A Model for Ultrasonic Transducers in a High-temperature Regime with Boundary Dynamics: An Evolutionary Equations Approach

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Declaration of Authenticity and Author's Rights

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

Signed: Michael Thomas Anthony Doherty

Date: 27/09/23

Dedicated to my parents,

Daniel P. Doherty and Charlotte J. Woods

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Abstract

In this thesis we formulate an abstract model to describe ultrasonic transducers, taking into account a high-temperature regime as well as dynamics at the boundary. We use an abstract boundary trace theory to extend from a known thermo-piezo-electromagnetic system, and encode boundary dynamics directly within our model. Using the theory of evolutionary equations invented by Rainer Picard, we establish the well-posedness of our system. Well-posedness in this context corresponds to both Hadamard well-posedness and causal dependence on given data. Moreover, we conduct a systematic investigation into different arrangements of complicated boundary dynamics which lead to a well-posed system. Motivated by a set of known piezo-electric boundary conditions, we formulate and consider novel generalised impedance like boundary conditions. Furthermore, we formulate and analyse a specific example of these generalised boundary dynamics, which account also for the influence of heat at the boundary. The resulting example pertains to all three physical aspects of our system, and thus harnesses the full generality afforded by our system.

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Chapter 1

Introduction

1.1 Evolutionary Equations

The field of *evolutionary equations* is a relatively fresh area of research in mathematical analysis. The world was first introduced to the theory back in 2009 with the publication of Rainer Picard's seminal paper [Pic09]. Although, the first stone in the path to evolutionary equations was paved some twenty years prior with key observations contained in [Pic89]. Since then it has been the sole focus of at least two research monographs, three PhD theses as well as two habilitation theses. The field played a supporting — but no less substantial — role in at least one other PhD thesis and research monograph. Moreover, there have been over fifty research articles published in the area. The exact routes taken by these articles are almost as varied as their total number might suggest. In a moment we will briefly outline some of these research trends.

At its core, the field of evolutionary equations concerns itself with problems of the form

$$(\partial_t M(\partial_t) + A)U = F. \tag{1.1}$$

We first regard this prototypical equation with a high-level view. The unbounded operator ∂_t denotes a specific realisation of the time derivative, whereas the operator $M(\partial_t)$ enjoys holomorphic and continuous properties. The role of this latter operator will be to encode the material parameters of the physical system being modelled. The operator A is unbounded, skew-selfadjoint and will account for the spatial aspects of the system. Lastly, the function U is our unknown and F is a given right-hand side. A suitable Hilbert space provides the setting. Later in Chapter 2 we will review these key components with a concise low-level mathematical view.

The cornerstone of the theory of evolutionary equations is established by *Picard's Theorem*. This result, first presented in [Pic09, Solution Theory, p. 1770], encapsulates the entire solution theory of the field. Under what amounts to very mild conditions, Picard's Theorem guarantees both standard Hadamard well-posedness as well as *causal dependence* on given data. Causal dependence pertains to a mathematical means of reflecting a given physical phenomenons bias and evolution in a set direction of time. We will recall this notion of *causality* in greater mathematical detail shortly (see Proposition 2.2.5 and Remark 2.2.6). As we will note, causality emerges as an indispensable ingredient for well-posedness in the setting of evolutionary equations. Moreover, this solution theory is very easy to apply, with much of the required work amounting to little more than simple algebra.

It turns out that many of the quintessential partial differential equations (PDEs) of mathematical physics can be reformulated to fit within the archetype of (1.1). For instance, it is well-known that the heat equation, wave equation and Maxwell's equations can each be regarded from the perspective of evolutionary equations. Furthermore, these phenomena and more can be combined via appropriate material couplings, and easily treated by the theory as a coupled system (see [Pic09, Section 4] and [STW22, Section 6.2, Chapter 7] for a plethora of such, and other, applications).

As mentioned, research trends in evolutionary equations are themselves incredibly diverse. For instance, work has been done to treat stochastic partial differential equations (SPDEs) with techniques from the field (c.f. [SW17]). Here, the authors demonstrated the power of the theory by successfully attacking the nonstandard stochastic Maxwell equations as well as some time fractional SPDEs. The subsequent article [PTW18] further developed this framework. Moreover, there have been considerable advances made in the area of asymptotic homogenisation. In [NVW21] the authors used evolutionary equations to establish a setting for handling the periodic and stochastic homogenisation of PDEs. The use of this framework was showcased by the authors for several ubiquitous equations including elliptic PDEs. Various other instances of the advances made in this direction can be found in [Wau16b], [CW18] and [CW19]. The notion of a nonlinear evolutionary equation has also been studied. Fundamental to this zone of investigation is the replacement of the skew-selfadjoint operator A in (1.1) by a *maximal monotone relation* (see for instance [STW22, Definition, p. 276]). This generalisation allows one to apply the theory of evolutionary equations to tackle certain nonlinear problems in mathematical physics (c.f. [Tro12], [Tro13], [Tro20] and [TW14]). The ubiquity of their application, coupled with an accessible solution theory and the ability to handle coupled systems, are some of the main draws to apply this theory to the problem of our thesis.

1.2 Ultrasonic Transducers and Motivation

Piezo-electric ultrasonic transducers are versatile measurement devices at the heart of many and ubiquitous applications. Ultrasonic transducers operate by emitting a wave towards a given material. The wave then proceeds to travel through the material being tested, before being received again by the transducer device. The resulting mechanical wave is then transformed into an electrical signal for analysis. Piezo-electric transducers enjoy usage across a range of diverse fields of study, although conventional applications of these devices mostly comprise medical imaging and non-destructive testing.

With regards to medical imaging, the most well-known example of these instruments lies in ultrasound technologies (see for instance [HAF83], [SZ96] and [WL11]). Piezo-electric ultrasonic transducers provide an attractive option for medical diagnostics on account of their cost-effectiveness, portability and non-invasiveness (c.f. [KYO11] and [LR17]). However, there exists a host of other clinical scenarios which rely upon the use of this technology, albeit to a variety of different ends (c.f. [ATH+95], [PIWS04], [LKBP15], [HC95], [KAC+12] and [BMJ11]).

Ultrasonic transducers are also frequently utilised in the area of non-destructive testing, as well as in the evaluation of safety critical systems. Instances of such systems can be found all around us. They include industrial and nuclear power plants (c.f. [Che12] and [KMC+99]), aerospace structures (c.f. [SH06] and [GS12]) as well as oil pipelines (c.f. [LZDZ12], [AAAR+15] and [LRL+13]). In the realm of non-destructive testing, ultrasonic instruments are used to assess the integrity and stability of a given system. Their use here can lead to the discovery of defects including cracks in the material of the structure (see for instance [HDW05], [TMG15], [TMLG15], [TGN+15] and [RJR95]).

The prospective modelling of such devices is clearly of practical and economic importance. When such models are used to attune material and manufacturing design parameters, the question of the associated systems well-posedness is paramount (c.f. [OMOH08a], [OMOH08b], [OMO+08], [MW11] and [WM16]). The establishment of a well-posed model allows one to consider the corresponding inverse problem, which is conducive to deducing those parameters (c.f. [Ohn90], [LON04], [KLMK06], [LKS08] and [FSC+20]). With ultrasonic transducers providing the backbone to a span of diverse and nontrivial applications, it is no wonder that they are the focus of much contemporary interdisciplinary interest.

Evolutionary equations have already been used to model ultrasonic transducer devices. The first effort to model ultrasonic transducers with this theory is contained in [MPTW16]. There, the authors constructed a coupled thermo-piezo-electromagnetic system and, among other objectives, addressed the question of its well-posedness under a set of homogeneous boundary conditions. The inclusion of a thermal aspect in the coupled system of [MPTW16] is by no means a trivial one, as it provides some motivation to consider the behaviour of an ultrasonic instrument under a high-temperature regime. Various industrial applications necessitate the usage of piezo-electric transducers at incredibly high temperatures. For instance, some nuclear power plants are cooled by heavy liquid metals including sodium, lead-bismuth and lead, which have melting temperatures of 97.99 °C, 123.5 °C and 327.5 °C, respectively (c.f. [KV21], [GPP+09] and [TPVV16]). Such considerations are of crucial importance to the fabrication of ultrasonic devices for use in non-destructive testing. This is because the piezo-electric transducer of interest could become damaged after continuous use under strenuous thermal conditions. This is particularly true for the use of piezo-electric transducers in the manufacturing process of molten materials including plastic. In this scenario, a piezo-electric transducer would be required to measure properties like liquid density at temperatures over 220 °C (c.f. [KSR+15] and [KŠM+13]). Besides adversely impacting on the accuracy of any measurement readings, the failure to adequately accommodate for a high-temperature regime can drive up production costs. Further references detailing the usage of ultrasonic transducers at high temperatures abound, and can be found for instance in [KMC+99], [KV21], [BPP79], [FWW89], [SKC07], [OJMS05], [HPH03] and [JLP00].

A second effort to model a piezo-electric transducer with the theory of evolutionary equations can be found in [Pic17]. This article focused on a coupled piezo-electric system without the influence of temperature. However, it did implement some complicated and interesting piezo-electromagnetic boundary dynamics, which we now recall. Presented in their original formulation (c.f. [AN11, Section 1] or [Pic17, Subsection 4.3.1]), the following piezo-electromagnetic *impedance boundary conditions*

$$\begin{aligned} n \times H - n \times \widetilde{Q}^{*}\partial_t u + n \times (E \times n) &= 0 \text{ on } \partial\Omega, \\ T \cdot n - \widetilde{Q}\,(n \times E) + \left(1 + \widetilde{\alpha}\partial_t^{-1}\right)\partial_t u &= 0 \text{ on } \partial\Omega, \end{aligned} \tag{1.2}$$

were considered. We clarify the meaning of these boundary conditions before discussing the meaning of an impedance boundary condition. Broadly speaking (see Chapter 4 for the precise details) E and H are the electric and magnetic fields, respectively, whereas u is the displacement of the underlying elastic body, Ω, and T is a suitable realisation of the accompanying stress tensor. Here, n is the outward unit normal, whereas $\widetilde{Q}$ and $\widetilde{\alpha}$ are given (continuous and linear) boundary mappings with

$$\widetilde{Q} : V_{\gamma_t} \to H^{1/2}(\partial\Omega)^3 \quad \text{and} \quad \widetilde{\alpha} : H^{1/2}(\partial\Omega)^3 \to H^{1/2}(\partial\Omega)^3.$$

The boundary traces and spaces V_{γ_t} and $H^{1/2}(\partial\Omega)^3$ are later recalled and examined in detail (c.f. Proposition 3.1.6), but in essence allow us to translate different types of boundary data between both elastic and electromagnetic parts of the system. Specific regularity assumptions are made in [AN11, Section 2, p. 4] which ensure that the boundary conditions (1.2) are well-defined as equations on $L_2(\partial\Omega)$.

Impedance boundary conditions are also known as *Leontovich boundary conditions*, with much of the initial groundwork being set by M. Leontovich in the early 1940s (c.f. [Leo44]). At its core, an impedance boundary condition is one which establishes a relationship between the tangential components of the electric and magnetic fields. The exact nuance of this connection is underpinned by an impedance function or operator, which depends on the electromagnetic properties of the underlying material (c.f. [Sen60, Section 1]). Leontovich boundary conditions are frequently implemented to more readily solve problems in electromagnetic scattering (see for instance [Ure14], [DAOC23] and [CC23]). More specifically, they enable one to ignore any internal complexity of the material being studied. Instead, one needs only to determine the electromagnetic fields on the surface of the medium (c.f. [Moh82] and [Hop95]). The inclusion of an impedance type boundary condition often looks to complicate the formulation of a problem. The specific impedance boundary conditions recalled above in (1.2) are mathematically stimulating since they are given as a separate PDE, posed on the boundary of the domain. For more details on electromagnetic impedance boundary conditions and the classical impedance operator, consult [LY12, Equation 1.40], [BK15, Subsection 4.1.5.1], [YI18, Chapter 1] and [BL22, Equation 1.31].

In order to accommodate for such complicated boundary dynamics, the author of [Pic17] deigned to use the mathematical framework provided by *abstract boundary data spaces* (c.f. [PTW16, Section 5.2], [PTW14, Section 4] and [Tro14, Subsection 2.2, Section 4]). This theory provides a means of treating boundary value problems which bypasses the need to assume any regularity of the boundary. This enables one to consider boundary value problems for arbitrary open sets. Far from being a purely academic exercise, the author of [Pic17] noted how their use of abstract boundary data space theory meant that the modelling of ultrasonic transducers with a fractal geometry (c.f. [OMOH08a], [MW11], [MMO+11], [AM15] and [BAM16]) was also covered by their system.

1.3 Research Aims

The overall goal of this thesis is to unite the disparate trajectories outlined in the previous section, and advance the modelling of ultrasonic transducers by evolutionary equations. More precisely, the aims of this thesis are to:

1. Extend the thermo-piezo-electromagnetic system used to model ultrasonic transducers presented in [MPTW16]. This will be achieved by applying the methodology and outlook of [PSTW16] and [Pic17]. In doing so, it is our aim to construct a model for ultrasonic transducers which takes into account *both* boundary dynamics (including impedance boundary conditions) and a high-temperature regime.

2. Identify those patterns of well-posed boundary behaviour covered by our extended system of thermo-piezo-electromagnetism. We will collate our findings into a catalogue, in an effort to make the identification of well-posed patterns of boundary dynamics clear and accessible to a broad audience.

3. Abstract from the piezo-electric impedance boundary conditions (1.2) to obtain a generalised impedance like boundary condition. We will also construct a novel example of nontrivial dynamic boundary conditions. These will involve the influence of heat at the boundary as well as the piezo-electromagnetic impedance effect of (1.2). As such, the example we have in mind will draw upon each of the thermal, elastic and electromagnetic aspects of our extended system.

As noted, the underlying example of impedance boundary conditions, (1.2), are of key interest to us. Indeed, they are interesting from a mathematical standpoint on account of their complexity, appearing as a separate PDE on the boundary. Moreover, there is very little in the way of literature which combines any notion of electromagnetic, elastic and thermal impedance. Indeed, the notion of thermal impedance exists, and its derivation mirrors that of electromagnetic impedance. Although, from applications it seems as if the form of any thermal impedance function is little more than a variation on the same scalar quotient (c.f. [BJRV$^+$21] and the references therein). As far as the author is aware, there is nothing in the way of literature which addresses thermal impedance like boundary conditions involving a combination of Dirichlet and normal componential boundary data. As such, we are influenced by the shape of the underlying piezo-electromagnetic boundary dynamics from [AN11, Section 1], and driven to formulate and consider our own novel, generalised impedance like boundary conditions.

We will employ a nonstandard approach to our modelling efforts. It is common to first construct a model to a given problem before embarking on its thorough analysis. We, however, are not setting out with a specific problem in mind. Instead, our aim is to build an abstract model and derive assumptions on the material parameters required to obtain a well-posed system. By doing so, we will divine the form that well-posed problems might assume. In some way then, our route to modelling travels in the direction opposite to that which is more frequently pursued. We perform the analysis first before asking what specific phenomenon our system might model.

1.4 Outline of Thesis

The thesis is organised as follows. We begin in Chapter 2 with a concise tour of the essential theory of evolutionary equations. Our aim is to recall the key results and mathematical constructions required to make sense of the prototypical evolutionary equation, (1.1). The first key idea is covered in Section 2.1 and focuses on the aforementioned realisation of the time derivative, which will be used throughout the setting. In Section 2.2 we recall how to make sense of functions of this operator, and thus justify the expression $M(\partial_t)$ appearing in (1.1). The spatial operators to appear in our study are then covered in Section 2.3. These operators will eventually be used to formulate our A in (1.1). Finally, in Section 2.4 we recall and re-present the central solution theory encapsulated by *Picard's Theorem*, which appears as Theorem 2.4.4 in this work.

Our recollection of the necessary preliminary material continues in Chapter 3. Starting with Section 3.1 we succinctly review the well-known classical theory of boundary traces and spaces. This will afford us a clearer recollection of their abstract analogues — the aforementioned abstract boundary data spaces — in Section 3.2. In Subsection 3.2.1 we compare both of the recalled classical and abstract perspectives on boundary traces, before setting the stage for their subsequent application in Subsection 3.2.2.

The aim of Chapter 4 is to formulate and present our extended model for thermo-piezo-electromagnetism with boundary dynamics and a high-temperature regime. In Section 4.1 we recall the last preliminaries on congruence transforms, which will be needed in the proof of our systems well-posedness. In Section 4.2 we recall and re-prove the well-posedness result for the underlying thermo-piezo-electromagnetic system with homogeneous boundary conditions from [MPTW16, Theorem 3.1]. In the current work, this is re-presented in Theorem 4.2.1. By doing so, we also establish the fundamental material parameters and constituent relations which will underpin our own extended model. Additionally, we will highlight key modelling assumptions which will provide a reference point when discussing our own model. In Section 4.3 we establish our extended model for thermo-piezo-electromagnetism with boundary dynamics and a high-temperature regime. The goal of Section 4.4 is to address the question of our systems well-posedness, and this we do in the proof of the central solution theory of this thesis, Theorem 4.4.6.

Chapter 5, is devoted to the catalogue of well-posed patterns of boundary behaviour mentioned in our research aims. The construction of this catalogue will follow a systematic investigation into those patterns of boundary behaviour which lead to a well-posed system. We first set the stage for our investigation in Section 5.1 with the recollection of some additional prerequisites. In Section 5.2 we consider the possible inclusion and recovery of Robin, Dirichlet and Neumann boundary conditions from within our model. In Sections 5.3 and 5.4 we continue our investigation with a thorough analysis of different and complicated patterns of boundary behaviour. We classify the corresponding variations in boundary dynamics by different *subcases*, and regard them in detail in Subsections 5.3.1 to 5.3.3 and Subsections 5.4.1 to 5.4.3, respectively. Finally, in Section 5.5 we extend the piezo-electric impedance boundary conditions (1.2) to a novel example of thermo-piezo-electromagnetic boundary dynamics. We will then address the question of well-posedness for this specific example.

The final Chapter 6 summarises the achievements of the thesis before concluding with an outline of several possible avenues of future research.

Chapter 2

A Brief Tour of Evolutionary Equations

The aim of this chapter is to collect the key components of the theory of evolutionary equations needed to establish and analyse our model for thermo-piezo-electro-magnetism. The main result of this chapter is *Picard's Theorem* (Theorem 2.4.4), which encapsulates the solution theory of evolutionary equations. On account of the open access publication of [STW22], we do not feel obliged to repeat the proofs of any of the results we now recall. The interested reader is invited to consult the proofs as indicated there and elsewhere. The seminal paper of Rainer Picard, [Pic09], is one such source that much of the material of this chapter can be traced back to. Other standard references for the field include [PM11] and [MPTW20]. For the most part we will follow the presentation of [STW22].

2.1 The Time Derivative

The first stop on our tour of evolutionary equations is the establishment of time differentiation as a normal operator. To that end we start by setting the mathematical stage of this thesis by recalling the Bochner–Lebesgue spaces. In what follows let (Ω, Σ, μ) be a σ-finite measure space. Standard references for these spaces include [Mik78], [DU77], [Yos95] as well as the more contemporary [ABHN11, Chapter 1] and [STW22, Section 3.1]. With the help of a particular Bochner–Lebesgue space, we will realise the time derivative as a normal operator. Sources for the time derivative in this context include [Pic89] and the more recent [STW22, Section 3.2]. We begin with two definitions (c.f. [STW22, Definitions p. 31]).

Definition 2.1.1. A function $f\colon \Omega \to X$ is called *simple* if $\operatorname{ran}(f)$ is finite and

$$f(t) = \sum_{x \in X} x \cdot \mathbb{1}_{A_{f,x}}(t)$$

where $A_{f,x} \coloneqq f^{-1}[\{x\}]$ is measurable and of finite measure if $x \in X \setminus \{0\}$. By $S(\mu; X)$ we denote the vector space of simple functions. $\Diamond$

Definition 2.1.2. A function $f\colon \Omega \to X$ is called *Bochner-measurable* if there exists a sequence $(f_n)_{n\in\mathbb{N}}$ in $S(\mu; X)$ such that $\lim_{n\to\infty} f_n(t) = f(t)$ for μ-almost every $t \in \Omega$. $\Diamond$

The next definition (c.f. [STW22, Definition p. 33]) introduces the spaces we alluded to above.

Definition 2.1.3. For $p \in [1, \infty]$ define

$$\mathcal{L}_p(\mu; X) \coloneqq \{f\colon \Omega \to X; f \text{ Bochner-measurable }, \|f\|_X \in \mathcal{L}_p(\mu)\}.$$

The *Bochner–Lebesgue* spaces are defined as $L_p(\mu; X) \coloneqq \mathcal{L}_p(\mu; X) /_{\sim}$ where $\sim$ denotes the equivalence relation of equality μ-almost everywhere. $\Diamond$

As a generalisation of the scalar Lebesgue spaces, it is not surprising that many familiar results prevail in the setting of Bochner–Lebesgue spaces (c.f. [Yos95], [DU77], [Mik78] as well as the accessible [ABHN11, Chapter 1]). When equipped with the norm defined by (c.f. [ABHN11, p. 14])

$$\|f\|_p \coloneqq \begin{cases} \left(\int_\Omega \|f(t)\|_X^p \,\mathrm{d}\mu(t)\right)^{\frac{1}{p}}, & \text{if } p < \infty, \\ \operatorname{ess-sup}_{t\in\Omega} \|f(t)\|_X, & \text{if } p = \infty, \end{cases}$$

the Bochner–Lebesgue spaces become normed vector spaces. For $p \in [1, \infty]$ they are in fact Banach spaces. Moreover, when $p = 2$ and $X = H$ is a Hilbert space, then so too is $L_2(\mu; H)$ with its inner product defined by

$$\langle f, g\rangle_{L_2(\mu;H)} \coloneqq \int_\Omega \langle f(t), g(t)\rangle_H \, \mathrm{d}\mu(t).$$

The proofs of the latter two claims follow in a manner directly analogous to that of their scalar counterparts (c.f. [STW22, Proposition 3.1.4]). Suppose now that $\nu \in \mathbb{R}$ and $A \in \mathcal{B}(\mathbb{R})$, the Borel-$\sigma$-algebra of $\mathbb{R}$. We introduce the measure

$$\mu_{2,\nu}(A) \coloneqq \int_A \mathrm{e}^{-2\nu t} \, \mathrm{d}\lambda(t)$$

and define the exponentially weighted space

$$L_{2,\nu}(\mathbb{R}; H) \coloneqq L_2(\mu_{2,\nu}; H).$$

It is not hard to see that the inclusion of a given function f in $L_{2,\nu}(\mathbb{R}; H)$ is granted if and only if both f is Bochner-measurable and

$$\int_\mathbb{R} \|f(t)\|_H^2 \, \mathrm{d}\mu_{2,\nu}(t) = \int_\mathbb{R} \|f(t)\|_H^2 \mathrm{e}^{-2\nu t} \, \mathrm{d}t$$

is finite. The space $L_{2,\nu}(\mathbb{R}; H)$ will allow us to realise the time derivative as a normal operator. To that end, we define a particular class of multiplication operators (c.f. [STW22, Definition p. 73]).

Definition 2.1.4. Let $V : \mathbb{R} \to \mathbb{K}$ be a measurable function. We define the *multiplication-by-V operator* as

$$\begin{aligned} V(\mathrm{m})\colon \operatorname{dom}(V(\mathrm{m})) \subseteq L_2(\mathbb{R}; H) &\to L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto V(t)f(t)) \end{aligned}$$

where

$$\operatorname{dom}(V(\mathrm{m})) \coloneqq \{f \in L_2(\mathbb{R}; H); (t \mapsto V(t)f(t)) \in L_2(\mathbb{R}; H)\}.$$

Moreover, whenever V is the identity operator on $\mathbb{R}$, then we will denote the operator $V(\mathrm{m})$ by m alone, and refer to it as the *multiplication-by-the-argument operator*. $\Diamond$

With this in mind, we continue by recalling the following proposition (c.f. [STW22, Corollary 3.2.5]) which deals with a specific multiplication operator.

Proposition 2.1.5. *Let $\nu \in \mathbb{R}$. The mapping*

$$\begin{aligned} \exp(-\nu\mathrm{m})\colon L_{2,\nu}(\mathbb{R};H) &\to L_2(\mathbb{R};H) \\ f &\mapsto \big(t \mapsto \mathrm{e}^{-\nu t} f(t)\big) \end{aligned}$$

is unitary.

Following the presentation of [STW22, Definitions pp. 43–44, 46], we introduce the time derivative operator via the definition of its inverse. In the following definition we denote by '$*$' the usual operation of convolution.

Definition 2.1.6. Let $\nu \neq 0$.

(i) We define the operator $I_\nu\colon L_{2,\nu}(\mathbb{R};H) \to L_{2,\nu}(\mathbb{R};H)$ by $I_\nu \coloneqq \mathbb{1}_{[0,\infty)}*$ if $\nu \in \mathbb{R}_{>0}$ and $I_\nu \coloneqq -\mathbb{1}_{(-\infty,0]}*$ if $\nu \in \mathbb{R}_{<0}$.

(ii) We define the *time derivative* on $L_{2,\nu}(\mathbb{R};H)$ by $\partial_{t,\nu} \coloneqq I_\nu^{-1}$.

(iii) Moreover, we define $\partial_{t,0} \coloneqq \exp(-\nu\mathrm{m})(\partial_{t,\nu} - \nu)\exp(-\nu\mathrm{m})^{-1}$. $\Diamond$

The properties of I_ν allow us to infer those of $\partial_{t,\nu}$. In particular, one readily obtains that $\|I_\nu\| \leq 1/|\nu|$, that I_ν is injective and that $C_\mathrm{c}^1(\mathbb{R};H) \subseteq \operatorname{ran}(I_\nu)$ (see [STW22, Proposition 3.2.3]). It then follows that $\partial_{t,\nu}$ is densely defined, closed and extends the action of the classical time derivative (for more details see [STW22, p. 45]). We could of course instead define the time derivative by a more rudimentary and standard means (c.f. [STW22, Proposition 4.1.1]).

Proposition 2.1.7. *Let $\nu \in \mathbb{R}$ and $f, g \in L_{2,\nu}(\mathbb{R};H)$. Then, $f \in \operatorname{dom}(\partial_{t,\nu})$ and $\partial_{t,\nu} f = g$ if and only if for all $\phi \in C_\mathrm{c}^\infty(\mathbb{R})$ we have*

$$-\int_\mathbb{R} \phi'(t) f(t)\,\mathrm{d}t = \int_\mathbb{R} \phi(t) g(t)\,\mathrm{d}t$$

for almost every $t \in \mathbb{R}$.

Before recalling the last result of this subsection, we first recall the formal definition of the adjoint of an unbounded linear operator (see [RS81, VIII.1, p. 252]).

Definition 2.1.8. Let $A\colon \operatorname{dom}(A) \subseteq H \to H$ be a densely defined linear operator on a Hilbert space H. Then, $y \in \operatorname{dom}(A^*)$ if and only if there exists $z \in H$ for all $x \in \operatorname{dom}(A)$ such that $\langle Ax, y\rangle = \langle x, z\rangle$. $\Diamond$

The next result (c.f. [STW22, Corollary 3.2.6]) establishes the most important properties of the time derivative.

Proposition 2.1.9. *Let $\nu \in \mathbb{R}$. Then, the adjoint of the time derivative is the operator $\partial_{t,\nu}^* = -\partial_{t,\nu} + 2\nu$. Moreover, $\partial_{t,\nu}$ is a normal operator and $\operatorname{Re} \partial_{t,\nu} = \nu$.*

The well-known spectral theorem for normal operators ensures that the time derivative is unitarily equivalent to a multiplication operator. The question as to the exact form of the unitary operator under which this equivalence is achieved provides us with the next stop on our tour.

2.2 The Fourier–Laplace Transformation and Material Law Operators

The (unitary) Fourier–Laplace transformation will allow us to obtain a spectral representation of the time derivative as a multiplication operator. By its usage, we will be able to assign a functional calculus to $\partial_{t,\nu}$. The class of functions we will apply to the time derivative is known as material law. The use of this class will allow us to encode physical properties within our eventual PDE system, and will encompass fundamental constitutive relations, specific material parameters and any underlying material couplings. The following use of the Fourier–Laplace transformation can be originally traced back to [Pic89], although, we will follow the presentation of [STW22, Sections 5.1 and 5.2]. The following material law and corresponding operator concepts can be found in [STW22, Section 5.3].

We recall the space $C_{\mathrm{b}}(\mathbb{R};H) \coloneqq \{f\colon \mathbb{R}\to H,\ \text{continuous and bounded}\}$ together with the $\|\cdot\|_\infty$-norm in preparation of the next definition (c.f. [STW22, Definition p. 67]).

Definition 2.2.1. Let H be a Hilbert space. We define the *Fourier transformation* on $L_2(\mathbb{R};H)$ as the unique continuous extension onto $L_2(\mathbb{R};H)$ of the operator

$$\begin{aligned}\mathcal{F}\colon L_1(\mathbb{R};H)\cap L_2(\mathbb{R};H) &\to C_{\mathrm{b}}(\mathbb{R};H)\\ f&\mapsto \left(s\mapsto \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-\mathrm{i}st} f(t)\,\mathrm{d}t\right).\end{aligned} \qquad \Diamond$$

We emphasise that $\mathcal{F}$ is unitary on $L_2(\mathbb{R};H)$ on account of *Plancherel's Theorem* (see [STW22, Theorem 5.1.4]). Armed with the above definition we now define the Fourier–Laplace transformation (c.f. [STW22, Definition p. 72]).

Definition 2.2.2. Let $\nu\in\mathbb{R}$. The *Fourier–Laplace transformation* is defined by

$$\begin{aligned}\mathcal{L}_\nu\colon L_{2,\nu}(\mathbb{R};H) &\to L_2(\mathbb{R};H)\\ f&\mapsto \mathcal{F}\exp(-\nu\mathrm{m})f.\end{aligned} \qquad \Diamond$$

As the composition of two unitary operators, it is clear that the Fourier–Laplace transformation is itself unitary. Moreover, for $\psi\in C_{\mathrm{c}}^\infty(\mathbb{R};H)\subseteq L_1(\mathbb{R};H)$ and $t\in\mathbb{R}$, we have

$$(\mathcal{L}_\nu\psi)(t) = \frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} \mathrm{e}^{-(\mathrm{i}t+\nu)s}\,\psi(s)\,\mathrm{d}s.$$

As such, the Fourier–Laplace transformation can be thought of as a shifted version of the Fourier transform recalled above. As was hinted at, the Fourier–Laplace transform crucially yields a spectral representation of the time derivative as a multiplication operator (c.f. [STW22, Theorem 5.2.3] and recall Definition 2.1.4).

Theorem 2.2.3. *Let $\nu\in\mathbb{R}$. Then $\partial_{t,\nu} = \mathcal{L}_\nu^*(\mathrm{im}+\nu)\mathcal{L}_\nu$ and $\sigma(\partial_{t,\nu}) = \{\mathrm{i}t+\nu\colon t\in\mathbb{R}\}$.*

In particular, the operator $\mathrm{im}+\nu$ acts as multiplication by $t\mapsto \mathrm{i}t+\nu$. We next define the function class of interest (c.f. [STW22, Definition, p. 74]).

Definition 2.2.4. We call a mapping $M\colon \operatorname{dom}(M)\subseteq\mathbb{C}\to L(H)$ a *material law* if

(i) $\operatorname{dom}(M) \subseteq \mathbb{C}$ open and M complex differentiable,

(ii) there exists some $\nu \in \mathbb{R}$ such that the half-plane $\mathbb{C}_{\mathrm{Re}>\nu} \subseteq \operatorname{dom}(M)$ and

$$\|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}} \coloneqq \sup_{z \in \mathbb{C}_{\mathrm{Re}>\nu}} \|M(z)\| < \infty.$$

Additionally, we define the *abscissa of boundedness of* M by

$$\mathrm{s_b}(M) \coloneqq \inf\{\nu \in \mathbb{R} \colon \mathbb{C}_{\mathrm{Re}>\nu} \subseteq \operatorname{dom}(M) \text{ and } \|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}} < \infty\}. \qquad \Diamond$$

We now apply the spectral representation of the time derivative to this function class. The resulting operator family, together with their key properties, is summarised in the next result (c.f. [Pic09, Theorem 2.10], [STW22, Proposition 5.3.2 and Proposition 8.1.4]).

Proposition 2.2.5. *Let $M \colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law and suppose that $\nu > \mathrm{s_b}(M)$. Then, the following statements hold true:*

(i) The operator

$$\begin{aligned} M(\mathrm{im} + \nu) \colon L_2(\mathbb{R}; H) &\to L_2(\mathbb{R}; H) \\ f &\mapsto (t \mapsto M(\mathrm{i}t + \nu) f(t)) \end{aligned}$$

is bounded.

(ii) The operator defined by

$$M(\partial_{t,\nu}) \coloneqq \mathcal{L}_\nu^* M(\mathrm{im} + \nu) \mathcal{L}_\nu$$

is continuous with $\|M(\partial_{t,\nu})\| \leq \|M\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}$.

(iii) The operator $M(\partial_{t,\nu})$ is causal *i.e. for all $a \in \mathbb{R}$ and for all $f, g \in L_{2,\nu}(\mathbb{R}; H)$ such that $f = g$ on $(-\infty, a]$, then $M(\partial_{t,\nu}) f = M(\partial_{t,\nu}) g$ on $(-\infty, a]$.*

(iv) The operator $M(\partial_{t,\nu})$ is autonomous *i.e. $M(\partial_{t,\nu}) \tau_h = \tau_h M(\partial_{t,\nu})$ for each $h \in \mathbb{R}$,*

where τ_h denotes the translation operator

$$\begin{aligned}\tau_h \colon L_{2,\nu}(\mathbb{R};H) &\to L_{2,\nu}(\mathbb{R};H)\\ f &\mapsto (t \mapsto f(t+h)).\end{aligned}$$

Remark 2.2.6. (i) The operator defined in the second item of Proposition 2.2.5 will be referred to in the sequel as *material law operator.*

(ii) As indicated in our introduction, causality (see also [Pic09, Definitions 2.8, 2.9]) is a notion which ensures the physical meaningfulness and relevancy of our solutions. Many physical processes exhibit and evolve according to some inherent direction of time [Wau15, Section 0]. As a mathematical notion, causality provides one such means of modelling this phenomenon. In essence, causality means that previous and current outputs do not depend on any future inputs [JP00, Section 2, p. 4]. It turns out that one cannot avoid holomorphy in the definition of a material law, and nor can it be exempted in the construction of causal operators (for more details see [STW22, Chapter 8] or [Wau16a, Chapter 2]). $\triangledown$

The next result (c.f. [STW22, Theorem 5.3.6]) demonstrates that the action of a material law operator is independent of the actual choice of ν. This result will help us to more readily prove the main result of this chapter in Section 2.4.

Theorem 2.2.7. *Let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law. Then, for $\nu, \mu > \mathrm{s_b}(M)$ and $f \in L_{2,\nu}(\mathbb{R};H) \cap L_{2,\mu}(\mathbb{R};H)$ we have $M(\partial_{t,\nu})f = M(\partial_{t,\mu})f$.*

2.3 Spatial Operators

We begin by defining the rudimentary spatial operators essential to our setting. We will later re-examine them when we come to consider boundary traces in Chapter 3 of this thesis. The standard definitions and results recalled here can be found in [STW22, Chapters 6 and 7]. A plethora of alternative sources for this material exist including the definitive references [Eva22], [Neč11], [Tem01], and [GR86].

2.3.1 Scalar Spatial Operators

We start by defining the scalar gradient, divergence and curl operators (c.f. [STW22, Definition p. 85]).

Definition 2.3.1. Let $\Omega \subseteq \mathbb{R}^d$ be open. We define the operators

$$\begin{aligned} \operatorname{grad}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega) \subseteq L_2(\Omega) &\to L_2(\Omega)^d \\ \phi &\mapsto (\partial_j \phi)_{j \in \{1,\dots,d\}} \quad \text{and} \\ \operatorname{div}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega)^d \subseteq L_2(\Omega)^d &\to L_2(\Omega) \\ (\phi_j)_{j \in \{1,\dots,d\}} &\mapsto \sum_{j \in \{1,\dots,d\}} \partial_j \phi_j \end{aligned}$$

and set

$$\operatorname{grad} \coloneqq -\operatorname{div}_{\mathrm{c}}^*, \quad \operatorname{div} \coloneqq -\operatorname{grad}_{\mathrm{c}}^*, \quad \operatorname{grad}_0 \coloneqq -\operatorname{div}^* \quad \text{and} \quad \operatorname{div}_0 \coloneqq -\operatorname{grad}^* .$$

Let $\Omega \subseteq \mathbb{R}^3$ be open. We define the operator

$$\begin{aligned} \operatorname{curl}_{\mathrm{c}} \colon C_{\mathrm{c}}^{\infty}(\Omega)^3 \subseteq L_2(\Omega)^3 &\to L_2(\Omega)^3 \\ (\phi_j)_{j \in \{1,2,3\}} &\mapsto \begin{pmatrix} \partial_2 \phi_3 - \partial_3 \phi_2 \\ \partial_3 \phi_1 - \partial_1 \phi_3 \\ \partial_1 \phi_2 - \partial_2 \phi_1 \end{pmatrix} \end{aligned}$$

and set

$$\operatorname{curl} \coloneqq \operatorname{curl}_{\mathrm{c}}^* \quad \text{and} \quad \operatorname{curl}_0 \coloneqq \operatorname{curl}^* . \qquad \Diamond$$

The next result establishes fundamental properties of these operators (c.f. [STW22, Proposition 6.1.1]).

Proposition 2.3.2. *The operators*

$$\operatorname{grad}, \quad \operatorname{grad}_0, \quad \operatorname{div}, \quad \operatorname{div}_0, \quad \operatorname{curl} \quad \textit{and} \quad \operatorname{curl}_0$$

are each densely defined and closed. As such, the domains of these operators each form *a Hilbert space when taken with their respective graph norm.*

Next, we clarify the notation we will use to specify the domains of these spatial operators. We define

$$\begin{aligned} H^1(\Omega) &\coloneqq \operatorname{dom}(\operatorname{grad}), \\ H^1_0(\Omega) &\coloneqq \operatorname{dom}(\operatorname{grad}_0), \\ H(\operatorname{div},\Omega) &\coloneqq \operatorname{dom}(\operatorname{div}), \\ H_0(\operatorname{div},\Omega) &\coloneqq \operatorname{dom}(\operatorname{div}_0), \\ H(\operatorname{curl},\Omega) &\coloneqq \operatorname{dom}(\operatorname{curl}) \quad \text{and} \\ H_0(\operatorname{curl},\Omega) &\coloneqq \operatorname{dom}(\operatorname{curl}_0). \end{aligned}$$

Each of these spaces will be familiar to the reader as the standard (spatial) Sobolev spaces. This point is made precise in the next result (c.f. [STW22, Theorem 6.1.2]).

Theorem 2.3.3. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Let $f \in L_2(\Omega)$, $g \in L_2(\Omega)^d$, $F \in L_2(\Omega)^3$ and $G \in L_2(\Omega)^3$. Then, the following statements hold true:*

(i) $f \in H^1(\Omega)$ *and* $g = \operatorname{grad} f$ *if and only if for all* $\phi \in C_c^\infty(\Omega)$ *and* $j \in \{1, \cdots, d\}$

$$-\int_\Omega f \cdot \partial_j \phi = \int_\Omega g_j \cdot \phi.$$

(ii) $f \in H^1_0(\Omega)$ *and* $g = \operatorname{grad}_0 f$ *if and only if there exists a sequence* $(f_k)_{k=1}^\infty$ *in* $C_c^\infty(\Omega)$ *such that* $f_k \to f$ *in* $L_2(\Omega)$ *and* $\operatorname{grad} f_k \to g$ *in* $L_2(\Omega)^d$ *as* $k \to \infty$.

(iii) $g \in H(\operatorname{div},\Omega)$ *and* $f = \operatorname{div} g$ *if and only if for all* $\phi \in C_c^\infty(\Omega)$

$$-\int_\Omega g \cdot \operatorname{grad} \phi = \int_\Omega f \cdot \phi.$$

(iv) $g \in H_0(\operatorname{div},\Omega)$ *and* $f = \operatorname{div}_0 g$ *if and only if there exists a sequence* $(g_k)_{k=1}^\infty$ *in* $C_c^\infty(\Omega)^d$ *such that* $g_k \to g$ *in* $L_2(\Omega)^d$ *and* $\operatorname{div} g_k \to f$ *in* $L_2(\Omega)$ *as* $k \to \infty$.

(v) $F \in H(\mathrm{curl}, \Omega)$ *and* $G = \mathrm{curl}\, F$ *if and only if for all* $\Phi \in C_c^\infty(\Omega)^3$

$$\int_\Omega F \cdot \mathrm{curl}\, \Phi = \int_\Omega G \cdot \Phi.$$

(vi) $F \in H_0(\mathrm{curl}, \Omega)$ *and* $G = \mathrm{curl}_0 F$ *if and only if there exists a sequence* $(F_k)_{k=1}^\infty$ *in* $C_c^\infty(\Omega)^3$ *such that* $F_k \to F$ *in* $L_2(\Omega)^3$ *and* $\mathrm{curl}\, F_k \to G$ *in* $L_2(\Omega)^3$ *as* $k \to \infty$.

Remark 2.3.4. (i) The second item in the statement of Theorem 2.3.3 highlights that $\mathrm{dom}(\mathrm{grad}_0)$ is precisely the closure of $C_c^\infty(\Omega)$ when computed with respect to the $H^1(\Omega)$-norm (i.e. the graph norm of grad c.f. [Eva22, Chapter 5, Theorem 2]). The fourth and sixth items above underline similar observations.

(ii) When $\partial\Omega$ enjoys sufficient regularity (e.g. when $\partial\Omega$ is Lipschitz) then membership of a function in either of $H_0^1(\Omega)$, $H_0(\mathrm{div}, \Omega)$ or $H_0(\mathrm{curl}, \Omega)$ necessitates the satisfaction of an appropriate homogeneous boundary condition (c.f. [Eva22, Chapter 5, Theorem 2] or [Tem01, Theorem 1.3]). More precisely, if $f \in H_0^1(\Omega)$ then f satisfies the Dirichlet boundary condition $f|_{\partial\Omega} = 0$ and thus vanishes at the boundary. Similarly, if $g \in H_0(\mathrm{div}, \Omega)$ then g satisfies the Neumann boundary condition $g|_{\partial\Omega} \cdot n = 0$ where n denotes the outward unit normal. In words, this means that the normal component of g vanishes at the boundary. If $F \in H_0(\mathrm{curl}, \Omega)$ then F satisfies the homogeneous boundary condition $F|_{\partial\Omega} \times n = 0$ with its tangential vector field vanishing at the boundary.

(iii) If Ω is assumed only to be an open subset of $\mathbb{R}^d$, without the imposition of any boundary regularity, then one can still construct each of the spatial operators $\mathrm{grad}_0, \mathrm{div}_0$ and curl_0 introduced above. On the one hand, that $H_0^1(\Omega)$, $H_0(\mathrm{div}, \Omega)$ and $H_0(\mathrm{curl}, \Omega)$ are the closures of the respective C_c^∞-type spaces continues to hold true. On the other hand, the homogeneous boundary conditions indicated in the last item cannot be realised without the imposition of sufficient boundary regularity. Indeed, for an arbitrary open subset $\Omega \subseteq \mathbb{R}^d$ the existence of the outward unit normal is not assured. In this case, the above homogeneous boundary conditions need to be taken in an abstract sense. This observation is the starting point for considering particular generalised boundary conditions, which do not require the imposition of any boundary regularity. We will revisit this in Chapter 3. $\triangledown$

2.3.2 The Symmetric Gradient and Row-wise Divergence

Elasticity theory necessitates certain extensions of the scalar gradient and divergence operators encountered above. Recall that $\mathbb{R}^{d\times d}$ denotes the space of real $d \times d$ square matrices. We start with two basic definitions (c.f. [STW22, Definition p. 103]).

Definition 2.3.5. We define the space of real symmetric square matrices by

$$\mathbb{R}^{d\times d}_{\mathrm{sym}} := \left\{ A \in \mathbb{R}^{d\times d} \colon A = A^T \right\} \subseteq \mathbb{R}^{d\times d}.$$

The subspace $\mathbb{R}^{d\times d}_{\mathrm{sym}} \subseteq \mathbb{R}^{d\times d}$ is closed. $\Diamond$

Definition 2.3.6. Let $\Omega \subseteq \mathbb{R}^d$ be open. We define the space of real symmetric matrices of compactly supported continuously differentiable functions by

$$\begin{aligned} C^\infty_{\mathrm{c}}(\Omega)^{d\times d}_{\mathrm{sym}} &:= C^\infty_{\mathrm{c}}\left(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}\right) \\ &= \left\{ (\Phi_{jk})_{j,k\in\{1,\ldots,d\}} \in C^\infty_{\mathrm{c}}(\Omega)^{d\times d} : \forall j,k \in \{1,\ldots,d\}\,,\ \Phi_{jk} = \Phi_{kj} \right\} \end{aligned}$$

and the space of real symmetric matrices of square-integrable functions by

$$\begin{aligned} L_2(\Omega)^{d\times d}_{\mathrm{sym}} &:= L_2\left(\Omega; \mathbb{R}^{d\times d}_{\mathrm{sym}}\right) \\ &= \left\{ (\Phi_{jk})_{j,k\in\{1,\ldots,d\}} \in L_2(\Omega)^{d\times d} : \forall j,k \in \{1,\ldots,d\}\,,\ \Phi_{jk} = \Phi_{kj} \right\}. \end{aligned}$$ $\Diamond$

We can now define the operators of interest (c.f. [STW22, Definition p. 104]).

Definition 2.3.7. Let $\Omega \subseteq \mathbb{R}^d$ be open. We define the *symmetric gradient* as

$$\begin{aligned} \mathrm{Grad}_{\mathrm{c}} \colon C^\infty_{\mathrm{c}}(\Omega)^d \subseteq L_2(\Omega)^d &\to L_2(\Omega)^{d\times d}_{\mathrm{sym}} \\ (\phi_j)_{j\in\{1,\ldots d\}} &\mapsto \frac{1}{2}\left(\partial_k\phi_j + \partial_j\phi_k\right)_{j,k\in\{1,\ldots d\}} \end{aligned}$$

and the *row-wise divergence* as

$$\begin{aligned}\operatorname{Div}_{\mathrm{c}}\colon C_{\mathrm{c}}^{\infty}\left(\Omega\right)_{\mathrm{sym}}^{d\times d} \subseteq L_2\left(\Omega\right)_{\mathrm{sym}}^{d\times d} &\to L_2\left(\Omega\right)^{d} \\ \left(\Phi_{jk}\right)_{j,k\in\{1,\ldots d\}} &\mapsto \left(\sum_{k=1}^{d}\partial_k\Phi_{jk}\right)_{j\in\{1,\ldots d\}}.\end{aligned}$$

Moreover, we define

$$\operatorname{Grad} \coloneqq -\operatorname{Div}_{\mathrm{c}}^{*}, \quad \operatorname{Div} \coloneqq -\operatorname{Grad}_{\mathrm{c}}^{*}, \quad \operatorname{Grad}_0 \coloneqq -\operatorname{Div}^{*} \quad \text{and} \quad \operatorname{Div}_0 \coloneqq -\operatorname{Grad}^{*}. \quad \lozenge$$

The next result establishes the basic properties of these extended spatial operators. In both statement and proof, it is entirely analogous to Proposition 2.3.2.

Proposition 2.3.8. *The operators*

$$\operatorname{Grad}, \quad \operatorname{Grad}_0, \quad \operatorname{Div}, \quad \textit{and} \quad \operatorname{Div}_0$$

are each densely defined and closed. As such, the domains of these operators each form a Hilbert space when taken with their respective graph norm.

As for the domains of these extended operators we define

$$\begin{aligned} H(\operatorname{Grad},\Omega) &\coloneqq \operatorname{dom}(\operatorname{Grad}), \\ H_0(\operatorname{Grad},\Omega) &\coloneqq \operatorname{dom}(\operatorname{Grad}_0), \\ H(\operatorname{Div},\Omega) &\coloneqq \operatorname{dom}(\operatorname{Div}) \quad \text{and} \\ H_0(\operatorname{Div},\Omega) &\coloneqq \operatorname{dom}(\operatorname{Div}_0). \end{aligned}$$

The inclusion of a function in either of $H_0(\operatorname{Grad},\Omega)$ or $H_0(\operatorname{Div},\Omega)$ requires the satisfaction of a suitable (abstract) homogeneous boundary condition (c.f. Remark 2.3.4).

2.4 Solution Theory

Before re-presenting the central well-posedness result of evolutionary equations, *Picard's Theorem* (Theorem 2.4.4), we need to make some additional preparations. We first recall the following result (c.f. [STW22, Exercise 6.5]), which we will use in the proof of Theorem 2.4.4.

Proposition 2.4.1. *Let H be a Hilbert space and $A\colon \operatorname{dom}(A) \subseteq H \to H$ be a skew-selfadjoint linear operator. Additionally, let $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be holomorphic and, for $z \in \operatorname{dom}(M)$ and some $c \in \mathbb{R}_{>0}$, such that $\operatorname{Re} M(z) \geq c$. Then, the mapping $\operatorname{dom}(M) \ni z \mapsto (M(z) + A)^{-1}$ is also holomorphic.*

For what follows, we will also require the following auxiliary result which is redolent of the well-known lemma of Lax–Milgram (see [STW22, Proposition 6.3.1 and Remark 6.3.2]).

Proposition 2.4.2. *Let H be a Hilbert space and $B\colon \operatorname{dom}(B) \subseteq H \to H$ be a densely defined and closed linear operator such that $\operatorname{dom}(B^*) \subseteq \operatorname{dom}(B)$. Assume that there exists $c \in \mathbb{R}_{>0}$ such that for all $\phi \in \operatorname{dom}(B)$ we have $\operatorname{Re}\langle \phi, B\phi\rangle_H \geq c\|\phi\|_H^2$. Then, $B^{-1} \in L(H)$ and $\|B^{-1}\| \leq 1/c$.*

Second, we clarify some notation and recall an additional result that we will use in the proof of Theorem 2.4.4. For a given Hilbert space H, $\nu \in \mathbb{R}$ and closed operator $A\colon \operatorname{dom}(A) \subseteq H \to H$ we define its corresponding lifted operator by

$$\begin{aligned} A_\mu \colon L_2(\mu; \operatorname{dom}(A)) \subseteq L_2(\mu; H) &\to L_2(\mu; H) \\ [\Omega \ni \omega \mapsto f(\omega) \in \operatorname{dom}(A)] &\mapsto [\Omega \ni \omega \mapsto Af(\omega) \in H]\,. \end{aligned}$$

The corresponding lifted operator is the extension of A to H-valued square-integrable functions. Moreover, for given σ-finite measure spaces $(\Omega_0, \Sigma_0, \mu_0)$ and $(\Omega_1, \Sigma_1, \mu_1)$ and unitary operator $\mathcal{F}\colon L_2(\mu_0) \to L_2(\mu_1)$, we define the (extended) measure-translating operator by

$$\mathcal{F}_H : L_2(\mu_0; H) \to L_2(\mu_1; H)$$
$$[\Omega_0 \ni \omega \mapsto f(\omega) \in H] \mapsto [\Omega_1 \ni \omega \mapsto \mathcal{F}f(\omega) \in H] .$$

With these preparations one readily obtains the following (c.f. [STW22, Exercise 6.2]).

Proposition 2.4.3. *Let H_0, H_1 be Hilbert spaces and $(\Omega_0, \Sigma_0, \mu_0), (\Omega_1, \Sigma_1, \mu_1)$ be σ-finite measure spaces. For $i \in \{0,1\}$ denote by A_{μ_i} the corresponding lifted operator and by $\mathcal{F}_{H_i}$ denote the measure-translating unitary operator. Then*

$$\mathcal{F}_{H_1} A_{\mu_0} \mathcal{F}^*_{H_0} = A_{\mu_1}.$$

In the sequel we will make no distinction between A and its corresponding lifted operator, A_μ. We will denote both by A, leaving it to context to reveal which is meant.

Our summary of evolutionary equations now culminates with a re-presentation of the main solution theory of the field (c.f. [Pic09, Solution Theory, p. 1770] or [STW22, Theorem 6.2.1]).

Theorem 2.4.4. *Let $\nu_0 \in \mathbb{R}$, H be a Hilbert space, $M\colon \operatorname{dom}(M) \subseteq \mathbb{C} \to L(H)$ be a material law such that $\mathrm{s_b}(M) < \nu_0$ and $A\colon \operatorname{dom}(A) \subseteq H \to H$ be skew-selfadjoint. For $z \in \mathbb{C}_{\operatorname{Re} \geq \nu_0}$ assume that $\operatorname{Re} zM(z) \geq c$ for some $c \in \mathbb{R}_{>0}$. Then, for all $\nu \geq \nu_0$ the operator $\partial_{t,\nu} M(\partial_{t,\nu}) + A$ is closable and*

$$S_\nu := \overline{(\partial_{t,\nu} M(\partial_{t,\nu}) + A)}^{-1} \in L\left(L_{2,\nu}(\mathbb{R}; H)\right).$$

Additionally, S_ν is such that $\|S_\nu\|_{L(L_{2,\nu}(\mathbb{R};H))} \leq 1/c$. Moreover, for all $F \in \operatorname{dom}(\partial_{t,\nu})$ we have $S_\nu F \in \operatorname{dom}(\partial_{t,\nu}) \cap \operatorname{dom}(A)$. Furthermore, S_ν is causal and eventually independent of ν i.e. for $\eta, \nu \geq \nu_0$ and $F \in L_{2,\nu}(\mathbb{R}; H) \cap L_{2,\eta}(\mathbb{R}; H)$ it follows that $S_\nu F = S_\eta F$.

Proof. Let $\nu \geq \nu_0$. First, we show that $\partial_{t,\nu} M(\partial_{t,\nu}) + A$ is closable for each choice of $\nu \geq \nu_0$. On account of unitary equivalence, we will instead establish that the operator $(\mathrm{im} +$

$\nu)M(\mathrm{im}+\nu)+A$ is closable. To that end, suppose $(u_n)_{n=1}^{\infty}$ in $\mathrm{dom}((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A)$ and $f \in L_2(\mathbb{R};H)$ such that $u_n \to 0$ and $((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A)u_n \to f$ in $L_2(\mathbb{R};H)$ as $n \to \infty$. For arbitrary $R \in \mathbb{R}_{>0}$, we have $\mathbb{1}_{[-R,R]}u_n \in \mathrm{dom}((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A)$. Moreover,

$$((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A)\mathbb{1}_{[-R,R]}u_n = \mathbb{1}_{[-R,R]}((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A)u_n$$

converges to $\mathbb{1}_{[-R,R]}f$ as $n \to \infty$. Compact support ensures that the operator $(\mathrm{im}+\nu)M(\mathrm{im}+\nu))$ is bounded on $L_2([-R,R];H)$. Thus $(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A$ is closed on $L_2([-R,R];H)$ and it follows that $\mathbb{1}_{[-R,R]}f = 0$. As $R \in \mathbb{R}_{>0}$ was chosen arbitrarily, it follows that $f = 0$. As such, the operator $(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A$ is closable on $L_2(\mathbb{R};H)$. Now, let $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$ and define $B(z) \coloneqq zM(z)+A$. It then follows that

$$\mathrm{dom}(B(z)) = \mathrm{dom}(zM(z)) \cap \mathrm{dom}(A) = \mathrm{dom}(A)$$

since we have assumed M to be a material law. As we have assumed A to be skew-selfadjoint, it follows that A^* is a densely defined linear operator (for details see [STW22, Lemma 2.2.7]). Additionally, this assumption allows us to compute $B(z)^* = (zM(z))^* - A$ and obtain

$$\mathrm{dom}(B(z)^*) = \mathrm{dom}((zM(z))^*) \cap \mathrm{dom}(A) = \mathrm{dom}(A),$$

from which it follows that $\mathrm{dom}\,(B(z)^*) = \mathrm{dom}\,(B(z))$. As such, $B(z)$ must also be densely defined and closed. Next, let $\phi \in \mathrm{dom}(B(z))$ and use the remaining statement assumption to compute

$$\mathrm{Re}\,\langle\phi, B(z)\phi\rangle_H = \mathrm{Re}\,\langle\phi, zM(z)\phi\rangle_H \geq c\langle\phi,\phi\rangle_H.$$

By Proposition 2.4.2, it follows that $B(z)^{-1} \in L(H)$ with $\|B(z)^{-1}\| \leq 1/c$. As such, the mapping

$$S\colon \mathbb{C}_{\mathrm{Re}\geq\nu} \ni z \mapsto B\,(z)^{-1}$$

assumes values in $L(H)$. Moreover, S is holomorphic (c.f. Proposition 2.4.1) and is hence also a material law with $\|S(\partial_{t,\nu})\| \leq 1/c$. Furthermore, $S(\partial_{t,\nu})$ is causal (c.f. Item (iii) of Proposition 2.2.5) and independent of the particular choice of ν (c.f. Theorem 2.2.7). Next, instead of showing that $S\left(\partial_{t,\nu}\right) = \overline{\left(\partial_{t,\nu}M(\partial_{t,\nu}) + A\right)}^{-1}$, we will instead equivalently (c.f. Proposition 2.4.3) show that

$$S\left(\mathrm{im}+\nu\right) = \overline{\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\right)}^{-1}.$$

We first show the inclusion $S\left(\mathrm{im}+\nu\right) \supseteq \overline{\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\right)}^{-1}$. Let $(f,u) \in \overline{\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\right)}^{-1}$. Since $\overline{\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\right)}^{-1}$ is closed, there exists a sequence $((f_n,u_n))_{n=1}^{\infty}$ in $((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A)^{-1}$ such that $(f_n,u_n) \to (f,u)$ in $L_{2,\nu}(\mathbb{R};H)^2$ as $n \to \infty$. For $n \in \mathbb{N}$ we have

$$f_n = (\mathrm{im}+\nu)M(\mathrm{im}+\nu)u_n + Au_n = B\left(\mathrm{im}+\nu\right)u_n,$$

from which it follows that $(f_n,u_n) \in B\left(\mathrm{im}+\nu\right)^{-1} = S\left(\mathrm{im}+\nu\right)$. Since $S(\mathrm{im}+\nu)$ is closed, it then follows that $\lim_{n\to\infty}(f_n,u_n) = (f,u) \in S(\mathrm{im}+\nu)$ which yields the claim. Now consider the remaining inclusion, $S(\mathrm{im}+\nu) \subseteq \overline{\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\right)}^{-1}$. Let $n \in \mathbb{N}$, $f \in L_2(\mathbb{R};H)$ and define

$$g_n(\cdot) \coloneqq S\left(\mathrm{im}+\nu\right)\mathbb{1}_{[-n,n]}(\cdot)f(\cdot).$$

As $S\left(\mathrm{im}+\nu\right)$ is continuous, it is clear that $\lim_{n\to\infty} g_n = S\left(\mathrm{im}+\nu\right)f$. Moreover, for $n \in \mathbb{N}$ we have $g_n \in \mathrm{dom}\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)\right)$. To see this, notice that $M(\mathrm{im}+\nu)g_n \in L_2(\mathbb{R};H)$ as $M(\mathrm{im}+\nu) \in L\left(L_2(\mathbb{R};H)\right)$. Recall that the operator $(\mathrm{im}+\nu)$ is a priori unbounded since the induced multiplication factor $(\mathrm{i}t+\nu)$ is unbounded in the limit as $t \to \infty$. As such, we need to confirm that

$$(\mathrm{im}+\nu)M(\mathrm{im}+\nu)g_n = \left[t \mapsto (\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)S\left(\mathrm{i}t+\nu\right)\mathbb{1}_{[-n,n]}(t)f(t)\right]$$

is well-defined in $L_2(\mathbb{R};H)$. To that end we compute

$$\begin{aligned}
&\int_{\mathbb{R}} \left|(\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)S\left(\mathrm{i}t+\nu\right)\mathbb{1}_{[-n,n]}(t)f(t)\right|^2 \mathrm{d}t \\
&= \int_{[-n,n]} \left|(\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)S\left(\mathrm{i}t+\nu\right)f(t)\right|^2 \mathrm{d}t,
\end{aligned}$$

which is finite on account of compact support. It also follows immediately from the definition of $S(\mathrm{im}+\nu)$ that $g_n(t) = S(\mathrm{i}t+\nu)\mathbb{1}_{[-n,n]}(t)f(t) \in \mathrm{dom}(A)$ for almost every $t \in \mathbb{R}$. With this in mind, we next establish that $g_n \in L_2\left(\mathbb{R};\mathrm{dom}(A)\right)$. We compute

$$\begin{aligned}
&\int_{\mathbb{R}} \|(\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)g_n(t) + Ag_n(t)\|_H^2 \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \|\left((\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)+A\right)g_n(t)\|_H^2 \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \|\left((\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)+A\right)S\left(\mathrm{i}t+\nu\right)\mathbb{1}_{[-n,n]}(t)f(t)\|_H^2 \,\mathrm{d}t \\
&= \int_{\mathbb{R}} \|\mathbb{1}_{[-n,n]}(t)f(t)\|_H^2 \,\mathrm{d}t \\
&\leq \|f\|_{L_2(\mathbb{R};H)}^2
\end{aligned}$$

from which it follows a posteriori that $g_n \in L_2(\mathbb{R};\mathrm{dom}(A))$. Putting this all together, we have $g_n \in \mathrm{dom}((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A)$ as well as that

$$\left((\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A\right)g_n(\cdot) = \mathbb{1}_{[-n,n]}(\cdot)f(\cdot)$$

almost everywhere. On passing to the limit on both sides of this equality, we obtain

$$\overline{(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A}\,S(\mathrm{im}+\nu)f = f.$$

This is equivalent to $(S(\mathrm{im}+\nu)f, f) \in \overline{(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A}$, which is in turn equivalent to $(f, S(\mathrm{im}+\nu)f) \in \overline{(\mathrm{im}+\nu)M(\mathrm{im}+\nu)+A}^{-1}$. Finally, we show that if $f \in \mathrm{dom}(\partial_{t,\nu})$ then $S_\nu f \in \mathrm{dom}(\partial_{t,\nu}) \cap \mathrm{dom}(A)$. Let $f \in \mathrm{dom}(\partial_{t,\nu})$. Then, by definition of the operators involved, it follows a posteriori that $(\mathrm{im}+\nu)\mathcal{L}_\nu f \in L_2(\mathbb{R};H)$. For

$t \in \mathbb{R}$ we compute

$$\begin{aligned} AS(\mathrm{i}t+\nu)\mathcal{L}_\nu f(t) &= A\left((\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)+A\right)^{-1}\mathcal{L}_\nu f(t) \\ &= \mathcal{L}_\nu f(t) - (\mathrm{i}t+\nu)M(\mathrm{i}t+\nu)S(\mathrm{i}t+\nu)\mathcal{L}_\nu f(t) \end{aligned}$$

from which it follows a posteriori that $S(\mathrm{im}+\nu)\mathcal{L}_\nu f \in L_2(\mathbb{R};\mathrm{dom}(A))$. By Proposition 2.4.3 it then follows that $S(\partial_{t,\nu})f \in L_{2,\nu}(\mathbb{R};\mathrm{dom}(A))$. It follows similarly that $(\mathrm{im}+\nu)S(\mathrm{im}+\nu)\mathcal{L}_\nu f \in L_2(\mathbb{R};H)$, from which we deduce that $S(\partial_{t,\nu})f \in L_{2,\nu}(\mathbb{R};\mathrm{dom}(\partial_{t,\nu}))$ which shows the claim. □

Remark 2.4.5. (i) We specify some nomenclature and notation for what is to follow. Let H be a Hilbert space and $T \in L(H)$. If there exists some $c \in \mathbb{R}_{>0}$ such that $\mathrm{Re}\, T \geq c$, then we call T *positive-definite*. We will also refer to such a T as *accretive*. As such, we will refer to the condition $\mathrm{Re}\, T \geq c$ as the corresponding *positive-definiteness* or *accretivity* condition. If $T \in L(H)$ is additionally selfadjoint, then it is immediate that the accretivity condition simply reads $T \geq c$. Irrespective of whether T is selfadjoint or not, should the exact value of $c \in \mathbb{R}_{>0}$ prove irrelevant to us, we will instead write $T \gg 0$ for the positive-definiteness condition.

(ii) The accretivity condition in Theorem 2.4.4 could instead be formulated more rigorously as requiring $\mathrm{Re}\langle \phi, zM(z)\phi\rangle_H \geq c\|\phi\|_H^2$ for $\phi \in H$ and $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$ (see [STW22, Definition, p. 89]). ▽

Chapter 3

Classical and Abstract Trace Spaces

In this chapter we consider the problem of boundary traces. We begin by recapping the well established theory for classical traces in the case of a bounded Lipschitz domain, $\Omega \subseteq \mathbb{R}^d$. This will allow us to properly motivate and introduce abstract boundary trace spaces which will prove to be an incredibly useful tool in what follows for the remainder of this thesis. In particular, they will allow us to formulate and discuss boundary value problems for arbitrary *open* subsets, $\Omega \subseteq \mathbb{R}^d$.

3.1 Classical Trace Spaces

In this section we recall the classical boundary traces for the spatial operators considered in Section 2.3. Even though we will formulate our thermo-piezo-electromagnetic model with tools from an abstract boundary trace theory, the classical perspective still serves as an important motivation. The notions and results recalled here are entirely standard and already familiar from the study of PDEs. As such, most of the results recalled here are done so without proof. Definitive references for this material include [Neč11], [Tar07], [Eva22, Section 5.5], [Tem01, Chapters 1, 2], [Soh12, Chapter 1] and [GR86, Chapters 1, 2]. Useful summaries of the key ideas used here can be found, for example, in [ABDG98, Section 2] and [TW09, Section 13.6]. For boundary traces relating to $H(\mathrm{curl}, \Omega)$, the key reference is [BCS02]. However, there are also the accessible presentations afforded by [PSTW16, Section 2] and [WS13, Section 4]. For boundary traces relating to the gradient and divergence, we base our present presentation in particular on that of the contemporary references [GR86], [Tem01] and [KA03].

We begin by recalling a density result (c.f. [GR86, Theorem 2.4], [Tem01, Theorem 1.1], [KA03, Theorem 3.6] or [STW22, Theorems 12.1.1, 12.2.1]).

Proposition 3.1.1. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. The set*

$$\mathcal{D} := \left\{ \phi \colon \Omega \to \mathbb{R} \colon \exists\, \psi \in C_c^\infty(\mathbb{R}^d) \colon \psi|_\Omega = \phi \right\}$$

is dense in $H^1(\Omega)$. Moreover, the set $\mathcal{D}^d$ is dense in $H(\mathrm{div}, \Omega)$.

We first recall the boundary trace of $H^1(\Omega)$ (c.f. [KA03, Theorem 3.6], [MQS21, Theorem A.12] or [BF12, Theorem III.2.19]).

Proposition 3.1.2. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. The operator*

$$\begin{aligned} \gamma \colon \mathcal{D} \subseteq H^1(\Omega) &\to L_2(\partial\Omega) \\ u &\mapsto u|_{\partial\Omega} \end{aligned}$$

is linear, densely defined and continuous. Thus, γ admits a unique continuous extension to $H^1(\Omega)$, again denoted by γ.

Remark 3.1.3. (i) The boundary trace of $H^1(\Omega)$ is more commonly referred to in the literature as the *Dirichlet trace*.

(ii) The Dirichlet trace (as presented) fails in general to be surjective. This is usually remedied by restricting its codomain to $\mathrm{ran}(\gamma) =: H^{1/2}(\partial\Omega)$. Doing so allows us to characterise the space of possible boundary values for $H^1(\Omega)$ functions. Indeed, it follows that $H^{1/2}(\partial\Omega)$ together with the norm

$$\|\gamma f\|_{H^{1/2}(\partial\Omega)} := \inf_{g \in H^1(\Omega)} \left\{ \|g\|_{H^1(\Omega)} \colon \gamma g = \gamma f \right\}$$

is isomorphic (by a quotient space argument for $H^1(\Omega)/\ker(\gamma)$) to $(\ker\gamma)^{\perp_{H^1(\Omega)}}$, and is thus itself a Hilbert space. For more details, see for instance [KA03, pp. 358–359].

(iii) It is not hard to see that $\ker(\gamma) = H_0^1(\Omega) = \mathrm{dom}(\mathrm{grad}_0)$. Indeed, this is precisely the space of those H^1-functions with vanishing trace, known also as trace-zero functions (c.f. Item (i) from Remark 2.3.4). $\triangledown$

In the following let n denote the outer unit normal. We next consider the boundary trace of $H(\mathrm{div}, \Omega)$, known more commonly as the *Neumann trace* (c.f. [KA03, p. 360, Theorem 6.13], [MQS21, Theorem A.14] or [BF12, p. 248]).

Proposition 3.1.4. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. The operator*

$$\gamma_{\cdot n}\colon \mathcal{D}^d \subseteq H\left(\mathrm{div}, \Omega\right) \to H^{1/2}\left(\partial\Omega\right)' =: H^{-1/2}\left(\partial\Omega\right)$$
$$q \mapsto \left(\gamma q\right) \cdot n$$

is linear, densely defined and continuous. Thus, $\gamma_{\cdot n}$ admits a unique continuous extension to $H(\mathrm{div}, \Omega)$, again denoted by $\gamma_{\cdot n}$. Additionally, $\gamma_{\cdot n}$ is surjective. Furthermore, for $f \in H^1(\Omega)$ and $q \in H(\mathrm{div}, \Omega)$ we have the integration by parts formula

$$\langle \mathrm{div}\, q, f\rangle_{L_2(\Omega)} + \langle q, \mathrm{grad}\, f\rangle_{L_2(\Omega)^d} = \left(\gamma_{\cdot n} q\right)\left(\gamma f\right). \tag{3.1}$$

It is well known that the boundary traces recalled for $H^1(\Omega)$ and $H(\mathrm{div}, \Omega)$ can even be used to establish suitable boundary traces for $H(\mathrm{Grad}, \Omega)$ and $H(\mathrm{Div}, \Omega)$ (c.f. [GSN86], [DD12, Chapter 7] or [BF12, pp. 248–249, Lemma IV.3.3]).

Proposition 3.1.5. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Then,*

$$H\left(\mathrm{Grad}, \Omega\right) \simeq H^1\left(\Omega\right)^d \quad \textit{and} \quad H\left(\mathrm{Div}, \Omega\right) \simeq H\left(\mathrm{div}, \Omega\right)^d \cap L_2\left(\Omega\right)^{d\times d}_{\mathrm{sym}} \subseteq H\left(\mathrm{div}, \Omega\right)^d.$$

Thus, the (d-dimensional) Dirichlet trace

$$\begin{aligned} \gamma\colon H\left(\mathrm{Grad},\Omega\right) &\to H^{1/2}\left(\partial\Omega\right)^d \\ (u_i)_{i=1}^d &\mapsto (u_i|_{\partial\Omega})_{i=1}^d \end{aligned}$$

is linear, continuous and surjective. Moreover, the (d-dimensional) Neumann trace

$$\begin{aligned} \gamma_{\cdot n}\colon H\left(\mathrm{Div},\Omega\right) &\to H^{-1/2}\left(\partial\Omega\right)^d \\ (\Phi_{i,j})_{i,j=1}^d &\mapsto \left(\left(\gamma\Phi_{i,j}\right)\cdot n\right)_{i,j=1}^d \end{aligned}$$

is linear, continuous and surjective. Furthermore, for $f \in H(\mathrm{Grad},\Omega)$ *and* $q \in H(\mathrm{Div},\Omega)$ *we have the integration by parts formula*

$$\langle \mathrm{Div}\, q, f\rangle_{L_2(\Omega)^d} + \langle q, \mathrm{Grad}\, f\rangle_{L_2(\Omega)^{d\times d}_{\mathrm{sym}}} = \sum_{i=1}^{d} \left(\gamma_{\cdot n} q\right)\left(\gamma f\right). \tag{3.2}$$

In preparation for the last boundary traces we will require, we introduce the space

$$L_2^\tau\left(\partial\Omega\right) \coloneqq \left\{f \in L_2\left(\partial\Omega\right)^3 : f\cdot n = 0\right\}$$

of tangential vector fields on the boundary, $\partial\Omega$. We consider two boundary traces of $H(\mathrm{curl},\Omega)$ (c.f. [PSTW16, Definition 2.15, Remark 2.4], [BCS02, Section 2] or [WS13, Section 4]).

Proposition 3.1.6. *Let* $\Omega \subseteq \mathbb{R}^3$ *be a bounded Lipschitz domain. We define the* tangential trace *and* tangential components trace *operators as*

$$\begin{aligned} \gamma_\tau\colon H^1\left(\Omega\right)^3 \subseteq H(\mathrm{curl},\Omega) &\to \mathrm{ran}(\gamma_\tau) \subseteq L_2^\tau\left(\partial\Omega\right) \\ q &\mapsto (\gamma q)\times n \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} \pi_\tau\colon H^1\left(\Omega\right)^3 \subseteq H(\mathrm{curl},\Omega) &\to \mathrm{ran}(\pi_\tau) \subseteq L_2^\tau\left(\partial\Omega\right) \\ q &\mapsto -n\times(n\times\gamma q), \end{aligned} \tag{3.4}$$

respectively. The image spaces $\operatorname{ran}(\gamma_\tau) =: V_\gamma$ *and* $\operatorname{ran}(\pi_\tau) =: V_\pi$ *are Hilbert spaces when considered together with the respective norms*

$$\|q\|_{V_\gamma} := \inf_{p \in H^1(\Omega)^3} \left\{ \|\gamma_\tau p\|_{H^{1/2}(\partial\Omega)^3} : \gamma_\tau p = q \right\}$$

and

$$\|q\|_{V_\pi} := \inf_{p \in H^1(\Omega)^3} \left\{ \|\pi_\tau p\|_{H^{1/2}(\partial\Omega)^3} : \pi_\tau p = q \right\}.$$

Both γ_τ *and* π_τ *are linear, surjective, densely defined and continuous and thus can be uniquely and continuously extended to operators*

$$\gamma_\tau : H(\operatorname{curl}, \Omega) \to V_\pi' \quad \textit{and} \quad \pi_\tau : H(\operatorname{curl}, \Omega) \to V_\gamma'.$$

Furthermore, for $p, q \in H^1(\Omega)^3$ *we have the integration by parts formula*

$$\langle \operatorname{curl} p, q \rangle_{L_2(\Omega)^3} + \langle p, \operatorname{curl} q \rangle_{L_2(\Omega)^3} = \langle \pi_\tau p, \gamma_\tau q \rangle_{L_2^\tau(\partial\Omega)}. \tag{3.5}$$

3.2 Abstract Boundary Data Spaces

The notion of an abstract boundary trace space was first introduced in [PTW16, Section 5.2], and provides a means of bypassing boundary regularity requirements when addressing boundary value problems. This is of course in direct contrast to the classical situation recalled in the previous section, where one had to at least assume $\partial\Omega$ Lipschitz. We now recall the main ideas and results from the theory of abstract boundary data spaces, which we will employ when formulating our own model for thermo-piezo-electromagnetism with boundary dynamics. Indeed, the importance that these spaces hold for us cannot be understated. Whilst the ideas and results considered here can be originally traced back to [PTW16, Section 5.2], they were also more recently treated in [Pic17, Section 4.1] and [STW22, Chapter 12]. Useful summaries can also be found in [PTW14, Section 4] and [Tro14, Subsection 2.2, Section 4].

We start with the recollection of an elementary result, which we will use frequently in the sequel (c.f. [PTW15, Lemma 3.2], [STW22, Lemma 11.3.3]).

Lemma 3.2.1. *Let H be a Hilbert space and $V \subseteq H$ a closed subspace. Let the operator*

$$\begin{aligned} \iota_V : V &\to H \\ x &\mapsto x, \end{aligned}$$

denote the canonical embedding of V into H. Then, $\iota_V \iota_V^* : H \to H$ is the orthogonal projection on V and $\iota_V^* \iota_V : V \to V$ is the identity on V.

Proof. First determine the form of ι_V^*. Let $x \in H$ with decomposition $x = y + z$ for $y \in V$ and $z \in V^{\perp}$. Suppose $v \in V$ and compute

$$\langle \iota_V v, x \rangle_H = \langle v, y \rangle_H = \langle v, \pi_V x \rangle_H,$$

where $\pi_V : H \to H$ denotes the *genuine* orthogonal projection on V. The action of ι_V^* can thus be identified with that of π_V. We next determine $\iota_V \iota_V^*$ and compute

$$\iota_V \iota_V^* x = \iota_V \pi_V x = y.$$

As ι_V is the canonical embedding of V into H, this computation realises y as an element of H and establishes that $\iota_V \iota_V^*$ is well-defined as a mapping from H to H. Boundedness follows immediately with $\|\iota_V \iota_V^*\| \leq 1$. Let x_1, $x_2 \in H$ with decompositions $x_1 = y_1 + z_1$ and $x_2 = y_2 + z_2$ for $y_1, y_2 \in V$ and $z_1, z_2 \in V^{\perp}$. Then, compute

$$\langle \iota_V \iota_V^* x_1, x_2 \rangle_H = \langle y_1, y_2 + z_2 \rangle_H = \langle y_1, y_2 \rangle_H = \langle y_1 + z_1, y_2 \rangle_H = \langle x_1, \iota_V \iota_V^* x_2 \rangle_H.$$

Hence $\iota_V \iota_V^*$ is selfadjoint. Idempotency follows from direct computation. The claim for $\iota_V^* \iota_V : V \to V$ also follows immediately from direct computation. □

Remark 3.2.2. The use of *genuine* in the preceding proof might seem superfluous, but it is important to distinguish between the true orthogonal projection and an operator whose action merely coincides with it. Indeed an orthogonal projection is defined as a bounded, selfadjoint idempotent operator. One immediately encounters issues upon endeavouring to establish these properties for ι_V^*. ▽

We highlight this latter point with an elementary example.

Example 3.2.1. Let H_1, H_2 and H_3 be Hilbert spaces and denote by $\mathcal{H}$ the Hilbert space formed by their direct sum. Suppose $T \in L(\mathcal{H})$ with block operator representation

$$T = \begin{pmatrix} 0 & 0 & 0 \\ 0 & t_{22} & 0 \\ 0 & 0 & t_{33} \end{pmatrix}, \tag{3.6}$$

for selfadjoint and invertible $t_{22} \in L(H_2)$ and $t_{33} \in L(H_3)$. Clearly T is selfadjoint and we can decompose $\mathcal{H} = \ker(T) \oplus \operatorname{ran}(T)$, with $\ker(T) = H_1$ and $\operatorname{ran}(T) = H_2 \oplus H_3$. In this case, $\iota_{\operatorname{ran}(T)}$ is the operator

$$\begin{aligned} \iota_{\operatorname{ran}(T)} = \begin{pmatrix} 0 & 0 \\ 1_{H_2} & 0 \\ 0 & 1_{H_3} \end{pmatrix} &: \operatorname{ran}(T) \to \mathcal{H} \\ \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} &\mapsto \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \end{aligned} \tag{3.7}$$

where 1_{H_2} and 1_{H_3} are the identity operators on H_2 and H_3, respectively. Its adjoint is

$$\begin{aligned} \iota^*_{\operatorname{ran}(T)} = \begin{pmatrix} 0 & 1_{H_2} & 0 \\ 0 & 0 & 1_{H_3} \end{pmatrix} &: \mathcal{H} \to \operatorname{ran}(T) \\ \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} &\mapsto \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}. \end{aligned} \tag{3.8}$$

From here we determine

$$\iota^*_{\operatorname{ran}(T)} \iota_{\operatorname{ran}(T)} = \begin{pmatrix} 1_{H_2} & 0 \\ 0 & 1_{H_3} \end{pmatrix}, \tag{3.9}$$

which is indeed the identity operator on $\operatorname{ran}(T)$, and

$$\iota_{\operatorname{ran}(T)}\iota^*_{\operatorname{ran}(T)} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1_{H_2} & 0 \\ 0 & 0 & 1_{H_3} \end{pmatrix}, \tag{3.10}$$

which is the genuine orthogonal projector along $\operatorname{ran}(T)$ in $\mathcal{H}$. Whilst the action of $\iota^*_{\operatorname{ran}(T)}$ coincides with that of $\iota_{\operatorname{ran}(T)}\iota^*_{\operatorname{ran}(T)}$, the operator $\iota^*_{\operatorname{ran}(T)}$ is neither selfadjoint nor idempotent. In fact, it cannot be selfadjoint and fails to be idempotent since the product $\iota^*_{\operatorname{ran}(T)}\iota^*_{\operatorname{ran}(T)}$ is undefined. △

Recall the spatial operators introduced in Section 2.3 together with their respective domains. We begin by detailing the following orthogonal complements (c.f. [PTW16, Lemma 5.1] or [STW22, Proposition 12.2.4]).

Proposition 3.2.3. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Then*

$$\begin{aligned} H_0^1(\Omega)^{\perp_{H^1(\Omega)}} &= \left\{u \in H^1(\Omega)\colon \operatorname{grad} u \in \operatorname{dom}(\operatorname{div}),\ \operatorname{div}\operatorname{grad} u = u\right\}, \\ H_0(\operatorname{div},\Omega)^{\perp_{H(\operatorname{div},\Omega)}} &= \{q \in H(\operatorname{div},\Omega) :\ \operatorname{div} q \in \operatorname{dom}(\operatorname{grad}),\ \operatorname{grad}\operatorname{div} q = q\}, \\ H_0(\operatorname{Grad},\Omega)^{\perp_{H(\operatorname{Grad},\Omega)}} &= \{u \in H(\operatorname{Grad},\Omega)\colon \operatorname{Grad} u \in \operatorname{dom}(\operatorname{Div}),\ \operatorname{Div}\operatorname{Grad} u = u\}, \\ H_0(\operatorname{Div},\Omega)^{\perp_{H(\operatorname{Div},\Omega)}} &= \{q \in H(\operatorname{Div},\Omega) :\ \operatorname{Div} q \in \operatorname{dom}(\operatorname{Grad}),\ \operatorname{Grad}\operatorname{Div} q = q\} \end{aligned}$$

and when $d = 3$

$$H_0(\operatorname{curl},\Omega)^{\perp_{H(\operatorname{curl},\Omega)}} = \{q \in H(\operatorname{curl},\Omega) :\ \operatorname{curl} q \in H(\operatorname{curl},\Omega),\ -\operatorname{curl}\operatorname{curl} q = q\}.$$

Proof. We only compute $H_0(\operatorname{curl},\Omega)^{\perp_{H(\operatorname{curl},\Omega)}}$ as the other characterisations follow analogously. Suppose $q \in H_0(\operatorname{curl},\Omega)^{\perp_{H(\operatorname{curl},\Omega)}}$. By definition of the orthogonal complement, it follows for all $p \in H_0(\operatorname{curl},\Omega)$ that

$$0 = \langle p, q\rangle_{H(\operatorname{curl},\Omega)} = \langle p, q\rangle_{L_2(\Omega)^3} + \langle \operatorname{curl}_0 p, \operatorname{curl} q\rangle_{L_2(\Omega)^3}.$$

Here, we have recalled $\operatorname{curl}|_{H_0(\operatorname{curl},\Omega)} = \operatorname{curl}_0$. We reformulate the above equality for all $p \in H_0(\mathrm{curl}, \Omega)$ as

$$\langle \mathrm{curl}_0\, p, \mathrm{curl}\, q \rangle_{L_2(\Omega)^3} = -\langle p, q \rangle_{L_2(\Omega)^3}.$$

Recall that $y \in \mathrm{dom}\,(\mathrm{curl}_0^*)$ if and only if there exists $z \in L_2(\Omega)^3$ for all $x \in \mathrm{dom}\,(\mathrm{curl}_0)$ such that $\langle \mathrm{curl}_0\, x, y \rangle_{L_2(\Omega)^3} = \langle x, z \rangle_{L_2(\Omega)^3}$ (c.f. Definition 2.1.8). This implies that $\mathrm{curl}\, q = y \in \mathrm{dom}\,(\mathrm{curl}_0^*)$ and that $-q = \mathrm{curl}_0^*\, \mathrm{curl}\, q$. Recalling that curl is closed (c.f. Proposition 2.3.2), together with $\mathrm{curl}_0 \coloneqq \mathrm{curl}^*$ (c.f. Definition 2.3.1), allows us to deduce the claim. □

Remark 3.2.4. The spaces formed from the orthogonal complements in Proposition 3.2.3 are precisely our abstract boundary data spaces. We introduce the notation

$$\begin{aligned}
H_0^1(\Omega)^{\perp_{H^1(\Omega)}} &=: \mathrm{BD}\,(\mathrm{grad})\,, \\
H_0(\mathrm{div}, \Omega)^{\perp_{H(\mathrm{div},\Omega)}} &=: \mathrm{BD}\,(\mathrm{div})\,, \\
H_0(\mathrm{Grad}, \Omega)^{\perp_{H(\mathrm{Grad},\Omega)}} &=: \mathrm{BD}\,(\mathrm{Grad})\,, \\
H_0(\mathrm{Div}, \Omega)^{\perp_{H(\mathrm{Div},\Omega)}} &=: \mathrm{BD}\,(\mathrm{Div}) \quad \text{and} \\
H_0(\mathrm{curl}, \Omega)^{\perp_{H(\mathrm{curl},\Omega)}} &=: \mathrm{BD}\,(\mathrm{curl})
\end{aligned}$$

where "BD" is naturally suggestive of "boundary data". Using the characterisations provided by Proposition 3.2.3, we might equivalently regard our boundary data spaces as the null spaces

$$\begin{aligned}
\mathrm{BD}\,(\mathrm{grad}) &= N\,(1 - \mathrm{div}\,\mathrm{grad})\,, \\
\mathrm{BD}\,(\mathrm{div}) &= N\,(1 - \mathrm{grad}\,\mathrm{div})\,, \\
\mathrm{BD}\,(\mathrm{Grad}) &= N\,(1 - \mathrm{Div}\,\mathrm{Grad})\,, \\
\mathrm{BD}\,(\mathrm{Div}) &= N\,(1 - \mathrm{Grad}\,\mathrm{Div}) \quad \text{and} \\
\mathrm{BD}\,(\mathrm{curl}) &= N\,(1 + \mathrm{curl}\,\mathrm{curl})\,.
\end{aligned}$$

In the literature the use of this notation is common, for instance, to [PTW16] and [Pic17]. ▽

The next result summarises what happens when we restrict our spatial operators to their corresponding abstract boundary data space (c.f. [PTW16, Theorem 5.2] or [STW22, Proposition 12.4.1]).

Proposition 3.2.5. *The mappings*

$$\begin{aligned}
\operatorname{grad}_{\mathrm{BD}}\colon \ \mathrm{BD}(\operatorname{grad}) &\to \mathrm{BD}(\operatorname{div})\\
u &\mapsto \operatorname{grad} u,\\
\operatorname{div}_{\mathrm{BD}}\colon \ \mathrm{BD}(\operatorname{div}) &\to \mathrm{BD}(\operatorname{grad})\\
q &\mapsto \operatorname{div} q,\\
\operatorname{Grad}_{\mathrm{BD}}\colon \ \mathrm{BD}(\operatorname{Grad}) &\to \mathrm{BD}(\operatorname{Div})\\
u &\mapsto \operatorname{Grad} u,\\
\operatorname{Div}_{\mathrm{BD}}\colon \ \mathrm{BD}(\operatorname{Div}) &\to \mathrm{BD}(\operatorname{Grad})\\
q &\mapsto \operatorname{Div} q \qquad \textit{and}\\
\operatorname{curl}_{\mathrm{BD}}\colon \ \mathrm{BD}(\operatorname{curl}) &\to \mathrm{BD}(\operatorname{curl})\\
q &\mapsto \operatorname{curl} q
\end{aligned}$$

are unitary with $\operatorname{div}_{\mathrm{BD}}^{*} = \operatorname{grad}_{\mathrm{BD}}$, $\operatorname{Div}_{\mathrm{BD}}^{*} = \operatorname{Grad}_{\mathrm{BD}}$ *and* $\operatorname{curl}_{\mathrm{BD}}^{*} = -\operatorname{curl}_{\mathrm{BD}}$.

Proof. We prove the assertion for $\operatorname{curl}_{\mathrm{BD}}$ only as the other assertions follow from an analogous reasoning. We first show that $\operatorname{curl}_{\mathrm{BD}}$ is well-defined and assumes values in $\mathrm{BD}(\operatorname{curl})$. Suppose $q \in \mathrm{BD}(\operatorname{curl})$. By Proposition 3.2.3 it then follows that $\operatorname{curl} q \in \operatorname{dom}(\operatorname{curl})$ and that $-\operatorname{curl}\operatorname{curl} q = q$. Applying the operator curl to the latter equality yields

$$-\operatorname{curl}\operatorname{curl}\operatorname{curl} q = \operatorname{curl} q.$$

As the right-hand side resides in $H(\operatorname{curl},\Omega)$, it follows a posteriori that the left-hand side must reside there also. Thus, we deduce that $\operatorname{curl}\operatorname{curl} q \in H(\operatorname{curl},\Omega)$ also. Renaming $\hat{q} \coloneqq \operatorname{curl} q$, we obtain

$$-\operatorname{curl}\operatorname{curl}\hat{q} = \hat{q}$$

with $\operatorname{curl}\hat{q} \in H(\operatorname{curl},\Omega)$, which yields the claim. We next show that $\operatorname{curl}_{\mathrm{BD}}$ is unitary.

We first establish that $\operatorname{curl}_{\mathrm{BD}}$ preserves the norm. Let $q \in \mathrm{BD}\,(\operatorname{curl})$ and compute

$$\begin{aligned}
\langle \operatorname{curl} q, \operatorname{curl} q\rangle_{\mathrm{BD}(\operatorname{curl})} &= \langle \operatorname{curl} q, \operatorname{curl} q\rangle_{L_2(\Omega)^3} + \langle \operatorname{curl}\operatorname{curl} q, \operatorname{curl}\operatorname{curl} q\rangle_{L_2(\Omega)^3} \\
&= \langle \operatorname{curl} q, \operatorname{curl} q\rangle_{L_2(\Omega)^3} + \langle -q, -q\rangle_{L_2(\Omega)^3} \\
&= \langle q, q\rangle_{\mathrm{BD}(\operatorname{curl})}.
\end{aligned}$$

In the second equality we have used the characterisation provided by the assumption that $q \in \mathrm{BD}\,(\operatorname{curl})$ (c.f. Proposition 3.2.3). The surjectivity of $\operatorname{curl}_{\mathrm{BD}}$ follows from the fact that it is in fact bijective. Indeed, noting that $\operatorname{curl}_{\mathrm{BD}}$ is defined everywhere on $\mathrm{BD}\,(\operatorname{curl})$ and such that $-\operatorname{curl}\operatorname{curl} q = q$ for $q \in \mathrm{BD}\,(\operatorname{curl})$, we infer that $\operatorname{curl}_{\mathrm{BD}}$ is bijective with $\operatorname{curl}_{\mathrm{BD}}^{-1} = -\operatorname{curl}_{\mathrm{BD}}$. Thus, $\operatorname{curl}_{\mathrm{BD}}$ is unitary. This in fact yields skew-selfadjointness as well. Since $\operatorname{curl}_{\mathrm{BD}}$ unitary, we have $\operatorname{curl}_{\mathrm{BD}}^{*} = \operatorname{curl}_{\mathrm{BD}}^{-1} = -\operatorname{curl}_{\mathrm{BD}}$. □

Remark 3.2.6. In [PTW16] and [Pic17] the notation $\overset{\bullet}{\operatorname{curl}}$ is encountered in place of $\operatorname{curl}_{\mathrm{BD}}$. We will also bear in mind the original (equivalent) formulation of the above operators as

$$\begin{aligned}
\operatorname{grad}_{\mathrm{BD}} &:= \iota_{\operatorname{div}}^{*} \operatorname{grad} \iota_{\operatorname{grad}}, \\
\operatorname{div}_{\mathrm{BD}} &:= \iota_{\operatorname{grad}}^{*} \operatorname{div} \iota_{\operatorname{div}}, \\
\operatorname{Grad}_{\mathrm{BD}} &:= \iota_{\operatorname{Div}}^{*} \operatorname{Grad} \iota_{\operatorname{Grad}}, \\
\operatorname{Div}_{\mathrm{BD}} &:= \iota_{\operatorname{Grad}}^{*} \operatorname{Div} \iota_{\operatorname{Div}} \text{ and} \\
\operatorname{curl}_{\mathrm{BD}} &:= \iota_{\operatorname{curl}}^{*} \operatorname{curl} \iota_{\operatorname{curl}}
\end{aligned}$$

as introduced in [PTW16, Section 5.2] and recalled, for instance, in [Pic17, Section 4.1]. Having such an explicit formulation to hand, in which one can work directly with the properties of the orthogonal projectors and canonical embeddings (c.f. Lemma 3.2.1) involved, will prove useful in the sequel. ▽

In the current abstract scenario we can even formulate integration by parts formulae analogous to (3.1), (3.2) and (3.5) (c.f. [STW22, Proposition 12.4.2]).

Proposition 3.2.7. *Let $\Omega \subseteq \mathbb{R}^d$. For $u \in H^1(\Omega)$ and $q \in H(\operatorname{div}, \Omega)$ we have*

$$
\begin{aligned}
\langle \operatorname{div} q, u \rangle_{L_2(\Omega)} + \langle q, \operatorname{grad} u \rangle_{L_2(\Omega)^d} &= \left\langle \operatorname{div}_{\mathrm{BD}} \iota^*_{\operatorname{div}} q, \iota^*_{\operatorname{grad}} u \right\rangle_{\mathrm{BD}(\operatorname{grad})} \\
&= \left\langle \iota^*_{\operatorname{div}} q, \operatorname{grad}_{\mathrm{BD}} \iota^*_{\operatorname{grad}} u \right\rangle_{\mathrm{BD}(\operatorname{div})} .
\end{aligned}
\tag{3.11}
$$

Moreover, for $u \in H(\operatorname{Grad}, \Omega)$ and $q \in H(\operatorname{Div}, \Omega)$ we have

$$
\begin{aligned}
\langle \operatorname{Div} q, u \rangle_{L_2(\Omega)^d} + \langle q, \operatorname{Grad} u \rangle_{L_2(\Omega)^{d \times d}_{\mathrm{sym}}} &= \langle \operatorname{Div}_{\mathrm{BD}} \iota^*_{\operatorname{Div}} q, \iota^*_{\operatorname{Grad}} u \rangle_{\mathrm{BD}(\operatorname{Grad})} \\
&= \langle \iota^*_{\operatorname{Div}} q, \operatorname{Grad}_{\mathrm{BD}} \iota^*_{\operatorname{Grad}} u \rangle_{\mathrm{BD}(\operatorname{Div})} .
\end{aligned}
\tag{3.12}
$$

If $d = 3$, then for $q, p \in H(\operatorname{curl}, \Omega)$ we have

$$
\begin{aligned}
\langle \operatorname{curl} q, p \rangle_{L_2(\Omega)^3} - \langle q, \operatorname{curl} p \rangle_{L_2(\Omega)^3} &= \langle \operatorname{curl}_{\mathrm{BD}} \iota^*_{\operatorname{curl}} q, \iota^*_{\operatorname{curl}} p \rangle_{\mathrm{BD}(\operatorname{curl})} \\
&= - \langle \iota^*_{\operatorname{curl}} q, \operatorname{curl}_{\mathrm{BD}} \iota^*_{\operatorname{curl}} p \rangle_{\mathrm{BD}(\operatorname{curl})} .
\end{aligned}
\tag{3.13}
$$

Proof. We prove the third assertion relating to curl. The remaining assertions follow by analogy. Suppose $p, q \in H(\operatorname{curl}, \Omega)$. Consider the decompositions $q = q_0 + q_1$ and $p = p_0 + p_1$ for $q_0, p_0 \in H_0(\operatorname{curl}, \Omega), q_1 = \iota^*_{\operatorname{curl}} q$ and $p_1 = \iota^*_{\operatorname{curl}} p$. By the action of the orthogonal projector, we have $q_1, p_1 \in \mathrm{BD}(\operatorname{curl})$ (c.f. Lemma 3.2.1). Compute

$$
\begin{aligned}
&\langle \operatorname{curl} q, p \rangle_{L_2(\Omega)^3} - \langle q, \operatorname{curl} p \rangle_{L_2(\Omega)^3} \\
&= \langle \operatorname{curl}_0 q_0, p \rangle_{L_2(\Omega)^3} + \langle \operatorname{curl} q_1, p \rangle_{L_2(\Omega)^3} - \langle q_0, \operatorname{curl} p \rangle_{L_2(\Omega)^3} - \langle q_1, \operatorname{curl} p \rangle_{L_2(\Omega)^3} \\
&= \langle \operatorname{curl} q_1, p \rangle_{L_2(\Omega)^3} - \langle q_1, \operatorname{curl} p \rangle_{L_2(\Omega)^3} \\
&= \langle \operatorname{curl} q_1, p_0 \rangle_{L_2(\Omega)^3} + \langle \operatorname{curl} q_1, p_1 \rangle_{L_2(\Omega)^3} - \langle q_1, \operatorname{curl}_0 p_0 \rangle_{L_2(\Omega)^3} - \langle q_1, \operatorname{curl} p_1 \rangle_{L_2(\Omega)^3} \\
&= \langle \operatorname{curl} q_1, p_1 \rangle_{L_2(\Omega)^3} - \langle q_1, \operatorname{curl} p_1 \rangle_{L_2(\Omega)^3} \\
&= \langle \operatorname{curl} q_1, p_1 \rangle_{L_2(\Omega)^3} + \langle \operatorname{curl} \operatorname{curl} q_1, \operatorname{curl} p_1 \rangle_{L_2(\Omega)^3} \\
&= \langle \operatorname{curl} q_1, p_1 \rangle_{\mathrm{BD}(\operatorname{curl})} \\
&= \langle \operatorname{curl}_{\mathrm{BD}} \iota^*_{\operatorname{curl}} q, \iota^*_{\operatorname{curl}} p \rangle_{\mathrm{BD}(\operatorname{curl})} .
\end{aligned}
$$

In the first and fourth equalities we have applied the adjoint (c.f. Definition 2.3.1). In the sixth equality we have used the corresponding boundary data space characterisation (c.f. Proposition 3.2.3). The outstanding equality follows from applying $\operatorname{curl}_{\mathrm{BD}}^{*}$ (c.f. Proposition 3.2.5). $\square$

Remark 3.2.8. Comparing the above abstract integration by parts formulae with their classical counterparts reveals something of a relation between the two perspectives. Take for instance the first integration by parts formula, (3.11). On comparison with (3.1), it could be argued that $\operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^{*}$ ought to be taken as the formal replacement of the Neumann trace, $\gamma_{\cdot n}$. At the same time, it would seem like $\iota_{\mathrm{grad}}^{*}$ should formally replace the Dirichlet trace, γ. On account of the latter observation, this would seem like a natural pair of replacements to make. At the same time however, (3.11) would also suggest that $\gamma_{\cdot n}$ could be replaced by ι_{div}^{*} and γ by $\operatorname{grad}_{\mathrm{BD}} \iota_{\mathrm{grad}}^{*}$. It is still unclear as to how precisely the abstract boundary traces introduced should be considered as generalisations of the classical ones recalled earlier. This is why we talk about *formal* replacements and not generalisations. We will take this observation further in the next subsection. $\triangledown$

3.2.1 Comparing Classical and Abstract Boundary Spaces

In this subsection we survey the deeper connection between the classical and abstract boundary trace spaces. Whilst Proposition 3.2.7 can be elementarily compared with each of (3.1), (3.2) and (3.5), we want to examine how exactly the abstract spaces considered can be regarded as an abstraction of the classical trace spaces regarded earlier. The next result clarifies this (c.f. [STW22, Theorem 12.4.3], [PTW14, Theorem 4.5] or [Tro14, Corollary 4.4]).

Theorem 3.2.9. *Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain. Then the operators*

$$\begin{aligned}
\gamma|_{\mathrm{BD(grad)}} &\colon \mathrm{BD}\,(\mathrm{grad}) \to H^{1/2}\,(\partial\Omega)\,,\\
\gamma_{\cdot n}|_{\mathrm{BD(div)}} &\colon \mathrm{BD}\,(\mathrm{div}) \to H^{-1/2}\,(\partial\Omega)\,,\\
\gamma|_{\mathrm{BD(Grad)}} &\colon \mathrm{BD}\,(\mathrm{Grad}) \to H^{1/2}\,(\partial\Omega)^d \quad \textit{and}\\
\gamma_{\cdot n}|_{\mathrm{BD(Div)}} &\colon \mathrm{BD}\,(\mathrm{Div}) \to H^{-1/2}\,(\partial\Omega)^d
\end{aligned}$$

are bounded and bijective. When $d = 3$, then the operators

$$\gamma_\tau|_{\mathrm{BD(curl)}} \colon \mathrm{BD}\,(\mathrm{curl}) \to V_\pi' \text{ and}$$
$$\pi_\tau|_{\mathrm{BD(curl)}} \colon \mathrm{BD}\,(\mathrm{curl}) \to V_\tau'$$

are bounded and injective.

Proof. We prove the assertion for the tangential trace. We first show that $\ker(\gamma_\tau) = H_0(\mathrm{curl}, \Omega)$. For the first inclusion, suppose $q \in H_0(\mathrm{curl}, \Omega)$. Then, there exists a sequence $(\phi_n)_{n=1}^\infty$ in $C_\mathrm{c}^\infty(\Omega)^3$ such that $\phi_n \to q$ in $H(\mathrm{curl}, \Omega)$ as $n \to \infty$ (c.f. Theorem 2.3.3). By continuity of γ_τ and Item (iii) from Proposition 3.1.2, $0 = \gamma_\tau \phi_n \to \gamma_\tau q$ as $n \to \infty$. For the second inclusion, assume $p, q \in H^1(\Omega)^3$ such that $\gamma_\tau q = 0$. Using the integration by parts formula (3.5) yields

$$\langle \mathrm{curl}\, p, q \rangle_{L_2(\Omega)^3} - \langle p, \mathrm{curl}\, q \rangle_{L_2(\Omega)^3} = \langle \pi_\tau p, \gamma_\tau q \rangle_{L_2^\tau(\partial\Omega)} = 0,$$

which we reformulate as

$$\langle \mathrm{curl}\, p, q \rangle_{L_2(\Omega)^3} = \langle p, \mathrm{curl}\, q \rangle_{L_2(\Omega)^3}.$$

This implies that $q \in \mathrm{dom}(\mathrm{curl}^*) = \mathrm{dom}(\mathrm{curl}_0)$ (c.f. Definition 2.1.8). By definition of $\mathrm{BD(curl)} = H_0(\mathrm{curl}, \Omega)^{\perp_{H(\mathrm{curl},\Omega)}}$ (c.f. Remark 3.2.4), it follows that $\ker(\gamma_\tau|_{\mathrm{BD(curl)}}) = \{0\}$. Hence $\gamma_\tau|_{\mathrm{BD(curl)}}$ injective. The continuity of the restricted trace $\gamma_\tau|_{\mathrm{BD(curl)}}$ follows from the continuity of γ_τ. The density of $H^1(\Omega)^3$ in $H(\mathrm{curl}, \Omega)$ allows us to deduce the claim for $p, q \in H(\mathrm{curl}, \Omega)$. The claim for the tangential components trace follows by direct analogy. Surjectivity of $\gamma|_{\mathrm{BD(grad)}}$ follows by definition of $H^{1/2}(\partial\Omega) = \mathrm{ran}(\gamma)$. Surjectivity of $\gamma_{\cdot n}|_{\mathrm{BD(div)}}$ follows from Proposition 3.1.4. The assertions for $\gamma|_{\mathrm{BD(Grad)}}$ and $\gamma_{\cdot n}|_{\mathrm{BD(Div)}}$ follow analogously. □

Remark 3.2.10. Both of the restricted tangential and tangential components traces can in fact be made bijective. This is done in [BCS02, Theorem 4.1] by replacing the target spaces V_τ' and V_π' with a particular pair of corresponding subspaces. The technical details are left to the interested reader to consider in the aforenoted reference. ▽

Theorem 3.2.9 affords us a particularly precise perspective on the connection between the classical and abstract boundary spaces introduced. Under sufficient boundary regularity assumptions, Theorem 3.2.9 emphasises that there is *no* difference between the boundary values obtained from either the classical or abstract perspective (at least with respect to the restricted Dirichlet and Neumann traces). A similar but more restricted view can be afforded to the boundary values of $H(\mathrm{curl}, \Omega)$ in light of Remark 3.2.10. In either case, it is interesting to observe how the above restrictions of the classical traces arise as the natural mappings between both spaces (and perspectives) of boundary values.

Taking this discussion further, let us consider the relation between the classical and abstract traces themselves. The following is based on (c.f. [Pic17, p. 11, (16)]). Let $\Omega \subseteq \mathbb{R}^d$ be a bounded Lipschitz domain, and suppose $f \in H^1(\Omega)$ and $q \in H(\mathrm{div}, \Omega)$. Recalling the integration by parts formula (3.11), we test against $\iota_{\mathrm{grad}}^* f$ in the inner product and compute

$$
\begin{aligned}
&\langle \iota_{\mathrm{grad}}^* f, \mathrm{div}_{\mathrm{BD}}\, \iota_{\mathrm{div}}^* q \rangle_{\mathrm{BD(grad)}} \\
&= \langle f, \mathrm{div}\, q \rangle_{L_2(\Omega)} + \langle \mathrm{grad}\, f, q \rangle_{L_2(\Omega)^d} \\
&= \langle \gamma f, \gamma_{\cdot n} q \rangle_{L_2(\partial\Omega)} \\
&= \langle (\gamma \iota_{\mathrm{grad}})\, \iota_{\mathrm{grad}}^* f, (\gamma_{\cdot n} \iota_{\mathrm{div}})\, \iota_{\mathrm{div}}^* q \rangle_{L_2(\partial\Omega)} \\
&= \langle (\gamma \iota_{\mathrm{grad}})\, \iota_{\mathrm{grad}}^* f, R_{H^{1/2}}^{-1} (\gamma_{\cdot n} \iota_{\mathrm{div}})\, \iota_{\mathrm{div}}^* q \rangle_{H^{1/2}(\partial\Omega)} \\
&= \langle \iota_{\mathrm{grad}}^* f, (\gamma \iota_{\mathrm{grad}})^* R_{H^{1/2}}^{-1} (\gamma_{\cdot n} \iota_{\mathrm{div}})\, \iota_{\mathrm{div}}^* q \rangle_{\mathrm{BD(grad)}}.
\end{aligned}
\tag{3.14}
$$

The second equality is the classical integration by parts formula (3.1). Having assumed sufficient boundary regularity, it follows that the two different formulae must coincide. In the fourth equality we follow the dual space perspective by invoking the Riesz mapping, $R_{H^{1/2}}^{-1}$. For arbitrary $q \in H(\mathrm{div}, \Omega)$ it follows that

$$
\begin{aligned}
&\mathrm{div}_{\mathrm{BD}}\, \iota_{\mathrm{div}}^* q = (\gamma \iota_{\mathrm{grad}})^* R_{H^{1/2}}^{-1} (\gamma_{\cdot n} \iota_{\mathrm{div}})\, \iota_{\mathrm{div}}^* q \\
\iff\ & R_{H^{1/2}} \left((\gamma \iota_{\mathrm{grad}})^*\right)^{-1} \mathrm{div}_{\mathrm{BD}}\, \iota_{\mathrm{div}}^* q = (\gamma_{\cdot n} \iota_{\mathrm{div}})\, \iota_{\mathrm{div}}^* q \\
&= \gamma_{\cdot n} q.
\end{aligned}
\tag{3.15}
$$

This can be taken as further motivation to replace the Neumann trace, $\gamma_{\cdot n}$, by the operator $\operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^{*}$ (c.f. Remark 3.2.8). Recalling that $\operatorname{ran}(\operatorname{div}_{\mathrm{BD}}) \subseteq \mathrm{BD}(\operatorname{grad})$ (c.f. Proposition 3.2.5), it would seem that the operator $(\gamma \iota_{\mathrm{grad}})^{*} R_{H^{1/2}}^{-1}$ arises to compensate for this fact. As such, it is hard to consider these formal replacements as proper generalisations of the corresponding classical traces. Computations similar to (3.14) are provided in [Pic17, Subsection 4.3.1] which further motivate the replacement by $\operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^{*}$ and $\operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^{*}$ of $\gamma_{\cdot n}$ and γ_{τ}, respectively.

There are other examples which highlight the need for such a compensation. In [STW22, Proposition 12.5.3] the Robin boundary condition $\gamma_{\cdot n} H = -\mathrm{i}\, \gamma u$, for $H \in H(\operatorname{div}, \Omega)$ and $u \in H^{1}(\Omega)$ on a bounded Lipschitz domain, is considered. However, this boundary condition turns out not to be the same as $\operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{BD(div)}}^{*} H = -\mathrm{i}\, \iota_{\mathrm{BD(grad)}}^{*} u$. For more insight, consider the examples regarded in [PTW16, Section 6], [Tro14, Section 5] and [PTW14, Section 5].

3.2.2 The Application of Abstract Boundary Traces

In this subsection we gather together the tools necessary to encode boundary dynamics directly within an evolutionary equation. To that end we will rely heavily on the notions and methodology introduced in [PSTW16, Sections 1, 2.3.2]. As we next outline, successfully encoding any boundary dynamics within an evolutionary system will precipitate an extension of the evolutionary equation of interest. Let $\Omega \subseteq \mathbb{R}^{d}$ be open. For a given evolutionary equation

$$\left(\partial_{t,\nu} M\left(\partial_{t,\nu}\right) + A\right) \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \in L_{2,\nu}\left(\mathbb{R}; H_{1} \oplus H_{2}\right)$$

we will consider the extended system

$$\left(\partial_{t,\nu} M\left(\partial_{t,\nu}\right) + A\right) \begin{pmatrix} u \\ \begin{pmatrix} v \\ \tau_{v} \end{pmatrix} \end{pmatrix} = \begin{pmatrix} f \\ \begin{pmatrix} g \\ h \end{pmatrix} \end{pmatrix} \in L_{2,\nu}\left(\mathbb{R}; H_{1} \oplus H_{2} \oplus H_{\mathrm{trace}}\right). \qquad (3.16)$$

Here, H_{trace} is an auxiliary Hilbert space upon which we will formulate boundary dynamics pertaining to the underlying evolutionary system. In the context of abstract boundary trace spaces, H_{trace} can be any of the boundary data spaces recalled in Proposition 3.2.3. Both the material law operator $M(\partial_{t,\nu})$ and spatial operator A will need to be suitably extended to accommodate for the introduction of H_{trace}, as well as the boundary condition to be formulated there. Whilst it is entirely possible to use classical boundary trace spaces to arrive at an analogous extension, we will focus solely on the application of abstract boundary data spaces to that end. In order to properly realise this extension, we first recall some additional preparations. In what follows, let X and Y be Hilbert spaces. We begin with a definition (c.f. [STW22, Definition, p. 133] or [PSTW16, Definition 1.3]).

Definition 3.2.11. Let $C\colon \operatorname{dom}(C) \subseteq X \to Y$ be a densely defined and closed linear operator. We define the operator $C^\diamond\colon Y \to \operatorname{dom}(C)'$ by $C^\diamond := C' \circ R_Y$ where C' denotes the dual to C. $\diamondsuit$

Remark 3.2.12. (i) By definition we have

$$\left(C^\diamond y\right)(x) = \big(C'\left(R_Y y\right)\big)(x) = \left(R_Y y\right)(Cx) = \langle y, Cx\rangle_Y$$

for $x \in \operatorname{dom}(C)$ and $y \in Y$.

(ii) Particular properties of the operator $C^\diamond$ follow immediately by definition or direct computation (c.f. [STW22, Proposition 9.2.2]). Indeed, it is not hard to see that $C^\diamond$ is linear, bounded and such that $C^* \subseteq C^\diamond$. Thus, the action of $C^\diamond$ in some way generalises that of C^*. The operator $(C^*)^\diamond\colon X \to \operatorname{dom}((C^*)')$ is called the *extrapolated operator* of C. One can also show that $C \subseteq (C^*)^\diamond$, so that the action of $(C^*)^\diamond$ similarly generalises that of C. $\triangledown$

In the following we specialise to when the target Hilbert space Y is given as a direct sum of Hilbert spaces. We consider $Y = \bigoplus_{i=1}^n Y_i$, where for $i \in \{1, \ldots, n\}$, Y_i is itself a Hilbert space. For $i \in \{1, \ldots, n\}$ we define $C_i \coloneqq \iota_{Y_i}^* C$ (c.f. Lemma 3.2.1), and obtain

$$Cx = C_1 x \oplus \cdots \oplus C_n x = \begin{pmatrix} C_1 x \\ \vdots \\ C_n x \end{pmatrix} = \begin{pmatrix} C_1 \\ \vdots \\ C_n \end{pmatrix} x \in \begin{pmatrix} Y_1 \\ \vdots \\ Y_n \end{pmatrix} = Y \tag{3.17}$$

for $x \in X$. The column operators arising here provide a hint as to the form of the extension to be employed in the formulation of the extended evolutionary equation (3.16). We next consider how to compute the adjoint of such an operator. This point is encapsulated in the next two vitally important results. The first can be found as [PSTW16, Theorem 1.6].

Theorem 3.2.13. *Let $C \colon \operatorname{dom}(C) \subseteq X \to Y$ be a densely defined and closed linear operator such that (3.17) holds for $x \in X$. Then*

$$\begin{aligned} C^* &= \begin{pmatrix} C_1^\diamond & \cdots & C_n^\diamond \end{pmatrix} \cap (Y \oplus X) \\ &= \left\{ ((y_1, \ldots, y_n), x) \in Y \oplus X : x = \sum_{i=1}^n C_i^\diamond y_i \in X \right\}. \end{aligned}$$

Proof. We first determine the form of $C^\diamond$ here. Let $x \in \operatorname{dom}(C)$ and $y \in Y = \bigoplus_{i=1}^n Y_i$. It follows by definition of $(\cdot)^\diamond$ that

$$\left(C^\diamond y\right)(x) = \langle y, Cx \rangle_Y = \sum_{i=1}^n \langle y_i, C_i x \rangle_{Y_i} = \sum_{i=1}^n \left(C_i^\diamond y_i\right)(x).$$

Thus, $C^\diamond y = \sum_{i=1}^n C_i^\diamond y_i$ with $C^\diamond = \begin{pmatrix} C_1^\diamond & \cdots & C_n^\diamond \end{pmatrix}$. As for the first inclusion of the remaining claim, it is clear by definition that both $C^* \subseteq C^\diamond$ and $C^* \subseteq Y \oplus X$. Thus,

$$C^* \subseteq \begin{pmatrix} C_1^\diamond & \cdots & C_n^\diamond \end{pmatrix} \cap (Y \oplus X).$$

For the second inclusion, suppose $(y, x) \in Y \oplus X$. Then, by definition of the adjoint (c.f. Definition 2.1.8),

$$\begin{aligned}(y,x)\in C^* &\iff \forall\phi\in\operatorname{dom}(C),\ \langle y, C\phi\rangle_Y = \langle x,\phi\rangle_X\\ &\iff \forall\phi\in\operatorname{dom}(C),\ (C^\diamond y)(\phi) = \langle x,\phi\rangle_X,\end{aligned}$$

from which it follows that $x = C^\diamond y = \sum_{i=1}^n C_i^\diamond y_i \in X$. $\square$

Our second crucial result can be found as [PSTW16, Corollary 1.8], and its proof follows immediately from the preceding theorem.

Corollary 3.2.14. *Let $C\colon \operatorname{dom}(C)\subseteq X\to Y$ be a densely defined and closed linear operator such that (3.17) holds for $x\in X$. Assume there exists a densely defined and closed linear operator $\mathring{C}_1$ such that*

$$\begin{pmatrix}\mathring{C}_1\\ 0\\ \vdots\\ 0\end{pmatrix}\subseteq\begin{pmatrix}C_1\\ C_2\\ \vdots\\ C_n\end{pmatrix} = C.$$

Then,

$$C^* = C^\diamond\cap\begin{pmatrix}\mathring{C}_1^* & 0 & \cdots & 0\end{pmatrix}\subseteq\begin{pmatrix}\mathring{C}_1^* & 0 & \cdots & 0\end{pmatrix}.$$

With these initial preparations to hand, we can now collect those results which will allow us to arrive at the extension indicated in (3.16). We first recall [PSTW16, Lemma 2.22], which provides characterisations of the operators $(\iota_{\mathrm{grad}}^*)^\diamond$, $(\iota_{\mathrm{Grad}}^*)^\diamond$ and $(\iota_{\mathrm{curl}}^*)^\diamond$.

Lemma 3.2.15. *We have*

$$\begin{aligned}\left(\iota_{\mathrm{grad}}^*\right)^\diamond &= (1+\mathrm{grad}^\diamond\,\mathrm{grad})\,\iota_{\mathrm{grad}},\\ (\iota_{\mathrm{Grad}}^*)^\diamond &= (1+\mathrm{Grad}^\diamond\,\mathrm{Grad})\,\iota_{\mathrm{Grad}}\ \textit{and}\\ (\iota_{\mathrm{curl}}^*)^\diamond &= (1+\mathrm{curl}^\diamond\,\mathrm{curl})\,\iota_{\mathrm{curl}}.\end{aligned}$$

Proof. We derive the first equality only, with the others following by direct analogy.

The assertion will follow immediately by definition of $(\cdot)^\diamond$ and the graph norm. For $\psi \in H^1(\Omega)$ and $\phi \in \mathrm{BD}\,(\mathrm{grad})$, compute

$$\begin{aligned}
\left(\iota_{\mathrm{grad}}^*\right)^\diamond (\phi)(\psi) &= \langle \phi, \iota_{\mathrm{grad}}^* \psi \rangle_{\mathrm{BD(grad)}} \\
&= \langle \iota_{\mathrm{grad}} \phi, \psi \rangle_{H^1(\Omega)} \\
&= \langle \iota_{\mathrm{grad}} \phi, \psi \rangle_{L_2(\Omega)} + \langle \mathrm{grad}\, \iota_{\mathrm{grad}} \phi, \mathrm{grad}\, \psi \rangle_{L_2(\Omega)^d} \\
&= (1 + \mathrm{grad}^\diamond\, \mathrm{grad})\, (\iota_{\mathrm{grad}} \phi)(\psi). \qquad \square
\end{aligned}$$

Our next result is an auxiliary one and will help us to more readily prove the main result of this subsection, Theorem 3.2.17.

Lemma 3.2.16. *Let $\Omega \subseteq \mathbb{R}^d$ be open. Suppose $\psi \in H\,(\mathrm{div}, \Omega)$. Then, the following statements are equivalent:*

(i) $(\mathrm{div} + \mathrm{grad}^\diamond)\, \psi = 0$,

(ii) $\psi \in H_0\,(\mathrm{div}, \Omega)$,

(iii) $\iota_{\mathrm{div}} \iota_{\mathrm{div}}^* \psi = 0$.

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(ii) $\psi \in H_0\,(\mathrm{Div}, \Omega)$,

(iii) $\iota_{\mathrm{Div}} \iota_{\mathrm{Div}}^* \psi = 0$.

Let $\Omega \subseteq \mathbb{R}^3$ be open and suppose $\psi \in \mathrm{dom}\,(\mathrm{curl})$. Then, the following statements are equivalent:

(i) $(\mathrm{curl}^\diamond - \mathrm{curl})\, \psi = 0$,

(ii) $\psi \in \mathrm{dom}\,(\mathrm{curl}_0)$,

(iii) $\iota_{\mathrm{curl}} \iota_{\mathrm{curl}}^* \psi = 0$.

Proof. We show the chain of equivalences for $\psi \in H(\mathrm{div}, \Omega)$ only. The other sets of equivalences follow by direct analogy. Suppose $\psi \in H(\mathrm{div}, \Omega)$ is such that

$$(\mathrm{div} + \mathrm{grad}^{\diamond})\, \psi = 0.$$

For $\phi \in H^1(\Omega)$, compute

$$\begin{aligned} 0 &= (\mathrm{div} + \mathrm{grad}^{\diamond})\, (\psi)(\phi) \\ &= (\mathrm{div}\, \psi)\,(\phi) + (\mathrm{grad}^{\diamond}\, \psi)\,(\phi) \\ &= \langle \mathrm{div}\, \psi, \phi \rangle_{L_2(\Omega)} + \langle \psi, \mathrm{grad}\, \phi \rangle_{L_2(\Omega)^d}, \end{aligned}$$

which we reformulate as

$$\langle \psi, \mathrm{grad}\, \phi \rangle_{L_2(\Omega)^d} = -\langle \mathrm{div}\, \psi, \phi \rangle_{L_2(\Omega)}.$$

We have $\psi \in \mathrm{dom}\,(\mathrm{grad}^*) = H_0\,(\mathrm{div}, \Omega)$ if and only if there exists $z \in L_2(\Omega)$ for all $x \in H^1(\Omega)$ such that $\langle \psi, \mathrm{grad}\, x \rangle_{L_2(\Omega)^d} = \langle z, x \rangle_{L_2(\Omega)}$ (c.f. Definition 2.1.8). Choosing $z := -\mathrm{div}\, \psi \in L_2(\Omega)$ and $x := \phi \in H^1(\Omega)$, it follows that $\psi \in H_0\,(\mathrm{div}, \Omega)$. The implication $(ii) \implies (i)$, on the other hand, follows immediately. Since $\iota_{\mathrm{div}} \iota_{\mathrm{div}}^*$ is the orthogonal projection onto $\mathrm{BD}\,(\mathrm{div})$ (c.f. Lemma 3.2.1), the remaining equivalence between (ii) and (iii) is also immediate. □

Our main result for this subsection is the following (c.f. [PSTW16, Theorem 2.24]).

Theorem 3.2.17. *We have the inclusion* $\begin{pmatrix} \mathrm{grad} \\ \iota_{\mathrm{grad}}^* \end{pmatrix}^* \subseteq (-\mathrm{div}\ \ 0)$*, and*

$$\mathrm{dom}\left(\begin{pmatrix} \mathrm{grad} \\ \iota_{\mathrm{grad}}^* \end{pmatrix}^*\right) = \left\{(q, \tau_q) \in \mathrm{dom}(\mathrm{div}) \times \mathrm{BD}(\mathrm{grad})\colon\, \iota_{\mathrm{div}}^* T + \mathrm{grad}_{\mathrm{BD}}\, \tau_q = 0\right\}.$$

Moreover, we have the inclusion $\begin{pmatrix} -\operatorname{Grad} \\ \iota^*_{\operatorname{Grad}} \end{pmatrix}^* \subseteq (\operatorname{Div}\ 0)$, *and*

$$\operatorname{dom}\left(\begin{pmatrix} -\operatorname{Grad} \\ \iota^*_{\operatorname{Grad}} \end{pmatrix}^*\right) = \{(T, \tau_T) \in \operatorname{dom}(\operatorname{Div}) \times \operatorname{BD}(\operatorname{grad})\colon \iota^*_{\operatorname{Div}} T - \operatorname{Grad}_{\operatorname{BD}} \tau_T = 0\}\,.$$

Furthermore, we have the inclusion $\begin{pmatrix} \operatorname{curl} \\ \iota^*_{\operatorname{curl}} \end{pmatrix}^* \subseteq (\operatorname{curl}\ 0)$, *and*

$$\operatorname{dom}\left(\begin{pmatrix} \operatorname{curl} \\ \iota^*_{\operatorname{curl}} \end{pmatrix}^*\right) = \{(H, \tau_H) \in \operatorname{dom}(\operatorname{curl}) \times \operatorname{BD}(\operatorname{curl})\colon \iota^*_{\operatorname{curl}} H + \operatorname{curl}_{\operatorname{BD}} \tau_H = 0\}\,.$$

Proof. We prove the assertion for the extended column operator relating to the scalar gradient and divergence only. The other assertions follow analogously. Recalling that $H^1_0(\Omega) \subseteq H^1(\Omega)$, it is clear that we have

$$\begin{pmatrix} \operatorname{grad}_0 \\ 0 \end{pmatrix} \subseteq \begin{pmatrix} \operatorname{grad} \\ \iota^*_{\operatorname{grad}} \end{pmatrix}.$$

Applying Corollary 3.2.14, we obtain

$$\begin{pmatrix} \operatorname{grad} \\ \iota^*_{\operatorname{grad}} \end{pmatrix}^* \subseteq \begin{pmatrix} \operatorname{grad}_0 \\ 0 \end{pmatrix}^* = \begin{pmatrix} -\operatorname{div} & 0 \end{pmatrix}.$$

On the other hand, from Theorem 3.2.13 it follows that $(q, \tau_q) \in \operatorname{dom}\left(\begin{pmatrix} \operatorname{grad} \\ \iota^*_{\operatorname{grad}} \end{pmatrix}^*\right)$ if and only if $q \in H(\operatorname{div}, \Omega)$, and

$$\begin{aligned} -\operatorname{div} q &= \operatorname{grad}^\diamond q + (\iota^*_{\operatorname{grad}})^\diamond \tau_q \\ &= \operatorname{grad}^\diamond q + (1 + \operatorname{grad}^\diamond \operatorname{grad}) \iota_{\operatorname{grad}} \tau_q. \end{aligned}$$

Here we have used Lemma 3.2.15 to obtain the second equality. Next, we reformulate the latter equality as

$$\begin{aligned}
-\operatorname{div} q - \operatorname{grad}^{\diamond} q &= (1 + \operatorname{grad}^{\diamond} \operatorname{grad})\iota_{\mathrm{grad}}\tau_q \\
&= (\operatorname{div}\operatorname{grad} + \operatorname{grad}^{\diamond} \operatorname{grad})\iota_{\mathrm{grad}}\tau_q \\
&= (\operatorname{div} + \operatorname{grad}^{\diamond}) \operatorname{grad} \iota_{\mathrm{grad}}\tau_q,
\end{aligned}$$

where in the second equality here we have applied Proposition 3.2.3. We reformulate the last equality so as to obtain that

$$\begin{aligned}
&(\operatorname{div} + \operatorname{grad}^{\diamond})q + (\operatorname{div} + \operatorname{grad}^{\diamond}) \operatorname{grad} \iota_{\mathrm{grad}}\tau_q = 0 \\
\iff &(\operatorname{div} + \operatorname{grad}^{\diamond})(q + \operatorname{grad} \iota_{\mathrm{grad}}\tau_q) = 0,
\end{aligned}$$

from which point our auxiliary result Lemma 3.2.16 yields the equivalent statement

$$\begin{aligned}
&\iota_{\mathrm{div}}\iota_{\mathrm{div}}^{*}(q + \operatorname{grad} \iota_{\mathrm{grad}}\tau_q) = 0 \\
\iff &\iota_{\mathrm{div}}(\iota_{\mathrm{div}}^{*} q + \operatorname{grad}_{\mathrm{BD}} \tau_q) = 0.
\end{aligned}$$

The injectivity of ι_{div} then yields the desired boundary condition. □

Remark 3.2.18. We emphasise that the conditions present in the respective domains of the adjoints of the extended column operators are a kind of abstract boundary condition. In particular, these boundary conditions are to be taken in addition to any boundary condition we might formulate on the respective H_{trace} space. ▽

In the next chapter we will apply the notions and results recalled here to arrive at our extended system of thermo-piezo-electromagnetism.

Chapter 4

Our Model for Thermo-Piezo-Electromagnetism with Boundary Dynamics

The aim of this chapter is to present the focal point of this thesis; our full thermo-piezo-electromagnetic model encoding dynamics on the boundary. The thermo-piezo-electromagnetic system studied in [MPTW16, Sections 2, 3] provides the basis we will use to formulate our own extended model. Using the theory of abstract boundary data spaces (see Chapter 3) and the methodology of [Pic17], we will extend the system of [MPTW16] and formulate individual boundary equations corresponding to the thermal, elastic and electromagnetic parts of the model. In particular, these boundary equations will be encoded directly within our model and will function as part of the system itself. We will then address the question of evolutionary well-posedness for our system. Our discussion will culminate in the presentation and proof of our own central solution result, Theorem 4.4.6. Besides presenting our main solution theory, the present chapter will serve to set the stage for us to explore those patterns of boundary behaviour which are evolutionarily well-posed and accomodated for by our model. We will explore this question in the subsequent and final chapter of this thesis. We begin with a slight detour, however, and recall the idea of congruence transforms as used extensively throughout [Pic17] and [MPTW16]. We also briefly recall a specific preliminary result.

4.1 Congruence Transforms

In this subsection we will briefly recall the notion of a *congruence transform*. At the same time we will use this opportunity to clarify the nomenclature and notation that we will employ in the sequel. As it will turn out, we will be concerned with applying Picard's Theorem (recall Theorem 2.4.4) to a large block operator system. Congruence transforms provide an elementary but no less useful tool which will allow us to distill key data from this system. The notion of a congruence is entirely standard in linear algebra, with the idea itself being adopted from there. Indeed, see [Pic17, Definition 2.4] or any standard linear algebra reference including [Had61, p. 253, Definition], [Her91, p. 352, Definition] or [SGL+20, Definition 8.3.1] .

Definition 4.1.1. Let H be a Hilbert space, $S, T \in L(H)$ and let $\mathcal{C} \in L(H)$ be bijective. We call T and S *congruent* if

$$\mathcal{C}\, T\, \mathcal{C}^* = S. \tag{4.1}$$

Here, $\mathcal{C}$ is called *congruence transformation.* ◊

Remark 4.1.2. (i) Following the nomenclature used in [Pic17] and [MPTW16], we will refer to the action of simultaneously composing an operator $T \in L(H)$ by $\mathcal{C}$ and $\mathcal{C}^*$ as a *symmetric Gauss step.*

(ii) We will also use the nomenclature *"T is congruent to the operator S under the congruence transform $\mathcal{C}$"* to mean that T and S are congruent. We will employ the preposition *under* when indicating that congruence with respect to which T is congruent to S, and will specify whether the congruence has been achieved under a symmetric Gauss step or permutation as a congruence transformation.

(iii) We will refer to (4.1) as *congruent form.* One might instead regard the (equivalent) congruent form

$$\mathcal{C}^{-1}\, S\, \left(\mathcal{C}^{-1}\right)^* = T.$$ ▽

In the next proposition we collect some facts relating to how congruent forms preserve certain operator properties. We omit their proof as they follow immediately by definition.

Proposition 4.1.3. *Let H be a Hilbert space and suppose $S, T \in L(H)$ are congruent under $\mathcal{C} \in L(H)$. Then the following statements hold true.*

(i) Additionally suppose T selfadjoint (skew-selfadjoint). Then S is also selfadjoint (skew-selfadjoint).

(ii) Additionally suppose T positive-definite (non-negative). Then S is also positive-definite (non-negative).

(iii) $\operatorname{Re} S = \mathcal{C} (\operatorname{Re} T) \mathcal{C}^*$.

As suggested by the language in Item (i) of Remark 4.1.2, congruence transforms will primarily play the role of a generalised form of Gaussian elimination. In this case, the form that the congruence should take will be immediate (following the intuition afforded by scalar Gaussian elimination). On the other hand, they might instead assume the role of a permutation block operator.

For a given 2×2 block operator we have the following factorisation under the assumption that one of the diagonal operator coefficients is continuously invertible. On account of the block structure, this can of course be extended beyond 2×2 block operator matrices. We omit the proof as it follows from direct computation. The notion of this decomposition is completely standard; see for instance [J+21, Theorem 2.B.1], [Gen07, Subsection 3.8.7], [Zha11, Section 7.3], [Zha04, Lemma 1.4], [HP14, Theorem 2.2] or [TB22, Exercise 20.3].

Proposition 4.1.4. *Let H_1, H_2 be Hilbert spaces. Let $A \in L(H_1)$ with continuous inverse, $B \in L(H_2, H_1)$ and $D \in L(H_2)$. Then*

$$\begin{pmatrix} 1 & 0 \\ -B^* A^{-1} & 1 \end{pmatrix} \begin{pmatrix} A & B \\ B^* & D \end{pmatrix} \begin{pmatrix} 1 & -A^{-1}B \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} A & 0 \\ 0 & D - B^* A^{-1} B \end{pmatrix}. \tag{4.2}$$

Remark 4.1.5. (i) The continuous invertibility of $A \in L(H_1)$ is guaranteed for instance in the situation of Proposition 4.1.6 when $A \gg 0$.

(ii) The operator

$$D - B^* A^{-1} B$$

obtained by this symmetric Gauss step is known as the *Schur complement of A in the block operator*

$$\begin{pmatrix} A & B \\ B^* & D \end{pmatrix}.$$

(iii) Of course, one could instead assume $D \in L(H_2)$ continuously invertible and analogously obtain the congruent block operator

$$\begin{pmatrix} A - BD^{-1}B^* & 0 \\ 0 & D \end{pmatrix}$$

under the symmetric Gauss step provided by the congruence transform

$$\begin{pmatrix} 1 & -BD^{-1} \\ 0 & 1 \end{pmatrix}.$$

(iv) In (4.2) the congruence transform is provided by the operator

$$\begin{pmatrix} 1 & 0 \\ -B^* A^{-1} & 1 \end{pmatrix}$$

which is clearly invertible. Its operator inverse is provided by

$$\begin{pmatrix} 1 & 0 \\ B^* A^{-1} & 1 \end{pmatrix}.$$

Such a congruence transform is known in the literature as a *Frobenius ma-*

trix or *Gauss matrix* on account of the fact that it generalises the action of Gaussian elimination (for the use of this nomenclature, see for instance [Pla10, Bermerkung 4.16, Wert 4], [FH07, p. 211] or [Lun10, p. 82]). $\triangledown$

Finally, in preparation for what is to follow in the remainder of this thesis, we recall the following result [STW22, Proposition 6.2.3 (b)]. It provides us with a useful criterion for continuous invertibility of bounded operators via accretivity (c.f. Remark 2.4.5).

Proposition 4.1.6. *Let* $T \in L(H)$, $c \in \mathbb{R}_{>0}$ *and assume that* $\operatorname{Re} T \geq c$. *Then* $T^{-1} \in L(H)$ *such that* $\|T^{-1}\| \leq \frac{1}{c}$ *and* $\operatorname{Re} T^{-1} \geq c\|T\|^{-2}$.

4.2 The Underlying Model

Our aim in this section is to recall the key components of [MPTW16, Sections 2, 3] which will serve as the basis for the construction of our own extended model. Our aim in this direction is twofold. We first remind ourselves of the formulation of the model for thermo-piezo-electromagnetism as an evolutionary equation with homogeneous boundary conditions considered there. Second, we will recall in detail the key evolutionary well-posedness result, [MPTW16, Theorem 3.1]. This recollection is not simply to the end of providing a complete display of the ideas behind the formulation of our own model however. It is in fact to the end of some interesting comparisons and discussions, which we take up later in Remark 4.4.2.

We first recall the basic equations presented in [MPTW16, Section 2] underpinning what will turn out to be a coupled thermo-piezo-electromagnetic system. The basic system is made up of the equation of elasticity, Maxwell's equations and the heat equation. In the following let $\Omega \subseteq \mathbb{R}^3$ be open and nonempty. We have the equation of elasticity

$$\partial_t^2 \rho_* u - \operatorname{Div} T = F_0, \tag{4.3}$$

where $u\colon \mathbb{R} \times \Omega \to \mathbb{R}^3$ denotes the displacement of the elastic body, Ω, and $T\colon \mathbb{R} \times \Omega \to \mathbb{R}^{3\times 3}_{\mathrm{sym}}$ (c.f. Definition 2.3.5) the stress tensor. The function $\rho_*\colon \Omega \to \mathbb{R}$ describes the density of Ω, and $F_0\colon \mathbb{R} \times \Omega \to \mathbb{R}^3$ is an external balancing force. Maxwell's equations are

$$\partial_t B + \operatorname{curl} E = F_3, \tag{4.4}$$

$$\partial_t D - \operatorname{curl} H = F_2 - \sigma E, \tag{4.5}$$

where $E, H, B, D\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$ are, respectively, the electric field, the magnetic field, the magnetic flux density, and the electrical displacement. The functions $F_2, F_3\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$ denote given current sources whereas $\sigma\colon \Omega \to \mathbb{R}$ describes the electrical resistance. The heat equation is

$$\partial_t \Theta_0 \eta + \operatorname{div} q = F_4, \tag{4.6}$$

where $\eta\colon \mathbb{R}\times\Omega \to \mathbb{R}$ is the entropy density, $q\colon \mathbb{R}\times\Omega \to \mathbb{R}^3$ describes the heat flux, $F_4\colon \mathbb{R}\times\Omega \to \mathbb{R}$ denotes a given external heat source, and $\Theta_0\colon \Omega \to \mathbb{R}$ is the reference temperature with $\Theta_0, \Theta_0^{-1} \in L^\infty(\Omega)$.

Coupling between the thermal, elastic and electromagnetic aspects of the problem will occur when the above equations are complemented by suitable constitutive material relations. In [MPTW16, Section 3] this was achieved by utilising the material relations described in [Min74], which we now recall. In [Min74, Section 2] there are first provided the material relations

$$T = C \operatorname{Grad} u - eE - \lambda\theta, \tag{4.7}$$

$$D = e^* \operatorname{Grad} u + \varepsilon E + p\theta, \tag{4.8}$$

$$B = \mu H, \tag{4.9}$$

$$\eta = \lambda^* \operatorname{Grad} u + p^* E + \alpha\Theta_0^{-1}\theta, \tag{4.10}$$

where $C \in L(L_2(\Omega)^{3\times 3}_{\mathrm{sym}})$ (c.f. Definition 2.3.6) denotes the elasticity tensor, $\varepsilon, \mu \in L(L_2(\Omega)^3)$ are respectively the permittivity and permeability, $\alpha := \rho_* c \in L(L_2(\Omega))$ describes the product of the mass density $\rho_* \in L^\infty(\Omega)$ and the specific heat capacity $c \in L(L_2(\Omega))$ whereas $\theta\colon \mathbb{R}\times\Omega \to \mathbb{R}$ denotes the temperature. Here, the operators $e \in L(L_2(\Omega)^3; L_2(\Omega)^{3\times 3}_{\mathrm{sym}}), \lambda \in L(L_2(\Omega); L_2(\Omega)^{3\times 3}_{\mathrm{sym}})$ and $p \in L(L_2(\Omega); L_2(\Omega)^3)$ act as coupling parameters.

Denoting the strain tensor by $\mathcal{E} := \operatorname{Grad} u$ and replacing the temperature, θ, by the relative temperature, $\Theta_0^{-1}\theta$, as the unknown temperature function, the material relations then become

$$T = C\mathcal{E} - eE - (\lambda\Theta_0)\,\Theta_0^{-1}\theta, \tag{4.11}$$

$$D = e^*\mathcal{E} + \varepsilon E + (p\Theta_0)\,\Theta_0^{-1}\theta, \tag{4.12}$$

$$B = \mu H, \tag{4.13}$$

$$\Theta_0\eta = (\Theta_0\lambda^*)\,\mathcal{E} + (\Theta_0 p^*)\,E + \gamma_0\Theta_0^{-1}\theta, \tag{4.14}$$

where the shorthand $\gamma_0 := \Theta_0\alpha$ has been introduced. In adapting the above material relations to be suitable for the evolutionary equation perspective, the authors of [MPTW16] solve for the strain tensor, $\mathcal{E}$, so that (4.11) through (4.14) become

$$\mathcal{E} = C^{-1}T + C^{-1}eE + C^{-1}(\lambda\Theta_0)\,\Theta_0^{-1}\theta, \tag{4.15}$$

$$D = e^*C^{-1}T + \left(\varepsilon + e^*C^{-1}e\right)E + \left(p\Theta_0 + e^*C^{-1}\lambda\Theta_0\right)\Theta_0^{-1}\theta, \tag{4.16}$$

$$B = \mu H, \tag{4.17}$$

$$\Theta_0\eta = \Theta_0\lambda^*C^{-1}T + \left(\Theta_0 p^* + \Theta_0\lambda^*C^{-1}e\right)E + \left(\gamma_0 + \Theta_0\lambda^*C^{-1}\lambda\Theta_0\right)\Theta_0^{-1}\theta. \tag{4.18}$$

Approaching our PDE problem with the evolutionary equation perspective in mind, we complement each of the given basic system equations with an appropriate constitutive relation. This is a key facet of the evolutionary equation approach. The (standard) constitutive relations used in the sequel can be found for instance in [STW22, Sections 6.2, 7.1, 7.2]. The equation of elasticity, (4.3), is usually complemented by a suitable strain-stress relation. In this case, it is already implicitly present in (4.7) above. Indeed, setting the coupling parameters $e = \lambda = 0$ here allows us to recover a version of *Hooke's Law*,

$$T = C\operatorname{Grad} u, \tag{4.19}$$

in the context of elasticity (c.f. Subsection 2.3.2). Maxwell's equations, (4.4) and (4.5), are usually complemented by three constitutive relations, the first two of which couple the electric displacement, D, with the electric field, E, and the magnetic field, H, with the magnetic flux density, B, respectively. The former of these is similarly already implicitly provided by (4.8), which can be more clearly recognised upon setting the coupling parameters $e = p = 0$. The latter is indeed precisely (4.9). The third constitutive relation underpinning Maxwell's equations is provided by *Ohm's law* which relates the electrical charge to the electric field via the electrical resistivity. This is precisely the $-\sigma E$ term present in (4.5) (c.f. [STW22, Section 6.2]). Finally, the authors of [MPTW16] complement the heat equation, (4.6), with a version of *Fourier's law*. In particular, they assume that the *Maxwell-Cattaneo-Vernotte modification* holds, which relates the heat flux and temperature via

$$\partial_t \kappa_1 q + \kappa_0^{-1} q + \operatorname{grad} \theta = 0 \tag{4.20}$$

for operators $\kappa_0, \kappa_1 \in L(L_2(\Omega)^3)$. It is clear that this is a generalisation of Fourier's law since one readily recovers the usual form of this constitutive relation upon setting $\kappa_1 = 0$.

With the above preparations made we can now present the full system. As an evolutionary equation we have

$$\left(\partial_{t,\nu} \widehat{M_0} + \widehat{M_1} + \widehat{A}\right) \widehat{U} = \widehat{F}$$

on $L_{2,\nu}(\mathbb{R}; \widehat{H})$ where

$$\widehat{H} \coloneqq L_2(\Omega)^3 \oplus L_2(\Omega)^{3\times 3}_{\text{sym}} \oplus L_2(\Omega)^3 \oplus L_2(\Omega)^3 \oplus L_2(\Omega) \oplus L_2(\Omega)^3, \tag{4.21}$$

with the operators $\widehat{M_0}, \widehat{M_1}$ and $\widehat{A}$ to be specified next. In this formulation, we have the material operators

$$\widehat{M_0} = \begin{pmatrix} \rho_* & 0 & 0 & 0 & 0 & 0 \\ 0 & C^{-1} & C^{-1}e & 0 & C^{-1}\lambda\Theta_0 & 0 \\ 0 & e^*C^{-1} & (\varepsilon + e^*C^{-1}e) & 0 & (p\Theta_0 + e^*C^{-1}\lambda\Theta_0) & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & \Theta_0\lambda^*C^{-1} & (\Theta_0 p^* + \Theta_0\lambda^*C^{-1}e) & 0 & (\gamma_0 + \Theta_0\lambda^*C^{-1}\lambda\Theta_0) & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_1 \end{pmatrix}, \tag{4.22}$$

$$\widehat{M}_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \kappa_0^{-1} \end{pmatrix}, \tag{4.23}$$

and the spatial operator

$$\widehat{A} = \begin{pmatrix} 0 & -\operatorname{Div} & 0 & 0 & 0 & 0 \\ -\operatorname{Grad}_0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\operatorname{curl} & 0 & 0 \\ 0 & 0 & \operatorname{curl}_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \operatorname{div}_0 \\ 0 & 0 & 0 & 0 & \operatorname{grad} & 0 \end{pmatrix}, \tag{4.24}$$

with unknown

$$\widehat{U} = \left(v, T, E, H, \Theta_0^{-1}\theta, q\right) \tag{4.25}$$

and given right-hand side

$$\widehat{F} = (F_0, 0, F_2, F_3, F_4, 0). \tag{4.26}$$

In (4.25) the function $v := \partial_t u$ is treated as the first unknown for the equation of elasticity, (4.3), instead of the displacement, u. This is a standard trick and has been employed so as to treat the equation of elasticity as a first order in time equation (c.f. [STW22, Sections 6.2, 7.1, 7.2]).

At this point we would like to remind ourselves, and the reader, about the boundary conditions satisfied by the spatial operator (4.24) above. The operator $\widehat{A}$ satisfies a set of abstract homogeneous boundary conditions. Considering the action of the spatial operator (4.24) on the unknown (4.25), it is clear that we need to additionally assume $v \in H_0(\operatorname{Grad}, \Omega)$, $E \in H_0(\operatorname{curl}, \Omega)$ and $q \in H_0(\operatorname{div}, \Omega)$ (see Item (ii) and Item (iii) from Remark 2.3.4). It is then clear that (4.24) is skew-selfadjoint by construction (c.f. Definition 2.3.1).

With the underlying model of thermo-piezo-electromagnetism with homogeneous boundary conditions from [MPTW16, Sections 2, 3] now recalled, we turn our attention to the second of our aims for this subsection. To recall in detail the statement and proof of the systems evolutionary well-posedness, [MPTW16, Theorem 3.1].

Theorem 4.2.1 (Theorem 3.1, [MPTW16])**.** *Let $\Omega \subseteq \mathbb{R}^d$ be open and $\widehat{H}$ as in (4.21). Additionally, let $\widehat{M_0}, \widehat{M_1} \in L(\widehat{H})$ be as in (4.22), (4.23) and $\widehat{A}$ as in (4.24). Assume ρ_*, ε, μ, C, γ_0, κ_1 selfadjoint and non-negative. Furthermore, assume ρ_*, μ, C, $\gamma_0 \gg 0$ as well as*

$$\nu\left(\varepsilon - \Theta_0 p \gamma_0^{-1} p^* \Theta_0\right) + \sigma \gg 0 \tag{4.27}$$

and

$$\nu\kappa_1 + \kappa_0^{-1} \gg 0 \tag{4.28}$$

for large enough $\nu \in \mathbb{R}_{>0}$. Then, for all $\nu \in \mathbb{R}_{>0}$ sufficiently large, the operator

$$\partial_{t,\nu}\widehat{M_0} + \widehat{M_1} + \widehat{A} \tag{4.29}$$

is densely defined and closable in $L_{2,\nu}(\mathbb{R};\widehat{H})$. The respective closure is continuously invertible with causal inverse being eventually independent of ν.

Proof. The assertion follows from applying Theorem 2.4.4 to the material law given by

$$M(z) \coloneqq \widehat{M_0} + z^{-1}\widehat{M_1} \tag{4.30}$$

and spatial operator $\widehat{A}$. In the discussion above we noted how $\widehat{A}$ is skew-selfadjoint by construction. As such, we need only focus on establishing

$$zM(z) \gg 0$$

uniformly in $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$ for large enough $\nu \in \mathbb{R}_{>0}$. By several of the statement assumptions it is sufficient to consider the question of positive-definiteness for the sub-block operator

$$\nu \begin{pmatrix} C^{-1} & C^{-1}e & C^{-1}\lambda\Theta_0 \\ e^*C^{-1} & \varepsilon + e^*C^{-1}e & p\Theta_0 + e^*C^{-1}\lambda\Theta_0 \\ \Theta_0\lambda^*C^{-1} & \Theta_0 p^* + \Theta_0\lambda^*C^{-1}e & \gamma_0 + \Theta_0\lambda^*C^{-1}\lambda\Theta_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.31}$$

Under a symmetric Gauss step provided by the congruence transform

$$\mathcal{C}_1 = \begin{pmatrix} 1 & 0 & 0 \\ -e^* & 1 & 0 \\ -\Theta_0\lambda^* & 0 & 1 \end{pmatrix}, \tag{4.32}$$

we see that (4.31) is congruent to the operator

$$\nu \begin{pmatrix} C^{-1} & 0 & 0 \\ 0 & \varepsilon & p\Theta_0 \\ 0 & \Theta_0 p^* & \gamma_0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.33}$$

From here, we need only consider the sub-block operator

$$\nu \begin{pmatrix} \varepsilon & p\Theta_0 \\ \Theta_0 p^* & \gamma_0 \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}. \tag{4.34}$$

A second symmetric Gauss step provided by the congruence transform

$$\mathcal{C}_2 = \begin{pmatrix} 1 & -p\Theta_0\gamma_0^{-1} \\ 0 & 1 \end{pmatrix}, \tag{4.35}$$

reveals that the operator (4.34) is congruent to

$$\nu \begin{pmatrix} \varepsilon - \Theta_0 p\gamma_0^{-1}p^*\Theta_0 & 0 \\ 0 & \gamma_0 \end{pmatrix} + \begin{pmatrix} \sigma & 0 \\ 0 & 0 \end{pmatrix}, \tag{4.36}$$

which is positive-definite by the remaining statement assumptions. $\square$

Remark 4.2.2. (i) The statement assumptions allow for the case obtained when

$$\varepsilon = \Theta_0 p \gamma_0^{-1} p^* \Theta_0,$$

provided that the electrical resistance, σ, is large enough to compensate in (4.27). Such a case wherein the effect of the dielectricity, ε, is neglected arises in the study of *eddy currents*. The resulting system obtained is referred to as the *eddy current approximation* (see for instance the standard reference [RV10], the more recent [PP17], [Wau16a, Section 5.3], [Pic09, Subsection 4.1.1] or [STW22, Section 6.2].

(ii) Both of the positive-definite conditions (4.27) and (4.28) could have instead (and equivalently) been presented as requiring

$$\exists\, c_0 \in \mathbb{R}_{>0} : \nu \left(\varepsilon - \Theta_0 p \gamma_0^{-1} p^* \Theta_0\right) + \operatorname{Re} \sigma \geq c_0$$

and

$$\exists\, c_1 \in \mathbb{R}_{>0} : \nu \kappa_1 + \operatorname{Re} \kappa_0^{-1} \geq c_1,$$

respectively. Indeed, recalling Remark 2.4.5 confirms the coincidence in notation.

$\triangledown$

4.3 Formulating Our Extended Model

Having recalled the underlying thermo-piezo-electromagnetic system under homogeneous boundary conditions from [MPTW16] in Section 4.2, the stage is set for the formulation of our own extended model. We will realise this extension by the application of tools from the abstract boundary data space framework, as reviewed earlier in Chapter 3. We will extend from the system in Section 4.2 by first introducing three auxiliary Hilbert spaces, one for each of the elastic, electromagnetic and thermal aspects of the model. Upon each of these three spaces we will formulate a boundary equation governing the boundary dynamics of the respective material part of the model. Concerning ourselves with the use of abstract boundary data spaces, the auxiliary Hilbert spaces will be familiar to us from Proposition 3.2.3. By encoding these additional spaces and equations from within our system itself, we will actually enlargen the underlying six dimensional model to a nine dimensional one. In doing so we apply the methodology used in [Pic17] which was based on the observations and findings of [PSTW16].

We first recall the evolutionary equation from Section 4.2. There we had

$$\left(\partial_{t,\nu}\widehat{M_0} + \widehat{M_1} + \widehat{A}\right)\begin{pmatrix} v \\ T \\ E \\ H \\ \Theta_0^{-1}\theta \\ q \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \\ F_2 \\ F_3 \\ F_4 \\ 0 \end{pmatrix} \in L_{2,\nu}(\mathbb{R};\widehat{H}) \tag{4.37}$$

with the Hilbert space

$$\begin{aligned} \widehat{H} = & L_2\left(\Omega\right)^3 \oplus L_2\left(\Omega\right)^{3\times 3}_{\mathrm{sym}} \oplus \\ & L_2\left(\Omega\right)^3 \oplus L_2\left(\Omega\right)^3 \oplus \\ & L_2\left(\Omega\right) \oplus L_2\left(\Omega\right)^3 . \end{aligned} \tag{4.38}$$

We extend the above system by formally replacing

$$T \in H(\mathrm{Div},\Omega), \quad H \in H(\mathrm{curl},\Omega) \quad \text{and} \quad q \in H(\mathrm{div},\Omega)$$

in the vector of unknowns (4.25) by

$$\begin{pmatrix} T \\ \tau_T \end{pmatrix} \in L_2\left(\Omega\right)^{3\times 3}_{\mathrm{sym}} \oplus \mathrm{BD}\left(\mathrm{Grad}\right), \tag{4.39}$$

$$\begin{pmatrix} H \\ \tau_H \end{pmatrix} \in L_2\left(\Omega\right)^3 \oplus \mathrm{BD}\left(\mathrm{curl}\right) \ \text{ and} \tag{4.40}$$

$$\begin{pmatrix} q \\ \tau_q \end{pmatrix} \in L_2\left(\Omega\right)^3 \oplus \mathrm{BD}\left(\mathrm{grad}\right), \tag{4.41}$$

respectively. In order to accommodate for the formal replacements in our vector of unknowns, the remaining parts in the evolutionary equation (4.37) need to be suitably amended. As our extended evolutionary equation we obtain

$$\left(\partial_{t,\nu}M_0 + M_1\left(\partial_{t,\nu}\right) + A\right)\begin{pmatrix} v \\ \begin{pmatrix} T \\ \tau_T \end{pmatrix} \\ E \\ \begin{pmatrix} H \\ \tau_H \end{pmatrix} \\ \Theta_0^{-1}\theta \\ \begin{pmatrix} q \\ \tau_q \end{pmatrix} \end{pmatrix} = \begin{pmatrix} F_0 \\ \begin{pmatrix} 0 \\ f_1 \end{pmatrix} \\ F_2 \\ \begin{pmatrix} F_3 \\ f_3 \end{pmatrix} \\ F_4 \\ \begin{pmatrix} 0 \\ f_5 \end{pmatrix} \end{pmatrix} \in L_{2,\nu}\left(\mathbb{R};\mathcal{H}\right), \tag{4.42}$$

where now we have the Hilbert space

$$\begin{aligned}\mathcal{H} =& L_2\left(\Omega\right)^3 \oplus L_2\left(\Omega\right)^{3\times 3}_{\mathrm{sym}} \oplus \mathrm{BD}\left(\mathrm{Grad}\right) \oplus \\ & L_2\left(\Omega\right)^3 \oplus L_2\left(\Omega\right)^3 \oplus \mathrm{BD}\left(\mathrm{curl}\right) \oplus \\ & L_2\left(\Omega\right) \oplus L_2\left(\Omega\right)^3 \oplus \mathrm{BD}\left(\mathrm{grad}\right),\end{aligned} \tag{4.43}$$

with the correspondingly extended operators A, M_0 and $M_1\left(\partial_{t,\nu}\right)$ to be defined in a moment. We point out that replacing T by (4.39), H by (4.40) and q by (4.41) is done without the loss of any generality. One could instead replace the unknowns v, E and $\Theta_0^{-1}\theta$ in an analogous manner. Any difference in the form of the resulting block structure of (4.42) would be purely formal, and could be undone via congruence transforms as permutation operators.

In defining each of A, M_0 and $M_1\left(\partial_{t,\nu}\right)$ in turn, we will take the time to specify any important properties that they exhibit. We will also note any pertinent mathematical and modelling perspectives underpinning their construction and formulation. We have as our spatial operator

$$A :=$$

$$\begin{pmatrix} 0 & -\begin{pmatrix} -\operatorname{Grad} \\ \iota_{\operatorname{Grad}}^* \end{pmatrix}^* & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -\operatorname{Grad} \\ \iota_{\operatorname{Grad}}^* \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & -\begin{pmatrix} \operatorname{curl} \\ \iota_{\operatorname{curl}}^* \end{pmatrix}^* & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \operatorname{curl} \\ \iota_{\operatorname{curl}}^* \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & -\begin{pmatrix} \operatorname{grad} \\ \iota_{\operatorname{grad}}^* \end{pmatrix}^* \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \operatorname{grad} \\ \iota_{\operatorname{grad}}^* \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}. \tag{4.44}$$

It is clear from construction that A is skew-selfadjoint. Recall that the domains of the nonzero column adjoint operators in (4.44) each provide an additional boundary condition (c.f. Theorem 3.2.17) and must be taken into account. As these boundary conditions are in some way inherent to the system, we will refer to them in the sequel as *inherent boundary conditions*. Descending the upper diagonal stair of adjoint operators in A, recall that the corresponding inherent boundary conditions are

$$\iota_{\operatorname{Div}}^* T - \operatorname{Grad}_{\mathrm{BD}} \tau_T = 0, \tag{4.45}$$

$$\iota_{\operatorname{curl}}^* H + \operatorname{curl}_{\mathrm{BD}} \tau_H = 0 \text{ and} \tag{4.46}$$

$$\iota_{\operatorname{div}}^* q + \operatorname{grad}_{\mathrm{BD}} \tau_q = 0, \tag{4.47}$$

respectively. We point out, upon recalling Theorem 3.2.17 that the action of A can actually be computed with the help of the restriction

$$A \subseteq \left(\begin{array}{cccccc} 0 & \begin{pmatrix} -\operatorname{Div} & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} -\operatorname{Grad} \\ \iota_{\mathrm{Grad}}^{*} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} -\operatorname{curl} & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \operatorname{curl} \\ \iota_{\mathrm{curl}}^{*} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} \operatorname{div} & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} \operatorname{grad} \\ \iota_{\mathrm{grad}}^{*} \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{array}\right). \tag{4.48}$$

We now turn our attention to the material operators M_0 and $M_1(\partial_{t,\nu})$. We begin by specifying some notation. We first introduce the blocks

$$\begin{aligned} M_{0,33} &:= \begin{pmatrix} C^{-1} & 0 \\ 0 & \alpha_{33} \end{pmatrix}, M_{0,36} := \begin{pmatrix} C^{-1}e & 0 \\ 0 & \alpha_{36} \end{pmatrix}, M_{0,39} := \begin{pmatrix} 0 & 0 \\ 0 & \alpha_{39} \end{pmatrix}, \\ M_{0,66} &:= \begin{pmatrix} \mu & 0 \\ 0 & \alpha_{66} \end{pmatrix}, M_{0,69} := \begin{pmatrix} 0 & 0 \\ 0 & \alpha_{69} \end{pmatrix}, M_{0,99} := \begin{pmatrix} \kappa_1 & 0 \\ 0 & \alpha_{99} \end{pmatrix}. \end{aligned} \tag{4.49}$$

We assume that the coefficients

$$\alpha_{33} \in L(\mathrm{BD}(\operatorname{Grad})),\ \alpha_{66} \in L(\mathrm{BD}(\operatorname{curl})),\ \alpha_{99} \in L(\mathrm{BD}(\operatorname{grad})) \tag{4.50}$$

are selfadjoint. Moreover we assume

$$\alpha_{36} \in L(\mathrm{BD}(\operatorname{curl}), \mathrm{BD}(\operatorname{Grad})),\ \alpha_{39} \in L(\mathrm{BD}(\operatorname{grad}), \mathrm{BD}(\operatorname{Grad})) \tag{4.51}$$

and

$$\alpha_{69} \in L(\mathrm{BD}(\operatorname{grad}), \mathrm{BD}(\operatorname{curl})). \tag{4.52}$$

Doing so we obtain the operator

$$M_0 \coloneqq$$
$$\begin{pmatrix} \rho_* & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,33} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,36} & \begin{pmatrix} C^{-1}\lambda\Theta_0 \\ 0 \end{pmatrix} & M_{0,39} \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & \varepsilon + e^* C^{-1} e & \begin{pmatrix} 0 & 0 \end{pmatrix} & p\Theta_0 + e^* C^{-1}\lambda\Theta_0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & {M_{0,36}}^* & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,66} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,69} \\ 0 & \begin{pmatrix} \Theta_0\lambda^* C^{-1} & 0 \end{pmatrix} & \Theta_0 p^* + \Theta_0\lambda^* C^{-1} e & \begin{pmatrix} 0 & 0 \end{pmatrix} & \gamma_0 + \Theta_0\lambda^* C^{-1}\lambda\Theta_0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & {M_{0,39}}^* & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & {M_{0,69}}^* & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,99} \end{pmatrix}. \tag{4.53}$$

By construction M_0 is selfadjoint. The operator coefficients α_{ij}, for $i, j \in \{3, 6, 9\}$, allow us to involve the time derivative explicitly when formulating dynamics on the boundary. Recall that for our model boundary dynamics are posed on the auxiliary Hilbert spaces introduced in our extension above. We consider $M_1(z)$ next. We define the block operator

$$M_1(z) \coloneqq$$
$$\begin{pmatrix} 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,33}(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,36}(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,39}(z) \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & \sigma & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,63}(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,66}(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,69}(z) \\ 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,93}(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,96}(z) & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{1,99}(z) \end{pmatrix} \tag{4.54}$$

and assume, for $i, j \in \{3, 6, 9\}$, that the mapping $z \mapsto M_{1,ij}(z)$ is holomorphic, and that $\|M_{1,ij}\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}} < \infty$. In our application, the sub-block operators $M_{1,ij}(z)$ assume the form

$$M_{1,ij}(z) \coloneqq \begin{pmatrix} 0 & 0 \\ 0 & K_{ij}(z) \end{pmatrix}, \tag{4.55}$$

when $i,j\in\{3,6,9\}$ with $(i,j)\neq(9,9)$, and

$$M_{1,99}\left(z\right):=\begin{pmatrix}\kappa_0^{-1} & 0\\ 0 & K_{99}\left(z\right)\end{pmatrix},\tag{4.56}$$

when $(i,j)=(9,9)$. We assume that the $K_{ij}(z)$ boundary coefficients are linear, bounded and individually map between the same abstract boundary data spaces as the corresponding α_{ij} coefficient does (recall the blocks (4.49) and the mappings (4.50), (4.51) and (4.52)). For $i,j\in\{3,6,9\}$, the mapping $z\mapsto K_{ij}(z)$ absorbs the previous holomorphicity assumption. In a similar way, we assume $\|K_{ij}\|_{\infty,\mathbb{C}_{\mathrm{Re}>\nu}}<\infty$. It is not hard to see that under these assumptions $M_1(z)$ is a material law. Hence, we can replace z by $\partial_{t,\nu}$ in the underlying block operator and obtain

$$\begin{aligned}&M_1\left(\partial_{t,\nu}\right)=\\&\begin{pmatrix}0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & M_{1,33}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,36}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,39}\left(\partial_{t,\nu}\right)\\ 0 & \begin{pmatrix}0 & 0\end{pmatrix} & \sigma & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & M_{1,63}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,66}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,69}\left(\partial_{t,\nu}\right)\\ 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & M_{1,93}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,96}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,99}\left(\partial_{t,\nu}\right)\end{pmatrix}\end{aligned}\tag{4.57}$$

where, for $i,j\in\{3,6,9\}$ and $(i,j)\neq(9,9)$, we have the block operators

$$M_{1,ij}\left(\partial_{t,\nu}\right)=\begin{pmatrix}0 & 0\\ 0 & K_{ij}\left(\partial_{t,\nu}\right)\end{pmatrix}\tag{4.58}$$

and, when $(i,j)=(9,9)$,

$$M_{1,99}\left(\partial_{t,\nu}\right)=\begin{pmatrix}\kappa_0^{-1} & 0\\ 0 & K_{99}\left(\partial_{t,\nu}\right)\end{pmatrix}.\tag{4.59}$$

This yields the full material law operator (c.f. Proposition 2.2.5)

$$M\left(\partial_{t,\nu}\right) \coloneqq M_0 + \partial_{t,\nu}^{-1} M_1\left(\partial_{t,\nu}\right) \tag{4.60}$$

appearing in (4.42). At this point, we would like to make a minute observation. In the formulation of our material law operator (4.60), one could dispense with the α_{ij} boundary coefficients present in M_0, and instead combine their purpose with that of the $K_{ij}(\partial_{t,\nu})$ coefficients present in $M_1\left(\partial_{t,\nu}\right)$. It is the authors opinion that this difference is one merely in notation and taste.

In summary our extended system (4.42) encodes the following equations for each aspect of our thermo-piezo-electromagnetic model. Starting with the equations related to the elastic part we have

$$\partial_t \rho^* v - \operatorname{Div} T = F_0, \tag{4.61}$$

$$\partial_t C^{-1} T + \partial_t C^{-1} e H + \partial_t C^{-1} \lambda \Theta_0 \left(\Theta_0^{-1}\theta\right) - \operatorname{Grad} v = 0, \tag{4.62}$$

$$\begin{aligned}\left(\partial_t \alpha_{33} + K_{33}\left(\partial_{t,\nu}\right)\right)\tau_T &+ \left(\partial_t \alpha_{36} + K_{36}\left(\partial_{t,\nu}\right)\right)\tau_H \\ &+ \left(\partial_t \alpha_{39} + K_{39}\left(\partial_{t,\nu}\right)\right)\tau_q + \iota_{\mathrm{Grad}}^* v = f_1.\end{aligned} \tag{4.63}$$

The equations corresponding to the electromagnetic aspect of the problem read

$$\partial_t \left(\varepsilon + e^* C^{-1} e\right) E + \partial_t \left(p\Theta_0 + e^* C^{-1} \lambda \Theta_0\right) \Theta_0^{-1}\theta + \sigma E - \operatorname{curl} H = F_2, \tag{4.64}$$

$$\partial_t e^* C^{-1} T + \partial_t \mu H + \operatorname{curl} E = F_3, \tag{4.65}$$

$$\begin{aligned}\left(\partial_t \alpha_{36}^* + K_{63}\left(\partial_{t,\nu}\right)\right)\tau_T &+ \left(\partial_t \alpha_{66} + K_{66}\left(\partial_{t,\nu}\right)\right)\tau_H \\ &+ \left(\partial_t \alpha_{69} + K_{69}\left(\partial_{t,\nu}\right)\right)\tau_q + \iota_{\mathrm{curl}}^* E = f_3.\end{aligned} \tag{4.66}$$

Lastly, the equations corresponding to the thermal part of the problem are

$$\begin{aligned}\partial_t\left(\Theta_0\lambda^*C^{-1}\right)T+\partial_t\left(\Theta_0p^*+\Theta_0\lambda^*C^{-1}e\right)E\\+\partial_t\left(\gamma_0+\Theta_0\lambda^*C^{-1}\lambda\Theta_0\right)\Theta_0^{-1}\theta+\operatorname{div}q=F_4,\end{aligned}\tag{4.67}$$

$$\left(\partial_t\kappa_1+\kappa_0^{-1}\right)q+\operatorname{grad}\left(\Theta_0^{-1}\theta\right)=0,\tag{4.68}$$

$$\begin{aligned}\left(\partial_t\alpha_{39}^*+K_{93}\left(\partial_{t,\nu}\right)\right)\tau_T+\left(\partial_t\alpha_{69}^*+K_{96}\left(\partial_{t,\nu}\right)\right)\tau_H\\+\left(\partial_t\alpha_{99}+K_{99}\left(\partial_{t,\nu}\right)\right)\tau_q+\iota_{\mathrm{grad}}^*\left(\Theta_0^{-1}\theta\right)=f_5.\end{aligned}\tag{4.69}$$

As noted earlier, one cannot forget to take into account the presence of the inherent boundary conditions (4.45), (4.46) and (4.47). We can re-present the model entirely in terms of the original unknowns by substituting these boundary conditions into (4.63), (4.66) and (4.69), respectively. That is to say, one can transcribe our system equations entirely bereft of the presence of the dummy boundary τ_T,τ_H and τ_q variables.

4.4 Evolutionary Well-posedness

In this section we prove that our extended model for thermo-piezo-electromagnetism with boundary dynamics is well-posed as an evolutionary equation. The heart of the proof lies in the application of Picard's Theorem (Theorem 2.4.4) to the block operator system (4.42). At this juncture we recall the importance of Picard's Theorem. Indeed the solution theory of evolutionary equations is entirely encapsulated by Theorem 2.4.4. In particular, its application will allow us to establish Hadamard well-posedness as well as causal dependence on given data for our system (c.f. Remark 2.4.5).

As we will see in our own solution result, Theorem 4.4.6, many of the assumptions required to apply Picard's Theorem are already satisfied by our system. Indeed this is by construction. However the assumptions required to ensure that the operator

$$\operatorname{Re}zM(z)=\operatorname{Re}z\left(M_0+z^{-1}M_1\left(z\right)\right)\tag{4.70}$$

is positive-definite are perhaps not so obvious. Here M_0 and $M_1\left(z\right)$ are as specified in (4.53) and (4.54), respectively. In particular, it is this positive-definite question that we address in a systematic manner in our proof. In Lemma 4.4.1 we initially identify an operator congruent to (4.70). The congruent form obtained will allow us to determine conditions sufficient to assure the required accretivity in two steps. In Lemma 4.4.3 we first establish the required conditions corresponding to the material part of our system independent of z. Then in Lemma 4.4.4 we ascertain the conditions needed for the remaining material part of our system dependent on z. Our solution theory, Theorem 4.4.6, combines all three auxiliary results together with some additional observations to establish our extended model's well-posedness as an evolutionary equation.

Lemma 4.4.1 (Congruent Form). *Let $\nu \in \mathbb{R}_{>0}$ and $z \in \mathbb{C}_{\mathrm{Re}>\nu}$. Let M_0 be as in (4.53), $M_1(z)$ as in (4.54) and assume $\mu - e^* C^{-1} e$ and $\widetilde{m_{0,55}}$ to be continuously invertible. Then the operator $\nu M_0 + \operatorname{Re} M_1(z)$ is congruent to*

$$\begin{aligned} &\nu \widetilde{M_0} + \operatorname{Re} \widetilde{M_1}(z) \\ &= \nu \begin{pmatrix} \rho_* & 0 & 0 & 0 \\ 0 & \widetilde{M_{0,22}} & 0 & 0 \\ 0 & 0 & \widetilde{M_{0,66}} & 0 \\ 0 & 0 & 0 & \kappa_1 \end{pmatrix} + \operatorname{Re} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \widetilde{M_{1,22}} & 0 & 0 \\ 0 & 0 & \widetilde{M_{1,66}}(z) & 0 \\ 0 & 0 & 0 & \kappa_0^{-1} \end{pmatrix} \end{aligned} \tag{4.71}$$

under symmetric Gauss steps and permutations as congruent transformations. Here we obtain the sub-block operator

$$\widetilde{M_{0,22}} = \begin{pmatrix} C^{-1} & 0 & 0 & 0 \\ 0 & \mu - e^* C^{-1} e & 0 & 0 \\ 0 & 0 & \widetilde{m_{0,44}} & 0 \\ 0 & 0 & 0 & \widetilde{m_{0,55}} \end{pmatrix} \tag{4.72}$$

where

$$\widetilde{m_{0,55}} = \gamma_0 - \Theta_0 \lambda^* C^{-1} e \left(\mu - e^* C^{-1} e\right)^{-1} e^* C^{-1} \lambda \Theta_0 \tag{4.73}$$

and

$$\widetilde{m_{0,44}} = \varepsilon + e^{*}C^{-1}e - \left(p\Theta_0 + e^{*}C^{-1}\lambda\Theta_0\right)^{*}\left(\widetilde{m_{0,55}}\right)^{-1}\left(p\Theta_0 + e^{*}C^{-1}\lambda\Theta_0\right). \tag{4.74}$$

In (4.4.1) *we also obtain the sub-block operators*

$$\widetilde{M_{0,66}} = \begin{pmatrix} \alpha_{33} & \alpha_{36} & \alpha_{39} \\ \alpha_{36}^{*} & \alpha_{66} & \alpha_{69} \\ \alpha_{39}^{*} & \alpha_{69}^{*} & \alpha_{99} \end{pmatrix}, \tag{4.75}$$

$$\widetilde{M_{1,22}} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{4.76}$$

and

$$\widetilde{M_{1,66}}(z) = \begin{pmatrix} K_{33}(z) & K_{36}(z) & K_{39}(z) \\ K_{63}(z) & K_{66}(z) & K_{69}(z) \\ K_{93}(z) & K_{96}(z) & K_{99}(z) \end{pmatrix}. \tag{4.77}$$

Proof. There are a total of six congruence transforms to be applied in order to arrive at the target congruent form indicated in the statement of Lemma 4.4.1. The first congruence is provided by an elementary permutation of the entire system. We have

$$\begin{aligned} \mathcal{C}_1\left(\nu M_0 + \operatorname{Re} M_1(z)\right)\mathcal{C}_1^{*} &= \nu\,\mathcal{C}_1 M_0 \mathcal{C}_1^{*} + \operatorname{Re}\mathcal{C}_1 M_1(z)\,\mathcal{C}_1^{*} \\ &= \nu\,\widetilde{\mathcal{N}_1} + \operatorname{Re}\widetilde{\mathcal{M}}(z) \end{aligned} \tag{4.78}$$

under the permutation

$$\mathcal{C}_1 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}, \tag{4.79}$$

with $\widetilde{\mathcal{N}_1}$ and $\widetilde{\mathcal{M}}$ to be specified next. In (4.78) we have

$$\widetilde{\mathcal{N}_1} = \begin{pmatrix} \rho_* & 0 & 0 & 0 \\ 0 & \widetilde{\mathcal{N}_1}' & 0 & 0 \\ 0 & 0 & \widetilde{M_{0,66}} & 0 \\ 0 & 0 & 0 & \kappa_1 \end{pmatrix} \tag{4.80}$$

with

$$\widetilde{\mathcal{N}_1}' = \begin{pmatrix} \varepsilon + e^* C^{-1} e & 0 & 0 & p\Theta_0 + e^* C^{-1} \lambda \Theta_0 \\ 0 & C^{-1} & C^{-1} e & C^{-1} \lambda \Theta_0 \\ 0 & e^* C^{-1} & \mu & 0 \\ \Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e & \Theta_0 \lambda^* C^{-1} & 0 & \gamma_0 + \Theta_0 \lambda^* C^{-1} \lambda \Theta_0 \end{pmatrix} \tag{4.81}$$

and where $\widetilde{M_{0,66}}$ is as it already appears in the statement of Lemma 4.4.1. We also have in (4.78)

$$\widetilde{\mathcal{M}}(z) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & \widetilde{\mathcal{M}}' & 0 & 0 \\ 0 & 0 & \widetilde{M_{1,66}}(z) & 0 \\ 0 & 0 & 0 & \kappa_0^{-1} \end{pmatrix} \tag{4.82}$$

where

$$\widetilde{\mathcal{M}}' = \begin{pmatrix} \sigma & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{4.83}$$

and $\widetilde{M_{1,66}}(z)$ is already of the form indicated in the statement of Lemma 4.4.1. As such, we need now only occupy ourselves with the task of determining the intermediary congruence transforms under which the sub-block operator $\widetilde{\mathcal{N}_1}'$ in (4.78) is congruent to $\widetilde{M_{0,22}}$ in (4.72). The first of these intermediary sub-block congruences is given by the permutation operator

$$\mathcal{C}_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{4.84}$$

under which we have

$$\mathcal{C}_2 \widetilde{\mathcal{N}_1}' \mathcal{C}_2^* = \widetilde{\mathcal{N}_2} \tag{4.85}$$

with

$$\widetilde{\mathcal{N}_2} = \begin{pmatrix} C^{-1} & 0 & C^{-1}e & C^{-1}\lambda\Theta_0 \\ 0 & \varepsilon + e^*C^{-1}e & 0 & p\Theta_0 + e^*C^{-1}\lambda\Theta_0 \\ e^*C^{-1} & 0 & \mu & 0 \\ \Theta_0\lambda^*C^{-1} & \Theta_0 p^* + \Theta_0\lambda^*C^{-1}e & 0 & \gamma_0 + \Theta_0\lambda^*C^{-1}\lambda\Theta_0 \end{pmatrix}. \tag{4.86}$$

From here we apply the second intermediary sub-block congruence, given by the operator

$$\mathcal{C}_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -e^* & 0 & 1 & 0 \\ -\Theta_0\lambda^* & 0 & 0 & 1 \end{pmatrix}, \tag{4.87}$$

under which

$$\mathcal{C}_3\,\widetilde{\mathcal{N}_2}\,\mathcal{C}_3^* = \widetilde{\mathcal{N}_3} \tag{4.88}$$

where we have

$$\widetilde{\mathcal{N}_3} = \begin{pmatrix} C^{-1} & 0 & 0 & 0 \\ 0 & \varepsilon + e^*C^{-1}e & 0 & p\Theta_0 + e^*C^{-1}\lambda\Theta_0 \\ 0 & 0 & \mu - e^*C^{-1}e & -e^*C^{-1}\lambda\Theta_0 \\ 0 & \Theta_0 p^* + \Theta_0\lambda^*C^{-1}e & -\Theta_0\lambda^*C^{-1}e & \gamma_0 \end{pmatrix}. \tag{4.89}$$

From here it is sufficient to consider the sub-block operator

$$\widetilde{\mathcal{N}_3}' = \begin{pmatrix} \varepsilon + e^*C^{-1}e & 0 & p\Theta_0 + e^*C^{-1}\lambda\Theta_0 \\ 0 & \mu - e^*C^{-1}e & -e^*C^{-1}\lambda\Theta_0 \\ \Theta_0 p^* + \Theta_0\lambda^*C^{-1}e & -\Theta_0\lambda^*C^{-1}e & \gamma_0 \end{pmatrix}. \tag{4.90}$$

Under the permutation as a congruence transform given by

$$\mathcal{C}_4 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{4.91}$$

we obtain the congruent form

$$\mathcal{C}_4\,\widetilde{\mathcal{N}_3}'\,\mathcal{C}_4^* = \widetilde{\mathcal{N}_4} \tag{4.92}$$

with

$$\widetilde{\mathcal{N}_4} = \begin{pmatrix} \mu - e^*C^{-1}e & 0 & -e^*C^{-1}\lambda\Theta_0 \\ 0 & \varepsilon + e^*C^{-1}e & p\Theta_0 + e^*C^{-1}\lambda\Theta_0 \\ -\Theta_0\lambda^*C^{-1}e & \Theta_0 p^* + \Theta_0\lambda^*C^{-1}e & \gamma_0 \end{pmatrix}. \tag{4.93}$$

At this point we recall $\widetilde{\mathcal{M}}'$ as specified in (4.83). It is at the symmetric application of the congruence transform

$$\widetilde{\mathcal{C}_4} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

that we simultaneously obtain

$$\widetilde{\mathcal{C}_4}\,\widetilde{\mathcal{M}}'\widetilde{\mathcal{C}_4^*} = \widetilde{M_{1,22}}, \tag{4.94}$$

with $\widetilde{M_{1,22}}$ as given in the statement of Lemma 4.4.1. With this, we have obtained the entirety of the congruent form $\widetilde{M_1}(z)$ as specified in (4.71). From (4.93) we then apply the symmetric Gauss step as a congruence transform provided by

$$\mathcal{C}_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \Theta_0\lambda^* C^{-1}e\left(\mu - e^* C^{-1}e\right)^{-1} & 0 & 1 \end{pmatrix}, \tag{4.95}$$

under which

$$\mathcal{C}_5\,\widetilde{\mathcal{N}_4}\,\mathcal{C}_5^* = \widetilde{\mathcal{N}_5} \tag{4.96}$$

where we have

$$\widetilde{\mathcal{N}_5} = \begin{pmatrix} \mu - e^* C^{-1}e & 0 & 0 \\ 0 & \varepsilon + e^* C^{-1}e & p\Theta_0 + e^* C^{-1}\lambda\Theta_0 \\ 0 & \Theta_0 p^* + \Theta_0\lambda^* C^{-1}e & \widetilde{m_{0,55}} \end{pmatrix} \tag{4.97}$$

with

$$\widetilde{m_{0,55}} = \gamma_0 - \Theta_0\lambda^* C^{-1}e\left(\mu - e^* C^{-1}e\right)^{-1} e^* C^{-1}\lambda\Theta_0. \tag{4.98}$$

From here we need only consider the sub-block

$$\widetilde{\mathcal{N}_5}' = \begin{pmatrix} \varepsilon + e^* C^{-1}e & p\Theta_0 + e^* C^{-1}\lambda\Theta_0 \\ \Theta_0 p^* + \Theta_0\lambda^* C^{-1}e & \widetilde{m_{0,55}} \end{pmatrix}. \tag{4.99}$$

The application of another symmetric Gauss step as a congruence transform given by

$$\mathcal{C}_6 = \begin{pmatrix} 1 & -\left(p\Theta_0 + e^* C^{-1}\lambda\Theta_0\right)\left(\widetilde{m_{0,55}}\right)^{-1} \\ 0 & 1 \end{pmatrix} \tag{4.100}$$

yields

$$\mathcal{C}_6 \widetilde{\mathcal{N}_5}' \mathcal{C}_6^* = \widetilde{\mathcal{N}_7} \tag{4.101}$$

in which we have

$$\widetilde{\mathcal{N}_7} = \begin{pmatrix} \widetilde{m_{0,44}} & 0 \\ 0 & \widetilde{m_{0,55}} \end{pmatrix} \tag{4.102}$$

where

$$\widetilde{m_{0,44}} = \varepsilon + e^* C^{-1} e - \left(p\Theta_0 + e^* C^{-1}\lambda\Theta_0\right)^* \left(\widetilde{m_{0,55}}\right)^{-1} \left(p\Theta_0 + e^* C^{-1}\lambda\Theta_0\right). \tag{4.103}$$

With this final congruence we arrive at $\widetilde{M_{0,22}}$ as specified in (4.72) which completes the proof. □

Remark 4.4.2. (i) On account of the additional rows (and corresponding columns) added to accomodate for boundary dynamics, the congruence transforms applied above appear slightly different to those employed in [Pic17, Theorem 3.2]. The displacement caused by the addition of these rows and columns is the reason for obtaining "distorted" versions of the block operators recalled in the proof of Theorem 4.2.1.

(ii) Recall Item (i) from Remark 4.2.2 on the eddy current approximation. It is important to note that the manner in which we have applied our congruence transforms above in Lemma 4.4.1 and Lemma 4.4.3 is nontrivial. Indeed, the pattern we have employed allows us to retain the option of eddy current approximation in our model for thermo-piezo-electromagnetism with dynamic boundary conditions. Recall the operator (4.90) obtained above in the proof of Lemma 4.4.1.

There we had

$$\widetilde{\mathcal{N}_3}' = \begin{pmatrix} \varepsilon + e^* C^{-1} e & 0 & p\Theta_0 + e^* C^{-1} \lambda \Theta_0 \\ 0 & \mu - e^* C^{-1} e & -e^* C^{-1} \lambda \Theta_0 \\ \Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e & -\Theta_0 \lambda^* C^{-1} e & \gamma_0 \end{pmatrix}. \tag{4.104}$$

For what we would like to elucidate in this remark, suppose we additionally assumed that the operator ε be continuously invertible (which happens, for instance, when $\varepsilon \gg 0$). If instead of applying the permutation as a congruence transform (4.91) we applied the operator

$$\mathcal{C}_4 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\left(\Theta_0 p^* + \Theta_0 \lambda^* C^{-1} e\right) \left(\varepsilon + e^* C^{-1} e\right)^{-1} & 0 & 1 \end{pmatrix} \tag{4.105}$$

then we would obtain the operator

$$\widetilde{\mathcal{N}_4} = \begin{pmatrix} \varepsilon + e^* C^{-1} e & 0 & 0 \\ 0 & \mu - e^* C^{-1} e & -e^* C^{-1} \lambda \Theta_0 \\ 0 & -\Theta_0 \lambda^* C^{-1} e & \widetilde{\gamma_0} \end{pmatrix} \tag{4.106}$$

as congruent to $\widetilde{\mathcal{N}_3}'$, where we have introduced

$$\widetilde{\gamma_0} := \gamma_0 - \left(p\Theta_0 + e^* C^{-1} \lambda \Theta_0\right)^* \left(\varepsilon + e^* C^{-1} e\right)^{-1} \left(p\Theta_0 + e^* C^{-1} \lambda \Theta_0\right). \tag{4.107}$$

From here it is sufficient to consider the sub-block operator

$$\widetilde{\mathcal{N}_4}' = \begin{pmatrix} \mu - e^* C^{-1} e & -e^* C^{-1} \lambda \Theta_0 \\ -\Theta_0 \lambda^* C^{-1} e & \widetilde{\gamma_0} \end{pmatrix} \tag{4.108}$$

which is in turn congruent to the operator

$$\widetilde{\mathcal{N}_5} = \begin{pmatrix} \mu - e^* C^{-1} e & 0 \\ 0 & \widetilde{\gamma_0}' \end{pmatrix} \tag{4.109}$$

where we have introduced

$$\widetilde{\gamma}_0' := \gamma_0 - \left(e^*C^{-1}\lambda\Theta_0\right)^* \left(\mu - e^*C^{-1}e\right)^{-1} e^*C^{-1}\lambda\Theta_0 \\ - \left(p\Theta_0 + e^*C^{-1}\lambda\Theta_0\right)^* \left(\varepsilon + e^*C^{-1}e\right)^{-1} \left(p\Theta_0 + e^*C^{-1}\lambda\Theta_0\right). \tag{4.110}$$

The approach outlined in this remark could still be employed to establish sufficient conditions under which our model be well-posed as an evolutionary equation. However, should one want to retain the possibility of an eddy current approximation (again, see Item (i) from Remark 4.2.2), then the alternative approach suggested would obscure such a possibility. Whilst it would still be possible to choose the operator ε 'close' to $e^*C^{-1}e$, the limit case $\varepsilon = -e^*C^{-1}e$ is excluded.

(iii) In the statement of Lemma 4.4.1 we assumed both $\mu - e^*C^{-1}e$ and $\widetilde{m_{0,55}}$ to be continuously invertible. With regards to the full model system however, this assumption will already be satisfied on account of a wider positive-definite assumption (c.f. Item (i) of Remark 4.1.5 and Lemma 4.4.3). $\triangledown$

Our second auxiliary result addresses the accretivity required for that part of the model independent of any boundary considerations. In other words, for that part of the model independent of z. Whilst this might be obvious from the previous congruent form, we should like to formalise it here.

Lemma 4.4.3. *Let $\nu \in \mathbb{R}_{>0}$ and let $\widetilde{M_{0,22}}$, $\widetilde{m_{0,55}}$, $\widetilde{m_{0,44}}$ and $\widetilde{M_{1,22}}$ be as in (4.72), (4.73), (4.74) and (4.76), respectively. Assume ρ_*, ε, μ, C, γ_0, κ_1 selfadjoint and non-negative. Furthermore, assume ρ_*, C, $\widetilde{m_{0,55}}$, $\mu - e^*C^{-1}e \gg 0$ as well as*

$$\nu\,\widetilde{m_{0,44}} + \sigma \gg 0 \tag{4.111}$$

and

$$\nu\kappa_1 + \kappa_0^{-1} \gg 0 \tag{4.112}$$

for large enough $\nu \in \mathbb{R}_{>0}$. Then the block operator

$$\nu \begin{pmatrix} \rho_* & 0 & 0 \\ 0 & \widetilde{M_{0,22}} & 0 \\ 0 & 0 & \kappa_1 \end{pmatrix} + \mathrm{Re} \begin{pmatrix} 0 & 0 & 0 \\ 0 & \widetilde{M_{1,22}} & 0 \\ 0 & 0 & \kappa_0^{-1} \end{pmatrix} \tag{4.113}$$

is accretive for all $\nu \in \mathbb{R}_{>0}$ sufficiently large.

Proof. Immediately it is clear that the first and last rows of (4.113) are accretive by assumption. The required accretivity for the remaining sub-block will follow after considering the congruent form obtained in Lemma 4.4.1. Indeed, we have

$$\begin{aligned} &\nu\widetilde{M_{0,22}} + \mathrm{Re}\,\widetilde{M_{1,22}} \\ &= \nu \begin{pmatrix} C^{-1} & 0 & 0 & 0 \\ 0 & \mu - e^* C^{-1} e & 0 & 0 \\ 0 & 0 & \widetilde{m_{0,44}} & 0 \\ 0 & 0 & 0 & \widetilde{m_{0,55}} \end{pmatrix} + \mathrm{Re} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \sigma & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{4.114}$$

which is likewise accretive by the statement assumptions. □

The third of our auxiliary results addresses the positive-definiteness of that sub-block of the model accounting for the boundary dynamics of the system. In other words, for the part of the system dependent on z. Recall from Lemma 4.4.1 that both $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}(z)$ handle the boundary dynamics of our system via the sub-block operator

$$\nu\widetilde{M_{0,66}} + \mathrm{Re}\,\widetilde{M_{1,66}}(z)\,. \tag{4.115}$$

Before considering if the block operator (4.115) is positive-definite, we introduce some notational simplifications. First of all recall that, when written out in full, (4.115) has the form

$$\nu \begin{pmatrix} \alpha_{33} & \alpha_{36} & \alpha_{39} \\ \alpha_{36}^* & \alpha_{66} & \alpha_{69} \\ \alpha_{39}^* & \alpha_{69}^* & \alpha_{99} \end{pmatrix} + \mathrm{Re} \begin{pmatrix} K_{33}(z) & K_{36}(z) & K_{39}(z) \\ K_{63}(z) & K_{66}(z) & K_{69}(z) \\ K_{93}(z) & K_{96}(z) & K_{99}(z) \end{pmatrix}. \tag{4.116}$$

Upon computing the real-part of $\widetilde{M_{1,66}}(z)$, the operator (4.116) coincides with the block operator

$$\begin{pmatrix} \nu\alpha_{33} + \operatorname{Re} K_{33}(z) & L_{36}(z) & L_{39}(z) \\ L_{36}(z)^* & \nu\alpha_{66} + \operatorname{Re} K_{66}(z) & L_{69}(z) \\ L_{39}(z)^* & L_{69}(z)^* & \nu\alpha_{99} + \operatorname{Re} K_{99}(z) \end{pmatrix} \tag{4.117}$$

where, for notational ease, we have introduced for $i, j \in \{3, 6, 9\}$

$$\kappa_{ij}(z) \coloneqq \frac{1}{2}\left(K_{ij}(z) + K_{ji}(z)^*\right). \tag{4.118}$$

Clearly, we have for all $i, j \in \{3, 6, 9\}$

$$\kappa_{ij}(z)^* = \kappa_{ji}(z). \tag{4.119}$$

In (4.117) we define for $i, j \in \{3, 6, 9\}$ the operator

$$L_{ij}(z) \coloneqq \nu\alpha_{ij} + \kappa_{ij}(z), \tag{4.120}$$

and similarly it is clear that for all $i, j \in \{3, 6, 9\}$ we have

$$L_{ij}(z)^* = L_{ji}(z). \tag{4.121}$$

Moreover, we define the operators

$$U_{11}(z) \coloneqq \nu\alpha_{33} + \operatorname{Re} K_{33}(z), \tag{4.122}$$

$$U_{22}(z) \coloneqq \nu\alpha_{66} + \operatorname{Re} K_{66}(z) - L_{36}(z)^* U_{11}^{-1}(z) L_{36}(z), \tag{4.123}$$

$$U_{33}(z) \coloneqq \nu\alpha_{99} + \operatorname{Re} K_{99}(z) - L_{39}(z)^* U_{11}^{-1}(z) L_{39}(z) - \mathbb{L}^*(z) U_{22}^{-1}(z) \mathbb{L}(z) \tag{4.124}$$

and finally introduce

$$\mathbb{L}(z) \coloneqq L_{69}(z) - L_{36}(z)^* U_{11}^{-1}(z) L_{39}(z). \tag{4.125}$$

With these notational simplifications to hand we provide our next auxiliary result.

Lemma 4.4.4. *Let $\nu \in \mathbb{R}_{>0}$ and $z \in \mathbb{C}_{\mathrm{Re}>\nu}$. Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}(z)$ be as in (4.76) and (4.77), respectively. Assume*

$$\nu\alpha_{33} + \operatorname{Re} K_{33}(z) \gg 0, \tag{4.126}$$

$$\nu\alpha_{66} + \operatorname{Re} K_{66}(z) - L_{36}(z)^* U_{11}^{-1}(z) L_{36}(z) \gg 0, \tag{4.127}$$

$$\nu\alpha_{99} + \operatorname{Re} K_{99}(z) - L_{39}(z)^* U_{11}^{-1}(z) L_{39}(z) - \mathbb{L}^*(z) U_{22}^{-1}(z) \mathbb{L}(z) \gg 0, \tag{4.128}$$

for $\nu \in \mathbb{R}_{>0}$ sufficiently large. Then the block operator

$$\begin{aligned}
&\nu \widetilde{M_{0,66}} + \operatorname{Re} \widetilde{M_{1,66}}(z) \\
&= \begin{pmatrix} \nu\alpha_{33} + \operatorname{Re} K_{33}(z) & L_{36}(z) & L_{39}(z) \\ L_{36}(z)^* & \nu\alpha_{66} + \operatorname{Re} K_{66}(z) & L_{69}(z) \\ L_{39}(z)^* & L_{69}(z)^* & \nu\alpha_{99} + \operatorname{Re} K_{99}(z) \end{pmatrix}
\end{aligned}$$

is accretive for all $\nu \in \mathbb{R}_{>0}$ sufficiently large.

Proof. The discussion prior to the statement of Lemma 4.4.4 establishes equality between the operator (4.115) and (4.117), the latter of which we take as our starting point. Our aim is to obtain a suitable congruent form to which we can apply the statement assumptions from above. The desired congruent form will be obtained after the application of two symmetric Gauss steps. The first symmetric Gauss step is provided by the block operator

$$\mathcal{C}_1(z) = \begin{pmatrix} 1 & 0 & 0 \\ -L_{36}(z)^* U_{11}(z)^{-1} & 1 & 0 \\ -L_{39}(z)^* U_{11}(z)^{-1} & 0 & 1 \end{pmatrix}. \tag{4.129}$$

Under the congruence transform $\mathcal{C}_1(z)$ we obtain the congruent form

$$\mathcal{C}_1(z)\left[\nu\widetilde{M_{0,66}} + \operatorname{Re} \widetilde{M_{1,66}}(z)\right] \mathcal{C}_1(z)^* = \widetilde{\mathcal{N}_1}(z), \tag{4.130}$$

where we have the block operator

$$\widetilde{\mathcal{N}_1}(z) = \begin{pmatrix} U_{11}(z) & 0 & 0 \\ 0 & \widetilde{\mathcal{N}_{22}}(z) & \widetilde{\mathcal{N}_{23}}(z) \\ 0 & \widetilde{\mathcal{N}_{23}}(z)^* & \widetilde{\mathcal{N}_{33}}(z) \end{pmatrix}, \tag{4.131}$$

wherein

$$\widetilde{\mathcal{N}_{22}}(z) := \nu\alpha_{66} + \operatorname{Re} K_{66}(z) - L_{36}(z)^* U_{11}(z)^{-1} L_{36}(z), \tag{4.132}$$

$$\widetilde{\mathcal{N}_{33}}(z) := \nu\alpha_{99} + \operatorname{Re} K_{99}(z) - L_{39}(z)^* U_{11}(z)^{-1} L_{39}(z), \tag{4.133}$$

$$\widetilde{\mathcal{N}_{23}}(z) := L_{69}(z) - L_{36}(z)^* U_{11}(z)^{-1} L_{39}(z). \tag{4.134}$$

From this point it suffices to consider the sub-block operator

$$\widetilde{\mathcal{N}_1}'(z) = \begin{pmatrix} \widetilde{\mathcal{N}_{22}}(z) & \widetilde{\mathcal{N}_{23}}(z) \\ \widetilde{\mathcal{N}_{23}}(z)^* & \widetilde{\mathcal{N}_{33}}(z) \end{pmatrix}. \tag{4.135}$$

The second congruence transform is provided by the block operator

$$\mathcal{C}_2(z) = \begin{pmatrix} 1 & 0 \\ -\mathbb{L}(z)^* \mathcal{N}_{22}(z)^{-1} & 1 \end{pmatrix}, \tag{4.136}$$

under which we obtain the congruent form

$$\mathcal{C}_2(z)\widetilde{\mathcal{N}_1}'(z)\,\mathcal{C}_2(z)^* = \widetilde{\mathcal{N}_2}(z), \tag{4.137}$$

where

$$\widetilde{\mathcal{N}_2}(z) = \begin{pmatrix} U_{22}(z) & 0 \\ 0 & U_{33}(z) \end{pmatrix}. \tag{4.138}$$

From here we can apply the remaining statement assumptions in order to derive the desired accretivity claim. □

Remark 4.4.5. (i) Notice that each of the operators first appearing on the main-diagonal of (4.117)

$$\nu\alpha_{33} + \operatorname{Re} K_{33}(z), \quad \nu\alpha_{66} + \operatorname{Re} K_{66}(z), \quad \nu\alpha_{99} + \operatorname{Re} K_{99}(z),$$

are selfadjoint. Indeed, that the α_{ii} are selfadjoint for $i \in \{3, 6, 9\}$ follows from the fact that (4.53) is selfadjoint by construction. Moreover, recall that for a Hilbert space H and $T \in L(H)$ that the *real-part of* T is defined as $\operatorname{Re} T := \frac{1}{2}(T + T^*)$. As such, it follows immediately by definition that the real-part of an operator is itself selfadjoint.

(ii) Assumption (4.126) guarantees the existence and continuity of the inverse operator $U_{11}(z)^{-1}$ by Proposition 4.1.6. An analogous statement holds for assumption (4.127) and the inverse $U_{22}(z)^{-1}$. As such, both the congruence transform (4.129) and congruent form obtained in (4.136) involving these inverses are well-defined.

(iii) In (4.123) there arises the operator

$$L_{36}(z)^* U_{11}^{-1}(z) L_{36}(z)$$

on account of the congruence transform applied. Similarly, in (4.124) there arises the operator

$$L_{39}(z)^* U_{11}^{-1}(z) L_{39}(z) + \mathbb{L}^*(z) U_{22}^{-1}(z) \mathbb{L}(z).$$

All of these are non-negative operators by definition. As such, the assumptions (4.127) and (4.128) could be restated so as to require

$$\begin{aligned}
&\nu\alpha_{66} + \operatorname{Re} K_{66}(z) \gg L_{36}(z)^* U_{11}^{-1}(z) L_{36}(z), \\
&\nu\alpha_{99} + \operatorname{Re} K_{99}(z) \gg L_{39}(z)^* U_{11}^{-1}(z) L_{39}(z) + \mathbb{L}^*(z) U_{22}^{-1}(z) \mathbb{L}(z),
\end{aligned}$$

uniformly in $z \in \mathbb{C}_{\operatorname{Re} \geq \nu}$ for sufficiently large $\nu \in \mathbb{R}_{>0}$.

(iv) It is perhaps interesting to note that the congruent form obtained in Lemma 4.4.4 coincides with that obtained by an LDL^* *decomposition* (see [GVL13, Theorem 4.1.3], [DR06, Satz 3.34] or any of the references mentioned at the beginning of Section 4.1). Indeed, recalling the operator (4.117), we have

$$\begin{aligned}
&\begin{pmatrix} \nu\alpha_{33}+\operatorname{Re}K_{33}\left(z\right) & L_{36}\left(z\right) & L_{39}\left(z\right) \\ L_{36}\left(z\right)^{*} & \nu\alpha_{66}+\operatorname{Re}K_{66}\left(z\right) & L_{69}\left(z\right) \\ L_{39}\left(z\right)^{*} & L_{69}\left(z\right)^{*} & \nu\alpha_{99}+\operatorname{Re}K_{99}\left(z\right) \end{pmatrix} \\
&= L\left(z\right)\begin{pmatrix} U_{11}\left(z\right) & 0 & 0 \\ 0 & U_{22}\left(z\right) & 0 \\ 0 & 0 & U_{33}\left(z\right) \end{pmatrix} L\left(z\right)^{*},
\end{aligned} \tag{4.139}$$

where $L\left(z\right)$ is the congruence transform provided by the block operator

$$L\left(z\right)=\begin{pmatrix} 1 & 0 & 0 \\ L_{36}\left(z\right)^{*}U_{11}^{-1} & 1 & 0 \\ L_{39}\left(z\right)^{*}U_{11}^{-1} & \mathbb{L}\left(z\right)^{*}U_{22}^{-1} & 1 \end{pmatrix}. \tag{4.140}$$

We have avoided using the means afforded by an LDL^{*} decomposition in the prequel as the operator L in the factorisation need not be an atomic matrix (see Item (iv) of Remark 4.1.5). With that in mind, such an operator L (or more particularly $L\left(z\right)$ above) *cannot* be regarded as a symmetric Gauss step, as we defined in Section 4.1. $\triangledown$

We now combine all three of our auxiliary results together with some additional observations to state and prove our main result of interest.

Theorem 4.4.6. *Let $\nu\in\mathbb{R}_{>0}$ and $z\in\mathbb{C}_{\operatorname{Re}>\nu}$. Let $\Omega\subseteq\mathbb{R}^{d}$ be open and $\mathcal{H}$ as in (4.43). Additionally, let M_0, $M_1(z)\in L(\mathcal{H})$ be as in (4.53), (4.54), and A as in (4.44). In particular, let $\widetilde{m_{0,44}}$ and $\widetilde{m_{0,55}}$ be as in (4.74) and (4.73), respectively. Moreover, let $U_{11}(z)$, $U_{22}(z)$ and $U_{33}(z)$ be as in (4.122), (4.123), and (4.124), respectively. Assume ρ_*, ε, μ, C, γ_0, κ_1 selfadjoint and non-negative. Furthermore, assume ρ_*, C, $\widetilde{m_{0,55}}$, $\mu-e^{*}C^{-1}e$, $U_{11}(z)$, $U_{22}(z)$, $U_{33}(z)\gg 0$ as well as*

$$\nu\,\widetilde{m_{0,44}}+\sigma\gg 0 \tag{4.141}$$

and

$$\nu\kappa_1 + \kappa_0^{-1} \gg 0 \tag{4.142}$$

for large enough $\nu \in \mathbb{R}_{>0}$*. Then, for all* $\nu \in \mathbb{R}_{>0}$ *sufficiently large, the operator*

$$\partial_{t,\nu} M_0 + M_1(\partial_{t,\nu}) + A \tag{4.143}$$

is densely defined and closable in $L_{2,\nu}(\mathbb{R}; \mathcal{H})$*. The respective closure is continuously invertible with causal inverse being eventually independent of* ν*.*

Proof. The assertion follows from applying Theorem 2.4.4 to the material law

$$M(z) := M_0 + z^{-1} M_1(z) \tag{4.144}$$

and spatial operator A. From the discussion in the presentation of our model in Section 4.3, it is clear that A is skew-selfadjoint by construction. As such, we need only focus on establishing

$$zM(z) \gg 0$$

uniformly in $z \in \mathbb{C}_{\mathrm{Re} \geq \nu}$ for large enough $\nu \in \mathbb{R}_{>0}$. However, this follows immediately from the congruent forms and sub-block positive-definite estimates provided by Lemma 4.4.1, Lemma 4.4.3 and Lemma 4.4.4. $\square$

Chapter 5

Catalogue of Boundary Behaviours Covered by the Model

In this chapter we will conduct a systematic investigation of the boundary dynamics captured by the thermo-piezo-electromagnetic model proposed by this thesis. Recalling the evolutionary equation behind our extended model, (4.42), it is clear (if not by design) that the boundary dynamics of our system are accounted for by the sub-block operator equation

$$\left(\partial_{t,\nu}\,\widetilde{M_{0,66}}+\widetilde{M_{1,66}}\left(\partial_{t,\nu}\right)\right)\begin{pmatrix}\tau_T\\ \tau_H\\ \tau_q\end{pmatrix}+\begin{pmatrix}\iota_{\mathrm{Grad}}^{*}v\\ \iota_{\mathrm{curl}}^{*}E\\ \iota_{\mathrm{grad}}^{*}\left(\Theta_0^{-1}\theta\right)\end{pmatrix}=\begin{pmatrix}f_1\\ f_3\\ f_5\end{pmatrix}. \tag{5.1}$$

Here, the fixed column operator of orthogonal projections can be traced back to the action of the extended spatial operator, A (c.f. (4.44)). There are however several degrees of freedom in the (arbitrary) operator coefficients of the sub-block operator

$$\begin{aligned} & \partial_{t,\nu} \widetilde{M_{0,66}} + \widetilde{M_{1,66}} \left(\partial_{t,\nu}\right) \\ & = \partial_{t,\nu} \begin{pmatrix} \alpha_{33} & \alpha_{36} & \alpha_{39} \\ \alpha_{36}^* & \alpha_{66} & \alpha_{69} \\ \alpha_{39}^* & \alpha_{69}^* & \alpha_{99} \end{pmatrix} + \begin{pmatrix} K_{33}\left(\partial_{t,\nu}\right) & K_{36}\left(\partial_{t,\nu}\right) & K_{39}\left(\partial_{t,\nu}\right) \\ K_{63}\left(\partial_{t,\nu}\right) & K_{66}\left(\partial_{t,\nu}\right) & K_{69}\left(\partial_{t,\nu}\right) \\ K_{93}\left(\partial_{t,\nu}\right) & K_{96}\left(\partial_{t,\nu}\right) & K_{99}\left(\partial_{t,\nu}\right) \end{pmatrix} \end{aligned} \tag{5.2}$$

derived from the material relation blocks M_0 and $M_1\left(\partial_{t,\nu}\right)$ as defined in (4.53) and (4.57), respectively. Our investigation will be characterised by the consideration of different *cases* corresponding to different patterns of choice in the operator coefficients of (5.2). Each pattern of choice will allow us to recover a different set and arrangement of boundary conditions for our model. These are catalogued below. We will not entirely reinvent our general well-posedness result Theorem 4.4.6 for each case. Instead we will offer an alternative proof to the auxiliary result Lemma 4.4.4 which was used in our solution theory to address the accretivity of the sub-block operator

$$\begin{aligned} & \nu \widetilde{M_{0,66}} + \operatorname{Re} \widetilde{M_{1,66}} (z) \\ & = \nu \begin{pmatrix} \alpha_{33} & \alpha_{36} & \alpha_{39} \\ \alpha_{36}^* & \alpha_{66} & \alpha_{69} \\ \alpha_{39}^* & \alpha_{69}^* & \alpha_{99} \end{pmatrix} + \operatorname{Re} \begin{pmatrix} K_{33}(z) & K_{36}(z) & K_{39}(z) \\ K_{63}(z) & K_{66}(z) & K_{69}(z) \\ K_{93}(z) & K_{96}(z) & K_{99}(z) \end{pmatrix}. \end{aligned} \tag{5.3}$$

Each of the cases catalogued here can be thought of as a kind of corollary to our main solution theory. However we have endeavoured to provide an alternative direct proof to Lemma 4.4.4 for each case with the ends of variety and accessibility in mind.

Furthermore, we close this chapter with the presentation and consideration of an example of particular interest. With this example we realise one of the aims of this thesis. Namely, to *further* extend the piezo-electromagnetic impedance boundary conditions originally considered (in terms of classical traces) in [AN11, Section 1]. These we recalled earlier in (1.2). The basis for our own extension is provided by that presented in [Pic17, Subsection 4.3.1] wherein the boundary conditions (1.2) were translated to the setting of abstract boundary data spaces. The focus of the extension presented here is to account for the influence of a high-temperature regime. We begin by establishing some additional notation.

5.1 Additional Preliminaries

Recall from (4.43) that our *full* thermo-piezo-electromagnetic model incorporating boundary dynamics is posed on the Hilbert space

$$\begin{aligned}\mathcal{H} =& L_2(\Omega)^3 \oplus L_2(\Omega)^{3\times 3}_{\mathrm{sym}} \oplus \mathrm{BD}(\mathrm{Grad}) \oplus \\ & L_2(\Omega)^3 \oplus L_2(\Omega)^3 \oplus \mathrm{BD}(\mathrm{curl}) \oplus \\ & L_2(\Omega) \oplus L_2(\Omega)^3 \oplus \mathrm{BD}(\mathrm{grad})\,.\end{aligned} \tag{5.4}$$

In what follows it will be convenient to specify the subspace of $\mathcal{H}$ upon which the systems boundary dynamics are formulated. As such, we define

$$\mathcal{H}_{\mathrm{BD}} := \mathrm{BD}(\mathrm{Grad}) \oplus \mathrm{BD}(\mathrm{curl}) \oplus \mathrm{BD}(\mathrm{grad}) \tag{5.5}$$

as the direct sum of the three auxiliary (abstract boundary data) Hilbert spaces. Recall sub-block operator equation (5.1). Each successive row in (5.1) corresponds to the respective boundary equation for the piezo, electromagnetic and thermal aspects of the system, with each governing the boundary dynamics for that part of the model. Different choices of the operator coefficients in (5.2) will allow us to recover correspondingly different boundary dynamics. To that purpose, we introduce some notation which will help us to more readily recover such varying boundary conditions. For $i \in \{3, 6, 9\}$ we consider a general boundary equation of the form

$$\left(\partial_{t,\nu}\widetilde{\alpha_{3i}} + K_{i3}(\partial_{t,\nu})\right)\tau_T + \left(\partial_{t,\nu}\widetilde{\alpha_{6i}} + K_{i6}(\partial_{t,\nu})\right)\tau_H + \left(\partial_{t,\nu}\alpha_{i9} + K_{i9}(\partial_{t,\nu})\right)\tau_q + \widetilde{\iota_i^*} = f \tag{5.6}$$

where f is the given right-hand side and we specify

$$\widetilde{\alpha_{3i}} = \begin{cases} \alpha_{33} & \text{for } i = 3, \\ \alpha_{36}^{*} & \text{for } i = 6, \\ \alpha_{39}^{*} & \text{for } i = 9, \end{cases} \quad \widetilde{\alpha_{6i}} = \begin{cases} \alpha_{36} & \text{for } i = 3, \\ \alpha_{66} & \text{for } i = 6, \\ \alpha_{69}^{*} & \text{for } i = 9, \end{cases} \quad \widetilde{\iota_i^{*}} = \begin{cases} \iota_{\mathrm{Grad}}^{*} v & \text{for } i = 3, \\ \iota_{\mathrm{curl}}^{*} E & \text{for } i = 6, \\ \iota_{\mathrm{grad}}^{*} \left(\Theta_0^{-1}\theta\right) & \text{for } i = 9. \end{cases}$$

Notice that when $i = 3$ in (5.6) we obtain the boundary equation relating to the equation of elasticity (c.f. (4.45)). Similarly, taking $i = 6$ or $i = 9$ in (5.6) yields the boundary equations relating to Maxwell's equations (c.f. (4.46)) or the heat equation (c.f. (4.47)), respectively. By definition it is clear that our general boundary equation, (5.6), is formulated on $\mathcal{H}_{\mathrm{BD}}$.

Finally, we recall the next result (c.f. [STW22, Proposition 7.1.4]) which provides us with a useful characterisation for accretivity (c.f. Remark 2.4.5) which we will use frequently in what follows.

Proposition 5.1.1. *Let H be a Hilbert space and $N_0, N_1 \in L(H)$ with N_0 selfadjoint. Assume that there exist $c_0, c_1 \in \mathbb{R}_{>0}$ such that $\langle x, N_0 x\rangle \geq c_0\|x\|^2$ for all $x \in \operatorname{ran}(N_0)$ and that $\operatorname{Re}\langle y, N_1 y\rangle \geq c_1\|y\|^2$ for all $y \in \ker(N_0)$. Then, for all $0 < c_1^{'} < c_1$ there exists $\nu_0 > 0$ such that for all $\nu \geq \nu_0$ we have $\nu N_0 + \operatorname{Re} N_1 \geq c_1^{'}$.*

5.2 Robin, Dirichlet and Neumann Boundary Behaviour

In this section we consider the problem of modelling *elementary* boundary dynamics. By elementary, we refer to those boundary conditions which are rudimentary in the common mathematical sense. In particular, we will consider how Robin-type boundary conditions can be recovered. We will also look at the "problem" of accounting for Dirichlet and Neumann boundary conditions within our extended system.

5.2.1 Recovering Robin Boundary Conditions

We claim that Robin-type boundary conditions are the most rudimentary type of boundary behaviour which can be accounted for by our extended system. They are rudimentary in that they are the mathematically simplest form of boundary condition that can be recovered directly from within our model by an elementary pattern of choice in the boundary coefficients of (5.2). We now demonstrate how straightforward this pattern of choice is, as well as how it can be applied to recover a Robin boundary condition from the general boundary equation (5.6).

We first consider the problem of recovering a Robin-type boundary condition for the thermal part of our extended system. As we will make frequent use of them from now on, we invite the reader to recall the inherent boundary conditions as presented in (4.45), (4.46) and (4.47). Fix $i = 9$ in (5.6) and set the given right-hand side and all remaining operator coefficients from $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}\left(\partial_{t,\nu}\right)$ equal to zero save for $K_{99}\left(\partial_{t,\nu}\right) = 1$. This results in the greatly simplified boundary equation

$$\tau_q + \iota_{\mathrm{grad}}^* \left(\Theta_0^{-1}\theta\right) = 0 \iff \iota_{\mathrm{grad}}^* \left(\Theta_0^{-1}\theta\right) = -\tau_q. \tag{5.7}$$

Whilst the equivalent reformulation in the latter part of (5.7) might seem redundant, it allows us to highlight an important observation. On account of the action of ι_{grad}^* as orthogonal projector on $\mathrm{BD}(\mathrm{grad})$ we can identify $\iota_{\mathrm{grad}}^*(\Theta_0^{-1}\theta)$ as the value of the relative temperature on the boundary. At the same time τ_q can be thought of as a dummy variable in $\mathrm{BD}(\mathrm{grad}) \subseteq H^1(\Omega)$. As such, (5.7) can be understood as the *specification* of the value of $\Theta_0^{-1}\theta$ on the boundary *as* $-\tau_q$. In other words is $-\tau_q$ the Dirichlet boundary value of $\Theta_0^{-1}\theta$. We then substitute (5.7) into the inherent boundary condition for the heat equation, (4.47), and obtain the boundary condition

$$\iota_{\mathrm{div}}^* q - \mathrm{grad}_{\mathrm{BD}} \left(\iota_{\mathrm{grad}}^* \left(\Theta_0^{-1}\theta\right)\right) = 0. \tag{5.8}$$

This is a Robin boundary condition. To see this note firstly, with a perspective entirely analogous to that above, that $\iota_{\mathrm{div}}^* q$ is the Dirichlet boundary term of the heat flux $q \in H(\mathrm{div}, \Omega)$. Secondly by Proposition 3.2.5 we have $\mathrm{grad}_{\mathrm{BD}}(\iota_{\mathrm{grad}}^*(\Theta_0^{-1}\theta)) \in \mathrm{BD}(\mathrm{div})$, which can be identified as the Neumann boundary term.

Indeed, there is nothing stopping us from applying an analogous pattern of choice to both the piezo and electromagnetic boundary equations to recover Robin boundary conditions there also. Fix $i = 3$ in (5.6) and set the right-hand side and all operator coefficients equal to zero except for $K_{33}(\partial_{t,\nu}) = 1$. This results in the simplified boundary equation

$$\tau_T + \iota_{\mathrm{Grad}}^* v = 0 \tag{5.9}$$

which yields the Robin boundary condition

$$\iota_{\mathrm{Div}}^* T + \mathrm{Grad}_{\mathrm{BD}} \left(\iota_{\mathrm{Grad}}^* v\right) = 0 \tag{5.10}$$

when substituted into the corresponding inherent boundary condition, (4.45). Similarly, fixing $i = 6$ in (5.6) and applying an analogous pattern of choice reduces the general boundary equation to

$$\tau_H + \iota_{\mathrm{curl}}^* E = 0. \tag{5.11}$$

When (5.11) is substituted into the matching inherent boundary condition, (4.46), we obtain

$$\iota_{\mathrm{curl}}^* H - \mathrm{curl}_{\mathrm{BD}} \left(\iota_{\mathrm{curl}}^* E\right) = 0 \tag{5.12}$$

as the corresponding Robin boundary condition.

Remark 5.2.1. (i) We emphasise the dichotomy between the boundary equation as it appears encoded in the block operator system and the corresponding Robin boundary condition. The boundary equation (any of (5.7), (5.9) or (5.11)) are what we recover *directly* from our extended model for thermo-piezo-electromagnetism. The presence of the inherent boundary condition (any of (4.47), (4.45) and (4.46)) allows us to recover the corresponding Robin boundary condition via substitution, even though the Robin boundary condition itself (any of (5.8), (5.10) or (5.12)) is *not* what appears *directly* in our extended system. The correspondence between the two however is unambiguous.

(ii) The unitary property of the restricted spatial operators (recall Proposition 3.2.5) involved in the formulation of the inherent boundary conditions means that each of the above Robin boundary conditions can be viewed from another (unitarily equivalent) perspective. We show this for the Robin boundary condition for the piezo aspect of the system. Notice that (5.10) is posed in BD (Div). After applying $\mathrm{Grad}_{\mathrm{BD}}^{*} = \mathrm{Div}_{\mathrm{BD}}$ to (5.10) one instead (and equivalently) considers the Robin boundary condition

$$\mathrm{Div}\, \iota_{\mathrm{Div}}^{*} T + \iota_{\mathrm{Grad}}^{*} v = 0 \tag{5.13}$$

framed now in BD (Grad). Likewise, the above Robin boundary conditions for both the thermal and electromagnetic parts can instead also be considered equivalently as

$$\mathrm{div}\, \iota_{\mathrm{div}}^{*} T - \iota_{\mathrm{grad}}^{*} v = 0 \tag{5.14}$$

and

$$-\,\mathrm{curl}_{\mathrm{BD}}\, \iota_{\mathrm{curl}}^{*} H + \iota_{\mathrm{curl}}^{*} E = 0, \tag{5.15}$$

respectively. ∇

5.2.2 Full Robin Boundary Behaviour

In this case we consider the boundary behaviour characterised by Robin boundary dynamics across all three of the piezo, electromagnetic and thermal components of the system. Dealing only with Robin-type boundary conditions, this will prove to be the simplest of all of our cases covered by the model. Applying the pattern of choice discussed in Subsection 5.2.1 to each of the boundary equations (recall (5.6)) corresponding to the piezo, electromagnetic and thermal parts of the system, we arrive at the following set of boundary conditions. In the abstract boundary data space framework, we have

$$\iota_{\mathrm{Div}}^{*} T + \mathrm{Grad}_{\mathrm{BD}} \left(\iota_{\mathrm{Grad}}^{*} v\right) = 0, \tag{5.16}$$

$$\iota_{\mathrm{curl}}^{*} H - \mathrm{curl}_{\mathrm{BD}} \left(\iota_{\mathrm{curl}}^{*} E\right) = 0, \tag{5.17}$$

$$\iota_{\mathrm{div}}^{*} q - \mathrm{grad}_{\mathrm{BD}} \left(\iota_{\mathrm{grad}}^{*} \left(\Theta_0^{-1} \theta\right)\right) = 0. \tag{5.18}$$

Using the inherent boundary conditions (recall (4.45), (4.46) and (4.47)), we can encode these boundary conditions as the block operator equation

$$\left(\partial_{t,\nu}\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\right)\begin{pmatrix}\tau_T\\\tau_H\\\tau_q\end{pmatrix}+\begin{pmatrix}\iota^*_{\mathrm{Grad}}v\\\iota^*_{\mathrm{curl}}E\\\iota^*_{\mathrm{grad}}\left(\Theta_0^{-1}\theta\right)\end{pmatrix}=\begin{pmatrix}0\\0\\0\end{pmatrix}. \tag{5.19}$$

The well-posedness of the corresponding thermo-piezo-electromagnetic system under these boundary dynamics as an evolutionary equation then depends on the accretivity of the operator

$$\begin{aligned} &z\,\widetilde{M_{0,66}}+\widetilde{M_{1,66}}\,(z)\\ &=z\begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}+\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}. \end{aligned}\tag{5.20}$$

Even though this material law is clearly positive-definite, we should like to formalise this fact for our catalogue, and do so with the next result.

Corollary 5.2.2. *Let $\nu\in\mathbb{R}_{>0}$, $z\in\mathbb{C}_{\mathrm{Re}>\nu}$ and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}\,(z)$ be as in (5.20). Then the operator $z\,\widetilde{M_{0,66}}+\widetilde{M_{1,66}}\,(z)$ is accretive for all $\nu\in\mathbb{R}_{>0}$ sufficiently large.*

Proof. The assertion follows immediately since

$$\mathrm{Re}\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}=\begin{pmatrix}1_{\mathrm{BD(Grad)}}&0&0\\0&1_{\mathrm{BD(curl)}}&0\\0&0&1_{\mathrm{BD(grad)}}\end{pmatrix}\tag{5.21}$$

is the identity operator on $\mathcal{H}_{\mathrm{BD}}$ and is positive-definite by definition. □

Remark 5.2.3. (i) Assuming it exists, denote by n the outward unit normal. In the setting of classical traces the above boundary conditions correspond formally to

$$T \cdot n + v = 0 \text{ on } \partial\Omega, \tag{5.22}$$

$$H_t - n \times E_t = 0 \text{ on } \partial\Omega, \tag{5.23}$$

$$q \cdot n - \Theta_0^{-1}\theta = 0 \text{ on } \partial\Omega. \tag{5.24}$$

In this classical perspective, both boundary equations (5.22) and (5.24) explicitly involve (versions of) the Neumann trace. This is because both of the operators $\mathrm{grad}_{\mathrm{BD}}$ and $\mathrm{Grad}_{\mathrm{BD}}$ arising in the corresponding abstract boundary data space formulation are unitary. To see how this arises for the thermal Robin boundary condition, recall the abstract boundary data space formulation and compute

$$\iota_{\mathrm{div}}^* q - \mathrm{grad}_{\mathrm{BD}}\left(\iota_{\mathrm{grad}}^*\left(\Theta_0^{-1}\theta\right)\right) = 0 \iff \mathrm{div}_{\mathrm{BD}}\, \iota_{\mathrm{div}}^* q - \iota_{\mathrm{grad}}^*\left(\Theta_0^{-1}\theta\right) = 0. \tag{5.25}$$

The latter equation here indeed formally corresponds to (5.24) in the classical setting. An analogous computation holds for identifying (5.22) from (5.16).

(ii) In this case, were we to additionally allow for the inclusion of the diagonal operator coefficients α_{33}, α_{66} and α_{99} in the sub-block operator equation (5.1) we would obtain

$$\left(\partial_{t,\nu}\begin{pmatrix}\alpha_{33} & 0 & 0\\ 0 & \alpha_{66} & 0\\ 0 & 0 & \alpha_{99}\end{pmatrix} + \begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)\begin{pmatrix}\tau_T\\ \tau_H\\ \tau_q\end{pmatrix} + \begin{pmatrix}\iota_{\mathrm{Grad}}^* v\\ \iota_{\mathrm{curl}}^* E\\ \iota_{\mathrm{grad}}^*\left(\Theta_0^{-1}\theta\right)\end{pmatrix} = \begin{pmatrix}0\\ 0\\ 0\end{pmatrix}. \tag{5.26}$$

The corresponding boundary equations would then read

$$\begin{aligned}(\partial_{t,\nu}\alpha_{33} + 1)\tau_q + \iota_{\mathrm{grad}}^*\left(\Theta_0^{-1}\theta\right) &= 0,\\ (\partial_{t,\nu}\alpha_{66} + 1)\tau_T + \iota_{\mathrm{Grad}}^* v &= 0,\\ (\partial_{t,\nu}\alpha_{99} + 1)\tau_H + \iota_{\mathrm{curl}}^* E &= 0.\end{aligned} \tag{5.27}$$

It is obvious (c.f. Lemma 4.4.4) that the corresponding system is well-posed as an evolutionary equation when for $i \in \{3, 6, 9\}$ one assumes $\nu\alpha_{ii} + 1 \gg 0$ for $\nu \in \mathbb{R}_{>0}$ sufficiently large. $\triangledown$

5.2.3 Homogeneous Dirichlet and Neumann Boundary Conditions

As was indicated in Item (ii) and Item (iii) from Remark 2.3.4, abstract homogeneous Dirichlet and Neumann boundary conditions are usually accommodated for by specific assumptions on the domains of the spatial operators in use. Indeed it is these assumptions which yield the skew-selfadjointness of the spatial operator (4.24) in Section 4.2 which is required to apply Picard's Theorem.

Unlike the Robin-type boundary conditions considered above, homogeneous Dirichlet and Neumann boundary conditions cannot be recovered *directly* from our extended system for thermo-piezo-electromagnetism. We have already seen above how Robin boundary conditions could be recovered directly from the sub-block operator (5.2) by a specific pattern of choice in its operator coefficients. A formal modification of the block structure of the extended model is however required in order to capture any homogeneous Dirichlet or Neumann boundary behaviour. We elucidate this point with an example.

Example 5.2.1. Recall the extended system proposed in Chapter 4. However, instead of modelling arbitrary inhomogeneous boundary dynamics for the elastic part of the system, consider homogeneous Dirichlet boundary data. This is usually modelled by assuming that $u \in \operatorname{dom}(\operatorname{Grad}_0)$ (c.f. Item (ii) of Remark 2.3.4). Modelling this boundary behaviour amounts simply to replacing the piezo block in our extended system (4.42) with the corresponding block already employed in the underlying system (4.37) (c.f. Section 4.2). Doing so we obtain the evolutionary equation

$$\left(\partial_{t,\nu}M_0 + M_1\left(\partial_{t,\nu}\right) + A\right)\begin{pmatrix} v \\ T \\ E \\ \begin{pmatrix} H \\ \tau_H \end{pmatrix} \\ \Theta_0^{-1}\theta \\ \begin{pmatrix} q \\ \tau_q \end{pmatrix} \end{pmatrix} = \begin{pmatrix} F_0 \\ 0 \\ F_2 \\ \begin{pmatrix} F_3 \\ f_3 \end{pmatrix} \\ F_4 \\ \begin{pmatrix} 0 \\ f_5 \end{pmatrix} \end{pmatrix} \in L_{2,\nu}\left(\mathbb{R};\mathcal{H}\right) \tag{5.28}$$

posed now on the Hilbert space

$$\begin{aligned} \mathcal{H} = & L_2\left(\Omega\right)^3 \oplus L_2\left(\Omega\right)^{3\times3}_{\mathrm{sym}} \oplus \\ & L_2\left(\Omega\right)^3 \oplus L_2\left(\Omega\right)^3 \oplus \mathrm{BD}\left(\mathrm{curl}\right) \oplus \\ & L_2\left(\Omega\right) \oplus L_2\left(\Omega\right)^3 \oplus \mathrm{BD}\left(\mathrm{grad}\right), \end{aligned} \tag{5.29}$$

with the appropriately amended constituent block operators to be specified next. Notice how dispensing with the auxiliary boundary data space for the elastic part of our system precipitates a drop in the dimension of the extended system by one. In this case, we have

$$M_0 = \begin{pmatrix} \rho_* & 0 & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} & 0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ 0 & C^{-1} & 0 & \begin{pmatrix} C^{-1}e & 0 \end{pmatrix} & C^{-1}\lambda\Theta_0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ 0 & 0 & \varepsilon + e^*C^{-1}e & \begin{pmatrix} 0 & 0 \end{pmatrix} & p\Theta_0 + e^*C^{-1}\lambda\Theta_0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} e^*C^{-1} \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,66} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,69} \\ 0 & \Theta_0\lambda^*C^{-1} & \Theta_0 p^* + \Theta_0\lambda^*C^{-1}e & \begin{pmatrix} 0 & 0 \end{pmatrix} & \gamma_0 + \Theta_0\lambda^*C^{-1}\lambda\Theta_0 & \begin{pmatrix} 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & {M_{0,69}}^* & \begin{pmatrix} 0 \\ 0 \end{pmatrix} & M_{0,99} \end{pmatrix} \tag{5.30}$$

as well as

$$M_1\left(\partial_{t,\nu}\right)=\begin{pmatrix} 0 & 0 & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ 0 & 0 & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ 0 & 0 & \sigma & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,66}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,69}\left(\partial_{t,\nu}\right)\\ 0 & 0 & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,96}\left(\partial_{t,\nu}\right) & \begin{pmatrix}0\\0\end{pmatrix} & M_{1,99}\left(\partial_{t,\nu}\right) \end{pmatrix} \tag{5.31}$$

and

$$A=\begin{pmatrix} 0 & -\operatorname{Div} & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ -\operatorname{Grad}_0 & 0 & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ 0 & 0 & 0 & -\begin{pmatrix}\operatorname{curl}\\ \iota_{\mathrm{curl}}^*\end{pmatrix}^* & 0 & \begin{pmatrix}0 & 0\end{pmatrix}\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}\operatorname{curl}\\ \iota_{\mathrm{curl}}^*\end{pmatrix} & \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix}\\ 0 & 0 & 0 & \begin{pmatrix}0 & 0\end{pmatrix} & 0 & -\begin{pmatrix}\operatorname{grad}\\ \iota_{\mathrm{grad}}^*\end{pmatrix}^*\\ \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0\\0\end{pmatrix} & \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} & \begin{pmatrix}\operatorname{grad}\\ \iota_{\mathrm{grad}}^*\end{pmatrix} & \begin{pmatrix}0 & 0\\0 & 0\end{pmatrix} \end{pmatrix}. \tag{5.32}$$

Of course, it is not hard to show that this amended system is also well-posed as an evolutionary equation. △

Remark 5.2.4. From our view the need to formally modify the block structure in order to accommodate for such boundary conditions is no real shortcoming. Indeed, the task of modifying the model in this way is an unlaborious task. We are much more concerned about the potential provision afforded by the model to cater to scenarios involving much more complicated boundary dynamics. It is also worth bearing in mind that, within the realm of evolutionary equations, both the consideration and implementation of homogeneous Dirichlet and Neumann boundary conditions are already very well-known. Indeed, consider any of the examples presented in [STW22, Chapters 6, 7]. ▽

5.3 Mixed Boundary Behaviour I

In this case we consider the first of two situations concerning mixed-type boundary behaviour. This particular scenario is characterised by non-standard inhomogeneous boundary behaviour for one part of the system, and Robin boundary dynamics for the remaining two parts. This first instance of mixed boundary behaviour consists of three subcases, with each corresponding to a different arrangement of the aforementioned boundary dynamics across the piezo, electromagnetic and thermal components of the system. With the difference across each subcase being (mathematically speaking) purely symbolic, it suffices to prove that the real-part condition holds for any one of the three subcases without the loss of any generality.

5.3.1 Subcase (i)

This subcase comprises non-standard inhomogeneous boundary behaviour for the piezo part of the system, and Robin boundary behaviour for the remaining electromagnetic and thermal parts. Allowing the operator coefficients in the piezo boundary equation (c.f. (5.6)) to be arbitrary, and applying the pattern of choice discussed in Subsection 5.2.1 to the electromagnetic and thermal boundary equations, we arrive at the following set of boundary conditions. In the abstract boundary data space setting we have

$$\begin{aligned}(\partial_{t,\nu}\alpha_{33} + K_{33}(\partial_{t,\nu}))\,\mathrm{Div}_{\mathrm{BD}}\,\iota^*_{\mathrm{Div}}T + K_{36}(\partial_{t,\nu})\,\mathrm{curl}_{\mathrm{BD}}\,\iota^*_{\mathrm{curl}}H&\\ -K_{39}(\partial_{t,\nu})\,\mathrm{div}_{\mathrm{BD}}\,\iota^*_{\mathrm{div}}q + \iota^*_{\mathrm{Grad}}v = f_1&,\end{aligned} \tag{5.33}$$

$$\mathrm{curl}_{\mathrm{BD}}\,\iota^*_{\mathrm{curl}}H + \iota^*_{\mathrm{curl}}E = 0, \tag{5.34}$$

$$-\,\mathrm{div}_{\mathrm{BD}}\,\iota^*_{\mathrm{div}}q + \iota^*_{\mathrm{grad}}\left(\Theta_0^{-1}\theta\right) = 0. \tag{5.35}$$

With the use of the inherent boundary conditions (recall (4.45), (4.46) and (4.47)), these boundary conditions can be encoded as the block operator equation

$$\left(\partial_{t,\nu}\begin{pmatrix}\alpha_{33} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}K_{33}\left(\partial_{t,\nu}\right) & K_{36}\left(\partial_{t,\nu}\right) & K_{39}\left(\partial_{t,\nu}\right)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\right)\begin{pmatrix}\tau_T\\ \tau_H\\ \tau_q\end{pmatrix}+ \\ +\begin{pmatrix}\iota_{\mathrm{Grad}}^{*}v\\ \iota_{\mathrm{curl}}^{*}E\\ \iota_{\mathrm{grad}}^{*}\left(\Theta_0^{-1}\theta\right)\end{pmatrix}=\begin{pmatrix}f_1\\ 0\\ 0\end{pmatrix}. \tag{5.36}$$

Evolutionary well-posedness of the corresponding thermo-piezo-electromagnetic system under these boundary dynamics depends only on the positive-definiteness of the operator

$$\begin{aligned} &z\,\widetilde{M_{0,66}}+\widetilde{M_{1,66}}\left(z\right)\\ &=z\begin{pmatrix}\alpha_{33} & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}+\begin{pmatrix}K_{33}\left(z\right) & K_{36}\left(z\right) & K_{39}\left(z\right)\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}\end{aligned} \tag{5.37}$$

and is established in the next result.

Corollary 5.3.1. *Let $\nu\in\mathbb{R}_{>0}$, $z\in\mathbb{C}_{\mathrm{Re}>\nu}$, and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}\left(z\right)$ be as in (5.37). Assume there exists $c_{33}\in\mathbb{R}_{>0}$ such that $\alpha_{33}\geq c_{33}$. Then the operator $z\,\widetilde{M_{0,66}}+\widetilde{M_{1,66}}\left(z\right)$ is accretive for all $\nu\in\mathbb{R}_{>0}$ sufficiently large.*

Proof. The assertion will follow from applying Proposition 5.1.1 to the material law given by

$$M(z):=\widetilde{M_{0,66}}+z^{-1}\widetilde{M_{1,66}}\left(z\right) \tag{5.38}$$

which will allow us to establish the positive-definiteness of $zM(z)$ uniformly in $z\in\mathbb{C}_{\mathrm{Re}\geq\nu}$ for large enough $\nu\in\mathbb{R}_{>0}$. In the situation of Proposition 5.1.1 we have $N_0=\widetilde{M_{0,66}}$ and $N_1=\widetilde{M_{1,66}}\left(z\right)$. We first show that the restriction of $\widetilde{M_{0,66}}$ to its range is positive-definite. This restriction will be realised by the application of operators of the form indicated in Lemma 3.2.1. To that end we introduce the operator

$$\begin{aligned} \iota_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} : \mathrm{ran}\left(\widetilde{M_{0,66}}\right) &\to \mathcal{H}_{\mathrm{BD}} \\ x_1 &\mapsto \begin{pmatrix} x_1 \\ 0 \\ 0 \end{pmatrix} \end{aligned} \tag{5.39}$$

which has the block operator representation

$$\iota_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 1_{\mathrm{BD(Grad)}} \\ 0 \\ 0 \end{pmatrix}. \tag{5.40}$$

Its adjoint is the operator with the block operator representation

$$\iota^*_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 1_{\mathrm{BD(Grad)}} & 0 & 0 \end{pmatrix}. \tag{5.41}$$

The restriction of $\widetilde{M_{0,66}}$ to $\mathrm{ran}\left(\widetilde{M_{0,66}}\right)$ is then realised as

$$\iota^*_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} \widetilde{M_{0,66}}\, \iota_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} = \alpha_{33}. \tag{5.42}$$

For $x \in \mathrm{ran}\left(\widetilde{M_{0,66}}\right)$ we then compute

$$\left\langle x, \iota^*_{\mathrm{ker}\left(\widetilde{M_{0,66}}\right)} \widetilde{M_{0,66}}\, \iota_{\mathrm{ker}\left(\widetilde{M_{0,66}}\right)} x \right\rangle = \langle x, \alpha_{33} x\rangle \geq c_{33}\|x\|^2, \tag{5.43}$$

where we have used our statement assumption. Thus $\widetilde{M_{0,66}}$ is positive-definite on $\mathrm{ran}\left(\widetilde{M_{0,66}}\right)$. Next we need to show that the restriction of $\widetilde{M_{1,66}}(z)$ to $\ker\left(\widetilde{M_{0,66}}\right)$ is also positive-definite. To that end we compute

$$\mathrm{Re}\begin{pmatrix} K_{33}(z) & K_{36}(z) & K_{39}(z) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} \mathrm{Re}\, K_{33}(z) & \frac{1}{2}K_{36}(z) & \frac{1}{2}K_{39}(z) \\ \frac{1}{2}K_{36}(z)^* & 1 & 0 \\ \frac{1}{2}K_{39}(z)^* & 0 & 1 \end{pmatrix}. \tag{5.44}$$

Just as we have done above, in order to realise the specific restriction of this operator we introduce

$$\begin{aligned} \iota_{\ker\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 0 & 0 \\ 1_{\mathrm{BD(curl)}} & 0 \\ 0 & 1_{\mathrm{BD(grad)}} \end{pmatrix} &: \ker\left(\widetilde{M_{0,66}}\right) \to \mathcal{H}_{\mathrm{BD}} \\ &\begin{pmatrix} y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ y_2 \\ y_3 \end{pmatrix} \end{aligned} \tag{5.45}$$

together with its adjoint

$$\iota^*_{\ker\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 0 & 1_{\mathrm{BD(curl)}} & 0 \\ 0 & 0 & 1_{\mathrm{BD(grad)}} \end{pmatrix}. \tag{5.46}$$

The restriction of $\widetilde{M_{1,66}}(z)$ to $\ker\left(\widetilde{M_{0,66}}\right)$ is then realised as the identity operator

$$\iota^*_{\ker\left(\widetilde{M_{0,66}}\right)} \widetilde{M_{1,66}}(z)\, \iota_{\ker\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 1_{\mathrm{BD(curl)}} & 0 \\ 0 & 1_{\mathrm{BD(grad)}} \end{pmatrix} \tag{5.47}$$

which is positive-definite by definition, with a bound provided by the positive unit. By Proposition 5.1.1 it follows that for all choices of the scalar $c_1^{'} \in (0,1)$ there exists $\nu_0 \in \mathbb{R}_{>0}$ such that for all $\nu \geq \nu_0$ the operator $zM(z)$ is accretive uniformly in $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$. □

Remark 5.3.2. Assuming it exists, denote by n the outward unit normal. The above boundary conditions would correspond to

$$\begin{aligned} (\partial_{t,\nu}\alpha_{33} + K_{33}(\partial_{t,\nu}))(T\cdot n) + K_{36}(\partial_{t,\nu})(n\times H_t) \\ -K_{39}(\partial_{t,\nu})(q\cdot n) + v = f_1 \text{ on } \partial\Omega, \end{aligned} \tag{5.48}$$

$$(n\times H_t) + E_t = 0 \text{ on } \partial\Omega, \tag{5.49}$$

$$-(q\cdot n) + \Theta_0^{-1}\theta = 0 \text{ on } \partial\Omega \tag{5.50}$$

in the setting of classical traces. $\triangledown$

As indicated and explained in the introduction to this section, the remaining two subcases for our first instance of mixed boundary behaviour are detailed without proof. Indeed, one can work in a manner analogous to that employed in the proof of Corollary 5.3.1.

5.3.2 Subcase (ii)

The second subcase consists of non-standard inhomogeneous boundary behaviour for the electromagnetic part of the system, as well as Robin boundary behaviour for the remaining piezo and thermal parts. A completely analogous pattern of choice of operator coefficients in the boundary equations yields the abstract boundary conditions

$$\mathrm{Div}_{\mathrm{BD}}\,\iota_{\mathrm{Div}}^{*}T+\iota_{\mathrm{Grad}}^{*}v=0, \tag{5.51}$$

$$\begin{aligned}K_{63}\left(\partial_{t,\nu}\right)\mathrm{Div}_{\mathrm{BD}}\,\iota_{\mathrm{Div}}^{*}T+\left(\partial_{t,\nu}\alpha_{66}+K_{66}\left(\partial_{t,\nu}\right)\right)\mathrm{curl}_{\mathrm{BD}}\,\iota_{\mathrm{curl}}^{*}H&\\+K_{69}\left(\partial_{t,\nu}\right)\mathrm{div}_{\mathrm{BD}}\,\iota_{\mathrm{div}}^{*}q+\iota_{\mathrm{curl}}^{*}E=f_{3},&\end{aligned} \tag{5.52}$$

$$-\,\mathrm{div}_{\mathrm{BD}}\,\iota_{\mathrm{div}}^{*}q+\iota_{\mathrm{grad}}^{*}\left(\Theta_{0}^{-1}\theta\right)=0 \tag{5.53}$$

which we encode as the block operator equation

$$\begin{aligned}\left(\partial_{t,\nu}\begin{pmatrix}0&0&0\\0&\alpha_{66}&0\\0&0&0\end{pmatrix}+\begin{pmatrix}1&0&0\\K_{63}\left(\partial_{t,\nu}\right)&K_{66}\left(\partial_{t,\nu}\right)&K_{69}\left(\partial_{t,\nu}\right)\\0&0&1\end{pmatrix}\right)\begin{pmatrix}\tau_{T}\\\tau_{H}\\\tau_{q}\end{pmatrix}+&\\+\begin{pmatrix}\iota_{\mathrm{Grad}}^{*}v\\\iota_{\mathrm{curl}}^{*}E\\\iota_{\mathrm{grad}}^{*}\left(\Theta_{0}^{-1}\theta\right)\end{pmatrix}=\begin{pmatrix}0\\f_{3}\\0\end{pmatrix}.&\end{aligned} \tag{5.54}$$

Evolutionary well-posedness of the matching thermo-piezo-electromagnetic system is determined by the accretivity of the block operator

$$\begin{aligned} &z\,\widetilde{M_{0,66}} + \widetilde{M_{1,66}}\,(z) \\ &= z \begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_{66} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ K_{63}\,(z) & K_{66}\,(z) & K_{69}\,(z) \\ 0 & 0 & 1 \end{pmatrix} \end{aligned} \tag{5.55}$$

and is encapsulated by the next result.

Corollary 5.3.3. *Let $\nu \in \mathbb{R}_{>0}$, $z \in \mathbb{C}_{\mathrm{Re}>\nu}$, and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}\,(z)$ be as in (5.55). Assume there exists $c_{66} \in \mathbb{R}_{>0}$ such that $\alpha_{66} \geq c_{66}$. Then the operator $z\,\widetilde{M_{0,66}} + \widetilde{M_{1,66}}\,(z)$ is accretive for all $\nu \in \mathbb{R}_{>0}$ sufficiently large.*

Remark 5.3.4. Assuming it exists, denote by n the outward unit normal. The set of boundary conditions regarded in this subcase would correspond to

$$(T \cdot n) + v = 0 \text{ on } \partial\Omega, \tag{5.56}$$

$$\begin{aligned} K_{63}\,(\partial_{t,\nu})\,(T \cdot n) + (\partial_{t,\nu}\alpha_{66} + K_{66}\,(\partial_{t,\nu}))\,(n \times H_t) \\ -K_{69}\,(\partial_{t,\nu})\,(q \cdot n) + E_t = f_3 \text{ on } \partial\Omega, \end{aligned} \tag{5.57}$$

$$-\,(q \cdot n) + \Theta_0^{-1}\theta = 0 \text{ on } \partial\Omega \tag{5.58}$$

in the setting of classical traces. $\triangledown$

5.3.3 Subcase (iii)

The third and final subcase is composed of non-standard inhomogeneous boundary dynamics for the thermal part of the system, and Robin boundary conditions for the outstanding piezo and electromagnetic parts. Again, an analogously systematic choice of operator coefficients yields the abstract boundary conditions

$$\operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^* T + \iota_{\mathrm{Grad}}^* v = 0, \tag{5.59}$$

$$\operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^* H + \iota_{\mathrm{curl}}^* E = 0, \tag{5.60}$$

$$\begin{aligned} &K_{93}\left(\partial_{t,\nu}\right) \operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^* T + K_{96}\left(\partial_{t,\nu}\right) \operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^* H \\ &\qquad - \left(\partial_{t,\nu}\alpha_{99} + K_{99}\left(\partial_{t,\nu}\right)\right) \operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^* q + \iota_{\mathrm{grad}}^* \left(\Theta_0^{-1}\theta\right) = f_5 \end{aligned} \tag{5.61}$$

which we encode as

$$\begin{aligned} &\left(\partial_{t,\nu} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_{99} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ K_{93}\left(\partial_{t,\nu}\right) & K_{96}\left(\partial_{t,\nu}\right) & K_{99}\left(\partial_{t,\nu}\right) \end{pmatrix}\right) \begin{pmatrix} \tau_T \\ \tau_H \\ \tau_q \end{pmatrix} + \\ &\qquad + \begin{pmatrix} \iota_{\mathrm{Grad}}^* v \\ \iota_{\mathrm{curl}}^* E \\ \iota_{\mathrm{grad}}^* \left(\Theta_0^{-1}\theta\right) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ f_5 \end{pmatrix}. \end{aligned} \tag{5.62}$$

Like before, the evolutionary well-posedness of the corresponding thermo-piezo-electro-magnetic system depends on the positive-definiteness of the block operator

$$\begin{aligned} &z\, \widetilde{M_{0,66}} + \widetilde{M_{1,66}}\,(z) \\ &= z \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \alpha_{99} \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ K_{93}\,(z) & K_{96}\,(z) & K_{99}\,(z) \end{pmatrix} \end{aligned} \tag{5.63}$$

and is formalised in the next result.

Corollary 5.3.5. *Let $\nu \in \mathbb{R}_{>0}$, $z \in \mathbb{C}_{\mathrm{Re}>\nu}$, and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}\,(z)$ be as in (5.63). Assume there exists $c_{99} \in \mathbb{R}_{>0}$ such that $\alpha_{99} \geq c_{99}$. Then the operator $z\,\widetilde{M_{0,66}} + \widetilde{M_{1,66}}\,(z)$ is accretive for all $\nu \in \mathbb{R}_{>0}$ sufficiently large.*

Remark 5.3.6. Assuming it exists, denote by n the outward unit normal. The boundary conditions considered above correspond to

$$(T \cdot n) + v = 0 \text{ on } \partial\Omega, \tag{5.64}$$

$$(n \times H_t) + E_t = 0 \text{ on } \partial\Omega, \tag{5.65}$$

$$\begin{aligned} &K_{93}\left(\partial_{t,\nu}\right)(T \cdot n) + K_{96}\left(\partial_{t,\nu}\right)(n \times H_t) \\ &\quad - \left(\partial_{t,\nu}\alpha_{99} + K_{99}\left(\partial_{t,\nu}\right)\right)(q \cdot n) + \Theta_0^{-1}\theta = f_5 \text{ on } \partial\Omega \end{aligned} \tag{5.66}$$

in the classical boundary trace setting. ∇

5.4 Mixed Boundary Behaviour II

In this case we consider the second of our mixed-type boundary behaviour scenarios. This instance is characterised by non-standard inhomogeneous boundary behaviour for two parts of the system, and Robin boundary dynamics for the remaining part. As such, this second instance of mixed boundary behaviour is seemingly more complicated than that considered previously. Like before however, this instance of mixed boundary behaviour consists of three subcases. Again, each subcase here corresponds to a different placement of the distinct boundary dynamics across the piezo, electromagnetic and thermal aspects of the system. Similarly, and without the loss of any generality, it suffices to prove accretivity for any one of these three subcases.

5.4.1 Subcase (i)

This subcase comprises non-standard inhomogeneous boundary behaviour for the electromagnetic and thermal parts of the system, and Robin boundary behaviour for the remaining piezo part. We proceed in a manner analogous to the preceding mixed boundary behaviour case. Allowing the operator coefficients in the electromagnetic and thermal boundary equations (c.f. (5.6)) to be arbitrary, and applying the pattern of choice discussed in Subsection 5.2.1 to the piezo boundary equation, we arrive at the following set of boundary conditions. In the setting of abstract boundary data spaces we have

$$\operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^{*} T+\iota_{\mathrm{Grad}}^{*} v=0, \tag{5.67}$$

$$\begin{aligned} K_{63}\left(\partial_{t,\nu}\right) \operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^{*} T+\left(\partial_{t,\nu} \alpha_{66}+K_{66}\left(\partial_{t,\nu}\right)\right) \operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^{*} H& \\ -\left(\partial_{t,\nu} \alpha_{69}+K_{69}\left(\partial_{t,\nu}\right)\right) \operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^{*} q+\iota_{\mathrm{curl}}^{*} E&=f_{3}, \end{aligned} \tag{5.68}$$

$$\begin{aligned} K_{93}\left(\partial_{t,\nu}\right) \operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^{*} T+\left(\partial_{t,\nu} \alpha_{69}^{*}+K_{96}\left(\partial_{t,\nu}\right)\right) \operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^{*} H& \\ -\left(\partial_{t,\nu} \alpha_{99}+K_{99}\left(\partial_{t,\nu}\right)\right) \operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^{*} q+\iota_{\mathrm{grad}}^{*}\left(\Theta_{0}^{-1} \theta\right)&=f_{5}. \end{aligned} \tag{5.69}$$

With the use of the inherent boundary conditions (recall (4.45), (4.46) and (4.47)), the above boundary conditions can be encoded as the block operator equation

$$\begin{aligned} &\left(\partial_{t,\nu}\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_{66} & \alpha_{69} \\ 0 & \alpha_{69}^{*} & \alpha_{99} \end{pmatrix}+\begin{pmatrix} 1 & 0 & 0 \\ K_{63}\left(\partial_{t,\nu}\right) & K_{66}\left(\partial_{t,\nu}\right) & K_{69}\left(\partial_{t,\nu}\right) \\ K_{93}\left(\partial_{t,\nu}\right) & K_{96}\left(\partial_{t,\nu}\right) & K_{99}\left(\partial_{t,\nu}\right) \end{pmatrix}\right)\begin{pmatrix} \tau_{T} \\ \tau_{H} \\ \tau_{q} \end{pmatrix}+ \\ &\qquad +\begin{pmatrix} \iota_{\mathrm{Grad}}^{*} v \\ \iota_{\mathrm{curl}}^{*} E \\ \iota_{\mathrm{grad}}^{*}\left(\Theta_{0}^{-1} \theta\right) \end{pmatrix}=\begin{pmatrix} 0 \\ f_{3} \\ f_{5} \end{pmatrix}. \end{aligned} \tag{5.70}$$

Evolutionary well-posedness of the corresponding thermo-piezo-electromagnetic system under these mixed-type boundary dynamics depends then only on the accretivity of the operator

$$\begin{aligned} &z\, \widetilde{M_{0,66}}+\widetilde{M_{1,66}}(z) \\ &=z\begin{pmatrix} 0 & 0 & 0 \\ 0 & \alpha_{66} & \alpha_{69} \\ 0 & \alpha_{69}^{*} & \alpha_{99} \end{pmatrix}+\begin{pmatrix} 1 & 0 & 0 \\ K_{63}(z) & K_{66}(z) & K_{69}(z) \\ K_{93}(z) & K_{96}(z) & K_{99}(z) \end{pmatrix} \end{aligned} \tag{5.71}$$

which is proven in the next result.

Corollary 5.4.1. *Let $\nu \in \mathbb{R}_{>0}$, $z \in \mathbb{C}_{\mathrm{Re}>\nu}$, and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}(z)$ be as in (5.71). Assume there exist $c_{66}, \widetilde{c}_{99} \in \mathbb{R}_{>0}$ such that $\alpha_{66} \geq c_{66}$, $\alpha_{99}-\alpha_{69}^{*} \alpha_{66}^{-1} \alpha_{69} \geq \widetilde{c}_{99}$. Then the operator $z\, \widetilde{M_{0,66}}+\widetilde{M_{1,66}}(z)$ is accretive for all*

$\nu \in \mathbb{R}_{>0}$ *sufficiently large.*

Proof. The proof follows the same line of reasoning as that of Corollary 5.3.1. Applying Proposition 5.1.1 to the material law given by

$$M(z) \coloneqq \widetilde{M_{0,66}} + z^{-1}\widetilde{M_{1,66}}\,(z) \tag{5.72}$$

will allow us to establish the desired accretivity of $zM(z)$ uniformly in $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$ for large enough $\nu \in \mathbb{R}_{>0}$. Once again, in the situation of Proposition 5.1.1 we have $N_0 = \widetilde{M_{0,66}}$ and $N_1 = \widetilde{M_{1,66}}\,(z)$. To that end we first show that the restriction of $\widetilde{M_{0,66}}$ to its range is positive-definite. To actualise this restriction we introduce the operator

$$\begin{aligned}\iota_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 0 & 0 \\ 1_{\mathrm{BD(curl)}} & 0 \\ 0 & 1_{\mathrm{BD(grad)}} \end{pmatrix} &: \mathrm{ran}\left(\widetilde{M_{0,66}}\right) \to \mathcal{H}_{\mathrm{BD}} \\ &\quad \begin{pmatrix} x_2 \\ x_3 \end{pmatrix} \mapsto \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix}\end{aligned} \tag{5.73}$$

together with its adjoint

$$\begin{aligned}\iota^*_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 0 & 1_{\mathrm{BD(curl)}} & 0 \\ 0 & 0 & 1_{\mathrm{BD(grad)}} \end{pmatrix} &: \mathcal{H}_{\mathrm{BD}} \to \mathrm{ran}\left(\widetilde{M_{0,66}}\right) \\ &\quad \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \begin{pmatrix} y_2 \\ y_3 \end{pmatrix}.\end{aligned} \tag{5.74}$$

The restriction of $\widetilde{M_{0,66}}$ to $\mathrm{ran}\left(\widetilde{M_{0,66}}\right)$ is then realised as

$$\iota^*_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)}\,\widetilde{M_{0,66}}\,\iota_{\mathrm{ran}\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} \alpha_{66} & \alpha_{69} \\ \alpha_{69}^* & \alpha_{99} \end{pmatrix} \tag{5.75}$$

which, via a rudimentary symmetric Gauss-step, is itself congruent to

$$\begin{pmatrix} \alpha_{66} & 0 \\ 0 & \alpha_{99} - \alpha_{69}^{*}\alpha_{66}^{-1}\alpha_{69} \end{pmatrix}. \tag{5.76}$$

For $x \in \operatorname{ran}\left(\widetilde{M_{0,66}}\right)$ we compute

$$\begin{aligned}
\left\langle x, \iota^{*}_{\operatorname{ran}\left(\widetilde{M_{0,66}}\right)} \widetilde{M_{0,66}}\, \iota_{\operatorname{ran}\left(\widetilde{M_{0,66}}\right)} x \right\rangle &= \left\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} \alpha_{66} & 0 \\ 0 & \alpha_{99} - \alpha_{69}^{*}\alpha_{66}^{-1}\alpha_{69} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\rangle \\
&= \langle x_1, \alpha_{66} x_1 \rangle + \langle x_2, \left(\alpha_{99} - \alpha_{69}^{*}\alpha_{66}^{-1}\alpha_{69}\right) x_2 \rangle \\
&\geq \min\left\{c_{66}, \widetilde{c}_{99}\right\} \|x\|^2,
\end{aligned}$$

where we have used both of our statement assumptions. Hence $\widetilde{M_{0,66}}$ is positive-definite on its range as required. Secondly, we show that the restriction of $\widetilde{M_{1,66}}(z)$ to $\ker\left(\widetilde{M_{0,66}}\right)$ is positive-definite. We compute

$$\operatorname{Re}\begin{pmatrix} 1 & 0 & 0 \\ K_{63}(z) & K_{66}(z) & K_{69}(z) \\ K_{93}(z) & K_{96}(z) & K_{99}(z) \end{pmatrix} = \begin{pmatrix} 1 & \frac{1}{2}K_{63}(z)^{*} & \frac{1}{2}K_{93}(z)^{*} \\ \frac{1}{2}K_{63}(z)^{*} & \operatorname{Re}K_{66}(z) & \kappa_{69}(z) \\ \frac{1}{2}K_{93}(z)^{*} & \kappa_{69}(z)^{*} & \operatorname{Re}K_{99}(z) \end{pmatrix} \tag{5.77}$$

where, to the end of simplicity, we have recalled the $\kappa_{ij}(z)$ notation for $i, j \in \{3, 6, 9\}$ from (4.118). The appropriate restriction will be realised after introducing the operator

$$\begin{aligned}
\iota_{\ker\left(\widetilde{M_{0,66}}\right)} : \ker\left(\widetilde{M_{0,66}}\right) &\to \mathcal{H}_{\mathrm{BD}} \\
y &\mapsto \begin{pmatrix} y \\ 0 \\ 0 \end{pmatrix}
\end{aligned} \tag{5.78}$$

which exhibits the block operator representation

$$\iota_{\ker\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 1_{\mathrm{BD(Grad)}} \\ 0 \\ 0 \end{pmatrix}. \tag{5.79}$$

Its adjoint is the operator with the block operator representation

$$\iota^*_{\ker\left(\widetilde{M_{0,66}}\right)} = \begin{pmatrix} 1_{\mathrm{BD(Grad)}} & 0 & 0 \end{pmatrix}. \tag{5.80}$$

The restriction of $\widetilde{M_{1,66}}(z)$ to $\ker\left(\widetilde{M_{0,66}}\right)$ is then given by

$$\iota^*_{\ker\left(\widetilde{M_{0,66}}\right)} \widetilde{M_{1,66}}(z)\ \iota_{\ker\left(\widetilde{M_{0,66}}\right)} = 1_{\mathrm{BD(Grad)}}. \tag{5.81}$$

For $y \in \ker\left(\widetilde{M_{0,66}}\right)$ we then compute

$$\left\langle y, \iota^*_{\ker\left(\widetilde{M_{0,66}}\right)} \operatorname{Re}\widetilde{M_{1,66}}(z)\ \iota_{\ker\left(\widetilde{M_{0,66}}\right)} y \right\rangle = \|y\|^2, \tag{5.82}$$

so that $\widetilde{M_{1,66}}(z)$ is positive-definite on $\ker\left(\widetilde{M_{0,66}}\right)$ with bound provided by the positive unit. By Proposition 5.1.1 it follows for all choices of $c_1^{'} \in (0,1)$ that there exists $\nu_0 \in \mathbb{R}_{>0}$ so that for all $\nu \geq \nu_0$ the operator $zM(z)$ is accretive uniformly in $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$. □

Remark 5.4.2. Assuming it exists, denote by n the outward unit normal. The boundary conditions above correspond to

$$T \cdot n + v = 0 \text{ on } \partial\Omega, \tag{5.83}$$

$$\begin{aligned} &K_{63}(\partial_{t,\nu})(T\cdot n) + (\partial_{t,\nu}\alpha_{66} + K_{66}(\partial_{t,\nu}))(n \times H_t) \\ &\qquad - (\partial_{t,\nu}\alpha_{69} + K_{69}(\partial_{t,\nu}))(q\cdot n) + E_t = f_3 \text{ on } \partial\Omega, \end{aligned} \tag{5.84}$$

$$\begin{aligned} &K_{93}(\partial_{t,\nu})(T\cdot n) + (\partial_{t,\nu}\alpha^*_{69} + K_{96}(\partial_{t,\nu}))(n \times H_t) \\ &\qquad - (\partial_{t,\nu}\alpha_{99} + K_{99}(\partial_{t,\nu}))(q\cdot n) + \Theta_0^{-1}\theta = f_5 \text{ on } \partial\Omega \end{aligned} \tag{5.85}$$

in the setting of classical traces. ▽

As noted and justified in the introduction to this section, the remaining two subcases in this scenario of mixed boundary behaviour are provided without proof. The above proof of Corollary 5.4.1 can be easily amended to fit either of the remaining subcases.

5.4.2 Subcase (ii)

Our second subcase consists of non-standard inhomogeneous boundary behaviour for the piezo and thermal parts of the system, as well as Robin boundary behaviour for the remaining electromagnetic part. A completely analogous pattern of choice in the operator coefficients of the corresponding boundary equations yields the abstract boundary conditions

$$\begin{aligned}\left(\partial_{t,\nu}\alpha_{33}+K_{33}\left(\partial_{t,\nu}\right)\right)\operatorname{Div}_{\mathrm{BD}}\iota_{\mathrm{Div}}^{*}T+K_{36}\left(\partial_{t,\nu}\right)\operatorname{curl}_{\mathrm{BD}}\iota_{\mathrm{curl}}^{*}H&\\ -\left(\partial_{t,\nu}\alpha_{39}+K_{39}\left(\partial_{t,\nu}\right)\right)\operatorname{div}_{\mathrm{BD}}\iota_{\mathrm{div}}^{*}q+\iota_{\mathrm{Grad}}^{*}v=f_{1},&\end{aligned}\tag{5.86}$$

$$\operatorname{curl}_{\mathrm{BD}}\iota_{\mathrm{curl}}^{*}H+\iota_{\mathrm{curl}}^{*}E=0,\tag{5.87}$$

$$\begin{aligned}\left(\partial_{t,\nu}\alpha_{39}^{*}+K_{93}\left(\partial_{t,\nu}\right)\right)\operatorname{Div}_{\mathrm{BD}}\iota_{\mathrm{Div}}^{*}T+K_{96}\left(\partial_{t,\nu}\right)\operatorname{curl}_{\mathrm{BD}}\iota_{\mathrm{curl}}^{*}H&\\ -\left(\partial_{t,\nu}\alpha_{99}+K_{99}\left(\partial_{t,\nu}\right)\right)\operatorname{div}_{\mathrm{BD}}\iota_{\mathrm{div}}^{*}q+\iota_{\mathrm{grad}}^{*}\left(\Theta_{0}^{-1}\theta\right)=f_{5}&\end{aligned}\tag{5.88}$$

which are encoded as the block operator equation

$$\begin{aligned}&\left(\partial_{t,\nu}\begin{pmatrix}\alpha_{33}&0&\alpha_{39}\\0&0&0\\\alpha_{39}^{*}&0&\alpha_{99}\end{pmatrix}+\begin{pmatrix}K_{33}\left(\partial_{t,\nu}\right)&K_{36}\left(\partial_{t,\nu}\right)&K_{39}\left(\partial_{t,\nu}\right)\\0&1&0\\K_{93}\left(\partial_{t,\nu}\right)&K_{96}\left(\partial_{t,\nu}\right)&K_{99}\left(\partial_{t,\nu}\right)\end{pmatrix}\right)\begin{pmatrix}\tau_{T}\\\tau_{H}\\\tau_{q}\end{pmatrix}+\\&\qquad+\begin{pmatrix}\iota_{\mathrm{Grad}}^{*}v\\\iota_{\mathrm{curl}}^{*}E\\\iota_{\mathrm{grad}}^{*}\left(\Theta_{0}^{-1}\theta\right)\end{pmatrix}=\begin{pmatrix}f_{1}\\0\\f_{5}\end{pmatrix}.\end{aligned}\tag{5.89}$$

Evolutionary well-posedness of the corresponding thermo-piezo-electromagnetic system is then determined solely by the positive-definiteness of the block operator

$$
\begin{aligned}
& z\,\widetilde{M_{0,66}} + \widetilde{M_{1,66}}\,(z) \\
& = z \begin{pmatrix} \alpha_{33} & 0 & \alpha_{39} \\ 0 & 0 & 0 \\ \alpha_{39}^{*} & 0 & \alpha_{99} \end{pmatrix} + \begin{pmatrix} K_{33}\,(z) & K_{36}\,(z) & K_{39}\,(z) \\ 0 & 1 & 0 \\ K_{93}\,(z) & K_{96}\,(z) & K_{99}\,(z) \end{pmatrix}
\end{aligned} \tag{5.90}
$$

and is encapsulated in the next result.

Corollary 5.4.3. *Let $\nu \in \mathbb{R}_{>0}$, $z \in \mathbb{C}_{\mathrm{Re}>\nu}$, and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}\,(z)$ be as in (5.90). Assume there exist $c_{33}, \widetilde{c}_{99} \in \mathbb{R}_{>0}$ such that $\alpha_{33} \geq c_{33}$, $\alpha_{99} - \alpha_{39}^{*}\alpha_{33}^{-1}\alpha_{39} \geq \widetilde{c}_{99}$. Then the operator $z\,\widetilde{M_{0,66}} + \widetilde{M_{1,66}}\,(z)$ is accretive for all $\nu \in \mathbb{R}_{>0}$ sufficiently large.*

Remark 5.4.4. Assuming it exists, denote by n the outward unit normal. The boundary dynamics regarded above coincide with the following boundary conditions

$$
\begin{aligned}
& \left(\partial_{t,\nu}\alpha_{33} + K_{33}\,(\partial_{t,\nu})\right)(T\cdot n) + K_{36}\,(\partial_{t,\nu})\,(n \times H_t) \\
& \qquad - \left(\partial_{t,\nu}\alpha_{39} + K_{39}\,(\partial_{t,\nu})\right)(q\cdot n) + v = f_1 \text{ on } \partial\Omega,
\end{aligned} \tag{5.91}
$$

$$
(n \times H_t) + E_t = 0 \text{ on } \partial\Omega, \tag{5.92}
$$

$$
\begin{aligned}
& \left(\partial_{t,\nu}\alpha_{39}^{*} + K_{93}\,(\partial_{t,\nu})\right)(T\cdot n) + K_{96}\,(\partial_{t,\nu})\,(n \times H_t) \\
& \qquad - \left(\partial_{t,\nu}\alpha_{99} + K_{99}\,(\partial_{t,\nu})\right)(q\cdot n) + \Theta_0^{-1}\theta = f_5 \text{ on } \partial\Omega
\end{aligned} \tag{5.93}
$$

in the classical setting. $\triangledown$

5.4.3 Subcase (iii)

Our final mixed-type subcase is composed of non-standard inhomogeneous boundary dynamics for the piezo and electromagnetic parts of the system, and Robin boundary conditions for the outstanding thermal part. Again, an analogously systematic choice of operator coefficients yields the following set of abstract boundary conditions

$$
\begin{aligned}
\left(\partial_{t,\nu}\alpha_{33} + K_{33}\left(\partial_{t,\nu}\right)\right) \operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^{*} T + \left(\partial_{t,\nu}\alpha_{36} + K_{36}\left(\partial_{t,\nu}\right)\right) \operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^{*} H& \\
-K_{39}\left(\partial_{t,\nu}\right) \operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^{*} q + \iota_{\mathrm{Grad}}^{*} v = f_1&,
\end{aligned}
\tag{5.94}
$$

$$
\begin{aligned}
\left(\partial_{t,\nu}\alpha_{36}^{*} + K_{63}\left(\partial_{t,\nu}\right)\right) \operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^{*} T + \left(\partial_{t,\nu}\alpha_{66} + K_{66}\left(\partial_{t,\nu}\right)\right) \operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^{*} H& \\
-K_{69}\left(\partial_{t,\nu}\right) \operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^{*} q + \iota_{\mathrm{curl}}^{*} E = f_3&,
\end{aligned}
\tag{5.95}
$$

$$
-\operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^{*} q + \iota_{\mathrm{grad}}^{*}\left(\Theta_0^{-1}\theta\right) = 0 \tag{5.96}
$$

which are encoded as

$$
\begin{aligned}
\left(\partial_{t,\nu}\begin{pmatrix} \alpha_{33} & \alpha_{36} & 0 \\ \alpha_{36}^{*} & \alpha_{66} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} K_{33}\left(\partial_{t,\nu}\right) & K_{36}\left(\partial_{t,\nu}\right) & K_{39}\left(\partial_{t,\nu}\right) \\ K_{63}\left(\partial_{t,\nu}\right) & K_{66}\left(\partial_{t,\nu}\right) & K_{69}\left(\partial_{t,\nu}\right) \\ 0 & 0 & 1 \end{pmatrix}\right) \begin{pmatrix} \tau_T \\ \tau_H \\ \tau_q \end{pmatrix} +& \\
+ \begin{pmatrix} \iota_{\mathrm{Grad}}^{*} v \\ \iota_{\mathrm{curl}}^{*} E \\ \iota_{\mathrm{grad}}^{*}\left(\Theta_0^{-1}\theta\right) \end{pmatrix} = \begin{pmatrix} f_1 \\ f_3 \\ 0 \end{pmatrix}&.
\end{aligned}
\tag{5.97}
$$

Like before, the evolutionary well-posedness of the corresponding thermo-piezo-electro-magnetic system depends on the accretivity of the block operator

$$
\begin{aligned}
&z\,\widetilde{M_{0,66}} + \widetilde{M_{1,66}}(z) \\
&= z \begin{pmatrix} \alpha_{33} & \alpha_{36} & 0 \\ \alpha_{36}^{*} & \alpha_{66} & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} K_{33}(z) & K_{36}(z) & K_{39}(z) \\ K_{63}(z) & K_{66}(z) & K_{69}(z) \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
\tag{5.98}
$$

and is the focus of the next result.

Corollary 5.4.5. *Let $\nu \in \mathbb{R}_{>0}$, $z \in \mathbb{C}_{\mathrm{Re}>\nu}$, and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Let $\widetilde{M_{0,66}}$ and $\widetilde{M_{1,66}}(z)$ be as in (5.98). Assume there exist $c_{33}, \widetilde{c}_{66} \in \mathbb{R}_{>0}$ such that $\alpha_{33} \geq c_{33}$, $\alpha_{66} - \alpha_{36}^{*}\alpha_{33}^{-1}\alpha_{36} \geq \widetilde{c}_{66}$. Then the operator $z\,\widetilde{M_{0,66}} + \widetilde{M_{1,66}}(z)$ is accretive for all*

$\nu \in \mathbb{R}_{>0}$ *sufficiently large.*

Remark 5.4.6. Assuming it exists, denote by n the outward unit normal. The boundary conditions considered correspond to

$$\begin{aligned}(\partial_{t,\nu}\alpha_{33} + K_{33}(\partial_{t,\nu}))(T\cdot n) + (\partial_{t,\nu}\alpha_{36} + K_{36}(\partial_{t,\nu}))(n\times H_t)&\\ -K_{39}(\partial_{t,\nu})(q\cdot n) + v = f_1 \text{ on } \partial\Omega,&\end{aligned} \tag{5.99}$$

$$\begin{aligned}(\partial_{t,\nu}\alpha_{36}^* + K_{63}(\partial_{t,\nu}))(T\cdot n) + (\partial_{t,\nu}\alpha_{66} + K_{66}(\partial_{t,\nu}))(n\times H_t)&\\ -K_{69}(\partial_{t,\nu})(q\cdot n) + E_t = f_3 \text{ on } \partial\Omega,&\end{aligned} \tag{5.100}$$

$$-(q\cdot n) + \Theta_0^{-1}\theta = 0 \text{ on } \partial\Omega \tag{5.101}$$

in the setting of classical boundary traces. ▽

5.5 An Abstract Example

We conclude this chapter with the presentation of an example of particular interest. Our example consists of a set of abstract boundary conditions which make full use of the scope for thermo-piezo-electromagnetic boundary data afforded by our solution result, Theorem 4.4.6. The piezo-electromagnetic impedance boundary conditions originally considered in [AN11, Section 1] (recalled earlier in (1.2)) form the basis for this example. With this example we realise one of the aims of this thesis, which is to extend these boundary conditions to accommodate for the influence of a high-temperature regime.

We begin this section by first recalling the translation of these boundary conditions to the language and setting of abstract boundary data spaces, as achieved in [Pic17, Subsection 4.3.1]. In this setting the boundary conditions (1.2) take the form

$$\begin{aligned}\operatorname{curl}_{\mathrm{BD}}\iota_{\mathrm{curl}}^* H - \operatorname{curl}_{\mathrm{BD}} Q^*\iota_{\mathrm{Grad}}^* v + \iota_{\mathrm{curl}}^* E &= 0,\\ \operatorname{Div}_{\mathrm{BD}}\iota_{\mathrm{Div}}^* T - Q\operatorname{curl}_{\mathrm{BD}}\iota_{\mathrm{curl}}^* E + \left(1+\alpha\partial_{t,\nu}^{-1}\right)\iota_{\mathrm{Grad}}^* v &= 0.\end{aligned} \tag{5.102}$$

In this formulation, the originally given boundary mappings $\widetilde{Q}$ and $\widetilde{\alpha}$ (c.f. (1.2)) have been replaced by the arbitrary (bounded) boundary operators

$$Q\colon \ \mathrm{BD}(\mathrm{curl}) \to \mathrm{BD}(\mathrm{Grad}) \quad \text{and} \quad \alpha\colon \ \mathrm{BD}(\mathrm{Grad}) \to \mathrm{BD}(\mathrm{Grad}),$$

respectively. In the case of a bounded Lipschitz domain however, the underlying boundary mappings $\widetilde{Q}$ and $\widetilde{\alpha}$ could be recovered via

$$\begin{aligned} Q\colon \ \mathrm{BD}(\mathrm{curl}) &\to \mathrm{BD}(\mathrm{Grad}) \\ H &\mapsto \gamma^{-1}\widetilde{Q}\gamma_t H \end{aligned} \tag{5.103}$$

and

$$\begin{aligned} \alpha\colon \ \mathrm{BD}(\mathrm{Grad}) &\to \mathrm{BD}(\mathrm{Grad}) \\ v &\mapsto \gamma^{-1}\widetilde{\alpha}\gamma v, \end{aligned} \tag{5.104}$$

respectively. This translation will serve as the starting point for the formulation of our own example. In doing so we will be careful to highlight important observations underpinning any modelling decisions made. Indeed, we collate these observations in Remark 5.5.1. We will also indicate the required choices in the operator coefficients of (5.2) needed to recover the boundary dynamics of this example from within our extended system. Just as was done with the preceding boundary cases investigation, we will offer an alternative and direct proof of the required positive-definiteness needed to establish evolutionary well-posedness of the extended system under these boundary conditions.

5.5.1 Formulating New Boundary Conditions

Extending the boundary conditions (5.102) in a particular manner will yield a novel set of impedance boundary conditions suitable for full thermo-piezo-electromagnetic boundary data. We will achieve this in part by augmenting the boundary conditions (5.102) with the addition of a new boundary condition for thermal data. In addition, the existing two equations for piezo and electromagnetic boundary data will be enhanced by the inclusion of new thermal boundary terms. These points will be explored further in Remark 5.5.1 below. Using (5.102) as the starting point, we arrive at the following set of novel boundary conditions. We present

$$\operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^* H - \operatorname{curl}_{\mathrm{BD}} Q^* \iota_{\mathrm{Grad}}^* v + \iota_{\mathrm{curl}}^* E + \beta \iota_{\mathrm{grad}}^* \Theta_0^{-1} \theta = 0, \tag{5.105}$$

$$\operatorname{Div}_{\mathrm{BD}} \iota_{\mathrm{Div}}^* T - Q \operatorname{curl}_{\mathrm{BD}} \iota_{\mathrm{curl}}^* E + \left(1 + \alpha \partial_{t,\nu}^{-1}\right) \iota_{\mathrm{Grad}}^* v + Q\beta \iota_{\mathrm{grad}}^* \Theta_0^{-1} \theta = 0, \tag{5.106}$$

$$-\operatorname{div}_{\mathrm{BD}} \iota_{\mathrm{div}}^* q - \beta^* Q^* \iota_{\mathrm{Grad}}^* v - \beta^* \iota_{\mathrm{curl}}^* E + \iota_{\mathrm{grad}}^* \Theta_0^{-1} \theta = 0, \tag{5.107}$$

where there has been introduced the arbitrary (bounded) boundary operator

$$\beta \colon \mathrm{BD}(\mathrm{grad}) \to \mathrm{BD}(\mathrm{curl}).$$

Like before, this new boundary operator could be traced back to an underlying (bounded and linear) boundary mapping $\widetilde{\beta} \colon H^{1/2}(\partial\Omega) \to V_{\gamma_t}$ via

$$\beta \colon \begin{cases} \mathrm{BD}(\mathrm{grad}) & \to & \mathrm{BD}(\mathrm{curl}) \\ u & \mapsto & \gamma_t^{-1} \widetilde{\beta} \gamma u \end{cases}$$

in the instance of a bounded Lipschitz domain. With the help of the inherent boundary conditions (recall (4.45), (4.46) and (4.47)) we can encode the new boundary conditions (5.105) as the block operator equation

$$\begin{pmatrix} \tau_q \\ \tau_H \\ \tau_T \end{pmatrix} + \begin{pmatrix} 1 & -\beta^* & -\beta^* Q^* \\ \beta & 1 & -\operatorname{curl}_{\mathrm{BD}} Q^* \\ Q\beta & -Q \operatorname{curl}_{\mathrm{BD}} & \left(1 + \alpha \partial_{t,\nu}^{-1}\right) \end{pmatrix} \begin{pmatrix} \iota_{\mathrm{grad}}^* \left(\Theta_0^{-1} \theta\right) \\ \iota_{\mathrm{curl}}^* E \\ \iota_{\mathrm{Grad}}^* v \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}. \tag{5.108}$$

The next remark is offered to contextualise the modelling decisions behind the formulation of the abstract boundary conditions just presented.

Remark 5.5.1. There are several key observations which justify the proposed extension of the boundary conditions (5.102). These we now outline.

(i) Firstly, notice that the original piezo and electromagnetic boundary conditions in (5.102) are posed on BD (Grad) and BD (curl), respectively. Indeed, one can see this more clearly upon recalling Proposition 3.2.5 and Lemma 3.2.1, being careful to note the action of the orthogonal projectors involved. Should one wish to extend these boundary conditions to account for the role of temperature, then there needs to be formulated an entirely new equation for boundary data pertaining to the thermal part of the system. In a manner analogous to the formulation of the underlying piezo and electromagnetic boundary conditions, any new thermal equation needs then to be framed within BD (grad). Indeed the last and entirely new equation in (5.105) is posed there.

(ii) Secondly, notice that each of the original, underlying boundary conditions in (5.102) involve *both* of the respective unknowns for the corresponding part of the system. In particular, the electromagnetic boundary condition explicitly involves both the electric field, E, and the magnetic field, H, whereas the piezo boundary condition explicitly involves the stress tensor, T, as well as the introduced unknown $v := \partial_t u$ (c.f. Section 4.2). As such, any new equation for the thermal part of the system should expressly involve the heat flux, q, and relative temperature, $\Theta_0^{-1}\theta$, which our new equation does.

(iii) Thirdly and finally, the original boundary conditions (5.102) need to be suitably modified in order to accommodate and couple with the newly implemented thermal boundary data. Indeed such a coupling already exists in the original boundary conditions between piezo and electromagnetic boundary data and is achieved by the action of the underlying boundary operators Q and α. To see this more clearly, consider the boundary spaces they map between as well as the action of the orthogonal projectors involved. For our extended set of boundary conditions, the newly introduced boundary operator β allows us to achieve this with the relative temperature, $\Theta_0^{-1}\theta$. Indeed, notice how in the first two equations of (5.105) β translates thermal boundary data to the respective realms of electromagnetic and piezo boundary data. Although in the latter of these cases one additionally needs to make use of Q in order to properly realise and justify the translation. $\triangledown$

For the sake of completeness we would like to indicate how our new boundary conditions can be recovered from the operator coefficients in the sub-block operator (5.2). Taking the block operator formulation of our boundary equations (5.108) as our starting point, we first compute and apply the inverse (the existence of which we later show in (5.120) with the help of Proposition 4.1.6 for large enough $\nu \in \mathbb{R}_{>0}$) to instead equivalently consider

$$\begin{pmatrix} 1 & -\beta^* & -\beta^* Q^* \\ \beta & 1 & -\operatorname{curl}_{\mathrm{BD}} Q^* \\ Q\beta & -Q \operatorname{curl}_{\mathrm{BD}} & \left(1+\alpha\partial_{t,\nu}^{-1}\right) \end{pmatrix}^{-1} \begin{pmatrix} \tau_T \\ \tau_H \\ \tau_q \end{pmatrix} + \begin{pmatrix} \iota_{\mathrm{Grad}}^* v \\ \iota_{\mathrm{curl}}^* E \\ \iota_{\mathrm{grad}}^* \left(\Theta_0^{-1}\theta\right) \end{pmatrix} = 0.$$

Here, the computed inverse

$$\begin{pmatrix} 1 & -\beta^* & -\beta^* Q^* \\ \beta & 1 & -\operatorname{curl}_{\mathrm{BD}} Q^* \\ Q\beta & -Q \operatorname{curl}_{\mathrm{BD}} & \left(1+\alpha\partial_{t,\nu}^{-1}\right) \end{pmatrix}^{-1} = \begin{pmatrix} K_{99}\left(\partial_{t,\nu}\right) & K_{96}\left(\partial_{t,\nu}\right) & K_{93}\left(\partial_{t,\nu}\right) \\ K_{69}\left(\partial_{t,\nu}\right) & K_{66}\left(\partial_{t,\nu}\right) & K_{63}\left(\partial_{t,\nu}\right) \\ K_{39}\left(\partial_{t,\nu}\right) & K_{36}\left(\partial_{t,\nu}\right) & K_{33}\left(\partial_{t,\nu}\right) \end{pmatrix}$$

has for diagonal coefficients

$$\begin{aligned} K_{33}\left(\partial_{t,\nu}\right) = \Big(&1 + Q\beta\left(Q\beta\right)^* + \alpha\partial_{t,\nu}^{-1} - \left[Q\beta\beta^* - Q\operatorname{curl}_{\mathrm{BD}}\right]\left(1+\beta\beta^*\right)^{-1} \\ &\cdot \left[\beta\left(Q\beta\right)^* - Q\operatorname{curl}_{\mathrm{BD}}\right]\Big)^{-1}, \end{aligned} \tag{5.109}$$

$$\begin{aligned} K_{66}\left(\partial_{t,\nu}\right) &= \left(1+\beta\beta^*\right)^{-1} + \left(1+\beta\beta^*\right)^{-1}\left[\beta\left(Q\beta\right)^* - \operatorname{curl}_{\mathrm{BD}} Q^*\right] K_{33}\left(\partial_{t,\nu}\right) \\ &\quad \cdot \left[Q\beta\beta^* - Q\operatorname{curl}_{\mathrm{BD}}\right]\left(1+\beta\beta^*\right)^{-1}, \end{aligned} \tag{5.110}$$

$$\begin{aligned} K_{99}\left(\partial_{t,\nu}\right) &= 1 + \Big[-\beta^*\left(1+\beta\beta^*\right)^{-1}\beta + \Big[\beta^*\left(1+\beta\beta^*\right)^{-1}\left[\beta\left(Q\beta\right)^* - \operatorname{curl}_{\mathrm{BD}} Q^*\right] \\ &\quad - \left(Q\beta\right)^*\Big] K_{33}\left(\partial_{t,\nu}\right)\Big[Q\beta - \left[Q\beta\beta^* - Q\operatorname{curl}_{\mathrm{BD}}\right]\left(1+\beta\beta^*\right)^{-1}\beta\Big]\Big], \end{aligned} \tag{5.111}$$

and for off-diagonal coefficients

$$K_{96}(\partial_{t,\nu}) = -\left[\left[(Q\beta)^{*} - \beta^{*}(1+\beta\beta^{*})^{-1}\left[\beta(Q\beta)^{*} - \mathrm{curl}_{\mathrm{BD}}\, Q^{*}\right] - \beta^{*}\right]\right. \tag{5.112}$$
$$\left.\cdot K_{33}(\partial_{t,\nu})\left[Q\beta\beta^{*} - Q\,\mathrm{curl}_{\mathrm{BD}}\right]\right](1+\beta\beta^{*})^{-1},$$

$$K_{69}(\partial_{t,\nu}) = -(1+\beta\beta^{*})^{-1}\left[\beta - \left[\beta(Q\beta)^{*} - \mathrm{curl}_{\mathrm{BD}}\, Q^{*}\right]\right. \tag{5.113}$$
$$\left.\cdot K_{33}(\partial_{t,\nu})\left[Q\beta - \left[Q\beta\beta^{*} - Q\,\mathrm{curl}_{\mathrm{BD}}\right](1+\beta\beta^{*})^{-1}\beta\right]\right],$$

and

$$K_{93}(\partial_{t,\nu}) = -\left[\beta^{*}(1+\beta\beta^{*})^{-1}\left[\beta(Q\beta)^{*} - \mathrm{curl}_{\mathrm{BD}}\, Q^{*}\right] - (Q\beta)^{*}\right] K_{33}(\partial_{t,\nu}), \tag{5.114}$$

$$K_{39}(\partial_{t,\nu}) = -K_{33}(\partial_{t,\nu})\left[Q\beta - \left[Q\beta\beta^{*} - Q\,\mathrm{curl}_{\mathrm{BD}}\right](1+\beta\beta^{*})^{-1}\beta\right], \tag{5.115}$$

as well as

$$K_{63}(\partial_{t,\nu}) = -(1+\beta\beta^{*})^{-1}\left[\beta(Q\beta)^{*} - \mathrm{curl}_{\mathrm{BD}}\, Q^{*}\right] K_{33}(\partial_{t,\nu}), \tag{5.116}$$

$$K_{36}(\partial_{t,\nu}) = -K_{33}(\partial_{t,\nu})\left[Q\beta\beta^{*} - Q\,\mathrm{curl}_{\mathrm{BD}}\right](1+\beta\beta^{*})^{-1}, \tag{5.117}$$

where the skew-symmetry of $\mathrm{curl}_{\mathrm{BD}}$ (recall Proposition 3.2.5) means that the inverse is only ever 'almost' symmetric. In this example there are only zero coefficients in the sub-block operator $\widetilde{M_{0,66}}$. Hence it can be ignored when addressing the question of accretivity. With these coefficients computed, the actual form of the material law operator $M(\partial_{t,\nu})$ in this instance can be fully realised.

5.5.2 Evolutionary Well-posedness

The next result summarises the evolutionary well-posedness of our extended thermo-piezo-electromagnetic system under the newly formulated set of boundary conditions. Much like in the preceding catalogue of evolutionarily well-posed cases, there is no need to completely reinvent our main solution result, Theorem 4.4.6. Like before, we simply provide an alternative proof to the auxiliary result Lemma 4.4.4, which addressed the accretivity of the sub-block operator governing boundary dynamics. As noted, since $\widetilde{M_{0,66}} = 0$ we need only concern ourselves with the accretivity of the block operator

$\widetilde{M_{1,66}}(z)$.

Corollary 5.5.2. *Let $\nu \in \mathbb{R}_{>0}$, $z \in \mathbb{C}_{\mathrm{Re}>\nu}$, and $\mathcal{H}_{\mathrm{BD}}$ be as in (5.5). Then the operator*

$$\widetilde{M_{1,66}}(z) = \begin{pmatrix} K_{33}(z) & K_{36}(z) & K_{39}(z) \\ K_{63}(z) & K_{66}(z) & K_{69}(z) \\ K_{93}(z) & K_{96}(z) & K_{99}(z) \end{pmatrix} \tag{5.118}$$

is accretive for all $\nu \in \mathbb{R}_{>0}$ sufficiently large.

Proof. We will use Proposition 4.1.6 to indirectly establish the positive-definiteness of $zM(z)$ uniformly in $z \in \mathbb{C}_{\mathrm{Re}\geq\nu}$ for large enough $\nu \in \mathbb{R}_{>0}$. Here the (simplified) material law is

$$M(z) := z^{-1}\widetilde{M_{1,66}}(z). \tag{5.119}$$

The indirect means employed here is convenient as it allows us to avoid making any recourse to the cumbersome block operator inverse computed above. For $x \in \mathrm{BD}(\mathrm{Grad})$ compute

$$\begin{aligned} \left\langle x, 1 + \mathrm{Re}\left(\alpha z^{-1}\right) x\right\rangle_{\mathrm{BD(Grad)}} &= \|x\|^2_{\mathrm{BD(Grad)}} + \left\langle x, \mathrm{Re}\left(\alpha z^{-1}\right) x\right\rangle_{\mathrm{BD(Grad)}} \\ &= \|x\|^2_{\mathrm{BD(Grad)}} + \mathrm{Re}\left\langle x, \left(\alpha z^{-1}\right) x\right\rangle_{\mathrm{BD(Grad)}} \\ &\geq \|x\|^2_{\mathrm{BD(Grad)}} - \|\alpha\|\,|z^{-1}|\|x\|^2_{\mathrm{BD(Grad)}} \\ &\geq \left(1 - \frac{\|\alpha\|}{\nu}\right)\|x\|^2_{\mathrm{BD(Grad)}} \end{aligned}$$

which in turn we use to compute

$$\begin{aligned} \mathrm{Re}\begin{pmatrix} 1 & -\beta^* & -\beta^* Q^* \\ \beta & 1 & -\operatorname{curl}_{\mathrm{BD}} Q^* \\ Q\beta & -Q\operatorname{curl}_{\mathrm{BD}} & \left(1+\alpha z^{-1}\right) \end{pmatrix} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \mathrm{Re}\left(1+\alpha z^{-1}\right) \end{pmatrix} \\ &\geq \min\left\{1, 1 - \frac{\|\alpha\|}{\nu}\right\} \\ &= 1 - \frac{\|\alpha\|}{\nu}. \end{aligned} \tag{5.120}$$

By Proposition 4.1.6 we can use this bound to estimate the real-part of the inverse, which is actually just the operator $\widetilde{M_{1,66}}(z)$. Indeed

$$
\begin{aligned}
&\operatorname{Re}\begin{pmatrix} 1 & -\beta^* & -\beta^* Q^* \\ \beta & 1 & -\operatorname{curl}_{\mathrm{BD}} Q^* \\ Q\beta & -Q\operatorname{curl}_{\mathrm{BD}} & \left(1+\alpha z^{-1}\right) \end{pmatrix}^{-1} \\
&\geq \left(1-\frac{\|\alpha\|}{\nu}\right)\left\|\begin{pmatrix} 1 & -\beta^* & -\beta^* Q^* \\ \beta & 1 & -\operatorname{curl}_{\mathrm{BD}} Q^* \\ Q\beta & -Q\operatorname{curl}_{\mathrm{BD}} & \left(1+\alpha z^{-1}\right) \end{pmatrix}\right\|^{-2}
\end{aligned}
\tag{5.121}
$$

which yields the desired positive-definiteness. □

Remark 5.5.3. (i) Assuming it exists, denote by n the outward unit normal. As was done in the preceding catalogue of cases, we conclude the consideration of this example by pointing out that in the classical setting these new boundary conditions correspond formally to

$$
\begin{aligned}
n \times H_t - n \times \widetilde{Q}^* v + E_t + \widetilde{\beta}\left(\Theta_0^{-1}\theta\right) &= 0 \text{ on } \partial\Omega, \\
T \cdot n - \widetilde{Q}\left(n \times E_t\right) + \left(1+\widetilde{\alpha}\partial_{t,\nu}^{-1}\right) v + \widetilde{Q}\widetilde{\beta}\left(\Theta_0^{-1}\theta\right) &= 0 \text{ on } \partial\Omega, \\
-q \cdot n + \widetilde{\beta}^* \widetilde{Q}^* v + \widetilde{\beta}^* E_t + \Theta_0^{-1}\theta &= 0 \text{ on } \partial\Omega.
\end{aligned}
$$

(ii) The same computation of the bound and real-part condition in the proof of Corollary 5.5.2 is retained when a simpler Robin-type boundary condition is considered instead. Indeed, in this case the corresponding block operator encoding boundary dynamics reads

$$
\begin{pmatrix} \tau_q \\ \tau_H \\ \tau_T \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & -\operatorname{curl}_{\mathrm{BD}} Q^* \\ 0 & -Q\operatorname{curl}_{\mathrm{BD}} & \left(1+\alpha\partial_{t,\nu}^{-1}\right) \end{pmatrix}\begin{pmatrix} \iota_{\mathrm{grad}}^*\left(\Theta_0^{-1}\theta\right) \\ \iota_{\mathrm{curl}}^* E \\ \iota_{\mathrm{Grad}}^* v \end{pmatrix} = 0,
$$

for which it is clear that the same real-part calculation holds. As such, the corresponding system continues to enjoy evolutionary well-posedness. This example falls in particular within the scope of the second instance of mixed boundary behaviours, as examined in Section 5.4. $\triangledown$

Chapter 6

Conclusion and Future Work

6.1 Summary

As we come to the end of the present work, let us recall what we have achieved. First, we used the setting provided by the theory of evolutionary equations to formulate and pose our own extended system for thermo-piezo-electromagnetism. We extended from a known thermo-piezo-electromagnetic system under homogeneous boundary conditions, first presented in [MPTW16, Sections 2, 3] (c.f. Section 4.2). Using the tools afforded to us by abstract boundary data space theory, we encoded boundary dynamics from within our model system. To this end we followed the methodology of [PSTW16, Sections 1, 2.3.2] and [Pic17, Subsections 4.1, 4.3] (recalled here in Section 3.2 and exhibited in Section 5.5, respectively). Armed with the workhorse solution theory of evolutionary equations, Picard's Theorem (c.f. Theorem 2.4.4), we established the well-posedness of our extended system in Theorem 4.4.6.

Second, we conducted a systematic investigation into different patterns and arrangements of boundary dynamics across the three physical aspects of our system. The results were disambiguated according to varying levels of formal complexity, and catalogued in Chapter 5. In Section 5.2 we argued how Robin boundary conditions were the most rudimentary type of boundary condition catered to by our model. We also addressed the 'problem' of directly recovering elementary homogeneous Dirichlet and Neumann boundary conditions from within our extended model. However, we did indicate how they can be recovered by a simple yet formal modification of our system. Then, in Sections 5.3 and 5.4, we ascertained the different placements of boundary dynamics which lead to our system being well-posed. The boundary dynamics considered comprised an arrangement of Robin and general impedance type boundary conditions. The arrangements regarded were ordered according to subcases, and detailed throughout Subsections 5.3.1 to 5.3.3 and 5.4.1 to 5.4.3. To the ends of variety and accessibility, we provided an alternative proof of well-posedness for each of the parent cases (c.f. Corollaries 5.3.1 and 5.4.1).

Thirdly and finally, in Section 5.5 we extended the impedance (Leontovich) boundary conditions first introduced in [AN11, Section 1] and later developed in [Pic17, Subsection 4.3.1]. Our particular extension provides an abstraction of these impedance boundary conditions. This abstraction accounts not only for the classical piezo-electromagnetic impedance boundary effect, but also for the influence of heat dynamics at the boundary. In Subsections 5.5.1 and 5.5.2 we addressed the most abstracted extension of the underlying boundary conditions. However, in Item (ii) of Remark 5.5.3 we indicated the form of the simplest generalisation of these impedance boundary conditions to the setting provided by our model.

Our modelling approach deviated from that of the norm. Usually, one models a phenomenon first before undertaking the corresponding analysis. Nonetheless, our approach sketched a blueprint for what potential systems might look like, and provided assumptions for material parameters which will lead immediately to a well-posed system.

6.2 Avenues of Future Research

Whilst the boundary conditions formulated in Section 5.5 are mathematically interesting in their own right, we emphasise that they are an abstract example. As was noted in Remark 5.5.1, the construction of these boundary conditions followed several connected observations on the shape of the underlying piezo-electric impedance boundary conditions from [AN11, Section 1] and [Pic17, Subsection 4.3.1]. The task of finding a physically relevant set of boundary conditions, which also fit within the schema afforded by our extended model, remains an open topic of research. To the best of the authors knowledge — and efforts — no comparable boundary conditions which fall within the scope of the generalised impedance type boundary condition harnessed by our model prevail in the literature. Should the search for inspiration from physically relevant applications continue to prove fruitless, then the task will turn to the formulation of potentially physically relevant boundary conditions. The use of "potentially physically relevant" is not offered as a euphemism for "abstract". Rather, it pertains to the conceptually sensible and physically meaningful formulation of boundary conditions, despite being done from a hypothetical vantage point. It is the authors expectation that effective collaboration and standardised nomenclature between applied and pure schools will bridge the gap, and catalyse such formulations.

Perhaps it goes without saying, but translating our model and its central well-posedness result, Theorem 4.4.6, from the language of abstract boundary data spaces to that of classical boundary traces, might help with this endeavour. Providing potential collaborators hailing from the realms of applied mathematics and engineering with findings in a common mathematical language could better engender fruitful collaborations. Despite being used effectively to address the well-posedness of a piezo-electric system with boundary dynamics in [Pic17], it cannot be assumed that abstract boundary trace theory be known by the wider ultrasonics community. Classical boundary traces, however, are much more widely known. As such, the reformulation of the key ideas of this thesis in terms of classical traces might abet the search for physically relevant boundary conditions.

However, as we recalled in the introduction to this thesis, there is a connection between the use of an abstract boundary data space formulation and piezo-electric transducers. In [Pic17] the author remarked how the use of abstract boundary traces covered the modelling of ultrasonic devices with a fractal boundary (c.f. [OMOH08a], [MW11], [MMO+11], [AM15] and [BAM16]). As is common throughout the modelling of ultrasonic devices, the systems employed in the papers cited here centre on a piezo-electric model. Perhaps there is scope then to extend these models to take into account the influence of a high-temperature regime, and apply the framework established in this thesis. Moreover, one could investigate the (potential, if not yet realised) use of such fractal piezo-electric transducer devices in harsh and corrosive environments. Environments not entirely unlike those of a nuclear or industrial plant, as indicated in the introduction. Indeed, much of the literature in this realm of application underlines the need to manufacture ultrasonic transducers which can operate at, and withstand, oppressive temperatures (again, see for instance [KMC+99], [KV21], [BPP79], [FWW89], [SKC07], [OJMS05], [HPH03] and [JLP00]).

It should be noted that the modelling of ultrasonic transducers is but one use case in the broader area of piezo-electric material modelling. As such, it could prove fruitful to consider the application of the system proposed in this thesis to areas related to, but beyond the sole scope of, piezo-electric transducers. Indeed, whilst the modelling of ultrasonic transducers focuses almost exclusively on their piezo-electric properties, the modelling of smart materials and composite structures often additionally includes temperature as standard. Smart materials are structures whose physical boundary and properties can be altered with changing temperature regimes, electrical input and physical stresses (c.f. [GT92] and [Sch08]). For instance, [CYO19] models a thermo-piezoelectric actuator device with an emphasis on the underlying thermo-piezo material coupling. One can find similar thermo-piezoelectric systems which additionally consider a mechanical wave (c.f. [KW12] and [LKW+13]). A wealth of related and seemingly promising applications prevail, and include [KWW12], [SCBCB+13], [MS14], [BGK15] as well as [KLW15].

Emerging trends in the modelling of piezo-electric and smart materials offer us promising future candidates for the boundary conditions we seek. However, it may still prove necessary to formulate theoretical but no less physically sensible and inspired boundary conditions, which fit within the framework afforded by our extended system of thermo-piezo-electromagnetism. An endeavour, no doubt, which will require no modest amount of creativity. And one which will benefit immensely from effective collaboration between manufacturers, engineers as well as applied and pure mathematicians.

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