gives the required cost of effort

$$\frac{\delta(\rho(\theta|e_L)(\rho(\theta|e_L)-2)+1)}{1-\delta\cdot\rho(\theta|e_L)}. \tag{B.78}$$

Note that the numerator is nonnegative as

$$\frac{1}{\rho(\theta|e_L)} + \rho(\theta|e_L) \geq 2 \;\forall\; \rho(\theta|e_L) < 1.$$

Define the value of $e_H$ that gives rise to (B.78) when mapped by $c(e)$ as $\hat{e}_H$. Hence

$$c(\hat{e}_H) \equiv \frac{\delta(\rho(\theta|e_L)(\rho(\theta|e_L)-2)+1)}{1-\delta\cdot\rho(\theta|e_L)}.$$

Now repeating the above procedure when there is a positive minimum level of retention, $\mu > 0$, I look for an cost of effort such that

$$u(e_H, \alpha^*(e_H); p^*(\alpha)) = u(e_L, \mu; p^*(\alpha)).$$

In this case, after substituting for functional forms and simplifying

$$\frac{\delta(\delta\rho(\theta|e_L)^2\alpha(e_L) - \rho(\theta|e_L)^2\alpha(e_L) + \rho(\theta|e_L)^2 + \rho(\theta|e_L)\alpha(e_L) - 2\rho(\theta|e_L) + 1)}{1-\delta\cdot\rho(\theta|e_L)}. \tag{B.79}$$

Defining analogously $\hat{e}_H(\mu)$ as the value of high effort, that when mapped by $c(e)$, gives rise to the disutility defined by (B.79) results in

$$c(\hat{e}_H(\mu))) \equiv \frac{\delta(\delta\rho(\theta|e_L)^2\alpha(e_L) - \rho(\theta|e_L)^2\alpha(e_L) + \rho(\theta|e_L)^2 + \rho(\theta|e_L)\alpha(e_L) - 2\rho(\theta|e_L) + 1)}{1-\delta\cdot\rho(\theta|e_L)}.$$

As we assume that the disutility is convex and monotonic the necessary condition for the originator's incentives for high effort to be improved after introducing Skin in the Game is

$$\hat{e}_H(\mu) > \hat{e}_H \Leftrightarrow c(\hat{e}_H(\mu)) > c(\hat{e}_H).$$

The latter condition can be written as

$$\delta\rho(\theta|e_L)^2\alpha(e_L) - \rho(\theta|e_L)^2\alpha(e_L) + \rho(\theta|e_L)^2 + \rho(\theta|e_L)\alpha(e_L) - 2\rho(\theta|e_L) + 1$$
$$>$$
$$\rho(\theta|e_L)^2 - 2\rho(\theta|e_L) + 1..$$

Which, after simplification, becomes

$$1 > \rho(\theta|e_L)(1-\delta). \tag{B.80}$$

Condition (B.80) automatically holds in the model as both $\delta$ and $\rho(\theta|e_L)$ are bounded above by one by Assumption 6 and 5. □

*Proof of Proposition 9.* With risk aversion the originator's expected payoff is given below by a modified (2.2) for high and low effort, respectively

$$u(e_H, \alpha_H; p_H, \sigma) = \left( -\sum_{i=1}^{k} l_i + \alpha_H \sum_{i=1}^{k} r_i + p_H \cdot (1-\alpha_H) \right)^{\sigma} - c(e_H),$$

and

$$\mathbb{E}[u(e_L, \alpha_L; p_L, \sigma)] =$$
$$\rho(\theta|e_L) \cdot \left( -\sum_{i=1}^{k} l_i + \alpha_L \sum_{i=1}^{k} r_i + p_L \cdot (1-\alpha_L) \right)^{\sigma} + (1-\rho(\theta|e_L)) \cdot \left( -\sum_{i=1}^{k} l_i + p_L \cdot (1-\alpha_L) \right)^{\sigma}.$$

To show that, in equilibrium, the risk-averse originator who has chosen low underwriting effort will still choose to sell the full security I show that this choice continues to maximise her payoff. Differentiating her low effort expected payoff in an effort-signalling equilibrium with respect to her choice of retention and setting the resulting first-order condition equal to zero obtains

$$\begin{aligned}&\mathbb{E}[u_\alpha(e_L,\alpha;p(\alpha(e_L)),\sigma)]=0\\&\Rightarrow \rho(\theta|e_L)\cdot\Big(-\sum_{i=1}^k l_i+\alpha_L\sum_{i=1}^k r_i+p(\alpha(e_L))\cdot(1-\alpha_L)\Big)^{\sigma-1}(\sigma)(1-p(\alpha(e_L)))\\&\quad+(1-\rho(\theta|e_L)\cdot\Big(-\sum_{i=1}^k l_i+p(\alpha(e_L))\cdot(1-\alpha_L)\Big)^{\sigma-1}(\sigma)(-p(\alpha(e_L)))=0.\end{aligned}\tag{B.81}$$

Re-arranging (B.81)

$$(\rho(\theta|e_L)\sigma-\rho(\theta|e_L)\sigma p(\alpha(e_L)))\cdot\Big(-\sum_{i=1}^k l_i+\alpha_L\sum_{i=1}^k r_i+p(\alpha(e_L))\cdot(1-\alpha_L)\Big)^{\sigma-1}$$
$$=$$
$$(p(\alpha(e_L))\sigma-\rho(\theta|e_L)\sigma p(\alpha(e_L)))\cdot\Big(-\sum_{i=1}^k l_i+p(\alpha(e_L))\cdot(1-\alpha_L)\Big)^{\sigma-1}.$$

Then employing equilibrium condition 3 one can substitute $p(\alpha(e_L))=\rho(\theta|e_L)$, which allows for further simplification. This final round of simplification gives $\alpha^*(e_L)=0$. Fixing the retention strategy that occur under low effort I know turn to the optimization problem of the risk-averse originator if she chooses high underwriting effort:

$$\max_{\alpha(e_H)\in[0,1]}\Big(-\sum_{i=1}^k l+\alpha_H\sum_{i=1}^k r+p(\alpha(e_H))\cdot(1-\alpha_H)\Big)^{\sigma}-c(e_H).\tag{B.82}$$

Subject to

$$\alpha(e_H)\geq\frac{1-\rho(\theta|e_L)}{1-\rho(\theta|e_L)^{1/\sigma}}\equiv\underline{\Phi}(\sigma).\tag{B.83}$$

With risk aversion one can ignore the upper-boundary of the incentive compatible set, as this condition is satisfied automatically by the primitives of the model. To see this the incentive compatibility constraint for high effort under risk aversion is given below

$$u(e_H,\alpha_H;p(\alpha(e_H)))\geq u(e_H,0;p(0)).$$

Substituting for functional forms

$$\Big(-\sum_{i=1}^k l_i+\alpha_H\sum_{i=1}^k r_i+p(\alpha(e_H))\cdot(1-\alpha_H)\Big)^{\sigma}-c(e_H)\geq\Big(-\sum_{i=1}^k l_i+p(\alpha(e_L))\Big)^{\sigma}-c(e_H).$$

The right hand side is the payoff to the originator from retaining some fraction of the security whilst the left hand side is the payoff from selling everything and receiving the lower price. Simplification gives

$$\alpha_H(1 - p(\alpha(e_H))) + p(\alpha(e_H)) \geq p(\alpha(e_L)),$$

which when one substitutes for equilibrium prices, again using equilibrium condition 3, becomes $1 \geq \rho(\theta|e_L)$. This condition holds with a strict inequality due to Assumption 5. To derive (B.83) one uses (IC1) under risk aversion, the incentive compatibility condition for a choice of low effort

$$u(e_L, 0; p(0), \sigma) \geq u(e_L, \alpha_H; p(\alpha(e_H)), \sigma).$$

Substituting for functional forms

$$\begin{gathered}\Big(-\sum_{i=1}^{k} l_i + p(\alpha(e_L))\Big)^{\sigma} \\ \geq \\ \rho(\theta|e_L) \cdot \Big(-\sum_{i=1}^{k} l_i + \alpha_H \sum_{i=1}^{k} r_i + p(\alpha(e_H)) \cdot (1 - \alpha_H)\Big)^{\sigma} \\ +(1 - \rho(\theta|e_L) \cdot \Big(-\sum_{i=1}^{k} l_i + p(\alpha(e_H)) \cdot (1 - \alpha_H)\Big)^{\sigma}.\end{gathered}$$

Simplification leads to

$$-\sum_{i=1}^{k} l_i + p(\alpha(e_L)) \geq \rho(\theta|e_L)^{1/\sigma} \cdot \Big(-\sum_{i=1}^{k} l_i + \alpha_H \sum_{i=1}^{k} r_i + p(\alpha(e_H)) \cdot (1 - \alpha_H)\Big) +$$
$$\Big(-\sum_{i=1}^{k} l_i + p(\alpha(e_H)) \cdot (1 - \alpha_H)\Big) - \rho(\theta|e_L)^{1/\sigma} \cdot \Big(-\sum_{i=1}^{k} l_i + p(\alpha(e_H)) \cdot (1 - \alpha_H)\Big) \geq 0.$$

Substituting for equilibrium prices and re-arranging gives (B.83):

$$\alpha_H \geq \frac{1 - \rho(\theta|e_L)}{1 - \rho(\theta|e_L)^{1/\sigma}} \equiv \underline{\Phi}(\sigma, p^*(\alpha)).$$

That the originator maximises her payoff by setting her choice of retention equal to this lower bound follows by the same logic of both the proof of Proposition 5 and Proposition 7. Despite her payoff now being concave-decreasing her optimal point is still at the boundary. Hence

$$\alpha^*(e_H; \sigma) = \underline{\Phi}(\sigma, p^*(\alpha)).$$

To derive the threshold level on the risk aversion parameter simply compare the two expressions and simplify

$$\frac{1 - \rho(\theta|e_L)}{1 - \rho(\theta|e_L)^{1/\sigma}} > \frac{1 - \rho(\theta|e_L)}{1 - \delta \cdot \rho(\theta|e_L)},$$

which gives

$$\sigma > \frac{\ln(\rho(\theta|e_L))}{\ln(\rho(\theta|e_L) \cdot \delta)}. \tag{B.84}$$

The properties of Proposition 9 when $\rho(\theta|e_L) = 1/2$ are by inspection. □

*Proof of Proposition 10.* To demonstrate that the qualitative properties of the effort-signalling equilibrium detailed in Proposition 5 and Proposition 9 continue to hold when the originator is risk-averse I need to show that introducing Skin in the Game continues to lead to a relatively larger effort-signalling retention and a greater range of values of high effort for which the originator prefers high effort in equilibrium. Recall that the effort-signalling level of retention with risk aversion is

$$\underline{\Phi}(\sigma, p^*(\alpha)) = \frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - \rho(\theta|e_L)^{1/\sigma}}.$$

Redefining the incentive compatibility constraints when the originator is risk-averse to accommodate Skin in the Game obtains the following effort-signalling level of retention, using arguments identical to those used in the proofs of Proposition 5 and Proposition 9

$$\underline{\Phi}(\sigma, p^*(\alpha), \mu) = \frac{p(\alpha(e_H)) - p(\alpha(e_L)) + \mu(p(\alpha(e_L)) - \rho(\theta|e_L)^{\sigma^{-1}})}{p(\alpha(e_H)) - \rho(\theta|e_L)^{1/\sigma}}.$$

Thus, to show the former property it is enough to show that

$$\underline{\Phi}(\sigma, p^*(\alpha), \mu) > \underline{\Phi}(\sigma, p^*(\alpha)),$$

equivalently

$$\frac{p(\alpha(e_H)) - p(\alpha(e_L)) + \mu(p(\alpha(e_L)) - \rho(\theta|e_L)^{\sigma^{-1}})}{p(\alpha(e_H)) - \rho(\theta|e_L)^{1/\sigma}} > \frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - \rho(\theta|e_L)^{1/\sigma}}. \tag{B.85}$$

Note that I am taking equilibrium prices to be equal to the true value of the assets, as the theorem is predicated on competitive investors, and the object of study is the effort-signalling equilibrium. Hence, one can substitute the equilibrium price values, as defined by equilibrium condition 3, into (B.85) to give

$$\frac{1-\rho(\theta|e_L)+\mu(\rho(\theta|e_L)-\rho(\theta|e_L)^{\sigma^{-1}})}{1-\rho(\theta|e_L)^{1/\sigma}} > \frac{1-\rho(\theta|e_L)}{1-\rho(\theta|e_L)^{1/\sigma}}.$$

This condition holds for all $\rho(\theta|e_L) < 1$ and $\sigma < 1$.

To show the second property, first define the disutility that equalizes the equilibrium payoffs under risk aversion, as defined by

$$u(e_H, \alpha^*(e_H); p^*(\alpha), \sigma) = u(e_L, 0; p^*(\alpha), \sigma).$$

This disutility is

$$c(e_H;\sigma) = (\alpha^*(e_H;\sigma) + p(\alpha(e_H)(1-\alpha^*(e_H;\sigma))))^{\sigma} - p(\alpha(e_L))^{\sigma}. \tag{B.86}$$

Denote the level of high effort that satisfies (B.86) by $\hat{e}_H(\sigma)$. I now find the analogous disutility after Skin in the Game has been introduced. In this case the disutility is given by

$$c(e_H;\sigma,\mu) = (\alpha^*(e_H;\sigma,\mu)+(1-(\alpha^*(e_H;\sigma,\mu)p(\alpha(e_H))^{\sigma}+\rho(\theta|e_L)((1-\mu)p(\alpha(e_L))^{\sigma} \\ -\rho(\theta|e_L)(\mu+(1-\mu)p(\alpha(e_L))^{\sigma}-((1-\mu)p(\alpha(e_L))^{\sigma}.$$

With associated high effort $\hat{e}_H(\sigma;\mu)$. Now to show that incentives for high effort continue to improve when Skin in the Game is introduced with a risk-averse originator the key condition is

$$\hat{e}_H(\sigma;\mu) > \hat{e}_H(\sigma) \Leftrightarrow c(e_H;\sigma,\mu) > c(e_H;\sigma).$$

As I assume that the functional form of the disutility is kept fixed, and monotonic. Thus we compare

$$c(e_H;\sigma,\mu) > c(e_H;\sigma),$$

which gives, after simplification

$$\alpha^*(e_H;\sigma,\mu)-\alpha^*(e_H;\sigma,\mu)p(\alpha(e_H))-\rho(\theta|e_L)^{1/\sigma}\mu+\mu p(\alpha(e_L)) > \alpha^*(e_H;\sigma)-\alpha^*(e_H;\sigma)p(\alpha(e_H)).$$

Collecting like terms

$$(1 - p(\alpha(e_H)))\alpha^*(e_H; \sigma, \mu) + \mu(p(\alpha(e_L)) - \rho(\theta|e_L)^{1/\sigma}) > (1 - p(\alpha(e_H)))\alpha^*(e_H; \sigma).$$

Substituting for the values of the effort-signalling prices as per equilibrium condition 3 gives $\mu(\rho(\theta|e_L) - \rho(\theta|e_L)^{1/\sigma} > 0$, which again holds for all $\rho(\theta|e_L) < 1$ and $\sigma < 1$. □

*Proof of Corollary 2.* The condition for existence under risk aversion is that $\underline{\Phi}(\sigma) < 1$, which is equivalent to

$$\frac{p(\alpha(e_H)) - p(\alpha(e_L))}{p(\alpha(e_H)) - \rho(\theta|e_L)^{1/\sigma}} < 1 \Leftrightarrow \rho(\theta|e_L) < p(\alpha(e_L))^{\sigma}. \tag{B.87}$$

In the effort-signalling equilibrium we have that $p(\alpha(e_L)) = \rho(\theta|e_L)$ thus (B.87) holds for all $\sigma < 1$, where $\sigma > 1$ can be interpreted as risk-seeking. □

## B.3 Chapter 3 Proofs

*Proof of Theorem 3.* Deriving the preliminary results of Mailath and von Thadden (2013) with my modified framework yields the following expression analogous to Mailath and von Thadden's (A5):

$$0 \geq g(\omega_0, \varphi(\omega), X(\omega)) \geq -(\omega - \omega_0)\Big\{\frac{1}{2}g_{11}([\omega;\lambda])_1(\omega - \omega_0) + g_{12}([\omega;\mu]_{23})(\varphi(\omega) - \varphi(\omega_0)) + g_{13}([\omega;\mu]_{23})(X(\omega) - X(\omega_0))\Big\}, \quad \text{(B.88)}$$

where

$$[\omega;\lambda]_1 \equiv (\lambda\omega_0 + (1-\lambda)\omega, \varphi(\omega), X(\omega)),$$

$$[\omega;\mu]_{23} \equiv (\omega_0, \mu\varphi(\omega_0) + (1-\mu)\omega, \mu X(\omega_0) + (1-\mu)X(\omega)),$$

and

$$g(\omega, \varphi(\hat{\omega}), x) \equiv V(\omega, \varphi(\hat{\omega}), x) - V(\omega, \varphi(\omega_0), X(\omega_0))$$

for fixed $\omega_0 \in \Omega$, arbitrary $\omega, \hat{\omega} \in \Omega$, $\lambda \in [0,1], \mu \in [0,1]$ and $x \in \mathcal{X}$. The simple yet key change in (B.88) relative to Mailath and von Thadden's (A5) is the presence of the informed agent's effort technology, which alters the $g_{12}$ term.

Performing a Taylor series expansion on $g(\omega_0, \varphi(\omega), X(\omega))$ around $(\omega_0, \varphi(\omega_0), X(\omega_0))$ and simplifying,

$$g(\omega_0, \varphi(\omega), X(\omega)) = g_2(\omega_0, \varphi(\omega_0), X(\omega_0))(\varphi(\omega) - \varphi(\omega_0)) + g_3(\omega_0, \varphi(\omega_0), X(\omega_0))(X(\omega) - X(\omega_0)) + \frac{1}{2}g_{22}([\omega;\gamma]_{23})(\varphi(\omega) - \varphi(\omega_0))^2 + \frac{1}{2}g_{33}([\omega;\gamma]_{23})(X(\omega) - X(\omega_0))^2 + g_{23}([\omega;\gamma]_{23})(\varphi(\omega) - \varphi(\omega_0))(X(\omega) - X(\omega_0)). \quad \text{(B.89)}$$

Substituting (B.89) into (B.88),

$$0 \geq g_2(\omega_0, \varphi(\omega_0), X(\omega_0))(\varphi(\omega) - \varphi(\omega_0)) + g_3(\omega_0, \varphi(\omega_0), X(\omega_0))(X(\omega) - X(\omega_0)) + \frac{1}{2}g_{22}([\omega;\gamma]_{23})(\varphi(\omega) - \varphi(\omega_0))^2 + \frac{1}{2}g_{33}([\omega;\gamma]_{23})(X(\omega) - X(\omega_0))^2 + g_{23}([\omega;\gamma]_{23})(\varphi(\omega) - \varphi(\omega_0))(X(\omega) - X(\omega_0)) \geq -(\omega - \omega_0)\Big\{\frac{1}{2}g_{11}([\omega;\lambda]_1)(\omega - \omega_0) + g_{12}([\omega;\mu]_{23})(\varphi(\omega) - \varphi(\omega_0)) + g_{13}([\omega;\mu]_{23})(X(\omega) - X(\omega_0))\Big\},$$

and dividing through by $(\omega-\omega_0)$,

$$
\begin{aligned}
0 \geq\ & g_2(\omega_0,\varphi(\omega_0),X(\omega_0))\frac{\varphi(\omega)-\varphi(\omega_0)}{\omega-\omega_0}+g_3(\omega_0,\varphi(\omega_0),X(\omega_0))\frac{X(\omega)-X(\omega_0)}{\omega-\omega_0}\\
&+\frac{1}{2}g_{22}([\omega;\gamma]_{23})\frac{(\varphi(\omega)-\varphi(\omega_0))^2}{\omega-\omega_0}+\frac{1}{2}g_{33}([\omega;\gamma]_{23})\frac{(X(\omega)-X(\omega_0))^2}{\omega-\omega_0}\\
&+g_{23}([\omega;\gamma]_{23})\frac{(\varphi(\omega)-\varphi(\omega_0))(X(\omega)-X(\omega_0))}{\omega-\omega_0}\\
\geq\ & -\frac{1}{2}g_{11}([\omega;\lambda]_1)(\omega-\omega_0)-g_{12}([\omega;\mu]_{23})(\varphi(\omega)-\varphi(\omega_0))-g_{13}([\omega;\mu]_{23})(X(\omega)-X(\omega_0)).
\end{aligned}
\tag{B.90}
$$

Finally, taking limits as $\omega$ approaches $\omega_0$ from above, $\omega\searrow\omega_0$, on (B.90)

$$
0 \geq g_2(\omega_0,\varphi(\omega_0),X(\omega_0))\lim_{\omega\searrow\omega_0}\left[\frac{\varphi(\omega)-\varphi(\omega_0)}{\omega-\omega_0}\right] + g_3(\omega_0,\varphi(\omega_0),X(\omega_0))\lim_{\omega\searrow\omega_0}\left[\frac{X(\omega)-X(\omega_0)}{\omega-\omega_0}\right]\geq 0,
$$

implying that $X$ is differentiable at $\omega_0$ and

$$
g_2(\omega_0,\varphi(\omega_0),X(\omega_0))\lim_{\omega\searrow\omega_0}\left[\frac{\varphi(\omega)-\varphi(\omega_0)}{\omega-\omega_0}\right] + g_3(\omega_0,\varphi(\omega_0),X(\omega_0))\lim_{\omega\searrow\omega_0}\left[\frac{X(\omega)-X(\omega_0)}{\omega-\omega_0}\right]=0.
$$

I have therefore shown that when $V_3(\omega_0,\varphi(\omega_0),X(\omega_0))\neq 0$ and when $X$ is continuous at $\omega_0$ that $X$ is differentiable at this point and the derivative satisfies

$$
X'(\omega_0)=-\frac{V_2(\omega_0,\varphi(\omega_0),X(\omega_0))\cdot\varphi'(\omega_0)}{V_3(\omega_0,\varphi(\omega_0),X(\omega_0))}.
$$

To then show that if $\omega\longrightarrow\omega_0$ then $V(\omega_0,\varphi(\omega_0),X(\omega))\longrightarrow V(\omega_0,\varphi(\omega_0),X(\omega_0))$ the proof follows Mailath and von Thadden's Lemma D and so will not be reproduced. The key point to note is that the proof continues to go through as I have assumed that $\varphi$ is continuous and therefore, for each $\epsilon>0$ and $\omega\in\Omega$, there is a $\delta>0$ such that

$$
\omega_0\in\Omega \text{ and } |\omega-\omega_0|<\delta\Rightarrow|\varphi(\omega)-\varphi(\omega_0)|<\epsilon,
$$

and hence I still have, as required,

$$|\omega - \omega_0| < \delta \implies |V(\omega_0, \varphi(\omega), x) - V(\omega_0, \varphi(\omega_0), x)| < \epsilon.$$

Having shown convergence, the final piece is the continuity of $X$ at $\omega_0$. For this Mailath and von Thadden's proof of Theorem 3 continues to apply where they employ the compactness of $\mathcal{X}$, the assumption on $V_3$, and the Bolzano-Weierstrass Theorem to find a convergent subsequence within $\mathcal{X}$ that demonstrates continuity of $X$ at $\omega_0$. □

*Proof of Theorem 4.* Before showing that, given (IC), the problem (EO) has a solution, I first demonstrate that (IC) continues to hold in this setting with endogenous effort. This first portion of the proof of Theorem 4 follows Mailath and von Thadden's (2013) Lemma H. Incentive compatibility requires that the following first order condition holds,

$$\begin{aligned}&\frac{\partial}{\partial x} V(\omega, \varphi(X^{-1}(x)), x)\\ &\quad = V_2(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x) + V_3(\omega, \varphi(X^{-1}(x)), x) = 0.\end{aligned} \tag{B.91}$$

By differentiating (B.91) I obtain the second-order sufficient condition

$$\begin{aligned}\frac{\partial^2}{\partial x^2} V(\omega, \varphi(X^{-1}(x)), x) = {}& V_{22}(\omega, \varphi(X^{-1}(x)), x) \cdot \left(\varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x)\right)^2\\ &+ V_2(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d^2}{dx^2} X^{-1}(x)\\ &+ V_2(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi''(X^{-1}(x)) \cdot \left(\frac{d}{dx} X^{-1}(x)\right)^2\\ &+ 2 \cdot V_{23}(\omega, \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x) + V_{33}(\omega, \varphi(X^{-1}(x)), x).\end{aligned} \tag{B.92}$$

To show that (B.92) is negative I first evaluate (B.91) at $\omega = X^{-1}(x)$,

$$V_2(X^{-1}(x), \varphi(X^{-1}(x)), x) \cdot \varphi'(X^{-1}(x)) \cdot \frac{d}{dx} X^{-1}(x) + V_3(X^{-1}(x), \varphi(X^{-1}(x)), x) = 0.$$

Differentiating this identity yields

$$\Big(V_{12}(X^{-1}(x),\varphi(X^{-1}(x)),x)+V_{22}(X^{-1}(x),\varphi(X^{-1}(x)),x)\cdot\varphi'(X^{-1}(x))\Big)\times$$
$$\times\left(\frac{d}{dx}X^{-1}(x)\right)^2\cdot\varphi'(X^{-1}(x))$$
$$+V_2(X^{-1}(x),\varphi(X^{-1}(x)),x)\cdot\left[\varphi''(X^{-1}(x))\cdot\left(\frac{d}{dx}x^{-1}(x)\right)^2+\varphi'(X^{-1}(x))\cdot\frac{d^2}{dx^2}X^{-1}(x)\right]$$
$$+\Big(V_{13}(X^{-1}(x),\varphi(X^{-1}(x)),x)+2\cdot V_{23}(X^{-1}(x),\varphi(X^{-1}(x)),x)\cdot\varphi'(X^{-1}(x))\Big)\cdot\frac{d}{dx}X^{-1}(x)$$
$$+V_{33}(X^{-1}(x),\varphi(X^{-1}(x)),x)=0. \quad \text{(B.93)}$$

Re-arranging (B.93), and dropping $(X^{-1}(x),\varphi(X^{-1}(x)),x)$ from the notation,

$$V_2\cdot\varphi''(X^{-1}(x))\cdot\left(\frac{d}{dx}X^{-1}(x)\right)^2+V_2\cdot\varphi'(X^{-1}(x))\cdot\frac{d^2}{dx^2}X^{-1}(x)+V_{33}$$
$$=$$
$$-\Big(V_{12}+V_{22}\cdot\varphi'(X^{-1}(x))\Big)\cdot\left(\frac{d}{dx}X^{-1}(x)\right)^2\cdot\varphi'(X^{-1}(x))$$
$$-\Big(V_{12}+2\cdot V_{23}\cdot\varphi'(X^{-1}(x))\Big)\cdot\frac{d}{dx}X^{-1}(x). \quad \text{(B.94)}$$

Evaluating (B.92) at $\omega=X^{-1}(x)$,

$$\frac{\partial^2}{\partial x^2}V(X^{-1}(x),\varphi(X^{-1}(x)),x)=V_{22}\cdot\left(\varphi'(X^{-1}(x))\cdot\frac{d}{dx}X^{-1}(x)\right)^2$$
$$+2\cdot V_{23}\cdot\varphi'(X^{-1}(x))\cdot\frac{d}{dx}X^{-1}(x)$$
$$+V_2\cdot\varphi''(X^{-1}(x))\cdot\left(\frac{d}{dx}X^{-1}(x)\right)^2+V_2\cdot\varphi'(X^{-1}(x))\cdot\frac{d^2}{dx^2}X^{-1}(x)+V_3. \quad \text{(B.95)}$$

Substituting (B.94) into (B.95) and simplifying gives,

$$\frac{\partial^2}{\partial x^2}V(X^{-1}(x),\varphi(X^{-1}(x)),x)=-V_{12}\cdot\left(\frac{d}{dx}X^{-1}(x)\right)^2\cdot\varphi'(X^{-1}(x))-V_{13}\cdot\frac{d}{dx}X^{-1}(x),$$
$$=-\left(\frac{d}{dx}X^{-1}(x)\right)^2\Big(V_{12}\cdot\varphi'(X^{-1}(x))+V_{13}\cdot X'(\omega)\Big). \quad \text{(A9)}$$

Therefore (B.92) is satisfied when $\varphi'(\cdot)$ is increasing, which I have assumed, as Mailath and von Thadden show that $V_{12}+V_{13}\cdot X'(\omega)>0$. I obtain (A9) by employing the property of the derivative of inverse functions,

$$\frac{d}{dx}X^{-1}(x) = \frac{1}{X'(\omega)} = X'(\omega) \cdot \frac{1}{X'(\omega)^2} = X'(\omega) \cdot \left(\frac{d}{dx}X^{-1}(x)\right)^2.$$

Finally, note that if $V_{12} = 0$, then (A9) is the same condition obtained as in the exogenous setting.

To now show that, given (IC), the problem (EO) has a solution I begin with the form suggested by (3.1), namely $V(\omega, \varphi(\hat{\omega}), x)$, which becomes

$$\mathcal{V}(\omega, \theta) \equiv V(\omega, \varphi(\omega), X(\omega)) = V(\omega, \varphi(\hat{\omega}), x)|_{\hat{\omega}=\omega, x=X(\omega)},$$

after (IC) based optimization. Hence, for a RI equilibrium to exist it is sufficient to check that

$$\mathcal{V}_{11}(\omega, \theta) = \frac{\partial^2}{\partial \omega^2} V(\omega, \varphi(\omega), X(\omega)) \leq 0 \ \forall \ \omega \in \Omega,$$

which yields the following derivative

$$\begin{aligned}
\mathcal{V}_{11}(\omega, \theta) = V_{11}(\omega, \varphi(\omega), X(\omega)) + V_2(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}\varphi(\omega) \\
+ V_{22}(\omega, \varphi(\omega), X(\omega)) \cdot \left(\frac{d}{d\omega}\varphi(\omega)\right)^2 \\
+ V_3(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}X(\omega) + V_{33}(\omega, \varphi(\omega), X(\omega)) \cdot \left(\frac{d}{d\omega}X(\omega)\right)^2 \\
+ 2 \cdot V_{12}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega}\varphi(\omega) + 2V_{13}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega}X(\omega) \\
+ 2 \cdot V_{23}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega}\varphi(\omega) \cdot \frac{d}{d\omega}X(\omega),
\end{aligned}$$

as by continuity $V_{12} = V_{21}$, $V_{13} = V_{31}$ and $V_{23} = V_{32}$. By the assumption of linearity in both the signal and the response of uninformed, and additive separability in the informed agent's effort and the response of the uninformed I have

$$\begin{aligned}\mathcal{V}_{11}(\omega,\theta) = &\underbrace{V_{11}(\omega,\varphi(\omega),X(\omega)) + V_2(\omega,\varphi(\omega),X(\omega)) \cdot \frac{d^2}{d\omega^2}\varphi(\omega)}_{<0} \\ &+ \underbrace{V_3(\omega,\varphi(\omega),X(\omega)) \cdot \frac{d^2}{d\omega^2}X(\omega)}_{<0} \\ &+ 2 \cdot V_{13}(\omega,\varphi(\omega),X(\omega)) \cdot \frac{d}{d\omega}X(\omega) + 2 \cdot V_{23}(\omega,\varphi(\omega),X(\omega)) \cdot \frac{d}{d\omega}\varphi(\omega) \cdot \frac{d}{d\omega}X(\omega)\end{aligned} \tag{B.96}$$

So far the conditions I have employed are relatively uncontroversial. Linearity in the signal has been a central assumption in the two applications considered in Section 3.3 and 3.4. Concavity of the informed agent's payoff, before optimization, holds in both applications as

$$\frac{\partial^2}{\partial\omega^2}V(\omega,\varphi(\hat{\omega}),x) = \delta\varphi''(\omega)(1-x) - \psi''(\omega) \leq 0,$$

in my extension of DeMarzo and Duffie and,

$$\frac{\partial^2}{\partial\omega^2}U(\omega,\varphi(\hat{\omega}),e) = e\frac{\varphi''(\omega)}{\varphi(\omega)^2} - 2e\frac{\varphi'(\omega)^2}{\varphi(\omega)^3} - \lambda \cdot \psi''(\omega) \leq 0,$$

in the extension of Spence. Moreover, additivity of the informed agent's payoff (3.1) is satisfied in effort and the response of the uninformed will be satisfied in general, and indeed is in our applications. For example, any model where the response of the uninformed takes the form of a wage or price etc will lead to (3.1) being increasing in this response and additively separable in effort and price/wage, which combined with concave/linear effort technology and our other statements leads to the first three terms of (B.96) being negative.

The set of conditions required for the theorem are somewhat more restrictive. To make the conclusion of existence one possible 'recipe' is to assume that $V_{23} = 0$, implying additive separability of (3.1) in signal and the response of the uninformed, which is satisfied in Section 3.4 but not in Section 3.3, which gives

$$\begin{aligned}\mathcal{V}_{11}(\omega,\theta) = &V_{11}(\omega,\varphi(\omega),X(\omega)) + V_2(\omega,\varphi(\omega),X(\omega)) \cdot \frac{d^2}{d\omega^2}\varphi(\omega) \\ &+ V_3(\omega,\varphi(\omega),X(\omega)) \cdot \frac{d^2}{d\omega^2}X(\omega) + 2 \cdot V_{13}(\omega,\varphi(\omega),X(\omega)) \cdot \frac{d}{d\omega}X(\omega).\end{aligned}$$

This simplifying assumption removes the complication of the product of terms

$\varphi'(\omega) \cdot X'(\omega)$ that may not be negative. Unfortunately, the following expression holds

$$V_{13}(\omega, \varphi(\omega), X(\omega) \cdot \frac{d}{d\omega} X(\omega) > 0, \tag{B.97}$$

for all signalling models. The reason is that $V_{13}$ captures increasing (decreasing) differences in the informed agent's objective in effort and signal, which in turn plays a role in determining whether the equilibrium signal is increasing (decreasing) in effort. Hence, if $V_{13} > 0$ then $X'(\omega) > 0$ and (B.97) holds, whilst if $V_{13} < 0$ then $X'(\omega) < 0$ and I am once again left with (B.97). Hence, using (3.2) I can conclude that

$$-\left\{V_{11}(\omega, \varphi(\omega), X(\omega)) + V_2(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2}\varphi(\omega) + V_3(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d^2}{d\omega^2} X(\omega)\right\}$$
$$> 2 \cdot V_{13}(\omega, \varphi(\omega), X(\omega)) \cdot \frac{d}{d\omega} X(\omega) \;\Rightarrow\; \mathcal{V}_{11}(\omega, \theta) < 0.$$

□

*Proof of Proposition 11.* To prove Proposition 11, I first show that, under Assumption 8, $\mathcal{V}(\omega, \delta)$ is concave in it's first argument, and, by demonstrating sufficiency, I apply the implicit function theorem. Prior to this however, I will derive $X : \Omega \to \mathcal{X}$. Beginning with the differential equation featured in the text:

$$X'(\omega) = -\frac{x \cdot \varphi'(\omega)}{\varphi(\hat{\omega}) - \delta\varphi(\omega)} = \frac{X(\omega) \cdot \varphi'(\omega)}{(\delta - 1)\varphi(\omega)}\bigg|_{\hat{\omega}=\omega, x=X(\omega)}.$$

After re-arranging I arrive at the following linear first order ordinary differential equation

$$X'(\omega)(1 - \delta)\varphi(\omega) + X(\omega)\varphi'(\omega) = 0. \tag{B.98}$$

To solve (B.98) I first separate the variables $\frac{X'(\omega)}{X(\omega)} = \frac{1}{\delta-1}\frac{\varphi'(\omega)}{\varphi(\omega)}$, and by integrating this expression one obtains

$$\int \frac{X'(\omega)}{X(\omega)}\, \mathrm{d}\omega = \frac{1}{\delta - 1}\int \frac{\varphi'(\omega)}{\varphi(\omega)}\, \mathrm{d}\omega.$$

By applying integration by parts one has

$$\ln(X(\omega)) = \ln(\varphi(\omega)^{\frac{1}{\delta-1}}) + c_1,$$

which can be solved for the equilibrium mapping by noting that

$$\ln(\varphi(\omega)^{\frac{1}{\delta-1}}) + c_1 = \ln(e^{\ln(\varphi(\omega)^{\frac{1}{\delta-1}} + c_1)}) = \ln(e^{\ln(\varphi(\omega)^{\frac{1}{\delta-1}})}e^{c_1}) = \ln(\varphi(\omega)^{\frac{1}{\delta-1}}e^{c_1}),$$

and hence

$$X(\omega) = \varphi(\omega)^{\frac{1}{\delta-1}} c_2 \quad \text{where} \quad c_2 = e^{c_1}. \tag{B.99}$$

To derive the boundary condition that will enable an explicit form for $c_2$, note that for $\omega = \omega_1$ the firm will choose $X(\omega_1) = 1$, the first best outcome: $X^{FB}(\omega_1)$. Inserting this into (B.99) yields $c_2 = \varphi(\omega_1)^{\frac{1}{1-\delta}}$, which is then substituted back into (B.99) to give

$$X(\omega) = \varphi(\omega)^{\frac{1}{\delta-1}} \varphi(\omega_1)^{\frac{1}{1-\delta}} = \left\{ \frac{\varphi(\omega_1)}{\varphi(\omega)} \right\}^{\frac{1}{1-\delta}}.$$

Now simplifying the firm's payoff in the second stage, which recall is given by (3.7),

$$\begin{aligned}
\mathcal{V}(\omega, \delta) &= \delta\varphi(\omega) + (1-\delta)\varphi(\omega) \left\{ \frac{\varphi(\omega_1)}{\varphi(\omega)} \right\}^{\frac{1}{1-\delta}} - \psi(\omega), \\
&= \delta\varphi(\omega) + (1-\delta)\varphi(\omega)^{1+\frac{1}{\delta-1}} \varphi(\omega_1)^{\frac{1}{1-\delta}} - \psi(\omega), \\
&= \delta\varphi(\omega) + (1-\delta)\varphi(\omega)^{\frac{\delta}{\delta-1}} \varphi(\omega_1)^{\frac{1}{1-\delta}} - \psi(\omega).
\end{aligned} \tag{B.100}$$

Partially differentiating (B.100) with respect to the firm's choice of effort obtains,

$$\begin{aligned}
\mathcal{V}_1(\omega, \delta) &= \delta\varphi'(\omega) + \frac{\delta}{\delta-1}(1-\delta)\varphi(\omega)^{\frac{\delta}{\delta-1}-1} \varphi(\omega_1)^{\frac{1}{1-\delta}} \varphi'(\omega) - \psi'(\omega), \\
&= \delta\varphi'(\omega) + \delta\left(\frac{1-\delta}{\delta-1}\right) \varphi(\omega)^{\frac{1}{\delta-1}} \varphi(\omega_1)^{\frac{1}{1-\delta}} \varphi'(\omega) - \psi'(\omega), \\
&= \delta\varphi'(\omega) - \delta\left(\frac{\delta-1}{\delta-1}\right) \left\{ \frac{\varphi(\omega_1)}{\varphi(\omega)} \right\}^{\frac{1}{1-\delta}} \varphi'(\omega) - \psi'(\omega), \\
&= \delta\varphi'(\omega) \left[ 1 - \left\{ \frac{\varphi(\omega_1)}{\varphi(\omega)} \right\}^{\frac{1}{1-\delta}} \right] - \psi'(\omega).
\end{aligned} \tag{B.101}$$

The first-order condition (B.101) is a necessary condition for the firm's choice of effort to satisfy (EO). To check sufficiency I partially differentiate again to obtain,

$$\mathcal{V}_{11}(\omega, \delta) = \delta\varphi''(\omega) - \delta\varphi(\omega_1)^{\frac{1}{1-\delta}} \times \left\{ \varphi''(\omega)\varphi(\omega)^{\frac{1}{\delta-1}} + \varphi'(\omega)\left(\frac{1}{\delta-1}\right)\varphi(\omega)^{\frac{1}{\delta-1}-1}\varphi'(\omega) \right\} - \psi''(\omega).$$

This second-order condition simplifies as follows

$$\mathcal{V}_{11}(\omega,\delta) = \delta\varphi''(\omega) - \delta\varphi''(\omega)\varphi(\omega_1)^{\frac{1}{1-\delta}}\varphi(\omega)^{\frac{1}{\delta-1}} + \frac{\delta}{1-\delta}(\varphi'(\omega))^2\varphi(\omega)^{\frac{1}{\delta-1}-1}\varphi(\omega_1)^{\frac{1}{1-\delta}} - \psi''(\omega),$$
$$= \delta\varphi''(\omega)\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right] + \varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) - \psi''(\omega). \tag{B.102}$$

From (B.102) the following preliminary result is immediate.

**Lemma 4.** *Suppose the effort production function is linear or convex, and the effort disutility is linear. Then, $\mathcal{V}(\omega,\delta)$ is convex in effort.*

*Proof of Lemma 4.* Linear production and disutility implies that (B.102) reduces to

$$\varphi'(\omega)\frac{\delta}{1-\delta}X(\omega) \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) > 0 \quad \text{for} \quad \varphi'(\omega) > 0,$$

whilst convex production implies that (B.102) is positive. □

Consider now the case with a concave effort production function whilst continuing to assume that effort disutility is linear, $\psi''(\omega) = 0$. In this case one has

$$\mathcal{V}_{11}(\omega,\delta) < 0 \iff \delta\varphi''(\omega)\left[\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} - 1\right] > \varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}} \times \frac{d}{d\omega}\ln[\varphi(\omega)](\omega),$$

as per Assumption 8. Finally note that I can write

$$\mathcal{V}_{11}(\omega,\delta) \overset{\text{sgn}}{=} (\delta-1)\left[\left(\frac{\varphi(\omega)}{\varphi(\omega_1)}\right)^{\frac{1}{1-\delta}} - 1\right]\mathcal{R}(\omega) + \frac{d}{d\omega}\ln[\varphi(\omega)](\omega) \quad \text{where} \quad \mathcal{R}(\omega) = -\frac{\varphi''(\omega)}{\varphi'(\omega)}.$$

Given that I have shown that Assumption 8 is sufficient for $\mathcal{V}(\omega,\delta)$ to be concave in effort, I can now apply the implicit function theorem to yield the comparative statics result. Define

$$Z(\omega(\delta),\delta) \equiv \delta\varphi'(\omega(\delta))\left[1 - \left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] - \psi'(\omega(\delta)) = 0.$$

Differentiating the identity $Z(\omega(\delta),\delta)$ yields

$$\frac{d}{d\delta}Z(\omega(\delta),\delta) = \frac{\partial}{\partial\omega}\left(\delta\varphi'(\omega(\delta))\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] - \psi'(\omega(\delta))\right)\frac{d}{d\delta}\omega(\delta)$$
$$+\varphi'(\omega)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega(\delta))}\right\}^{\frac{1}{1-\delta}}\right] = 0,$$

hence

$$\begin{aligned}\frac{d}{d\delta}\omega(\delta) &= -\frac{\frac{\partial}{\partial\delta}\left(\delta\varphi'(\omega)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right]\right)}{\frac{\partial}{\partial\omega}\left(\delta\varphi'(\omega)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right]-\psi'(\omega)\right)},\\
&= -\frac{\varphi'(\omega)-\left[\varphi'(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}+\delta\varphi'(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\ln\left(\frac{\varphi(\omega_1)}{\varphi(\omega)}\right)\frac{d}{d\delta}\frac{1}{1-\delta}\right]}{\delta\varphi''(\omega)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right]+\varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\frac{d}{d\omega}\ln[\varphi(\omega)](\omega)-\psi''(\omega)},\\
&= -\frac{\varphi'(\omega)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right]-\frac{\delta}{(1-\delta)^2}\varphi'(\omega)\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\left[\ln(\varphi(\omega_1))-\ln(\varphi(\omega))\right]}{\delta\varphi''(\omega)\left[1-\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\right]+\varphi'(\omega)\frac{\delta}{1-\delta}\left\{\frac{\varphi(\omega_1)}{\varphi(\omega)}\right\}^{\frac{1}{1-\delta}}\frac{d}{d\omega}\ln[\varphi(\omega)](\omega)-\psi''(\omega)}.\end{aligned}$$
(B.103)

I can express this in the more compact form

$$\frac{d}{d\delta}\omega(\delta) = \frac{\varphi'(\omega)\Big[X(\omega)-1\Big]+\sigma(\delta)\varphi'(\omega)X(\omega)\Big[\ln(\varphi(\omega_1))-\ln(\varphi(\omega))\Big]}{\delta\varphi''(\omega)X(\omega)+(1-\delta)\sigma(\delta)\varphi(\omega)X(\omega)\times\frac{d}{d\omega}\ln[\varphi(\omega)](\omega)},$$

$$\text{where}\quad \sigma(\delta) = \frac{\delta}{(1-\delta)^2}.$$

Note that (B.103) implies

$$\mathcal{V}_{12}(\omega,\delta) = \varphi'(\omega)\Big[1-X(\omega)\Big]-\sigma(\delta)\varphi'(\omega)X(\omega)\Big[\ln(\varphi(\omega_1))-\ln(\varphi(\omega))\Big], \quad \text{(B.104)}$$

which to sign note that I have two cases: the first is when $0<\varphi(\omega_1)<\varphi(\omega)<1$ and the second is when $\varphi(\omega)>1$. In the first case I have that

$$\ln(\varphi(\omega_1))-\ln(\varphi(\omega))<0 \quad\text{as}\quad \ln(\varphi(\omega_1))<\ln(\varphi(\omega))<0,$$

and in the second assume that $\varphi(\omega_1) = 1$, without loss of generality as $\ln(\cdot)$ is increasing, so that

$$\ln(1) - \ln(\varphi(\omega)) < 0 \quad \text{as} \quad \ln(1) = 0 \quad \text{and} \quad \ln(\varphi(\omega)) > 0 \text{ for } \varphi(\omega) > 1.$$

Hence I can conclude that (B.104) is positive as required for supermodularity and Lemma 3, $\mathcal{V}_{12}(\omega, \delta) > 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$. □

*Proof of Proposition 12.* Under the conditions of Proposition 12 with $c(e, \varphi(\omega) = e \cdot \varphi(\omega)^{-1}$, the worker's payoff is given by (3.11), which is

$$\mathcal{U}(\omega, \lambda) = \varphi(\omega) - \frac{1}{2}\left\{\frac{\varphi(\omega)^2 - \varphi(\omega_1)^2}{\varphi(\omega)}\right\} - \lambda\psi(\omega).$$

If the worker chooses zero effort, and I assume that zero effort yields zero productivity, then $\varphi(\omega_1) = 0$. Thus,

$$\begin{aligned}
\mathcal{U}(\omega, \lambda) &= \varphi(\omega) - \frac{1}{2}\frac{\varphi(\omega)^2}{\varphi(\omega)} - \lambda\psi(\omega), \\
&= \varphi(\omega) - \frac{\varphi(\omega)}{2} - \lambda\psi(\omega), \\
&= \frac{\varphi(\omega)}{2} - \lambda\psi(\omega).
\end{aligned}$$

It is easy to see that under the conditions postulated in Proposition 12, where the effort production function is concave and disutility strictly convex, so that $\varphi''(\omega) \leq 0$ and $\psi''(\omega) > 0$, the function $\mathcal{U}(\omega, \lambda)$ is now the sum of a concave function and a strictly concave function, and is therefore strictly concave. Explicitly, the second partial derivatives are,

$$\mathcal{U}_{11}(\omega, \lambda) = \frac{\varphi''(\omega)}{2} - \lambda\psi''(\omega) < 0 \quad \text{when} \quad \varphi''(\omega) < 0,$$

and

$$\mathcal{U}_{11}(\omega, \lambda) = -\lambda\psi''(\omega) < 0 \quad \text{when} \quad \varphi''(\omega) = 0,$$

for $\lambda > 0$. To obtain the comparative statics result, I first define an identity, using the first-order necessary condition,

$$\mathcal{U}_1(\omega, \lambda) = 0 \;\Rightarrow\; \frac{\varphi'(\omega)}{2} = \lambda\psi'(\omega),$$

that implicitly defines optimal effort $\omega(\lambda)$, which is continuous by continuity of

$\varphi$, $\psi$ and $\lambda > 0$,

$$Z(\omega(\lambda), \lambda) \equiv \frac{\varphi'(\omega(\lambda))}{2} - \lambda\psi'(\omega(\lambda)) = 0,$$

to which one can apply the implicit function theorem, as the strict concavity of $\mathcal{U}$ in $\omega$ has been demonstrated, to yield

$$\frac{d}{d\lambda}\omega^*(\lambda) = -\frac{-\psi'(\omega(\lambda))}{\frac{\varphi''(\omega(\lambda))}{2} - \lambda\psi''(\omega(\lambda))} = \frac{\psi'(\omega(\lambda))}{\frac{\varphi''(\omega(\lambda))}{2} - \lambda\psi''(\omega(\lambda))} < 0 \text{ where } \varphi''(\omega) < 0,$$

and

$$\frac{d}{d\lambda}\omega^*(\lambda) = -\frac{-\psi'(\omega(\lambda))}{-\lambda\psi''(\omega(\lambda))} = \frac{\psi'(\omega(\lambda))}{-\lambda\psi''(\omega(\lambda))} < 0 \text{ where } \varphi''(\omega) = 0,$$

as $\psi'(\omega) > 0$ for each $\omega \in \Omega \setminus \{\omega_1\}$.

□

*Proof of Proposition 13.* I will show submodularity of $\mathbb{E}[\mathcal{U}(\omega, \tau)]$ in $(\omega, \tau)$ by first showing showing supermodularity of $\int \Psi(\omega, \lambda)dF(\lambda|\tau)$ in $(\omega, \tau)$. Note that $\Psi_1(\omega, \lambda) = \lambda\psi'(\omega) > 0$. The required inequality for $\omega' > \omega$ and $\tau' > \tau$ is

$$\int \Psi(\omega', \lambda)dF(\lambda|\tau') - \int \Psi(\omega, \lambda)dF(\lambda|\tau') \geq \int \Psi(\omega', \lambda)dF(\lambda|\tau) - \int \Psi(\omega, \lambda)dF(\lambda|\tau),$$
$$\int [\Psi(\omega', \lambda) - \Psi(\omega, \lambda)]dF(\lambda|\tau') \geq \int [\Psi(\omega', \lambda) - \Psi(\omega, \lambda)]dF(\lambda|\tau). \tag{B.105}$$

Note that

$$[\Psi(\omega', \lambda) - \Psi(\omega, \lambda)] = \lambda\psi(\omega') - \lambda\psi(\omega) > 0,$$

as $\psi$ is increasing. Therefore as $F(\lambda|\tau') < F(\lambda|\tau)$ by assumption (B.105) holds by first-order stochastic dominance. To show submodularity of the objective in $(\omega, \tau)$ I multiply (B.105) by $-1$, as the rest of the objective in independent of $\tau$, to give

$$-\int [\Psi(\omega', \lambda) - \Psi(\omega, \lambda)]dF(\lambda|\tau') \leq -\int [\Psi(\omega', \lambda) - \Psi(\omega, \lambda)]dF(\lambda|\tau).$$

□

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