# Induction and Coinduction Schemes in Category Theory

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# Declaration

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

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# Abstract

This thesis studies induction and coinduction schemes from the point of view of category theory. We start from the account of inductive and coinductive types given by initial algebra semantics and final coalgebra semantics, respectively. We then use fibrations as a generic setting describing a logic for a type theory to study induction and coinduction. As our starting point we consider the seminal work of Hermida and Jacobs [Her93, HJ98], who pioneered the fibrational approach. We extend their induction and coinduction schemes to give provably sound generic induction and coinduction schemes for arbitrary inductive and coinductive types. To achieve this we introduce the notion of a quotient category with equality (QCE) which i) abstracts the standard notion of a fibration of relations constructed from a given fibration and ii) gives us the correct structure to compare induction and coinduction from a categorical perspective. This allows us to broaden the applications of the coinduction scheme, as well as present the duality between coinduction and induction in a systematic way. Finally, we consider induction and coinduction schemes in the more general setting of fibred fibrations which is used to give sound, generic indexed induction and coinduction schemes for indexed inductive and coinductive types.

# Contents

# Introduction

Iteration operators provide a uniform way to express common and naturally occurring patterns of recursion over inductive data types. Expressing recursion via iteration operators makes code easier to read, write, and understand; facilitates code reuse; guarantees properties of programs such as totality and termination; and supports optimising program transformations such as fold fusion and short cut fusion.

Categorically, iteration operators arise from initial algebra semantics: the signature of an inductive data type is given by a functor $F$, the constructors of the data type are modelled as the structure map of the initial $F$-algebra $\mathit{in} : F(\mu F) \to \mu F$, the data type itself is modelled as the carrier $\mu F$ of the initial $F$-algebra, and the iteration operator $\mathit{fold} : (FA \to A) \to \mu F \to A$ for $\mu F$ maps an $F$-algebra $h : FA \to A$ to the unique $F$-algebra morphism from $\mathit{in}$ to $h$. Initial algebra semantics provides a comprehensive theory of iteration that is

- *principled*, in that it ensures that programs have rigorous mathematical foundations that can be used to give them meaning and prove their soundness;
- *expressive*, in that it is applicable to *all* inductive types — i.e., all types that are carriers of initial algebras — rather than just to syntactically defined classes of data types such as polynomial ones; and
- *sound*, in that it is valid in any model — set-theoretic, domain-theoretic, realisability, etc. — interpreting data types as carriers of initial algebras.

Final coalgebra semantics gives an equally comprehensive understanding of coinductive types: the signature of a coinductive data type is given by a functor

$F$, the destructors of the data type are modelled as the structure map of the final $F$-coalgebra $out : \nu F \to F(\nu F)$, the data type itself is modelled as the carrier $\nu F$ of the final $F$-coalgebra, and the coiteration operator $unfold : (A \to FA) \to A \to \nu F$ for $\nu F$ maps an $F$-coalgebra $k : A \to FA$ to the unique $F$-coalgebra morphism from $k$ to $out$. Final coalgebra semantics thus provides a theory of coiteration which is as principled, expressive, and sound as for iteration.

Induction is often used to prove properties of functions defined by iteration, and the soundness of induction schemes is often established by reducing it to that of iteration. Since induction and iteration are closely linked, we might expect initial algebra semantics to give a theory of induction that is as principled, expressive, and sound as the theory of iteration it provides. Unfortunately, most theories of induction for inductive data types — i.e., for data types of the form $\mu F$ for an endofunctor $F$ on a base category $\mathcal{B}$ — are only sound under significant restrictions on the category $\mathcal{B}$, the functor $F$, or the property to be established. However, Hermida and Jacobs recently made a conceptual breakthrough in the theory of induction [Her93,HJ98]. They start with a category $\mathcal{B}$, typically thought of as a category of types, and show how to lift an arbitrary functor $F$ on $\mathcal{B}$, to a functor $\hat{F}$ on a category of predicates on $\mathcal{B}$. They then take the premise of an induction scheme for $\mu F$ to be an $\hat{F}$-algebra. Their main theorem about induction shows that if $\hat{F}$ preserves truth predicates then the resulting induction scheme is sound. They then show that for any polynomial functors $F$ the lifting $\hat{F}$ preserves truth predicates.

In the same way, coinduction and coiteration are closely linked, so we might expect final coalgebra semantics to give a principled, expressive, and sound theory of coinduction. However, most theories of coinduction suffer from the same drawbacks that theories of induction do. In [HJ98], Hermida and Jacobs complemented their theory of induction with a theory of coinduction. While induction deals with predicates, coinduction deals with relations, so this time Hermida and Jacobs show how to lift an arbitrary functor $F$ on a base category $\mathcal{B}$, again typically thought of as a category of types to a functor $\check{F}$ on a category of binary predicates on $\mathcal{B}$. (Binary predicates are the traditional representation of relations in a type-theoretic setting). Hermida and Jacobs take the premise of a coinduction scheme for $\nu F$ to be an $\breve{F}$-coalgebra. Their main theorem about induction dualises to show that the if $\breve{F}$ preserves equality the resulting coinduction scheme is sound. They then show that for any polynomial functors $F$ the lifting $\breve{F}$ preserves equality.

To formally capture the informal notions of a category of types and a category of predicates above these types, Hermida and Jacobs worked in a fibrational setting. Fibrations support a uniform, axiomatic approach to induction and coinduction that is widely applicable, and that abstracts over the specific choices of category, functor, and predicate. This is advantageous because:

- the semantics of data types in languages involving recursion, corecursion and other effects usually involves categories other than Set;
- in such circumstances, the standard set-based interpretations of predicates are no longer germane;
- in any setting, there can be more than one reasonable notion of predicate;
- fibrations allow induction and coinduction schemes for many classes of data types to be obtained by instantiation of a single, generic theory rather than developing an ad hoc, case-by-case basis.

Thus, Hermida and Jacobs overcome two of the aforementioned limitations on induction schemes, namely the restrictions on the base category $\mathcal{B}$ and the restriction on the properties that can be established. They also provide a formal setting in which to study induction and coinduction schemes. But since they only prove the soundness of their induction and coinduction schemes for polynomial data types, the limitation on the functors treated remains in their work. Their theory therefore needs to be extended whenever a non-polynomial data type is considered. Among the class of non-polynomial data types are the inductive data types of rose trees and of hereditary sets as well as the coinductive data type of non-deterministic automata.

In this thesis, we extend the existing body of work on fibrational induction and coinduction in the following ways.

- We show how to remove the restriction to polynomial functors and derive a sound, generic induction scheme that can be instantiated to *every* inductive type. This is important because it provides a counterpart for induction to the existence of an iteration operator for every inductive type. We take Hermida and Jacobs' approach as our point of departure and show that, under slightly different assumptions about the fibration involved, we can lift *any* functor on the base category of a fibration to a functor on the total category of the fibration that preserves truth predicates. Since we define our lifting uniformly for all functors, the induction scheme to which it gives rise is completely generic as well.

- We show how to remove the restriction to polynomial functors and derive a sound, generic coinduction scheme that can be instantiated to *every* coinductive data type, i.e., to every type that is the carrier of a final coalgebra. This is equally important because it provides a counterpart for coinduction to the existence of a coiteration operator for every coinductive type. Again, here we start from Hermida and Jacobs' approach and lift *any* functor on the base category of a fibration to a functor on the total category of the fibration that preserves equality relations. Since we define our lifting uniformly for all functors, the coinduction scheme to which it gives rise is completely generic, just as our induction scheme is.

- We show how to remove a restriction appearing in [HJ98] on the notion of relation involved in the coinduction scheme. We thereby derive a coinduction scheme that treats an abstract notion of relation that is sufficiently general to include binary predicates. This is important for two reasons. First, by accommodating a more abstract notion of relation, it encompasses a wider class of fibrations than fibrations of binary predicates. This allows our coinduction scheme to be instantiated in new settings. Second, our more abstract notion of relation gives rise to a coinduction scheme whose level of abstraction reflects that of our induction scheme. This allows us to present coinduction as the dual of induction in a very natural way.

- We derive sound, generic induction schemes for inductive indexed types. Data types arising as initial algebras of endofunctors on a category of types are fairly simple. More sophisticated data types — e.g., untyped lambda terms and red-black trees — are often modelled as inductive indexed types, arising as initial algebras of functors on slice categories, presheaf categories and similar structures. We show how to derive a sound, generic indexed induction scheme to reason about such data types with a version of our generic induction scheme instantiated in the more general context of fibred fibration. Such a setting allows us to consider a logic above another fibration, this gives us the possibility to consider any indexing of types that is captured by a fibration.

- Since we can derive sound, generic induction schemes for inductive types and inductive indexed types, and sound, generic coinduction schemes for coinductive types, we might expect to be able to derive sound, generic coinduction schemes for coinductive indexed types, as well. We confirm that this is indeed possible, and give a number of examples of such schemes.

This thesis consists in part of work done jointly with my supervisors Patricia Johann and Neil Ghani. This work was published in [GJF10,FGJ11,GJF12], and the journal version [GJF] of [FGJ11] was invited for submission to the CALCO 2011 special issue of LMCS. As is natural during the course of a PhD, my contributions to these papers grew chronologically:

- In [GJF10] and its journal version [GJF12], my contributions were mainly to the treatment of induction in arbitrary fibrations, particularly the use of Lawvere fibrations to obtain a modular presentation of a generic induction scheme. My contributions elaborated and built upon the concrete treatment of induction and the preliminary presentation of a fibrational treatment of generic induction schemes, both of which were reasonably well settled prior to my involvement in this research.

- In [FGJ11] and its extended version [GJF], my main contributions were to define the notion of quotient category with equality and to ascertain its relevance for coinduction and indexed coinduction. I also developed the detailed treatment of indexed induction and coinduction in a fibred fibrational setting. A generalisation of the notion of QCE is currently being used as the technical basis of a categorical study of parametricity by Patricia Johann and Neil Ghani.

The following description of the chapters of the thesis specifies the relations between these papers and the thesis.

The thesis is structured in the following way. In Chapter 1 we introduce fibrations and some basic results about them, and describe how they can be used to model different types of logics above type theories. No new results are introduced in this chapter — we only recall sufficient preliminary results for a reader not familiar with fibrations to enable them to follow the thesis. We do, however, assume basic knowledge of category theory.

In Chapter 2 we will concentrate on two different kinds of fibrations, namely comprehension categories with unit and quotient categories with equality. The notion of comprehension category with unit appears when modelling dependent type theory with category in [Ehr88a, Ehr88b] under the name D-category and is studied as an instance of the notion of comprehension category in [Jac93]. This is the setting in which we will study induction. We borrow most of the results in this chapter from these papers. The new notion of quotient category with equality arises from our intention to present coinduction as dual to induction, and will play a role in the study of coinduction similar to the role of comprehension category with unit for the study of induction. The section of this chapter which introduces quotient categories with equality is based on results presented in [FGJ11] and [GJF].

Chapter 3 studies the notion of lifting of a functor with regard to a fibration. For a fibration, the lifting of a functor on the base category is a functor on the total category that represents, in some way, the first functor. The notion of lifting is at the heart of the treatment of induction and coinduction in fibrations. We present two ways to construct a canonical lifting for an arbitrary functor, one in the setting in which we study induction and the other in the setting in which we study coinduction. We conclude the chapter with different properties of these canonical liftings as well as their relationship with arbitrary liftings. This Chapter is based on [GJF10, GJF12] for the study of general liftings and canonical liftings in the setting in which we study induction, as well as in [FGJ11] for the study of canonical liftings in the setting in which we study coinduction.

Chapter 4 details our fibrational approach to derive sound induction and coinduction schemes for arbitrary inductive and coinductive types respectively. This chapter start with some well-known knowledge on initial algebra and final coalgebra semantics. We then use the different notions introduced in the previous chapters to, first, define the notion of induction and coinduction schemes in fibration. And second, show how to derive, in a generic way, such schemes in the setting of comprehension categories with unit and quotient categories with equality, respectively. This chapter is based on [GJF10, GJF12] for the study of induction, and on [FGJ11] for the study of coinduction.

Chapter 5 studies sound induction for arbitrary indexed inductive types. It demonstrates how we can exploit the generality of the fibrational setting in which we have chosen to work to obtain interesting results about a specific setting of interest. In this chapter we consider the notion of fibred fibration which allows us to consider a fibration of a logic above a fibration of indexed types. We look at the fibred notion of comprehension category with unit in which we study induction for indexed types. This chapter is based on [GJF].

Chapter 6 studies sound coinduction for arbitrary indexed coinductive types. We show how the notion of quotient category with equality is generalised in the setting of fibred fibration and how this allows us to study coinduction schemes for indexed coinductive types. This chapter is based on [GJF].

# Chapter 1

# Theory of fibrations

This chapter introduces the basic notions of the theory of fibrations. In Section 1.1 we give the definition of a fibration. We then look at some examples of standard fibrations, such as the codomain, syntactic, and simple fibrations. The syntactic fibration will be of particular interest, as it highlights the relationship between fibrations, type theory, and logic in order to make the connection between our fibrational approach to induction and coinduction and the standard approach.

In Section 1.2 we present the basic tools we use to create and manipulate fibrations and fibred structures. In particular, we investigate categorical structure in the fibrational context, the construction of new fibrations from existing ones, and the conditions under which these constructions preserve this categorical structure. The main tool defining structure in the fibrational context is the notion of a fibred adjunction, and the main tool for constructing fibrations from existing ones is change of base, i.e., the pullback of a fibration along a functor.

The content of this chapter is well-known; none of the results presented here are new, unless stated otherwise. Mots of the examples that we present in this chapter, as well as the other chapters of this thesis, are description of standard results of the theory of fibration. We will in most cases not reproduce the associated proofs and refer to standard documents like [Jac99], from which we borrow most of the content in this chapter, for the ones we do not provide.

## 1.1 Fibrations

In this subsection we recall the notion of a fibration and the dual notion of opfibration. Fibrations and opfibrations are special kinds of functors $p : \mathcal{E} \to \mathcal{B}$ which capture the idea that $\mathcal{E}$ is continuously indexed by $\mathcal{B}$. Our interest in fibrations lies in the fact that they provide models of a logic of predicates above a type theory. In such models, objects of $\mathcal{E}$ are thought of as predicates, objects of $\mathcal{B}$ are thought of as types, and $p$ is thought to map each predicate $P$ in $\mathcal{E}$ to the type $pP$ about which it is a predicate. The notion of opfibrations arises in our work when we want to express the duality between induction and coinduction. More details about fibrations, opfibrations and their relations with logic and type theory can be found in [Jac99, Pav90] for example.

Before introducing any definitions, note that we can already see any functor $p : \mathcal{E} \to \mathcal{B}$ as an indexing of the category $\mathcal{E}$ by the category $\mathcal{B}$: for $X$ (resp., $f$) in $\mathcal{B}$, an object $P$ (resp., morphism $\alpha$) of $\mathcal{E}$ has index, or is *above*, $X$ if $pP = X$ (resp., $p\alpha = f$). The point of view of fibrations is to index morphisms of $\mathcal{E}$ by objects of $\mathcal{B}$ instead of morphisms of $\mathcal{B}$: a morphism $\alpha : P \to Q$ in $\mathcal{E}$ has index $X$ in $\mathcal{B}$ if $p\alpha = id_X$, such a morphism is called *vertical*. For a non-vertical morphism $\alpha : P \to Q$ above $f : X \to Y$ we then ask that $\alpha$ is uniquely determined by $f$ and another morphism with index $X$ (or $Y$ for $p$ to be an opfibration). The idea being that $\alpha$ has index $X$ up to *reindexing* its codomain $Q$ from $Y$ to $X$ *along* $f$ (or the dual for opfibration). The key concept to ensure this property is the notion of *cartesian morphisms* in $\mathcal{E}$. The definition of a cartesian morphism above a morphism $f$ guarantees that every morphism above $f$ can be decomposed as a vertical morphism followed by the cartesian morphism. Cartesian morphisms can be thought of as only containing indexing information from $\mathcal{B}$, and they are the direct opposite of vertical morphisms. Dually, there is the notion of an opcartesian morphism above $f$ which guarantees that every morphism above $f$ decomposes as a vertical morphism preceded by the opcartesian morphism.

The key notions of cartesian and opcartesian morphism are defined as follows.

**Definition 1.1.1.** Let $p : \mathcal{E} \to \mathcal{B}$ be a functor.

(i) A morphism $g : Q \to P$ in $\mathcal{E}$ is *cartesian* above a morphism $f : X \to Y$ in

$\mathcal{B}$ if $pg = f$, and for every $g' : Q' \to P$ in $\mathcal{E}$ for which $pg' = f \circ v$ for some $v : pQ' \to X$ there exists a unique $h : Q' \to Q$ in $\mathcal{E}$ such that $ph = v$ and $g \circ h = g'$.

(ii) A morphism $g : P \to Q$ in $\mathcal{E}$ is *opcartesian* above a morphism $f : X \to Y$ in $\mathcal{B}$ if $pg = f$, and for every $g' : P \to Q'$ in $\mathcal{E}$ for which $pg' = v \circ f$ for some $v : Y \to pQ'$ there exists a unique $h : Q \to Q'$ in $\mathcal{E}$ such that $ph = v$ and $h \circ g = g'$.

We can capture cartesian and opcartesian morphisms diagrammatically as follows (the left and right parts, respectively).

$$\begin{tikzcd}
 & Q' \arrow[ddr, dotted, "h"'] \arrow[ddrr, bend left=15, "g'"] & & & & & Q' \\
\mathcal{E} \arrow[dd, "p"'] & & & & & & \\
 & & Q \arrow[r, "g"'] & P & P \arrow[r, "g"'] \arrow[uurr, bend left=15, "g'"] & Q \arrow[uur, dotted, "h"'] & \\
\mathcal{B} & pQ' \arrow[r, "v"'] & X \arrow[r, "f"'] & Y & X \arrow[r, "f"'] & Y \arrow[r, "v"'] & pQ'
\end{tikzcd}$$

Note that in this thesis we follow the convention of drawing elements above one another in a diagram when they are "above" one another in the categorical sense.

Cartesian morphisms (opcartesian morphisms) are the essence of fibrations (resp., opfibrations). We introduce both fibrations and their duals now since the latter will prove useful later in our development. Below we speak primarily of fibrations, with the understanding that the dual observations hold for opfibrations.

**Definition 1.1.2.** Let $p : \mathcal{E} \to \mathcal{B}$ be a functor.

(i) The functor $p$ is a *fibration* if for every object $P$ of $\mathcal{E}$, and every morphism $f : X \to pP$ in $\mathcal{B}$ there is a cartesian morphism $g : Q \to P$ in $\mathcal{E}$ above $f$.

(ii) Dually, the functor $p$ is an *opfibration* if for every object $P$ of $\mathcal{E}$, and every morphism $f : pP \to Y$ in $\mathcal{B}$ there is an opcartesian morphism $g : P \to Q$ in $\mathcal{E}$ above $f$.

(iii) The functor $p$ is a *bifibration* if it is simultaneously a fibration and an opfibration.

For $p : \mathcal{E} \to \mathcal{B}$ an (op/bi)fibration, we call $\mathcal{E}$ the *total category* and $\mathcal{B}$ the *base category* of $p$. For any object $X$ of $\mathcal{B}$, we write $\mathcal{E}_X$ for the *fibre above* $X$, i.e., the subcategory of $\mathcal{E}$ comprising objects above $X$ and vertical morphisms (morphisms above $id_X$). In this thesis we will also use the notion of a fibre for functors that are not necessarily fibrations, indeed this notion is well defined for any functor.

Rather than reasoning about fibrations and opfibrations separately, we will use duality to reason about them simultaneously. The crucial lemma is the following:

**Lemma 1.1.3.** *A functor* $p : \mathcal{E} \to \mathcal{B}$ *is an opfibration if and only if the opposite functor* $p^{op} : \mathcal{E}^{op} \to \mathcal{B}^{op}$ *is a fibration.*

The proof follows from the duality between cartesian and opcartesian morphisms.

It is not hard to see that a cartesian morphism $g$ above a morphism $f$ with codomain $pP$ is unique up to isomorphism, and similarly for opcartesian morphisms. It is often very useful to be able to choose a specific cartesian morphism given a morphism $f$ in $\mathcal{B}$ and $P$ above the codomain of $f$. We call this specific choice a *cartesian lifting* of $f$ of codomain $P$. A fibration which comes with a choice of cartesian liftings, and dually an opfibration which comes with a choice of opcartesian liftings, are said to be *cloven*. In the rest of this thesis all the (op/bi)fibrations that we consider will be assumed to be cloven. The main point for assuming fibrations and opfibrations to be cloven is to be able to derive the following two constructions.

For $p : \mathcal{E} \to \mathcal{B}$ a fibration, $f : X \to Y$ a morphism in $\mathcal{B}$ and $P$ an object of $\mathcal{E}_Y$, we write $f^{\S}_P : f^*P \to P$ for the cartesian lifting of $f$ of codomain $P$. When clear from the context, we might drop $P$ and simply write $f^{\S}$. The function mapping each object $P$ of $\mathcal{E}_Y$ to $f^*P$, the domain of $f^{\S}_P$, extends to a functor $f^* : \mathcal{E}_Y \to \mathcal{E}_X$. The functor $f^*$ then maps a morphism $k : P \to P'$ in $\mathcal{E}_Y$, to the unique morphism

$f^*k$ making the following diagram commute

$$\begin{tikzcd}
f^*P \arrow[r, "f^{\S}_P"] \arrow[d, dotted, "f^*k"'] & P \arrow[d, "k"] \\
f^*P' \arrow[r, "f^{\S}_{P'}"'] & P'
\end{tikzcd}$$

The universal property of $f^{\S}_{P'}$ ensures the existence and uniqueness of $f^*k$. We call the functor $f^*$ the *reindexing functor along* $f$.

For $p : \mathcal{E} \to \mathcal{B}$ an opfibration, $f : X \to Y$ a morphism in $\mathcal{B}$ and $Q$ an object of $\mathcal{E}_X$, we write $f_{\S}^{Q} : Q \to \Sigma_f Q$ for the opcartesian lifting of $f$ of domain $Q$. When clear from the context, we might drop $Q$ and simply write $f_{\S}$. Dually to reindexing, the function mapping each object $Q$ in $\mathcal{E}_X$ to $\Sigma_f Q$, the codomain of $f_{\S}^{Q}$, extends to a functor $\Sigma_f : \mathcal{E}_X \to \mathcal{E}_Y$. We call the functor $\Sigma_f$ the *opreindexing functor along* $f$.

We now illustrate the notions of fibration, opfibration, and bifibration with some examples.

**Example 1.1.4.** Let $\mathcal{B}$ be an arbitrary category. Then the identity functor on $\mathcal{B}$ is a fibration, called the *identity fibration* above $\mathcal{B}$. Each fibre has exactly one object, and the cartesian lifting of a morphism $f$ is $f$ itself. Moreover, the identity fibration is an opfibration and hence also a bifibration.

The following syntactic example illustrates our use of fibrations for induction. If we have a type theory modelled by a category $\mathcal{B}$ and a logic $\mathcal{E}$ to reason about the type theory, then the logic is modelled by a fibration with base category $\mathcal{B}$ and total category $\mathcal{E}$.

**Example 1.1.5.** Consider a type theory such as the simply typed lambda calculus. We can model this type theory with a category $\mathcal{B}$ whose objects are the types of the type theory and whose morphisms are ($\beta\eta$-equivalence classes of) terms of the type theory.

Now suppose we have a predicate logic on this type theory. Such a logic is given by a collection of predicates, i.e., by a collection of propositions, each of which is parameterised above (at most) a single type. Concretely, writing

$x : X \vdash P : Prop$ for a proposition $P$ whose only possible free variable is $x : X$, we view $P$ as a predicate on $X$. Then we can construct a category $\mathcal{E}$ to model this predicate logic by taking the objects of $\mathcal{E}$ to be predicates, and a morphism of $\mathcal{E}$ from a predicate $x : X \vdash P : Prop$ to a predicate $y : Y \vdash Q : Prop$ to be a term $t$ such that $x : X \vdash t : Y$ together with a logical derivation of the entailment $x : X, P \vdash Q[y \leftarrow t]$.

If our predicate logic is closed under substitution — i.e., if $y : Y \vdash P : Prop$ and if $x : X \vdash t : Y$, then $x : X \vdash P[y \leftarrow t] : Prop$ — then the functor $p : \mathcal{E} \to \mathcal{B}$ mapping a predicate $x : X \vdash Q : Prop$ to the type $X$ of its free variable(s) is a fibration. Indeed, given a term $t$ in $\mathcal{B}$ such that $x : X \vdash t : Y$ and a predicate $y : Y \vdash Q : Prop$, the substitution of $y$ in $Q$ by $t$ defines reindexing of $Q$ along $t$, and the cartesian lifting of $t$ at $Q$ is the morphism of $\mathcal{E}$ from $x : X \vdash Q[y \leftarrow t] : Prop$ to $y : Y \vdash Q : Prop$ given by $t$ itself together with the obvious derivation of the entailment $x : X, Q[y \leftarrow t] \vdash Q[y \leftarrow t]$. The fibration defined in this way is called the *syntactic fibration* for the type theory under consideration.

Furthermore, if the logic has the additional property that for any term $x : X \vdash t : Y$ and predicate $x : X \vdash P : Prop$ there is a predicate $y : Y \vdash \exists a : X.\ t[x \leftarrow a] = y.\ P[x \leftarrow a] : Prop$ (with equality as described in Section 10.1 in [Jac99]) then the associated fibration is a bifibration. Indeed the adjunction $\Sigma_u \dashv u^*$, for $u$ that captures a term $x : X \vdash t : Y$ and any $x : X \dashv P : Prop$ and $y : Y \dashv Q : Prop$, amounts then to the correspondence

$$\frac{y : Y,\ \big(\exists a : X.\ t[x \leftarrow a] = y.\ P[x \leftarrow a]\big) \vdash Q}{x : X,\ P \vdash Q[y \leftarrow t]}$$

In the next example we look at the fibration of set-indexed sets. This fibration will help us develop intuitions on most of the fibrational constructions that we will see in this thesis.

**Example 1.1.6.** The category Fam(Set) has as objects pairs $(X, P)$ with $X$ a set and $P : X \to \text{Set}$ a function. We call $X$ the *domain* of $(X, P)$, and write $P$ for $(X, P)$ when convenient. A morphism from $P : X \to \text{Set}$ to $P' : X' \to \text{Set}$

is a pair $(f, f^{\sim})$ of functions $f : X \to X'$ and $f^{\sim} : \forall x \in X. P\,x \to P'(f\,x)$. The functor $p : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Set}$ mapping $(X, P)$ to $X$ and $(f, f^{\sim})$ to $f$ is called the *family fibration* of Set. If $f : X \to Y$ is a morphism in Set and $P : Y \to \mathrm{Set}$, then reindexing of $P$ along $f$ is defined by $f^*(Y, P) = (X, P \circ f)$, and the cartesian lifting of $f$ at $P$ is $(f, \lambda x.\ id)$. In fact, this fibration is actually a bifibration: if $f : X \to Y$ is a morphism in Set and $P : X \to \mathrm{Set}$, then opreindexing of $P$ along $f$ is given by $\Sigma_f(X, P) = (Y, \lambda y.\ \bigcup_{\{x \in X | fx = y\}} Px)$, and the opcartesian lifting of $f$ at $P$ is $\big(f, \lambda x \in X, \lambda p \in Px.(x, p)\big)$.

In the following example we describes a construction that associates any category $\mathcal{C}$ with a fibration of set-indexed elements of $\mathcal{C}$. The fibration of set-indexed sets presented in the previous example is an substantiation of this construction. We keep both examples separate since the following example provides a nice intermediary setting between the intuitive fibration of set-indexed sets and arbitrary (abstract) fibrations.

**Example 1.1.7.** For $\mathcal{C}$, a category, consider the category $\mathrm{Fam}(\mathcal{C})$ of set-indexed families of objects of $\mathcal{C}$. An object of $\mathrm{Fam}(\mathcal{C})$ is a pair $(X, P)$ with $X$ a set of indices and $P : X \to \mathcal{C}$ a function mapping to each index the corresponding indexed element of $\mathcal{C}$. We write $P$ for $(X, P)$ when convenient. A morphism from $P : X \to \mathcal{C}$ to $P' : X' \to \mathcal{C}$ is a pair $(f, f^{\sim})$ of a function $f : X \to X'$ and a family of morphisms $f^{\sim}_x : Px \to P'(f\,x)$ in $\mathcal{C}$, for $x$ in $X$.

The category $\mathrm{Fam}(\mathcal{C})$ is fibred above Set with the functor $p : \mathrm{Fam}(\mathcal{C}) \to \mathrm{Set}$ sending a family $(X, P)$ to the index $X$ and a map $(f, f^{\sim})$ to the function $f$. For $f : X \to Y$ in Set and $P : Y \to \mathcal{C}$ in $\mathrm{Fam}(\mathcal{C})_Y$, define the reindexing of $P$ along $f$, $f^*P : X \to \mathcal{C}$, as $P \circ f$. The cartesian lifting $f^{\S}_P : f^*P \to P$ is then defined as the pair $(f, \lambda x \in X.\ id_{P(fx)})$. This fibration is called the *family fibration of* $\mathcal{C}$.

In case the category $\mathcal{C}$ has set-indexed coproducts $\coprod$, the family fibration of $\mathcal{C}$ is a bifibration. For $f : X \to Y$ in Set and $P : X \to \mathcal{C}$ in $\mathrm{Fam}(\mathcal{C})_X$, define $\Sigma_f P : Y \to \mathcal{C}$ as $\lambda y.\ \coprod_{x \in (f^{-1}y)} Px$, where $f^{-1}y$ denote the set $\{x \in X | fx = y\}$. The opcartesian lifting $f^P_{\S} : P \to \Sigma_f P$ is then defined as the pair $(f, f^{\S})$ where $f^{\S}_x$ is the injection of $Px$ in $\coprod_{y \in (f^{-1}(fx))} Py$, for all $x \in X$.

**Example 1.1.8.** The *arrow category* of $\mathcal{B}$, denoted $\mathcal{B}^{\rightarrow}$, has morphisms of $\mathcal{B}$ as its objects. A morphism from $f : X \to Y$ to $f' : X' \to Y'$ in $\mathcal{B}^{\rightarrow}$ is a pair of morphisms $\alpha_1 : X \to X'$ and $\alpha_2 : Y \to Y'$ in $\mathcal{B}$ such that the following diagram commutes.

$$\begin{tikzcd}
X \arrow[r, "\alpha_1"] \arrow[d, "f"'] & X' \arrow[d, "g"] \\
Y \arrow[r, "\alpha_2"'] & Y'
\end{tikzcd}$$

The *codomain functor cod* : $\mathcal{B}^{\rightarrow} \to \mathcal{B}$ maps an object $f : X \to Y$ to $Y$ and a morphism $(\alpha_1, \alpha_2)$ to $\alpha_2$. Its fibre above an object $Y$ is the slice category $\mathcal{B}/Y$. The codomain functor is an opfibration: given an object $g : X \to Y$ in $\mathcal{B}^{\rightarrow}$ above $Y$ and a morphism $f : Y \to Z$ in $\mathcal{B}$, the pair $(id_X, f \circ g)$ gives an opcartesian morphism above $f$ of domain $g$. Furthermore, if $\mathcal{B}$ has pullbacks then *cod* is a bifibration called the *codomain fibration* above $\mathcal{B}$. Indeed, given an object $f : X \to Y$ in $\mathcal{B}/Y$ and a morphism $f' : X' \to Y$ in $\mathcal{B}$, the pullback of $f$ along $f'$ gives a cartesian morphism above $f'$.

**Example 1.1.9.** Let $\mathcal{B}$ be a category and, dually to the previous example, consider the *domain functor dom* : $\mathcal{B}^{\rightarrow} \to \mathcal{B}$, which maps an object $f : X \to Y$ to $X$ and a morphism $(\alpha_1, \alpha_2)$ to $\alpha_1$. The domain functor is a fibration, with cartesian morphisms given by composition. Its fibre above an object $Y$ in $\mathcal{B}$ is the coslice category $Y/\mathcal{B}$. Furthermore, if $\mathcal{B}$ has pushouts then *dom* is a bifibration, called the *domain fibration* above $\mathcal{B}$. Indeed, the opcartesian lifting of a morphism $g : X \to X'$ of domain $f : X \to Y$ is given by the pushout of $f$ along $g$ in $\mathcal{B}$.

We can consider restrictions of the arrow category that provide interesting fibrations such as the fibration of monos, epis, subobjects, and so on. We only describe the fibration of monos:

**Example 1.1.10.** If $\mathcal{B}$ is a category, then the category of monos of $\mathcal{B}$, denoted $Mono(\mathcal{B})$, has monomorphisms in $\mathcal{B}$ as its objects. A morphism in $Mono(\mathcal{B})$ from $f : X \hookrightarrow Y$ to $f' : X' \hookrightarrow Y'$ is a pair of morphisms $(\alpha_1, \alpha_2)$ in $\mathcal{B}$ such that $\alpha_2 \circ f = f' \circ \alpha_1$.

The map $p : Mono(\mathcal{B}) \to \mathcal{B}$ mapping a monomorphism $f : X \hookrightarrow Y$ to $Y$ extends to a functor, which we will also call $p$. If $\mathcal{B}$ has pullbacks, then $p$ is a fibration, called the *mono fibration* above $\mathcal{B}$ (note that pullbacks do indeed give cartesian morphisms since the pullback of a mono along any morphism is again a mono). The fibre of $Mono(\mathcal{B})$ above an object $Y$ of $\mathcal{B}$ has as objects the monomorphisms of codomain $Y$. A morphism in $Mono(\mathcal{B})_Y$ from $f : X \hookrightarrow Y$ to $f' : X' \hookrightarrow Y$ is a map $\alpha_1 : X \to X'$ in $\mathcal{B}$ such that $f = f' \circ \alpha_1$.

The following example is a standard construction to derive a setting of relations from a pre-existing fibration.

**Example 1.1.11.** Given a fibration $p : \mathcal{E} \to \mathcal{B}$, where $\mathcal{B}$ has products, we can consider the subcategory $Rel(\mathcal{E})$ of $\mathcal{E}$ whose objects are those objects of $\mathcal{E}$ above a product of the form $A \times A$. The morphisms of $Rel(\mathcal{E})$ are the morphisms of $\mathcal{E}$ above a product of the form $f \times f$. The functor $p' : Rel(\mathcal{E}) \to \mathcal{B}$ mapping an object of $\mathcal{E}$ above $A \times A$ to the object $A$ is a fibration, and is called the *relations fibration* for $p$. If $f : A \to B$ is a morphism in $\mathcal{B}$ and $P$ is above $B$ in $Rel(\mathcal{E})$, then the cartesian lifting of $f$ at $P$ with respect to $p'$ is given by the cartesian lifting of $f \times f$ at $P$ with respect to $p$.

It is not hard to see that if $p$ is a bifibration in Example 1.1.11, then $p'$ is also a bifibration. This example provides one way to construct a new fibration from an already existing one. We will see techniques for doing this generally in the next section.

**Example 1.1.12.** Let $\mathcal{B}$ be a category with cartesian products. The category $s(\mathcal{B})$ has as objects pairs $(I, X)$ of objects of $\mathcal{B}$, and as morphisms from $(I, X)$ to $(J, Y)$ pairs of morphisms $(u, f)$ in $\mathcal{B}$ where $u$ is a morphism from $I$ to $J$ and $f$ is a morphism from $I \times X$ to $Y$. The first projection functor $s(\mathcal{B}) \to \mathcal{B}$ is a fibration, called the *simple fibration* on $\mathcal{B}$. Indeed, for any morphism $f : I \to J$ in $\mathcal{B}$ and object $(J, X)$ in $s(\mathcal{B})_J$, the cartesian lifting of $f$ at $(J, X)$ is $(f, \pi_2)$. Note that this fibration is not an opfibration, and therefore is not a bifibration.

We now look at some basic properties of cartesian morphisms. Bear in mind that these results dualise for opcartesian morphisms.

**Lemma 1.1.13.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration, the following properties hold:*

*(i) the cartesian lifting of an isomorphism is an isomorphism.*

*(ii) all isomorphisms in $\mathcal{E}$ are cartesian.*

*(iii) for any $f : X \to Y$ in $\mathcal{B}$ and $P$, $Q$ in $\mathcal{E}$, $\mathcal{E}_f(P, Q) \cong \mathcal{E}_X(P, f^*Q)$.*

*(iv) for $\alpha : P \to Q$ and $\beta : Q \to R$ in $\mathcal{E}$, if $\alpha$ and $\beta$ are cartesian, so is $\beta \circ \alpha$. If $\beta$ and $\beta \circ \alpha$ are cartesian, so is $\alpha$.*

Here, we write $\mathcal{E}_f(P, Q)$ for the subclass of morphisms from $P$ to $Q$ in $\mathcal{E}$ above $f$.

We then have the following very useful property on the relationship between cartesian and opcartesian morphisms:

**Lemma 1.1.14.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration. Then $p$ is a bifibration iff, for every morphism $f : X \to Y$ in $\mathcal{B}$, $f^*$ has a left adjoint $\Sigma_f$.*

*Proof.* We only describe the different constructions involved in the proof, for a complete proof see [Jac93].

Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration and $f : X \to Y$ be a morphism in $\mathcal{B}$. The isomorphism associated to the adjunction $\Sigma_f \dashv f^*$ is determined by the following diagram,

$$\begin{tikzcd}
P \arrow[r, "f_\S"] \arrow[d, "\alpha"'] & \Sigma_P \arrow[d, "\beta"] \\
f^*P \arrow[r, "f^\S"'] & P \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

where $\beta$ uniquely determine $\alpha$ by the universal property of $f^\S$ and $\alpha$ uniquely determine $\beta$ by the universal property of $f_\S$.

Let us now assume $p : \mathcal{E} \to \mathcal{B}$ a fibration with, for every morphism $f : X \to Y$ in $\mathcal{B}$, a left adjoint $\Sigma_f$ of $f^*$. For $Q$ in $\mathcal{E}_X$, define $f_\S : Q \to \Sigma_f Q$ as the composition $Q \xrightarrow{\eta_Q} \Sigma_f f^* Q \xrightarrow{\Sigma_f(f^\S)} \Sigma_f Q$, where $\eta$ is the unit of the adjunction $\Sigma_f \dashv f^*$. □

We finish this section with the definition of the Beck-Chevalley condition for bifibrations.

**Definition 1.1.15.** Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration. We say that $p$ *satisfies the Beck-Chevalley condition* if, for any pullback square:

$$\begin{tikzcd} A \arrow[r, "t"] \arrow[d, "s"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "f"] \\ C \arrow[r, "g"'] & D \end{tikzcd}$$

in $\mathcal{B}$, the canonical natural transformation $\Sigma_s t^* \dot{\to} g^*\Sigma_f$ defined as

$$\Sigma_s t^* \xrightarrow{\Sigma_s t^* \eta^f} \Sigma_s t^* f^* \Sigma_f \xrightarrow{\cong} \Sigma_s s^* g^* \Sigma_f \xrightarrow{\epsilon^s g^* \Sigma_f} g^* \Sigma_f$$

is an isomorphism, with $\eta^f$ the unit of the adjunction $\Sigma_f \dashv f^*$ and $\epsilon^s$ the counit of the adjunction $\Sigma_s \dashv s^*$.

Among the previous examples, we can check that the bifibrations of Examples 1.1.4, 1.1.6 and 1.1.8 satisfy the Beck-Chevalley condition, and the construction of Example 1.1.11 preserves the Beck-Chevalley condition.

We are particularly interested in the following consequence of the Beck-Chevalley condition.

**Lemma 1.1.16.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration that satisfies the Beck-Chevalley condition. For any mono $f : X \to Y$ in $\mathcal{B}$ and any $P$ above $X$, the unit $\eta : P \to f^*\Sigma_f P$ is an isomorphism. Or, equivalently, for any mono $f : X \to Y$ in $\mathcal{B}$ the functor $\Sigma_f : \mathcal{E}_X \to \mathcal{E}_Y$ is full and faithful.*

*Proof.* Since $f : X \to Y$ is mono, the square $\begin{tikzcd} X \arrow[r, "id"] \arrow[d, "id"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$ is a pullback. Therefore, by the Beck-Chevalley condition, this implies that the composition:

$$\Sigma_{id} id^* \xrightarrow{\Sigma_{id} id^* \eta_f} \Sigma_{id} id^* f^* \Sigma_f \xrightarrow{=} \Sigma_{id} id^* f^* \Sigma_f \xrightarrow{\epsilon_{id} f^* \Sigma_f} f^* \Sigma_f$$

is an isomorphism. Now, remembering that the (op)cartesian lifting of an isomorphism is an isomorphism, the functors $id^*$ and $\Sigma_{id}$ are full and faithful hence the counit $\epsilon_{id}$ is an isomorphism. We then have that $\Sigma_{id} id^* \eta_f$ is an isomorphism thus, the unit $\eta_f : id_{\epsilon_X} \dot{\to} f^* \Sigma_f$ is an isomorphism, making $\Sigma_f$ a full and faithful functor. □

## 1.2 Fibred category theory

Now that we have the notion of a fibration at our disposal, we present two different ways to construct new fibrations from existing ones, namely, *change of base* and *composition*. We then consider fibred structure, i.e., categorical structure within a fibration. Categorical structure is given in terms of adjunctions, which themselves are given in term of functors and natural transformations. We find the exact same situation with fibrations: there are notions of fibred functors, fibred natural transformations, and, derived from these, fibred adjunctions. In this section, we present these fibred notions. We then conclude this section by considering how the change of base construction preserves fibred structure.

We first consider one standard way to construct fibrations. If $\mathcal{C}$ is a category and if $f : X \to Z$ and $g : Y \to Z$ are morphisms, then write $(f \times_Z g, f^*g, g^*f)$ for the pullback of $g$ along $f$ in $\mathcal{C}$.

**Lemma 1.2.1.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration and $F : \mathcal{B}' \to \mathcal{B}$ a functor. The pullback*

$$\begin{tikzcd}
F \times_{\mathcal{B}} p \arrow[r, "p^*F"] \arrow[d, "F^*p"'] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}' \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}$$

*of $p$ along $F$ in* **Cat** *defines a new fibration $F^*p : F \times_{\mathcal{B}} p \to \mathcal{B}$. We say that the fibration $F^*p$ is constructed by* change of base *of $p$ along $F$.*

The fibration $F^*p$ is described concretely in the following way. Following the standard construction of pullbacks in **Cat**, the total category $F \times_{\mathcal{B}} p$ has as objects pairs $(Y, P)$, where $Y$ is an object of $\mathcal{B}'$, $P$ is an object of $\mathcal{E}$, and $FY = pP$, and morphisms are given similarly. The functor $F^*p$ is then the first projection. Now, to check that $F^*p$ is a fibration, let $f : X \to Y$ be a morphism in $\mathcal{B}'$, and let $(Y, P)$ be an element of $(F \times_{\mathcal{B}} p)_Y$. Then reindexing of $(Y, P)$ along $f$ is given by $(X, (Ff)^*P)$, and the cartesian lifting of $f$ at $(Y, P)$ with respect to $F^*p$ is the cartesian lifting of $Ff$ at $P$ with respect to $p$. Note that a morphism $f$ is cartesian with respect to $F^*p$ if and only if the morphism $p^*Ff$ is cartesian with respect to $p$.

We then have, using the duality between fibration and opfibration (Lemma 1.1.3), the following corollary:

**Corollary 1.2.2.** *Change of base preserves opfibrations and thus bifibrations.*

It is not hard to see that the relations fibration $p' : \mathit{Rel}(\mathcal{E}) \to \mathcal{B}$ for a fibration $p : \mathcal{E} \to \mathcal{B}$ presented in Example 1.1.11 can be constructed as a change of base of $p$ along the endofunctor $\Delta : \mathcal{B} \to \mathcal{B}$ that maps an object $X$ of $\mathcal{B}$ to the product $X \times X$.

As we will see in the following lemma, another way to construct a fibration is by composition.

**Lemma 1.2.3.** *Let $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{B} \to \mathcal{A}$ be fibrations, then the composition $p' \circ p : \mathcal{E} \to \mathcal{A}$ is again a fibration.*

This time a cartesian lifting of a morphism $f : X \to Y$ in $\mathcal{A}$ at $P$ is given by successively lifting the morphism $f$ with respect to $p'$ and then with respect to $p$. More specifically, if $P$ is such that $p'pP = Y$ then the cartesian lifting of $f$ at $P$ with respect to $p' \circ p$ is the cartesian lifting at $P$, with respect to $p$, of the cartesian lifting of $f$ at $pP$ with respect to $p'$. In symbols, it is $(f^{\S}_{pP})^{\S}_{P}$, where the inner cartesian morphism is taken with respect to $p'$ and the outer one is taken with respect to $p$.

We can derive from the change of base and the composition operations more complex constructions like the product of two fibrations with the same base category. We have

**Corollary 1.2.4.** *Let $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{E}' \to \mathcal{B}$ be two fibrations. The functor $p \times p'$ defined as follows is a fibration.*

$$\begin{tikzcd}
p \times_{\mathcal{B}} p' \arrow[r, "p'^{*}p"] \arrow[d, "p^{*}p'"'] \arrow[dr, "p \times p'" description] & \mathcal{E}' \arrow[d, "p'"] \\
\mathcal{E} \arrow[r, "p"'] & \mathcal{B}
\end{tikzcd}$$

Here the resulting fibration has as fibre $(p \times_{\mathcal{B}} p')_X \cong \mathcal{E}_X \times \mathcal{E}'_X$ for $X$ in $\mathcal{B}$.

Let us now consider fibred structure. We begin with the notion of a fibred functor.

**Definition 1.2.5.** Given two fibrations, $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{E}' \to \mathcal{B}'$, a *fibred functor* from $p$ to $p'$ is given by a pair of functors $F : \mathcal{B} \to \mathcal{B}'$ and $H : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes and $H$ preserves cartesian morphisms.

$$\begin{tikzcd}
\mathcal{E} \arrow[r, "H"] \arrow[d, "p"'] & \mathcal{E}' \arrow[d, "p'"] \\
\mathcal{B} \arrow[r, "F"'] & \mathcal{B}'
\end{tikzcd}$$

In this situation we write $(H, F) : p \to p'$.

**Example 1.2.6.** Any functor $F : \mathcal{B} \to \mathcal{B}'$ induces a fibred functor $(F^{\to}, F)$ between the two domain fibrations $dom : \mathcal{B}^{\to} \to \mathcal{B}$ and $dom' : \mathcal{B}'^{\to} \to \mathcal{B}'$, where $F^{\to}$ is the obvious extension of $F$ to the arrow categories $\mathcal{B}^{\to}$ and $\mathcal{B}'^{\to}$. Indeed, the cartesian morphisms in $dom$ are pairs $(\alpha_1, \alpha_2)$ of morphisms in $\mathcal{B}$ where $\alpha_1$ is an isomorphism (see Example 1.1.9). Hence, $F^{\to}$ preserves cartesian morphisms since (any functor) $F$ preserves isomorphisms.

**Example 1.2.7.** If $\mathcal{B}$ and $\mathcal{B}'$ are two categories with pullbacks, and $F$ is a functor which preserves pullbacks, then $(F^{\to}, F)$ is a fibred functor between the codomain fibrations $cod : \mathcal{B}^{\to} \to \mathcal{B}$ and $cod' : \mathcal{B}'^{\to} \to \mathcal{B}'$. In this case, cartesian morphisms are pullback squares (see Example 1.1.8) and $F$ preserves them by assumption.

Another fibred functor arises via change of base.

**Example 1.2.8.** If $F^*p$ is the fibration obtained by pulling $p$ back along $F$, then $(p^*F, F) : F^*p \to p$ is a fibred functor.

Note that fibrations and fibred functors form a category **Fib**. In fact, the category **Fib** is fibred above **Cat**. Indeed, the functor $U$ mapping a fibration $p : \mathcal{E} \to \mathcal{B}$ to its base category $\mathcal{B}$ is a fibration, and reindexing and cartesian lifting are given by change of base. That is, if $p : \mathcal{E} \to \mathcal{B}$ is a fibration and $F : \mathcal{B} \to \mathcal{B}'$ is a functor, then the cartesian lifting of $F$ at $p$ is given by $(p^*F, F)$.

With the notion of fibred functor comes the following notion of a fibred natural transformation.

**Definition 1.2.9.** Given two fibred functors $(H, F)$ and $(L, G)$ from $p : \mathcal{E} \to \mathcal{B}$ to $p' : \mathcal{E}' \to \mathcal{B}'$, a *fibred natural transformation* from $(H, F)$ to $(L, G)$ is given by a pair of natural transformations $\eta : F \to G$ and $\gamma : H \to L$ such that every component $\gamma_P$ is above the component $\eta_{(pP)}$.

It is straightforward to check that any natural transformation $\eta : F \to G$ induces a fibred natural transformation $(\eta^{\to}, \eta) : (F^{\to}, F) \to (G^{\to}, G)$ in the domain fibration (following Example 1.2.6), as well as in the codomain fibration as soon as both $F$ and $G$ preserve pullbacks (following Example 1.2.7).

Now that we have notions of fibred functors and fibred natural transformations, we can introduce the notion of a fibred adjunction.

**Definition 1.2.10.** Given two fibred functors $(H, F) : p \to p'$ and $(L, G) : p' \to p$, we say that $(H, F)$ is a *fibred left adjoint* of $(L, G)$ if $F$ is left adjoint to $G$, $H$ is left adjoint to $L$, and the unit, or equivalently the counit, of the adjunction $H \dashv L$ is above the unit (resp., counit) of the adjunction $F \dashv G$.

**Example 1.2.11.** Any adjunction $F \dashv G$ can be seen as a fibred adjunction between the extensions of the functors $F$ and $G$ to their corresponding domain fibrations.

In this thesis we will mainly use fibred functors $(H, F) : p \to p'$ where $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{E}' \to \mathcal{B}$ have the same base category $\mathcal{B}$ and $F$ is taken to be the identity. In this case a fibred functor from $p$ to $p'$ is simply a functor $H : \mathcal{E} \to \mathcal{E}'$ such that the following diagram commutes and $H$ preserves cartesian morphisms,

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "H"] \arrow[dr, "p"'] & & \mathcal{E}' \arrow[dl, "p'"] \\
& \mathcal{B} &
\end{tikzcd}$$

We say that the functor $H$ is *fibred above* $\mathcal{B}$, or simply *fibred*, if $\mathcal{B}$ is clear from context. This construction determines a category $\mathbf{Fib}(\mathcal{B})$ of fibrations with base category $\mathcal{B}$ and fibred functors above $\mathcal{B}$. In fact, $\mathbf{Fib}(\mathcal{B})$ is the fibre above $\mathcal{B}$ in the fibration from $\mathbf{Fib}$ to $\mathbf{Cat}$ described after Example 1.2.8. Notice then that every fibration $p : \mathcal{E} \to \mathcal{B}$ is also a fibred functor:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, "p"] \arrow[dr, "p"'] & & \mathcal{B} \arrow[dl, "id_{\mathcal{B}}"] \\
& \mathcal{B} &
\end{tikzcd}$$

and that this makes the identity of $\mathcal{B}$ the terminal object of $\mathbf{Fib}(\mathcal{B})$. Furthermore, the product fibration defined in the Corollary 1.2.4 is a cartesian product in $\mathbf{Fib}(\mathcal{B})$.

If we restrict ourselves to fibred functors in $\mathbf{Fib}(\mathcal{B})$, the notions of a fibred natural transformation and of a fibred adjunction are correspondingly simplified. We then have that a fibred natural transformation above $\mathcal{B}$ is a natural transformation between fibred functors above $\mathcal{B}$ whose components are vertical. A fibred adjunction above $\mathcal{B}$ is an adjunction between fibred functors above $\mathcal{B}$ such that the (components of its) unit, or equivalently counit, are vertical.

Notice then that, for any $X$ in $\mathcal{B}$, we can restrict a fibred functor $F : p \to p'$ above $\mathcal{B}$ to a functor $F_X$ between the fibres $\mathcal{E}_X$ and $\mathcal{E}'_X$. Similarly, we can restrict a fibred natural transformation $\alpha : F \to G$ above $\mathcal{B}$ to a natural transformation $\alpha_X : F_X \to G_X$, for any $X$ in $\mathcal{B}$. As adjunctions are described in terms of functors and natural transformations, there is a similar result for fibred adjunctions. The next lemma shows how we can use these restrictions in order to have a fibrewise presentation of fibred adjunction.

We will make good use of the following lemma in the remainder of this thesis.

**Lemma 1.2.12.** *Let $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{E}' \to \mathcal{B}$ be fibrations, and $G : p \to p'$ be a fibred functor above $\mathcal{B}$. Then $G$ has a fibred left (resp., right) adjoint above $\mathcal{B}$ if and only if the following two conditions hold:*

*(i) For any $b$ in $\mathcal{B}$, $G_b$ has a left (resp., right) adjoint $F_b$.*

*(ii) The* Beck Chevalley condition *holds, i.e., for every map $u : a \to d$ in $\mathcal{B}$ and every pair of reindexing functors $u^* : \mathcal{E}_d \to \mathcal{E}_a$ and $u^\# : \mathcal{E}'_d \to \mathcal{E}'_a$, the canonical natural transformation $F_a u^\# \dot{\to} u^* F_d$ (resp., $u^* F_d \dot{\to} F_a u^\#$) obtained as the transpose of the composition $u^\# \xrightarrow{u^\# \eta} u^\# G_d F_d \overset{\cong}{\to} G_a u^* F_d$ (resp., $G_a u^* F_d \overset{\cong}{\to} u^\# G_d F_d \xrightarrow{u^\# \epsilon} u^\#$) is an isomorphism.*

Now that we have a notion of a fibred adjunction, let us look at some specific structures that we can define using it.

We first use fibred adjunctions to define the notion of a terminal object functor. Terminal object functors capture the notion of truth in context when modelling logic with fibration (see the following examples). This notion is then fundamental for the presentation of induction schemes in fibration.

**Definition 1.2.13.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration, we say that $p$ has a *terminal object functor* if the unique fibred functor from $p$ to $id_{\mathcal{B}}$ has a fibred right adjoint. Diagrammatically we have:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "p"] \arrow[rr, phantom, "\perp"] \arrow[dr, "p"'] & & \mathcal{B} \arrow[ll, bend left, "\mathbf{1}"] \arrow[dl, "id_{\mathcal{B}}"] \\
& \mathcal{B} &
\end{tikzcd}$$

We denote the terminal object functor for $p$ by $\mathbf{1}_p$, or simply by $\mathbf{1}$ when $p$ is clear from the context.

Since the fibration $id_{\mathcal{B}}$ is the terminal object of $\mathbf{Fib}(\mathcal{B})$, the terminal object functor can be understood as a fibred terminal object.

We can deduce the following properties from this definition.

**Lemma 1.2.14.** *Let p be a fibration with a terminal object functor. The terminal object functor is full and faithful. Moreover, any full and faithful right adjoint to p is a terminal object functor for p.*

*Proof.* The key observation is that the only vertical morphisms in the fibration $id_{\mathcal{B}}$ are the identities. We then know that, first, whenever there is a fibred adjunction $p \dashv \mathbf{1}$ the components of the counit are necessarily identities, i.e., $\mathbf{1}$ is full and faithful, and second, that this is a sufficient condition to have a fibred right adjoint to $p$. □

Using Lemma 1.2.12 on Definition 1.2.13, the terminal object functor can be given in a fibrewise form.

**Corollary 1.2.15.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration. Assume further that, for every object $X$ of $\mathcal{B}$, the fibre $\mathcal{E}_X$ has a terminal object $\mathbf{1}X$ such that, for any $f : X' \to X$*

*in $\mathcal{B}$, $f^*(\mathbf{1}X) \cong \mathbf{1}X'$. Then the assignment mapping each object $X$ in $\mathcal{B}$ to $\mathbf{1}X$ in $\mathcal{E}$, and each morphism $f : X' \to X$ in $\mathcal{B}$ to the composition $\mathbf{1}X' \cong f^*\mathbf{1}X \xrightarrow{f^\S} \mathbf{1}X$ in $\mathcal{E}$, defines the terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$.*

**Example 1.2.16.** (Example 1.1.5, continued) If the predicate logic on a type theory has a constantly true proposition **true**, then the syntactic fibration for that type theory has a terminal object functor. It maps any type in the type theory to the constantly **true**-valued predicate.

**Example 1.2.17.** (Example 1.1.6, continued) The family fibration on Set has a terminal object functor. It maps a set $X$ to $(X, \lambda x.1)$, where 1 is the one point set.

**Example 1.2.18.** (Example 1.1.7, continued) For $\mathcal{C}$, a category with a terminal object $\top$, the family fibration of $\mathcal{C}$ has a terminal object functor $\mathbf{1}$ : Set → Fam($\mathcal{C}$) defined as $\mathbf{1}\ X = (X, \lambda x \in X.\ \top)$.

**Example 1.2.19.** (Example 1.1.8, continued) The codomain fibration on a category $\mathcal{B}$ has a terminal object functor. It maps an object $X \in \mathcal{B}$ to the identity morphism $id_X$.

**Example 1.2.20.** (Example 1.1.12, continued) If a category $\mathcal{B}$ has cartesian products and terminal object 1, then simple fibration on it has a terminal object functor. It maps an object $X$ of $\mathcal{B}$ to $(X, 1)$.

**Example 1.2.21.** The functor mapping a category $\mathcal{B}$ to the identity fibration above $\mathcal{B}$ is the terminal object functor for the fibration **Fib** above **Cat**.

Before looking at another structure constructed from a fibred adjunction we present the notion of section of a fibration. The sections of a fibration will be used to model (an abstract form of) equality in the setting of coinduction.

**Definition 1.2.22.** A *section* of a fibration $p : \mathcal{E} \to \mathcal{B}$ is a functor $e : \mathcal{B} \to \mathcal{E}$ right inverse of $p$, i.e., such that $p \circ e = id_{\mathcal{B}}$.

Note that, as any right inverse functor, a section is necessarily faithful. Also, note that the terminal object functor of a fibration $p$ is a full, fibred section, as well as the terminal object of the category of sections of $p$ and vertical natural transformations between them.

We then have the following property.

**Lemma 1.2.23.** *Let* $p : \mathcal{E} \to \mathcal{B}$ *be a fibration with a (full) section* $e : \mathcal{B} \to \mathcal{E}$ *and* $F : \mathcal{A} \to \mathcal{B}$ *be a (full) functor. The fibration* $F^*p$ *obtained by a change of base of* $p$ *along* $F$ *has a (full) section.*

*Proof.* Construct the section $e' : \mathcal{A} \to \mathcal{E}'$ of $F^*p$ as the unique morphism making the following diagram commute:

$$\begin{tikzcd}
\mathcal{A} \arrow[dr, "e'"] \arrow[drr, bend left, "e\circ F"] \arrow[ddr, bend right, "id_{\mathcal{A}}"'] & & \\
& \mathcal{E}' \arrow[r] \arrow[d, "F^*p"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
& \mathcal{A} \arrow[r, "F"'] & \mathcal{B}
\end{tikzcd}$$

Concretely, the functor $e'$ sends an object $X$ in $\mathcal{A}$ to the pair $(X, eFX)$. Hence, if $e$ and $F$ are full it follows that $e'$ is full. $\square$

Next we use fibred adjunctions to define a notion of fibred products.

**Definition 1.2.24.** A fibration $p : \mathcal{E} \to \mathcal{B}$ *has products* if the diagonal (fibred) functor $\Delta : \mathcal{E} \to \mathcal{E} \times \mathcal{E}$ above $\mathcal{B}$ has a fibred right adjoint, i.e., if

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\Delta"] \arrow[rr, phantom, "\top"] \arrow[dr, "p"'] & & \mathcal{E} \times \mathcal{E} \arrow[ll, bend left, "\times"] \arrow[dl, "p\times p"] \\
& \mathcal{B} &
\end{tikzcd}$$

As with fibred terminal object functors, we can construct fibred products fibrewise.

**Corollary 1.2.25.** *A fibration* $p : \mathcal{E} \to \mathcal{B}$ *has products iff for every object* $X$ *in* $\mathcal{B}$*, the fibre* $\mathcal{E}_X$ *has a cartesian product* $\times_X$*, and for every* $f : X \to Y$ *in* $\mathcal{B}$ *and* $P, Q$ *in* $\mathcal{E}_Y$ *the canonical map* $< f^*\pi_1, f^*\pi_2 >: f^*(P \times_Y Q) \to f^*P \times_X f^*Q$ *is an isomorphism.*

**Example 1.2.26.** (Example 1.1.5, continued) Conjunction of predicates stable under substitution define a fibred product in the syntactic fibration of Example 1.1.5.

**Example 1.2.27.** (Example 1.2.21, continued) The product of the two fibrations defined in Corollary 1.2.4 define a fibred product for the fibration of **Fib** above **Cat**.

We now turn our attention to some properties of the change of base operation. To begin with, it is worth noting the following:

**Lemma 1.2.28.** *The operation of change of base preserves fibred adjunctions.*

The change of base construction allows us to lift a natural transformation to a fibred functor and a fibred natural transformation.

**Lemma 1.2.29.** *Let $K, L : \mathcal{A} \to \mathcal{B}$ be two functors, $\sigma : K \to L$ be a natural transformation between them, and $p : \mathcal{E} \to \mathcal{B}$ be a fibration. Then $\sigma$ lifts to a fibred functor $\langle\sigma\rangle : L\times_{\mathcal{B}} p \to K\times_{\mathcal{B}} p$, as well as to a fibred natural transformation $\bar{\sigma} : p^*K \circ \langle\sigma\rangle \to p^*L$. Diagrammatically, we have*

$$\begin{tikzcd}
K\times_{\mathcal{B}} p \arrow[rr, "p^*K", ""{name=T, below}] \arrow[dd, "K^*p"'] & & \mathcal{E} \arrow[dd, "p"] \\
& L\times_{\mathcal{B}} p \arrow[ul, "\langle\sigma\rangle"] \arrow[ur, "p^*L"', ""{name=P}] \arrow[dl, "L^*p"] & \\
\mathcal{A} \arrow[rr, bend left, "K", ""{name=U, below}] \arrow[rr, bend right, "L"', ""{name=D}] & & \mathcal{B}
\arrow[Rightarrow, from=T, to=P, "\bar{\sigma}"]
\arrow[Rightarrow, from=U, to=D, "\sigma"]
\end{tikzcd}$$

*Proof.* (Sketch) An object of $L\times_{\mathcal{B}} p$ consists of a pair $(a, e)$ such that $La = pe$, and an object of $K\times_{\mathcal{B}} p$ consists of a pair $(a', e')$ such that $Ka' = pe'$. We can therefore define the fibred functor $\langle\sigma\rangle$ by $\langle\sigma\rangle(a, e) = (a, \sigma^*_a e)$. The fibred natural transformation $\bar{\sigma}$ can be defined by $\bar{\sigma}_{(a,e)} = (\sigma_a)^{\S}_e$. See [Jac99] for a complete proof. □

This allows the derivation of the following lemma, from [Her99].

**Lemma 1.2.30.** *If a functor $F : A \to \mathcal{B}$ has a right adjoint $G$, the functor $q^*F$ obtained from pulling $F$ back along a fibration $q : \mathcal{C} \to \mathcal{B}$ has a right adjoint $\underline{G}$.*

$$\begin{tikzcd}
q \times_{\mathcal{B}} F \arrow[r, "F^*q"] \arrow[d, bend right, "q^*F"'] \arrow[d, phantom, "\dashv"] & A \arrow[d, bend right, "F"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{C} \arrow[r, "q"'] \arrow[u, bend right, "\underline{G}"'] & \mathcal{B} \arrow[u, bend right, "G"']
\end{tikzcd}$$

*Proof.* Here we describe the construction of the adjunction $q^*F \dashv \underline{G}$. For a complete proof see [Her99]. Applying Lemma 1.2.29 to the fibration $q$ and $\epsilon$ the counit for $F \dashv G$ gives the following diagram:

$$\begin{tikzcd}
\cdot \arrow[r, "G^*(F^*q)"] \arrow[d, "(F^*q)^*G"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "G"] \arrow[dd, bend left=60, "id", ""{name=I}] \\
\cdot \arrow[r, "F^*q"] \arrow[d, "q^*F"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{A} \arrow[d, "F"] \\
\mathcal{C} \arrow[r, "q"'] \arrow[uu, bend left=60, "\langle\epsilon\rangle"] & \mathcal{B}
\arrow[Rightarrow, from=2-2, to=I, "\epsilon"]
\end{tikzcd}$$

The right adjoint $\underline{G}$ to $q^*F$ is then defined by $\underline{G} = (F^*q)^*G \circ \langle\epsilon\rangle$. The adjunction $q^*F \dashv \underline{G}$ has unit $\overline{\eta}\langle\epsilon F\rangle$ and counit $\overline{\epsilon}$. □

We will need the following new elaboration on Lemma 1.2.30:

**Corollary 1.2.31.** *In Lemma 1.2.30, if $G$ is full and faithful then so is $\underline{G}$. In this case $\langle\epsilon\rangle$, the lifting of $\epsilon$ the counit of $F \dashv G$, is an isomorphism, and so $\underline{G}$ can be defined to be $(F^*q)^*G$. Similarly, if $F$ is full and faithful then so is the functor $q^*F$.*

*Proof.* In the situation of Lemma 1.2.30, let $\epsilon'$ and $\eta'$ be the unit and counit of the adjunction $q^*F \dashv \underline{G}$, and $\epsilon$ and $\eta$ be the unit and counit of the adjunction $F \dashv G$. Notice, from the description of $\epsilon'$ and $\eta'$ in the proof of the Lemma, that the components of $\epsilon'$ and $\eta'$ are cartesian liftings of the components of $\epsilon$ and $\eta$ respectively. Therefore, since every cartesian morphism above an isomorphism is itself an isomorphism, if $\epsilon$ (resp., $\eta$) is a natural isomorphism, so is $\epsilon'$ (resp., $\eta'$). □

We finish this chapter with the following proposition about adjunctions and functors preserving (op)cartesian morphisms. This proposition will be of use in the remainder of this thesis (in particular, to prove Lemmas 5.1.4 and 6.1.2). We are not aware of any previous publication of this result.

**Proposition 1.2.32.** *Let* $H : \mathcal{E} \to \mathcal{A}$, $L : \mathcal{B} \to \mathcal{A}$, $F : \mathcal{E} \to \mathcal{B}$ *and* $G : \mathcal{B} \to \mathcal{E}$ *be functors such that* $H = L \circ F$ *and* $L = H \circ G$. *If* $F \dashv G$ *with vertical unit (or equivalently, counit) then the functor* $F$ *preserves opcartesian morphisms and the functor* $G$ *preserves cartesian morphisms.*

*Proof.* We only prove that $G$ preserves cartesian morphisms, the second result is then obtained by dualisation. Write $\phi : \dfrac{FX \to Y}{X \to GY}$ for the natural isomorphism characterising the adjunction $F \dashv G$ and note that we can restrict this adjunction to adjunctions between the fibres since the unit is vertical. Let $f : X \to Y$ be a morphism in $\mathcal{A}$ and $u : Q \to P$ be a cartesian morphism above $f$ in $\mathcal{B}$. We want to prove that $Gu$ is cartesian above $f$ in $\mathcal{E}$. Let $l : R \to GP$ be a morphism in $\mathcal{E}$ above $f \circ g$ for some $g$ in $\mathcal{A}$. We then have a unique morphism $v : FR \to Q$ in $\mathcal{B}$ above $g$ such that $u \circ v = \phi^{-1} l$ since $u$ is cartesian. This gives us a unique morphism $\phi v : R \to GQ$ in $\mathcal{E}$ above $g$ such that $Gu \circ \phi v = l$ by naturality of $\phi$. □

# Chapter 2

# Comprehension and quotient

In this chapter we introduce the notions of a *comprehension category with unit* and a *quotient category with equality*. These two notions are central to our work. In Section 2.1 we recall the notion of a comprehension category with unit, or *CCU* for short. It was introduced in [Ehr88a, Ehr88b] under the name D-category and studied as an instance of the more general notion of *comprehension category* in [Jac93]. This notion is fundamental in the fibrational treatment of induction: It is used as a sufficient condition to guarantee the existence of induction schemes for polynomial data types in [HJ98] and will be used to construct our canonical liftings in the next chapter. Comprehension categories with unit where introduced to capture the operation of context extension when using a fibration to model a dependent type theory, as we will see in Example 2.1.7. In addition, if we see a fibration as a model of a logic above a type theory then a CCU captures a notion of *comprehension types*, as we will see in Example 2.1.2. We conclude the section by considering which of the constructions on fibrations preserve CCUs.

In Section 2.2, we introduce our new notion of a quotient category with equality, or *QCE* for short. These fibrations play a role in coinduction that is similar to that played by CCUs in induction. In [Jac94] the notion of a fibration having quotients is introduced in order to model quotient types in a simple type theory. It was later generalised to model quotient types in a dependent type theory in [Jac99]. The notion of QCE is an abstraction of the notion of quotients for a fibration. The idea of quotient types is that, in a predicate logic on a type theory (see Example 1.1.5), we can construct a new type from a relation (i.e., binary predicate) on an old type by identifying related elements of the old type. Since objects of a total category are sometimes considered to be relations, quotients define a functor from the total category of a fibration to its base. Equality refers to the presence of a functor that maps an object of a category to a relation representing equality on that object. As for CCUs, we look at instances of QCEs in logic and type theory, and complete the section by considering which of our constructions preserve QCEs.

## 2.1 Comprehension categories with unit

We start with the definition of comprehension category with unit.

**Definition 2.1.1.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$. We say that $p$ *admits comprehension* if $\mathbf{1}$ has a right adjoint. We write this adjoint $\{-\}$ and refer to it as the *comprehension functor* for $p$.

If $p$ admits comprehension, then there is a functor $\pi : \mathcal{E} \to \mathcal{B}^{\to}$ that maps an object $P$ of $\mathcal{E}$ to $p\epsilon_P$, where $\epsilon$ is the counit of the adjunction $\mathbf{1} \dashv \{-\}$. We call $\pi$ a *comprehension category with unit* (CCU) for $p$. We say that $p$ *admits full comprehension* if $\pi$ is full and faithful.

Note that, since for any fibration $p$ with a terminal object functor $\mathbf{1}$, $\mathbf{1}$ is full and faithful, we have that if $p$ admits comprehension then the unit of the adjunction $\mathbf{1} \dashv \{-\}$ is an isomorphism with inverse given by $\pi\mathbf{1}$.

**Example 2.1.2.** (Example 1.2.16, continued) The syntactic fibration admits comprehension if the type theory under consideration has *comprehension types* for all predicates, i.e., if, for every predicate $P$, there exists the type $\{P\}$ comprising pairs $(x, t)$ such that $x \in pP$ and $t$ is a proof that $P$ holds for $x$. In this case, the comprehension functor maps a predicate $P$ to the type $\{P\}$ and the CCU maps a predicate $P$ to the term in context $(x, t) : \{P\} \vdash x : X$.

If the logic of $\mathcal{E}$ is proof irrelevant — that is, if there is only one proof of any true proposition — then comprehension gives subset types. Indeed, the comprehension of a predicate $P$ is the subtype of $pP$ consisting of those terms that satisfy $P$.

**Example 2.1.3.** (Example 1.2.17, continued) The family fibration on Set admits comprehension. The comprehension functor maps an object $(X, P)$ of Fam(Set) to the set $\{(X, P)\} = \{(x, p) | x \in X, p \in P\, x\}$. The CCU maps an object $(X, P)$ of Fam(Set) to the first projection $\pi_1 : \{(X, P)\} \to X$ in $\mathrm{Set}^{\to}$.

**Example 2.1.4.** (Example 1.2.18, continued) Let $\mathcal{C}$ be a category with a terminal object $\top$ and small homsets $\mathcal{C}(\top, X)$. The family fibration $p : \mathrm{Fam}(\mathcal{C}) \to \mathrm{Set}$ admits comprehension. The comprehension functor $\{-\} : \mathrm{Fam}(\mathcal{C}) \to \mathrm{Set}$ maps an object $(X, P)$ to the disjoint union $\coprod_{x \in X} \mathcal{C}(\top, Px)$.

**Example 2.1.5.** (Example 1.2.19, continued) The fibration *cod* provides the canonical example of CCU. The comprehension functor is $dom : \mathcal{B}^{\to} \to \mathcal{B}$, and the CCU is the identity functor on $\mathcal{B}^{\to}$.

**Example 2.1.6.** (Example 1.2.20, continued) Let $\mathcal{B}$ be a category with product $\times$ and terminal object 1. The simple fibration on $\mathcal{B}$ admits comprehension. The comprehension functor maps $(X, Y)$ to $(X \times Y)$. The CCU maps an object $(X, Y)$ of $\mathcal{E}$ to the first projection $\pi_1 : X \times Y \to X$ in $\mathcal{B}^{\to}$.

CCUs capture the operation of context extension in dependent type systems. This operation can be represented by the inference rule

$$\frac{\Gamma \vdash \sigma : \mathit{Type}}{\overline{\Gamma, \sigma : \mathit{Context}}}$$

This is explored in the following example. Please refer to [Ehr88a] or [Jac99] for further detail.

**Example 2.1.7.** Given a calculus with type dependency and a unit type we can form a full comprehension category with unit in the following way. (As in Example 1.1.5 we deal with $\beta\eta$-equivalence classes of terms, types, contexts, etc.) The objects of $\mathcal{B}$ are contexts $\Gamma$. A morphism from $\Gamma$ to $\Delta$, where $\Delta = y_1 : \tau_1, \ldots, y_n : \tau_n$, is an $n$-tuple of terms $M_1, \ldots, M_n$ such that $\Gamma \vdash M_i : \tau_i[y_1 \leftarrow$

$M_1, \ldots, y_{i-1} \leftarrow M_{i-1}]$. Objects of the category $\mathcal{E}$ are type judgments $\Gamma \vdash \sigma : \mathit{Type}$. A morphism of $\mathcal{E}$ from $\Gamma \vdash \sigma : \mathit{Type}$ to $\Delta \vdash \tau : \mathit{Type}$ is a pair $(\vec{M}, N)$ with $\vec{M} : \Gamma \to \Delta$ in $\mathcal{B}$ and $\Gamma, x : \sigma \vdash N : \tau[\vec{y} \leftarrow \vec{M}]$. The functor $\mathbf{1}$ then maps an object $\Gamma$ to $\Gamma \vdash 1 : \mathit{Type}$, where 1 is the unit type. The comprehension functor maps a type judgment $\Gamma \vdash \sigma : \mathit{Type}$ to the context $\Gamma, x : \sigma$. The associated adjoint correspondence is then given by:

$$\frac{(\Gamma \vdash 1 : \mathit{Type}) \to (\Delta \vdash \tau : \mathit{Type})}{\overline{\Gamma \to (\Delta, x : \tau)}}$$

This amounts to the correspondence between a context morphism from $\Gamma$ to $\Delta, x : \tau$ and the pair of a context morphism $\vec{M}$ from $\Gamma$ to $\Delta$ and a term $\Gamma, x : 1 \vdash N : \tau[\vec{y} \leftarrow \vec{M}]$ (straightforward from the definition of context morphisms). The CCU maps a type judgment $\Gamma \vdash \sigma : \mathit{Type}$ to the morphism of context $\Gamma, x : \sigma \to \Gamma$ that forgets the type $\sigma$, i.e., if $\Gamma$ is the context $x_1 : \tau_1, \ldots, x_n : \tau_n$ then $\pi_{(\Gamma \vdash \sigma : \mathit{Type})}$ is given by the $n$-tuple $x_1, \ldots, x_n$.

Example 2.1.2 shows that comprehension can be seen as a type constructor. Comprehension categories with unit therefore capture constructive logic by showing how to represent collections of proofs as types. Concretely, the proofs of a predicate $P$ can be seen as terms of the type $\{P\}$. This point of view is the one that we take in this thesis, but note that Example 2.1.7 illustrates an alternative point of view, namely, that CCUs capture the notion of context extension.

We can see from Examples 2.1.2 and 2.1.7 that comprehension can be used to model both logical predicates and context extension in a dependent type theory. Of course, in some fibrations, comprehension can be (intuitively) thought of as playing both roles. For example, using the propositions-as-types metaphor, we can think of an object of the total category of the simple fibration as representing both a type and a proposition. An object $(I, X)$ of the fibre above $I$ can then be thought of as the proposition $X$ in context $I$, with comprehension mapping this object to the type $I \times X$. Alternatively, we may think of an object $(I, X)$ of the fibre above $I$ as the type $X$ definable in context $I$. In this case, comprehension maps $(I, X)$ to the extended context $I \times X$. A similar analysis is possible for the codomain fibration on a locally cartesian closed category, the family fibration on Set, and so on.

We now look at some properties and structures of CCUs. We start with a notion of morphism that we can associate to CCUs, following [Jac91]:

**Definition 2.1.8.** Let $p : \mathcal{E} \to \mathcal{B}$ and $p' : \mathcal{E}' \to \mathcal{B}'$ admit comprehension with $p \dashv \mathbf{1} \dashv \{-\}$ and $p' \dashv \mathbf{1}' \dashv \{-\}'$. A *morphism of comprehension categories with unit* from $p$ to $p'$ is a fibred functor $(H, F) : p \to p'$ preserving the terminal object functor such that the canonical map $F\{-\} \to \{-\}'H$ is an isomorphism. The latter is obtained as the transpose of the composition $\mathbf{1}'F\{-\} \cong H\mathbf{1}\{-\} \xrightarrow{F\epsilon} H$ with $\epsilon$ the counit of the adjunction $\mathbf{1} \dashv \{-\}$.

This notion will be relevant in Lemma 5.1.4.

The following result from [Jac93] shows that every CCU is a fibred functor.

**Lemma 2.1.9.** *Let $p : \mathcal{E} \to \mathcal{B}$ admit comprehension. Then, the associated CCU $\pi : \mathcal{E} \to \mathcal{B}^{\to}$ sends cartesian morphisms to pullback squares.*

*Proof.* Dual of Lemma 2.2.8. □

We now consider how comprehension categories with unit behave under the change of base operation.

**Lemma 2.1.10.** *Let $p : \mathcal{E} \to \mathcal{B}$ and $q : \mathcal{C} \to \mathcal{B}$ be fibrations. If $p$ admits comprehension, then so does the fibration $q^*p$ obtained by change of base of $p$ along $q$. Furthermore, if $p$ admits full comprehension, $q^*p$ also admits full comprehension.*

*Proof.* Let $p : \mathcal{E} \to \mathcal{B}$, $\mathbf{1}$, and $\{-\}$ provide a comprehension category with unit $\mathbf{1}$, and let $q : \mathcal{C} \to \mathcal{B}$ be a fibration. By Lemma 1.2.30, the change of base of $p$ along $q$ yields an adjunction $q^*p \dashv \underline{\mathbf{1}}$, and by Corollary 1.2.31, $\underline{\mathbf{1}} = (p^*q)^*\mathbf{1}$ is full and faithful.

$$\begin{tikzcd}
q \times_{\mathcal{B}} p \arrow[r, "p^*q"] \arrow[d, bend right, "q^*p"'] \arrow[d, phantom, "\dashv"] & \mathcal{E} \arrow[d, bend right, "p"'] \arrow[d, phantom, "\dashv"] \\
\mathcal{C} \arrow[r, "q"'] \arrow[u, bend right, "\underline{\mathbf{1}}"'] & \mathcal{B} \arrow[u, bend right, "\mathbf{1}"']
\end{tikzcd}$$

Since $p^*q$ is a fibration, we can use Lemma 1.2.30 again to conclude that the change of base of $\mathbf{1}$ along $p^*q$ yields an adjunction $\underline{\mathbf{1}} \dashv \underline{\{-\}}$.

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, bend right=40, "\underline{\mathbf{1}}"'] \arrow[d, phantom, "\dashv"] & \mathcal{B} \arrow[d, bend right=40, "\mathbf{1}"'] \arrow[d, phantom, "\dashv"] \\
q \times_{\mathcal{B}} p \arrow[r, "p^*q"'] \arrow[u, bend right=40, "\underline{\{-\}}"'] & \mathcal{E} \arrow[u, bend right=40, "\{-\}"']
\end{tikzcd}$$

We therefore have the following situation:

$$\begin{tikzcd}
q \times_{\mathcal{B}} p \arrow[d, bend right=60, "q^*p"'] \arrow[d, bend left=60, "\underline{\{-\}}"] \\
\mathcal{C} \arrow[u, "\dashv\underline{\mathbf{1}}\dashv" description]
\end{tikzcd}$$

Since $\underline{\mathbf{1}}$ is full and faithful, Lemma 1.2.14 ensures that $\underline{\mathbf{1}}$ is a terminal object functor for $q^*p$. The comprehension functor $\underline{\{-\}}$ maps an object $(c, e)$ in $q \times_{\mathcal{B}} p$ to $(\pi_e)^*c$ in $\mathcal{C}$, where $\pi$ is the CCU for $p$. The CCU for $q^*p$ maps an object $(c, e)$ in $q \times_{\mathcal{B}} p$ to $(\pi_e)^{\S}_c : (\pi_e)^*c \to c$ in $\mathcal{C}^{\to}$.

See [Jac93] for the sketch of an alternative proof of the lemma and a sketch proof of preservation of fullness. □

Next we will introduce the notion of Lawvere fibration. This notion comes from the notion of *hyperdoctrines that satisfy the comprehension scheme* introduced by Lawvere for the first description of comprehension in category theory [Law70]. This notion of hyperdoctrines was then translated in the theory of fibration in [Jac93]. Lawvere fibrations will be important when defining liftings of functors.

**Definition 2.1.11.** A fibration $p : \mathcal{E} \to \mathcal{B}$ is a *Lawvere fibration* if it admits comprehension and is a bifibration.

The following equivalent presentation, which appears as Result (i) in [Jac93], highlights a useful structure of Lawvere fibrations.

**Lemma 2.1.12.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration. Then, $p$ is a Lawvere fibration iff it has a terminal object functor $\mathbf{1}$, and the functor $\phi : \mathcal{B}^{\to} \to \mathcal{E}$ mapping an object $f : X \to Y$ of $\mathcal{B}^{\to}$ to $\Sigma_f \mathbf{1} X$ has a right adjoint $\pi$ with $cod \circ \pi = p$ and* *vertical counit. In this case we have that $\pi = p\epsilon$, where $\epsilon$ is the counit of the adjunction $\mathbf{1} \dashv \{-\}$, and is a CCU.*

If $\pi$ is a full CCU, then we call $p$ a *full Lawvere fibration.* In this case, the counit of the adjunction $\phi \dashv \pi$ is an isomorphism. Let $(\alpha, \beta) : f \to g$ be a morphism in $\mathcal{B}^{\to}$.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "f"'] & Z \arrow[d, "g"] \\
Y \arrow[r, "\beta"'] & T
\end{tikzcd}$$

The morphism part of the functor $\phi$ maps $(\alpha, \beta)$ to the unique morphism above $\beta$, making the following diagram commute:

$$\begin{tikzcd}
\mathbf{1}X \arrow[r, "\mathbf{1}\alpha"] \arrow[d, "f_\S^{\mathbf{1}X}"'] & \mathbf{1}Z \arrow[d, "g_\S^{\mathbf{1}Z}"] \\
\Sigma_f \mathbf{1}X \arrow[r, dotted, "\phi(\alpha,\beta)"'] & \Sigma_g \mathbf{1}Z
\end{tikzcd}$$

The morphism $\phi(\alpha, \beta)$ exists and is unique by the universal property of the opcartesian morphism $f_\S^{\mathbf{1}X}$.

**Example 2.1.13.** (Example 2.1.3, continued) We have seen that the family fibration on Set is a bifibration that admits comprehension, and is therefore a Lawvere fibration. The functor $\phi$ maps a function $f : X \to Y$ to the inverse image family $(Y, f^{-1})$.

**Example 2.1.14.** (Example 2.1.4, continued) Let $\mathcal{C}$ be a category with set-indexed coproducts, a terminal object $\top$ and small homsets $\mathcal{C}(\top, X)$. The family fibration of $\mathcal{C}$ is a Lawvere fibration (keep in mind Example 1.1.7 for the bifibred structure and Example 2.1.4 for the CCU structure). The functor $\phi : \mathrm{Set}^{\to} \to \mathrm{Fam}(\mathcal{C})$ then maps a function $f : X \to Y$ to the family $(Y, \lambda y. \coprod_{x \in (f^{-1}y)} \top)$.

**Example 2.1.15.** (Example 2.1.5, continued) The fibration *cod* is a Lawvere fibration by Examples 1.1.8, 1.2.19 and 2.1.5. It is the canonical example of a Lawvere fibration. In this case, both the functor $\pi$ and the functor $\phi$ are the identity functor on $\mathcal{B}^{\to}$.

As a direct consequence of the preservation of bifibrations by change of base (see Corollary 1.2.2), together with the preservation of CCUs by change of base (see Lemma 2.1.10), we have the following preservation property of Lawvere fibrations.

**Corollary 2.1.16.** *Lawvere fibrations are stable under change of base along a fibration. Furthermore, if the original Lawvere fibration is full, so is the one obtained from the change of base.*

## 2.2 Quotient categories with equality

Just as Lawvere fibrations provide us with sufficient structure to establish sound induction schemes, so quotient category with equality (QCE for short) provide the structure needed to give sound coinduction schemes. We define QCEs in the following way.

**Definition 2.2.1.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a full (and necessarily faithful) section $e : \mathcal{B} \to \mathcal{E}$. If $e$ has a left adjoint $Q$, we say that $p$ *admits e-quotients*, or simply *admits quotients* if $e$ is clear from the context.

$$\begin{tikzcd}
 & \mathcal{E} \arrow[dl, bend right, "Q"'] \arrow[d, "p"] \\
\mathcal{B} \arrow[ur, "e"'] \arrow[ur, phantom, shift left=2, "\dashv"] \arrow[r, "id_{\mathcal{B}}"'] & \mathcal{B}
\end{tikzcd}$$

We call $Q$ the *quotient functor* for $p$. If $p$ admits quotients, then there are functors $\rho : \mathcal{E} \to \mathcal{B}^{\to}$ and $\psi : \mathcal{B}^{\to} \to \mathcal{E}$ defined by $\rho P = p\eta_P$, where $\eta$ is the unit of the adjunction $e \vdash Q$, and $\psi\,(f : X \to Y) = f^*eY$. We call $\rho$ the *quotient category with equality* (QCE) for $p$.

Intuitively, the functor $e$ is thought of as an abstract equality functor. Note that any fibration $p$ with a terminal object functor $\mathbf{1}$ trivially admits $\mathbf{1}$-quotients with quotient functor $p$. If the functor $\rho : \mathcal{E} \to \mathcal{B}^{\to}$ is full and faithful we say that $p$ admits *full quotients*. We stress that this notion of fullness of quotient will usually imply that any relation is equivalent to a kernel relation (see Example 2.2.5 for example) and in particular, is an equivalence relation.

We now describe the dual construction of $tC$-opfibrations, which will also be useful.

**Definition 2.2.2.** Let $p : \mathcal{E} \to \mathcal{B}$ be an opfibration with a full (and necessarily faithful) section $t : \mathcal{B} \to \mathcal{E}$ and with $C$ a right adjoint to $t$.

$$\begin{tikzcd}
 & \mathcal{E} \arrow[d, "p"] \arrow[dl, bend right, "C"'] \\
\mathcal{B} \arrow[ur, "t"'] \arrow[ur, phantom, shift left=3, "\vdash"] \arrow[r, "id_{\mathcal{B}}"'] & \mathcal{B}
\end{tikzcd}$$

We say that $p$ is a *$tC$-opfibration.* If $p$ is a $tC$-opfibration, then there are functors $\pi : \mathcal{E} \to \mathcal{B}^{\to}$ and $\phi : \mathcal{B}^{\to} \to \mathcal{E}$ defined by $\pi P = p\epsilon_P$, where $\epsilon$ is the counit of the adjunction $t \dashv C$, and $\phi(f : X \to Y) = \Sigma_f tX$.

If $p$ is a $tC$-opfibration and a bifibration we will call $p$ a *$tC$-bifibration.* If the functor $\pi : \mathcal{E} \to \mathcal{B}^{\to}$ is full and faithful we call $p$ a *full $tC$-opfibration.*

As apparent, while we present QCEs as the counterpart of CCUs for coinduction, the notion of QCE is not dual to the notion of CCU (but of $tC$-opfibration). As we will see in the next chapter, while QCEs have enough structure to derive our coinduction schemes, CCUs do not have enough structure to derive our induction schemes and we need to consider Lawvere fibrations. In fact, by duality it is enough to have the structure of a $tC$-opfibration, of which a Lawvere fibration is a specific instance (see Corollary 2.2.7). The lack of symmetry in our treatment of induction and coinduction is due to the following:

First, for our study of induction, while Lawvere fibrations are less general than $tC$-opfibrations, they are known to capture a relevant setting between a logic and a type theory (See [Jac93]). In particular they have a terminal object functor when $tC$-opfibrations only have a full section. Strengthening the condition on the full section guarantees the presence of a notion of truth in the logic.

On the other hand, for our study of coinduction we consider QCEs where the relation between fibrations and their associated section is more lax. This is because strengthening the condition on the equality functor (the section) to be fibred adjoint to the fibration implies that equality relations are initial in fibres, we can then only consider fibration of reflexive relations.

We could of course regain symmetry in the presentation of induction and coinduction by, either treating $tC$-opfibrations as the basic setting for induction (instead of Lawvere fibrations), or asking the equality functor to be fibred (left) adjoint to the fibration in the definition of a QCE. Note however that, while the presentation is asymmetric, the duality between QCEs and $tC$-opfibrations is behind most of our results (using the key fact that a Lawvere fibration is a $tC$-opfibration).

We now look at different examples of QCEs:

**Example 2.2.3.** (Example 1.1.9, continued) The fibration $dom : \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$ on a category $\mathcal{B}$ is the canonical example of QCE. Here, the full section is given by the functor that maps an object $X \in \mathcal{B}$ to the identity morphism $id_X$. The quotient functor is given by the codomain functor $cod : \mathcal{B}^{\rightarrow} \rightarrow \mathcal{B}$. Furthermore, both $\rho$ and $\psi$ are the identity functor on $\mathcal{B}^{\rightarrow}$. To develop intuitions, consider $\mathcal{B}$ as a category of set like objects: We can then understand an element $f : X \rightarrow Y$ of $\mathcal{B}^{\rightarrow}$ as the kernel relation of $f$ on $X$, i.e., two elements $x$ and $x'$ of $X$ are in relation iff they have the same image under $f$. Hence the identity only relates equal elements. The quotient (codomain) of $f$ then contains the actual quotient of the kernel relation of $f$ (the image of $f$), while the elements in $Y$ that are not in the image of $f$ can be understood as empty classes of equivalence.

To formalise this intuition using the internal language of $\mathcal{B}$ we can say that, for $f : X \rightarrow Y$ an element of $\mathcal{B}^{\rightarrow}_X$ two terms of type $X$ in context $A$, given by $u, v : A \rightarrow X$ in $\mathcal{B}$, are related by $f$ if $f \circ u = f \circ v$. We then find back the above mentioned intuition when $\mathcal{B}$ is Set. Note that if $\mathcal{B}$ has cartesian products then two elements are related by $f \times g$ iff their first projections are related by $f$ and second projections are related by $g$ (from the universal property of cartesian products). However, $\mathcal{B}$ needs to have coproducts with the additional property that any map $f : X \rightarrow Y + Y'$ is either of the form $X \rightarrow Y \xrightarrow{inl} Y + Y'$ or $X \rightarrow Y' \xrightarrow{inl} Y + Y'$ (with $inl$ and $inr$ the two injections of the coproduct) in order for $f + g$ to capture a well behaving sum of the relations $f$ and $g$.

**Example 2.2.4.** (Example 1.1.11, continued) Let $\mathcal{B}$ be a category with cartesian products and $p : \mathcal{E} \rightarrow \mathcal{B}$ be a bifibration satisfying the Beck-Chevalley condition with terminal object functor $\mathbf{1}$. We consider the relations bifibration $p' : Rel(\mathcal{E}) \to \mathcal{B}$ for $p$. Define the functor $Eq : \mathcal{B} \to Rel(\mathcal{E})$ by $X \mapsto \Sigma_{\delta_X} 1X$, where $\delta_X : X \to X \times X$ is the diagonal morphism (for intuitions on this definition look at how opreindexing is defined in Example 1.1.6 ). Note that by Lemma 1.2.14 and Lemma 1.1.16 the functor $Eq$ is full and faithful. The fibration $p$ is said to have quotients if $p'$ admits $Eq$-quotients (See Definition 4.1 in [Jac94] or Definition 4.8.1 in [Jac99]).

**Example 2.2.5.** (Example 1.1.6, continued) We now consider the relations bifibration of the family fibration on Set from Example 1.1.6. The category of Set-indexed relations, $Rel(\text{Fam}(\text{Set}))$, has as its objects the objects $(X, P)$ of Fam(Set) whose first components are products of the form $A \times A$ for some set $A$. As its morphisms it has the morphisms $(f, f^{\sim})$ of Fam(Set) whose first components are products of the form $g \times g$ for some function $g$ in Set. We can then see an object $(X \times X, P)$ as a binary relation $P$ on $X$ where two elements $x_1$ and $x_2$ are in $P$ if $Px_1x_2$ is not the empty set. As in Example 2.2.4, we can define the functor $Eq : \mathcal{B} \to Rel(\text{Fam}(\text{Set}))$ mapping $X$ to $\Sigma_{\delta_X} 1X$, i.e. to the equality relation on $X$. The functor $Eq$ has a left adjoint $Q$, which maps a relation $(X \times X, R)$ to $X/R$, the quotient set of $X$ by the least equivalence relation containing $R$. Therefore, the relations fibration for the family fibration admits $Eq$-quotients. In such a case, we have that the QCE $\rho$ maps a relation $R$ above $X$ to the quotient map $c_R : X \to X/R$ that maps an element of $X$ to its equivalence class under the equivalence closure of $R$. The functor $\psi$ maps a map $f : X \to Y$ into its kernel relation $ker(f)$, i.e., $(x, y) \in ker(f)$ iff $f\,x = f\,y$.

**Example 2.2.6.** (Example 1.1.7, continued) Let $\mathcal{C}$ be a category with set-indexed coproducts $\coprod$ and a terminal object $\top$, such that the initial object $\bot = \coprod_{\varnothing} \top$ is strict, i.e., any morphism $A \to \bot$ is an isomorphism. We then have that the relation fibration of the family fibration of $\mathcal{C}$ admits quotients with the following section.

The section $Eq : \text{Set} \to Rel\big(\text{Fam}(\mathcal{C})\big)$ is given by $Eq\,X = \lambda(x, x') \in X \times X. \begin{cases} \top & \text{if } x = x' \\ \bot & \text{otherwise.} \end{cases}$. Then, for $R : X \times X \to \mathcal{C}$ in $Rel\big(\text{Fam}(\mathcal{C})\big)$, consider the relation $\bar{R} = \{(x, x') \in X \times X | R(x, x') \not\cong \bot\}$. Define the quotient $Q : \mathit{Rel}\big(\mathrm{Fam}(\mathcal{C})\big) \to \mathrm{Set}$ of the QCE as mapping $R$ to $X/R$ the quotient set of $X$ by the least equivalence relation containing $\bar{R}$.

Indeed, a map $\alpha : R \to Eq\,Y$ above $f : X \to Y$ implies, by definition of $Eq$ and since the only maps into $\bot$ are isomorphisms, that if for $(x, x') \in X \times X$, $fx \neq fx'$ then $R(x, x') \cong \bot$. Hence $\bar{R}$ is a sub-relation of the kernel relation of $f$, and so is the smallest equivalence relation containing $\bar{R}$. Thus, $f$ extends naturally to a function $g : X/R \to Y$ such that $f = g \circ c_R$. Now, given a function $g : X/R \to Y$, associate the function $(g', h)$, where $g' = g \circ c_R$ and $h$ is the uniquely defined family of morphisms in $\mathcal{C}$ mapping $R(x, x')$ to $\top$ if $g'x = g'y$ and $\bot$ otherwise. The section is full since the counit $Q(Eq\,X) \to X$ is an isomorphism.

Note that Lawvere fibrations and QCEs are not dual, but QCEs are slightly more abstract than the duals of Lawvere fibrations (see corollary 2.2.7 below). In particular, in a QCE, the section $e$ is not required to be left adjoint to the fibration, whereas in a Lawvere fibrations the terminal object functor is required to be right adjoint to the fibration. As we will see in the next chapter, QCEs form the basis of our treatment of coinduction, and Lawvere fibrations form the basis of our treatment of induction. In fact, we will prove that QCEs and $tC$-opfibrations are enough to allow us to derive valid induction and coinduction schemes. It is often necessary that the section associated to a QCE is not adjoint to the fibration in order to derive interesting coinduction schemes. The analogous relaxation for $t$ is not, however, necessary in the inductive setting. For this reason, we derive our induction schemes with respect to Lawvere fibrations rather than $tC$-opfibrations.

Nevertheless, the concept of a $tC$-opfibration still permits us to treat induction and coinduction as formal duals, and so any results obtained at this level of abstraction are directly valid for both induction and coinduction. We have

**Corollary 2.2.7.** *Every Lawvere fibration is a $tC$-opfibration.*

*Proof.* Let $p$ be a Lawvere fibration, and take $t$ to be the terminal object functor and $C$ to be the comprehension functor for $p$. □

We now look at some properties of QCEs. We start with a couple of results highlighting the relationship between the notion of CCU and the notion of QCE.

In order to see this, we recall that behind the notion of CCU is the notion of *comprehension category* (see [Jac93]) which can be summarised as: a functor $\mathcal{P} : \mathcal{E} \to \mathcal{B}^{\to}$ is a comprehension category if the functor $cod \circ \mathcal{P} : \mathcal{E} \to \mathcal{B}$ is a fibration and $\mathcal{P}$ sends cartesian morphisms to pullback squares. Lemma 2.1.9 insures then that CCUs are indeed comprehension categories. Similarly, the following results shows that QCEs are automatically opfibred (hence one could imagine a notion of *quotient category* dual to the notion of comprehension category, of which QCEs would be a specific instance).

**Lemma 2.2.8.** *Let $p : \mathcal{E} \to \mathcal{B}$ admit quotients. Then, the associated QCE $\rho : \mathcal{E} \to \mathcal{B}^{\to}$ sends opcartesian morphisms to pushout squares.*

The proofs of Lemma 2.2.8, as well as Corollary 2.2.9 and Lemma 2.2.10 are the exact dual of the proofs of similar results for CCUs in [Jac93] (among which there is Lemma 2.1.9 and Lemma 2.1.12). We reproduce them here in order to check that weakening the hypothesis (namely not asking for $e$ to be adjoint to $p$) does not affect the proofs.

*Proof.* Let $e$ denote the full section of $p$ and $Q$ for the quotient functor. Let $f : A \to B$ be a morphism in $\mathcal{B}$, $R$ be an element of $\mathcal{E}_A$ and $l$ be an opcartesian morphism above $f$. The image of $l$ by $\rho$ is then given by the following square in $\mathcal{B}$:

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\rho R"'] & B \arrow[d, "\rho R'"] \\
QR \arrow[r, "Ql"'] & Q(R')
\end{tikzcd} \tag{*}$$

Let $\Phi$ denote the natural isomorphism associated to the adjunction $Q \dashv e$ and $\eta : id_{\mathcal{E}} \to eQ$ for its unit. In order to prove that the square (*) is a pushout, let us assume $g : B \to X$ and $h : QR \to X$ two morphisms in $\mathcal{B}$ such that $g \circ f = h \circ \rho R$. We have a morphism $\Phi h : R \to eX$ such that $\Phi h = eh \circ \eta_R$ by naturality of $\Phi$. Hence $\Phi h$ is above the composition $h \circ \rho R$ which is equal to $g \circ f$. Therefore, by the universal property of the opcartesian morphism $l$, there exists a unique morphism $\gamma : R' \to eX$ above $g$, such that $\Phi h = \gamma \circ l$. We then have the unique morphism making the square (*) a pushout square given by $\Phi^{-1}\gamma : Q(R') \to X$. Indeed, the following holds by naturality of $\Phi$:

$$\begin{aligned} \Phi^{-1}\gamma \circ Q(l) &= \epsilon_X \circ Q\gamma \circ Q(l) \\ &= \epsilon_X \circ Q(\Phi h) \\ &= \Phi^{-1}(\Phi h) \\ &= h \end{aligned}$$

as well as the following, by naturality of $\Phi$ and $\eta$:

$$\begin{aligned} \Phi^{-1}\gamma \circ \rho(R') &= \epsilon_X \circ Q\gamma \circ p(\eta_{(R')}) \\ &= p(e\epsilon_X \circ eQ\gamma \circ \eta_{(R')}) \\ &= p(e\epsilon_X \circ \eta_{eX} \circ \gamma) \\ &= p\gamma \\ &= g \end{aligned}$$

To conclude, the uniqueness of $\Phi^{-1}\gamma$ is obtained from the uniqueness of $\gamma$. □

We can then deduce the following corollary:

**Corollary 2.2.9.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration that admit quotients with section $e : \mathcal{B} \to \mathcal{E}$, quotient functor $Q : \mathcal{E} \to \mathcal{B}$ and QCE $\rho : \mathcal{E} \to \mathcal{B}^{\to}$. For all $R$ in $\mathcal{E}$ and $f : pR \to X$ in $\mathcal{B}$, there exists the following pushout square in $\mathcal{B}$:*

$$\begin{tikzcd} pR \arrow[r, "f"] \arrow[d, "\rho R"'] & X \arrow[d, "\rho\Sigma_f R"] \\ QR \arrow[r, "Qf_{\S}^{R}"'] \arrow[ur, phantom, "\ulcorner", very near end] & Q(\Sigma_f R) \end{tikzcd}$$

We can now present the following result that characterises bifibrations that admit quotients.

**Lemma 2.2.10.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration with a full section $e : \mathcal{B} \to \mathcal{E}$. The bifibration $p$ admits $e$-quotients with quotient functor $Q : \mathcal{E} \to \mathcal{B}$ iff the functor $\psi : \mathcal{B}^{\to} \to \mathcal{E}$ mapping an object $f : X \to Y$ of $\mathcal{B}^{\to}$ to $f^*eY$ has a left adjoint $\rho$*

*with $dom \circ \rho = p$ and vertical unit. In such a case, we have that $\rho$ is the associated QCE, i.e., $\rho = p\eta$ where $\eta$ is the unit of the adjunction $Q \dashv e$.*

*Proof.* Let $p : \mathcal{E} \to \mathcal{B}$ a bifibration with full section $e$. Assume first that the functor $\psi : \mathcal{B}^{\to} \to \mathcal{E}$ has a left adjoint $\rho$ with $dom \circ \rho = p$ and vertical unit. The functor $I : \mathcal{B} \to \mathcal{E}$ that sends an object $A$ in $\mathcal{B}$ to $\psi id_A$ in $\mathcal{E}$ is isomorphic to $e$, since $I\,A = id_A^* eA \cong eA$. Furthermore, $I$ has a left adjoint $Q = cod \circ \rho$ by composition of adjoints: $I = \psi \circ id_{(\_)} \vdash cod \circ \rho = Q$. Hence, we also have $Q \dashv e$. Now, we have that the unit of the adjunction $Q \dashv e$ is, by construction, $\eta = \psi\bar{\eta}\rho.\eta'$ with $\bar{\eta}$ the unit of $cod \dashv id_{(\_)}$ and $\eta'$ the unit of $\rho \dashv \psi$. Hence, considering that the unit of the adjunction $\psi \vdash \rho$ is vertical, we have that $p\eta = p(\psi\bar{\eta}\rho) = dom(\bar{\eta}\rho)$ and since $dom\,\bar{\eta} = id$ we can conclude that $p\eta = \rho$.

For the other direction of the equivalence, assume that $p$ admits quotients with quotient functor $Q : \mathcal{E} \to \mathcal{B}$. For $f : A \to B$ in $\mathcal{B}$, denoting $A/\mathcal{B}$ for the coslice category with respect to $A$, we have:

$$\begin{aligned}
\mathcal{B}^{\to}(\rho R, f) &\cong \bigcup_{u : pR \to A} A/\mathcal{B}(\rho(\Sigma_u R), f) && (1)\\
&\cong \bigcup_{u : pR \to A} B/\mathcal{B}\Big(\rho\big(\Sigma_f(\Sigma_u R)\big), id_B\Big) && (2)\\
&\cong \bigcup_{u : pR \to A} \mathcal{E}_B\big(\Sigma_f(\Sigma_u R), eB\big) && (3)\\
&\cong \bigcup_{u : pR \to A} \mathcal{E}_A(\Sigma_u R, f^* eB) && (4)\\
&\cong \mathcal{E}(R, \psi f) && (5)
\end{aligned}$$

When (1) and (2) come from Corollary 2.2.9, (4) comes from the adjunction $\Sigma_f \dashv f^*$ and (5) comes from the universal property of opcartesian morphisms. Now for (3), it is easy to check that if we restrict the isomorphism associated to the adjunction $Q \dashv e$ to the fibre $\mathcal{E}_B$ we have, for any $P$ in $\mathcal{E}_B$, the isomorphism $\mathcal{E}_B(P, eB) \cong B/\mathcal{B}(\rho R, id_B)$. It is easy to check from its construction that the unit of $Q \dashv e$ is vertical. □

Note that Lemma 2.1.12 can be presented as a corollary of the dual of Lemma 2.2.10.

We now look at two preservation properties of QCEs.

**Lemma 2.2.11.** *QCEs are stable under change of base of the underlying fibration along an opfibration.*

*Proof.* Let $p$, $e$ and $Q$ provide a QCE, and let $q : \mathcal{E}' \to \mathcal{B}$ be an opfibration. Consider the following pullback diagram.

$$\begin{tikzcd}
\mathcal{B} \arrow[d, bend left, "e"] \arrow[dd, bend right=60, "id_{\mathcal{B}}"'] & \cdot \arrow[l] \arrow[d, "(p^*q)^*e"] \arrow[dl, phantom, very near start, "\llcorner"] \\
\mathcal{E} \arrow[u, bend left, "Q"] \arrow[u, phantom, "\dashv"] \arrow[d, "p"] & \cdot \arrow[l, "p^*q"'] \arrow[d, "q^*p"] \arrow[dl, phantom, very near start, "\llcorner"] \\
\mathcal{B} & \mathcal{E}' \arrow[l, "q"]
\end{tikzcd}$$

We have that $p \circ e = id$ implies $q^*p \circ (p^*q)^*e = id$, and Corollary 1.2.2 ensures that change of base preserves opfibrations, we also have that $p^*q$ is an opfibration. The dual of Lemma 1.2.30 therefore ensures that the functor $(p^*q)^*e$ has a left adjoint $\underline{Q}$, and the dual of Corollary 1.2.31 ensures that $(p^*q)^*e$ is full and faithful. $\square$

**Lemma 2.2.12.** *Let $p : \mathcal{E} \to \mathcal{B}$, $e : \mathcal{B} \to \mathcal{E}$, and $Q : \mathcal{E} \to \mathcal{B}$ provide a QCE. Let $I$ be an object of $\mathcal{B}$, and let $p/I : \mathcal{E}/eI \to \mathcal{B}/I$ be the functor that maps an object $\alpha : R \to eI$ to $p\alpha : pR \to I$. Then $p/I$ admits $e/I$-quotients with quotient functor $Q/I = \Phi$, where $\Phi$ is the natural isomorphism characterising the adjunction $Q \dashv e$.*

*Proof.* First, $p/I$ is a fibration. Indeed, let $h : f \to g$ be a morphism of $\mathcal{B}/I$.

$$\begin{tikzcd}
X \arrow[rr] \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& I &
\end{tikzcd}$$

and let $\alpha : P \to eI$ be a morphism above $g$. The cartesian lifting of $h$ at $P$ with respect to $p/I$ is the $\mathcal{E}/eI$-morphism

$$\begin{tikzcd}
h^*P \arrow[rr, "h^\S_P"] \arrow[dr, "\alpha \circ h^\S_P"'] & & P \arrow[dl, "\alpha"] \\
& eI &
\end{tikzcd}$$

Here, $h^\S_P$ is the cartesian lifting of $h$ at $P$ with respect to $p$. Now, since $p \circ e = id$, we have that $p/I \circ e/I = id$. Moreover, since $e$ is full and faithful, so is $e/I$. To see that $Q/I \dashv e/I$, observe that the following two diagrams are (naturally) isomorphic by naturality of $\Phi$:

$$\begin{tikzcd}
QP \arrow[r, "h"] \arrow[dr, "Q/I\alpha"'] & X \arrow[d, "f"] \\
& I
\end{tikzcd}
\qquad
\begin{tikzcd}
P \arrow[r, "\Phi^{-1}h"] \arrow[dr, "\alpha"'] & eX \arrow[d, "e/If"] \\
& eI
\end{tikzcd}$$

□

# Chapter 3

# Lifting

A central aspect of Hermida and Jacobs' approach to induction and coinduction is to show how an endofunctor $F$ acting on types can be lifted to either an endofunctor $\widehat{F}$ acting on predicates or, an endofunctor $\check{F}$ acting on relations. These liftings make it possible to derive an induction scheme for the initial algebra of $F$ or a coinduction scheme for the final coalgebra of $F$. Before looking at the derivation of induction and coinduction schemes, which will be the subject of the next chapter, in this chapter we concentrate on the operation of lifting.

In Section 3.1 we introduce a general notion of *lifting of a functor* with regard to a fibration. From there, we derive the two notions of **1**-*preserving* and *e*-*preserving* liftings. We then show how to construct a canonical **1**-preserving lifting of an arbitrary functor in Lawvere fibrations as well as a canonical $e$-preserving lifting of an arbitrary functor in QCEs. In Section 3.2 we look at how the canonical liftings behave with regard to the algebraic properties of the lifted functors. We will then conclude by linking our canonical liftings with the lifting operation of Hermida and Jacobs.

## 3.1 Definitions and canonical liftings

We start with the definition of a lifting of a functor in a fibration:

**Definition 3.1.1.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration and $F$ be a functor on $\mathcal{B}$. A *lifting* of $F$ with respect to $p$ is a functor $\bar{F} : \mathcal{E} \to \mathcal{E}$ such that the following diagram commutes:

$$\begin{tikzcd}\mathcal{E} \arrow[r, "\bar{F}"] \arrow[d, "p"'] & \mathcal{E} \arrow[d, "p"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{B}\end{tikzcd}$$

Categorically, a lifting can be understood as a weak notion of fibred endofunctor. Indeed, for any fibred endofunctor $(H, F)$, the functor $H$ is a lifting of $F$. Furthermore, similarly to fibred functors, we can restrict a lifting $\bar{F} : \mathcal{E} \to \mathcal{E}$ of a functor $F : \mathcal{B} \to \mathcal{B}$ to a functor $\bar{F}_X : \mathcal{E}_X \to \mathcal{E}_{FX}$ between fibres, for every $X$ in $\mathcal{B}$.

With induction schemes in mind, the idea is that endofunctors on the base category of a fibration are understood as defining the structure of (potential) inductive and coinductive types (see Section 4.1). Therefore, a lifting of a functor $F$ can be seen as a predicate transformer that follows the structure defined by $F$.

**Example 3.1.2.** (Example 2.1.13, continued) Consider the family fibration on Set and the functor $F$ : Set → Set defined as $F\,X = 1 + A \times X$ with 1 the one point set. Then the functor $\bar{F}\,(Y, P) = \mathbf{1}1 + \mathbf{1}A \times (Y, P)$ is a lifting of $F$.

**Example 3.1.3.** (Example 2.2.5, continued) Consider the relations bifibration of the family fibration on Set and the functor $F$ : Set → Set defined as $F\,X = 1{+}A{\times}X$ with 1 the one point set and $A$ some set. Then the functor $\bar{F}\,(X{\times}X, R) = Eq1 + EqA \times (X{\times}X, R)$ is a lifting of $F$.

The next example introduces the lifting operation of Hermida and Jacobs, as defined in [HJ98] (refer to this paper for further detail). This gives a first link between their lifting operation and the liftings presented in this chapter. Note that the liftings introduced in the two previous examples are instantiations of the following.

**Example 3.1.4.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a terminal object functor $\mathbf{1}$, and where $\mathcal{E}$ is bicartesian above $\mathcal{B}$, i.e., $\mathcal{E}$ and $\mathcal{B}$ are bicartesian categories and $p$ preserves the bicartesian structure. Consider a polynomial functor $T : \mathcal{B} \to \mathcal{B}$, i.e., a functor built from the identity, constants, and finite products and coproducts. Then:

(i) We can construct the (polynomial) functor $Pred(T) : \mathcal{E} \to \mathcal{E}$ by induction on the structure of $T$. The bicartesian structure of $\mathcal{B}$ used in $T$ is replaced by the bicartesian structure of $\mathcal{E}$ in $Pred(T)$, and every constant $A$ in $\mathcal{B}$ occurring in $T$ is replaced by the constant $\mathbf{1}A$ in $Pred(T)$.

(ii) If $p$ is a bifibration we can construct the functor $Rel(T) : Rel(\mathcal{E}) \to Rel(\mathcal{E})$ by induction on the structure of $T$. The bicartesian structure of $\mathcal{B}$ in $T$ is replaced by the bicartesian structure of $\mathcal{E}$ in $Rel(T)$, and every constant $A$ in $\mathcal{B}$ occurring in $T$ is replaced by the constant $Eq\,A$ in $Rel(T)$ (remember Example 2.2.4).

It is straightforward to check that the functor $Pred(T)$ and $Rel(T)$ are indeed liftings of $T$.

**Example 3.1.5.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$. Given any functor $F : \mathcal{B} \to \mathcal{B}$ the functor $\bar{F} = \mathbf{1} \circ F \circ p$ is a lifting of $F$.

For a given fibration, we can group the associated liftings into a category:

**Definition 3.1.6.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration. The *category of liftings with respect to* $p$, written $\mathcal{L}p$, has as objects pairs $(\bar{F}, F)$ where $\bar{F} : \mathcal{E} \to \mathcal{E}$ is a lifting of $F : \mathcal{B} \to \mathcal{B}$ with regard to $p$, and has morphisms from $(\bar{F}_1, F_1)$ to $(\bar{F}_2, F_2)$ pairs $(\alpha, \beta)$ where $\alpha : \bar{F}_1 \to \bar{F}_2$ is a natural transformation above the natural transformation $\beta : F_1 \to F_2$.

In fact, while not useful to the present work, it is interesting to note that we can organise liftings in a fibration as described in the following lemma.

**Lemma 3.1.7.** *Let* $p : \mathcal{E} \to \mathcal{B}$ *be a fibration, and write* $[\mathcal{B}, \mathcal{B}]$ *for the category of endofunctors on* $\mathcal{B}$. *The functor* $l : \mathcal{L}p \to [\mathcal{B}, \mathcal{B}]$ *that sends an object* $(\bar{F}, F)$ *to* $F$ *is a fibration.*

*Proof.* We only describe the construction: the reindexing of the fibration is done pointwise, i.e., let $\bar{F}$ be a lifting of $F$ and $\sigma : G \to F$ be a natural transformation, define $\sigma^*(\bar{F}, F)$ by $(\bar{G}, G)$ where $\bar{G}\,P = (\sigma_{pP})^*(\bar{F}\,P)$, and similarly, define $\sigma^{\S}(\bar{F}, F)$ by $(\sigma', \sigma)$ where $\sigma'_P = \sigma^{\S}_{(\bar{F}P)}$. □

Note that the fibre above a functor $F$ is then (isomorphic to) the category of liftings of $F$ and vertical natural transformations between them. Furthermore, in the case where $p : \mathcal{E} \to \mathcal{B}$ has a terminal object functor, it is straightforward to check that the lifting presented in Example 3.1.5 defines a terminal object functor for $l$. Note also that this reasoning dualises in the case of $p$ being an opfibration, and thus, if $p$ is a bifibration, so is $l$.

The notion of lifting is quite general and we will need to restrict it to two subclasses in order to use them for induction and coinduction:

**Definition 3.1.8.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$, let $F$ be a functor on $\mathcal{B}$, and let $\bar{F} : \mathcal{E} \to \mathcal{E}$ be a lifting of $F$. We say that $\bar{F}$ is a $\mathbf{1}$-*preserving* lifting of $F$ if we have $\bar{F} \circ \mathbf{1} \cong \mathbf{1} \circ F$.

And, similarly:

**Definition 3.1.9.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with full section $e : \mathcal{B} \to \mathcal{E}$, let $F$ be an endofunctor on $\mathcal{B}$ and $\bar{F}$ be a lifting of $F$. We say that $\bar{F}$ is an $e$-*preserving* lifting of $F$ if we have $\bar{F} \circ e \cong e \circ F$.

While there is clearly redundancy in the above definitions (since a terminal object functor is a full section) we find that distinguishing both concepts helps to clarify the remaining of the thesis. Furthermore, a lifting can be $e$-preserving in different ways (there can be different isomorphisms characterising the preservation of $e$), it is $\mathbf{1}$-preserving in a canonical way. Indeed, for a $\mathbf{1}$-preserving lifting $\bar{F}$ of $F$, the isomorphism $\bar{F} \circ \mathbf{1} \cong \mathbf{1} \circ F$ is unique since $\mathbf{1}$ maps objects to terminal objects of fibres. Hence, saying that a lifting is $e$-preserving identifies an additional structure while saying that a lifting is $\mathbf{1}$-preserving is a property (See Definition 3.1.10).

It is easy to check that the liftings from Examples 3.1.5 and 3.1.2, and from part (i) of Example 3.1.4, are all $\mathbf{1}$-preserving, and that the liftings of Example 3.1.3 and part (ii) of Example 3.1.4 are $e$-preserving.

For a given a fibration, we can group the associated $e$-preserving liftings as well as $\mathbf{1}$-preserving liftings in categories:

**Definition 3.1.10.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration.

- If $p$ has a terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$, define the *category of* $\mathbf{1}$-*preserving liftings with respect to* $p$, written $\mathbf{1}\mathcal{L}p$, as the obvious full subcategory of $\mathcal{L}p$.

- If $p$ has a full section $e : \mathcal{B} \to \mathcal{E}$, define the *category of e-preserving liftings with respect to p*, written $e\mathcal{L}p$, as the category whose objects are triples $(\bar{F}, F, \alpha)$ with $(\bar{F}, F)$ a lifting and $\alpha$ a natural isomorphism $\bar{F} \circ e \cong e \circ F$. A morphism from $(\bar{F}, F, \alpha)$ to $(\bar{G}, G, \beta)$ is a morphism of lifting $(\gamma, \delta) : (\bar{F}, F) \to (\bar{G}, G)$ such that the following diagram commutes:

$$\begin{tikzcd}
\bar{F}e \arrow[r, "\alpha"] \arrow[d, "\gamma e"'] & eF \arrow[d, "e\delta"] \\
\bar{G}e \arrow[r, "\beta"'] & eG
\end{tikzcd}$$

Note that the category $\mathbf{1}\mathcal{L}p$ is equivalent to the category $e\mathcal{L}p$ when we choose the associated full section $e$ to be the terminal object functor. Indeed, the unicity of the $\mathbf{1}$-preserving isomorphism will guarantee that the condition on morphisms of $e$-preserving liftings is always valid. Furthermore, since reindexing preserves terminal objects, the fibration of liftings restricts to a fibration $l^{\mathbf{1}} : \mathbf{1}\mathcal{L}p \to [\mathcal{B}, \mathcal{B}]$ of $\mathbf{1}$-preserving liftings. While there is no similar result for $e$-preserving liftings, we will have to consider the following subcategory of $e\mathcal{L}p$:

**Definition 3.1.11.** For $p : \mathcal{E} \to \mathcal{B}$ a fibration and $F : \mathcal{B} \to \mathcal{B}$ a functor, let $e\mathcal{L}p_F$ be the category whose objects are pairs $(G, \alpha)$ such that $(G, F, \alpha)$ is a $e$-preserving lifting, and where a morphism $\gamma$ from $(G, \alpha)$ to $(H, \beta)$ is a vertical natural transformation $\gamma : G \to H$ such that the following diagram commutes

$$\begin{tikzcd}
Ge \arrow[r, "\alpha"] \arrow[d, "\gamma e"'] & eF \\
He \arrow[ur, "\beta"'] &
\end{tikzcd}$$

Note that if we choose $e$ to be the terminal object functor of $p$ then the category $e\mathcal{L}p_F$ is equivalent to the fibre $\mathbf{1}\mathcal{L}p_F$ of the above mentioned fibration of $\mathbf{1}$-preserving liftings w.r.t. $p$.

Now that we have the notion of $\mathbf{1}$-preserving and $e$-preserving liftings, we turn our attention to the task of constructing such canonical liftings. Remember that the idea is to construct a functor on the total category which behaves like the one on the base category. First, we show how to construct a $\mathbf{1}$-preserving lifting for any endofunctor on the base category of a Lawvere fibration $p : \mathcal{E} \to \mathcal{B}$. For this, recall from Lemma 2.1.12 that we have the two functors $\pi : \mathcal{E} \to \mathcal{B}^{\to}$ and $\phi : \mathcal{B}^{\to} \to \mathcal{E}$ which intuitively translate elements of the total category to arrows in the base category. We can then use the morphism part of the functor that we want to lift and these two functors to derive a lifting which would behave in the same way on the total category:

**Definition 3.1.12.** Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration and $F : \mathcal{B} \to \mathcal{B}$ be an arbitrary endofunctor. Define the following endofunctor on $\mathcal{E}$:

$$\begin{aligned} \widehat{F} &: \mathcal{E} \to \mathcal{E} \\ \widehat{F} &= \phi F^{\to} \pi \end{aligned}$$

We have:

**Theorem 3.1.13.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration. For any functor $F$ on $\mathcal{B}$, the functor $\widehat{F}$ is a $\mathbf{1}$-preserving lifting of $F$.*

*Proof.* This is a consequence of Corollary 2.2.7 and Lemma 3.1.19 below. □

If $F$ is an endofunctor on the base category of a Lawvere fibration, we will call the functor $\widehat{F}$, the *canonical $\mathbf{1}$-preserving lifting* of $F$.

**Example 3.1.14.** (Example 3.1.2, continued) Recall from Example 2.1.3 and 2.1.13 that the family fibration on Set is a Lawvere fibration, that $\pi : \mathrm{Fam(Set)} \to \mathrm{Set}^{\to}$ maps an object $(X, P)$ to the first projection $\pi(X, P) : \{(X, P)\} \to X$, and that $\phi : \mathrm{Set}^{\to} \to \mathrm{Fam(Set)}$ maps a function $f : X \to Y$ to the inverse image family $(Y, f^{-1})$. Considering the functor $F : \mathrm{Set} \to \mathrm{Set}$ defined as $F\,X = 1 + A \times X$, its canonical $\mathbf{1}$-preserving lifting is then:

$$\begin{aligned} \widehat{F}\,(X, P) &= \phi(F\pi(X, P)) \\ &= \big(F\,X,\ (id_1 + id_A \times \pi_1)^{-1}\big) \\ &\cong \big(F\,X,\ \mathbf{1}1 + \mathbf{1}A \times P\big) \end{aligned}$$

Indeed, it is straightforward to see that $(id_A)^{-1} \cong \mathbf{1}A$ and $\big(\pi(X, P)\big)^{-1} \cong (X, P)$.

We stress that, to define our lifting, the codomain functor above the base $\mathcal{B}$ of the Lawvere fibration does not need to be a fibration, i.e., $\mathcal{B}$ need not have pullbacks. Also, note that this definition of our lifting is equivalent to the definition given in [GJF10], namely, $\hat{F}\,P = \Sigma_{F\pi_P}\mathbf{1}F\{P\}$.

We now show how to construct an $e$-preserving lifting for any endofunctor on the base category of a fibration $p : \mathcal{E} \to \mathcal{B}$ that admits quotients. For this, recall from Definition 2.2.1 that we have the QCE $\rho : \mathcal{E} \to \mathcal{B}^{\to}$ and the functor $\psi : \mathcal{B}^{\to} \to \mathcal{E}$. This two functors will be used as translator from the total category to arrows in the base category. We can then once again use the morphism part of a functor and these two functors to define a lifting:

**Definition 3.1.15.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration that admits quotients and $F : \mathcal{B} \to \mathcal{B}$ be an arbitrary endofunctor. Define the following endofunctor on $\mathcal{E}$:

$$\check{F} : \mathcal{E} \to \mathcal{E}$$
$$\check{F} = \psi F^{\to} \rho$$

We have:

**Theorem 3.1.16.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with full section $e : \mathcal{B} \to \mathcal{E}$ that admits e-quotients. For any functor $F : \mathcal{B} \to \mathcal{B}$, the functor $\check{F}$ is an e-preserving lifting of $F$.*

*Proof.* To prove $p\check{F} = Fp$, note that, for each $P$ in $\mathcal{E}$, the morphism $\rho P$ has domain $pP$, hence $dom\, F^{\to}\rho = Fp$. Also note that $p\psi = dom$, therefore we have $p\check{F} = p\psi F^{\to}\rho = Fp$. To prove $\check{F}e \cong eF$, we first assume that i) for every $X$ in $\mathcal{B}$, $\rho eX$ is an isomorphism in $\mathcal{B}$, and ii) for every isomorphism $f$ in $\mathcal{B}$, $\psi\, f \cong e(dom\, f)$. Then since $pe = Id_{\mathcal{B}}$, i) and ii) imply that $\check{F}e = \psi F^{\to}\rho e \cong e\, dom\, F^{\to}\rho e = eFpe = eF$. To discharge i), note that since $e$ is full and faithful, $\eta e : e \to eQe$ is $e\kappa$ for a natural transformation $\kappa : Id_{\mathcal{E}} \to Qe$, where each $\kappa_X$ is an isomorphism with inverse $\epsilon_X$ and $\epsilon$ is the counit of $Q \dashv e$. Then $\rho eX = p\eta_{eX} = pe\kappa_X = \kappa_X$, so that $\rho eX$ is indeed an isomorphism. To discharge ii), let $f$ be an isomorphism in $\mathcal{B}$. Since cartesian morphisms above isomorphisms are isomorphisms, we have $\psi f = f^*(e\,(cod f)) \cong e\,(cod\, f) \cong e\,(dom\, f)$. Here, the first isomorphism is witnessed by $f^{\S}$ and the second by $ef^{-1}$. $\square$

If $F$ is an endofunctor on the base category of a QCE, we call the functor $\check{F}$ the *canonical e-preserving lifting* of $F$.

**Example 3.1.17.** (Example 3.1.3, continued) Recall from the previous example on the relations bifibration of the family fibration on Set that this bifibration admits quotients where the QCE $\rho$ maps a relation $R$ above $X$ to the quotient map $c_R : X \to X/R$ and the functor $\psi$ maps a map $f : X \to Y$ into its kernel relation $ker(f)$. Now, let us consider the functor $F$ : Set → Set defined as $F\,X = 1 + A \times X$, its canonical $e$-preserving lifting is then:

$$\begin{aligned}\check{F}\,(X \times X, R) &= \psi(F(\rho(X \times X, R))) \\ &= \big(FX \times FX, ker(id_1 + id_A \times c_R)\big) \\ &\cong Eq1 + EqA \times (X \times X, R)\end{aligned}$$

Indeed, it is straightforward to see that $ker(id_A) \cong EqA$ and $ker(c_R) \cong (X{\times}X, R)$.

**Example 3.1.18.** We now consider the relations fibration for the family fibration above Class, the category of classes, i.e., $p : Rel(\text{Class}) \to \text{Class}$ where $Rel(\text{Class})$ is the category whose objects are relations of the form $R : X \times X \to \text{Class}$ with $X$ a class. This fibration admits quotients in the same way as the relations fibration for the family fibration above Set. We then have the functor $\rho$ that maps a relation $R$ above $X$ to the quotient map $c_R : X \to X/R$ and the functor $\psi$ maps a map $f : X \to Y$ into its kernel relation $ker(f)$.

Now consider the canonical $e$-preserving lifting $\check{\mathscr{P}}$ of the power set functor $\mathscr{P} : \text{Class} \to \text{Class}$. We have that $\check{\mathscr{P}}$ maps a relation $R : A \times A \to \text{Class}$ to the relation $\check{\mathscr{P}}R : \mathscr{P}A \times \mathscr{P}A \to \text{Class}$ defined by $\check{\mathscr{P}}R = \psi(\mathscr{P}(\rho R))$. Thus, if $X$ and $Y$ are subsets of $A$, then $(X, Y) \in \check{\mathscr{P}}R$ iff $\mathscr{P}\rho_R X = \mathscr{P}\rho_R Y$. Since the action of $\mathscr{P}$ on a morphism $f$ maps any subset of the domain of $f$ to its image under $f$, the relation $\check{\mathscr{P}}R$ is defined as $(X, Y) \in \check{\mathscr{P}}R$ iff $(\forall x \in X).(\exists y \in Y).\, xRy \;\wedge\; (\forall y \in Y).(\exists x \in X).\, xRy$.

As mentioned in the previous chapter, the dual of Theorem 3.1.16 covers Lawvere fibrations and thus permits us to derive Theorem 3.1.13 as a corollary. The lifting $\breve{F}$ has as its dual the lifting $\widehat{F}$ generalised to the setting of $tC$-opfibrations:

**Lemma 3.1.19.** *Let* $p : \mathcal{E} \to \mathcal{B}$, $t$ *and* $C$ *provide a* $tC$*-opfibration and* $F : \mathcal{B} \to \mathcal{B}$ *be a functor. Define the functor* $\widehat{F}$ *by*

$$\widehat{F} : \mathcal{E} \to \mathcal{E}$$

$$\widehat{F} = \phi\, F^{\to}\, \pi$$

*Then* $\widehat{F}$ *is a* $t$*-preserving lifting of* $F$*, i.e.,* $p \circ \widehat{F} = F \circ p$ *and* $\widehat{F} \circ t \cong t \circ F$*.*

*Proof.* By dualisation of Theorem 3.1.16. The setting on the left below with $p$ an opfibration is equivalent to the setting on the right with $p$ a fibration.

$$\begin{tikzcd}
& \mathcal{E} \arrow[ld, bend right, "C"'] \arrow[d, "p"] & & & \mathcal{E}^{op} \arrow[ld, bend right, "C"'] \arrow[d, "p"] \\
\mathcal{B} \arrow[ru, "\vdash", "t"'] \arrow[r, "Id_{\mathcal{B}}"'] & \mathcal{B} & & \mathcal{B}^{op} \arrow[ru, "\dashv", "t"'] \arrow[r, "Id_{\mathcal{B}^{op}}"'] & \mathcal{B}^{op}
\end{tikzcd}$$

□

## 3.2 An algebra of liftings

We have proved that in any Lawvere fibration $p : \mathcal{E} \to \mathcal{B}$, every endofunctor $F$ on $\mathcal{B}$ has a canonical $\mathbf{1}$-preserving lifting $\widehat{F}$ on $\mathcal{E}$, and that in any fibration $p : \mathcal{E} \to \mathcal{B}$ that admits $e$-quotients, every functor $F$ on $\mathcal{B}$ has a canonical $e$-preserving lifting $\breve{F}$ on $\mathcal{E}$. In this section we ask what kinds of algebraic properties the two lifting operations have. We organise this section by first presenting the results about the preservation properties of the two canonical liftings and then present the results about the relationship between the two canonical liftings and other $\mathbf{1}$-preserving (resp., $e$-preserving) liftings.

We start with the canonical $\mathbf{1}$-preserving lifting of constant functors in Lawvere fibrations.

**Lemma 3.2.1.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration and let $X$ be an object of $\mathcal{B}$. If $F_X$ is the constantly $X$-valued functor on $\mathcal{B}$, then $\widehat{F_X}$ is isomorphic to the constantly $\mathbf{1}X$-valued functor on $\mathcal{E}$.*

*Proof.* For any object $P$ of $\mathcal{E}$ we have

$$\widehat{F_X}P = (\phi(F_X)^{\rightarrow}\pi)P = \phi(F_X\pi_P) = \Sigma_{F_X\pi_P}\mathbf{1}F_X\{P\} = \Sigma_{id}\mathbf{1}X \cong \mathbf{1}X$$

The isomorphism holds because $id^* \cong Id$ and $\Sigma_{id} \dashv id^*$. □

We have a similar result for the canonical $e$-preserving liftings in QCEs.

**Lemma 3.2.2.** *Let $p : \mathcal{E} \to \mathcal{B}$, $e$ and $Q$ provide a QCE and let $X$ be an object of $\mathcal{B}$. If $F_X$ is the constantly $X$-valued functor on $\mathcal{B}$, then $\widecheck{F_X}$ is isomorphic to the constantly $eX$-valued functor on $\mathcal{E}$.*

*Proof.* This is dual of Lemma 3.2.1. □

We now show that the canonical $\mathbf{1}$-preserving lifting operation in Lawvere fibrations preserves coproducts.

**Lemma 3.2.3.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration and let $F$ and $G$ be functors on $\mathcal{B}$. Then $\widehat{F+G} \cong \widehat{F} + \widehat{G}$.*

*Proof.* We have

$$\begin{aligned}
(\widehat{F+G})P &= \phi((F+G)^{\rightarrow}\pi_P) \\
&= \phi(F^{\rightarrow}\pi_P + G^{\rightarrow}\pi_P) \\
&\cong \phi(F^{\rightarrow}\pi_P) + \phi(G^{\rightarrow}\pi_P) \\
&= \widehat{F}P + \widehat{G}P
\end{aligned}$$

The isomorphism holds because $\phi$ is a left adjoint (Lemma 2.1.12) and so preserves coproducts. □

Note that the statement of Lemma 3.2.3 does not assert the existence of either of the two coproducts mentioned, but rather that, whenever both do exist, they must be equal. Also, note that the lemma generalises to any colimit of functors.

We have a dual result for the canonical $e$-preserving lifting operation in QCEs.

**Lemma 3.2.4.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration that admits quotients, and let $F$ and $G$ be functors on $\mathcal{B}$. Then $\widecheck{F \times G} \cong \check{F} \times \check{G}$.*

*Proof.* This is dual of Lemma 3.2.3. ☐

Again, here the statement of Lemma 3.2.4 does not assert the existence of either of the two products mentioned, but rather that, whenever both do exist, they must be equal. And once again, note that the lemma generalises to any limit of functors.

We do not know if the canonical lifting of a product is a product in any lawvere fibration (and hence, if the canonical lifting of a sum is a sum in any QCE). It has however been proved in [AGJJ12] that the property holds under the additional condition that the Lawvere fibration has *very strong sums* (this amounts to ask for the comprehension category $\pi$ to be opfibred). The proof dualises to show that the canonical lifting of a sum is a sum if the QCE is fibred. Under the additional hypothesis of fullness we know that canonical liftings in a Lawvere fibration preserve identity:

**Lemma 3.2.5.** *In any full Lawvere fibration, $\widehat{Id} \cong Id$.*

*Proof.* By Lemma 2.1.12 we have the adjunction $\phi \dashv \pi$. Since $\pi$ is full and faithful, the counit $\epsilon$ of this adjunction is an isomorphism, i.e., $\epsilon : \phi \circ \pi \xrightarrow{\cong} Id$. We therefore have that

$$Id \cong \phi \circ \pi = (\phi \, Id^{\rightarrow} \, \pi) = \widehat{Id}$$

☐

This is similar for canonical liftings in a full QCE.

**Lemma 3.2.6.** *In any bifibration that admits full quotients, $\widecheck{Id} \cong Id$.*

*Proof.* This is dual of Lemma 3.2.5 ☐

The last results of this section consider whether or not there is anything fundamentally special about the canonical liftings we have constructed. In the next chapter we will argue why these liftings are the "right" liftings for deriving induction and coinduction schemes. But other **1**-preserving (resp., $e$-preserving) liftings might also exist and, if this is the case, then we might hope our liftings satisfy some universal property. Unfortunately at this level of generality we are only able to prove the following two properties.

We now look at the relationship between the canonical **1**-preserving lifting and other **1**-preserving liftings in Lawvere fibrations.

**Lemma 3.2.7.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration and let $F$ be a functor on $\mathcal{B}$. The canonical $\mathbf{1}$-preserving lifting $\widehat{F}$ is weakly initial in the category $\mathbf{1}\mathcal{L}p_F$ of $\mathbf{1}$-preserving liftings of $F$ and vertical natural transformations between them.*

*Proof.* Let $\bar{F}$ be a **1**-preserving lifting of $F$. We can then construct a morphism $t_P : \widehat{F}P \to \bar{F}P$ for any $P$ with the following diagram:

$$\begin{tikzcd}
\bar{F}\mathbf{1}\{P\} \arrow[r, "\bar{F}\epsilon_P"] \arrow[d, dash, "\cong"'] & \bar{F}P \\
\mathbf{1}F\{P\} \arrow[r, "(F\pi_P)_\S"'] & \widehat{F}P \arrow[u, dotted, "t_P"']
\end{tikzcd}$$

above $F\{P\} \xrightarrow{F\pi_P} F(pP)$ , where $t_P$ exists by the universal property of the op-cartesian morphism $(F\pi_P)_\S$, and is above the identity. The naturality condition comes from a straightforward diagram chasing. □

We now look at the relationship between the canonical $e$-preserving lifting and other $e$-preserving liftings in QCEs.

**Lemma 3.2.8.** *Let $p : \mathcal{E} \to \mathcal{B}$, $e$ and $Q$ provide a QCE and let $F$ be a functor on $\mathcal{B}$. The canonical $e$-preserving lifting $\check{F}$ is a weakly terminal object in $e\mathcal{L}p_F$.*

*Proof.* Consider $F : \mathcal{B} \to \mathcal{B}$ on the base category of a fibration, $(\check{F}, \alpha)$ the canonical lifting of $F$ and $(\bar{F}, \beta)$ a $e$ preserving lifting of $F$. Let $\gamma : \bar{F} \to \check{F}$ be the natural transformation defined at $R$ as the unique vertical morphism $\gamma_R$ making the following diagram commutes.

$$\begin{tikzcd}
\bar{F}eQR \arrow[d, dash, "\beta_R"', "\cong"] & \bar{F}R \arrow[l, "\bar{F}\eta_R"'] \arrow[d, dotted, "\gamma_R"] \\
eFQR & \check{F}R \arrow[l, "(F\rho_R)^\S"]
\end{tikzcd}$$

In order to obtain a map of $e$-preserving lifting from $\bar{F}$ to $\check{F}$ we need to check if $\beta = \alpha \circ \gamma e$. For this, note that by construction of $\alpha$ the following diagram commutes:

$$\begin{tikzcd}
\bar{F}eQeX & \bar{F}eX \arrow[l, "\bar{F}\eta_{eX}"'] \arrow[d, dash, "\beta_X"', "\cong"] \arrow[rdd, dotted, bend left, "\gamma_{eX}"] & \\
eFQeX \arrow[u, phantom] & eFX \arrow[l, "eF\rho eX"] \arrow[rd, dotted, "\alpha_X"'] & \\
& & \check{F}eX \arrow[llu, bend left, "(F\rho_{eX})^{\S}"]
\end{tikzcd}$$

$$\begin{tikzcd}
\bar{F}eQeX \arrow[d, dash, "\beta_{eX}"', "\cong"] \\
eFQeX
\end{tikzcd}$$

$\square$

These two results will provide a correctness criterion for our constructions of canonical liftings with regard to induction and coinduction. This will be discussed after Corollary 4.3.3 and Corollary 4.3.10 respectively.

In fact, through some additional hypotheses we can show that our lifting is the only $\mathbf{1}$-preserving (resp., $e$-preserving) lifting. Our proof uses a line of reasoning which appears in Remark 2.13 in [HJ98].

**Lemma 3.2.9.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a full Lawvere fibration and let $\Box F$ be a $\mathbf{1}$-preserving lifting of a functor $F$ on $\mathcal{B}$. If $\Box F$ is opfibred — i.e., if $(\Box F)(\Sigma_f P) \cong \Sigma_{Ff}(\Box F)P$ — then $\Box F \cong \widehat{F}$.*

*Proof.* We have

$$\begin{aligned}
(\Box F)P &\cong (\Box F)(\widehat{Id}P) \\
&\cong (\Box F)(\Sigma_{\pi_P}\mathbf{1}\{P\}) \\
&\cong \Sigma_{F\pi_P}(\Box F)\mathbf{1}\{P\} \\
&\cong \Sigma_{F\pi_P}\mathbf{1}F\{P\} \\
&= \widehat{F}P
\end{aligned}$$

$\square$

This is similar for liftings in QCE.

**Lemma 3.2.10.** *Let* $p : \mathcal{E} \to \mathcal{B}$ *admits full e-quotients and let* $\square F$ *be a e-preserving lifting of a functor* $F$ *on* $\mathcal{B}$*. If* $\square F$ *is fibred — i.e., if* $(\square F)(f^*P) \cong (Ff)^*(\square F)P$ *— then* $\square F \cong \check{F}$*.*

*Proof.* This is dual of Lemma 3.2.9. □

Note that Lemma 3.2.9 tells us that the canonical lifting of $F$ is opfibred as soon as there is an opfibred lifting of $F$. We do not know if a canonical lifting is necessarily opfibred otherwise (the dual remark applies for the Lemma 3.2.10). However, any canonical lifting is opfibred if the Lawvere fibration has very strong sums [AJG11, AGJJ12]. Briefly, the very strong sums property amounts to ask that the comprehension category $\pi$ is opfibred, we then have from Proposition 1.2.32 that $\phi$ (the left adjoint of $\pi$) is opfibred as well as $F^{\to}$ (see Example 1.2.6), hence by composition $\widehat{F} = \phi \circ F^{\to} \circ \pi$ is opfibred. The dual result tells us that canonical liftings in QCE are fibred if the QCE itself is fibred.

Finally, we can return to the question of the relationship between the liftings of polynomial functors given by Hermida and Jacobs (reproduced in Example 3.1.4) and the canonical liftings derived by our methods. We have seen that for constant functors, the identity functor, coproducts of functors in Lawvere fibrations and products of functors in QCEs, our constructions agree. Moreover, as already observed in [HJ98], if Hermida and Jacobs' liftings preserve $\Sigma$-types then Lemma 3.2.9 guarantees that in a full Lawvere fibration their lifting (and hence their lifting for products) coincides with ours. Lemma 3.2.10 provides the dual result for the lifting of coproducts in a full QCE.

# Chapter 4

# Induction and coinduction

In this chapter we present induction and coinduction in category theory. In Section 4.1 we quickly go through induction and coinduction as definition principles for types. This consists of well-known results on initial algebras and final coalgebras of endofunctors on a category of types. In Section 4.2, we take a detailed look at the relationship between the inductive definition and induction scheme of the familiar case of natural numbers. This will develop some intuition that we will then use for the main subject of this thesis. In Section 4.3 we present induction and coinduction schemes in fibration for initial algebras and final coalgebras respectively.

## 4.1 Inductive and coinductive definitions in category theory

As previously mentioned, an endofunctor on the base category $\mathcal{B}$ of a fibration specifies the signature of an inductive or a coinductive type. In this section we recall sufficient notions of initial algebra and final coalgebra semantics to formalise this. We begin the section with initial algebras and inductive types. We then look at final coalgebras and coinductive types and we finish the section with a result that links relations between the two functors and relations between the algebras (resp., coalgebras) of the functors.

We start with the notion of algebras associated to a functor and of the morphisms between them, which together form the category of algebras of a functor:

**Definition 4.1.1.** Let $F : \mathcal{B} \to \mathcal{B}$ be an endofunctor on $\mathcal{B}$. The *category of $F$-algebras*, written $Alg_F$, as the category whose objects are pairs $(X, \alpha : FX \to X)$, and whose morphisms between two objects $(X, \alpha : FX \to X)$ and $(Y, \beta : FY \to Y)$ are morphisms $f : X \to Y$ in $\mathcal{B}$ such that the following square commutes in $\mathcal{B}$.

$$\begin{tikzcd}
FX \arrow[r, "Ff"] \arrow[d, "\alpha"'] & FY \arrow[d, "\beta"] \\
X \arrow[r, "f"'] & Y
\end{tikzcd}$$

For $F$ an endofunctor, we call an object $(X, \alpha : FX \to X)$ of $Alg_F$ an *algebra* of $F$ or *$F$-algebra*, where $X$ is called the *carrier* of the algebra and $\alpha$ is called the *structure map* of the algebra. We might refer to an algebra by its structure map. Intuitively, if an endofunctor is the signature of a structure, the carrier of an algebra is an object that carries this structure, and the structure map provides operations to build elements of the carrier using the structure. Algebra morphisms are then morphisms that respect the structure defined by the functor. See the following examples:

**Example 4.1.2.** Let $\mathcal{B}$ be a category that is a model of a simple type theory (See Example 1.1.5), and consider the endofunctor $NX = 1 + X$ on $\mathcal{B}$.

The structure map of an $N$-algebra $(X, \alpha : 1 + X \to X)$ is equivalent to two maps, $\alpha_1 : 1 \to X$ and $\alpha_2 : X \to X$. Such an algebra corresponds in the type theory to a type $X$ that contains a specific element $\alpha_1$ and has a unary operation $\alpha_2$. Looking now at the condition on maps of algebra, a map between two $N$-algebras, $(X, \alpha : 1 + X \to X)$ and $(Y, \beta : 1 + Y \to Y)$, is a function $f$ from $X$ to $Y$ such that $f \circ \alpha = \beta \circ (Ff)$, i.e., $f\alpha_1 = \beta_1$ and $f(\alpha_2 x) = \beta_2(fx)$.

**Example 4.1.3.** Let $\mathcal{B}$ be a category that is a model of a simple type theory and let $L : \mathcal{B} \to \mathcal{B}$ be the functor defined as $LX = 1 + A \times X$. This time the structure map of an $L$-algebra $(X, \alpha : 1 + A \times X \to X)$ is equivalent to two maps, $\alpha_1 : 1 \to X$ and $\alpha_2 : A \times X \to X$. Hence, a $L$-algebra corresponds in the type theory to a type $X$ that contains a specific element $\alpha_1$ and that has a "type action" of $A$ on $X$ given by $\alpha_2$. A map between two $L$-algebras, $(X, \alpha : 1 + A \times X \to X)$ and

$(Y, \beta : 1 + A \times Y \to Y)$, is then a function $f$ from $X$ to $Y$ such that $f\alpha_1 = \beta_1$ and $f\big(\alpha_2(a, x)\big) = \beta_2(a, fx)$.

Among the algebras of a functor, we are particularly interested in the *initial algebra.* The initial algebra of a functor $F : \mathcal{B} \to \mathcal{B}$ is the initial object of the category $Alg_F$. Explicitly it is a $F$-algebra, that we write $(\mu F, in : F\mu F \to \mu F)$, such that for any $F$-algebra $(X, \alpha : FX \to X)$ there exists a unique morphism $(\!|\alpha|\!) : \mu F \to X$ making the following diagram commute

$$\begin{tikzcd}
F\mu F \arrow[r, "F(\!|\alpha|\!)"] \arrow[d, "in"'] & FX \arrow[d, "\alpha"] \\
\mu F \arrow[r, "(\!|\alpha|\!)"'] & X
\end{tikzcd}$$

These unique morphisms are sometimes referred to as *catamorphisms.* With the point of view that an $F$-algebra consists of an object that carries the structure defined by $F$, the initial $F$-algebra consists of the *smallest* object carrying the structure defined by $F$. In the category Set, this corresponds to the "smallest set closed by ..." construction, i.e., free structures. The catamorphism $(\!|\alpha|\!)$ assigns then to an element $x \in \mu F$ its interpretation in $X$. More generally, a key observation due to Lambek is that for any category $\mathcal{B}$ and endofunctor $F$ on $\mathcal{B}$, the structure map *in* of the initial $F$-algebra is an isomorphism. We can therefore see $\mu F$ as the least fixed point of $F$. Following this reasoning, it is not surprising to find that, for $\mathcal{B}$ a category of types, inductive types correspond to initial algebras $(\mu F, in : F\mu F \to \mu F)$, where the carrier $\mu F$ is the actual type, the structure map *in* provides the constructors of the type, and iteration operators correspond to catamorphisms (See [JR97] for example).

We present two examples in order to illustrate this correspondence.

**Example 4.1.4.** (Example 4.1.2, continued) Let $\mathcal{B}$ be a category that is a model of a simple type theory, and consider the functor $NX = 1 + X$ on $\mathcal{B}$. The carrier of the initial algebra of $N$ corresponds to the type of natural numbers in the type theory. Indeed, first observe that the constructors of natural numbers, namely the number *Zero* and the successor operation $\mathit{Succ} : \mathsf{Nat} \to \mathsf{Nat}$, define a $N$-algebra. Furthermore, the iteration operator is given by the following term in the type theory:

$$\begin{array}{lcl} \mathit{foldNat} & : & X \to (X \to X) \to \mathsf{Nat} \to X \\ \mathit{foldNat}\, z\, s\, \mathit{Zero} & = & z \\ \mathit{foldNat}\, z\, s\, (\mathit{Succ}\, n) & = & s\, (\mathit{foldNat}\, z\, s\, n) \end{array}$$

This gives for any $N$-algebra $(X, \alpha : NX \to X)$, the catamorphism $\mathit{foldNat}\, \alpha_1\, \alpha_2$ of codomain $\alpha$.

**Example 4.1.5.** (Example 4.1.3, continued) Let $\mathcal{B}$ be a category that is a model of a simple type theory, and consider the functor $LX = 1 + A \times X$ on $\mathcal{B}$. The carrier of the initial algebra of $L$ corresponds to the type of lists of elements of type $A$ in the type theory. Indeed, first observe that the constructors of lists, namely the empty list *Nil* and the concatenation $\mathit{Con} : A \times \mathit{List} \to \mathit{List}$, define a $L$-algebra. Furthermore, the iteration operator is given by the following term in the type theory:

$$\begin{array}{lcl} \mathit{foldList} & : & X \to (A \times X \to X) \to \mathit{List} \to X \\ \mathit{foldList}\, n\, f\, \mathit{Nil} & = & n \\ \mathit{foldList}\, n\, f\, (\mathit{Con}\, a\, l') & = & f\, a\, (\mathit{foldList}\, n\, f\, l') \end{array}$$

This gives for any algebra $(X, \alpha : LX \to X)$, the catamorphism $\mathit{foldList}\, \alpha_1\, \alpha_2$ of codomain $\alpha$.

Note that, as a category does not necessarily have an initial object, an endofunctor does not necessarily have an initial algebra. In fact, the question of the existence of an initial algebra can be a difficult one. However, the literature already contains abstract results guaranteeing the existence of initial algebras for different classes of functors, see [LS81, SP77] for example, where it is shown how to construct initial algebras in a recursive fashion provided that the category under consideration has all the colimits of countable chains and that the functor preserves them. For this reason, in the remainder of this thesis we assume that the functors we are dealing with have an initial algebra. This assumption does not result in a loss of generality but, on the contrary, it allows us to exploit the power of abstraction of the concept of initial algebra. We will make a similar assumption for the existence of final coalgebras.

We will now look at the notion dual to algebra: *coalgebra*. This notion has the same role for coinductive types that the notion of algebra has for inductive types.

**Definition 4.1.6.** Let $F : \mathcal{B} \to \mathcal{B}$ be a functor on $\mathcal{B}$. The *category of $F$-coalgebra*, written $coAlg_F$, has as objects pairs $(X, \alpha : X \to FX)$. A morphism between $(X, \alpha : X \to FX)$ and $(Y, \beta : Y \to FY)$ is given by a morphism $f : X \to Y$ in $\mathcal{B}$ such that the following square commutes in $\mathcal{B}$.

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "\alpha"'] & Y \arrow[d, "\beta"] \\
FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$

For $F$ an endofunctor, we call an object $(X, \alpha : X \to FX)$ of $coAlg_F$ a *coalgebra* of $F$ or *$F$-coalgebra*, where $X$ is called the *carrier* of the coalgebra and $\alpha$ the *structure map*. We might refer to a coalgebra by its structure map. Intuitively, if an endofunctor is the signature of a structure, as for algebras, a coalgebra $(X, \alpha : X \to FX)$ consists of an object $X$ with a structure defined by $F$, but dually to algebras, the structure map does not tell us how to construct elements following the structure, but rather how to destruct the elements to observe the structure. This can be seen in the following example:

**Example 4.1.7.** Let $\mathcal{B}$ be a category that is a model of a simple type theory. Consider the endofunctor $LX = 1 + A \times X$ on $\mathcal{B}$.

A coalgebra $(X, \alpha : X \to 1 + A \times X)$ consists of a type of $X$, such that for any element $x$ of $X$, we can destruct $x$ to either $\star$, the unique element of 1, or a pair $(a, x')$ where $a$ is some observation in $A$ that we can do from $x$, and $x'$ is the remainder of $x$. Notice that by successively applying $\alpha$ we can observe a potentially infinite sequence of elements of type $A$.

A morphism of coalgebra from $(X, \alpha : X \to 1 + A \times X)$ to $(X, \beta : Y \to 1 + A \times Y)$ is then a function $f : X \to Y$ such that if an element $x$ destructs to $\star$, so does $fx$, and if an element $x$ destructs to a pair $(a, x')$ then the element $fx$ destructs to $(a, fx')$.

**Example 4.1.8.** Consider the functor $SX = \mathscr{P}(A \times X)$ on the category of classes, where $\mathscr{P}(A \times X)$ is the powerset of $A \times X$. A $S$-coalgebra $(X, \alpha : X \to \mathscr{P}(A \times X))$ represents a $A$-labelled non-deterministic automaton. Indeed, we can see $X$ as the set of states of an automaton and $\alpha$ as the transition relation, i.e., there is a transition $x \xrightarrow{a} x'$ in the automaton iff $(a, x')$ is an element of $\alpha x$.

We can add $B$ valued observations to the automaton by considering the functor $TX = \mathscr{P}(A \times X) \times B$. This time, a $T$-coalgebra $(X, \alpha : X \to \mathscr{P}(A \times X) \times B)$ represents a set of states $X$, a transition relation $\alpha_1 : X \to \mathscr{P}(A \times X)$ and an observation function $\alpha_2 : X \to B$. We then have $T = S$ if the set of observations is the one point set.

A morphism of $T$-coalgebras from $(X, \alpha : X \to \mathscr{P}(A \times X) \times B)$ to $(Y, \beta : Y \to \mathscr{P}(A \times Y) \times B)$ is then a morphism, $f : X \to Y$, between the set of states that respects the transitions and observations, i.e., $x \xrightarrow{a} x'$ imply $fx \xrightarrow{a} fx'$ and $\alpha_2 x = \beta_2(fx)$.

Dually to algebra, among the coalgebras of a functor we are particularly interested in the *final coalgebra*. The final coalgebra of a functor $F : \mathcal{B} \to \mathcal{B}$ is the terminal object of the category $coAlg_F$. We write $(\nu F, out : \nu F \to F\nu F)$ to denote the final coalgebra of $F$. Explicitly, the final coalgebra is a $F$-coalgebra such that for any $F$-coalgebra $(X, \alpha : X \to FX)$ there exists a unique morphism $[\![(\alpha)]\!] : X \to \nu F$ making the following diagram commute

$$\begin{tikzcd}
FX \arrow[r, "F[\![(\alpha)]\!]"] & F\nu F \\
X \arrow[u, "\alpha"] \arrow[r, "{[\![(\alpha)]\!]}"'] & \nu F \arrow[u, "out"']
\end{tikzcd}$$

This unique morphisms are sometime referred to as *anamorphisms*. We saw that a $F$-coalgebra consists of an object on which we can observe the structure defined by $F$. So, the final $F$-coalgebra consists of the *largest* such object, which intuitively corresponds to the object of all the possible observations. Intuitively, the anamorphism $[\![(\alpha)]\!]$ maps then elements of $\nu F$ to the possible observations that can be done from them. As for initial algebras, for any endofunctor $F$ the structure map *out* is an isomorphism, thus we can see $\nu F$ as the greatest fixed point of $F$. Also, dually to initial algebras, for $\mathcal{B}$ a category of types, coinductive types correspond to final coalgebras $(\nu F, out : \nu F \to F\nu F)$, where the carrier $\nu F$ is the actual type, the structure map *out* gives the destructors of the type, and coiteration operators correspond to anamorphisms (See [JR97, Jac] for example).

We present two examples to illustrate this correspondence.

**Example 4.1.9.** (Example 4.1.7, continued) Let $\mathcal{B}$ be a category that is a model of a simple type theory, and consider the functor $LX = 1 + A \times X$ on $\mathcal{B}$.

The final coalgebra of $L$ is the type of possibly infinite lists, or colists, of elements of $A$. Indeed, the destructors of colists, namely the map that sends a colist $l$ to, either $\star$ the only element of 1 if $l$ is the empty colist, or a pair $(a, l')$ with $a : A$ and $l'$ the remainder of the colist, defines an $L$-coalgebra. Also, given a coalgebra $(X, \alpha : X \to LX)$, the coiteration operator $unfold\,\alpha$ that maps any element $x$ to the element that deconstructs into, either $\star$ if $\alpha x = \star$, or $(a, unfold_\alpha x')$ if $\alpha x = (a, x')$, defines the anamorphisms.

**Example 4.1.10.** (Example 4.1.8, continued) While the functor $TX = \mathscr{P}(A \times X) \times B$ does not have a final coalgebra in the category of sets, $T$ has a final coalgebra in Class, the category of classes. Remember that a $T$-coalgebra is a non-deterministic automaton, the final coalgebra $(\nu T, out : \nu T \to \mathscr{P}(A \times \nu T) \times B)$ is the non-deterministic automaton of all possible transitions labelled by $A$ and observations in $B$. The unique homomorphism $[\![(\alpha)]\!] : Q \to \nu T$ assigns to every state $q$ the class of the automata that correspond to $q$'s nodes.

We now look at the relationship between two categories of algebras (resp. coalgebras) induced by the relationship between the corresponding functors. The following theorem is from [HJ98]. We will use this to link the algebras of a functor and the algebras of a lifting of this functor.

**Theorem 4.1.11.** *Let $F : \mathcal{B} \to \mathcal{B}$, $G : \mathcal{A} \to \mathcal{A}$, and $S : \mathcal{B} \to \mathcal{A}$ be functors. A natural transformation $\alpha : GS \to SF$, i.e., a natural transformation $\alpha$ such that*

$$\begin{tikzcd}
\mathcal{A} \arrow[r, "G"] & \mathcal{A} \\
\mathcal{B} \arrow[u, "S"] \arrow[r, "F"'] \arrow[ur, phantom, "\overset{\alpha}{\searrow}"] & \mathcal{B} \arrow[u, "S"']
\end{tikzcd}$$

*induces a functor*

$$\begin{tikzcd} Alg_F \arrow[r, "S\text{-}Alg"] & Alg_G \end{tikzcd}$$

*given by S-Alg* $(f : FX \to X) = Sf \circ \alpha_X$. *Moreover, if* $\alpha$ *is an isomorphism, then a right adjoint* $T$ *to* $S$ *induces a right adjoint*

$$\begin{tikzcd} Alg_G \arrow[r, bend left, "T\text{-}Alg"] \arrow[r, phantom, "\top"] & Alg_F \arrow[l, bend left, "S\text{-}Alg"] \end{tikzcd}$$

*given by T-Alg*$(g : GX \to X) = Tg \circ \beta_X$, *where* $\beta : FT \to TG$ *is the image of* $G\epsilon \circ \alpha_T^{-1} : SFT \to G$ *under the adjunction isomorphism* $Hom(SX, Y) \cong Hom(X, TY)$, *and* $\epsilon : ST \to id$ *is the counit of this adjunction.*

We spell out the dual of this theorem as a corollary since it will be of equal importance.

**Corollary 4.1.12.** *Let* $F : \mathcal{B} \to \mathcal{B}$, $G : \mathcal{A} \to \mathcal{A}$, *and* $S : \mathcal{A} \to \mathcal{B}$ *be functors. A natural transformation* $\alpha : SG \to FS$, *i.e., a natural transformation* $\alpha$ *such that*

$$\begin{tikzcd} \mathcal{A} \arrow[r, "G"] \arrow[d, "S"'] & \mathcal{A} \arrow[d, "S"] \arrow[dl, phantom, "\overset{\alpha}{\swarrow}"] \\ \mathcal{B} \arrow[r, "F"'] & \mathcal{B} \end{tikzcd}$$

*induces a functor*

$$\begin{tikzcd} CoAlg_G \arrow[r, "S\text{-}CoAlg"] & CoAlg_F \end{tikzcd}$$

*given by S-CoAlg* $(g : X \to GX) = \alpha_X \circ Sg$. *Moreover, if* $\alpha$ *is an isomorphism, then a left adjoint* $T$ *to* $S$ *induces a left adjoint*

$$\begin{tikzcd} CoAlg_G \arrow[r, bend left, "S\text{-}CoAlg"] \arrow[r, phantom, "\top"] & CoAlg_F \arrow[l, bend left, "T\text{-}CoAlg"] \end{tikzcd}$$

Note that a first consequence of this theorem is that, for $\bar{F}$ a lifting of $F$ with respect to a fibration $p : \mathcal{E} \to \mathcal{B}$, there is a functor *p-Alg* : $Alg_{\bar{F}} \to Alg_F$ and a functor *p-CoAlg* : $CoAlg_{\bar{F}} \to CoAlg_F$. Furthermore, if $p$ has a terminal object functor $\mathbf{1}$, $p\text{-}Alg$ has a right adjoint given by $\mathbf{1}\text{-}Alg : Alg_F \to Alg_{\bar{F}}$. Hence in such a case, if a lifting $\bar{F}$ has an initial algebra $(\mu_{\bar{F}}, in)$ then $F$ has an initial algebra, which is given by $p\text{-}Alg\,(\mu_{\bar{F}}, in)$.

## 4.2 A familiar induction scheme

In this section we look at the relationship between the iteration operator and the induction scheme of natural numbers. The goal is to develop intuitions in order to motivate the definition of induction schemes in fibrations given in the next section. In order to simplify this a step further, we place ourselves in the setting where the types are sets.

Consider the inductive type Nat of natural numbers and the associated iteration operator *foldNat*, both defined in Example 4.1.4. The iteration operator *foldNat* can be used to derive the standard induction scheme for Nat which coincides with the standard induction scheme for natural numbers, i.e., with the familiar principle of mathematical induction. This scheme says that if a property $P$ holds for 0, and if $P$ holds for $n + 1$ whenever it holds for a natural number $n$, then $P$ holds for all natural numbers. Representing each property of natural numbers as a predicate $P : \mathsf{Nat} \to \mathrm{Set}$ mapping each term $n : \mathsf{Nat}$ to the set of proofs that $P$ holds for $n$, we wish to represent this scheme at the object level as a function *indNat* with type

$$\forall (P : \mathsf{Nat} \to \mathrm{Set}). \quad P\,Zero \to (\forall n : \mathsf{Nat}.\, P\,n \to P\,(Succ\,n)) \to (\forall n : \mathsf{Nat}.\, P\,n)$$

Code fragments such as those above, which involve quantification over sets, properties, or functors, are to be treated as "categorically inspired". This is because quantification over such higher-kinded objects cannot be interpreted in Set. In order to give a formal interpretation to code fragments like the ones above, we would need to work in a category such as that of modest sets. While the ability to work with functors over categories other than Set is one of the motivations for working in the general fibrational setting, formalising the semantics of such code fragments would obscure the central message of this thesis. Our decision to treat such fragments as categorically inspired is justified in part by the fact that the use of category theory to suggest computational constructions has long been regarded as fruitful within the functional programming community (see [BdM96, BM98, Mog91] for example).

A function *indNat* with the above type takes as input the property $P$ to be proved, a proof $\phi$ that $P$ holds for *Zero*, and a function $\psi$ mapping each $n : \mathsf{Nat}$ and each proof that $P$ holds for $n$ to a proof that $P$ holds for $\mathit{Succ}\, n$, and returns a function mapping each $n : \mathsf{Nat}$ to a proof that $P$ holds for $n$, i.e., to an element of $P\, n$. We can write *indNat* in terms of *foldNat* — and thus reduce induction for $\mathsf{Nat}$ to iteration for $\mathsf{Nat}$ — as follows. First note that *indNat* cannot be obtained by instantiating the type $X$ in the type of *foldNat* to a type of the form $Pn$ for a specific $n$ because *indNat* returns elements of the types $P\, n$ for different values $n$ and these types are, in general, distinct from one another. We therefore need a type containing all of the elements of $P\, n$ for every $n$. Such a type can informally be thought of as the union over $n$ of $Pn$, and is formally given by the dependent type $\Sigma n : \mathsf{Nat}.\, P\, n$ comprising pairs $(n, p)$ where $n : \mathsf{Nat}$ and $p : P\, n$.

The standard approach to defining *indNat* is thus to apply *foldNat* to an $N$-algebra with carrier $\Sigma n : \mathsf{Nat}.\, P\, n$. Such an algebra has components $\alpha : \Sigma n : \mathsf{Nat}.\, P\, n$ and $\beta : \Sigma n : \mathsf{Nat}.\, P\, n \to \Sigma n : \mathsf{Nat}.\, P\, n$. Given $\phi : P\, \mathit{Zero}$ and $\psi : \forall n.\, P\, n \to P\, (\mathit{Succ}\, n)$, we choose $\alpha = (\mathit{Zero}, \phi)$ and $\beta\, (n, p) = (\mathit{Succ}\, n, \psi\, n\, p)$ and note that $\mathit{foldNat}\, \alpha\, \beta : \mathsf{Nat} \to \Sigma n : \mathsf{Nat}.\, P\, n$. We tentatively take $\mathit{indNat}\ P\ \phi\ \psi\ n$ to be $p$, where $\mathit{foldNat}\, \alpha\, \beta\, n = (m, p)$. But in order to know that $p$ actually gives a proof for $n$ itself, we must show that $m = n$. Fortunately, this follows on easily from the uniqueness of $\mathit{foldNat}\, \alpha\, \beta$. Indeed, we have that

$$\begin{tikzcd}
1 + \mathsf{Nat} \arrow[r] \arrow[d, "in"'] & 1 + \Sigma n : \mathsf{Nat}.\, P\, n \arrow[r] \arrow[d, "{[\alpha,\beta]}"'] & 1 + \mathsf{Nat} \arrow[d, "in"] \\
\mathsf{Nat} \arrow[r, "\mathit{foldNat}\ \alpha\ \beta"'] & \Sigma n : \mathsf{Nat}.\, P\, n \arrow[r, "\lambda(n,p).\, n"'] & \mathsf{Nat}
\end{tikzcd}$$

commutes and, by initiality of *in*, that $(\lambda(n, p).\, n) \circ (\mathit{foldNat}\, \alpha\, \beta)$ is the identity map. Thus

$$n = (\lambda(n,p).\, n)(\mathit{foldNat}\,\alpha\,\beta\,n) = (\lambda(n,p).\, n)(m,p) = m$$

Letting $\pi'_P$ be the second projection on dependent pairs involving the predicate $P$, the induction scheme for $\mathsf{Nat}$ is thus

$$\begin{array}{lcl} \mathit{indNat} & : & \forall(P : \mathsf{Nat} \to \mathrm{Set}).\, P\,\mathit{Zero} \to (\forall n : \mathsf{Nat}.\, P\,n \to P\,(\mathit{Succ}\,n)) \\ & & \quad \to (\forall n : \mathsf{Nat}.\, P\,n) \\ \mathit{indNat}\ P\ \phi\ \ \psi & = & \pi'_P\ \circ (\mathit{foldNat}\ \ (\mathit{Zero}, \phi)\ \ (\lambda(n,p).\,(\mathit{Succ}\,n, \psi\,n\,p))) \end{array}$$

As expected, this induction scheme states that, for every property $P$, to construct a proof that $P$ holds for every $n : \mathsf{Nat}$, it suffices to provide a proof that $P$ holds for *Zero*, and to show that, for any $n : \mathsf{Nat}$, if there is a proof that $P$ holds for $n$, then there is also a proof that $P$ holds for $\mathit{Succ}\,n$.

The use of dependent types is fundamental to this formalisation of the induction scheme for Nat, but this is only possible because the properties to be proved are taken to be set-valued functions. In the next section we look at how to use fibrations in order to generalise the above treatment of induction to arbitrary functors and arbitrary properties which are suitably fibred above the category whose objects interpret types.

## 4.3 Induction and coinduction schemes in fibrations

In this Section we give the definitions of induction and coinduction schemes in fibrations. We start from the intuition from Section 4.2 in order to introduce the definition of induction schemes in fibration. We then illustrate the definition by looking at different examples. We finish with coinduction schemes in fibrations.

In Section 4.2 we saw that we can derive the induction scheme on natural numbers from the iteration operator *foldNat* when working with sets as types and set-indexed sets as predicates. In order to deduce generic induction schemes from these intuitions we need to abstract away in two directions: arbitrary setting, and arbitrary inductive type. As we saw in Chapter 1, fibrations give us the right tool to consider the setting in a generic way. In addition, as we saw in Section 4.1 of this chapter, initial algebra semantics gives us the right tool to consider inductive types in a generic way.

In order to abstract the reasoning of Section 4.2, we begin by considering what we might naively expect an induction scheme for an inductive type $\mu F$ to look like in the setting of sets. The derivation for `Nat` suggests that, in general, it should look something like this:

$$ind \ : \ \forall P : \mu\, F \to \text{Set}. \ \ ??? \ \to \ \forall\, x : \mu F.\, P\, x$$

But what should the premises — denoted ??? here — of the generic induction scheme *ind* be? Since we want to construct, for any term $x : \mu F$, a proof term of type $P\,x$ from proof terms for $x$'s substructures, and since the functionality of the iteration operator for $\mu F$ is precisely to compute a value for $x : \mu F$ from the values for $x$'s substructures, it is natural to try to equip $P$ with an $F$-algebra structure that can be input to the iteration operator to yield a mapping of each $x : \mu F$ to an element of $P\,x$. But a predicate $P$ is not a set and so $F$ cannot be applied to $P$ as it is needed to equip $P$ with an $F$-algebra structure.

In fact, note that the setting used in Section 4.2 corresponds to the family fibration $p : \text{Fam(Set)} \to \text{Set}$ described in Example 1.1.6, where the types and predicates described correspond to the elements of the base category and the total category of $p$ respectively. We then clearly see that we can't directly apply an endofunctor on the base category of the fibration to elements of the total category: this is where we need liftings. Indeed, if we can find a functor $\bar{F}$ that behaves in the same way as $F$ but this time, on predicates, we can then consider $\bar{F}$-algebras of carrier $P$ as a candidate for the premises of $\mu F$'s induction scheme. For $\bar{F}$ to be a correct candidate we need to be able, given a $\bar{F}$-algebra $\gamma : \bar{F}P \to P$, to produce a proof that the predicate $P$ holds for all elements of $\mu F$. Remember from Example 1.2.16, that such a proof corresponds to a morphism $t : \mathbf{1}\mu F \to P$ above the identity. Hermida and Jacobs' analysis is that asking for the initial algebra of the lifting $\bar{F}$ to be the image of the initial algebra of $F$ by $\mathbf{1}$ is the correct way to fulfil this condition. Indeed, if this is the case then for any $\bar{F}$-algebra $\gamma : \bar{F}P \to P$ the catamorphism $(\!|\gamma|\!)$ is from $\mu\bar{F}$ to $P$, i.e., from $\mathbf{1}\mu F \to P$.

We then have the following definition of induction schemes in fibrations which is a straightforward generalisation to our setting of Hermida and Jacobs' definition (Definition 3.1 in [HJ98]):

**Definition 4.3.1.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$ and $F : \mathcal{B} \to \mathcal{B}$ be a functor. We say that a $\mathbf{1}$-preserving lifting $\bar{F}$ of $F$ *defines an induction scheme for* $\mu F$ *in* $p$ if the functor $\mathbf{1}\text{-}Alg : Alg_F \to Alg_{\bar{F}}$ that sends an $F$-algebra $FX \xrightarrow{\alpha} X$ to the $\bar{F}$-algebra $\bar{F}\mathbf{1}X \cong \mathbf{1}FX \xrightarrow{\mathbf{1}\alpha} \mathbf{1}X$ preserves the initial object.

If $\bar{F}$ is the canonical lifting $\widehat{F}$ of $F$, we speak of the *canonical induction scheme.* The induction scheme is then given by the catamorphisms of $\mu\bar{F}$, i.e., $\bar{F}$-algebras are premises of the induction scheme, and the resulting proof is the catamorphism of codomain the given algebra. Note that if a functor $\bar{F}$ defines an induction scheme for $F$ in $p$, the unique map from the initial algebra of $\bar{F}$ to a $\bar{F}$-algebra $\beta$ is above the unique map from the initial algebra of $F$ to the $F$-algebra $\alpha = p\beta$. This is important to ensure that the proof done by induction speaks about the correct term (see Example 4.3.4). We can present the induction scheme in a logical fashion with the following inference rule (where all arrows are vertical):

$$\frac{\bar{F}P \to \alpha^* P}{\mathbf{1}\mu F \to (\!|\alpha|\!)^* P}$$

Which, when $\alpha$ is the initial $F$ algebra *in* boils down to:

$$\frac{\bar{F}P \to in^* P}{\mathbf{1}\mu F \to P}$$

From this definition, we can make the following observation for induction schemes in CCUs.

**Lemma 4.3.2.** *Let* $p : \mathcal{E} \to \mathcal{B}$ *be a CCU. For any functor* $F : \mathcal{B} \to \mathcal{B}$*, any* $\mathbf{1}$*-preserving lifting* $\bar{F}$ *of* $F$ *defines an induction scheme for* $\mu F$ *in* $p$*.*

*Proof.* This is dual of Lemma 4.3.9 which is proved below. □

We can then derive from this lemma the fact that any endofunctor on the base category of a Lawvere fibration has a canonical induction scheme:

**Corollary 4.3.3.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere Fibration. For any functor $F : \mathcal{B} \to \mathcal{B}$, the canonical $\mathbf{1}$-preserving lifting $\widehat{F}$ defines a canonical induction scheme for $\mu F$.*

As we saw in Lemma 3.2.7, for $F : \mathcal{B} \to \mathcal{B}$ a functor on the base category of a Lawvere fibration $p$ we can construct for any $\mathbf{1}$-preserving lifting $\bar{F}$ of $F$ a vertical natural transformation $t : \widehat{F} \to \bar{F}$. This means that for any algebra $\alpha : \bar{F}P \to P$ above $\beta : F(pP) \to pP$ we can construct an $\widehat{F}$-algebra $\alpha \circ t_{pP}$ above $\beta$. Now we have that the inductive proofs on $\mu F$ are done by providing an algebra of a $\mathbf{1}$-preserving lifting of $F$. So, Lemma 3.2.7 provides an argument of correctness for the canonical induction scheme by ensuring that any proof done by induction on $\mu F$ can be done with the canonical induction scheme.

We now have the promised sound generic fibrational induction scheme for every functor $F$ on the base of a Lawvere fibration. To demonstrate the flexibility of this scheme, we now look at different instances of these canonical induction schemes. The first example shows that the induction scheme on the natural numbers discussed in Section 4.2 is an instance of Definition 4.3.1.

**Example 4.3.4.** (Example 4.1.4, continued) We consider the family fibration $p : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Set}$ and the type of natural numbers, $\mu N$ where $N$ is the functor on Set defined by $N\,X = 1 + X$.

Now consider the following endofunctor $\bar{N}$ on Fam(Set) given by

$$\begin{array}{rcl} \bar{N}P\,(\mathit{inl}\,\cdot) & = & 1 \\ \bar{N}P\,(\mathit{inr}\,n) & = & P\,n \end{array}$$

Since it is obtained following the method described in Example 3.1.4, it is a $\mathbf{1}$-preserving lifting of $N$. By Lemma 4.3.2 we then have that $\bar{N}$ provides an induction scheme for natural numbers in $p$. In fact, using similar reasoning to that in Example 3.1.14, we can show that the lifting $\bar{N}$ is also the canonical $\mathbf{1}$-preserving lifting $\widehat{N}$ of $N$.

An $\bar{N}$-algebra with carrier $P : \mathsf{Nat} \to \mathrm{Set}$ can be given by $in : 1 + \mathsf{Nat} \to \mathsf{Nat}$ and $in^{\sim} \ : \ \forall t : 1 + \mathsf{Nat}.\ \bar{N} P\, t \to P(in\, t)$. Since $in\,(inl\,\cdot) = 0$ and $in\,(inr\,n) = n+1$, we see that $in^{\sim}$ consists of an element $h_1 : P\,0$ and a function $h_2 : \forall n : \mathsf{Nat}.\ P\,n \to P\,(n+1)$. Thus, the second component $in^{\sim}$ of an $\bar{N}$-algebra with carrier $P : \mathsf{Nat} \to \mathrm{Set}$ and the first component $in$ gives the premises of the familiar induction scheme, as described in Section 4.2.

Induction schemes in fibration can be instantiated to familiar schemes for polynomial types, as well as to ones we would expect for types such as rose trees, finite hereditary sets and hyperfunctions. While these types do not directly lie within the scope of Hermida and Jacobs' method as described in [HJ98], there exist extensions of their method to cover them (see [HJ97] for example). The induction schemes for Rose trees and finite hereditary sets are instantiated in the family fibration on Set while the induction scheme for hyperfunctions need to be instantiated with CPOs so that hyperfunctions can be formalised.

**Example 4.3.5.** We consider the family fibration $p : \mathrm{Fam}(\mathrm{Set}) \to \mathrm{Set}$. The type of rose trees is given in Haskell-like syntax by

$$\mathit{data\ Rose} = \mathit{Node}(\mathit{Int}, \mathit{List\ Rose})$$

The functor underlying *Rose* is $FX = \mathit{Int} \times \mathit{List}\, X$ and its induction scheme is

$$\begin{aligned} \mathit{indRose} \quad : \quad & \forall\ (P : X \to \mathrm{Set})\ ((k, k^{\sim}) : (\widehat{F}P \to P)). \\ & \forall (x : X).\, P\,(\llparenthesis k \rrparenthesis\, x) \end{aligned}$$

As we saw in Example 3.1.14, the canonical $\mathbf{1}$-preserving lifting in the family fibration on Set is given by $\widehat{F}P = (F\pi_P)^{-1} : F\,X \to \mathrm{Set}$. Then, writing $xs\,!!\,k$ for the $k^{th}$ component of a list $xs$ and bearing in mind that $\pi_P^{-1} \cong P$, we have that

$$\begin{array}{cl}
 & \widehat{F}\, P\, (i, rs) \\
= & \{z : F\{P\} \mid F\pi_p z = (i, rs)\} \\
= & \{(j, cps) : Int \times List\,\{P\} \mid F\pi_P(j, cps) = (i, rs)\} \\
= & \{(j, cps) : Int \times List\,\{P\} \mid (id \times List\,\pi_P)(j, cps) = (i, rs)\} \\
= & \{(j, cps) : Int \times List\,\{P\} \mid j = i \;and\; List\;\pi_P\;cps = rs\} \\
= & \{(j, cps) : Int \times List\,\{P\} \mid j = i \;and\; \forall k < length\,cps.\, \pi_P\,(cps\,!!\,k) = rs\,!!\,k\} \\
= & \bigcup_{k < length\,rs} P(rs\,!!\,k)
\end{array}$$

An $\widehat{F}$-algebra whose underlying $F$-algebra is $k : Int \times List\,X \to X$ is thus a pair of functions $(k, k^{\sim})$, where $k^{\sim}$ has type

$$\forall i : Int.\, \forall rs : List\,X.\; \big(\forall k < length\,rs.\, P(rs\,!!\,k)\big) \to P\,\big(k\,(i, rs)\big)$$

We can then rewrite the induction scheme on rose trees as:

$$\begin{array}{ll}
indRose \;:\; \forall\,(P : X \to \mathrm{Set}) & \\
\qquad\qquad (k : Int \times List\,X \to X) & \\
\qquad\qquad \Big(\forall i : Int.\, \forall rs : List\,X.\; \big(\forall k < length\,rs.\, P(rs\,!!\,k)\big) \to P\,\big(k\,(i, rs)\big)\Big). & \\
\qquad\qquad\quad \forall (x : X).\, P\,\big((\!|k|\!)\,x\big) &
\end{array}$$

We now look at finite hereditary sets, which, although defined in terms of quotients, and thus lie outside the scope of previously known methods, can be considered with ours.

**Example 4.3.6.** Consider the family fibration $p : \mathrm{Fam(Set)} \to \mathrm{Set}$. Hereditary sets are sets whose elements are themselves sets, as are the core data structures within set theory. The type $HS$ of finitary hereditary sets is $\mu\mathscr{P}_f$ for the finite powerset functor $\mathscr{P}_f$. We can derive an induction scheme for finite hereditary sets as follows. If $P : X \to \mathrm{Set}$, then $\mathscr{P}_f\pi_P : \mathscr{P}_f(\Sigma x : X.Px) \to \mathscr{P}_f X$ maps each set $\{(x_1, p_1), \ldots, (x_n, p_n)\}$ to the set $\{x_1, \ldots, x_n\}$, so that $(\mathscr{P}_f\pi_P)^{-1}$ maps a set $\{x_1, \ldots, x_n\}$ to the set $Px_1 \times \ldots \times Px_n$. A $\widehat{\mathscr{P}_f}$-algebra with carrier $P : HS \to \mathrm{Set}$

and first component *in* therefore has as its second component a function of type

$$\forall(\{s_1, \ldots, s_n\} : \mathscr{P}_f(HS)).\, Ps_1 \times \ldots \times Ps_n \to P(in\{s_1, \ldots, s_n\})$$

The induction scheme for finite hereditary sets is thus

$$\begin{aligned} indHS :: (&\forall(\{s_1, \ldots, s_n\} : \mathscr{P}_f(HS)).\, Ps_1 \times \ldots \times Ps_n \to P(in\{s_1, \ldots, s_n\})) \\ &\to \forall(s : HS).P\, s \end{aligned}$$

We now derive an induction scheme for a type and properties on it that cannot be modelled in Set. Being able to derive induction schemes for fixed points of functors in categories other than Set is a key motivation for working in a general fibrational setting.

**Example 4.3.7.** The fixed point $Hyp = \mu F$ of the functor $FX = (X \to Int) \to Int$ is the type of hyperfunctions. Since $F$ has no fixed point in Set, we interpret it in the category $\omega CPO_\perp$ of $\omega$-cpos with $\perp$ and strict continuous monotone functions. In this setting, a property of an object $X$ of $\omega CPO_\perp$ is an admissible sub-$\omega CPO_\perp$ $A$ of $X$. Admissibility means that the bottom element of $X$ is in $P$ and $P$ is closed under least upper bounds of $\omega$-chains in $X$. This structure forms a Lawvere fibration [Jac93, Jac99]. In particular, $\Sigma_f P$ is constructed for a continuous map $f : X \to Y$ and an admissible predicate $P \subseteq X$, as the intersection of all admissible $Q \subseteq Y$ with $P \subseteq f^{-1}Q$. The terminal object functor maps $X$ to $X$, and comprehension $\widehat{F}$ maps a sub-$\omega CPO_\perp$ $P$ of $X$ to $P$. The lifting $F$ maps a sub-$\omega CPO_\perp$ $P$ of $X$ to the least admissible predicate on $FX$ containing the image of $FP$. Finally, the derived induction scheme states that if $P$ is an admissible sub-$\omega CPO_\perp$ of $Hyp$, and if $\widehat{F}P \subseteq P$, then $P = Hyp$.

Now that we have presented the induction schemes in fibration, let us look at the dual: coinduction schemes in fibration.

**Definition 4.3.8.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a full section $e : \mathcal{B} \to \mathcal{E}$ and let $F : \mathcal{B} \to \mathcal{B}$ be a functor. We say that a $e$-preserving lifting $\bar{F}$ of $F$ *defines a coinduction scheme for $\nu F$ in $p$* if the functor $e\text{-}CoAlg : CoAlg_F \to CoAlg_{\bar{F}}$

that sends an $F$-coalgebra $X \xrightarrow{\alpha} FX$ to the $\bar{F}$-coalgebra $eX \xrightarrow{e\alpha} eFX \cong \bar{F}eX$ preserves the terminal object.

If $\bar{F}$ is the canonical lifting $\check{F}$ of $F$, we speak of the *canonical coinduction scheme*. The coinduction scheme is then given by the anamorphisms of $\nu\bar{F}$, i.e., $\bar{F}$-coalgebras are premises of the coinduction scheme, and the resulting proof is the anamorphism whose domain is the given coalgebra. Note that if a functor $\bar{F}$ defines a coinduction scheme for $\nu F$ in $p$, the unique map into the final coalgebra of $\bar{F}$ from a $\bar{F}$-coalgebra $\beta$ is above the unique map into the final coalgebra of $F$ from the $F$-coalgebra $\alpha = p\beta$. We can present the coinduction scheme in a logical fashion with the following inference rule (where all arrows are vertical):

$$\frac{R \to \alpha^* \bar{F} R}{R \to [\![(\alpha)]\!]^* e\nu F}$$

Which, when $\alpha$ is the final $\bar{F}$ coalgebra *out* boils down to:

$$\frac{R \to out^* \bar{F} R}{R \to \nu F}$$

From this definition, we can make the following observation for coinduction schemes in QCEs.

**Lemma 4.3.9.** *Let $p : \mathcal{E} \to \mathcal{B}$ admits e-quotients. For any functor $F : \mathcal{B} \to \mathcal{B}$, any e-preserving lifting $\bar{F}$ of $F$ defines a coinduction scheme for $\nu F$ in $p$.*

*Proof.* Let $p$, $e$ and $Q$ provide a QCE. We then have that $e$ is right adjoint to $Q$. Now, since $\bar{F}$ is an $e$-preserving lifting, we can use Corollary 4.1.12 to deduce that the functor *e-CoAlg* : $CoAlg_F \to CoAlg_{\bar{F}}$ has a right adjoint, and thus preserves the terminal object, i.e., the final coalgebra of $\check{F}$ is given by $e$ *out* where *out* is the final coalgebra of $F$. □

Thus, from this lemma we can derive that (the final coalgebra of) any endofunctor on the base category of a QCE has a canonical coinduction scheme:

**Corollary 4.3.10.** *Let $p : \mathcal{E} \to \mathcal{B}$ admits e-quotients. For any functor $F : \mathcal{B} \to \mathcal{B}$, the canonical e-preserving lifting $\check{F}$ defines a canonical coinduction scheme for $\nu F$.*

As for the canonical induction scheme, the canonical coinduction scheme comes with an argument of correctness based on Lemma 3.2.8. Dually from the inductive case: For $F : \mathcal{B} \to \mathcal{B}$ a functor on the base category of a fibration $p$ that admits $e$-quotients, for any $e$-preserving lifting $\bar{F}$ of $F$ we can construct a vertical natural transformation $t : \check{F} \to \bar{F}$. Hence we can construct a $\check{F}$ coalgebra above $\beta : X \to FX$ from any $\bar{F}$ coalgebra above $\beta$. This ensure that any proof done by coinduction on $\nu F$ can be done with the canonical coinduction scheme. We now have the promised sound generic fibrational coinduction scheme for every functor $F$ on the base category of a QCE. To demonstrate the flexibility of this scheme, we now look at different instances of this canonical coinduction scheme. We start with a coinduction scheme for possibly infinite lists.

**Example 4.3.11.** (Example 4.1.9continued) Consider the fibration of relation on Set, $p : \mathit{Rel}(\mathrm{Fam}(\mathrm{Set})) \to \mathrm{Set}$ and remember from Example 2.2.5 that it admits quotients. Consider then the type of colists, $\nu L$ where $L$ is the functor on Set defined by $L\,X = 1 + A \times X$.

Now consider the following endofunctor $\bar{L}$ on $\mathit{Rel}(\mathrm{Set})$ given by $\bar{L}R = e1 + eA \times R$. Since it is obtained following the method described in Example 3.1.4, it is a $e$-preserving lifting of $L$. By Lemma 4.3.9 we then have that $\bar{L}$ provides a coinduction scheme for streams in $p$. In fact, we saw in Example 3.1.17 that the lifting $\bar{L}$ is also the canonical $e$-preserving lifting $\check{L}$ of $L$ in the family fibration on Set.

Let $R$ be a relation in $\mathcal{E}$ and the carrier of a $\bar{L}$-coalgebra $(R, \alpha : R \to e1 + eA \times R)$ above a coalgebra $(X, \beta : X \to 1 + A \times X)$. This mean that in the logic we have a proof that whenever two elements $x$, $y$ of $X$ are in relation by $R$, either $fx = fy = \star$, or $fx = (a, x')$, $fy = (b, y')$, with $a = b$ and $x'$ and $y'$ are again in relation by $R$. Now, the coinduction scheme says that whenever we have such a proof, we can deduce that for any two elements $x$ and $y$ ind $X$, if $xRy$ then $[\![(f)]\!]x = [\![(f)]\!]y$. In particular, if $f$ is the final coalgebra *out* of $L$, this provides a way to prove the equality of two (potentially infinite) streams.

**Example 4.3.12.** (Examples 4.1.9, 1.1.9 continued) Consider, $\mathit{dom} : \mathcal{B}^{\to} \to \mathcal{B}$, the domain fibration on a bicartesian category $\mathcal{B}$ and remember that it admits quotients where both $\rho$ and $\psi$ are the identity functor on $\mathcal{B}^{\to}$.

In this setting, every canonical lifting is given by the morphism part of a functor, indeed $\breve{F} = \psi F^{\to}\rho = F^{\to}$. We then have that a $\breve{F}$-coalgebra is given by the following commuting square in $\mathcal{B}$:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, "u_1"'] & Y \arrow[d, "u_2"] \\
FX \arrow[r, "Ff"'] & FY
\end{tikzcd}$$

using the intuition given in Example 2.2.3, this coalgebra can be understood as a function from $X$ to $FX$ such that any elements in relation by $f$ are mapped into elements in relation by $Ff$.

The resulting anamorphism is then given by the following commuting square:

$$\begin{tikzcd}
X \arrow[rrr, dotted, "{[\![(u_1,u_2)]\!]_1}"] \arrow[dr, "f"] \arrow[ddd, "u_1"'] & & & \nu F \arrow[dl, "id"'] \arrow[ddd, "out"] \\
& Y \arrow[r, dotted] \arrow[d, "u_2"] & \nu F \arrow[d, "out"] & \\
& FY \arrow[r, dotted] & F(\nu F) & \\
FX \arrow[ur, "Ff"] \arrow[rrr, dotted, "{F[\![(u_1,u_2)]\!]_1}"'] & & & F\nu F \arrow[ul, "id"']
\end{tikzcd}$$

the intuition being that we have a map $[\![(u_1, u_2)]\!]_1$ that maps any elements in relation by $f$ to equal elements in $\nu F$ and such that $out \circ [\![(u_1, u_2)]\!] = F[\![(u_1, u_2)]\!]_1 \circ u_1$.

In particular, setting $F$ to be the functor $LX = 1+A\times X$, the lifting of $L$ maps $f : X \to Y$ to $id_1 + id_A \times f$. Using the interpretation given in Example 2.2.3, $\widehat{L}f$ relates two elements of $1+A\times X$ iff they are either both the unique element of 1, or pairs $(a, x)$ and $(a, x')$ with $x$ and $x'$ in relation by $f$. Hence we can understand the associated coinduction scheme as: given $f : X \to Y$ a relation on $X$, if there is a function $u : X \to 1 + A \times X$ with a proof that $u$ maps elements related by $f$ to elements related by $id_1 + id_A \times f$, then there is a function $[\![(u)]\!] : X \to \nu L$ such that $out \circ [\![(u)]\!] = L[\![(u)]\!] \circ u$ and a proof that $[\![(u)]\!]$ maps elements related by $f$ to equal elements in $\nu L$.

We now look at the coinduction scheme for the power set functor $\mathscr{P}$. Since $\mathscr{P}$ is not polynomial, it lies outside the scope of [HJ98], but it is important, since a number of canonical coalgebras are built from it, as we saw in Example 4.1.8.

**Example 4.3.13.** (Example 3.1.18, continued) We now consider the relations fibration for the family fibration above Class, $p : \mathit{Rel}(\text{Class}) \to \text{Class}$.

Consider the power set functor $\mathscr{P} : \text{Class} \to \text{Class}$ and its canonical $e$-preserving lifting functor $\breve{\mathscr{P}} : \mathit{Rel}(\text{Class}) \to \mathit{Rel}(\text{Class})$ that maps a relation $R : A \times A \to \text{Class}$ to the relation $\breve{\mathscr{P}}R : \mathscr{P}A \times \mathscr{P}A \to \text{Class}$ defined by $X(\breve{\mathscr{P}}R)Y \iff (\forall x \in X).(\exists y \in Y).\, xRy \wedge (\forall y \in Y).(\exists x \in X).\, xRy$. We can then look at the resulting coinduction scheme. It has as its premises a $\breve{\mathscr{P}}$-coalgebra, i.e., a relation $R : A \times A \to \text{Class}$ and a map from $R$ to $\breve{\mathscr{P}}R$ in $\mathit{Rel}(\text{Class})$. A morphism in $\mathit{Rel}(\text{Class})$ from $(X, R)$ to $(X', R')$ consists of a morphism $\phi : X \to X'$ in Class and a morphism $\phi^{\sim} : \forall(x, y) \in X \times X.\ xRy \to (\phi x)R'(\phi y)$. Thus, a $\breve{\mathscr{P}}$-coalgebra consists of a function $\alpha : A \to \mathscr{P}A$ together with a function $\alpha^{\sim} : (\forall a, a' \in A).\, aRa' \to (\alpha a)\ \breve{\mathscr{P}}R\ (\alpha a')$. If we regard $\alpha : A \to \mathscr{P}A$ as a transition relation, i.e., if we define $a \to b$ iff $b \in \alpha a$, then $\alpha^{\sim}$ captures the condition that $R$ is a bisimulation above $\alpha$. The coinduction scheme thus asserts that any two bisimilar states have the same interpretation in the final coalgebra.

**Example 4.3.14.** (Examples 3.1.18, 1.1.9 continued) Consider the domain fibration on the category of classes, $p : \text{Class}^{\to} \to \text{Class}$.

As seen in Example 4.3.12, any canonical lifting is given by the morphism part of the lifted functor, we then have $\breve{\mathscr{P}} = \mathscr{P}^{\to}$. Now since relations are interpreted in the domain fibration using kernel relations, we have for $f : A \to B$ a relation on $A$, writing $|f|$ for the kernel relation on $f$:

$$\begin{aligned} A_1\, |\breve{\mathscr{P}}f|\, A_2 &\leftrightarrow (\breve{\mathscr{P}}f)\, A_1 = (\breve{\mathscr{P}}f)\, A_2 \\ &\leftrightarrow \forall a \in A_1 \exists a' \in A_2.\ a\,|f|\,a' \wedge \forall a \in A_2 \exists a' \in A_1.\ a\,|f|\,a' \end{aligned}$$

Hence, a $\breve{\mathscr{P}}$-coalgebra of carrier $f$ is a proof that $|f|$ is a bisimulation and the coinduction scheme associates to such a proof a map from $A$ to $\nu\mathscr{P}$ with a proof that any elements related by $|f|$ are mapped to two equal elements in $\nu\mathscr{P}$.

In summary, we have a sound generic induction scheme for Lawvere fibrations and a sound generic coinduction scheme for QCEs. Both are valid for arbitrary functor $F$ on the base category. We derive a sound induction scheme for $\mu F$ from the canonical $\mathbf{1}$-preserving liftings $\widehat{F}$, and a sound coinduction scheme for $\nu F$ from the canonical $e$-preserving lifting $\breve{F}$.

# Chapter 5

# Indexed induction

Data types arising as initial algebras and final coalgebras on traditional semantic categories such as Set and $\omega cpo_\perp$ are of limited expressivity. More sophisticated data types arise as initial algebras of functors on their indexed versions. To build intuition about the resulting *inductive indexed types*, first consider the inductive type $\mathsf{List}\,X$ of lists of $X$. It is clear that the definition of $\mathsf{List}\,X$ does not require an understanding of $\mathsf{List}\,Y$ for any $Y \neq X$. Since, each type $\mathsf{List}\,X$ is in isolation inductive, the type $\mathsf{List}$ can be considered a *family of inductive types*. By contrast, consider the inductive definition of the $\mathsf{Nat}$-indexed type $\mathsf{Fin} : \mathsf{Nat} \to \mathrm{Set}$ of finite sets given by

$$\frac{}{fz : \mathsf{Fin}\,(n+1)} \qquad\qquad \frac{x : \mathsf{Fin}\,n}{fs\;x : \mathsf{Fin}\,(n+1)}$$

and $\mathsf{Lam} : \mathsf{Nat} \to \mathrm{Set}$ of untyped $\lambda$-terms up to $\alpha$-equivalence with free variables in $\mathsf{Fin}\,n$ given by

$$\frac{i : \mathsf{Fin}\,n}{Var\,i : \mathsf{Lam}\,n} \qquad \frac{f : \mathsf{Lam}\,n \qquad a : \mathsf{Lam}\,n}{App\,f\;a : \mathsf{Lam}\,n} \qquad \frac{b : \mathsf{Lam}\,(n+1)}{Abs\,b : \mathsf{Lam}\,n}$$

The intuition is that for a term $n : \mathsf{Nat}$, the type $\mathsf{Fin}\,n$ is a type with exactly $n$-elements and $\mathsf{Lam}\,n$ is the type of untyped $\lambda$-terms up to $\alpha$-equivalence with free variables in $\mathsf{Fin}\,n$. Unlike $\mathsf{List}\,X$, the types $\mathsf{Fin}\,n$ and $\mathsf{Lam}\,n$ cannot be defined in isolation using only the elements of $\mathsf{Fin}\,n$ and $\mathsf{Lam}\,n$ that have already been constructed. Indeed, elements of $\mathsf{Fin}\,n$ are needed to construct elements of $\mathsf{Fin}\,(n+$

1), and elements of $\mathsf{Lam}\,(n+1)$ are needed to construct elements of $\mathsf{Lam}\,n$ so that, in effect, all of the types $\mathsf{Fin}\,n$ and $\mathsf{Lam}\,n$ must be inductively constructed simultaneously. Each of the inductive indexed types $\mathsf{Fin}$ and $\mathsf{Lam}$ are thus an *inductive family of types*, rather than a family of inductive types.

If types are interpreted in a category $\mathcal{B}$, and if $I$ is a set of indices considered as a discrete category, then an inductive $I$-indexed type can be modelled by the initial algebra of a functor on the functor category $[I, \mathcal{B}]$. Alternatively, indices can be modelled by objects $I$ of $\mathcal{B}$, and inductive $I$-indexed types can be modelled by initial algebras of functors on slice categories $\mathcal{B}/I$. Similarly, coinductive indexed types can be modelled by final colagebras of functors on functor categories or functors on slice categories.

Initial algebra semantics for inductive indexed types has been developed extensively [Dyb94, GH04, AM09]. Pleasingly, no fundamentally new insights were required: the standard initial algebra semantics only needed to be instantiated to categories such as $\mathcal{B}/I$. By contrast, the theory of induction for inductive indexed types has received comparatively little attention.

In this chapter we will derive sound induction schemes for such types by similarly instantiating the fibrational framework to appropriate categories. We will then look at different examples of fibrations in which we can instantiate our results as well as ways to derive a fibration for indexed induction from a fibration for non-indexed induction. We finish the chapter by looking at some properties of the different structure introduced.

## 5.1 The setting

In this section we look at induction schemes in an indexed setting. In order to do this we first present how fibrations extend to a setting for reasoning about indexed types. We then use the results from Chapter 4 to come up with the definition of an induction scheme in this new setting.

First, bear in mind that as previously mentioned in Chapters 1 and 2, besides their use for modelling a logic above a type theory, fibrations can be used to model dependent types. In fact, since fibrations capture indexing closed by substitution, we can use them to capture any such indexing of types by considering a fibration whose total category is a category of types and whose base category is a category of objects indexing these types. For $\mathcal{B}$ a category of types, traditional fibrations for representing indexed types would be the family fibration of $\mathcal{B}$ (Example 1.1.7) that captures set-indexed types, but also the codomain fibration on $\mathcal{B}$ (Example 1.1.8) that captures type-indexed types (i.e., dependent types, see [See84, Hof94]). Given a fibration $r : \mathcal{B} \to \mathcal{A}$, of $\mathcal{A}$-indexed types, an *inductive indexed type* with index $a$ in $\mathcal{A}$ is given by the initial algebra of an endofunctor $F_a : \mathcal{B}_a \to \mathcal{B}_a$ on the fibre $\mathcal{B}_a$. The two standard approaches mentioned at the beginning of the chapter are recovered by setting $r$ to be the family fibration $\mathrm{Fam}(\mathcal{B}) \to \mathrm{Set}$ and the codomain fibration $cod : \mathcal{B}^{\to} \to \mathcal{B}$, respectively. Indeed, a fibre above a set $I$ of the family fibration is equivalent to the functor category $[I, \mathcal{B}]$, and, for $A$ in $\mathcal{B}$ a fibre above $A$ in *cod* is the slice category $\mathcal{B}/A$.

Now that we are fixed on our setting for indexed types, we need to consider a logic above it. For $r : \mathcal{B} \to \mathcal{A}$ a fibration of types, we represent a logic on these types with a second fibration $p : \mathcal{E} \to \mathcal{B}$ above the total category of the fibration of types. This seems to be the most natural setting since, even when indexed, the types are objects of the category $\mathcal{B}$. Another possibility is to consider the fibration of predicates $p$ as another fibration on the base category $\mathcal{A}$. The latter approach particularly makes sense if $\mathcal{A}$ is a category of contexts because the fibration $r$ is then understood as a fibration of types in context and $p$ is understood as a fibration of propositions in type context. This approach is notably used in [Jac99] to model higher order predicate logic above a dependent type theory with *DPL-structure* and its extensions. We will see how the two approaches are related in Example 5.2.4.

We then work in a setting where we have a fibration of predicates $p : \mathcal{E} \to \mathcal{B}$ above the total category of a fibration of ($\mathcal{A}$-indexed) types $r : \mathcal{B} \to \mathcal{A}$. Since we still have a fibration of predicates, we would like to directly apply our theory from Chapter 4. However, the difference with non-indexed induction schemes is that we are not dealing with (initial algebra of) endofunctors on the base category $\mathcal{B}$ of the fibration of predicates, but with endofunctors on fibres of $r$ i.e., subcategories of $\mathcal{B}$. In this situation we cannot expect to lift a functor $F : \mathcal{B}_a \to \mathcal{B}_a$, for $a$ in $\mathcal{A}$, to the (whole) category of predicates $\mathcal{E}$. On the other hand, we might be able to lift $F$ to a subcategory of $\mathcal{E}$, provided that the fibration $p$ restricts to a fibration above $\mathcal{B}_a$ with sufficient structure to lift $F$. Remember that $r \circ p$ is a fibration by Lemma 1.2.3. We then have the following well-known lemma (Theorem 4.1 in [Str99] for example) which gives us a first result in that direction.

**Lemma 5.1.1.** *Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be two fibrations. For any object $a$ in $\mathcal{A}$ the fibration $p$ restricts to a fibration $p_a : \mathcal{E}_a \to \mathcal{B}_a$ between $\mathcal{E}_a$, the fibre above $a$ of the fibration $r \circ p$, and $\mathcal{B}_a$ the fibre above $a$ of the fibration $r$. Furthermore, if $p$ is a bifibration so is the fibration $p_a$, and if $p$ has a terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$, the terminal object functor restricts to a terminal object functor $\mathbf{1}_a : \mathcal{B}_a \to \mathcal{E}_a$ for the fibration $p_a$.*

*Proof.* The key observation is that we can obtain the fibration $p_a : \mathcal{E}_a \to \mathcal{B}_a$ by a change of base of $p$ along the inclusion functor $i_a : \mathcal{B}_a \to \mathcal{B}$ for any $a$ in $\mathcal{A}$. Furthermore, since a change of base preserves bifibrations and terminal object functors (Corollary 1.2.2 and Lemma 1.2.28) $p_a$ is a bifibration and has a terminal object functor if $p$ has the corresponding structure. □

This Lemma shows that the basic structure of a logic (reindexing and terminal object functors) above a fibration of indexed types restricts to a corresponding logic above types with a specific index.

Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be two fibrations with $\mathbf{1} : \mathcal{B} \to \mathcal{E}$ a terminal object functor of $p$. For $a$ an object of $\mathcal{A}$ and $F : \mathcal{B}_a \to \mathcal{B}_a$ a functor, the fibration $p_a$ gives us a setting to consider liftings and $\mathbf{1}_a$-preserving liftings of a functor $F$ to the subcategory $\mathcal{E}_a$ of $\mathcal{E}$. We are back to a setting that corresponds to the non-indexed induction case, i.e., a fibration $p_a : \mathcal{E}_a \to \mathcal{B}_a$ and a functor $F : \mathcal{B}_a \to \mathcal{B}_a$ on the base category. Furthermore, since the terminal object functor $\mathbf{1}_a$ is a restriction of $\mathbf{1}$, we know that any proof of the form $\mathbf{1}_a X \to P$ in $\mathcal{E}_a$ is the same as a proof $\mathbf{1} X \to P$ in $\mathcal{E}$ above $a$. We can then adapt our definition of induction schemes in fibrations to this new setting:

**Definition 5.1.2.** Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be fibrations with $\mathbf{1} : \mathcal{B} \to \mathcal{E}$ the terminal object functor of $p$. For $a$ in $\mathcal{A}$ and $F : \mathcal{B}_a \to \mathcal{B}_a$ a functor, we say that a

$\mathbf{1}_a$-preserving lifting $\bar{F} : \mathcal{E}_a \to \mathcal{E}_a$ of $F$ *defines an $a$-indexed induction scheme for $\mu F$ in $p$* if the functor $\mathbf{1}\text{-}Alg : Alg_F \to Alg_{\bar{F}}$ that sends an $F$-algebra $FX \xrightarrow{\alpha} X$ in $\mathcal{B}_a$ to the $\bar{F}$-algebra $\bar{F}1X \cong 1FX \xrightarrow{1\alpha} 1X$ in $\mathcal{E}_a$, preserves the initial object.

Equivalently, the functor $\bar{F}$ defines an $a$-indexed induction scheme for $\mu F$ in $p$ if it defines an induction scheme for $\mu F$ in $p_a$.

Note that this definition generalises Definition 4.3.1, since setting the fibration $r$ to be the unique functor from $\mathcal{B}$ to the one object (one morphism) category gives us the definition of (non-indexed) induction schemes in fibrations.

As for the non-indexed definition, the induction scheme is given by the catamorphisms associated to the initial algebra of $\bar{F}$. Note that in the indexed setting, since $\bar{F}$ is a functor on $\mathcal{E}_a$, we can only apply the induction scheme to predicates with the same index as $\mu F$. For $P$ in $\mathcal{E}_b$ we can however use reindexing provided that we have a morphism $f : a \to b$ in $\mathcal{A}$, as this would let us produce proof of the form $\mathbf{1}\mu F \to f^*P \xrightarrow{f^\S} P$.

Now, let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be two fibrations such that $p$ has a terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$. The next step is to consider what structures on $p$ and $r$ guarantee indexed induction schemes for any indexed inductive types. We know from the previous chapter that for $a$ in $\mathcal{A}$, a functor $F : \mathcal{B}_a \to \mathcal{B}_a$ and a $\mathbf{1}_a$-preserving lifting $\bar{F} : \mathcal{E}_a \to \mathcal{E}_a$, $\bar{F}$ defines an $a$-indexed induction scheme for $\mu F$ as soon as $p_a$ admits comprehension. Therefore, in order to have an $a$-indexed induction scheme from any $\mathbf{1}_a$-preserving functor, a minimal condition to instantiate our results is to ask for any $a$ in $\mathcal{A}$ that the fibrations $p_a$ admits comprehension. Furthermore, to have a canonical indexed induction scheme for any indexed inductive types in $\mathcal{B}$, a minimal condition to instantiate our results is to ask for any $a$ in $\mathcal{A}$ that $p_a$ is a Lawvere fibration. However, the fact that no coherence conditions are required between the different comprehension structures of each $p_a$ might pose a problem at the level of types: if a fibration admits comprehension this implies that for any proof $\gamma : \mathbf{1}X \to P$ at the predicate level there is a morphism $u : X \to \{P\}$ such that $\gamma$ is above $\pi_P \circ u$. This morphism $u$ is understood as mapping any term $x$ of $X$ a proof that $P$ holds at $x$. Now, if we use the $a$-indexed inductions scheme for $\mu F$ an inductive type dependent of $a$

with a predicate $Q$ depending on $b$ through a morphism $f : a \to b$, while we would obtain a proof $\mathbf{1}\mu F \xrightarrow{\gamma} f^*Q \xrightarrow{f^{\S}} Q$, nothing guarantees the existence of an arrow $u : \mu F \to \{Q\}$, and indeed, we don't have that $p$ itself admits comprehension.

More generally, if we only ask that for any $a$ in $\mathcal{A}$, the fibrations $p_a$ admits comprehension, we don't have that $p$ admits comprehension, not even that $p$ is a bifibration. The question of the corresponding logic structure of $p$ is then not evident. On the other hand, we can not always restrict a comprehension category with unit to CCUs between the fibres, i.e., $p$ admits comprehension does not imply that $p_a$ does. We are then looking for a notion of comprehension category with unit above a fibration which restricts to the fibres. This notion already exists and is denoted a *fibred comprehension category with unit* (Definition 4.4.5 in [Jac91]):

**Definition 5.1.3.** Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be two fibrations with $\mathbf{1} : \mathcal{B} \to \mathcal{E}$ the terminal object functor of $p$. We say that $p$ *admits comprehension above* $r$ if $\mathbf{1}$ has a fibred right adjoint $\{-\} : r \circ p \to r$:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\{-\}"] \arrow[rr, phantom, "\top"] \arrow[dr, "r \circ p"'] & & \mathcal{B} \arrow[ll, bend left, "\mathbf{1}"] \arrow[dl, "r"] \\
& \mathcal{A} &
\end{tikzcd}$$

That $\mathbf{1}$ is a fibred functor from $r$ to $r \circ p$ follows from the fact that it is fibred from $id_{\mathcal{B}}$ to $p$.

A first consequence of this definition is that, if a fibration $p : \mathcal{E} \to \mathcal{B}$ admits comprehension above a fibration $r : \mathcal{B} \to \mathcal{A}$, $p$ clearly admits comprehension. If $p$ admits comprehension above $r$, we say that the CCU associated to $p$ is *fibred above* $r$. Furthermore, we have from Lemma 1.2.12 that if $p$ admits comprehension above $r$, for every $a$ in $\mathcal{A}$ the fibration $p_a$ admits comprehension. In fact we have the following correspondence:

**Lemma 5.1.4.** *Let* $p : \mathcal{E} \to \mathcal{B}$ *and* $r : \mathcal{B} \to \mathcal{A}$ *be fibrations. The following are equivalent:*

*i The fibration* $p$ *admits comprehension above* $r$

*ii For every $a$ in $\mathcal{A}$ the fibration $p_a$ admits comprehension and, for every $u : a \to a'$ in $\mathcal{A}$ and $u^* : \mathcal{B}_{a'} \to \mathcal{B}_a$, there is a $u^\# : \mathcal{E}_{a'} \to \mathcal{E}_a$ forming a morphism of CCU $p_{a'} \to p_a$.*

*iii The fibration $p$ admits comprehension with terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$ and comprehension functor $\{-\} : \mathcal{E} \to \mathcal{B}$, and for every $a$ in $\mathcal{A}$, the adjunction $\mathbf{1} \dashv \{-\}$ restricts to an adjunction $\mathbf{1}_a \dashv \{-\}_a$ such that $p$ admits comprehension with comprehension functor $\{-\}_a : \mathcal{E}_a \to \mathcal{B}_a$, the restriction of $\{-\}$.*

*Proof.* To verify that (*iii*) ⇒ (*i*), let $p$ admit comprehension with terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$ and comprehension functor $\{-\} : \mathcal{E} \to \mathcal{B}$, and let $p$ be such that for every $a$ in $\mathcal{A}$, the fibration $p_a : \mathcal{E}_a \to \mathcal{B}_a$ admits comprehension with comprehension functor $\{-\}_a : \mathcal{E}_a \to \mathcal{B}_a$ given by restricting $\{-\}$ to the corresponding fibres. First, by the dual of Proposition 1.2.32 we have that $\{-\}$ is fibred from $r \circ p$ to $r$. Then, since the adjunction $\{-\} \vdash \mathbf{1}$ restricts to the adjunctions $\{-\}_a \vdash \mathbf{1}_a$, the unit of $\{-\} \vdash \mathbf{1}$ is vertical with respect to $r$.

(*i*) ⇒ (*iii*)is straightforward.

For (*i*) ⇔ (*ii*) see Lemma 4.4.3 in [Jac91] □

This result shows that the notion of CCU above a fibration is the one we were looking for for indexed induction. We have a structure that characterises a collection of CCUs above each fibre of a fibration $r$ which, taken together, gives a CCU above the total category of $r$.

Definition 5.1.3 straightforwardly extends to Lawvere fibrations:

**Definition 5.1.5.** Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be fibrations. We say that $p$ is a *Lawvere fibration above $r$* if $p$ admits comprehension above $r$ and is a bifibration.

The next two corollaries are immediate:

**Corollary 5.1.6.** *Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be fibrations. The fibration $p$ is a Lawvere fibration above $r$ iff $p$ is a Lawvere fibration and for every $a$ in $\mathcal{A}$, the structure restricts between $\mathcal{E}_a$ and $\mathcal{B}_a$, making $p_a$ a Lawvere fibration.*

**Theorem 5.1.7.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration above $r : \mathcal{B} \to \mathcal{A}$. For any $a$ in $\mathcal{A}$ and $F : \mathcal{B}_a \to \mathcal{B}_a$, a $\mathbf{1}_a$-preserving lifting $\bar{F} : \mathcal{E}_a \to \mathcal{E}_a$ of $F$ defines a sound $a$-indexed induction scheme for $\mu F$ in $p$.*

*In particular, the canonical $\mathbf{1}_a$-preserving lifting $\widehat{F}$ defines a sound canonical $a$-indexed induction scheme for $\mu F$.*

### 5.1.1 More thought about fibred structures above a fibration

In this chapter, we started by discussing a fibration $p$ above another fibration $r$. The fact that this notation is similar to the notation of a fibration above a base category $\mathcal{B}$ is not accidental: a fibration $p : \mathcal{E} \to \mathcal{B}$ can be characterised by a collection of category $\mathcal{E}_b$ for each $b$ in $\mathcal{B}$ and reindexing functor $f^* : \mathcal{E}_b \to \mathcal{E}_a$ for each morphism $f : a \to b$ in $\mathcal{B}$ (see the equivalence between fibrations and indexed categories in [Jac99] for example). In fact we have a similar description for a fibration above a fibration, as shown in the following Lemma:

**Lemma 5.1.8.** *Consider two fibrations $q : \mathcal{E} \to \mathcal{A}$ and $r : \mathcal{B} \to \mathcal{A}$, and a fibred functor $p : q \to r$ above $\mathcal{A}$. The following statements are equivalent:*

*(i) $p$ is itself a fibration.*

*(ii) for each $a$ in $\mathcal{A}$, the restriction $p_a : \mathcal{E}_a \to \mathcal{B}_a$ of $p$ is a fibration, and for each morphism $f : a \to b$ in $\mathcal{A}$ and reindexing functor $f^* : \mathcal{B}_b \to \mathcal{B}_a$ (with regard to $r$), there is a reindexing functor $f^{\#} : \mathcal{E}_b \to \mathcal{E}_a$ (with regard to $q$) forming a fibred functor $(f^{\#}, f^*) : p_b \to p_a$.*

In fact, it is well-known ( [Jac99,Str99] for example) that the two points of the previous lemma are also equivalent to the definition of a fibration in the 2-category $\mathbf{Fib}(\mathcal{A})$ (using the definition of fibrations in 2-category due to [Str74, Str80]).

Similarly, we have the notion of fibred structure above the base category of a fibration $p$ to denote structures on the total category of $p$ that restrict to fibres and are stable under reindexing (as illustrated in Lemma 1.2.12 for example). And for $p$ a fibration above $r : \mathcal{B} \to \mathcal{A}$, we have the notion of fibred structures above $r$ to denote structures that restrict to the fibrations $p_a$ and are stable under the reindexing functor described in Lemma 5.1.8. For example, we have that a terminal object functor of $p$ implies a fibred terminal object functor, as shown in the following lemma:

**Lemma 5.1.9.** *Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be two fibrations. The fibration $p$ has a terminal object functor above $r$ iff*

- *for every $a$ in $\mathcal{A}$, $p_a$ has a terminal object functor $\mathbf{1}_a$*
- *for every $u : b \to a$ in $\mathcal{A}$ and every reindexing functor $u^* : \mathcal{B}_a \to \mathcal{B}_b$, there is a reindexing functor $u^\# : \mathcal{E}_b \to \mathcal{E}_a$ such that $u^\# \circ \mathbf{1}_a \cong \mathbf{1}_b \circ u^*$.*

*Proof.* Note that the natural isomorphism $u^\# \circ \mathbf{1}_b \cong \mathbf{1}_a \circ u^*$ is necessarily the canonical map since $\mathbf{1}_a$ is terminal, hence we can use Lemma 1.2.12 and the fact that the counit of the adjunction $\mathbf{1} \vdash p$ corresponds to the counit of the adjunctions $\mathbf{1}_a \vdash p_a$ to conclude the proof. □

The notion of fibred CCU above $r$ can also be described in this fashion, see [Jac91].

Another interesting fibred construction is the notion of arrow fibration:

**Definition 5.1.10.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration, and denote $V(E)$ for the full subcategory of $\mathcal{E}^\to$ with vertical arrows as objects. The functor $p^\to : V(\mathcal{E}) \to \mathcal{B}$, defined as $p^\to = p \circ cod = p \circ dom$, is again a fibration.

In fact, we have that the arrow fibration is the arrow object of the corresponding fibration in the 2-category **Fib**($\mathcal{B}$) (See Lemma 9.4.2 in [Jac99]).

Dually to fibration, there is also a notion of opfibration fibred above a fibration:

**Definition 5.1.11.** Let $q : \mathcal{E} \to \mathcal{A}$ and $r : \mathcal{B} \to \mathcal{A}$ be fibrations and $p : q \to r$ be a fibred functor. We say that $p$ is an *opfibration fibred above* $r$ iff

(i) for any $a$ in $\mathcal{A}$ the restriction $p_a : \mathcal{E}_a \to \mathcal{B}_a$ of $p$ is an opfibration

(ii) for any $u : b \to a$ and $Q$ in $\mathcal{E}_a$, the unique map making the following diagram commute is an isomorphism.

$$\begin{tikzcd}
u^*Q \arrow[d, "(u^\# f)_\S"'] \arrow[dr, "u^*(f^Q_\S)"] & \\
\Sigma_{u^\# f} u^*Q \arrow[r, dotted] & u^*(\Sigma_f Q)
\end{tikzcd}$$

This definition straightforwardly implies a notion of bifibration fibred above a fibration. Note then that the Definition 5.1.5 of a Lawvere fibration above a fibration does *not* correspond to the definition of a Lawvere fibration *fibred* above a fibration (since we are not asking for a bifibration fibred above a fibration but just a bifibration).

## 5.2 Examples and properties

**Example 5.2.1.** (Example 2.1.5, continued) As codomain functors provide canonical examples of CCUs, *fibred codomain functors*, i.e.,

$$\begin{tikzcd}
V(\mathcal{E}) \arrow[rr, "cod_p"] \arrow[dr, "p^\to"'] & & \mathcal{E} \arrow[dl, "p"] \\
& \mathcal{B} &
\end{tikzcd}$$

provide canonical examples of fibred CCUs. Similarly to the non-fibred case, in order for $cod_p$ to be a fibration we need to have that for any vertical morphism $\alpha : B \to A$ and arbitrary morphism $f : X \to A$ in $\mathcal{E}$ there is a pullback square

$$\begin{tikzcd}
\cdot \arrow[r] \arrow[d, "f^*\alpha"'] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "\alpha"] \\
X \arrow[r, "f"'] & A
\end{tikzcd}$$

such that $f^*\alpha$ is again vertical. It is then straightforward to check that $cod_p$ is a fibration and the following fibred adjunctions $cod_p \dashv id_p \dashv dom_p$ hold, making $cod_p$ a fibred CCU above $p$.

Note then that we can then associate to any category $\mathcal{B}$ with pullbacks the following fibred CCU:

$$\begin{tikzcd}
V(\mathcal{B}^{\to}) \arrow[rr, bend left, "dom_{cod}"] \arrow[rr, bend right, "cod_{cod}"'] \arrow[rr, phantom, shift left=2, "\top"] \arrow[rr, phantom, shift right=2, "\top"] \arrow[dr, "cod^{\to}"'] & & \mathcal{B}^{\to} \arrow[ll, "id_{cod}" description] \arrow[dl, "cod"] \\
& \mathcal{B} &
\end{tikzcd}$$

**Example 5.2.2.** (Example 2.1.2 continued) Consider a fibration $r : \mathcal{B} \to \mathcal{A}$ that captures a theory of indexed types. More specifically, $\mathcal{A}$ is a category of indices, and $\mathcal{B}$ a category of indexed types where each type $X$ has as index $rX$, and types are closed under reindexing. Furthermore, any term $x : X \vdash t : Y$ above a morphism of index $\alpha : I \to J$ is equivalent to a term $x : X \vdash t' : \alpha^* Y$ which is then said to be of index $I$. For example, think about a set-indexed type theory captured by a family fibration $\mathrm{Fam}(\mathcal{C}) \to \mathrm{Set}$ (see Example 1.1.7).

Consider now a predicate logic above the theory given by a fibration $p : \mathcal{E} \to \mathcal{B}$. Then $p$ admits comprehension above $r$ if for every predicate $P$ above a type $X$ of index $I$ there is a comprehension type $\{P\}$ (see Example 2.1.2) of index $I$, and comprehension types are stable under reindexing, i.e., for any morphism $u : J \to I$ in $\mathcal{A}$, $u^*\{P\} \cong \{u^*P\}$ (or equivalently the associated correspondence $\begin{array}{c}(y : Y \vdash 1 : \mathit{Prop}) \to (x : X \vdash P : \mathit{Prop}) \\ \hline\hline Y \to \{P\} : \mathit{Type}\end{array}$ restricts to a specific index).

If we have in addition that the fibration $p$ is a bifibration (see Example 1.1.5 for the corresponding logical structure) for $I$ in $\mathcal{A}$, $X$ in $\mathcal{B}_I$, $P$ in $\mathcal{E}_X$ and $F : \mathcal{B}_I \to \mathcal{B}_I$, we can describe the canonical lifting of $F$ at $P$. Using that $\widehat{F}\,P = \Sigma_{F(\pi P)} 1 F\{P\}$, this definition captures the predicate of index $I$

$$x : F\,X \vdash \exists a : F\{P\}.\ (F(\pi P))\,a = x : \mathit{Prop}$$

Of course, the term $(F(\pi P))a$ depends on the definition of $F$ and the structure of $\mathcal{E}$. We then have that a $\widehat{F}$-algebra of carrier $P$ above an algebra $\alpha : FX \to X$ is given by the following entailment of index $I$:

$$x' : FX,\ \exists a : F\{P\}.\ (F(\pi P))\,a = x \vdash P[x \leftarrow (\alpha x')]$$

The induction scheme provides then the following rule, where everything is of the same index:

$$\frac{\begin{array}{l} x : X \vdash P : \mathit{Prop} \\ x : FX \vdash \alpha : X \\ x' : FX,\ \exists a : F\{P\}.\ (F(\pi P))\, a = x \vdash P[x \leftarrow (\alpha x')] \end{array}}{x' : \mu F \vdash P[x \leftarrow (\!|\alpha|\!) x']}$$

The next example specialises the previous syntactic model to the case where the indexed type theory is a dependent type theory:

**Example 5.2.3.** (Example 2.1.2 continued) In this example we consider a predicate logic over a dependent type theory[1]. We know from Example 2.1.2 that a dependent type theory is captured by a fibration $r : \mathcal{B} \to \mathcal{A}$ that admits comprehension. We then have $\mathcal{A}$ a category of dependent type context $\Gamma$ and $\mathcal{B}$ a category of dependent types in context $\Gamma \vdash t : \mathit{Type}$. A predicate over this type theory is understood as a proposition in a dependent type context $\Gamma \vdash P : \mathit{Prop}$. Similarly to Example 1.1.5, such a predicate logic is captured by another fibration $p : \mathcal{E} \to \mathcal{A}$.

We can find back a setting similar to the one describe in the previous example by setting predicates above a type $X$ in $\mathcal{B}_\Gamma$ to be the predicates in $\mathcal{E}_{\{X\}}$. This construction is described more generally in Example 5.2.4. If we have in addition that the fibration $p$ is a bifibration, we can describe canonical liftings in this setting using the description made in previous example: For $F : \mathcal{B}_\Gamma \to \mathcal{B}_\Gamma$ and $P$ in $\mathcal{B}_\Gamma$, $\widehat{F}P$ captures the following predicate

$$\Gamma,\ x : F\,X \vdash \exists a : F\{P\}.\ (F(\pi P))a = x : \mathit{Prop}$$

Hence a $\widehat{F}$-algebra of carrier $P$ above an algebra $\alpha : FX \to X$ is given by the following entailment:

$$\Gamma,\ x' : FX,\ \exists a : F\{P\}.\ (F(\pi P))\, a = x \vdash P[x \leftarrow (\alpha x')]$$

---

[1] or *dependent predicate logic*, see 11.1 in [Jac99]

The induction scheme provides then the following logical rule:

$$\frac{\begin{array}{l}\Gamma,\ x : X \vdash P : \mathit{Prop} \\ \Gamma,\ x : FX \vdash \alpha : X \\ \Gamma,\ x' : FX,\ \exists a : F\{P\}.\ (F(\pi P))\, a \!=\! x \vdash P[x \leftarrow (\alpha x')]\end{array}}{\Gamma,\ x' : \mu F \vdash P[x \leftarrow (\!|\alpha|\!) x']}$$

We start with an example that looks at another possible setting in which we can study indexed induction and then discuss how it relates to the one we considered for Definition 5.1.2.

**Example 5.2.4.** We will now take a look at a class of settings that are more common from the perspective of dependent type theory: consider two fibrations $p : \mathcal{E} \to \mathcal{A}$ and $r : \mathcal{B} \to \mathcal{A}$ where $r$ admits comprehension with comprehension functor $\{-\}_r : \mathcal{B} \to \mathcal{A}$ and $p$ has a terminal object functor $\mathbf{1}_p : \mathcal{A} \to \mathcal{E}$. The idea is that $r$ captures a dependent type theory and $p$ a logic, i.e., $\mathcal{A}$ is thought of as a category of (dependent) contexts, $\mathcal{B}$ is thought of as a category of types in context and $\mathcal{E}$ as a category of propositions in (type) context. In order to express induction schemes for inductive dependent types, i.e. initial algebra of endofunctors $F : \mathcal{B}_a \to \mathcal{B}_a$ for some $a$ in $\mathcal{A}$, consider the fibration $p' = \{-\}_r^* p$ above $r$ where $\{-\}_r$ is the comprehension functor of $r$. Diagrammatically we have:

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B} \arrow[r, "\{-\}_r"'] \arrow[d, "r"'] & \mathcal{A} \\
\mathcal{A} &
\end{tikzcd}$$

Remember (Example 2.1.7) that the comprehension functor of a fibration of dependent types performs context extension, that is, for $B$ a type in context $\Gamma$, $\{B\}_r$ is the context $\Gamma, B$. Therefore, a predicate in $p'$ above $B$ is a proposition in the context $\Gamma, B$ and the terminal object functor $\mathbf{1}' : \mathcal{B} \to \mathcal{E}'$ maps the type $B$ to the true proposition in context $\Gamma, B$. We now have a setting in which we can apply our definition of indexed induction schemes. The question of the existence of such schemes remains open since we do not know if $\mathbf{1}'$ has a fibred right adjoint (by Lemma 2.1.10, a sufficient condition is for the functor $\{-\}_r$ to be a fibration). Note that the existence of such an adjoint $\{-\}' : \mathcal{E}' \to \mathcal{B}$ would correspond to the presence of a notion of *dependent comprehension type* in $\mathcal{B}$ (as a straightforward generalisation of Example 2.1.2). Indeed, by using the internal language of the structure the adjunction is characterised by a correspondence between the pair of a term $\Gamma, B \vdash t : B'$ (in $\mathcal{B}$) and a proof $\Gamma, B \vdash p : t^*P$ (in $\mathcal{E}$) and a term $\Gamma, B \vdash a : \{P\}_r$ (in $\mathcal{B}$).

As mentioned in the introduction, structure similar to this one have been investigated in [Jac99], in particular with the notion of DPL-structure where the fibration $p$ describes a (complete) dependent type theory and the fibration $r$ describes a higher order (proof irrelevant) predicate logic. The notion of dependent comprehension type restricts in this case to a notion of *dependent subset-type* (see Definition 11.2.3 in [Jac99]). We then have that any DPL-structure with $p$, a bifibration, and with dependent subset types has arbitrary indexed induction schemes.

The next example shows how we can take any Lawvere fibration whose base category has pullbacks and associate it to a fibred fibration obtained though the method described in the previous example.

**Example 5.2.5.** Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration of predicates above types where $\mathcal{B}$ has pullbacks. We know from Example 2.1.5 that there is in fact already enough information to speak about dependent types in this setting using the codomain fibration. We then found ourselves in a setting similar to the one in Example 5.2.4: a fibration of predicates $p : \mathcal{E} \to \mathcal{B}$ and a fibration of dependent types $cod : \mathcal{B}^{\to} \to \mathcal{B}$. The point of this example is that in this particular situation $\mathcal{B}^{\to}$ has dependent comprehension types, and hence, arbitrary indexed induction schemes.

To see this, let $q : \mathcal{E}' \to \mathcal{B}^{\to}$ be the fibration above *cod* obtained from the change of base of $p$ along the comprehension functor of *cod*. See the following diagram:

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}^{\rightarrow} \arrow[r, "dom"'] \arrow[d, "cod"'] & \mathcal{B} \\
\mathcal{B} &
\end{tikzcd}$$

We know that *dom* is also a fibration (Example 1.1.9), and hence by Corollary 2.1.16, that $q$ is a Lawvere fibration. Now bear in mind that for any $I$ in $\mathcal{B}$ the fibration $q_I$ can be obtained with the following change of base:

$$\begin{tikzcd}
\mathcal{E}/I \arrow[r] \arrow[d, "q_I"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}' \arrow[d, "q"] \\
\mathcal{B}/I \arrow[r, "i_I"'] & \mathcal{B}^{\rightarrow}
\end{tikzcd}$$

Here, $i_I$ is the inclusion functor of the slice category into the arrow category. Now, since the composition $\mathcal{B}/I \xrightarrow{i_I} \mathcal{B}^{\rightarrow} \xrightarrow{dom} \mathcal{B}$ is equal to the functor $dom_I : \mathcal{B}/I \to \mathcal{B}$, which is straightforwardly a fibration, we have by Corollary 2.1.16 that for any $I$ in $\mathcal{B}$ the fibration $q_I$ is a Lawvere fibration. Hence, by Corollary 5.1.6 $q$ is a Lawvere fibration above *cod*.

We conclude this chapter with two preservation properties of change of base for fibred CCUs and fibred Lawvere fibrations, as well as a preservation property with composition.

**Lemma 5.2.6.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration (admits comprehension) above $r : \mathcal{B} \to \mathcal{A}$ and let $q : \mathcal{B}' \to \mathcal{B}$ be a fibration. Let $p' : \mathcal{E}' \to \mathcal{B}'$ be the fibration obtained by change of base of $p$ along $q$. In a diagram:*

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}' \arrow[r, "q"] \arrow[dr, "r \circ q"'] & \mathcal{B} \arrow[d, "r"] \\
& \mathcal{A}
\end{tikzcd}$$

*The fibration $p'$ is a Lawvere fibration (resp., admits comprehension) above $r \circ q$.*

*Proof.* First, by Lemma 2.1.10 we know that $p'$ admits comprehension. Then, remark that for any $a$ in $\mathcal{A}$ we have $p'_a = (q_a)^*(p_a)$, hence $p'_a$ also admits comprehension. We conclude with Lemma 5.1.4. It is straightforward to extend this proof to cover Lawvere fibrations. ☐

**Lemma 5.2.7.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration (admits comprehension) above $r : \mathcal{B} \to \mathcal{A}$ and let $s : \mathcal{A}' \to \mathcal{A}$ be a functor. We then have the following situation:*

$$\begin{tikzcd}
\mathcal{E}' \arrow[r] \arrow[d, "p'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}' \arrow[r, "r^*s"] \arrow[d, "r'"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{B} \arrow[d, "r"] \\
\mathcal{A}' \arrow[r, "s"'] & \mathcal{A}
\end{tikzcd}$$

*Here, the fibration $p'$ is a Lawvere fibration (resp., admits comprehension) above the fibration $r'$.*

*Proof.* Indeed, since change of base preserves fibred adjunction we have that

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\{-\}"] \arrow[rr, bend right, "p"'] \arrow[rr, phantom, shift left=2, "\top"] \arrow[rr, phantom, shift right=2, "\top"'] \arrow[dr, "r \circ p"'] & & \mathcal{B} \arrow[ll, "\mathbf{1}" description] \arrow[dl, "r"] \\
& \mathcal{A} &
\end{tikzcd}
\quad \text{implies} \quad
\begin{tikzcd}
\mathcal{E}' \arrow[rr, bend left, "\{-\}'"] \arrow[rr, bend right, "p'"'] \arrow[rr, phantom, shift left=2, "\top"] \arrow[rr, phantom, shift right=2, "\top"'] \arrow[dr, "r' \circ p'"'] & & \mathcal{B}' \arrow[ll, "\mathbf{1}'" description] \arrow[dl, "r'"] \\
& \mathcal{A}' &
\end{tikzcd}$$

☐

**Lemma 5.2.8.** *Let $p : \mathcal{E} \to \mathcal{B}$ be a Lawvere fibration (admits comprehension) above $r : \mathcal{B} \to \mathcal{A}$ and let $t : \mathcal{A} \to \mathcal{D}$ be fibration. The fibration $p$ is a Lawvere fibration (resp., admits comprehension) above $t \circ r$.*

*Proof.* Indeed, we have the following straightforward implication:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\{-\}"] \arrow[rr, phantom, "\top"] \arrow[dr, "r\circ p"'] & & \mathcal{B} \arrow[ll, bend left, "\mathbf{1}"] \arrow[dl, "r"] \\
& \mathcal{A} \arrow[d, "t"] & \\
& \mathcal{D} &
\end{tikzcd}
\quad \text{implies} \quad
\begin{tikzcd}
\mathcal{E} \arrow[rr, bend left, "\{-\}"] \arrow[rr, phantom, "\top"] \arrow[ddr, "t\circ r\circ p"'] & & \mathcal{B} \arrow[ll, bend left, "\mathbf{1}"] \arrow[ddl, "t\circ r"] \\
& & \\
& \mathcal{D} &
\end{tikzcd}$$

□

# Chapter 6

# Indexed coinduction

In this chapter we derive coinduction schemes for coinductive indexed types. Examples of such types are infinitary versions of inductive indexed types, such as infinitary untyped lambda terms and the interaction structures of Hancock and Hyvernat [HH06]. As for indexed induction, we will first introduce the definition of indexed coinduction in fibrations. We will then look at how to index the notion of QCE in order to derive a setting admitting coinduction scheme for arbitrary indexed coinductive types. We conclude this chapter by looking at examples and properties of this new setting.

## 6.1 The setting

As in Chapter 5, the setting that we are considering is given by a fibred fibration: we have a fibration $r : \mathcal{B} \to \mathcal{A}$, where we think of an object of $\mathcal{B}$ as a type indexed by an object of $\mathcal{A}$, and a fibration $p : \mathcal{E} \to \mathcal{B}$ where we think of the objects of $\mathcal{E}$ as relations above the indexed types. Our aim is to investigate sound coinduction schemes for final coalgebras of functors $F : \mathcal{B}_a \to \mathcal{B}_a$, where $a$ is any object of $\mathcal{A}$.

Before giving a definition of indexed coinduction in this setting, first note that by Lemma 1.2.23 a (full) section of the fibration $p$ restricts to a (full) section of the fibrations $p_a$. Therefore, for any $a$ in $\mathcal{A}$, any proof of the form $R \to e_a X$ in $\mathcal{E}_a$ is the same as a proof $R \to eR$ in $\mathcal{E}$ above $a$.

We can then adapt our definition of coinduction schemes in fibrations to this new setting with the following definition:

**Definition 6.1.1.** Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with full section $e : \mathcal{B} \to \mathcal{E}$ and $r : \mathcal{B} \to \mathcal{A}$ be a fibration. For $a$ in $\mathcal{A}$ and $F : \mathcal{B}_a \to \mathcal{B}_a$ a functor, we say that a $e_a$-preserving lifting $\bar{F} : \mathcal{E}_a \to \mathcal{E}_a$ of $F$ *defines an a-indexed coinduction scheme for* $\nu F$ *in* $p$ if the functor *e-CoAlg* : $CoAlg_F \to CoAlg_{\bar{F}}$ that sends an $F$-coalgebra $X \xrightarrow{\alpha} FX$ to the $\bar{F}$-coalgebra $eX \xrightarrow{e\alpha} eFX \cong \bar{F}eX$ preserves the terminal object.

Equivalently, the functor $\bar{F}$ defines an $a$-indexed coinduction scheme for $\nu F$ in $p$ if it defines a coinduction scheme for $\nu F$ in $p_a$.

As for induction, the definition of indexed coinduction schemes generalises Definition 4.3.8 by setting $r$ to be the trivial fibration from $\mathcal{B}$ to the terminal category.

As with the non-indexed definition, the coinduction scheme is given by the anamorphisms associated to the final coalgebra of $\bar{F}$. We observe a difference with the indexed inductive case for $\bar{F} : \mathcal{E}_a \to \mathcal{E}_a$, for $R$ in $\mathcal{E}_b$ and a morphism $f : a \to b$ in $\mathcal{A}$: If we use the reindexing to apply the coinduction scheme to $f^*R$ we can't deduce a proof of the form $R \to e\nu F$ above $f$ since we are in a situation where the coinduction scheme gives a morphism $\gamma : f^*R \to e\nu F$, while the cartesian lifting of $f$ goes from $f^*R$ to $R$. To obtain such a proof we can ask that $r \circ p$ is an opfibration and consider $R \xrightarrow{f_\S} \Sigma_f R \xrightarrow{\gamma} R$.

Let $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a full section $e : \mathcal{B} \to \mathcal{E}$ and $r : \mathcal{B} \to \mathcal{A}$ be a fibration. We now look for settings that provide indexed coinduction schemes for arbitrary indexed coinductive types. We know from Chapter 4 that in order to have arbitrary indexed coinduction schemes we need that for any $a$ in $\mathcal{A}$ the fibration $p_a : \mathcal{E}_a \to \mathcal{B}_a$ admits quotients. As in Chapter 5, we want to highlight the uniformity connecting the different fibrations $p_a$ but requiring that each fibration $p_a$ admits quotients does not automatically imply that $p$ admits quotients. However, by contrast with the situation in the inductive case, if $p$ has a full section $e : \mathcal{B} \to \mathcal{E}$ fibred with regard to $r$, requiring that each fibration $p_a$ admits $e_a$-quotients does ensure that $p$ admits quotients. Indeed, we have the following result:

**Lemma 6.1.2.** *Let* $r : \mathcal{B} \to \mathcal{A}$ *be a fibration, and* $q : \mathcal{E} \to \mathcal{A}$ *and* $e : \mathcal{B} \to \mathcal{E}$ *be functors such that* $q \circ e = r$. *The functor* $e$ *has a left adjoint* $Q : \mathcal{E} \to \mathcal{B}$

*with vertical unit (resp., counit) iff $e$ preserves cartesian morphisms and for each object $a$ in $\mathcal{A}$ the restriction $e_a : \mathcal{B}_a \to \mathcal{E}_a$ of $e$ to the fibres has a left adjoint $Q_a$.*

*Proof.* Let us assume a collection of left adjoint $Q_a : \mathcal{E}_a \to \mathcal{B}_a$ to the restriction $e_a : \mathcal{B}_a \to \mathcal{E}_a$ of $e$. The proof then follows the proof of Lemma 1.8.9 of [Jac99]: we will prove that for each $a$ in $\mathcal{A}$ and $R$ in $\mathcal{E}_a$ the unit component $\eta_R : R \to eQ_aR$ is a universal map from $R$ to $e$ (and not just to $e_a$). Let us assume a morphism $l : R \to eY$ above $h : a \to b$, we then have $l = e(h^{\S}_Y) \circ u$ for a unique vertical morphism $u : R \to e(h^*Y)$ using the fact that $e(h^{\S}_Y)$ is cartesian. Now, since $u$ is in $\mathcal{E}_a$ we can use the universal property of $\eta_R$ to deduce a unique morphism $g : Q_aR \to h^*Y$ in $\mathcal{B}_a$ such that $u = eg \circ \eta_R$. Therefore, we have a unique morphism $f = h^{\S}_Y \circ g$ such that $l = ef \circ \eta_R$.

Conversely assume a functor $Q$, left adjoint to $e$ with vertical unit. We directly obtain a collection of adjunctions between the fibres and that $e$ preserves cartesian morphisms from Proposition 1.2.32. □

This Lemma can be compared with Lemma 1.2.12.

We then deduce the following setting.

**Definition 6.1.3.** Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be two fibrations with $e : \mathcal{B} \to \mathcal{E}$ a full section of $p$. We say that $p$ *admits quotients above* $r$ if either of the following holds:

(i) $e$ has a left adjoint $Q : \mathcal{E} \to \mathcal{B}$ with unit (or equivalently counit) vertical with regard to $r$.

(ii) $e$ is fibred above $r$ and, for any $a$ in $\mathcal{A}$, $p_a$ admits $e_a$-quotients.

Lemma 6.1.2 ensures that the two points are equivalent.

This definition corresponds to the definition of fibred comprehension in the sense that it gives us a structure that characterises a collection of QCEs above each fibre of a fibration $r$ which, taken together, gives a QCE above the total category of $r$. We would however not call it a fibred QCE since the definition does not imply stability under reindexing of the quotients. If we want quotients to be stable under reindexing we need to consider the following definition:

**Definition 6.1.4.** Let $p : \mathcal{E} \to \mathcal{B}$ and $r : \mathcal{B} \to \mathcal{A}$ be two fibrations with $e : \mathcal{B} \to \mathcal{E}$ a full section of $p$. We say that $p$ *admits fibred quotients above* $r$ if $e$ has a left adjoint $Q : \mathcal{E} \to \mathcal{B}$ and the adjunction is fibred above $\mathcal{A}$:

$$\begin{tikzcd}
\mathcal{E} \arrow[rr, bend right, "Q"'] \arrow[rr, phantom, "\top"] \arrow[dr, "r \circ p"'] & & \mathcal{B} \arrow[ll, bend right, "e"'] \arrow[dl, "r"] \\
& \mathcal{A} &
\end{tikzcd}$$

Note that if $p$ admits fibred quotients above $r$, $p$ trivially admits quotients above $r$.

We then have the following corollary:

**Corollary 6.1.5.** *Let $p : \mathcal{E} \to \mathcal{B}$ admits e-quotients above $r : \mathcal{B} \to \mathcal{A}$. For any $a$ in $\mathcal{A}$ and $F : \mathcal{B}_a \to \mathcal{B}_a$, a $e_a$-preserving lifting $\bar{F} : \mathcal{E}_a \to \mathcal{E}_a$ of $F$ defines an $a$-indexed coinduction scheme for $\nu F$ in $p$.*

*In particular, the canonical $e_a$-preserving lifting $\check{F}$ defines a canonical $a$-indexed coinduction scheme for $\nu F$.*

## 6.2 Examples and properties

**Example 6.2.1.** (Example 2.2.3, continued) As domain functors provide canonical examples of QCEs, *fibred domain functors*, i.e.,

$$\begin{tikzcd}
V(\mathcal{E}) \arrow[rr, "dom_p"] \arrow[dr, "p^{\to}"'] & & \mathcal{E} \arrow[dl, "p"] \\
& \mathcal{B} &
\end{tikzcd}$$

provide canonical examples of fibred QCEs. Similarly to the non-fibred case $dom_p$ is systematically a fibration, indeed reindexing is then given by the following diagram:

$$\begin{tikzcd}
X \arrow[r, "f"] \arrow[d, dotted, "f^*\alpha"'] & A \arrow[d, "\alpha"] \\
(pf)^*B \arrow[r, "(pf)^{\S}"'] & B
\end{tikzcd}$$

It is then straightforward to check that the following fibred adjunctions $cod_p \dashv id_p \dashv dom_p$ holds, making $dom_p$ a fibred QCE above $p$.

Note that we can then associate to any category $\mathcal{B}$ a fibred QCE:

$$\begin{tikzcd}
V(\mathcal{B}^{\to}) \arrow[rr, bend left, "dom_{dom}"] \arrow[rr, bend right, "cod_{dom}"'] \arrow[rr, phantom, shift left=2.5, "\top"] \arrow[rr, phantom, shift right=2.5, "\top"] \arrow[dr, "dom^{\to}"'] & & \mathcal{B}^{\to} \arrow[ll, "id_{dom}" description] \arrow[dl, "dom"] \\
& \mathcal{B} &
\end{tikzcd}$$

We start the section with an example which describes a construction similar to the family fibration (Example 1.1.7), this time however we index a whole fibration and present a way to index it by a category which is not necessarily Set.

**Example 6.2.2.** Let $\mathcal{C}$ be a category with a terminal object 1, and $p : \mathcal{E} \to \mathcal{B}$ be a fibration with a full section $e : \mathcal{B} \to \mathcal{E}$ and quotient functor $Q : \mathcal{E} \to \mathcal{B}$. For $X$ an object of $\mathcal{C}$, write $GX$ for the discrete category whose objects are the morphisms $x : 1 \to X$ in $\mathcal{C}$. Remember from Example 1.1.5 that if $\mathcal{C}$ is seen as a category of types, the morphisms from 1 to $X$ represent the closed terms of type $A$, i.e., $GX$ is to be understood as the discrete category of closed terms of type $X$ (also if $\mathcal{C}$ is Set, $GX \cong X$ for any set $X$).

Let $\mathrm{Fam}_{\mathcal{C}}(\mathcal{E})$ be the category with objects, pairs $(X, P)$ where $X$ is an object of $\mathcal{C}$ and $P$ a functor from $GX$ to $\mathcal{E}$. A morphism of $\mathrm{Fam}_{\mathcal{C}}(\mathcal{E})$ from $(X, P)$ to $(Y, Q)$ is a pair $(f, f^{\sim})$ with $f : X \to Y$ a morphism in $\mathcal{C}$ and $f^{\sim}$ a collection of morphisms $f^{\sim}_x : Px \to P(f \circ x)$ in $\mathcal{E}$, for every $1 \xrightarrow{x} X$ in $GX$ (or equivalently, $f^{\sim}$ is a natural transformation from $P$ to $P \circ f$ where $f$ is seen as a functor from $GX$ to $GY$). Let $\mathrm{Fam}_{\mathcal{C}}(\mathcal{B})$ be the category obtained with a similar construction on $\mathcal{B}$.

Consider the functor $p' : \mathrm{Fam}_{\mathcal{C}}(\mathcal{E}) \to \mathrm{Fam}_{\mathcal{C}}(\mathcal{B})$ that maps an object $(X, P)$ to $(X, p \circ P)$ and a morphism $(f, f^{\sim})$ to $(f, p \circ f^{\sim})$. The functor $p'$ is a fibration:

For a $(X, P)$ in $\mathrm{Fam}_{\mathcal{C}}(\mathcal{E})$ and $(f, f^{\sim}) : (Y, A) \to (X, p \circ P)$ a morphism in $\mathrm{Fam}_{\mathcal{C}}(\mathcal{B})$, define $(f, f^{\sim})^*(X, P)$ to be the object $(Y, P')$ in $\mathrm{Fam}_{\mathcal{C}}(\mathcal{E})_{(Y,A)}$ where $P' : GY \to \mathcal{E}$ maps $y$ to $(f^{\sim}_y)^* P(f \circ y)$. Define then the cartesian lifting $(f, f^{\sim})^{\S}_{(X,P)}$ to be the morphism $(f, l) : (Y, P') \to (X, P)$ with $l_y = (f^{\sim}_y)^{\S}$.

Consider the functor $r : \mathrm{Fam}_{\mathcal{C}}(\mathcal{B}) \to \mathcal{C}$ that maps an object $(X, A)$ to $X$. The functor $r$ is a fibration: for $(X, A)$ in $\mathrm{Fam}_{\mathcal{C}}(\mathcal{B})$ and $f : Y \to X$ in $\mathcal{C}$, define $f^*(X, A)$ as the pair $(Y, A \circ f)$. The cartesian lifting $f^{\S}_{(X,A)}$ is given by the pair $(f, f')$ with $f'_y = id_{A(f \circ y)}$, for every $y$ in $GY$. Note that since every cartesian morphism $(f, f^{\sim})$ is isomorphic to a cartesian lifting, all the morphisms of the collection $f^{\sim}$ are necessarily isomorphisms ($f^{\sim}$ is a natural isomorphism).

The fibration $p'$ has a full section $e' : \mathrm{Fam}_{\mathcal{C}}(\mathcal{B}) \to \mathrm{Fam}_{\mathcal{C}}(\mathcal{E})$ fibred above $\mathcal{C}$. The section $e'$ is given by $e'\,(X, A) = (X, e \circ A)$. For a cartesian morphism $(h, h^{\sim}) : (Y, B) \to (X, A)$ above $h$, since we have that $h^{\sim}_y$ is an isomorphism for every $y$ in $GY$ then $e(h^{\sim}_y)$ is an isomorphism above $h^{\sim}_y$, and since every isomorphism is a cartesian morphism then $e(h^{\sim}_y)$ is cartesian above $h^{\sim}_y$. This makes $e'(h, h^{\sim})$ cartesian above $h$. Note that this argument can be used to lift any functor between $\mathcal{E}$ and $\mathcal{B}$ to a fibred functor between $\mathrm{Fam}_{\mathcal{C}}(\mathcal{E})$ and $\mathrm{Fam}_{\mathcal{C}}(\mathcal{B})$.

Finally, we can define a fibred quotient functor above $r$ for $p'$ to be $Q' : \mathrm{Fam}_{\mathcal{C}}(\mathcal{E}) \to \mathrm{Fam}_{\mathcal{C}}(\mathcal{B})$ given by $Q'(X, P) = (X, Q \circ P)$. Indeed, $Q'$ is fibred using the argument in the previous paragraph, and we have $Q' \dashv e'$ since for any $X$ in $\mathcal{C}$, the following adjunction holds $Q'_X \dashv e'_X$ by a pointwise construction and by Lemma 6.1.2.

Notice that it is straightforward to index a CCU with the method presented in this example, however, indexing a Lawvere fibration poses a problem when indexing the opreindexing structure.

The next example generalises the construction of a fibration of relations presented in Example 1.1.11 for fibred fibrations. As for the non-indexed case, we can derive a fibred fibration of indexed relations from a fibration of indexed predicates in the following way:

**Example 6.2.3.** (Example 2.2.4, continued) Let $p : \mathcal{E} \to \mathcal{B}$ be a bifibration with a terminal object functor $\mathbf{1} : \mathcal{B} \to \mathcal{E}$ that satisfies the Beck-Chevalley condition, and let $r : \mathcal{B} \to \mathcal{A}$ be a fibration with fibred cartesian products, i.e., products in the fibres preserved by reindexing. Let $\Delta_r : r \to r$ be the fibred diagonal functor sending an object $X$ to $X \times X$ in each fibres. The fibration of indexed relations (above $r$) $Rel_r(p) : Rel_r(\mathcal{E}) \to \mathcal{B}$ is obtained by a change of base of $p$ along $\Delta_r$.

It comes equipped with a full and faithful equality functor $Eq_r : \mathcal{B} \to Rel_r(\mathcal{E})$, mapping an object $X$ of $\mathcal{B}$ to $\Sigma_{\delta_r}\mathbf{1}X$, where $\delta_r : Id_{\mathcal{B}_r} \to \Delta_r$ is the diagonal fibred natural transformation (full and faithfulness comes from Lemma 1.1.16).

Furthermore, the equality functor $Eq_r$ is fibred from $r$ to $r \circ Rel_r(p)$: First, note that for any cartesian natural transformation $\eta : F \to G$ and cartesian morphism $l$ the naturality square $\eta_l$ is a pullback square (it is in fact straightforward to show that any square with two parallel vertical morphisms and two parallel cartesian morphisms is a pullback). Then since $\delta_r$ is a fibred natural transformation we have that for $B$ an object of $\mathcal{B}$ and $f : X \to rB$ in $\mathcal{A}$, the following square is a pullback square in $\mathcal{B}$

$$\begin{tikzcd}
f^*A \arrow[r, "f^\S"] \arrow[d, "\delta_r f^*A"'] \arrow[dr, phantom, very near start, "\lrcorner"] & A \arrow[d, "\delta_r A"] \\
f^*A \times f^*A \arrow[r, "f^\S"'] & A \times A
\end{tikzcd}$$

Then, from the Beck-Chevalley condition we have that for any $P$ in $\mathcal{E}$ above $A$, $\Sigma_{\delta_r f^*A} f^*P \cong f^*\Sigma_{\delta_r A}P$. Hence, we have the following canonical isomorphism:

$$\begin{aligned}
Eq_r(f^*A) &= \Sigma_{\delta_r f^*A}\mathbf{1}f^*X \\
&\cong \Sigma_{\delta_r f^*A} f^*\mathbf{1}X \\
&\cong f^*\Sigma_{\delta_r A}\mathbf{1}X \\
&= f^*(Eq_r A)
\end{aligned}$$

We then have that $Rel_r(p)$ has arbitrary coinduction schemes as soon as $Eq_r$ has a left adjoint, i.e., as soon as $Rel_r(p)$ admits (fibred) $Eq_r$-quotients.

**Example 6.2.4.** (Example 5.2.2, continued) Consider a logic of predicates $p : \mathcal{E} \to \mathcal{B}$ above a theory of indexed types $r : \mathcal{B} \to \mathcal{A}$ as described in Example 5.2.2. Then $Rel(p)$ admits quotient above $r$ if for every relation $R$ above a type $X$ of index $I$ there is a quotient type $QR$ (See Example 2.2.4) of index $I$, and quotient are stable under reindexing, i.e., for any morphism $u : J \to I$ in $\mathcal{A}$, $u^*QR \cong Q(u^*R)$ (or equivalently the associated correspondence $\begin{array}{c} (y, y' : Y \vdash R : \mathit{Prop}) \to (x, x' : X \vdash Eq\,X : \mathit{Prop}) \\ \hline\hline QR \to X : \mathit{Type} \end{array}$ restricts to a specific index).

Remembering Example 5.2.2, we can then describe the lifting $\check{F}$ of a functor $F : \mathcal{B}_\Gamma \to \mathcal{B}_\Gamma$ with the following relation

$$\Gamma,\, x, x' : F\,X \;\vdash Eq_{F(QR)}\, \big((F(\rho R))x, (F(\rho R))x'\big)$$

A $\check{F}$-coalgebra of carrier $R$ above the a coalgebra $\alpha : X \to FX$ is then given by the following entailment:

$$\Gamma,\, x, x' : X,\; R\;(x, x') \vdash Eq_{F(QR)}\, \big((F(\rho R))(\alpha\, x), (F(\rho R))(\alpha\, x')\big)$$

The coinduction scheme provides then the following logical rule:

$$\frac{\begin{array}{l}\Gamma,\, x, x' : X \vdash R : \mathit{Prop}\\ \Gamma,\, x : X \vdash \alpha : FX\\ \Gamma,\, x, x' : X,\; R\;(x, x') \vdash Eq_{F(QR)}\, \big((F(\rho R))(\alpha\, x), (F(\rho R))(\alpha\, x')\big)\end{array}}{\Gamma,\, x, x' : X,\; R\;(x, x') \vdash Eq_{\nu F}([\![\alpha)\!]x, [\![\alpha)\!]x')}$$

The next example combines the results from Chapter 5 on indexed induction with Example 6.2.3.

**Example 6.2.5.** (Example 5.2.4, continued) Assume $p : \mathcal{E} \to \mathcal{A}$ and $r : \mathcal{B} \to \mathcal{A}$, two fibrations of predicates and types respectively. Furthermore, assume that $r$ admits comprehension with comprehension functor $\{-\} : \mathcal{B} \to \mathcal{A}$ and has fibred products, and $p$ is a bifibration that satisfies the Beck-Chevalley condition and has a terminal object functor $\mathbf{1} : \mathcal{A} \to \mathcal{E}$. We can then use the method presented in Example 5.2.4 to derive a fibred fibration from this setting and then apply the method of Example 6.2.3 to derive a fibration of relations fibred above $r$. Diagrammatically we have:

$$\begin{tikzcd}
Rel_r(\mathcal{E}') \arrow[r] \arrow[d, "Rel_r(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B} \arrow[r, "\Delta"'] \arrow[dr, "r"'] & \mathcal{B} \arrow[r, "\{-\}"'] \arrow[d, "r"'] & \mathcal{A} \\
 & \mathcal{A} &
\end{tikzcd}$$

With a fibred equality functor $Eq_r : r \to r \circ Rel_r(q)$. (We implicitly used the fact that change of base along a pullback-preserving functor preserves the Beck-Chevalley condition, and $\{-\}$, as a right adjoint is such a functor).

We then have a setting to which we can apply our definition of indexed coinduction schemes. Again, the question of the existence of such schemes remains open. Also, note that the existence of fibred quotients above $r$, given by a functor $Q_r : rq \to r$ would correspond to the presence of a notion of a *dependent quotient type* in $\mathcal{B}$. Indeed, using the internal language of the structure, the adjunction $Q_r \dashv Eq_r$ is characterised by a correspondence between the pair of a term $\Gamma, B \vdash t : B'$ in $\mathcal{B}$ and a proof $\Gamma, (x : B), (y : B), R\,x\,y \vdash Eq_r\,tx\,ty$, and a term $\Gamma, a : Q_r B \vdash u : B'$.

Again, here the notion of DPL-structure gives us an instance of the current setting. The DPL-structures also have a corresponding notion of dependent quotient types (Definition 11.2.5 in [Jac99]). We then have that any DPL-structure with dependent quotient types has arbitrary indexed coinduction schemes.

We will now look at a construction that is similar to Example 5.2.5 and see how we can associate any QCE above a base category with pullbacks to a QCE above the codomain fibration. As this construction turned out to be complex, we present this next example as a section of this chapter.

### 6.2.1 Lifting a QCE above the codomain fibration

Consider a bifibration $p : \mathcal{E} \to \mathcal{B}$ with quotients, i.e., let $\mathcal{B}$ have products and $p$ be a bifibration with terminal object functor $\mathbf{1}$, such that the equality functor $Eq : \mathcal{B} \to Rel(\mathcal{E})$ (given by $Eq = \Sigma_\delta \mathbf{1}$) has a left adjoint $Q : \mathcal{B} \to Rel(\mathcal{E})$. Additionally, we assume that $\mathcal{B}$ has pullbacks and $p$ satisfies the Beck-Chevalley condition. In this section we will show that the fibration $Rel_{cod}(q)$ above the codomain fibration, as described in the following diagram, admits arbitrary indexed coinduction schemes.

$$\begin{tikzcd}
Rel_{cod}(\mathcal{E}') \arrow[r] \arrow[d, "Rel_{cod}(q)"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E}' \arrow[r] \arrow[d, "q"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{E} \arrow[d, "p"] \\
\mathcal{B}^{\to} \arrow[r, "\Delta^{\to}"'] \arrow[dr, "cod"'] & \mathcal{B}^{\to} \arrow[r, "dom"'] \arrow[d, "cod"'] & \mathcal{B} \\
 & \mathcal{B} &
\end{tikzcd}$$

Here, since products are given by pullbacks in slices of $\mathcal{B}$, the fibred functor $\Delta^{\to}$ maps an object $f : X \to I$ in $\mathcal{B}^{\to}$ to the composition $f \circ i = f \circ j$ where $i$ and $j$ are the projections of the pullback of $f$ by itself. We will note $f^2$ for $\Delta^{\to} f$ and $X_f X$ for the domain of $f^2$.

In order to develop some intuitions on the fibration $Rel_{cod}(q)$, note that in Set the object $X_f X$ corresponds to the subset of $X \times X$, $\{(x, x') \in X \times X \mid f\,x = f\,x'\}$. Then, since $f$ represents the family $f^{-1} : I \to$ Set of elements of $X$ indexed by $I$, an indexed relation $R$ on $f$ corresponds to a relation on $X$ that only compares elements with the same index, i.e., a family of relations $R_i$ on $f^{-1} i$ for $i$ in $I$.

We now fix an object $I$ of $\mathcal{B}$ and consider the fibration $q_I : \mathcal{E}'_I \to \mathcal{B}/I$ and its fibration of relations $Rel(q_I) : Rel(\mathcal{E}'_I) \to \mathcal{B}/I$. A first remark is that $Rel(q_I) = Rel_{cod}(q)_I$ and $Rel(\mathcal{E}'_I) = Rel_{cod}(\mathcal{E}')_I$. Concretely, an object of $Rel(\mathcal{E}'_I)$ above $f : X \to I$ is an object of $\mathcal{E}'_I$ above $f^2$ with respect to $q_I$. This is, in turn, equivalent to an object $R$ of $\mathcal{E}$ above $X_f X$ with respect to $p$. Also, note that $q_I$ is obtained by change of base of $p$ along $dom_I : \mathcal{B}/I \to \mathcal{B}$, hence we have that $q_I$ is a bifibration and has a terminal object functor $\mathbf{1}_I : \mathcal{B}/I \to \mathcal{E}'_I$. Therefore, $q_I$ has an equality functor $Eq_{q_I} : \mathcal{B}/I \to Rel(\mathcal{E}'_I)$ defined as $Eq_{q_I} = \Sigma_{\delta_I} \mathbf{1}_I$. Concretely, note that the component of the diagonal natural transformation $\delta_I : Id \to \Delta^{\to}_I$ at $f : X \to I$ is given by the following diagram on the left. Thus, $Eq_{q_I}$ maps an object $f : X \to I$ of $\mathcal{B}/I$ to the unique morphism above $f^2$ in the diagram on the right induced by the opcartesian map $m$ above $(\delta_I)_f$:

$$\begin{tikzcd}
X \arrow[rrd, bend left, "id"] \arrow[rdd, bend right, "id"'] \arrow[rd, dotted, "(\delta_I)_f"'] & & \\
& X_fX \arrow[r, "j"] \arrow[d, "i"'] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
& X \arrow[r, "f"'] & I
\end{tikzcd}
\qquad
\begin{tikzcd}
& \mathbf{1}I \\
\mathbf{1}X \arrow[ur, "\mathbf{1}f"] \arrow[r, "m"'] & \Sigma_{(\delta_I)_f}\mathbf{1}X \arrow[u, dotted, "Eq_{q_I}f"']
\end{tikzcd}$$

Or, considering an object of $Rel(\mathcal{E}'_I)$ above $f : X \to I$ as an element of $\mathcal{E}$ above $X_fX$, the functor $Eq_{q_I} : B/I \to Rel(\mathcal{E}_I)$ is defined as $Eq_{q_I}f = \Sigma_{(\delta_I)_f}\mathbf{1}(domf)$.

We now need to link back the equality $Eq_{q_I} : \mathcal{B}/I \to Rel(\mathcal{E}'_I)$ of $q_I$ to the equality $Eq : \mathcal{B} \to Rel(\mathcal{E})$ of $p$ so that we can use the adjunction $Eq \vdash Q$ to derive the quotient functor for $q_I$. In order to do this, note that for $f : X \to I$ in $\mathcal{B}$, $i$ and $j$ the projections for the pullback square defining $X_fX$, there is a morphism $v_f$ given by

$$\begin{tikzcd}
& X_fX \arrow[dl, "i"'] \arrow[dr, "j"] \arrow[d, dotted, "v_f"'] & \\
X & X \times X \arrow[l, "\pi_1"'] \arrow[r, "\pi_2"] & X
\end{tikzcd}$$

It is routine to check that $v_f$ is the equalizer of the parallel arrows $f\circ\pi_1, f\circ\pi_2 : X\times X \to I$, and that it extends to a natural transformation $v : dom\circ\Delta_I^{\to} \stackrel{.}{\to} \Delta\circ dom$.

We can then prove the following lemma.

**Lemma 6.2.6.** *For $f : X \to I$ in $\mathcal{B}$, $Eq_{q_I}f \cong (v_f)^*Eq\,X$ in $\mathcal{E}$.*

*Proof.* First, since $v_f$ is the equalizer of $f\circ\pi_1$ and $f\circ\pi_2$, we have that $\delta_X = v_f\,(\delta_I)_f$ and that $v_f$ is mono. Since $v_f$ is mono its pullback along itself is given by the identity on $X$, hence by the Beck-Chevalley condition, the unit $\eta'$ of the adjunction $\Sigma_{v_f} \dashv (v_f)^*$ is an isomorphism. We can then deduce that $Eq_{q_I}(f : X \to I) = \Sigma_{\delta_I}\mathbf{1}X \cong (v_f)^*\Sigma_{v_f}\Sigma_{\delta_I}\mathbf{1}X \cong (v_f)^*\Sigma_\delta\mathbf{1}X = (v_f)^*EqX$. □

Now, notice that for any $f : X \to I$ in $\mathcal{B}$, the following diagram

$$\begin{tikzcd}
X_f X \arrow[r, "v_f"] \arrow[d, "f^2"'] & X \times X \arrow[d, "f \times f"] \\
I \arrow[r, "\delta_I"'] & I \times I
\end{tikzcd}$$

commutes by the universal property of the product $I \times I$ since $\pi_n \circ f \times f \circ v_f = f^2 = \pi_n \circ \delta_I \circ f^2$ for $n \in \{1; 2\}$. We can then define for any object $R$ above $f$ in $\mathit{Rel}(\mathcal{E}'_I)$ a morphism $h : \Sigma_{v_f} R \to Eq\, I$ above $f \times f$ in $\mathcal{E}$ by the universal property of the opcartesian morphism $(v_f)_\S$:

$$\begin{tikzcd}
R \arrow[r, "(v_f)_\S^R"] \arrow[d, "!R \circ \mathbf{1} f^2"'] & \Sigma_{v_f} R \arrow[d, dotted, "h"] \\
\mathbf{1} I \arrow[r, "(\delta_I)_\S^{\mathbf{1} I}"'] & Eq\, I
\end{tikzcd}$$

From this we can deduce that:

**Theorem 6.2.7.** *Let* $\Phi$ *be the isomorphism associated to the adjunction* $Eq \vdash Q$*, we then have the following adjunction* $Eq_{q_I} \vdash \Phi h$.

*Proof.* Let $R$ be an element of $\mathit{Rel}(\mathcal{E}'_I)$ above $f : X \to I$, let $g : Y \to I$ be in $\mathcal{B}/I$ and for $\alpha : f \to g$ in $\mathcal{B}/I$ write $\bar{\alpha}$ for $\mathit{dom}(\Delta_I^{\to} \alpha) : X_f X \to Y_g Y$. We then have:

$$\begin{aligned}
\mathit{Rel}(\mathcal{E}'_I)(R, Eq_{q_I} g) &= \bigcup_{\alpha : f \to g} \mathcal{E}_{\bar{\alpha}}(R, {v_g}^* EqY) && (1)\\
&\cong \bigcup_{\alpha : f \to g} \mathcal{E}_{X_f X}(R, \bar{\alpha}^* {v_g}^* EqY) && (2)\\
&\cong \bigcup_{\alpha : f \to g} \mathcal{E}_{Y_g Y}(\Sigma_{v_g} \Sigma_{\bar{\alpha}} R, EqY) && (3)\\
&\cong \bigcup_{\alpha : f \to g} \mathcal{E}_{Y_g Y}(\Sigma_{\alpha \times \alpha} \Sigma_{v_f} R, EqY) && (4)\\
&\cong \bigcup_{\alpha : f \to g} \mathcal{E}_{\alpha \times \alpha}(\Sigma_{v_f} R, EqY) && (5)\\
&\cong \mathit{Rel}(\mathcal{E})/EqI(h, Eqg) && (6)\\
&\cong B/I(\Phi h, g) && (7)
\end{aligned}$$

Here, (1) comes from the definition of $Rel(\mathcal{E}'_I)$ and $Eq_{q_I}$ (2) and (5) from the universal property of (op)cartesian morphisms, (3) from the adjunctions $\bar{\alpha}^* \vdash \Sigma_{\bar{\alpha}}$ and ${v_f}^* \vdash \Sigma_{v_f}$ (4) from the fact that the following square $$\begin{tikzcd} X_f X \arrow[r, "\bar{\alpha}"] \arrow[d, "v_f"'] & Y_g Y \arrow[d, "v_g"] \\ X{\times}X \arrow[r, "\alpha\times\alpha"'] & Y{\times}Y \end{tikzcd}$$ commutes (since $v$ is a natural transformation), (6) from the fact that for any morphism $\gamma : R \to Eq_{q_I} g$ in $Rel_{cod}(\mathcal{E}')_I$ above $\beta : f \to g$ we have the following commuting diagram in $\mathcal{E}$: $$\begin{tikzcd} R \arrow[rrr, "v_{f_\S}"] \arrow[dr, "\gamma"] \arrow[dd] & & & \Sigma_{v_f} R \arrow[dl, dotted, "l"'] \arrow[dd, dotted, "h"] \\ & Eq_{q_I} g \arrow[r, "{v_g}^\S"'] \arrow[dl] & EqY \arrow[dr, "Eqg"'] & \\ \mathbf{1}I \arrow[rrr, "\delta_\S"'] & & & EqI \end{tikzcd}$$ where the left triangle commutes by definition of morphism in $Rel(\mathcal{E}'_I)$. The square on the top commutes by the universal property of $v_{f_\S}$ where $l$ is the image of $\gamma$ through the process (1)-(5). The square on the bottom commutes since ${v_g}^\S$ is opcartesian (since $\Sigma_{v_g}$ is full and faithful) and $Eq$ is full and faithful. The outer square commutes by construction of $h$. The left triangle commutes by unicity of $h$ and (7) by naturality of $\Phi$. □

Hence, defining $Q_{q_I}$ to send $R$ to $\Phi h$ makes $Rel(q_I) : Rel(\mathcal{E}'_I) \to \mathcal{B}/I$ a QCE.

Now that we know that for any $I$ in $\mathcal{B}$ $Rel(q_I)$ admits quotients, we can conclude with the following lemma:

**Lemma 6.2.8.** *The fibration of indexed relation* $Rel_{cod}(q) : Rel_{cod}(\mathcal{E}') \to \mathcal{B}^\to$ *of* $q : \mathcal{E}' \to \mathcal{B}^\to$ *admits quotients above cod.*

*Proof.* Notice that the equality functor $Eq^\to : \mathcal{B}^\to \to Rel_{cod}(\mathcal{E}')$ defined as $\Sigma_{\delta^\to} K^\to$ restricts to the $Eq_{q_I}$ since $\delta_I$ is the restriction of $\delta^\to$ to the corresponding fibres. Thus, the conclusion follows from Lemma 6.1.2. □

# Chapter 7

# Conclusions

We have investigated sound induction and coinduction schemes in category theory. We started with the work of Hermida and Jacobs [Her93, HJ98] which combines initial algebra semantics of inductive types (resp., final coalgebra semantics of coinductive types) and the theory of fibration to model a logic above a type theory. Our aim was to maximise the use of the abstraction power of initial algebra (resp., final coalgebra) semantics and the theory of fibration, in order to cover as many logics, type theories and classes of inductive (resp., coinductive) types as possible. We showed how the Lawvere fibrations provide a minimal structure to associate any functor on its base category to a terminal object preserving lifting and hence to guarantee sound induction schemes for any inductive types. We introduced the notion of quotient category with equality, a generalisation of the property of having quotients for a fibration, and showed how it provides a minimal structure to associate any functor on its base category to a section preserving functor and hence to guarantee sound coinduction schemes for any coinductive types. We then investigated a version of Lawvere fibration and QCE above a fibration in order to derive sound indexed induction and coinduction schemes for indexed inductive and coinductive types respectively.

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