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Economic Aspects of Intelligent Network Selection: A Game-Theoretic Approach

by

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Declaration

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Abstract

Mobile communications has become an indispensable part of our everyday lives, with increasingly more people owning a smartphone, and being given access to a plethora of wireless access technologies: WiFi, 3G, and 4G. In an environment of such diversity, where each wireless access technology has its own distinct characteristics, network selection mechanisms provide an efficient way of handling communications services by matching the services' required quality with the characteristics of a particular access technology.

This thesis explores the economic aspects of intelligent network selection in the context of Digital Marketplace—a theoretical market-based framework where network operators compete in a procurement auction-based setting for the right to transport the user's requested service over their infrastructure. It investigates the suitability of a first-price sealed-bid auction as a network selection mechanism. Since this auction-based mechanism constitutes the main trading mechanism of the Digital Marketplace, the results reported herein affect its feasibility as a market for trading wireless communications services of the future. Since it lacks extensive and rigorous economic analysis, this thesis addresses this deficiency by providing an extensive game theoretic analysis of the network selection mechanism.

This thesis creates an economic model of the network selection mechanism, and is the first to characterise the equilibrium bidding behaviour for an arbitrary number of network operators participating in the Digital Marketplace. It proposes three novel numerical methods that allow for numerical derivation of the equilibrium bidding behaviour: forward shooting method (FSM), polynomial projection method (PPM), and extended FSM (EFSM). The FSM and PPM methods allow for numerically approximating equilibrium bidding behaviour for a subset of all possible bidding scenarios, while the EFSM method enables computation of the numerical solution to all bidding scenarios. Finally, since the EFSM method becomes numerically unstable for large number of network operators, a novel methodology for approximating the network selection mechanism with an auction format for which there exist many well-defined and extensively studied numerical solutions is discussed.

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Contents

List of Publications

[1] J. Konka, C. Michie, I. Andonovic, and R. Atkinson, "Approximating Equilibrium in the Digital Marketplace," in *Vehicular Technology, 2015. VTC2015-Spring. The 81st IEEE Conference on*, 11–14 May 2015.

[2] J. Konka, C. Michie, I. Andonovic, and R. Atkinson, "Auction-based Network Selection in a Market-based Framework for Trading Wireless Communications Services," *Vehicular Technology. IEEE Transactions on*, vol. 63, pp. 1365–1377, March 2014.

[3] J. Konka, J. Irvine, and R. Atkinson, "Economics of Intelligent Selection of Wireless Access Networks in a Market-Based Framework: A Game-Theoretic Approach," *Advances in Networks and Services. IARIA International Journal on*, vol. 5, no. 3 & 4, pp. 210–224, 2012.

[4] J. Konka, R. Atkinson, and J. Irvine, "Economic Aspects of Intelligent Network Selection: A Game-Theoretic Approach," in *Mobile Ubiquitous Computing, Systems, Services and Technologies, 2011. UBICOMM'11. The Fifth International Conference on*, 20-25 Nov. 2011.

Chapter 1

Introduction

This thesis explores the economic aspects of intelligent network selection in the context of Digital Marketplace. In this chapter, the motivation for research into the economics of intelligent network selection is explained, the problem under consideration is defined, and the main contributions are stated.

1.1 Research Objectives

Mobile communications has become an indispensable part of our everyday lives. According to Ofcom [1], 51% of all adults in the UK own a smartphone, and approximately 24% of all UK households own a tablet. Furthermore, one in five adults declares they would miss their smartphone most if it were taken away. It should be noted that these numbers continue to rise, and with each year the penetration of mobile communications will increase.

Parallel to this, mobile users (henceforth referred to as subscribers) are given access to a plethora of wireless access technologies: from WiFi, through 3G, to the latest 4G. Cities throughout the UK are now offering free WiFi hotspots [2]. Furthermore, according to Ofcom, while 3G already covers 98% of the UK population indoors, this figure is promised to be at least matched by the 4G mobile services by the end of 2017 at the latest [3]. In an environment of such diversity and heterogeneity, where each wireless access technology has its own distinct characteristics, network selection mechanisms provide an efficient way of handling communications services by matching the services' required quality with the characteristics of a particular access technology [4]. The importance of these mechanisms is emphasised by the fact that multimode smartphones (iPhones, Android phones, BlackBerry phones) and tablets (iPads, Android tablets) currently dominate the market thus enabling subscribers to connect to many of the available wireless access technologies.

This diversity opens exciting, new possibilities in both the technological and economic sense. The exclusive one-to-one mapping between network operators and subscribers need no longer hold; when requesting a service (for example, making a phone call, or checking the email), the network selection mechanism will be responsible for selecting the network operator (access technology) that best matches the required quality requirements of the service. From the subscribers' perspective, this permits the ability to seamlessly connect at any time, at any place, and to the technology offering the highest quality available for the best price: a paradigm referred to as *Always Best Connected* [5]. From the network operators' perspective, the integration of wireless access technologies will allow for more efficient usage of network resources, and more importantly, improved revenue generation. In other words, it might be the most economic way of providing both universal coverage and broadband access [4].

However, there also exists the possibility of a "tussle" since there are many different actors with opposing interests involved [6]. For example, it is in the best interest of subscribers to obtain the highest quality of the service for the lowest price. Network operators, on the other hand, aim to maximise their profit, and perform efficient load balancing. Furthermore, the situation may become even more complex should the service provision be decoupled from the network operators; that is, if the service provision is handled by a separate entity, service provider, while network operators are left with handling of the transport provision [7]. Therefore, the problem of network selection, which was considered to be technologically difficult, can also be considered to be the problem of economics where wireless access, traded on a per connection basis, is an electronic good that is sold to the subscribers.

This thesis explores the economic aspects of intelligent network selection. The problem is studied within the context of Digital Marketplace—a theoretical market-based framework for trading wireless communications services. It was first proposed by Irvine *et al.* in 2000 [8, 9], and it was developed with the heterogeneous wireless communications environment in mind, where the subscribers have the ability to select a network operator that reflects their preferences on a per service basis. Since the Digital Marketplace was created with free market (or "perfect" competition) in mind, it is particularly well-suited towards the management of future wireless environment where wireless access is traded on a per service basis. It is for this reason that this research explores the problem of network selection within the context of Digital Marketplace.

More specifically, the main research objectives of the work reported in this thesis are to:

1. review existing approaches to intelligent network selection by other researchers;

2. understand the fundamental assumptions about the operation of the Digital Mar-

ketplace;

3. create an economic model of the network selection mechanism described in the Digital Marketplace;

4. apply game theory, and in particular auction theory, to study the model and derive equilibrium bidding strategies for the network operators.

The work conducted in this thesis will investigate the suitability of a first-price sealed-bid auction as a network selection mechanism, and it will explore its economic merits. Since this auction-based mechanism constitutes the main trading mechanism of the Digital Marketplace, the results reported herein will affect the feasibility of the Digital Marketplace as a market for trading wireless communication services of the future.

1.2 Main Contributions

As briefly outlined in the previous section, in order to fully utilise the wealth and heterogeneity of the future wireless access environment, intelligent network selection is a necessary condition: without it, subscribers will only be able to achieve a suboptimal ratio of price to quality of the received services, while network operators will struggle to maximise their profit and usage of their resources. Over the last decade, several different approaches have been proposed as possible solutions to the technologically advanced problem of network selection. Tools as disparate as supervised machine learning algorithms [10], fuzzy logic [11], and Markov decision process were utilised [12].

Intelligent network selection on its own, however, is not an immediately obvious sufficient condition for fully unlocking the potential of the future wireless access environment. A landscape of such diversity and with so many different actors involved will inevitably lead to a "tussle". Therefore, a careful economic analysis of the problem is needed. Several researchers have employed the tools of economics to study the problem of network selection, and to a degree, considered the economic implications of the problem. Those approaches included utility theory [13], multiple attribute decision making [14], and game theory [15].

The work reported in this thesis complements the existing research base through the following main contributions:

1. This thesis presents an economic analysis of the network selection mechanism in the context of Digital Marketplace. The work reported in this thesis complements the work of other researchers on intelligent network selection by proposing an auction-based approach which is based on a procurement first-price sealed-bid auction. In this way, since the proposed mechanism is based on an auction, it

embraces the uncertainties that exist in a competitive environment—an element that the vast majority of the existing literature has ignored. It should be noted, however, that the analysis and results presented in this thesis are easily extrapolated from the context of the Digital Marketplace. As a matter of fact, one of the main aims of this research was to keep the analysis as generic as possible so that the results can easily be applied elsewhere.

2. This thesis fills the gap in the research on the Digital Marketplace. The Digital Marketplace lacks comprehensive economic analysis of the network selection mechanism, and this thesis address this problem by providing such an analysis. An analysis from the economic perspective is important in order to verify whether the mechanism performs as expected in economic terms. For example, thanks to the economic analysis presented in this thesis, it is verified that the network selection mechanism in the Digital Marketplace maximised the expected utilities of the network operators, and from the subscriber's perspective, the network operator who matches the subscriber's preferences in terms of requested price and quality of service is selected. Indeed, with the results presented in this thesis, the Digital Marketplace can now be considered a framework of choice for the management of wireless access networks of the future.

3. Finally, this thesis adapts numerical algorithms for approximating first-price sealed-bid auction with asymmetric bidders to the bidding problem posed by the network selection mechanism in the Digital Marketplace. In this case, the asymmetry of the bidders is unusual and no existing numerical methods are directly applicable. This thesis addresses that deficiency by proposing an extended numerical method that tackles the unusual aspect of the network selection in the Digital Marketplace. To the best of the author's knowledge, the proposed method is the only numerical algorithm in existence capable of solving the bidding problem of the Digital Marketplace.

1.3 Thesis Outline

This thesis is organised as follows. In Chapter 2, the concept and importance of intelligent network selection in future wireless access networks is explained. To this end, the chapter outlines the concepts of heterogeneous wireless access network and Always Best Connected paradigm, and outlines the role of network selection. Then, a summary of previous research on intelligent network selection is given. Finally, the contributions of the research presented in this thesis to the problem of intelligent network selection are outlined.

Chapter 3 introduces the concept of Digital Marketplace and its fundamental assumptions. It starts with description of the principles of operation of the Digital Marketplace, and then moves onto a brief overview of auction theory. The overview is necessary to understand the fundamental assumptions of the network selection mechanism employed by the Digital Marketplace which are subsequently outlined. Furthermore, Digital Marketplace is compared with the wholesale electricity market, and it is shown that due to the similarities between the two markets, Digital Marketplace is more likely to be adopted by the industry as a commodity market for trading wireless communications services of the future. Finally, the chapter concludes with a summary of previous research on Digital Marketplace, and outlines how the research work reported in this thesis complements the work of other researchers.

In Chapter 4, the network selection mechanism employed by the Digital Marketplace is cast into the framework of game theory. The mechanism is then directly analysed in three special cases: 1) when only reputation ratings of the network operators decide on the winning network operator, 2) when only the monetary bids of the network operators matter in the selection of the winner, and finally, 3) when all network operators are characterised by the same reputation rating. In essence, those special cases correspond to the extremes of the studied bidding problem, and in all those cases, the equilibrium bidding strategies are derived. The chapter then concludes with the analysis of the mechanism in a restricted case with only two network operators, for which the equilibrium bidding strategies are derived. They are shown, however, to be suboptimal as they allow the network operators to submit a negative monetary bid.

In Chapter 5, the studied problem is transformed into a problem that has already been researched by the economic community, and hence, there exist results that are applicable to the problem at hand. The chapter then proceeds to characterising the equilibrium bidding strategies (their existence and uniqueness) in the generic case; that is, with an arbitrary number of network operators and arbitrary distributions of costs. The equilibrium bidding strategies are then explicitly derived in the restricted case; that is, with the number of network operators restricted to two and costs uniformly distributed. Finally, the chapter concludes with the presentation of three numerical methods: forward shooting method (FSM), polynomial projection method (PPM), and extended FSM (EFSM). The methods can be used to numerically approximate the equilibrium bidding strategies in the case of more than two network operators characterised by uniform distributions of costs. The first two of the presented methods, FSM and PPM, allow for numerically approximating equilibrium bidding strategies for a subset of all possible bidding scenarios resulting in nontrivial equilibria, while the third method

(EFSM) enables computation of the numerical solution to all bidding scenarios.

The EFSM method becomes numerically unstable for large number of bidders. Therefore, Chapter 6 explores whether an auction format represented by the network selection mechanism employed in the Digital Marketplace can be modelled as an auction with common prior. In an auction with common prior, the range the costs can vary is the same for each bidder. For this type of problem, there are many well-defined and extensively studied numerical solutions. In the first instance, the assumptions governing an auction with common prior are described, and the existence and uniqueness of the equilibrium bidding strategies is formally defined. Following that a numerical method tailored specifically to the auction with common prior is presented. Having derived the numerical method for approximating the equilibrium in the auction with common prior, the methodology for casting the original problem into the auction with common prior is discussed. Finally, the methodology for quantifying the accuracy of the approximation is presented, and the chapter concludes with the presentation of approximation results for four bidding scenarios with two, three, four and five bidders respectively.

Finally, Chapter 7 draws final conclusions, and discusses future work. Furthermore, in Appendix A, mathematical proofs of all propositions included in the thesis are presented, while in Appendix B, the mathematical notation used in this thesis is explained, and an overview of the more important mathematical concepts necessary to understand the work reported in this thesis is provided.

Chapter 2

Intelligent Network Selection

This chapter explains the concept and importance of intelligent network selection in future wireless access networks. To this end, firstly, the concepts of heterogeneous wireless access network and Always Best Connected paradigm are described, and the role of network selection is outlined. Secondly, the chapter summarises previous research work on intelligent network selection. Finally, the contributions of the research presented in this thesis are outlined.

2.1 The Problem of Intelligent Network Seletion

The aim of this section is to explain the concept and importance of intelligent network selection in the future wireless access networks. To this end, this section starts with an explanation of the concept of a heterogeneous wireless access network. It then moves onto a discussion of the Always Best Connected paradigm and the role of intelligent network selection in fulfilling its assumptions. Finally, the importance of economic aspects of intelligent network selection is highlighted.

2.1.1 Heterogeneous Wireless Access Network

Over the last decade, the world of wireless and mobile communications has witnessed several major improvements [5]. The evolution of traditional 2nd Generation (2G) cellular systems (such as GSM), through 3rd Generation (3G) systems (such as UMTS or CDMA2000), into 4th Generation (4G) systems (such as LTE), has drastically improved the cellular coverage worldwide, and provided mobile Internet access [4, 16]. At the same time, IEEE 802.11-based Wireless Local Area Network (WLAN; commonly referred to as WiFi) solutions have emerged as the predominant high-speed wireless Internet access at airports, in hotels, or even at home.

With the introduction of smartphones (iPhones, Android phones, BlackBerry phones)

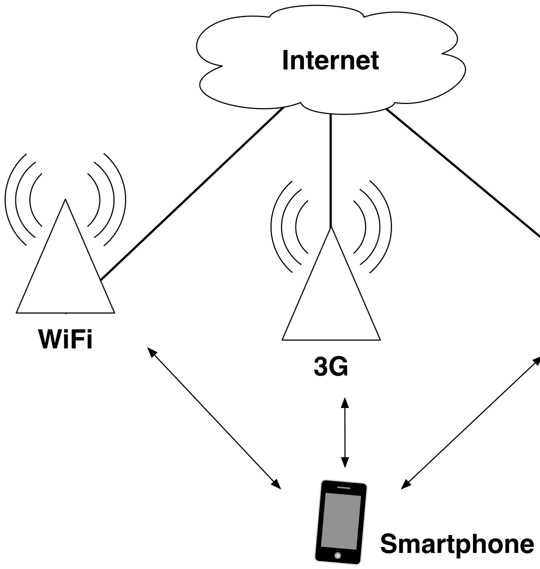


Figure 2.1
Heterogeneous wireless access network (adapted from [4])

and tablets (iPads, Android tablets), the subscribers are finally able to take advantage of both the coverage offered by 3G/4G cellular access network and the high-speed Internet access offered by WiFis. Whenever the smartphone/tablet is in close proximity to a WiFi hot spot, it automatically switches from 3G/4G to WiFi mode for faster data access. However, this only works when either the WiFi hot spot provides free access, or is within the subscriber's subscription; for example, as part of the monthly data allowance plan with a local wireless access network operator. Moreover, this solution lacks the support for session continuity, and does not provide any intelligence when switching from one access network to another. For instance, although the WiFi hot spot is by definition deemed to offer faster data rates, this does not necessarily translate into higher quality of service (QoS). In fact, it might be just the contrary, especially in a very crowded hot spot area where the subscribers run very bandwidth intensive applications such as video or music streaming, or even on-line gaming. For example, under such circumstances, trying to make a Skype call can be nearly impossible [17]. Therefore, the decision to switch from one network to another should not only consider the availability of a particular wireless access network, but also the QoS offered for the best user experience.

Concurrently, the industry is driving for an all-IP-based platform which enables integration of diverse access networks in a common scalable framework [16]. As a result, wide range of multimedia services can be extended to subscribers over *heterogeneous wireless access networks*. The heterogeneous wireless access network spans different wireless access technologies integrated into one network to provide subscribers with the requested multimedia services and QoS. It will take full advantage of the multimodality offered

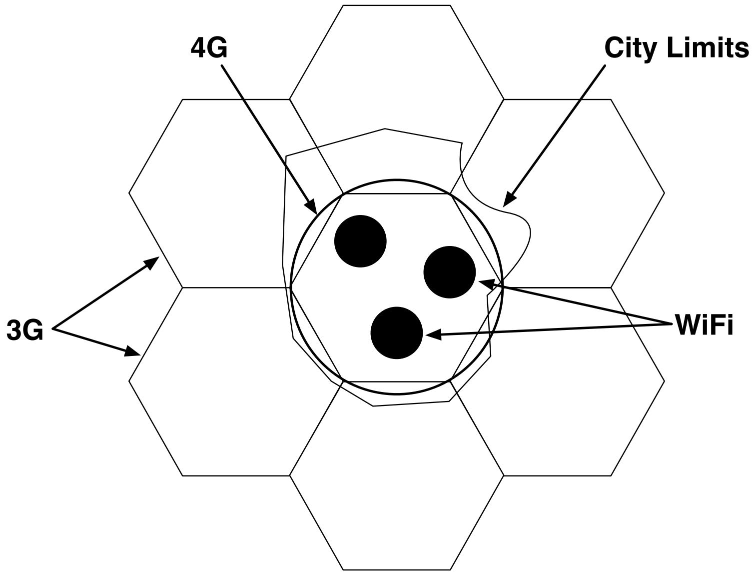


Figure 2.2
Typical distribution of wireless access technologies in a modern-day city

by the smartphones by having the device connected to all wireless access technologies at all times. This is depicted in Figure 2.1.

The heterogeneous wireless access network will possess many advantages over the contemporary wireless networking solution. From the subscribers' perspective, different coverage and QoS characteristics of each of the included wireless access technologies will lead to the ability to seamlessly connect at any time, at any place, and to the access technology which offers the most optimal quality available. This is referred to as *Always Best Connected* paradigm [5], and will be introduced in more detail in the subsequent section. From the network operators' perspective, on the other hand, the integration of wireless access technologies will allow for improved revenue generation, more efficient usage of the network resources, and might be the most economic way of providing both universal coverage and broadband access [4].

Figure 2.2 depicts a typical distribution of wireless access technologies in a modern-day city. In the example, WiFi hot spots are used as a localised very high-speed Internet access; 4G covers nearly the ³/₄ of the city area, and provides high-speed Internet access; and 3G delivers medium speed wireless access inside as well as outside the city. There is a high overlap of different wireless access technologies within the city. With the adoption of a heterogeneous wireless access network, this overlap could be utilised to its full potential by providing better network resources management, and high-speed and high quality Internet access for the subscribers inside as well as outside the city limits.

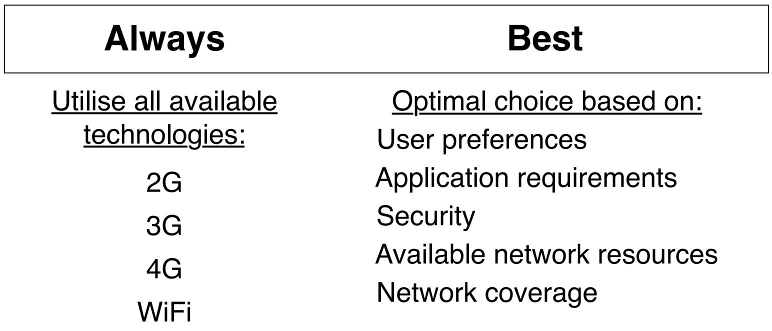


Figure 2.3
The essence of ABC networking paradigm

2.1.2 Always Best Connected and Intelligent Network Selection

The *Always Best Connected (ABC)* paradigm assumes that a subscriber is: (1) "always" connected to the Internet, and (2) uses the "best" access technology available [5]. "Always" should be interpreted as being able to utilise all wireless access technologies available at any time, while "best" implies that when a particular technology is being chosen, several factors such as subscriber's preferences, application requirements, network coverage, etc., are considered in order to make the most optimal selection possible (see Figure 2.3). The mechanism responsible for implementing the ABC principles is referred to as *intelligent network selection*.

Furthermore, the paradigm emphasises seamless information delivery and extensive mobility support. In other words, the changes in the communications environment should affect the subscriber as little as possible, even when they are "on the move". Therefore, should the subscriber move from the coverage area of one access technology to another, the switch should be as non-disrupting for the subscriber as possible; i.e., the session continuity should be maintained at all times, regardless of the access technology currently used. Thus, it is clear that intelligent network selection plays a vital role in the successful operation of the ABC solution.

2.1.3 Economic Aspects of Intelligent Network Selection

Since there are many different actors with opposing interests involved, there also exists the possibility of a "tussle" [6]. For example, it is in the best interest of the subscribers to obtain the highest quality of the service for the lowest price. Network operators, on the other hand, aim to maximise their profit, and perform efficient load balancing. Furthermore, the situation may become even more complex should the service provision be decoupled from the network operators; that is, if the service provision is handled by a separate entity, service provider, while network operators are left with handling of the transport provision [7]. Therefore, the problem of intelligent network selection, which as pointed out in the previous section, was considered to be technologically difficult, can also be considered to be the problem of economics where wireless access, traded on a per connection basis, is an electronic good that is sold to the subscribers.

In this section, the core concepts and the importance of intelligent network selection in the wireless access networks of the future were outlined. The next section summarises the research efforts of the researchers on the problem of intelligent network selection.

2.2 Intelligent Network Selection in the Literature

Over the last decade, several papers have explored the problem of intelligent network selection in heterogeneous wireless access networks utilising concepts from economics. It is the aim of this section to summarise the research work of other researchers on the problem of intelligent network selection. To this end, the approaches are grouped based on the economic theory they utilise; that is, utility theory-based approaches are discussed first, followed by multiple attribute decision making, and game theory. Finally, approaches to network selection that are not economics oriented are summarised in a seperate section entitled miscellaneous. It is worth noting that the grouping is inspired by an excellent survey paper by Wang and Kuo [18].

2.2.1 Utility Theory

The utility theory approach to network selection is based on the concepts of classical demand theory of microeconomics. Classical demand theory assumes that each consumer is characterised by a preference relation that can be captured by a mapping into real numbers—the utility function [19]. For example, suppose a subscriber prefers a monthly subscription contract to wireless services over a yearly one. Then, their utility of the former is greater than the utility of the latter.

When applied to the problem of intelligent network selection, the utility function measures the level of subscriber's satisfaction with a set of characteristics offered by an access network [20]. Thus, the main challenge is to capture subscriber's preferences for different attributes/characteristics of a wireless service in the form of a utility function. Some common attributes include but are not limited to: bandwidth required by the service, price of the service, bit error rate, delay, etc. In the context of network selection, utility function is defined as a mapping from the set of all attributes of a wireless service to the set of real numbers. Formally, let $\mathcal{X}$ denote the set of all attributes of a wireless service. Then, the utility function is a function $U : \mathcal{X} \to \mathbb{R}$. Therefore, it encodes user preferences for different attributes of a particular service as real numbers. The network selection mechanism uses the derived utility function to select network/access technology that yields the highest overall utility to the subscriber. The main shortcoming of this approach is to decide on a correct functional form of the utility function for each attribute of a wireless service. The most common ones include: linear piecewise, logarithmic, exponential and sigmoidal [20]. The question remains, however, whether those functional forms are representative of the reality. In fact, there are as many characterisations of the attributes and associated utility functions as there are researchers working on the problem of utility theory-based network selection (cf. Table III in [18]).

A more interesting approach to network selection that is not a pure utility theory-based approach but uses the concept of utility function is by Ormond *et al.* [21, 22, 13]. The authors propose to use the concept of consumer surplus to drive network selection. In economic terminology, consumer surplus refers to a measure of net benefit from consuming a good, and is equal to the difference between the utility the consumer extracts from consuming the good and its price [19, 23, 24]. In their work, Ormond *et al.* focus mainly on non real-time data services, and for this case, derive a nonincreasing utility function which captures subscriber's willingness-to-pay versus their willingness-to-wait. The network that achieves the highest consumer surplus for the service is then selected. However, the approach is suffering from the same problem as all utility theory-based approaches; namely, how to capture utility of a subscriber so that it is representative of the reality.

2.2.2 Multiple Attribute Decision Making

Multiple attribute decision making (MADM) is a branch of multiple criteria decision making (MCDM), a subdiscipline of operations research [25]. The main aim of MADM is to aid a decision-maker in making a complex decision that often depends on multiple, possibly conflicting attributes.

MADM-based network selection is closely related to utility-based approach as it effectively studies methods of combining utilities for different attributes in the most optimal way [26]. As such, the main challenge of this approach is similar to that of utility-based approach; that is, capturing subscriber's preference per attribute per scenario.

The MADM-based network selection usually involves [14]:

1. gathering and quantifying values for each considered attribute for each candidate network. For example, candidate network A might have a price attribute of 0.5, while candidate network B of 0.25.

2. calculating weights for each attribute based on subscriber's preferences. For example, a subscriber might prefer a higher price but a lower delay, and hence, a

higher weight is assigned to the price attribute.

3. utilising a MADM algorithm to score each candidate network based on the combination of attributes and weights. It is interesting to note that the score is equivalent to the combined utility function in the utility-based network selection. The score can be computed, for example, as a simple sum of all the attributes weighted by the weights, or alternatively, the attributes can be raised to the power of the corresponding weights and multiplied together. The former approach is referred to as simple additive weighting (SAW) [27], while the latter as multiplicative exponential weighting (MEW) [20]. Other popular algorithms include: grey relational analysis (GRA) [28], and technique for order preference by similarity to an ideal solution (TOPSIS) [29].

4. selecting candidate network which achieves the highest score.

The MADM-based network selection algorithms suffer from two issues: 1) since they utilise the concept of subscriber's utility, similarly to the utility theory-based approaches, it is difficult to correctly capture subscriber's preference per attribute per scenario; 2) different MADM methods often produce inconsistent ranking outcomes for the same problem, and therefore, establishing the validity of the proposed MADM-based scheme proves prohibitive [30, 31].

2.2.3 Game Theory

Game theory deals with the analysis of mathematical models of conflict and cooperation between two or more intelligent individuals [32, 33, 34]. In game theoretic terminology, a game refers to any social (and possibly conflictual) situation involving two or more individuals who are referred to as players. The players are assumed to be rational decision-makers; i.e., they will always strive to make the best decision possible in pursuit of their own interests. Furthermore, each player is characterised by a set of strategies and payoffs/utilities for choosing a particular strategy. For each player, payoff depends not only on their chosen strategy, but also on the strategies chosen by other players. The aim of game theoretic analysis is then to characterise an equilibrium (typically, Nash equilibrium); that is, a set of strategies that if played by all the players, guarantee the highest utility given the strategy choices of other players.

There exist two distinct approaches to game theory, classical and evolutionary game theory, and both were extensively applied to the problem of intelligent network selection [18]. While classical game theory concentrates on the analysis of possible outcomes of a game between N players, in evolutionary game theory the focus is put on the dynamics of the whole population (or a group) of decision-makers. In other words, classical game theory concentrates on characterising an equilibrium (if exists) to an N-player game. Evolutionary game theory, on the other hand, examines how a particular decision made by the whole population changes over time in response to the decisions made by all players individually.

When applied to network selection, the problem is modelled as either a game between the subscribers, or a game between the network operators. There has been some nonextensive research carried out that permitted a third possibility of a game between subscribers and network operators [18, 35]. In particular, in [35], the authors model the network selection problem as a noncooperative game where subscribers select network operators who maximise the requested services' requirements, while the network operators select the subscribers who maximise their revenue and allow for optimal load balancing. The mechanism proposed by the authors is limited due to the fact it is not guaranteed to converge on an equilibrium solution; in fact, in their formulation of the game, Nash equilibrium is not guaranteed to exist. The authors provide an alternative solution concept, called *suboptimal solution*, however, this solution concept is not rigorously shown to constitute an equilibrium of the game. In other words, in practice, it might happen that the players (subscribers and network operators) will deviate from it significantly; therefore, putting in question the proposed network selection scheme. This result demonstrates the complexity of the problem of network selection, and the fact that modelling it as a game between the subscribers and network operators might not be viable. For this reason, this thesis concentrates on the former two, more widespread approaches.

2.2.3.1 Games Between Subscribers

Modelling network selection as a game between the subscribers aims at arriving at an (equilibrium) distribution of the subscribers between the available networks, and as a result, avoiding network congestion and performance degradation. In other words, limited wireless resources are shared in the most optimal way between the subscribers, and at the same time, the subscribers select the network that matches their preferences in the best possible way [36]. In what follows, a few noteworthy examples of this modelling approach described in the literature are outlined.

Niyato and Hossain [36] use evolutionary game theory to study dynamics of competition among groups of subscribers in different service areas. The authors propose two algorithms for deriving an equilibrium to the problem: population evolution and reinforcement learning. The first algorithm uses centralised controller (e.g., a base station) to maintain payoff information for all subscribers. In the second approach, each subscriber tries different networks, observes the size of the allocated bandwidth and price from the chosen network, and changes the network selection if necessary; in other words, each subscriber learns and adopts independently based on the past events. The authors simulate the proposed algorithms in a scenario with three different wireless access technologies, and examine the performance of the algorithms (such as the speed of convergence on the equilibrium). Addtionally, in a different paper [37], the same authors model the subscriber churning behavior in heterogeneous wireless access networks using evolutionary game theory, and use the derived evolutionary equilibrium to study two different pricing schemes for the wireless providers: noncooperative and cooperative. Finally, Zhu *et al.* [38] build upon the work reported in [36], and use Bayesian evolutionary game theory to derive and study the dynamics of the equilibrium in an environment where subscribers have only limited (incomplete) information about each others preferences.

To recap, modelling network selection as a game between subscribers assumes that the subscribers are the decision-makers, and that they will distribute themselves between the available networks in a way that avoids network congestion and performance degradation. While this is appealing from the perspective of economic usage of scarce network resources, network operators are more likely to prioritise revenue generation over optimal load balancing. Thus, it might be challenging to convince the network operators to adopt this approach. Furthermore, since this approach relies on modelling the payoffs/utilities of the subscribers, it suffers from the same problem as utility theory-based and MADM-based approaches: difficulty in correctly capturing subscribers' preferences.

2.2.3.2 Games Between Network Operators

In a heterogeneous wireless access network, network operators will witness a more severe competition for the subscribers, since the subscribers will be given the freedom to choose a network operator at any time. Therefore, it is crucial for network operators to understand the implications of the increased competition and the behaviour of the competing network operators in order to ensure they stay competitive in this market of the future and attract as many subscribers as possible [18]. This is the aim of the second game theoretical approach to network selection, that is, modelling the problem as a game between network operators. This approach is different from the previous one in the sense that it indirectly guides the subscribers into selecting the most optimal network by prescribing guidelines to the competing network operators. In other words, with this approach, the network operators are prescribed a set of guidelines that lead to an equilibrium in which their networks are selected by subscribers with matching preferences. Several researchers have considered this problem, and the most interesting of the approaches are briefly outlined below.

Niyato and Hossain [15] explore the competitive pricing in a heterogeneous wireless access network. To this end, they study a scenario consisting of three network operators, and assume each network operator offers two types of connections: premium and best-effort connections. The former have a fixed price, while the latter are dynamically priced and depend on the competitive or cooperative behaviour of the network operators. The authors model the problem in three different ways: as a simulatenous-move noncooperative game (i.e., prices are offered to the subscribers at the same time), leader-follower Stackelberg game (i.e., network operator may offer their price before other network operators), and a cooperative pricing game (i.e., network operators cooperate in order to maximise their total revenue across all network operators). In [39], Antoniou and Pitsillides model the problem as a noncooperative game between wireless access networks with the aim of obtaining the best possible trade-off between the efficiency and the available capacity of networks, while, at the same time, satisfying the requested quality by the subscribers. Charilas *et al.* [40, 41] extend the work reported in [39] by focusing on the computation of payoffs for each competing network—the authors employ GRA to compute the payoffs. Chang *et al.* [42] propose a scheme that combines utilty-based approach with game theory approach. Their approach involves the following three stages: 1) subscriber calculates utility value for each candidate network; 2) the competition between candidate networks is modelled as a cooperative game; and 3) the network that maximises linear combination of utility and the resulting equilibrium payoffs of stage 2) is selected.

All of the proposed solutions discussed thus far suffer from the same shortcoming: in all cases, the authors assume that the network operators have complete knowledge of the cost structure, etc., of their opponents, i.e., other network operators. This assumption is unrealistic as there is always some uncertainty present especially in a competitive setting such as this one [23]. Some researchers have attempted to address this issue by modelling the network selection mechanism as an auction. To elaborate further, the main use case for auctions is a scenario where one party (usually the seller) is uncertain how valuable the object being traded is to the other party (usually the potential buyers; see Section 3.2.1 for a detailed overview of auctions). Khan *et al.* [43, 44, 45] model the problem as a procurement second-price sealed-bid auction where network operators bid for the right to service the subscriber's request. While their results are theoretically appealing, second-price sealed-bid auctions are rarely used in practice due to several inherent weaknesses, such as vulnerability to collusion by a coalition of losing bidders, or vulnerability to the use of multiple bidding identities by a single bidder [46].

Finally, Irvine *et al.* [8, 47, 48] propose a theoretical market-based framework called the Digital Marketplace where network operators compete in a variant of a procurement first-price sealed-bid auction for the right to transport the subscriber's requested service over their infrastructure. However, the authors do not verify whether the proposed bidding strategies constitute an equilibrium of the proposed auction. This is a major flaw from the economic viewpoint as otherwise it is unclear whether the mechanism induces rational behaviour in participants; for example, whether it maximises the network operators' expected utilities.

2.2.4 Miscellaneous

Thus far, those approaches to intelligent network selection that utilise concepts from economics to drive the mechanism were scrutinised. Since this thesis studies the economic aspects of the problem, they are deemed as directly relevant; however, the research on network selection abounds, and there exist many different approaches that are not economics oriented. The most notable of those approaches are outlined in this section.

Espi *et al.* [10] present a machine learning approach to network selection; in particular, the authors utilise a Hopfield neural network to solve the underlying optimisation problem. Hopfield neural network is an example of a supervised learning approach based on the artifical neural network algorithm. In their paper, the authors model the cost function for each subscriber as a ratio between the bandwidth required by the service to the total available bandwidth of a network operator. They further demonstrate that their approach achieves better bandwidth allocation than two other similar approaches of other researchers. Since the authors model the subscribers' preferences consisting of only the required bandwidth, they fail to capture the true complexity of the problem of network selection. That is, other technical parameters such as packet delay, jitter, etc. are left out. Furthermore, no accounting of subscribers' monetary preferences and commitments is given. If included, it would strengthen the applicability of the algorithm to real-life scenarios. Khaleel *et al.* [49], on the other hand, propose a mechanism based on k-nearest neighbour ($k$nn) classfication algorithm. They model the subscriber as being characterised by three parameters: cost, mobility and energy. Energy is equivalent to the current level of the phone's battery. Mobility specifies how mobile the subscriber anticipates to be while contracting the service. Cost is assumed to represent the subscriber's willingness to pay for the service; it can take on one of three possible labels: not important, avoid, very important. While the approach proposed by the authors tries to capture the subscriber's monetary preferences to a degree, restricting it to three possible labels is prohibite and not representative of reality. Liu *et al.* [50] propose an algorithm for optimal network selection which mainly aims at optimising energy consumption of the subscriber equipment. Since the proposed algorithm prioritises energy efficiency above all else, their solution suffers from the same shortcoming as the previous two approaches: it does not capture subscriber's monetary preferences, and therefore, under-represents the complexity of the problem of network selection.

Hou and O'Brien [11] cast the problem of network selection into the framework of fuzzy logic. Fuzzy logic generalises traditional (binary) logic to situations that are neither true nor false; rather, to a degree, they can be both at the same time. The authors advocate the usefulness of fuzzy logic in the context of network selection due to its ability to solve uncertainty and contradiction embedded within the problem; for example, achieving both high throughput and zero delay. The authors model the problem using three inputs: probability of interruption, failure probability of handover, and size of unsent messages. While their approach is innovative, it concentrates solely on technical aspects of network selection, ignoring the economics of the problem. Finally, Stevens-Navarro *et al.* [12] model the problem as a Markov decision process (MDP). MDP is used in modelling of decision making processes that contain some degree of randomness, and yet are at least partially under the control of the decision-maker. Applied to the problem of network selection, the decision-maker is the subscriber, and the process corresponds to selecting network under uncertainty of receiving the desired QoS. Similarly to the previous papers mentioned in this section, the authors focus on the technical aspects of the network selection, and ignore the important economic aspect.

This section provided an overview of the literature on intelligent network selection. In the next section, the contributions of this research to intelligent network selection are summarised.

2.3 Contributions of this Research to Intelligent Network Selection

As outlined in the previous section, substantial work has been carried into the modelling of network selection from the economic perspective. The approaches can be categorised into 3 major classes: utility theory-based, MADM-based, and game theory-based approaches. Each class of approaches is facing some challenges that need to be addressed if it is to be implemented in reality. To summarise, utility theory-based and MADM-based approaches require a specification of a correct functional form of the utility functions for the subscribers that is representative of reality; not an easily accomplished task. The game theory-based approaches can be further subdivided into two categories: 1) games between subscribers where the subscribers distribute themselves in the most optimal way between the available networks; and 2) games between network operators where the network operators compete for the right to provide the subscribers with a service. The former approach faces two major challenges. Firstly, since it relies on modelling of the utilities of the subscribers, similarly to utility theory-based and MADM-based approaches, it requires a correct specification of the utility functions for the subscribers. Secondly, this approach prioritises economic usage of network resources over revenue generation for the network operators, and as such, it might not be appealing to the network operators. In the latter approach, on the other hand, the vast majority of the authors assume that the network operators have complete knowledge of the cost structure, etc., of their opponents. This assumption is unrealistic as there is always some uncertainty present especially in a competitive setting such as this one [23]. Some researchers have attempted to address this issue by modelling the network selection problem as an auction. In particular, first-price and second-price sealed-bid auctions were advocated. While the theoretical properties of the second-price sealed-bid auction are appealing, the auction itself is rarely used in practice due the several inherent weaknesses. The first-price sealed-bid auction-based mechanism was first proposed by Le Bodic *et al.* [8] in their theoretical framework, Digital Marketplace. While the use of the first-price sealed-bid auction as a network selection mechanism is very appealing, the mechanism itself was not subjected to extensive economic analysis. Therefore, fundamental questions such as whether the proposed bidding strategies by the authors constitute an equilibrium of the proposed auction remain unanswered.

The main contribution of the research reported in this thesis to the problem of intelligent network selection is an extensive economic and game theoretic analysis of the problem of network selection in the context of the Digital Marketplace (see Chapter 3 for a detailed description of the Digital Marketplace). It should be noted, however, that the analysis and results presented in this thesis are easily extrapolated away from the context of the Digital Marketplace. In fact, one of the main aims of this research was to analyse the bidding problem in the context of the Digital Marketplace, but in a manner such that the results are stated in a form generic enough to be applied elsewhere. In this way, this thesis contributes to the problem of intelligent network selection by proposing an auction-based network selection mechanism which is based on a procurement first-price sealed-bid auction. Therefore, since the mechanism is based on an auction, it embraces the uncertainties that exist between the network operators in a competitive environment.

2.4 Summary

In this chapter, the concept and importance of intelligent network selection in heterogeneous wireless access network and Always Best Connected paradigm were explained. Furthermore, research work on the concept of network selection of other researchers was summarised, and the contributions of the research work documented in this thesis were outlined.

Chapter 3

Digital Marketplace

This chapter introduces the concept of Digital Marketplace (DMP) and its fundamental assumptions. Firstly, the principles of operation of the DMP are described. Then, a brief overview of auction theory is provided, and the fundamental assumptions of network selection mechanism employed by the DMP are outlined. Furthermore, DMP is compared with the wholesale electricity market, and it is shown that due to the similarilities between the two markets and the fact that the wholesale electricity market is used in practice, DMP is more likely to be adopted by the industry as a commodity market for trading wireless communications services of the future. Finally, the chapter concludes with a summary of previous research on DMP, and outlines how the research work reported in this thesis complements the work of other researchers.

3.1 Principles of Operation

The DMP is a theoretical market-based framework for trading wireless communications services. It was first proposed by Irvine *et al.* in 2000 [8, 9], and it was developed with the heterogeneous wireless communications environment in mind, where the subscribers (of communications services) have the ability to select a network operator that reflects their preferences on a per service basis.

In its basic form, there are three main groups of economic agents involved in the operation of the DMP: *subscribers*, *network operators*, and *market provider*. This is depicted in Figure 3.1. The subscribers are the end-users of the communications services, and act as the buyers in the DMP. The network operators, on the other hand, act as the sellers/bidders, and are responsible for providing the subscribers with services and facilitating network resources required to transport said services. In networking terminology, network operators are equivalent to mobile network operators (MNOs); for example, O2 or Vodafone in the UK. Lastly, the market provider is tasked with operating the

DMP; thus providing common platform for all agents involved. It is left open-ended who should be the market provider; however, one of the following three choices is the most likely: a regulatory body, a consortium of network operators, or a single network operator on behalf of the regulatory body [8, 48].

It should be noted that the original specification of the DMP found for example in [8, 48, 51] differentiates between service and network providers. Thus, the network operator as an entity is decoupled into service and network providers as shown in Figure 3.2. According to this model, the service provider is responsible for providing communications services to the end-user, while network provider facilitates (physical) network resources required to transport said services. A good example of a service provider is that of a mobile virtual network operator (MVNO). An MVNO, such as Giffgaff in the UK, provides services to the end-users, but does not necessarily own a physical network infrastructure; instead, they enter into a contract with an MNO, and use their network to transport users' services. In the UK, Giffgaff has such a contractual agreement with O2.

While the original specification of the DMP advocates decoupling of service and network provision, the research reported in this thesis concentrates on the basic business model; i.e., service and network provision is handled by one entity, a network operator. The basic business model is seen as the most appropriate since any additional aspects of the decoupled case can later be incorporated into the mathematical model of the negotiation process developed in this thesis without affecting the results reported herein. The applicability of the results reported in this thesis to the decoupled case deserves a more elaborate explanation. In the decoupled case, the subscribers do not enter into direct negotiation with the network providers. Instead, they are represented by the service providers who conduct the negotiation on their behalf; hence, acting as the buyers. While the results presented in this thesis are still applicable to this case, the overall business model of the DMP is significantly complicated by the fact that the service providers act as intermediaries between the subscribers and the network providers. For example, for the subcribers characterised by similar preferences, the service providers might enter into negotiation with the network providers only once rather than negotiate on behalf of each subscriber separately. Furthermore, the service providers might reach an agreement with the network providers that the winner of the negotiation will supply their services to a number of subscribers for a specified period of time, such as a day, a week, etc. The model developed in this thesis should incorporate these possibilities if it was extended to the decoupled case.

In this section, the principles of operation of the DMP were outlined. The next section provides a conceptual overview of the negotation process between the subscriber

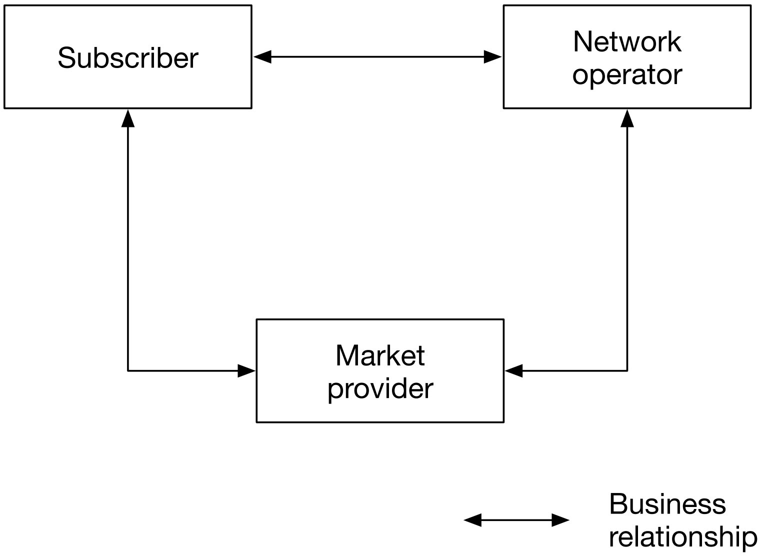


Figure 3.1
The business model of Digital Marketplace (adapted from [48])

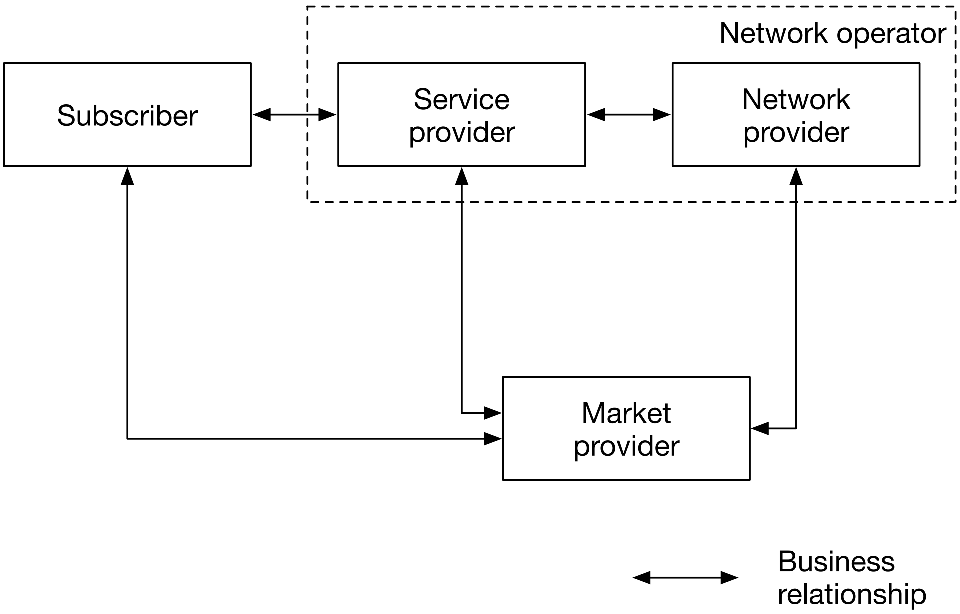


Figure 3.2
Decoupling of network operator into service and network providers (adapted from [48])

and the network operators. This process is termed *network selection mechanism*, and is the means for the subscriber to select the network operator who reflects their service preferences best.

3.2 Network Selection Mechanism

This section is organised as follows. Firstly, a high-level overview of auction theory is given, which is necessary to understand the fundamental assumptions governing the network selection mechanism in the DMP. Then, a conceptual overview of the network selection mechanism is provided.

3.2.1 Primer in Auction Theory

As argued by many economists [52, 53], auction theory is one of the most prominent branches of economics. Examples of auctions being used in real life abounds: purchasing rare items of considerable value such as paintings, buying a house, or simply shopping on eBay. Another very good example of the popularity of auctions in real life is the well-known case of spectrum auctions which were used by both the US and the UK governments to sell the radio spectrum licenses to network operators. It is the purpose of this section to provide a high-level overview of the most important models and assumptions of auction theory.

3.2.1.1 Common Auction Formats

There are many different auction formats reported in the auction theory literature; however, four are particularly popular, and are explored in this section [54]. Those are *English* (also known as open ascending-price), *Dutch* (also known as open descending-price), *first-price sealed-bid*, and *second-price sealed-bid* auctions.

In an English auction, the person who conducts the sale, i.e., the auctioneer, calls out bids in an increasing fashion until there is only one interested bidder left [55]. For example, the sale of a painting (or other work of art) would traditionally be facilitated by the mechanism of an English auction. In such an auction, the auctioneer would set the base price for the object to be sold, x say. Suppose further that some bidder A registered their interest (by raising their hand, or otherwise) in obtaining the object for the price of x. Then, bidder A would be proclaimed the highest bidder, and the auctioneer would call out another price, y, such that it is higher from the previous one, $x < y$ (hence, the alternative name of ascending-price auction). If other bidder, bidder B say, registered their interest in obtaining the object for the new price of y, then bidder B would become the highest bidder, and the auctioneer would further increase the price. And so on, until no further interest was observed. The object would then go to the current highest bidder.

Similarly to an English auction, in a Dutch auction, the prices of the object for sale change in a sequential manner; however, in a Dutch auction, the price is decreasing [55]. The auction is conducted in the following way. The auctioneer starts at a price x, say. If no bidder registers interest within the given time limit set by the auctioneer, then the price is decreased to y, say, such that $y < x$. And so on, until a bidder registers interest. Then, the object is sold for that price to that bidder. It is worth noting that English and Dutch auctions are an example of open auctions since every bidder observes the bids of all the other bidders. This is in contrast to first-price and second-price sealed-bid auctions where bidders submit their bids simultaneously in sealed envelopes.

In a first-price sealed-bid auction, the bidders submit their bids simultaneously in sealed envelopes to the auctioneer [55]. The bidder who submitted the highest bid, wins the auction and pays what they bid. Second-price sealed-bid auction is very similar to a first-price sealed-bid auction with this difference that the highest-bidding bidder wins the auction, but pays the amount equal to the second-highest bid. That is, suppose the highest and second-highest bids are denoted by b_1 and b_2 respectively such that $b_1 > b_2 > b_i$ for all i such that $i > 2$. Then, in a first-price sealed-bid auction, bidder who submitted the highest bid b_1 wins and pays that amount. In a second-price sealed-bid auction, on the other hand, bidder who submitted the highest bid b_1 wins, but pays the amount equal to the second-highest bid b_2.

3.2.1.2 Bidder Valuations

The main reason for a seller of an object to use an auction mechanism is because they are uncertain how valuable the object is to the potential bidder. Otherwise, if the bidders' valuations were known, the seller would simply discriminate by offering the object to the bidder who is willing to pay the most [54]. This uncertainty in bidders' valuations is categorised into three distinct models: *private values*, *common values*, and *interdependent values* model.

In private values model, each bidder knows the value of the object to himself at the time of bidding. Furthermore, each bidder can only make an intelligent guess about the valuations of other bidders. Finally, the knowledge of other bidders' valuations would not affect the bidder's own valuation for the object.

In common values models, the value of the object is the same for all bidders. It is, however, unknown to them at the time of bidding except for some estimates of the true value.

The interdependent values model generalises the common values model in the sense that the value of the object is not necessarily the same for all bidders. The bidders, however, still possess only estimates of the true value of the object at the time of bidding. Furthermore, in interdependent values model, the information available to other bidders, i.e., their estimates of the value, may influence the value of the remaining bidders if known to them.

To put the models in context, a scenario where a painting is sold at an auction and the bidders do not intend to resell the object in the resale market, that is, they assign values to a painting only on the basis of how much utility they would derive from possessing it, can accurately be approximated by private values model. Common values and interdependent values model, on the other hand, is a good approximation to an auction where land with an unknown amount of oil is being sold; bidders may possess different estimates of the amount of oil, but the final value depends directly on the future sales of the extracted oil.

3.2.1.3 Standard versus Procurement Auctions

So far, only the assumptions and models of auction theory where a number of bidders is contesting for the right to buy an object were discussed. However, auctions can also be used to sell goods. In other words, in such a scenario, the auctioneer is trying to purchase an item for the lowest price from a set of sellers/bidders. This type of auctions is referred to as *procurement auctions*.

It is important to realise that, from a game-theoretical perspective, procurement auctions are equivalent to standard auctions in the same setting [54]. Therefore, the abundance of results on standard auctions applies to procurement auctions with only certain small, conceptual differences; for example, standard auctions consider the maximum bid, while procurement auctions the minimum bid. This fact is exploited in this thesis, and proofs of the results not already covered in the literature on auctions in general are provided. This is since if the result is proved in one case (be it either for standard or procurement auctions), it can immediately be adapted to the other case.

To provide an example, consider a procurement first-price sealed-bid auction. First-price auction is discussed since, as evident in the subsequent section, the network selection mechanism in the DMP is based on a procurement first-price sealed-bid auction. Similarly to the standard auction, the bidders (sellers of a good) submit their bids (prices) simultaneously in sealed envelopes to the auctioneer (buyer). Suppose the lowest and second-lowest bids are denoted by b_1 and b_2 respectively such that $b_1 < b_2 < b_i$ for all i such that $i > 2$. Then, the bidder who submitted the lowest bid b_1 wins the auction, and sells the good to the auctioneer for the price equivalent to their bid, i.e., b_1.

3.2.2 Conceptual Overview

The process of negotiation (or the network selection mechanism) in the DMP is based on a procurement first-price sealed-bid auction. Unlike in a procurement first-price sealed-bid auction, however, the winning bid is a weighted (convex) combination of both the network operator's monetary bid and their reputation rating; henceforth referred to as the *compound bid*. The network operator is elected as the winner of the auction if their compound bid is the lowest in value, and accrues their monetary bid minus the cost of supporting the service. The monetary bid is equivalent to the price of supporting the service by the network operator. The precise definition of the price is left open-ended; one possibility, for example, would be to charge the buyer per unit of bandwidth. The weights in the compound bid are set by the subscriber before each auction, and are announced to the network operators. This effectively gives the subscriber the freedom to choose any combination ranging from: a low price for the service but also poor quality; to a high quality but for a high price [51].

It is important to note that, out of sequential-bid and sealed-bid auctions, a procurement first-price sealed-bid auction was chosen due to the following reasons. Firstly, given the timing constraints in the DMP (e.g., the waiting time of the subscriber for the service to be admitted), and the difficulty in predicting the number of bids placed until the winner is selected in a sequential-bid auction, sealed-bid auctions were deemed as the most appropriate [51]. Secondly, the rules governing a second-price sealed-bid auction may appear as counter-intuitive to the subscriber; that is, as mentioned in the previous section, the lowest bid secures the auction but the price paid equals the second-lowest bid. Lastly, since the subscribers not only base their network selection strategy on the offered price, but also on reputation, a first-price sealed-bid auction is the best fit to such a requirement.

Furthermore, since the communications services are traded on an individual service level, it might be difficult for the subscriber to judge the overall quality of the services supplied by a particular network operator [48]. Therefore, one of the fundamental assumptions governing the operation of the DMP is that, by registering in the DMP, network operators agree to report on their contract fulfillments to the market provider; that is, they agree to report a binary value denoting the success in delivering the service to the subscriber within the agreed QoS bounds [8]. The value of 0 denotes a failure, while the value of 1 a success. The latest d ($d > 1$) reports are then used to compute the reputation rating of the network operator which will be used when a new service request arrives in the marketplace. Hence, assuming network operator i admitted t service requests, the formula for computing a reputation rating update is as follows (cf. Section 3.2

in [8])

$$r_i^{t+1} = \sum_{k=t-d}^{d} \frac{1-\rho_i^k}{d}, \tag{3.1}$$

where ρ_i^k denotes the k^{th} binary report of the network operator i. It is important to notice that the reputation rating relates to the network operator rather than to the type of service offered by the network operator, such as phone call, email or web browsing requests [51]. Note further that Equation (3.1) implies $r_i^{t+1} = 0$ if the network operator i has successfully delivered d services to the subscriber, while $r_i^{t+1} = 1$ if has failed in all d attempts. Furthermore, Equation (3.1) implies that if the operator is consistently unreliable, their performance is reflected accordingly by their reputation rating history. Whilst, similarly, one failure in delivering the service does not immediately render a network operator unreliable; rather, it marginally affects their updated reputation rating. At the same time, at the end of each contract, the subscriber may report on their satisfaction (or Quality of Experience, QoE) with the service, for example, by submitting a mean opinion score in case of real-time services, and achieved throughput for non-real-time ones. The reputation rating update formula in Equation (3.1) could then be modified to incorporate QoE, for instance, by taking an appriopriately weighted composition of both network operator's and subscriber's reports. The literature on the concept of QoE abounds. For example, Kilkki discusses the conceptual differences between QoS and QoE, and how the concepts fit into the communications ecosystem [56]. He further defines QoE to be a set of metrics, such as mean opinion score, that capture the experience of the users with a particular service. At the same time, he reserves QoS to consist of metrics capturing the quality of the service at the technical level between network and application; e.g., bit rates, delay properties, and packet loss rates. Brooks and Hestnes, on the other hand, discuss different ways of measuring QoE both subjectively and objectively [57]. Furthermore, Fiedler *et al.* propose a quantitative formula for relating QoS with QoE [58], and similarly, Shaikh *et al.* provide insight into the correlations of QoS and QoE; hence, capturing the relationships between the two metrics [59].

In this section, a conceptual overview of the network selection mechanism was provided. The next section draws a comparison between the DMP and the wholesale electricity market, argumenting that through the similarilities of the two markets and the fact that the wholesale electricity market is used in practice, DMP is more likely to be adopted by the industry.

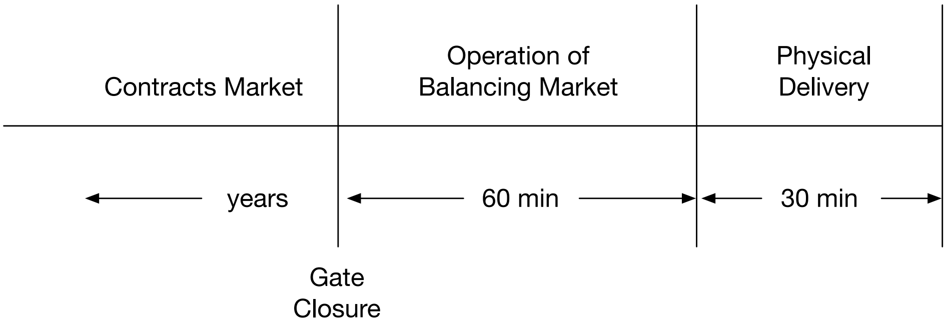


Figure 3.3
Time line for wholesale electricity trading in the UK (adapted from [61])

3.3 Digital Marketplace as a Commodity Market

In this section, the DMP is considered as a commodity market and compared with an existing market where a commodity of similar characteristics is traded; namely, electricity market. It is further demonstrated using the example of the electricity market that DMP is a feasible framework for trading wireless communications services.

DMP is an example of a commodity market where the traded commodity are wireless communications services. There are in existence markets which trade commodities of very similar characteristics to wireless communications services. The most notable example is that of the electricity markets. When it comes to tradability, electricity and wireless services share several features; for example, there is a huge consumer demand, and both commodities do not take the form of physical raw materials [60]. A good example of a functioning electricity market is the UK wholesale electricity market. The operation of the market is characterised by three main time periods: 1) bilateral trading via contracts (also known as forward) market, 2) operation of the balancing market, and 3) physical delivery [61]. This is depicted in Figure 3.3.

The majority of electricity is traded through bilateral, long-term contracts in the contracts market between the bulk electricity generators (sellers) and the distributors of electricity to final consumers (buyers) [62]. Electricity is traded in 30 minute blocks referred to as *settlement periods*, and each trading day consists of 48 settlement periods [63]. The participants of the market are allowed to trade until an hour before a particular settlement period is being realised, i.e., when physical delivery of electricity occurs. This period is referred to as *gate closure* (see Figure 3.3).

In the period between the gate closure and the start of the physical delivery, the market operator (e.g., National Grid in England and Wales) is balancing the supply and demand in real-time to avoid potential breaches of electrity system's limits [62]. This is depicted as the operation of the balancing market in Figure 3.3. The matching of supply and demand is performed via an auction where the generators and distributors submit bids and offers in order to decrease generation or increase demand, or *vice versa* [64]. It is important to note that the bids and offers need to be submitted by the market participants before the gate closure. During the balancing period, the market operator is effectively executing the auction, determining the winners of the submitted bids and offers, and as a result, balances supply and demand [65].

In the context of the DMP, the bulk electricity generators correspond to network providers, while the distributors to service providers. Furthermore, the balancing mechanism corresponds to the network selection mechanism in the DMP. It is encouraging to realise that the theoretical concept of the DMP is very closely related to the existing wholesale electrecity market. This fact speaks favourably in support of adoption of the DMP as a market for trading wireless communications services of the future.

This section compared the DMP with an existing wholesale electricity market, demonstrating its feasibility. The next section outlines the contributions of the research reported in this thesis to the DMP.

3.4 Previous Research on the Digital Marketplace

In this section, previous research on the DMP is summarised.

Over the last two decades, several papers have explored both the economic and technical challenges of the DMP. In their seminal paper, Le Bodic *et al.* [8] discuss how the DMP can be used in a 3G environment to increase the competition between network operators by using a marketplace approach. Furthermore, they describe the business and technical underpinnings of the DMP. Finally, the operation and the fundamental assumptions of the network selection mechanism are outlined, and a bidding strategy based on a tatonnement process is proposed. The tatonnement process is based on the principle that each network operator analyses past auctions in order to determine the market price they have to offer to remain competitive. It is important to realise that the authors do not verify whether the proposed bidding strategy constitutes an equilibrium of the network selection mechanism; rather, it is provided as is. From the economic perspective, this is major flaw in the analysis as otherwise it is unclear whether the mechanism induces rational behaviour in participants; for example, whether it maximises the network operators' expected utilities. The authors also simulate DMP with the proposed bidding strategy as the equilibrium bidding strategy, and the evolution of the market equilibrium is analysed under different simulated scenarios. It is worth noting that the ideas presented in Le Bodic's paper [8] are substantially extended and elaborated upon in Le Bodic's PhD dissertation [51].

Irvine *et al.* [47] explore the problem of interconnection between DMPs. More specifically, they examine how, in a hybrid environment featuring a variety of wireless access technologies, service requests are handled between multiple DMPs, e.g., call interconnections. Simulation results for four interconnected DMPs are presented, and the bidding mechanism proposed in [8] is simulated. It is concluded that with that bidding strategy in force, network operator should accept as many calls as possible, and let the quality of the service degrade. The paper then concludes with the statement that since the DMP is a complex system, many different bidding strategies may be used, but complex bidding strategies are not necessarily beneficial.

In [48], Irvine describes in detail the business model for the DMP, and presents the fundamental requirements for a market to operate freely. He then highlights how the DMP fulfils all the requirements thus establishing DMP a free market. Finally, he summarise the operation of the DMP, discusses how DMP can be used as a management platform for services beyond 3G, and draws a comparison between DMP and other similar solutions.

Mathur *et al.* [66] propose a method for accurate estimation of reputation ratings. They derive a 3D surface graph which allows the network operators to deduce how many users they can accomodate in the network before they reach limits in capacity, and as a result, risk degrading their reputation. The 3D surface is derived through the means of simulation modelling, and encapsulates three categories of users: voice heavy users, data (mainly web browsing) heavy users, and video heavy users.

In [67], McDiarmid *et al.* extend the reputation system of the DMP by introducing a new parameter called commitment level as a means of securing the DMP. This parameter is meant to encapsulate the commitment of the network operators to successfully providing the required QoS levels of a service as requested by the subscriber. That is, if a network operator sets their commitment level at 90%, then they declare that they will strive to fulfil at least 90% of all the contracts (within a time window) to their required QoS levels. The authors simulate their proposed reputation system, and conclude that it is fair towards network operators; that is, it rewards providers who exceed their commitment level, and punishes those who fail.

Finally, in [7], Bush *et al.* extend the ideas presented in [48] to next generation wireless networks; that is, a wireless environment featuring a plethora of wireless access technologies, such as 3G, 4G and beyond. The authors argue that the DMP can be used to manage the effect of a 'tussle' in the next generation wireless networks. They predict the tussle between different economic agents involved, and especially between the end-users of wireless service, service and network providers.

In this section, previous research on the DMP was summarised. The next section outlines the contributions of this research to the DMP.

3.5 Contributions of this Research to Digital Marketplace

Out of six papers presented in the previous section, only two papers [8, 47] touch upon the issue of the best bidding strategy for the network operators. In both cases, the bidding strategies are given as is rather than being a result of an in-depth economic/game-theoretical analysis. The work reported in this thesis addresses this issue by providing a comprehensive analysis of the network selection mechanism from the economic viewpoint.

It is important to analyse the network selection mechanism from the economic point of view (using game theory or otherwise) in order to verify whether the mechanism performs as expected in economic terms. For example, thanks to the economic analysis presented in this thesis, it is verified that the DMP network selection mechanism maximises the expected utilities of the network operators, and from the subscriber's perspective, the network operator who matches the subscriber's preferences in terms of requested price and quality of service wins the auction. In particular, this thesis characterises expected bidding behaviour that constitutes an equilibrium of the auction upon which the network selection mechanism is based. In this context, the equilibrium bidding strategies are equivalent to the best bidding strategies network operators can undertake. Furthermore, it is shown that the equilibrium exists and it is unique, and can be approximated numerically.

3.6 Summary

In this chapter, the concept of the DMP was introduced, and an outline of the fundamental assumptions and operation of the network selection mechanism employed by the DMP was provided. Furthermore, DMP was compared with the wholesale electricity market, and it was shown that due to the similarilities between the two markets and the fact that the wholesale electricity market is used in practice, DMP is more likely to be adopted by the industry as a market for trading wireless communications services of the future. The chapter then concluded with a summary of previous research on DMP, and an explanation of how the research work reported in this thesis contributes to the DMP framework.

Chapter 4

Direct Analysis of Network Selection Mechanism

This chapter formally defines the network selection mechanism in the DMP, and casts it into the framework of game theory. The mechanism is then directly analysed in three special cases: 1) when only reputation ratings of the network operators decide on the winning network operator, 2) when only the monetary bids of the network operators matter in the selection of the winner, and finally, 3) when all network operators are characterised by the same reputation rating. In essence, those special cases correspond to the extremes of the studied bidding problem, and in all those cases, the equilibrium bidding strategies are derived. The chapter then concludes with the analysis of the mechanism in a restricted case with only two network operators, for which the equilibrium bidding strategies are derived. They are shown, however, to be suboptimal as they allow the network operators to submit a negative monetary bid.

4.1 Problem Definition and Assumptions

The game-theoretic description of the network selection mechanism employed in the DMP is as follows. The model constitutes a version of procurement first-price sealed-bid auction (henceforth, referred to as FPA). To recap, in FPA, bidders submit their bids simultaneously [33]. Furthermore, each bidder knows their own cost of selling the good to the buyer but does not know any other bidder's type; i.e., the costs are private knowledge. The bidder who submitted the highest bid, wins the auction and sells the good to the buyer for the price equivalent to their bid. Since the costs are private knowledge, the bidders are uncertain about another bidders' utility functions. Thus, FPA and the network selection mechanism represent Bayesian games of incomplete information (see Section B.4, Appendix B for the formal definition of a Bayesian game of incomplete information).

There are n network operators who bid for the right to sell their product to the subscriber such that $n = |N|$ where N denotes the set of all network operators. Let $\beta : \mathbb{R}_+ \times [0,1] \to \mathbb{R}_+$, defined by

$$\beta(b_i, r_i) = w_{price} \cdot b_i + w_{penalty} \cdot r_i \quad \text{for all } i \in N, \tag{4.1}$$

denote the compound bid. Each network operator i is characterised by the utility function u_i such that

$$u_i(b, c, r) = \begin{cases} b_i - c_i & \text{if } \beta(b_i, r_i) < \min_{j \neq i} \beta(b_j, r_j), \\ 0 & \text{if } \beta(b_i, r_i) > \min_{j \neq i} \beta(b_j, r_j), \end{cases} \tag{4.2}$$

where $b = (b_i, b_{-i})$ represents the monetary bid (or offered price) vector, $c = (c_i, c_{-i})$ the cost vector, and $r = (r_i, r_{-i})$ the reputation rating vector. In this notation, b_{-i} is a shorthand notation for a vector containing all elements with the b_i element excluded; that is, $b_{-i} = (b_1, \ldots, b_{i-1}, b_{i+1}, \ldots, b_n)$. Furthermore, as stated in Section 3.2.2, Chapter 3, the monetary bid is equivalent to the price of supporting the service by the network operator. The precise definition of the price is left open-ended; one possibility, for example, would be to charge the buyer per unit of bandwidth.

The cost of each network operator is assumed to represent the minimum price for transporting the service request under consideration. The winner of the auction is determined as the network operator whose compound bid is the lowest one; i.e., network operator i is the winner if

$$\beta(b_i, r_i) < \min_{j \neq i} \beta(b_j, r_j). \tag{4.3}$$

In the event that there is a tie

$$\beta(b_i, r_i) = \min_{j \neq i} \beta(b_j, r_j), \tag{4.4}$$

the winner is randomly selected with equal probability.

It is, moreover, assumed that the price and reputation weights $(w_{price}, w_{penalty})$ are announced by the subscriber to all network operators before the auction. They are specific to the subscriber and the service they requested. In other words, it is envisaged that the same subscriber might use different weights for subsequently requested services during their participation in the DMP. Since the weights are announced by the subscriber to all network operators before the auction, there is no uncertainty in knowing how much the subscriber values the offered price of the service over the reputation of the network operator (or vice versa). Furthermore,

$$w_{price} + w_{penalty} = 1, \quad 0 \leq w_{price}, w_{penalty} \leq 1. \tag{4.5}$$

In order to simplify the notation, it is assumed throughout the rest of this thesis that $w = w_{price}$. This simplifies the definition of the compound bid in Equation (4.1) to

$$\beta(b_i, r_i) = wb_i + (1 - w)r_i \quad \text{for all } i \in N. \tag{4.6}$$

Note, however, that since w is assumed to be common knowledge, it could potentially lead to a situation where network operators manipulate the knowledge of w to increase their profits by overcharging the subscriber. Therefore, in order to circumvent such an eventuality, the subscriber would only consider offers such that

$$v \geq wb_i + (1 - w)r_i \quad \text{for all } i \in N \tag{4.7}$$

where $v \in (0, 1]$ is the subscriber's valuation, and it is private knowledge [8]. The subscriber's valuation is effectively equivalent to a secret (or hidden) reserve price [68, 69, 70]. This creates an additional uncertainty about the auction that each network operator needs to incorporate into their equilibrium bidding strategies. To elaborate further, by setting a secret reserve valuation, the subscriber creates a phantom bidder characterised by a reputation rating r_0 and submitting a bid b_0 such that $v = \beta(b_0, r_0)$, and therefore, risks not obtaining a service offer from any network operator if $v < \beta(b_i, r_i)$ for all $i \in N$. Therefore, the solution presented in this and the following chapters would have to be modified by including the phantom bidder in the derivation of the equilibrium bidding strategies. However, in order to keep the analysis tractable, this assumption is not incorporated in this research.

Following the standard assumptions from the auction literature [54], the set of network operators, N, is finite and the network operators are risk neutral; that is, they seek to maximise their expected profits. Furthermore, the subscriber is risk neutral and does not have any budget constraints; that is, the subscriber is prepared to accept any offer from the network operators. In reality, it is unlikely for the network operators and the buyer to be risk neutral, and the buyer to not have any budget constraints. However, in order for the problem to be tractable, those assumptions are enforced in this research. The implications of relaxing those assumptions on the analysis of an auction are explored by Krishna [54].

The costs c_i for each network operator i are private knowledge. Thus, they are particular realisations of the random variables (r.v.) C_i for each i. Furthermore, it is assumed that each C_i is identically and independently distributed (i.i.d.) over the interval $[0, 1]$ according to some (absolutely-) continuous probability distribution which admits a distribution function F_C and an associated density function f_C such that f_C is locally bounded away from zero over the interval $[0, 1]$. In order to keep the specification generic, the costs are not explicitly decomposed into the underlying components such as interconnection charges, infrastructure fixed costs, etc. The reader is referred to [71, 72, 73] for an in-depth coverage of the problem. Furthermore, it is recognised that various network operators will have different architecture, topology, infrastructure, etc., and that this impacts cost to support the service. In order to capture the variation in costs, an i.i.d. random variable is employed to assign a cost to a particular network operator at the time of the service request. This further aids generality of the results and subsequent analysis presented in this research.

The reputation ratings r_i for each network operator i are common knowledge. It is assumed that each $r_i \in [0, 1]$ such that the higher the reputation, the lower the rating r_i. Initially, it was assumed that ratings are private knowledge. However, after analysis, it was concluded that this would contradict its purpose. The reputation of each network operator, in order to be meaningful, must be freely available to everyone, including the competitors of the network operators. For example, in the Amazon.com Marketplace, the buyers have the right to rate the seller they buy from on a scale from one to five (with five being the best), and these ratings are publicly available [74]. Similarly, on eBay, the buyers can leave sellers feedback (negative, neutral, or positive) which over time is viewed as reputation, and is also publicly available [75].

The bidding strategy functions $b_i : [0, 1] \to \mathbb{R}_+$ are nonnegative in value for all i. The aim is to solve the game for pure-strategy Bayesian Nash equilibrium(-a) as defined in Equation (B.6), Section B.4.1, Appendix B. It is further assumed that the network operators will bid at least their cost (unless explicitly stated otherwise). The problem of bidding below cost or negatively deserves a more elaborate explanation. There are two fundamental assumptions governing game theory [32]: 1) economic agents are rational decision-makers; that is, they make decisions consistently in pursuit of their own objectives; and 2) their objective is to maximise the expected value of their own utility. In the light of those assumptions, network operators are implied to bid at least their cost as they would always strive to maximise their expected utility. However, the real behaviour of the network operators might be different in the sense that they might, in the view of game theory, behave irrationally by bidding below their cost to secure the contract with the subscriber. In fact, if the temporal aspect of the DMP is considered, network operators will interact by engaging in the DMP auction more than once. Then, it might prove beneficial for them to bid below their cost trading positive utility for securing the win—this is a well-known pricing strategy in economics called "loss leader" pricing strategy [76, 77]. The idea behind the strategy is to sell a good at a price below its market cost to increase the store traffic and encourage sales of other, possibly more profitable goods. Applied back to the DMP, a network operator could, in principle, willingly incur cost by bidding below their cost to encourage more subscribers to use their services, or to improve their reputation rating by serving more subscribers. While the fact that situations like this can occur in reality is appreciated, this thesis follows the fundamental assumptions of game theory; that is, in the rest of this thesis (unless explicitly stated otherwise), it is assumed that all economic agents involved in the DMP are rational decision-makers and strive to maximise their expected utility. Otherwise, the mathematical treatment of the problem would prove impossible [78].

The problem is divided into two cases: generic and restricted case. In the former case, no additional assumptions about the game than those already stated in this section will be made, and the discussion will concentrate on finding a symmetric equilibrium. In the latter case, on the other hand, the problem will be simplified by considering only two network operators, letting costs be drawn from the uniform distribution, and focusing on (possibly different) bidding strategy functions which are linear functions of cost.

4.2 Generic Case

Suppose that all network operators use the same strictly increasing in c_i bidding strategy function; i.e., $b_i = b_i(c_i) = b(c_i)$ for all $i \in N$. In this case, the equilibrium profile $(b^*, \ldots, b^*)$ is called *symmetric*. In its generic form, the problem proves to be too complicated for the analytical solution using the existing methods of solving auctions. It would seem that since the problem is a modified version of the standard FPA, the standard analytical approach, found for example in [54, 55, 79, 80], should apply. However, this is not the case. To see why, note that each network operator i faces an optimisation problem

$$\max_{b_i} E\left[b_i - c_i \;\middle|\; wb_i + (1-w)r_i < \min_{j \neq i}(wb(C_j) + (1-w)r_j) \right]. \tag{4.8}$$

Note that

$$\min_{j\neq i}(wb(C_j) + (1-w)r_j) \geq w \min_{j\neq i} b(C_j) + (1-w) \min_{j\neq i} r_j. \tag{4.9}$$

Substituting the inequality (4.9) into the identity (4.8), yields for all $w \neq 0$

$$\max_{b_i} E\left[b_i - c_i \,\middle|\, b^{-1}\left(b_i + \frac{1-w}{w}(r_i - \min_{j\neq i} r_j)\right) < \min_{j\neq i} C_j\right], \tag{4.10}$$

where the fact that b is increasing was used, and hence, it is invertible (cf. Corollary B.1) and $\min_x b(x) = b(\min_x x)$ for all x.

Let $C_{1:n-1} = \min_{j\neq i} C_j$ be the lowest order statistic of an i.i.d. random sample C_j for all $j \neq i$ with the distribution function $F_{C_{1:n-1}}$. Hence, the identity (4.10) becomes

$$\begin{aligned}&\max_{b_i}\left(b_i - c_i\right)\left(1 - F_{C_{1:n-1}}\left(b^{-1}\left(b_i + \frac{1-w}{w}(r_i - \min_{j\neq i} r_j)\right)\right)\right)\\ =&\max_{b_i}\left(b_i - c_i\right)\left(1 - F_C\left(b^{-1}\left(b_i + \frac{1-w}{w}(r_i - \min_{j\neq i} r_j)\right)\right)\right)^{n-1},\end{aligned} \tag{4.11}$$

where the fact was used that the distribution function of an i^{th} order statistic of an i.i.d. random sample is defined as in Equation (B.2), Section B.3.1, Appendix B.

Finally, recalling that at a symmetric equilibrium $b_i = b(c_i)$ and letting $k = \frac{(1-w)}{w}(r_i - \min_{j\neq i} r_j)$, the identity (4.11) becomes

$$\begin{aligned}&\frac{d}{dc_i} b\left(b^{-1}(b(c_i)+k)\right)\left[1 - F_C(b^{-1}(b(c_i)+k))\right]^{n-1}\\ =&(n-1)(b(c_i) - c_i)\left[1 - F_C(b^{-1}(b(c_i)+k))\right]^{n-2} f_C(b^{-1}(b(c_i)+k)).\end{aligned} \tag{4.12}$$

It is difficult (if possible) to derive a closed-form solution for the resulting ordinary differential equation in Equation (4.12). Therefore, it can be concluded that even significant simplification of the problem is not enough to heuristically derive an optimal bidding strategy function for each network operator i. It is possible, however, to derive the optimal bidding strategies in a handful of special cases: $w = 0$, $w = 1$, and $r_i = r_j$ for all $i, j \in N$ such that $i \neq j$. This is the subject of the next three sections.

4.2.1 Special Case $w = 0$

In one of the extreme cases, however, when $w = 0$, the problem becomes simpler. For then, the utility function simplifies to

$$u_i(b, c, r) = \begin{cases} b_i - c_i & \text{if } r_i < \min_{j\neq i} r_j, \\ 0 & \text{if } r_i > \min_{j\neq i} r_j. \end{cases} \tag{4.13}$$

Since the reputation ratings, r_i, are common knowledge, the probability of winning, i.e., the probability of the event such that $r_i < \min_{j\neq i} r_j$ for all i, is either 0 or 1, and does not depend on the value of the bid, b_i. In other words, each network operator knows in advance whether they won, tied, or lost based on their own and their opponents reputation ratings since these are deterministic in nature. Hence, it is clear that the network operator with the lowest reputation rating will have an incentive to bid abnormally high since they are guaranteed a win regardless of the value of their bid. The remaining network operators, on the other hand, will be indifferent to the value of the submitted bids as it is impossible for them to win regardless of the values of their bids. In case of a tie, i.e., in case there is more than one network operator with the lowest reputation rating, each has an equal probability of winning the auction, and this probability is independent of the values of their bids. Hence, in this case, the network operators also have an incentive to bid abnormally high. Formally,

Proposition 4.1. *Suppose c_i is i.i.d. over the interval $[0, 1]$ for all $i \in N$ and $r_i \in [0, 1]$ for all $i \in N$ is common knowledge. Let $N_0 \subseteq N$ be the set of all those network operators with the lowest reputation rating. If $w = 0$, then every network operator $j \in N_0$ will have an incentive to bid abnormally high, i.e., $b_j \to \infty$, while every remaining network operator $k \in N \setminus N_0$ will be indifferent to the value of their bid.*

The formal proof of Proposition 4.1 as well as any other proposition included in this thesis is given in Appendix A.

4.2.2 Special Case $w = 1$

When $w = 1$, on the other hand, the problem becomes that of standard FPA auction. The utility of each network operator i becomes

$$u_i(b, c, r) = \begin{cases} b_i - c_i & \text{if } b_i < \min_{j \neq i} b_j, \\ 0 & \text{if } b_i > \min_{j \neq i} b_j. \end{cases} \tag{4.14}$$

Network operator i conjecturing that other network operators follow b symmetric bidding strategy and submit their costs truthfully, tries to solve

$$\begin{aligned} &\max_{b_i} E\left[b_i - c_i \,\middle|\, b_i < \min_{j \neq i} b(C_j)\right] \\ =&\max_{b_i} E\left[b_i - c_i \,\middle|\, b^{-1}(b_i) < \min_{j \neq i} C_j\right] \\ =&\max_{b_i} E\left[b_i - c_i \,\middle|\, b^{-1}(b_i) < C_{1:n-1}\right] \\ =&\max_{b_i} \int_{b^{-1}(b_i)}^{1} (b_i - c_i) dF_{C_{1:n-1}}(t) \end{aligned}$$

$$= \max_{b_i} (b_i - c_i)(1 - F_{C_{1:n-1}}(b^{-1}(b_i))), \tag{4.15}$$

where, as before, $C_{1:n-1} = \min_{j \neq i} C_j$ is the lowest order statistic of an i.i.d. random sample C_j for all $j \neq i$ with the distribution function $F_{C_{1:n-1}}$, and associated density $f_{C_{1:n-1}}$. The first-order condition yields

$$1 - F_{C_{1:n-1}}(b^{-1}(b_i)) - (b_i - c_i)\frac{f_{C_{1:n-1}}(b^{-1}(b_i))}{\frac{d}{db_i}b(b^{-1}(b_i))} = 0. \tag{4.16}$$

Recalling that at a symmetric equilibrium $b_i = b(c_i)$, the identity (4.16) becomes

$$\frac{d}{dc_i}b(c_i) - b(c_i)\frac{f_{C_{1:n-1}}(c_i)}{1 - F_{C_{1:n-1}}(c_i)} = -c_i\frac{f_{C_{1:n-1}}(c_i)}{1 - F_{C_{1:n-1}}(c_i)}, \tag{4.17}$$

or equivalently,

$$\frac{d}{dc_i}(b(c_i)(1 - F_{C_{1:n-1}}(c_i))) = -c_i f_{C_{1:n-1}}(c_i). \tag{4.18}$$

Since $c_i \in [0, 1]$ for all $i \in N$, it follows $b(1) = 1$. To see this, suppose network operator 1 is characterised by cost $c_1 = 1$. Then, they would never submit a bid higher than their cost $c_1 = 1$ since they would never win. That is, the competing network operator, network operator 2 say, regardless of their cost, could just bid $b(c_2) = c_1 = 1$ and win the auction. Furthermore, network operator 1 would never submit a bid lower than their cost $c_1 = 1$ since they would make a loss if they were to win the auction. Therefore, it must be that $b(1) = 1$. It follows then

$$\begin{aligned} b(c_i) &= \frac{1}{1 - F_{C_{1:n-1}}(c_i)} \int_{c_i}^{1} t dF_{C_{1:n-1}}(t) \\ &= \frac{n-1}{(1 - F_C(c_i))^{n-1}} \int_{c_i}^{1} t(1 - F_C(t))^{n-2} f_C(t) dt. \end{aligned} \tag{4.19}$$

Thus, the symmetric bidding strategy in Equation (4.19) is the most likely candidate for a symmetric pure-strategy Bayesian Nash equilibrium at $w = 1$.

Proposition 4.2. *Suppose c_i is i.i.d. over the interval $[0, 1]$ for all $i \in N$ and $r_i \in [0, 1]$ for all $i \in N$ is common knowledge. If $w = 1$, then the symmetric equilibrium bidding strategy function of the standard procurement first-price sealed-bid auction,*

$$b^*_{FPA}(c_i) = \frac{1}{1 - F_{C_{1:n-1}}(c_i)} \int_{c_i}^{1} t dF_{C_{1:n-1}}(t) \quad \textit{for all } i \in N, \tag{4.20}$$

constitutes a symmetric pure-strategy Bayesian Nash equilibrium of the DMP variant of a procurement *first-price sealed-bid auction.*

4.2.3 Special Case $r_i = r_j$

In the last extreme case, when all network operators are characterised by the same reputation rating, i.e., when $r_i = r_j$ for all $i \neq j$, and when $w \neq 0$, it can be easily verified that the problem simplifies to the special case $w = 1$. To see why, let $r = r_i$, for all network operators i. Then, for all $i \in N$ and $w \neq 0$

$$\begin{aligned} & \beta(b_i, r) < \min_{j\neq i} \beta(b_j, r) \\ \iff & \frac{1}{w}\left(b_i + \frac{1-w}{w}r\right) < \frac{1}{w}\min_{j\neq i}\left(b_j + \frac{1-w}{w}r\right) \\ \iff & b_i + \frac{1-w}{w}r < \min_{j\neq i} b_j + \frac{1-w}{w}r \\ \iff & b_i < \min_{j\neq i} b_j. \end{aligned} \tag{4.21}$$

Hence, the utility of each network operator i simplifies to

$$u_i(b, c, r) = \begin{cases} b_i - c_i & \text{if } b_i < \min_{j\neq i} b_j, \\ 0 & \text{if } b_i > \min_{j\neq i} b_j. \end{cases} \tag{4.22}$$

Formally,

Corollary 4.3. *Suppose c_i is i.i.d. over the interval $[0, 1]$ for all $i \in N$ and $r_i \in [0, 1]$ for all $i \in N$ is common knowledge. Suppose $r_i = r_j$ for all $i \neq j$, and $w \neq 0$. Then, the problem simplifies to the special case $w = 1$, and hence, b^*_{FPA} is the symmetric equilibrium bidding strategy (Proposition 4.2).*

In this section, it was shown that, in the generic case, it is difficult to derive a closed-form solution to the bidding problem. Furthermore, the optimal bidding strategy was characterised in three special cases: $w = 0$, $w = 1$, $r_i = r_j$ for all $i, j \in N$ such that $i \neq j$.

The problem is considerably simplified by restricting the number of network operators to $n = 2$, and letting the costs be drawn from the uniform distribution. In this case, it is possible to derive bidding strategies for both bidders, and this is the subject of the following section.

4.3 Restricted Case $n = 2$

In this section, the discussion will be restricted to only two network operators. Since the problem in its generic form proved intractable to be solved analytically, this section will explore whether in a much simplified scenario it is possible to find a closed-form solution. To this end, let $n = 2$. The utility function for each network operator $i \in \{1, 2\}$ thus becomes

$$u_i(b, c, r) = \begin{cases} b_i - c_i & \text{if } \beta(b_i, r_i) < \beta(b_j, r_j), \\ \frac{1}{2}(b_i - c_i) & \text{if } \beta(b_i, r_i) = \beta(b_j, r_j), \\ 0 & \text{otherwise.} \end{cases} \tag{4.23}$$

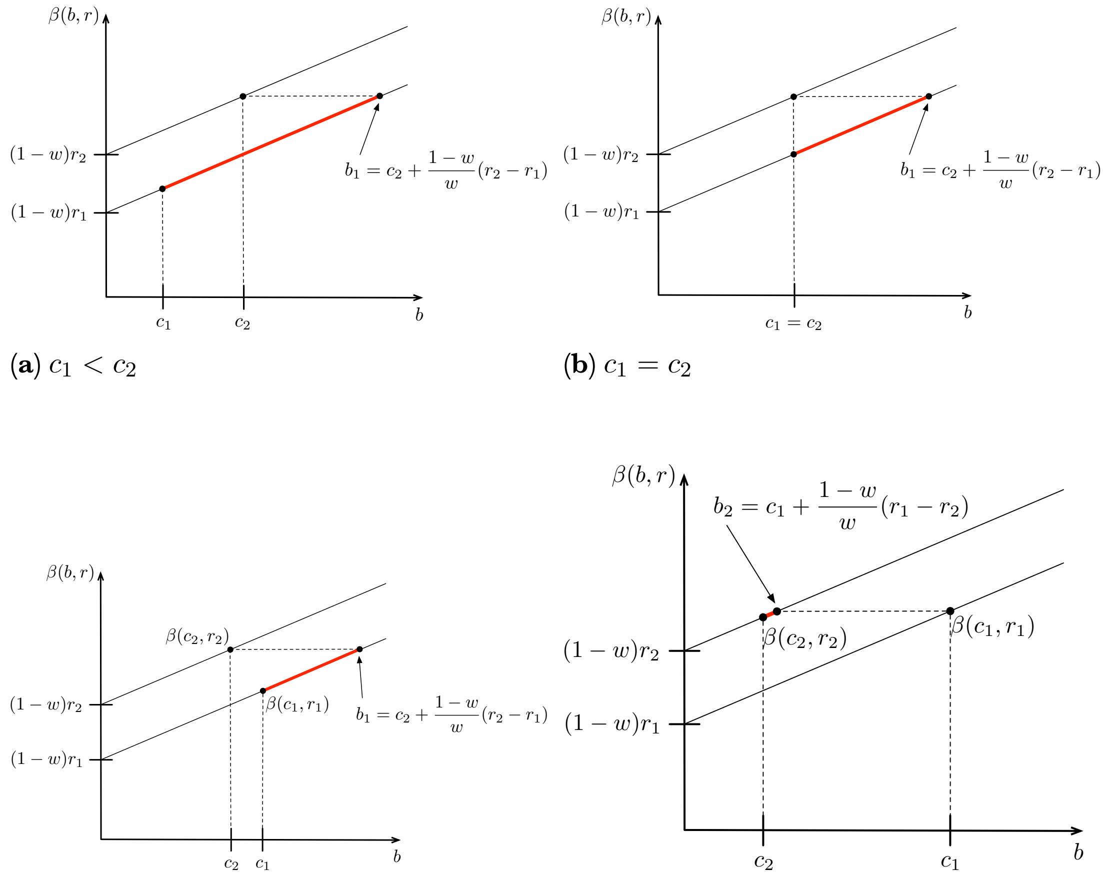
Furthermore, the assumption concerning the symmetric equilibrium profile is relaxed; that is, network operators are permitted to use differing bidding strategies.

The analysis is conducted in two steps. Firstly, it is assumed that information is complete; that is, that each network operator not only knows their own cost and reputation, but also those of their opponent's. Secondly, the standard case is considered; that is, that the reputation ratings of the network operators are assumed to be known, while the costs are private knowledge.

4.3.1 Complete Information

Here, it is assumed that information is complete; i.e., that each network operator knows their own and their opponent's cost and reputation rating. In total, there are 7 different bidding scenarios to consider as described below.

Figure 4.1 shows the first 4 cases for which $r_1 < r_2$. (Note that exactly the same reasoning applies to the situation when $r_1 > r_2$.) If $c_1 < c_2$, network operator 1 is guaranteed a victory and a positive profit as long as they bid within the highlighted part of the $\beta(b, r)$ curve depicted in Figure 4.1a. Thus, their optimal bidding strategy would be to bid slightly less than their opponent's compound bid evaluated at their opponent's cost, $\beta(c_2, r_2)$; that is, $b_1 = c_2 + \frac{1-w}{w}(r_2 - r_1) - \epsilon$ where $\epsilon > 0$ is very small. Network operator 2, on the other hand, should find it optimal to bid $b_2 = c_2$. To see why, suppose network operator 2 bids $\hat{b}_2 > c_2$. Since network operator 1's reputation rating and cost are strictly lower than those of network operator 2's, they can undercut the network operator 2's bid by a small amount so that $\hat{b}_1 < \hat{b}_2$ and still make positive profit. But, in response, network operator 2 will find it optimal to lower their bid so that it undercuts that of network operator 1's; that is, $\hat{b}_2 < \hat{b}_1$. This process will continue until one of the network operators is forced to bid their cost. Since network operator 1's reputation rating and cost are strictly lower than those of network operator 2's, it can be concluded



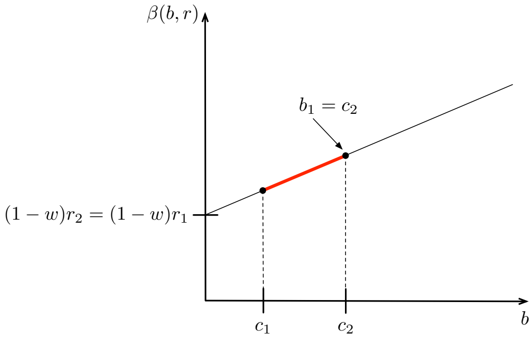
(**a**) $c_1 < c_2$

(**b**) $c_1 = c_2$

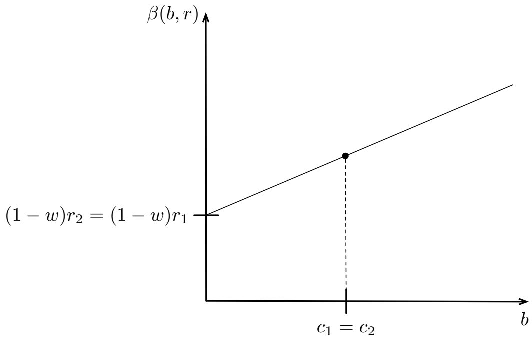
(**c**) $c_1 > c_2$ and $\beta(c_1, r_1) < \beta(c_2, r_2)$

(**d**) $c_1 > c_2$ and $\beta(c_1, r_1) \geq \beta(c_2, r_2)$

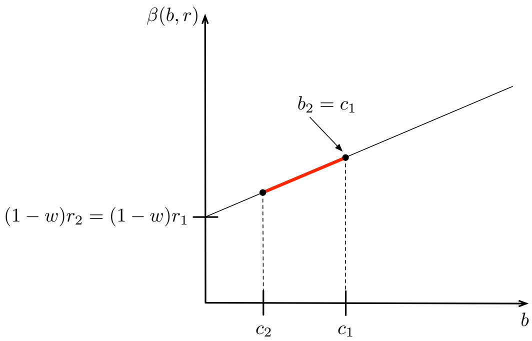
Figure 4.1
Different bidding scenarios for $r_1 < r_2$



(**a**) $c_1 < c_2$



(**b**) $c_1 = c_2$



(**c**) $c_1 > c_2$

Figure 4.2
Different bidding scenarios for $r_1 = r_2$

that $b_2 = c_2$ and $b_1 = c_2 + \frac{1-w}{w}(r_2 - r_1) - \epsilon$ where $\epsilon > 0$ is very small.

If $c_1 = c_2$, arguing in the similar manner as previously, network operator 1's optimal bidding strategy would be to bid $b_1 = c_2 + \frac{1-w}{w}(r_2 - r_1) - \epsilon$ where $\epsilon > 0$ is very small; while network operator 2 should bid $b_2 = c_2$ (see Figure 4.1b).

If $c_1 > c_2$, there are two cases to consider. If $\beta(c_1, r_1) < \beta(c_2, r_2)$, then network operator 1 still has some room for manoeuvre, and should find it optimal to bid $b_1 = c_2 + \frac{1-w}{w}(r_2 - r_1) - \epsilon$ where $\epsilon > 0$ is very small; while network operator 2 to bid $b_2 = c_2$ (see Figure 4.1c). If $\beta(c_1, r_1) \geq \beta(c_2, r_2)$, on the other hand, the roles are reversed, and network operator 2 should find it optimal to bid $b_2 = c_1 + \frac{1-w}{w}(r_1 - r_2) - \epsilon$ where $\epsilon > 0$ is very small; while network operator 1 to bid $b_1 = c_1$ (see Figure 4.1d).

Figure 4.2 depicts the remaining 3 cases for which $r_1 = r_2$. If $c_1 < c_2$, network operator 1's optimal bidding strategy would be to bid $b_1 = c_2 - \epsilon$ where $\epsilon > 0$ is very small; while network operator 2 should bid $b_2 = c_2$ (see Figure 4.2a).

If $c_1 = c_2$, both network operators should bid their costs; that is, $b_1 = c_1$ and $b_2 = c_2$ (see Figure 4.2b).

If $c_1 > c_2$, network operator 2's optimal bidding strategy would be to bid $b_2 = c_1 - \epsilon$ where $\epsilon > 0$ is very small; while network operator 1 should bid $b_1 = c_1$ (see Figure 4.2c).

It can be concluded that the bidding strategies depend only on costs if $r_1 = r_2$. In the remaining cases, they are asymmetric in the sense that the winning network operator is characterised by

$$b_1 = c_2 + \frac{1-w}{w}(r_2 - r_1) - \epsilon \quad \text{with } \epsilon > 0 \text{ being very small,} \tag{4.24}$$

while the losing network operator by bidding their own cost

$$b_2 = c_2. \tag{4.25}$$

Hence, when dealing with incomplete information, these results will be exploited by concentrating on equilibrium bidding strategies which are linear functions of cost.

4.3.2 Incomplete Information

Here, contrary to previous section, the standard case is assumed; that is, that reputation rating values for both network operators $i \in \{1, 2\}$ are known at the time of bidding; however, their costs are private knowledge. Suppose that the network operators use a strategy function $b_i : [0, 1] \to \mathbb{R}$ defined by the rule

$$b_i(c_i) = \zeta_i + \eta_i c_i, \quad \text{for all } \zeta_i \in \mathbb{R}, \eta_i > 0, i \in \{1, 2\} \tag{4.26}$$

and costs are independently drawn from the uniform distribution over the interval $[0, 1]$. In other words, (although somewhat counter-intuitive) negative bids from the network operators are allowed. The motivation for such an assumption will be explained in detail later on in the section. Note, moreover, that the strategy function is assumed to be linear in cost. Network operator 1 faces an optimisation problem

$$\max_{b_1} E\left[b_1 - c_1 \mid wb_1 + (1-w)r_2 < w(\zeta_2 + \eta_2 C_2) + (1-w)r_2\right] \tag{4.27}$$

If $w = 0$, then the result described in Proposition 4.1, Section 4.2.1, holds. Otherwise, for $0 < w \leq 1$, network operator 1 solves

$$\begin{aligned}
&\max_{b_1} E\left[b_1 - c_1 \,\middle|\, \frac{1}{\eta_2}\left(b_1 + \frac{1-w}{w}(r_1 - r_2) - \zeta_2\right) < C_2\right] \\
&= \max_{b_1} \int_{\frac{1}{\eta_2}(b_1 + \frac{1-w}{w}(r_1 - r_2) - \zeta_2)}^{1} (b_1 - c_1) dF_C(t) \\
&= \max_{b_1} \left(b_1 - c_1\right)\left(1 - \frac{1}{\eta_2}b_1 - \frac{1}{\eta_2}\left(\frac{1-w}{w}(r_1 - r_2) - \zeta_2\right)\right).
\end{aligned} \tag{4.28}$$

The first-order condition yields

$$\begin{aligned}
&1 - \frac{2}{\eta_2}b_1 + \frac{1}{\eta_2}c_1 - \frac{1}{\eta_2}\left(\frac{1-w}{w}(r_1 - r_2) - \zeta_2\right) = 0 \\
\iff\ & b_1 = \frac{\eta_2}{2} - \frac{1}{2}\left(\frac{1-w}{w}(r_1 - r_2) - \zeta_2\right) + \frac{1}{2}c_1.
\end{aligned} \tag{4.29}$$

(Note that the second-order condition is satisfied; i.e., $\frac{d^2}{db_1^2}E[\cdot|\cdot] = -\frac{2}{\eta_2} < 0$ since $\eta_2 > 0$.) Similar argument for network operator 2 yields

$$b_2 = \frac{\eta_1}{2} - \frac{1}{2}\left(\frac{1-w}{w}(r_2 - r_1) - \zeta_1\right) + \frac{1}{2}c_2. \tag{4.30}$$

Thus, it follows

$$\begin{cases}
\eta_1 &= \eta_2 = \frac{1}{2}, \\
\zeta_1 &= \frac{\eta_2}{2} - \frac{1}{2}\left(\frac{1-w}{w}(r_1 - r_2) - \zeta_2\right), \\
\zeta_2 &= \frac{\eta_1}{2} - \frac{1}{2}\left(\frac{1-w}{w}(r_2 - r_1) - \zeta_1\right).
\end{cases} \tag{4.31}$$

Solving the above equations simultaneously yields the equilibrium bidding strategies for both bidders

$$b_1(c_1) = \frac{1}{2} - \frac{1-w}{3w}(r_1 - r_2) + \frac{1}{2}c_1, \tag{4.32}$$

$$b_2(c_2) = \frac{1}{2} - \frac{1-w}{3w}(r_2 - r_1) + \frac{1}{2}c_2. \tag{4.33}$$

Formally,

Proposition 4.4. *Let there be $n = 2$ network operators. For all $i \in \{1, 2\}$, suppose c_i is independently drawn from uniform distribution over the interval $[0, 1]$, and $r_i \in [0, 1]$ is common knowledge. Then the equilibrium bidding strategies for all $w \in (0, 1]$ are given by*

$$b_1(c_1) = \frac{1}{2} - \frac{1-w}{3w}(r_1 - r_2) + \frac{1}{2}c_1, \tag{4.34}$$

$$b_2(c_2) = \frac{1}{2} - \frac{1-w}{3w}(r_2 - r_1) + \frac{1}{2}c_2. \tag{4.35}$$

Observe that the pair of strategies (b_1, b_2) does not constitute a symmetric equilibrium.

By way of example, Table 4.1 depicts a particular set of cost-reputation pairs of two network operators. Figure 4.3 shows the value of the compound bid, β, for different values of w for both network operators, while Figure 4.4 depicts the value of the monetary bid (or offered price), b_i, for different values of w for both network operators. The numerical data in Table 4.1 suggests that network operator 2 should be the winner for the values of $w \to 1$ since network operator 2's cost is strictly lower than that of their opponent's. On the other hand, network operator 1 should be winner for the values of $w \to 0$ since network operator 1's reputation rating is strictly lower than that of their opponent's (which implies that network operator 1's reputation is in fact strictly higher than that of their opponent's). This prediction agrees with the numerical output shown in Figure 4.3. Let w_c denote the value of w for which an intersection between the compound bids of both network operators occurs (if it exists). In Figure 4.3, $w_c = 0.4$. Hence, network operator 2 wins the auction for the values of $w \in (w_c, 1]$, while network operator 1 for the values of $w \in [0, w_c)$. Note, moreover, that network operator 2 bids

Table 4.1
An exemplary set of cost-reputation pairs of two network operators

	Cost, c_i	**Reputation rating**, r_i
Network operator 1	0.75	0.25
Network operator 2	0.25	0.75

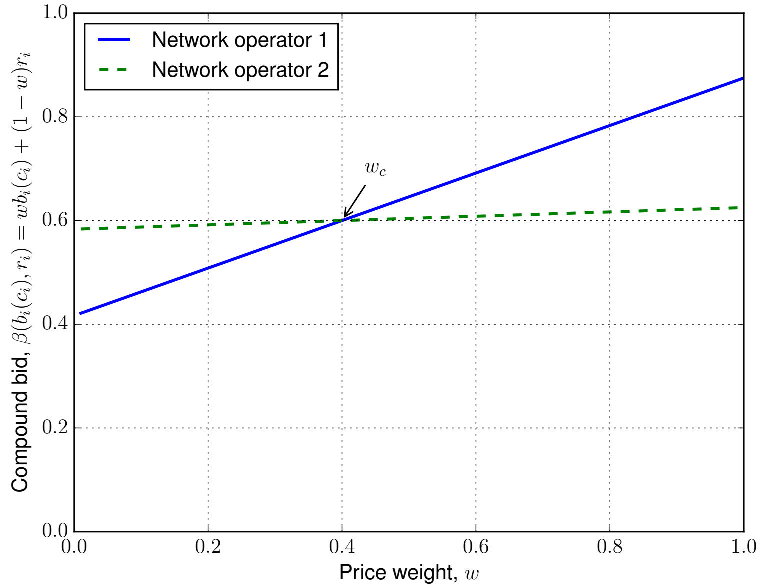


Figure 4.3
Compound bid plotted against the price weight

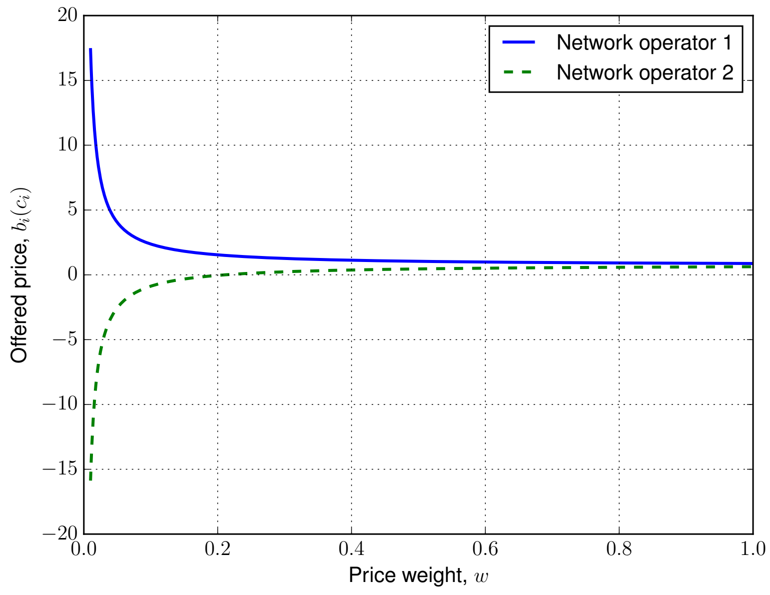


Figure 4.4
Offered prices (bids) plotted against the price weight

below their cost for values of $w < w_c$ (see Figure 4.4). However, this does not necessarily disqualify the equilibrium bidding strategies given by Equations (4.34) and (4.35). The following observations show why. Firstly,

Proposition 4.5. *Suppose both network operators bid according to b_i bidding strategies in Equations* (4.34) *and* (4.35)*. Then they are guaranteed nonnegative profit in case of winning (or a draw).*

Even though the prediction suggests that one of the network operators may bid negatively, they will not win the auction, and hence, are guaranteed profit at worst equal to zero.

Secondly, let $(\mathbf{Q}, \mathbf{M})$ be the direct mechanism induced by the equilibrium bidding strategies, b_i, in Equations (4.34) and (4.35) where $\mathbf{Q} = (Q_1, Q_2)$ and $\mathbf{M} = (M_1, M_2)$. Here, Q_i represents the allocation rule for each network operator $i \in \{1, 2\}$ defined by

$$Q_1(c_1, c_2) = \begin{cases} 1 & \text{if } \beta(b_1(c_1), r_1) < \beta(b_2(c_2), r_2), \\ \frac{1}{2} & \text{if } \beta(b_1(c_1), r_1) = \beta(b_2(c_2), r_2), \\ 0 & \text{otherwise,} \end{cases} \tag{4.36}$$

for network operator 1, and

$$Q_2(c_1, c_2) = \begin{cases} 1 & \text{if } \beta(b_2(c_2), r_2) < \beta(b_1(c_1), r_1), \\ \frac{1}{2} & \text{if } \beta(b_2(c_2), r_2) = \beta(b_1(c_1), r_1), \\ 0 & \text{otherwise,} \end{cases} \tag{4.37}$$

for network operator 2. M_i for all $i \in \{1, 2\}$, on the other hand, denotes the payment rule, and is defined by

$$M_1(c_1, c_2) = Q_1(c_1, c_2) b_1(c_1) \tag{4.38}$$

for network operator 1, and

$$M_2(c_1, c_2) = Q_2(c_1, c_2) b_2(c_2). \tag{4.39}$$

for network operator 2. Suppose network operator 2 reveals their cost truthfully. The equilibrium payoff function for network operator 1 characterized by cost c_1 but revealing c_1' is

$$\begin{aligned} \tilde{\tilde{u}}_1(c_1') &= E\left[M_1(c_1', C_2) - c_1 Q_1(c_1', C_2)\right] \\ &= E\left[(b_1(c_1') - c_1) Q_1(c_1', C_2)\right] \\ &= E\left[b_1(c_1') - c_1 \mid \beta(b_1(c_1'), r_1) < \beta(b_2(C_2), r_2)\right]. \end{aligned} \tag{4.40}$$

It turns out that it is in network operator 1's best interest to reveal their cost truthfully as well; i.e., $c_1' = c_1$. Moreover, both network operators cannot be better off by not participating in the auction; i.e., their equilibrium payoff function is nonnegative, $\tilde{\tilde{u}}_i(c_i) \geq 0$ for all $i \in \{1, 2\}$. Formally,

Proposition 4.6. *The direct mechanism* $(\mathbf{Q}, \mathbf{M})$ *where* $\mathbf{Q} = (Q_1, Q_2)$ *and* $\mathbf{M} = (M_1, M_2)$ *satisfies both the IC and IR constraints.*

This strengthens the fact that even though the network operators may bid negatively, the auction is still attractive to them.

Thirdly, suppose that economic agents are computers who bid on behalf of the network operators. This assumption is reasonable since there currently are estimated 6.1 billion mobile subscribers around the world [81]. In other words, bidding on a per call basis would have to be automated by the network operators in order to make the process manageable. One way of achieving such an automation would be to utilise the concept of a direct mechanism. In a direct mechanism, economic agents submit their costs (which need not be truthful) directly to the mechanism which then computes the bids and chooses the winner on their behalf. By the Revelation Principle, for every mechanism and an equilibrium for that mechanism, there exists an incentive compatible direct mechanism which yields the same outcomes as in the given equilibrium of the original mechanism (see Section B.4.2, Appendix B for the definition of the Revelation Principle). In this case, the direct mechanism $(\mathbf{Q}, \mathbf{M})$ is the direct representation of the DMP variant of an FPA. Since it is incentive compatible, economic agents will not lie about their costs. Since it is individually rational, they will find it beneficial to participate in the mechanism. Therefore, the possibility of one of the network operators bidding below their cost or negatively will not matter to any of the network operators and will not lead to an outcome in which the service is sold for a negative price.

4.4 Summary

In this chapter, game-theoretical model for the DMP network selection mechanism was formally defined. Several simplifying assumptions were made in order to keep the analysis mathematically tractable. For example, the network operators and the subscriber are risk neutral, and the subscriber does not have any budget constraints. Despite the fact that those assumptions are not entirely representative of the reality, following in the footsteps of von Neumann and Morgenstern, the mathematical theory of an economic phenomenon should be rigorous and developed gradually [78]. Therefore, the simplifying assumptions made in this chapter serve as a starting point for the rigorous, gradual development of the economic theory of operation of the DMP network selection mechanism before it can embark on capturing the reality to a high degree.

This chapter further demonstrated that for the price weight of $w = 1$, and equal reputation ratings for all network operators, $r_i = r_j$ for all $i \neq j$, the DMP auction reduces to the standard, symmetric FPA (Proposition 4.2 and Corollary 4.3). In this case, the abundance of theoretical results and economic insight from the auction literature applies, found, for example, in Krishna [54]. For the price weight of $w = 0$, however, it was shown that the network operators would engage in abnormally high bidding (Proposition 4.1). Hence, charging the subcriber the maximum they are prepared to pay for the service. While this result sounds like a potential design flaw, in reality, the subscribers will necessarily be budget constrained, and therefore, abnormally high bidding of the network operators will translate into charging the subscribers a premium price for the service that is within the limits of their respective budgets.

Finally, the chapter concluded with the specification of an analytical solution to the restricted case of two network operators $n = 2$ (Proposition 4.4). The solution is suboptimal in the sense that the derived equilibrium bidding strategies permit the network operators to bid negatively. In the view of game theory, this would imply that the network operators are not rational decision-makers. However, it was also shown that negative bidding does not lead to negative profit for either network operator (Proposition 4.5). Concurrently, it was proved that the network operators would not find it beneficial not to participate in the auction if they were to bid according to the strategies summarised in Proposition 4.4 (Proposition 4.6). It should further be noted that the real behaviour of the network operators might be dictated by the need to secure the contract with the subscriber first and foremost, and hence, lead to negative bidding; a strategy akin to the "loss leader" pricing strategy. However, since the ultimate aim of this thesis is to gradually develop rigorous economic theory of operation of the DMP auction, it is assumed throughout this thesis that network operators will bid at least their cost.

Chapter 5

Indirect Analysis of Network Selection Mechanism

In this chapter, the bidding problem described in Section 4.1 is transformed from a bidding problem with symmetric cost (or type) distributions into a bidding problem with asymmetric cost distributions. This type of bidding problems has already been researched by the economic community, both in a very specific setting (two bidders, specific cost distributions) [82, 83], and in a very general setting (n bidders, arbitrary cost distributions) [84, 85], and hence there exist results that are applicable to the problem at hand.

In the first instance, it is showed how the problem can be restated into a bidding problem with asymmetric cost distributions. The discussion then proceeds to characterising the equilibrium bidding strategies (their existence and uniqueness) in the generic case; that is, with an arbitrary number of network operators and an arbitrary distribution of costs. The equilibrium bidding strategies are then explicitly derived in the restricted case; that is, with the number of network operators restricted to two and the costs uniformly distributed. Finally, the chapter concludes with the presentation of three numerical methods that can be used to numerically approximate the equilibrium bidding strategies in the case of more than two network operators characterised by uniform distributions of costs.

5.1 Problem Restatement

In order to transform the problem, recall the utility function for each network operator i

$$u_i(b,c,r) = \begin{cases} b_i - c_i & \text{if } wb_i + (1-w)r_i < \min\limits_{j\neq i}[wb_j + (1-w)r_j], \\ 0 & \text{if } wb_i + (1-w)r_i > \min\limits_{j\neq i}[wb_j + (1-w)r_j], \end{cases} \tag{5.1}$$

and let

$$\hat{b}_i = wb_i + (1-w)r_i \quad \text{for all } i \in N. \tag{5.2}$$

Solving Equation (5.2) for b_i yields

$$b_i = \frac{\hat{b}_i - (1-w)r_i}{w}, \quad w \neq 0. \tag{5.3}$$

Substituting Equation (5.3) back into the utility function yields

$$u_i(\hat{b}, c, r) = \begin{cases} \dfrac{1}{w}\left[\hat{b}_i - (wc_i + (1-w)r_i)\right] & \text{if } \hat{b}_i < \min\limits_{j \neq i} \hat{b}_j, \\ 0 & \text{if } \hat{b}_i > \min\limits_{j \neq i} \hat{b}_j. \end{cases} \tag{5.4}$$

Further let

$$\hat{c}_i = wc_i + (1-w)r_i \quad \text{for all } i \in N, \tag{5.5}$$

then the utility function simplifies to

$$u_i(\hat{b}, \hat{c}) = \begin{cases} \dfrac{1}{w}\left(\hat{b}_i - \hat{c}_i\right) & \text{if } \hat{b}_i < \min\limits_{j \neq i} \hat{b}_j, \\ 0 & \text{if } \hat{b}_i > \min\limits_{j \neq i} \hat{b}_j. \end{cases} \tag{5.6}$$

The rest of this thesis concentrates on the utility function with the scaling factor, $1/w$, omitted; that is, let

$$\hat{u}_i(\hat{b}, \hat{c}) = w \cdot u_i(\hat{b}, \hat{c}) \tag{5.7}$$

for all $i \in N$. In fact, it can be noted that the pure-strategy Bayesian Nash equilibrium for the auction with utility function in Equation (5.7) constitutes an equilibrium for the auction with utility function in Equation (5.6). Formally,

Proposition 5.1. *Suppose $(\hat{b}_1^*, \ldots, \hat{b}_n^*)$ is a pure-strategy Bayesian Nash equilibrium profile for an auction with the utility function*

$$\hat{u}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) = \begin{cases} \left(\hat{b}_i - \hat{c}_i\right) & \textit{if} \quad \hat{b}_i < \min\limits_{j \neq i} \hat{b}_j, \\ 0 & \textit{if} \quad \hat{b}_i > \min\limits_{j \neq i} \hat{b}_j. \end{cases} \tag{5.8}$$

Then, the same profile constitutes an equilibrium for an auction with the utility function

$$u_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) = \frac{1}{w} \cdot \hat{u}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}). \tag{5.9}$$

In order to avoid ambiguity, $\hat{c}_i$ will be referred to as cost-hat and $\hat{b}_i$ as bid-hat, while c_i will still be referred to as cost and b_i as bid. Note, moreover, that since both w and r_i are assumed to be given to the network operators (i.e., they cannot directly modify their values), the costs-hat and bids-hat are simply convex (and hence, linear) combinations involving costs and bids respectively (Equations (5.2) and (5.5)). Therefore, a network operator bidding their cost-hat is equivalent to bidding their cost.

As a result of this transformation, the costs-hat, $\hat{c}_i$, for each network operator i are distributed over the interval

$$\hat{c}_i \in [(1-w)r_i, (1-w)r_i + w] = [\underline{\hat{c}}_i, \bar{\hat{c}}_i] \tag{5.10}$$

since $c_i \in [0, 1]$ for all $i \in N$. Note, moreover, that for all $i \in N$

$$[\underline{\hat{c}}_i, \bar{\hat{c}}_i] \subset [0, 1] \tag{5.11}$$

since $w \in (0, 1)$ and $r_i \in [0, 1]$, and in particular, if $w = 1$

$$[\underline{\hat{c}}_i, \bar{\hat{c}}_i] = [0, 1]. \tag{5.12}$$

Therefore, in terms of costs-hat, the network operators are *ex ante* asymmetric; that is, due to differing domains of costs-hat between the network operators, the probability distributions will have differing supports.

With these results at hand, the discussion can proceed with the characterisation of the equilibrium bidding strategies in the generic case, which is the subject of the next section.

5.2 Generic Case

In the generic case, with arbitrary probability distributions of costs and $n \geq 2$ network operators, recall that: if $w = 0$, then Proposition 4.1 holds; if $w = 1$, then Proposition 4.2 holds; and if $r_i = r_j$ for all $i, j \in N$ such that $i \neq j$, then Corollary 4.3 holds. Therefore, it suffices to consider only the case when $w \in (0, 1)$.

Firstly, note that under the generic assumptions specified in Section 4.1 and $w \in (0, 1]$, the problem satisfies the following regularity conditions.

Proposition 5.2 (Regularity Conditions). *Let F_i be the distribution function of $\hat{c}_i$ for all $i \in N$, and suppose $w \in (0, 1]$. Then,*

1. *the support of F_i is an interval $[\underline{\hat{c}}_i, \bar{\hat{c}}_i]$;*

2. *F_i is differentiable over $(\underline{\hat{c}}_i, \bar{\hat{c}}_i]$ with a derivative f_i locally bounded away from zero over this interval; and*

3. *F_i is atomless.*

The regularity conditions in Proposition 5.2 correspond to the regularity assumptions on type distributions put forward by Lebrun [85] (cf. Assumptions A.1 in [85]). Therefore, since the problem satisfies Lebrun's assumptions, his results are applicable.

Further assume that

Assumptions 5.1. Assume that

1. $w \in (0, 1)$;

2. there exists $i \in N$ such that $r_i \neq r_j$ for all $i \neq j$ and $j \in N$; and

3. without loss of generality, let network operator 1 be characterized by the lowest reputation rating; that is, $r_1 \leq r_i$ for all $i \in N$ such that $i \neq 1$. If there exists $j \in N$ such that $j \neq 1$ and $r_1 = r_j$, then it is further assumed that there exists $\delta > 0$ such that F_i is strictly log-concave over $(\bar{\hat{c}}_1 - \delta, \bar{\hat{c}}_1) \cap (\underline{\hat{c}}_i, \bar{\hat{c}}_i)$ for all $i \in N$.

In equilibrium, the bids of each network operator equal $\hat{b}_i = \hat{b}_i(\hat{c}_i)$, where $\hat{b}_i$ is the equilibrium bidding function. Denote by $\hat{c}_i(\hat{b}_i) = \hat{b}_i^{-1}(\hat{b}_i)$ an inverse equilibrium bidding function for each network operator $i \in N$. Therefore, the expected utility for each network operator $i \in N$ can be written as

$$\begin{aligned} \Pi_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) &= (\hat{b}_i - \hat{c}_i) P\{\text{winning} \mid \hat{b}_i\} \\ &= (\hat{b}_i - \hat{c}_i) Q_i(\hat{b}_i), \end{aligned} \tag{5.13}$$

where

$$Q_i(\hat{b}_i) = \prod_{j \neq i} \Big(1 - F_j(\hat{c}_j(\hat{b}_i))\Big) \tag{5.14}$$

is the probability that network operator i is the lowest bidder.

The first order condition for maximising network operator i's expected utility is

$$\frac{d}{d\hat{b}_i} \Pi_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) = Q_i(\hat{b}_i) + (\hat{b}_i - \hat{c}_i) \cdot \frac{d}{d\hat{b}_i} Q_i(\hat{b}_i) = 0, \tag{5.15}$$

where

$$\frac{d}{d\hat{b}_i} Q_i(\hat{b}_i) = (-1) \sum_{j \neq i} f_j(\hat{c}_j(\hat{b}_i)) \frac{d}{d\hat{b}_i} \hat{c}_j(\hat{b}_i) \prod_{k \neq j} \left(1 - F_k(\hat{c}_k(\hat{b}_i))\right). \tag{5.16}$$

Noting that in equilibrium $\hat{c}_i = \hat{c}_i(\hat{b}_i)$, letting $\hat{b}_i = b$, and rearranging terms in Equation (5.15) yields

$$\begin{aligned} \frac{1}{b - \hat{c}_i(b)} &= \frac{\sum_{j \neq i} f_j(\hat{c}_j(b)) \frac{d}{db} \hat{c}_j(b) \prod_{k \neq j} (1 - F_k(\hat{c}_k(b)))}{\prod_{j \neq i} (1 - F_j(\hat{c}_j(b)))} \\ &= \sum_{j \neq i} \frac{f_j(\hat{c}_j(b))}{1 - F_j(\hat{c}_j(b))} \cdot \frac{d}{db} \hat{c}_j(b). \end{aligned} \tag{5.17}$$

Summing Equation (5.17) over all n network operators yields

$$\frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{b - \hat{c}_i(b)} = \sum_{i=1}^{n} \frac{f_i(\hat{c}_i(b))}{1 - F_i(\hat{c}_i(b))} \cdot \frac{d}{db} \hat{c}_i(b). \tag{5.18}$$

Subtracting Equation (5.17) from (5.18) yields

$$\frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{b - \hat{c}_i(b)} - \frac{1}{b - \hat{c}_i(b)} = \frac{f_i(\hat{c}_i(b))}{1 - F_i(\hat{c}_i(b))} \cdot \frac{d}{db} \hat{c}_i(b) \tag{5.19}$$

which leads to the system of nonlinear ordinary differential equations (ODE)

$$\frac{d}{db} \hat{c}_i(b) = \frac{1 - F_i(\hat{c}_i(b))}{f_i(\hat{c}_i(b))} \left[\frac{1}{n-1} \sum_{i=1}^{n} \frac{1}{b - \hat{c}_i(b)} - \frac{1}{b - \hat{c}_i(b)} \right] \tag{5.20}$$

for $i = 1, 2, \ldots, n$. As will be shown briefly, there exists a unique set of inverse bidding functions that satisfy the system and constitute a pure-strategy Bayesian Nash equilibrium where network operators submit at least their costs-hat. Firstly, few concepts need to be defined (cf. Definitions 1, 2 and 3 in Lebrun [85]).

Definition 5.1 (Upper bound on bids)**.** Let Assumptions 5.1 be satisfied. Then, the upper bound on bids is defined as follows

$$\bar{\hat{b}} = \min \arg \max_{b \in [\bar{\hat{c}}_1, \bar{\hat{c}}_2]} (b - \bar{\hat{c}}_1) \prod_{i > 1} (1 - F_i(b)). \tag{5.21}$$

Definition 5.2 (Feasible bidders)**.** Let Assumptions 5.1 be satisfied. Let J denote the set of feasible bidders. Then, J is a subset of N such that

$$J = \left\{ j \;\middle|\; 1 \leq j \leq n \text{ and } \underline{\hat{c}}_j < \bar{\hat{b}} \right\}. \tag{5.22}$$

Furthermore, let $n' = |J|$.

Definition 5.3 (Characterisation of lower bound on bids)**.** Let Assumptions 5.1 be satisfied. Then,

1. For all $\underline{\hat{b}} \in (\underline{\hat{c}}_2, \bar{\hat{b}})$, there exists one and only one $k(\underline{\hat{b}}) \in \{2, \ldots, n\}$ such that $\underline{\hat{c}}_{k(\underline{\hat{b}})} < \underline{\hat{b}}$ and

$$\frac{1}{\underline{\hat{b}} - \underline{\hat{c}}_{k(\underline{\hat{b}})}} \leq \frac{1}{k(\underline{\hat{b}}) - 1} \sum_{i=1}^{k(\underline{\hat{b}})} \frac{1}{\underline{\hat{b}} - \underline{\hat{c}}_i}, \tag{5.23}$$

   and if $\underline{\hat{c}}_{k(\underline{\hat{b}})+1} < \underline{\hat{b}}$ (and $k(\underline{\hat{b}}) < n$)

$$\frac{1}{k(\underline{\hat{b}}) - 1} \sum_{i=1}^{k(\underline{\hat{b}})} \frac{1}{\underline{\hat{b}} - \underline{\hat{c}}_i} < \frac{1}{\underline{\hat{b}} - \underline{\hat{c}}_{k(\underline{\hat{b}})+1}}. \tag{5.24}$$

   The proof of this assertion can be found in Lebrun [86] (see Lemma A4.1, Appendix 4).

2. For all $\underline{\hat{b}} \in (\underline{\hat{c}}_2, \bar{\hat{b}})$, let $\hat{c}(\underline{\hat{b}})$ be defined as follows

$$\hat{c}(\underline{\hat{b}}) = \underline{\hat{b}} - \left(k(\underline{\hat{b}}) - 1\right) \Big/ \left(\sum_{i=1}^{k(\underline{\hat{b}})} \frac{1}{\underline{\hat{b}} - \underline{\hat{c}}_i} \right). \tag{5.25}$$

Note that Definition 5.3 implies

$$\begin{aligned} & \underline{\hat{c}}_{k(\underline{\hat{b}})} \leq \hat{c}(\underline{\hat{b}}) < \underline{\hat{c}}_{k(\underline{\hat{b}})+1} && \text{if} \quad k(\underline{\hat{b}}) < n, \\ \text{and} \quad & \underline{\hat{c}}_{k(\underline{\hat{b}})} \leq \hat{c}(\underline{\hat{b}}) && \text{if} \quad k(\underline{\hat{b}}) = n. \end{aligned} \tag{5.26}$$

With those definitions at hand, the discussion can proceed with the characterisation of the equilibrium which is due to Lebrun [85].

Proposition 5.3 (Characterisation of the Equilibrium)**.** *Let Assumptions 5.1 be satisfied. There exists one and only one pure-strategy Bayesian Nash equilibrium where network operators submit at least their costs-hat. In every such equilibrium, network operator $i \in J$ follows a bid function $\hat{b}_i$, for all $1 \leq i \leq n$. Moreover, there exists $\underline{\hat{b}} \in (\underline{\hat{c}}_2, \bar{\hat{b}})$ such that, for all $i \in J$, there exists a continuous extension of $\hat{b}_i$ to the interval $\left[\min\{\underline{\hat{c}}_i, \hat{c}(\underline{\hat{b}})\}, \bar{\hat{b}}\right]$ that is differentiable with a strictly positive derivative* *everywhere over this interval, except possibly at $\underline{\hat{c}}_i$ or when its value is equal to $\bar{\hat{b}}$, and such that the inverse bid functions $\hat{c}_i$ for all $i \in J$ of these extensions, where differentiable, satisfy the following system of differential equations*

$$\frac{d}{db}\hat{c}_i(b) = \frac{1 - F_i(\hat{c}_i(b))}{f_i(\hat{c}_i(b))}\left[\frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{b - \hat{c}_k(b)} - \frac{1}{b - \hat{c}_i(b)}\right] \tag{5.27}$$

for all $1 \leq i \leq n$, with the following lower boundary condition

$$\hat{c}_i(\underline{\hat{b}}) = \min\left\{\underline{\hat{c}}_i, \hat{c}(\underline{\hat{b}})\right\} \quad \textit{for all } i \in J \tag{5.28}$$

and the upper boundary condition

$$\hat{c}_i(\bar{\hat{b}}) = \bar{\hat{b}} \tag{5.29}$$

for all, except possibly one, $1 \leq i \leq n$.

It is worth noting that, to the best of the author's knowledge, the system of ODEs in Equation (5.27) with boundary conditions (5.28) and (5.29) is unique to the domain of auction theory. This can be attributed to the fact that the derivation of the system involves the inverses of the equilibrium bidding strategy functions (as opposed to the equilibrium bidding strategy functions themselves), and unknown *a priori* lower bound on bids, $\underline{\hat{b}}$.

The intuition behind the upper boundary condition in Equation (5.29) is that the network operator bids their cost-hat when their probability of winning is zero. Ignoring the minimum operator, the intuition behind the lower boundary condition in Equation (5.28) on the other hand, is that the lowest bid-hat of each network operator is reached for their lowest cost-hat.

Since both w and r_i are assumed to be given to the network operators (i.e., they cannot directly modify their values), the costs-hat and bids-hat are simply convex (and hence, linear) combinations involving costs and bids respectively (Equations (5.2) and (5.5)). Therefore, a network operator bidding their cost-hat is equivalent to bidding their cost, and the following corollary can immediately be deduced.

Corollary 5.4. *Let Assumptions 5.1 be satisfied. There exists one and only one pure-strategy Bayesian Nash equilibrium where network operators submit at least their costs.*

Even though it is guaranteed that there exists a unique equilibrium to the problem, the establishment of a closed-form solution to the system of ODEs for $n \geq 2$ network operators and arbitrary cost distribution is very difficult (if possible), and for $n > 2$ is not possible [85, 54]. However, it is possible to explicitly derive the equilibrium bidding strategy functions in a much restricted setting with two network operators characterised by uniform distributions of costs. This is explored in the next section.

5.3 Restricted Case $n = 2$

Let $n = 2$ network operators, and assume costs, c_i, for both network operators are drawn from the uniform distribution. Furthermore, let Assumptions 5.1 be satisfied. Without loss of generality, suppose $r_1 < r_2$, which implies $\underline{\hat{c}}_1 < \underline{\hat{c}}_2$ and $\bar{\hat{c}}_1 < \bar{\hat{c}}_2$. The utility function for each $i \in \{1, 2\}$ is

$$u_i(\hat{b}, \hat{c}) = \begin{cases} \frac{1}{w}\left(\hat{b}_i - \hat{c}_i\right) & \text{if } \hat{b}_i < \hat{b}_j, \\ \frac{1}{2w}\left(\hat{b}_i - \hat{c}_i\right) & \text{if } \hat{b}_i = \hat{b}_j, \\ 0 & \text{if } \hat{b}_i > \hat{b}_j. \end{cases} \tag{5.30}$$

Since the distribution of costs, c_i, for each network operator i is uniform with support $[0, 1]$, the distribution of costs-hat, $\hat{c}_i$, for each network operator i is uniform with support $[\underline{\hat{c}}_i, \bar{\hat{c}}_i] = [(1-w)r_i, (1-w)r_i + w]$. Therefore, the distribution function of costs-hat satisfies the regularity conditions specified in Proposition 5.2, and by Corollary 5.4, it can be concluded that the pure-strategy Bayesian Nash equilibrium where network operators submit at least their costs exists and is unique.

The derivation of the equilibrium involves three stages: 1) deriving equilibrium inverse bidding strategy functions using the procedure described by Kaplan and Zamir [82]; 2) numerically estimating the equilibrium bidding strategy functions by inverting the inverses; and 3) transforming the problem back to the original domain (from costs-hat and bids-hat back to costs and bids).

First, note that, by Definition 5.1, the upper bound on bids is equal to

$$\bar{\hat{b}} = \min \arg \max_{b \in [\bar{\hat{c}}_1, \bar{\hat{c}}_2]} \frac{(b - \bar{\hat{c}}_1)(\bar{\hat{c}}_2 - b)}{\bar{\hat{c}}_2 - \underline{\hat{c}}_2} = \frac{\bar{\hat{c}}_1 + \bar{\hat{c}}_2}{2}, \tag{5.31}$$

where the fact that F_2 is the distribution function of the uniform distribution with support $[\underline{\hat{c}}_2, \bar{\hat{c}}_2]$ was used.

If $\bar{\hat{b}} \leq \underline{\hat{c}}_2 \iff \bar{\hat{c}}_1 \leq 2\underline{\hat{c}}_2 - \bar{\hat{c}}_2$, then, by Definition 5.2, only network operator 1 is a feasible bidder. In this case, any pure-strategy Bayesian Nash equilibrium must have network operator 1 always bidding $\underline{\hat{c}}_2$, and hence, always winning the auction at price $\underline{\hat{c}}_2$ [82]. Henceforth, this case will be referred to as trivial. Otherwise, in the nontrivial case, by Proposition 5.3, the inverse equilibrium bidding functions are determined by the system

$$
\begin{cases}
\dfrac{d}{db}\hat{c}_1(b) & = \dfrac{\bar{\hat{c}}_1 - \hat{c}_1(b)}{b - \hat{c}_2(b)} \\
\dfrac{d}{db}\hat{c}_2(b) & = \dfrac{\bar{\hat{c}}_2 - \hat{c}_2(b)}{b - \hat{c}_1(b)}
\end{cases}
\tag{5.32}
$$

with boundary conditions (cf. boundary conditions in [82]) $\hat{c}_1(\underline{\hat{b}}) = \underline{\hat{c}}_1$ and $\hat{c}_1(\bar{\hat{b}}) = \bar{\hat{c}}_1$ for network operator 1, and $\hat{c}_2(\underline{\hat{b}}) = \underline{\hat{c}}_2$ and $\hat{c}_2(\bar{\hat{b}}) = \bar{\hat{b}}$ for network operator 2.

Note that, since $n = 2$, $k(\underline{\hat{b}}) = 2$ by Definition 5.3. Hence, by (5.26), $\underline{\hat{c}}_2 \leq \hat{c}(\underline{\hat{b}})$, and since $\underline{\hat{c}}_1 < \underline{\hat{c}}_2$, this reduces the lower boundary condition in Equation (5.28) to $\hat{c}_i(\underline{\hat{b}}) = \underline{\hat{c}}_i$ for all $i \in \{1, 2\}$.

Integrating the system (5.32) bounded by the aforementioned boundary conditions results in the derivation of the equilibrium inverse bidding strategy functions. The derivation procedure is fully described in Kaplan and Zamir [82]; hence, only the final result is provided.

Proposition 5.5. *Let there be $n = 2$ network operators, and suppose c_i is independently drawn from uniform distribution over the interval $[0, 1]$ for all $i \in \{1, 2\}$. Furthermore, let Assumptions 5.1 be satisfied. The equilibrium inverse bidding strategy functions are given by*

$$
\hat{c}_1(b) = \bar{\hat{c}}_1 + \frac{(\bar{\hat{c}}_2 - \bar{\hat{c}}_1)^2}{(\bar{\hat{c}}_2 + \bar{\hat{c}}_1 - 2b)d_1 \exp\left(\dfrac{\bar{\hat{c}}_2 - \bar{\hat{c}}_1}{\bar{\hat{c}}_2 + \bar{\hat{c}}_1 - 2b}\right) + 4(\bar{\hat{c}}_2 - b)}, \tag{5.33}
$$

$$
\hat{c}_2(b) = \bar{\hat{c}}_2 + \frac{(\bar{\hat{c}}_1 - \bar{\hat{c}}_2)^2}{(\bar{\hat{c}}_1 + \bar{\hat{c}}_2 - 2b)d_2 \exp\left(\dfrac{\bar{\hat{c}}_1 - \bar{\hat{c}}_2}{\bar{\hat{c}}_1 + \bar{\hat{c}}_2 - 2b}\right) + 4(\bar{\hat{c}}_1 - b)}, \tag{5.34}
$$

where

$$
d_1 = \frac{\dfrac{(\bar{\hat{c}}_2 - \bar{\hat{c}}_1)^2}{\underline{\hat{c}}_1 - \bar{\hat{c}}_1} + 4(\underline{\hat{b}} - \bar{\hat{c}}_2)}{-2(\underline{\hat{b}} - \bar{\hat{b}})} \exp\left(\frac{\bar{\hat{c}}_2 - \bar{\hat{c}}_1}{2(\underline{\hat{b}} - \bar{\hat{b}})}\right), \tag{5.35}
$$

$$
d_2 = \frac{\dfrac{(\bar{\hat{c}}_1 - \bar{\hat{c}}_2)^2}{\underline{\hat{c}}_2 - \bar{\hat{c}}_2} + 4(\underline{\hat{b}} - \bar{\hat{c}}_1)}{-2(\underline{\hat{b}} - \bar{\hat{b}})} \exp\left(\frac{\bar{\hat{c}}_1 - \bar{\hat{c}}_2}{2(\underline{\hat{b}} - \bar{\hat{b}})}\right), \tag{5.36}
$$

and

$$\underline{\hat{b}} = \frac{\underline{\hat{c}}_1\underline{\hat{c}}_2 - \dfrac{(\bar{\hat{c}}_1 + \bar{\hat{c}}_2)^2}{4}}{\underline{\hat{c}}_1 - \bar{\hat{c}}_1 + \underline{\hat{c}}_2 - \bar{\hat{c}}_2}, \quad \bar{\hat{b}} = \frac{\bar{\hat{c}}_1 + \bar{\hat{c}}_2}{2}. \tag{5.37}$$

It is worth noting that, if the network operators are assumed to submit at least their costs, then Proposition 4.4 is ruled out by Proposition 5.3 combined with Proposition 5.5 since the latter establishes an analytical solution to the bidding problem while Proposition 5.3 makes this solution unique. However, if the assumption that network operators submit at least their costs is relaxed, then, as shown by Kaplan and Zamir [87], Proposition 5.3 need no longer hold, and hence, there may exist multiple equilibria in the first-price sealed-bid auction bidding problem. As a result, both equilibria summarised in Proposition 5.5 as well as Proposition 4.4 are valid. Kaplan and Zamir [87], who classify equilibria like the one specified in Proposition 4.4 as non-standard, further argue that such equilibria are important and should not be neglected since they may result in different network operators winning the auction, and as a result, different expected prices. In other words, relaxing the assumption about network operators submitting at least their cost might help in understanding the "deviations" from the predicted equilibrium bidding behaviour prescribed in Proposition 5.5 should these occur in reality. Here, those "deviations" may be captured by non-standard equilibria.

The equilibrium inverse bidding strategy functions are inconvenient to work with: for a particular bid-hat value, they map into a particular cost-hat for either network operator. It would be more intuitive to work with their inverses, where for a particular cost-hat, a particular bid-hat is obtained. Since inverting the equilibrium inverse bidding strategy functions in Equations (5.33) and (5.34) is analytically intractable, numerical method is proposed that can be employed to estimate the inverses for a particular set of cost-reputation pairs with respect to the price weights for both network operators.

Listing 5.1 depicts the pseudo-code of the proposed method. The steps of the algorithm can be summarised as follows:

1. For a particular price weight w and reputation ratings r_1 and r_2, calculate the costs-hat supports for both network operators; that is, the endpoints of the interval $[\underline{\hat{c}}_1, \bar{\hat{c}}_1]$ for network operator 1, and $[\underline{\hat{c}}_2, \bar{\hat{c}}_2]$ for network operator 2 (lines 1–4).

2. If $\bar{\hat{c}}_1 \leq 2\underline{\hat{c}}_2 - \bar{\hat{c}}_2$, then the equilibrium is trivial. Network operator 1 bids the lower endpoint of the cost-hat support of network operator 2; that is, network operator 1 bids $\underline{\hat{c}}_2$ for all $\hat{c} \in [\underline{\hat{c}}_1, \bar{\hat{c}}_1]$. Network operator 2, on the other hand, bids their cost-hat; that is, network operator 2 bids $\hat{c}$ for all $\hat{c} \in [\underline{\hat{c}}_2, \bar{\hat{c}}_2]$ (lines 6–9).

3. If $\bar{\hat{c}}_1 > 2\underline{\hat{c}}_2 - \bar{\hat{c}}_2$, then the equilibrium is nontrivial. Hence,

Algorithm 5.1 Inverse of equilibrium inverse bidding strategy functions

Input: $w \in (0,1]; r_1, r_2 \in [0,1]$ such that $r_1 \leq r_2$
Output: Tabulation of functions $\hat{b}_i(\hat{c}_i)$ for all $i \in \{1,2\}$: $(\hat{c}_1, \hat{b}_1)$ and $(\hat{c}_2, \hat{b}_2)$

$\underline{\hat{c}}_1 \leftarrow (1-w)r_1$
$\bar{\hat{c}}_1 \leftarrow (1-w)r_1 + w$

$\underline{\hat{c}}_2 \leftarrow (1-w)r_2$
$\bar{\hat{c}}_2 \leftarrow (1-w)r_2 + w$

if $\bar{\hat{c}}_1 \leq 2\underline{\hat{c}}_2 - \bar{\hat{c}}_2$ **then**
    $\hat{c}_1 \leftarrow \{\underline{\hat{c}}_1, \ldots, \bar{\hat{c}}_1\}$
    $\hat{b}_1 \leftarrow \{\underline{\hat{c}}_2, \ldots, \underline{\hat{c}}_2\}$

    $\hat{c}_2 \leftarrow \{\underline{\hat{c}}_2, \ldots, \bar{\hat{c}}_2\}$
    $\hat{b}_2 \leftarrow \{\underline{\hat{c}}_2, \ldots, \bar{\hat{c}}_2\}$
else
    $\underline{\hat{b}} \leftarrow$ compute using (5.37)
    $\bar{\hat{b}} \leftarrow$ compute using (5.37)

    $\hat{b}_1 \leftarrow \{\underline{\hat{b}}, \ldots, \bar{\hat{b}}\}$
    $\hat{c}_1 \leftarrow$ compute using (5.33) for all $b \in \hat{b}_1$

    $\hat{b}_2 \leftarrow \{\underline{\hat{b}}, \ldots, \bar{\hat{c}}_2\}$
    $\hat{c}_2 \leftarrow$ compute using (5.34) for all $b \in \hat{b}_2$

(a) Calculate the common bids-hat support $[\underline{\hat{b}}, \bar{\hat{b}}]$ using Equation (5.37) (lines 11–12).

(b) For all $\hat{b} \in [\underline{\hat{b}}, \bar{\hat{b}}]$, calculate the corresponding costs-hat for both network operators using Equations (5.33) and (5.34). Since by assumption $r_1 < r_2$, it follows that $\bar{\hat{c}}_1 \leq \bar{\hat{c}}_2$, and hence, $\bar{\hat{b}} \leq \bar{\hat{c}}_2$. Thus, network operator 2 bids their cost-hat, $\hat{c}_2(\hat{b}) = \hat{c}_2$ for all $\hat{b} \in [\bar{\hat{b}}, \bar{\hat{c}}_2]$ (lines 13–16).

The result of the steps described above is the tabulation of the costs-hat and their corresponding equilibrium bids-hat for a particular price weight w, and reputation ratings r_1 and r_2 for both network operators, in the ranges $[\underline{\hat{c}}_1, \bar{\hat{c}}_1]$ for network operator 1 and $[\underline{\hat{c}}_2, \bar{\hat{c}}_2]$ for network operator 2.

Denote by

$$\hat{b}_1(\hat{c}_1) = \hat{b}_1 \quad \text{for all } \hat{c}_1 \in [\underline{\hat{c}}_1, \bar{\hat{c}}_1], \tag{5.38}$$

and

$$\hat{b}_2(\hat{c}_2) = \hat{b}_2 \quad \text{for all } \hat{c}_2 \in [\underline{\hat{c}}_2, \bar{\hat{c}}_2] \tag{5.39}$$

the resultant equilibrium bidding strategy functions. The problem can be transformed back into the original domain by substituting Equations (5.2) and (5.5) into Equations (5.38) and (5.39); that is,

$$\hat{b}_1(\hat{c}_1) = \hat{b}_1 \iff b_1 = \frac{\hat{b}_1(wc_1 + (1-w)r_1) - (1-w)r_1}{w} \tag{5.40}$$

for all $c_1 \in [0, 1]$, and

$$\hat{b}_2(\hat{c}_2) = \hat{b}_2 \iff b_2 = \frac{\hat{b}_2(wc_2 + (1-w)r_2) - (1-w)r_2}{w} \tag{5.41}$$

for all $c_2 \in [0, 1]$. Keeping costs and reputation ratings fixed, one can then estimate the equilibrium bidding strategy functions with respect to the price weights by sliding the value of $w \in (0, 1)$.

By way of example, the equilibrium bidding strategy functions were estimated for the set of cost-reputation pairs depicted in Table 5.1. Figure 5.1 shows the value of the compound bid, $\beta(b_i, r_i)$, for different values of w for both network operators, while Figure 5.2 depicts the value of the monetary bid (or offered price), b_i, for different values of w for both network operators. The numerical data in Table 5.1 suggests that network operator 2 should be the winner for the values of $w \to 1$ since network operator 2's cost

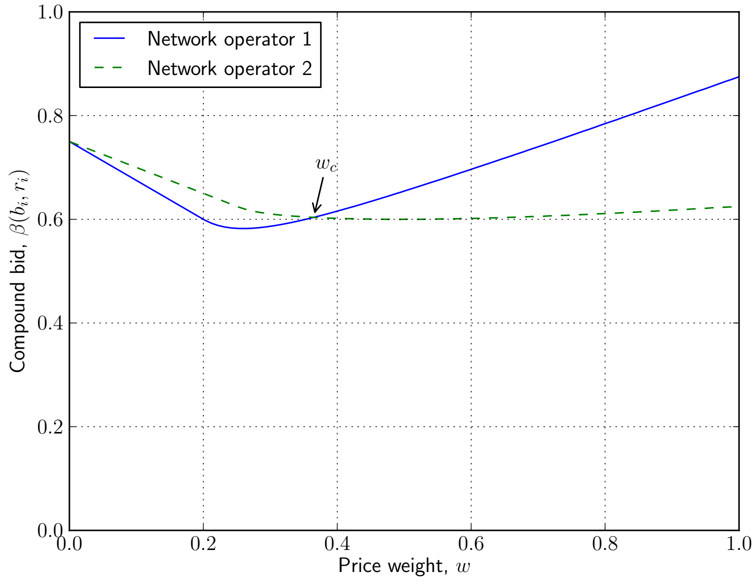


Figure 5.1
Compound bid plotted against the price weight

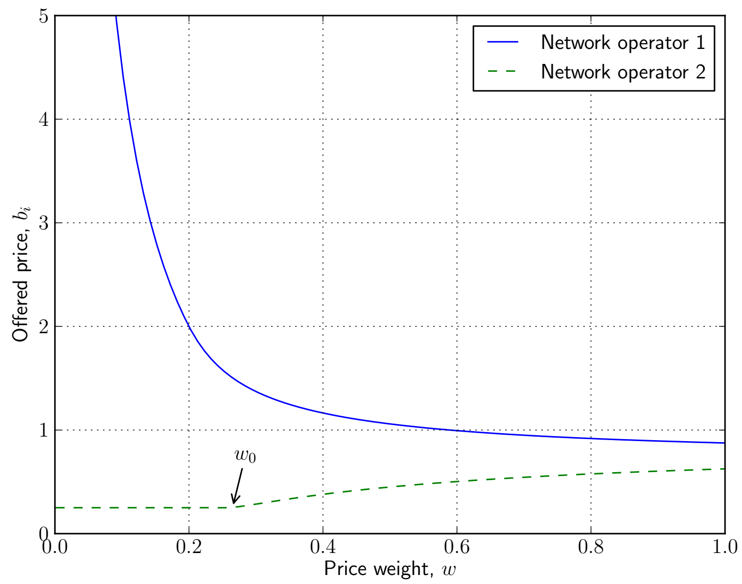


Figure 5.2
Offered prices (bids) plotted against the price weight

Table 5.1
An exemplary set of cost-reputation pairs of two network operators

	Cost, c_i	**Reputation rating**, r_i
Network operator 1	0.75	0.25
Network operator 2	0.25	0.75

is strictly lower than that of their opponent's. On the other hand, network operator 1 should be winner for the values of $w \to 0$ since network operator 1's reputation rating is strictly lower than that of their opponent's (which implies that network operator 1's reputation is in fact strictly higher than that of their opponent's). This prediction agrees with the numerical output shown in Figure 5.1. Let w_c denote the value of w for which an intersection between the compound bids of both network operators occurs (if it exists). In Figure 5.1, $w_c \approx 0.365$. Hence, network operator 2 wins the auction for the values of $w_c < w < 1$, while network operator 1 for the values of $0 < w < w_c$.

Note, furthermore, that since it was explicitly required for the network operators to bid their own costs when their probability of winning is zero, the monetary bid of network operator 2 is capped at their cost, $b_2 = 0.25$, for the values of $0 < w \leq w_0$ where $w_0 \approx 0.265$ (see Figure 5.2). In the same range of w, as w decreases, network operator 1's bid increases in an exponential-like fashion, to finally culminate in $b_1 \to \infty$ at $w = 0$ in accordance with Proposition 4.1. As $w \to 1$, on the other hand, the monetary bids of both network operators tend to the values specified in Proposition 4.2, that is, $b_1 = 0.875$ and $b_2 = 0.625$, to finally attain those values at $w = 1$.

Having derived the equilibrium bidding strategy functions, it is possible to examine the expected prices the subscriber will have to pay for different values of the price weight given the reputation ratings of the network operators. This is examined next.

5.3.1 Subscriber's Perspective: Expected Prices

Suppose there are two network operators, and costs are uniformly distributed over the interval $[0, 1]$. The expected price is equivalent to the expected value of the winning bid; that is,

$$E[p](w, r_1, r_2) = E[b_i \mid \arg \min_{i \in \{1,2\}} \beta(w, b_i, r_i)], \tag{5.42}$$

where b_i is the equilibrium bid, and $\beta(w, b_i, r_i) = \beta(b_i, r_i)$ evaluated for a particular value of w for all $i \in \{1, 2\}$.

If both network operators have equal reputation ratings, $r = r_1 = r_2$ say, then

Corollary 4.3 holds for all $w \in [0,1]$. Therefore, regardless of the choice of the price weight, the subscriber expects to pay the price of

$$E[p^*] = E[p](w,r,r) = E\left[\min_{i\in\{1,2\}} \frac{1+c_i}{2}\right] \quad \text{for all } w \in [0,1], \tag{5.43}$$

which is equivalent to Equation (4.20) evaluated at $n = 2$. In particular, for costs, c_i, uniformly distributed over the interval $[0,1]$, $E[p^*] = \frac{2}{3}$.

If, on the other hand, both network operators are characterised by different reputation ratings, then an analytical derivation of the expected prices for each value of the price weight given a pair of reputation ratings is cumbersome. This is due to the fact that network operators bid according to a pair of inverse equilibrium bidding functions specified in Proposition 5.5, which are not easily invertible. Hence, numerical method is used to estimate average (sample mean) prices for selected values of the price weight given a pair of reputation ratings.

To this end, for any given pair of reputation ratings, the costs are pseudo-randomly drawn from the uniform distribution over the discretised interval $[0,1]$. For each selected price weight, the average price is averaged over 10,000 i.i.d. observations. The Strong Law of Large Numbers implies that as the number of observations tends to infinity, the average (sample mean) of the observations approaches the real mean of the distribution of the r.v. in question (see Section B.3.2, Appendix B for the definition of the Strong Law of Large Numbers). Furthermore, it was empirically established that averaging over more than 10,000 observations does not drastically improve the results; that is, the already narrow 95% confidence intervals do not get narrower as the number of observations increases beyond 10,000. In other words, 10,000 is large enough a sample size, and therefore, an average of 10,000 observations of the price for each selected price weight should provide a reasonable approximation of the expected price for that price weight. Without loss of generality, suppose further that $r_1 \leq r_2$. Figure 5.3 shows the result of the estimation for four pairs of reputation ratings: $(r_1, r_2) = (0.25, 0.25)$, $(0.25, 0.5)$, $(0.25, 0.75)$, and $(0.25, 1.0)$.

It can be observed that regardless of the values of the reputation ratings, the expected prices, $E[p](w, r_1, r_2)$, are bounded from below by $E[p^*]$ for each price weight; this is depicted in Figure 5.3. Hence, it can be concluded that regardless of the values of the reputation ratings, the lowest expected price is achieved for $w = 1$, and will not decrease as w decreases; in fact, it can only either increase or remain constant.

Furthermore, as the difference $(r_2 - r_1)$ increases, the expected prices, $E[p](w, r_1, r_2)$, increase as the price weight decreases; this is depicted in Figure 5.3. Therefore, it can be hypothesised that the smaller the difference $(r_2 - r_1)$, the less (expected) price sen-

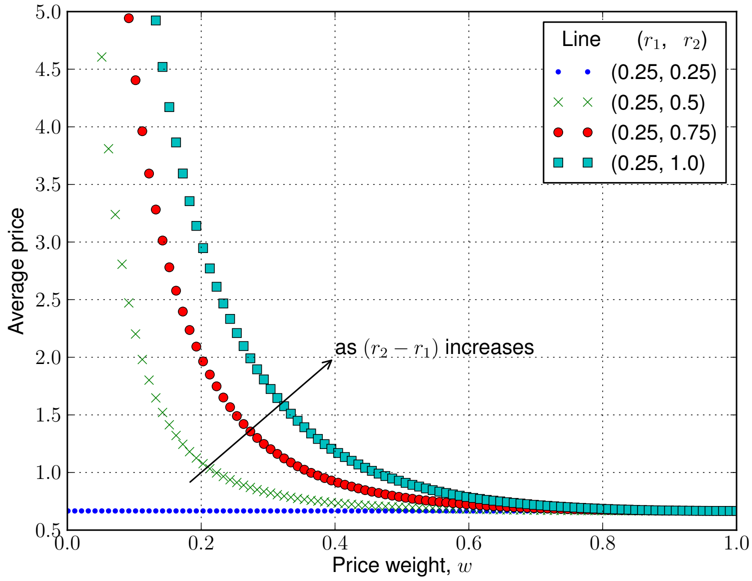


Figure 5.3
Average prices plotted against the price weight for different pairs of reputation ratings

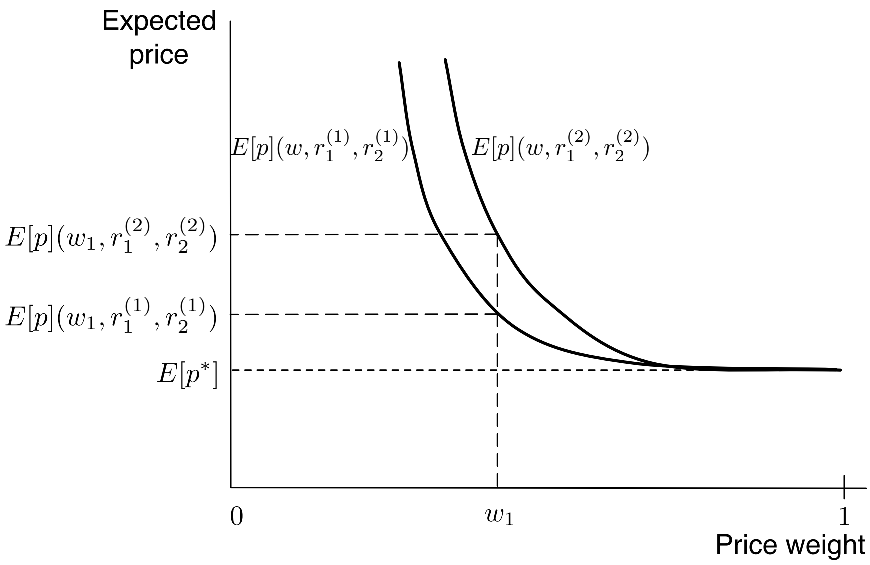


Figure 5.4
Sensitivity of the price weight to the expected prices

sitive the price weight; that is, for any $w_1 \in [0, 1]$, if $(r_2^{(2)} - r_1^{(2)}) > (r_2^{(1)} - r_1^{(1)})$ for all $r_1^{(1)}, r_2^{(1)}, r_1^{(2)}, r_2^{(2)} \in [0, 1]$, then $E[p](w_1, r_1^{(2)}, r_2^{(2)}) \geq E[p](w_1, r_1^{(1)}, r_2^{(1)})$ (see Figure 5.4). In other words, for any expected price, as the difference $(r_2 - r_1)$ between the reputation ratings of the network operators increases, the price weight has to increase (or remain constant) in order to keep the expected price fixed. This observation carries very serious implications on the operation of the DMP, as the subscriber is effectively given the ability to influence the expected prices by an appropriate choice of the price weight. To illustrate, suppose there are 2 network operators characterised by reputation ratings (r_1, r_2). Suppose further that the subscriber paid the price of p_1 at some point in the past for some type of service, and they request the same service again. Therefore, in order to pay the expected price of at most p_1, the subscriber solves

$$p_1 \geq E[p_1](w, r_1, r_2) \tag{5.44}$$

for the price weight w. In this way, the subscriber is guaranteed the expected price of at most p_1.

As already mentioned in the previous section, the bidding problem does not possess a closed-form solution in the case of more than two network operators. However, it is possible to approximate the solution numerically, as it will be discussed in the next section. It should further be noted that the equilibrium bidding strategies derived in this section will be used to verify the correctness of the numerical methods presented in the next section.

5.4 Numerical Analysis

Precisely because the system of ODEs (5.27) together with the lower and upper boundary conditions (5.28) and (5.29) does not possess any known closed-form solution in a generic setting, and especially when $n > 2$, there exists a considerable research base studying methods for numerical approximation of the solution to the system of ODEs in question. See Hubbard and Paarsch [88] for an excellent overview of the subject.

The literature is mostly concerned with asymmetric first-price auctions in which the bidders are characterised by different probability distributions sharing a common support; that is, $F_i(x) \neq F_j(x)$ for at least one $i \in N$ such that $i \neq j, j \in N$, and for all $x \in [\underline{\hat{c}}, \bar{\hat{c}}]$ where $[\underline{\hat{c}}, \bar{\hat{c}}] = [\underline{\hat{c}}_i, \bar{\hat{c}}_i]$ for all $i \in N$. This is not true in this case. Therefore, the methods described in Hubbard and Paarsch [88] have to be adapted before they can be applied to the problem under investigation here.

Firstly, it should be noted that there exist many methods for numerically approximating the solutions to a system of (ordinary or partial) differential equations; but, finite-difference methods, such as Euler or Runge-Kutta methods, are particularly well suited to solving systems of ODEs [89, 90]. In the problem at hand, since the system of ODEs satisfies the Lipschitz condition of continuity at the lower boundary condition $\underline{\hat{b}}$, if $\underline{\hat{b}}$ was known, standard finite-difference methods would apply [91] (see Section B.2.2, Appendix B for the definition of Lipschitz condition). However, this is not true in either scenario, the one considered in the literature and the one at hand: the common lower bound on bids, $\underline{\hat{b}}$, is unknown *a priori* [88]. On the other hand, since the common upper bound on bids, $\bar{\hat{b}}$, is known *a priori*, it would seem that the finite-difference methods could be applied to the system of ODEs in Equation (5.27) with the upper bound on bids as a starting point (the so-called *terminal value problem* as opposed to the more common *initial value problem*). As shown by Hubbard and Paarsch [88], the system does not satisfy the Lipschitz condition as the solution approaches the upper bound on bids, $\bar{\hat{b}}$. Therefore, much of the theory of ordinary differential equations no longer applies. In practice, this effectively means that the numerical solution obtained using a finite-difference method applied to the terminal value problem will quickly diverge. This is depicted in Figure 5.5. In this scenario, there are 2 network operators characterised by reputation ratings $r_1 = 0.25$ and $r_2 = 0.75$, and the price weight is set to $w = 0.5$. The 4$^{\text{th}}$-order backwards Runge-Kutta method is used to numerically solve the terminal value problem. It is clear in the figure that the numerical solution tracks the analytical equilibrium path only for $\hat{b} \in [0.72, 0.75]$, while it diverges for all the remaining values of bid-hat, $\hat{b} \in [0.515625, 0.72)$. Note, further, that after the solution diverges, it never recovers rendering the approximation useless.

In this section, two numerical algorithms are considered which overcome the aforementioned problem: the forward shooting method (FSM), and the polynomial projection method (PPM), both of which were first proposed by Bajari [91]. It is worth noting that the problem naturally fits into the framework of the FSM method since, as discussed in the previous paragraph, if the lower boundary condition, $\underline{\hat{b}}$, was known, standard finite-difference methods could be used to a great success in finding a numerical solution to the system of ODEs. As demonstrated in the subsequent section, the FSM method tries iteratively to guess the lower boundary condition, and for each guess, it then uses finite-difference methods to numerically solve the system of ODEs. Furthermore, the thesis focuses on the FSM and PPM methods since they are the simplest out of all of the available algorithms described in Hubbard and Parsch [88], and hence, they pose the least technical difficulties when adapting to the problem at hand, and yet yield numerical results of acceptable quality to permit conlusions to be drawn. To this end, let costs, c_i, be drawn from a uniform distribution for all network operators as in Section 5.3. Again, this implies that the distribution of costs-hats, $\hat{c}_i$, is uniform with

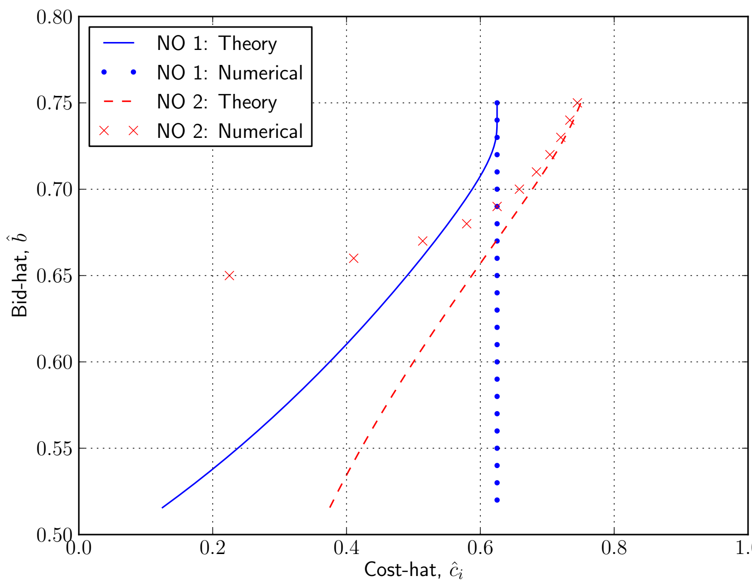


Figure 5.5
Violation of Lipschitz condition leads to divergent numerical solution

supports $[\underline{\hat{c}}_i, \bar{\hat{c}}_i] = [(1-w)r_i, (1-w)r_i + w]$ for all $i \in N$. The discussion concentrates only on cases in which $J = N$; that is, $\underline{\hat{c}}_i < \bar{\hat{b}}$ for all $i \in N$. In particular, it is required $w \in (0.5, 1)$ which immediately implies $\underline{\hat{c}}_i < \bar{\hat{b}}$ for all $i \in N$. To see this, without loss of generality, suppose $r_1 \leq \cdots \leq r_n$ with at least one inequality strict. Since $\bar{\hat{b}} \in [\bar{\hat{c}}_1, \bar{\hat{c}}_2]$ by Definition 5.1, and in particular, if $\underline{\hat{c}}_n < \bar{\hat{c}}_1$, then $\underline{\hat{c}}_n < \bar{\hat{b}}$. Thus, it is required $\underline{\hat{c}}_n = (1-w)r_n < (1-w)r_1 + w = \bar{\hat{c}}_1$. This is equivalent to $1 - \frac{1}{1+r_n-r_1} < w$. Since $(r_n - r_1) \in (0, 1]$, then $1 - \frac{1}{1+r_n-r_1} \in (0, 0.5]$. Therefore, if $0.5 < w$, then $1 - \frac{1}{1+r_n-r_1} < w$ for all $(r_n - r_1) \in (0, 1]$.

Furthermore, note that assumption 3 in Assumptions 5.1 is satisfied even if there exists two or more network operators characterised by the lowest reputation rating. To see this, let, without loss of generality, network operators 1 and 2 be characterised by the lowest reputation rating. Then, $\bar{\hat{c}}_1 = \bar{\hat{c}}_2$. Recall that $\bar{\hat{b}} \in [\bar{\hat{c}}_1, \bar{\hat{c}}_2]$ by Definition 5.1. Thus, $\bar{\hat{b}} = \bar{\hat{c}}_1$. For assumption 3 in Assumptions 5.1 not to hold, it is required $\bar{\hat{b}} = \bar{\hat{c}}_1 \leq \underline{\hat{c}}_i$ for any $i \in N$ and any $\delta > 0$. But this means network operator i is not a feasible bidder; that is, $i \notin J$ by Definition 5.2. A contradiction since it was assumed $J = N$. Now, let $\delta = \bar{\hat{b}} - \underline{\hat{c}}_n > 0$ where $n = |N|$; that is, $r_1 \leq r_n$. Then,

$$(\bar{\hat{b}} - \delta, \bar{\hat{b}}) \cap (\underline{\hat{c}}_i, \bar{\hat{c}}_i) = (\underline{\hat{c}}_n, \bar{\hat{b}}) \cap (\underline{\hat{c}}_i, \bar{\hat{c}}_i) = (\underline{\hat{c}}_n, \bar{\hat{b}}) \tag{5.45}$$

for all $i \in N$. The resulting set is convex, and since $(\underline{\hat{c}}_n, \bar{\hat{b}}) \subset [\underline{\hat{c}}_i, \bar{\hat{c}}_i]$ for all $i \in N$, this implies that F_i is strictly log-concave over $(\bar{\hat{b}} - \delta, \bar{\hat{b}}) \cap (\underline{\hat{c}}_i, \bar{\hat{c}}_i)$ for all $i \in N$.

Finally, the discussion will concentrate on cases such that

$$\underline{\hat{c}}_i \leq \hat{c}(\underline{\hat{b}}) \quad \text{for all } i \in N. \tag{5.46}$$

This requirement simplifies the problem so that it is numerically tractable using existing numerical methods. More specifically, it reduces the lower boundary condition in Equation (5.28) to

$$\hat{c}_i(\underline{\hat{b}}) = \underline{\hat{c}}_i \quad \text{for all } i \in N. \tag{5.47}$$

The ultimate aim of the numerical analysis is to obtain a numerical approximation to the equilibrium bidding strategies for all bidding scenarios that involve feasible bidders (cf. Definition 5.2). However, as shown below, the assumption (5.46) restricts the choice of the price weight and the reputation ratings to a subset of all possible bidding scenarios involving feasible bidders. Without this assumption, as explained by Lebrun [85], there might exist i such that

$$\hat{c}(\underline{\hat{b}}) < \underline{\hat{c}}_i < \bar{\hat{b}} \tag{5.48}$$

which forces the bid function $\hat{b}_i$ to be extended to the interval $[\hat{c}(\underline{\hat{b}}), \bar{\hat{b}}]$, which is strictly larger than the actual support, truncated at $\bar{\hat{b}}$, $[\underline{\hat{c}}_i, \bar{\hat{b}}]$ of network operator i's cost. This result is somewhat confusing since even though $F_i(\hat{c}_i) = 0$ for all $\hat{c}_i \in [\hat{c}(\underline{\hat{b}}), \underline{\hat{c}}_i]$, $\hat{b}_i(\hat{c}_i)$ is still network operator i's best response for all $\hat{c}_i \in [\hat{c}(\underline{\hat{b}}), \underline{\hat{c}}_i]$. The main difficulty when considering such cases stems from the fact that for all $i \in I$, where $I = \left\{ i \in N \;\middle|\; \hat{c}(\underline{\hat{b}}) < \underline{\hat{c}}_i < \bar{\hat{b}} \right\}$, the system of ODEs in Equation (5.27) reduces to

$$0 = \frac{1}{n-1} \sum_{k=1}^{n} \frac{1}{b - \hat{c}_k(b)} - \frac{1}{b - \hat{c}_i(b)}. \tag{5.49}$$

As further explained by Lebrun [85], the (inverse) equilibrium bidding functions are then determined by

$$\frac{d}{db}\hat{c}_j(b) = \frac{1 - F_j(\hat{c}_j(b))}{f_j(\hat{c}_j(b))} \left[\frac{1}{k(\underline{\hat{b}}) - 1} \sum_{\substack{k \in N \\ k \notin I}} \frac{1}{b - \hat{c}_k(b)} - \frac{1}{b - \hat{c}_j(b)} \right] \tag{5.50}$$

for network operators $j \in N, j \notin I$, and by the system in Equation (5.49) for network operators $i \in I$. Both systems are combined for all $b > \underline{\hat{b}}$ until the common function $\hat{c}_i$, $i \in I$, takes as its value the smallest lower extremity strictly smaller than $\hat{c}(\underline{\hat{b}})$. At the bid where this next smallest lower extremity is reached, the functions $\hat{c}_i$ of the network operators with this lower extremity of their supports are added to the system in Equation (5.50). This process is repeated until $\hat{c}_i$ for all $i \in N$ are included in (5.50). This procedure is not easily accommodated by any of the numerical methods described in the literature; hence, the assumption

$$\underline{\hat{c}}_i \leq \hat{c}(\underline{\hat{b}}) \quad \text{for all } i \in N. \tag{5.51}$$

Note, however, that this assumption cannot be enforced *a priori* since $\underline{\hat{b}}$ is unknown. The (approximate) probability of enforcing this condition can be maximised by reasoning as follows. Without loss of generality, suppose $r_1 \leq \cdots \leq r_n$ with at least one inequality strict. If $k(\underline{\hat{b}}) = n$, then, by Definition 5.3, $\underline{\hat{c}}_n \leq \hat{c}(\underline{\hat{b}})$ and $\underline{\hat{c}}_n < \underline{\hat{b}}$. Furthermore, since $\underline{\hat{b}} \in (\underline{\hat{c}}_2, \bar{\hat{b}})$, then as the distance $(\underline{\hat{c}}_n - \underline{\hat{c}}_2) \to 0$, $k(\underline{\hat{b}}) \to n$. Thus, the probability that $\underline{\hat{b}} \in (\underline{\hat{c}}_n, \bar{\hat{b}})$ (assuming $\underline{\hat{b}}$ is distributed uniformly over $(\underline{\hat{c}}_2, \bar{\hat{b}})$) can be quantified as follows

$$P\left\{\underline{\hat{b}} \in (\underline{\hat{c}}_n, \bar{\hat{b}})\right\} = 1 - P\left\{\underline{\hat{b}} \in (\underline{\hat{c}}_2, \underline{\hat{c}}_n)\right\} = 1 - \frac{\underline{\hat{c}}_n - \underline{\hat{c}}_2}{\bar{\hat{b}} - \underline{\hat{c}}_2} = \frac{\bar{\hat{b}} - \underline{\hat{c}}_n}{\bar{\hat{b}} - \underline{\hat{c}}_2}. \tag{5.52}$$

For example, for the probability of at least 0.9, it is required

$$\frac{\bar{\hat{b}} - \underline{\hat{c}}_n}{\bar{\hat{b}} - \underline{\hat{c}}_2} \geq 0.9 \iff \bar{\hat{b}} \geq 10\underline{\hat{c}}_n - 9\underline{\hat{c}}_2. \tag{5.53}$$

Since $\bar{\hat{c}}_1 \leq \bar{\hat{b}}$ by Definition 5.1, it follows that

$$\bar{\hat{c}}_1 \geq 10\underline{\hat{c}}_n - 9\underline{\hat{c}}_2 \iff w \geq 1 - \frac{1}{10r_n - 9r_2 - r_1 + 1}. \tag{5.54}$$

Note that, since it was assumed $r_1 \leq \cdots \leq r_n$ with at least one inequality strict, the denominator is always strictly greater than 1 ($10r_n - 9r_2 - r_1 + 1 > 1$), hence guaranteeing the right-hand side of the inequality to be smaller than 1.

To conclude, the rest of this section will concentrate on cases such that

$$w \in (0.5, 1) \quad \text{and} \quad w \geq 1 - \frac{1}{10r_n - 9r_2 - r_1 + 1} \tag{5.55}$$

for all $r_1, r_2, r_n \in [0, 1]$ such that $r_1 \leq \cdots \leq r_n$ with at least one inequality strict.

In what follows, firstly the inner workings of the FSM and PPM methods are outlined, and then the numerically approximated equilibrium bidding strategies are presented for two bidding scenarios generated using both algorithms. In both cases, it is assumed without loss of generality that network operator 1 is characterised by the lowest reputation rating, network operator 2 by the second lowest, and so on. Hence, $\underline{\hat{c}}_1 \leq \underline{\hat{c}}_2 \leq \underline{\hat{c}}_i$ and $\bar{\hat{c}}_1 \leq \bar{\hat{c}}_2 \leq \bar{\hat{c}}_i$ for all $i \geq 2$.

5.4.1 Forward Shooting Method

The idea behind the FSM is to find the best approximation of the lower bound on bids, $\underline{\hat{b}}'$ say, by successively picking a value from the feasible interval $(\underline{\hat{c}}_2, \bar{\hat{b}})$, and verifying whether a numerical solution to the initial value problem

$$\begin{aligned} \frac{d}{db}\hat{c}_i(b) &= \frac{1 - F_i(\hat{c}_i(b))}{f_i(\hat{c}_i(b))}\left[\frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{b - \hat{c}_k(b)} - \frac{1}{b - \hat{c}_i(b)}\right] \\ \hat{c}_i(\underline{\hat{b}}') &= \underline{\hat{c}}_i \end{aligned} \tag{5.56}$$

for all $i \in N$ satisfies the following three conditions: 1) it is a function mapping $[\underline{\hat{b}}', \bar{\hat{b}}]$ into $[\underline{\hat{c}}_i, \bar{\hat{c}}_i]$, that is,

$$s_i : [\underline{\hat{b}}', \bar{\hat{b}}] \to [\underline{\hat{c}}_i, \bar{\hat{c}}_i]; \tag{5.57}$$

2) it is monotonically increasing everywhere except possibly at $\bar{\hat{b}}$, that is,

$$b_1 < b_2 \implies s_i(b_1) < s_i(b_2) \text{ for all } b_1, b_2 \in [\underline{\hat{b}}', \bar{\hat{b}}); \tag{5.58}$$

and 3) each function value is strictly lower than its argument except possibly at $\bar{\hat{b}}$, that is,

$$s_i(b) < b \text{ for all } b \in [\underline{\hat{b}}', \bar{\hat{b}}). \tag{5.59}$$

This specification of the problem is a modified version of the First Algorithm in Bajari [91] (cf. Section 3.3 in [91]) that accommodates for different lower and upper extremities in the supports of bidders' costs.

The pseudo-code for the FSM is depicted in listing Algorithm 5.2. For any given tolerance, $\epsilon \in \left(0, \bar{\hat{b}} - \underline{\hat{c}}_2\right)$, the algorithm aims at finding the interval $LH = [low, high] \subseteq [\underline{\hat{c}}_2, \bar{\hat{b}}]$ such that (approximately) $\underline{\hat{b}} \in LH$ and $high - low < \epsilon$. The approximation to the lower bound on bids is then found to be $\underline{\hat{b}}' = 0.5 \cdot (low + high)$.

The initial guess supplied to the algorithm is the interval $[\underline{\hat{c}}_2, \bar{\hat{b}}]$. In every iteration, the guessed value for the lower bound on bids is the midpoint of the interval; that is, $guess = 0.5 \cdot (low + high)$. The algorithm then uses this value as the new initial condition for the system in (5.56). If the solution to the system lies within the set S_i for all $i \in N$, then $guess$ becomes the new upper endpoint of the interval LH; that

Algorithm 5.2 Forward shooting method

Input: $\epsilon \in (0, \bar{\hat{b}} - \underline{\hat{c}}_2)$; $low, high \in [\underline{\hat{c}}_2, \bar{\hat{b}}]$ such that $low \leq high$
Output: Approximation to $\underline{\hat{b}}$

```
low ← ĉ₂ (underlined)
high ← b̂ (barred)

while high − low > ε do
    guess ← 0.5 · (low + high)
    bids ← [guess, b̂ (barred)]
    (costs_1, ..., costs_n) ← solve (5.56) with initial value b̂' (underlined) = guess
                                 evaluated at points b ∈ bids
    if (bids, costs_i) satisfies (5.57), (5.58) and (5.59) for all i ← 1 to n then
        high ← guess
    else
        low ← guess

b̂' (underlined) ← 0.5 · (low + high)
```

is, $LH = [low, guess]$. Otherwise, it becomes the new lower endpoint; that is, $LH = [guess, high]$. This procedure is repeated until the length of the interval is smaller than ϵ.

In each step, the system of ODEs in (5.56) can be solved numerically using any type of finite-difference methods, such as Euler or Runge-Kutta methods. The results presented in this section were obtained using the GNU Scientific Library (GSL) implementation of the Embedded Runge-Kutta-Fehlberg (4, 5) method [92].

Furthermore, in the implementation of the FSM, it was assumed that

$$F_i(x) = \frac{x - \underline{\hat{c}}_i}{\bar{\hat{c}}_i - \underline{\hat{c}}_i} \quad \text{and} \quad f_i(x) = \frac{1}{\bar{\hat{c}}_i - \underline{\hat{c}}_i} \tag{5.60}$$

for all $i \in N$ and $x \in \mathbb{R}$. This assumption reduces the original initial value problem in (5.56) to

$$\begin{aligned} \frac{d}{db}\hat{c}_i(b) &= \left[\bar{\hat{c}}_i - \hat{c}_i(b)\right] \cdot \left[\frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{b - \hat{c}_k(b)} - \frac{1}{b - \hat{c}_i(b)}\right], \\ \hat{c}_i(\underline{\hat{b}}') &= \underline{\hat{c}}_i, \end{aligned} \tag{5.61}$$

and it is there mainly to avoid possible divisions by zero which are not handled properly by the GSL library. Situations like this may arise due to the nature of finite-difference methods, and the fact that the system of ODEs features a fraction

$$\frac{1 - F_i(x)}{f_i(x)} \tag{5.62}$$

which is undefined for all $x \in \mathbb{R}$ such that $x < \underline{\hat{c}}_i$ and $x > \bar{\hat{c}}_i$ (since $f_i(x) = 0$ for all $x < \underline{\hat{c}}_i$ and $x > \bar{\hat{c}}_i$). On the other hand, by enforcing this assumption, it is implicitly assumed that the algorithm always operates in the feasible region; that is,

$$\hat{c}_i(b) \in [\underline{\hat{c}}_i, \bar{\hat{c}}_i] \quad \text{for all } i \in N \text{ and } b \in \left[\underline{\hat{b}}', \bar{\hat{b}}\right]. \tag{5.63}$$

This, of course, cannot be guaranteed for all choices of $\underline{\hat{b}}'$ made by the algorithm, and therefore, it might skew the final result away from the actual value of the lower bound on bids, $\underline{\hat{b}}$.

5.4.2 Polynomial Projection Method

The PPM method assumes that the inverse equilibrium bidding function for each network operator $i \in N$ can be approximated by a K^{th} order polynomial of the form

$$\hat{c}_i(b; \underline{\hat{b}}, \alpha_i) = \underline{\hat{c}}_i + \sum_{k=1}^{K} \alpha_{i,k}(b - \underline{\hat{b}})^k, \tag{5.64}$$

where $K \in \mathbb{N}_+$ and $K \geq 2$, and $\alpha_i = (\alpha_{i,1}, \ldots, \alpha_{i,K})^T$ is a vector of K unknown polynomial coefficients such that $\alpha_i \in \mathbb{R}^K$. The idea behind the method is then to employ a nonlinear optimisation technique, such as the Nelder-Mead method, to find an approximation to the lower bound on bids and the set of polynomial coefficients that best satisfy the system of ODEs (5.27) with lower and upper boundary conditions (5.28) and (5.29) in the least squares sense. That is, minimise the least squares objective function

$$H(\underline{\hat{b}}, \alpha_1, \ldots, \alpha_n) = \sum_{i \in N} \sum_{b \in B} G_i(b; \underline{\hat{b}}, \alpha_1, \ldots, \alpha_n)^2 + |B| \cdot \sum_{i \in N} (\bar{\hat{b}} - \hat{c}_i(\bar{\hat{b}}; \underline{\hat{b}}, \alpha_i))^2, \tag{5.65}$$

where $B = \{\underline{\hat{b}}, \ldots, \bar{\hat{b}}\}$ is a (finite) grid of points uniformly spaced between $\underline{\hat{b}}$ and $\bar{\hat{b}}$, $|B| < \infty$ denotes the cardinality of B, and G_i captures network operator i's first-order condition for profit maximisation, and is defined as follows

$$\begin{aligned} G_i(b; \underline{\hat{b}}, \alpha_1, \ldots, \alpha_n) &= \frac{d}{db}\hat{c}_i(b; \underline{\hat{b}}, \alpha_i) \\ &- \frac{1 - F_i(\hat{c}_i(b; \underline{\hat{b}}, \alpha_i))}{f_i(\hat{c}_i(b; \underline{\hat{b}}, \alpha_i))} \left[\frac{1}{n-1} \sum_{j=1}^{n} \frac{1}{b - \hat{c}_j(b; \underline{\hat{b}}, \alpha_j)} - \frac{1}{b - \hat{c}_i(b; \underline{\hat{b}}, \alpha_i)}\right]. \end{aligned} \tag{5.66}$$

Note that $G_i(b; \underline{\hat{b}}, \alpha_1, \ldots, \alpha_n) = 0$ corresponds to the ODE in Equation (5.27) for each network operator $i \in N$. Furthermore, note that the objective function, H, incorporates only the upper boundary condition (5.29); that is,

$$\sum_{i \in N} (\bar{\hat{b}} - \hat{c}_i(\bar{\hat{b}}; \underline{\hat{b}}, \alpha_i))^2. \tag{5.67}$$

The lower boundary condition (5.28) is omitted since it is always satisfied through the definition of the inverse equilibrium bidding function in Equation (5.64). To see this, recall that the lower boundary condition implies

$$\underline{\hat{c}}_i - \hat{c}_i(\underline{\hat{b}}; \underline{\hat{b}}, \alpha_i) = \underline{\hat{c}}_i - \underline{\hat{c}}_i - \sum_{k=1}^{K} \alpha_{i,k} (\underline{\hat{b}} - \underline{\hat{b}})^2 = 0 \tag{5.68}$$

for all $i \in N$. Therefore, it is unnecessary to include the lower boundary condition in the objective function, as it is always equal to 0.

This specification of the problem is a modified version of the Third Algorithm in Bajari [91] (cf. Section 3.5 in [91]) that accommodates for different lower and upper extremities in the supports of network operators' costs.

Algorithm 5.3 Polynomial projection method

Input: $k \leq K$ where $K \geq 3$; $\alpha_i \in \mathbb{R}^k$ for all $i \in N$; $\underline{b} \in (\underline{\hat{c}}_2, \bar{\hat{b}})$
Output: Approximate $\underline{\hat{b}}$; $\alpha_i \in \mathbb{R}^K$ for all $i \in N$

```
k ← 3
K ← 8
for i ← 1 to n do
    α_i ← create k-element vector of 10^{-2}
b̲ ← ĉ̲_2

repeat
    bids ← B
    (b̲, α_1, ..., α_n) ← minimise (5.65) with initial values (b̲, α_1, ..., α_n)
                             evaluated at points b ∈ B
    k ← k + 1
    for i ← 1 to n do
        α_i ← (α_{i,1}, ..., α_{i,k}, 10^{-6})
until k ≤ K
```

The pseudo-code for the PPM is shown in listing Algorithm 5.3. The algorithm aims at refining the solution to the minimisation problem in Equation (5.65) by successively increasing the order of approximating polynomials. In each such iteration, the output from the previous run of the algorithm is used as the input to the next run. That is, suppose $(\underline{b}^k, \alpha_1^k, \ldots, \alpha_n^k)$ is the output from the algorithm where k^{th} order polynomials were used. Then, this output is used as the input (and a starting point for the minimisation problem in Equation (5.65)) to the next stage of the algorithm where $(k+1)^{\text{th}}$ order polynomials are used. This procedure is repeated until the approximating polynomials are of the desired order. A similar approach is used by Katzwer [93] in his AuctionSolver software, however, with the difference that he advocates the use of Chebyshev polynomials rather than ordinary polynomials.

In each step of the algorithm, the nonlinear optimisation problem in (5.65) can be solved numerically using any nonlinear optimisation technique, such as Nelder-Mead or Broyden-Fletcher-Goldfarb-Shanno methods. The results presented in this section were obtained using the GSL implementation of the Nelder-Mead method [92]. A brief overview of the most basic form of the Nelder-Mead simplex method can be found for example in [94].

Furthermore, in the implementation of the PPM, the same simplification was made as in the implementation of the FSM method; that is,

$$F_i(x) = \frac{x - \underline{\hat{c}}_i}{\bar{\hat{c}}_i - \underline{\hat{c}}_i} \quad \text{and} \quad f_i(x) = \frac{1}{\bar{\hat{c}}_i - \underline{\hat{c}}_i} \tag{5.69}$$

for all $i \in N$ and $x \in \mathbb{R}$. This assumption reduces the definition of G_i to

$$\begin{aligned} G_i(b; \underline{\hat{b}}, \alpha_1, \ldots, \alpha_n) &= \frac{d}{db}\hat{c}_i(b; \underline{\hat{b}}, \alpha_i) \\ &- \left[\bar{\hat{c}}_i - \hat{c}_i(b; \underline{\hat{b}}, \alpha_i)\right] \cdot \left[\frac{1}{n-1}\sum_{j=1}^{n} \frac{1}{b - \hat{c}_j(b; \underline{\hat{b}}, \alpha_j)} - \frac{1}{b - \hat{c}_i(b; \underline{\hat{b}}, \alpha_i)}\right], \end{aligned} \tag{5.70}$$

and it is there mainly to avoid possible divisions by zero which are not handled properly by the GSL library. It is important to realise, however, that by enforcing this assumption, it is implicitly assumed that the algorithm always operates in the feasible region; that is,

$$\underline{\hat{b}} \in (\underline{\hat{c}}_2, \bar{\hat{b}}) \quad \text{and} \quad \hat{c}_i(b; \underline{\hat{b}}, \alpha_i) \in [\underline{\hat{c}}_i, \bar{\hat{c}}_i] \tag{5.71}$$

for all $b \in [\underline{\hat{b}}, \bar{\hat{b}}]$, $\alpha_i \in \mathbb{R}^K$, and $i \in N$. This, of course, cannot be guaranteed since the minimisation problem in (5.65) is treated as an unconstrained optimisation problem, and therefore, there exists a possibility that the algorithm will reach a minimum that is outside of the feasible range of values. At the same time, note that certain precautionary measures may be employed as to reduce the probability of such an event; for example, the unconstrained optimisation problem can be translated into a constrained optimisation problem using a penalty function approach as described in [94]. Alternatively, a

Table 5.2
Bidding scenario with 2 network operators

	Price weight, w	**Reputation rating**, r_i
Network operator 1	0.5	0.25
Network operator 2		0.75

constrained optimisation technique, such as the Constrained Optimisation BY Linear Approximation (COBYLA) method [95], could be used in place of the Nelder-Mead method.

5.4.3 Approximation Results

In this subsection, the approximation results for two bidding scenarios with 3 and 4 network operators are presented. The source code of all algorithms presented in this thesis is available upon request from the author. It is also envisaged that, in the future, the source code will be publicly available on the author's Github website: `https://github.com/kubkon`.

Before analysing numerical results, both algorithms were tested for correct implementation. To this end, the numerically approximated equilibrium for $n = 2$ network operators was compared with the closed-form solution derived in Proposition 5.5, Section 5.3. The parameters for the test bidding scenario are shown in Table 5.2. Figure 5.6 depicts the results of the comparison for the FSM, while Figure 5.7 for the PPM. It is clear from Figure 5.6 that the result produced by FSM matches the (theoretical) closed-form solution perfectly. In case of PPM, on the other hand, the numerical result approaches the closed-form solution; however, due to the nature of the approximating polynomials, the match is not ideal. The accuracy of the solution could be increased by increasing the order of the approximating polynomials, or by utilising a basis of approximating functions which is better suited to the least squares optimisation, for example, employing Chebyshev polynomials in place of the polynomials in Equation (5.64) [88, 96]. Nevertheless, the results demonstrate that both algorithms were implemented correctly.

Note, however, that this does not prove that the algorithms will provide correct results for any number of network operators. On the contrary, it merely suggests that the implementation of the algorithms passed basic sanity check. Therefore, in what follows, for each produced output, it will be verified whether the numerically approximated bidding strategies satisfy the sufficiency condition for an equilibrium; that is, whether the numerically derived bidding strategy for each network operator is a best response to the

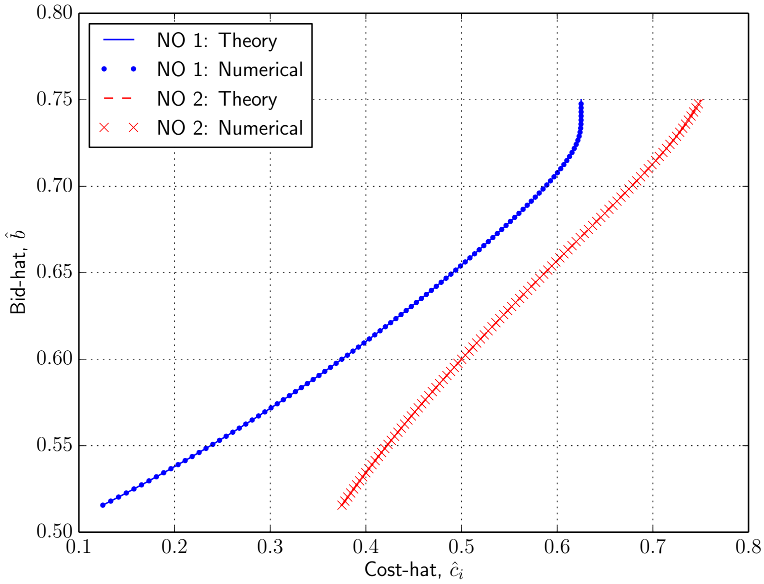


Figure 5.6
FSM solution to the bidding problem characterised by: $w = 0.5$, $r_1 = 0.25$, and $r_2 = 0.75$ agreeing with the closed-form solution

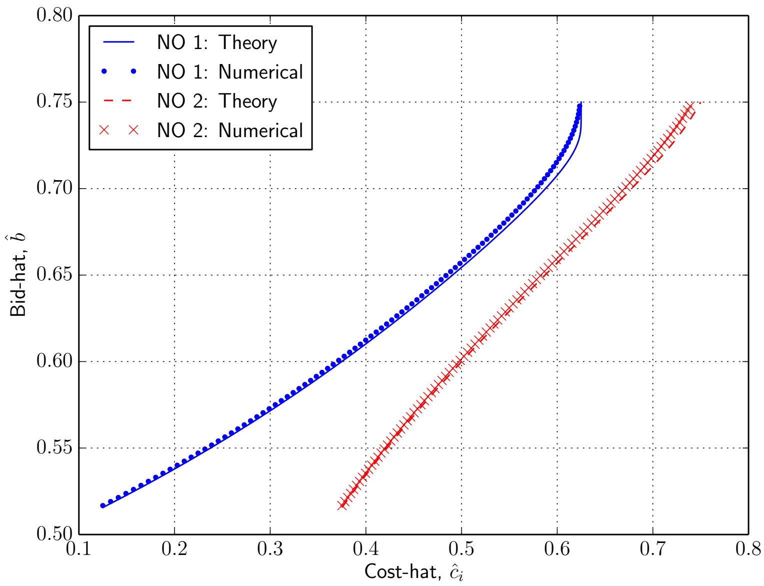


Figure 5.7
PPM solution to the bidding problem characterised by: $w = 0.5$, $r_1 = 0.25$, and $r_2 = 0.75$ agreeing with the closed-form solution

Table 5.3
Bidding scenario with 3 network operators

	Price weight, w	**Reputation rating**, r_i
Network operator 1		0.25
Network operator 2	0.75	0.5
Network operator 3		0.75

bidding strategies of the remaining network operators. If the derived bidding strategies constitute best responses that are mutually consistent, then they constitute an approximate Bayesian Nash equilibrium.

In the first examined scenario, there are 3 network operators characterised by reputation ratings as summarised in Table 5.3. Furthermore, the price weight is set to 0.75. Figures 5.8 and 5.9 depict the results of the approximation generated by the FSM and the PPM methods respectively. Both approximation methods yield virtually the same estimate of the lower bound on bids, $\hat{\underline{b}} \approx 0.375$. In the FSM case, the approximation diverges in the very near proximity of $\hat{\bar{b}}$, and therefore, the approximation satisfies the sufficiency only until b reaches a close neighbourhood of $\hat{\bar{b}}$. This is due to the fact that the system of ODEs does not satisfy Lipschitz condition as b approaches $\hat{\bar{b}}$. The PPM method, on the other hand, eliminates this problem entirely, and the approximation satisfies sufficiency for all $b \in [\hat{\underline{b}}, \hat{\bar{b}}]$.

In the second examined scenario, there are 4 network operators characterized by reputation ratings as summarized in Table 5.4. Furthermore, the price weight is set to 0.85. Figures 5.10 and 5.11 depict the results of the approximation generated by the FSM and PPM methods respectively. Both approximation methods yield virtually the same estimate of the lower bounds on bids, $\hat{\underline{b}} \approx 0.38$. Similarly to the bidding scenario with 3 network operators, in the FSM method case, the approximation diverges in the very near proximity of $\hat{\bar{b}}$, and therefore, the approximation satisfies the sufficiency only until b reaches a close neighbourhood of $\hat{\bar{b}}$. In the PPM method case, the approximation satisfies sufficiency for all $b \in [\hat{\underline{b}}, \hat{\bar{b}}]$.

In this section, the FSM and PPM numerical algorithms were described, and two exemplary bidding scenarios were analysed for which the equilibrium bidding strategies were generated using the aforementioned algorithms. It should be noted, however, that one of the key assumptions of this section was $\hat{\underline{c}}_i \leq \hat{c}(\hat{\underline{b}})$ for all network operators $i \in N$, which restricted the choice of the price weight and the reputation ratings to a set satisfying $w \geq 1 - 1/(10r_n - 9r_2 - r_1 + 1)$. In the following section, this assumption is relaxed by allowing cases such that $\hat{c}(\hat{\underline{b}}) < \hat{\underline{c}}_i < \hat{\bar{b}}$ for at least one network operator

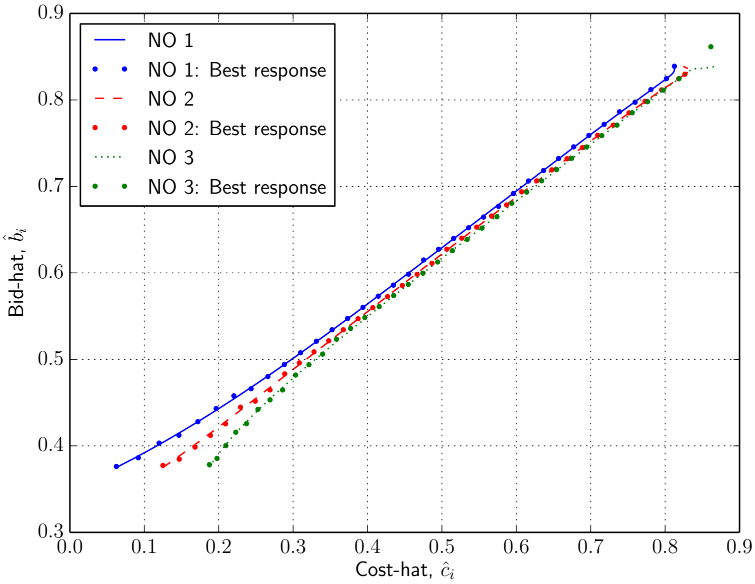


Figure 5.8
FSM solution to the bidding problem characterised by: $w = 0.75$, $r_1 = 0.25$, $r_2 = 0.5$, and $r_3 = 0.75$. The solution satisfies the sufficiency condition for all $\hat{b}$ except $\hat{b}$'s in the close neighbourhood of $\bar{\hat{b}}$.

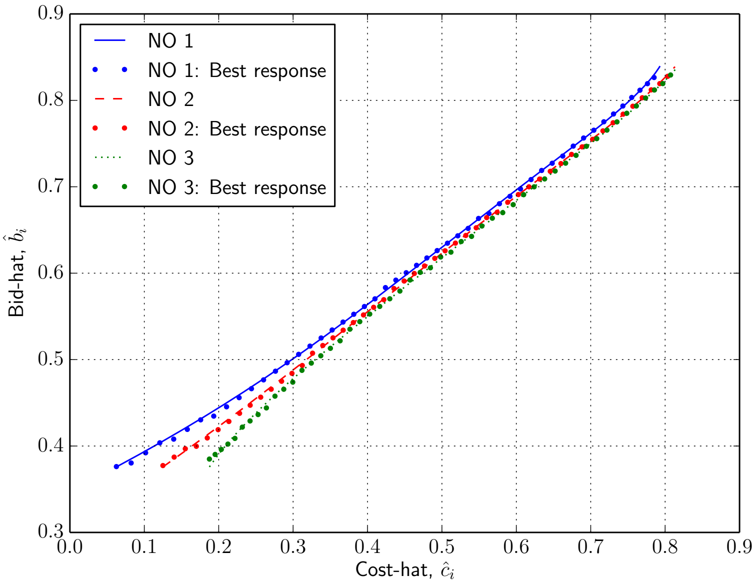


Figure 5.9
PPM solution to the bidding problem characterised by: $w = 0.75$, $r_1 = 0.25$, $r_2 = 0.5$, and $r_3 = 0.75$. The solution satisfies the sufficiency condition for all $\hat{b}$.

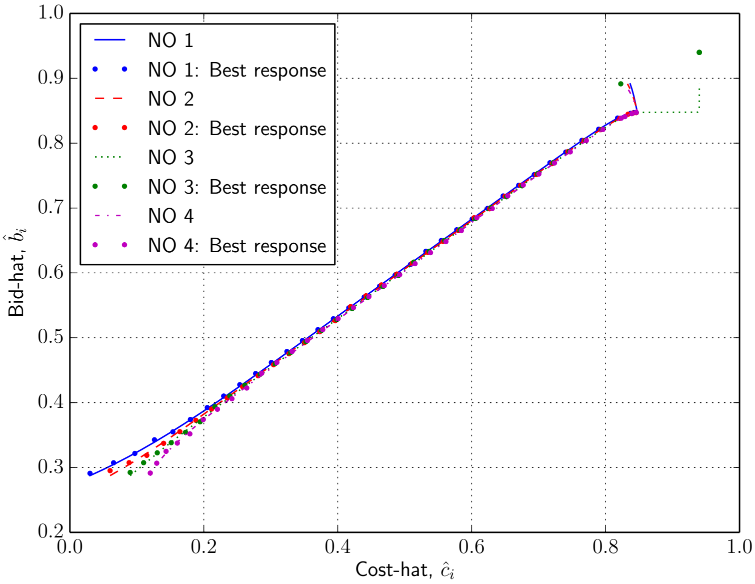


Figure 5.10
FSM solution to the bidding problem characterised by: $w = 0.85$, $r_1 = 0.2$, $r_2 = 0.4$, $r_3 = 0.6$, and $r_4 = 0.8$. The solution satisfies the sufficiency condition for all $\hat{b}$ except $\hat{b}$'s in the close neighbourhood of $\bar{\hat{b}}$.

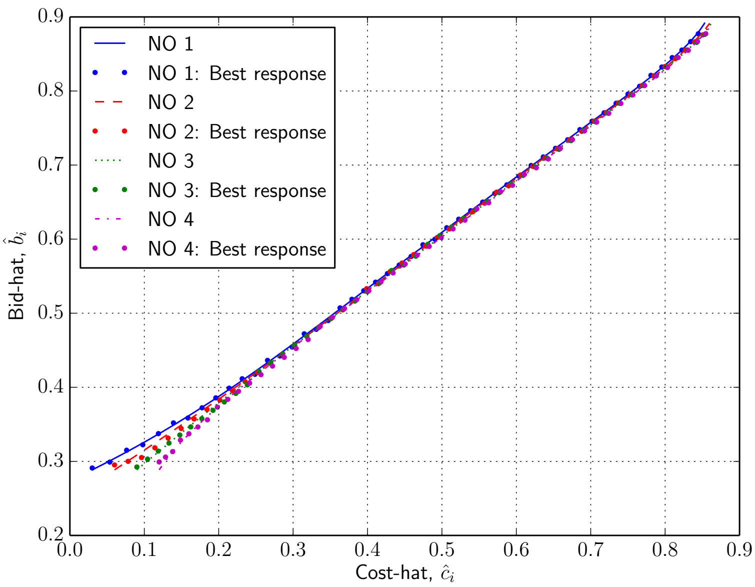


Figure 5.11
PPM solution to the bidding problem characterised by: $w = 0.85$, $r_1 = 0.2$, $r_2 = 0.4$, $r_3 = 0.6$, and $r_4 = 0.8$. The solution satisfies the sufficiency condition for all $\hat{b}$.

Table 5.4
Bidding scenario with 4 network operators

	Price weight, w	**Reputation rating**, r_i
Network operator 1		0.2
Network operator 2	0.85	0.4
Network operator 3		0.6
Network operator 4		0.8

$i \in N$. In other words, all nontrivial equilibria characterised by Proposition 5.3 are considered. Furthermore, a numerical algorithm which is able to generate equilibrium bidding strategies under the relaxed assumption is presented.

5.5 Extended Numerical Analysis

Suppose $\hat{c}(\underline{\hat{b}}) < \underline{\hat{c}}_i < \bar{\hat{b}}$ for at least one $i \in N$, and let all the remaining assumptions of Section 5.4 hold. To be more specific, let $w \in (0.5, 1.0)$ for all $r_i \in [0, 1], i \in N$ such that $r_1 \leq \cdots \leq r_n$ with at least one inequality strict.

In what follows, a numerical algorithm which improves upon the algorithms considered in Section 5.4 is presented: the extended forward shooting method (EFSM). Furthermore, the numerically approximated equilibrium bidding strategies for two bidding scenarios generated using the algorithm are analysed. It is, furthermore, assumed without loss of generality that network operator 1 is characterised by the lowest reputation rating, network operator 2 by the second lowest, and so on. Hence, $\underline{\hat{c}}_1 \leq \underline{\hat{c}}_2 \leq \underline{\hat{c}}_i$ and $\bar{\hat{c}}_1 \leq \bar{\hat{c}}_2 \leq \bar{\hat{c}}_i$ for all $i \geq 2$.

5.5.1 Extended Forward Shooting Method

The EFSM method extends the FSM method by effectively implementing the reasoning behind the bidding extension characterised by Lebrun [85] (see Section 5.4 for a description of the extension). To the best of the author's knowledge, the EFSM method developed in this thesis is the only numerical algorithm in existence that considers all nontrivial equilibria to the system of ODEs in Equation (5.27) with lower and upper boundary conditions in Equations (5.28) and (5.29) respectively.

Similarly to the FSM method, the EFSM aims at finding the best approximation of the lower bound on bids, $\underline{\hat{b}}'$ say, by successively picking a value from the feasible interval

$(\underline{\hat{c}}_2, \bar{\hat{b}})$, and verifying whether a numerical solution to the initial value problem

$$\begin{aligned} \frac{d}{db}\hat{c}_i(b) &= \frac{1 - F_i(\hat{c}_i(b))}{f_i(\hat{c}_i(b))}\left[\frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{b - \hat{c}_k(b)} - \frac{1}{b - \hat{c}_i(b)}\right] \\ \hat{c}_i(\underline{\hat{b}}') &= \min\{\underline{\hat{c}}_i, \hat{c}(\underline{\hat{b}}')\} \end{aligned} \tag{5.72}$$

for all $i \in N$ satisfies the following three conditions: 1) it is a function mapping $[\underline{\hat{b}}', \bar{\hat{b}}]$ into $[\min\{\underline{\hat{c}}_i, \hat{c}(\underline{\hat{b}}')\}, \bar{\hat{c}}_i]$, that is,

$$s_i : [\underline{\hat{b}}', \bar{\hat{b}}] \rightarrow [\min\{\underline{\hat{c}}_i, \hat{c}(\underline{\hat{b}}')\}, \bar{\hat{c}}_i]; \tag{5.73}$$

2) it is monotonically increasing everywhere except possibly at $\bar{\hat{b}}$, that is,

$$b_1 < b_2 \implies s_i(b_1) < s_i(b_2) \text{ for all } b_1, b_2 \in [\underline{\hat{b}}', \bar{\hat{b}}); \tag{5.74}$$

and 3) each function value is strictly lower than its argument except possibly at $\bar{\hat{b}}$, that is,

$$s_i(b) < b \text{ for all } b \in [\underline{\hat{b}}', \bar{\hat{b}}). \tag{5.75}$$

The pseudo-code for the EFSM is shown in listing Algorithm 5.4. The flow of the algorithm is almost exactly the same as for the FSM. The main differences are twofold: the algorithm estimates $k(\underline{\hat{b}}')$ and $\hat{c}(\underline{\hat{b}}')$ (line 6); and the algorithm solves the system (5.72) using the reasoning behind the bidding extension characterised by Lebrun [85] and described below (line 8).

Listing Algorithm 5.5 depicts the pseudo-code for the function 'estimateKC' which estimates $k(\underline{\hat{b}}')$ and $\hat{c}(\underline{\hat{b}}')$. It takes as an input an estimate of the lower bound on bids, $\underline{\hat{b}}'$, and the set of lower extremities $\{\underline{\hat{c}}_i\}$ for all $i \in N$. The function then iterates over $k \in \{2, \dots, n\}$, and for each k it computes $\hat{c}(\underline{\hat{b}}')$ according to Equation (5.25). If $k < n$ and $\hat{c}(\underline{\hat{b}}')$ satisfies (5.26), then the function returns that particular pair of values $(k, \hat{c}(\underline{\hat{b}}'))$ such that $2 \leq k < n$. Otherwise, $k = n$ is returned which reduces system (5.72) to (5.56), and EFSM to FSM.

In order to describe how the algorithm solves the system (5.72), suppose there are $n = 4$ network operators. The following argument can easily be adapted to the case of $n = 3$ or $n > 4$ network operators. Furthermore, suppose that

$$\underline{\hat{c}}_1 < \underline{\hat{c}}_2 < \hat{c}(\underline{\hat{b}}) < \underline{\hat{c}}_3 < \underline{\hat{c}}_4. \tag{5.76}$$

The method comprises three stages as depicted in Figure 5.12.

Algorithm 5.4 Extended forward shooting method

Input: $\epsilon \in (0, \bar{\hat{b}} - \hat{\underline{c}}_2)$; $low, high \in [\hat{\underline{c}}_2, \bar{\hat{b}}]$ such that $low \leq high$
Output: Approximation to $\hat{\underline{b}}$

$low \leftarrow \hat{\underline{c}}_2$
$high \leftarrow \bar{\hat{b}}$

while $high - low > \epsilon$ **do**
  $guess \leftarrow 0.5 \cdot (low + high)$
  $(k, \hat{c}) \leftarrow$ estimateKC$(guess, \hat{\underline{c}}_1, \ldots, \hat{\underline{c}}_n)$
  $bids \leftarrow [guess, \bar{\hat{b}})$
  $(costs_1, \ldots, costs_n) \leftarrow$ solve (5.72) with initial value $\hat{\underline{b}}' = guess$ evaluated at points $b \in bids$, and $k(\hat{\underline{b}}) = k$ and $\hat{c}(\hat{\underline{b}}) = \hat{c}$
  if $(bids, costs_i)$ satisfies (5.73), (5.74) and (5.75) for all $i \leftarrow 1$ to n **then**
    $high \leftarrow guess$
  else
    $low \leftarrow guess$

$\hat{\underline{b}}' \leftarrow 0.5 \cdot (low + high)$

Algorithm 5.5 Function for estimating $k(\hat{\underline{b}})$ and $\hat{c}(\hat{\underline{b}})$

Input: Estimate of $\hat{\underline{b}}$; $\hat{\underline{c}}_i$ for all $i \in N$
Output: $k(\hat{\underline{b}}) \in \{2, \ldots, n\}$; $\hat{c}(\hat{\underline{b}})$ computed according to (5.25)

function estimateKC$(\hat{\underline{b}}, \hat{\underline{c}}_1, \ldots, \hat{\underline{c}}_n)$
  for $k = 2 \rightarrow n$ **do**
    $\hat{c}(\hat{\underline{b}}) \leftarrow$ compute using (5.25) with $k(\hat{\underline{b}}) = k$
    if $k < n$ **then**
      if $\hat{\underline{c}}_k \leq \hat{c}(\hat{\underline{b}}) \wedge \hat{c}(\hat{\underline{b}}) < \hat{\underline{c}}_{k+1}$ **then**
        break
  return $(k, \hat{c}(\hat{\underline{b}}))$

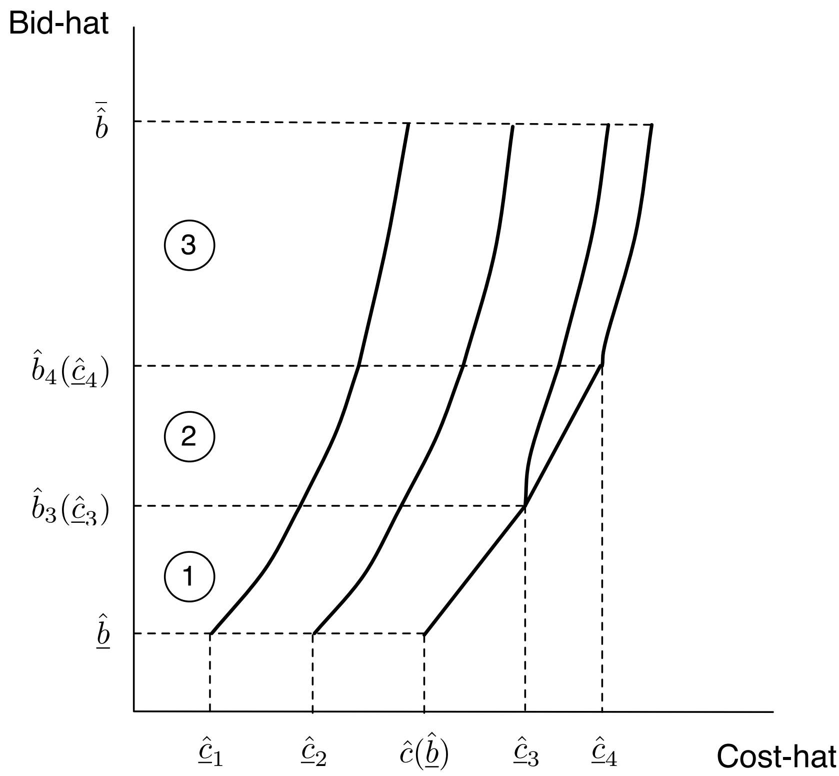


Figure 5.12
The reasoning behind the bidding extension characterised by Lebrun [85] captured by the EFSM method

Stage 1 Since $\underline{\hat{c}}_1 < \underline{\hat{c}}_2 < \hat{c}(\underline{\hat{b}}) < \underline{\hat{c}}_3 < \underline{\hat{c}}_4$, then $k(\underline{\hat{b}}) = 2$, and hence, network operators 1 and 2 solve

$$\begin{aligned} \frac{d}{db}\hat{c}_i(b) &= \frac{1 - F_i(\hat{c}_i(b))}{f_i(\hat{c}_i(b))} \left[\frac{1}{k(\underline{\hat{b}}) - 1} \sum_{k=1}^{k(\underline{\hat{b}})} \frac{1}{b - \hat{c}_k(b)} - \frac{1}{b - \hat{c}_i(b)} \right], \quad (5.77) \\ \hat{c}_i(\underline{\hat{b}}) &= \underline{\hat{c}}_i \end{aligned}$$

while network operators 3 and 4 solve

$$\hat{c}_i(b) = b - \frac{k(\underline{\hat{b}}) - 1}{\sum_{j=1}^{k(\underline{\hat{b}})} \frac{1}{b - \hat{c}_j(b)}}. \quad (5.78)$$

The system (5.77) is solved for all $b \in [\underline{\hat{b}}, \bar{\hat{b}}]$, and the approximation is then used to solve (5.78). However, only bid values such that $\underline{\hat{b}} \le b \le \hat{b}_3(\underline{\hat{c}}_3)$ are kept, where $\hat{c}_3(\hat{b}_3(\underline{\hat{c}}_3)) = \underline{\hat{c}}_3$ maps into network operator's 3 cost-hat.

Note that the solution to (5.78) depends only on $k(\underline{\hat{b}})$ and the solution to (5.77). Therefore, both network operator 3 and 4 are characterised by the same set of values (Stage 1, Figure 5.12).

Stage 2 Next, $k(\hat{\underline{b}})$ is incremented by 1 over the previous value, and the procedure in Stage 1 is repeated with this difference that now network operators 1, 2 and 3 solve system (5.77) while network operator 4 solves (5.78). The system (5.77) is solved for $b \in [\hat{b}_3(\hat{\underline{c}}_3), \bar{\hat{b}}]$, but only bid values such that $\hat{b}_3(\hat{\underline{c}}_3) \leq b \leq \hat{b}_4(\hat{\underline{b}}_4)$ are kept (Stage 2, Figure 5.12).

Stage 3 Finally, all network operators solve the system (5.77) for bid values $b \in [\hat{b}_4(\hat{\underline{c}}_4), \bar{\hat{b}}]$ (Stage 3, Figure 5.12).

In each stage, the system of ODEs in Equation (5.77) can be solved numerically using any type of finite-difference methods, such as Euler or Runge-Kutta methods. The results presented in this section were obtained using the GSL implementation of the Embedded-Runge-Kutta-Fehlberg (4, 5) method [92].

Furthermore, similarly to the implementation of the FSM and PPM methods, in the implementation of EFSM, it was assumed that

$$F_i(x) = \frac{x - \hat{\underline{c}}_i}{\bar{\hat{c}}_i - \hat{\underline{c}}_i} \quad \text{and} \quad f_i(x) = \frac{1}{\bar{\hat{c}}_i - \hat{\underline{c}}_i} \tag{5.79}$$

for all $i \in N$ and $x \in \mathbb{R}$. The discussion of the consequences arising from this simplifying assumption can be found in Section 5.4.1.

5.5.2 Approximation Results

This subsection presents the approximation results for two bidding scenarios with 3 and 4 network operators. Similarly to the algorithms presented in Section 5.4, the EFSM method was tested for correct implementation using the same procedure as presented in Section 5.4. However, since the verification results match exactly those presented in Figure 5.6, their discussion is omitted from this section.

In the first examined scenario, there are 3 network operators characterised by reputation ratings summarised in Table 5.5. Furthermore, the price weight is set to 0.55. Figure 5.13 depicts the results of the approximation generated by the EFSM method. The method yields an estimate of the lower bound on bids of $\hat{\underline{b}} \approx 0.407$, and satisfies the sufficiency for all $b \in [\hat{\underline{b}}, \bar{\hat{b}}]$. It is worth noting that, as expected, the bidding scenario comprises two stages (cf. Figure 5.12): stage 1 such that $\hat{b} \in [0.407, 0.43]$ where network operators 1 and 2 are competing against each other only, and network operator 3 is characterised by the bidding extension given by Equation (5.78); and stage 2 such that $\hat{b} \in [0.43, 0.71]$ where all network operators are competing against each other, and are solving the system in Equation (5.77).

Table 5.5
Bidding scenario with 3 network operators

	Price weight, w	**Reputation rating**, r_i
Network operator 1		0.25
Network operator 2	0.55	0.5
Network operator 3		0.75

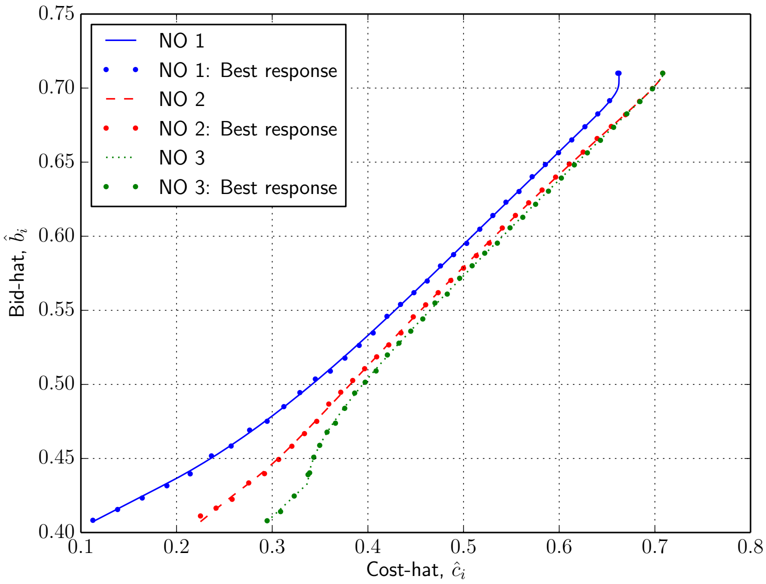


Figure 5.13
EFSM solution to the bidding problem characterised by: $w = 0.55$, $r_1 = 0.25$, $r_2 = 0.5$, and $r_3 = 0.75$. The solution satisfies the sufficiency for all $\hat{b}$.

In the second examined scenario, there are 4 network operators characterised by reputation ratings summarised in Table 5.6. Furthermore, the price weight is set to 0.55. Figure 5.14 depicts the results for the approximation generated by EFSM. The estimate of the lower bound on bids is $\underline{\hat{b}} \approx 0.353$, and the solution satisfies the sufficiency for all $b \in [\underline{\hat{b}}, \bar{\hat{b}}]$. It is worth noting that, as expected, the bidding scenario comprises three stages (cf. Figure 5.12): stage 1 such that $\hat{b} \in [0.353, 0.3625]$ where network operators 1 and 2 are competing against each other, while network operators 3 and 4 are characterised by the bidding extension given by Equation (5.78); stage 2 such that $\hat{b} \in [0.3625, 0.43]$ where network operators 1, 2 and 3 are competing against each other, and network operator 4 is still characterised by the bidding extension, albeit different compared to the bidding extension established in stage 1; and stage 3 such that $\hat{b} \in [0.43, 0.675]$ where all network operators are competing against each other.

Table 5.6
Bidding scenario with 4 network operators

	Price weight, w	**Reputation rating**, r_i
Network operator 1		0.2
Network operator 2	0.55	0.4
Network operator 3		0.6
Network operator 4		0.8

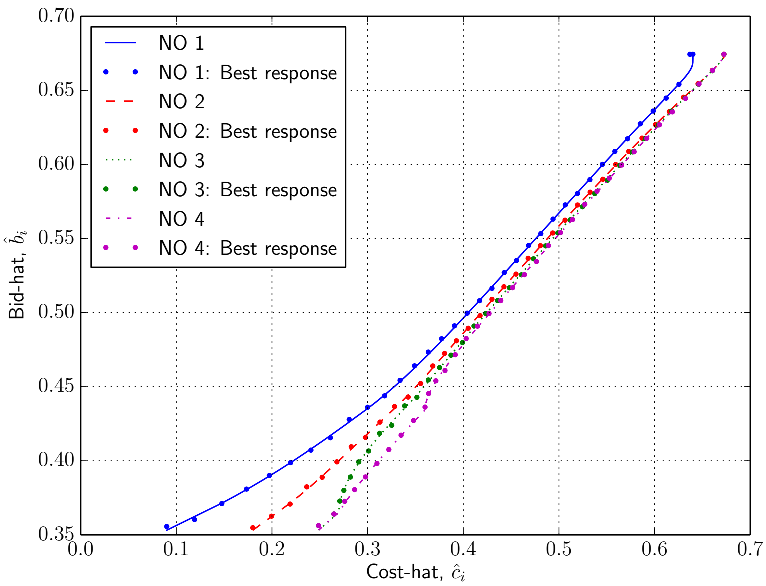


Figure 5.14
EFSM solution to the bidding problem characterised by: $w = 0.55$, $r_1 = 0.2$, $r_2 = 0.4$, $r_3 = 0.6$, and $r_4 = 0.8$. The solution satisfies the sufficiency for all $\hat{b}$.

In this section, the EFSM numerical algorithm was outlined, and two exemplary bidding scenarios were analysed for which the equilibrium bidding strategies were generated by the aforementioned algorithms. The method "completes" the FSM and PPM methods in the sense that it permits for cases such that $\hat{c}(\underline{\hat{b}}) < \underline{\hat{c}}_i < \bar{\hat{b}}$ for at least one network operator $i \in N$, and hence, considers all nontrivial equilibria characterised by Proposition 5.3. It is an important improvement since otherwise only a very restricted subset of all possible bidding scenarios resulting in nontrivial equilibria could be considered and numerically approximated.

5.6 Summary

In this chapter, the bidding problem with symmetric cost distributions, as defined in Chapter 4, was transformed into a bidding problem with asymmetric cost distributions. Following the transformation, the equilibrium bidding strategies for the generic case of n network operators were formally characterised; that is, it was shown that the pure strategy Bayesian Nash equilibrium exists and is unique (Proposition 5.3 and Corollary 5.4). This is an important result as it proves that the DMP network selection mechanism is economically well-behaved since the equilibrium exists.

When restricted to $n = 2$ network operators, the equilibrium bidding strategies were analytically derived. To aid in the derivation, it was necessary to assume that costs for the network operators were uniformly distributed. Given the lack of knowledge of the way the costs are distributed, it is standard practice to assume the probability of each cost to be uniform [97]. Nonetheless, such an assumption is limiting and it is highly likely it will not be fully representative of the reality. Furthermore, in the case of $n = 2$ network operators, the expected prices for the subscriber were analysed, and it was shown that, for any expected price, as the difference between the reputation ratings of the network operators increases, the price weight has to increase (or remain constant) in order to keep the expected price fixed. This observation carries very serious implications on the operation of the DMP, as the subscriber is effectively given the ability to influence the expected prices by an appropriate choice of the price weight.

Finally, for the case of $n \geq 2$ network operators, three numerical algorithms for approximating the equilibrium bidding strategies were proposed: FSM (Algorithm 5.2), PPM (Algorithm 5.3), and EFSM (Algorithm 5.4). When developing the algorithms, similarly to the restricted case with $n = 2$ network operators, it was assumed that costs for the network operators were uniformly distributed. Therefore, the same limitations apply. However, generalising algorithms to nonuniform distributions should not prove difficult since other researchers have successfully employed similar numerical methods for studying problems where distributions were nonuniform [88]. The algorithms were further verified for correct implementation, and, for each approximated scenario, the derived equilibrium bidding strategies were tested for sufficiency condition for a pure strategy Bayesian Nash equilibrium. The FSM and PPM methods allow for numerically approximating equilibrium bidding strategies for a subset of all possible bidding scenarios resulting in nontrivial equilibria, while the EFSM method enables computation of the numerical solution to all bidding scenarios. Since, as shown in Section 5.2, the analytical derivation of the equilibrium bidding strategies in the case of more than two network operators is not possible, the existence of algorithms capable of numerically approximating the solutions is a major step forward in the development of the economic theory of operation of the DMP network selection mechanism. Furthermore, the algorithms constitute a tool that the network operators participating in the DMP can use to formulate their own bidding strategies and understand the bidding behaviours of other network operators.

Chapter 6

Casting Network Selection Mechanism into Common Prior Setting

This chapter presents a methodology for approximating the DMP network selection mechanism with an asymmetric FPA auction with common prior. It is further argued that this methodology constitutes a possible resolution to the potential problem of numerical instability of the FSM and EFSM methods.

Fibich and Gavish [98] showed that the FSM method and its derivatives, such as the EFSM method, become numerically unstable for large numbers of bidders. The issue has not impacted the results presented in this thesis thus far due to the fact that only the scenarios with as many as 4 network operators were considered. However, it is important to acknowledge the fact that the issue exists and, sooner or later, for large number of network operators, it will affect the numerical solutions generated by FSM and, more importantly, EFSM methods. Therefore, it is vital to address the issue on a proactive rather than reactive basis.

The most obvious way of addressing the issue would be to employ a different numerical method in place of the EFSM method. However, to the best of the author's knowledge, the EFSM method is the only numerical algorithm in existence that considers all nontrivial equilibria to the system of ODEs in Equation (5.27) with lower and upper boundary conditions in Equations (5.28) and (5.29) respectively. Furthermore, it is not immediately obivious how the EFSM method would have to be modified to be based entirely on methods that are not FSM derivatives, and hence, do not possess numerical instability issues.

In this chapter, an alternative approach is presented. It is explored whether an auction format represented by the DMP network selection mechanism can be modelled as an asymmetric FPA auction with common prior (henceforth, referred to as CP auction). In a CP auction, the range the costs can vary is the same for each bidder. More formally, the cost distributions for each bidder share the same support. By modelling the DMP network selection mechanism as a CP auction, the numerical solution methods (other than the FSM-based methods) presented in Hubbard and Parsch [88], and extensively studied by the economic community, could be used to approximate the solution to the DMP auction. This would allow network operators to consider a simpler bidding problem for which there are many well-defined numerical solutions. As a result, presented with a DMP auction, network operators could bid according to the equilibrium bidding strategies of the corresponding CP auction while approximately retaining the expected utility if bidding according to the equilibrium bidding strategies of the DMP auction, and hence, avoiding the need to use the EFSM method to solve the DMP auction.

Modelling of the DMP auction as a CP auction assumes that the network operators will use the equilibrium bidding strategies of the CP auction (CP strategies) as bidding strategies in the DMP auction. However, by Proposition 5.3, the CP strategies do not constitute an equilibrium to the DMP auction; they are merely used as *approximations* to the actual equilibrium bidding strategies of the DMP auction (equilibrium strategies). Therefore, there exists possibility that a network operator might exploit this fact by bidding according to the equilibrium strategies while other network operators will bid according to the CP strategies. Concurrently, however, since the equilibrium strategies can only be derived using the EFSM method, it is likely that the derivation might fail due to the numerical instability of the algorithm. All in all, each network operator faces a tradeoff: bid according to the equilibrium strategies but risk lack of convergence, or bid according to CP strategies but risk other network operators bidding according to the equilibrium strategies. Of course, the magnitude of the problem decreases dramatically as the number of network operators involved in the DMP increases. For then the numerical instability will lead to the divergence of the EFSM algorithm and render the derivation of the equilibrium strategies impossible. Hence, the network operators will be forced to rely on the CP strategies.

The analysis is organised as follows. In the first instance, the assumptions governing the CP auction are described, and the existence and uniqueness of the equilibrium bidding strategies is formally defined. Following that the FSM numerical method tailored specifically to the CP auction setting is presented. It is worth noting at this point that the CP version of the FSM algorithm corresponds to the original FSM algorithm first presented by Bajari [91] (cf. Algorithm 1 in [91]). Having derived the numerical method for approximating the equilibrium in the CP auction setting, the methodology for casting the DMP bidding scenario into a CP auction setting is discussed. That is, it is showed how a DMP auction can be approximated as a CP equivalent. Furthermore, the methodology for quantifying the accuracy of the approximation is presented.

Finally, the chapter concludes with the presentation of approximation results for four bidding scenarios with two, three, four and five bidders respectively.

6.1 Mathematical Description

Following the notation of Chapter 5, let each bidder i be characterised by the utility function

$$u_i(b, c) = \begin{cases} b_i - c_i & \text{if } b_i < \min_{j \neq i} b_j, \\ 0 & \text{if } b_i > \min_{j \neq i} b_j, \end{cases} \tag{6.1}$$

where, as before, $b = (b_1, \ldots, b_n)$, and $c = (c_1, \ldots, c_n)$. In the CP auction, it is assumed that each bidder i draws their cost from common support across all bidders; i.e., let

$$c_i \in [\underline{c}, \bar{c}] \quad \text{for all } i \in N \text{ such that } [\underline{c}, \bar{c}] \subseteq [0, 1]. \tag{6.2}$$

Let F_i be the distribution function of c_i for all $i \in N$. Note that the distribution functions between bidders need not be equal, and hence, the problem is that of an asymmetric FPA.

It is further assumed that

Assumptions 6.1. Assume that

1. F_i is differentiable over $(\underline{c}, \bar{c}]$ with a derivative f_i locally bounded away from zero over this interval;
2. F_i is atomless; and
3. $F_i(c) > 0$ for all $c \in [\underline{c}, \bar{c}]$ and $i \in N$.

These assumptions correspond to Assumptions A.1 and Theorem U.1 in Lebrun [85], and, as shown by Lebrun, with these assumptions satisfied, there exists one and only one pure-strategy Bayesian Nash equilibrium where bidders engage in serious bidding; that is, bid at least their cost. Formally,

Proposition 6.1 (Characterisation of the Equilibrium in Common Prior Setting)**.** *Let Assumptions 6.1 be satisfied. There exists one and only one pure-strategy Bayesian Nash equilibrium where bidders submit at least their costs. In every such equilibrium, bidder $i \in N$ follows a bid function b_i, for all $1 \leq i \leq n$ such that its inverse, $c_i = b_i^{-1}$, satisfy the following system of* *differential equations*

$$\frac{d}{db}c_i(b) = \frac{1 - F_i(c_i(b))}{f_i(c_i(b))}\left[\frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{b - c_k(b)} - \frac{1}{b - c_i(b)}\right] \tag{6.3}$$

for all $1 \le i \le n$, *with the following lower boundary condition*

$$c_i(\underline{b}) = \underline{c} \tag{6.4}$$

and the upper boundary condition

$$c_i(\bar{c}) = \bar{c} \tag{6.5}$$

for all $1 \le i \le n$.

In effect, Proposition 6.1 is a special case of Proposition 5.3. That is, the equilibrium bidding functions still have to satisfy the system of nonlinear ODEs given by Equation (5.27); however, in this case, the lower boundary condition reduces to

$$c_i(\underline{b}) = \underline{c}, \tag{6.6}$$

and the upper boundary condition to

$$c_i(\bar{c}) = \bar{c}, \tag{6.7}$$

i.e., the bids never exceed the upper extremity of the common support range.

It should be noted that, even though the bidding problem is considerably simpler than the original one discussed in this thesis (cf. Chapter 5), it still involves finding the lower bound on bids, and hence, the closed-form solution exists only in a handful of special cases [54, 88]. However, as presented by Hubbard and Paarsch [88], the problem can be approximated using numerical methods, which is discussed in the next section.

6.2 Numerical Solutions

In this section, a CP auction is approximated using the FSM method already introduced in Section 5.4, Chapter 5, but tailored to the problem at hand. The FSM method was chosen due to its relatively low implementation complexity (compared to the PPM method), and the fact that it was also used to approximate the DMP bidding problem. Therefore, in terms of the numerical accuracy and stability, the numerical solutions to the DMP and CP auctions should be of comparable quality. Furthermore, since the discussion concentrates on a relatively small number of bidders, the FSM (and the EFSM) method is still well-behaved numerically.

6.2.1 Forward Shooting Method

To briefly recap, the FSM method was first proposed by Bajari [91] (cf. Algorithm 1 in [91]). The method aims at finding the best approximation of the lower bound on bids, $\underline{b}$, by successively picking a value from the feasible interval, $(\underline{c}, \bar{c})$, and verifying whether a numerical solution to the initial value problem

$$\begin{aligned} \frac{d}{db}c_i(b) &= \frac{1 - F_i(c_i(b))}{f_i(c_i(b))}\left[\frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{b - c_k(b)} - \frac{1}{b - c_i(b)}\right] \\ c_i(\underline{b}) &= \underline{c} \end{aligned} \tag{6.8}$$

for all $i \in N$ satisfies the following three conditions: 1) it is a function mapping $[\underline{b}, \bar{c}]$ into $[\underline{c}, \bar{c}]$, that is,

$$s : [\underline{b}, \bar{c}] \to [\underline{c}, \bar{c}]; \tag{6.9}$$

2) it is monotonically increasing everywhere except possibly at $\underline{c}$, that is,

$$b_1 < b_2 \implies s(b_1) < s(b_2) \text{ for all } b_1, b_2 \in [\underline{b}, \bar{c}); \tag{6.10}$$

and 3) each function value is strictly lower than its argument except possibly at $\bar{c}$, that is,

$$s(b) < b \text{ for all } b \in [\underline{b}, \bar{c}). \tag{6.11}$$

The pseudo-code for the FSM is depicted in listing Algorithm 6.1. Note that the algorithm is almost identical to the FSM version tailored to the DMP auction (cf. Algorithm 5.2), with only differences being the definition of the set of permissible functions, S, and the algorithm's search region delimited by low and $high$ variables. Hence, the discussion of the algorithm is omitted, and the reader is referred to Section 5.4.1.

Similarly to the implementation of the FSM (and EFSM) algorithm for the DMP auction, the approximation results presented in this chapter have been derived using the GSL implementation of the Embedded Runge-Kutta-Fehlberg (4,5) method.

Algorithm 6.1 Forward shooting method (common prior version; Bajari [91])

Input: $\epsilon \in (0, \bar{c} - \underline{c})$; $low, high \in [\underline{c}, \bar{c}]$ such that $low \leq high$
Output: Approximation to $\underline{b}$

$low \leftarrow \underline{c}$
$high \leftarrow \bar{c}$

while $high - low > \epsilon$ **do**
    $guess \leftarrow 0.5 \cdot (low + high)$
    $bids \leftarrow [guess, \bar{c})$
    $(costs_1, \ldots, costs_n) \leftarrow$ solve (6.8) with initial value $\underline{b} = guess$ evaluated at points $b \in bids$
    if $(bids, costs_i)$ satisfies (6.9), (6.10) and (6.11) for all $i \leftarrow 1$ to n **then**
        $high \leftarrow guess$
    else
        $low \leftarrow guess$

$\underline{b} \leftarrow 0.5 \cdot (low + high)$

6.2.2 Verification

Before proceeding with the modelling and analysis, the FSM algorithm was tested for correct implementation. The bidding scenario used to verify the algorithm is taken from the Bajari's paper [91]. There are three bidders, and each is characterised by a truncated normal distribution but with different mean and standard deviation parameters (see Table 6.1). Furthermore, each bidder draws their cost from common costs' range, $c_i \in [2, 8]$.

Figure 6.1 depicts the numerically approximated solution to the problem. It is clear that the approximation agrees with that of Bajari's [91] (cf. Figure 1 in [91]). Furthermore, in Figure 6.2, the numerical solution is verified whether it satisfies the sufficiency condition for an equilibrium; that is, whether the numerically derived bidding strategy for each bidder is a best response to the bidding strategies of the remaining bidders. As expected, the solution satisfies the sufficiency condition, and hence, it is concluded that the algorithm was implemented correctly.

In this section, version of the FSM algorithm tailored to the CP auction was presented, and verified for correct implementation. The next section explores the methodology for approximating a DMP auction with a CP auction.

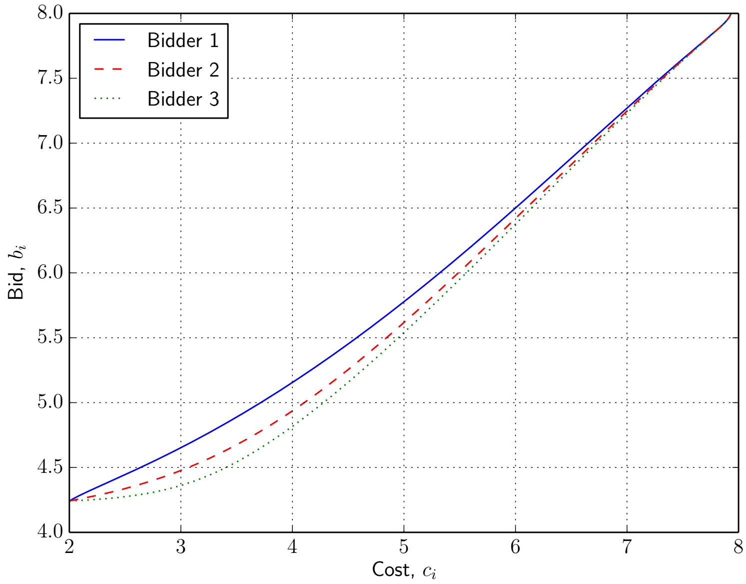


Figure 6.1
FSM solution to the test common prior bidding problem

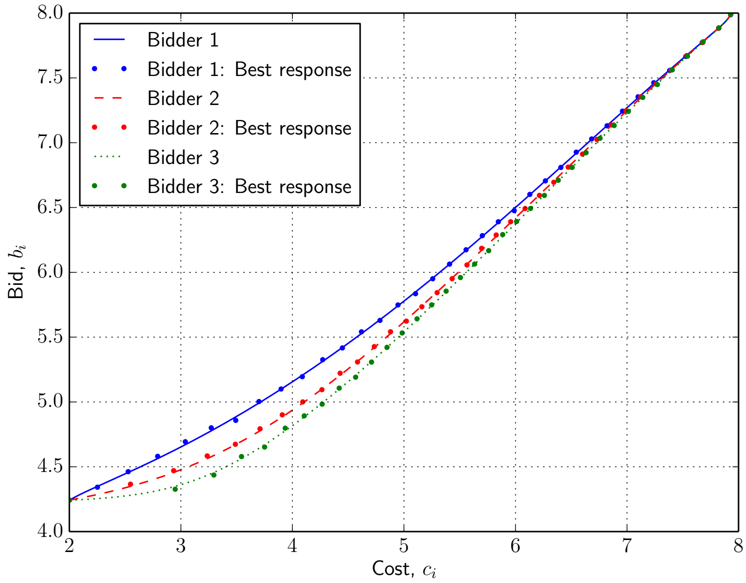


Figure 6.2
FSM solution satisfies sufficiency condition for an equilibrium

Table 6.1
Test bidding scenario

	Mean, μ_i	**Standard deviation**, σ_i
Bidder 1	4	1.5
Bidder 2	5	1.5
Bidder 3	6	1.5

6.3 Network Selection Mechanism Cast into Common Prior Setting

In this section, the DMP auction is firstly modelled as a CP auction where bidders are characterised by costs distributed according to a truncated normal distribution. Then, the methodology used to quantify the accuracy of approximations is outlined.

6.3.1 Modelling using Truncated Normal Distribution

Recall from Chapter 5 that, in the DMP auction, each bidder i draws their cost from a uniform distribution with the support

$$[(1-w)r_i, (1-w)r_i + w] = [\underline{c}_i, \bar{c}_i] \subset [0,1]. \tag{6.12}$$

In order to simplify the exposition of the concepts presented in this chapter, the costs-hat, $\hat{c}_i$, introduced in Chapter 5 will be referred to as costs, c_i. Therefore, in the general case, unless the bidders are characterised by the same reputation rating, that is $r_i = r_j$ for all $i, j \in N$, their distributions' supports will not overlap fully; i.e.,

$$[\underline{c}_i, \bar{c}_i] \neq [\underline{c}_j, \bar{c}_j], \quad i \neq j \text{ and } i, j \in N. \tag{6.13}$$

Recall further that, in a CP auction, every bidder is characterised by a distribution (of costs) with common support across all bidders. Hence, in order to model any DMP bidding scenario, firstly, it needs to be agreed on a support that is common to every bidder and, at the same time, encompasses the supports of every individual bidder from the original (DMP) auction. The smallest such support is

$$[\underline{c}, \bar{c}] = \left[\min_{i\in N}\{\underline{c}_i\}, \max_{i\in N}\{\bar{c}_i\}\right] \subset [0,1]. \tag{6.14}$$

To see this, recall that, for any given $w \in (0,1)$, assuming $r_1 \leq \cdots \leq r_n$ with at

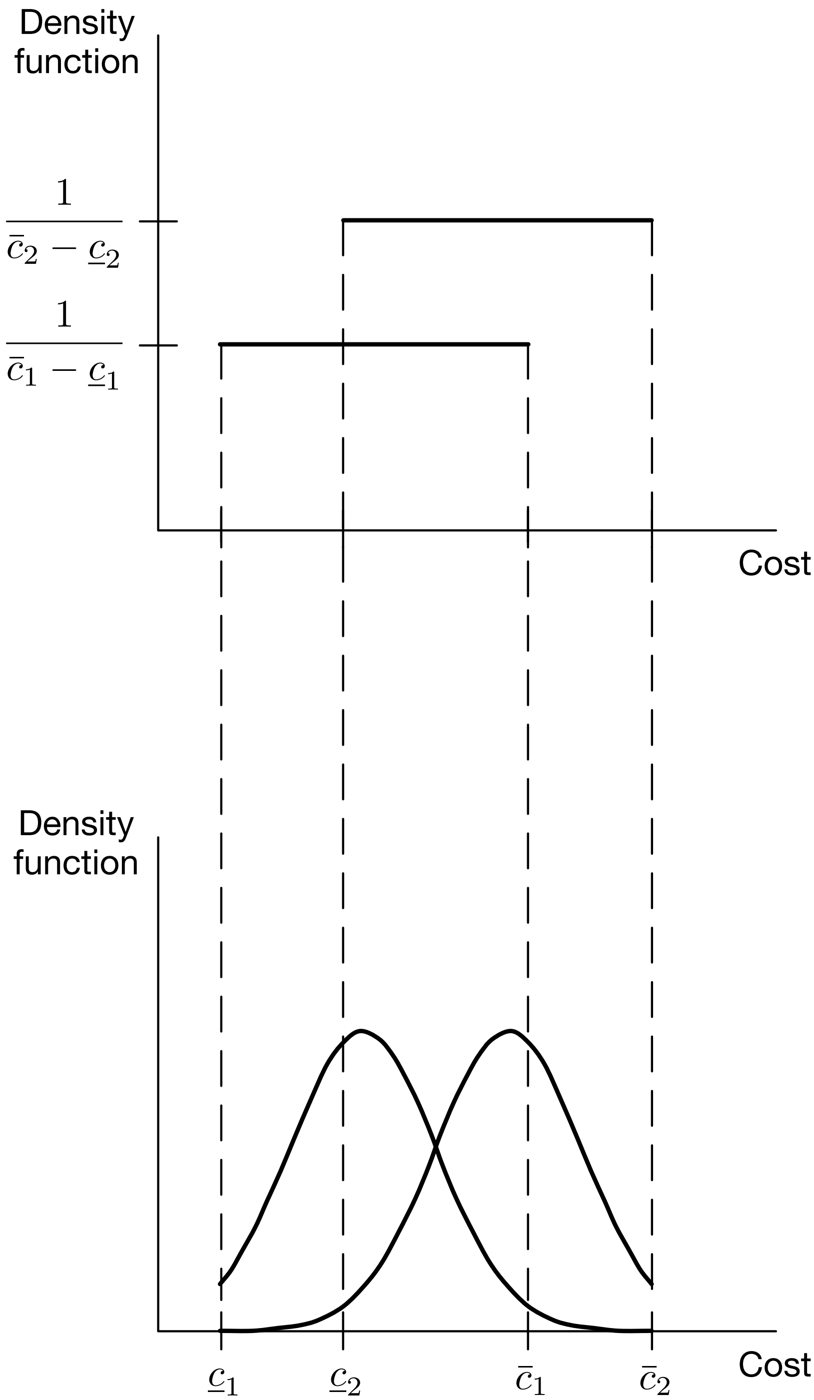


Figure 6.3
Mapping probability distributions from the DMP auction into truncated normal distributions with common support

least one inequality strict, it follows $\underline{c}_1 \leq \cdots \leq \underline{c}_n$ and $\bar{c}_1 \leq \cdots \leq \bar{c}_n$ with at least one inequality strict. Further let $C_i = [\underline{c}_i, \bar{c}_i]$; then $C = \bigcup_{i \in N} C_i$ is the smallest set containing all sets C_i for all $i \in N$. Since C_i is closed for all $i \in N$, it follows that C is closed, and $C = [\underline{c}, \bar{c}]$ such that $\underline{c} \leq \underline{c}_i$ and $\bar{c}_i \leq \bar{c}$ for all $i \in N$, which is equivalent to $[\min_{i \in N}\{\underline{c}_i\}, \max_{i \in N}\{\bar{c}_i\}]$.

All that remains is to then select a family of distributions which captures the numerical ranges of the original supports as closely as possible. To provide an illustrative example, let there be 2 bidders such that $\underline{c}_1 < \underline{c}_2 < \bar{c}_1 < \bar{c}_2$. Each bidder is characterised by a uniform distribution. The common support in this case equals $[\underline{c}, \bar{c}] = [\underline{c}_1, \bar{c}_2]$. Firstly, recall that the chosen distirbutions have to satisfy Assumptions 6.1. Thus, uniform distributions considered over the common support cannot be chosen since they violate those assumptions. To see this, let F_1 be the cumulative distribution function (cdf) of the uniform distribution with the support $[\underline{c}_1, \bar{c}_1]$. Extended to the common support $[\underline{c}, \bar{c}]$, the derivative of F_1, the probability density function (pdf), is zero over the interval $[\bar{c}_1, \bar{c}] = [\bar{c}_1, \bar{c}_2]$, and hence, it is not locally bounded away from zero over the common support. As a result, it is necessary to choose distributions such that they satisfy Assumptions 6.1, and at the same time, possess the shape characteristics similar to the uniform distribution, such as symmetry about the mean. One possible way of casting this scenario into common prior setting is to model the distributions of both bidders as truncated normal distributions truncated to the interval $[\underline{c}_1, \bar{c}_2]$, and with differing mean and standard deviation parameters. This is depicted in Figure 6.3.

In order to describe the truncated normal distribution, firstly recall the pdf of standard normal distribution

$$\phi(c) = \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{1}{2}c^2\right\}, \tag{6.15}$$

and cdf

$$\Phi(c) = \int_{-\infty}^{c} \phi(c)dc = \frac{1}{2}\left[1 + \operatorname{erf}\left(\frac{c}{\sqrt{2}}\right)\right] \tag{6.16}$$

for all $c \in \mathbb{R}$. The pdf of the truncated normal distribution, truncated to the interval $c \in [\underline{c}, \bar{c}]$, can then be described in terms of the pdf of the standard normal distribution as follows

$$f(c; \mu, \sigma, \underline{c}, \bar{c}) = \frac{\frac{1}{\sigma}\phi\left(\frac{c-\mu}{\sigma}\right)}{\Phi\left(\frac{\bar{c}-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{c}-\mu}{\sigma}\right)} \tag{6.17}$$

where $\mu \in \mathbb{R}$ is the mean (or location) of the distribution, and $\sigma^2 \geq 0$ is the variance (or squared scale) [99, 100]. Similarly, the cdf of the truncated normal distribution can

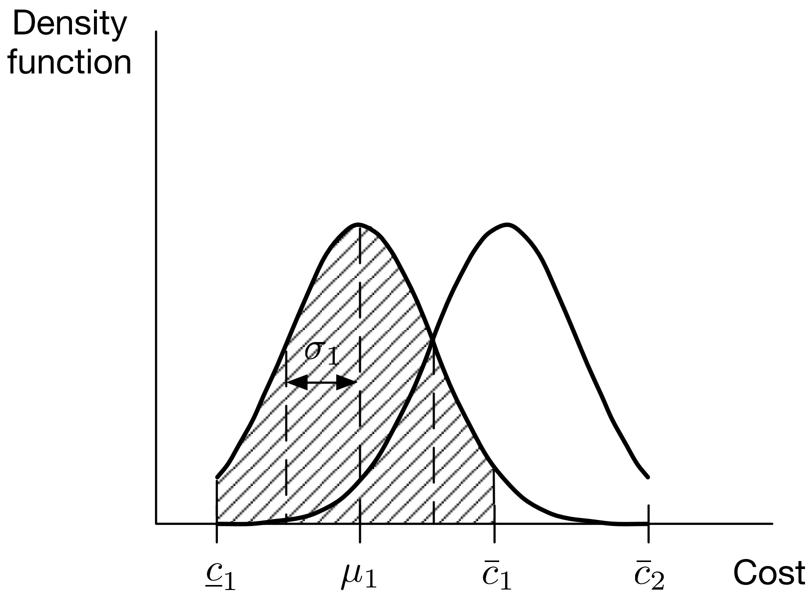


Figure 6.4
Choosing parameters for the truncated normal distributions of the bidders

be defined as follows

$$F(c;\mu,\sigma,\underline{c},\bar{c}) = \int_{-\infty}^{c} f(c;\mu,\sigma,\underline{c},\bar{c})dc = \frac{\Phi\left(\frac{c-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{c}-\mu}{\sigma}\right)}{\Phi\left(\frac{\bar{c}-\mu}{\sigma}\right) - \Phi\left(\frac{\underline{c}-\mu}{\sigma}\right)}. \tag{6.18}$$

Before moving on to discussing the methodology for quantifying the accuracy of the approximations, consider bidding scenario summarized in Table 5.2 in Chapter 5. Suppose this scenario was cast into common prior setting where bidders are characterised by truncated normal distributions. Firstly, it can be noted that the supports for both bidders are

$$[\underline{c}_1, \bar{c}_1] = [0.125, 0.625] \tag{6.19}$$

for bidder 1, and

$$[\underline{c}_2, \bar{c}_2] = [0.375, 0.875] \tag{6.20}$$

for bidder 2, while the common support is given by

$$[\underline{c}, \bar{c}] = [\underline{c}_1, \bar{c}_2] = [0.125, 0.875]. \tag{6.21}$$

Secondly, the distribution specific parameters (mean and standard deviation) need to be specified for each bidder. The choice of the parameters is motivated by the shape

Table 6.2
Numerical values of the chosen truncated normal distribution parameters

	Mean, μ_i	**Standard deviation**, σ_i
Bidder 1	0.375	0.125
Bidder 2	0.625	0.125

of the normal distribution. Therefore, the midpoints of the original supports are picked as means for both bidders, that is,

$$\mu_i = \underline{c}_i + \frac{\bar{c}_i - \underline{c}_i}{2} = \underline{c}_i + \frac{w}{2}. \tag{6.22}$$

Furthermore, noting that, in the case of normal distribution, 95% of all the values falls within 2 standard deviations away from the mean [99], the standard deviations are selected to be equal to the quarter of the length of the original supports, that is,

$$\sigma_i = \frac{\bar{c}_i - \underline{c}_i}{4} = \frac{w}{4}. \tag{6.23}$$

In this way, for each bidder, 95% of all the costs falls within the interval $[\underline{c}_i, \bar{c}_i]$, and therefore, the probability of drawing cost outside this interval is minimised. With this choice of parameters, the truncated normal distributions are effectively imitating uniform distributions with support $[\underline{c}_i, \bar{c}_i]$ for each bidder. This is depicted in Figure 6.4 as the shaded region under the bell curve.

Table 6.2 summarises the numerical values of the described parameters, while the resultant pdfs are depicted in Figure 6.5, and Figure 6.6 shows the resultant equilibrium bidding strategies for both bidders. It is worth noting that the pdfs match the illustrative example shown in Figure 6.3. Furthermore, note that the pdfs for both bidders, as intended, are centred around the midpoints of their original supports respectively, and they tail off to zero as the bounds of the supports are reached.

6.3.2 Methodology for Quantifying Accuracy of the Approximations

There are two fundamental questions that need to be addressed when it comes to quantifying accuracy of the approximations. First, how can the predictions (in terms of the equilibrium bidding strategies) produced by both auction types be compared, and second, how such a comparison can be quantified to allow for a programmatic treatment of the problem (thus, removing the possibility of human error when visually comparing the results). Two metrics will be considered: buyer's expected price, and *ex ante* expected

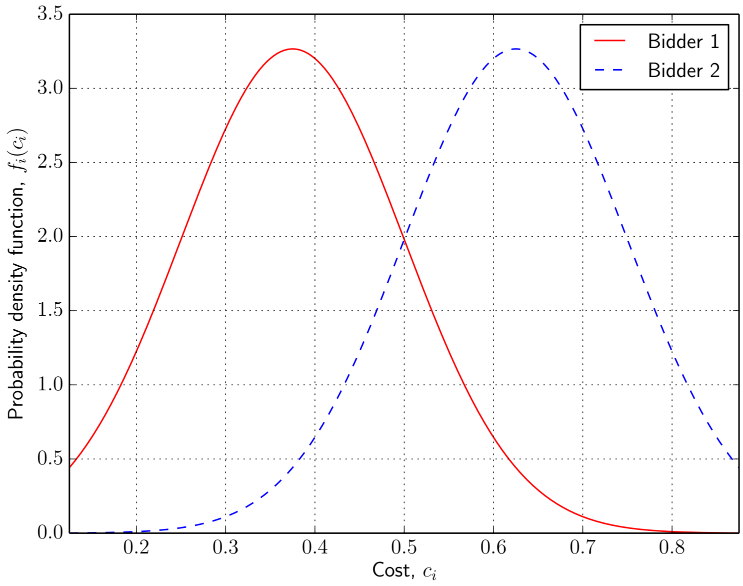


Figure 6.5
Pdfs of the truncated normal distributions from the CP bidding problem characterised by: $\underline{c} = \underline{c}_1 = 0.125$, $\bar{c} = \bar{c}_2 = 0.875$, and $\mu_1 = 0.375$ and $\sigma_1 = 0.125$ for bidder 1, and $\mu_2 = 0.625$ and $\sigma_2 = 0.125$ for bidder 2

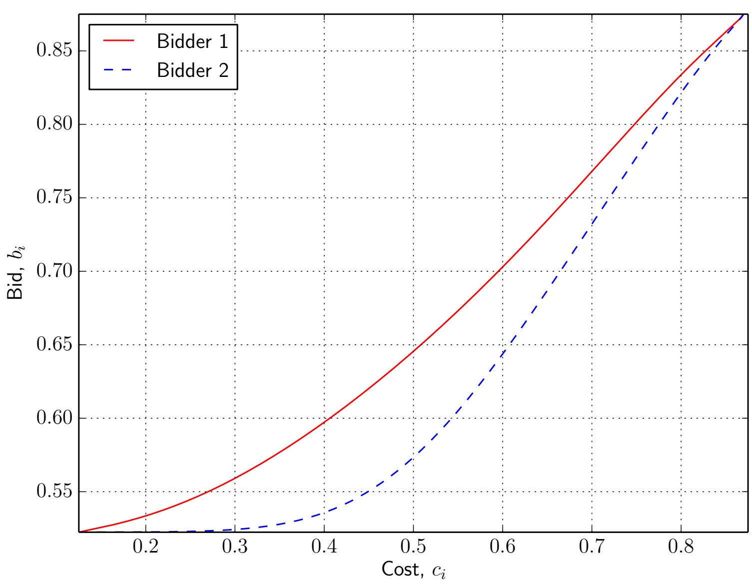


Figure 6.6
FSM solution to the CP bidding problem characterised by: $\underline{c} = \underline{c}_1 = 0.125$, $\bar{c} = \bar{c}_2 = 0.875$, and $\mu_1 = 0.375$ and $\sigma_1 = 0.125$ for bidder 1, and $\mu_2 = 0.625$ and $\sigma_2 = 0.125$ for bidder 2

utility for each bidder. In this way, an indicator of how better off (or worse off) is the buyer and each of the bidders is obtained; that is, all agents involved in the auction are considered.

The buyer's expected price is equivalent to the expected value of the winning bid; that is,

$$p = E[b_i(c_i) \mid b_i(c_i) < \min_{j \neq i} b_j(c_j)], \tag{6.24}$$

where b_i is the equilibrium bidding function for all $i \in N$. Since an analytical derivation of the closed-form solution is not straightforward, similarly to the analysis presented in Section 5.3.1, Chapter 5, the buyer's expected price is estimated numerically. That is, for each considered bidding scenario, the costs for each bidder are pseudo-randomly drawn from uniform distribution, the corresponding equilibrium bids are computed, and the minimum is chosen as the winning bid (price). This procedure is repeated 1000 times, yielding 1000 i.i.d. observations of the price which are then averaged to give an estimate of the expected price (consequence of the Strong Law of Large Numbers; see Section B.3.2, Appendix B).

In order to define the bidder's *ex ante* expected utility, with some abuse of notation, the expected utility function for each bidder $i \in N$ as defined in Equation (5.13), Chapter 5 is restated here:

$$\Pi_i(c_i) = (b_i(c_i) - c_i) \cdot \prod_{j \neq i} \left(1 - F_j(b_j^{-1}(b_i(c_i)))\right) \tag{6.25}$$

where b_i is the equilibrium bidding function, and F_i is the distribution function of costs for bidder i. The *ex ante* expected utility is then equivalent to the expected value of the expected utility; that is,

$$\Pi_i = E[\Pi_i(c_i)] = \int_{\underline{c}_i}^{\bar{c}_i} \Pi_i(t) dF_i(t) \tag{6.26}$$

for all $i \in N$. In other words, the *ex ante* expected utility can be thought of as the average expected utility for each bidder for each considered bidding scenario, and it follows from the definition of *ex ante* expected payments in a standard first-price auction put forward by Krishna [54] (cf. Section 2.4 Revenue Comparison in [54]).

The way the aforementioned metrics are actually computed deserves a more elaborate explanation. The numerical derivation of equilibrium in CP auction relies on approximating the bidders' distributions of costs with truncated normal distributions with common support, as discussed in Section 6.3.1. When computing the expected

Table 6.3
Expected prices and *ex ante* expected utilities for the considered bidding scenario

	Expected price, p	***ex ante* expected utility**, Π_i **Bidder 1**	**Bidder 2**
DMP	0.583	0.183	0.030
CP	0.573	0.176	0.026

price and *ex ante* expected utilities for all bidders in the CP auction, it is assumed, however, that the bidders draw their costs from their actual (uniform) distributions but use the equilibrium bidding strategies derived for the CP auction with truncated normal distributions to compute their bids. In this way, when computing the expected price and *ex ante* expected utilities, the bidders' distributions of costs are not misrepresented, and hence, ensure the comparison results of casting the DMP auction into CP auction setting are as realistic as possible. To see this, suppose that, in the CP auction, the bidders' costs are drawn from the truncated normal distributions but in reality they come from uniform distributions. Let F_i^{CP} denote the truncated normal distribution function with support $[\underline{c}, \bar{c}]$ (according to Equation (6.14)), and let F_i^{DMP} denote the uniform distribution function with support $[\underline{c}_i, \bar{c}_i]$ for all $i \in N$. Then, as shown in the previous section, $[\underline{c}_i, \bar{c}_i] \subset [\underline{c}, \bar{c}]$. Hence, there exists $c \in [\underline{c}, \bar{c}]$ such that $c \in [\underline{c}_i, \bar{c}_i]$ for some $i \in N$; that is, a bidder is allowed to submit a cost lying outside their actual support.

By way of example, consider the numerical example from the previous section. Table 6.3 presents the resulting expected prices and *ex ante* expected utilities for both bidders for both auctions. It is difficult to judge by the values of expected prices and *ex ante* expected utilities how erroneous the approximation for each bidder is. To account for this fact, the relative error in expected prices is defined as

$$\eta_p = \left| \frac{p^{DMP} - p^{CP}}{p^{DMP}} \right| \tag{6.27}$$

and the relative error in *ex ante* expected utilities as

$$\eta_{\Pi_i} = \left| \frac{\Pi_i^{DMP} - \Pi_i^{CP}}{\Pi_i^{DMP}} \right| \tag{6.28}$$

for all $i \in N$, where p^{DMP} and p^{CP} denote the expected prices for DMP and CP auction respectively, and Π_i^{DMP} and Π_i^{CP} denote the *ex ante* expected utilities for bidder i for DMP and CP auction respectively. For the values of expected prices and *ex ante* expected utilities depicted in Table 6.3, the (percentage) relative errors are summarised

Table 6.4
Percentage relative errors in expected prices and *ex ante* expected utilities for the considered bidding scenario

	Expected price, p	***ex ante* expected utility**, Π_i **Bidder 1**	**Bidder 2**
Percentage relative error, $\eta \cdot 100\%$	1.72%	3.83%	13.33%

in Table 6.4.

In this section, it was shown how a DMP bidding scenario can be cast into CP setting by approximating bidders' cost distributions with truncated normal distributions with common support. Furthermore, expected prices and *ex ante* expected utilities were suggested as metrics for quantifying the accuracy of approximating DMP auction with CP auction. In what follows, the proposed metrics are used to study approximation results in four different bidding scenarios with two, three, four and five bidders respectively.

6.4 Approximation Results

This section analyses the results for four bidding scenarios: with $n = 2$, $n = 3$, $n = 4$ and $n = 5$ bidders respectively. The discussion concentrates on only up to five bidders due to the following three reasons. Firstly, the time required to simulate the problem increases exponentially with each additional bidder. It should be noted that the simulations were run on a 12-core Xeon processor, and were fully parallelised (i.e., each repetition was run in a separate process, and up to 20 processes were running at any one time). The time required to complete each simulation run took approximately: 1.6 hours for $n = 2$ bidders, 25.6 hours for $n = 3$ bidders, 76.2 hours for $n = 4$ bidders, and 271.9 hours for $n = 5$ bidders. Figure 6.7 depicts results of fitting an exponential function of the form

$$f(x) = ae^{bx} + c, \quad \text{where } a, b, c \in \mathbb{R}, \text{ and } x \in \mathbb{R}_+ \tag{6.29}$$

to the data. Assuming the growth rate of simulation times with each additional bidder will be at least as big as inferred from the measured simulation times, simulating for more than five bidders is impractical. For example, a predicted simulation time for ten bidders is approximately 124,255 hours, which equates to more than 14 years.

Secondly, as shown by Fibich and Gavish [98], FSM method becomes numerically unstable for large numbers of bidders (cf. Corrolary 3.2 in [98]). It applies to EFSM

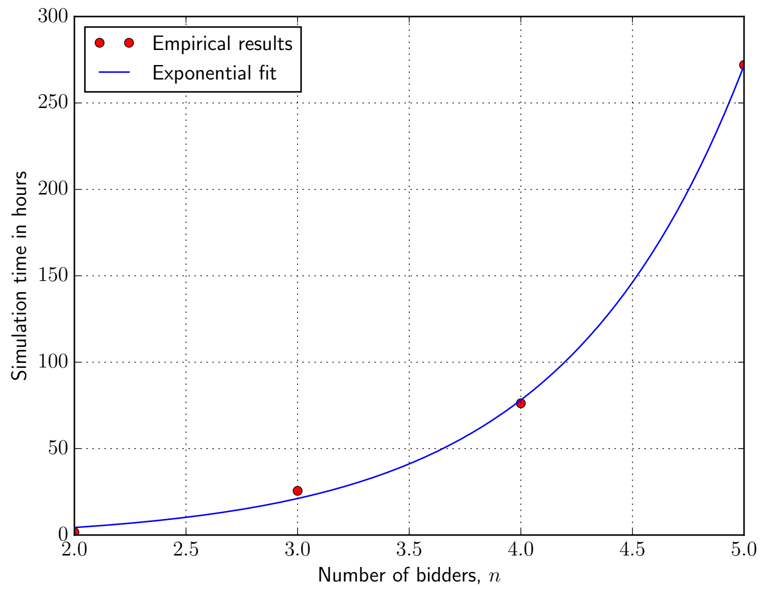


Figure 6.7
Exponential function fitted to the simulation time data

method since it is based on the FSM method.

Finally, it can be noted that since the UK market is currently dominated by an oligopoly of four incumbent network operators (bidders) who own their infrastructure (EE, Vodafone, O2, and Three), solving the problem for up to five bidders is directly relevant.

The procedure for generating the approximation results is as follows:

1. For each chosen value of price weight, generate 100 reputation ratings vectors, $(r_1, \ldots, r_n)$. Each vector is ordered; that is, $r_1 < r_2 < \cdots < r_n$. Therefore, in what follows, bidder 1 is characterised by the lowest reputation rating, bidder 2 by the second lowest, and so on. By ordering individual reputation ratings within the vectors, the mean relative errors in *ex ante* expected utilities can be explored for individual bidders characterised by the lowest reputation rating, second lowest, etc. In other words, if a bidder is characterised by the lowest reputation rating, the mean relative error in *ex ante* expected utility the bidder is going to incur by bidding according to the equilibrium bidding strategies prescribed by the CP auction is quantified. Without this assumption, the mean relative error curves would converge on the same value for all bidders, and thus, some valuable insight into the extent of the mean relative errors in *ex ante* expected utilities would be lost. It is worth noting, however, that the mean relative error in expected price is unaffected by ordering of the reputation ratings.

Furthermore, each r_i for each bidder i is drawn from a uniform distribution over the range $(0, 1)$. It should be noted that reputation ratings have to be unique: if (r_1, r_2), then $r_1 \neq r_2$; and if $r = (r_1, r_2)$ and $g = (g_1, g_2)$ are two consecutively generated reputation rating vectors, then it is required $r \neq g$. By Assumptions 5.1, there exists at least one $r_i \neq r_j$ for all $1 \leq i, j \leq n$ such that $i \neq j$. This immediately rules out the possibility of bidders having equal reputation ratings in case of 2 bidders. In case of 3 or more bidders, Assumptions 5.1 permit for 2 or more bidders (but not all) to be characterised by equal reputation ratings. In order to keep the analysis numerically tractable, however, the bidding scenarios with bidders characterised by equal reputation ratings in case of 3 or more bidders are not considered.

2. For each reputation ratings vector, evaluate relative errors in expected price and *ex ante* expected utility per bidder using Equations (6.27) and (6.28).

3. Evaluate mean relative errors in expected price and *ex ante* expected utility per bidder, and associated 95% confidence intervals. The confidence interval for the mean is computed using the formula described in [101]; that is, given a random sample of size k with unknown mean and standard deviation, the confidence interval is defined as

$$\mathrm{ci} = \bar{X} \pm t_{1-\alpha/2, k-1} \frac{s}{\sqrt{k}}, \tag{6.30}$$

where $\bar{X}$ is the sample mean, s is the sample standard deviation, and $t_{1-\alpha/2,k-1}$ is the upper $1-\alpha/2$ critical value for the t-distribution with $k-1$ degrees of freedom. It is worth noting that for 95% confidence interval, $\alpha = 0.05$.

4. Repeat for price weight values ranging from 0.55 to 0.99. Since only feasible bidders are considered, it is required that $w \in (0.5, 1)$ which was shown to be sufficient to warrant feasible bidding in Section 5.4, Chapter 5.

6.4.1 $n = 2$ Bidders

The approximation results for two bidders are depicted in Figure 6.8. It is worth observing that as the price weight increases, the confidence intervals for the mean relative errors decrease. This is a direct consequence of the fact that as the price weight approaches 1, the actual values of the reputation ratings of the bidders do not significantly influence the mean relative errors in expected price and *ex ante* expected utilities for both bidders. To see this, recall from Equation (6.14) the common support $[\min_i \underline{c}_i, \max_i \bar{c}_i] = [(1-w)\min_i r_i, (1-w)\max_i r_i + w]$. As $w \to 1$, this reduces to $[\lim_{w\to 1}(1-w)\min_i r_i, \lim_{w\to 1}(1-w)\max_i r_i + w] = [0,1]$. Hence, as the price weight increases, the less significant the effect of the reputation ratings on the common support.

Another interesting observation is that, as the price weight approaches 1, the mean relative errors in *ex ante* expected utilities for both bidders start to converge. This is due to the fact that, as w approaches 1 and in particular at $w = 1$, the DMP auction becomes a standard FPA auction with all bidders characterised by uniform distributions which are overlapping to a high degree; i.e., with some abuse of notation, $F_i(x) \approx F_j(x)$ for all x, $i \neq j$ and $i, j \in N$. The same is true for the CP auction with this difference that all bidders are characterised by almost equal truncated normal distributions. Furthermore, in both auctions, the bidders are characterised by symmetric, albeit different across auctions, equilibrium bidding strategies. This is due to the fact that at a symmetric equilibrium the support becomes identical in both auctions, and hence, uniform distribution of costs and truncated normal distribution of costs will result in different equilibrium bidding strategies. This in turn leads to almost equal mean relative errors in *ex ante* expected utilities for all bidders.

The error bounds are explored next. The mean error in expected prices is approximately linearly increasing in price weight, and is bounded from above by 8% and from below by 3%. The mean error in *ex ante* expected utility for bidder 1 also linearly increasing in price weight, and is bounded from above by 15% and from below by 7%. For bidder 2, however, the relationship between the price weight and the mean error is nonlinear, with the error attaining its maximum of approximately 15.5% for the price weight of $w \approx 0.8$. It is bounded from above by 15.5% and from below by 13%. It is clear that bidder 1 who is characterised by lower reputation rating is experiencing overall smaller mean error for all values of the price weight. However, as $w \to 1$ and as explained in the previous paragraph, the mean error converges on the same value of approximately 15% for both bidders.

To summarise, for all analysed values of price weight, the mean relative error in expected prices is relatively small compared to the mean errors in *ex ante* expected utilities for both bidders. It is important to notice that the mean relative errors for both bidders are bounded from above by the same mean relative error of 15%. In terms of the lower bound, however, bidder 2 who is characterised by higher reputation rating is characterised by much higher mean relative error (13% in contrast to *only* 7% for bidder 1).

6.4.2 $n = 3$ Bidders

Figure 6.9 depicts the approximation results for three bidders. First of all, it should be noted that the first two observations pointed out in case of two bidders also apply to the

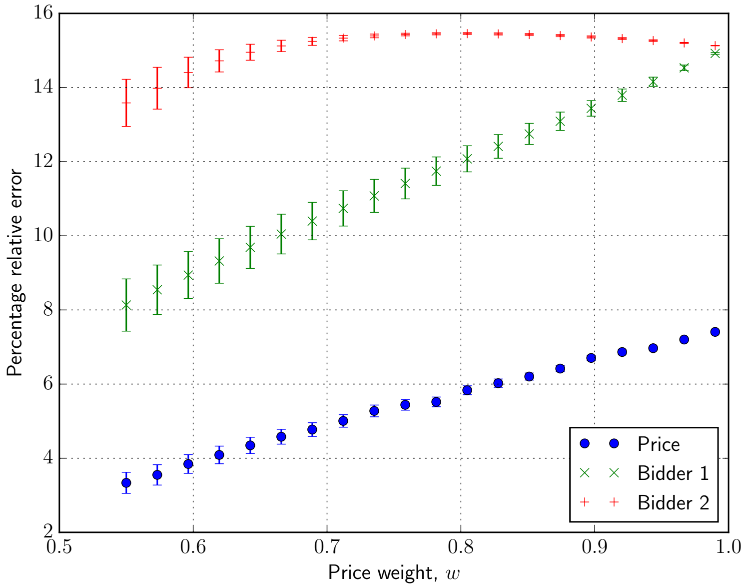


Figure 6.8
Approximation results for two bidders

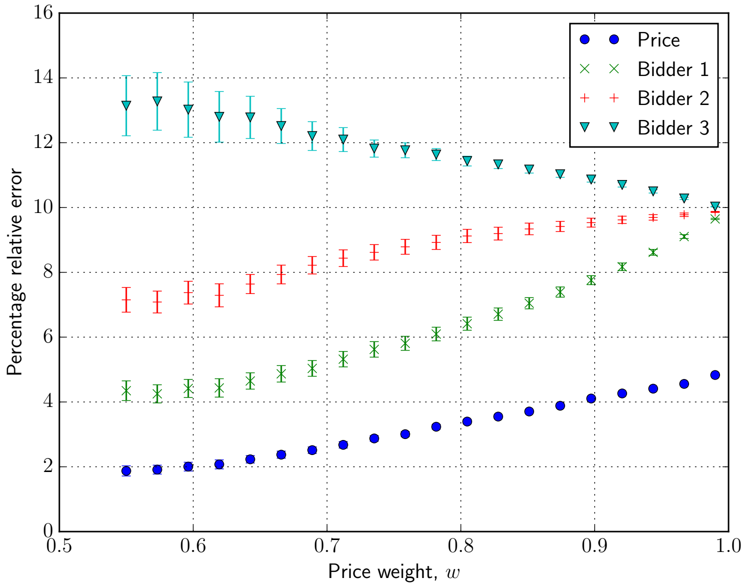


Figure 6.9
Approximation results for three bidders

current case. More specifically, as the price weight increases, the confidence intervals for the mean relative errors decrease, and, as the price weight approaches 1, the mean relative errors in *ex ante* expected utilities for all bidders start to converge.

All mean relative errors, unlike in the case of two bidders, exhibit clear nonlinearity in price weight. Furthermore, the mean relative error in expected prices is nondecreasing as the price weight increases, and achieves its maximum at $w = 0.99$. It is bounded from above by 5% and from below by approximately 1.8%. The mean relative error in *ex ante* expected utilities for bidder 1 is bounded from above by 10% and from below by 4.5%. The mean relative error in *ex ante* expected utilities for bidder 2 is also bounded from above by 10%, but it is bounded from below by 7%. It is worth noting that the shape of the mean relative error curve for bidder 2 resembles that of the mean relative error curve for bidder 1 translated in y-direction. Finally, the mean relative error in *ex ante* expected utilities for bidder 3 is bounded from above by 15% and from below by 10%.

As expected, bidder 3 who is characterised by the highest reputation rating experiences the highest mean relative error in *ex ante* expected utilities for all values of the price weight out of all bidders. In fact, the lower bound for bidder 3 is the same as the upper bound for the remaining bidders. This agrees with the conclusion drawn for the case of two bidders, where bidder 2 was the bidder characterised by the highest reputation rating and experienced the highest mean relative error out of all bidders.

6.4.3 $n = 4$ Bidders

Figure 6.10 depicts the approximation results for four bidders. Firstly, it should be noted that, similarly to the previous two scenarios, as the price weight approaches 1, the mean relative errors in *ex ante* expected utilities for all bidders start to converge. Furthermore, as the price weight increases, the confidence intervals for the mean relative errors decrease.

In terms of shape, similarly to the case of three bidders, all mean relative errors exhibit nonlinearity in price weight. Furthermore, the mean relative error in expected prices is bounded from above by approximately 0.7%, and from below by approximately 0.1%. The mean relative error in *ex ante* expected utilities for bidder 1 is bounded from above by 2%, and from below by approximately 0.1%. The mean relative error in *ex ante* expected utilities for bidder 2 is bounded from above by 2.5%, and from below by 0.1%. It is worth noting that, for the values of price weight $w \in [0.65, 0.9]$, the mean relative error for bidder 2 is actually smaller than for bidder 1, even though bidder 1 is characterised by the lowest reputation rating. The mean relative error in *ex ante* expected utilities for bidder 3 is bounded from above by 6.1%, and from below by 1.8%. Finally,

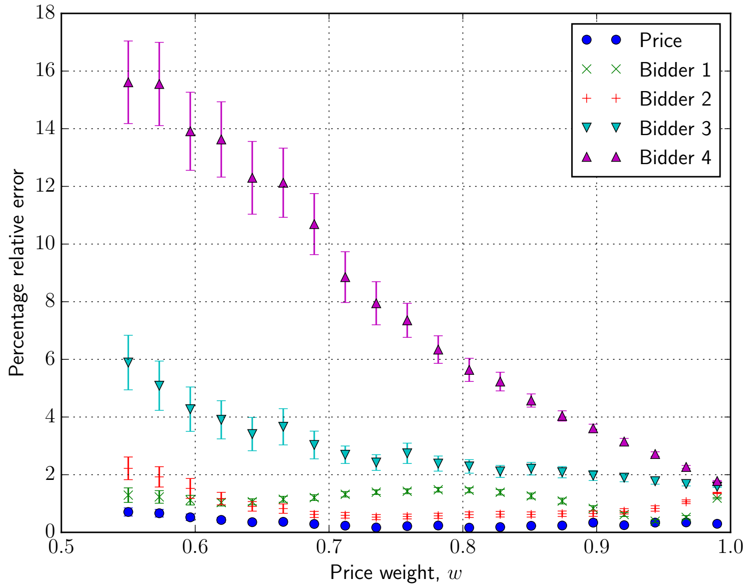


Figure 6.10
Approximation results for four bidders

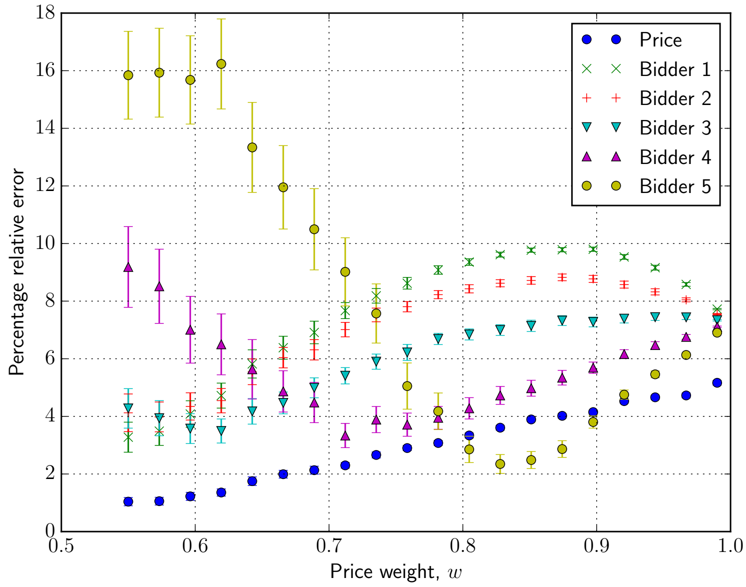


Figure 6.11
Approximation results for five bidders

the mean relative error for bidder 4 is bounded from above by 16%, and from below by 1.5%.

As expected, bidder 4 who is characterised by the highest reputation rating experiences the highest mean relative error in *ex ante* expected utilities for all values of the price weight out of all bidders. This agrees with the conclusion drawn for the previous two bidding scenarios, where bidder who was characterised by the highest reputation rating, experienced the highest mean relative error out of all bidders. It should further be noted that the range of values the mean relative error takes is much larger than it was the case for bidders characterised by the highest reputation rating in the previous two bidding scenarios.

6.4.4 $n = 5$ Bidders

Figure 6.11 depicts the approximation results for five bidders. Similarly to the previous three scenarios, as the price weight approaches 1, the mean relative errors in *ex ante* expected utilities for all bidders start to converge. Furthermore, as the price weight increases, the confidence intervals for the mean relative errors decrease.

All mean relative errors, similarly to the case of three and four bidders, exhibit clear nonlinearity in price weight. Furthermore, the mean relative error in expected prices is bounded from above by approximately 6%, and from below by approximately 1%. The mean relative error in *ex ante* expected utilities is bounded from above and below by: 10% and 3% respectively for bidder 1; 9% and 3.5% for bidder 2; 8% and 3% for bidder 3; 9.5% and 3% for bidder 4; 16.5% and 2% for bidder 5.

Similarly to the previous scenarios, bidder 5 who is characterised by the highest reputation rating experiences the highest mean relative error in *ex ante* expected utilities; however, unlike in the previously considered scenarios, it no longer holds for *all* values of the price weight. In particular, while for the values of price weight $w \in [0.55, 0.7]$ bidder 5 is indeed characterised by the highest error, for $w \in [0.8, 1)$ the error is the lowest. Interestingly, it is bidder 1 who is characterised by the highest relative error for $w \in [0.8, 1)$. This is an unexpected result as it contradicts the conclusions drawn from the previously considered scenarios. At the same time, it is a positive result for bidder 5 as it means that being the bidder characterised by the highest reputation rating does not necessarily entails experiencing the highest error for all values of the price weight. Finally, similarly to previous scenarios, bidder 5 is still characterised by largest range of mean relative errors out all bidders.

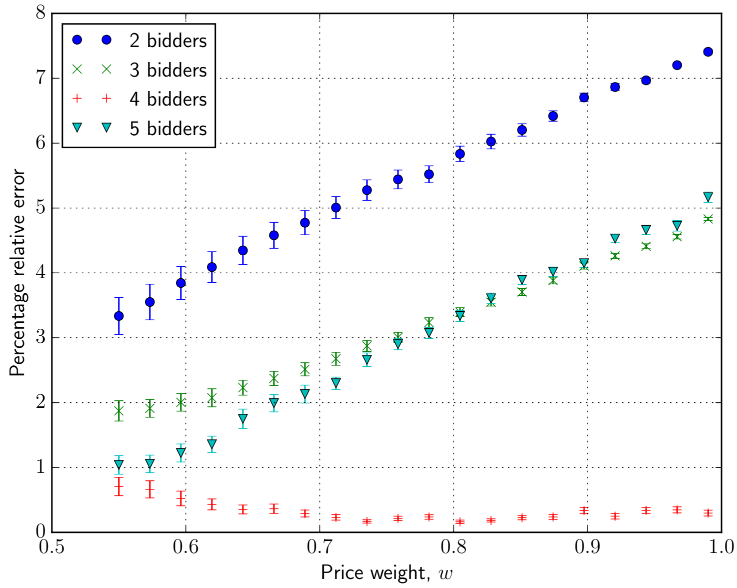


Figure 6.12
Mean relative error in expected prices across all bidding scenarios

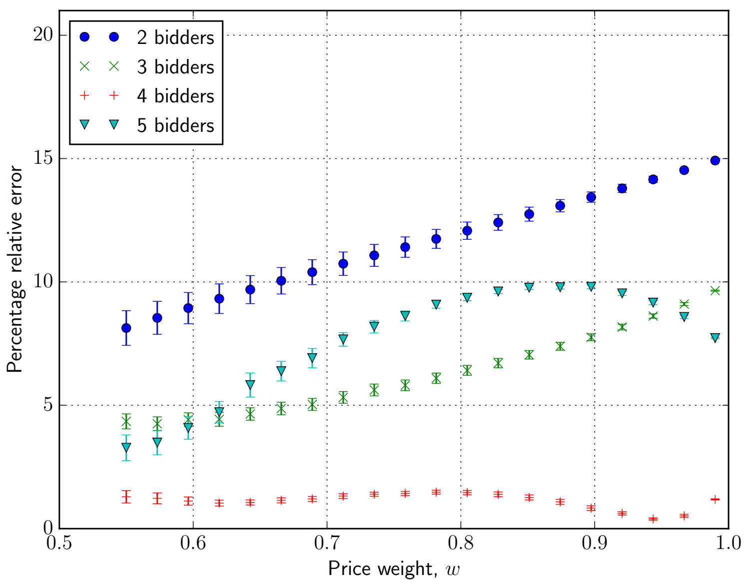


Figure 6.13
Mean relative error in *ex ante* expected utilities for bidder 1 across all bidding scenarios

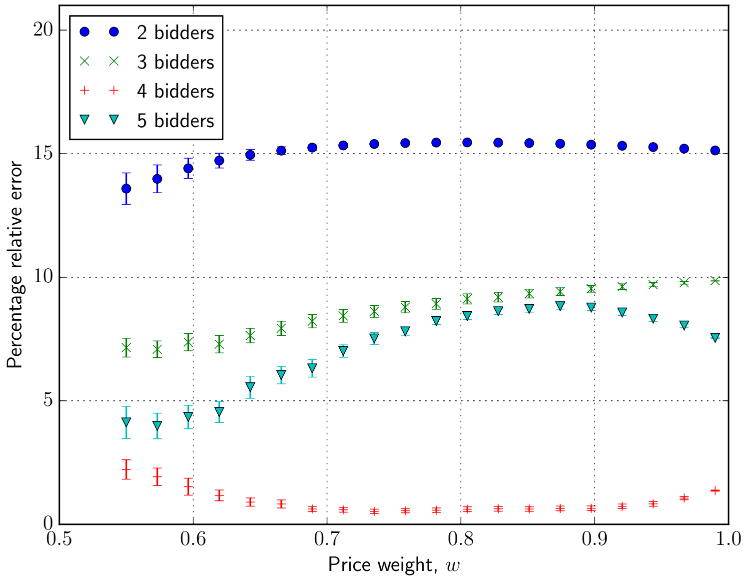


Figure 6.14
Mean relative error in *ex ante* expected utilities for bidder 2 across all bidding scenarios

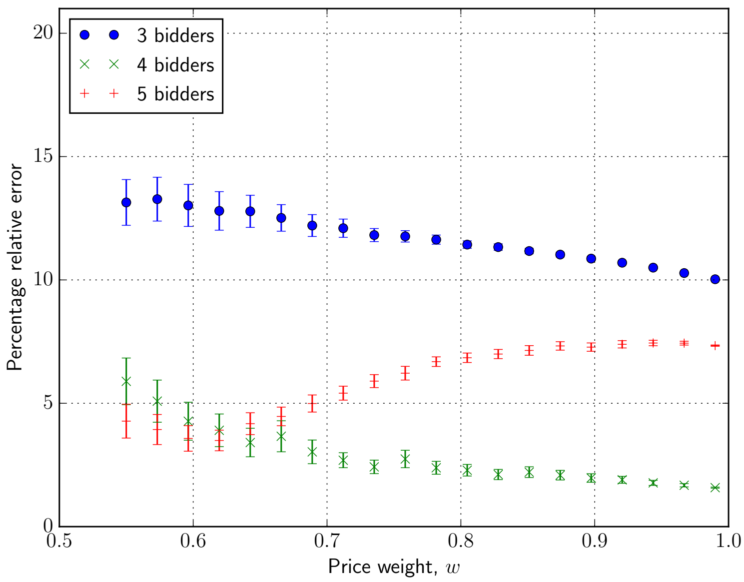


Figure 6.15
Mean relative error in *ex ante* expected utilities for bidder 3 across all bidding scenarios

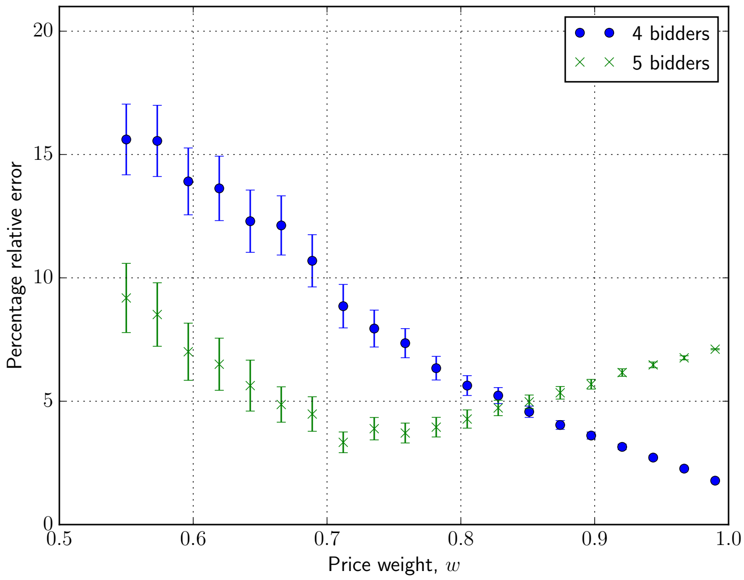


Figure 6.16
Mean relative error in *ex ante* expected utilities for bidder 4 across all bidding scenarios

6.4.5 Discussion

Considering all bidding scenarios together, the relationship between the number of bidders and mean relative errors in prices and *ex ante* expected utilities for each bidder is nonlinear. Furthermore, there is no clear tendency between the number of bidders and mean relative errors; that is, the mean relative errors do not necessarily decrease with each additional bidder. To see this, it can be noted that for all values of the price weight, the mean relative error in expected prices decreases for 2 and 3 bidders, achieves its minimum for 4 bidders, and then increases for 5 bidders (see Figure 6.12). The same applies to the mean relative errors in *ex ante* expected utilities for bidder 1 and 2 (see Figures 6.13 and 6.14). However, the situation is more complicated for bidder 3 and bidder 4. In the former case, for the price weight values $w \in [0.55, 0.62]$, the mean relative error decreases with each additional bidder, while for $w \in (0.62, 1)$ it behaves similarly to the case of bidder 1 and 2 (see Figure 6.15). In the latter case, the situation is almost exactly the same: the mean relative error decreases with each additional bidder for $w \in [0.55, 0.82]$, and behaves similarly to the case of bidder 1 and 2 for $w \in (0.82, 1)$ (see Figure 6.16).

It can be concluded, however, that approximating the network selection mechanism employed by the DMP with a CP auction consitutes a valid alternative, and as such, even though not perfectly accurate (mean relative errors as large as 16% for all bidders), it might be a more desirable option for the network operators due to the wealth of numerical methods available that have been extensively studied by the researchers [88]. The same cannot be said about the EFSM method presented in this thesis (Section 5.5, Chapter 5), which, first of all, becomes numerically unstable for large number of bidders [98], and secondly, to the best of the author's knowledge, has not yet been considered by the economic community.

6.5 Summary

It is a well-known fact that the FSM method and its derivatives, such as the EFSM method, become numerically unstable for large number of bidders [98]. While this issue did not impact the results presented in the thesis (due to the fact that only as many as five network operators were considered), it is important to acknowledge the fact that the issue exists. Therefore, in order to address the problem of numerical instability, in this chapter, it was explored whether the DMP auction can be approximated with a CP auction. To this end, the notion of the CP auction was introduced and formally defined. It was shown that the pure strategy Bayesian Nash equilibrium exists and is unique, provided the cost distributions for each bidder satisfy certain assumptions (see Proposition 6.1).

Furthermore, this chapter presented a numerical algorithm, first proposed by Bajari [91], for numerically approximating equilibrium bidding strategies to the CP auction (see Algorithm 6.1). The method was verified for correct implementation in two ways: by comparing the resultant equilibrium bidding strategy functions with those presented by Bajari [91]; and by testing the equilibrium bidding strategy functions for sufficiency condition for a pure strategy Bayesian Nash equilibrium.

Finally, the DMP auction was modelled as a CP auction where each bidder drew their costs from a truncated normal distribution with common support but differing parameters. A formal methodology for comparing the results generated by the CP auction with those of the DMP auction was presented. The methodology is based on two metrics: expected price and *ex ante* expected utilities for all bidders. The chapter culminated with the analysis of approximation errors in four bidding scenarios: with $n = 2$, $n = 3$, $n = 4$ and $n = 5$ bidders. It was concluded that, even though not perfectly accurate (approximation errors as large as 16% for all bidders), approximating the original DMP auction with the CP auction might be a more desirable option for the network operators. This is emphasised by the fact that there exists an abundance of numerical methods for solving a CP auction that have been extensively studied by the economic community, which are less prone to numerical errors than the FSM method and its derivatives such as the EFSM method.

Chapter 7

Conclusions and Further Work

7.1 Conclusions

The world of mobile communications is becoming increasingly diverse in terms of different wireless access technologies available: WiFi, 3G, and the cutting-edge 4G are gradually being rolled out in many countries across the world. In an environment of such diversity and heterogeneity, where each wireless access technology has its own distinct characteristics, intelligent network selection provides a resource efficient way of handling communications services by matching the services' required quality with the characteristics of a particular access technology.

To make full use of this increasingly diverse environment and increase the competition between network operators even further, the one-to-one mapping between network operators and subscribers need no longer hold. This allows the subscribers to seamlessly switch not only between different wireless access technologies belonging to one particular network operator, but also between network operators themselves. In this way, the subscriber, when requesting a service, is given the option to select a network operator and a wireless access technology that best matches the required quality requirements of the service. It is not only to the benefit of the subscribers, however, since the integration of wireless access technologies will allow network operators for improved revenue generation, and more efficient usage of network resources.

This thesis explored the economic aspects of intelligent network selection. The problem was studied within the context of Digital Marketplace—a theoretical market-based framework for trading wireless communications services. It was first proposed by Irvine *et al.* in 2000 [8, 9], and it was developed with the heterogeneous wireless communications environment in mind, where the subscribers have the ability to select a network operator that reflects their preferences on a per service basis. Since the Digital Marketplace was created with free market (or "perfect" competition) in mind, it is particularly well-suited towards the management of future wireless environment where wireless access is traded on a per service basis. It is for this reason that this research explored the problem of network selection within the context of Digital Marketplace.

The network selection mechanism advocated by the Digital Marketplace lacked extensive and rigorous economic analysis. With the game theoretic analysis presented in this thesis, this deficiency has been addressed. More specifically, in Chapter 4, a game-theoretic model of the network selection mechanism was formally defined. Several simplifying assumptions were made in order to keep the analysis mathematically tractable. For example, the network operators and the subscriber are risk neutral, and the subscriber does not have any budget constraints. Despite the fact that those assumptions are not entirely representative of the reality, following in the footsteps of von Neumann and Morgenstern, the mathematical theory of an economic phenomenon should be rigorous and developed gradually [78]. Therefore, the simplifying assumptions made in this chapter and thesis serve as a starting point for the rigorous, gradual development of the economic theory of operation of the network selection mechanism in the Digital Marketplace, before it can embark on capturing the reality to a high degree.

Furthermore, in the chapter, the equilibrium bidding strategies were derived for three special/extreme cases. In the first case, when only reputation ratings of the network operators decide on the winning network operator, it was shown that network operators will find it beneficial to submit abnormally high bids, since their bid is independent of the probability of winning the auction. While this result sounds like a potential design flaw, in reality, the subscribers will necessarily be budget constrained, and therefore, abnormally high bidding of the network operators will translate into charging the subscribers a premium price for the service that is within the limits of their respective budgets. In the second case, when only the monetary bids of the network operators matter in the selection of the winner, it was shown that the problem reduces to a standard first-price sealed-bid auction with symmetric bidders, and therefore, the symmetric equilibrium bidding strategies of the standard first-price sealed-bid auction applies. Similarly, the third case, when all network operators are characterised by the same reputation rating, was shown to be a special case of the second case. In both cases, the abundance of theoretical results and economic insight from the auction literature applies, found, for example, in Krishna [54].

Finally, the equilibrium bidding strategies for only two network operators were analytically derived. It was shown that although the derived equilibium bidding strategies allows for negative bids, it does not lead to negative profit in case of winning (or a tie) of either network operator. In fact, it was established that the direct mechanism representation of the Digital Marketplace auction satisfies both individual rationality and incentive compatibility constraints. As a result, this proved that the network operators would find it in their best interest to participate in the auction, and they would reveal their costs truthfully. It was further noted that the real behaviour of the network operators might be dictated by the need to secure the contract with the subscriber first and foremost, and hence, lead to negative bidding; a strategy akin to the "loss leader" pricing strategy. However, since the ultimate aim of this thesis was gradual development of rigorous economic theory of the operation of the network selection mechanism within the Digital Marketplace, network operators were always assumed to behave rationally from the perspective of game theory, and bid at least their cost.

In Chapter 5, by mathematically transforming the problem into an alternate form, it was shown that the equilibrium bidding strategies exist and are unique. This was an important result from the perspective of the development of rigorous theory of the operation of the network selection mechanism as it proved that the mechanism is economically well-behaved since the equilibrium exists. Furthermore, the equilibrium bidding strategies were explicitly derived in the case of two network operators and their costs assumed to be uniformly distributed. The assumption of uniform distribution of costs for the network operators was argumented by the fact that it is a standard practice when there is a lack of knowledge of the actual type of the distributions [97]. Nevertheless, it was noted that such an assumption is limiting, and it is highly likely it will not be fully representative of the reality. Furthermore, in the case of two network operators, the expected prices the subscribers will have to pay for different values of the price weight were examined. It was shown that, for any expected price, as the difference between the reputation ratings of the network operators increases, the price weight has to increase (or remain constant) in order to keep the expected price fixed. It was noted that this observation carries very serious implications on the operation of the Digital Marketplace, as the subscriber is effectively given the ability to influence the expected prices by an appropriate choice of the price weight.

Finally, three numerical methods for numerically approximating the equilibrium bidding strategies in the case of more than two network operators were proposed: forward shooting method (FSM), polynomial projection method (PPM), and extended FSM (EFSM). When developing the algorithms, similarly to the restricted case with two network operators, it was assumed that costs for the network operators were uniformly distributed; therefore, the same limitations applied. However, generalising the algorithms to nonuniform distributions should not prove difficult since other researchers have successfully employed similar numerical methods for studying problems where distributions were nonuniform [88]. The FSM and PPM methods allow for numerically approximating equilibrium bidding strategies for a subset of all possible bidding scenarios resulting in nontrivial equilibria, while the EFSM method enables computation of the numerical solution to all bidding scenarios. It should be noted at this point that the development of the EFSM method constitutes an indirect contribution of this thesis. The method allows for approximating solution to first-price sealed-bid auction with asymmetric bidders posed by the Digital Marketplace bidding problem, and to the best of the author's knowledge, this type of auctions has not yet been solved numerically by the economic community. Since the analytical derivation of the equilibrium bidding strategies in the case of more than two network operators is not possible, the existence of algorithms capable of numerically approximating the solutions is a major step forward in the development of the economic theory of operation of the network selection mechanism in the Digital Marketplace. Furthermore, the algorithms constitute a tool that the network operators participating in the Digital Marketplace can use to formulate their own bidding strategies, and understand the bidding behaviours of other network operators.

It is a well-known fact that the FSM method and its derivatives, such as the EFSM method, become numerically unstable for large number of bidders [98]. While this issue did not impact the results presented in the thesis (due to the fact that only as many as five network operators were considered), it is important to acknowledge the fact that the issue exists. Therefore, in order to address the problem of numerical instability, Chapter 6 explored whether an auction format represented by the network selection mechanism employed in the Digital Marketplace can be modelled as an auction with common prior. In an auction with common prior, the range the costs can vary is the same for each bidder. To this end, the methodology for casting the bidding problem posed by the Digital Marketplace into the auction with common prior was presented, and the methodology for quantifying the accuracy of the approximation was outlined. Finally, the chapter concluded with the presentation of approximation results for four bidding scenarios with two, three, four and five bidders respectively. It was shown that approximating the network selection mechanism employed by the Digital Marketplace with an auction with common prior consitutes a valid alternative, and as such, even though not perfectly accurate (mean relative errors as large as almost 16% for all bidders), it might be a more desirable option for the network operators due to the wealth of numerical methods available that have been extensively studied by the researchers.

To conclude, the work reported in this thesis constitutes the first step towards the development of rigorous economic theory of the operation of the network selection mechanism in the Digital Marketplace. As such, the participating network operators can use the results derived in this thesis to formulate their pricing strategies and understand the bidding behaviour of their opponents. The subscribers, on the other hand, are in the position to understand the prices they will be required to pay depending on their preferences for the requested service (in terms of the price weight) and the reputation ratings of the participating network operators. However, due to the fact that many simplifying assumptions had to be made in order to develop the theory, it will not be fully representative of the reality, and therefore, the results presented herein should be taken "with a grain of salt".

7.2 Further Work

There are many aspects of the research presented in this thesis that can further be elaborated upon. The most important future directions are as follows.

7.2.1 Dynamic Aspect of the Network Selection Mechanism

This thesis did not consider the temporal aspect of the network selection mechanism. In game-theoretic terms, the game was assumed to be static as opposed to dynamic. In other words, it was assumed that if the network operators were to interact in an auction more than once, they would discount any previous, historic interactions. This limits the applicability of the results presented in this thesis to real-life scenarios for two main reasons.

Firstly, treating the game as static ignores the forces of supply and demand that necessarily exist in the market, and ignores the possibility of existence of market equilibrium (regardless of whether it is actually achievable) [19]. Indeed, the physical medium (i.e., the radio frequency spectrum) which facilitates distribution of wireless communications services has finite capacity, and therefore, on many occassions, supply may outweigh demand, or in less populated areas, demand may outweigh supply [72]. While the problem of matching supply and demand in wireless communications markets is not as critical as in electricity markets where imbalance may damage the entire electricity network, it is important to consider as it will affect the subscribers' experience with the services received [62].

Secondly, as discussed by Figliozzi *et al.*, repeated interaction of bidders in an auction will inevitably lead to information revelation and learning by the bidders [102]. For example, if it was assumed that the bids are disclosed to all network operators after the auction concludes, this would inevitably lead to network operators inferring the cost structure of other network operators, and hence, induce bidding strategies that diverge from the equilibrium bidding strategies derived in this thesis.

7.2.2 Relaxing Fundamental Assumptions

The results presented in this thesis are based on the following fundamental assumptions: 1) subscriber does not have any budget constraints; 2) subscriber and network operators are risk neutral; and 3) network operators are characterised by symmetric cost distributions (see Section 4.1, Chapter 4). It is generally agreed that these assumptions do not reflect the real world to a high degree [54].

To elaborate further, by enforcing budget constraints on the subscriber, the theory developed in this thesis would be in a better position to rigorously put a limit on abnormally high bidding of network operators stated in Proposition 4.1, which otherwise had to be speculated. Secondly, the assumption of risk neutrality is theoretically desirable as it implies that the expected payoff of a bidder is additively separable, and hence, easier to handle mathematically [54]. However, risk aversion, which assumes that bidders are more likely to accept a certain outcome rather than embrace the uncertainty and gamble, may be a more credible assumption [103]. Finally, the last assumption supposes that the network operators are *ex ante* identical, and, as argued by Guth *et al.* for example, it is violated in many real-life auction environments. Thus, it should be dropped in favour of more credible assumption that the network operators are characterised by asymmetric cost distributions.

It is important to notice, however, that the reason those assumptions were incorporated in this research in the first place was to retain the mathematical tractability. With those assumptions relaxed, this is no longer guaranteed, and hence, it gives rise to a tradeoff between mathematical tractability and applicability to real-world scenarios.

7.2.3 Subscriber's Perspective: Expected Prices

This thesis examined the expected prices only in the case of two network operators (see Section 5.3.1, Chapter 5). Since the UK market is currently dominated by an oligopoly of four incumbent network operators who own their infrastructure (EE, Vodafone, O2, and Three), examining the expected prices for more than two network operators is directly relevant.

It should be noted that deriving the expected prices should be fairly straightforward to execute. All that is required is to treat the numerically approximated equilibrium bidding strategies (generated using EFSM method, or otherwise) as the input to Algorithm 5.1, and proceed according to the methodology presented in Section 5.3.1, Chapter 5.

An analysis of expected prices in scenarios with more than two network operators would be of benefit not only to the subscribers involved in the Digital Marketplace, but also to the system designer/market provider. With the characterisation in place, it would then be possible to verify whether the subscribers are still able to influence the expected prices by an appropriate choice of the price weight like in the case of only two network operators (see Section 5.3.1, Chapter 5).

7.2.4 Generalising Extended Forward Shooting Method

The implementation of the EFSM method assumes the bidders to be characterised by uniform distributions of costs (see Section 5.5.1, Chapter 5). As discussed in the thesis, uniform distributions are an appropriate assumption given the lack of knowledge of the way the costs are distributed. However, should the Digital Marketplace and the network selection mechanism presented in this thesis be used in practice, the market participants may acquire knowledge that will conflict with the assumption of uniformly distributed costs [97]. Therefore, it will strengthen the generality of the results presented in this thesis if the EFSM method is implemented for bidders characterised by nonuniform distributions as well.

Furthermore, generalising the EFSM method to nonuniform distributions would make it applicable to problems beyond the Digital Marketplace. To the best of the author's knowledge, the EFSM method is the only numerical algorithm in existence that numerically solves the unusual first-price sealed-bid auction with asymmetric bidders posed by the Digital Marketplace bidding problem. As such, it would constitute a major contribution to the field of computational auction theory.

7.2.5 Different Distributions in Common Prior Auction

It is a well-known fact that the FSM method and its derivatives, such as the EFSM method, become numerically unstable for large number of bidders [98]. In order to address the problem of numerical instability, this thesis explored whether an auction format represented by the network selection mechanism employed in the Digital Marketplace can be modelled as an auction with common prior.

Casting the auction format represented by the network selection mechanism employed in the Digital Marketplace into an auction with common prior assumes that the latter is based on truncated normal distributions (see Section 6.3, Chapter 6). However, this assumption generated an approximation error as large as 16%. It should be explored whether utilisation of different distributions would decrease the error. A natural starting point would be to test truncated normal distributions with different mean and standard deviation parameters. But the investigation is not limited to this particular family of distributions since there exist many distributions that could be used in their place; for example, truncated log-normal distributions.

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Appendix A

Mathematical Proofs

In this chapter, mathematical proofs of all propositions are presented.

Proposition 4.1. *Suppose c_i is i.i.d. over the interval $[0,1]$ for all $i \in N$ and $r_i \in [0,1]$ for all $i \in N$ is common knowledge. Let $N_0 \subseteq N$ be the set of all those network operators with the lowest reputation rating. If $w = 0$, then every network operator $j \in N_0$ will have an incentive to bid abnormally high, i.e., $b_j \to \infty$, while every remaining network operator $k \in N \setminus N_0$ will be indifferent to the value of their bid.*

Proof. Let $m = |N_0|$ be the number of network operators with the lowest reputation rating such that $m \in \mathbb{Z}_+$. Since $n = |N|$ is finite and $N_0 \subseteq N$, then $m \leq n$. Now, each $j \in N_0$ is facing a maximisation problem

$$\max_{b_j} \frac{1}{m}\left(b_j - c_j\right), \quad \text{for all } j \in N_0. \tag{A.1}$$

Since $1 \leq m \leq n$, and since $b_j \in \mathbb{R}_+$ and $\mathbb{R}_+$ is not bounded from above, this implies that the maximisation problem is unbounded; that is, $b_j \to \infty$ for all $j \in N_0$.

The remaining network operators $k \in N \setminus N_0$ will try to solve

$$\max_{b_k} 0, \quad \text{for all } k \in N \setminus N_0, \tag{A.2}$$

since $r_k > r_j = \min_{i \in N} r_i$. Hence, each network operator $k \in N \setminus N_0$ is indifferent to the value of their bid, which concludes the proof. ■

Proposition 4.2. *Suppose c_i is i.i.d. over the interval $[0,1]$ for all $i \in N$ and $r_i \in [0,1]$ for all $i \in N$ is common knowledge. If $w = 1$, then the symmetric equilibrium bidding strategy function of the standard procurement first-price sealed-bid auction,*

$$b^*_{FPA}(c_i) = \frac{1}{1 - F_{C_{1:n-1}}(c_i)} \int_{c_i}^{1} t\, dF_{C_{1:n-1}}(t) \quad \textit{for all } i \in N, \tag{A.3}$$

constitutes a symmetric pure-strategy Bayesian Nash equilibrium of the DMP variant of a procurement first-price sealed-bid auction.

Proof. The proof is analogous to the proof of Proposition 2.2 in Krishna [54]. ■

Proposition 4.4. *Let there be $n = 2$ network operators. For all $i \in \{1, 2\}$, suppose c_i is independently drawn from uniform distribution over the interval $[0, 1]$, and $r_i \in [0, 1]$ is common knowledge. Then the equilibrium bidding strategies for all $w \in (0, 1]$ are given by*

$$b_1(c_1) = \frac{1}{2} - \frac{1-w}{3w}(r_1 - r_2) + \frac{1}{2}c_1, \tag{A.4}$$

$$b_2(c_2) = \frac{1}{2} - \frac{1-w}{3w}(r_2 - r_1) + \frac{1}{2}c_2. \tag{A.5}$$

Proof. Suppose there are two network operators: network operator 1 and 2 with cost-reputation pairs (c_1, r_1) and (c_2, r_2) respectively. Suppose that network operator 2 follows b_2 equilibrium bidding strategy. It will be argued that it is optimal for network operator 1 to follow b_1 equilibrium bidding strategy. First, note that b_1 is strictly increasing and continuous function of cost (similarly is b_2). Suppose that network operator 1 bids an amount b_1. Since b_1 is strictly increasing, it is bijective. Therefore, there exists unique cost c_1' such that $c_1' = {b_1}^{-1}(b_1)$. Network operator 1's expected utility from bidding $b_1(c_1')$ is

$$\begin{aligned}
&\tilde{u}_1(b_1(c_1'), c_1) \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{(A.6)}\\
&= E\left[b_1(c_1') - c_1 \mid wb_1(c_1') + (1-w)r_1 < wb_2(c_2) + (1-w)r_2\right]\\
&= \frac{1}{2}\left(1 - \frac{2}{3}\cdot\frac{1-w}{w}(r_1 - r_2) + c_1' - 2c_1\right)\left(1 - c_1' - \frac{2}{3}\cdot\frac{1-w}{w}(r_1 - r_2)\right).
\end{aligned}$$

Thus, it follows

$$\tilde{u}_1(b_1(c_1), c_1) - \tilde{u}_1(b_1(c_1'), c_1) = \frac{1}{2}(c_1 - c_1')^2 \geq 0 \tag{A.7}$$

regardless of whether $c_1' \geq c_1$ or $c_1' \leq c_1$. It was thus argued that if network operator 2 follows b_2, network operator 1 with a cost c_1 cannot benefit by bidding anything other than $b_1(c_1)$. Similar argument can be used to show that it is optimal for network operator 2 to follow b_2 while network operator 1 is following b_1. Hence, (b_1, b_2) constitutes a Bayesian-Nash equilibrium profile. ■

Proposition 4.5. *Suppose both network operators bid according to b_i bidding strategies in Equations* (4.34) *and* (4.35). *Then they are guaranteed nonnegative profit in case of winning (or a draw).*

Proof. Let there be two network operators: network operator 1 and 2 with cost-reputation pairs (c_1, r_1) and (c_2, r_2) respectively. Suppose that both network operators follow the equilibrium bidding strategy, $b_i(c_i)$. It needs to be shown that network operator 1's bid is always at least as high as their cost whenever they win or draw with network operator 2; that is, $b_1(c_1) \geq c_1$.

First of all, note that if $r_1 \leq r_2$,

$$b_1(c_1) = \frac{1}{2} - \frac{1-w}{3w}(r_1 - r_2) + \frac{1}{2}c_1 \geq \frac{1}{2}(1+c_1) \geq c_1, \quad \text{for all } c_1 \in [0,1]. \tag{A.8}$$

Thus, the case when $r_1 > r_2$ needs only to be considered.

Suppose $r_1 > r_2$. If $c_1 > c_2$, and since $b_1(c_2)$ is strictly increasing in c_1, network operator 1 will lose for all values of $w \in (0, 1]$. If $c_1 = c_2$, network operator 1 will lose for all values of $w \in (0, 1)$, except at $w = 1$ when there will be a draw. But at $w = 1$, network operator 1's bid is at least as high as her cost; i.e.,

$$b_1(c_1) = \frac{1}{2}(1 + c_1) \geq c_1, \quad \text{for all } c_1 \in [0,1]. \tag{A.9}$$

If $c_1 < c_2$, it is sufficient to show that the intersection of $b_1(c_1)$ and c_1 in terms of w can never occur before the intersection of $\beta(b_1(c_1), r_1)$ and $\beta(b_2(c_2), r_2)$. First of all, it needs to be checked that both intersections do occur; that is,

$$b_1(c_1) = c_1 \iff w = \frac{1}{1 + \frac{3}{2} \cdot \frac{1-c_1}{r_1-r_2}}. \tag{A.10}$$

Similarly,

$$\beta(b_1(c_1), r_1) = \beta(b_2(c_2), r_2) \iff w = \frac{1}{1 + \frac{3}{2} \cdot \frac{c_2-c_1}{r_1-r_2}}. \tag{A.11}$$

Since $r_1 > r_2$ and $c_1 < c_2$, it follows that $0 < r_1 - r_2 \leq 1$ and $0 < c_2 - c_1 \leq 1$. Therefore, this implies

$$0 < w = \frac{1}{1 + \frac{3}{2} \cdot \frac{1-c_1}{r_1-r_2}} \leq 1, \tag{A.12}$$

and

$$0 < w = \frac{1}{1 + \frac{3}{2} \cdot \frac{c_2-c_1}{r_1-r_2}} \leq 1. \tag{A.13}$$

Now, suppose that the intersection of $b_1(c_1)$ and c_1 occurs before that of $\beta(b_1(c_1), r_1)$

and $\beta(b_2(c_2), r_2)$. It must thus follow

$$\frac{1}{1+\frac{3}{2}\cdot\frac{c_2-c_1}{r_1-r_2}} < \frac{1}{1+\frac{3}{2}\cdot\frac{1-c_1}{r_1-r_2}} \iff \frac{1-c_2}{r_1-r_2} < 0. \tag{A.14}$$

But since $c_2 \in [0,1]$ and $r_1 > r_2$ by assumption,

$$0 < \frac{1-c_2}{r_1-r_2}; \tag{A.15}$$

a contradiction, which concludes the proof. ■

Proposition 4.6. *The direct mechanism* $(\mathbf{Q}, \mathbf{M})$ *where* $\mathbf{Q} = (Q_1, Q_2)$ *and* $\mathbf{M} = (M_1, M_2)$ *satisfies both the IC and IR constraints.*

Proof. Let there be two network operators: network operator 1 and 2 with cost-reputation pairs (c_1, r_1) and (c_2, r_2) respectively. Suppose that both network operators participate in the direct mechanism $(\mathbf{Q}, \mathbf{M})$. Firstly, it is shown that the mechanism is incentive compatible. Without loss of generality, suppose that network operator 2 truthfully submits their cost to the mechanism. It is argued that it is optimal for network operator 1 to also submit their cost truthfully. Suppose to the contrary; that is, that network operator 1 has an incentive not to reveal their cost truthfully by submitting c_1'. Thus, their expected utility becomes

$$\begin{aligned}\tilde{\tilde{u}}_1(c_1') &= E\left[b_1(c_1') - c_1 \,\middle|\, 2b_1(c_1') - 1 + \frac{4}{3}\cdot\frac{1-w}{w}(r_1-r_2) < C_2\right] \\ &= \left(\frac{1}{2} - \frac{1}{3}\cdot\frac{1-w}{w}(r_1-r_2) + \frac{1}{2}c_1' - c_1\right)\left(1 - c_1' - \frac{2}{3}\cdot\frac{1-w}{w}(r_1-r_2)\right).\end{aligned} \tag{A.16}$$

The first-order condition yields $c_1' = c_1$ and the second-order condition is satisfied. Hence, this shows that $(\mathbf{Q}, \mathbf{M})$ is incentive compatible.

Secondly, it is shown that $(\mathbf{Q}, \mathbf{M})$ is individually rational. Since the mechanism is incentive compatible, each network operator reveals their cost truthfully. Hence, for all c_1

$$\begin{aligned}\tilde{\tilde{u}}_1(c_1) &= \left(\frac{1}{2} - \frac{1}{3}\cdot\frac{1-w}{w}(r_1-r_2) - \frac{1}{2}c_1\right)\left(1 - c_1 - \frac{2}{3}\cdot\frac{1-w}{w}(r_1-r_2)\right) \\ &= \frac{1}{2}\left(1 - c_1 - \frac{2}{3}\cdot\frac{1-w}{w}(r_1-r_2)\right)^2 \geq 0.\end{aligned} \tag{A.17}$$

Therefore, $(\mathbf{Q}, \mathbf{M})$ is individually rational. ■

Proposition 5.1. *Suppose* $(\hat{b}_1^*, \ldots, \hat{b}_n^*)$ *is a pure-strategy Bayesian Nash equilibrium profile for* *an auction with the utility function*

$$\hat{u}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) = \begin{cases} \left(\hat{b}_i - \hat{c}_i\right) & \text{if} \quad \hat{b}_i < \min_{j \neq i} \hat{b}_j, \\ 0 & \text{if} \quad \hat{b}_i > \min_{j \neq i} \hat{b}_j. \end{cases} \tag{A.18}$$

Then, the same profile constitutes an equilibrium for an auction with the utility function

$$u_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) = \frac{1}{w} \cdot \hat{u}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}). \tag{A.19}$$

Proof. Let

$$\hat{\Pi}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) = \hat{u}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) P\{\text{winning} \mid \hat{b}_i\} \tag{A.20}$$

and

$$\Pi_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) = u_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}, \hat{c}_{-i}) P\{\text{winning} \mid \hat{b}_i\} \tag{A.21}$$

be the expected utilities corresponding to utility functions $\hat{u}_i$ and u_i respectively. By definition of the pure-strategy Bayesian Nash equilibrium [33], for all i and $\hat{c}_i$,

$$\hat{\Pi}_i(\hat{b}_i^*, \hat{c}_i, \hat{b}_{-i}^*, \hat{c}_{-i}) \geq \hat{\Pi}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}^*, \hat{c}_{-i}) \tag{A.22}$$

for all $\hat{b}_i$. Since $w \in (0, 1)$, both sides of the inequality may be multiplied by $\frac{1}{w} > 0$, which yields, for all i and $\hat{c}_i$

$$\begin{aligned} & \frac{1}{w}\hat{\Pi}_i(\hat{b}_i^*, \hat{c}_i, \hat{b}_{-i}^*, \hat{c}_{-i}) \geq \frac{1}{w}\hat{\Pi}_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}^*, \hat{c}_{-i}) \\ \iff & \frac{1}{w}\hat{u}_i(\hat{b}^*, \hat{c}) P\{\text{winning} \mid \hat{b}_i^*\} \geq \frac{1}{w}\hat{u}_i(\hat{b}_i, \hat{b}_{-i}^*, \hat{c}) P\{\text{winning} \mid \hat{b}_i\} \\ \iff & u_i(\hat{b}^*, \hat{c}) P\{\text{winning} \mid \hat{b}_i^*\} \geq u_i(\hat{b}_i, \hat{b}_{-i}^*, \hat{c}) P\{\text{winning} \mid \hat{b}_i\} \\ \iff & \Pi_i(\hat{b}_i^*, \hat{c}_i, \hat{b}_{-i}^*, \hat{c}_{-i}) \geq \Pi_i(\hat{b}_i, \hat{c}_i, \hat{b}_{-i}^*, \hat{c}_{-i}) \end{aligned} \tag{A.23}$$

for all $\hat{b}_i$. Hence, it was just shown that $(\hat{b}_1^*, \ldots, \hat{b}_n^*)$ constitutes a pure-strategy Bayesian Nash equilibrium of the auction with utility function u_i. ■

Proposition 5.2. *Let F_i be the distribution function of $\hat{c}_i$ for all $i \in N$, and suppose $w \in (0, 1]$. Then,*

5. *the support of F_i is an interval $[\underline{\hat{c}}_i, \bar{\hat{c}}_i]$;*

6. *F_i is differentiable over $(\underline{\hat{c}}_i, \bar{\hat{c}}_i]$ with a derivative f_i locally bounded away from zero over this*

interval; and

7. *F_i is atomless.*

Proof. Proof of 1) is trivial. To prove 2) and 3), note that for all $x \in [(1-w)r_i, (1-w)r_i + w]$,

$$F_i(x) = P\{\hat{C}_i \leq x\} \tag{A.24}$$
$$= P\{wC + (1-w)r_i \leq x\}$$
$$= P\left\{C \leq \frac{x-(1-w)r_i}{w}\right\}$$

since $\hat{c}_i = wc_i + (1-w)r_i$ and $w \neq 0$. Hence,

$$F_i(x) = F_C\left(\frac{x-(1-w)r_i}{w}\right) \tag{A.25}$$

and

$$\frac{x-(1-w)r_i}{w} \in [0,1] \tag{A.26}$$

for all $x \in [(1-w)r_i, (1-w)r_i + w]$. Therefore, since F_C is differentiable over $(0,1]$ with a derivative f_C locally bounded away from zero over this interval, by extension, F_i is differentiable over $((1-w)r_i, (1-w)r_i + w]$ with a derivative f_i locally bounded away from zero over this interval, and this proves 2). Moreover, since F_C is absolutely-continuous, it is atomless (see [104, 105] for definition of atomless probability distribution), and by extension, F_i is atomless, which proves 3). ■

Proposition 5.3. *Let Assumptions 5.1 be satisfied. There exists one and only one pure-strategy Bayesian Nash equilibrium where network operators submit at least their costs-hat. In every such equilibrium, network operator $i \in J$ follows a bid function $\hat{b}_i$, for all $1 \leq i \leq n$. Moreover, there exists $\hat{\underline{b}} \in (\hat{\underline{c}}_2, \hat{\bar{b}})$ such that, for all $i \in J$, there exists a continuous extension of $\hat{b}_i$ to the interval $\left[\min\{\hat{\underline{c}}_i, \hat{c}(\hat{\underline{b}})\}, \hat{\bar{b}}\right]$ that is differentiable with a strictly positive derivative everywhere over this interval, except possibly at $\hat{\underline{c}}_i$ or when its value is equal to $\hat{\bar{b}}$, and such that the inverse bid functions $\hat{c}_i$ for all $i \in J$ of these extensions, where differentiable, satisfy the following system of differential equations*

$$\frac{d}{db}\hat{c}_i(b) = \frac{1-F_i(\hat{c}_i(b))}{f_i(\hat{c}_i(b))}\left[\frac{1}{n-1}\sum_{k=1}^{n}\frac{1}{b-\hat{c}_k(b)} - \frac{1}{b-\hat{c}_i(b)}\right] \tag{A.27}$$

for all $1 \leq i \leq n$, with the following lower boundary condition

$$\hat{c}_i(\hat{\underline{b}}) = \min\left\{\hat{\underline{c}}_i, \hat{c}(\hat{\underline{b}})\right\} \quad \textit{for all } i \in J \tag{A.28}$$

and the upper boundary condition

$$\hat{c}_i(\bar{\hat{b}}) = \bar{\hat{b}} \tag{A.29}$$

for all, except possibly one, $1 \leq i \leq n$.

Proof. To prove existence, note that Lebrun [85] proves the existence of a pure-strategy Bayesian Nash equilibrium where network operators submit at least their costs-hat (cf. C.5 Characterization with Possibly Different Lower and Upper Extremities in [85]).

To prove uniqueness, without loss of generality, let network operator 1 be characterised by the lowest reputation rating, network operator 2 by the second lowest, and so on; that is, let $r_1 \leq r_2 \leq \cdots \leq r_n$. This implies $\underline{\hat{c}}_1 \leq \underline{\hat{c}}_2 \leq \cdots \leq \underline{\hat{c}}_n$ and $\bar{\hat{c}}_1 \leq \bar{\hat{c}}_2 \leq \cdots \leq \bar{\hat{c}}_n$. Since Assumptions 5.1 are satisfied, then at least one inequality is strict. Two cases need to be considered: 1) $r_1 < r_2$, and 2) $r_1 = r_2$. When 1) holds, then $\bar{\hat{c}}_1 < \bar{\hat{c}}_2$, implying that the additional condition (ii) in Theorem 1 in Lebrun [85] holds. Otherwise, if 2) holds, then the additional condition (iii) in Theorem 1 in Lebrun [85] is satisfied. Thus, the considered first-price auction has one and only one pure-strategy Bayesian Nash equilibrium where network operators bid at least their costs-hat. ■

Proposition 5.5. *Let there be* $n = 2$ *network operators, and suppose* c_i *is independently drawn from uniform distribution over the interval* $[0, 1]$ *for all* $i \in \{1, 2\}$. *Furthermore, let Assumptions 5.1 be satisfied. The equilibrium inverse bidding strategy functions are given by*

$$\hat{c}_1(b) = \bar{\hat{c}}_1 + \frac{(\bar{\hat{c}}_2 - \bar{\hat{c}}_1)^2}{(\bar{\hat{c}}_2 + \bar{\hat{c}}_1 - 2b)d_1 \exp\left(\dfrac{\bar{\hat{c}}_2 - \bar{\hat{c}}_1}{\bar{\hat{c}}_2 + \bar{\hat{c}}_1 - 2b}\right) + 4(\bar{\hat{c}}_2 - b)}, \tag{A.30}$$

$$\hat{c}_2(b) = \bar{\hat{c}}_2 + \frac{(\bar{\hat{c}}_1 - \bar{\hat{c}}_2)^2}{(\bar{\hat{c}}_1 + \bar{\hat{c}}_2 - 2b)d_2 \exp\left(\dfrac{\bar{\hat{c}}_1 - \bar{\hat{c}}_2}{\bar{\hat{c}}_1 + \bar{\hat{c}}_2 - 2b}\right) + 4(\bar{\hat{c}}_1 - b)}, \tag{A.31}$$

where

$$d_1 = \frac{\dfrac{(\bar{\hat{c}}_2 - \bar{\hat{c}}_1)^2}{\underline{\hat{c}}_1 - \bar{\hat{c}}_1} + 4(\underline{\hat{b}} - \bar{\hat{c}}_2)}{-2(\underline{\hat{b}} - \bar{\hat{b}})} \exp\left(\frac{\bar{\hat{c}}_2 - \bar{\hat{c}}_1}{2(\underline{\hat{b}} - \bar{\hat{b}})}\right), \tag{A.32}$$

$$d_2 = \frac{\dfrac{(\bar{\hat{c}}_1 - \bar{\hat{c}}_2)^2}{\underline{\hat{c}}_2 - \bar{\hat{c}}_2} + 4(\underline{\hat{b}} - \bar{\hat{c}}_1)}{-2(\underline{\hat{b}} - \bar{\hat{b}})} \exp\left(\frac{\bar{\hat{c}}_1 - \bar{\hat{c}}_2}{2(\underline{\hat{b}} - \bar{\hat{b}})}\right), \tag{A.33}$$

and

$$\underline{\hat{b}} = \frac{\underline{\hat{c}}_1 \underline{\hat{c}}_2 - \dfrac{(\bar{\hat{c}}_1 + \bar{\hat{c}}_2)^2}{4}}{\underline{\hat{c}}_1 - \bar{\hat{c}}_1 + \underline{\hat{c}}_2 - \bar{\hat{c}}_2}, \quad \bar{\hat{b}} = \frac{\bar{\hat{c}}_1 + \bar{\hat{c}}_2}{2}. \tag{A.34}$$

Proof. The proof is analogous to the proof of Proposition 1 in Kaplan and Zamir [82]. ■

Proposition 6.1. *Let Assumptions 6.1 be satisfied. There exists one and only one pure-strategy Bayesian Nash equilibrium where bidders submit at least their costs. In every such equilibrium, bidder $i \in N$ follows a bid function b_i, for all $1 \leq i \leq n$ such that its inverse, $c_i = b_i^{-1}$, satisfy the following system of differential equations*

$$\frac{d}{db} c_i(b) = \frac{1 - F_i(c_i(b))}{f_i(c_i(b))} \left[\frac{1}{n-1} \sum_{k=1}^{n} \frac{1}{b - c_k(b)} - \frac{1}{b - c_i(b)} \right] \tag{A.35}$$

for all $1 \leq i \leq n$, with the following lower boundary condition

$$c_i(\underline{b}) = \underline{c} \tag{A.36}$$

and the upper boundary condition

$$c_i(\bar{c}) = \bar{c} \tag{A.37}$$

for all $1 \leq i \leq n$.

Proof. The proposition is just a restatement of the Theorems C.1 Characterization of the Equilibria and U.1 Uniqueness of the Equilibrium in Lebrun [85], and hence, the reader is referred to that paper for proofs. ■

Appendix B

Mathematical Notation and Preliminaries

This chapter, firstly, introduces mathematical notation used in this thesis. It then provides an overview of the more important mathematical concepts necessary to understand the work reported in this thesis.

B.1 Notation

Following the standard notation used in real analysis literature, the set of all real numbers is denoted by $\mathbb{R}$. An open subset of $\mathbb{R}$ is denoted by $(a, b) \subset \mathbb{R}$ such that $a < b$ and $a, b \in \mathbb{R}$. Similarly, $[a, b] \subset \mathbb{R}$ denotes a closed subset of $\mathbb{R}$, $(a, b]$ a half-open (from the left) subset, and $[a, b)$ a half-open (from the right) subset. The set of all positive (negative) real numbers, however, is denoted by $\mathbb{R}_+$ ($\mathbb{R}_-$).

B.2 Mathematical Analysis

B.2.1 Invertibility of a Function

Let $f : X \to Y$ be a function mapping set X into Y.

Theorem (Inverse of a Function). *The function $f : X \to Y$ has an inverse $f^{-1} : Y \to X$ if and only if f is one-to-one and onto.*

Proof. Suppose f has an inverse f^{-1}. Since f^{-1} is a function, then for all $y \in Y$, there exists $x \in X$ such that $f^{-1}(y) = x$. Hence, f is onto. Suppose there exist $x, x' \in X$ and $y \in Y$ such that $f(x) = y$ and $f(x') = y$. Then, $x = f^{-1}(y)$ and $x' = f^{-1}(y)$. But f^{-1} is a function; hence, $x = x'$ and f is one-to-one.

Conversely, suppose f is one-to-one and onto. Let $f^{-1} : Y \to X$ so that for all $y \in Y$, there exists $x \in X$ such that $f^{-1}(y) = x$. Since f is onto, for all $y \in Y$, there exists $x \in X$ such that $f(x) = y$. Furthermore, since $f^{-1}(y) = x$ for all $y \in Y$ by definition, then $f^{-1}(f(x)) = x$. Since f is one-to-one, for all $x, x' \in X$, $f(x) = y$ and $f(x') = y$ implies $x = x'$. Since $f^{-1}(f(x)) = x$ by the previous assertion, then $f^{-1}(f(x)) = f^{-1}(f(x'))$. Hence, f^{-1} is an inverse of f. ■

Let f be an increasing function. That is, for all $x, y \in X$ such that $x < y$, it follows that $f(x) < f(y)$.

Corollary B.1. *If f is increasing, then it is invertible.*

Proof. Every increasing function is one-to-one and onto. ■

B.2.2 Lipschitz Condition

Let f be a function such that $f : \mathbb{R}^n \to \mathbb{R}^n$, and let E be a subset of $\mathbb{R}^n$. Then, f is said to satisfy Lipschitz condition on E if there exists $M > 0$ such that

$$\|f(\mathbf{x}) - f(\mathbf{y})\| \leq M\|\mathbf{x} - \mathbf{y}\| \tag{B.1}$$

for all $\mathbf{x}, \mathbf{y} \in E$, where $\|\cdot\|$ denotes a norm on $\mathbb{R}^n$ [88, 106].

B.3 Probability Theory and Statistics

Let X denote an (absolutely-) continuous random variable (r.v.) with the support $[a, b] \subset \mathbb{R}$. Let F_X denote a cumulative distribution function of the X r.v.; therefore, for any $x \in \mathbb{R}$, $F_X(x) = P\{X \leq x\}$, where $P\{X \leq x\}$ denotes the probability of the event such that $X \leq x$. If F_X admits a density function, it shall be denoted by $f_X = \frac{d}{dx}F_X$. If it is clear from the context which variable is considered random, the subscript will be dropped; that is, $F_X = F$.

The expected value of X, denoted by $E[X]$, is defined as $E[X] = \int_{-\infty}^{\infty} x dF(x)$. Similarly, if u is a function of X, then the expected value of $u(X)$ is defined as $E[u(X)] = \int_{-\infty}^{\infty} u(x)dF(x)$.

B.3.1 Order Statistics

Let $X_1, \ldots, X_n$ be independent continuous r.v.s with distribution function F and density function $f = \frac{d}{dx}F$. Let $X_{i:n}$ denote the ith smallest of these r.v.s; then $X_{1:n}, \ldots, X_{n:n}$ are called the *order statistics* [107, 108]. In the event that the r.v.s are independently and identically distributed (i.i.d.), the distribution of $X_{i:n}$ is

$$F_{X_{i:n}}(x) = \sum_{k=i}^{n} \binom{n}{k} (F(x))^k (1 - F(x))^{n-k}, \tag{B.2}$$

while the density of $X_{i:n}$ can be obtained by differentiating Equation (B.2) with respect to x [109]. Hence,

$$f_{X_{i:n}}(x) = \frac{n!}{(n-i)!(i-1)!} f(x)(F(x))^{i-1}(1-F(x))^{n-i}. \tag{B.3}$$

B.3.2 Strong Law of Large Numbers

Let $X_1, \ldots, X_n$ be i.i.d. r.v.s with finite mean μ. Let

$$\bar{X}(n) = \frac{\sum_{i=1}^{n} X_i}{n}. \tag{B.4}$$

Then, for sufficiently large n, $\bar{X}(n)$ provides a reasonable approximation of μ [101]. This result is known as the Strong Law of Large Numbers,

Theorem (Strong Law of Large Numbers)**.** $\bar{X}(n) \to \mu$ *with probability* 1 *as* $n \to \infty$.

Proof. For proof of this theorem, the reader is referred to Chung's "A Course in Probability Theory" [110]. ■

B.4 Game Theory

B.4.1 Static Games with Incomplete Information

Let $\Gamma^B = [N, \{S_i\}, \{u_i\}, \Theta, F]$ be a *Bayesian game with incomplete information*. Formally, in this type of games, each player $i \in N$ has a utility function $u_i(s_i, s_{-i}, \theta_i)$, where $s_i \in S_i$ denotes player i's action, $s_{-i} \in S_{-i} = \times_{j \neq i} S_j$ denotes actions of all other players different from i, and $\theta_i \in \Theta_i$ represents the type of player i. Letting $\Theta = \times_{i \in N} \Theta_i$, the joint probability distribution of the $\theta \in \Theta$ is given by $F(\theta)$, which is assumed to be common knowledge among the players. For a more in-depth treatment of the theory of games see for example [32, 33, 19].

In game Γ^B, a *pure strategy* for player i is a function $\psi_i : \Theta_i \to S_i$, where for each type $\theta_i \in \Theta_i$, $\psi_i(\theta_i)$ specifies the action from the feasible set S_i that type θ_i would choose. Therefore, player i's pure strategy set Ψ_i is the set of all such functions.

Player i's *expected utility* given a profile of pure strategies $(\psi_1, \ldots, \psi_{|N|})$ is given by

$$\tilde{u}_i(\psi_1, \ldots, \psi_{|N|}) = E[u_i(\psi_1(\theta_1), \ldots, \psi_{|N|}(\theta_{|N|}), \theta_i)], \tag{B.5}$$

where the expectation is taken over the realisations of the players' types, $\theta \in \Theta$. Now, in game Γ^B, a strategy profile $(\psi_1^*, \ldots, \psi_{|N|}^*)$ is a *pure-strategy Bayesian Nash equilibrium* if it constitutes a Nash equilibrium of game $\Gamma^N = [N, \{\Psi_i\}, \{\tilde{u}_i\}]$; that is, if for each player

$i \in N$,

$$\tilde{u}_i(\psi_i^*, \psi_{-i}^*) \geq \tilde{u}_i(\psi_i, \psi_{-i}^*) \tag{B.6}$$

for all $\psi_i \in \Psi_i$, where $\tilde{u}_i(\psi_i, \psi_{-i})$ is defined as in Equation (B.5).

Alternatively, a strategy profile $(\psi_1^*, \ldots, \psi_{|N|}^*)$ constitutes a pure-strategy Bayesian Nash equilibrium in game Γ^B if and only if, for all $i \in N$ and all $\hat{\theta}_i \in \Theta_i$ occurring with positive probability

$$E[u_i(\psi_i^*(\hat{\theta}_i), \psi_{-i}^*(\theta_{-i}), \hat{\theta}_i) \mid \hat{\theta}_i] \geq E[u_i(s_i', \psi_{-i}^*(\theta_{-i}), \hat{\theta}_i) \mid \hat{\theta}_i] \tag{B.7}$$

for all $s_i' \in S_i$, where the expectation is taken over realisations of the other players' types, θ_{-i}, conditional on player i's realisation of his type, $\hat{\theta}_i$. In other words, each type of player i can be thought of as a separate player who maximises his payoff given his conditional probability distribution over the strategy choices of his rivals.

B.4.2 Mechanism Design Theory

In economics, a system where economic transactions take place and goods are allocated is called an *allocation mechanism*; for example, an auction is an allocation mechanism. This section summarises the most important concepts of mechanism design theory. For a more in-depth treatment, see for example [111, 54, 112, 113, 114, 115].

Let $(\mathcal{B}, \pi, \mu)$ be a mechanism representing any given auction. In this notation: $\mathcal{B}$ is a set of all possible bids; $\pi : \mathcal{B} \to \Delta$ is an *allocation rule*, where Δ is a set of all probability distributions over the set of bidders N; and $\mu : \mathcal{B} \to \mathbb{R}^n$ is a *payment rule* where $n = |N|$. The allocation rule quantifies as a function of all n bids the probability that bidder i receives the good. The payment rule determines as a function of all n bids the expected payment that bidder i must make. For example, if $\mathbf{b} = (b_i, b_{-i})$ is the vector of all bids submitted to the mechanism, the probability that bidder i receives the good is $\pi_i(\mathbf{b})$, while the expected payment is $\mu_i(\mathbf{b})$.

Every mechanism can be viewed as a game with incomplete information between n bidders. For each bidder i, let $b_i(\cdot) : \Theta_i \to \mathcal{B}_i$ be the pure strategy where as before Θ_i is the set of all possible valuations of bidder i. The *equilibrium* of the mechanism is hence defined as a vector of strategies $(b_i(\cdot), b_{-i}(\cdot))$ if for all i and for all $\theta_i \in \Theta_i$, $b_i(\theta_i)$ maximises bidder i's expected payoff.

If $\mathcal{B}_i = \Theta_i$ for all i, then the mechanism becomes the so-called *direct mechanism*. In a direct mechanism, bidders are effectively submitting their valuations rather than bids to the mechanism. In general, direct mechanisms tend to be smaller and simpler than generic mechanisms, and therefore are easier to analyse while still being able to model the scenario accurately. Formally, a direct mechanism is defined as a pair $(\mathbf{Q}, \mathbf{M})$ with an allocation rule defined as $\mathbf{Q} : \Theta \to \Delta$, and a payment rule defined as $\mathbf{M} : \Theta \to \mathbb{R}^n$. Note that in direct mechanism bidders' valuations are directly used to determine the outcome of the mechanism.

A direct mechanism $(\mathbf{Q}, \mathbf{M})$ is said to satisfy *incentive compatibility* (IC) constraint if for all $i \in N$, for all $\theta_i \in \Theta_i$, and for all $\hat{\theta}_i \in \Theta_i$,

$$\tilde{\tilde{u}}_i(\theta_i) = q_i(\theta_i)\theta_i - m_i(\theta_i) \geq q_i(\hat{\theta}_i)\theta_i - m_i(\hat{\theta}_i), \tag{B.8}$$

where

$$q_i(\hat{\theta}_i) = E[Q_i(\hat{\theta}_i, \theta_{-i})], \tag{B.9}$$

and

$$m_i(\hat{\theta}_i) = E[M_i(\hat{\theta}_i, \theta_{-i})]. \tag{B.10}$$

In both cases, the expectation is taken over the realisations of all but player i types, $\theta_{-i} \in \Theta_{-i}$.

A direct mechanism $(\mathbf{Q}, \mathbf{M})$ is said to satisfy *individual rationality* (IR) constraint if for all $i \in N$, and for all $\theta_i \in \Theta_i$,

$$\tilde{\tilde{u}}_i(\theta_i) \geq 0. \tag{B.11}$$

This thesis also utilises the very powerful Revelation Principle theorem due to Myerson which states the link between any generic mechanism and a direct mechanism [114, 54]:

Theorem (Revelation Principle)**.** *Given a mechanism and an equilibrium for that mechanism, there exists a direct mechanism in which (1) it is an equilibrium for each buyer to report his or her value truthfully and (2) the outcomes are the same as in the given equilibrium of the original mechanism.*

Proof. For proof of this theorem, the reader is referred to Krishna's "Auction Theory" [54]. ■