

A Categorical Approach to the Foundations of Quantum Theory

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Abstract

In this work we pursue two goals: the first is to bridge a gap between various projects within the field of categorical quantum theory, namely: topos quantum theory as initiated by Butterfield and Isham; monoidal quantum theory, as initiated by Abramsky and Coecke; and the sheaf theoretic approach to contextuality and non-locality, as initiated by Abramsky and Brandenburger. We show connections between these projects on the level of mathematical formalisms, and also on the level of physical interpretations. The central element is a generalisation of the topos theoretic structures of Butterfield and Isham, which incorporates the categorical structures considered within monoidal quantum theory, resulting in structures that can be embedded into the general sheaf-formalism of Abramsky and Brandenburger.

The second main thrust of this work is in providing a framework within which to consider the foundations of quantum theory, specifically within the context of a pragmatic metaphysical interpretation. This is done using similar mathematical structures as topos quantum theory, but using quite different metaphysical presuppositions. In particular, we consider presheaves on posets of commutative subsemialgebras which can be naturally associated with a physical system as represented in categorical quantum mechanics. The main distinction between our work and topos quantum theory is that we do not pursue a realist metaphysical interpretation of these structures. Rather we pursue a pragmatic interpretation of metaphysics which is consistent with the sheaf-formalism of Abramsky and Brandenburger

From this categorical perspective, we consider some of the fundamental features of quantum foundations, including a derivation of the Born rule, contextuality/non-locality, the meaning of quantum probabilities, and model toy quantum-like theories. We also present a new outlook on quantum metaphysics which seeks to lend clarity to some of the conceptual problems which exist in quantum foundations and which draws a connection with the Bohrification programme of Heunen, Spitters and Landsman.

Declaration

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

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Signed:

Date:

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Chapter 1

Introduction

Quantum foundations is known for being abstract, obscure, and of little relevance to experimental physics, whereas category theory is known for being abstract, obscure, and of little relevance to applied mathematics. Presented with “a categorical approach to the foundations of quantum theory” one might anticipate an exponential speed-up in abstractness, obscurity, and irrelevance in one’s mathematical physics, however, on the contrary, the central premise guiding the development of this work is that the language and techniques of category theory can lend clarity to the conceptual issues of quantum foundations.

While they may not be of concern to experimental physicists, it is our belief that quantum foundations are undoubtedly of practical importance for physics, particularly in reconciling quantum theory with other highly successful physical theories, most notably, general relativity. A better understanding of the foundations of quantum theory will play a key role in the development of quantum computation and quantum information.

As for the question of why we should use category theory, we refer to a description by Leinster [150, p. 1]:

Category theory takes a bird’s eye view of mathematics. From high in the sky, details become invisible, but we can spot patterns that were impossible to detect from ground level.

The “patterns” which a categorical perspective expose will give a new insight into some of the conceptual issues of quantum theory. The tendency for a categorical perspective to dissolve the details of the target subject has led to its reputation as ‘abstract nonsense’, and our work is related to a particularly infamous branch of category theory, described by Johnstone [128, p. xvi] as follows:

The average mathematician, who regards category theory as “generalized abstract nonsense”, tends to regard topos theory as generalized abstract category theory.

However, despite these dangers of excessive abstraction, our reformulation of quantum foundations is rooted in a deeply pragmatic approach to quantum theory. The categorical framework we develop is more general than the traditional Hilbert space formulation of quantum theory, but it is certainly not the most general level on which we could present our results, and we certainly have no interest in abstractness for the sake of abstractness. Guiding our entire approach are the words of Mac Lane, one of the founders of category theory [154, p. 108]:

good general theory does not search for the maximum generality, but for the right generality.

While the framework we develop is strictly more general than the Hilbert space formalism, we aim to achieve the “right generality”: we want to be able to introduce and exploit new mathematical structures and techniques, but we do not want to lose relevance to the traditional Hilbert space formulation of quantum theory.

We use *categorical quantum theory* as an umbrella term that encompasses any formulation of quantum theory that uses the language of category theory. The framework we present is closely related to three existing projects within categorical quantum theory, namely: *topos quantum theory*, as initiated by Butterfield and Isham [124]; *monoidal quantum theory*, initiated by Abramsky and Coecke [7]; and the sheaf-theoretic formulation of *abstract empirical models*

due to Abramsky and Brandenburger [5]. While all three of these projects fall within categorical quantum theory, each focuses on a different aspect of quantum theory, and makes use of quite different mathematical structures and techniques.

The application of topos theory to quantum theory goes back to Isham [122], while *topos quantum theory* in the form we use was initiated in a series of papers by Isham and Butterfield [124, 33, 100, 34], and developed further by Doering and Isham [63]. We use the monograph of Flori [82] as our main reference for topos quantum theory. The principal aim of topos quantum theory is to resolve the fundamental conceptual issues of quantum theory, in particular questions of ontology and realism. The topos quantum theory of Butterfield, Isham and Doering is closely related to another project based on the language of topos theory, namely *Bohrification*, due to Heunen, Spitters and Landsman [114, 116, 115].

Monoidal quantum theory, introduced by Abramsky and Coecke [7], applies the theory of symmetric monoidal categories and their associated graphical calculi to quantum theory. Monoidal quantum theory can be viewed as a diagrammatic theory of *abstract processes* [47]. In contrast with topos quantum theory, the monoidal approach is far less concerned with conceptual issues, but rather aims to provide a powerful and intuitive calculus with which to reason about quantum systems.

The sheaf-theoretic *abstract empirical models* of Abramsky and Brandenburger [5] provide a general theory of systems of measurements or observations. This framework provides a setting in which one studies the empirical data from experimental observations as a mathematical object. Empirical models provide a language in which the concepts of contextuality and non-locality can be formulated and analysed on an abstract level which subsumes, but is vastly more general than, quantum theory.

We bridge the gap between these three formulations of quantum theory by introducing the theory of *abstract spectral presheaves*. This framework is defined in terms of *commutative \mathbb{S}^* -semialgebras* which generalise commutative C^* -algebras. Figure 1.1 gives a rough schematic of the relationship between our

spectral presheaf framework and the three formalisms discussed above.

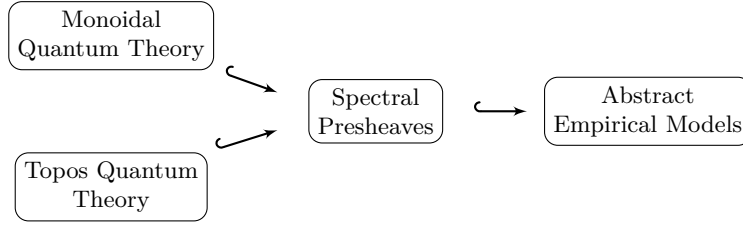


Figure 1.1: Relationships between categorical formalisms.

The spectral presheaf framework directly generalises topos quantum theory in such a way as to incorporate the structures typically considered within monoidal quantum theory; it can be seen as taking the “ingredients” of monoidal quantum theory but following the “recipe” of topos quantum theory. However, beyond some initial definitions, the framework we present makes a clear departure from the topos quantum theory of Butterfield and Isham. This divergence in mathematical formalism stems from underlying differences in the interpretation of quantum theory; we do not pursue anything resembling a *realist* interpretation of quantum theory, which is the central aim of topos quantum theory [63, §2.2.3], but instead, we view quantum theory in purely pragmatic terms as an *operational* theory which only describes the observed outcomes of measurements. Adhering strictly to a pragmatic interpretation of quantum theory will allow us to make the connection with the sheaf-theoretic empirical models of Abramsky and Brandenburger.

We use the spectral presheaf framework as a setting in which to formulate the foundations of quantum theory. In Figure 1.2 we list several phenomena and features of quantum theory that any reasonable foundations of quantum theory ought to account for.

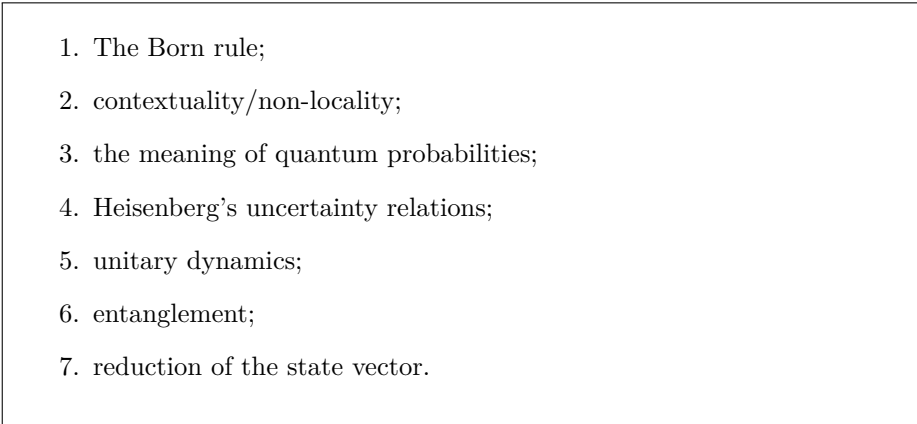
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1. The Born rule;
 2. contextuality/non-locality;
 3. the meaning of quantum probabilities;
 4. Heisenberg's uncertainty relations;
 5. unitary dynamics;
 6. entanglement;
 7. reduction of the state vector.

Figure 1.2: *Some of the fundamental aspects of quantum theory.*

We do not claim that Figure 1.2 is an exhaustive list of features relevant to the foundations of quantum theory; these are just the phenomena that we touch upon within our framework. Even if this list were exhaustive, there is more to quantum foundations than just giving an account of the phenomena we list in Figure 1.2. In particular, we understand the term “foundation” in the sense expressed by Lawvere [144, p. 235]:

A foundation makes explicit the essential general features, ingredients, and operations of a science as well as its origins and general laws for development. The purpose of making these explicit is to provide a guide for learning, use, and further development of the science. A “pure” foundation that forgets this purpose and pursues a speculative “foundation” for its own sake is clearly a nonfoundation.

Our formulation of quantum foundations serves to guide the use and development of physics, and we will describe two specific ways in which it does so:

1. our framework provides a unified setting in which to formalise quantum theory alongside various *foil theories* or *toy theories*;

2. our framework provides insight into the relationship between some of the fundamental conceptual problems of quantum theory.

We will expand each of these points in turn. A foil theory is a hypothetical theory not intended as a viable competitor to quantum theory in the sense of being an alternative candidate for explaining reality, but rather is created purely to contrast with quantum theory to highlight distinctive aspects of quantum theory [39, pp. 4-5]. The philosophy behind using foil theories to tease apart features of quantum theory is expressed by Jennings and Leifer [127, p. 2]:

If a phenomenon of quantum physics also occurs within a classical statistical physics setting, perhaps with minor additional assumptions that don't violently clash with our everyday conceptions, then it should not be viewed as an intrinsically quantum mechanical phenomenon.

Foil theories are typically presented in a somewhat informal way, and hence any comparison with quantum theory will not be fully rigorous. Having a single, unified mathematical framework in which to present quantum theory alongside its various foil theories allows us to make the comparisons between these theories precise mathematical statements. Our framework is general enough to incorporate quantum theory and foil theories, in particular we give a comprehensive account of *Spekkens' Toy Theory* [198] in Chapter 7. The generality afforded by the language of category theory plays an essential role in the versatility of the framework.

Another way in which our formulation of quantum foundations contributes to the development of the quantum theory is in how it elucidates the relationships between some of the fundamental conceptual issues of quantum theory. Isham describes four fundamental conceptual issues of quantum theory [123, Chap. 8], which we list in Figure 1.3:

1. The meaning of probability.
2. The role of measurement.
3. The reduction of the state vector.
4. Quantum entanglement.

Figure 1.3: *The fundamental conceptual problems of quantum theory, according to Isham.*

Isham describes these fundamental conceptual problems as follows [123, p. 150]:

These four problems are tightly linked together, with each depending strongly on the other three. The explicit manifestation, and general significance, of each varies greatly according to the overall interpretive scheme that is being adopted.

Our framework takes no preconceived stance on any of these foundational issues. In the next section, we outline the pragmatic interpretation of quantum theory which we adhere to. Such an interpretation makes no claims about the meaning of probabilities, or the role of measurement. We remain completely agnostic regarding the reduction of the state vector, and we make minimal assumptions about composite systems, using the tensor product very sparingly in our proofs, and never mentioning entanglement explicitly. By remaining neutral with respect to these conceptual issues we are in a position to learn something about how they relate to one another. We will illustrate this with an explicit example. Talking about the four fundamental problems, Isham asserts the following [123, p. 219]:

The central issue in all this is really the phenomenon of quantum entanglement, and its striking contrast with the reductionist concepts of Western philosophy.

Within our framework we will reformulate two recent results which can be interpreted to say something about the meaning of the probabilities that appear in quantum theory, namely, the Colbeck-Renner theorem [51], and the PBR theorem [185]. These results show that an epistemic view of quantum probabilities is untenable – that is, one cannot view the probabilities as reflecting a ‘lack of knowledge’ of an objective underlying state. There is good reason to believe that entanglement plays a fundamental role in these results as the original proofs of both the Colbeck-Renner theorem and the PBR theorem hinged on an argument based on the tensor product of Hilbert spaces. However, our reformulation of these theorems makes no use of composite systems or entanglement except in recovering the case for two-dimensional Hilbert spaces. For every other dimension, infinite and even nonseparable systems, the tensor product plays no role whatsoever. Hence, within our framework one can begin to probe just how ‘tightly linked’ these four fundamental problems are, and examine to what extent they depend on one another.

Our perspective on the foundations of quantum mechanics makes no attempt to provide *the* definitive account of foundations, or close the book once and for all on the old questions of quantum theory. Rather, we aim to provide a practical framework within which new and old questions surrounding quantum theory can be articulated and rendered transparent.

The practical and versatile nature of our framework is the result of two conscious choices we have made: our choice of underlying mathematics; and our choice of overarching metaphysics. The mathematical formalism we develop is highly flexible and expressive, and we only make a handful of minimal pragmatic metaphysical assumptions.

1.1 The Role of Metaphysics

When considering foundational aspects of physics, the distinction between physics and metaphysics becomes important. We take our understanding of the fundamental role of physics from Bohr [181, p. 12]:

It is wrong to think that the task of physics is to find out how nature is. Physics concerns what we can say about nature.

According to Bohr, questions about what *is* do not belong to physics, but rather, belong to metaphysics, and in particular, belong the branch of metaphysics called *ontology*. For most branches of physics this distinction between what is, and what can in principle be known is of no practical importance, however, in quantum theory this distinction is brought into sharp focus.

A physical theory must be equipped with an overarching metaphysical interpretation. Quantum theory is perhaps unique among the major branches of physics in the lack of consensus of how to interpret its fundamental elements. Bohr's interpretation of quantum theory can be characterised as a *pragmatic interpretation* of quantum theory. Isham gives a set of minimal criteria for an interpretation of quantum theory to qualify as a pragmatic interpretation which we show in Figure 1.4:

1. Quantum theory is viewed as a scheme for predicting the probabilistic distribution of the outcomes of *measurements* made on suitably-prepared copies of a system.
2. The probabilities are interpreted in a statistical way as referring to the *relative frequencies* with which various results are obtained if the measurements are repeated a sufficiently large number of times.
3. No claims are made about whether the invocation of the concept of 'measurement', or the emphasis placed on the relative frequency interpretation of probability, are to be regarded as *fundamental* ingredients in the theory, or if they are merely a pragmatic reflection of what physicists actually do when carrying out their professional duties. In particular, nothing is said about whether the system 'possesses' values for the physical quantities concerned before the measurements are made.

Figure 1.4: *Minimum criteria for a pragmatic interpretation of quantum theory [123, p. 80].*

The pragmatic interpretation is the one adopted by most practising physicists, and the one we adopt also. However, the pragmatic interpretation is insufficient for a comprehensive foundations of quantum theory. As noted by Isham [123, p. 80]:

With its reluctance to address issues like the status of physical quantities, the [pragmatic] approach cannot be regarded as a full *interpretation* of the quantum formalism. On the other hand, it is undoubtedly the view adopted by many practicing physicists.

There are countless possible full interpretations of quantum theory, but here we consider just two extreme positions, namely: an *instrumentalist interpretation*, and a *realist interpretation*.

According to Isham the minimal criteria for an interpretation of quantum theory to qualify as an *instrumentalist interpretation* or an *anti-realist interpretation* are those shown in Figure 1.5.

1. The notion of an individual physical system ‘having’ or ‘possessing’ values for all its physical quantities is *inappropriate* in the context of quantum theory.
2. The concept of ‘measurement’ is fundamental in the sense that the scope of quantum theory is *intrinsically* restricted to predicting the probabilistic spread of results of repeated measurements made on systems that have been prepared in precisely specified ways.
3. The relative-frequency interpretation of probability is a *fundamental* ingredient of quantum theory. In particular, the spread in the results of measurements on identically-prepared systems must *not* be interpreted as reflecting a ‘lack of knowledge’ of some objectively existing state of affairs.

Figure 1.5: *Minimum criteria for an instrumentalist interpretation of quantum theory [123, p. 81].*

Hawking advocates a pragmatic or instrumentalist approach to physics in the following assertion [105, p. 121]:

I don't demand that a theory correspond to reality because I don't know what it is. Reality is not a quality you can test with litmus paper. All I'm concerned with is that the theory should predict the results of measurements.

Pragmatic and instrumentalist interpretations of quantum theory, such as that of Bohr or Hawking, have been criticised for simply ignoring the deeper questions of quantum theory. Mermin characterised the Copenhagen interpretation of quantum theory, of which Bohr was a principle founder, by the slogan "shut up and calculate" [161, p. 9], suggesting that the pragmatic and instrumentalist approaches interpretations of physics reject or avoid metaphysics and the question of meaning altogether. However, Penrose responds to the above statement of Hawking as follows [178, p. 785]:

My own position, on the other hand, is that the issue of ontology is crucial to quantum mechanics, though it raises some matters that are far from being resolved at the present time.

To view quantum theory as an ontological theory requires some form of *realist interpretation* of quantum theory. In Figure 1.6 we outline Isham's criteria for a realist interpretation of quantum theory.

1. It *is* appropriate in quantum theory to say that an individual system possesses values for its physical quantities. In this context, ‘appropriate’ signifies that propositions of this type can be handled using standard propositional logic.
2. Quantum theory is a framework for predicting the probabilistic distribution of these possessed values: it is *not* just a theory of the results of measurements. In particular, the concept of ‘measurement’ plays no *fundamental* role in the theory.
3. Quantum-theoretical probabilities can be interpreted as a reflection of our lack of *knowledge* of what is actually the case: this is the so-called *epistemic* interpretation of probability.

Figure 1.6: *Minimum criteria for a realist interpretation of quantum theory [123, p. 82].*

The many no-go theorems of quantum theory cast serious doubt on the plausibility of a realist interpretation of quantum theory, at least in the sense described in Figure 1.6.

Topos quantum theory of Butterfield, Doering and Isham seeks to revive realism in the form of *neo-realism* [68, 79], in order to address the issues of ontology in quantum theory, explicitly taking on Heidegger’s [107] question: what is a thing? The following quote of Heidegger appears at the beginning of [63]:

From the range of the basic questions of metaphysics we shall here ask this one question: “What is a thing?” The question is quite old. What remains ever new about it is merely that it must be asked again and again.

The neo-realist interpretation of quantum theory modifies each of the criteria in Figure 1.6. The ambition of topos quantum theory and neo-realism is to provide a precise way to make statements or propositions about the properties of a quantum system, and then to formulate a working logic with which to reason about these statements, in a way that the *quantum logic* of von Neumann and

Birkhoff was largely unsuccessful [23]. We will go into neo-realism in greater depth in Chapter 2.

The main difference between our formalism and topos quantum theory is that topos quantum theory sets as its end goal a neo-realist interpretation of quantum theory. Our perspective, however, takes the pragmatic interpretation of quantum theory as our *starting* point, and we let the mathematics unfold itself, recovering familiar structures like density operators, or well-known results like Bell's theorem along the way. We now want to argue that these two approaches need not be seen as incompatible, and that a comprehensive understanding of quantum theory could incorporate both aspects. This is discussed at length in Chapter 9.

Relative to full interpretations like realism or instrumentalism, the incomplete pragmatic interpretation is characterised by Isham as follows [123, pp. 80-81]:

In many respects [the pragmatic interpretation] can be thought of as the 'safe', fall-back position: moving towards a full-blown interpretation then resembles looking over a parapet towards the enemy lines whilst reserving the option to duck one's head at the first signs of fire!

By adopting the pragmatic position we do not want to give the reader the impression that we are some kind of conscientious objector, or that we are in the business of avoiding controversy; our position on quantum metaphysics will be fleshed out into a fully fledged interpretation, however, we begin with only the pragmatic assumptions of Figure 1.4, and subsequently derive an instrumentalist interpretation from the formalism.

We summarise our approach to metaphysics by referring to a quote of Mermin discussing the Kochen-Specker theorem [163, p. 1] and we cut out the details to make a general template:

Of course elementary metaphysics insists [X]; the point of [theorem Y] is to extract [X] directly from the quantum-mechanical formalism, rather than merely appealing to precepts enunciated by the founders.

We will view this quote of Mermin in its original context when we consider the Kochen-Specker theorem in Chapter 5. In the original context of the quote, “phenomenon X” is a statement about the contextual nature of quantum theory, and “theorem Y” is “the Kochen-Specker theorem”, but this provides a template for our entire approach to metaphysics: we want to extract metaphysical truth from a formalism based on a set of minimal metaphysical assumptions.

In particular, we will extract a fundamentally instrumentalist interpretation of quantum theory, as described in Figure 1.5, from a formalism based on the pragmatic interpretation described in Figure 1.4.

We make the minimum of strictly pragmatic metaphysical assumptions, and beyond this we are only interested in what metaphysical truths can be extracted from the formalism. There are obvious advantages to deriving metaphysical truths as mathematical theorems, rather than relying on the metaphysical intuition, as our intuition has proven to be an extremely unreliable tool for understanding the universe at the atomic scale, for example, the EPR paradox [77] and Schrödinger’s cat thought-experiment [189] were both presented as intuitive arguments *against* the established formulation of quantum theory. Bohr rejected such arguments based on intuition: when Einstein made the assertion that “God does not play dice with the universe” [78, p. 88], Bohr is reported to have responded that we “ought not speak for what Providence can or cannot do”, [197, p. 84].

For this reason – the inadequacy of intuition – we advocate that the physicist’s relationship with metaphysics should be like that between the mathematician and metamathematics; the mathematician is concerned with metamathematics only insofar as he can prove theorems about metamathematics. Similarly, the physicist should only engage with metaphysics insofar as he can prove theorems about metaphysics. In other words, our entire approach to metaphysics can be summarised: shut up and calculate!

1.2 The Role of Mathematics

We characterise mathematics as the thing one ends up doing when trying to talk about something in a completely precise way. Hence we give the following definition of mathematics:

Mathematics is the precise use of language.

We derive this conceptualisation of mathematics from the view of Grothendieck who emphasises the position of language in mathematics [98], see Zalamea [216, pp. 152-153] for an English translation:

The structure of a [mathematical] thing is not in any way something we can ‘invent’ [rather] we are lead constantly to *‘invent’ the language* that can express, ever more finely, the intimate structure of the mathematical thing.

Mathematics is surely the study of ‘mathematical things’, and hence, according to Grothendieck, mathematics is really about ‘inventing’ new languages. For example: the desire to talk about symmetry in a precise way leads one to invent group theory; trying to talk about computation in a precise way leads one to invent Turing machines; talking about motion leads one to invent differential calculus. This common theme in the development of mathematical ideas is not limited to applied mathematics, for example: there is an obvious difference between a torus and a sphere – the torus ‘has a hole’ whereas the sphere does not. In order to make this a precise statement one is led to invent the language of homology.

Even category theory follows this general pattern, according to Leinster [150, p. 9]:

it was the desire to formalize the notion of natural transformation that led to the birth of category theory. By the early 1940’s, researchers in algebraic topology had started to use the phrase ‘natural transformation’, but only in an informal way. Two mathematicians,

Samuel Eilenberg and Saunders Mac Lane, saw that a precise definition was needed. But before they could define natural transformation, they had to define functor; and before they could define functor, they had to define category. And so the subject was born.

Our definition of mathematics is a pragmatic one, reflecting the way mathematics is *done* by mathematicians, so it will make a suitable companion for our pragmatic interpretation of physics. If we take Bohr seriously and view physics as the study of what we can *say* about nature, then we cannot overstate the importance of language. Moreover, if we insist on saying things in a precise way, then we see how mathematics becomes an essential part of physics. From this perspective, the following maxim of Wittgenstein [212, p. 74] takes on an almost literal truth:

The limits of my language mean the limits of my world.

There are two complementary aspects to any field in mathematics: there is the material aspect, the definitions and core theorems; and there is the cultural aspect, a style of thinking, which complements those definitions. The culture of a mathematical discipline reflects the way that problems are formulated, and the kind of reasoning typical to that field.

The style of reasoning typical of category theory is described by one of its pioneers, Grothendieck, who gives the analogy of a theorem being like a nut which can be opened in one of two ways: one can use a hammer and chisel to crack the nut open by force; or, alternatively one can soak the nut in water for weeks or months, and eventually, the softened shell will peel off in one's hand [98], for an English translation see McLarty [160, p. 301].

Grothendieck suggests that a problem be “submerged and dissolved by some more or less vast theory” [160, p. 301], which is very much the manner in which we use category theory; we certainly do not use the most sophisticated definitions, nor the most powerful theorems of category theory to “crack” the hard problems of quantum theory. Almost all of the definitions and theorems we invoke are

basic, very much bread-and-butter category theory, nothing exotic that would not appear within the first few chapters of an introductory textbook on category theory. The concepts we make most essential use of are natural transformations and adjunctions. We already saw that making “natural transformation” a precise concept was the major driving force behind the development of category theory, and since then adjunctions have become *the* fundamental concept in category theory, at least according to Mac Lane who described the role of adjunctions as follows [154, p. 107]:

The multiple examples, here and elsewhere, of adjoint functors tend to show that adjoints occur almost everywhere in many branches of Mathematics. It is the thesis of this book that a systematic use of all these adjunctions illuminates and clarifies these subjects.

The real effort – and our main technical contribution – comes in showing that concepts from quantum theory can be expressed in terms of natural transformations, and noticing when and where adjunctions emerge and can be exploited to illuminate and clarify mathematical structures and physical concepts. We do not invoke proofs by “general abstract nonsense”, but rather, using the language of category theory, many of the subtleties and complexities of quantum theory can be baked directly into the definitions, and with these ideas better organised we can make conceptually simple arguments.

The emphasis on language is highly reminiscent of the broader philosophy behind monoidal quantum theory: although the concept of monoidal category is fairly advanced, not covered by most introductory books on category theory, the underlying approach of monoidal quantum theory is not to blast the problems of quantum theory with powerful theorems from the literature. Rather, once one has formulated quantum theory in the language of monoidal categories, one has a new language with which to reason about quantum theory, and there is a great problem-solving power in that language. This is very much the conviction of the founders of monoidal quantum theory who emphasise the monoidal approach as *diagrammatic reasoning* [43, 47].

1.3 Outline and Summary

In Chapter 3 we introduce the spectral presheaf framework which can be seen as a generalisation of topos quantum theory which encompasses the types of \dagger -symmetric monoidal category typically considered within monoidal quantum theory. Moreover, we show that those internal algebra structures which represent observables in monoidal quantum theory lift naturally to classical measurement contexts in the general spectral presheaf framework. Hence, Chapter 3 can be seen as defining the spectral presheaf framework and demonstrating the left-hand inclusion arrows shown in Figure 1.1.

In Chapter 4 we depart from topos quantum theory as formulated by Butterfield and Isham by defining *operational states*, motivated by Bohr’s doctrine of classical concepts. We give a complete characterisation of these operational states, showing that they correspond precisely with density operators. We frame this characterisation of the operational states to give a derivation of the Born rule from a set of operational axioms. We then discuss how this derivation of the Born rule compares to those in the literature.

In Chapter 5 we show how the spectral presheaf framework inherits a notion of *contextuality* from topos quantum theory. We then show how the spectral presheaf framework naturally embeds into Abramsky and Brandenburger’s abstract formulation of empirical models, as depicted in Figure 1.1. We then show how our operational states inherit a notion of local hidden-variable model from the Abramsky-Brandenburger formalism. We then prove a correspondence between contextuality and local hidden-variable models on the most general level. This allows us to derive a version of Bell’s theorem [19] which applies for hidden-variable models over non-standard probabilities, for example, negative probabilities.

In Chapter 6 we introduce Spekkens’ and Harrigan’s notion of *ontological model* [104] into our framework and then reformulate both the Colbeck-Renner theorem [51] and the PBR theorem [185]. Our reformulation can be seen as a strengthening of these results as it applies to systems with arbitrary dimension,

not just those with finite dimension.

In Chapter 7 we show how Spekkens' Toy Theory can be modelled using the spectral presheaf framework. While Spekkens' Toy Theory is presented informally as a “classical” theory, we use our model to *derive* the classical properties of Spekkens' Toy Theory as theorems – in the words of Mermin, we extract them from the formalism. We also compare our spectral presheaf model of Spekkens' Toy Theory to a monoidal quantum theory model due to Coecke and Edwards [44, 75]. We will see that modelling Spekkens' Toy Theory with spectral presheaves has several advantages over the monoidal category model.

In Chapter 8 we discuss future directions for this project. In particular, we show initial steps towards incorporating unitary dynamics and the uncertainty relations into our framework. We then sketch some ideas on how one might incorporate spatiotemporal structure into our framework, and a view on taking composite systems and entanglement. We explore some concepts relating to classical mechanics, in particular: how one can incorporate quantization and *quasi-quantization* into the spectral presheaf framework; and we construct a toy theory reminiscent of Spekkens' Toy Theory, but based on a richer structure of Hamiltonian mechanics.

In Chapter 9 we discuss the implications of our work for some of the conceptual issues of quantum theory. We discuss the realist and instrumentalist interpretations of physical theories and we expand on the idea of extracting metaphysics from the mathematical formalism to give a new perspective on the relationship between metaphysical interpretations and mathematical models of a physical theory. In particular, we give what we identify as the fundamental problem of quantum metaphysics.

Chapter 2

Background

This chapter is devoted to background material from physics and mathematics. In Section 2.1 we review some necessary category theory. In Section 2.2 review topos quantum theory, and in Section 2.3, we review monoidal quantum theory.

2.1 Categorical Preliminaries

We assume the reader is familiar with category theory to the extent of understanding the definitions of category, functor, and natural transformation.

Toposes and Presheaf Categories

A topos is category which resembles the category **Set** of sets and functions in the sense that it provides a setting in which one can axiomatise and reason about mathematical structures. In particular, every topos comes equipped with an internal logic, see [155, Chap. VI] or [128, Chap. 5]. The logic associated with the topos **Set** is classical logic, whereas the logic associated with other toposes is typically non-Boolean, but rather *intuitionistic* in general.

A *topos* is a category which: has all finite limits; is Cartesian closed; and has a subobject classifier. We do not labour this definition because all of the toposes we consider are presheaf categories, which we will work with directly. A *presheaf*

category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ has as objects functors of the form

$$\mathcal{C}^{\text{op}} \xrightarrow{P} \mathbf{Set}$$

for some fixed category \mathcal{C} , and morphisms natural transformations.

Remark 2.1.1. Both the topos quantum theory of Butterfield, Isham and Doering, as well as the Bohrification program of Heunen, Spitters and Landsman make extensive use of the logical aspects of topos theory. Although our main categories of interest are toposes, our approach – at this stage of its development – is not really topos-theoretic in nature. In Chapters 8 and 9 we discuss how we might incorporate more explicitly topos theoretic structures and techniques.

A *subobject* of an object A in a category \mathcal{C} consists of an equivalence class of monomorphisms $m : B \rightarrow A$, where m and m' are equivalent if and only if there exists an isomorphism f such that the diagram

$$\begin{array}{ccc} B & \xrightarrow{m} & A \\ f \downarrow & \nearrow m' & \\ B' & & \end{array}$$

commutes.

For example, subobjects in the category of sets are simply subsets. In a presheaf category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$, subobjects are defined as follows: a subobject of the presheaf

$$\mathcal{C}^{\text{op}} \xrightarrow{P} \mathbf{Set}$$

consists of a presheaf

$$\mathcal{C}^{\text{op}} \xrightarrow{Q} \mathbf{Set}$$

such that for each object C in \mathcal{C} we have an inclusion of sets $Q(C) \subseteq P(C)$ and such that for every morphism $f : C \rightarrow D$ in \mathcal{C} the function

$$Q(D) \xrightarrow{Q(f)} Q(C)$$

is the restriction of the function $P(f) : P(D) \rightarrow P(C)$.

We denote the *constant presheaf* on the set X by

$$\mathcal{C}^{\text{op}} \xrightarrow{C_X} \mathbf{Set}$$

which sends every object in \mathcal{C} to the set X , and every morphism to the identity on X . The constant presheaf $C_{\{*\}} : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ on the singleton set $\{*\}$ is the terminal object in the category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$, and we will use the less unwieldy notation $C_{\{*\}} = \mathbf{1}$.

For any category \mathcal{C} we have the *constant embedding functor*

$$\mathbf{Set} \xrightarrow{C_{(-)}} \mathbf{Set}^{\mathcal{C}^{\text{op}}}$$

which sends a set X to the constant functor $C_X : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$.

Definition 2.1.2. A *global section* (or *global element*) of a presheaf P consists of a natural transformation

$$\begin{array}{ccc} & P & \\ \mathcal{C}^{\text{op}} & \begin{array}{c} \curvearrowright \\ \uparrow \alpha \\ \curvearrowleft \end{array} & \mathbf{Set} \\ & \mathbf{1} & \end{array}$$

The *global sections functor* takes a presheaf P to its set of global sections.

$$\begin{array}{ccc} \mathbf{Set}^{\mathcal{C}^{\text{op}}} & \xrightarrow{\Gamma} & \mathbf{Set} \\ P \longmapsto & \Gamma & \rightarrow \text{Hom}(\mathbf{1}, P) \end{array}$$

We call $\Gamma(P)$ the *global sections* of P .

Monads, Lawvere Theories, and Adjunctions

Monads provide a general language with which to formulate algebraic concepts. We recall the definition of a monad and its corresponding Eilenberg-Moore

category [154, Chap. 6].

Definition 2.1.3. A *monad* on \mathcal{C} consists of a functor

$$\mathcal{C} \xrightarrow{\mathcal{T}} \mathcal{C}$$

equipped with a pair of natural transformations

$$\mathcal{T} \circ \mathcal{T} \xrightarrow{\mu} \mathcal{T} \quad \text{id} \xrightarrow{\eta} \mathcal{T}$$

such that the following diagrams commute

$$\begin{array}{ccc} \mathcal{T} \circ \mathcal{T} \circ \mathcal{T} & \xrightarrow{\mu_{\mathcal{T}}} & \mathcal{T} \circ \mathcal{T} \\ \mathcal{T}\mu \downarrow & & \downarrow \mu \\ \mathcal{T} \circ \mathcal{T} & \xrightarrow{\mu} & \mathcal{T} \end{array}$$

$$\begin{array}{ccccc} & \eta_{\mathcal{T}} & & \mathcal{T}\eta & \\ \mathcal{T} & \xrightarrow{\quad} & \mathcal{T} \circ \mathcal{T} & \xleftarrow{\quad} & \mathcal{T} \\ & \text{id}_{\mathcal{T}} \searrow & \downarrow \mu & \swarrow \text{id}_{\mathcal{T}} & \\ & & \mathcal{T} & & \end{array}$$

Definition 2.1.4. Let $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ be a monad. The *Eilenberg-Moore category* $\mathcal{EM}(\mathcal{T})$ of \mathcal{T} is the category in which an object consists of an *Eilenberg-Moore algebra* of \mathcal{T} , that is, an object A in \mathcal{C} , together with a morphism in \mathcal{C} of the form

$$\mathcal{T}(A) \xrightarrow{a} A$$

such that the diagrams

$$\begin{array}{ccc}
 \mathcal{T}\mathcal{T}(A) & \xrightarrow{\mathcal{T}(a)} & \mathcal{T}(A) \\
 \mu_A \downarrow & & \downarrow a \\
 \mathcal{T}(A) & \xrightarrow{a} & A
 \end{array}
 \qquad
 \begin{array}{ccc}
 A & \xrightarrow{\eta_A} & \mathcal{T}(A) \\
 & \searrow \text{id}_A & \downarrow a \\
 & & A
 \end{array}$$

commute.

A morphism $f : (A, a) \rightarrow (B, b)$ in $\mathcal{EM}(\mathcal{T})$ consists of a morphism $f : A \rightarrow B$ in \mathcal{C} such that the diagram

$$\begin{array}{ccc}
 \mathcal{T}(A) & \xrightarrow{\mathcal{T}(f)} & \mathcal{T}(B) \\
 a \downarrow & & \downarrow b \\
 A & \xrightarrow{f} & B
 \end{array}$$

commutes.

For any monad $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ there is a *forgetful functor*

$$\mathcal{EM}(\mathcal{T}) \xrightarrow{U} \mathcal{C}$$

which sends each Eilenberg-Moore algebra (A, a) to the underlying object A , and any morphism of algebras to the underlying morphism in \mathcal{C} .

Example 2.1.5. For $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ a monad, each object of the form $\mathcal{T}(A) \in \mathcal{C}$ determines an Eilenberg-Moore algebra $(\mathcal{T}(A), \mu_A)$. These are called the *free Eilenberg-Moore algebras*.

Theorem 2.1.6. For any monad $\mathcal{T} : \mathbf{Set} \rightarrow \mathbf{Set}$, and \mathcal{C} a category, the functor

$$\begin{array}{ccc}
 \mathbf{Set}^{\mathcal{C}} & \xrightarrow{\tilde{\mathcal{T}}} & \mathbf{Set}^{\mathcal{C}} \\
 F \dashv & \xrightarrow{\quad} & \mathcal{T} \circ F
 \end{array}$$

is a monad.

Proof. The monad structure of $\tilde{\mathcal{T}}$ is inherited in the obvious way: we define the natural transformation

$$\tilde{\mathcal{T}}\tilde{\mathcal{T}} \xrightarrow{\tilde{\mu}} \tilde{\mathcal{T}}$$

as follows. The natural transformation $\tilde{\mu}$ is determined by its components, that is, for each functor $F : \mathcal{C} \rightarrow \mathbf{Set}$ it is enough to define the morphism

$$\tilde{\mathcal{T}}\tilde{\mathcal{T}}F \xrightarrow{\tilde{\mu}_F} \tilde{\mathcal{T}}F$$

for each F , which is a morphism of the form

$$\mathcal{T}\mathcal{T}F \xrightarrow{\tilde{\mu}_F} \mathcal{T}F$$

This morphism is itself a natural transformation and hence is determined by its components, hence it is enough to specify the components

$$\mathcal{T}\mathcal{T}F(C) \xrightarrow{\tilde{\mu}_{FC}} \mathcal{T}F(C)$$

Which we define as $\tilde{\mu}_{FC} = \mu_{F(C)}$, where $\mu : \mathcal{T}\mathcal{T} \rightarrow \mathcal{T}$ is the multiplication map of the underlying monad \mathcal{T} . We define $\tilde{\eta}$ analogously. The natural transformations $\tilde{\mathcal{T}}$ and $\tilde{\eta}$ satisfy the conditions of Definition 2.1.3 because μ and η do. \square

Lemma 2.1.7. *Let \mathcal{C} be a category and let $\mathcal{T} : \mathbf{Set} \rightarrow \mathbf{Set}$ be a monad and*

$$\mathbf{Set}^{\mathcal{C}^{\text{op}}} \xrightarrow{\tilde{\mathcal{T}}} \mathbf{Set}^{\mathcal{C}^{\text{op}}}$$

the corresponding monad (in sense of Theorem 2.1.6). There is an equivalence of categories

$$\mathcal{EM}(\tilde{\mathcal{T}}) \simeq \mathcal{EM}(\mathcal{T})^{\mathcal{C}^{\text{op}}}$$

Proof. Consider a functor

$$\mathcal{C}^{\text{op}} \xrightarrow{A} \mathcal{EM}(\mathcal{T})$$

then for each $C \in \mathcal{C}$ we have a map

$$\mathcal{T}F(C) \xrightarrow{A_C} F(C)$$

satisfying the conditions of Definition 2.1.4, that is, such that the diagram

$$\begin{array}{ccc} \mathcal{T}\mathcal{T}F(C) & \xrightarrow{\mathcal{T}(A_C)} & \mathcal{T}F(C) \\ \mu_{F(C)} \downarrow & & \downarrow A_C \\ \mathcal{T}F(C) & \xrightarrow{A_C} & F(C) \end{array}$$

commutes. This is precisely the data of a natural transformation

$$\mathcal{T}F \xrightarrow{A_{(-)}} F$$

such that

$$\begin{array}{ccc} \mathcal{T}\mathcal{T}F & \xrightarrow{\mathcal{T}(A)} & \mathcal{T}F \\ \tilde{\mu}_F \downarrow & & \downarrow A \\ \mathcal{T}F & \xrightarrow{A} & F \end{array}$$

where $\tilde{\mu}$ is defined as in the proof of Theorem 2.1.6. One makes the same argument for η and this is precisely the data of an Eilenberg-Moore algebra for the monad $\tilde{\mathcal{T}}$, as required. \square

There is another way to represent algebraic structures in the language of category theory, namely with *Lawvere theories*. Lawvere theories were first introduced by Lawvere [143], but here we follow the presentation of Borceux [28, Chap. 3]. Note that Borceux uses the term *algebraic theory* for a Lawvere theory.

Definition 2.1.8. A *Lawvere theory* consists of a category \mathcal{L} with finite products, and an object A such that every object in \mathcal{L} is the n -fold product A^n of A for some $n \in \mathbb{N}$.

For a category with finite products \mathcal{C} , a *\mathcal{C} -valued model* of \mathcal{L} consists of a finite-product preserving functor

$$\mathcal{L} \xrightarrow{M} \mathcal{C}$$

Morphisms between \mathcal{C} -valued models of \mathcal{L} are natural transformations. The category of \mathcal{C} -valued models of \mathcal{L} is denoted $\mathcal{C}\text{-Mod}_{\mathcal{L}}$.

Each morphism $f : A^n \rightarrow A^m$ in the category \mathcal{L} is completely characterised by morphisms of the form $o : A^n \rightarrow A$, and hence we think of \mathcal{L} as axiomatising a class of n -ary operations. Indeed, a **Set**-valued model of \mathcal{L} consist precisely of a set $M(A)$ together with a family of n -ary operations on $M(A)$.

Lawvere theories let us interpret algebraic structures in any category with finite products. For example, there is a Lawvere theory \mathcal{L} characterising groups, in the sense that the category of groups and group homomorphisms is equivalent to the category of **Set**-valued models of \mathcal{L} . The models of \mathcal{L} in the category of topological spaces and continuous functions are exactly the topological groups.

The following theorem gives a natural correspondence between Lawvere theories and a certain class of monads on **Set**, namely *finitary monads* [10, §3.18] – monads which preserve directed colimits. In particular there is a natural equivalence between **Set**-valued models of a Lawvere theory and the Eilenberg-Moore category of the corresponding finitary monad. We refer to Borceux [28, Proposition 4.6.2] for a proof (note that Borceux refers to a finitary monad as a *monad with finite rank*).

Theorem 2.1.9. *There is a one-to-one correspondence between Lawvere theories and finitary monads on **Set**. In particular, for each Lawvere theory \mathcal{L} , there is a finitary monad $\mathcal{T} : \mathbf{Set} \rightarrow \mathbf{Set}$ such that*

$$\mathcal{EM}(\mathcal{T}) \simeq \mathbf{Set}\text{-Mod}_{\mathcal{L}}$$

Whether we want to reason about algebraic structures using a Lawvere theoretic approach or a monadic approach depends on the circumstances. We will find that the ‘big picture’ is most elegantly stated in terms of Lawvere theories, however when it comes to proving theorems, working concretely in terms of Eilenberg-Moore algebras will be quite useful. We will pass between the two points of view using the following theorem, which is a refinement of [10, Theorem 3.18].

Theorem 2.1.10. *Let \mathcal{L} be a Lawvere theory and let $\mathcal{T} : \mathbf{Set} \rightarrow \mathbf{Set}$ be the corresponding monad, in the sense of Theorem 2.1.9. The category of $(\mathbf{Set}^{\mathcal{C}^{\text{op}}})$ -valued models of \mathcal{L} is equivalent to the Eilenberg-Moore category of the monad*

$$\mathbf{Set}^{\mathcal{C}^{\text{op}}} \xrightarrow{\tilde{\mathcal{T}}} \mathbf{Set}^{\mathcal{C}^{\text{op}}}$$

with $\tilde{\mathcal{T}}$ as defined in Theorem 2.1.6.

Proof. A $(\mathbf{Set}^{\mathcal{C}^{\text{op}}})$ -valued model M of the Lawvere theory \mathcal{L} is by definition a product preserving functor

$$\mathcal{L} \xrightarrow{M} \mathbf{Set}^{\mathcal{C}^{\text{op}}}$$

That is, a functor satisfying $M(A) \times M(A) \cong M(A^2)$. Products are computed point-wise in presheaf categories, hence this is equivalent to saying that for every object $C \in \mathcal{C}$ we have

$$M(A)(C) \times M(A)(C) \cong M(A^2)(C) \tag{2.1}$$

There is a correspondence between functors of the form

$$\mathcal{L} \longrightarrow \mathbf{Set}^{\mathcal{C}^{\text{op}}}$$

and functors of the form

$$\mathcal{L} \times \mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set}$$

and functors of the form

$$\mathcal{C}^{\text{op}} \longrightarrow \mathbf{Set}^{\mathcal{L}}$$

Under this equivalence, the functor

$$\mathcal{L} \xrightarrow{M} \mathbf{Set}^{\mathcal{C}^{\text{op}}}$$

maps to a functor

$$\begin{aligned} \mathcal{C}^{\text{op}} &\longrightarrow \mathbf{Set}^{\mathcal{L}} \\ C &\longmapsto M(-)(C) \end{aligned}$$

Note that if M is product preserving then by the isomorphism (2.1) we see that each of these functors

$$\mathcal{L} \xrightarrow{M(-)(C)} \mathbf{Set}$$

is product preserving – that is, a \mathbf{Set} -valued model of \mathcal{L} . Hence we have shown the first equivalence

$$\begin{aligned} (\mathbf{Set}^{\mathcal{C}^{\text{op}}})\text{-Mod}_{\mathcal{L}} &\simeq (\mathbf{Set}\text{-Mod}_{\mathcal{L}})^{\mathcal{C}^{\text{op}}} \\ &\simeq \mathcal{EM}(\mathcal{T})^{\mathcal{C}^{\text{op}}} \\ &\simeq \mathcal{EM}(\tilde{\mathcal{T}}) \end{aligned}$$

while the second follows from Theorem 2.1.9, and the third follows from Lemma 2.1.7, as required. \square

The concept of an *adjunction* between categories is closely related to that of monads and Lawvere theories. Adjunctions make it possible to transport the

structure from one category to another, a technique we will make extensive use of in our proofs.

Definition 2.1.11. Let \mathcal{C} and \mathcal{D} be categories. An *adjunction* between \mathcal{C} and \mathcal{D} , denoted

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

consists of a pair of functors F and G , together with a pair of natural transformations called the *unit*

$$\text{id}_{\mathcal{D}} \xrightarrow{\eta} G \circ F$$

and the *counit*

$$F \circ G \xrightarrow{\varepsilon} \text{id}_{\mathcal{C}}$$

which satisfy the so-called *triangle identities*, that is, the diagrams

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow \text{id}_F & \downarrow \varepsilon_F \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta_G} & GFG \\ & \searrow \text{id}_G & \downarrow G\varepsilon \\ & & G \end{array}$$

commute.

Here we say that F is *left adjoint* to G , and that G is *right adjoint* to F , which we denote $F \dashv G$.

Example 2.1.12. If categories \mathcal{C} and \mathcal{D} are posets (considered as categories) then an adjunction between \mathcal{C} and \mathcal{D} is precisely a *Galois connection* of posets, see [154, Chap. 4 §5].

There is an equivalent characterisation of adjunctions given by the following result, see for example [150, Chap. 2].

Theorem 2.1.13. For a pair of categories \mathcal{C} and \mathcal{D} , and a pair of functors

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

then $F \dashv G$ if and only if for all objects $A \in \mathcal{C}$ and $B \in \mathcal{D}$ there is an isomorphism

$$\mathrm{Hom}(F(A), B) \cong \mathrm{Hom}(A, G(B))$$

natural in A and B .

Theorem 2.1.14. Given a pair of categories \mathcal{C} , \mathcal{D} and an adjunction

$$\begin{array}{ccc} & F & \\ \mathcal{C} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathcal{D} \\ & G & \end{array}$$

then the corresponding functors on the presheaf categories also form an adjunction

$$\begin{array}{ccc} & F^* & \\ \mathbf{Set}^{\mathcal{D}^{\mathrm{op}}} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \mathbf{Set}^{\mathcal{C}^{\mathrm{op}}} \\ & G^* & \end{array}$$

where these functors are defined on objects $F^*(P) = PF$, while for a morphism $f : P \rightarrow Q$ we define $F^*(f) : F^*(P) \rightarrow F^*(Q)$ to be the natural transformation with components $F^*(f)_A = f_{F(A)}$. The functor G^* is defined in the same way.

Proof. Let η and ε be the unit and counit of the adjunction $F \dashv G$. We need to define a unit

$$\mathrm{id} \xrightarrow{\bar{\eta}} G^* \circ F^*$$

and a counit

$$F^* \circ G^* \xrightarrow{\bar{\varepsilon}} \mathrm{id}$$

We define $\bar{\varepsilon}$ component-wise, that is, for each P we define the component $\bar{\varepsilon}_P$ to be the morphism

$$PFG \xrightarrow{P\varepsilon} P$$

and we define $\bar{\eta}$ in the same way. To check that F^* and G^* form an adjunction we need to check that $\bar{\eta}$ and $\bar{\varepsilon}$ satisfy to triangle identities of Definition 2.1.11.

Checking that the first of these diagrams commute it is enough to check that for each $P \in \mathbf{Set}^{\mathcal{D}^{\text{op}}}$ the diagram

$$\begin{array}{ccc} F^*(P) & \xrightarrow{F^*\bar{\eta}_P} & F^*G^*F^*(P) \\ & \searrow \text{id}_{F^*(P)} & \downarrow \bar{\varepsilon}_{F^*(P)} \\ & & F^*(P) \end{array}$$

commutes. Note that these morphisms making up this diagram are themselves natural transformations and hence can be computed component-wise, that is for each $A \in \mathcal{D}$ we need to check that

$$\begin{array}{ccc} PF(A) & \xrightarrow{PF\eta_{(A)}} & PFGF(A) \\ & \searrow \text{id}_{PF(A)} & \downarrow P\varepsilon_{F(A)} \\ & & PF(A) \end{array} \tag{2.2}$$

commutes. Note that by assumption the diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{F\eta_A} & FGF(A) \\ & \searrow \text{id}_{F(A)} & \downarrow \varepsilon_{F(A)} \\ & & F(A) \end{array} \tag{2.3}$$

commutes, and since $\text{id}_{PF(A)} = P\text{id}_{F(A)}$, we see that diagram (2.2) is the image of the diagram (2.3) under P , and hence the diagram (2.2) commutes also. The

same argument applies to the second triangle identity, as required. \square

Eilenberg-Moore Categories and the Global Sections Adjunction

An important example of an adjunction is the global sections functor Γ – Definition 2.1.2 – and constant presheaf embedding $C_{(-)}$. This is a special case of a result shown in [155, p. 350].

Theorem 2.1.15. *There is an adjunction*

$$\begin{array}{ccc} & C_{(-)} & \\ \text{Set} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \text{Set}^{\mathcal{C}^{\text{op}}} \\ & \Gamma & \end{array}$$

We call this adjunction the *global sections adjunction*. In this section we show that the global sections adjunction of Theorem 2.1.15 lifts to Eilenberg-Moore categories.

Lemma 2.1.16. *For \mathcal{C} a category and $\mathcal{T} : \mathbf{Set} \rightarrow \mathbf{Set}$ a monad, the global sections functor*

$$\mathbf{Set}^{\mathcal{C}^{\text{op}}} \xrightarrow{\Gamma} \mathbf{Set}$$

lifts to a functor $\widehat{\Gamma}$ on the category of Eilenberg-Moore algebras such that the diagram

$$\begin{array}{ccc} \mathcal{EM}(\widetilde{\mathcal{T}}) & \xrightarrow{\widehat{\Gamma}} & \mathcal{EM}(\mathcal{T}) \\ \widetilde{U} \downarrow & & \downarrow U \\ \mathbf{Set}^{\mathcal{C}^{\text{op}}} & \xrightarrow{\Gamma} & \mathbf{Set} \end{array}$$

commutes, where U and \widetilde{U} are the respective forgetful functors.

Proof. First we must give a definition for the functor $\widehat{\Gamma}$. For an Eilenberg-Moore algebra (P, a) in $\mathcal{EM}(\widetilde{\mathcal{T}})$ we define $\widehat{\Gamma}(P, a)$ to be the set of algebra

homomorphisms from the algebra $(\tilde{\mathcal{T}}(\mathbf{1}), \mu_{\tilde{\mathcal{T}}(\mathbf{1})})$ – the free Eilenberg-Moore algebra on the terminal presheaf $\mathbf{1}$ – to (P, a) .

We now show that the above diagram commutes. Note that the forgetful functor U is right adjoint to the functor which generates the free \mathcal{T} -algebra and hence we have

$$\begin{aligned} \widehat{\Gamma}(P, a) &= \text{Hom}((\tilde{\mathcal{T}}(\mathbf{1}), \mu_{\tilde{\mathcal{T}}(\mathbf{1})}), (P, a)) \\ &\cong \text{Hom}(\mathbf{1}, U(P, a)) \\ &= \text{Hom}(\mathbf{1}, P) \end{aligned}$$

as required. \square

The following result can be found in [28, Theorem 4.5.6].

Lemma 2.1.17. *For monads $\mathcal{T} : \mathcal{C} \rightarrow \mathcal{C}$ and $\mathcal{T}' : \mathcal{D} \rightarrow \mathcal{D}$ and functors G and K such that the diagram*

$$\begin{array}{ccc} \mathcal{EM}(\mathcal{T}) & \xrightarrow{K} & \mathcal{EM}(\mathcal{T}') \\ U \downarrow & & \downarrow U' \\ \mathcal{C} & \xrightarrow{G} & \mathcal{D} \end{array}$$

commutes, if G admits a left adjoint F , then K admits a left adjoint L and the diagram

$$\begin{array}{ccc} \mathcal{EM}(\mathcal{T}) & \xleftarrow{L} & \mathcal{EM}(\mathcal{T}') \\ U \downarrow & & \downarrow U' \\ \mathcal{C} & \xleftarrow{F} & \mathcal{D} \end{array}$$

commutes, where U and U' are the forgetful functors.

We can now show that the global sections adjunction lifts to an adjunction on the Eilenberg-Moore categories of a monad and its induced counterpart.

Theorem 2.1.18. *Let $\mathcal{T} : \mathbf{Set} \rightarrow \mathbf{Set}$ be a monad, let \mathcal{C} be a category, and let*

$$\mathbf{Set}^{\mathcal{C}^{\text{op}}} \xrightarrow{\tilde{\mathcal{T}}} \mathbf{Set}^{\mathcal{C}^{\text{op}}}$$

be the corresponding monad (in the sense of Theorem 2.1.6), then the global sections adjunction lifts to the corresponding Eilenberg-Moore categories

$$\begin{array}{ccc}
 & \widehat{\mathcal{C}}_{(-)} & \\
 & \curvearrowright & \\
 \mathcal{EM}(\mathcal{T}) & \perp & \mathcal{EM}(\tilde{\mathcal{T}}) \\
 & \curvearrowleft & \\
 & \widehat{\Gamma} & \\
 U \downarrow & \mathcal{C}_{(-)} & \downarrow \tilde{U} \\
 \mathbf{Set} & \perp & \mathbf{Set}^{\mathcal{C}^{\text{op}}} \\
 & \curvearrowleft & \\
 & \Gamma &
 \end{array} \tag{2.4}$$

Proof. This follows immediately from Lemma 2.1.16, Theorem 2.1.15, and Lemma 2.1.17. □

2.2 Topos Quantum Theory

Before discussing quantum theory we will discuss an algebraic formulation of classical mechanics due to Nestruev [172]. We do this as it will provide a physical intuition for algebras and their associated structures which carries over to the quantum setting, and will be central to our entire approach.

After discussing the algebraic formulation of classical mechanics we will review a standard formulation of quantum theory – namely, the Hilbert space formalism – and we frame aspects of the Hilbert space formalism in terms of the algebraic structures introduced in the context of classical mechanics.

Once we have established what we mean by “quantum theory” we will present Butterfield and Isham’s topos quantum theory, and we will do so using the

algebraic vocabulary for classical mechanics of Nestruev.

Classical Physics and Commutative Algebras

Classical mechanics is traditionally formulated using the language of differential geometry, in which the *state space* of a physical system forms a smooth manifold, typically equipped with some extra structure – for example, a symplectic form, or Poisson bracket – while the *dynamics* of such a system are described by a Hamiltonian vector field on that manifold.

Here we review a dual perspective on classical mechanics which is due to Nestruev [172], in which classical mechanics is constructed entirely in the language of commutative algebra. The Nestruev book is a collaborative project written under the collective singular pseudonym ‘Jet Nestruev’. We adopt the same grammatical convention surrounding Bourbaki and refer to Nestruev as a single entity. The book grew out of a seminar aimed at understanding the mathematical language of quantum field theory, where, in the words of Nestruev [172, p. x]:

it became apparent that the difficulties of quantum field theory come from the fact that physicists express their ideas in an inadequate language, and that an adequate language simply does not exist.

This observation led the group to reformulate classical mechanics – in particular, the differential calculus – in algebraic terms, a project which matured into the book. The role of algebraic structures in differential geometry was known before the work of Nestruev – for example, see [137, Chap. VIII] – the main contribution of Nestruev is the physical interpretation of these structures from the perspective of classical mechanics, which Nestruev describes as follows [172, p. viii]:

what is really new in this book is the motivation of the algebraic approach to smooth manifolds. It is based on the fundamental notion of *observable*, which comes from physics. It is this notion that creates

an intuitively clear environment for the introduction of the main definitions and constructions. The concepts of *state of a physical system* and *measuring device* endow the very abstract notions of *point of the spectrum* and *element of the algebra* with very tangible physical meanings.

It is from this perspective that we will interpret algebraic structures when we move to the quantum setting. In particular, the intuition provided by Nestruev [172, p. xi], as shown in Figure 2.1, will guide and inform all of our constructions.

Physics lab	Commutative unital \mathbb{R} -algebra \mathfrak{A}
Measuring device	Element of the algebra $A \in \mathfrak{A}$
State of the observed physical system	Homomorphism of unital \mathbb{R} -algebras $\rho : \mathfrak{A} \rightarrow \mathbb{R}$
Output of the measuring device	Value of this function $\rho(A)$, $A \in \mathfrak{A}$

Figure 2.1: Algebraic formulation of classical mechanics.

We denote the set of algebra homomorphisms $\rho : \mathfrak{A} \rightarrow \mathbb{R}$ by $\text{Spec}(\mathfrak{A})$, which we call the *spectrum of \mathfrak{A}* .

Nestruev stresses that the choice of ground ring is somewhat unimportant to this construction and interpretation, however, given that in classical physics most of the quantities we want to measure – length, energy, time, etc. – can be represented by real numbers, the field \mathbb{R} is a reasonable choice. In quantum mechanics one typically takes scalar values in \mathbb{C} , but one can take any ring, or, as we will see, any semiring in its place and the physical interpretation of Figure 2.1 remains valid.

We will now explain the origins of the physical interpretations depicted in Figure 2.1. Consider a physical system \mathcal{S} represented by the manifold M , that is, the points on M represent the *states* of a physical system. A *measurement* corresponds to a function

$$M \xrightarrow{f} \mathbb{R}$$

for example f might correspond with measuring momentum, where $f(x)$ is the momentum of the physical system in the state x .

We do not consider arbitrary functions of this type, but typically restrict to only the *smooth*, that is, infinitely differentiable functions. We denote the collection of smooth functions by $C^\infty(M)$. The collection of smooth functions forms a commutative \mathbb{R} -algebra – where the operations of addition, multiplication and scalar multiplication defined on functions point-wise. Hence, associated with a physical system represented by the manifold M is a commutative algebra $C^\infty(M)$, and moreover the algebra $C^\infty(M)$ completely characterises the manifold M . Consider an element of $\text{Spec}(C^\infty(M))$, that is, an algebra homomorphism

$$C^\infty(M) \xrightarrow{\rho} \mathbb{R}$$

which assigns a real value to each measurement on M . Intuitively, simultaneously knowing the outcomes of all measurements should allow us to reconstruct the underlying state.

Nestruev identifies the algebras $C^\infty(M)$, where M is a manifold, as examples of *smooth \mathbb{R} -algebras* [172, p. 37] – note that we omit the full definition of smooth algebra as it requires several new concepts which are not necessary for the discussion here.

Given any smooth algebra \mathfrak{A} , the spectrum $\text{Spec}(\mathfrak{A})$ comes equipped with a smooth manifold structure, and moreover, for each smooth manifold M the spectrum $\text{Spec}(C^\infty(M))$ we have $M \cong \text{Spec}(C^\infty(M))$, see for example [172, Chap. 7] or [137, Chap. VIII]. Nestruev presents this result as showing an equivalence of categories

$$\begin{array}{ccc} & \text{Spec}(-) & \\ \curvearrowright & & \curvearrowleft \\ (\mathbf{SmthAlg})^{\text{op}} & & \mathbf{SmthMan} \\ \curvearrowleft & & \curvearrowright \\ & C^\infty(-) & \end{array}$$

where $\mathbf{SmthMan}$ is the category of compact smooth manifolds and $\mathbf{SmthAlg}$

is the category of smooth algebras.

Remark 2.2.1. The equivalence between the category of smooth manifold and the category of smooth algebras is an example of a *Stone-type duality*, which is the name given to a family of contravariant equivalences between some category of algebraic objects and their homomorphisms on the one hand, and a category of geometric or topological objects and their continuous maps on the other, for a general presentation of such dualities see [129]. The name *Stone duality* goes back to Stone's representation theorem for Boolean algebras [202], which we will see in Chapter 4. Another Stone-type duality we will make extensive use of is *Gelfand duality*.

This Stone-type duality between smooth manifolds and smooth algebras means that for every smooth manifold M we have $M \cong \text{Spec}(C^\infty(M))$, and that for every smooth algebra \mathfrak{A} we have $\mathfrak{A} \cong C^\infty(\text{Spec}(\mathfrak{A}))$, but the fact that these correspondences are natural tells us something more: suppose we have access to only a limited collection of measurement devices on our physical system that are unable to distinguish certain of states from one another; that is, there are certain states the system could inhabit x_1, x_2, \dots such that for every measurement device A we have at our disposal we have $A(x_i) = A(x_j)$ for all i and j . We can place an equivalence relation on the manifold M where $x \sim y$ if and only if we are unable to distinguish the states x and y using the measurements available to us. With this we can define a quotient

$$M \xrightarrow{q} M/\sim$$

where we identify the points of M which our restricted set of measurements are unable to distinguish. Under the Stone-type duality of smooth manifolds and smooth algebras a quotient of manifolds $q : M \rightarrow M/\sim$ corresponds with a subalgebra

$$\mathfrak{B} \xhookrightarrow{i} C^\infty(M)$$

in particular, if we view the algebra $C^\infty(M)$ as modelling the collection of

measurement devices on the system, then the elements of the subalgebra \mathfrak{B} are precisely those measurements $f : M \rightarrow \mathbb{R}$ for which $f(x) = f(y)$ for $x \sim y$, that is, those measurements which cannot distinguish those states satisfying $x \sim y$. That is, \mathfrak{B} is the algebra defined by the following condition: for points $x, y \in M$ we have

$$x \sim y \quad \text{if and only if} \quad f(x) = f(y) \text{ for all } f \in \mathfrak{B}$$

Hence, for an inclusion of subalgebras $\mathfrak{B} \hookrightarrow \mathfrak{A}$, the algebra \mathfrak{B} can be seen as a *coarse-graining* of the context \mathfrak{A} .

Nestruiev goes to great lengths to show that we *can* reformulate classical mechanics in terms of commutative algebra, but it is not clear that we ought to. There are two reasons one might prefer the algebraic perspective: for mathematical reasons there are technical advantages in being able to solve problems using the language of algebra as opposed to geometry; and also for conceptual physical reasons the algebraic formulation takes “measurement” to be a fundamental notion, based on what Nestruiev calls the *observability principle* [172, p. viii] which states:

One of the fundamental principles of contemporary physics asserts
that which exists is only that which can be observed.

We will see in the next section that the algebraic language is more appropriate in making the connection with quantum theory on the level of mathematical formalism, and also that the observability principle which motivates this approach is completely consistent with our pragmatic interpretation of quantum theory. We discuss this concept in greater depth in Chapter 9.

Remark 2.2.2. It is noted by Penrose [179, p. 23] that within physics the adjective “classical” is typically taken to mean “non-quantum”. We use the term positively, in that we take “classical” to mean “admits a description in terms of classical mechanics”.

Quantum Physics and Non-Commutative Algebras

In the previous section we saw how commutative algebras appear naturally in classical mechanics. Now we will see how non-commutative algebras appear naturally in the Hilbert space formulation of quantum theory. Hilbert space quantum theory is due to von Neumann [173, 174], but here we broadly follow the presentation of Isham [123]. In particular, the axioms shown in Figure 2.2 are essentially those presented by Isham [123, Chap. 5 §1.2].

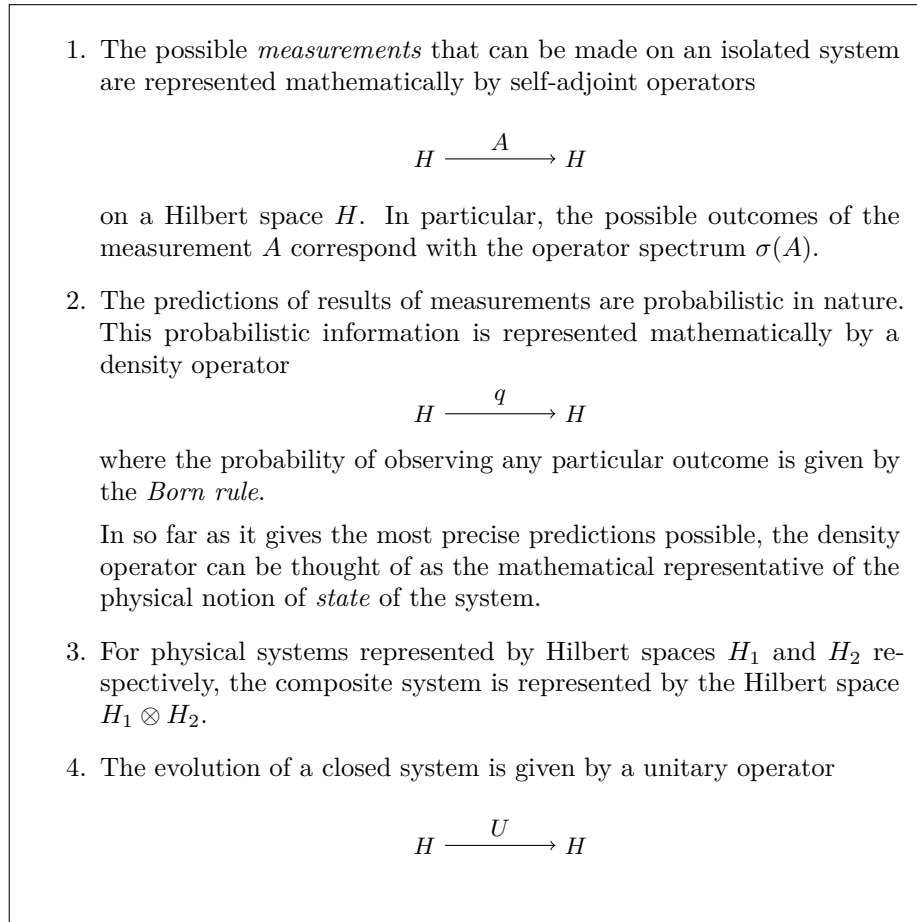


Figure 2.2: A standard axiomatisation of quantum theory.

Definition 2.2.3. For a linear operator $T : H \rightarrow H$, the *operator spectrum*

$\sigma(T)$ of T is a subset of \mathbb{C} consisting of those values $\lambda \in \mathbb{C}$ for which the operator

$$H \xrightarrow{T - (\lambda \cdot \text{id}_H)} H$$

is not invertible, that is, does not have a two-sided inverse.

If H is a finite-dimensional Hilbert space then the operator spectrum coincides exactly with the eigenvalues of T . If H is a finite-dimensional Hilbert space, then the spectral theorem for self-adjoint operators states that every self-adjoint operator $A : H \rightarrow H$ for H can be written in the form

$$A = \sum_{a_i \in \sigma(A)} a_i P_i$$

where each a_i is a real scalar and where each $P_i : H \rightarrow H$ is a self-adjoint projector, which form a pair-wise orthogonal family and sum to the identity operator, $\sum_i P_i = \text{id}_H$. These values a_i are the eigenvalues of A , where each P_i projects onto the corresponding eigenspace, and so Axiom 1 in Figure 2.2 asserts that the possible outcomes which we might observe when performing the measurement A are the values $a_i \in \sigma(A)$.

The *Born rule* goes back to Born [29], but the presentation we follow is due to von Neumann [173, 174, §III.1]. For a discussion of the Born rule see [142]. A *density operator* is a bounded linear operator

$$H \xrightarrow{q} H$$

which: is self-adjoint; satisfies $\text{tr}(q) = 1$; and satisfies the condition that for all vectors $|\psi\rangle \in H$ we have $\langle \psi | q | \psi \rangle \geq 0$. If the “state” of the system is represented by the density operator $q : H \rightarrow H$ then the *Born rule* states that probability of observing the outcome a_i when performing the measurement A – which we denote $\text{Prob}(A = a_i | q)$ – is given by

$$\text{Prob}(A = a_i | q) = \text{tr}(q P_i)$$

Remark 2.2.4. In order to understand the Born rule for operators $A : H \rightarrow H$ with continuous operator spectrum $\sigma(A)$ we require the continuous formulation of the spectral theorem which states that each self-adjoint operator $A : H \rightarrow H$ can be written as an integral

$$A = \int_{\sigma(A)} a dP_a.$$

In this case the Born rule still applies, however it is no longer appropriate to ask the probability of observing any particular outcome, but rather the probability that the outcome lies in some interval $[r_1, r_2] \subseteq \mathbb{R}$. One can define a projector $P_{[r_1, r_2]}$ associated with this interval

$$P_{[r_1, r_2]} = \int_{r_1}^{r_2} dP_a$$

which projects a state onto the subspace of H for which the corresponding values of $\sigma(A)$ lie within the interval $[r_1, r_2]$. In this case, the Born rule states that the probability of observing a result that lies within some interval $[r_1, r_2]$ is given by

$$\text{Prob}(a \in [r_1, r_2] | A) = \text{tr}(q P_{[r_1, r_2]})$$

We will discuss the Born rule in greater depth in Chapter 4, where we give a formal derivation of the Born rule from an alternative set of axioms to those shown in Figure 2.2.

We will now discuss how non-commutative algebras – in particular C^* -algebras – appear quite naturally in this formulation of quantum theory. Note that there are formulations of quantum theory which take C^* -algebras as the fundamental concept, for example: *algebraic quantum field theory* [99]; or certain formulations of quantum information theory [135]. However, this is not what we do here; rather, C^* -algebras emerge naturally within traditional Hilbert space based quantum theory. This connection between the Hilbert space formalism and C^* -algebras is explicit in some presentations of the Hilbert space formulation quantum theory, for example, that of HeinoSaari and Ziman [109].

We recall some basic definitions and concepts associated with C^* -algebras, for which [60] is a standard reference.

Definition 2.2.5. A *Banach algebra* consists of a complex vector space \mathfrak{A} together with an associative bilinear multiplication map which distributes over vector addition

$$\mathfrak{A} \times \mathfrak{A} \longrightarrow \mathfrak{A}$$

and a norm satisfying

$$\|a \cdot b\| \leq \|a\| \|b\|$$

for all $a, b \in \mathfrak{A}$, such that \mathfrak{A} is closed with respect to this norm.

A Banach algebra is *unital* if it has a multiplicative unit $1_{\mathfrak{A}}$ satisfying $\|1_{\mathfrak{A}}\| = 1$.

Definition 2.2.6. A *unital C^* -algebra* consists of a unital Banach algebra \mathfrak{A} with an *involution*

$$\mathfrak{A} \xrightarrow{(-)^*} \mathfrak{A}$$

which is a linear map satisfying:

1. $a^{**} = a$
2. $(a \cdot b)^* = b^* \cdot a^*$
3. $\|a^*\| = \|a\|$
4. $\|a^* \cdot a\| = \|a\|^2$

A *C^* -algebra homomorphism* $f : \mathfrak{A} \rightarrow \mathfrak{B}$ is a linear map which preserves involution, multiplication and the unit – that is, $f(1_{\mathfrak{A}}) = 1_{\mathfrak{B}}$.

Note that we always assume C^* -algebras are unital, and that C^* -subalgebras contain the multiplicative unit.

Example 2.2.7. Let **Hilb** be the category of Hilbert spaces and bounded linear maps. The collection $\text{Hom}(H, H)$ of bounded linear operators

$$H \xrightarrow{f} H$$

on a Hilbert space H forms a C^* -algebra, where addition and scalar multiplication can be computed point-wise and where multiplication is given by morphism composition. The involution is taken to be the hermitian adjoint $A^* = A^\dagger$.

Remark 2.2.8. In this work we only consider C^* -algebras of the form $\text{Hom}(H, H)$, as in Example 2.2.7 and their subalgebras. The C^* -algebras of the form $\text{Hom}(H, H)$ are precisely the type I von Neumann algebras. It should be noted that many of our technical results hold for wider classes of von Neumann algebras than the type I von Neumann algebras, that is, for much of this work one could consider a more general construction where $\text{Hom}(H, H)$ is replaced by an arbitrary von Neumann algebra \mathfrak{X} , however we do not have a physical motivation for doing so.

We say a C^* -algebra \mathfrak{A} is *finite-dimensional* if it is isomorphic to $\text{Hom}(H, H)$ for H a finite-dimensional Hilbert space.

In the standard axiomatisation of quantum theory, given in Figure 2.2, the measurements of a quantum system represented by self-adjoint operators $A : H \rightarrow H$ on some Hilbert space, which are elements of the C^* -algebra $\text{Hom}(H, H)$. In the algebraic characterisation of classical mechanics of Nestruev, the measurements on a system are also represented by elements of an algebra. The crucial distinction between these classical and quantum algebras is that the C^* -algebras $\text{Hom}(H, H)$ are always non-commutative, except for the trivial case when $\dim(H) = 1$.

While for all H with $\dim(H) > 1$ the C^* -algebra $\text{Hom}(H, H)$ is not commutative, by the discussion of the previous section, a commutative subalgebra $\mathfrak{A} \subset \text{Hom}(H, H)$ can be interpreted as a classical physical system. While measurements – that is, self-adjoint operators – belong to a non-commutative algebra $\text{Hom}(H, H)$, they can be grouped together into commutative subalgebras which we think of as *classical contexts*. Viewing the commutative subalgebras as classical contexts is further justified by the following result.

Theorem 2.2.9. *For self-adjoint operators $A : H \rightarrow H$ and $B : H \rightarrow H$ the following statements are equivalent:*

1. there is a commutative C^* -subalgebra of $\text{Hom}(H, H)$ containing both A and B ;
2. the operators A and B commute;
3. the composite operator AB is self-adjoint.

Proof. Suppose AB is self-adjoint, then we have

$$\begin{aligned} AB &= (AB)^\dagger \\ &= B^\dagger A^\dagger \\ &= BA \end{aligned}$$

hence 3 implies 2.

Conversely, suppose A and B commute, then we have

$$\begin{aligned} (AB)^\dagger &= (BA)^\dagger \\ &= A^\dagger B^\dagger \\ &= AB \end{aligned}$$

and hence 2 implies 3.

It is obvious that 1 implies 2. Conversely, if $AB = BA$ then the algebra generated by A and B under the algebraic operations will be commutative, as required. \square

The three equivalent conditions of Theorem 2.2.9 can be interpreted respectively as:

1. There is a classical context which contains both measurements A and B ;
2. the order in which the measurements A and B are performed is irrelevant;
3. the sequential composition of measurements A and B is itself a well-defined measurement.

Remark 2.2.10. We are hesitant to say that these conditions are equivalent to measurements A and B being *simultaneously measurable*, as there is a notion of simultaneous measurement which applies to more general classes of measurements, not just those which commute [108]. A more general notion of compatibility for measurements can be formulated using *positive operator-valued measurements*, or *POVMs* which we consider in Chapter 8.

We saw in the previous section that a commutative algebra is an appropriate setting in which to formulate classical mechanics, and hence we can consider each commutative $\mathfrak{A} \hookrightarrow \text{Hom}(H, H)$ as a classical system in its own right. Just as with the commutative algebras we considered in the previous section we have a notion of *spectrum* associated with commutative C^* -algebras.

Definition 2.2.11. the *Gelfand spectrum* of a commutative C^* -algebra \mathfrak{A} is the set

$$\text{Spec}_{\mathbb{C}}(\mathfrak{A}) = \{ \rho : \mathfrak{A} \rightarrow \mathbb{C} \mid \rho \text{ a } C^*\text{-algebra homomorphism} \}$$

We call the elements of $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ *characters* of \mathfrak{A} .

Remark 2.2.12. The *prime spectrum* $\text{Spec}_{\mathbb{P}}(\mathfrak{A})$ of a commutative C^* -algebra \mathfrak{A} is defined to be the set of prime ideals of \mathfrak{A} , and is naturally isomorphic to the Gelfand spectrum. The correspondence comes from the fact that an ideal $J \subset \mathfrak{A}$ is prime if and only if it is the kernel of a character $\rho : \mathfrak{A} \rightarrow \mathbb{C}$. The prime spectrum of a commutative C^* -algebra is also equivalent to the *maximal spectrum*, taken to be the collection of maximal ideals. In Chapter 3 we will consider these different notions of spectra for more general algebraic structures where they do not coincide.

The Gelfand spectrum of a commutative C^* -algebra comes naturally equipped with a compact Hausdorff topology. This topology can be defined directly, inherited from the topology on \mathbb{C} , or it can be seen as a special case of the more general *Zariski topology* which we consider in Chapter 3.

For every compact Hausdorff topological space X , the set $C(X)$ of continuous

functions

$$X \xrightarrow{f} \mathbb{C}$$

forms a commutative C^* -algebra, with all operations defined point-wise.

The Gelfand spectrum gives rise to a Stone-type duality called *Gelfand duality*, much in the same way as the spectrum for smooth algebras we considered in the previous section, as discussed in Remark 2.2.1.

Theorem 2.2.13. *There is an equivalence of categories*

$$\begin{array}{ccc} & \text{Spec}_G & \\ \curvearrowright & & \curvearrowleft \\ (\mathbf{Com-C^*-Alg})^{\text{op}} & & \mathbf{CompHaus} \\ \curvearrowleft & & \curvearrowright \\ & C(-) & \end{array}$$

Remark 2.2.14. Gelfand duality totally breaks down for non-commutative C^* -algebras. In particular, there is no obvious non-commutative analogue of the Gelfand spectrum. A pursuit of the correct geometric analogue for non-commutative C^* -algebras led Connes to develop *non-commutative geometry* [54].

We have our own perspective on how one ought to generalise the Stone-type duality between algebras of measurements and spaces of states from classical mechanics to the quantum setting, which we discuss in Chapter 9.

We will now make a connection between the “classical state space” – $\text{Spec}_G(\mathfrak{A})$ – associated with each “classical context” \mathfrak{A} and the axioms for quantum theory we outlined in Figure 2.2. In particular, the following result will show us a relationship between the Gelfand spectrum and the operator spectrum in the finite-dimensional setting. Like in our discussion of the Born rule – Remark 2.2.4 – this picture remains morally true when we move to infinite-dimensional systems, but becomes more subtle, and requires a more sophisticated machinery. We present the finite-dimensional case here in depth to provide an intuition.

Definition 2.2.15. Let \mathfrak{A} be a C^* -algebra. An element $P \in \mathfrak{A}$ is called a

projection if it is self-adjoint and idempotent, that is, satisfies $P = P^*$ and $PP = P$.

A projection P in \mathfrak{A} is said to be *primitive in \mathfrak{A}* if for all projections $P_a, P_b \in \mathfrak{A}$ such that $P_a P_b = 0$, then $P_a + P_b = P$ implies that $P_a = P$ or $P_b = P$.

The following lemma follows directly from the spectral theorem for normal operators.

Lemma 2.2.16. *Let \mathfrak{A} be a finite-dimensional commutative C^* -algebra. Every element $A \in \mathfrak{A}$ can be written uniquely as*

$$A = \sum_i a_i P_i$$

where P_i are the primitive projections in \mathfrak{A} .

Lemma 2.2.17. *For \mathfrak{A} a finite-dimensional commutative C^* -algebra, there is a natural isomorphism between $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ and $\{P_j\}_{j \in J}$, the set of primitive projections in \mathfrak{A} . The correspondence between $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ and $\{P_j\}_{j \in J}$ is as follows: for each $\rho \in \text{Spec}_{\mathbb{C}}(\mathfrak{A})$ there is exactly one primitive projection P_k such that $\rho(P_k) = 1$, while $\rho(P_i) = 0$ for all primitive projections with $i \neq k$.*

Proof. Recall, the collection of primitive idempotents sum to the identity $\sum_{j \in J} P_j = 1_{\mathfrak{A}}$. Since ρ is a homomorphism we have

$$\begin{aligned} 1 &= \rho(1_{\mathfrak{A}}) \\ &= \rho\left(\sum_{j \in J} P_j\right) \\ &= \sum_{j \in J} \rho(P_j) \end{aligned}$$

and so there must be at least one P_k such that $\rho(P_k) \neq 0$.

Suppose there are two primitive projections P_k, P_l , with $P_k \neq P_l$, such that $\rho(P_k) \neq 0$ and $\rho(P_l) \neq 0$. This implies $\rho(P_k P_l) \neq 0$, however $P_k P_l = 0$, and

hence we have

$$\begin{aligned}\rho(P_k P_l) &= \rho(0) \\ &= 0\end{aligned}$$

a contradiction, and hence there is exactly one P_k such that $\rho(P_k) \neq 0$. Moreover, from the equation $\sum_{j \in J} \rho(P_j) = 1$ we can conclude that $\rho(P_k) = 1$.

Conversely, for any primitive projection P_k , one can simply define the homomorphism $\rho(P_k) = 1$ and $\rho(P_j) = 0$ for all other primitive projections. By Lemma 2.2.16 this extends to a homomorphism $\rho : \mathfrak{A} \rightarrow \mathbb{C}$. \square

We can now show the relationship between the Gelfand spectrum and the operator spectrum.

Theorem 2.2.18. *Let $A : H \rightarrow H$ be a self-adjoint operator on a finite-dimensional Hilbert space H . Let $\mathfrak{A} \subseteq \text{Hom}(H, H)$ be a commutative C^* -algebra with $A \in \mathfrak{A}$, then we have*

$$\sigma(A) = \{ \rho(A) \mid \rho \in \text{Spec}_G(\mathfrak{A}), \text{ where } A \in \mathfrak{A} \}$$

where $\sigma(A)$ is the operator spectrum of A .

Proof. By Lemma 2.2.16, for each $A \in \mathfrak{A}$ we have

$$A = \sum_i a_i P_i.$$

where each P_i is a primitive projection in \mathfrak{A} , and a_i are the elements of the operator spectrum $\sigma(A)$.

Conversely, by Lemma 2.2.17, each element $\rho \in \text{Spec}_G(\mathfrak{A})$ corresponds with

exactly one primitive projector P_k in \mathfrak{A} , and hence we have

$$\begin{aligned}\rho(A) &= \rho\left(\sum_i a_i P_i\right) \\ &= \sum_i a_i \rho(P_i) \\ &= a_k\end{aligned}$$

hence for each $\rho \in \text{Spec}_G(\mathfrak{A})$, the value $\rho(A)$ is an element of the operator spectrum $\sigma(A)$. \square

In summary, Theorem 2.2.18 tells us that there is an agreement between the traditional Hilbert space formalism interpretation of measurement outcomes, and the algebraic interpretation of Nestruev.

According to the interpretation of Nestruev – as depicted in Figure 2.1 – the values $\rho(A)$ for the elements $\rho \in \text{Spec}_G(\mathfrak{A})$ represent the possible output values of the measurement A . According to the axioms of quantum theory we presented in Figure 2.2 the possible outcomes of a measurement A correspond with the elements of the operator spectrum $\sigma(A)$, and hence Theorem 2.2.18 shows us that these interpretations agree.

Topos Representations of a Physical System

Topos representations of a quantum system can be motivated concretely by Bohr’s *doctrine of classical concepts*, which he expressed in the following quotation [27, p. 209]:

It is decisive to recognize that, however far the phenomena transcend the scope of classical physical explanation, the account of all evidence must be expressed in classical terms.

According to Bohr’s interpretation of quantum theory, although physical reality is by nature quantum, as classical beings conducting experiments in our labs we only have access to the “classical snapshots” of a quantum system.

Regardless of the true nature of reality, this “classical snapshots” perspective is certainly how reality appears to us, and appears to be the limit within which we can interact with reality.

This “classical snapshots” view of a quantum system can be made precise as follows: while a physical system is represented by a non-commutative algebra $\text{Hom}(H, H)$, it can at best be understood as a collection of classical subsystems, that is, the commutative subalgebras $\mathfrak{A} \subseteq \text{Hom}(H, H)$. It is not immediately obvious which commutative subalgebras of $\text{Hom}(H, H)$ we should consider. By Theorem 2.2.9 a commutative C^* -subalgebra \mathfrak{A} is one for which given any pair of measurements $A, B \in \mathfrak{A}$, the composition AB is also a measurement – that is, AB is also a self-adjoint operator, and hence we choose to consider the C^* -subalgebras, that is, those subalgebras which are themselves C^* -algebras.

Topos quantum theory of Butterfield Isham and Doering places further restriction on which subalgebras are considered; topos quantum theory considers only the commutative *von Neumann* C^* -subalgebras of $\text{Hom}(H, H)$. We discuss the implications of this below.

The collection of commutative C^* -subalgebras of $\text{Hom}(H, H)$ has the structure of a poset, which we denote $\mathbf{Hilb-Alg}(H)$. The reason for this somewhat cumbersome notation will become clear in Chapter 3 when we generalise away from Hilbert spaces and C^* -algebras.

Remark 2.2.19. The posets of commutative subalgebras of a C^* -algebra have been studied extensively in their own right [151, 118].

While there is no notion of Gelfand spectrum associated with the C^* -algebra $\text{Hom}(H, H)$, we can consider the Gelfand spectrum $\text{Spec}_G(\mathfrak{A})$ for each classical context $\mathfrak{A} \in \mathbf{Hilb-Alg}(H)$. Note that for a pair of classical contexts where $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$, functoriality of the Gelfand spectrum gives a map on the respective spectra given by restriction, that is, we obtain the map

$$\begin{array}{ccc} \text{Spec}_G(\mathfrak{A}) & \xrightarrow{i^*} & \text{Spec}_G(\mathfrak{B}) \\ \rho \vdash & \longrightarrow & \rho|_{\mathfrak{B}} \end{array}$$

and hence the Gelfand spectrum defines a presheaf on the category $\mathbf{Hilb-Alg}(H)$,

$$\mathbf{Hilb-Alg}(H)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

where we consider the poset as a category with at most one morphism between any pair of objects. The spectral presheaf Spec_G encodes all of the information of the “state spaces” associated with each “classical subsystem” or “classical context”, and it will be useful to study this Spec_G as an object of the presheaf category $\mathbf{Set}^{\mathbf{Hilb-Alg}(H)^{\text{op}}}$.

Remark 2.2.20. There is another formulation of quantum theory based on the language of topos theory, namely *Bohrification* due to Heunen, Landsman and Spitters [114, 116, 115] which instead considers the category of *covariant* functors $\mathbf{Set}^{\mathbf{Hilb-Alg}(H)}$. For a detailed comparison of the covariant and contravariant approaches see [213]. While we follow a contravariant formalism, we believe that there is a connection to be made between our approach and Bohrification, which we explore in Chapter 9.

We have already mentioned that in topos quantum theory of Butterfield, Isham and Doering, one does not consider all of the commutative C^* -subalgebras of $\text{Hom}(H, H)$ but restricts to just the commutative von Neumann C^* -subalgebras, which we will now define. Von Neumann algebras go back to Murray and von Neumann [167, 168, 169, 170], and can be defined in a number of ways; we follow an algebraic characterisation in terms of the *commutant*.

Definition 2.2.21. Let M be a monoid and let $B \subseteq M$ be a subset. The set

$$B' = \{ x \in M \mid xy = yx \text{ for all } y \in B \}$$

is called the *commutant* of B .

Definition 2.2.22. A C^* -subalgebra $\mathfrak{A} \subseteq \text{Hom}(H, H)$ is said to be a *von Neumann algebra* if it satisfies $\mathfrak{A}'' = \mathfrak{A}$.

Remark 2.2.23. We will make two remarks regarding this definition of von Neumann algebra: firstly, the definition seems to only apply to those C^* -algebras which are a subalgebras of $\text{Hom}(H, H)$ for some Hilbert space H , however, as we noted in Remark 2.2.8, every C^* -algebra can be faithfully represented by one of this form, and hence there is no loss of generality; and secondly, the definition of von Neumann algebra is typically given to be those C^* -algebras closed with respect to a certain topology. Von Neumann’s *Bicommutant theorem* shows the equivalence between the topological definition and the algebraic characterisation, see, for example, [56, Theorem 12.3]. For our purposes the algebraic characterisation of von Neumann algebras is more appropriate as it will still apply when we move to algebraic structures more general than C^* -algebras.

The commutative von Neumann subalgebras of $\text{Hom}(H, H)$ form a subposet of $\mathbf{Hilb-Alg}(H)$ which we denote

$$\mathbf{Hilb-Alg}_{\text{vN}}(H) \xhookrightarrow{i} \mathbf{Hilb-Alg}(H)$$

and topos quantum theory represents a physical system using the the category $\mathbf{Set}^{\mathbf{Hilb-Alg}_{\text{vN}}(H)^{\text{op}}}$.

Since presheaf categories are toposes, representing a quantum system by the category $\mathbf{Set}^{\mathbf{Hilb-Alg}(H)^{\text{op}}}$, $\mathbf{Set}^{\mathbf{Hilb-Alg}_{\text{vN}}(H)^{\text{op}}}$, or $\mathbf{Set}^{\mathbf{Hilb-Alg}(H)}$ opens up quantum theory to an analysis using the techniques of topos theory. In particular, both the topos quantum theory of Butterfield, Doering and Isham and the Bohrification program seeks to use the rich logical structures of topos theory to give a working quantum logic, which can be seen as a direct generalisation of logical structures that appear in classical physics. We give a brief discussion of the problems that these logical structures are being used to overcome; a more in-depth discussion can be found in [114].

The logical structure of the state space in classical mechanics is discussed at length by Isham [123, §4.3]. In particular, the “state space” in classical mechanics is represented by a manifold M , and *propositions* about the system can be represented by subsets of M . This allows one to introduce logic into the

description of classical systems, as described by Isham [123, p. 77]:

The crucial observation is that, by associating propositions in this way with subsets of [the state space], the logical structure of the *propositions* about the physical properties of the system is identified with the standard Boolean algebra structure on the *subsets* of the space of states.... This is the precise way in which normal logical thinking (i.e., Boolean logic) becomes implemented in the mathematical structure of physics.

Taking the dual, algebraic perspective of classical mechanics of Nestruev, the state space is represented by the set $\text{Spec}_G(\mathfrak{A})$ for some smooth algebra \mathfrak{A} , and hence the logical structure of classical mechanics can be viewed as the logical structure associated with the collection of subsets of $\text{Spec}_G(\mathfrak{A})$. In a direct generalisation of this observation, topos quantum theory considers the *subobjects* of the presheaf

$$\mathbf{Hilb}\text{-}\mathbf{Alg}_{vN}(H)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

viewed as an object in the category $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}_{vN}(H)^{\text{op}}}$.

While the collection of subsets of a set carry the structure of a Boolean algebra, the collection of subobjects of an object in a topos carries the structure of Heyting algebra. Boolean algebras are the formal mathematical structures one needs to talk about classical logic in a rigorous and precise way, and Heyting algebras play the same role but for *intuitionistic logic*, see [128, Chap. 5] or [155, Chap. VI].

Recall the criteria for a realist interpretation of quantum theory according to Isham that we showed in Figure 1.6. The first realist criterion states that we are right to think of a quantum system as possessing values for physical quantities, and that propositions about the physical properties of a system should obey the laws of classical logic. Isham and Doering adopt a interpretation of *neo-realism*, see [68, 79], which modifies this criterion only in that the propositions

about the physical properties of system need not obey the classical logic, but rather intuitionistic logic, encapsulated by the Heyting algebra structure on the subobjects of the presheaf

$$\mathbf{Hilb}\text{-}\mathbf{Alg}_{\text{vN}}(H)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

Neo-realism essentially agrees with the second realist criterion from Figure 1.6, while the third criterion is subsumed by *daseinisation*, see [62], or [63, §5].

The word “daseinisation” is derived from the German “dasein”, which means “being” or “existence”, and is a fundamental concept in the philosophy of Heidegger [106]. The process of daseinisation in topos quantum theory describes the coming into being of physical properties of a quantum system.

Our approach resembles topos quantum theory in that we consider a category of contravariant presheaves, but are two main differences between the formalism that we develop and the topos quantum theory of Butterfield, Isham and Doering. We will briefly discuss these differences in mathematical structures, and explain the underlying physical and metaphysical reasons for these differences:

1. As with the program of Bohrification, we consider the full posets of commutative C^* -subalgebras $\mathbf{Hilb}\text{-}\mathbf{Alg}(H)$, not just the von Neumann algebras $\mathbf{Hilb}\text{-}\mathbf{Alg}_{\text{vN}}(H)$. In topos quantum theory the choice of commutative von Neumann subalgebras as opposed to all of the commutative C^* -subalgebras is for purely technical reasons, see [63, p. 56]. The choice of von Neumann algebras seems to lack physical motivation, and is rather a technical necessity. We believe that there is good physical motivation to consider the full poset of C^* -subalgebras, see Remark 4.2.18 below.

Note that we will make extensive use of the properties of commutative von Neumann subalgebras, but our final results will be carried over into the more general setting.

2. We have a different metaphysical conceptualisation of the “state space” and hence we end up with a different mathematical representation of the “state

space”. In particular, we follow a strictly pragmatic or instrumentalist interpretation, rather than a realist or neo-realist interpretation. This is discussed at length in Chapter 9.

Remark 2.2.24. Topos quantum theory can be seen as a special case of an even more general topos theoretic foundation of physical theories, due to Doering and Isham [64, 65, 66, 67]. In this work we do not attempt to formulate our framework at this level of generality, instead focussing on the concrete motivations behind our constructions.

2.3 Monoidal Quantum Theory

Monoidal quantum theory, initiated by Abramsky and Coecke [7], reformulates aspects of quantum theory using the mathematical language of symmetric monoidal categories. In doing so it exploits the diagrammatic calculi associated with symmetric monoidal categories, which can reduce complex calculations to manipulations of simple diagrams [41, 42]. These graphical techniques have been applied to quantum computation, in particular: verifying quantum protocols [49]; measurement based quantum computing [69, 71]; and topological quantum computing [120].

In this approach aspects from quantum theory such as “observables” are encoded in terms of the monoidal structure of the category of Hilbert spaces. One typically does this by considering internal algebra structures.

In contrast with the topos approach of Butterfield, Doering and Isham the monoidal approach does not address the foundational aspects of quantum theory. The overarching philosophy of the monoidal approach is described in [43, p. 310]:

This approach is informed by techniques used in computer science, logic, and the branch of mathematics called category theory, however its roots can be traced to Schrödinger’s conviction that the essential characteristic of quantum theory is the manner in which systems compose.

In practice, the monoidal category formulation of quantum theory is entirely consistent with a “shut up and calculate” approach to quantum theory, but radically reconceptualises what it means to “calculate” – reducing calculation to intuitive manipulations of string diagrams. In this sense, monoidal quantum theory also qualifies as a pragmatic approach to quantum theory.

Symmetric Monoidal Categories and Graphical Calculi

First we give the definition for a symmetric monoidal category [154, Chap. VII].

Definition 2.3.1. A *monoidal category* $(\mathcal{A}, \otimes, I)$ consists of a category \mathcal{A} together with a functor

$$\mathcal{A} \times \mathcal{A} \xrightarrow{- \otimes -} \mathcal{A}$$

and a natural isomorphism called the *associator*

$$A \otimes (B \otimes C) \xrightarrow{\alpha_{A,B,C}} (A \otimes B) \otimes C$$

satisfying the pentagon diagram

$$\begin{array}{ccccc} A \otimes (B \otimes (C \otimes D)) & \xrightarrow{\alpha} & (A \otimes B) \otimes (C \otimes D) & \xrightarrow{\alpha} & ((A \otimes B) \otimes C) \otimes D \\ \text{id}_A \otimes \alpha \downarrow & & & & \downarrow \alpha \otimes \text{id}_D \\ A \otimes ((B \otimes C) \otimes D) & \xrightarrow{\alpha} & & & (A \otimes (B \otimes C)) \otimes D \end{array}$$

and an object I called the *monoidal unit*, together with natural isomorphisms called the *unit isomorphisms*

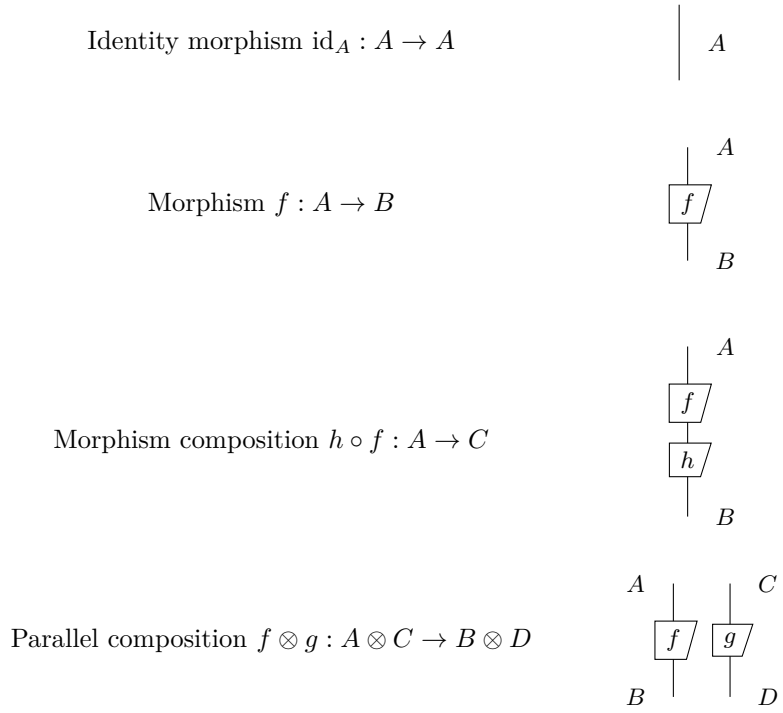
$$I \otimes A \xrightarrow{\lambda_A} A \qquad A \otimes I \xrightarrow{\rho_A} A$$

satisfying the triangle diagram

$$\begin{array}{ccc}
 A \otimes (I \otimes B) & \xrightarrow{\alpha_{A,I,B}} & (A \otimes I) \otimes B \\
 \text{id}_A \otimes \lambda_B \searrow & & \nearrow \rho_A \otimes \text{id}_B \\
 & A \otimes B &
 \end{array}$$

For objects A and B we call the object $A \otimes B$ the *tensor product* of A and B .

A monoidal category is said to be *strict* if the associator and unit isomorphisms are identities, and the well-known *coherence theorem* of Mac Lane [154, Chap. XI. §3. Theorem 1] states that every symmetric monoidal category is monoidally equivalent to a strict symmetric monoidal category. Joyal and Street [130] showed that strict monoidal categories admit a graphical calculus, which we now briefly review, a more detailed summary of these graphical calculi can be found in [192]. Note, we read our diagrams from the top of the page downwards. The morphisms in a strict symmetric monoidal category can be represented in the following way:



The monoidal unit is depicted by the empty diagram and hence a morphism of type $x : I \rightarrow X$ is depicted



A monoidal category is said to be *braided* if there is a natural isomorphism

$$A \otimes B \xrightarrow{\sigma_{A,B}} B \otimes A$$

and is *symmetric* if it has a braiding which satisfies $\sigma_{A,B}^{-1} = \sigma_{B,A}$. For a symmetric monoidal category the symmetric braid maps are represented in the graphical

calculus by the crossing of wires

$$\sigma_{A,B} = \begin{array}{ccc} A & & B \\ & \searrow & / \\ & X & \\ & / & \searrow \\ B & & A \end{array}$$

Definition 2.3.2. A \dagger -category (*involutive category* or **-category*) consists of a category \mathcal{A} together with a functor

$$\mathcal{A}^{\text{op}} \xrightarrow{\dagger} \mathcal{A}$$

which is the identity on objects, and satisfies $\dagger \circ \dagger = \text{id}_{\mathcal{A}}$.

A morphism f in a \dagger -category is said to be *self-adjoint* if $f^\dagger = f$. An isomorphism f in a \dagger -category is said to be *unitary* if $f^\dagger = f^{-1}$.

Definition 2.3.3. A \dagger -symmetric monoidal category consists of a symmetric monoidal category $(\mathcal{A}, \otimes, I)$ such that: \mathcal{A} is a \dagger -category; \dagger is a strict monoidal functor; and the symmetric braiding, associator and unit isomorphisms are unitary.

Definition 2.3.4. A \dagger -symmetric monoidal category $(\mathcal{A}, \otimes, I)$ is said to be *monoidally well-pointed* if for any pair of morphisms $f, g : X \otimes Y \rightarrow Z$ we have $f \circ (x \otimes y) = g \circ (x \otimes y)$ for all $x : I \rightarrow X$ and $y : I \rightarrow Y$ implies $f = g$.

As well as the \dagger -symmetric monoidal structure, the categories typically studied within monoidal quantum theory have some additional structures.

A *zero-object* in a category is an object which is both initial and terminal. In a category with a zero-object 0 , for every pair of objects X and Y we call the unique map $X \rightarrow 0 \rightarrow Y$ the *zero-morphism*, which we denote by $0_{X,Y} : X \rightarrow Y$, or simply $0 : X \rightarrow Y$. We say that a pair of composable morphisms f and g are *orthogonal* if $f \circ g = 0$.

A category \mathcal{A} is said to have *finite biproducts* if it has a zero object 0 , and if for each pair of objects X_1 and X_2 there exists an object $X_1 \oplus X_2$ which is both the coproduct and the product of X_1 and X_2 . If \mathcal{A} is a \dagger -category with

finite biproducts such that the coprojections $\kappa_i : X_i \rightarrow X_1 \oplus X_2$ and projections $\pi_i : X_1 \oplus X_2 \rightarrow X_i$ are related by $\kappa_i^\dagger = \pi_i$, then we say \mathcal{A} has *finite \dagger -biproducts*

Definition 2.3.5. For a category with finite biproducts each hom-set $\text{Hom}(X, Y)$ is equipped with a commutative monoid operation [164, Lemma 18.3] which we call *biproduct convolution*, where for $f, g : X \rightarrow Y$ we define $f + g : X \rightarrow Y$ by the composition

$$X \xrightarrow{\Delta_X} X \oplus X \xrightarrow{f \oplus g} Y \oplus Y \xrightarrow{\nabla_Y} Y$$

where $\Delta_X = \langle \text{id}_X, \text{id}_X \rangle$, and $\nabla_Y = [\text{id}_Y, \text{id}_Y]$. The additive unit for this monoid structure is given by the zero-morphism $0_{X,Y} : X \rightarrow Y$.

A category \mathcal{A} with finite biproducts admits a matrix calculus [164, Chap. I. §17] characterised as follows: for $X = \bigoplus_{j=1}^n X_j$ and $Y = \bigoplus_{i=1}^m Y_i$ a morphism $f : X \rightarrow Y$ is determined completely by the morphisms $f_{i,j} : X_j \rightarrow Y_i$, and so we represent f by the matrix

$$f = \begin{pmatrix} f_{1,1} & f_{1,2} & \cdots & f_{1,m} \\ f_{2,1} & f_{2,2} & \cdots & f_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ f_{n,1} & f_{n,2} & \cdots & f_{n,m} \end{pmatrix}$$

and morphism composition is computed by matrix multiplication. If \mathcal{A} has finite \dagger -biproducts, then given the matrix representation of f , then f^\dagger has the corresponding matrix representation

$$f^\dagger = \begin{pmatrix} f_{1,1}^\dagger & f_{1,2}^\dagger & \cdots & f_{n,1}^\dagger \\ f_{1,2}^\dagger & f_{2,2}^\dagger & \cdots & f_{n,2}^\dagger \\ \vdots & \vdots & \ddots & \vdots \\ f_{1,m}^\dagger & f_{2,m}^\dagger & \cdots & f_{n,m}^\dagger \end{pmatrix}$$

Biproduct convolution $f + g$ for given matrix representations $f_{i,j}$ and $g_{i,j}$ is given by point-wise biproduct convolution $(f + g)_{i,j} = f_{i,j} + g_{i,j}$.

We now give some examples of \dagger -monoidal categories typically considered within monoidal quantum theory.

Examples 2.3.6.

- The category **Hilb** of Hilbert spaces and bounded linear maps. The dagger is given by Hermitian adjoint, biproducts are given by direct sums and the monoidal tensor is given by tensor product of Hilbert spaces with monoidal unit the ground field \mathbb{C} .
- The category **fdHilb** of finite-dimensional Hilbert spaces, with all structures defined as for **Hilb**.
- The category **Rel** whose objects are sets and whose morphisms are relations. Dagger is given by relational converse, biproducts are disjoint union and symmetric monoidal structure is given by Cartesian product with the monoidal unit the singleton set $\{*\}$.
- The category **Rel_Q** whose objects are sets and whose morphisms are quantale-valued relations over a fixed quantale Q (see Chapter 3 Section 3.2). Here biproducts and monoidal tensor are as for **Rel**. Note that if we take the quantale Q to be the two-element Boolean algebra **2** then **Rel_Q** is equivalent to the category **Rel**.
- For any \dagger -symmetric monoidal category \mathcal{A} the categories **CP***(\mathcal{A}) and **CPM**(\mathcal{A}), following the CP* construction [46] and CPM-construction of [191], are again \dagger -symmetric monoidal categories. Note that the category **CP***(**Hilb**) does not have finite biproducts, however it does retain an additive structure on its hom sets which is enough to apply the general construction we develop below.

We will require some degree of compatibility between the monoidal structure and the biproduct structure. Monoidal quantum theory initially generalised the categorical structure of the category **fdHilb** of finite-dimensional Hilbert spaces, which is a *closed monoidal category*. A *closed monoidal category* is a monoidal category \mathcal{A} where if for each object $X \in \mathcal{A}$ the functor

$$\mathcal{A} \xrightarrow{X \otimes -} \mathcal{A}$$

admits a left adjoint. It follows immediately from the fact that right adjoints preserve limits that for all $X, Y, Z \in \mathcal{A}$ we have

$$X \otimes (Y \oplus Z) \cong (X \otimes Y) \oplus (X \otimes Z) \quad (2.5)$$

and also

$$X \otimes 0 \cong 0 \quad (2.6)$$

However, we are interested in categories that are not necessarily monoidal closed, for example the category of **Hilb** of Hilbert spaces. Although for **Hilb** the functors $H \otimes -$ are not adjunctions, they do preserve finite biproducts and the zero-object, that is, we have $H \otimes (J \oplus K) \cong (H \otimes J) \oplus (H \otimes K)$ and $H \otimes 0 \cong 0$ for all Hilbert spaces H, J and K .

Definition 2.3.7. For a monoidal category \mathcal{A} with biproducts satisfying (2.5), we say that \mathcal{A} is a monoidal category with *distributive biproducts*. For a monoidal category \mathcal{A} with a zero-object satisfying (2.6) we say \mathcal{A} , we say that \mathcal{A} is a monoidal category with an *absorbing zero-object*.

Note that if \mathcal{A} has distributive biproducts then its zero-object is necessarily absorbing, that is, (2.5) implies (2.6).

In summary, the general form of monoidal category that we are interested in is locally small \dagger -symmetric monoidal with finite distributive \dagger -biproducts.

Internal Algebras as Observables: Finite Dimensions

For a \dagger -symmetric monoidal category \mathcal{A} , an *algebra in \mathcal{A}* consists of a *carrier* object X , and a *multiplication* morphism $\mu : X \otimes X \rightarrow X$, denoted in the graphical calculus



Dually, a *coalgebra in \mathcal{A}* consists of a *carrier* object X , and a *comultiplication* morphism $\delta : X \rightarrow X \otimes X$, denoted in the graphical calculus



An *algebra-coalgebra pair* consists of a carrier object X with given multiplication and comultiplication maps. Note that in a locally small \dagger -symmetric monoidal category each algebra (X, μ) also defines a coalgebra (X, μ^\dagger) and hence every algebra in a locally small \dagger -symmetric monoidal category forms an algebra-coalgebra pair. An algebra-coalgebra pair of this type will be referred to as a \dagger -*algebra*.

Remark 2.3.8. Algebras of this type cannot be described by Lawvere theories in general, but can be characterised using the more general notion of a *PROP* which goes back to MacLane [157, §24]. This is largely because in this setting the monoidal tensor is not the categorical product. One can formulate the internal algebras relevant to monoidal quantum theory in terms of PROPs [70], but we do not need this level of generality here.

Monoidal quantum theory encodes the notion of observable in terms of Frobenius algebras.

Definition 2.3.9. A *unital special commutative Frobenius algebra* in a \dagger -

symmetric monoidal category \mathcal{A} is a \dagger -algebra satisfying axioms

$$\exists \begin{array}{c} \circ \\ | \end{array} \text{ such that } \begin{array}{c} \circ \\ \diagup \diagdown \\ | \end{array} = \begin{array}{c} \circ \\ \diagdown \diagup \\ | \end{array} = \begin{array}{c} | \end{array} \quad (\text{U})$$

$$\begin{array}{c} \circ \\ \circ \\ | \end{array} = \begin{array}{c} | \end{array} \quad (\text{S})$$

$$\begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \\ \diagup \diagdown \\ | \end{array} = \begin{array}{c} \circ \\ \diagdown \diagup \\ \circ \\ \diagup \diagdown \\ | \end{array} \quad (\text{A})$$

$$\begin{array}{c} \diagup \diagdown \\ \circ \\ \diagup \diagdown \\ | \end{array} = \begin{array}{c} \circ \\ \diagup \diagdown \\ | \end{array} \quad (\text{C})$$

$$\begin{array}{c} \diagup \diagdown \\ \circ \\ \diagup \diagdown \\ | \end{array} = \begin{array}{c} \circ \\ \diagup \diagdown \\ \circ \\ \diagup \diagdown \\ | \end{array} = \begin{array}{c} \circ \\ \diagdown \diagup \\ \circ \\ \diagup \diagdown \\ | \end{array} \quad (\text{F})$$

Note that since we are considering \dagger -algebras we get the corresponding inverted equations (U), (A) and (C) for the comultiplication morphism.

Example 2.3.10. Let H be a finite-dimensional Hilbert space with $\dim(H) = D$ and let $\{|i\rangle\}_{i \in J}$ be an orthonormal basis. There is a Frobenius algebra with comultiplication defined by

$$\begin{aligned} H &\xrightarrow{\delta} H \otimes H \\ |i\rangle &\longmapsto |i\rangle \otimes |i\rangle \end{aligned}$$

and unit $|\eta\rangle = \frac{1}{D} \sum_{i \in J} |i\rangle$.

The following theorem of Coecke, Pavlovic and Vicary [48, Theorem 5.1] shows that the Frobenius algebras described in Example 2.3.10 characterise all the Frobenius algebras in **fdHilb**, the category of finite-dimensional Hilbert spaces.

Theorem 2.3.11. *Every unital special commutative Frobenius algebra in \mathbf{fdHilb} is of the form*

$$\begin{aligned} H &\xrightarrow{\delta} H \otimes H \\ |i\rangle &\longmapsto |i\rangle \otimes |i\rangle \end{aligned}$$

for some orthonormal basis $\{|i\rangle\}_{i \in J}$.

For a \dagger -algebra (X, μ) the *set-like elements* (or *copyable elements*, *group-like elements*, or *classical elements*) are the morphisms $\alpha : I \rightarrow X$ satisfying

$$\begin{array}{c} \triangle \alpha \\ \circ \\ \cup \end{array} = \begin{array}{c} \triangle \alpha \\ \uparrow \end{array} \begin{array}{c} \triangle \alpha \\ \uparrow \end{array}$$

Under the interpretation of (X, μ) as an observable, one typically views the set-like elements as corresponding with the observable outcomes or states associated with that observable.

Set-like elements are said to be *normal* if

$$\begin{array}{c} \triangle \alpha \\ \circ \\ \downarrow \downarrow \end{array} = \begin{array}{c} \triangle \alpha \\ \downarrow \end{array} \quad \text{and} \quad \begin{array}{c} \triangle \alpha \\ \downarrow \triangle \alpha \end{array} = \text{id}_I$$

For the category \mathbf{fdHilb} we can see by Theorem 2.3.11 that the set-like elements are precisely those basis elements $|i\rangle : \mathbb{C} \rightarrow H$. There is a one-to-one correspondence between the orthonormal bases of a Hilbert space H , and the special commutative unital Frobenius algebra structures that one can endow H with. The correspondence is that every orthonormal basis of H forms the set-like elements of some special commutative unital Frobenius algebra (H, μ) . Note that this correspondence holds only if the underlying Hilbert space is finite-dimensional.

One consequence of reformulating concepts using only the monoidal structure of the category \mathbf{fdHilb} is that one can now examine these algebraic structures in other categories, for example the category \mathbf{Rel} , whose objects are sets and morphisms are relations.

For a relation $R : A \rightarrow B$ we use the notation $a \sim_R b$ if R relates the element $a \in A$ to $b \in B$. The following result is shown in [176, Theorem 4.4].

Theorem 2.3.12. *Every Frobenius algebra in **Rel***

$$A \times A \xrightarrow{\mu} A$$

$$\{*\} \xrightarrow{\eta} A$$

is of the form $A = \bigsqcup_i A_i$, a disjoint union of abelian groups, where

$$(a, b) \sim_{\mu} ab$$

where a and b both belong to one of the groups A_i and ab denotes group multiplication.

Let $e_i \in A_i$ be the unit of each respective group. The relation η is defined

$$* \sim e_i$$

for each i .

There is a set-like element $\alpha_i : \{*\} \rightarrow A$ for each abelian group A_i making up the Frobenius algebra A . These set-like elements are precisely the relations

$$* \sim_{\alpha_i} a$$

for all $a \in A_i$.

This fact is exploited by Coecke and Edwards in [44, 75] where Spekkens' Toy Theory is modelled in **Rel**, with Frobenius algebras representing observables. Equipped with this general algebraic formulation of “observable” in terms of Frobenius algebras, Spekkens' Toy Theory can then closely compared with a quantum theory in a mathematically precise way. We discuss Spekkens' Toy Theory and the Frobenius algebra model of Coecke and Edwards in detail in Chapter 7.

Internal Algebras as Observables: Infinite Dimensions

The Frobenius algebra description of orthonormal bases fails for Hilbert spaces of infinite dimension, as the unit $|\eta\rangle$ defined in Example 2.3.10 cannot exist if the underlying Hilbert space has infinite dimension.

An infinite-dimensional alternative has been proposed by Abramsky and Heunen in the form of algebraic structures closely related to Frobenius algebras: abstract H^* -algebras [9].

A *concrete H^* -algebra*, defined by Ambrose [13], consists of a Hilbert space H , equipped with the structure of a (not necessarily unital) Banach algebra such that for each element $x \in H$ there exists an element $\tilde{x} \in H$ such that

$$\langle x \cdot y | z \rangle = \langle y | \tilde{x} \cdot z \rangle$$

for all $y, z \in H$.

The interpretation of H^* -algebras as orthonormal bases stems from the following theorem [13, Corollary 4.1].

Theorem 2.3.13. *Any proper commutative H^* -algebra is isomorphic to a Hilbert space direct sum of one-dimensional algebras.*

The following result [9, Proposition 9] shows how concrete H^* -algebras can be axiomatised in terms of the monoidal structure of the category of Hilbert spaces as an associative special algebra satisfying the (H) axiom.

Theorem 2.3.14. *A \dagger -algebra in **Hilb** satisfying (A), (S), and*

$$\forall \begin{array}{c} \triangle \\ \uparrow \\ x \end{array} \exists! \begin{array}{c} \downarrow \\ \tilde{x} \\ \triangle \end{array} \text{ such that } \begin{array}{c} \triangle \\ x \\ \circ \\ \downarrow \end{array} = \begin{array}{c} \downarrow \\ \tilde{x} \\ \triangle \\ \circ \\ \downarrow \end{array} \quad (H)$$

is a H^ -algebra, and conversely any H^* -algebra is a \dagger -algebra in **Hilb** satisfying (A), (S) and (H).*

Remark 2.3.15. Note that Heunen and Abramsky [9] introduce the axiom (H) without the uniqueness condition. The uniqueness of \tilde{x} is equivalent to the

H^* -algebra being *proper*, see [9] for details.

It is shown in [9, Lemma 5.5] that every unital Frobenius algebra (in any category) satisfies (H), and hence the authors propose H^* -algebras as an axiomatisation for “observable” in infinite-dimensional quantum theory in the monoidal approach to quantum theory.

Definition 2.3.16. A *special commutative H^* -algebra* in a \dagger -symmetric monoidal category \mathcal{A} is a \dagger -algebra satisfying the following axioms

$$\begin{array}{l}
 \begin{array}{c} \circ \\ | \\ \circ \end{array} = \begin{array}{c} | \\ | \\ | \end{array} \quad (S) \\
 \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \\ | \end{array} = \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \\ | \end{array} \quad (A) \\
 \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \\ | \end{array} = \begin{array}{c} \circ \\ \swarrow \quad \searrow \\ \circ \\ | \end{array} \quad (C) \\
 \forall \begin{array}{c} \triangle \\ | \\ \triangle \end{array} \quad \exists! \begin{array}{c} \triangle \\ | \\ \triangle \end{array} \quad \text{such that} \quad \begin{array}{c} \triangle \\ \swarrow \quad \searrow \\ \circ \\ | \end{array} = \begin{array}{c} \triangle \\ \swarrow \quad \searrow \\ \circ \\ | \end{array} \quad (H)
 \end{array}$$

The (H) axiom is closely related to the Frobenius axiom (F) as the following pair of results show [9, Lemma 5, Lemma 6].

Lemma 2.3.17. *In any \dagger -monoidal category (F) and (U) imply (H).*

Lemma 2.3.18. *In a monoidally well-pointed \dagger -monoidal category, (H) and (A) imply (F).*

This motivates the idea that H^* -algebras are the appropriate generalisation of Frobenius algebras to a possibly infinite-dimensional setting.

The Phase Group

The *phase group* is an important piece of structure associated with Frobenius algebras or H^* -algebras in the monoidal approach to quantum theory.

Definition 2.3.19. Let $(\mathcal{C}, I, \otimes)$ be a \dagger -symmetric monoidal category, and let (X, μ) be a \dagger -algebra in \mathcal{C} . A morphism $U : X \rightarrow X$ is called a *phase* for (X, μ) if it is unitary and satisfies

$$\begin{array}{c} \text{cup} \\ \circ \\ \text{---} \\ \square U \\ \text{---} \\ \circ \\ \text{cap} \end{array} = \begin{array}{c} \text{---} \\ \square U \\ \text{---} \\ \circ \\ \text{cup} \end{array} \quad \text{and} \quad \begin{array}{c} \text{---} \\ \square U \\ \text{---} \\ \circ \\ \text{cap} \end{array} = \begin{array}{c} \text{cup} \\ \circ \\ \text{---} \\ \square U \\ \text{---} \\ \circ \\ \text{cap} \end{array}$$

The collection of phases $G(\mu) \subseteq \text{Hom}(X, X)$ of μ forms a group called the *phase group*.

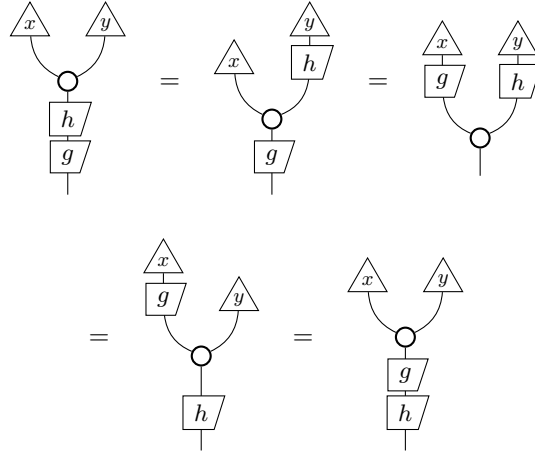
Remark 2.3.20. If (X, μ) is unital Frobenius algebra, that is satisfies (U) and (F) from Definition 2.3.9, then it is enough to assume only one of the equations of Definition 2.3.19, with the other being derivable.

As the name suggests, the phase group of an algebra is a group, and the following result shows that these phase groups are very often abelian groups.

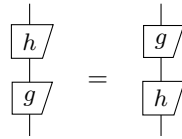
Theorem 2.3.21. *Let $(\mathcal{C}, I, \otimes)$ be a locally small \dagger -symmetric monoidal category, and let (X, μ) be a \dagger -algebra in \mathcal{C} . If any of the three conditions are satisfied then the phase group of (X, μ) is an abelian group.*

1. *The category \mathcal{C} is monoidally well-pointed and the algebra (X, μ) is commutative, that is, satisfies (C);*
2. *the algebra (X, μ) is special and commutative, that is, satisfies (S) and (C);*
3. *the algebra is associative, unital and commutative, that is, satisfies (A), (U) and (C).*

Proof. Supposing that \mathcal{C} is monoidally well-pointed. For all x, y we have

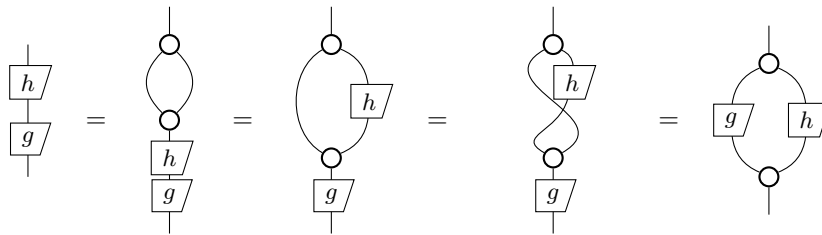


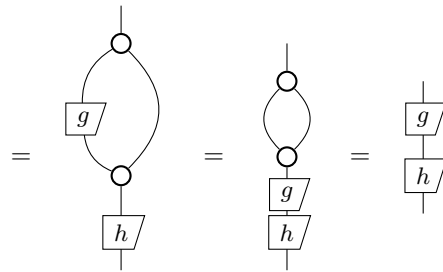
and hence, by monoidal well-pointedness we conclude



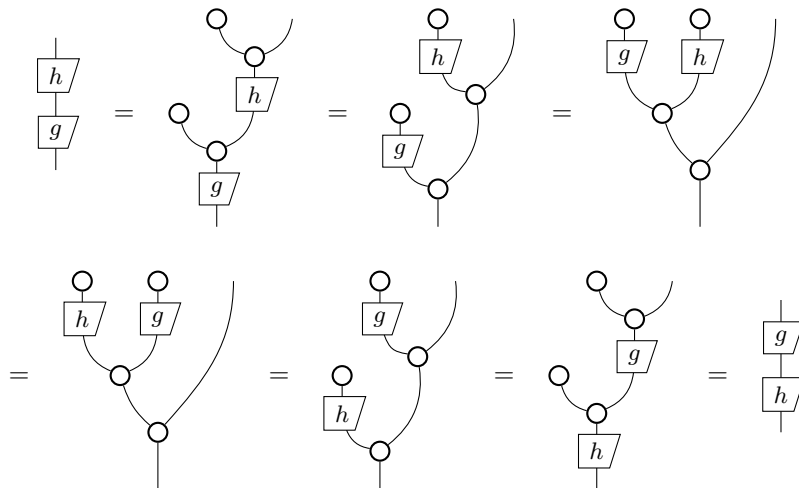
as required.

Suppose (X, μ) satisfies the special axiom (S), then we can use essentially the same argument as above, that is, for phases f and g we have



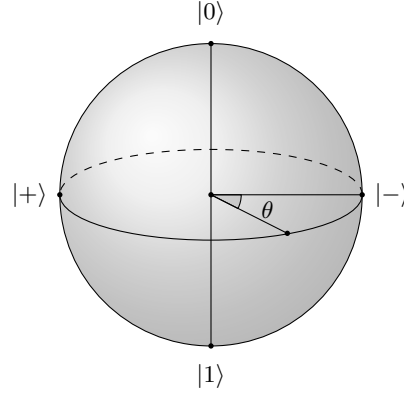


Now suppose (X, μ) has a unit, then we have



as required. □

Example 2.3.22. Consider the Bloch sphere representation of the qubit – that is, a two-dimensional Hilbert space. Let μ be the Frobenius algebra corresponding with the basis $|0\rangle, |1\rangle$.



Let $U_\theta : H \rightarrow H$ be the rotations of the Bloch sphere which fix $|0\rangle$ and $|1\rangle$.

$$U_\theta = \begin{pmatrix} e^{-\frac{i\theta}{2}} & 0 \\ 0 & e^{\frac{i\theta}{2}} \end{pmatrix}$$

More generally, for a Hilbert space H of dimension n the phase group of any Frobenius algebra on H is isomorphic to an n -fold direct sum of one-parameter unitary groups.

Example 2.3.23. For Frobenius algebras in **Rel** the phase groups are structured as follows. By Theorem 2.3.12 a Frobenius algebra $\mu : A \times A \rightarrow A$ in **Rel** is of the form $A = \bigsqcup_i A_i$ for a family of abelian groups A_i . The phase group for μ is isomorphic to $\bigoplus_i A_i$.

In what follows we will see how these internal algebras and their associated structures correspond with elements of topos quantum theory and their generalisations.

Chapter 3

The Spectral Presheaf Framework

We now present our *spectral presheaf framework*, which we will see is a direct extension of topos quantum theory, as described in Chapter 2. Even though the categories we consider are toposes, we do not employ the techniques typical of topos theory and so we do not call our approach “a topos approach”. In Chapter 8 and Chapter 9 we discuss ways in which more topos-theoretic structures and techniques might be incorporated into our framework.

We give the basic definitions in Section 3.1, which are a direct generalisation of that of $\mathbf{Hilb}\text{-}\mathbf{Alg}(H)$ and the functor $\text{Spec}_{\mathbf{G}}$, in particular we define the poset

$$\mathcal{A}\text{-}\mathbf{Alg}(X)$$

for any object X in a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts \mathcal{A} .

In Section 3.2 we take an in-depth view at an example where $\mathcal{A} = \mathbf{Rel}_Q$, the category of quantale-valued relations over a fixed quantale Q . This category is more general than \mathbf{Rel} , which is commonly considered as a non-standard model for concepts in monoidal quantum theory. We will use categories of this form to

provide concrete examples of the structures and concepts we develop.

In Section 3.3 we show how the notion of “observable” internal to a \dagger -symmetric monoidal category discussed in the previous chapter – Frobenius or H^* -algebras – naturally corresponds with the notion of measurement context in the more general von Neumann semialgebra framework. In particular, we show the relationship between the set-like elements of H^* -algebras correspond with elements of the generalised Gelfand spectrum.

3.1 Defining the Framework

The definition of the poset $\mathbf{Hilb-Alg}(H)$ relies on the rich algebraic structure associated with the set $\text{Hom}(H, H)$, which is a C^* -algebra. In order to generalise this approach to incorporate the more abstract machinery of monoidal quantum theory we need an adequate generalisation of this C^* -algebra structure. Here we introduce the notion of an \mathbb{S}^* -semialgebra, based on the language of semirings and semimodules, and show that hom-sets of \dagger -symmetric monoidal categories with distributive \dagger -biproducts come equipped with this structure.

Semirings, Semimodules, and Semialgebras

In this section we review semirings, semimodules, for which [94] is a standard reference, and we introduce the notion of an \mathbb{S}^* -semialgebra. Recall, a *semiring*, or *rig*, is a ring “without negatives”, that is, a structure identical to a ring but for which the addition operation forms a commutative monoid and not necessarily an abelian group.

Definition 3.1.1. A *semiring* $(R, \cdot, 1, +, 0)$ consists of a set R equipped with a commutative monoid operation $+$: $R \times R \rightarrow R$ with unit $0 \in R$, and a monoid operation \cdot : $R \times R \rightarrow R$, with unit $1 \in R$, such that for all $r, s, t \in R$

1. $t \cdot (r + s) = t \cdot r + t \cdot s$;
2. $(r + s) \cdot t = r \cdot t + s \cdot t$;

$$3. 0 \cdot s = s \cdot 0 = 0.$$

A semiring is called *commutative* if \cdot is commutative. A **-semiring*, or *involutive semiring* is one equipped with an operation $(-)^* : R \rightarrow R$ satisfying

$$4. (s^*)^* = s$$

$$5. (s + t)^* = s^* + t^*$$

$$6. 0^* = 0$$

$$7. (s \cdot t)^* = t^* \cdot s^*$$

$$8. 1^* = 1$$

As the notation suggests we will refer to the monoid operations of a semiring as *addition* and *multiplication* respectively. We say that a semiring R is *zero-divisor free (ZDF)* if for all $s, t \in R$ we have $s \cdot t = 0$ implies $s = 0$ or $t = 0$.

Many structures and properties associated with rings can be lifted directly to the level of semirings in the obvious way, for example homomorphisms and kernels.

Definition 3.1.2. Let $(R, \cdot, 1, +, 0)$ be a commutative semiring, an *R-semimodule* consists of a commutative monoid $+_M : M \times M \rightarrow M$, with unit 0_M , together with a *scalar multiplication* $\bullet : R \times M \rightarrow M$ such that for all $r, s \in R$ and $m, n \in M$:

$$1. s \bullet (m +_M n) = s \bullet m +_M s \bullet n ;$$

$$2. (r \cdot s) \bullet m = r \bullet (s \bullet m) ;$$

$$3. (r + s) \bullet m = (r \bullet m) +_M (s \bullet m);$$

$$4. 0 \bullet m = s \bullet 0_M = 0_M;$$

$$5. 1 \bullet m = m.$$

Definition 3.1.3. An R -semialgebra $(M, \cdot_M, 1_M, +_M, 0_M)$ consists of an R -semimodule $(M, +_M, 0_M)$ equipped with a monoid operation $\cdot_M : M \times M \rightarrow M$, with unit 1_M , such that $(M, \cdot_M, 1_M, +_M, 0_M)$ forms a semiring, and where scalar multiplication obeys $s \bullet (m \cdot_M n) = (s \bullet m) \cdot_M n = m \cdot_M (s \bullet n)$. An R -semialgebra is called *commutative* if \cdot_M is commutative.

Definition 3.1.4. Let R be a $*$ -semiring. An R^* -semialgebra consists of an R -semialgebra $(M, \cdot_M, 1_M, +_M, 0_M)$, such that M , considered as a semiring is a $*$ -semiring with involution $(-)^* : M \rightarrow M$ and R have compatible involutions, that is, one that satisfies $(s \bullet m)^* = s^* \bullet m^*$.

Notice that every $*$ -semiring R is an R^* -semialgebra, where the scalar multiplication of R is taken to be the usual multiplication in R .

We define a *unital subsemialgebra* $i : N \hookrightarrow M$ of M is a subset N containing 0_M and 1_M closed under all the algebraic operations. A *subsemialgebra* $N \subseteq M$ is a subset N containing 0_M which is closed under multiplication and which is a semialgebra in its own right, though may have a different unit from M . A (*unital*) $*$ -subsemialgebra of a $*$ -semialgebra is a (*unital*) subsemialgebra closed under taking involutions.

Non-zero elements s, t of a semialgebra are *orthogonal* if $s \cdot t = 0$.

Definition 3.1.5. A *subunital idempotent* in a semialgebra is an idempotent element p such that there is an orthogonal idempotent q where $p + q = 1_M$. A *primitive subunital idempotent* is a subunital idempotent p such that there exist no non-trivial subunital idempotents s and t with $s + t = p$. A \dagger -idempotent is one satisfying $p^\dagger = p$.

Example 3.1.6. The \dagger -idempotents $P : H \rightarrow H$ for H a Hilbert space are exactly the self-adjoint projectors. In **Hilb** every \dagger -idempotent is a subunital idempotent.

Example 3.1.7. In **Rel** a relation $R : X \rightarrow X$ is \dagger -idempotent if and only if for every pair $x, y \in X$ such that $x \sim_R y$ then $y \sim_R x$, that is, if it satisfies $R \circ R = R$ and is symmetric. The subunital \dagger -idempotents are those $R : X \rightarrow X$

of the form $\text{id}_Y : X \rightarrow X$ which is the identity relation on some subset $Y \subseteq X$ and the null relation on the complement of Y in X .

The Generalised Topos Representations

We now show that just as the set of bounded linear endomorphisms $\text{Hom}(H, H)$ on a Hilbert space H carry the structure of a C^* -algebra, the set of endomorphisms $\text{Hom}(X, X)$ for any locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts carries the structure of an \mathbb{S}^* -semialgebra in a canonical way. The results in this section appear in [72].

The following lemma states that morphism composition distributes over biproduct convolution, this result goes back to [156, §19], or see [164, Proposition 18.4].

Lemma 3.1.8. *Let \mathcal{A} be a category with finite biproducts. For morphisms $f, g : X \rightarrow Y$, $h_1 : X \rightarrow X$, and $h_2 : Y \rightarrow Y$ we have $(f+g) \circ h_1 = (f \circ h_1) + (g \circ h_1)$ and $h_2 \circ (f + g) = (h_2 \circ f) + (h_2 \circ g)$.*

To each locally small \dagger -symmetric monoidal category there is an associated commutative $*$ -semiring of *abstract scalars*, described in the following theorem. The following theorem has been shown by Abramsky and Coecke [7, §6], and a proof can be found in [112, Corollary 4.1].

Theorem 3.1.9. *For a locally small \dagger -symmetric monoidal category \mathcal{A} with finite distributive \dagger -biproducts the set $\mathbb{S} = \text{Hom}(I, I)$ is a commutative $*$ -semiring.*

Proof. From Lemma 3.1.8 we see that biproducts give the set $\text{Hom}(I, I)$ the structure of a semiring. It is shown in [133, Proposition 6.1] that the multiplicative operation – morphism composition – is commutative.

The claim is that the dagger gives us the required involution. To see this we need to verify the remaining equations of Definition 3.1.1: Equations 7 and 8 follow from functoriality; Equation 5 holds since \dagger preserves biproducts; Equation 4 follows from the equation $\dagger \circ \dagger = \text{Id}_{\mathcal{A}}$; and since the functor \dagger acts

as the identity on objects, it preserves the zero-object and therefore preserves zero-morphisms, and hence equation 6 holds. \square

Example 3.1.10. For **Hilb** the abstract scalars are the set of linear maps $\text{Hom}(\mathbb{C}, \mathbb{C})$, which is canonically isomorphic to the field \mathbb{C} . Involution is given by complex conjugation.

Example 3.1.11. For **Rel** the abstract scalars correspond with the set of relations on the singleton set $\{*\}$. There are only two relations from the one-element set to itself, and the $*$ -semiring structure on this set is isomorphic to the two element Boolean algebra **2** where addition is taken to be meet \vee , and multiplication is taken to be join \wedge , and where the involution is trivial.

Abramsky has observed [3, §2] that the abstract scalars $\mathbb{S} = \text{Hom}(I, I)$ in a monoidal closed category \mathcal{A} act on the hom-sets of \mathcal{A} . The following theorem shows that if \mathcal{A} is a monoidal category with finite distributive biproducts then the action of these scalars on hom-sets gives each $\text{Hom}(X, Y)$ the structure of an \mathbb{S} -semimodule. This result can be found in [112, §4].

Theorem 3.1.12. *Let \mathcal{A} be a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts and absorbing zero object, and let $\mathbb{S} = \text{Hom}(I, I)$. For any pair of objects the set $\text{Hom}(X, Y)$ is an \mathbb{S} -semimodule.*

Proof. By Theorem 3.1.9 we have a semiring of scalars $\mathbb{S} = \text{Hom}(I, I)$, and biproduct convolution – Definition 2.3.5 – defines an additive monoid structure on $\text{Hom}(X, Y)$. We need to show that there is a scalar action

$$\mathbb{S} \times \text{Hom}(X, Y) \xrightarrow{- \bullet -} \text{Hom}(X, Y)$$

and that the scalar action is compatible with the additive structure on $\text{Hom}(X, Y)$, that is, we need to show that the five equations of Definition 3.1.2 are satisfied.

For a morphism $f : X \rightarrow Y$ the scalar action $s \bullet f$ for $s : I \rightarrow I$ is defined

$$X \xrightarrow{\rho_X^{-1}} X \otimes I \xrightarrow{f \otimes s} Y \otimes I \xrightarrow{\rho_Y} Y$$

What remains is to check that the five equations of Definition 3.1.2. Note that since \mathcal{A} is strict monoidal, the following diagram commutes

$$\begin{array}{ccccccc}
 X & \xrightarrow{\rho_X^{-1}} & X \otimes I & \xrightarrow{f \otimes \text{id}_I} & Y \otimes I & \xrightarrow{\text{id}_Y \otimes s} & Y \otimes I \\
 \downarrow f & & \downarrow f \otimes \text{id}_I & & \downarrow \text{id}_Y \otimes s & & \downarrow \rho_Y \\
 Y & \xrightarrow{\rho_Y^{-1}} & Y \otimes I & \xrightarrow{\text{id}_Y \otimes s} & Y \otimes I & \xrightarrow{\rho_Y} & Y
 \end{array}$$

Tracing the exterior of the diagram clockwise from the top left-hand corner is equal to $s \bullet f$. Hence this diagram commuting states that

$$s \bullet f = f \circ (s \bullet \text{id}_Y) \quad (3.1)$$

To see Equation 1 from Definition 3.1.2, consider

$$\begin{aligned}
 s \bullet (f + g) &= (f + g) \circ (s \bullet \text{id}_Y) \\
 &= (f \circ (s \bullet \text{id}_Y)) + (g \circ (s \bullet \text{id}_Y)) \\
 &= (s \bullet f) + (s \bullet g)
 \end{aligned}$$

The first equality is an application of (3.1), the second equality follows directly from Lemma 3.1.8, the third equality is another application of (3.1).

To see Equation 2 from Definition 3.1.2, consider

$$\begin{aligned}
 (s \circ r) \bullet f &= f \circ ((s \circ r) \bullet \text{id}_Y) \\
 &= f \circ (s \bullet \text{id}_Y) \circ (r \bullet \text{id}_Y) \\
 &= s \bullet (r \bullet f)
 \end{aligned}$$

which is simply multiple applications of (3.1).

Taking $s = 1_{\mathbb{S}}$ in Equation (3.1) gives $1_{\mathbb{S}} \bullet f = f$, that is Equation 5 from Definition 3.1.2.

To see Equation 4 from Definition 3.1.2, let $f : X \rightarrow Y$ be a morphism, and consider $0_{\mathbb{S}} \bullet f$ the scalar multiplication by the zero element $0_{\mathbb{S}} \in \mathbb{S}$, which is defined to be the composition

$$\begin{array}{ccc}
 X & \xrightarrow{0_{\mathbb{S}} \bullet f} & Y \\
 \wr \downarrow & & \uparrow \wr \\
 X \otimes I & \xrightarrow{f \otimes 0_{\mathbb{S}}} & Y \otimes I
 \end{array}$$

however, the morphism $f \otimes 0_{\mathbb{S}}$ factors through the object $X \otimes 0$, and hence by $X \otimes 0 \cong 0$ the morphism $0_{\mathbb{S}} \bullet f$ factors through 0, which is the definition of the zero map $0_{XY} : X \rightarrow Y$, and hence we obtain Equation 4.

To see Equation 3 consider the following diagram. Tracing the left hand side of the diagram is the morphism $(s \bullet f) + (r \bullet f)$, while tracing the right hand side of the diagram is the morphism $(s + r) \bullet f$.

$$\begin{array}{ccc}
 X & \xrightarrow{\rho_X^{-1}} & X \otimes I \\
 \Delta_X \downarrow & & \downarrow \text{id}_X \otimes \Delta_I \\
 X \oplus X & & X \otimes (I \oplus I) \\
 \rho_X^{-1} \oplus \rho_X^{-1} \downarrow & \nearrow [\text{id}_X \otimes \kappa_I, \text{id}_X \otimes \kappa_I] & \downarrow f \otimes (r \oplus s) \\
 (X \otimes I) \oplus (X \otimes I) & & Y \otimes (I \oplus I) \\
 (f \otimes r) \oplus (f \otimes s) \downarrow & \nearrow [\text{id}_Y \otimes \kappa_I, \text{id}_Y \otimes \kappa_I] & \downarrow \text{id}_Y \otimes \nabla_I \\
 (Y \otimes I) \oplus (Y \otimes I) & \xrightarrow{\nabla_{Y \otimes I}} & Y \otimes I \\
 \rho_Y \oplus \rho_Y \downarrow & & \downarrow \rho_Y \\
 Y \oplus Y & \xrightarrow{\nabla_Y} & Y
 \end{array}$$

hence, verifying Equation 3 amounts to checking that this diagram commutes. The upper most and lower most segments commute by the properties of biproducts, while the central segments follow due to the fact that the biproducts are assumed to be distributive. \square

Theorem 3.1.13. *Let \mathcal{A} be a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts, and let $\mathbb{S} = \text{Hom}(I, I)$. For any object X the set $\text{Hom}(X, X)$ is a \mathbb{S}^* -semialgebra.*

Proof. By Theorem 3.1.12 $\text{Hom}(X, X)$ is a semimodule. We define the multiplication to be morphism composition. Multiplication then distributes over addition by Lemma 3.1.8. The scalar action being compatible with multiplication

follows from the coherence conditions of symmetric monoidal categories, while the dagger provides the necessary involution. \square

Example 3.1.14. For H an object in **Hilb** the semialgebra structure on $\text{Hom}(H, H)$ is precisely the C^* -algebra structure. The scalar action is that of usual complex scalar multiplication, while the dagger is the usual Hermitian adjoint involution.

We are now in a position to directly generalise the definition of the poset $\mathbf{Hilb}\text{-Alg}(H)$.

Definition 3.1.15. For \mathcal{A} a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts and X an object, we define the category $\mathcal{A}\text{-Alg}(X)$ to be the category with objects commutative unital \mathbb{S}^* -subsemialgebras

$$\mathfrak{A} \hookrightarrow \text{Hom}(X, X)$$

and arrows inclusion of subalgebras.

Recall Definition 2.2.21: the commutant B' of a subset $B \subseteq \text{Hom}(X, X)$ is the set

$$B' = \{ f : X \rightarrow X \mid f \circ g = g \circ f \text{ for all } g \in B \}$$

and recall the definition of a von Neumann algebra, that is a C^* -algebra satisfying $\mathfrak{A}'' = \mathfrak{A}$. We generalise this to \mathbb{S}^* -semialgebras.

Definition 3.1.16. Let $\mathfrak{A} \hookrightarrow \text{Hom}(X, X)$ be an \mathbb{S}^* -subsemialgebra. We say that \mathfrak{A} is a *von Neumann subsemialgebra* if it satisfies $\mathfrak{A} = \mathfrak{A}''$.

We define the full subcategory of *von Neumann \mathbb{S}^* -subsemialgebras*

$$\mathcal{A}\text{-Alg}_{\text{vN}}(X) \hookrightarrow \mathcal{A}\text{-Alg}(X)$$

to have objects those unital \mathbb{S}^* -subsemialgebras \mathfrak{A} which satisfy $\mathfrak{A} = \mathfrak{A}''$.

Note that if H is a finite-dimensional Hilbert space then every C^* -subalgebra $\mathfrak{A} \subseteq \text{Hom}(H, H)$ is a von Neumann algebra, and hence the inclusion

$$\mathbf{Hilb}\text{-}\mathbf{Alg}_{\text{vN}}(H) \xhookrightarrow{i} \mathbf{Hilb}\text{-}\mathbf{Alg}(H)$$

is the identity if H is a finite-dimensional. In contrast, for the category \mathbf{Rel} the inclusion

$$\mathbf{Rel}\text{-}\mathbf{Alg}_{\text{vN}}(A) \xhookrightarrow{i} \mathbf{Rel}\text{-}\mathbf{Alg}(A)$$

can be proper, even for finite sets A .

Theorem 3.1.17. *The inclusion*

$$\mathbf{Rel}\text{-}\mathbf{Alg}_{\text{vN}}(A) \xhookrightarrow{i} \mathbf{Rel}\text{-}\mathbf{Alg}(A)$$

is proper if A has two or more elements.

Proof. To see this, consider a two-element set A and consider the following relations on A

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

This set of relations contains the multiplicative and additive units, and it is easy to verify directly that it is closed under composition, addition and involution. If we denote this collection of relations by \mathfrak{A} then it is enough to show that $\mathfrak{A} \neq \mathfrak{A}'$.

First consider \mathfrak{A}' . Since every relation commutes with both the identity relation and the zero relation \mathfrak{A}' is completely characterised by those relations which commute with the only other relation in \mathfrak{A} , that is those relations g such that

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}$$

and hence we see that \mathfrak{A}' consists of precisely those relations of the form

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}$$

for $g_1, g_4 \in \mathbf{2}$. Using a similar argument we can compute \mathfrak{A}'' and see that $\mathfrak{A}' = \mathfrak{A}''$. Note that \mathfrak{A}'' contains the relation

$$\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and hence $\mathfrak{A}'' \neq \mathfrak{A}$, as required. \square

We attach no physical significance to measurement contexts satisfying the condition $\mathfrak{A}'' = \mathfrak{A}$, however, for purely technical reasons the poset of commutative von Neumann semialgebras $\mathcal{A}\text{-Alg}_{\mathbf{vN}}(X)$ will be extremely important in proving our results.

We now recall some properties of the commutant which we will make extensive use of. All of these results exist in the literature, but for the sake of completeness we state and prove them all here.

Lemma 3.1.18. *Let B and A be subsets of $\text{Hom}(X, X)$, then the following hold:*

1. B' is a unital subsemialgebra of $\text{Hom}(X, X)$;
2. if B is closed under \dagger then so is B' ;
3. if $A \subseteq B$ then $B' \subseteq A'$;
4. all elements of B commute if and only if $B \subseteq B'$;
5. $B \subseteq B''$;
6. $B' = B'''$.

Proof. 1. Let $g, h \in B'$, then for all $x \in B$ we have $gx = xg$ and $hx = xh$. Since $x(g + h) = xg + xh = gx + hx = (g + h)x$ we see B' is closed under addition.

Since $xgh = gxh = ghx$ we see B' is closed under multiplication. Since for all $s \in S$ we have $(s \bullet g)x = s \bullet (gx) = s \bullet (xg) = x(s \bullet g)$ we see that B' is closed under scalar multiplication. Clearly $1x = x1$ and $0x = x0$ and hence B' is a subsemialgebra, as required.

2. Suppose B is closed under involutions. Let $g \in B'$, hence for all $x \in B$ we have $gx = xg$. It follows that $x^\dagger g^\dagger = g^\dagger x^\dagger$ for all $x \in B$. Since B is closed under \dagger this is equivalent to the statement $g^\dagger y = yg^\dagger$ for all $y \in B$ and hence $g^\dagger \in B'$, as required.

3. Suppose $A \subseteq B$ and let $g \in B'$. For all $x \in B$ we have $xg = gx$, and in particular this holds for for all $x \in A$, and hence $g \in A'$.

4. Suppose $B \subseteq B'$, then for each $x \in B$ we have $xy = yx$ for all $y \in B$, and hence all elements of B commute. Conversely, suppose all elements of B commute. Then for each $x \in B$ we have $xy = yx$ for all $y \in B$, which means $x \in B'$, as required.

5. Let $x \in B$, then by definition of B' we have $xy = yx$ for all $y \in B'$ and therefore $x \in B''$, as required.

6. Note that by 5 we have $B \subseteq B''$ and therefore by 3 we have $B''' \subseteq B'$. We need to show the reverse inclusion $B' \subseteq B'''$, which also follows from 5 as $(B') \subseteq (B')''$, as required. \square

The following lemma follows directly from Lemma 3.1.18.

Lemma 3.1.19. *For a commutative \mathbb{S}^* -subsemialgebra $\mathfrak{A} \subseteq \text{Hom}(X, X)$, the set $\mathfrak{A}'' \subseteq \text{Hom}(X, X)$:*

1. *is a commutative von Neumann subsemialgebra;*
2. *and there is a semialgebra inclusion $\mathfrak{A} \hookrightarrow \mathfrak{A}''$.*

Proof. By Lemma 3.1.18.1 & 2, \mathfrak{A}'' is a \mathbb{S}^* -subsemialgebra of $\text{Hom}(X, X)$, and by Lemma 3.1.18.5 there is an inclusion $\mathfrak{A} \hookrightarrow \mathfrak{A}''$.

It remains to show that \mathfrak{A}'' is commutative, and that \mathfrak{A}'' is a von Neumann semialgebra. To see that \mathfrak{A}'' is a von Neumann semialgebra we need to show that $(\mathfrak{A}'')'' = \mathfrak{A}''$, which can be seen by applying Lemma 3.1.18.6

To see that \mathfrak{A}'' is commutative, by Lemma 3.1.18.4 it is enough to show that $\mathfrak{A}'' \subseteq (\mathfrak{A}'')'$. Since \mathfrak{A} is commutative then by Lemma 3.1.18.4 we have $\mathfrak{A} \subseteq \mathfrak{A}'$ and applying Lemma 3.1.18.3 we have $\mathfrak{A}'' \subseteq \mathfrak{A}'$, and applying Lemma 3.1.18.3 again we have $\mathfrak{A}'' \subseteq (\mathfrak{A}'')'$, as required. \square

The double commutant defines the following functor on the posets of sub-semialgebras.

Definition 3.1.20. Let v be the functor

$$\begin{array}{ccc} \mathcal{A}\text{-Alg}(X) & \xrightarrow{v} & \mathcal{A}\text{-Alg}_{\text{vN}}(X) \\ \mathfrak{A} \vdash & \longrightarrow & \mathfrak{A}'' \end{array}$$

We need to check that v does in fact define a functor. Lemma 3.1.19.1 shows that v is defined on objects – that \mathfrak{A}'' is a von Neumann semialgebra. Checking the action on morphisms amounts to checking that for $\mathfrak{B} \hookrightarrow \mathfrak{A}$ we have $\mathfrak{B}'' \hookrightarrow \mathfrak{A}''$, which follows directly from two applications of Lemma 3.1.18.3.

We are not interested in von Neumann semialgebras directly, however, the following theorem shows that the functor v is an adjoint, and the existence of this adjunction will allow us to exploit the nicer properties of von Neumann semialgebras, results which will carry across the adjunction. This is not only true for C^* -algebras, but applies to \mathbb{S}^* -semialgebras on the most general level. In particular, in Section 3.2 when we consider semialgebras of quantale-valued relations we will make extensive use of the double commutant property, but using the adjunctions described below our results will apply to the full poset of \mathbb{S}^* -subsemialgebras.

Theorem 3.1.21. *There is an adjunction*

$$\begin{array}{ccc} & i & \\ \mathcal{A}\text{-Alg}_{\text{vN}}(X) & \begin{array}{c} \curvearrowright \\ \top \\ \curvearrowleft \end{array} & \mathcal{A}\text{-Alg}(X) \\ & v & \end{array}$$

Proof. We need to define a pair of natural transformations: the unit

$$\mathrm{id} \xrightarrow{\eta} i \circ v$$

and the counit

$$v \circ i \xrightarrow{\varepsilon} \mathrm{id}$$

Recall, Lemma 3.1.19 shows that for each commutative \mathbb{S}^* -semialgebra \mathfrak{A} , there is an inclusion $\mathfrak{A} \hookrightarrow \mathfrak{A}''$, and we can take the component $\eta_{\mathfrak{A}}$ to be this inclusion.

For a pair of commutative \mathbb{S}^* -semialgebras \mathfrak{B} and \mathfrak{A} such that $\mathfrak{B} \hookrightarrow \mathfrak{A}$, by Lemma 3.1.18.3 we have $\mathfrak{B}'' \hookrightarrow \mathfrak{A}''$ and hence the diagram

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\eta_{\mathfrak{B}}} & \mathfrak{B}'' \\ \downarrow & & \downarrow \\ \mathfrak{A} & \xrightarrow{\eta_{\mathfrak{A}}} & \mathfrak{A}'' \end{array}$$

commutes, and therefore η is natural.

To define the counit, for each von Neumann \mathbb{S}^* -semialgebra we need to define

$$\mathfrak{A}'' \xrightarrow{\varepsilon_{\mathfrak{A}}} \mathfrak{A}$$

Recall, a von Neumann \mathbb{S}^* -semialgebra is one satisfying $\mathfrak{A}'' = \mathfrak{A}$, and hence we can take $\varepsilon_{\mathfrak{A}}$ to be the identity on \mathfrak{A} . Note that for a pair of von Neumann \mathbb{S}^* -semialgebras $\mathfrak{B} \hookrightarrow \mathfrak{A}$ the diagram

$$\begin{array}{ccc} \mathfrak{B}'' & \xrightarrow{\varepsilon_{\mathfrak{B}}} & \mathfrak{B} \\ \downarrow & & \downarrow \\ \mathfrak{A}'' & \xrightarrow{\varepsilon_{\mathfrak{A}}} & \mathfrak{A} \end{array}$$

commutes, and hence ε is natural.

It remains to check the triangle diagrams of Definition 2.1.11, that is, we need to check the following diagram commute

$$\begin{array}{ccc}
 v & \xrightarrow{v\eta} & v \circ i \circ v \\
 & \searrow \text{id}_v & \downarrow \varepsilon_v \\
 & & v
 \end{array}
 \qquad
 \begin{array}{ccc}
 i & \xrightarrow{\eta_i} & i \circ v \circ i \\
 & \searrow \text{id}_i & \downarrow i\varepsilon \\
 & & i
 \end{array}$$

Checking the left hand diagram amounts to checking that for each commutative \mathbb{S}^* -semialgebra \mathfrak{A} the diagram

$$\begin{array}{ccc}
 \mathfrak{A}'' & \xrightarrow{v\eta_{\mathfrak{A}}} & \mathfrak{A}'''' \\
 & \searrow \text{id}_{\mathfrak{A}''} & \downarrow \varepsilon_{\mathfrak{A}''} \\
 & & \mathfrak{A}''
 \end{array}$$

commutes. By Lemma 3.1.18.6 we have $\mathfrak{A}'' = \mathfrak{A}''''$ and the diagram commutes.

Checking the right hand triangle identity amounts to checking that for each von Neumann \mathbb{S}^* -semialgebra \mathfrak{A} , the diagram

$$\begin{array}{ccc}
 \mathfrak{A} & \xrightarrow{\eta_{i\mathfrak{A}}} & \mathfrak{A}'' \\
 & \searrow \text{id}_{\mathfrak{A}} & \downarrow \varepsilon_{\mathfrak{A}} \\
 & & \mathfrak{A}
 \end{array}$$

commutes. Since \mathfrak{A} is assumed to be von Neumann, morphism $\eta_{i\mathfrak{A}} : \mathfrak{A} \hookrightarrow \mathfrak{A}''$ is the identity, and $\varepsilon_{\mathfrak{A}}$ is the identity morphism by definition, and therefore the diagram commutes, and hence we have $v \dashv i$, as required. \square

Recall, Theorem 2.1.14 showed that adjunctions lift the level of presheaf categories. We will now apply this result to the adjunction described in Theorem 3.1.21.

Theorem 3.1.22. *The adjunction described in Theorem 3.1.21 lifts to an adjunction*

$$\begin{array}{ccc} & i^* & \\ \text{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} & \xrightarrow{\quad} & \text{Set}^{\mathcal{A}\text{-Alg}_{\text{vN}}(X)^{\text{op}}} \\ & \xleftarrow{\quad} & \\ & v^* & \end{array}$$

with i^* and v^* defined as in Theorem 2.1.14.

Proof. This is a direct application of Theorem 3.1.21 and Theorem 2.1.14. \square

We will make extensive use of Theorem 3.1.22, in particular it will allow us to prove results using the nicer properties of von Neumann semialgebras and then transport them across the adjunction and show that they apply in the more general setting. To this end we will use Theorem 3.1.22 in conjunction with the following lemma.

Lemma 3.1.23. *The functor v^* as defined in Theorem 3.1.22 preserves the terminal object.*

Proof. Let $C_{\{*\}} : \mathcal{A}\text{-Alg}_{\text{vN}}(X)^{\text{op}} \rightarrow \mathbf{Set}$ be the terminal presheaf, that is the presheaf which sends every object to the singleton $\{*\}$ and every morphism to the identity on $\{*\}$. The presheaf $v^*(C_{\{*\}})$ is defined on objects

$$\begin{aligned} v^*(C_{\{*\}})(\mathfrak{A}) &= C_{\{*\}}(v(\mathfrak{A})) \\ &= C_{\{*\}}(\mathfrak{A}'') \\ &= \{*\} \end{aligned}$$

and hence $v^*(C_{\{*\}})$ is the constant presheaf on $\{*\}$ in the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$, as required. \square

The Generalised Gelfand Spectrum

We now generalise the Gelfand spectrum of a commutative C^* -algebra to an \mathbb{S}^* -semialgebra. The following definition was first introduced in [72].

Definition 3.1.24. Let \mathcal{A} be a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts, let \mathbb{S} be the semiring $\text{Hom}(I, I)$, and let X be an object in \mathcal{A} . The *generalised Gelfand spectrum* characterises the presheaf

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\text{Spec}_{\mathbb{G}}} \mathbf{Set}$$

defined on objects to be the set of *characters*

$$\text{Spec}_{\mathbb{G}}(\mathfrak{A}) = \{ \rho : \mathfrak{A} \rightarrow \mathbb{S} \mid \rho \text{ an } \mathbb{S}^*\text{-semialgebra homomorphism} \}$$

while the action on morphisms is given by restriction.

We use the term *spectral presheaf framework* to describe those representations of physical systems by presheaf categories of the form $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$, where the presheaf characterised by the Gelfand spectrum $\text{Spec}_{\mathbb{G}}$ plays a central role.

In Remark 2.2.12 we mentioned that there are other notions of spectrum aside from the Gelfand spectrum which coincide for commutative C^* -algebras. For more general commutative semialgebras these different notions of spectrum will not necessarily coincide, and it will be useful for us to consider them separately.

To generalise the prime spectrum we will need the notion of an ideal for a semiring.

Definition 3.1.25. Let R be a commutative semiring. A proper subset $J \subset R$ is called an *ideal* if it contains 0, is closed under addition, and for all $s \in R$ and $a \in J$, $as \in J$.

An ideal is called *prime* if $st \in J$ implies $s \in J$ or $t \in J$. A *k-ideal* (or *subtractive ideal*) is an ideal J such that if $a \in J$ and $a + b \in J$ then $b \in J$. A *k*-ideal* of a $*$ -semiring is a k -ideal closed under involutions.

The k -ideals of a semiring are the most natural generalisation of the ideals of a ring, since they are the ideals by which one can form a quotient. It is easy to check that for any ring considered as a semiring every ideal is a k -ideal.

The ideals and k -ideals of an S -semialgebra M are defined to be the ideals and k -ideals of $(M, \cdot_M, 1_M, +_M, 0_M)$ as a semiring.

Example 3.1.26. An *idempotent semiring* is a semiring R where $x + x = x$ for all $x \in R$. Idempotent semirings naturally carry a partial order where we define $x \leq y$ if and only if $x + y = y$. The k -ideals of an idempotent semiring are precisely those ideals which are downward closed with respect to this partial order.

Definition 3.1.27. The *prime spectrum* $\text{Spec}_P(\mathfrak{A})$ of a commutative S^* -semialgebra \mathfrak{A} is defined to be the set of prime k^* -ideals of \mathfrak{A} .

The prime spectrum characterises a spectral presheaf, defined as follows.

Definition 3.1.28. Let \mathcal{A} be a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts, and X an object. Let

$$\mathcal{A}\text{-}\mathbf{Alg}_{\text{vN}}(X)^{\text{op}} \xrightarrow{\text{Spec}_P} \mathbf{Set}$$

be the presheaf defined on objects

$$\text{Spec}_P(\mathfrak{A}) = \{ J \subset \mathfrak{A} \mid J \text{ a prime } k^*\text{-ideal} \}$$

while for $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ the action on morphisms

$$\text{Spec}_P(\mathfrak{A}) \xrightarrow{i^*} \text{Spec}_P(\mathfrak{B})$$

is defined $i^*(K) = \{ x \in \mathfrak{B} \mid i(x) \in K \}$.

To see that Spec_P is functorial we must check that $i^*(K)$ is a prime k^* -ideal. We omit the proof, but one can see this following a similar argument to that of Golan [94, Proposition 6.13].

Remark 3.1.29. An element of the Gelfand spectrum can be interpreted as a state because it can be seen as the simultaneous assignment of outcome values to each measurement, however this picture no longer necessarily holds for elements of the prime spectrum. However, Nestruev discusses how these other notions of spectrum still have a meaningful interpretation as “states” of a physical system,

see [172, Chap. 8]. We will see an explicit example of the relationship between different notions of spectra when we consider semialgebras of quantale-valued relations in Section 3.2.

Remark 3.1.30. One can also define a functor which assigns to each \mathfrak{A} the collection of all prime ideals, not just the prime k^* -ideals – for details see [94, Chap. 6] – although for the purposes of this work k^* -ideals are a more natural choice, as there is a closer connection with the generalised Gelfand spectrum.

One can define the *maximal spectrum* for an arbitrary semialgebra or semiring, taken to be the set of maximal ideals, although this fails to be functorial in general, see [195, Chap. 2. §5] for example, and so does not define a presheaf.

In Remark 2.2.12 we discussed that for **Hilb** the prime spectrum and Gelfand spectrum coincide. In earlier work [72] we showed that the same is true for the category of sets and relations **Rel**, although we will see in Example 3.2.21 that this is not the case in general.

Theorem 3.1.31. *Consider the category of sets and relations **Rel** and let X be a set. Let $\mathfrak{A} \subset \text{Hom}(X, X)$ be a commutative $\mathbf{2}^*$ -subsemialgebra. There is an isomorphism of the Gelfand spectrum and the prime spectrum, that is*

$$\text{Spec}_{\mathbf{G}}(\mathfrak{A}) \cong \text{Spec}_{\mathbf{P}}(\mathfrak{A})$$

where $J \subset \mathfrak{A}$ is prime k^* -ideal if and only if it is the kernel of a semialgebra homomorphism $\rho : \mathfrak{A} \rightarrow \mathbf{2}$.

Theorem 3.1.31 follows as a special case of Theorem 3.2.16.

Topologising the Gelfand Spectrum

Recall, Gelfand duality – which we considered in Theorem 2.2.13 – states that the Gelfand spectrum of a commutative C^* -algebra comes naturally equipped with the structure of a compact Hausdorff topological space. We will now show that the more general Gelfand and prime spectra of an \mathbb{S}^* -semialgebra comes

naturally equipped with the structure of a topological space. The topology which these spectra carry is the *Gelfand topology*, or more generally the *Zariski topology*.

Definition 3.1.32. Let \mathcal{A} be a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts. Let X be some object, and let \mathfrak{A} be an object in $\mathcal{A}\text{-Alg}(X)$. For each ideal $J \subset \mathfrak{A}$ define the set

$$\mathbb{V}_P(J) = \{ K \in \text{Spec}_P(\mathfrak{A}) \mid J \subset K \}$$

and take these sets $\mathbb{V}_P(J)$ to be a basis of closed sets for the *Zariski topology* on $\text{Spec}_P(\mathfrak{A})$.

Similarly, for each ideal $J \subset \mathfrak{A}$ define the set

$$\mathbb{V}_G(J) = \{ \rho \in \text{Spec}_G(\mathfrak{A}) \mid J \subset \ker(\rho) \}$$

and take these sets $\mathbb{V}_G(J)$ to be a basis of closed sets for the *Gelfand topology* on $\text{Spec}_G(\mathfrak{A})$.

Recall, a topological space is said to be T_0 if all points are *topologically distinguishable*, that is, for every pair of points x and y there is at least one open set containing one but not both of these points. The generalised Gelfand spectrum of a commutative \mathbb{S}^* -semialgebra equipped with a Zariski topology need not even be T_0 , as we will see in Example 3.2.23. It follows from general results about semirings that the prime spectrum equipped with the Zariski topology is always T_0 [94, Proposition 6.14].

It also follows from general results about semirings that both $\text{Spec}_G(\mathfrak{A})$ and $\text{Spec}_P(\mathfrak{A})$ are compact topological spaces when equipped with the Gelfand and Zariski topology respectively. This can be seen following the argument in [94, Proposition 6.14].

Theorem 3.1.33. *For a pair of commutative \mathbb{S}^* -semialgebras $\mathfrak{A}, \mathfrak{B}$ such that*

$i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ the corresponding map

$$\mathrm{Spec}_P(\mathfrak{A}) \xrightarrow{i^*} \mathrm{Spec}_P(\mathfrak{B})$$

as defined in Definition 3.1.28, is continuous with respect to the Zariski topology, and the function

$$\mathrm{Spec}_G(\mathfrak{A}) \xrightarrow{i^*} \mathrm{Spec}_G(\mathfrak{B})$$

as defined in Definition 3.1.24, is continuous with respect to the Zariski topology.

Proof. The proofs for these two statements are very similar. Consider $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ with corresponding $i^* : \mathrm{Spec}_P(\mathfrak{A}) \rightarrow \mathrm{Spec}_P(\mathfrak{B})$. We will show that for a closed set $\mathbb{V}_P(J) \subset \mathrm{Spec}_G(\mathfrak{A})$, the preimage $i^{*-1}(\mathbb{V}_P(J))$ is a closed set of the form $\mathbb{V}_P(K)$ for $K \subset \mathfrak{B}$ an ideal.

We have

$$i^{*-1}(\mathbb{V}_P(J)) = \{ Q \subset \mathfrak{A} \mid J \subset i^*(Q), \text{ for } Q \text{ a } k^*\text{-prime ideal} \}$$

We let $K \subset \mathfrak{A}$ be the ideal generated by the image of J , and we will show that $i^{*-1}(\mathbb{V}_P(J)) = \mathbb{V}_P(K)$.

First we will show $\mathbb{V}_P(K) \subseteq i^{*-1}(\mathbb{V}_P(J))$. Let $Q \in \mathbb{V}_P(K)$ – that is, $K \subset Q$ – then since $i(J) \subset K$ we have $Q \in i^{*-1}(\mathbb{V}_G(J))$, and therefore $\mathbb{V}_P(K) \subseteq i^{*-1}(\mathbb{V}_P(J))$.

Now we show the reverse inclusion $i^{*-1}(\mathbb{V}_P(J)) \subseteq \mathbb{V}_P(K)$ suppose $Q \in i^{*-1}(\mathbb{V}_P(J))$ – that is, $J \subseteq i^*(Q)$. Since $i^*(Q)$ is the collection of all elements $x \in \mathfrak{B}$ such that $i(x) \in Q$ for all $a \in J$ we have $i(a) \in Q$. So for $x \in K$ we have $x = a_1x_1 + \dots + a_nx_n$ with each $a_j \in J$, and since J is an ideal, each $a_nx_n \in J$, and hence $x \in J$ and therefore $i^*(x) \in Q$, and therefore $i^{*-1}(\mathbb{V}_P(J)) \subseteq \mathbb{V}_P(K)$, as required.

Consider $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ with corresponding $i^* : \mathrm{Spec}_G(\mathfrak{A}) \rightarrow \mathrm{Spec}_G(\mathfrak{B})$. We will show that for a closed set $\mathbb{V}_G(J) \subseteq \mathrm{Spec}_G(\mathfrak{A})$, the preimage $i^{*-1}(\mathbb{V}_G(J))$ is a

closed set of the form $\mathbb{V}_G(K)$ for $K \subset \mathfrak{B}$ an ideal.

$$i^{*-1}(\mathbb{V}_G(J)) = \{ \rho : \mathfrak{A} \rightarrow S \mid J \subset \ker(\rho \circ i) \}$$

Let $K \subset \mathfrak{A}$ be the ideal generated by the image of J , that is $x \in K$ if $x = a_1x_1 + \dots + a_nx_n$ for $a_j \in J$ and $x_k \in \mathfrak{A}$. The claim is $i^{*-1}(\mathbb{V}_G(J)) = \mathbb{V}_G(K)$.

Let $\rho \in \mathbb{V}_G(K)$ – that is, $\rho : \mathfrak{A} \rightarrow S$ with $K \subseteq \ker(\rho)$, then since $J \subseteq K$ we have $\rho \circ i(J) = 0$ and hence $\rho \in i^{*-1}(\mathbb{V}_G(J))$.

Conversely, suppose $\gamma : \mathfrak{A} \rightarrow S$ belongs to $i^{*-1}(\mathbb{V}_G(J))$, then for all $a \in J$ we have $\gamma(a) = 0$. Let $x \in K$ – that is, $x = a_1x_1 + \dots + a_nx_n$ – then for $a_j \in J$ and $x_k \in \mathfrak{A}$. Therefore

$$\begin{aligned} \gamma(x) &= \gamma(a_1)\gamma(x_1) + \dots + \gamma(a_n)\gamma(x_n) \\ &= 0 \end{aligned}$$

and hence $K \subseteq \ker(\gamma)$, as required. \square

Theorem 3.1.33 states the the prime spectrum and Gelfand spectrum define functors of the form

$$\mathcal{A}\text{-Alg}(A)^{\text{op}} \xrightarrow{\text{Spec}_P} \mathbf{ComTop} \qquad \mathcal{A}\text{-Alg}(A)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{ComTop}$$

where \mathbf{ComTop} is the category of compact topological spaces and continuous functions. In the case when $\mathcal{A} = \mathbf{Hilb}$ we have a natural isomorphism between these two spectral presheaves and the topology is Hausdorff. In the more general case, the topology on $\text{Spec}_P(\mathfrak{A})$ need not be Hausdorff but will be at least T_0 , while we will see in the next section an example where the topology on $\text{Spec}_G(\mathfrak{A})$ need not even be T_0 .

3.2 Semialgebras of Quantale-Valued Relations

We now turn our attention to the category of quantale-valued relations over a fixed quantale Q . This will give us an in-depth look at an example of our framework other than the standard quantum setting in which $\mathcal{A} = \mathbf{Hilb}$.

This section contains technical results which are not essential to the broader narrative of this work and could be safely skipped on first reading.

In later chapters we will use these results in illustrating the concepts we develop for these non-standard models, in particular: in Chapter 5 we will show that such models admit local hidden-variable models; and in Chapter 7 we will use relations – a special case of quantale-valued relations – to model Spekkens’ Toy Theory.

Most of the results in this section appeared in [74], and a standard reference for quantales is [187].

Quantales and Quantale-Valued Relations

We now define quantales and the category of sets and quantale-valued relations, which generalises the category \mathbf{Rel} of sets and relations.

Definition 3.2.1. A *join-semilattice* consists of a poset L together with finite joins, that is for every pair of elements $x, y \in L$ the join or greatest lower bound $x \vee y \in L$ exists.

A *complete* join-semilattice is one such that for arbitrary subsets of $K \subseteq L$ the join or greatest lower bound $\bigvee K \in L$ exists.

Definition 3.2.2. A *lattice* consists of a poset L such that every pair of elements $x, y \in L$ has both a join or greatest lower bound $x \vee y \in L$, and a meet or least upper bound $x \wedge y \in L$.

A lattice is said to be *distributive* if it satisfies one of the following (equivalent)

conditions:

$$\begin{aligned}x \vee (y \wedge z) &= (x \vee y) \wedge (x \vee z); \\x \wedge (y \vee z) &= (x \wedge y) \vee (x \wedge z).\end{aligned}$$

A *complete* lattice is one for which arbitrary meets and joins exist.

Definition 3.2.3. A distributive lattice B is called *Boolean* or a *Boolean algebra* if it has a top element 1 and a bottom element 0 , and comes equipped with an *orthocomplement*, that is, a function

$$L \xrightarrow{(-)^\perp} L$$

that for all $x, y \in L$:

1. $(x^\perp)^\perp = x$;
2. $x \leq y$ if and only if $y^\perp \leq x^\perp$;
3. $x \wedge x^\perp = 0$;
4. $x \vee x^\perp = 1$.

A *complete* Boolean algebra is one that is complete as a lattice. An *atom* in a Boolean algebra is an element $a \in B$ such that $a \neq 0$ but for any $x \in B$ such that $x \leq a$ we have $x = a$. A Boolean algebra is said to be *atomic* if for all $x \in B$ we have $x = \bigvee_{i \in I} a_i$, for some collection of atoms a_i .

A homomorphism of Boolean algebras is a function which preserves, 0 , 1 , \vee , \wedge , and $(-)^{\perp}$.

Definition 3.2.4. A *quantale* $(Q, \vee, \cdot, 1_Q)$ is a complete join-semilattice (Q, \vee) equipped with a monoid operation $\cdot : Q \times Q \rightarrow Q$ with unit 1_Q such that for any $x \in Q$ and $P \subseteq Q$

$$x \cdot \left(\bigvee_{y \in P} y \right) = \bigvee_{y \in P} (x \cdot y) \quad \text{and} \quad \left(\bigvee_{y \in P} y \right) \cdot x = \bigvee_{y \in P} (y \cdot x)$$

An *involutive quantale* is one equipped with an involution map $*$: $Q \rightarrow Q$ which is a semilattice homomorphism which is an involution $(x^*)^* = x$ satisfying $(x \cdot y)^* = y^* \cdot x^*$ and $1_Q^* = 1_Q$. A *commutative quantale* is one for which the monoid operation is commutative. A *subquantale* is a subset of Q closed under all joins and the monoid operation and containing 1_Q .

Example 3.2.5. Every complete Boolean algebra can be seen as a commutative quantale where one takes multiplication to be the meet \wedge .

We are primarily interested in involutive commutative quantales, however, note that every commutative quantale can be equipped with the trivial involution. A quantale has a least element \perp , defined to be the join of the empty set, and this is an absorbing element – that is, for all $x \in Q$ we have $x \cdot \perp = \perp$. We assume all quantales are non-trivial, that is, $\perp \neq \top$, where $\top = \bigvee_{x \in Q} x$.

Remark 3.2.6. An involutive quantale Q is a $*$ -semiring with addition given by the join and multiplication given by the monoid operation. The bottom element \perp is the zero element of the semiring and will hence be denoted 0 . We say a quantale is *zero-divisor free (ZDF)* if it is zero-divisor free as a semiring.

Example 3.2.7. The intervals $[0, 1]$ and $[0, \infty]$ are quantales when equipped with the usual multiplication, and where $\bigvee S = \sup S$.

The following definition generalises the category **Rel** of sets and relations.

Definition 3.2.8. For a commutative involutive quantale Q , the category of *quantale-valued relations* **Rel** $_Q$ has sets as objects and morphisms $f : X \rightarrow Y$ consist of functions $f : X \times Y \rightarrow Q$. For $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ composition is defined where $g \circ f : X \times Z \rightarrow Q$ by

$$g \circ f(x, z) = \bigvee_{y \in Y} f(x, y) \cdot g(y, z)$$

We say that a morphism $f : X \rightarrow Y$ in **Rel** $_Q$ *relates* $x \in X$ to $y \in Y$ if $f(x, y) \neq 0$.

The following result is folklore, see for example [9, §5.2].

Theorem 3.2.9. *The category \mathbf{Rel}_Q is a locally small \dagger -symmetric monoidal category with distributive \dagger -biproducts with: the monoidal product is given by the Cartesian product, with unit the one element set; the biproduct is given by disjoint union; and the dagger is given by reordering and pointwise application of the involution $f^\dagger(y, x) = f(x, y)^*$.*

Remark 3.2.10. The category \mathbf{Rel}_Q can also be defined as the category of relations over the underlying category of Q -valued sets [84, Chap II].

Example 3.2.11. Any complete Boolean algebra is a quantale. In particular the two-element Boolean algebra $\mathbf{2} = \{0, 1\}$, where the corresponding category \mathbf{Rel}_2 is just the category \mathbf{Rel} .

A Structure Theorem for Semialgebras of Quantale-Valued Relations

We now turn our attention to the category $\mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(X)$ for a set X . The monoidal unit for \mathbf{Rel}_Q is the singleton set $I = \{*\}$, and the set of morphisms $\text{Hom}(I, I)$, that is the set of functions

$$\{*\} \times \{*\} \xrightarrow{f} Q$$

is clearly isomorphic to the set Q , and moreover – viewing the quantale Q as a semiring – there is an isomorphism of quantales $\text{Hom}(I, I) \cong Q$, and hence an isomorphism as semirings.

Viewing Q as a semiring, then for each set X , the Q -semialgebra $\text{Hom}(X, X)$ is in fact a quantale, with the join given pointwise and multiplication given by morphism composition. That is, for a set $S \subset \text{Hom}(X, X)$ of Q -valued relations, we define $\bigvee S : X \rightarrow X$ to be the Q -relation defined for each pair of elements $x, y \in X$ by

$$\bigvee S(x, y) = \bigvee_{f \in S} f(x, y)$$

that is, the join is defined pointwise by the join in Q . Since morphism composition can be computed pointwise, it distributes over arbitrary joins.

We will now show that for a commutative Q -subsemialgebra $\mathfrak{A} \subset \text{Hom}(X, X)$, if \mathfrak{A} is a von Neumann semialgebra, that is, it satisfies $\mathfrak{A}'' = \mathfrak{A}$, then \mathfrak{A} is in fact a subquantale of $\text{Hom}(X, X)$.

Theorem 3.2.12. *Each commutative Q^* -subsemialgebra \mathfrak{A} in $\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)$ is a commutative subquantale of $\text{Hom}(X, X)$.*

Proof. By definition \mathfrak{A} is a subsemiring, we need to show that \mathfrak{A} is closed under arbitrary joins. Let $B \subseteq \mathfrak{A}$ be any subset, we need to show that $\bigvee_{x \in B} x \in \mathfrak{A}$. Let $g \in \mathfrak{A}'$ then for all $x \in B$ we have $g \cdot x = x \cdot g$. So we have

$$\begin{aligned} g \cdot \left(\bigvee_{x \in B} x \right) &= \bigvee_{x \in B} (g \cdot x) \\ &= \bigvee_{x \in B} (x \cdot g) \\ &= \left(\bigvee_{x \in B} x \right) \cdot g \end{aligned}$$

and hence $\bigvee_{x \in B} x \in \mathfrak{A}''$, and since \mathfrak{A} is von Neumann $\bigvee_{x \in B} x \in \mathfrak{A}$, as required. \square

Note that the converse to Theorem 3.2.12 fails, in particular, for a X a two-element set one can consider the subsemialgebra

$$\mathfrak{A} = \left\{ \begin{pmatrix} q & 0 \\ 0 & 0 \end{pmatrix} \mid q \in Q \right\}$$

which is a quantale isomorphic to Q , however, assuming Q is ZDF one can compute \mathfrak{A}'' to be

$$\mathfrak{A}'' = \left\{ \begin{pmatrix} q & 0 \\ 0 & p \end{pmatrix} \mid q, p \in Q \right\}$$

and hence \mathfrak{A} is a subquantale of $\text{Hom}(X, X)$ but not von Neumann, as $\mathfrak{A} \neq \mathfrak{A}''$.

We now give an important structure theorem for these von Neumann semialgebras. Recall the definition of a primitive subunital idempotent in a semiring, Definition 3.1.5. The subunital idempotents $f : X \rightarrow X$ in the category \mathbf{Rel}_Q

are those relations where for a given subset $Y \subseteq X$ we have $f(x, y) = 1_Q$ if and only if $x = y$ and $x \in Y$, that is, f is a subunital idempotent if it is the identity on some subset of X , and acts as the zero relation on all other elements.

The following lemma follows directly from, for example, [91, Chap. 14, Theorem 8].

Lemma 3.2.13. *Let $\mathcal{P}(X)$ be the Boolean algebra of subsets of X , and let $B \subseteq \mathcal{P}(X)$ be a Boolean subalgebra. If B is complete – in the sense of Definition 3.2.3 – then B is atomic.*

Theorem 3.2.14. *Let $(Q, \leq, \bigvee, \perp, \cdot, 1_Q)$ be a commutative ZDF quantale and let $\mathfrak{A} \in \mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)$. There are orthogonal primitive subunital idempotents $\{e_i\}$ such that*

$$\mathfrak{A} = \prod_i e_i \mathfrak{A}$$

which denotes the direct product of \mathbb{S}^* -semialgebras – that is, every element $f \in \mathfrak{A}$ can be written in the form

$$f = \bigvee_i f \circ e_i$$

Proof. Let $f : X \rightarrow X$ be a Q -relation. Let $\text{supp}(f) \subseteq X$, the *support* of f be the set of elements x such that there exists $y \in X$ such that f relates x to y . Let $\text{cosupp}(f) \subseteq X$, the *cosupport* of f be the set of elements x such that there exists $y \in X$ such that f relates y to x . Note that each Q^* -subsemialgebra $\mathfrak{A} \subset \text{Hom}(X, X)$ is assumed to be closed under \dagger , and hence for any $f \in \mathfrak{A}$ we have $f \circ f^\dagger = f^\dagger \circ f$. We claim that for Q -relations satisfying $f \circ f^\dagger = f^\dagger \circ f$ we have $\text{supp}(f) = \text{cosupp}(f)$. Suppose $x \in \text{supp}(f)$ then if Q is ZDF then $f^\dagger \circ f$ relates x to itself. However, if $x \notin \text{cosupp}(f)$ then clearly $f \circ f^\dagger$ cannot relate x to any other element – in the sense that $f \circ f^\dagger(x, y) = 0_Q$ for all $y \in X$ – and hence $x \in \text{supp}(f)$ if and only if $x \in \text{cosupp}(f)$. So $X = \text{supp}(f) \sqcup \overline{\text{supp}(f)}$, where $\overline{\text{supp}(f)}$ denotes the complement of $\text{supp}(f)$ in X , and f has a corresponding matrix representation $f = \begin{pmatrix} f_1 & 0 \\ 0 & 0 \end{pmatrix}$. For each $f \in \mathfrak{A}$ let $f_{\text{supp}} = \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix}$ be the relation which is the identity on the support of f and zero otherwise.

Let $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in \mathfrak{A}'$, then in particular $g \circ f = f \circ g$ and hence

$$\begin{pmatrix} g_1 f_1 & 0 \\ g_3 f_1 & 0 \end{pmatrix} = \begin{pmatrix} f_1 g_1 & f_1 g_2 \\ 0 & 0 \end{pmatrix}$$

and so if Q is ZDF then $g_2 = 0$ and $g_3 = 0$, and hence $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}$. Then clearly

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix} \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} \text{id} & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}$$

that is, $f_{\text{supp}} \circ g = g \circ f_{\text{supp}}$, and hence we have shown that $f_{\text{supp}} \in \mathfrak{A}''$, and hence by the assumption that \mathfrak{A} is von Neumann we have $f_{\text{supp}} \in \mathfrak{A}$. By a similar argument $f_{\overline{\text{supp}}} = \begin{pmatrix} 0 & 0 \\ 0 & \text{id} \end{pmatrix}$ also belongs to \mathfrak{A} .

Consider the collection of elements f_{supp} for all $f \in \mathfrak{A}$. Each f_{supp} corresponds with a subset of X and hence this collection forms a Boolean subalgebra of $\mathcal{P}(X)$, the powerset of X . By Theorem 3.2.12 \mathfrak{A} has all joins and hence this collection of subunital maps forms a complete Boolean subalgebra of $\mathcal{P}(X)$ which by Lemma 3.2.13 is atomic. The atoms e_i of this Boolean algebra are the primitive subunital idempotents of \mathfrak{A} , and $1_{\mathfrak{A}} = \bigvee e_i$. For every element $f \in \mathfrak{A}$ we have $f = \bigvee f \circ e_i$ for pairwise orthogonal subunital idempotents, and hence \mathfrak{A} is the direct product of the subalgebras $e_i \mathfrak{A}$. \square

Remark 3.2.15. This structure theorem follows from a more general structure theorem for a more general class of \dagger -symmetric monoidal categories with \dagger -kernels, which we showed in [73].

This structure theorem – Theorem 3.2.14 – will be extremely useful in characterising the spectra of these semialgebras. We now give a characterisation of the prime spectrum for semialgebras of quantale-valued relations.

Theorem 3.2.16. *For Q a commutative involutive ZDF quantale considered as a semialgebra, and \mathfrak{A} an object in $\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)$, then $J \subset \mathfrak{A}$ is a k^* -prime ideal if and only if it is the kernel of some Q^* -semialgebra homomorphism $\gamma : \mathfrak{A} \rightarrow \mathbf{2}$.*

Proof. Let $\gamma : \mathfrak{A} \rightarrow \mathbf{2}$ be an \mathbb{S}^* -semialgebra homomorphism. We need to show that $\ker(\gamma)$ satisfies the following: that for all $A, B \in \mathfrak{A}$ we have $AB \in \ker(\gamma)$ implies that $A \in \ker(\gamma)$ or $B \in \ker(\gamma)$; that if $A \in \ker(\gamma)$ and $A + B \in \ker(\gamma)$, then $B \in \ker(\gamma)$; and that if $A \in \ker(\gamma)$ then $A^\dagger \in \ker(\gamma)$.

To show the first condition suppose $AB \in \ker(\gamma)$, that is $\gamma(AB) = 0$, then we have $\gamma(A)\gamma(B) = 0$, and since the semialgebra $\mathbf{2}$ is ZDF it follows that either $\gamma(A) = 0$ or $\gamma(B) = 0$, that is, either $A \in \ker(\gamma)$ or $B \in \ker(\gamma)$.

Next, suppose $A \in \ker(\gamma)$ and $A + B \in \ker(\gamma)$, that is $\gamma(A) = 0$ and $\gamma(A + B) = 0$. Then we have

$$\begin{aligned}\gamma(B) &= \gamma(A) + \gamma(B) \\ &= \gamma(A + B) \\ &= 0\end{aligned}$$

and hence $B \in \ker(\gamma)$.

Finally, if $A \in \ker(\gamma)$, then $\gamma(A) = 0$. Since γ preserves the involution we have

$$\begin{aligned}\gamma(A^\dagger) &= \gamma(A)^* \\ &= 0^* \\ &= 0\end{aligned}$$

and hence $A^\dagger \in \ker(\gamma)$, and therefore $\ker(\gamma)$ is a prime k^* -ideal.

Conversely, let $J \subset \mathfrak{A}$, and define $\gamma : \mathfrak{A} \rightarrow \mathbf{2}$ as $\gamma(A) = 0$ if $A \in J$ and $\gamma(A) = 1$ otherwise. We need to check that γ defines a Q^* -semialgebra homomorphism, that is, we need to check: $\gamma(AB) = \gamma(A)\gamma(B)$ for all $A, B \in \mathfrak{A}$; $\gamma(A + B) = \gamma(A) + \gamma(B)$ for all $A, B \in \mathfrak{A}$; and $\gamma(A)^* = \gamma(A^\dagger)$ for all $A \in \mathfrak{A}$.

Since J is assumed to be prime $AB \in J$ if and only if $A \in J$ or $B \in J$, and hence we have $\gamma(AB) = 0$ if and only if $\gamma(A) = 0$ or $\gamma(B) = 0$. We also have $AB \notin J$ if and only if we have $A \notin J$ and $B \notin J$, and hence $\gamma(AB) = 1$ if and only if $\gamma(A) = 1$ and $\gamma(B) = 1$. Hence we see that $\gamma(AB) = \gamma(A)\gamma(B)$.

Since the addition of elements of \mathfrak{A} is given by the join operation of the underlying quantale, the additive structure is idempotent, that is for all $A \in \mathfrak{A}$ we have $A + A = A$. Now suppose $A + B \in J$, then $A + (A + B) \in J$, and since J is a k -ideal it follows that $A \in J$, and by the same argument $B \in J$. Hence we have $A + B \in J$ if and only if $A \in J$ and $B \in J$, and hence it follows that $\gamma(A + B) = 0$ if and only if $\gamma(A) = 0$ and $\gamma(B) = 0$, hence $\gamma(A + B) = \gamma(A) + \gamma(B)$.

Finally, since J is assumed to be a k^* -ideal we have $A \in J$ if and only if $A^\dagger \in J$, hence $\gamma(A) = 0$ if and only if $\gamma(A^\dagger) = 0$, as required. \square

By Theorem 3.2.12 we can consider a refinement of the prime spectrum which uses the fact that the von Neumann semialgebras are in fact quantales.

Definition 3.2.17. For Q a quantale and X a set, let

$$\mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\text{vN}}(X)^{\text{op}} \xrightarrow{\text{Spec}_Q} \mathbf{Set}$$

be the presheaf defined on objects

$$\text{Spec}_Q(\mathfrak{A}) = \{ \rho : \mathfrak{A} \rightarrow \mathbf{2} \mid \rho \text{ an involutive quantale homomorphism} \}$$

with the action on morphisms given by restriction.

Note that quantale homomorphisms preserve arbitrary joins, whereas a semialgebra homomorphism need only preserve finite joins.

The following lemma lets us see this quantale spectrum as a refinement of the prime spectrum.

Lemma 3.2.18. *For Q a ZDF quantale, and X a set, there is a monomorphic natural transformation*

$$\begin{array}{ccc} & \text{Spec}_P & \\ & \curvearrowright & \\ \mathbf{Rel}_Q\text{-}\mathbf{Alg}_{\text{vN}}(X)^{\text{op}} & \uparrow m & \mathbf{Set} \\ & \curvearrowleft & \\ & \text{Spec}_Q & \end{array}$$

Proof. If Q is an involutive ZDF quantale then $\mathbf{2}$ can be considered a Q -semialgebra, with $q \bullet 1 = 1$ for all $q \neq 0_Q$. Hence, every involutive quantale homomorphism $\rho : \mathfrak{A} \rightarrow \mathbf{2}$ can be viewed as a Q^* -semialgebra homomorphism. Hence, by Theorem 3.2.16 we have an inclusion $\text{Spec}_Q(\mathfrak{A}) \subseteq \text{Spec}_P(\mathfrak{A})$ for each \mathfrak{A} . \square

In fact, if the underlying set X is finite the quantale spectrum and prime spectrum will coincide.

We can give a complete characterisation of the quantale spectrum $\text{Spec}_Q(\mathfrak{A})$. This characterisation will be useful in Chapter 5 when we consider contextuality. Also, this result is remarkably similar to the case of commutative von Neumann C^* -algebras.

Theorem 3.2.19. *For $\mathfrak{A} \in \mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)$ with decomposition*

$$\mathfrak{A} = \prod_i e_i \mathfrak{A}$$

there is a one-to-one correspondence between the primitive subunital idempotents e_i and the elements of $\text{Spec}_Q(\mathfrak{A})$.

Proof. Let $\gamma \in \text{Spec}_Q(\mathfrak{A})$. First we show there is at most one primitive idempotent e_a such that $\gamma(e_a) = 1$. Suppose there is another e_b such that $\gamma(e_b) = 1$. Since e_a and e_b are orthogonal we have

$$\begin{aligned} 1_Q &= \gamma(e_a)\gamma(e_b) \\ &= \gamma(e_a e_b) \\ &= \gamma(0) \\ &= 0_Q \end{aligned}$$

a contradiction, and hence there is at most one e_a such that $\gamma(e_a) = 1$. Suppose there are no primitive idempotents which map to 1. We still have $\gamma(1_{\mathfrak{A}}) = \gamma(\bigvee e_i) = 1$. Since γ is a quantale homomorphism we have $\gamma(\bigvee e_i) = \bigvee \gamma(e_i)$,

a contradiction, and hence there is exactly one primitive idempotent satisfying $\gamma(e_a) = 1$. \square

Theorem 3.2.20. *For Q a commutative ZDF quantale there are natural transformations $\xi : \text{Spec}_G \rightarrow \text{Spec}_P$ and $\tau : \text{Spec}_P \rightarrow \text{Spec}_G$ such that $\xi \circ \tau \cong \text{id}$.*

Proof. For Q a quantale there is exactly one quantale homomorphism $! : \mathbf{2} \rightarrow Q$. For Q a ZDF quantale there is at least one homomorphism $w : Q \rightarrow \mathbf{2}$, which sends all non-zero elements to 1. Since Spec_P can be characterised by the collection of homomorphisms $\gamma : \mathfrak{A} \rightarrow \mathbf{2}$ let $\tau(\gamma) = ! \circ \gamma$. Similarly for $\rho : \mathfrak{A} \rightarrow Q$ define $\xi(\rho) = w \circ \rho$. Naturality is easy to check and clearly $w \circ ! \circ \gamma = \gamma$, as required. \square

Example 3.2.21. Let Q be the commutative involutive quantale $[0, 1]$ with usual multiplication, trivial involution, and where $\bigvee S = \sup S$. Let X be a two element set and consider \mathfrak{A} the von Neumann Q -semialgebra

$$\mathfrak{A} = \left\{ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mid p, q \in Q \right\} \cong Q \oplus Q$$

There are four elements of $\text{Spec}_P(\mathfrak{A})$:

$$J_1 = \left\{ \begin{pmatrix} p & 0 \\ 0 & 0 \end{pmatrix} \mid p \in Q \right\} \quad J_2 = \left\{ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mid p \in Q, q < 1 \right\}$$

$$K_1 = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & q \end{pmatrix} \mid q \in Q \right\} \quad K_2 = \left\{ \begin{pmatrix} p & 0 \\ 0 & q \end{pmatrix} \mid q \in Q, p < 1 \right\}$$

There are three semialgebra homomorphisms from Q to itself (viewing the quantale Q as a semialgebra): $u : Q \rightarrow Q$ defined as $u(x) = 1$ for all $x \neq 0$; $d : Q \rightarrow Q$ defined as $d(x) = 0$ for all $x < 1$; and the identity $\text{id} : Q \rightarrow Q$. Hence there are six homomorphisms

$$\varphi_1 = \langle d, 0 \rangle : Q \oplus Q \rightarrow Q \quad \varphi_2 = \langle u, 0 \rangle : Q \oplus Q \rightarrow Q \quad \varphi_3 = \langle \text{id}, 0 \rangle : Q \oplus Q \rightarrow Q$$

$$\theta_1 = \langle 0, d \rangle : Q \oplus Q \rightarrow Q \quad \theta_2 = \langle 0, u \rangle : Q \oplus Q \rightarrow Q \quad \theta_3 = \langle 0, \text{id} \rangle : Q \oplus Q \rightarrow Q$$

corresponding to the six elements of $\text{Spec}_G(\mathfrak{A})$.

There are only two elements of $\text{Spec}_Q(\mathfrak{A})$ corresponding with the two prime ideals J_1 and K_1 . The semialgebra homomorphisms $j_2, k_2 : \mathfrak{A} \rightarrow \mathbf{2}$ corresponding with prime ideals J_2 and K_2 do not satisfy the equation

$$j\left(\bigvee_{x \in B} x\right) = \bigvee_{x \in B} j(x)$$

The Topology of the Spectra of Semialgebras of Quantale-Valued Relations

We now show how the relationship between the prime and Gelfand spectra for semialgebras of quantale-valued relations can be interpreted from the topological perspective. In particular, we see that the correspondence described in Theorem 3.1.33 is compatible with the respective topologies of the Gelfand and prime spectra.

Theorem 3.2.22. *For Q a ZDF quantale and \mathfrak{A} in $\mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(X)$, each*

$$\text{Spec}_G(\mathfrak{A}) \xrightarrow{\xi_{\mathfrak{A}}} \text{Spec}_P(\mathfrak{A})$$

as defined in Theorem 3.2.20, is a quotient of topological spaces where $\rho_1 \sim \rho_2$ if and only if ρ_1 and ρ_2 are not distinguishable by the Zariski topology.

Proof. Clearly $\xi_{\mathfrak{A}}$ identifies those characters which have the same kernel. Recall a closed set A in a topological space is said to be *irreducible* if for any closed sets A_1 and A_2 such that $A \subseteq A_1 \cup A_2$ we have $A \subseteq A_1$ or $A \subseteq A_2$. Being T_0 is equivalent to the statement that no irreducible closed set is the closure of more than one singleton subset. Clearly the closure of a singleton is an irreducible closed set.

Let $\gamma \in \text{Spec}_G(\mathfrak{A})$ and consider $\overline{\{\gamma\}} \subseteq \text{Spec}_G(\mathfrak{A})$ be the intersection of all closed sets containing γ . Since the sets $\mathbb{V}_G(J)$ form a basis for the topology the closed set $\overline{\{\gamma\}}$ is some union and intersection of closed sets of the form $\mathbb{V}_G(J)$.

Since $\overline{\{\gamma\}}$ is irreducible it follows that $\overline{\{\gamma\}}$ is in fact just an intersection of sets of the form $\mathbb{V}_G(J)$ which contain γ , that is $\overline{\{\gamma\}} = \bigcap_{J \subseteq \ker(\gamma)} \mathbb{V}_G(J)$.

Hence, if $\ker(\gamma) = \ker(\rho)$ then

$$\begin{aligned} \overline{\{\gamma\}} &= \bigcap_{J \subseteq \ker(\gamma)} \mathbb{V}_G(J) \\ &= \bigcap_{J \subseteq \ker(\rho)} \mathbb{V}_G(J) \\ &= \overline{\{\rho\}} \end{aligned}$$

and hence we have an irreducible closed set which is both the closure of $\{\gamma\}$ and of $\{\rho\}$ and hence $\text{Spec}_G(\mathfrak{A})$ is not T_0 .

Finally we show that characters with the same kernels are the only indistinguishable points, that is if $\ker(\gamma) \neq \ker(\rho)$ then there exists an open set in $\text{Spec}_G(\mathfrak{A})$ which contains exactly one of γ or ρ . Suppose $\ker(\gamma) \neq \ker(\rho)$ but $\ker(\gamma) \subseteq \ker(\rho)$. Consider the closed set $\mathbb{V}_G(\ker(\gamma))$. This closed set does not contain ρ , and hence the complement of $\mathbb{V}_G(\ker(\gamma))$ is an open set containing ρ but not γ . Now suppose neither $\ker(\gamma) \subseteq \ker(\rho)$ nor $\ker(\rho) \subseteq \ker(\gamma)$, then again we have $\rho \notin \mathbb{V}_G(\ker(\gamma))$, and hence the complement of $\mathbb{V}_G(\ker(\gamma))$ is an open set containing ρ but not γ and hence these points are topologically distinguishable. \square

The map $\xi_{\mathfrak{A}}$ identifies those characters which have the same kernel, which are precisely those characters which the Zariski topology on $\text{Spec}_G(\mathfrak{A})$ cannot distinguish.

Theorem 3.2.22 allows us to think of $\text{Spec}_P(\mathfrak{A})$ as a coarse-graining of the state space $\text{Spec}_G(\mathfrak{A})$ of our physical system. To illustrate this we revisit Example 3.2.21.

Example 3.2.23. Let \mathfrak{A} be as in Example 3.2.21. The Zariski topology on $\text{Spec}_P(\mathfrak{A})$ has a basis consisting of the closed sets

$$\{J_1, J_2\}, \quad \{K_1, K_2\}, \quad \{J_1\}, \quad \{K_1\}$$

It is easy to check that this topology is T_0 but that it is not T_1 and therefore not Hausdorff. For $\text{Spec}_G(\mathfrak{A})$ the Zariski topology has a basis consisting of the closed sets

$$\{\varphi_1, \varphi_2, \varphi_3\}, \quad \{\varphi_1\}, \quad \{\theta_1, \theta_2, \theta_3\}, \quad \{\theta_1\}$$

It is easy to check that there is no open set distinguishing φ_2 and φ_3 from one another, nor θ_2 from θ_3 , as these respective pairs of characters have the same kernels and hence $\text{Spec}_G(\mathfrak{A})$ fails even to be T_0 .

Note that $\xi_{\mathfrak{A}}(\varphi_2) = \xi_{\mathfrak{A}}(\varphi_3) = K_1$ and $\xi_{\mathfrak{A}}(\theta_2) = \xi_{\mathfrak{A}}(\theta_3) = J_1$ (as defined in Theorem 3.2.20) and hence the topologically indistinguishable points in the generalised Gelfand spectrum are identified by the map $\xi_{\mathfrak{A}}$.

3.3 Semialgebras and Monoidal Quantum Theory

In this section we show how the notion of observable in the monoidal approach to quantum theory - a commutative H^* -algebra (X, μ) in \mathcal{A} - lifts naturally to the notion of observable in the spectral presheaf approach - a commutative von Neumann \mathbb{S}^* -semialgebra $\mathfrak{A} \subset \text{Hom}(X, X)$.

Definition 3.3.1. An element in a poset $p \in \mathbf{P}$ is said to be *maximal* if for any $q \in \mathbf{P}$ such that $p \leq q$ we have $p = q$.

Lemma 3.3.2. *The semialgebra \mathfrak{A} is a maximal element of $\mathcal{A}\text{-Alg}(X)$ if and only if $\mathfrak{A} = \mathfrak{A}'$.*

Proof. Suppose \mathfrak{A} is a maximal element. Since \mathfrak{A} is commutative, then by Lemma 3.1.18.4 we have $\mathfrak{A} \hookrightarrow \mathfrak{A}'$, and hence by maximality of \mathfrak{A} we have $\mathfrak{A} = \mathfrak{A}'$.

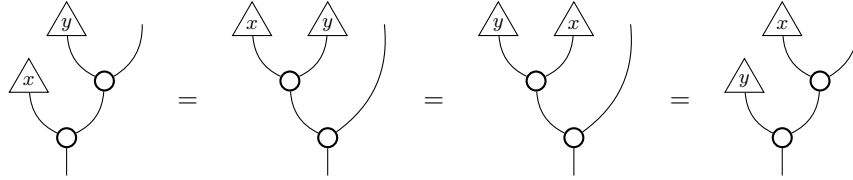
Conversely, suppose $\mathfrak{A} = \mathfrak{A}'$. Suppose $\mathfrak{A} \hookrightarrow \mathfrak{B}$ for some $\mathfrak{B} \in \mathcal{A}\text{-Alg}(X)$. By Lemma 3.1.18.3 we have $\mathfrak{B}' \hookrightarrow \mathfrak{A}'$ and hence $\mathfrak{B}' \hookrightarrow \mathfrak{A}$. Since \mathfrak{B} is commutative, then by Lemma 3.1.18.4 we have $\mathfrak{B} \hookrightarrow \mathfrak{B}'$ and therefore $\mathfrak{B} \hookrightarrow \mathfrak{A}$, as required. \square

Theorem 3.3.3. *Let \mathcal{A} be a monoidally well-pointed locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts, and let (X, μ) be a commutative H^* -algebra in \mathcal{A} . Consider the set of endomorphisms on X defined by*

$$R(\mu) = \left\{ R_x = \begin{array}{c} \triangle x \\ \curvearrowright \\ \circ \\ \downarrow \end{array} \mid \text{for all points } \begin{array}{c} \triangle x \\ \uparrow \end{array} \right\}$$

The subset $R(\mu)' \subseteq \text{Hom}(X, X)$ is a maximal commutative von Neumann \mathbb{S}^* -semialgebra. Moreover, if (X, μ) is unital – that is, satisfies (U) from Definition 2.3.9 – then $R(\mu) = R(\mu)'$.

Proof. First we show that the elements of $R(\mu)$ commute with one another. Consider $R_x \circ R_y$

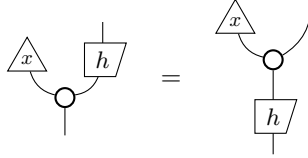


The first equality follows from the axiom (A), the second from (C), and the third from (A), and hence $R_x \circ R_y = R_y \circ R_x$, as required.

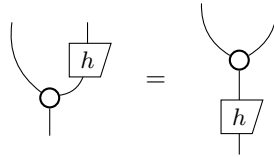
Since $R(\mu)$ is a commutative family, then by Lemma 3.1.18.4. $R(\mu) \subseteq R(\mu)'$. By the (H) axiom $R(\mu)$ is closed under \dagger and by Lemma 3.1.18.2. so is $R(\mu)'$. By Lemma 3.1.18.1. $R(\mu)'$ is closed under the algebraic operations and hence $R(\mu)'$ is a commutative \mathbb{S}^* -semialgebra.

By Lemma 3.3.2 the set $R(\mu)'$ is a maximal commutative von Neumann \mathbb{S}^* -semialgebra if and only if $R(\mu)' = R(\mu)''$. Since $R(\mu)$ is commutative, Lemma 3.1.18.4. implies $R(\mu) \subseteq R(\mu)'$, and therefore by Lemma 3.1.18.3. $R(\mu)'' \subseteq R(\mu)'$, and hence to prove maximality of $R(\mu)'$ it is enough to show $R(\mu)' \subseteq R(\mu)''$, which by Lemma 3.1.18.4. is equivalent to $R(\mu)'$ being a commutative family.

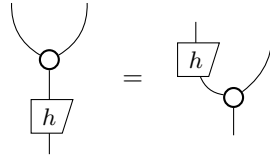
Consider $h \in R(\mu)'$, that is, for all \triangleup_x we have



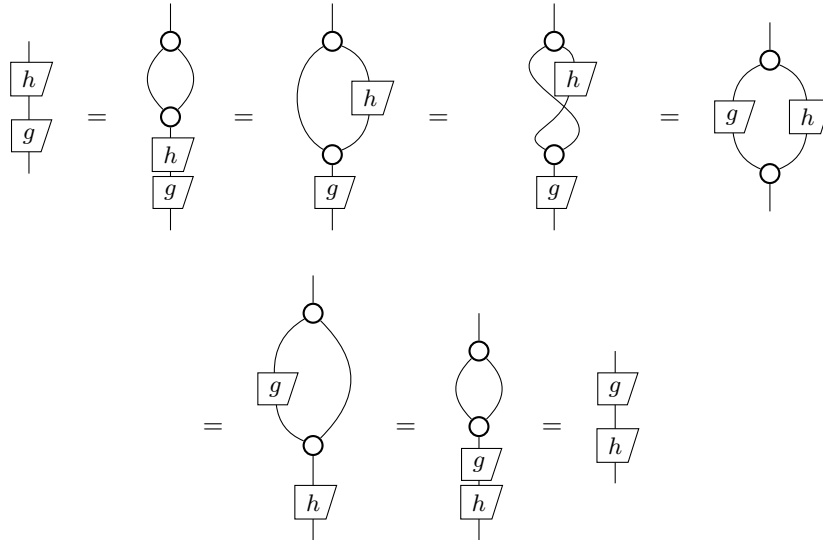
then, by monoidal well-pointedness we have



and by (C) we have



Hence for g and h in $R(\mu)'$ we have



and hence $R(\mu)' \subseteq R(\mu)''$, as required.

If (X, μ) is unital then for each $h \in R(\mu)'$ we have

and hence $h \in R(\mu)$, and therefore $R(\mu) = R(\mu)'$, as required. \square

Definition 3.3.4. Given an H^* -algebra (X, μ) we say that $R(\mu)'$ is the \mathbb{S}^* -semialgebra *generated* by (X, μ) .

Hence an “observable” in the monoidal approach – an H^* -algebra – gives rise to an “observable” system in the topos approach. Next we show that the notion of states in the former – set-like elements – determine states in the latter – elements of the Gelfand spectrum.

Remark 3.3.5. This approach of considering representations of the internal algebra is similar to [206] where every finite-dimensional C^* -algebra is a (non-commutative) Frobenius algebra in the category **fdHilb** of finite-dimensional Hilbert spaces.

It is natural to ask if the converse to Theorem 3.3.3 holds, that is, are all maximal von Neumann semialgebras generated by some H^* -algebra. It is clear that for the case of finite-dimensional Hilbert spaces this is the case as every maximal commutative von Neumann subalgebra of $\text{Hom}(H, H)$ is generated by mutually orthogonal one-dimensional projectors $P_i : H \rightarrow H$, for which we can define the corresponding orthogonal basis $|e_i\rangle$ which by Theorem 2.3.13 corresponds with a H^* -algebra structure on H . However, this correspondence does not hold for infinite-dimensional Hilbert spaces in general.

Remark 3.3.6. The converse of Theorem 3.3.3 does not hold in general, for example, taking $\mathcal{A} = \mathbf{Rel}$ there are maximal commutative \mathbb{S}^* -semialgebras that are not generated by H^* -algebra in **Rel**. For example, for X a two-element set, consider the semialgebra \mathfrak{A} with the following elements:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

It is straightforward to verify that this collection is closed under all of the algebraic operations, and moreover one can check directly that $\mathfrak{A}' = \mathfrak{A}$, and hence by Lemma 3.3.2, \mathfrak{A} is maximal and is a von Neumann semialgebra. Note that the only H^* -algebra structures in **Rel** are of the form of groupoids, but the only groupoid structures which X can be equipped with are \mathbb{Z}_2 and the disjoint union of trivial groups, neither of which generate the semialgebra \mathfrak{A} . Hence we see that there are maximal von Neumann semialgebras for **Rel** which do not arise from H^* -algebras in **Rel**.

The following theorem shows how the set-like elements naturally form a subset of the spectrum.

Theorem 3.3.7. *Let (X, μ) be an H^* -algebra with normal set-like elements and \mathfrak{A} the von Neumann semialgebra it generates. Each set-like element α of (X, μ) determines an \mathbb{S}^* -semialgebra homomorphism $\rho_\alpha : \mathfrak{A} \rightarrow \mathbb{S}$ defined*

$$\begin{array}{c} \square \\ | \\ f \\ | \\ \square \end{array} \mapsto \begin{array}{c} \triangle \\ | \\ \alpha \\ | \\ \square \\ | \\ f \\ | \\ \square \\ | \\ \alpha \\ | \\ \triangle \end{array}$$

Proof. It is easy to check ρ_α preserves zero, since if f factors through the zero-object then so does $\alpha^\dagger \circ f \circ \alpha$. To see that ρ_α preserves the multiplicative unit, note that the multiplicative unit is id_X and since the set-like element α is assumed to be normal we have $\alpha^\dagger \circ \alpha = \text{id}_I$.

Recall the involution in both \mathfrak{A} and \mathbb{S} is defined by the functor \dagger . Note that

we have

$$\begin{aligned}
 \rho_\alpha(f)^* &= (\alpha^\dagger \circ f \circ \alpha)^\dagger \\
 &= \alpha^\dagger \circ f^\dagger \circ \alpha^{\dagger\dagger} \\
 &= \alpha^\dagger \circ f^\dagger \circ \alpha \\
 &= \rho_\alpha(f^*)
 \end{aligned}$$

and hence we see that ρ_α preserves the involution.

To see that ρ_α preserved addition we note that

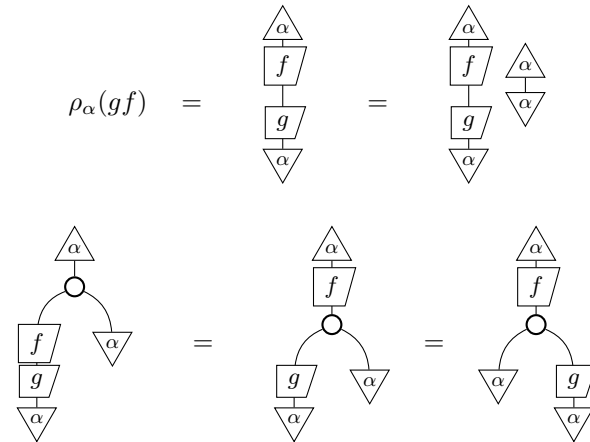
$$\rho_\alpha(f + g)^* = \alpha^\dagger \circ (f + g) \circ \alpha$$

and since addition is defined by biproduct convolution, by Lemma 3.1.8 we have

$$\begin{aligned}
 \alpha^\dagger \circ (f + g) \circ \alpha &= (\alpha^\dagger \circ f \circ \alpha) + (\alpha^\dagger \circ g \circ \alpha) \\
 &= \rho_\alpha(f) + \rho_\alpha(g)
 \end{aligned}$$

and hence ρ_α preserves addition.

To see ρ_α preserves multiplication, we have



$$\begin{aligned}
 &= \begin{array}{c} \triangle \alpha \\ \circ \\ \swarrow \quad \searrow \\ \square f \quad \square g \\ \triangle \alpha \quad \triangle \alpha \end{array} = \begin{array}{c} \triangle \alpha \quad \triangle \alpha \\ \square f \quad \square g \\ \triangle \alpha \quad \triangle \alpha \end{array} = \rho_\alpha(g)\rho_\alpha(f)
 \end{aligned}$$

as required. □

For H^* -algebras in **Hilb** and **Rel** the set-like elements of the H^* -algebra completely determine the spectrum of the corresponding semialgebra, that is, the subset of the spectrum identified with the set-like elements is in fact the entire spectrum. For **Rel** this is illustrated in Chapter 7.

The Generalised Phase Group Object

Recall the definition of the phase group of an internal algebra in a monoidal category – Definition 2.3.19. Phase groups represent the physical transformations compatible with a particular observable. In this section we show how this notion of phase group associated with an internal algebra structure can be lifted to the more general spectral presheaf approach; in particular, we will define the *phase group object* and show how it relates to the phase group.

Let $\mathcal{U}(X) \subseteq \text{Hom}(X, X)$ be the subset of unitary morphisms, that is those isomorphisms satisfying $U^\dagger = U^{-1}$. This is a group and there is a group action

$$\begin{aligned}
 \mathcal{U}(X) \times \text{Hom}(X, X) &\longrightarrow \text{Hom}(X, X) \\
 (U, A) &\longmapsto UAU^\dagger
 \end{aligned} \tag{3.2}$$

and this group action preserves all of the semialgebra structure associated with $\text{Hom}(X, X)$.

Definition 3.3.8. For a semialgebra $\mathfrak{A} \in \mathcal{A}\text{-Alg}(X)$ we define the *generalised phase group* for \mathfrak{A} to be the stabiliser $\text{stab}_{\mathfrak{A}} \subseteq \mathcal{U}(X)$ of the set \mathfrak{A} under the group action (3.2), where $U \in \text{stab}_{\mathfrak{A}}$ if

$$UAU^\dagger = A$$

for all $A \in \mathfrak{A}$.

There is an equivalent formulation of this generalised phase group: a unitary $U : X \rightarrow X$ belongs to $\text{stab}_{\mathfrak{A}}$ if and only if $U \in \mathfrak{A}'$. This definition can be turned into an object in the presheaf category.

Definition 3.3.9. The *phase presheaf* is the presheaf

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\Phi} \mathbf{Set}$$

defined

$$\Phi(\mathfrak{A}) = \{ U \in \mathfrak{A}' \mid U^\dagger U = U U^\dagger = \text{id}_X \}$$

To see that Φ is a functor recall from Lemma 3.1.18.3 that if $\mathfrak{B} \hookrightarrow \mathfrak{A}$ then $\mathfrak{A}' \hookrightarrow \mathfrak{B}'$, and hence the unitaries in \mathfrak{A}' are a subset of the unitaries in \mathfrak{B}' – that is, for $\mathfrak{B} \hookrightarrow \mathfrak{A}$ we have $\Phi(\mathfrak{A}) \subseteq \Phi(\mathfrak{B})$.

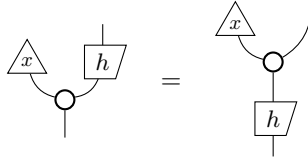
Theorem 3.3.10. Let (X, μ) be a H^* -algebra in a monoidally well-pointed category \mathcal{A} , and let \mathfrak{A} be the von Neumann semialgebra generated by (X, μ) (in the sense of Theorem 3.3.3). The phase group $G(\mu)$ of (X, μ) coincides exactly with the generalised phase group $\Phi(\mathfrak{A})$ of \mathfrak{A} .

Proof. Recall \mathfrak{A} is by definition $R(\mu)'$ where

$$R(\mu) = \{ R_x : X \rightarrow X \mid \text{for all } x : I \rightarrow X \}$$

Clearly then if $U \in G(\mu)$ then U commutes with every map of the form R_x and hence $U \in \mathfrak{A}$, and hence $U \in \mathfrak{A}'$, as required.

Conversely, suppose $h \in \mathfrak{A}'$, that is h commutes with all elements of $R(\mu)'$. Since $R(\mu) \subset R(\mu)'$, then in particular, h commutes with all elements of the form R_x , that is we have



for all $x : I \rightarrow X$. If \mathcal{A} is monoidally well-pointed then this equation implies that h is in fact a phase for (X, μ) , as required. \square

Next we recall the definition of a group object in a cartesian category [154, Chap. III. §6].

Definition 3.3.11. Let \mathcal{C} be a category with finite products and a terminal object. An *group object* in \mathcal{C} consists of an object G together with morphisms

$$\begin{aligned} \mathbf{1} &\xrightarrow{e} G \\ G \times G &\xrightarrow{m} G \\ G &\xrightarrow{(-)^{-1}} G \end{aligned}$$

such that the diagrams

$$\begin{array}{ccc} G \times G \times G & \xrightarrow{m \times \text{id}_G} & G \times G \\ \text{id}_G \times m \downarrow & & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccc} G & \xrightarrow{e \times \text{id}_G} & G \times G \\ \text{id}_G \times e \downarrow & \searrow \text{id}_G & \downarrow m \\ G \times G & \xrightarrow{m} & G \end{array}$$

$$\begin{array}{ccccc}
 & & G \times G & \xrightarrow{\text{id}_G \times (-)^{-1}} & G \times G & & \\
 & \nearrow \Delta & & & & \searrow m & \\
 G & & & & & & G \\
 & \searrow \Delta & & \xrightarrow{\text{id}_G} & & & \\
 & & G \times G & \xrightarrow{(-)^{-1} \times \text{id}_G} & G \times G & & \\
 & & & & & \nearrow m &
 \end{array}$$

commute.

Theorem 3.3.12. *The phase presheaf*

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\Phi} \mathbf{Set}$$

is a group object in the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$.

Proof. We define the natural transformations e , m and $(-)^{-1}$ component-wise – that is, for each von Neumann semialgebra \mathfrak{A} we need to specify functions

$$\mathbf{1}(\mathfrak{A}) \xrightarrow{e_{\mathfrak{A}}} \Phi(\mathfrak{A})$$

$$\Phi(\mathfrak{A}) \times \Phi(\mathfrak{A}) \xrightarrow{m_{\mathfrak{A}}} \Phi(\mathfrak{A})$$

$$\Phi(\mathfrak{A}) \xrightarrow{(-)^{-1}_{\mathfrak{A}}} \Phi(\mathfrak{A})$$

which obey the axioms of a group.

Each $\Phi(\mathfrak{A})$ is a subgroup of the group of unitaries $\text{Hom}(X, X)$, we simply define the component-wise group operations via the group structure inherited by each component. This defines natural transformations satisfying the commutative diagrams of Definition 3.3.11. \square

Remark 3.3.13. In the proof of Theorem 3.3.12 we essentially showed that the phase presheaf Φ is a subobject of the constant presheaf $\mathcal{C}_{\mathcal{U}(X)}$ where $\mathcal{U}(X)$ is the set of unitary endomorphisms of X , and moreover the corresponding morphism

$m : \Phi \rightarrow C_{\mathcal{U}(X)}$ is a homomorphism of group objects. This is discussed more in Chapter 8.

Chapter 4

The Quantum State Space

So far, the framework we have introduced is a direct generalisation of the topos quantum theory of Butterfield, Isham and Doering. In this chapter we make a clear departure from the approach of topos quantum theory, in particular, we take a very different approach towards how we conceptualise the *state space* of a quantum system.

Since we are adhering to a pragmatic approach to quantum theory, our definition of “state” is going to be rooted in the observed outcomes of measurements in the context of performing experiments. Bohr describes the role of experiments as follows [27, p. 209]:

By the word “experiment” we refer to a situation where we can tell others what we have done and what we have learned and that, therefore, the account of the experimental arrangement and of the results of the observations must be expressed in unambiguous language with suitable application of the terminology of classical physics.

The structure of an experimental procedure is described by Heinosaari and Ziman [109, p. 46]:

[In an experimental set-up] there is a collection of possible preparations and a collection of possible measurements and that any prepa-

ration and any measurement can be combined to form an experiment leading to a probability distribution of measurement outcomes.

This description of the experimental process is depicted in Figure 4.1.

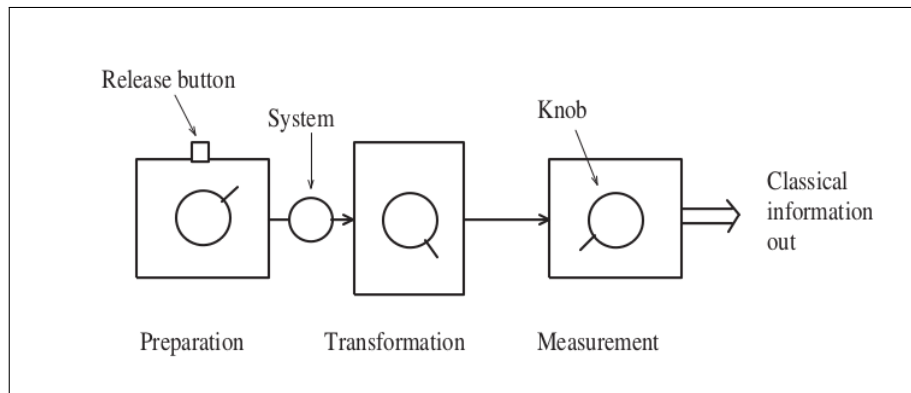


Figure 4.1: A schematic of a laboratory set-up. Image taken from [102, Fig. 1].

The concept of *state* enters into this picture in the manner described by Hardy [103, p. 1]:

The *state* associated with a particular preparation is defined to be (that thing represented by) any mathematical object that can be used to determine the probability associated with the outcomes of any measurement that may be performed on a system prepared by the given preparation.

The “state” corresponding with a given preparation is merely some mathematical representation of the empirical data that can be extracted from the system. We can make a tautologous definition of the state and simply define the state *as* that recorded data. This recorded data certainly can be used to determine the recorded data and so it does satisfy the definition of “state”. This is reflected in the definition of Heinosaari and Ziman [109, p. 48]:

We define a *state* of the system to be a collection of equivalent preparation procedures.

The “equivalent preparation procedures” are characterised as follows [109, p. 48]:

Two preparation procedures can be superficially quite different and yet lead to the same probability distribution in any chosen measurement. If this is the case we say that the two preparations are equivalent.

This instrumental or pragmatic definition of state is also used in *operational probabilistic theories*, see, for example [36, Definition 10].

We want a better representation of the state, one which exposes deeper structure that can be exploited to give us a better understanding of the properties of the system. We can make an instrumentalist definition of “state” for classical mechanics, but representing the state as a point on a smooth manifold allows to use powerful mathematical tools in our analysis of classical systems – most notably, the differential calculus.

In standard presentations of quantum theory – for example the axioms we showed in Figure 2.2 – the states of a quantum system are represented by density operators, and the empirical data one obtains from making measurements is computed via the Born rule. Density operators provide a much more compact and useful representation for the probabilities associated with measurement outcomes than just defining the state as that collection of probabilities. It is not obvious a priori that density operators are a reasonable representation of the quantum state; we start with the instrumentalist definition of state and derive the density operator representation from first principles. This amounts to a derivation of the Born rule.

In Section 4.1 we give a general definition of *operational states* within the general spectral presheaf framework that we introduced in Chapter 3. We then show that density operators give rise to such operational states for the special case when $\mathcal{A} = \mathbf{Hilb}$.

In Section 4.2 we give a complete characterisation of the operational states for $\mathcal{A} = \mathbf{Hilb}$, in particular, we show that operational states are in one-to-

one correspondence with density operators; that is we show that our notion of operational state is equivalent to the usual notion of quantum state. This characterisation of the operational states can be framed as a derivation of the Born rule from a set of operational axioms, which we present in Figure 4.2.

In Section 4.3 we show how the operational states inherit a convex structure analogous to the convex on the set of density operators on a Hilbert space. This general convex structure will play an important role in Chapters 5, 6, and 7.

4.1 Defining Operational States

The axiomatisation of quantum theory we showed in Figure 2.2 asserts that the outcomes of measurements are probabilistic. We interpret this to mean that if we repeat an experiment multiple times under identical circumstances we do not expect to see identical outcomes. If we fix a measurement context \mathfrak{A} , then repeating the experiment we will obtain a probabilistic spread of results for each measurement in \mathfrak{A} .

In order to talk about probabilities in a precise way we need to introduce some concepts from measure theory. Note that many terms in measure theory take on slightly different meanings in the literature. For the purposes of consistency we use Bogachev [25, 26] as our reference for all measure-theoretic definitions and results.

Definition 4.1.1. A σ -algebra Σ on a set X consists of a collection of subsets of X which:

1. contains the empty set \emptyset and X ;
2. is closed under complementation, that is, if $A \in \Sigma$ then $X \setminus A \in \Sigma$;
3. is closed under countable unions, that is, $\bigcup A_i \in \Sigma$ for any countable family $A_i \in \Sigma$;
4. is closed under countable intersections, that is, $\bigcap A_i \in \Sigma$ for any countable family $A_i \in \Sigma$.

A *subalgebra* of a σ -algebra Σ consists of a subset $\mathcal{A} \subseteq \Sigma$ which contains the empty set, is closed under complementation, and is closed under finite unions and finite intersections. Note that a subalgebra of a σ -algebra need not be a σ -algebra.

Definition 4.1.2. A *measurable space* consists of a set X together with σ -algebra Σ on X . We call the elements of Σ the *measurable subsets* of X .

A *measure* μ on a measurable space (X, Σ) is a function

$$\Sigma \xrightarrow{\mu} [0, \infty]$$

which satisfies $\mu(\emptyset) = 0$, and for any countable family $A_1, A_2, \dots \in \Sigma$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$ we have

$$\mu\left(\bigcup_i A_i\right) = \sum_i \mu(A_i)$$

A measure on X is called *finite* if $\mu(X) < \infty$ and is called a *probability measure* if $\mu(X) = 1$. A *measure space* (X, Σ, μ) consists of a measurable space (X, Σ) equipped with a measure μ .

Definition 4.1.3. Given a pair of measurable spaces (X, Σ_X) and (Y, Σ_Y) a *measurable function* $f : (X, \Sigma_X) \rightarrow (Y, \Sigma_Y)$ consists of a function $f : X \rightarrow Y$ such that for all $T \in \Sigma_Y$ the set $f^{-1}(T)$ belongs to Σ_X – that is, a function is measurable if and only if the preimage of a measurable set is measurable.

Given a pair of measurable spaces (X, Σ_X) and (Y, Σ_Y) and a measurable function $f : X \rightarrow Y$, a measure μ on (X, Σ_X) defines a measure ν on (Y, Σ_Y) , where we define

$$\nu(A) = \mu(f^{-1}(A))$$

called the *pushforward measure* of μ along f .

Every topological space comes equipped with the structure of a measurable space, called the *Borel σ -algebra*. The Borel σ -algebra on a topological space X

is the smallest σ -algebra on X which contains every open subset of X . Most of the measurable spaces we consider are of this form.

Definition 4.1.4. For X a topological space the *Borel σ -algebra* on X is the σ -algebra consisting of those subsets of X which can be formed from the open sets under the operations of countable union, countable intersection, and complementation. An element of the Borel σ -algebra is called a *Borel set*.

For X a topological space and Σ the Borel σ -algebra, a measure μ on (X, Σ) is called a *Borel measure*.

The following lemma shows that continuous maps between topological spaces are measurable functions, when we consider those topological spaces as measurable spaces equipped with their respective Borel σ -algebras. This result can be found in Bogachev [26, Lemma 6.2.2].

Lemma 4.1.5. *Let X and Y be topological spaces, equipped with their respective Borel σ -algebras. Any continuous map of topological spaces $f : X \rightarrow Y$ is a measurable function between X and Y when considered measurable spaces.*

Generally we are interested in a subclass of the Borel measures known as the *Radon measures*. There are various ways of defining Radon measures common in the literature. We follow the presentation of Bogachev [26, Definition 7.1.1].

Definition 4.1.6. For X a topological space with Borel σ -algebra Σ . A Borel measure μ on (X, Σ) is called a *Radon measure* if for every Borel set $U \in \Sigma$ and for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subseteq X$ such that $K_\varepsilon \subseteq U$ and $\mu(U \setminus K_\varepsilon) < \varepsilon$.

The following lemma states that for Hausdorff topological spaces Radon measures are preserved under the pushforward measure along those measurable functions which correspond with continuous functions – in the sense of Lemma 4.1.5. This result can be found in Bogachev [26, Theorem 9.1.1].

Lemma 4.1.7. *Let X and Y be Hausdorff topological spaces and let $f : X \rightarrow Y$ be a continuous map. If μ is a Radon measure on X then the pushforward measure of μ along f is a Radon measure on Y .*

Recall, Gelfand duality states that for a commutative C^* -algebra \mathfrak{A} , the set $\text{Spec}_G(\mathfrak{A})$ is naturally equipped with a compact Hausdorff topology, and hence $\text{Spec}_G(\mathfrak{A})$ can be viewed as a measurable space equipped with the Borel σ -algebra, and hence we can define the following presheaf.

Definition 4.1.8. The presheaf

$$\mathbf{Hilb-Alg}(H)^{\text{op}} \xrightarrow{\Psi} \mathbf{Set}$$

is defined on objects:

$$\Psi(\mathfrak{A}) = \{ \mu \mid \mu \text{ a Radon measure on } \text{Spec}_G(\mathfrak{A}) \}$$

while for each $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ define

$$\Psi(\mathfrak{A}) \xrightarrow{\Psi(i)} \Psi(\mathfrak{B}) \tag{4.1}$$

to be the map which sends each Radon measure μ on $\text{Spec}_G(\mathfrak{A})$ to the pushforward of μ along $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$.

There are two things that we must verify in order to check that Definition 4.1.8 defines a functor: first we must check that the map $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$ is a measurable function; and second, we must check that the pushforward measure of a Radon measure along i^* is a Radon measure.

By Gelfand duality, for any commutative C^* -algebra, the topological space $\text{Spec}_G(\mathfrak{A})$ is Hausdorff and each $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$ is a continuous map, and hence, by Lemma 4.1.5 the map $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$ is a measurable function, and by Lemma 4.1.7, i^* maps Radon measures to Radon measures, and hence Definition 4.1.8 does in fact define a functor

Given an experimental preparation procedure ψ , and a fixed classical measurement context \mathfrak{A} , then repeatedly making the preparation ψ followed by one of the measurements in $A \in \mathfrak{A}$ sufficiently many times for each $A \in \mathfrak{A}$, we obtain a probability measure on the possible outcomes of measurements in \mathfrak{A} – that

is, a probability measure on $\text{Spec}_G(\mathfrak{A})$, which is an element $\psi_{\mathfrak{A}} \in \Psi(\mathfrak{A})$. The collection of all such $\psi_{\mathfrak{A}}$ across all contexts is the collection of all the experimental data we can extract from the quantum system when prepared according to the procedure ψ . By the discussion of what constitutes a “state” from the beginning of this chapter, we can call this family of probability measures the *operational state* associated with that preparation.

Given that there are relations between the contexts – for example, given a pair of contexts $\mathfrak{A}, \mathfrak{B}$ we might have $\mathfrak{B} \hookrightarrow \mathfrak{A}$ – we expect there to be some relation between the corresponding probabilities $\psi_{\mathfrak{A}}$ and $\psi_{\mathfrak{B}}$.

We will assume that this collection of probability measures $\psi_{\mathfrak{A}} \in \Psi(\mathfrak{A})$ is natural in \mathfrak{A} , that is, for $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$, the probability measure $\psi_{\mathfrak{B}} \in \Psi(\mathfrak{B})$ is equal to the pushforward measure of $\psi_{\mathfrak{A}}$ along the map

$$\text{Spec}_G(\mathfrak{A}) \xrightarrow{i^*} \text{Spec}_G(\mathfrak{B})$$

that is, ψ forms a global section

$$\begin{array}{ccc} & \Psi & \\ \text{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\ & \uparrow \psi & \\ & \mathbf{1} & \end{array}$$

The naturality condition on the collection of measures $\psi_{\mathfrak{A}}$ is a form of *no-signalling*, analogous to Abramsky and Brandenburger’s approach to abstract empirical models [5]. Abramsky and Brandenburger’s conceptualisation of no-signalling generalises the notion of Ghirardi, Rimini and Weber [89]. Intuitively, naturality corresponds with no-signalling in the following way: Alice’s and Bob’s choice of measurement each correspond with a subalgebra of $\text{Hom}(H, H)$. Unpacking the naturality condition for these subalgebras, the probability distribution associated with Alice’s measurement corresponds with the marginalisation over Bob’s measurement outcomes, which is precisely the usual notion of no-signalling. We describe this explicitly in our discussion of *parameter independence*

in Chapter 6, and for an in-depth discussion of no-signalling in these terms see [5, §2.5].

In the context of Bell’s theorem, no-signalling is interpreted as the condition of no faster-than-light communication between spacelike separated measurement sites. Operational probabilistic theories typically consider a closely related notion of *causality* [38, §4.1.1] which states roughly that no signal can be sent from the future to the past. However, our framework has neither a notion of space nor time built into it, although in Chapter 8 we discuss ways of incorporating a spatiotemporal structure into the formalism.

Abramsky and Brandenburger note that the generalised no-signalling condition subsumes the spatiotemporal notion as a special case [5, p. 25]:

[No-signalling, as a property of the] tensor product structure can be retrieved automatically as a special case of the general situation of commuting operators on a single space. Thus the special form of representation for Bell-type scenarios is not really necessary, although it is the one which is standardly used.

Moreover, Abramsky and Brandenburger show in [5, §9] that (modulo details) the converse of this statement also holds: that is, the general form of no-signalling for arbitrary families of commuting operators can be retrieved from the form of no-signalling strictly in terms of the tensor product structure. We reproduce this discussion in Chapter 6 when we consider a generalised form of the condition of *parameter independence*, which can be viewed as a form of no-signalling for the ontic states of an ontological model.

We call this natural family of probability measures the *operational state*, associated with the particular experimental preparation ψ . Hence, we have the following definition.

Definition 4.1.9. For a quantum system represented by a Hilbert space H , the

operational states of that system are the global sections

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

Note that we are building the no-signalling condition into the definition of operational state. In principle one might consider operational states which do not necessarily satisfy no-signalling, but we are not interested in such cases.

Remark 4.1.10. Our definition of operational state is highly reminiscent of the notion of *empirical model* from the sheaf-theoretic formalism of Abramsky and Brandenburger [5]. We will consider empirical models in depth in Chapter 5, where we will make an explicit connection between operational states and empirical models.

Remark 4.1.11. We think of the presheaf Ψ as the “state space object” – the object playing the role of state space. We arrive at this definition from following purely pragmatic principles. In topos quantum theory, one takes $\text{Spec}_{\mathbb{C}}$ to be the “state space object”, which is more of an aesthetic choice, driving the analogue with classical mechanics. In Section 4.2 we will see a connection between these two conceptualisations of the state space; in particular we will show that an operational state determines an element of the subobject lattice of $\text{Spec}_{\mathbb{C}}$.

We want to give a complete characterisation of the operational states of a quantum system – as defined in Definition 4.1.8. Recall the adjunction of Theorem 3.1.22

$$\begin{array}{ccc}
 & u^* & \\
 \text{Set}^{\text{Hilb-Alg}(H)^{\text{op}}} & \xrightarrow{\quad} & \text{Set}^{\text{Hilb-Alg}_{\text{vN}}(H)^{\text{op}}} \\
 & \uparrow \top & \\
 & v^* &
 \end{array}$$

induced by the inclusion map $u : \text{Hilb-Alg}_{\text{vN}}(H) \hookrightarrow \text{Hilb-Alg}(H)$, and the

map $v : \mathbf{Hilb-Alg}(H) \rightarrow \mathbf{Hilb-Alg}_{\text{vN}}(H)$, which sends a C^* -algebra \mathfrak{A} to the von Neumann algebra \mathfrak{A}'' .

The presheaf

$$\mathbf{Hilb-Alg}_{\text{vN}}(H)^{\text{op}} \xrightarrow{u^*(\Psi)} \mathbf{Set}$$

is the restriction of Ψ to those C^* -algebras which are von Neumann algebras, that is, for each commutative von Neumann algebra \mathfrak{A} we have $u^*(\Psi)(\mathfrak{A}) = \Psi(\mathfrak{A})$.

The following theorem shows that using this adjunction we can reduce the problem of characterising operational states to just the case where we only consider the commutative von Neumann algebras, and not the arbitrary commutative C^* -subalgebras of $\text{Hom}(H, H)$. This is significance of this result is that allows us to prove results about operational states considering only the commutative von Neumann algebras, which have much richer structure than commutative C^* -algebras.

Theorem 4.1.12. *There is a natural one-to-one correspondence between the global sections of the presheaf*

$$\mathbf{Hilb-Alg}(H)^{\text{op}} \xrightarrow{\Psi} \mathbf{Set}$$

and the global sections of the presheaf

$$\mathbf{Hilb-Alg}_{\text{vN}}(H)^{\text{op}} \xrightarrow{u^*(\Psi)} \mathbf{Set}$$

Proof. Lemma 3.1.23 states that $v^*(\mathbf{1})$ is the terminal object in the category $\mathbf{Set}^{\mathbf{Hilb-Alg}(H)^{\text{op}}}$, that is $v^*(\mathbf{1}) = \mathbf{1}$. The set of global sections of Ψ is equal to the set $\text{Hom}(v^*(\mathbf{1}), \Psi)$. Hence, by the adjunction of Theorem 3.1.22, we have

$$\begin{aligned} \text{Hom}(\mathbf{1}, u^*(\Psi)) &\cong \text{Hom}(v^*(\mathbf{1}), \Psi) \\ &= \text{Hom}(\mathbf{1}, \Psi) \end{aligned}$$

as required. □

Operational States from Density Operators

We will now show that the notion of “state” in the usual Hilbert space formalism – density operators – give rise to operational states, as defined in Definition 4.1.9. This can be seen as a generalisation of a result of Abramsky and Brandenburger [5, Proposition 9.2] which shows that density operators satisfy a form of generalised no-signalling.

Recall the definition of a Boolean algebra, Definition 3.2.3, and consider the following examples. Boolean algebras occur naturally in the study of commutative C^* -algebras, in particular, the collection of projections $\text{Proj}(\mathfrak{A})$ in a commutative C^* -algebra \mathfrak{A} has the structure of a Boolean algebra [186, Proposition 4.16]. This association can be functorial in the sense of the following definition.

Definition 4.1.13. There is a functor

$$\mathbf{Hilb-Alg}(H) \xrightarrow{\text{Proj}} \mathbf{BoolAlg}$$

which takes each commutative C^* -subalgebra \mathfrak{A} to its Boolean algebra of projections $\text{Proj}(\mathfrak{A})$. The action of Proj on each inclusion $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ is just the restriction to the inclusion of Boolean algebras $\text{Proj}(\mathfrak{B}) \hookrightarrow \text{Proj}(\mathfrak{A})$.

For any topological space X , the set of subsets which are both open and closed – the clopen sets – determines a Boolean algebra which we denote $\mathbf{Clop}(X)$. This process characterises a functor, as described in the following definition.

Definition 4.1.14. There is a functor

$$\mathbf{Top}^{\text{op}} \xrightarrow{\mathbf{Clop}} \mathbf{BoolAlg}$$

which sends a topological space to the Boolean algebra $\mathbf{Clop}(X)$ of clopen subsets, and which sends a continuous $f : X \rightarrow Y$ to the Boolean algebra homomorphism

$$\begin{aligned} \mathbf{Clop}(Y) &\xrightarrow{f^{-1}} \mathbf{Clop}(X) \\ U &\longmapsto f^{-1}(U) \end{aligned}$$

We see then that we have a second method of obtaining a Boolean algebra from a commutative C^* -algebra \mathfrak{A} : we can first take the Gelfand spectrum $\text{Spec}_G(\mathfrak{A})$, considered as a topological space, and then take the Boolean algebra $\mathbf{Clop}(\text{Spec}_G(\mathfrak{A}))$ of clopen subsets of $\text{Spec}_G(\mathfrak{A})$. The two methods of obtaining a Boolean algebra from a commutative C^* -algebra \mathfrak{A} , that is, the Boolean algebra of projections $\text{Proj}(\mathfrak{A})$, and the Boolean algebra of clopens $\mathbf{Clop}(\text{Spec}_G(\mathfrak{A}))$ are equivalent in the sense that there is an isomorphism

$$\begin{aligned} \text{Proj}(\mathfrak{A}) &\xrightarrow{\sim} \mathbf{Clop}(\text{Spec}_G(\mathfrak{A})) \\ P &\longmapsto U_P = \{ \rho \in \text{Spec}_G(\mathfrak{A}) \mid \rho(P) = 1 \} \end{aligned}$$

see, for example [140, Theorem C. 168].

The following theorem shows that this isomorphism of Boolean algebras is natural in \mathfrak{A} .

Theorem 4.1.15. *For \mathfrak{A} a commutative C^* -algebra, there is a natural isomorphism*

$$\begin{array}{ccc} \mathbf{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\text{Proj}} & \mathbf{BoolAlg}^{\text{op}} \\ & \searrow \text{Spec}_G & \downarrow \alpha \\ & & \mathbf{Top} \\ & & \nearrow \mathbf{Clop} \end{array}$$

Proof. For each \mathfrak{A} we have an isomorphism $\alpha_{\mathfrak{A}}$ given by

$$\begin{aligned} \text{Proj}(\mathfrak{A}) &\xrightarrow{\alpha_{\mathfrak{A}}} \mathbf{Clop}(\text{Spec}_G(\mathfrak{A})) \\ P &\longmapsto U_P = \{ \rho \in \text{Spec}_G(\mathfrak{A}) \mid \rho(P) = 1 \} \end{aligned}$$

To show that this family of isomorphisms is natural we need to show that

for each $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ the diagram

$$\begin{array}{ccc} \text{Proj}(\mathfrak{B}) & \xrightarrow{\alpha_{\mathfrak{B}}} & \mathbf{Clop}(\text{Spec}_G(\mathfrak{B})) \\ \downarrow i & & \downarrow i^{*-1} \\ \text{Proj}(\mathfrak{A}) & \xrightarrow{\alpha_{\mathfrak{A}}} & \mathbf{Clop}(\text{Spec}_G(\mathfrak{A})) \end{array}$$

commutes.

Note that for each projector $P \in \text{Proj}(\mathfrak{B})$, the projector $i(P) \in \text{Proj}(\mathfrak{A})$ is precisely same map $P : H \rightarrow H$. Let $\alpha_{\mathfrak{B}}(P)$ be that clopen subset

$$\alpha_{\mathfrak{B}}(P) = \{ \gamma \in \text{Spec}_G(\mathfrak{B}) \mid \gamma(P) = 1 \}$$

and note that we have

$$\begin{aligned} i^{*-1}(\alpha_{\mathfrak{B}}(P)) &= \{ \rho \in \text{Spec}_G(\mathfrak{A}) \mid \rho|_{\mathfrak{B}} \in \alpha_{\mathfrak{B}}(P) \} \\ &= \{ \rho \in \text{Spec}_G(\mathfrak{A}) \mid \rho(P) = 1 \} \\ &= \alpha_{\mathfrak{A}}(P) \end{aligned}$$

as required. □

As we already mentioned in Theorem 4.1.12, we will characterise the operational states, that is, the global sections of Ψ by considering just the restriction to the von Neumann subalgebras. Our proof will exploit the topological properties of on each Gelfand spectrum $\text{Spec}_G(\mathfrak{A})$ when \mathfrak{A} is a von Neumann algebra. Recall Gelfand duality state that the spectrum of a C^* -algebra is a compact Hausdorff topological space. If \mathfrak{A} is a von Neumann algebra then the spectrum carries the structure of a *Stone space*, which we review now.

A *connected component* of a topological space is an open set that cannot be expressed as a union of two non-empty disjoint open subsets. A *totally disconnected* topological space is one for which the only connected components are singleton subsets.

Definition 4.1.16. A *Stone space* is compact Hausdorff space which is totally disconnected. We denote the full subcategory of the category of topological spaces and continuous maps whose objects are Stone spaces by **Stone**.

The following result can be found in [140, Theorem C.167].

Theorem 4.1.17. For \mathfrak{A} a commutative von Neumann C^* -algebra, the topological space $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ is a Stone space.

Recall, for X a topological space, a *base* for a topology on X is a family of subsets $B = \{V_i\}_{i \in I}$ such that every open set $U \subset X$ is a union $U = \bigcup_i V_i$ of sets in B .

The following lemma is a well-known result, see for example [129, Theorem 4.2].

Lemma 4.1.18. For X a Stone space, the collection of clopen sets $\mathbf{Clop}(X)$ form a base for the topology on X .

Remark 4.1.19. Stone's representation theorem for Boolean algebras [202] states that every Boolean algebra is isomorphic to the algebra of clopen sets of a Stone space. In other words, every Boolean algebra is isomorphic to one of the form described in Definition 4.1.14. Moreover, restricting to the subcategory of stone spaces $\mathbf{Stone} \leftrightarrow \mathbf{Top}$ the functor \mathbf{Clop} defines an equivalence of categories, known as *Stone duality*, and is the original *Stone-type duality* which we discussed in Remark 2.2.1.

In order to show that density operators determine operational states – Theorem 4.1.29 – we will need a sequence of definitions and results from Bogachev [26]. The following can be found in [26, Definition 7.2.1].

Definition 4.1.20. A Borel measure μ on a topological space X is said to be τ -additive if for every increasing net of open sets $(U_\lambda)_{\lambda \in \Lambda}$ in X then

$$\mu\left(\bigcup_{\lambda \in \Lambda} U_\lambda\right) = \lim_{\lambda} \mu(U_\lambda)$$

The following lemma can be found in [26, Proposition 7.2.2].

Lemma 4.1.21. *Every Radon measure is τ -additive.*

The following lemma can be found in [26, Corollary 7.2.3].

Lemma 4.1.22. *Let μ and ν be τ -additive Borel measures on a topological space X . Let $\mathcal{A} \subseteq \Sigma$ be a collection of subsets that contains a base of the topology such that \mathcal{A} is closed under finite intersections. If μ and ν agree on all sets in \mathcal{A} then $\mu = \nu$.*

The following definition can be found in [26, Definition 7.1.5].

Definition 4.1.23. Let X be a topological space, let \mathcal{A} be some collection of subsets of X . A function μ

$$\mathcal{A} \xrightarrow{\mu} \mathbb{R}^+$$

is said to be *regular* if for every $A \in \mathcal{A}$ and for every $\varepsilon > 0$ there exists a closed set $F_\varepsilon \subseteq X$ such that $F_\varepsilon \subseteq A$, and $A \setminus F_\varepsilon \in \mathcal{A}$, such that $\mu(A \setminus F_\varepsilon) < \varepsilon$.

Definition 4.1.24. Let X be a topological space, let \mathcal{A} be some collection of subsets of X . A function μ

$$\mathcal{A} \xrightarrow{\mu} \mathbb{R}^+$$

is said to be *additive* if

$$\mu\left(\bigcup_{i \in I} A_i\right) = \sum_{i \in I} \mu(A_i)$$

for every collection of disjoint $A_i \in \mathcal{A}$.

The following result can be found in [26, Theorem 7.3.2].

Lemma 4.1.25. *Let (X, Σ) be a compact Hausdorff space equipped with the Borel σ -algebra seen as a measurable space, and let \mathcal{A} be a collection of subsets of X which contains a base of the topology. A regular, additive function (in the*

sense of Definitions 4.1.23 and 4.1.24)

$$\mathcal{A} \xrightarrow{\mu} \mathbb{R}^+$$

can be uniquely extended to a Radon measure on X

$$\Sigma \xrightarrow{\widehat{\mu}} \mathbb{R}^+$$

Remark 4.1.26. Lemma 4.1.25 as stated by Bogachev [26, Theorem 7.3.2] requires an additional property of *tightness*, however, it is noted [26, p. 69] that if the underlying topological space X is compact then tightness holds automatically.

Also, in Bogachev's statement of Lemma 4.1.25, the measure μ is required to have *bounded variation*, a property of *signed* measures (those that can take possibly negative values). We only consider positive measures, and positive measures always satisfy the condition of bounded variation.

Lemma 4.1.27. *Let X be a topological space and let $\mathbf{Clop}(X)$ be the collection of clopen sets. Then any function*

$$\mathbf{Clop}(X) \xrightarrow{\mu} \mathbb{R}^+$$

satisfying $\mu(\emptyset) = 0$ is regular, in the sense of Definition 4.1.23.

Proof. Consider $A \in \mathbf{Clop}(X)$, then A is a closed set and for each $\varepsilon > 0$ we can simply take $F_\varepsilon = A$. We have $F_\varepsilon \subseteq A$, and $A \setminus F_\varepsilon = \emptyset$ and since $\mu(\emptyset) = 0$ we have $\mu(A \setminus F_\varepsilon) = 0 < \varepsilon$ for every $\varepsilon > 0$. \square

The following crucial lemma shows that a Radon measure on $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$, for \mathfrak{A} a commutative von Neumann algebra, is uniquely determined by its values on the clopen subsets of $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$.

Lemma 4.1.28. *Let \mathfrak{A} be a commutative von Neumann algebra. Any Radon*

measure on $\text{Spec}_G(\mathfrak{A})$

$$\Sigma \xrightarrow{\mu} \mathbb{R}^+$$

where Σ is the Borel σ -algebra, is uniquely determined by the values it takes on the clopen subsets of $\text{Spec}_G(\mathfrak{A})$, that is, μ is completely characterised by the restriction

$$\mathbf{Clop}(\text{Spec}_G(\mathfrak{A})) \xrightarrow{\mu|_{\mathbf{Clop}}} \mathbb{R}^+$$

Proof. Let μ and ν be a pair of Radon measures on $\text{Spec}_G(\mathfrak{A})$ which coincide on all clopen subsets of $\text{Spec}_G(\mathfrak{A})$. By Lemma 4.1.21 μ and ν are τ -additive.

By Theorem 4.1.17 $\text{Spec}_G(\mathfrak{A})$ is a Stone space, and hence by Lemma 4.1.18, the clopen sets of $\text{Spec}_G(\mathfrak{A})$ forms a base for the topology because. The set of clopen sets is closed under taking finite intersections and hence, by Lemma 4.1.22, we conclude that $\mu = \nu$. \square

We are now in a position to show that density operators determine operational states, as defined in Definition 4.1.9.

Theorem 4.1.29. *Let H be a Hilbert space. Each density operator $q : H \rightarrow H$ uniquely determines a global section*

$$\begin{array}{ccc} & \Psi & \\ \curvearrowright & & \curvearrowleft \\ \mathbf{Hilb-Alg}(H)^{\text{op}} & \uparrow \tilde{q} & \mathbf{Set} \\ \curvearrowleft & & \curvearrowright \\ & \mathbf{1} & \end{array}$$

Proof. By Theorem 4.1.12, it is enough to show that each density operator q uniquely determines a global section

$$\begin{array}{ccc} & u^*(\Psi) & \\ \curvearrowright & & \curvearrowleft \\ \mathbf{Hilb-Alg}_{\text{vN}}(H)^{\text{op}} & \uparrow \tilde{q} & \mathbf{Set} \\ \curvearrowleft & & \curvearrowright \\ & \mathbf{1} & \end{array}$$

For each \mathfrak{A} define the map

$$\begin{aligned} \text{Proj}(\mathfrak{A}) &\xrightarrow{\widehat{q}_{\mathfrak{A}}} [0, 1] \\ P &\longmapsto \text{tr}(qP) \end{aligned}$$

Under the correspondence of Theorem 4.1.15, we can transfer $\widehat{q}_{\mathfrak{A}}$ to a map

$$\mathbf{Clop}(\text{Spec}_{\mathbb{C}}(\mathfrak{A})) \xrightarrow{\tilde{q}_{\mathfrak{A}}} [0, 1] \quad (4.2)$$

If we can verify that this map is additive (in the sense of Definition 4.1.24) and regular (in the sense of Definition 4.1.23) then using Lemma 4.1.25 we can extend the map $\tilde{q}_{\mathfrak{A}}$ to a Radon measure

$$\Sigma_{\mathfrak{A}} \xrightarrow{\tilde{q}_{\mathfrak{A}}} [0, 1]$$

where $\Sigma_{\mathfrak{A}}$ is the Borel σ -algebra on $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$.

Since, under the correspondence of Theorem 4.1.15, the empty set $\emptyset \subseteq \text{Spec}_{\mathbb{C}}(\mathfrak{A})$ corresponds with the zero projection P_0 we have

$$\begin{aligned} \tilde{q}_{\mathfrak{A}}(\emptyset) &= \text{tr}(qP_0) \\ &= 0 \end{aligned}$$

and hence regularity follows directly from Lemma 4.1.27.

Let U_{P_i} be a family of pairwise disjoint clopen subsets of $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ with corresponding projectors $P_i \in \text{Proj}(\mathfrak{A})$, then we have

$$\begin{aligned}
\tilde{q}_{\mathfrak{A}}\left(\bigcup U_{P_i}\right) &= \operatorname{tr}\left(q\left(\sum_i P_i\right)\right) \\
&= \sum_i \operatorname{tr}(qP_i) \\
&= \sum_i \tilde{q}_{\mathfrak{A}}(U_{P_i})
\end{aligned}$$

The first equality is the definition of $\tilde{q}_{\mathfrak{A}}$, and the second equality follows because the trace function is completely additive on orthogonal projections, and hence we have shown that the map (4.2) is additive.

Since the function $\tilde{q}_{\mathfrak{A}}$ is additive and regular, then by Lemma 4.1.25 the function $\tilde{q}_{\mathfrak{A}}$ can be uniquely extended to a Radon measure on $\operatorname{Spec}_{\mathbb{C}}(\mathfrak{A})$.

It remains to check naturality, that is, that for each $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ the measure $\tilde{q}_{\mathfrak{B}}$ coincides with the pushforward measure of $\tilde{q}_{\mathfrak{A}}$ along $i^* : \operatorname{Spec}_{\mathbb{C}}(\mathfrak{A}) \rightarrow \operatorname{Spec}_{\mathbb{C}}(\mathfrak{B})$. Since \mathfrak{A} and \mathfrak{B} are von Neumann algebras, then by Lemma 4.1.28 it is enough to check that the two coincide on the clopen subsets of $\operatorname{Spec}_{\mathbb{C}}(\mathfrak{B})$. That is, for each clopen set $V_P \in \operatorname{Spec}_{\mathbb{C}}(\mathfrak{A})$ with corresponding projector $P \in \operatorname{Proj}(\mathfrak{A})$, we need to check that

$$\tilde{q}_{\mathfrak{B}}(V_P) = \tilde{q}_{\mathfrak{A}}(i^{*-1}(V_P))$$

Recall from Theorem 4.1.15 that the projector in $\operatorname{Proj}(\mathfrak{A})$ corresponding with the clopen set $i^{*-1}(V_P) \subseteq \operatorname{Spec}_{\mathbb{C}}(\mathfrak{A})$ is that same projection P viewed as an element of \mathfrak{A} . Hence we have

$$\begin{aligned}
\tilde{q}_{\mathfrak{B}}(V_P) &= \operatorname{tr}(qP) \\
&= \tilde{q}_{\mathfrak{A}}(i^{*-1}(V_P))
\end{aligned}$$

as required. \square

Theorem 4.1.29 asserts that density operators satisfy no-signalling, which is

similar to a result by Abramsky and Brandenburger [5, Proposition 9.2], which generalises the no-signalling result of Ghirardi, Rimini and Weber [89], which asserts that density operators can in no way facilitate superluminal signalling. In the case of Ghirardi, Rimini and Weber, no-signalling is understood to be a restriction on the transmission of information between two spacelike separated measurement sites, while in the discussion of their generalised no-signalling result [5, Proposition 9.2], similar to Theorem 4.1.29 Abramsky and Brandenburger make the following assertion [5, p. 27]:

Thus we see that quantum mechanics obeys a general form of no-signalling, which applies to compatible families of observables in general, not just those represented as operating on different factors of a tensor product. This form of no-signalling says that, at the level of distributions, the statistics obtained for a measurement on a given state are independent of the context of other compatible measurements which may also have been performed.

Theorem 4.1.29 can be seen as an extension of Abramsky and Brandenburger's result, encompassing systems of arbitrary dimension, not just those with finite dimension.

Generalised Operational States

We want to generalise the notion of operational state away from **Hilb**, for more general \dagger -symmetric monoidal categories as considered in Chapter 3. Let X be an object in \mathcal{A} , a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts. Recall, by Theorem 3.1.33 for any commutative \mathbb{S}^* -subsemialgebra $\mathfrak{A} \subseteq \text{Hom}(X, X)$ the generalised Gelfand spectrum $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$ comes equipped with the Zariski topology, and hence for each \mathfrak{A} we can consider the set of Radon measures on $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$. As for the Hilbert space setting we can denote the set of Radon measures on $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$ by $\Psi(\mathfrak{A})$, however, this fails to

be functorial in the sense of defining a presheaf

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\Psi} \mathbf{Set}$$

as the topology on $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$ is not Hausdorff in general, and hence we cannot use the same argument for functoriality as we did for Definition 4.1.8 – in particular we cannot apply Lemma 4.1.7.

We could consider the set of all Borel measures on $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$ but given the pathological nature of these topologies in general there seems to be little that can be said at this level of generality, and so in this more general case we restrict to a discrete setting, analogous to finite-dimensional quantum theory; that is, we restrict to probability distributions as opposed to probability measures. We make explicit use of such operational states for $\mathcal{A} = \mathbf{Rel}$ in Chapter 7 where we model Spekkens' Toy Theory.

Definition 4.1.30. A *finite probability distribution* on a set X consists of a function

$$X \xrightarrow{d} \mathbb{R}^+$$

such that

$$\sum_{x \in X} d(x) = 1$$

which has *finite support*, that is $d(x) > 0$ for only finitely many $x \in X$.

While considering probability distributions as opposed to probability measures might seem like a restriction, it does allow us to consider more general forms of distributions, not just over the positive reals. These more general distributions can be taken over an arbitrary semiring R , for example: taking $R = \mathbf{2}$, the two-element Boolean algebra considered as a semiring allows us to consider *possibilistic distributions*. Taking $R = \mathbb{R}$ allows us to consider *negative probabilities*, discussed at length in [5, 6, 11].

Definition 4.1.31. For a commutative semiring R , define the *R -distribution*

functor

$$\mathbf{Set} \xrightarrow{\mathcal{D}_R} \mathbf{Set}$$

which assigns to a set U the set of finitely supported R -distributions over U

$$\mathcal{D}_R(U) = \{ d : U \rightarrow R \mid \sum_{x \in X} d(x) = 1_R, d \text{ has finite support} \}$$

Just as before, the action on morphisms $f : U \rightarrow V$ is given by

$$\mathcal{D}_R(U) \xrightarrow{f_*} \mathcal{D}_R(V)$$

For $d : U \rightarrow R$ an R -distribution on U , the R -distribution $f_*(d) : V \rightarrow R$ is defined as follows

$$\begin{array}{ccc} V & \xrightarrow{f_*(d)} & R \\ v & \longmapsto & \sum_{\substack{u \in U \\ f(u)=v}} d(u) \end{array}$$

Note that we can represent an R -distribution $d : X \rightarrow R$ as a formal sum

$$\sum_{x \in X} p_x \cdot \underline{x}$$

where the coefficient $p_x \in R$ of the element $x \in X$ is defined to be the value $d(x)$, and if an element has the coefficient 0_R it can be omitted from the sum, for details see [86]. Under this formal sum notation, for a function $f : X \rightarrow Y$ the corresponding R -distribution $f_*(d)$ on Y represented by the formal sum

$$\sum_{x \in X} p_{f(x)} \cdot \underline{f(x)}$$

while all elements of Y not in the image of f are assigned the coefficient 0_R .

Remark 4.1.32. When we take $R = \mathbb{R}^+$ the semiring of positive real numbers under the usual addition and multiplication operations this functor $\mathcal{D}_{\mathbb{R}^+}$ is called

the *finitary Giry monad*. This functor is a discrete version of one originally introduced by Giry in [90]. As the name suggests, the Giry monad is a monad, and we will see in Section 4.3 that the functor \mathcal{D}_R is a monad for arbitrary R .

Definition 4.1.33. We define the *operational R -state functor*

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\Delta_R} \mathbf{Set}$$

as the composition of $\mathcal{D}_R \circ \text{Spec}_G$.

We define an *operational R -state* to be a global section of Δ_R .

Operational R -states are the subject of Chapter 5 where we consider a general notion local hidden-variable model.

The operational R -states can be seen as generalising the notion of operational state as defined in Definition 4.1.9 in the following sense: for H a finite-dimensional Hilbert space, the presheaves

$$\mathbf{Hilb}\text{-Alg}(H)^{\text{op}} \xrightarrow{\Delta_{\mathbb{R}^+}} \mathbf{Set}$$

and

$$\mathbf{Hilb}\text{-Alg}(H)^{\text{op}} \xrightarrow{\Psi} \mathbf{Set}$$

coincide. For each \mathfrak{A} the set $\text{Spec}_G(\mathfrak{A})$ is a finite set equipped with the discrete topology, hence the Radon measures on $\text{Spec}_G(\mathfrak{A})$ are precisely the probability distributions on $\text{Spec}_G(\mathfrak{A})$. That is, the operational R -states generalise the operational states from finite-dimensional quantum theory. Note that for infinite-dimensional Hilbert spaces there is a monomorphism

$$\Delta_{\mathbb{R}^+} \hookrightarrow \Psi$$

which simply takes the inclusion of the discrete probability measures into the collection of all Radon measures for each spectrum.

4.2 Operational States in Quantum Theory

In this section we will give a complete characterisation of the global sections

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

for all Hilbert spaces H such that $\dim(H) \geq 3$. In particular we show that every operational state is of the form \tilde{q} as in Theorem 4.1.29, for some density operator $q : H \rightarrow H$.

We prove the result for von Neumann algebras, that is, we give a complete characterisation of the global sections

$$\begin{array}{ccc}
 & u^*(\Psi) & \\
 \text{Hilb-Alg}_{\text{vN}}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

which, by Theorem 4.1.12 gives a full characterisation for more general C^* -algebras.

The result will hold if H is a separable Hilbert space, and will hold for nonseparable Hilbert spaces under the additional assumption of the *Continuum Hypothesis*. Recall, a Hilbert space H is said to be *separable* if it admits a countable orthogonal basis. A Hilbert space is *nonseparable* if it is not separable.

The Continuum Hypothesis asserts that there is no set A with cardinality that lies strictly between that of the integers and reals

$$|\mathbb{Z}| < |A| < |\mathbb{R}|$$

The Continuum Hypothesis is known to be independent of the axioms of set theory [50], that is, it can neither be proven nor its negation proven from the

axioms of ZFC.

It is certainly not the case that every operational state corresponds with a density operator when $\dim(H) = 2$ for the underlying Hilbert space H ; in fact, any collection of probability measures $\psi_{\mathfrak{A}} \in \Psi(\mathfrak{A})$ will trivially satisfy naturality. However, we will show that under some mild additional assumptions we can rule out all but those operational states which do correspond with density operators and thus recover the case $\dim(H) = 2$. We will frame this result as a derivation of the Born rule from a set of operational axioms, which parallel the axiomatisation we presented in Chapter 2 in Figure 2.2.

Characterising Operational Quantum States

Our characterisation of operational states in quantum theory will rely on Gleason's theorem, and some generalisations of Gleason's theorem which we will review now. Let $\text{Proj}(H)$ denote the set of self-adjoint projectors on a Hilbert space H .

Definition 4.2.1. A function

$$\text{Proj}(H) \xrightarrow{\mu} [0, 1]$$

is said to be:

1. *finitely additive* if for every finite family of pairwise orthogonal projectors $\{P_i\}_{i \in I}$ the equation

$$\mu\left(\sum_{i \in I} P_i\right) = \sum_{i \in I} \mu(P_i) \tag{4.3}$$

2. *countably additive* if equation (4.3) holds for every countable family of pairwise orthogonal projectors $\{P_i\}_{i \in I}$.
3. *completely additive* if equation (4.3) holds for any family of pairwise orthogonal projectors $\{P_i\}_{i \in I}$.

The following result is Gleason's theorem [92, Theorem 4.1] in its original form.

Theorem 4.2.2. *Let H be a separable Hilbert space with $\dim(H) \geq 3$. For each countably additive function*

$$\text{Proj}(H) \xrightarrow{\mu} [0, 1]$$

satisfying $\mu(\text{id}_H) = 1$ there exists a unique density operator $q : H \rightarrow H$ such that

$$\mu(P) = \text{tr}(qP)$$

for all $P \in \text{Proj}(H)$.

Eilers and Horst [76, Proposition 2] show that assuming the Continuum Hypothesis holds, Gleason's theorem can be extended to the nonseparable case.

Theorem 4.2.3. *Let H be a Hilbert space with $\dim(H) \geq 3$. Assuming the Continuum Hypothesis, for each countably additive function*

$$\text{Proj}(H) \xrightarrow{\mu} [0, 1]$$

satisfying $\mu(\text{id}_H) = 1$, there exists a unique density operator $q : H \rightarrow H$ such that

$$\mu(P) = \text{tr}(qP)$$

for all $P \in \text{Proj}(H)$.

We are now in a position to prove the converse of Theorem 4.1.29, that is, to give a complete characterisation of the operational states for quantum theory.

Theorem 4.2.4. *Let H be a Hilbert space with $\dim(H) \geq 3$. If H is separable or if we assume the Continuum Hypothesis, then every operational state is of the*

form

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \tilde{q} & \\
 & \mathbf{1} &
 \end{array}$$

for some density operator $q : H \rightarrow H$, and where \tilde{q} is defined as in Theorem 4.1.29.

Proof. By Theorem 4.1.12, to characterise the operational states it is enough to characterise the global sections

$$\begin{array}{ccc}
 & u^*(\Psi) & \\
 \text{Hilb-Alg}_{\text{vN}}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

Let ψ be such a global section, then for each commutative von Neumann subalgebra $\mathfrak{A} \subseteq \text{Hom}(H, H)$ we have a Radon measure $\psi_{\mathfrak{A}}$ on the measurable space $(\text{Spec}_{\mathbb{G}}(\mathfrak{A}), \Sigma_{\mathfrak{A}})$. By Lemma 4.1.28 such a measure $\psi_{\mathfrak{A}}$ is completely determined by the values it takes on the clopen subsets of $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$. Restricting the measure $\psi_{\mathfrak{A}}$ to the subset of the σ -algebra consisting of just clopen subsets of $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$, we get a map

$$\mathbf{Clop}(\text{Spec}_{\mathbb{G}}(\mathfrak{A})) \xrightarrow{\psi_{\mathfrak{A}}|_{\mathbf{Clop}}} [0, 1]$$

under the correspondence of Theorem 4.1.15, gives a function

$$\begin{array}{ccc}
 \text{Proj}(\mathfrak{A}) & \xrightarrow{\tau_{\mathfrak{A}}} & [0, 1] \\
 P & \longmapsto & \psi_{\mathfrak{A}}(U_P)
 \end{array}$$

where $U_P \subseteq \text{Spec}_{\mathbb{G}}(\mathfrak{A})$ is the clopen set corresponding with the projection P under the isomorphism of Theorem 4.1.15.

We extend this collection of functions $\tau_{\mathfrak{A}}$ to a function

$$\text{Proj}(H) \xrightarrow{\mu} [0, 1]$$

as follows: for a projector $P \in \text{Hom}(H, H)$ let $\mathfrak{B} = \{P\}''$ be the von Neumann algebra generated by P and define

$$\mu(P) = \psi_{\mathfrak{B}}(U_P)$$

where $U_P \subseteq \text{Spec}_G(\mathfrak{B})$ is the clopen subset corresponding with the projection $P \in \mathfrak{B}$, in the sense of Theorem 4.1.15. Theorem 4.1.15 describes isomorphisms of Boolean algebras, under which disjoint clopen subsets map to orthogonal projections. Since $\psi_{\mathfrak{A}}$ is a measure, it satisfies countable additivity, that is, for a countable family of disjoint measurable sets U_j we have $\psi_{\mathfrak{A}}(\bigcup_j U_j) = \sum_j \psi_{\mathfrak{A}}(U_j)$. In particular, this holds for disjoint families of clopen sets. Hence, for a countable family of orthogonal projections $\{P_j \mid j \in I\}$, and $\mathfrak{A} = \{P_j \mid j \in I\}''$ the von Neumann algebra generated by this family, we have

$$\tau_{\mathfrak{A}}\left(\bigvee_{j \in I} P_j\right) = \sum_{j \in I} \tau_{\mathfrak{A}}(P_j) \quad (4.4)$$

We will now show that this function μ we have defined is countably additive, in the sense of Definition 4.2.1. This will follow from naturality of the family $\psi_{\mathfrak{A}}$, but we need to unpack what this naturality condition means for the family $\tau_{\mathfrak{A}}$. For $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ naturality of ψ means that for each clopen set $U \subseteq \text{Spec}_G(\mathfrak{B})$ we have

$$\psi_{\mathfrak{B}}(U) = \psi_{\mathfrak{A}}(i^{*-1}(U)) \quad (4.5)$$

Transporting across the isomorphism described in Theorem 4.1.15, Equation (4.5) implies that

$$\tau_{\mathfrak{B}}(P) = \tau_{\mathfrak{A}}(i(P))$$

where $i(P)$ denotes the inclusion of the projector P into $\text{Proj}(\mathfrak{A})$, and so as

elements of $\text{Hom}(H, H)$ we have $P = i(P)$. The naturality condition can then be stated as $\tau_{\mathfrak{B}}(P) = \tau_{\mathfrak{A}}(P)$ for all $P \in \mathfrak{B}$.

Let $\{P_j \mid j \in I\} \subseteq \text{Hom}(H, H)$ be a countable family of orthogonal projectors and denote their sum $\bigvee_{j \in I} P_j$. Let $\mathfrak{B} = \{\bigvee_{j \in I} P_j\}''$ be the algebra generated by the projector $\bigvee_{j \in I} P_j$. Let $\mathfrak{A} = \{P_j \mid j \in I\}''$ be algebra generated by the set of projectors $\{P_j \mid j \in I\}$, note that we have $\mathfrak{B} \hookrightarrow \mathfrak{A}$. Let $\mathfrak{P}_j = \{P_j\}''$ be the von Neumann algebra generated by each P_j . For each $j \in I$ we have $\mathfrak{P}_j \hookrightarrow \mathfrak{A}$. Now consider

$$\begin{aligned}
 \mu\left(\bigvee_j P_j\right) &= \tau_{\mathfrak{B}}\left(\bigvee_j P_j\right) && \text{by definition of } \mu \\
 &= \tau_{\mathfrak{A}}\left(\bigvee_j P_j\right) && \text{by naturality} \\
 &= \sum_j \tau_{\mathfrak{A}}(P_j) && \text{by Equation (4.4)} \\
 &= \sum_j \tau_{\mathfrak{P}_j}(P_j) && \text{by naturality} \\
 &= \sum_j \mu(P_j) && \text{by definition of } \mu
 \end{aligned}$$

and hence μ is a countably additive function on $\text{Proj}(H)$, in the sense of Definition 4.2.1.

If H is separable then by Gleason's theorem – Theorem 4.2.2 – we conclude that μ is uniquely determined by a density operator $q : H \rightarrow H$ such that for each $P \in \text{Proj}(H)$ we have

$$\begin{array}{ccc}
 \text{Proj}(H) & \xrightarrow{\mu} & [0, 1] \\
 P & \longmapsto & \text{tr}(qP)
 \end{array}$$

If H is not separable, but we assume the Continuum Hypothesis then we can apply Eilers and Horst's generalisation of Gleason's theorem – Theorem 4.2.3 –

and arrive at the same conclusion.

Hence, for each commutative von Neumann subalgebra \mathfrak{A} , by Lemma 4.1.28 the Radon measure $\psi_{\mathfrak{A}}$, is completely determined by the values it takes on the clopen subsets of $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$. By Theorem 4.1.15 then $\psi_{\mathfrak{A}}$ is completely determined by a function

$$\text{Proj}(\mathfrak{A}) \xrightarrow{\tau_{\mathfrak{A}}} [0, 1]$$

which by Gleason's theorem is uniquely determined by a density operator q and the trace formula, that is $\psi = \tilde{q}$, as required. \square

Remark 4.2.5. It might seem surprising to rely so heavily on the properties of von Neumann algebras only to then claim that the result holds for the full poset of commutative C^* -subalgebras. We can provide some intuition as to why this is not as great a leap as it might seem. Note that every commutative C^* -subalgebra is contained in a maximal one, and hence, to specify a global section of Ψ it would be enough to define the components $\psi_{\mathfrak{A}}$ for just those C^* -subalgebras which are maximal. Recall from Lemma 3.3.2 that a commutative C^* -subalgebra is maximal if and only if $\mathfrak{A} = \mathfrak{A}'$, and hence maximal C^* -subalgebras certainly satisfy $\mathfrak{A} = \mathfrak{A}''$, and are therefore von Neumann algebras. Therefore, to specify a global section of Ψ it is enough to define the components on just a collection of von Neumann algebras.

Suppose we do not assume the Continuum Hypothesis, then the proof of Theorem 4.2.4 still gives us a useful result. To see this we require a slightly more general notion of *state* commonly used within quantum theory: a state of a C^* -algebra.

Definition 4.2.6. Let \mathfrak{X} be a (not necessarily commutative) C^* -algebra. A *state* ω of \mathfrak{X} consists of a linear map

$$\mathfrak{X} \xrightarrow{\omega} \mathbb{C}$$

satisfying:

1. $\omega(1_{\mathfrak{X}}) = 1$;
2. and $\omega(AA^*) \geq 0$ for all $A \in \mathfrak{X}$.

The notion of state of a C^* -algebra generalises the concept of density operator as every density operator $q : H \rightarrow H$ determines a state ω_q of the C^* -algebra $\text{Hom}(H, H)$ as follows:

$$\begin{array}{ccc} \text{Hom}(H, H) & \xrightarrow{\omega_q} & \mathbb{C} \\ A & \longmapsto & \text{tr}(qA) \end{array} \quad (4.6)$$

Not every state of the C^* -algebra $\text{Hom}(H, H)$ is of the form (4.6) – those which are are precisely the normal states of the C^* -algebra $\text{Hom}(H, H)$.

Definition 4.2.7. States of the C^* -algebra $\text{Hom}(H, H)$ of the form (4.6) are called *normal*.

Remark 4.2.8. Note that Definition 4.2.7 is not the usual definition of *normal state*, but is equivalent to the usual definition, see, for example Landsman [140, Theorem 4.12].

If H is not finite-dimensional then there are many states of the C^* -algebra $\text{Hom}(H, H)$ that are not normal, which we will consider later in this chapter, and in Chapter 6.

There is weaker form of Gleason's theorem that applies to C^* -algebras, see for example [140, Theorem 4.29].

Theorem 4.2.9. *Let H be a Hilbert space with $\dim(H) \geq 3$. For each finitely additive function (in the sense of Definition 4.2.1)*

$$\text{Proj}(H) \xrightarrow{\mu} [0, 1]$$

satisfying $\mu(\text{id}_H) = 1$ there exists a unique state

$$\text{Hom}(H, H) \xrightarrow{\omega} \mathbb{C}$$

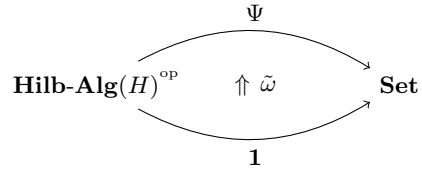
such that

$$\mu(P) = \omega(P)$$

for all $P \in \text{Proj}(H)$.

Assuming the Continuum Hypothesis, countable additivity of the function $\mu : \text{Hom}(H, H) \rightarrow [0, 1]$ is enough to show that the C^* -algebra state ω determined by the μ is normal, in the sense of Definition 4.2.7. But without the Continuum Hypothesis we cannot assume the state is normal, though we still obtain a state.

Theorem 4.2.10. *Let H be a Hilbert space with $\dim(H) \geq 3$. Every operational state is of the form*



for a C^* -algebra state

$$\text{Hom}(H, H) \xrightarrow{\omega} \mathbb{C}$$

Proof. In the proof of Theorem 4.2.4 we showed that an operational state is completely determined by a countably additive function $\mu : \text{Proj}(H) \rightarrow [0, 1]$. A countably additive function is clearly finitely additive, and hence by Theorem 4.2.9 we conclude that there is a unique state of $\text{Hom}(H, H)$, as required. \square

Deriving the Born Rule

Theorem 4.2.4 allows us to recover density operators and the trace formula from the no-signalling property of operational states. We will now frame this result as a derivation of the Born rule, but first we have to specify the axioms from which are deriving the Born rule. Figure 4.2 shows a set of operational axioms for quantum theory.

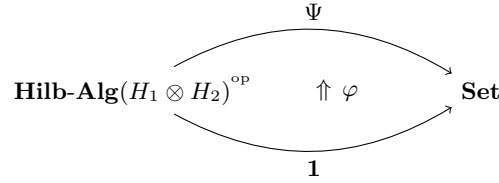
1. A quantum system is represented by a Hilbert space H . In particular the *measurements* on a quantum system are represented by elements of the C^* -algebra $\text{Hom}(H, H)$.

These measurements are grouped into commutative C^* -subalgebras $\mathfrak{A} \subseteq \text{Hom}(H, H)$, *classical contexts*.

2. The outcomes of measurements are probabilistic in nature: if one repeats the same preparation and measurements from a given classical context \mathfrak{A} , one obtains a probability measure on $\text{Spec}_G(\mathfrak{A})$.

For a given preparation, we call the collection of probability measures ranging over all classical contexts the *operational state* associated with that preparation. These probabilities are assumed to satisfy no-signalling.

3. For systems represented by Hilbert spaces H_1 and H_2 , the composite system is represented by the Hilbert space $H_1 \otimes H_2$. In particular, given an operational state ψ on H_1 and an operational state ϕ on H_2 , then there exists an operational state of the composite system



such that for classical contexts of the form $\mathfrak{A} \otimes \mathfrak{B}$ for $\mathfrak{A} \subseteq \text{Hom}(H_1, H_1)$ and $\mathfrak{B} \subseteq \text{Hom}(H_2, H_2)$ we have

$$\varphi_{\mathfrak{A} \otimes \mathfrak{B}}(A \otimes B) = \psi_{\mathfrak{A}}(A) \cdot \phi_{\mathfrak{B}}(B)$$

for every $A \otimes B \in \mathfrak{A} \otimes \mathfrak{B}$.

Figure 4.2: Operational axioms of quantum theory.

We are justified in calling the commutative subalgebras *classical contexts* as they correspond with classical mechanical systems in the sense of Nestruiev, as discussed in Chapter 2. Associated with each measurement context there is the classical notion of *state space* $\text{Spec}_G(\mathfrak{A})$.

Note that in the standard axioms of quantum theory – Figure 2.2 – we

included a fourth axiom which specified how the physical system evolves with time. A complete set of axioms would specify the dynamics of the system, but this is not needed for our derivation of the Born rule. However, we do discuss unitary dynamics in Chapter 8.

As a measurement is performed on a quantum mechanical system, the system is understood to undergo a so-called *reduction of the state vector*, or *collapse of the wavefunction* [123, §8.5]. The manner in which the act of measurement affects the system is one of the fundamental problems in quantum metaphysics – one of Isham’s four fundamental conceptual issues we list in Figure 1.2. We remain completely agnostic towards reduction of the state vector; we assume nothing about the system after measurement, not even that the system exists any longer as in many cases the measurement completely destroys the system, for example a photon being absorbed by a Geiger counter. The traditional axioms we presented in Figure 2.2 also said nothing about the state post-measurement, but we are emphasising this point because we will derive the Born rule from these axioms and some derivations of the Born rule rely on assumptions about the ‘collapse of the wavefunction’; we discuss this further below.

The following theorem can be seen as a derivation of the Born rule from the operational axioms shown in Figure 4.2. Note that Theorem 4.2.4 has already done almost all of the work, and that all that remains is to account for the case when $\dim(H) = 2$.

Theorem 4.2.11. *Assuming the axioms of Figure 4.2, and assuming the Continuum Hypothesis every operational state is uniquely determined by a density operator and the Born rule.*

Proof. For $\dim(H) \geq 3$ the result follows immediately from Theorem 4.2.4. To recover the result for $\dim(H) = 2$ we will use Axiom 3.

Let H be a Hilbert space with $\dim(H) = 2$ and let K be a Hilbert space

with $\dim(K) > 1$. Let

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}_{\text{vN}}(H)^{\text{op}} & \begin{array}{c} \curvearrowright \\ \uparrow \psi \\ \curvearrowleft \end{array} & \text{Set} \\
 & \mathbf{1} &
 \end{array}$$

and

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}_{\text{vN}}(K)^{\text{op}} & \begin{array}{c} \curvearrowright \\ \uparrow \phi \\ \curvearrowleft \end{array} & \text{Set} \\
 & \mathbf{1} &
 \end{array}$$

be a pair of operational states. Let

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}_{\text{vN}}(H \otimes K)^{\text{op}} & \begin{array}{c} \curvearrowright \\ \uparrow \varphi \\ \curvearrowleft \end{array} & \text{Set} \\
 & \mathbf{1} &
 \end{array}$$

be the operational state which, as per Axiom 3, satisfies

$$\varphi_{\mathfrak{A} \otimes \mathfrak{B}}(A \otimes B) = \psi_{\mathfrak{A}}(A) \cdot \phi_{\mathfrak{B}}(B)$$

For $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$. Now since $\dim(H \otimes K) \geq 3$, by Theorem 4.2.4 we see that there is a density operator

$$H \otimes K \xrightarrow{q} H \otimes K$$

such that $\varphi = \tilde{q}$. Now we must show that there exists a density operator

$$H \xrightarrow{r} H$$

such that $\psi = \tilde{r}$. Define $r : H \rightarrow H$ to be the partial trace of q , tracing out K . Now let $\mathfrak{B} \subseteq \text{Hom}(K, K)$ be the trivial algebra, which has only one element

$\tau \in \text{Spec}_G(\mathfrak{B})$ corresponding with the projector $\text{id}_K : K \rightarrow K$. Now we have the following

$$\begin{aligned}
 \psi_{\mathfrak{A}}(\rho) &= \psi_{\mathfrak{A}}(\rho) \cdot 1 \\
 &= \psi_{\mathfrak{A}}(\rho) \cdot \psi_{\mathfrak{B}}(\tau) \\
 &= \varphi_{\mathfrak{A} \otimes \mathfrak{B}}(\rho \cdot \tau) \\
 &= \text{tr}(q(P_\rho \otimes \text{id}_K)) \\
 &= \text{tr}(rP_\rho) \\
 &= \tilde{r}_{\mathfrak{A}}(\rho)
 \end{aligned}$$

and hence ψ is determined by a density operator, as required. \square

Remark 4.2.12. Recovering the case of $\dim(H) = 2$ can be achieved in a number of ways. In Theorem 4.2.11 we used Axiom 3, an assumption about being able to combine systems using the tensor product. But that is not to say that the Born rule necessarily relies on the tensor product in this way. We suspect there are numerous ways one could recover dimension 2, with different assumptions suiting one's needs. For example, we can give an abstract definition of *purity*, see Chapter 8, and so one might envisage an alternative to Axiom 3 which states that every operational state is the partial trace of a pure state on a larger system. Such purification axioms are considered in approaches to quantum theory based on *operational probabilistic theories* [38, 37].

There are many derivations of the Born rule from various set of axioms and metaphysical perspectives in the literature. Here we will give a brief comparison of some of these derivations of the Born rule to ours:

- The derivation of the Born rule of Deutsch [57] and Wallace [209, 210, 211] is based on the *many-worlds hypothesis*, which is a rather strong metaphysical assumption about the nature of reality, while our approach is underpinned by a minimal, pragmatic metaphysics, which makes no prior assumptions about the nature of reality or the meaning of probabilities.

- The approach of Zurek [218] hinges on the assumption of the principle of *environment-assisted invariance*, which is a symmetry property of entangled states. Entanglement plays a central role in the proof, as does the representation of dynamics with unitary operators. Our proof makes no reference to dynamics in general or unitary operators in particular, and entanglement plays only a minor role in recovering the special case of $\dim(H) = 2$. Unlike ours, Zurek’s proof only applies in the finite-dimensional case.
- Galley and Masanes [88] derive the Born rule from the assumption of *bit symmetry*, which requires introducing the notions of *pure state*, and of unitary operators capturing dynamics. Again, the derivation of Galley and Masanes also only applies in the finite-dimensional setting.
- Our derivation of the Born rule bears some resemblance to the approach of Logiurato and Smerzi [152]. Logiurato and Smerzi identify a principle of *non-contextual probability* motivated from a principle of no-faster-than-light signalling which allows them to derive a condition similar to the no-signalling condition we use. However, Logiurato and Smerzi start with more axioms and also require collapse of the state vector after measurement, however, we make no assumptions about the state of the system post-measurement. The Logiurato and Smerzi proof also only applies in the finite-dimensional setting.
- There is a class of derivations of the entire Hilbert space formalism, including the Born rule, from some minimal set of axioms, for example: Hardy’s five reasonable axioms [102]; Masanes and Müller’s derivation from information-theoretic principles [166]; and Chiribella, D’Ariano and Perinotti’s derivation from operational-probabilistic theories [38]. Each of these programs derives the entire Hilbert space formalism from some set of axioms or principles with a strong flavour of probability theory and information theory. The key distinction between these sets of axioms and our axioms of Figure 4.2 is that we axiomatise the mathematical structures representing the *measurements* of a quantum system, whereas their

formulation of quantum theory axiomatises the representation of the *states* of a system; Hardy explicitly states that he does not motivate his axioms from experimental observations [102, p. 3]. This distinction is explored further in Chapter 9.

Remark 4.2.13. The appearance of the Continuum Hypothesis in deriving the Born rule seems somewhat surprising. We have tried as far as possible to justify our mathematical assumptions from physical principles, but we can offer no such justification for the assumption of the Continuum Hypothesis. Of course, if one is only interested in separable Hilbert spaces – as is often the case in physics – then the Continuum Hypothesis is not required. We end our discussion of the Continuum Hypothesis with the following observation of Penrose [178, §16.7]:

It is perhaps remarkable, in view of the close relationship between mathematics and physics, that issues of such basic importance in mathematics as transfinite set theory and computability have as yet had a very limited impact on our description of the physical world.

Perhaps it is possible to conceive of an experiment which can falsify the Continuum Hypothesis? This marks a possible connection with a project of Nestruev to rethink the foundations of mathematics in terms of the concept of “observability”, [172, p. 209], which we will discuss further in Chapter 9.

Operational States and Topos Quantum Theory

While we have emphasised that our notion of state diverges from the topos quantum theory view on the state space, there is a relationship between these two notions. In particular, we will show that any operational state determines a subobject of the presheaf $\text{Spec}_{\mathcal{G}}$. This relationship is analogous to the way that the support of a probability measure μ on a measure space X determines a subset of X . Recall from Definition 4.1.30 the support of a probability distribution $d : X \rightarrow \mathbb{R}^+$ consists of those elements $x \in X$ where $d(x) > 0$. The *support* of a Radon probability measure generalises this notion, and is defined by Bogachev as follows [26, p. 77].

Definition 4.2.14. Let μ be a Radon probability measure on X . The *support* $\text{supp}(\mu) \subseteq X$ of μ is defined to be the intersection of all closed sets $C \subseteq X$ such that $\mu(C) = 1$.

Remark 4.2.15. It is noted by Bogachev [26, Proposition 7.2.9] that while the support of a Radon probability measure is well-behaved, the support of a more general Borel measure need not even satisfy $\mu(\text{supp}(\mu)) = 1$. For more general measure spaces there is not even a reasonable candidate for the a general definition for the “support” of a measure.

In order to show how operational states determine subobjects of the presheaf Spec_G we need the following well-known result in topology, sometimes called the *closed map lemma*, which can be found in most introductory texts on topology, for example, [125, Proposition 6.15].

Lemma 4.2.16. *If X is a compact topological space and Y is Hausdorff then for any continuous map $f : X \rightarrow Y$, if $C \subseteq X$ is closed in X then the image $f(C) \subseteq Y$ is closed in Y .*

Theorem 4.2.17. *Every operational state*

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

determines a subobject

$$\begin{array}{ccc}
 & \text{Spec}_G & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \xi & \\
 & \text{supp} &
 \end{array}$$

where for each \mathfrak{A} the component $\text{supp}_{\mathfrak{A}} \subseteq \text{Spec}_G(\mathfrak{A})$ is the support of the Radon measure $\psi_{\mathfrak{A}}$.

Proof. For each \mathfrak{A} we define $\text{supp}(\psi_{\mathfrak{A}})$ to be the support of $\psi_{\mathfrak{A}}$ in the sense of Definition 4.2.14, and we define the component of the natural transformation

$$\text{supp}_{\mathfrak{A}} \xrightarrow{\xi_{\mathfrak{A}}} \text{Spec}_G(\mathfrak{A})$$

to be subset inclusion.

We need to specify how supp is defined on the morphisms $i : \mathfrak{B} \rightarrow \mathfrak{A}$ and to show that for each such morphism the diagram

$$\begin{array}{ccc} \text{supp}(\psi_{\mathfrak{A}}) & \xrightarrow{\text{supp}(i)} & \text{supp}(\psi_{\mathfrak{B}}) \\ \xi_{\mathfrak{A}} \downarrow & & \downarrow \xi_{\mathfrak{B}} \\ \text{Spec}_G(\mathfrak{A}) & \xrightarrow{i^*} & \text{Spec}_G(\mathfrak{B}) \end{array}$$

commutes.

We will show that $i^*(\text{supp}(\psi_{\mathfrak{A}})) \subseteq \text{supp}(\psi_{\mathfrak{B}})$ and therefore we will be able to define the map $\text{supp}(i)$ as the restriction of i^* to the subset $\psi_{\mathfrak{A}}$, and since the components of ξ are just inclusion maps the above diagram will commute trivially.

Recall that the naturality condition in the definition of operational state specifies that for $i : \mathfrak{B} \rightarrow \mathfrak{A}$ the measure $\psi_{\mathfrak{B}}$ is equal to the pushforward of $\psi_{\mathfrak{A}}$ along the measurable function $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$. That is, for all Borel sets $V \subseteq \text{Spec}_G(\mathfrak{B})$ we have $\psi_{\mathfrak{B}}(V) = \psi_{\mathfrak{A}}(i^{*-1}(V))$.

Let $U \subseteq \text{Spec}_G(\mathfrak{A})$ be a closed subset with $\psi_{\mathfrak{A}}(U) = 1$. By Lemma 4.2.16 the subset $i^*(U) \subseteq \text{Spec}_G(\mathfrak{B})$ is also closed. Note that $U \subseteq i^{*-1}i^*(U)$ and so $\psi_{\mathfrak{A}}(U) \leq \psi_{\mathfrak{A}}(i^{*-1}i^*(U))$, but we assumed $\psi_{\mathfrak{A}}(U) = 1$, the maximum value $\psi_{\mathfrak{A}}$ attains, and hence $\psi_{\mathfrak{A}}(i^{*-1}i^*(U)) = 1$. That is, $i^*(U)$ is a closed subset of $\text{Spec}_G(\mathfrak{B})$ with $\psi_{\mathfrak{B}}(i^*(U)) = 1$.

Moreover, every closed $V \subseteq \text{Spec}_G(\mathfrak{B})$ is of the form $V = i^*(U)$ for $U \subseteq \text{Spec}_G(\mathfrak{A})$ as we can take $U = i^{*-1}(V)$. Note that since i^* is continuous, if V is closed then so is $i^{*-1}(V)$, hence if $V \subseteq \text{Spec}_G(\mathfrak{B})$ is a closed set such that

$\psi_{\mathfrak{B}}(V) = 1$ then $U = i^{*-1}(V) \subseteq \text{Spec}_G(\mathfrak{A})$ is a closed set satisfying $\psi_{\mathfrak{A}}(U) = 1$.

Hence we have shown equality of the set

$$\{i^*(U) \subseteq \text{Spec}_G(\mathfrak{B}) \mid U \subseteq \text{Spec}_G(\mathfrak{A}) \text{ and } \psi_{\mathfrak{A}}(U) = 1\}$$

and the set

$$\{V \subseteq \text{Spec}_G(\mathfrak{B}) \mid \psi_{\mathfrak{B}}(V) = 1\}$$

We can now show that $i^*(\text{supp}(\psi_{\mathfrak{A}})) \subseteq \text{supp}(\psi_{\mathfrak{B}})$. Consider the following

$$\begin{aligned} i^*(\text{supp}(\psi_{\mathfrak{A}})) &= i^*\left(\bigcap\{U \mid U \subseteq \text{Spec}_G(\mathfrak{A}) \text{ and } \psi_{\mathfrak{A}}(U) = 1\}\right) \\ &\subseteq \bigcap\{i^*(U) \mid U \subseteq \text{Spec}_G(\mathfrak{A}) \text{ and } \psi_{\mathfrak{A}}(U) = 1\} \\ &= \{V \subseteq \text{Spec}_G(\mathfrak{B}) \mid \psi_{\mathfrak{B}}(V) = 1\} \\ &= \text{supp}(\psi_{\mathfrak{B}}) \end{aligned}$$

The first equality is simply the definition of support, Definition 4.2.14. The inequality is a well-known fact about the intersection of the image of a function. The next equality is from the equality of sets we just showed, and the final equality is again just the definition of support, as required. \square

Note that the morphism ξ as defined in Theorem 4.2.17 is not a monomorphism, as there are different measures which have the same support.

A result similar to Theorem 4.2.4 is shown by Doering in [61, Theorem IV.1]. Doering shows that the density operators $q : H \rightarrow H$ naturally correspond with *abstract measures* on the so-called *clopen subobjects* of the presheaf

$$\mathbf{Hilb}\text{-}\mathbf{Alg}_{vN}(H)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

As with our characterisation of operational states this result relies heavily on Gleason's theorem.

Doering extracts density operators from the definition of abstract measure, but this definition is not clearly motivated physically, whereas we extract density

operators from the definition of operational state which is motivated by a purely pragmatic interpretation of quantum theory. Our result can be framed as a derivation of the Born rule from a set of axioms, but the result of Doering is not presented in this way.

Furthermore, the result of Doering relies on restricting to the poset of von Neumann subalgebras, which we already discussed does not have a clear physical motivation.

Remark 4.2.18. There is a reason why we should not restrict to just those commutative C^* -algebras that are von Neumann algebras. In the traditional presentation, classical mechanical systems are represented by manifolds. A compact manifold is necessarily Hausdorff, but is only a Stone space if it is a finite set equipped with the discrete topology – that is, a collection of points locally homeomorphic to \mathbb{R}^0 . Hence, classical mechanical systems are generally not represented by Stone spaces, and therefore if restricting the “classical subsystems” of a quantum system means restricting only to those subsystems corresponding with von Neumann algebras then we rule out the classical subsystems which resemble *actual* classical mechanical systems.

The commutative von Neumann subalgebras have been deemed inadequate from the perspective of Bohrfication as well, albeit for quite different reasons, which we will not go into here [115, §5.3].

4.3 The Convex Structure of the State Space

In this section we consider the convex structure on operational R -states associated with a system. We will make extensive use of this convex structure in the chapters that follow.

Traditionally, in quantum theory, one typically takes the collection of density operators

$$H \xrightarrow{q} H$$

to be the state space of the system. The set of density operators is a convex

space; we can take convex combinations of density operators in the obvious way, for $r \in [0, 1]$ and density operators q_1 and q_2 it is easy to check that the operator

$$r \cdot q_1 + (1 - r) \cdot q_2$$

is also a density operator.

In our framework the object which plays the role of state space is the presheaf Ψ , or more generally Δ_R . In this section we will show how these presheaves similarly carry a convex structure which can be characterised using the language of monads.

First, we recall an algebraic characterisation of a *convexity* due to Stone [203], although we follow the presentation of [86].

Definition 4.3.1. A *convex space* consists of a set X together with a family of convex combination operations

$$X \times X \xrightarrow{\nu_\lambda} X \quad \text{for each } \lambda \in [0, 1]$$

that satisfies the following conditions

1. The unit law:

$$\nu_0(x, y) = y \text{ for all } x, y \in X$$

2. Idempotency:

$$\nu_\lambda(x, x) = x \text{ for all } x \in X$$

3. Parametric commutativity:

$$\nu_\lambda(x, y) = \nu_{(1-\lambda)}(y, x)$$

4. Deformed parametric associativity:

$$\nu_\lambda(\nu_\kappa(x, y), z) = \nu_{\lambda'}(x, \nu_{\kappa'}(y, z))$$

where

$$\lambda' = \lambda\kappa \quad \text{and} \quad \kappa' = \begin{cases} \frac{\lambda(1-\kappa)}{(1-\lambda\kappa)} & \text{when } \lambda\kappa \neq 1 \\ \text{arbitrary} & \text{when } \lambda = \kappa = 1 \end{cases}$$

An *affine map* is a function $f : X \rightarrow Y$ between a pair of convex spaces that preserves convex combinations.

The collection of convex spaces and affine maps forms a category.

Example 4.3.2. For a set X the set of probability distributions on X has the structure of a convex space. For probability distributions d_1 and d_2 , and $r_1, r_2 \in [0, 1]$ such that $r_1 + r_2 = 1$ we define a probability distribution

$$r_1 \cdot d_1 + r_2 \cdot d_2$$

called the *convex sum* of d_1 and d_2 .

More generally, for (X, Σ) a measurable space, the set of probability measures on (X, Σ) is a convex space, with convex structure defined as follows: for a pair of probability measures μ_1 and μ_2 , and $r_1, r_2 \in \mathbb{R}^+$ such that $r_1 + r_2 = 1$ then we can define the convex sum of measures $r_1 \cdot \mu_1 + r_2 \cdot \mu_2$, which is defined

$$(r_1 \cdot \mu_1 + r_2 \cdot \mu_2)(U) = r_1\mu_1(U) + r_2\mu_2(U)$$

for each $U \in \Sigma$.

Algebraic Characterisations of Convexity

This algebraic description of convexity shown in Definition 4.3.1 can be characterised in terms of a monad and a Lawvere theory, in the same way that the axioms of a group or a ring can. This will be an essential step in making the notion of *convexity* in more general presheaf categories a precise one.

Recall the definition of \mathcal{D}_R , the R -distribution functor – Definition 4.1.31. For the special case where $R = \mathbb{R}^+$ this functor is known as the *finitary Giry*

monad, and is due to Giry [90]. We will now show that the functor \mathcal{D}_R is a monad for an arbitrary semiring R , not just the case when $R = \mathbb{R}^+$.

Theorem 4.3.3. *For a semiring R , the R -distribution functor*

$$\mathbf{Set} \xrightarrow{\mathcal{D}_R} \mathbf{Set}$$

is a monad.

Proof. To explain the monad structure of this functor we use the following notation: we view a probability distribution on a set X as a formal sum

$$\sum_i p_i \cdot \underline{x}_i$$

where $x_i \in X$, and $p_i \in R$ such that all but a finite number of $p_i = 0_R$ such that $\sum_i p_i = 1_R$.

If $p_i = 0$ then we omit it from the sum and hence we can represent each distribution as a finite term

$$p_{i_1} \cdot \underline{x}_{i_1} + \dots + p_{i_n} \cdot \underline{x}_{i_n}$$

For $p \in R$ we can define the multiplicative action on formal sums

$$p \cdot \left(\sum_i p_i \cdot \underline{x}_i \right) = \sum_i p p_i \cdot \underline{x}_i$$

Now define the natural transformations

$$\mathcal{D}_R \circ \mathcal{D}_R \xrightarrow{\mu} \mathcal{D}_R \quad \text{id} \xrightarrow{\eta} \mathcal{D}_R$$

as follows

$$\begin{array}{ccc} \mathcal{D}_R \circ \mathcal{D}_R(X) & \xrightarrow{\mu_X} & \mathcal{D}_R(X) \\ \sum_j p_j \cdot \left(\sum_i p_i \cdot \underline{x}_i \right) & \longmapsto & \sum_j p_j \cdot \left(\sum_i p_i \cdot \underline{x}_i \right) \end{array}$$

and

$$\begin{array}{ccc} X & \xrightarrow{\eta_X} & \mathcal{D}_R(X) \\ x & \longmapsto & 1_R \cdot \underline{x} \end{array}$$

It is straightforward to verify that μ and η satisfy the conditions of Definition 2.1.3. \square

The finitary Giry monad $\mathcal{D}_{\mathbb{R}^+} : \mathbf{Set} \rightarrow \mathbf{Set}$ gives us an algebraic characterisation of convex spaces in the sense of the following theorem [204].

Theorem 4.3.4. *The category of convex spaces and affine maps is equivalent to the Eilenberg-Moore category $\mathcal{EM}(\mathcal{D}_{\mathbb{R}^+})$.*

Using the language of formal sums from the proof of Theorem 4.3.3 we see that an Eilenberg-Moore algebra of $\mathcal{D}_{\mathbb{R}^+}$ consists a function

$$\mathcal{D}_{\mathbb{R}^+}(A) \xrightarrow{a} A$$

which assigns to each formal convex sum of elements of A a true element of A . These Eilenberg-Moore algebras correspond with convex spaces defined concretely in Definition 4.3.1. A full discussion of this can be found in [86], for example.

We are interested in not just probability distributions but more general R -distributions. One can take convex combinations with values in any semiring R , not just \mathbb{R}^+ . In particular, an element of $\mathcal{D}_R(X)$ can be seen as a convex sum of elements of X . This prompts us to define a more general notion of R -convex space. In light of Theorem 4.3.4 we make the definition as follows.

Definition 4.3.5. The category of R -convex spaces is defined to be the Eilenberg-Moore category of the monad

$$\mathbf{Set} \xrightarrow{\mathcal{D}_R} \mathbf{Set}$$

Hence an R -convex space is defined to be set A equipped with a map

$$\mathcal{D}_R(A) \xrightarrow{a} A$$

such that the necessary diagrams of Definition 2.1.4 commute.

Convex Objects in Presheaf Categories

The monad which characterises convex spaces – namely, the finitary Giry monad – has a corresponding Lawvere theory $\mathcal{L}_{\mathbb{R}^+}$ (in the sense of Theorem 2.1.9). For the explicit details of this Lawvere theory for convex spaces see Fritz [86, 87].

Similarly, for the monad \mathcal{D}_R there is a corresponding Lawvere theory \mathcal{L}_R , and hence we can make the following definition.

Definition 4.3.6. Let \mathcal{L}_R be the Lawvere theory corresponding with the monad \mathcal{D}_R . An R -convex object in a category \mathcal{C} is an \mathcal{C} -valued model of \mathcal{L}_R .

The main purpose of Definition 4.3.6 is to show that the presheaf

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\Delta_R} \mathbf{Set}$$

is an R -convex object in the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$.

We need the language of Lawvere theories in order to state the definition of an R -convex object in a presheaf category, but we will now show that we can reason about these objects using the language of monads. In particular, we define the R -convex objects in the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$ as the Eilenberg-Moore algebras of the monad

$$\begin{array}{ccc} \mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} & \xrightarrow{\widetilde{\mathcal{D}}_R} & \mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} \\ F \dashv & \xrightarrow{\quad\quad\quad} & \mathcal{D}_R \circ F \end{array}$$

as defined in Theorem 2.1.6.

Theorem 4.3.7. *The R -convex objects in $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$ are precisely the Eilenberg-Moore algebras of the monad $\widetilde{\mathcal{D}}_R$.*

Proof. This is an immediate application of Theorem 2.1.10. \square

Theorem 4.3.8. *The R -operational state presheaf*

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\Delta_R} \mathbf{Set}$$

is an R -convex object in the category of presheaves $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$.

Proof. This is true essentially by definition, since we have

$$\begin{aligned} \Delta_R &= \mathcal{D}_R \circ \text{Spec}_G \\ &= \widetilde{\mathcal{D}}_R(\text{Spec}_G) \end{aligned}$$

and so Δ_R is a free Eilenberg-Moore algebra for $\widetilde{\mathcal{D}}_R$, and hence by Theorem 4.3.7 the presheaf Δ_R is an R -convex object. \square

Remark 4.3.9. The very general proof of Theorem 4.3.8 does not quite apply to the presheaf

$$\mathbf{Hilb}\text{-Alg}_{\text{vN}}(H)^{\text{op}} \xrightarrow{\Psi} \mathbf{Set}$$

as Ψ is not defined as the composition $\mathcal{D}_{\mathbb{R}^+} \circ \text{Spec}_G$, however there is a straightforward proof of convexity. Showing that Ψ is a convex object is equivalent – by Theorem 4.3.7 – to giving an Eilenberg-Moore algebra map

$$\mathcal{D}_{\mathbb{R}^+} \Psi \xrightarrow{a} \Psi$$

This can be defined component-wise as each set $\Psi(\mathfrak{A})$ is a convex space, as we saw in Example 4.3.2.

The set of operational R -states is defined to be the set of global sections $\Gamma(\Delta_R)$. Now recall Theorem 2.1.18 which states that for a monad \mathcal{T} on \mathbf{Set} the

global sections adjunction

$$\begin{array}{ccc}
 & C_{(-)} & \\
 \text{Set} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & \text{Set}^{\mathscr{A}\text{-Alg}(X)^{\text{op}}} \\
 & \Gamma &
 \end{array}$$

lifts to an adjunction on the Eilenberg-Moore categories $\mathcal{EM}(\mathcal{T})$ and $\mathcal{EM}(\tilde{\mathcal{T}})$.

Applying Theorem 2.1.18 for the case when $\mathcal{T} = \mathcal{D}_R$, we see that the set $\Gamma(\Delta_R)$ is an R -convex space. We characterise this convex structure explicitly with the following result.

Theorem 4.3.10. *Let $r_1, \dots, r_n \in R$ be a set of elements such that $\sum_{i=1}^n r_i = 1_R$ and let ψ^1, \dots, ψ^n be a family of operational R -states. There is an operational R -state ψ defined to be the weighted sum of the ψ^i defined on each \mathfrak{A}*

$$\begin{array}{ccc}
 \text{Spec}_G(\mathfrak{A}) & \xrightarrow{\psi_{\mathfrak{A}}} & R \\
 \rho \vdash & \longrightarrow & \sum_{i=1}^n r_i \cdot \psi_{\mathfrak{A}}^i(\rho)
 \end{array}$$

Proof. We need to check that ψ determines an operational R -state, that is, we need to check that ψ defines a natural transformation. Let $\mathfrak{B} \hookrightarrow \mathfrak{A}$. We need to check that the diagram

$$\begin{array}{ccc}
 & & \Delta_R(\mathfrak{A}) \\
 \{*\} & \begin{array}{c} \nearrow \psi_{\mathfrak{A}} \\ \searrow \psi_{\mathfrak{B}} \end{array} & \downarrow i^* \\
 & & \Delta_R(\mathfrak{B})
 \end{array}$$

commutes. By definition we have

$$\begin{array}{ccc}
 \text{Spec}_G(\mathfrak{B}) & \xrightarrow{i^*(\psi_{\mathfrak{A}})} & R \\
 \gamma \vdash & \longrightarrow & \sum_{\substack{\rho \in \text{Spec}_G(\mathfrak{A}) \\ \rho|_{\mathfrak{B}} = \gamma}} \psi_{\mathfrak{A}}(\rho)
 \end{array}$$

and

$$\begin{aligned}
\sum_{\substack{\rho \in \text{Spec}_G(\mathfrak{A}) \\ \rho|_{\mathfrak{B}} = \gamma}} \psi_{\mathfrak{A}}(\rho) &= \sum_{\substack{\rho \in \text{Spec}_G(\mathfrak{A}) \\ \rho|_{\mathfrak{B}} = \gamma}} \left(\sum_{i=1}^n r_i \cdot \psi_{\mathfrak{A}}^i(\rho) \right) \\
&= \sum_{i=1}^n r_i \cdot \left(\sum_{\substack{\rho \in \text{Spec}_G(\mathfrak{A}) \\ \rho|_{\mathfrak{B}} = \gamma}} \psi_{\mathfrak{A}}^i(\rho) \right) \\
&= \sum_{i=1}^n r_i \cdot \psi_{\mathfrak{B}}^i(\gamma) \\
&= \psi_{\mathfrak{B}}(\gamma)
\end{aligned}$$

as required. \square

In light of Theorem 4.3.10 we can make the following definition.

Definition 4.3.11. We call an operational R -state ψ of the form described in Theorem 4.3.10 an R -convex sum of the operational R -states ψ^i and denote

$$\psi = \sum_{i=1}^n r_i \cdot \psi^i$$

Let H be a Hilbert space with $\dim(H) \geq 3$. We can now relate the set of density operators $q : H \rightarrow H$ with the presheaf

$$\mathbf{Hilb-Alg}(X)^{\text{op}} \xrightarrow{\Psi} \mathbf{Set}$$

as convex objects. In particular, Theorem 4.2.4 shows a natural correspondence between density operators and operational states; we will now show that this correspondence is not only an isomorphism of sets, but an isomorphism of convex spaces.

Theorem 4.3.12. *Let H be a Hilbert space with $\dim(H) \geq 3$, then assuming the Continuum Hypothesis there is an isomorphism of convex sets*

$$\{q : H \rightarrow H \mid q \text{ a density operator}\} \cong \Gamma(\Psi)$$

That is, for a convex sum of density operators

$$q = \sum_{i=1}^n r_i \cdot q_i$$

we have

$$\tilde{q} = \sum_{i=1}^n r_i \cdot \tilde{q}_i$$

a convex sum of operational states, in the sense of Definition 4.3.11, and where \tilde{q} is defined as in Theorem 4.1.29.

Proof. This is not difficult to check directly, where everything is computed component-wise, but the result holds for completely abstract reasons, following immediately from Theorem 4.3.7 and Theorem 2.1.18, which shows that the global sections adjunction preserves the R -convex structure because it is characterised by a monad. \square

Note that if we do not assume the Continuum Hypothesis then we do not get an isomorphism between the operational states and the density operators. Each operational state corresponds with a state ω of the C^* -algebra $\text{Hom}(H, H)$, however it is important to note that this correspondence is not an isomorphism – for example, singular states of the the C^* -algebra do not correspond with operational states. Despite not being an isomorphism, this identification of operational states with C^* -algebra states still respects the convex structure. This will be important in Chapter 6.

Recall, those states of the C^* -algebra determined by a density operator are called normal, Definition 4.2.7. Another important class of states are the *singular states*, which are characterised as follows, see for example [140, Proposition 4.20].

Definition 4.3.13. A state ω of the C^* -algebra $\text{Hom}(H, H)$ is *singular* if $\omega(P) = 0$ for every projection $P : H \rightarrow H$ that has a finite-dimensional image.

The collection of states on a C^* -algebra carries the structure of a convex space, with convex sums being defined point-wise in the obvious way. The

following theorem states that every state of the C^* -algebra $\text{Hom}(H, H)$ can be written uniquely as a convex sum of a normal state and a singular state, see, for example, [140, p. 112].

Theorem 4.3.14. *Every state of the C^* -algebra $\text{Hom}(H, H)$ can be written uniquely as a convex sum $t \cdot \omega_n + (1 - t) \cdot \omega_s$ for ω_n a normal state, ω_s a singular state, and $t \in [0, 1]$.*

It follows from Theorem 4.3.14 and Theorem 4.2.10 that every operational state can be expressed as a convex sum of an operational state which corresponds with a normal state, and one that corresponds with a singular state.

Theorem 4.3.15. *For H a Hilbert space with $\dim(H) \geq 3$, every operational state is of the form $t \cdot \tilde{q} + (1 - t) \cdot \tilde{\sigma}$ for $t \in [0, 1]$, where $q : H \rightarrow H$ is a density operator, and $\sigma : \text{Hom}(H, H) \rightarrow \mathbb{C}$ is a singular C^* -algebra state.*

Chapter 5

Contextuality and Hidden-Variables

In this chapter we give a general account of the phenomena of *contextuality* – in the sense expressed by the Kochen-Specker theorem [136]; and *non-locality*, in the sense expressed by Bell’s theorem [19].

The principle of non-contextuality states that the outcome of a particular measurement should not depend on the *context* in which that measurement is performed – that is, the outcome should not depend on which other measurements are made simultaneously. Classical physics is typically formulated as non-contextual [123, Chap. 4], however the Kochen-Specker theorem demonstrates that for measurements in quantum theory, the context in which the measurement is performed matters. Mermin describes this theorem as follows [163, p. 1]:

The Kochen-Specker theorem demonstrates that it is, in general, impossible to ascribe to an individual quantum system a definite value for each of a set of observables not all of which commute. Of course elementary metaphysics insists that we cannot assign definite values to noncommuting observables; the point of the Kochen-Specker theorem is to extract this directly from the quantum-mechanical

formalism, rather than merely appealing to precepts enunciated by the founders.

If quantum theory were non-contextual then it would be possible to consistently assign a definite outcome to each measurement across all contexts in which that measurement appears. The Kochen-Specker theorem is typically proven through the construction of an explicit obstruction to such a consistent assignment of outcomes: the original proof by Kochen and Specker [136] requires describing 117 vectors in a three-dimensional Hilbert space which cannot be consistently assigned outcome values. There have been many subsequent refinements of essentially the same argument which require fewer vectors, for example [134, 35].

The Kochen-Specker theorem has played a fundamental role in the development of topos quantum theory, indeed, a series of papers titled *a topos perspective on the Kochen-Specker theorem* [124, 33, 100, 34] makes up a substantial portion of the topos quantum theory literature.

Bell's theorem asserts that any hidden-variable model of quantum theory would fail to satisfy the condition of *local realism*, and hence quantum theory is *non-local*. Bell's original argument involves constructing an explicit state, and measurements such that the statistics of the measurement outcomes would violate a statistical property that a hidden-variable model satisfying the condition of local-realism would have to satisfy, namely *Bell's inequality*.

There have been many refinements of Bell's theorem, for example: by Clauser, Horne, Shimony and Holt give a refinement of Bell's inequality, the so-called *CHSH inequality* [40]. Greenberger, Horne and Zeilinger [96], Mermin [162], and Hardy [101] all give proofs of Bell's theorem which are not probabilistic in that they do not involve the violation of some equality that a hidden-variable model would have to satisfy, but like Bell's theorem they rely on the explicit construction of a state and a collection of measurements which can be shown to be incompatible with a hidden-variable model.

On the level of metaphysical interpretation, the concepts of contextuality and

non-locality are very similar – Mermin asserts that both are arguments against the existence of hidden-variable models [163, p. 1] – but mathematically they have quite a different flavour, and are treated as distinct concepts [180, Chap. 6 & Chap. 7].

In Section 5.1 we give a general definition of *Kochen-Specker contextuality* for the spectral presheaf framework based on a generalisation of the Kochen-Specker theorem as formulated in topos quantum theory.

In Section 5.2 we introduce the sheaf theoretic framework of contextuality and non-locality of Abramsky and Brandenburger [5].

In Section 5.3 we show that by an extremely general categorical construction we can lift the operational R -states – as defined in Chapter 4 – the setting of the Abramsky-Brandenburger formalism, and thus apply the techniques of that formalism, in particular, we can use their notion of local hidden-variable model and apply it to our notion of operational state.

Once we have done this we can relate these general local hidden-variable models to the notion of Kochen-Specker contextuality on the level of abstract spectral presheaves. This result has implications for the special case of Hilbert spaces, allowing us to prove a version of Bell’s theorem which, although arguably weaker than the original, can be used to rule out a wider class of hidden-variable models, including those with possibly negative probabilities which have been considered in the literature, see for example [5, §5], or [11].

5.1 Kochen-Specker Contextuality

There is an elegant presentation of the Kochen-Specker theorem in topos quantum theory going back to Isham and Butterfield [124]. We present this result – Theorem 5.1.1 – in a later form due to Hamilton, Isham and Butterfield [100, §2.2.1], or see [82, Chap. 3]. For a careful treatment and discussion of the Kochen-Specker theorem in its traditional form see, for example, [123, Chap. 9] or [180, Chap. 7].

Theorem 5.1.1. *For a Hilbert space H such that $\dim(H) \geq 3$, the presheaf*

$$\mathbf{Hilb-Alg}(H)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

has no global sections.

Proof. Suppose there exists a global section α . Note that there is a natural transformation $\text{Spec}_G \rightarrow \Psi$, which, for each \mathfrak{A} , sends an element of $\text{Spec}_G(\mathfrak{A})$ to the point-distribution on that element. By composing natural transformations, a global section of $\text{Spec}_G(\mathfrak{A})$ gives a global section of Ψ . By Theorem 4.2.4 this operational state is determined by some density operator $q : H \rightarrow H$, however the operational state determined by α determines a point distribution on each $\text{Spec}_G(\mathfrak{A})$, and no such density operator can realise this family of probability distributions. To see this, consider some \mathfrak{A} whose primitive projections are all one-dimensional. Then there exists precisely one of those projections P such that $\text{tr}(Pq) = 1$, which implies that $P = q$. Now consider some other \mathfrak{B} whose primitive projections are one-dimensional, but which has no one-dimensional projections in common with \mathfrak{A} . By the same argument for \mathfrak{A} must be equal to one of the one-dimensional projections in \mathfrak{B} , a contradiction. \square

There is a formulation of the Kochen-Specker theorem similar to Theorem 5.1.1 in the Bohrifaction program, see [115, Theorem 4.10].

Remark 5.1.2. The proof of the Kochen-Specker theorem – Theorem 5.1.1 – we present above is not conceptually new as it is well known that the Kochen-Specker theorem follows as a corollary of Gleason’s theorem, see [140, Theorem 4.32], for example. Our proof of the Kochen-Specker theorem invokes Theorem 4.2.4, which relies on Gleason’s theorem.

Defining Kochen-Specker Contextuality

Based on the topos quantum theory characterisation of the Kochen-Specker theorem – Theorem 5.1.1 – we make the following general definition of *Kochen-*

Specker contextuality.

Definition 5.1.3. Let \mathcal{A} be a locally small \dagger -symmetric monoidal category with finite distributive \dagger -biproducts. A physical system represented by the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$ is said to be *Kochen-Specker contextual* if the spectral presheaf

$$\mathcal{A}\text{-Alg}(X)^{\text{op}} \xrightarrow{\text{Spec}_{\mathbf{G}}} \mathbf{Set}$$

has no global sections. We say that the system is *Kochen-Specker non-contextual* if it is not Kochen-Specker contextual, that is, if a global section does exist.

It is not obvious that Definition 5.1.3 is a meaningful one. We must ask whether this a reasonable generalisation of the Kochen-Specker theorem. We offer two pieces of evidence justifying this generalisation.

1. We will be able to relate this notion of contextuality to a notion of *hidden-variable model* for the spectral presheaf framework, and prove a correspondence between these two concepts on this level of generality. When specialised to the setting of quantum mechanics this correspondence will allow us to prove a strengthened form of Bell's theorem.
2. When we model Spekkens' Toy Theory, in Chapter 7, we will see that the global sections of $\text{Spec}_{\mathbf{G}}$ correspond precisely with the *ontic states* of Spekkens' Toy Theory, and the existence of such states can be seen as a fundamental difference between this essentially classical theory, and quantum theory. Spekkens' Toy Theory is "classical by construction", in an informal sense. We model Spekkens' Toy Theory in the spectral presheaf framework, and equipped with our more general notion of hidden-variable model we can derive the classical properties of Spekkens' Toy Theory as theorems; we extract the classical properties from the formalism, rather than appealing to the elementary metaphysics of the theory.

Recall an operational R -state is a global section of the presheaf

$$\mathcal{A}\text{-Alg}_{\mathbf{vN}}(X)^{\text{op}} \xrightarrow{\Delta_R} \mathbf{Set}$$

where $\Delta_R = \mathcal{D}_R \circ \text{Spec}_G$ – as defined in Definition 4.1.33.

We can naturally view the global sections of Spec_G – if they exist – as a subset of the operational R -states. In particular, they correspond with the globally-deterministic operational R -states, which we now define.

Definition 5.1.4. A *globally-deterministic operational R -state* is an operational R -state

$$\begin{array}{ccc} & \Delta_R & \\ & \curvearrowright & \\ \mathcal{A}\text{-Alg}_{\mathbf{vN}}(X)^{\text{op}} & \uparrow \psi & \mathbf{Set} \\ & \curvearrowleft & \\ & \mathbf{1} & \end{array}$$

such that for each \mathfrak{A} the R -distribution $\psi_{\mathfrak{A}}$ is a point-distribution on $\text{Spec}_G(\mathfrak{A})$. That is, for exactly one element of $\rho \in \text{Spec}_G(\mathfrak{A})$ we have $\psi_{\mathfrak{A}}(\rho) = 1_R$ and $\psi_{\mathfrak{A}}(\gamma) = 0_R$ for all $\gamma \neq \rho$.

The correspondence between globally-deterministic operational R -states and global sections of Spec_G is shown in the following lemma.

Lemma 5.1.5. *There is a one-to-one correspondence between the globally-deterministic operational R -states and the global sections of $\text{Spec}_G(\mathfrak{A})$.*

Proof. Let $\chi : \mathbf{1} \rightarrow \text{Spec}_G$ be a global section. For each \mathfrak{A} we have an element $\chi_{\mathfrak{A}} \in \text{Spec}_G(\mathfrak{A})$. Define the operational state $\tilde{\chi} : \mathbf{1} \rightarrow \Delta_R$ such that for each \mathfrak{A} , $\tilde{\chi}_{\mathfrak{A}}$ corresponds with the R -distribution $1_R \cdot \chi_{\mathfrak{A}}$. We need to show that $\tilde{\chi}_{\mathfrak{A}}$ is natural in \mathfrak{A} , that is, for $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$, with corresponding restriction map $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$ we have $\mathcal{D}_R(i^*)(\tilde{\chi}_{\mathfrak{A}}) = \tilde{\chi}_{\mathfrak{B}}$. By definition of \mathcal{D}_R we have $\mathcal{D}_R(i^*)(\tilde{\chi}_{\mathfrak{A}}) = 1_R \cdot i^*(\chi_{\mathfrak{A}})$. Since χ is natural we have $i^*(\chi_{\mathfrak{A}}) = \chi_{\mathfrak{B}}$ and

hence

$$\begin{aligned} 1_R \cdot i^*(\chi_{\mathfrak{A}}) &= 1_R \cdot \chi_{\mathfrak{B}} \\ &= \tilde{\chi}_{\mathfrak{B}} \end{aligned}$$

and $\tilde{\chi}$ is natural and every $\tilde{\chi}_{\mathfrak{A}}$ is a point-distribution, as required.

In the opposite direction, let $\psi : \mathbf{1} \rightarrow \Delta_R$ be a globally-deterministic operational state. For each \mathfrak{A} the corresponding R -distribution $\psi_{\mathfrak{A}}$ is of the form $1_R \cdot \omega_{\mathfrak{A}}$ for some $\omega_{\mathfrak{A}} \in \text{Spec}_G(\mathfrak{A})$. This defines a global section $\omega : \mathbf{1} \rightarrow \text{Spec}_G$. We need to check that ω is natural, that is, for $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ we have $\omega_{\mathfrak{A}}|_{\mathfrak{B}} = \omega_{\mathfrak{B}}$. Since ψ is natural we have $\mathcal{D}_R(i^*)(\psi_{\mathfrak{A}}) = \psi_{\mathfrak{B}}$, hence $1_R \cdot \omega_{\mathfrak{A}}|_{\mathfrak{B}} = 1_R \cdot \omega_{\mathfrak{B}}$, and therefore $\omega_{\mathfrak{A}}|_{\mathfrak{B}} = \omega_{\mathfrak{B}}$, as required.

It is straightforward to verify that these processes are inverse to one another. Given $\tilde{\chi}$ as defined above for some χ , this operational state defines a unique point-distribution $1_R \cdot \chi_{\mathfrak{A}}$ on $\text{Spec}_G(\mathfrak{A})$ for each \mathfrak{A} , and from these point distributions the global section of Spec_G we recover is precisely χ .

Conversely, a globally-deterministic operational state ϕ by definition determines a point-distribution $1_R \cdot \omega_{\mathfrak{A}}$ for each \mathfrak{A} . Taking the corresponding global section ω of Spec_G , we see that $\tilde{\omega}$ is characterised by giving the point-distribution $1_R \cdot \omega_{\mathfrak{A}}$ for each \mathfrak{A} , which is precisely how ϕ was defined in the first place, as required. \square

Remark 5.1.6. As a result of Lemma 5.1.5, for any semirings R and R' there is a one-to-one correspondence between the operational R -states and the R' -operational states, and hence we can talk unambiguously about *the* globally-deterministic operational states.

Hence we see that the condition of Kochen-Specker contextuality is equivalent to the system not admitting any globally-deterministic operational states. In particular, the Kochen-Specker theorem asserts that for $\mathcal{A} = \mathbf{Hilb}$ there are no globally-deterministic operational states. In the next section we will see an example where globally-deterministic operational states do exist.

Non-Contextuality for Quantale-Valued Relations

Recall the definition of the category of quantale-valued relations, which we defined in Chapter 3. We now show that every physical system represented by the category

$$\mathbf{Set}^{\mathbf{Rel}_Q\text{-Alg}(X)^{\text{op}}}$$

for a commutative ZDF quantale Q , is Kochen-Specker non-contextual. The results of this section are not needed for any subsequent chapters, and are given here for the purposes of providing a detailed account of one model other than the Hilbert space case, and hence this section can be safely skipped. However, for the reader not necessarily interested in the category \mathbf{Rel}_Q , we would like to highlight the role of the adjunction described in Theorem 3.1.22 plays in the results we show here; we seem to get a surprising mileage out of this adjunction, even in a case totally removed from von Neumann algebras in the traditional sense of C^* -algebras.

In order to show that there exist global sections of the presheaf

$$\mathbf{Rel}_Q\text{-Alg}(X)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

we can use the same trick we used for \mathbf{Hilb} – that is, Theorem 3.1.22 and Lemma 3.1.23 – which show that it is enough to show the existence of global sections of the presheaf

$$\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

As with C^* -algebras, the von Neumann condition on semialgebras of quantale-valued relations will be very useful in proofs.

The existence of these global sections will become meaningful when we model Spekkens' Toy Theory in Chapter 7.

Lemma 5.1.7. *Let Q be a ZDF quantale, and X a set. If the presheaf*

$$\mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(X)^{\text{op}} \xrightarrow{\text{Spec}_Q} \mathbf{Set}$$

as defined in Definition 3.2.17 has a global section, then so does

$$\mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(X)^{\text{op}} \xrightarrow{\text{Spec}_G} \mathbf{Set}$$

Proof. This follows directly from Lemma 3.2.18 and Theorem 3.2.20. \square

Hence, to show that a system represented by a category

$$\mathbf{Set}^{\mathbf{Rel}_Q\text{-Alg}(X)^{\text{op}}}$$

is Kochen-Specker non-contextual it is enough to show the existence of a global section of the presheaf

$$\begin{array}{ccc} & \text{Spec}_Q & \\ \curvearrowright & & \curvearrowleft \\ \mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(X)^{\text{op}} & \uparrow \xi & \mathbf{Set} \\ \curvearrowleft & & \curvearrowright \\ & \mathbf{1} & \end{array}$$

We prove this non-contextuality result by giving a complete characterisation of the global sections of Spec_Q . This characterisation makes essential use of the structure theorem for the commutative von Neumann semialgebras of quantale-valued relations – Theorem 3.2.14.

Theorem 5.1.8. *For Q a commutative ZDF quantale, and X a set, each element $x \in X$ uniquely determines a global section*

$$\begin{array}{ccc} & \text{Spec}_Q & \\ \curvearrowright & & \curvearrowleft \\ \mathbf{Rel}_Q\text{-Alg}(X)^{\text{op}} & \uparrow \tilde{x} & \mathbf{Set} \\ \curvearrowleft & & \curvearrowright \\ & \mathbf{1} & \end{array}$$

Proof. By Theorem 3.1.22 and Lemma 3.1.23 it is enough to show that each $x \in X$ determines a global section

$$\begin{array}{ccc}
 & \text{Spec}_Q & \\
 \text{Rel}_Q\text{-Alg}_{\text{vN}}(X)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \tilde{x} & \\
 & \mathbf{1} &
 \end{array}$$

the case when we restrict to only commutative von Neumann semialgebras.

By the structure theorem for von Neumann semialgebras of quantale-valued relations – Theorem 3.2.14 – each semialgebra \mathfrak{A} in $\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)$ has a decomposition $\mathfrak{A} = \prod_i e_i \mathfrak{A}$ for primitive subunital idempotents e_i . These subunital idempotents form a partition of the underlying set X and hence the element $x \in X$ lies in the image of exactly one of the primitive subunital idempotents, which we will denote e_x .

Now define the component

$$\mathfrak{A} \xrightarrow{\tilde{x}_{\mathfrak{A}}} \mathbf{2}$$

to be the map defined by $\tilde{x}_{\mathfrak{A}}(q) = 1$ if and only if $e_x q \neq 0$, for each $q \in \mathfrak{A}$. that a semialgebra homomorphism of this type is completely determined by the values it takes on the primitive subunital idempotents.

We need to check that this family $\tilde{x}_{\mathfrak{A}}$ is natural in \mathfrak{A} . Suppose we have $\mathfrak{B} \hookrightarrow \mathfrak{A}$. We need to check that $\tilde{x}_{\mathfrak{A}}|_{\mathfrak{B}} = \tilde{x}_{\mathfrak{B}}$.

By the structure theorem – Theorem 3.2.14 – we have $\mathfrak{B} = \prod_j d_j \mathfrak{B}$. Since these maps are completely determined by their values on primitive subunital idempotents it is enough to check that $\tilde{x}_{\mathfrak{A}}(d_x) = 1$ where d_x is the primitive subunital idempotent in \mathfrak{B} such that $\tilde{x}_{\mathfrak{B}}(d_x) = 1$.

Note that since both of the relations d_x and e_x relate x to itself, the composition $d_x \circ e_x$ must also relate x to itself, and therefore $d_x \circ e_x \neq 0$. In particular $d_x \notin \overline{e_x \mathfrak{A}}$, and hence $\tilde{x}_{\mathfrak{A}}(d_x) = 1$, as required. \square

We can use Theorem 5.1.8 to show that physical systems represented by categories of this form are Kochen-Specker non-contextual.

Theorem 5.1.9. *For Q a ZDF quantale and X a non-empty set, any physical system represented by*

$$\mathbf{Set}^{\mathbf{Rel}_Q\text{-Alg}(X)^{\text{op}}}$$

is Kochen-Specker non-contextual.

Proof. Theorem 5.1.8 shows the existence of global sections of Spec_Q . By Lemma 3.2.18 this implies the existence of global sections for Spec_P , which, by Lemma 5.1.7 implies the existence of global sections of Spec_G , as required. \square

In the interest of giving a full account of \mathbf{Rel}_Q , we will show a partial converse of Theorem 5.1.8, that is, that every global section of Spec_Q naturally corresponds with an element of the underlying set X . To show this we will need the following lemmas.

Lemma 5.1.10. *For a set X let $\mathfrak{E} = \{ q \bullet \text{id}_X \mid q \in Q \}$, we have $\mathfrak{E} = \mathfrak{E}''$.*

Proof. Clearly $\mathfrak{E}' = \text{Hom}(X, X)$, and hence an element in \mathfrak{E}'' is a morphism $f : X \rightarrow X$ such that for all $g : X \rightarrow X$ we have $f \circ g = g \circ f$.

Let $p_x : X \rightarrow X$ be the identity on x and zero otherwise, that is, for all $y, z \in X$

$$p_x(y, z) = \begin{cases} 1_Q & \text{if } x = y = z \\ 0_Q & \text{otherwise} \end{cases}$$

The set X can be written as a biproduct $\{x\} \sqcup (X \setminus \{x\})$, and the relation p_x has corresponding matrix representation $p_x = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$.

Let f be a relation such that $p_x \circ f = f \circ p_x$. f has matrix representation $f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$ and hence

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

which implies

$$\begin{pmatrix} f_1 & f_2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} f_1 & 0 \\ f_3 & 0 \end{pmatrix}$$

and hence under the assumption that Q is ZDF we have $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \end{pmatrix}$. This implies that if $f(x, y) \neq 0_Q$ then $x = y$. Note that this applies for all $x \in X$, and hence for all $x, y \in X$, if $f(x, y) \neq 0$, then $x = y$.

It remains to show that $f(x, x) = f(y, y)$ for all $x, y \in X$. To see this consider $\sigma_{x,y} : X \rightarrow X$ defined for all $u, v \in X$

$$\sigma_{x,y}(u, v) = \begin{cases} 1_Q & \text{if } x = u \text{ and } y = v \\ 1_Q & \text{if } x = v \text{ and } y = u \\ 0_Q & \text{otherwise} \end{cases}$$

Letting $X = \{x\} \sqcup \{y\} \sqcup (X \setminus \{x, y\})$ the relation $\sigma_{x,y}$ has corresponding matrix representation

$$\sigma_{x,y} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Now suppose $\sigma_{x,y} \circ f = f \circ \sigma_{x,y}$, that is

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix} = \begin{pmatrix} f_1 & 0 & 0 \\ 0 & f_2 & 0 \\ 0 & 0 & f_3 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

and therefore $f_1 = f_2$. That is, $f(x, x) = f(y, y)$ for all x and $y \in X$, as required. \square

We call \mathfrak{E} , as defined in Lemma 5.1.10 the *trivial semialgebra on X* . Clearly there is an inclusion $\mathfrak{E} \hookrightarrow \mathfrak{A}$ for every \mathfrak{A} in $\mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(X)$, and hence \mathfrak{E} is the bottom element of $\mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(X)$.

Lemma 5.1.11. *If $\mathfrak{A} = e_1\mathfrak{A} \oplus e_2\mathfrak{A}$ belongs to $\mathbf{Rel}_Q\text{-Alg}_{\mathbf{vN}}(A)$ where e_1 is the*

identity morphism on some subset $E \subseteq A$ then $e_1\mathfrak{A}$ viewed as a subsemialgebra $e_1\mathfrak{A} \subseteq \text{Hom}(E, E)$ belongs to $\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(E)$.

Proof. The relation e_1 has matrix representation $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$. Let $g \in \mathfrak{A}'$ with matrix representation $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$. By assumption $g \circ e_1 = e_1 \circ g$, that is

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$$

and hence

$$\begin{pmatrix} g_1 & 0 \\ g_3 & 0 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ 0 & 0 \end{pmatrix}$$

Since Q is assumed to be ZDF we conclude that $g = \begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}$, and moreover, $g_1 \in (e_1\mathfrak{A})'$.

Now consider $h_1 \in (e_1\mathfrak{A})''$, and let $h = \begin{pmatrix} h_1 & 0 \\ 0 & 0 \end{pmatrix}$. Since by assumption we have $g_1h_1 = h_1g_1$ then it follows that $gh = hg$, and therefore $h \in \mathfrak{A}''$. Since \mathfrak{A} is assumed to be von Neumann we have $h \in \mathfrak{A}$, and therefore $h_1 \in e_1\mathfrak{A}$, as required. \square

Lemma 5.1.12. *Suppose $\mathfrak{A} \subseteq \text{Hom}(A, A)$ belongs to $\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(A)$ and suppose $\mathfrak{B} \subseteq \text{Hom}(B, B)$ belongs to $\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(B)$ then*

$$\mathfrak{A} \oplus \mathfrak{B} \subseteq \text{Hom}(A \sqcup B, A \sqcup B)$$

belongs to $\mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(A \sqcup B)$.

Proof. For $f \in \text{Hom}(A \sqcup B, A \sqcup B)$, since \sqcup is a biproduct, we have a matrix representation $f = \begin{pmatrix} f_1 & f_2 \\ f_3 & f_4 \end{pmatrix}$.

The morphisms $f \in \mathfrak{A} \oplus \mathfrak{B}$ are exactly those with matrix representations $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \end{pmatrix}$ where $f_1 \in \mathfrak{A}$ and $f_4 \in \mathfrak{B}$.

Let $g = \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \in (\mathfrak{A} \oplus \mathfrak{B})'$. Then in particular

$$\begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} g_1 & g_2 \\ g_3 & g_4 \end{pmatrix}$$

and hence

$$\begin{pmatrix} g_1 & 0 \\ g_3 & 0 \end{pmatrix} = \begin{pmatrix} g_1 & g_2 \\ 0 & 0 \end{pmatrix}$$

and therefore $g_3 = 0$ and $g_2 = 0$. We have shown that g has matrix representation $\begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}$. Let $f \in \mathfrak{A} \oplus \mathfrak{B}$ with matrix representation $f = \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \end{pmatrix}$. Then clearly

$$\begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix} \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \end{pmatrix} = \begin{pmatrix} f_1 & 0 \\ 0 & f_4 \end{pmatrix} \begin{pmatrix} g_1 & 0 \\ 0 & g_4 \end{pmatrix}$$

if and only if $f_1 \circ g_1 = g_1 \circ f_1$ and $f_4 \circ g_4 = g_4 \circ f_4$, and hence $(\mathfrak{A} \oplus \mathfrak{B})' = \mathfrak{A}' \oplus \mathfrak{B}'$.

Now consider $h = \begin{pmatrix} h_1 & h_2 \\ h_3 & h_4 \end{pmatrix} \in (\mathfrak{A} \oplus \mathfrak{B})'' = (\mathfrak{A}' \oplus \mathfrak{B}')'$. Then in particular $h \circ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \circ h$ and $h \circ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \circ h$ and hence, by a similar argument as above, $h = \begin{pmatrix} h_1 & 0 \\ 0 & h_4 \end{pmatrix}$ with $h_1 \in \mathfrak{A}''$ and $h_2 \in \mathfrak{B}''$ and since \mathfrak{A} and \mathfrak{B} are von Neumann we have $h \in \mathfrak{A} \oplus \mathfrak{B}$, as required. \square

We now show a partial converse of Theorem 5.1.8.

Theorem 5.1.13. *For Q a commutative ZDF quantale, every global section*

$$\begin{array}{ccc} & \text{Spec}_Q & \\ \curvearrowright & & \curvearrowleft \\ \mathbf{Rel}_Q\text{-Alg}(X)^{\text{op}} & \uparrow \chi & \mathbf{Set} \\ \curvearrowleft & & \curvearrowright \\ & \mathbf{1} & \end{array}$$

uniquely picks an element $x \in X$.

Proof. Using Theorem 3.1.22 Lemma 3.1.23 it is enough to prove the result considering only the von Neumann Q^* -semialgebras, that is to show that each global section

$$\begin{array}{ccc} & \text{Spec}_Q & \\ \curvearrowright & & \curvearrowleft \\ \mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)^{\text{op}} & \uparrow \chi & \mathbf{Set} \\ \curvearrowleft & & \curvearrowright \\ & \mathbf{1} & \end{array}$$

uniquely picks an element $x \in X$.

Let $\mathfrak{A}, \mathfrak{B} \in \mathbf{Rel}_Q\text{-Alg}_{\text{vN}}(X)$. By Theorem 3.2.14 these semialgebras admit decompositions $\mathfrak{A} = \prod_i e_i \mathfrak{A}$, and by Theorem 3.2.19 there is one primitive idempotent element e_a such that $\chi_{\mathfrak{A}}(e_a) = 1$. For $\mathfrak{B} = \prod_j d_j \mathfrak{B}$ there is one d_b such that $\chi_{\mathfrak{B}}(d_b) = 1$. We claim that for e_a and d_b we have $e_a \circ d_b \neq 0$.

Let $E_a = \text{supp}(e_a)$ and $E_b = \text{supp}(d_b)$, and suppose $e_a \circ d_b = 0$, which implies that $E_a \cap E_b = \emptyset$. Let \mathfrak{C}_1 be the trivial semialgebra defined on the set $X \setminus (E_a \sqcup E_b)$. Let \mathfrak{C}_2 be the trivial semialgebra on $X \setminus E_a$ and \mathfrak{C}_3 be the trivial semialgebra on $X \setminus E_b$. By Lemma 5.1.10 these trivial semialgebras are von Neumann semialgebras. By Lemma 5.1.11 $e_a \mathfrak{A}$ and $d_b \mathfrak{B}$ are von Neumann semialgebras and by Lemma 5.1.12 so are the direct sums of these semialgebras, and hence we have subsemialgebra inclusions

$$\begin{array}{ccccc}
 & & e_a \mathfrak{A} \oplus d_b \mathfrak{B} \oplus \mathfrak{C}_1 & & \\
 & \swarrow & & \searrow & \\
 \mathfrak{A} & & & & \mathfrak{B} \\
 & \swarrow & & \searrow & \\
 & & e_a \mathfrak{A} \oplus \mathfrak{C}_2 & & d_b \mathfrak{B} \oplus \mathfrak{C}_3
 \end{array}$$

By naturality, if $\chi_{\mathfrak{A}}(e_a) = 1$ then $\chi_{e_a \mathfrak{A} \oplus e_b \mathfrak{B} \oplus \mathfrak{C}_1}(e_a) = 1$ which implies that $\chi_{e_a \mathfrak{A} \oplus e_b \mathfrak{B} \oplus \mathfrak{C}_1}(e_b) = 0$, which in turn implies that $\chi_{\mathfrak{B}}(e_b) = 0$, which is a contradiction, and hence it cannot be the case that $e_a \circ d_b = 0$.

Now consider the von Neumann semialgebra \mathfrak{A} generated by the projections $P_{\{x\}}$ for each $x \in X$. This semialgebra is isomorphic to the X -fold product of Q , $\mathfrak{A} \cong \prod_{x \in X} Q$. The primitive idempotents in the decomposition $\mathfrak{A} = \prod_i e_i \mathfrak{A}$ are precisely those $P_{\{x\}}$ and hence, by Theorem 3.2.19, there is a one-to-one correspondence between the elements of $\text{Spec}_Q(\mathfrak{A})$ and the elements of the set X . For every other von Neumann semialgebra $\mathfrak{B} = \prod_j d_j \mathfrak{B}$, for each e_i there is exactly one d_j such that $e_i \circ d_j \neq 0$, and hence a global section $\chi : \mathbf{1} \rightarrow \text{Spec}_Q$ is completely determined by the component $\chi_{\mathfrak{A}}$, which corresponds to precisely one element $x \in X$. \square

5.2 The Abramsky-Brandenburger Formalism

Here we outline the approach to abstract empirical models of Abramsky and Brandenburger [5]. This is a general framework in which one can formulate contextuality and non-locality at a completely general level. The framework can be applied to quantum measurements, but also more general models which are not quantum-realizable, for example, PR boxes [183], as well as phenomena totally unrelated to physics, for example: modelling constraints in classical computation [8], or querying databases [4]. How we present empirical models is similar to that of Karvonen [132].

In the Abramsky-Brandenburger formalism one considers a set \mathcal{M} of *measurements* on a system and \mathcal{O} the set of possible outcomes of those measurements. Let \mathcal{C} be a family of subsets $C \subseteq \mathcal{M}$ such that $\bigcup_{C \in \mathcal{C}} C = \mathcal{M}$. This family \mathcal{C} defines the set of *maximal measurement contexts*. Any set of measurements in the downward closure of \mathcal{C} , that is, the set

$$\downarrow \mathcal{C} = \{ U \subseteq \mathcal{M} \mid \text{there exists } C \in \mathcal{C} \text{ with } U \subseteq C \}$$

is a *measurement context*. Measurement contexts are those collections of measurements which are compatible – those which can be performed simultaneously.

Note that we are assuming no additional structure on these sets of measurements and outcomes, in particular, we are not assuming that the measurements come equipped with any algebraic structure.

Definition 5.2.1. Call such a triple $(\mathcal{M}, \mathcal{C}, \mathcal{O})$ an *abstract measurement scenario*.

The Abramsky-Brandenburger formalism considers the category of presheaves on the poset $\mathcal{P}(\mathcal{M})$ – the powerset of \mathcal{M} ordered by subset inclusion.

In operational approaches to quantum theory one often talks in terms of *events*, described by Heinosaari and Ziman [109, p. 48] as follows:

An *elementary event* in any physical experiment is of the form

‘The recorded measurement outcome is x .’

This notion is described generalised in the Abramsky-Brandenburger formalism as follows.

Definition 5.2.2. For $(\mathcal{M}, \mathcal{C}, \mathcal{O})$ and abstract measurement scenario. Given a context $U \in \downarrow \mathcal{C}$, an *event* is an assignment – that is, a function

$$U \xrightarrow{\alpha} \mathcal{O}$$

which specifies an outcome for each measurement in the context U .

Note the similarity between these abstract events and elements $\rho \in \text{Spec}_{\mathbb{G}}(\mathfrak{A})$ for some commutative algebra, which is an assignment of an outcome value for each measurement in \mathfrak{A} , as discussed in Figure 2.1.

Definition 5.2.3. Let $\mathcal{P}(\mathcal{M})$ be the powerset of \mathcal{M} . The *event sheaf* is the functor

$$\mathcal{P}(\mathcal{M})^{\text{op}} \xrightarrow{\mathcal{E}} \mathbf{Set}$$

which is defined on objects $\mathcal{E}(U) = \{ s : U \rightarrow \mathcal{O} \}$, the set of functions assigning an outcome value to each measurement in U . The action on morphisms is given by function restriction.

Recall the definition of the R -distribution functor $\mathcal{D}_R : \mathbf{Set} \rightarrow \mathbf{Set}$ – Definition 4.1.31. We compose this functor with the event sheaf to give the presheaf

$$\mathcal{P}(\mathcal{M})^{\text{op}} \xrightarrow{\mathcal{D}_R \mathcal{E}} \mathbf{Set}$$

We give a concrete description of the action of $\mathcal{D}_R \mathcal{E}$ on morphisms: for $i : V \hookrightarrow U$ the corresponding morphism of R -distributions is given by the *marginalisation* of R -distributions

$$\mathcal{D}_R \mathcal{E}(U) \xrightarrow{\mathcal{D}_R i^*} \mathcal{D}_R \mathcal{E}(V)$$

where for an R -distribution $d : \mathcal{E}(U) \rightarrow R$ we define the R -distribution

$$\begin{array}{ccc} \mathcal{E}(V) & \xrightarrow{\mathcal{D}_R i^*(d)} & R \\ \alpha \vdash & \longrightarrow & \sum_{\substack{\beta \in \mathcal{E}(U) \\ \beta|_V = \alpha}} d(\beta) \end{array}$$

and we denote the distribution $\mathcal{D}_R i^*(d) = d|_V$.

Definition 5.2.4. Let $(\mathcal{M}, \mathcal{C}, \mathcal{O})$ be an abstract measurement scenario. An R -valued empirical model for this measurement scenario consists of a family of R -distributions

$$e_C \in \mathcal{D}_R \mathcal{E}(C)$$

for each context $C \in \downarrow \mathcal{C}$. An empirical model is said to be *no-signalling* if this family of distributions satisfies

$$e_C|_{C \cap C'} = e_{C'}|_{C \cap C'}$$

for all C and C' in $\downarrow \mathcal{C}$.

As with operational states, we will only consider empirical models which satisfy no-signalling,

Remark 5.2.5. In [6] the generalised no-signalling condition is shown to be equivalent to a condition the authors call *free choice of measurements*.

Recall, in Chapter 4, the naturality condition in the definition of operational states was related to the notion of no-signalling. This formulation of no-signalling is taken from the Abramsky-Brandenburger framework. To see this, we will repackage the definition of empirical model as follows.

Definition 5.2.6. Let \mathbf{S} be the subpresheaf of the terminal presheaf

$$\begin{array}{ccc}
 & \mathbf{1} & \\
 \mathcal{P}(\mathcal{M})^{\text{op}} & \xrightarrow{\quad} & \mathbf{Set} \\
 & \uparrow & \\
 & \mathbf{S} &
 \end{array}$$

defined by

$$\begin{array}{ccc}
 \mathcal{P}(\mathcal{M})^{\text{op}} & \xrightarrow{\quad \mathbf{S} \quad} & \mathbf{Set} \\
 U \vdash & \longrightarrow & \begin{cases} \{*\} & \text{if } U \in \downarrow \mathcal{C} \\ \emptyset & \text{otherwise} \end{cases}
 \end{array} \tag{5.1}$$

We now show that the empirical models correspond with the natural transformations $e : \mathbf{S} \rightarrow \mathcal{D}_R\mathcal{E}$.

Theorem 5.2.7. *There is a one-to-one correspondence between no-signalling empirical models and the natural transformations*

$$\begin{array}{ccc}
 & \mathcal{D}_R\mathcal{E} & \\
 \mathcal{P}(\mathcal{M})^{\text{op}} & \xrightarrow{\quad} & \mathbf{Set} \\
 & \uparrow e & \\
 & \mathbf{S} &
 \end{array}$$

Proof. Consider a family $e_U \in \mathcal{D}_R\mathcal{E}(U)$ that satisfies the no-signalling property. Now we need to show that for $V \subseteq U$ we have $e_{U|_V} = e_V$. Suppose $V \subseteq U$ then we have

$$\begin{aligned}
 e_{U|_V} &= e_{U|_{U \cap V}} && \text{since } U \cap V = V \\
 &= e_{V|_{U \cap V}} && \text{by no-signalling} \\
 &= e_{V|_V} \\
 &= e_V
 \end{aligned}$$

and hence the family form a natural transformation.

Conversely, suppose there is a family $e_U \in \mathcal{D}_R\mathcal{E}(U)$ which is natural, then we need to show that for arbitrary U and V we have $e_U|_{U \cap V}$. Since $U \cap V \hookrightarrow U$, then by naturality we have $e_U|_{U \cap V} = e_{U \cap V}$, and also since $U \cap V \hookrightarrow V$ then by naturality we have $e_V|_{U \cap V} = e_{U \cap V}$, and therefore

$$\begin{aligned} e_U|_{U \cap V} &= e_{U \cap V} \\ &= e_V|_{U \cap V} \end{aligned}$$

as required. □

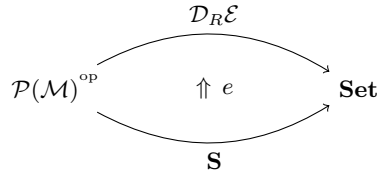
In the form shown in Theorem 5.2.7 it is easy to see how our definition of *operational state* – Definition 4.1.9 – is directly inspired by Abramsky and Brandenburger’s definition of empirical model.

Local Hidden-Variable Models

Abramsky and Brandenburger [5, §3] give a general definition of a *local hidden-variable model* for an empirical model, which subsumes the notion of locality formulated by Bell [19].

Definition 5.2.8. Let $(\mathcal{M}, \mathcal{C}, \mathcal{O})$ be an abstract measurement scenario and let e be an R -valued empirical model for some semiring R . The empirical model e is said to admit a *local hidden-variable model* if there exists an event $e_{\mathcal{M}} \in \mathcal{D}_R\mathcal{E}(\mathcal{M})$ such that $e_{\mathcal{M}}|_U = e_U$ for all $U \in \downarrow \mathcal{C}$.

Equivalently, an empirical model



admits a local hidden-variable model if there exists a global section

$$\begin{array}{ccc}
 & \mathcal{D}_R\mathcal{E} & \\
 \mathcal{P}(\mathcal{M})^{\text{op}} & \xrightarrow{\quad} & \mathbf{Set} \\
 & \uparrow \bar{e} & \\
 & \mathbf{1} &
 \end{array}$$

such that the diagram

$$\begin{array}{ccc}
 \mathbf{S} & \xrightarrow{e} & \mathcal{D}_R\mathcal{E} \\
 \downarrow & \nearrow \bar{e} & \\
 \mathbf{1} & &
 \end{array}$$

commutes.

It is not immediately clear what Definition 5.2.8 has to do with locality or local hidden-variable models in the traditional sense of Bell [19]. This is discussed at length in [5, §8], but we give a condensed version of this explanation here.

Definition 5.2.9. Let e^i be a finite family of R -valued empirical models and let $r_i \in R$ be a collection of elements such that $\sum_i r_i = 1_R$. Define the R -convex sum of empirical model $e = \sum_i r_i \cdot e^i$ as follows: for each $U \in \downarrow \mathcal{C}$, and for each event $\alpha \in \mathcal{E}(U)$, each empirical model e^i assigns a value $e^i_U(\alpha) \in R$ and. Let $e_U(\alpha)$ be the value $e_U(\alpha) = \sum_i r_i \cdot e^i_U(\alpha)$.

Recall, the convex structure that the operational R -state presheaf Δ_R inherits, which we described in Chapter 4. By the same argument the presheaf $\mathcal{D}_R\mathcal{E}$ inherits an R -convex structure which we describe in the following definition.

Definition 5.2.10. Let e be an R -valued empirical model that admits a local hidden-variable model. Suppose e admits a factorisation

$$e = \sum_{\lambda} r_{\lambda} \cdot e^{\lambda}$$

such that for each $U \in \mathcal{C}$ the distribution $e^{\lambda}_U \in \mathcal{D}_R\mathcal{E}(U)$ is a point-distribution. We call such a factorisation a *hidden-variable factorisation* of e , and we call each

λ a *hidden-variable*.

The following theorem is essentially the same as a result of Abramsky and Brandenburger [5, Theorem 8.1], which justifies the definition of local hidden-variable models as global extensions of empirical models.

Theorem 5.2.11. *An R -valued empirical model*

$$\begin{array}{ccc}
 & \mathcal{D}_R\mathcal{E} & \\
 & \curvearrowright & \\
 \mathcal{P}(\mathcal{M})^{\text{op}} & \uparrow e & \mathbf{Set} \\
 & \curvearrowleft & \\
 & \mathbb{S} &
 \end{array}$$

admits a local hidden-variable model if and only if it admits a local hidden-variable factorisation $e = \sum_{\lambda} r_{\lambda} \cdot e^{\lambda}$ where for each λ and for each $U \in \downarrow \mathcal{C}$, the R -distribution $e_U^{\lambda} \in \mathcal{D}_R\mathcal{E}(U)$ is a point-distribution.

Proof. Suppose e admits a local hidden-variable model \bar{e} . Let $\alpha_i \in \mathcal{E}(\mathcal{M})$ be those events such that $\bar{e}_{\mathcal{M}}(\alpha_i) \neq 0_R$. For each such α_i , let $\bar{e}^i : \mathbf{1} \rightarrow \mathcal{D}_R\mathcal{E}$ be the global section uniquely determined by the component $\bar{e}_{\mathcal{M}}^i(\alpha_i) = 1_R$. Letting $r_i = \bar{e}_{\mathcal{M}}(\alpha_i)$, we have $\bar{e} = \sum_i r_i \cdot \bar{e}^i$. Since $\bar{e}_{\mathcal{M}}^i$ is a point-distribution, so is each $\bar{e}_{\mathcal{M}}^i|_U$, for every $U \in \mathcal{P}(\mathcal{M})$, that is, the R -distribution $\bar{e}_U^i \in \mathcal{D}_R\mathcal{E}(U)$ is the point-distribution on the event $\alpha_i|_U : U \rightarrow \mathbb{S}$. Now let $e^i : \mathbb{S} \rightarrow \mathcal{D}_R\mathcal{E}$ be the natural transformation defined $e_U^i = \bar{e}_U^i$. Then clearly $e = \sum_i r_i \cdot e^i$, as required.

Conversely, suppose the empirical model e admits a factorisation $e = \sum_i r_i \cdot e^i$ where for each $U \in \downarrow \mathcal{C}$ the component e_U^i is a point-distribution. We will show that $e^i : \mathbb{S} \rightarrow \mathcal{D}_R\mathcal{E}$ admits a global extension $\bar{e}^i : \mathbf{1} \rightarrow \mathcal{D}_R\mathcal{E}$. It is enough to define the component $\bar{e}_{\mathcal{M}}^i$, and to then show that $\bar{e}_{\mathcal{M}}^i|_U = e_U^i$. Note that for each $A \in \mathcal{M}$, the singleton $\{A\}$ belongs to $\downarrow \mathcal{C}$. We define the following event

$$\begin{array}{ccc}
 \mathcal{M} & \xrightarrow{\alpha^i} & R \\
 A & \longmapsto & e_{\{A\}}^i(A)
 \end{array}$$

and take $\bar{e}_{\mathcal{M}}^i \mathcal{D}_R \mathcal{E}(\mathcal{M})$ to be the point-distribution where $\bar{e}_{\mathcal{M}}^i(\alpha^i) = 1$. It remains to show that $\bar{e}_{\mathcal{M}}^i|_U = e_U^i$ for all $U \in \downarrow \mathcal{C}$. Let $\beta \in \mathcal{D}_R \mathcal{E}(U)$ be the event such that $e^i(\beta) = 1$, we need to show that $\beta = \alpha_i|_U$. Note that

$$\begin{array}{ccc} U & \xrightarrow{\alpha^i|_U} & R \\ A & \longmapsto & e_{\{A\}}^i(A) \end{array}$$

and by naturality of e^i we have $e_{\{A\}}^i(A) = e_U^i(A)$ for all A , and hence $\beta = \alpha_i|_U$, and hence each e^i admits a global extension, and therefore, by Lemma 5.3.16, so does e . \square

Remark 5.2.12. It is natural to ask in exactly what sense this definition of local hidden-variable model is *local*. In Bell's theorem, locality is understood to be a condition on a system consisting of two spacelike separated measurement sites, represented by measurements on different factors of a tensor product of Hilbert spaces, but the definition of local hidden-variables, or hidden-variable factorisation makes no reference to tensor products or spacelike separation. Abramsky and Brandenburger discuss how if one specialises this condition to "Bell-type" scenarios then this notion reduces to Bell's notion of locality [5, §3].

In Chapter 6 we can make this distinction precise by introducing the notions of *local parameter independence* and *global parameter independence*, which are properties of ontological models, structures which generalise the concept of hidden-variable models. In particular, the traditional notion of locality amounts to local parameter independence, a no-signalling condition between two measurement sites, whereas global parameter independence amounts to the generalised notion of no-signalling for hidden-variables, which is precisely the statement that for each hidden-variable λ the family of probability distributions e_U^λ is natural in U . Abramsky and Brandenburger address the relationship between these two conditions: global parameter independence subsumes local parameter independence, and Abramsky and Brandenburger discuss an argument for the converse statement [5, §9], based on a result by Tsirelson which we discuss in greater

depth in Chapter 6. Of course, one might argue from physical principles that the argument of Tsirelson does not hold, and that global parameter independence is strictly stronger than local parameter independence, however, in Chapter 6. However, with Theorem 6.2.4 we show that under the additional assumption of preparation independence, we can show that every local hidden-variable model satisfying local parameter independence must also satisfy global parameter independence. Of course one can argue that that preparation independence is too strong an assumption, and the Bell's theorem can be proven with weaker assumptions [111]. Such considerations might be interesting if one has a formalised notion of spacetime in the theory, which might make such distinctions more meaningful. In Chapter 8 we discuss how one might incorporate such notions of spacetime and composite systems into our mathematical framework. Until then discussions about the metaphysics of hidden-variables feels like paranormal debunkers arguing over the precise nature of ghosts.

Remark 5.2.13. Although we have a lot more to say about the metaphysics of quantum mechanics – see Chapter 9 – we have little more to say on the metaphysics of “hidden-variables”, however one wants to interpret that term. Our final remark on the subject we will be to point out that the formulation of local hidden-variable model we adopt seems to be highly reminiscent of that considered by von Neumann in his widely discredited no-go theorem for hidden-variable models [174, Chap. 4]. Each term e^λ in a hidden-variable factorisation satisfies the property of being a *dispersion free state*, see [109, §2.1.6], or [32, p. 3]. Bub gives a re-appraisal of von Neumann's theorem, which he believes to be misunderstood, summarising it as follows [32, p. 8]:

What von Neumann's proof excludes, then, is the class of hidden variable theories in which (i) dispersion free (deterministic) states are the extremal states, and (ii) the beables of the hidden variable theory correspond to the physical quantities represented by the Hermitian operators of quantum mechanics.

In light of Remark 5.2.12 then it might be worth revisiting von Neumann's ‘no-

hidden variables' proof, particularly his underlying metaphysical assumptions, however, we do not pursue this further here.

5.3 Empirical Models From Operational States

There is a clear similarity between features of this formalism and the definition of operational states from Chapter 4; our definition of operational state is directly inspired by Abramsky and Brandenburger's notion of empirical model. Figure 5.1 outlines the correspondence between features of our spectral presheaf framework and the Abramsky-Brandenburger framework.

Collection of measurements	$\text{Hom}(X, X)$	\mathcal{M}
Measurement context	$\mathfrak{A} \in \mathcal{A}\text{-Alg}_{\text{vN}}(X)$	$U \in \downarrow \mathcal{C}$
Measurement outcomes	$\text{Hom}(I, I)$	\mathcal{O}
Assignments of outcomes to measurements	$\text{Spec}_{\mathbb{G}}(\mathfrak{A})$	$\mathcal{E}(U)$
States	Operational states	Empirical models

Figure 5.1: Correspondence between features of respective formalisms and their interpretations.

We can turn the correspondence shown in Figure 5.1 into precise statements, and show that if a physical system can be represented by the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$, then there is an abstract measurement scenario corresponding to the system.

Definition 5.3.1. Consider the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$. We define the *underlying measurement scenario* for this category as follows: the family of measurements \mathcal{M} be defined

$$\mathcal{M} = \bigcup_{\mathfrak{A} \in \mathcal{A}\text{-Alg}_{\text{vN}}(X)} \mathfrak{A}$$

and the set of measurement contexts \mathcal{C} is taken to be those subsets

$$\mathcal{C} = \{ \mathfrak{A} \subseteq \mathcal{M} \mid \mathfrak{A} \in \mathcal{A}\text{-}\mathbf{Alg}(X) \}$$

and let \mathcal{O} be the scalar semiring $\text{Hom}(I, I) = \mathbb{S}$.

The underlying abstract measurement scenario is the result of forgetting the algebraic structure associated with measurement contexts, outcomes and events. Note that for $\mathcal{A} = \mathbf{Hilb}$ and a fixed Hilbert space H , then \mathcal{M} defined in this way is a proper subset of $\text{Hom}(H, H)$. Clearly any element $A \in \text{Hom}(H, H)$ which is not *normal*, that is, does not satisfy $AA^\dagger = A^\dagger A$ cannot possibly belong to any commutative C^* -subalgebra. Since there are non-normal elements $A, B : H \rightarrow H$ such that $AB : H \rightarrow H$ is not normal we see that \mathcal{M} as a subalgebra of $\text{Hom}(H, H)$ is not closed under multiplication and hence is not a C^* -algebra, and hence we must consider it merely a set.

Remark 5.3.2. Forgetting the algebraic structure on the measurement contexts drastically changes the properties of these two presheaf representations of a system, in particular, while $\text{Spec}_{\mathbb{C}}$ and \mathcal{E} play a similar role in their respective formalisms, there are important differences between them. For example, when $\mathcal{A} = \mathbf{Hilb}$ the Kochen-Specker theorem asserts that $\text{Spec}_{\mathbb{C}}$ admits no global sections, however the presheaf \mathcal{E} always admits global sections. For example, to define a global section

$$\begin{array}{ccc}
 & \mathcal{E} & \\
 & \curvearrowright & \\
 \mathcal{P}(\mathcal{M})^{\text{op}} & \uparrow \zeta & \mathbf{Set} \\
 & \curvearrowleft & \\
 & \mathbf{1} &
 \end{array}$$

it is enough to define the component $\zeta_{\mathcal{M}} \in \mathcal{E}(\mathcal{M})$. For any element $o \in \mathcal{O}$ we

can simply define the event

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\zeta_{\mathcal{M}}} & \mathcal{O} \\ A & \longmapsto & o \end{array}$$

and hence we have a global section of \mathcal{E} .

Recall from Chapter 2 that within the algebraic formulation of classical mechanics, the *state space* of a classical system represented by the commutative algebra \mathfrak{A} is $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$, the collection of algebra homomorphisms $\rho : \mathfrak{A} \rightarrow \mathbb{C}$, assigning an outcome value to each measurement in. The fact that these assignments are not arbitrary functions, but respect the algebraic structure of \mathfrak{A} is extremely important, and is central to the formulation of classical mechanics as commutative algebra of Nestruev [172]. If we abandon the algebraic constraints and accept arbitrary functions assigning outcomes to measurements we lose the meaningful physical interpretation of Nestruev.

In the general Abramsky-Brandenburger framework there is no assumed restriction on abstract events, that is one can consider arbitrary assignments of measurements to outcomes, and so we make the following distinction.

Definition 5.3.3. Suppose a physical system is represented by the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$, and consider the underlying abstract measurement scenario $(\mathcal{M}, \mathcal{C}, \mathcal{O})$, in the sense of Definition 5.3.1. For $\mathfrak{A} \in \mathcal{C}$ we call an event $\alpha \in \mathcal{E}(\mathfrak{A})$ a *physical event* if it belongs to $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$.

The physical events are precisely those which have a meaningful interpretation as states in the sense of Nestruev, which we discussed in Chapter 2.

We will now see how given an operational R -state one can construct an R -valued empirical model in the abstract measurement scenario.

There is an inclusion map of posets

$$\mathcal{A}\text{-Alg}(X) \xleftarrow{u} \mathcal{P}(\mathcal{M})$$

which gives a map on the corresponding presheaf categories

$$\begin{array}{ccc} \mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}} & \xrightarrow{u^*} & \mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} \\ F \dashv & \longrightarrow & F \circ u \end{array}$$

Below, we will show that u^* admits a right adjoint

$$\begin{array}{ccc} & u^* & \\ & \curvearrowright & \\ \mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} & \perp & \mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}} \\ & \curvearrowleft & \\ & u! & \end{array}$$

and we will use this adjunction to lift the operational states from the spectral presheaf framework to empirical models of the Abramsky-Brandenburger formalism, and hence we will be able to apply the techniques Abramsky and Brandenburger to our notion of operational state.

For each $\mathfrak{A} \in \mathcal{C}$ we have the subset $\text{Spec}_{\mathbb{G}}(\mathfrak{A}) \subseteq \mathcal{E}(\mathfrak{A})$ of physical events. This collection of subsets extends to a monomorphic natural transformation

$$\begin{array}{ccc} & u^*(\mathcal{E}) & \\ & \curvearrowright & \\ \mathcal{A}\text{-Alg}(X)^{\text{op}} & \uparrow & \mathbf{Set} \\ & \curvearrowleft & \\ & \text{Spec}_{\mathbb{G}} & \end{array}$$

Note that we have

$$\begin{aligned} \mathcal{D}_R \circ (u^* \mathcal{E}) &= \mathcal{D}_R \circ \mathcal{E} \circ u \\ &= u^*(\mathcal{D}_R \mathcal{E}) \end{aligned}$$

and hence we also have a monomorphism

$$\begin{array}{ccc}
 & u^*(\mathcal{D}_R\mathcal{E}) & \\
 \mathcal{A}\text{-Alg}(X)^{\text{op}} & \begin{array}{c} \curvearrowright \\ \uparrow m \\ \curvearrowleft \end{array} & \mathbf{Set} \\
 & \Delta_R &
 \end{array}$$

An operational R -state is by definition a global section $\psi : \mathbf{1} \rightarrow \Delta_R$. By composing with $m : \Delta_R \rightarrow u^*(\mathcal{D}_R\mathcal{E})$ we obtain a global section of $u^*(\mathcal{D}_R\mathcal{E})$, and hence the operational R -states naturally form a subset

$$\text{Hom}(\mathbf{1}, \Delta_R) \subseteq \text{Hom}(\mathbf{1}, u^*(\mathcal{D}_R\mathcal{E}))$$

In particular, in Lemma 5.3.9 we show that for the terminal presheaf $\mathbf{1} \in \mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$, the presheaf $u_!(\mathbf{1}) = \mathbf{S}$, and hence we can show the following.

Theorem 5.3.4. *The operational R -states naturally correspond with a subset of the R -valued empirical models.*

Proof. By the adjunction we have $\text{Hom}(\mathbf{1}, u^*(\mathcal{D}_R\mathcal{E})) \cong \text{Hom}(u_!(\mathbf{1}), \mathcal{D}_R\mathcal{E})$, and by Lemma 5.3.9 we have $u_!(\mathbf{1}) = \mathbf{S}$, and therefore $\text{Hom}(\mathbf{1}, u^*(\mathcal{D}_R\mathcal{E})) \cong \text{Hom}(\mathbf{S}, \mathcal{D}_R\mathcal{E})$. Since the operational R -states form a subset of $\text{Hom}(\mathbf{1}, u^*(\mathcal{D}_R\mathcal{E}))$ they can be identified with a subset of $\text{Hom}(\mathbf{S}, \mathcal{D}_R\mathcal{E})$, as required. \square

Lifting Operational States Concretely

Due to the existence of the adjunction which we will construct in the next section, we will be able to lift an operational R -state, to a corresponding R -valued empirical model. However, in this section we show how to construct an empirical model from an operational state directly, without appealing to the adjunction. We do this because the explicit construction will be needed for our later results, and also because computing the adjunction is quite lengthy and complicated. In the next section we compute the adjunction and show that unpacking this adjunction for the relevant presheaves one obtains the construction described in the following definition.

Definition 5.3.5. We define a map

$$\mathrm{Hom}(\mathbf{1}, \Delta_R) \xrightarrow{\widehat{(\cdot)}} \mathrm{Hom}(\mathbf{S}, \mathcal{D}_R \mathcal{E})$$

where for each operational R -state $\psi : \mathbf{1} \rightarrow \Delta_R$, we define the R -valued empirical model $\widehat{\psi} : \mathbf{S} \rightarrow \mathcal{D}_R \mathcal{E}$ as follows: for $\mathfrak{A} \in \mathcal{C}$, define $\widehat{\psi}_{\mathfrak{A}} \in \mathcal{D}_R \mathcal{E}(\mathfrak{A})$ to be the extension of the R -distribution $\psi_{\mathfrak{A}}$ on $\mathrm{Spec}_G(\mathfrak{A}) \subset \mathcal{E}(\mathfrak{A})$ to all of $\mathcal{E}(\mathfrak{A})$, that is, the R -distribution defined

$$\widehat{\psi}_{\mathfrak{A}}(\alpha) = \begin{cases} \psi_{\mathfrak{A}}(\alpha) & \text{if } \alpha \in \mathrm{Spec}_G(\mathfrak{A}) \\ 0_R & \text{otherwise} \end{cases}$$

It remains to show how $\widehat{\psi}_U$ is defined for $U \in \downarrow \mathcal{C}$. Note that for $U \in \downarrow \mathcal{C}$, all of the elements of U commute, and hence, the double commutant U'' of this set is a commutative semialgebra, that is, $U'' \in \mathcal{C}$, and we define $\widehat{\psi}_U = \widehat{\psi}_{U''}|_U$. We call $\widehat{\psi}$ the R -valued empirical model *generated by* ψ .

We need to check that $\widehat{\psi}_U$ defined in this way is natural in U , that is for $i : V \hookrightarrow U$ we need to show that

$$\begin{array}{ccc} & & \mathcal{D}_R \mathcal{E}(U) \\ & \nearrow \widehat{\psi}_U & \downarrow i^* \\ \{*\} & & \mathcal{D}_R \mathcal{E}(V) \\ & \searrow \widehat{\psi}_V & \end{array}$$

commutes. To see this consider

$$\begin{array}{ccc} \mathcal{E}(V) & \xrightarrow{i^* \widehat{\psi}_U} & R \\ \beta \vdash & \longrightarrow & \sum_{\substack{\rho \in \mathrm{Spec}_G(U'') \\ \rho|_V = \beta}} \psi_{U''}(\rho) \end{array}$$

Since we have the inclusion of semialgebras $V'' \hookrightarrow U''$ naturality of ψ means

that

$$\psi_{V''}(\gamma) = \sum_{\substack{\rho \in \text{Spec}_G(U'') \\ \rho|_{V''} = \gamma}} \psi_{U''}(\rho)$$

and hence

$$\begin{aligned} \widehat{\psi}_V(\beta) &= \sum_{\substack{\gamma \in \text{Spec}_G(V'') \\ \gamma|_V = \beta}} \psi_{V''}(\gamma) \\ &= \sum_{\substack{\gamma \in \text{Spec}_G(V'') \\ \gamma|_V = \beta}} \left(\sum_{\substack{\rho \in \text{Spec}_G(U'') \\ \rho|_{V''} = \gamma}} \psi_{U''}(\rho) \right) \\ &= \sum_{\substack{\rho \in \text{Spec}_G(U'') \\ \rho|_V = \beta}} \psi_{U''}(\rho) \\ &= i^* \widehat{\psi}_U(\beta) \end{aligned}$$

and hence the map $\widehat{(\cdot)}$ of Definition 5.3.5 is well-defined.

The next result is analogous to Theorem 4.3.8, where we showed that Δ_R is an R -convex object in $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$.

Theorem 5.3.6. *The presheaf*

$$\mathcal{P}(\mathcal{M})^{\text{op}} \xrightarrow{\mathcal{D}_R \mathcal{E}} \mathbf{Set}$$

is an R -convex object (in the sense of Definition 4.3.6) in the category $\mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}}$.

Proof. By Theorem 4.3.7 the R -convex objects in $\mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}}$ are the Eilenberg-Moore algebras of the functor $\widetilde{\mathcal{D}}_R : \mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}} \rightarrow \mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}}$, which is defined on objects

$$\widetilde{\mathcal{D}}_R(F) = \mathcal{D}_R F.$$

Hence the presheaf $\mathcal{D}_R \mathcal{E}$ is equal to $\widetilde{\mathcal{D}}_R(\mathcal{E})$ and is therefore an Eilenberg-Moore algebra of $\widetilde{\mathcal{D}}_R$, as required. \square

An immediate consequence of Theorem 5.3.6 is that the set of empirical models inherits an R -convex structure in much the same way as operational R -states do.

The following result shows that the $\widehat{(\cdot)}$ -construction respects the convex structure on operational states and empirical models, that is $\widehat{(\cdot)}$ is not just a map of sets, but is an affine map of R -convex spaces.

Lemma 5.3.7. *The $\widehat{(\cdot)}$ -construction preserves convexity, that is, for an operational R -state of the form $\psi = \sum_i r_i \cdot \psi^i$, we have $\widehat{\psi} = \sum_i r_i \cdot \widehat{\psi}^i$.*

Proof. We prove the result for binary products, that is, suppose $\widehat{\theta} = r \cdot \widehat{\psi} + s \cdot \widehat{\phi}$, where $r + s = 1_R$. For each $U \in \downarrow \mathcal{C}$ and $\alpha \in \mathcal{E}(U)$ we have

$$\begin{aligned}
\widehat{\theta}_U(\alpha) &= \sum_{\substack{\rho \in \text{Spec}_{\mathbf{G}}(U'') \\ \rho|_V = \beta}} \theta_{U''}(\rho) \\
&= \sum_{\substack{\rho \in \text{Spec}_{\mathbf{G}}(U'') \\ \rho|_V = \beta}} (r \cdot \psi_{U''}(\rho) + s \cdot \phi_{U''}(\rho)) \\
&= r \cdot \left(\sum_{\substack{\rho \in \text{Spec}_{\mathbf{G}}(U'') \\ \rho|_V = \beta}} \psi_{U''}(\rho) \right) + s \cdot \left(\sum_{\substack{\rho \in \text{Spec}_{\mathbf{G}}(U'') \\ \rho|_V = \beta}} \phi_{U''}(\rho) \right) \\
&= \widehat{\psi}_U(\alpha) + \widehat{\phi}_U(\alpha)
\end{aligned}$$

and hence $\widehat{\theta}$ is the convex sum of $\widehat{\psi}$ and $\widehat{\phi}$, as required. \square

Lifting Operational States Abstractly

We have already seen how the right adjoint is defined, it is the functor induced directly from the inclusion of posets

$$\mathcal{A}\text{-Alg}(X) \xleftarrow{u} \mathcal{P}(\mathcal{M})$$

and is defined

$$\begin{array}{ccc} \mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}} & \xrightarrow{u^*} & \mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} \\ F \dashv & \xrightarrow{\quad} & F \circ u \end{array}$$

We now show that u^* admits a left adjoint $u_!$

$$\begin{array}{ccc} & u_! & \\ & \curvearrowright & \\ \mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} & \perp & \mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}} \\ & \curvearrowleft & \\ & u^* & \end{array}$$

We will now show how the $\widehat{(\cdot)}$ -construction of Definition 5.3.5 follows from unpacking this adjunction.

For all of the results in later sections involving the $\widehat{(\cdot)}$ -construction it is enough to use the construction as described in Definition 5.3.5, that is, none of this section is necessary for the rest of this work and can be safely avoided by those with an aversion to abstract nonsense. We include it only to demonstrate the $\widehat{(\cdot)}$ -construction of Definition 5.3.5 is a canonical construction.

The left adjoint $u_!$ exists for extremely general reasons, but is quite difficult to compute. Here we compute $u_!$ using the construction of Mac Lane and Moerdijk [155, Chap. VII. §2], for which we need to introduce the following machinery.

Let \mathcal{C} be a category. For a pair of functors

$$\mathcal{C}^{\text{op}} \xrightarrow{P} \mathbf{Set} \qquad \mathcal{C} \xrightarrow{A} \mathbf{Set}$$

Define the set $P \otimes_{\mathcal{C}} A$ to be the coequaliser of the pair of arrows

$$\bigsqcup_{C, C' \in \mathcal{C}} P(C) \times \text{Hom}(C', C) \times A(C') \xrightarrow[\tau]{\theta} \bigsqcup_{D \in \mathcal{C}} P(D) \times A(D)$$

where for $p \in P(C)$, $f : C' \rightarrow C$ and $a \in A(C')$ we define

$$\theta(p, f, a) = (P(f)p, a)$$

where $P(f)p \in P(C')$, and

$$\tau(p, f, a) = (p, A(f)a)$$

where $A(f)a \in A(C)$.

For a locally small category \mathcal{D} define the functor

$$\begin{aligned} \mathcal{D} \times \mathcal{D}^{\text{op}} &\xrightarrow{\bullet \mathcal{D} \bullet} \mathbf{Set} \\ (D, D') &\longmapsto \text{Hom}(D', D) \end{aligned}$$

For locally small categories \mathcal{C} and \mathcal{D} with a functor $\phi : \mathcal{C} \rightarrow \mathcal{D}$ define the functor

$$\begin{aligned} \mathcal{C} \times \mathcal{D}^{\text{op}} &\xrightarrow{\phi \mathcal{D} \bullet} \mathbf{Set} \\ (C, D') &\longmapsto \text{Hom}(D', \phi(C)) \end{aligned}$$

and for each object $D \in \mathcal{D}$ we have the functor

$$\begin{aligned} \mathcal{C} &\xrightarrow{\phi \mathcal{D}^D} \mathbf{Set} \\ C &\longmapsto \text{Hom}(D, \phi(C)) \end{aligned} \tag{5.2}$$

The following theorem is shown in [155, Chap. VII. §2. Theorem 2].

Theorem 5.3.8. *For \mathcal{C} and \mathcal{D} locally small categories and a functor $\phi : \mathcal{C} \rightarrow \mathcal{D}$, the corresponding functor*

$$\begin{aligned} \mathbf{Set}^{\mathcal{D}^{\text{op}}} &\xrightarrow{\phi^*} \mathbf{Set}^{\mathcal{C}^{\text{op}}} \\ P &\longmapsto P \circ \phi \end{aligned}$$

admits a left adjoint $\phi_!$ defined

$$\begin{aligned} \mathbf{Set}^{\mathcal{C}^{\text{op}}} &\xrightarrow{\phi_!} \mathbf{Set}^{\mathcal{D}^{\text{op}}} \\ F &\longmapsto F \otimes_{\mathcal{C}} \phi \mathcal{D} \bullet \end{aligned}$$

where $F \otimes_{\mathcal{C}} \phi \mathcal{D}^\bullet$ is the presheaf defined by

$$\begin{array}{ccc} \mathcal{D}^{\text{op}} & \xrightarrow{F \otimes_{\mathcal{C}} \phi \mathcal{D}^\bullet} & \mathbf{Set} \\ D \vdash & \longrightarrow & F \otimes_{\mathcal{C}} \phi \mathcal{D}^D \end{array}$$

We are interested in the case where $\mathcal{C} = \mathcal{A}\text{-Alg}(X)$ and $\mathcal{D} = \mathcal{P}(\mathcal{M})$ with $\phi : \mathcal{C} \rightarrow \mathcal{D}$ the inclusion functor

$$\mathcal{A}\text{-Alg}(X) \hookrightarrow \mathcal{P}(\mathcal{M}) \xleftarrow{u}$$

Theorem 5.3.8 shows the existence of the functor

$$\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}} \xrightarrow{u!} \mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}}$$

left adjoint to u^* .

We now need to show that this adjunction give us the inclusion

$$\text{Hom}(\mathbf{1}, \Delta_R) \hookrightarrow \text{Hom}(\mathbf{S}, \mathcal{D}_R \mathcal{E})$$

and that this inclusion corresponds with the $\widehat{(\cdot)}$ -construction of Definition 5.3.5.

Lemma 5.3.9. *There is natural isomorphism $u_!(\mathbf{1}) \cong \mathbf{S}$ in the category $\mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}}$*

Proof. Since the category $\mathcal{A}\text{-Alg}(X)$ is a poset, the set $\text{Hom}(\mathfrak{B}, \mathfrak{A})$ is a singleton if and only if $\mathfrak{B} < \mathfrak{A}$, and the empty set otherwise, and hence the set

$$\bigsqcup_{\mathfrak{A}, \mathfrak{B} \in \mathcal{A}\text{-Alg}(X)} \mathbf{1}(\mathfrak{A}) \times \text{Hom}(\mathfrak{B}, \mathfrak{A}) \times_u \mathcal{P}(\mathcal{M})^U(\mathfrak{B})$$

is isomorphic to the set

$$\bigsqcup_{\substack{\mathfrak{A}, \mathfrak{B} \in \mathcal{A}\text{-Alg}(X) \\ \mathfrak{B} < \mathfrak{A}}} \mathcal{P}(\mathcal{M})^U(\mathfrak{B})$$

which, by definition of ${}_u\mathcal{P}(\mathcal{M})^U(\mathfrak{B})$ is isomorphic to

$$\bigsqcup_{\substack{\mathfrak{A}, \mathfrak{B} \in \mathcal{A}\text{-Alg}(X) \\ U \subseteq u(\mathfrak{B}) \subseteq u(\mathfrak{A})}} \text{Hom}(U, u(\mathfrak{B}))$$

Note that if there is no \mathfrak{B} such that $U \subseteq \mathfrak{B}$ then this set is empty. The corresponding functions

$$\bigsqcup_{\substack{\mathfrak{A}, \mathfrak{B} \in \mathcal{A}\text{-Alg}(X) \\ U \subseteq u(\mathfrak{B}) \subseteq u(\mathfrak{A})}} \text{Hom}(U, u(\mathfrak{B})) \xrightarrow[\tau]{\theta} \bigsqcup_{\mathfrak{D} \in \mathcal{A}\text{-Alg}(X)} \text{Hom}(U, u(\mathfrak{D}))$$

are defined as follows

$$\theta(U \hookrightarrow u(\mathfrak{B})) = (U \hookrightarrow u(\mathfrak{B})) \in \text{Hom}(U, u(\mathfrak{B}))$$

and

$$\tau(U \hookrightarrow u(\mathfrak{B})) = (U \hookrightarrow u(\mathfrak{A})) \in \text{Hom}(U, u(\mathfrak{A}))$$

Hence taking the coequaliser of θ and τ equivalent to taking a quotient of the set

$$\bigsqcup_{\mathfrak{D} \in \mathcal{A}\text{-Alg}(X)} \text{Hom}(U, u(\mathfrak{D}))$$

under the relation generated by

$$(U \hookrightarrow u(\mathfrak{D})) \sim (U \hookrightarrow u(\mathfrak{E})) \quad \text{iff} \quad \mathfrak{D} < \mathfrak{E}$$

Let \mathfrak{B} be the smallest commutative \mathbb{S}^* -semialgebra containing U (this can be defined as the intersection of all commutative \mathbb{S}^* -semialgebras containing U).

If there exists some \mathfrak{D} such that $U \subseteq u(\mathfrak{D})$ then

$$\bigsqcup_{\mathfrak{D} \in \mathcal{A}\text{-Alg}(X)} \text{Hom}(U, u(\mathfrak{D})) / \sim$$

has precisely one element, as for all \mathfrak{D} such that $U \subseteq u(\mathfrak{D})$ we have

$$(U \hookrightarrow u(\mathfrak{D})) \sim (U \hookrightarrow u(\mathfrak{B}))$$

Hence we have shown

$$u_!(\mathbf{1})(U) = \begin{cases} [(U \hookrightarrow u(U''))]_{\sim} & \text{if } U \subseteq \mathfrak{D} \text{ for some } \mathfrak{D} \in \mathcal{A}\text{-Alg}(X) \\ \emptyset & \text{if } U \not\subseteq \mathfrak{D} \text{ for all } \mathfrak{D} \in \mathcal{A}\text{-Alg}(X) \end{cases}$$

and since $[(U \hookrightarrow u(U''))]_{\sim}$ is a singleton set we see that this is precisely how \mathbf{S} is defined. \square

The following result gives a characterisation of the presheaf $u_!(\Delta_R)$, an object in $\mathbf{Set}^{\mathcal{P}(\mathcal{M})^{\text{op}}}$.

Lemma 5.3.10. *There is a natural isomorphism $u_!(\Delta_R) \cong \mathcal{S}$ where for $U \in \downarrow \mathcal{C}$*

$$\mathcal{S}(U) = \{ \psi_{U''|_U} : \mathcal{E}(U) \rightarrow R \mid \psi_{U''} \in \Delta_R(U'') \text{ for } \psi \text{ an operational state} \}$$

and where $\mathcal{S}(U) = \emptyset$ otherwise.

Proof. For each U , the set

$$\bigsqcup_{\mathfrak{A}, \mathfrak{B} \in \mathcal{A}\text{-Alg}(X)} \Delta_R(\mathfrak{A}) \times \text{Hom}(\mathfrak{B}, \mathfrak{A}) \times_u \mathcal{P}(\mathcal{M})^U(\mathfrak{B})$$

is isomorphic to the set

$$\bigsqcup_{\substack{\mathfrak{A}, \mathfrak{B} \in \mathcal{A}\text{-Alg}(X) \\ U \subseteq u(\mathfrak{B}) \subseteq u(\mathfrak{A})}} \Delta_R(\mathfrak{A}) \times \text{Hom}(\mathfrak{B}, \mathfrak{A}) \times_u \mathcal{P}(\mathcal{M})^U(\mathfrak{B})$$

Under the coequalizer maps we identify pairs $(\psi_{\mathfrak{A}} \in \Delta_R(\mathfrak{A}), U \hookrightarrow u(\mathfrak{A}))$ and $(\psi_{\mathfrak{B}} \in \Delta_R(\mathfrak{B}), U \hookrightarrow u(\mathfrak{B}))$, and hence the pair $(\psi_{\{U\}''} \in \Delta_R(U''), U \hookrightarrow u(U''))$

uniquely determines each equivalence class. Hence we have shown

$$u_! \Delta_R(U) \cong \{ [\psi_{U''} \in \Delta_R(U''), U \hookrightarrow u(U'')] \}$$

which is precisely the same data as

$$\mathcal{S}(U) = \{ \psi_{U''|_U} : \mathcal{E}(U) \rightarrow R \mid \psi_{U''} \in \Delta_R(U'') \text{ for } \psi \text{ an operational state} \}$$

as required. It is straightforward to show that this isomorphism is natural. \square

Lemma 5.3.11. *The presheaf $u_! u^*(\mathcal{D}_R \mathcal{E}) : \mathcal{P}(\mathcal{M})^{\text{op}} \rightarrow \mathbf{Set}$ is naturally isomorphic to the presheaf \mathcal{T} where for $U \in \downarrow \mathcal{C}$*

$$\mathcal{T}(U) = \{ \alpha_{U''|_U} : \mathcal{E}(U) \rightarrow R \mid \text{for } \alpha : \mathbf{1} \rightarrow u^* \mathcal{D}_R \mathcal{E} \}$$

and where $\mathcal{T}(U) = \emptyset$ otherwise.

Proof. Uses the same argument as Lemma 5.3.10. \square

We have not given an explicit description of the unit and counit for the adjunction, as we only require the component of the counit ε for the object $\mathcal{D}_R \mathcal{E}$, that is the map

$$u_! u^*(\mathcal{D}_R \mathcal{E}) \xrightarrow{\varepsilon_{\mathcal{D}_R \mathcal{E}}} \mathcal{D}_R \mathcal{E}$$

which is characterised as follows: by Lemma 5.3.11 we have an isomorphism $u_! u^*(\mathcal{D}_R \mathcal{E}) \cong \mathcal{T}$, and hence $\varepsilon_{\mathcal{D}_R \mathcal{E}}$ is characterised by a natural transformation

$$\mathcal{T} \xrightarrow{\tau} \mathcal{D}_R \mathcal{E}$$

which is much easier to see: if $U \in \downarrow \mathcal{C}$ then we take $\tau_U : \mathcal{T}(U) \rightarrow \mathcal{D}_R \mathcal{E}(U)$ to be the identity map, while if $U \notin \downarrow \mathcal{C}$ then we have the trivial inclusion $\tau_U : \emptyset \rightarrow \mathcal{D}_R \mathcal{E}(U)$.

Note that since $\mathbf{S}(U)$ is only non-empty for $U \in \downarrow \mathcal{C}$ the natural transformations $\gamma : \mathbf{S} \rightarrow \mathcal{T}$ contain precisely the same information as the natural

transformations $\delta : \mathbf{S} \rightarrow \mathcal{D}_R\mathcal{E}$.

Let $\kappa : \mathbf{1} \rightarrow u^*(\mathcal{D}_R\mathcal{E})$ be a global section now consider the image of this morphism under the functor $u_!$:

$$u_!(\mathbf{1}) \xrightarrow{u_!(\kappa)} u_!u^*(\mathcal{D}_R\mathcal{E})$$

This is a natural transformation of the form

$$\mathbf{S} \xrightarrow{u_!(\kappa)} u_!u^*(\mathcal{D}_R\mathcal{E})$$

which, by Lemma 5.3.11, for each $U \in \downarrow \mathcal{C}$ is defined $u_!(\kappa)_U = \kappa_{U''}|_U$. Such a natural transformation is precisely the data of a natural transformation $\widehat{\kappa} : \mathbf{S} \rightarrow \mathcal{D}_R\mathcal{E}$ where for each $U \in \downarrow \mathcal{C}$ we define $\widehat{\kappa}_U = u_!(\kappa)_U$. This coincides with the $(\widehat{\cdot})$ -construction of Definition 5.3.5. Implicit in Definition 5.3.5 was that we only consider those $\kappa : \mathbf{1} \rightarrow u^*(\mathcal{D}_R\mathcal{E})$ of the form

$$\mathbf{1} \xrightarrow{\psi} \Delta_R \rightarrow u^*(\mathcal{D}_R\mathcal{E})$$

for some operational R -state ψ , but the $(\widehat{\cdot})$ -construction realises the whole isomorphism

$$\mathrm{Hom}(\mathbf{1}, u^*(\mathcal{D}_R\mathcal{E})) \xrightarrow{(\widehat{\cdot})} \mathrm{Hom}(u_!(\mathbf{1}), \mathcal{D}_R\mathcal{E})$$

induced by the adjunction.

Hidden-Variable Models for Operational States

Given a physical system represented by the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\mathrm{op}}}$, Section 5.2 showed that we also have a corresponding abstract measurement scenario $(\mathcal{M}, \mathcal{C}, \mathcal{O})$, and that we can lift operational states to empirical models. Therefore we can apply the techniques of the Abramsky-Brandenburger formalism to the operational states of our theory.

Definition 5.3.12. An operational R -state

$$\begin{array}{ccc}
 & \Delta_R & \\
 \mathcal{A}\text{-Alg}(X)^{\text{op}} & \xrightarrow{\quad} & \mathbf{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

is said to admit a *local hidden-variable model* if the corresponding R -valued empirical model

$$\begin{array}{ccc}
 & \mathcal{D}_R \mathcal{E} & \\
 \mathcal{P}(\mathcal{M})^{\text{op}} & \xrightarrow{\quad} & \mathbf{Set} \\
 & \uparrow \hat{\psi} & \\
 & \mathbf{S} &
 \end{array}$$

admits a local hidden-variable model (in the sense of Definition 5.2.8), where $\hat{\psi}$ is defined as in Definition 5.3.5.

Recall the definition of physical event Definition 5.3.3.

Definition 5.3.13. Suppose ψ is an operational R -state that admits a local hidden-variable model. We say that this local hidden-variable model is *physically supported* if for each e^λ in the hidden-variable factorisation $\hat{\psi} = \sum_{\lambda} r_{\lambda} \cdot e^{\lambda}$, the unique event $\alpha \in \mathcal{E}(\alpha)$ such that $e^{\lambda}(\alpha) = 1_R$ is a physical event, that is $\alpha \in \text{Spec}_{\mathbb{G}}(\alpha)$.

The following lemma will show that physically supported R -valued hidden-variable models generalise the usual notion of probabilistic local hidden-variable models.

Lemma 5.3.14. *If an operational \mathbb{R}^+ -state admits an \mathbb{R}^+ -valued hidden-variable model then the hidden-variable factorisation is physically supported.*

Proof. Let ψ be an Abramsky-Brandenburger local state with hidden-variable factorisation $\hat{\psi} = \sum_{\lambda} r_{\lambda} \cdot e^{\lambda}$. Let $\mathfrak{A} \in \mathcal{C}$ and suppose $e_{\mathfrak{A}}^{\lambda}(\alpha) = 1$ for some $\alpha : \mathfrak{A} \rightarrow \mathbb{S}$. Then $\hat{\psi}_{\mathfrak{A}}(\alpha) \neq 0$. By the definition of $\hat{\psi}_{\mathfrak{A}}$ – Definition 5.3.5 – $\hat{\psi}_{\mathfrak{A}}$ is

the extension of the R -distribution $\psi_{\mathfrak{A}}$ on $\text{Spec}_G(\mathfrak{A})$ to all of $\mathcal{E}(\mathfrak{A})$, and hence for any $\alpha \in \mathcal{E}(\mathfrak{A}) \setminus \text{Spec}_G(\mathfrak{A})$ we have $\widehat{\psi}_{\mathfrak{A}}(\alpha) = 0_R$, and therefore if $\widehat{\psi}_{\mathfrak{A}}(\alpha) \neq 0_R$ then $\alpha \in \text{Spec}_G(\mathfrak{A})$, as required. \square

Note that the proof of Lemma 5.3.14 certainly does not apply if we instead considered \mathbb{R} -valued empirical models, as if $e_{\mathfrak{A}}^\lambda(\alpha) \neq 0$ we cannot conclude that $\sum_{\lambda} r_{\lambda} \cdot e_{\mathfrak{A}}^\lambda(\alpha) \neq 0$ as there is the possibility for negative coefficients to cancel this term.

By Lemma 5.3.14 we see that the physically supported R -valued local hidden-variable models generalise the traditional probabilistic notion of local hidden-variable model.

Lemma 5.3.15. *If an operational R -state*

$$\begin{array}{ccc}
 & \Delta_R & \\
 \mathcal{A}\text{-Alg}(X)^{\text{op}} & \xrightarrow{\quad} & \mathbf{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

is globally-deterministic – in the sense of Definition 5.1.4 – then the corresponding empirical model

$$\begin{array}{ccc}
 & \mathcal{D}_R \mathcal{E} & \\
 \mathcal{P}(\mathcal{M})^{\text{op}} & \xrightarrow{\quad} & \mathbf{Set} \\
 & \uparrow \widehat{\psi} & \\
 & \mathbf{S} &
 \end{array}$$

admits a local hidden-variable model.

Proof. We need to define $\overline{\psi} : \mathbf{1} \rightarrow \mathcal{D}_R \mathcal{E}$ such that $\overline{\psi}$ is a global extension of $\widehat{\psi} : \mathbf{S} \rightarrow \mathcal{D}_R \mathcal{E}$, that is, such that for all $U \in \downarrow \mathcal{C}$ we have $\overline{\psi}_U = \widehat{\psi}_U$. To define the global section $\overline{\psi}$ it is enough to define the component $\overline{\psi}_{\mathcal{M}}$.

Suppose ψ is globally-deterministic, then by Lemma 5.1.5 there is a global section of $\chi : \mathbf{1} \rightarrow \text{Spec}_G$ such that for each \mathfrak{A} the R -distribution $\psi_{\mathfrak{A}} \in \Delta_R$ is the point-distribution on the element $\chi_{\mathfrak{A}}$. Let $\alpha \in \mathcal{E}(\mathcal{M})$ be the event defined:

$$\begin{array}{ccc} \mathcal{M} & \xrightarrow{\alpha} & \mathbb{S} \\ A \vdash & \longrightarrow & \chi_{\{A\}''}(A) \end{array}$$

and define $\bar{\psi}_{\mathcal{M}}$ to be the point-distribution $1_R \cdot \underline{\alpha}$, that is, $\bar{\psi}_{\mathcal{M}}(\alpha) = 1_R$, and $\bar{\psi}_{\mathcal{M}}(\beta) = 0_R$ for all $\beta \neq \alpha$. By specifying $\bar{\psi}_{\mathcal{M}}$ we have defined a global section, as we can define $\bar{\psi}_U = \bar{\psi}_{\mathcal{M}}|_U$ for all $U \in \mathcal{P}(\mathcal{M})$. It remains to show that this global section is an extension of $\hat{\psi}$; we need to check that $\hat{\psi}_U = \bar{\psi}_U$ for all $U \in \downarrow \mathcal{C}$.

The R -distribution $\bar{\psi}_U$ is the point-distribution on the event $\alpha|_U$. The R -distribution $\hat{\psi}_U$ is the point-distribution on the event:

$$\begin{array}{ccc} U & \xrightarrow{\beta} & \mathbb{S} \\ A \vdash & \longrightarrow & \chi_{U''}(A) \end{array}$$

Now since we have an inclusion of commutative von Neumann semialgebras $\{A\}'' \hookrightarrow U''$, by naturality of χ we have $\chi_{\{A\}''}(A) = \chi_{U''}(A)$ for all $A \in U$, and hence $\alpha|_U = \beta$, and therefore we have $\hat{\psi}_U = \bar{\psi}_U$, as required. \square

Lemma 5.3.16. *Let $e : \mathbf{S} \rightarrow \mathcal{D}_R \mathcal{E}$ be an R -valued empirical model of the form $e = \sum_i r_i \cdot e^i$. If each e^i admits a local hidden-variable model then so does e .*

Proof. Let $e : \mathbf{S} \rightarrow \mathcal{D}_R \mathcal{E}$ be an R -valued empirical model that is a convex sum $e = \sum_i r_i \cdot e^i$ of R -valued empirical models e^i which admit R -valued local hidden-variable models \bar{e}^i . In order to define a local hidden-variable model \bar{e} , it is enough to characterise the component $\bar{e}_{\mathcal{M}}$, which we define $\bar{e}_{\mathcal{M}} = \sum_i r_i \cdot \bar{e}_{\mathcal{M}}^i$. We need to check that for $U \in \downarrow \mathcal{C}$ that $\bar{e}_{\mathcal{M}}|_U = e_U$. Since for each i we have

$\bar{e}_{\mathcal{M}}^i|_U = e_U^i$, for each $U \in \downarrow \mathcal{C}$ we have

$$\begin{aligned} \bar{e}_U &= \bar{e}_{\mathcal{M}}|_U \\ &= \sum_i r_i \cdot \bar{e}_{\mathcal{M}}^i|_U \\ &= \sum_i r_i \cdot e_U^i \\ &= e_U \end{aligned}$$

as required. \square

The following result shows that R -valued hidden-variable models can be completely characterised in terms of globally-deterministic operational R -states.

Theorem 5.3.17. *An operational R -state $\psi : \mathbf{1} \rightarrow \Delta_R$ admits a physically supported local hidden-variable model if and only if ψ is an R -convex sum of globally-deterministic states.*

Proof. Suppose $\psi = \sum_{\lambda} r_{\lambda} \cdot \psi^{\lambda}$ such that each ψ^{λ} is globally-deterministic. Then by Lemma 5.3.15 each $\widehat{\psi}^{\lambda}$ admits a local hidden-variable model. By Lemma 5.3.7 we have $\widehat{\psi} = \sum_{\lambda} r_{\lambda} \cdot \widehat{\psi}^{\lambda}$, and since each $\widehat{\psi}^{\lambda}$ admits a local hidden-variable model, then by Lemma 5.3.16 so does $\widehat{\psi}$, and hence ψ admits a local hidden-variable model.

Conversely, suppose that $\widehat{\psi}$ admits a local hidden-variable model, and let $\widehat{\psi} = \sum_{\lambda} r_{\lambda} \cdot e^{\lambda}$ be the hidden-variable factorisation. Consider those $\mathfrak{A} \in \mathcal{C}$. If the hidden-variable model is physically supported then for each e^{λ} the support of the R -distribution $e_{\mathfrak{A}}^{\lambda}$ is in $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$, and hence we can define the unique R -distribution ψ^{λ} on $\text{Spec}_{\mathbb{G}}(\mathfrak{A})$ which extends to e^{λ} on $\mathcal{E}(\mathfrak{A})$ – that is, such that $\widehat{\psi}^{\lambda} = e^{\lambda}$. Naturality of ψ^{λ} follows from naturality of e^{λ} and hence we have an operational R -state $\psi^{\lambda} : \mathbf{1} \rightarrow \Delta_R$, and moreover $\psi = \sum_{\lambda} r_{\lambda} \cdot \psi^{\lambda}$. Note that since each $e_{\mathfrak{A}}^{\lambda}$ is a point-distribution, the operational state ψ^{λ} is globally deterministic, as required. \square

Theorem 5.3.17 allows us to relate physically supported local hidden-variable models and Kochen-Specker contextuality on the most general level.

Theorem 5.3.18. *For a system represented by the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}(X)^{\text{op}}}$, and R a commutative semiring, the following statements are equivalent:*

1. *The system is Kochen-Specker contextual.*
2. *There are no operational R -states that admit physically supported local hidden-variable models.*

Proof. Suppose the system is not Kochen-Specker contextual – that is, suppose $\text{Spec}_{\mathbb{G}}$ admits a global section. Then by Lemma 5.1.5 the system admits a globally-deterministic operational R -state for any choice of R , and therefore, by Lemma 5.3.15, the system has an operational R -state that admits a local hidden-variable model.

Conversely suppose the system has an operational R -state that admits a physically supported local hidden-variable model. By Theorem 5.3.17 this state must be an R -convex sum of globally-deterministic states, which implies that the system admits globally-deterministic states. By Lemma 5.1.5 this implies that $\text{Spec}_{\mathbb{G}}$ has global sections, as required. \square

We have good reasons to present contextuality and non-locality on a more general level than Hilbert spaces and quantum theory, for example, in Chapter 7 we show that Spekkens' Toy Theory can be modelled using this framework, where one takes the underlying category $\mathcal{A} = \mathbf{Rel}$, and the operational states of the model correspond with the *epistemic states* of the toy theory. The globally-deterministic states in this model correspond precisely with the *ontic states* of the theory.

Bell's Theorem and Negative Probabilities

We now consider the implications of Theorem 5.3.18 for the special case of Hilbert space models of quantum theory. In particular, we will show that Theorem

5.3.18 allows us to prove a version of Bell’s theorem [19]. This version of Bell’s theorem, which we state as Bell does [20, p. 65], is arguably weaker in one sense, and arguably stronger in another sense, and it also follows a very different conceptual argument.

Theorem 5.3.19. *No physical theory of local hidden-variables can reproduce all of the predictions of quantum mechanics.*

Proof. We prove this theorem assuming only the operational axioms we outlined in Figure 4.2. The proof can be broken down into three steps:

1. By “the predictions of quantum mechanics” we mean the probabilities associated with measurement outcomes across all contexts, which by assumption satisfy no-signalling, that is, form operational states, in the sense of Definition 4.1.9;
2. Since local hidden-variable models for \mathbb{R}^+ -valued empirical models are always physically supported, by Theorem 5.3.18, if such an operational state was to admit a local hidden-variable model the spectral presheaf $\text{Spec}_{\mathbb{G}}$ would admit global sections;
3. By the Kochen-Specker theorem no such global sections exist.

and hence no local hidden-variable models exist for any quantum systems with dimension greater than 2. \square

Recovering Bell’s theorem in this way gives us a proof that is of a fundamentally different character to the usual proofs for the non-locality of quantum theory, for example: those of Mermin [162]; Greenberger, Horne and Zeilinger [96]; Hardy [101]; or Abramsky and Brandenburger [5, Proposition 4.2], which all follow the same pattern of “proof by counterexample”. That is, they explicitly construct a state and a collection of measurements that cannot be consistent with a local hidden-variable model. Mermin “unifies” Bell’s theorem and the Kochen-Specker theorem by giving a counter example which simultaneously

demonstrated both results [162], however this does not give a logical relationship between Bell's theorem and the Kochen-Specker theorem; the same object appearing in two proofs says nothing about the relationship between the statements.

Our proof is not by counterexample, but is direct, and moreover, we see that the Kochen-Specker theorem *implies* Theorem 5.3.19. This illustrates a formal, mathematical connection between the phenomena of contextuality and non-locality, and on a level strictly more general than Hilbert space-based quantum theory. We will see the close interplay of the concepts of Kochen-Specker contextuality and local hidden-variable models when we model Spekkens' Toy Theory in Chapter 7.

Theorem 5.3.19 is in a sense stronger than Bell's theorem: note that in the proof of Theorem 5.3.19 we do not assume that the probabilities associated with measurement outcomes are computed via a density operator and the Born rule, in fact, we do not mention probabilities explicitly at all, hence, we can extend the no-go theorem to modifications of quantum theory where the outcomes are not probabilistic, but are represented by some more general R -distribution.

For any semiring R such that there is an inclusion of semirings $\mathbb{R}^+ \hookrightarrow R$, the \mathbb{R}^+ -valued empirical models form a subset of the R -valued empirical models. Although quantum states naturally correspond with a class of \mathbb{R}^+ -valued empirical models, they can be considered as R -valued empirical models for any such R . While Bell's theorem rules out the possibility of a local hidden-variable model for $\hat{\psi}$, we can ask the following question: if we view $\hat{\psi}$ as an R -valued empirical model, does it admit a local hidden-variable model? With the following theorem we show that if such a local hidden-variable model were to exist it cannot be physically supported, in the sense of Definition 5.3.13.

Theorem 5.3.20. *Let R be a commutative semiring with $\mathbb{R}^+ \hookrightarrow R$. No theory of physically supported R -valued local hidden-variable models can reproduce all of the predictions of quantum mechanics.*

Proof. We can use the same argument as in the proof of Theorem 5.3.19. The

only difference is that we must assume the local hidden-variable models are physically supported. \square

If we take $R = \mathbb{R}$ then Theorem 5.3.20 gives us some insight into the nature of hidden-variable models with “negative probabilities”, considered at length by Abramsky and Brandenburger [5, §5], and also by Al-Safi and Short [11], and even by Feynman [81] and Dirac [59, p 8].

Dirac seems to suggest that we take such a possibility seriously [59, p 8]:

Negative energies and probabilities should not be considered as nonsense... [they] should be considered simply as things which do not appear in experimental results.

Moreover, Abramsky and Brandenburger give credibility to the idea with the following result [5, Theorem 5.9].

Theorem 5.3.21. *Every \mathbb{R}^+ -valued empirical model, when viewed as an \mathbb{R} -valued empirical model, admits a local hidden-variable model.*

This result of Abramsky and Brandenburger seems to suggest that quantum theory could be characterised by a theory of local hidden-variables with possibly negative probabilities, however, Theorem 5.3.20 demonstrates that such local hidden-variable models cannot be physically supported, which has some significant implications for the conceptual nature of such models.

Suppose a quantum state, when viewed as an \mathbb{R} -valued empirical model, admits a local hidden-variable model with hidden-variable factorisation $\hat{\psi} = \sum_{\lambda} r_{\lambda} \cdot e^{\lambda}$. One interpretation of negative probabilities is the possibility that there is a physical event $\alpha \in \text{Spec}_{\mathbb{C}}(\mathfrak{A})$ that “does not appear in experimental results”, that is, $\hat{\psi}_{\mathfrak{A}}(\alpha) = 0$, but where $e_{\mathfrak{A}}^{\lambda}(\alpha) = 1$ for some hidden-variable λ . That is, the event occurs within the hidden-variable model, but it is not observed experimentally because the possibly negative probabilities are cancelled in the sum. However, Theorem 5.3.20 demonstrates that this cannot be the whole story, as such a hidden-variable model would be physically supported. For a hidden-variable model to not be physically supported means the following:

for each e^λ there must be a context \mathfrak{A} such that the unique event $\alpha \in \mathcal{E}(\mathfrak{A})$ satisfying $e_{\mathfrak{A}}^\lambda(\alpha) = 1$ is not a physical event, that is, does not belong to the subset $\text{Spec}_G(\mathfrak{A}) \subseteq \mathcal{E}(\mathfrak{A})$. We have a clear interpretation of the elements of $\text{Spec}_G(\mathfrak{A})$ as the states of a classical subsystem represented by the commutative algebra \mathfrak{A} , but it is not clear at all how we ought to interpret these more general events.

In summary, the introduction of negative probabilities cannot resolve the issues of non-locality without introducing new conceptual issues.

Chapter 6

The Colbeck-Renner and PBR Theorems

In the previous chapter we gave a proof of Bell's theorem which asserts that the probabilistic nature of quantum mechanics cannot reflect an underlying local hidden-variable model. In this chapter we examine deeper the question of how we ought to interpret the probabilistic nature of quantum theory. Probabilistic outcomes of measurements arise naturally in classical physics, and are usually interpreted in the following manner described by Isham [123, p. 151]:

The most common meaning attached to probability in classical physics is an epistemic one [in which] probabilities measure the extent to which an observer knows the properties of the system. Our ignorance in this matter arises from the great complexity of the system (for example, a box of gas) but the underlying assumption is that, at least in principle, this uncertainty can be made arbitrarily small with the aid of more precise measurements of the actual properties of the system.

The interpretation of probabilities as reflecting a lack of knowledge of some underlying “true state” is an intuitive one, and was reflected in the view of

Heisenberg [110, pp. 53-54]:

These uncertainties [i.e. probabilities] may be called objective in so far as they are simply a consequence of the description in the terms of classical physics and do not depend on any observer. They may be called subjective in so far as they refer to our incomplete knowledge of the world.

However, two recent results cast serious doubt on the validity of such an interpretation: the *PBR theorem*, by Pusey, Barrett and Rudolph [185]; and the *Colbeck-Renner theorem* by Colbeck and Renner [51, 52, 53].

These results show that this “incomplete knowledge” of an underlying objective state is inconsistent with quantum theory. Pusey, Barrett and Rudolph describe their theorem as follows [185, p. 4]:

[The PBR theorem is] a no-go theorem, which – modulo assumptions – shows that models in which the quantum state is interpreted as mere information about an objective physical state of a system cannot reproduce the predictions of quantum theory.

While Colbeck and Renner summarise their result with a stronger statement [51, p. 1]:

[The Colbeck-Renner theorem shows] there [is no] extension of quantum theory [...] that would convey any additional information about the outcomes of future measurements.

We will reformulate both the PBR theorem and the Colbeck-Renner theorem in the language of spectral presheaves. We do this through incorporating the language of *ontological models* due to Harrigan and Spekkens [104], which we review in Section 6.1. We will then reformulate ontological models in the language of spectral presheaves.

In Section 6.2 we review the Colbeck-Renner theorem as it is presented in the literature and give our reformulation of the theorem. We actually prove

a stronger version of the Colbeck-Renner theorem which applies to systems of arbitrary dimension, not just those with finite dimension. The Colbeck-Renner theorem as we state it more closely resembles the formulations of Landsman [139] and of Leegwater [148]. We will discuss the significance of the differences in the character of our proof from those of Leegwater and Landsman.

Similarly, in Section 6.3 we review the PBR theorem, before reformulating the PBR theorem in our framework. This proof is very similar to the proof of the Colbeck-Renner theorem. In fact, the PBR theorem can be seen as a corollary of the Colbeck-Renner theorem, but we believe it is worth discussing separately because, again, we will discuss the significance of the ways in which our proof differs from that of Pusey, Barrett and Rudolph.

6.1 Ontological Models

Before establishing ontological models in the spectral presheaf framework we will review the original presentation of Harrigan and Spekkens [104, Definition 1].

Recall from Chapter 4 we consider an experimental paradigm where we have a collection of possible preparation procedures – which we will denote Π – and a collection of possible measurements.

For each preparation $\psi \in \Pi$, we obtain the probabilistic spread of outcomes for measurements, that is, for each measurement A we obtain a probability distribution on the outcomes of A , that is we have a collection of probabilities

$$\text{Prob}(A = a_j | \psi)$$

where recall from Chapter 2 this denotes the probability of observing the outcome a_j when measuring A when the system is prepared as ψ .

An ontological model consists of a measurable space Λ of *ontic states*. These ontic states represent the true state of the system, which may be inaccessible to us due to our limited measurement apparatus.

When we perform the preparation procedure $\psi \in \Pi$, the process is actually

picking one of these underlying ontic states, but in a probabilistic way; that is – each preparation $\psi \in \Pi$ actually corresponds with drawing from a probability measure μ_ψ on the measurable space ontic states.

If we were able to prepare the system in the precise ontic state $\lambda \in \Lambda$ we would obtain measurement statistics corresponding with this state, that is, for each measurement A we would obtain probabilities

$$\text{Prob}(A = a_j | \lambda)$$

The observed probabilities derive from the fact that our preparation procedure draws from the probability distribution μ_ψ of ontic states, and the statistics corresponding with the preparation procedure $\psi \in \Pi$ correspond precisely with the integral

$$\text{Prob}(A = a_j | \psi) = \int_{\Lambda} \text{Prob}(A = a_j | \lambda) d\mu_\psi$$

Example 6.1.1. Suppose we have an experimental set-up consisting of a box with two buttons, one labelled ‘prepare’, and the other labelled ‘measure’; and two light bulbs, one green and one red.

There is only one possible preparation procedure which corresponds with pressing the ‘prepare’ button, and one possible measurement which we perform by pressing the ‘measure’ button. Upon performing a measurement on a prepared system we observe exactly one of the bulbs flashing on. After numerous runs of preparation followed by measurement we observe that the red and green lights are illuminated with equal probability. That is, the operational state corresponding with the preparation p can be represented by the probability distribution on the outcomes the measurement M

$$\text{Prob}(M = \textit{green} | p) = \frac{1}{2} \quad \text{and} \quad \text{Prob}(M = \textit{red} | p) = \frac{1}{2}$$

By constructing an ontological model we are hypothesising what might be happening inside the box. We hypothesise that the system inside the box might be capable of existing in other states apart from just p . In particular, we

hypothesise that inside the box there is in fact a tiny man who flips a coin every time we press the ‘prepare’ button. When we press the ‘measure’ button the tiny man illuminates the green bulb if the coin-toss came up heads, and the red bulb if it came up tails.

That is, we are hypothesising a set of ontic states

$$\Lambda = \{ heads, tails \}$$

which the system may *really* be in after we press ‘prepare’. By flipping the coin the tiny man is drawing from the probability distribution which we denote μ_p

$$\frac{1}{2} \cdot heads + \frac{1}{2} \cdot tails$$

that is, pressing the ‘prepare’ corresponds with drawing from the probability distribution μ_p .

If we were able to prepare the system in the ontic state *heads* the tiny man would always illuminate the green bulb when we press ‘measure’, and if we could prepare the system in the ontic state *tails* he would always illuminate the red bulb. These two purely hypothetical preparations correspond with the operational state *heads* characterised by the probability distribution

$$\text{Prob}(M = green | heads) = 1 \quad \text{and} \quad \text{Prob}(M = red | heads) = 0$$

and the operational state *tails*, characterised by the probability distribution

$$\text{Prob}(M = green | tails) = 0 \quad \text{and} \quad \text{Prob}(M = red | tails) = 1$$

In practice we are unable to make such delicate preparations, as the coin is really very tiny.

This ontological model is consistent with the observed outcomes of the

experiment, as we have

$$\begin{aligned}
 \text{Prob}(M = \textit{green} | p) &= \text{Prob}(\textit{heads} | \mu_p) \cdot \text{Prob}(M = \textit{green} | \textit{heads}) \\
 &\quad + \text{Prob}(\textit{tails} | \mu_p) \cdot \text{Prob}(M = \textit{green} | \textit{tails}) \\
 &= \frac{1}{2} \cdot 1 + \frac{1}{2} \cdot 0 \\
 &= \frac{1}{2}
 \end{aligned}$$

Of course we have not proven anything about what is really happening inside the box; the existence of the ontological model shows that the outcomes we see are consistent with the belief that the probabilities reflect a lack of knowledge of an objective underlying state.

The ontological model described in Example 6.1.1 satisfies a property called being ψ -epistemic introduced by Harrián and Spekkens [104]. We will consider ψ -epistemic ontological models in Section 6.3.

Remark 6.1.2. Note that we do not necessarily assume that for each ontic state $\lambda \in \Lambda$ the corresponding distribution

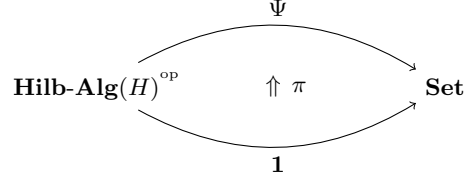
$$\text{Prob}(A = a_j | \lambda)$$

is a point-distribution, as is the case in Example 6.1.1. For quantum theory, such an ontological model would be equivalent to the existence of local hidden-variable models, ruled out by Bell's theorem. Hence, ontological models can be seen as a generalisation of local hidden-variable models.

We now give a slightly different formulation of ontological models, which is similar to that of Mansfield [158], who gives a formulation of ontological models in terms of the sheaf-theoretic empirical models of Abramsky and Brandenburger [5].

In the spectral presheaf formulation of quantum theory, as we outlined in the axioms of Figure 4.2, from each preparation $\pi \in \Pi$ we obtain an operational

state



and recall, by Theorem 4.2.10, each such π corresponds with some state of the C^* -algebra $\text{Hom}(H, H)$.

The Colbeck-Renner and PBR theorems consider ontological models for quantum theory where the set of possible preparations is restricted to just the pure states of a quantum system. That is, we assume that for each preparation $\psi \in \Pi$, there is some $|\psi\rangle \in H$, such that the operational state ψ is of the form $\psi = \tilde{q}$ for $q = |\psi\rangle\langle\psi|$, in the sense of Theorem 4.1.29.

Definition 6.1.3. Consider a quantum mechanical system represented by a Hilbert space H , with a given set of preparations Π , which correspond with pure states. An *ontological model for pure states on H* consists of the following:

- a measurable space Λ of *ontic states* such that for each preparation $\psi \in \Pi$ there exists a probability measure μ_ψ on Λ – that is, there is a function

$$\Pi \xrightarrow{\mu(-)} \mathcal{M}(\Lambda)$$

where $\mathcal{M}(\Lambda)$ is the set of probability measures on Λ ;

- for each ontic state $\lambda \in \Lambda$ there are corresponding operational probabilities, that is, there is a probability measure δ_λ^λ on the measurable space of states $\text{Spec}_G(\mathfrak{A})$ for each classical context \mathfrak{A} , that is for each $\lambda \in \Lambda$ and for each \mathfrak{A} we have $\delta_\lambda^\lambda \in \Psi(\mathfrak{A})$;
- for each \mathfrak{A} the operational probabilities $\psi_\mathfrak{A} \in \Psi(\mathfrak{A})$ corresponding with the preparation $\psi \in \Pi$ are determined by

$$\psi_\mathfrak{A} = \int_\Lambda \delta_\mathfrak{A}^\lambda d\mu_\psi$$

What in [5, §8] is called λ -independence states that the probability μ_ψ on Λ corresponding with preparation $\psi \in \Pi$ does not depend on choice of measurements to be made. This can be easily motivated by the principle that we can make the preparation ψ before deciding which measurement, or indeed which measurement context we will use. The principle of λ -independence is introduced in [58]. Abramsky, Brandenburger and Savochnik observe that dropping the condition of λ -independence allows one to trivially reproduce any empirical behaviour, as one can construct an ontological model for which the ontic states determine which measurements can be made [6]. Abramsky, Brandenburger and Savochnik show that the notion of λ -independence is equivalent to the principle of *free choice of measurement* [6].

The condition of λ -independence is implicitly assumed in the proof of the PBR theorem [185], and is stated explicitly in the proofs of the Colbeck-Renner theorem of both Landsman [139, p. 3 fn. 5] and of Leegwater [148, p. 6 fn. 5].

The condition of λ -independence is implicit in our assumption that there is a function

$$\Pi \xrightarrow{\mu(-)} \mathcal{M}(\Lambda)$$

We will use the notion of *Lebesgue integral*. We only deal with positive functions, that is, those which take values in the positive reals. We are also only considering functions which have a finite integral. Given a measure space (X, Σ, μ) and a function

$$X \xrightarrow{f} \mathbb{R}^+$$

the function is *integrable* if it has a well-defined *Lebesgue integral*, which we denote

$$\int_X f d\mu$$

which is (in our case) a finite real value. We will not show how the Lebesgue integral is defined (see [25, Chap. 2], for example) but we will need the following properties of the Lebesgue integral.

Definition 6.1.4. For (X, Σ, μ) a measure space. A pair of functions f and g

$$X \xrightarrow{f, g} \mathbb{R}$$

are said to be *equal almost everywhere* if there exists a set $A \subseteq X$ with $\mu(A) = 0$ such that for all $x \in X \setminus A$ we have $f(x) = g(x)$.

Clearly two functions that are equal are equal almost everywhere, as one can take $A = \emptyset$.

The following properties of the Lebesgue integral can be found [25, Theorem 2.5.1]

Theorem 6.1.5. *Let (X, Σ, μ) be a measure space. The Lebesgue integral satisfies the following conditions:*

1. For f a bounded integrable function

$$\int_X f \, d\mu \leq \sup_{x \in X} \{ f(x) \cdot \mu(X) \}$$

2. for f and g integrable functions then

$$\int_X (f + g) \, d\mu = \int_X f \, d\mu + \int_X g \, d\mu$$

3. For $X = A \cup B$ for disjoint sets A and B , then for any integrable function f we have

$$\int_X f \, d\mu = \int_A f \, d\mu + \int_B f \, d\mu$$

4. Integrable functions satisfy the Chebyshev inequality (also known as the Markov inequality), that is, for integrable functions f and g such that $f(x) \leq g(x)$ almost everywhere we have

$$\int_X f \, d\mu \leq \int_X g \, d\mu$$

The following lemma follows immediately from the Chebyshev inequality and can be found in [25, Corollary 2.5.4].

Lemma 6.1.6. *For (X, Σ, μ) a measure space and*

$$X \xrightarrow{f} [0, 1]$$

an integrable function such that

$$\int_X f(x) d\mu = 0$$

then $f(x) = 0$ almost everywhere.

The Parameter Independence Condition

Parameter independence is a concept first introduced in the study of hidden-variable theories [31, Chap. 2. §5]. In this section we will see that there are two notions of parameter independence, which we term *local parameter independence* and *global parameter independence*. Landsman and Leegwater use local parameter independence in their formulations of the Colbeck-Renner theorem, however, for our formulation of this result we will require global parameter independence.

There is clear a parallel between global parameter independence vs. local parameter independence, and no signalling vs. generalised no-signalling in the sense proposed by Abramsky and Brandenburger – see Definition 4.1.9. In both cases, the traditional form makes explicit use of the tensor product structure on the Hilbert spaces, but global parameter independence, just like generalised no-signalling has nothing inherently to do with this tensor product structure. In fact, we can view parameter independence as ‘no-signalling at the level of ontic states’.

What we call *global parameter independence* is equivalent to *parameter independence* as formulated by Abramsky and Brandenburger [5, §8], or by Mansfield [158, Definition 3.6].

Definition 6.1.7. An ontological model is said to satisfy *global parameter independence* if for each $\lambda \in \Lambda$ the family of probability distributions

$$\delta_{\mathfrak{A}}^\lambda \in \Psi(\mathfrak{A})$$

are natural in \mathfrak{A} , that is, they form a global section

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \begin{array}{c} \curvearrowright \\ \uparrow \delta^\lambda \\ \curvearrowleft \end{array} & \text{Set} \\
 & \mathbf{1} &
 \end{array}$$

The data of a parameter independent ontological model can therefore be combined to give a natural transformation

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \begin{array}{c} \curvearrowright \\ \uparrow \delta^{(-)} \\ \curvearrowleft \end{array} & \text{Set} \\
 & C_\Lambda &
 \end{array}$$

where C_Λ is the constant functor on the set of ontic states.

Remark 6.1.8. Note that the notion of ontological model satisfying global parameter independence subsumes the notion of local hidden-variable model as defined in Definition 5.2.8; for a local hidden-variable model each $\delta_{\mathfrak{A}}^\lambda$ is assumed to be a point-distribution.

The formulation of parameter independence used by both Landsman and Leegwater is what we will call *local parameter independence*. Local parameter independence is stated as a property of a system represented by a Hilbert space $H \cong H_1 \otimes H_2$ where the system is a composite consisting of two separate systems, H_1 and H_2 . Alice makes measurements on the system represented by the Hilbert space H_1 , and Bob makes measurements on the physical system represented by the Hilbert space H_2 . Hence, Alice’s measurements are represented by operators

of the form

$$H_1 \otimes H_2 \xrightarrow{A \otimes \text{id}_{H_2}} H_1 \otimes H_2$$

and Bob's measurements are represented by operators of the form

$$H_1 \otimes H_2 \xrightarrow{\text{id}_{H_1} \otimes B} H_1 \otimes H_2$$

An ontological model satisfies *local parameter independence* if the outcome Alice's measurement on H_1 is independent of Bob's choice of measurement on H_2 – if Bob chooses to make a measurement at all.

Definition 6.1.9. Let H be a Hilbert space with decomposition $H \cong H_1 \otimes H_2$. An ontological model on $H \cong H_1 \otimes H_2$ satisfies *local parameter independence* if for all measurements X on H_1 and for all measurements Y on H_2 we have

$$\text{Prob}(X \otimes \text{id}_{H_2} = x_j | \lambda) = \sum_y \text{Prob}(X \otimes \text{id}_{H_2} = x_j, \text{id}_{H_1} \otimes Y = y_k | \lambda)$$

where the sum is taken over all possible outcomes of the measurement Y .

We see that local parameter independence is a condition that applies to observables which act on separate factors of a tensor product, while global parameter independence applies to arbitrary families of commutative observables. Since Alice's measurements $X \otimes \text{id}_{H_2}$ will always commute with Bob's $\text{id}_{H_1} \otimes Y$, local parameter independence follows immediately from global parameter independence, as the following result shows.

Theorem 6.1.10. *Global parameter independence implies local parameter independence.*

Proof. Let $H \cong H_1 \otimes H_2$, and consider measurements represented by operators $X : H_1 \rightarrow H_1$ and $Y : H_2 \rightarrow H_2$. Let $\mathfrak{B} = \{X \otimes \text{id}_{H_2}\}''$ and $\mathfrak{A} = \{X \otimes \text{id}_{H_2}, \text{id}_{H_1} \otimes Y\}''$. Note that $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$. Now suppose we have a global section δ of Ψ , then in particular we have probability distributions $\delta_{\mathfrak{B}} \in \Psi(\mathfrak{B})$ and $\delta_{\mathfrak{A}} \in \Psi(\mathfrak{A})$ such that $\delta_{\mathfrak{B}} = i^*(\delta_{\mathfrak{A}})$ under the map $i^* : \Psi(\mathfrak{A}) \rightarrow \Psi(\mathfrak{B})$.

Let $\gamma \in \text{Spec}_G(\mathfrak{B})$ such that $\gamma(X \otimes \text{id}_{H_2}) = x_i$. Then naturality states that the probability $\delta_{\mathfrak{B}}(\gamma)$ is equal to the sum of probabilities for all $\rho \in \text{Spec}_G(\mathfrak{A})$ such that $\rho(X \otimes \text{id}_{H_2}) = x_i$.

Note that $X \otimes Y = (X \otimes \text{id}_{H_2}) \circ (\text{id}_{H_1} \otimes Y)$ and hence

$$\rho(X \otimes Y) = \rho(X \otimes \text{id}_{H_2}) \cdot \rho(\text{id}_{H_1} \otimes Y)$$

which demonstrates that assigning outcomes of the measurement $X \otimes Y$ is equivalent to making a pair of assignments, for X and Y respectively. Those $\rho \in \text{Spec}_G(\mathfrak{A})$ such that $\rho|_{\mathfrak{B}} = \gamma$ are precisely those such that $\rho(X \otimes \text{id}_{H_2}) = x_i$. Hence the naturality condition implies that the probability assigned to x_i , is equal to the sum for all j of the pairs (x_i, y_j) , which is precisely the condition described in Definition 6.1.9. \square

It seems that global parameter independence is a stronger condition than local parameter independence, however, due to a result by Tsirelson, there is an argument to be made that for a system represented by a finite-dimensional Hilbert space the conditions of global and local parameter independence are in fact equivalent, as expressed in the following quote [205, p. 1]:

Quantum Bell-type inequalities are defined in terms of two (or more) subsystems of a quantum system. The subsystems may be treated either via (local) Hilbert spaces, – tensor factors of the given (global) Hilbert space, or via commuting (local) operator algebras. [...] If the given (global) Hilbert space H is finite-dimensional and only two subsystems are dealt with, then the two approaches are equivalent.

Going back to Bell [19], the tensor product structure of a system is typically interpreted as representing some kind of spatio-temporal condition on measurement sites. The arguments of Landsman and Leegwater make no reference to spacelike separation, indeed Leegwater states [148, p. 8 fn. 9]:

No requirement is imposed on the spatiotemporal relation between the two measurements constituting the join measurement: they are

not necessarily spacelike separated.

Since a Hilbert space of the form $H \cong H_1 \otimes H_2$ has many other possible decompositions as a tensor product of two Hilbert spaces, and since we are attaching no physical significance to these representations should the condition described in Definition 6.1.9 not apply to all such decompositions?

That is, given a pair of observables $A, B : H \rightarrow H$, then if it is the case that there exists Hilbert spaces H_1, H_2 such that $H \cong H_1 \otimes H_2$ and such that A and B take the form $A = A_1 \otimes \text{id}_{H_2}$ and $B = \text{id}_{H_1} \otimes B_2$ for some pair of operators $A_1 : H_1 \rightarrow H_1$ and $B_2 : H_2 \rightarrow H_2$, then should the condition of local parameter independence apply to this pair?

Remark 6.1.11. The question of when a pair of measurements can be viewed in tensor form is addressed in [217].

Given that there is no notion of spacelike separation, or any physical significance attached to the tensor structure, it is completely arbitrary to require that local parameter independence only apply to certain decompositions.

If we accept this argument, that the condition of local parameter independence ought to apply to arbitrary tensor decompositions then the following result of Tsirelson [205] shows that local parameter independence implies global parameter independence.

A full proof of Tsirelson’s theorem can be found in [188, Theorem 1].

Theorem 6.1.12. *Let $\{X_i\}_i$ and $\{Y_j\}_j$ be sets of operators on a Hilbert space H such that for all i and j we have $X_i Y_j = Y_j X_i$, and such that the sets $\{X_i\}_i$ and $\{Y_j\}_j$ generate finite-dimensional von Neumann subalgebras of $\text{Hom}(H, H)$. There exists a Hilbert space H' which can be decomposed $H' = H_1 \otimes H_2$ such that we can faithfully embed the families of operators $\{X_i\}_i \subseteq \text{Hom}(H_1, H_1)$ and $\{Y_j\}_j \subseteq \text{Hom}(H_2, H_2)$.*

This result is summarised by Scholz and Werner [188, pp. 4–5] as

Hence, in finite dimension, every quantum correlation function de-

rived from commuting observables can also be represented by observables having tensor product form.

Our discussion here echoes that of Abramsky and Brandenburger who discuss Tsirelson's theorem within the context of no-signalling [5, §9].

Tsirelson's theorem, as stated above only applies to the finite-dimensional setting, however, Scholz and Werner [188, Theorem 3] have extended the result beyond the finite-dimensional setting for families of operators on an infinite-dimensional Hilbert space satisfying an additional property of having *approximately finite dimension* [188, Definition 2]. Scholz and Werner observe that the property of having approximately finite dimension is held by most physical models, including: any fermionic system; quantum spin systems; the CHSH case; and usual models from quantum field theory. Hence, for such systems the same argument for the equivalence of local and global parameter independence also applies. However, the general case for Hilbert spaces with infinite dimension remains an open problem, known as *Tsirelson's problem*, and is equivalent to *Connes' embedding problem* [131], an open problem in the study of operator algebras.

For further discussion of the physical plausibility of global parameter independence, see Section 6.2.

The Preparation Independence Condition

Preparation independence concerns the way in which we can combine systems, and is discussed at length in [149, §7.3]. According to the argument in [149, §7.3] the Cartesian product assumption is a weakening of *Einstein Separability*, defined in [121, p. 173] as the principle that:

Two spatially separated systems possess their own separate real states.

Given a pair of measurable spaces Λ_A and Λ_B with respective σ -algebras Σ_A and Σ_B then $\Lambda_A \times \Lambda_B$ can be seen as a measurable space with σ -algebra

$\Sigma_A \otimes \Sigma_B$ which is the σ -algebra defined as the collection of subsets of $\Lambda_A \times \Lambda_B$ of the form $U_A \times U_B$ where $U_A \in \Lambda_A$ and $U_B \in \Lambda_B$.

Definition 6.1.13. A *product measure* on the measurable space $(\Lambda_A \times \Lambda_B, \Sigma_A \otimes \Sigma_B)$ is a measure

$$\Sigma_A \otimes \Sigma_B \xrightarrow{\mu} [0, \infty]$$

where $\mu(U_A \times U_B) = \mu_A(U_A)\mu_B(U_B)$ for some pair of measures μ_A and μ_B on Λ_A and Λ_B respectively.

If we have two physical systems represented by sets of preparations Π_A and Π_B , with corresponding ontological models Λ_A and Λ_B , then there ought to be a physical system which is simply considering A and B side by side. This system will have a set of preparations Π_{AB} . Implicit in the assumption of preparation independence is the assumption that a preparation on A and B separately should constitute a valid preparation in the composite system – that is

$$\Pi_A \times \Pi_B \subseteq \Pi_{AB}$$

We state the definition of preparation independence as in [149, Definition 7.3], where it is split into two separate conditions.

Definition 6.1.14. Consider a pair of physical systems with sets of preparations Π_A and Π_B with corresponding spaces of ontic states Λ_A and Λ_B , and consider the composite physical system with preparations Π_{AB} . This pair of systems is said to satisfy *preparation independence* if the following conditions are satisfied:

1. The *Cartesian Product Assumption*: for two systems with preparations Π_A and Π_B and corresponding sets of ontic states Λ_A and Λ_B the set of ontic states for the corresponding composite system is

$$\Lambda_A \times \Lambda_B \subseteq \Lambda_{AB}$$

and moreover preparations on the composed system of the form $(\psi_A, \psi_B) \in \Pi_A \times \Pi_B$ correspond with measures on this subset of ontic $\Lambda_A \times \Lambda_B$ states.

2. The *No-Correlation Assumption*: the probability measure $\mu_{(\psi_A, \psi_B)}$ on $\Lambda_A \times \Lambda_B$ corresponding with a preparation $(\psi_A, \psi_B) \in \Pi_A \times \Pi_B$ is equal to the product measure $\mu_{\psi_A} \times \mu_{\psi_B}$.

Our formulation of Preparation Independence is unpacked in the proof of Theorem 6.2.3.

6.2 The Colbeck-Renner Theorem

We frame our reformulation of the Colbeck-Renner theorem alongside those of Landsman and Leegwater for two main reasons: because the original proof of Colbeck and Renner [51] was not fully rigorous; and because their assumptions can be more readily translated into our framework. In particular, the original argument by Colbeck and Renner relies on the assumption of *freedom of choice*, which has been deemed controversial. According to Leegwater [148, p. 2]

[The Colbeck-Renner Theorem] crucially hinges on an assumption dubbed ‘Freedom of Choice’. As the name suggests, this assumption is meant to be about the freedom of experimenters to choose their measurement settings. From this assumption, [Colbeck and Renner] derive ‘no-signalling’, which is essentially equal to [local parameter independence]. Nevertheless, when inspecting the way ‘Freedom of Choice’ is defined, it becomes apparent that [local parameter independence] is in fact part of this assumption.

Both the Landsman and Leegwater proofs replace freedom of choice by the condition of *local parameter independence*.

Here we present the formulation of the Colbeck-Renner theorem by Leegwater [148, Theorem 1]. The relationship between this version of the theorem and that of the original [51] is discussed at length by Leegwater [148].

Theorem 6.2.1. *Let H be a finite-dimensional Hilbert space. Suppose there exists an ontological model for pure states on H with ontic states Λ satisfying*

local parameter independence and preparation independence. For each preparation $\psi \in \Pi$ we have

$$\psi = \delta^\lambda$$

for almost all $\lambda \in \Lambda$, with respect to the measure μ_ψ .

Remark 6.2.2. The statement of Theorem 6.2.1 is almost identical to Landsman's result [139, Theorem 4.1], except that Landsman includes several additional assumptions. These additional assumptions have no clear physical motivation, as remarked by Landsman [139, p. 3]

Unfortunately, [these] assumptions are purely technical and have solely been invented to carry out certain steps in the proof.

The proofs of Leegwater and Landsman differ significantly, but both are highly technical. Although requiring more assumptions, the proof of Landsman is far shorter than that of Leegwater, and follows essentially the same argument as Colbeck and Renner in their original proof.

Reformulating the Colbeck-Renner Theorem

We will now give a reformulation of the Colbeck-Renner theorem using the language of spectral presheaves.

Compared to the proofs of Landsman or Leegwater our proof will hold in arbitrary dimension, even nonseparable Hilbert spaces and not just the finite-dimensional case. We will see that preparation independence is only required to cover the case for $\dim(H) = 2$.

Theorem 6.2.3. *Consider an ontological model Λ for the pure states on a quantum system represented by a Hilbert space H . Consider a preparation ψ :*

1. *If $\dim(H) \geq 3$, and assuming global parameter independence then for almost all $\lambda \in (\Lambda, \mu_\psi)$ we have $\delta^\lambda = \psi$.*
2. *If $\dim(H) = 2$, and assuming global parameter independence and preparation independence, then for almost all $\lambda \in (\Lambda, \mu_\psi)$ we have $\delta^\lambda = \psi$.*

Proof. Suppose $\dim(H) \geq 3$. Assuming global parameter independence, Theorem 4.2.10 states that for each $\lambda \in \Lambda$ the natural transformation δ^λ is of the form

$$\delta^\lambda = \tilde{\omega}$$

where ω is a state of the C^* -algebra $\text{Hom}(H, H)$ – in the sense of Definition 4.2.6.

Recall, if we assume the Continuum Hypothesis, then Theorem 4.2.4 asserts that

$$\delta^\lambda = \tilde{q}$$

for some density operator q , however, in general, a state of a C^* -algebra does not correspond with a density operator – that is, is not necessarily normal – however every state of a C^* -algebra is a convex sum of a normal state and a singular state. Therefore for each $\lambda \in \Lambda$ we have

$$\delta^\lambda = p_1 \cdot \tilde{q}^\lambda + p_2 \cdot \tilde{\sigma}^\lambda$$

where q^λ is a density operator, where σ^λ is a singular state of $\text{Hom}(H, H)$, and where $p_1 + p_2 = 1$.

Now by assumption the preparation ψ corresponds with some unit vector $|\psi\rangle \in H$. Let P_ψ be the corresponding one-dimensional projector, and let \mathfrak{A} be the commutative von Neumann subalgebra generated by P_ψ , that is, $\mathfrak{A} = \{P_\psi\}''$.

We have

$$\begin{aligned} \int_{\Lambda} \delta_{\mathfrak{A}}^\lambda(P_\psi) \mu_\psi &= \psi_{\mathfrak{A}}(P_\psi) \\ &= 1 \end{aligned}$$

and hence

$$\int_{\Lambda} (p_1 \cdot \tilde{q}_{\mathfrak{A}}^\lambda(P_\psi) + p_2 \cdot \tilde{\sigma}_{\mathfrak{A}}^\lambda(P_\psi)) \mu_\psi = 1$$

By Theorem 6.1.5.2 we have

$$(p_1 \cdot \int_{\Lambda} \tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi}) \mu_{\psi}) + (p_2 \cdot \int_{\Lambda} \tilde{\sigma}_{\mathfrak{A}}^{\lambda}(P_{\psi}) \mu_{\psi}) = 1$$

Recall – Definition 4.3.13 – that the singular states are characterised by the property that $\sigma(P) = 0$ for all finite-dimensional projections P , and since P_{ψ} is a one-dimensional projection, hence by 6.1.5.1 we have

$$p_2 \cdot \int_{\Lambda} \tilde{\sigma}_{\mathfrak{A}}^{\lambda}(P_{\psi}) \mu_{\psi} = 0$$

and therefore

$$p_1 \cdot \int_{\Lambda} \tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi}) \mu_{\psi} = 1$$

and hence we conclude that $p_1 = 1$, and therefore $p_2 = 0$. That is, we have shown that for each λ we have

$$\delta^{\lambda} = \tilde{q}^{\lambda}$$

Note that this equation holds automatically if we assume the Continuum Hypothesis, however, we have seen that it holds even if we do not assume the Continuum Hypothesis.

Note that we have $0 \leq \tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi}) \leq 1$ for all $\lambda \in \Lambda$. Hence $1 - \tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi})$ is always positive, therefore, by Lemma 6.1.6 the equation

$$\int_{\Lambda} (1 - \tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi})) \mu_{\psi} = 0$$

implies that $1 - \tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi}) = 0$ for almost all $\lambda \in \Lambda$, that is $\tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi}) = 1$ for almost all $\lambda \in \Lambda$, and recall that $\tilde{q}_{\mathfrak{A}}^{\lambda}(P_{\psi}) = 1$ if and only if q^{λ} is equal to the one-dimensional projector P_{ψ} . Hence we have shown that $\delta^{\lambda} = \psi$ almost everywhere, with respect to the measure μ_{ψ} , for each preparation ψ corresponding with a pure state, that is, a one-dimensional projection. Note that it follows immediately from this that the probability distributions associated with each δ^{λ} are computed via the trace

formula almost everywhere, that is

$$\delta_{\mathfrak{A}}^\lambda(P) = \text{tr}(\psi P) \quad \text{almost everywhere}$$

It remains to show the case for $\dim(H) = 2$. In order to show this we will make essential use of the argument above combined with the condition of Preparation Independence – Definition 6.1.14.

Let H be a Hilbert space with $\dim(H) = 2$ corresponding with a physical system with preparations Π_H and let K be a Hilbert space with $\dim(K) = 3$ corresponding with a physical system with preparations Π_K . Let Λ_H and Λ_K be the corresponding sets of ontic states.

Now consider a preparation $\theta = (\psi, \varphi) \in \Pi_H \times \Pi_K$ on the composite system $H \otimes K$. Note that by the above argument, since $\dim(H \otimes K) > 2$ we have for each \mathfrak{A} and $P : H \otimes K \rightarrow H \otimes K$ we have

$$\delta_{\mathfrak{A}}^\lambda(P) = \text{tr}(\theta P) \quad \text{almost everywhere}$$

The Cartesian Product Assumption requires that for P_1 a projector on H and P_2 a projector on K , with \mathfrak{A}_1 the algebra generated by P_1 and \mathfrak{A}_2 the algebra generated by P_2 we have

$$\delta_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}^{(\lambda_1, \lambda_2)}(P_1 \otimes P_2) = \delta_{\mathfrak{A}_1}^{\lambda_1}(P_1) \cdot \delta_{\mathfrak{A}_2}^{\lambda_2}(P_2)$$

for each $(\lambda_1, \lambda_2) \in \Lambda_H \times \Lambda_K$.

Note that since $\theta = \psi \otimes \varphi$ we have

$$\text{tr}(\theta(P_1 \otimes P_2)) = \text{tr}(\psi P_1) \cdot \text{tr}(\varphi P_2) \tag{6.1}$$

Also, note that by the No Correlation Assumption we have $\mu_\theta = \mu_\psi \times \mu_\varphi$,

and hence we have the following

$$\begin{aligned}
 \text{tr}(\psi P_1) \cdot \text{tr}(\varphi P_2) &= \text{tr}(\theta(P_1 \otimes P_2)) \\
 &= \delta_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}^{(\lambda_1, \lambda_2)}(P_1 \otimes P_2) \\
 &= \delta_{\mathfrak{A}_1}^{\lambda_1}(P_1) \cdot \delta_{\mathfrak{A}_2}^{\lambda_2}(P_2) \\
 &= \delta_{\mathfrak{A}_1}^{\lambda_1}(P_1) \cdot \text{tr}(\varphi P_2)
 \end{aligned}$$

The first equality is (6.1). The second equality holds almost everywhere since $\dim(H \otimes K) > 2$. The third equality is the Cartesian Product Assumption, and the final equality holds almost everywhere since $\dim(K) > 2$. Hence we have

$$\text{tr}(\psi P_1) = \delta_{\mathfrak{A}_1}^{\lambda_1}(P_1) \quad \text{almost everywhere}$$

as required. □

The Role of Parameter Independence

The proof of Theorem 6.2.3 hinges critically on Theorem 4.2.10, which does most of the heavy lifting. We can only apply Theorem 4.2.10 once we have introduced the concept of global parameter independence, which is probably the most controversial aspect of our reformulation of the Colbeck-Renner theorem.

One might be unconvinced by the argument based on Tsirelson’s result that global and local parameter independence are equivalent, as discussed in Section 6.1. Furthermore, one might object to global parameter independence, insisting that it is too strong an assumption for quantum theory. However, the following theorem shows that such objections seem somewhat futile, as even if one rejects global parameter independence as an *assumption*, it seems inescapable as a *conclusion* of quantum theory.

Theorem 6.2.4. *Local parameter independence and preparation independence imply global parameter independence holds almost everywhere.*

Proof. Consider an ontological models for pure state quantum theory with ontic

states Λ , satisfying local parameter independence and preparation independence.

By the Colbeck-Renner theorem – Theorem 6.2.1 – for each ψ we have

$$\delta^\lambda = \psi \quad \text{for almost all } \lambda \in \Lambda$$

Hence, because ψ satisfies naturality – Theorem 4.1.29 – then δ^λ satisfies naturality for almost all $\lambda \in \Lambda$, which is precisely the condition that the ontological model satisfies global parameter independence – Definition 6.1.7. \square

With this observation we draw another parallel between our discussion of parameter independence and the discussion of no-signalling of Abramsky and Brandenburger who observe [6, p. 2]:

There are additional grounds for doubting that [no-signalling] is essentially relativistic in nature. In particular, it is satisfied by ordinary quantum mechanics, with a classical background. Indeed, it can be seen to arise purely as a property of families of commuting sets of observables, without any consideration of tensor product structure or any other reflection of spacelike separation.

We are not suggesting that local parameter independence be rejected, however we believe that it is appropriate to reintroduce the notions of local parameter independence, and preparation independence once a spatiotemporal structure has been introduced into the formalism. We discuss ways in which this might be approached in Chapter 8.

6.3 The PBR Theorem

The PBR theorem is similar to the Colbeck-Renner theorem, but is stated in terms of the *support* of the measures μ_ψ on Λ , recall Definition 4.2.14. In Remark 4.2.15 we noted that the support of a measure need not be a well-defined notion for measure spaces in general, and in proving Theorem 6.2.3 we made no additional assumptions about Λ beyond being a measurable space.

In the original statement of the PBR theorem [185] it is assumed that the probability measures μ_ψ on Λ corresponding with each preparation ψ are in fact probability distributions, and note that probability distributions have a well-defined notion of support: the support of a probability distribution μ on Λ is simply those elements $\lambda \in \Lambda$ such that $\mu(\lambda) > 0$. We will also restrict from arbitrary probability measures to just the probability distributions.

Definition 6.3.1. An ontological model is said to be ψ -ontic if for every pair of preparations ψ and ϕ the probability distributions μ_ψ and μ_ϕ do not overlap, that is, for any $\lambda \in \Lambda$ such that $\mu_\psi(\lambda) > 0$ we have $\mu_\phi(\lambda) = 0$.

An ontological model is said to be ψ -epistemic if it is not ψ -ontic.

The point of making the distinction between ψ -ontic and ψ -epistemic ontological models is to establish whether the ‘lack of information’ interpretation of probabilities is tenable. Recall Example 6.1.1, in this example we saw that it was reasonable to posit that the observed probability distribution of outcomes reflected some underlying lack of knowledge of the true state. The ontological model we described in Example 6.1.1 is ψ -epistemic.

The PBR theorem asserts that quantum theory does not admit a ψ -epistemic ontological model, and hence the probability distributions cannot be viewed as reflections of a lack of knowledge, but must be considered as elements of reality.

Theorem 6.3.2 (PBR). *Any ontological model of pure state quantum theory (in finite dimensions) satisfying preparation independence and local parameter independence is ψ -ontic.*

Our proof of the PBR theorem will follow directly from our formulation of the Colbeck-Renner theorem. In their standard presentations, there is a close connection between the Colbeck-Renner theorem and the PBR theorem. This was considered by Colbeck and Renner [52] who give a different argument for the same result, and is discussed by Leegwater [148, §9].

Reformulating the PBR Theorem

The following result is a reformulation of the PBR theorem.

Theorem 6.3.3. *Consider an ontological model Λ for the pure states on a quantum system represented by a Hilbert space H , where for each preparation ψ the corresponding probability measure μ_ψ is a probability distribution. Consider a pair of preparations ψ, ϕ :*

1. *If $\dim(H) \geq 3$, and assuming global parameter independence, then $\psi \neq \phi$ implies $\text{supp}(\mu_\psi) \cap \text{supp}(\mu_\phi) = \emptyset$.*
2. *If $\dim(H) = 2$, and assuming global parameter independence and preparation independence, then $\psi \neq \phi$ implies $\text{supp}(\mu_\psi) \cap \text{supp}(\mu_\phi) = \emptyset$.*

Proof. The support of a probability distribution μ on Λ consists precisely those elements $\lambda \in \Lambda$ such that $\mu(\lambda) > 0$.

For μ a probability distribution on Λ considered as a measure, then if some property applies almost everywhere then in particular it must apply on $\text{supp}(\mu)$.

Theorem 6.2.3 states that for each ψ and corresponding probability distribution μ_ψ we have

$$\delta^\lambda = \psi \quad \text{for almost all } \lambda \in \Lambda$$

and hence for all $\lambda \in \text{supp}(\mu_\psi)$ we have $\delta^\lambda = \psi$.

Hence if there exists some λ in the support of both μ_ψ and μ_ϕ we have both $\delta^\lambda = \psi$ and $\delta^\lambda = \phi$. Hence, if the supports of μ_ψ and μ_ϕ overlap, we must have $\psi = \phi$, equivalently, $\psi \neq \phi$ implies $\text{supp}(\mu_\psi) \cap \text{supp}(\mu_\phi) = \emptyset$, as required. \square

Since our proof of Theorem 6.3.3 follows directly from our proof of the Colbeck-Renner theorem, the role of parameter independence in our proof is essentially identical to that of the Colbeck-Renner theorem. Hence, for a discussion of the role of parameter independence in this proof we refer to the discussion at the end of Section 6.2.

The Role of Preparation Independence

In our formulation of the PBR theorem preparation independence is required only for systems represented by Hilbert spaces with dimension 2, while in the original proof preparation independence is one of the fundamental ingredients. It is already known that preparation independence is required for the case when $\dim(H) = 2$, see for example, the discussion in [149, §7.5]. The necessity of preparation independence for $\dim(H) = 2$ can be seen through the existence of a ψ -epistemic ontological model for the qubit due to Kochen and Specker [136], or see for example [149, Example 4.6].

Note that a model which violates the conditions of the PBR theorem would also violate the conditions of the Colbeck-Renner theorem, and hence preparation independence is also a necessary condition for the Colbeck-Renner theorem to hold in dimension 2.

Although our discussion here applies to both the Colbeck-Renner theorem and the PBR theorem we focus on the PBR theorem because the role of preparation independence in this result has already been considered at length in the literature; for a discussion see [149, §7.5]. In particular, there have been two approaches to the PBR theorem which do not require preparation independence:

- Mansfield conjectures that the condition of preparation independence can be weakened to a condition of *no-preparation-signalling* [158], and provides partial results in this direction [159].
- Barrett, Cavalcanti, Lal and Maroney have proven a weaker version of the PBR Theorem which does not assume preparation independence [18, Theorem 4]. The statement is weaker, not ruling out all ψ -epistemic ontological models, but just the so-called *maximally epistemic* ontological models [18, Definition 3], and it only applies in the case $\dim(H) \geq 3$.

Our formulation of the PBR theorem can be seen as a strengthening of the result of Barrett, Cavalcanti, Lal and Maroney: for $\dim(H) \geq 3$ our proof rules out *all* ψ -epistemic ontological models, not just the maximally epistemic ones.

Recall, in Chapter 1 we discussed Isham's four fundamental conceptual problems of quantum theory, listed in Figure 1.3. We argued that our formulation of the Colbeck-Renner and PBR theorems brought into question how closely linked the phenomenon of entanglement is with the question of the meaning of quantum probabilities. Our proofs of the Colbeck-Renner theorem and PBR theorem do not rely on any notion of tensor product or entanglement in any dimension except 2.

Chapter 7

Modelling Spekkens' Toy Theory

Spekkens' Toy Theory [198] is a hypothetical physical theory based on a classical system for which there is an in-built restriction on knowledge we can have about the system. Spekkens' Toy Theory is particularly interesting because the restriction of knowledge seems to reproduce some phenomena typically thought of as quantum – for example: interference; non-commutative measurements; entanglement; no-cloning; and teleportation – despite the theory being essentially classical.

In the original formulation, Spekkens' Toy Theory is presented in an informal way. It is highly desirable to have a rigorous mathematical framework in which we can model both quantum theory and Spekkens' Toy Theory, as having the same type of mathematical objects representing the states, measurements and dynamics of the respective models allows for a much more precise comparison of the properties and features of those physical systems.

Spekkens' Toy Theory has been successfully modelled using the monoidal approach to quantum theory [44, 75], in particular the *observables* of the theory can be modelled using Frobenius algebras in the category of sets and relations

Rel.

In this chapter we show how Spekkens' Toy Theory can also be modelled using the structures we have introduced in the preceding chapters. In Section 7.1 we outline Spekkens' Toy Theory itself. In Section 7.2 we show how this physical theory can be modelled using commutative semialgebras, and the associated structures introduced in the preceding chapters. This serves as an in-depth case study: an illustration of features of the spectral presheaf framework developed in the preceding chapters.

In Section 7.3 we compare our spectral presheaf model of Spekkens' Toy Theory to the Frobenius algebra model of Coecke and Edwards [44, 75].

7.1 Spekkens' Toy Theory

Spekkens' Toy Theory describes a physical system consisting of a particle p which can occupy one of four boxes labelled as follows:

a	b	c	d
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There is a measuring device which can observe the contents of the boxes, however, the device has a “maximum resolution” of two boxes – that is, the device cannot detect with certainty that the particle p is in box a , but can detect whether or not p lies in box a or box b . The measuring device can cover the set of boxes in the following ways.

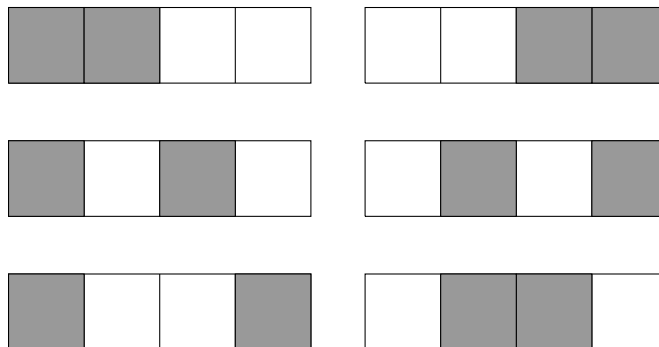


Figure 7.1: Possible measurement configurations.

Each measurement corresponds with the Yes/No question “is p in one of these two boxes”. Despite the limited resolution of a single measurement, the position of p could be determined exactly by knowing the outcomes of certain pairs of measurements, for example: if we know the answer to “is p in a or b ?” and the answer to “is p in b or c ?”, then we can determine the precise location of p .

Being a Yes/No question means that the outcome of any measurement is specified by a single bit of information. As there are four possible positions that p can occupy, specifying the position of p requires exactly two bits of information.

The Knowledge-Balance Principle

The physical system described by Spekkens’ Toy Theory is subject to the *knowledge balance principle*, which we state as:

At any time we can know at most one bit of information regarding the position of p .

Imposing the knowledge balance principle has several effects on the system. For example, suppose we ask “is p in a or b ?” and obtain the outcome “Yes”, this constitutes one bit of information. If we then ask “is p in b or c ?” and obtain the outcome “Yes”, then we have acquired a second bit of information. If

we are allowed to hold these two bits of information simultaneously then we can deduce that p is in box b , however, the knowledge balance principle prohibits this. In effect, by obtaining the answer to the question “is p in b or c ?” we must lose knowledge of the answer to the question “is p in a or b ?”.

If we know that p is in a or b , and then we ask “is p in b or c ?” and obtain the answer “Yes”, we can deduce that in the moment before measuring the particle must have been in b , however now, after measuring, all we know is that the particle may be in b or c .

This effect of losing the information content of previous measurements could be attributed to physical disturbance of the system introduced by the measurement process.

Definition 7.1.1. The position of the particle is called the *ontic state* of the system. Our state of knowledge about the position of the particle is called the *epistemic state*. The ontic states are represented by the elements of X , and the epistemic states correspond with subsets $E \subseteq X$.

The knowledge balance principle requires that for an epistemic state $E \subseteq X$ we have $|E| \geq 2$.

If we have just performed the measurement is “ p in a or b ?” and obtain the answer “Yes”, then the epistemic state of the system is represented by the subset $\{a, b\}$. Note that an epistemic state, as described in Definition 7.1.1 constitutes a single bit of information.

Measurements are not deterministic in general, for example, if we prepare the system in the epistemic state $\{a, b\}$ and we ask the question “is the p in the box a or c ?” we would not consistently observe a particular outcome. This could be modelled in different ways, for example, via probability, in which case we would obtain the answer “Yes” with probability one half.

Since a measurement of the system may alter the state of the system, the order in which we perform the measurements will effect the possible outcomes of those measurements. However, a measurement need not necessarily effect outcomes of another. For example, knowing the answer to the question “is p in

a or b ?” constitutes the same knowledge as knowing the answer to the question “is p in c or d ?”. That is, with only one bit of information we can answer both of these questions. We say that such collections of measurements are *epistemically equivalent*.

Definition 7.1.2. Call a pair of measurements m_1 and m_2 *epistemically equivalent* if the outcome of m_1 uniquely determines the outcome of m_2 and vice versa.

Recall, the knowledge balance principle places a restriction on the amount of information that we can know about the system at any given moment. If we currently hold one bit of information regarding the location of p , and then perform a measurement which will extract a second bit of information, then we must lose the first bit we held. One immediate implication of this fact is that the order in which we perform measurements will affect the final state of the system post-measurement.

However, if a pair of measurements are epistemically equivalent then this type of disturbance does not occur, as the knowledge balance principle is not violated by simultaneously holding the outcomes of both measurements.

If two measurements are epistemically equivalent then, if we perform one after another there is no loss of information, and the order in which the measurements are performed does not affect the final state of the system, or our state of knowledge of the system. We can organise measurements into their pairs of epistemically equivalent measurements:

$$\begin{aligned} & \{ \text{“}p \text{ is in } a \text{ or } b\text{”, “}p \text{ is in } c \text{ or } d\text{”} \} \\ & \{ \text{“}p \text{ is in } a \text{ or } c\text{”, “}p \text{ is in } b \text{ or } d\text{”} \} \\ & \{ \text{“}p \text{ is in } a \text{ or } d\text{”, “}p \text{ is in } b \text{ or } c\text{”} \} \end{aligned}$$

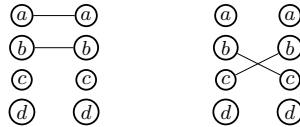
Remark 7.1.3. We can introduce a notion of “eigenstate” for the measurements of Spekkens’ Toy Theory, where the generalised eigenstates of a measurement would be those epistemic states for which the measurement has a deterministic

outcome. For example, if we prepare epistemic state $\{a, b\}$ then making the measurement “is p in a or b ?” has a deterministic outcome – that is, the answer is always “Yes” – and our state of knowledge of the system remains unchanged. The same is true for the measurement “is p in c or d ?”. The notion of epistemically equivalent measurements could also be stated in terms of generalised eigenstates, whereby two measurements are epistemically equivalent if they share the same eigenstates. Compare this notion to the case of quantum theory where two self-adjoint operators share the same eigenstates if and only if they commute, that is, if and only if the order in which they are performed does not alter the possible outcome of those measurements.

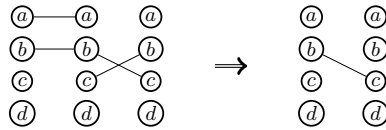
7.2 A Spectral Presheaf Model

We now demonstrate that Spekkens' Toy Theory can be modelled using commutative semialgebras of relations $\mathbf{Rel}\text{-}\mathbf{Alg}(X)$ where $X = \{a, b, c, d\}$. Moreover, we will show that the epistemic states of Spekkens' Toy Theory are modelled by operational states, as introduced in Chapter 4, and the ontic states of Spekkens' Toy Theory can be seen to correspond with the hidden-variables of a local hidden-variable model (as introduced in Chapter 5), or to correspond with the ontic states of an ontological model (as introduced in Chapter 6).

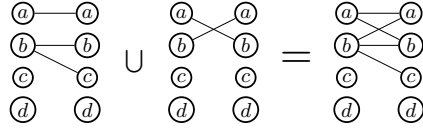
We use a simple graphical representation of relations $R : X \rightarrow X$ on a set $X = \{a, b, c, d\}$ where a line connecting an element a on the left to b on the right if and only if $a \sim_R b$. The composition of a pair of relations, for example



is given by concatenating the diagram and tracing paths



The union of relations can easily be seen by laying these pictures on top of one another, for example:



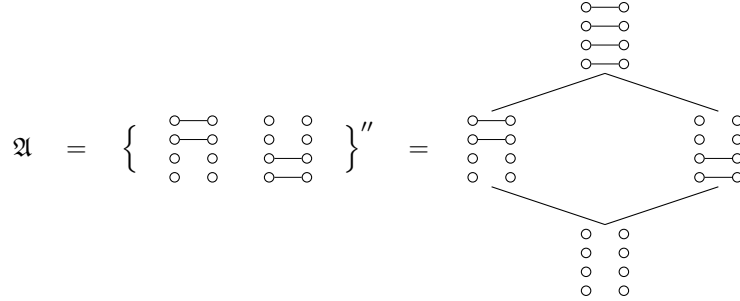
Note that the union of relations is an idempotent operation and hence for sets X and Y , the set $\text{Hom}(X, Y)$ has the structure of a poset where $R_1 < R_2$ if and only if $R_1 + R_2 = R_2$. Clearly the zero relation is the bottom element of each such poset. When depicting semialgebras of relations we will represent this ordered structure using Hasse diagrams.

Each measurement corresponds with a two-element subset $Y \subseteq X$, and is paired with an epistemically equivalent measurement, which corresponds with the two-element subset $X \setminus Y$. Our mathematical representation of measurements will be slightly different, not representing measurements as subsets $Y \subset X$, but rather by the relations $P_Y : X \rightarrow X$, where for $Y \subseteq X$ we define $P_Y : X \rightarrow X$ to be the relation for which $y \sim y$ for $y \in Y$. We associate the measurement $P_{\{a,b\}}$ with the question “is the particle in box a or b ”, and likewise we associate $P_{\{c,d\}}$ with the question “is the particle in box c or d ”.

We want to consider the commutative **2**-subsemialgebras $\mathfrak{A} \subseteq \text{Hom}(X, X)$ generated by pairs of epistemically equivalent measurements. For example, the pair

$$\{P_{\{a,b\}}, P_{\{c,d\}}\} = \left\{ \begin{array}{cc} \circ - \circ & \circ \quad \circ \\ \circ - \circ & \circ \quad \circ \\ \circ \quad \circ & \circ - \circ \\ \circ \quad \circ & \circ - \circ \end{array} \right\}$$

generates the $\mathbf{2}^*$ -semialgebra



Note that the zero relation P_\emptyset is interpreted as the question “is p in none of the boxes?”, while the identity relation P_X is interpreted as “is p in any of the boxes?”.

By assumption p is always in one of the boxes, and so the answer to the question P_\emptyset is always “No”, and the answer to P_X is always “Yes”.

The knowledge balance principle states that we cannot simultaneously assign outcomes to arbitrary pairs of measurements, however, we can always assign outcomes to pairs of epistemically equivalent measurements, and hence we can simultaneously assign outcomes to all of the measurements in \mathfrak{A} .

Note that “Yes” and “No” are not arbitrary labels, but carry logical meaning. We can encode this by associating “Yes” and “No” with the elements of 1 and 0 of the two-element Boolean algebra $\mathbf{2}$, where we interpret the operations \wedge and \vee as the logical operations “and”, and “or” respectively. We can consider this Boolean algebra as a semialgebra where we take \wedge to be multiplication and \vee to be addition. Hence, an assignment of outcomes to the measurements in \mathfrak{A} corresponds with a function

$$\mathfrak{A} \xrightarrow{\rho} \mathbf{2}$$

which can be seen as a function between a pair of $\mathbf{2}$ -semialgebras.

We will now show that the only such assignments that are consistent with the metaphysics of Spekkens’ Toy Theory are those which are semialgebra

homomorphisms. Let $\rho : \mathfrak{A} \rightarrow \mathbf{2}$ be an assignment of outcomes, then the overarching metaphysics of Spekkens' Toy Theory demands the following:

1. The answer to the question “is the particle in one of the boxes a or b , OR in one of the boxes c or d ?” is always “Yes”, that is, we have

$$\begin{aligned} \rho(P_{\{a,b\}}) \vee \rho(P_{\{c,d\}}) &= \rho(P_{\{a,b\}} + P_{\{c,d\}}) \\ &= \rho(P_{\{a,b,c,d\}}) \\ &= \rho(\text{id}_X) \\ &= 1 \end{aligned}$$

2. The answer to the question “is p in none of the boxes a , b , c nor d ?” is always “No”, that is, we have

$$\rho(0) = 0$$

3. If the answer to the question “is the particle in box a or b ?” is Yes, then the answer to the question “is the particle in box c or d ?” must be “No”, that is, we have

$$\begin{aligned} \rho(P_{\{a,b\}}) \wedge \rho(P_{\{c,d\}}) &= \rho(P_{\{a,b\}}P_{\{c,d\}}) \\ &= \rho(0) \\ &= 0 \end{aligned}$$

That is, we have shown that the only assignments of measurements to measurement outcomes that we should consider \mathfrak{A} are those which are $\mathbf{2}$ -homomorphisms, that is, elements of $\text{Spec}_G(\mathfrak{A})$.

The generalised Gelfand spectrum $\text{Spec}_G(\mathfrak{A})$ consists of the semialgebra

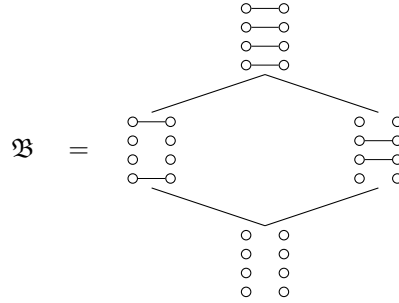
homomorphisms

$$\mathfrak{A} \xrightarrow{\rho} \mathbf{2}$$

There are precisely two elements of the spectrum $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$, which are completely determined by the values they take on the elements $P_{\{a,b\}}$ and $P_{\{c,d\}}$.

$$\begin{array}{ccc} \mathfrak{A} & \xrightarrow{\rho_1} & \mathbf{2} \\ P_{\{a,b\}} & \longmapsto & 1 \\ P_{\{c,d\}} & \longmapsto & 0 \end{array} \qquad \begin{array}{ccc} \mathfrak{A} & \xrightarrow{\rho_2} & \mathbf{2} \\ P_{\{a,b\}} & \longmapsto & 0 \\ P_{\{c,d\}} & \longmapsto & 1 \end{array}$$

There are two other pairs of epistemically equivalent measurements, for example the semialgebra \mathfrak{B} , generated by $P_{\{a,d\}}$ and $P_{\{b,c\}}$



The elements of the spectrum of \mathfrak{B} are characterised as follows

$$\begin{array}{ccc} \mathfrak{B} & \xrightarrow{\gamma_1} & \mathbf{2} \\ P_{\{a,c\}} & \longmapsto & 1 \\ P_{\{b,d\}} & \longmapsto & 0 \end{array} \qquad \begin{array}{ccc} \mathfrak{B} & \xrightarrow{\gamma_2} & \mathbf{2} \\ P_{\{a,c\}} & \longmapsto & 0 \\ P_{\{b,d\}} & \longmapsto & 1 \end{array}$$

If we prepare the system in the epistemic state $\{a, b\}$ and perform measurements from the context \mathfrak{A} we will find that we obtain outcome of “Yes” for ρ_1 , and the answer “No” for ρ_2 . However, for the context \mathfrak{B} we will see some non-deterministic behaviour. This non-determinism can be handled in a number of ways, but we will use the language of probabilities. That is, we find the answer “Yes” to γ_1 with probability one half, and the answer “Yes” to γ_2 with

probability one half. Instead of a probabilistic picture using \mathbb{R}^+ -distributions we could use $\mathbf{2}$ -distributions; a possibilistic view of measurement outcomes. We are going to use the language of probabilities since it is a closer parallel to quantum theory, and because $\mathbf{2}$ has featured heavily in the construction of the model so far.

For each epistemic state $\{x, y\}$ we can prepare, we can apply the measurements across all measurement contexts, and we will observe a probabilistic spread of outcomes for each measurement context \mathfrak{A} ; for each \mathfrak{A} we will obtain a probability distribution on $\text{Spec}_G(\mathfrak{A})$.

This behaviour can be encoded using the language of operational states, in particular, the notion of an R -operational state – Definition 4.1.33 – where we take $R = \mathbb{R}^+$. The data we obtain from repeated experiments for a given preparation is encoded in a family of \mathbb{R}^+ -distributions

$$\psi_{\mathfrak{A}} \in \Delta_{\mathbb{R}^+}(\mathfrak{A})$$

and we will assume that these distributions satisfy no-signalling, that is, that they form a global section

$$\begin{array}{ccc}
 & \Delta_{\mathbb{R}^+} & \\
 \text{Rel-Alg}(X)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

Remark 7.2.1. Note that characterising the epistemic state as an operational \mathbb{R}^+ -state in this way implicitly assumes that every commutative subsemialgebra of $\text{Hom}(X, X)$ constitutes a valid measurement context. We might consider instead a subposet of $\mathbf{Rel-Alg}(X)$. Below we discuss the necessity of such an assumption, within the context of local hidden-variable models and ontological models.

A Hidden-Variable Model for Spekkens' Toy Theory

We will now show that Spekkens' Toy Theory admits a local hidden-variable model, in the sense introduced in Chapter 5. This can also be framed in the language of ontological models – as discussed in Chapter 6 – which we describe in the next section.

The main difference between how we present the hidden variable model and the ontological model are in how we present the underlying metaphysical assumptions; there are two, dual, perspectives we can take:

1. The particle inhabits precisely one box at any given moment, but our measurement apparatus is restricted to viewing two-element subsets at a time. We can interpret this as saying that in *principle* there could exist measurements determining the precise location of the particle but these measurements are not available to us.
2. We are limited to making preparations of states containing one bit of information. That is, we can prepare the system in the states of the form “ p is in x or y ” where $x \neq y$. However, in *principle* there could exist apparatus that could make precise preparations.

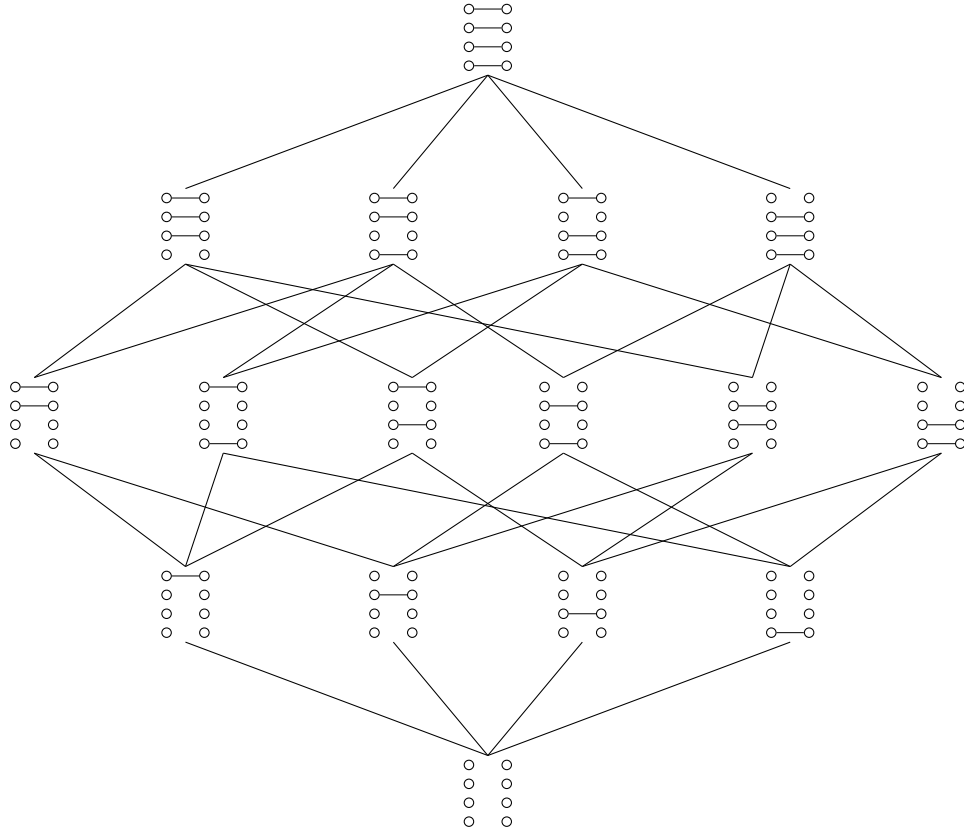
From the first set of assumptions we will construct a hidden-variable model, in the sense of Definition 5.2.8, which means that each operational state ψ admits a decomposition

$$\psi = \sum_{\lambda} r_{\lambda} \cdot e^{\lambda}$$

where each e^{λ} corresponds with a globally-deterministic operational state – Definition 5.1.4, that is, one for which each probability distribution $e_{\mathfrak{A}}^{\lambda}$ is a point-distribution.

In the next section we construct an ontological model, in the sense of Definition 6.1.3. Ontological models subsume local hidden-variable models in general. In particular the ontological model we present is essentially equivalent to the local hidden-variable model, however, the ontological model can be motivated and conceptualised differently.

Following the discussion above, a set of hypothetical sharp measurements would be represented by the relations $P_{\{a\}}$, $P_{\{b\}}$, $P_{\{c\}}$ and $P_{\{d\}}$. These four relations generate the following commutative semialgebra, which we denote \mathfrak{D} :



The corresponding generalised Gelfand spectrum $\text{Spec}_G(\mathfrak{D})$ contains four elements, specified below, note that it is enough to determine the values of these assignments on the primitive idempotent elements:

$$\begin{array}{ccc}
 \mathfrak{D} & \xrightarrow{\phi_a} & \mathbf{2} & & \mathfrak{D} & \xrightarrow{\phi_b} & \mathbf{2} \\
 P_{\{a\}} & \longmapsto & 1 & & P_{\{a\}} & \longmapsto & 0 \\
 P_{\{b\}} & \longmapsto & 0 & & P_{\{b\}} & \longmapsto & 1 \\
 P_{\{c\}} & \longmapsto & 0 & & P_{\{c\}} & \longmapsto & 0 \\
 P_{\{d\}} & \longmapsto & 0 & & P_{\{d\}} & \longmapsto & 0 \\
 \\
 \mathfrak{D} & \xrightarrow{\phi_c} & \mathbf{2} & & \mathfrak{D} & \xrightarrow{\phi_d} & \mathbf{2} \\
 P_{\{a\}} & \longmapsto & 0 & & P_{\{a\}} & \longmapsto & 0 \\
 P_{\{b\}} & \longmapsto & 0 & & P_{\{b\}} & \longmapsto & 0 \\
 P_{\{c\}} & \longmapsto & 1 & & P_{\{c\}} & \longmapsto & 0 \\
 P_{\{d\}} & \longmapsto & 0 & & P_{\{d\}} & \longmapsto & 1
 \end{array}$$

We interpret the assignment ϕ_a answering “Yes” to the question “is the particle in box a ”, and answering “No” to the others.

Clearly there is a one-to-one correspondence between the elements of the set X and the elements of $\text{Spec}_{\mathbb{G}}(\mathfrak{D})$, under the map $x \mapsto \phi_x$. Hence, we can identify the ontic states of the theory with the elements of $\text{Spec}_{\mathbb{G}}(\mathfrak{D})$. We will now show that every element of $\text{Spec}_{\mathbb{G}}(\mathfrak{D})$ uniquely determines a global section

$$\begin{array}{ccc}
 & \Delta_{\mathbb{R}^+} & \\
 & \curvearrowright & \\
 \text{Rel-Alg}(X)^{\text{op}} & \uparrow \phi & \text{Set} \\
 & \curvearrowleft & \\
 & \mathbf{1} &
 \end{array}$$

and moreover that every global section is determined in this way. That is, we will show that we can identify the ontic states of Spekkens' Toy Theory with the global sections of $\text{Spec}_{\mathbb{G}}$.

Lemma 7.2.2. *Every global section ϕ of $\text{Spec}_{\mathbb{G}}$ is uniquely determined by the component $\phi_{\mathfrak{D}} \in \text{Spec}_{\mathbb{G}}(\mathfrak{D})$.*

Proof. For any \mathfrak{A} let $i : B(\mathfrak{A}) \hookrightarrow \mathfrak{A}$ be the subsemialgebra consisting only of subunital idempotents. Since a homomorphism $\rho : \mathfrak{A} \rightarrow \mathbf{2}$ is determined completely by the values it takes on its subunital idempotents this is equivalent to the statement that the restriction map

$$\mathrm{Spec}_G(\mathfrak{A}) \xrightarrow{i^*} \mathrm{Spec}_G(B(\mathfrak{A})) \quad (7.1)$$

is an isomorphism.

Note that for all \mathfrak{A} we have $B(\mathfrak{A}) \hookrightarrow \mathfrak{D}$ and hence specifying $\phi_{\mathfrak{D}}$ completely determines $\phi_{B(\mathfrak{A})}$ for all \mathfrak{A} , and since the map (7.1) is an isomorphism, $\phi_{\mathfrak{D}}$ therefore completely determines $\phi_{\mathfrak{A}}$ for all \mathfrak{A} . \square

In our model we may not wish to consider all possible commutative semialgebras of $\mathrm{Hom}(X, X)$ as valid measurement contexts, but it follows from Lemma 7.2.2 that if we are to include \mathfrak{D} among the measurement contexts we wish to consider then any operational state of that theory is completely determined by the probability distribution on $\mathrm{Spec}_G(\mathfrak{D})$. Moreover any probability distribution $\psi_{\mathfrak{D}} \in \Delta_{\mathbb{R}^+}(\mathfrak{D})$ is of the form

$$\psi_{\mathfrak{D}} = \sum_{x \in X} r_x \cdot e_{\mathfrak{D}}^x$$

where $e_{\mathfrak{D}}^x \in \Delta_{\mathbb{R}^+}(\mathfrak{D})$ is the point-distribution picking out $\phi^x \in \mathrm{Spec}_G(\mathfrak{D})$. By a similar argument as in the proof of Lemma 7.2.2 we see that each $e_{\mathfrak{D}}^x$ can be extended uniquely to a globally deterministic-operational state e^x – that is, one which defines a point-distribution at every component – and we have

$$\psi = \sum_{x \in X} r_x \cdot e^x$$

which is the definition of a local hidden-variable model, Definition 5.2.8.

That is, if we suppose the existence of a set of sharp measurements, even if these measurements are not accessible to us directly, then any operational state must admit a local hidden-variable model, where the hidden-variables correspond

with the underlying elements of X ; the ontic states. Echoing Mermin's assertion about the Kochen-Specker theorem, of course this conclusion can be reached by the "elementary metaphysics" of Spekkens' Toy Theory; the point of this section is to extract this fact from the formalism. Furthermore, this provides a 'sanity check' that our generalisation of local hidden-variable model is reasonable one: that it corresponds with the obvious thing for Spekkens' Toy Theory.

The Frobenius algebra model of Spekkens' Toy Theory, which we consider in Section 7.3, can formalise a notion of non-locality, namely *Mermin non-locality* [93], and in particular it is shown that Spekkens' Toy Theory does not exhibit this kind of non-locality. Mermin locality shows that certain types of obstructions to the existence of global sections of empirical models do not exist. In our framework we show that no obstructions exist whatsoever.

An Ontological Model for Spekkens' Toy Theory

In the previous section, our justification for the existence of a local hidden-variable model for Spekkens' Toy Theory hinged crucially on the existence of a hypothetical set of sharp measurements. Even without this assumption, we can still extract the essentially classical nature of Spekkens' Toy Theory, using a slightly different line of reasoning.

Rather than supposing that there are hypothetical measurements that are sharper than those that we have available to us, let us suppose instead that there are hypothetical states that the system might exist in, but which we cannot prepare directly. Such states are explicit in the model: they correspond with the actual position of the particle – the ontic states.

With this in mind then we will define an ontological model in the sense of Definition 6.1.3, where we take $\Lambda = X$ – that is, the ontic states of Spekkens' Toy Theory correspond precisely with the ontic states of the ontological model. Recall, the elements of $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ determine a partition on the set X . For each measurement context \mathfrak{A} we have available to us – perhaps not every commutative subsemialgebra of $\text{Hom}(X, X)$, and in particular, perhaps not \mathfrak{D} , as defined in

the previous section – we define $\delta_{\mathfrak{A}}^x$ to be the point-distribution on $\text{Spec}_G(\mathfrak{A})$ that picks out the element $\rho \in \text{Spec}_G(\mathfrak{A})$ corresponding with the element of the partition of X that contains x .

The preparation of an epistemic state, for example placing p among $\{a, b\}$, corresponds with a probability distribution on Λ , that is a probability distribution on X , in this case $\frac{1}{2} \cdot a + \frac{1}{2} \cdot b$. This corresponds with the operational probabilities given by

$$\frac{1}{2} \cdot \delta_{\mathfrak{A}}^a + \frac{1}{2} \cdot \delta_{\mathfrak{A}}^b$$

for each available measurement context \mathfrak{A} , which gives us precisely the probabilities observed in Spekkens' Toy Theory.

This ontological model is ψ -epistemic – in the sense of Definition 6.3.1 – and hence we can interpret the probabilistic nature of the measurement outcomes associated with Spekkens' Toy Theory as reflecting our lack of knowledge of a true underlying state. Of course, again echoing Mermin, this is essentially true by construction for Spekkens' Toy Theory, but what we have shown here validates the definition of ontological model in general, and our formulation of the concept in particular, and invites application of the concepts for other non-trivial toy models.

7.3 A Comparison of Models

Spekkens' Toy Theory has been studied extensively from the perspective of the monoidal approach to quantum theory by Coecke and Edwards [44, 75]. In particular, Coecke and Edwards model the observables in Spekkens' Toy Theory using special commutative Frobenius algebras in the \dagger -symmetric monoidal category **Rel**. We will make a comparison between our spectral presheaf model of Spekkens' Toy Theory with the Frobenius algebra model of Coecke and Edwards.

To make our comparison of these frameworks precise we will make use of a series of results from Chapter 3, in which we showed how to lift the internal

algebras and associated structures from monoidal quantum theory to the spectral presheaf framework. We can summarise the correspondence between these models in Figure 7.2.


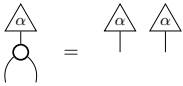
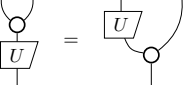
Spekkens' Toy Theory	Monoidal Category Model	Presheaf Model
Measurement context		$\mathfrak{A} \in \mathbf{Rel}\text{-}\mathbf{Alg}_{\mathbf{vN}}(X)$
Measurement outcomes		$\rho \in \text{Spec}_{\mathbf{G}}(\mathfrak{A})$
Dynamics		$\Phi(\mathfrak{A}) \subseteq \text{Hom}(X, X)$
Epistemic state	$E \subset X$	Operational state $\psi : \mathbf{1} \rightarrow \Delta_{\mathbb{R}^+}$
Ontic state	$x \in X$	Global section $\omega : \mathbf{1} \rightarrow \text{Spec}_{\mathbf{G}}$

Figure 7.2: Comparing features of models of Spekkens' Toy Theory.

Recall Theorem 2.3.12 which states that Frobenius algebras in **Rel** are abelian groupoids. In the monoidal category model of Spekkens' Toy Theory of Coecke and Edwards [44, 75] observables are represented by Frobenius algebras on the set $X = \{a, b, c, d\}$, that is, groupoids, of the form $\mathbb{Z}_2 \sqcup \mathbb{Z}_2$. Let (X, μ) be the Frobenius algebra in **Rel** which is defined by the groupoid isomorphic to $\mathbb{Z}_2 \sqcup \mathbb{Z}_2$ with multiplication table

	a	b	c	d
a	a	b	\emptyset	\emptyset
b	b	a	\emptyset	\emptyset
c	\emptyset	\emptyset	c	d
d	\emptyset	\emptyset	d	c

The unit of this Frobenius algebra is the relation $\eta : \{*\} \rightarrow X$ defined

$$* \sim_\eta a \quad \text{and} \quad * \sim_\eta c$$

and there are precisely two set-like elements $\alpha, \beta : \{*\} \rightarrow X$ where α is defined

$$* \sim_\alpha a \quad \text{and} \quad * \sim_\alpha b$$

and β is defined

$$* \sim_\beta c \quad \text{and} \quad * \sim_\beta d$$

Hence we see that the set-like elements of this Frobenius algebra naturally correspond with the partition $\{\{a, b\}, \{c, d\}\}$.

There are three ways that we can partition the set X as $X \cong \mathbb{Z}_2 \sqcup \mathbb{Z}_2$, which correspond with the three pairs of compatible measurements in Spekkens' Toy Theory. Coecke and Edwards use three Frobenius algebras corresponding with the three partitions on the set X which correspond with the pairs of compatible measurements. Hence we see each Frobenius algebra playing the role of measurement context, as represented in Figure 7.2.

However, such a choice of Frobenius algebra structure for a given partition is not unique; there are four Frobenius algebras which determine the same partition structure and which have the same set-like elements. These correspond with the following multiplication tables

$$\mu_1 = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline a & a & b & \emptyset & \emptyset \\ \hline b & b & a & \emptyset & \emptyset \\ \hline c & \emptyset & \emptyset & c & d \\ \hline d & \emptyset & \emptyset & d & c \end{array} \qquad \mu_2 = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline a & a & b & \emptyset & \emptyset \\ \hline b & b & a & \emptyset & \emptyset \\ \hline c & \emptyset & \emptyset & d & c \\ \hline d & \emptyset & \emptyset & c & d \end{array}$$

$$\mu_3 = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline a & b & a & \emptyset & \emptyset \\ \hline b & a & b & \emptyset & \emptyset \\ \hline c & \emptyset & \emptyset & c & d \\ \hline d & \emptyset & \emptyset & d & c \end{array} \qquad \mu_4 = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline a & b & a & \emptyset & \emptyset \\ \hline b & a & b & \emptyset & \emptyset \\ \hline c & \emptyset & \emptyset & d & c \\ \hline d & \emptyset & \emptyset & c & d \end{array}$$

This essentially reduces to the fact that there are two ways to endow a two element set $\{a, b\}$ with the structure of the group \mathbb{Z}_2 , and two ways to endow the set $\{c, d\}$ with the group structure \mathbb{Z}_2 , and hence four groupoids of the form $\mathbb{Z}_2 \sqcup \mathbb{Z}_2$ which partitions X in this way.

Remark 7.3.1. It has been observed that the precise abelian groupoid structure is not important when considering Frobenius algebras in **Rel** as observables [80], rather, what matters is how the groupoid partitions the underlying set. The four groupoids μ_1, μ_2, μ_3 and μ_4 are isomorphic as groupoids, and all partition X in the same way. This means they are different representations of the same observable in the underlying theory, and we call them *observationally equivalent*.

There is another way that we can see these groupoids as “the same”. For G a group, one can view the underlying set as a set equipped with the group action of G . This group action will satisfy some particularly nice properties: it will be both *transitive* and *free*. A group action

$$G \times X \xrightarrow{\quad \cdot \quad} X$$

is said to be *transitive* if for all $x, y \in X$ there exists $g \in G$ such that $g \cdot x = y$, and is said to be *free* if $g \cdot x = h \cdot x$ for all $x \in X$ implies $g = h$. A set equipped with a free and transitive action is called a *principle homogeneous space* or a *torsor*. Any group G acting on its own underlying set determines a torsor by taking the group action to be the group multiplication map. To recover the group structure from such a torsor amounts to picking a point in the set to act as the group identity. For example, consider the set $\{a, b\}$ equipped with the

group action by \mathbb{Z}_2 , where 0 is the identity, and 1 permutes a and b . This action is free and transitive, and hence this structure defines a torsor. To turn this torsor into a group isomorphic to \mathbb{Z}_2 requires choosing either a or b to be the group element 0. Hence, for the groupoid given by the disjoint union of groups $\mathbb{Z}_2 \sqcup \mathbb{Z}_2$ considered as a disjoint union of torsors, there are precisely four ways to pick identity elements to turn this disjoint union of torsors back into a groupoid. In this sense, moving from a group perspective to a torsor perspective can be seen as forgetting the identity elements of the groupoid.

Hence, as disjoint unions of groups, the four Frobenius algebras are different, however viewing them as disjoint unions of torsors – that is, if we forget the identity elements – then we see identical structures, and it is our claim that the torsor structure is the essential structure for the interpretation as observables.

Frobenius Algebras and Spectral Presheaves

Consider the Frobenius algebra isomorphic to $\mathbb{Z}_2 \sqcup \mathbb{Z}_2$ as above

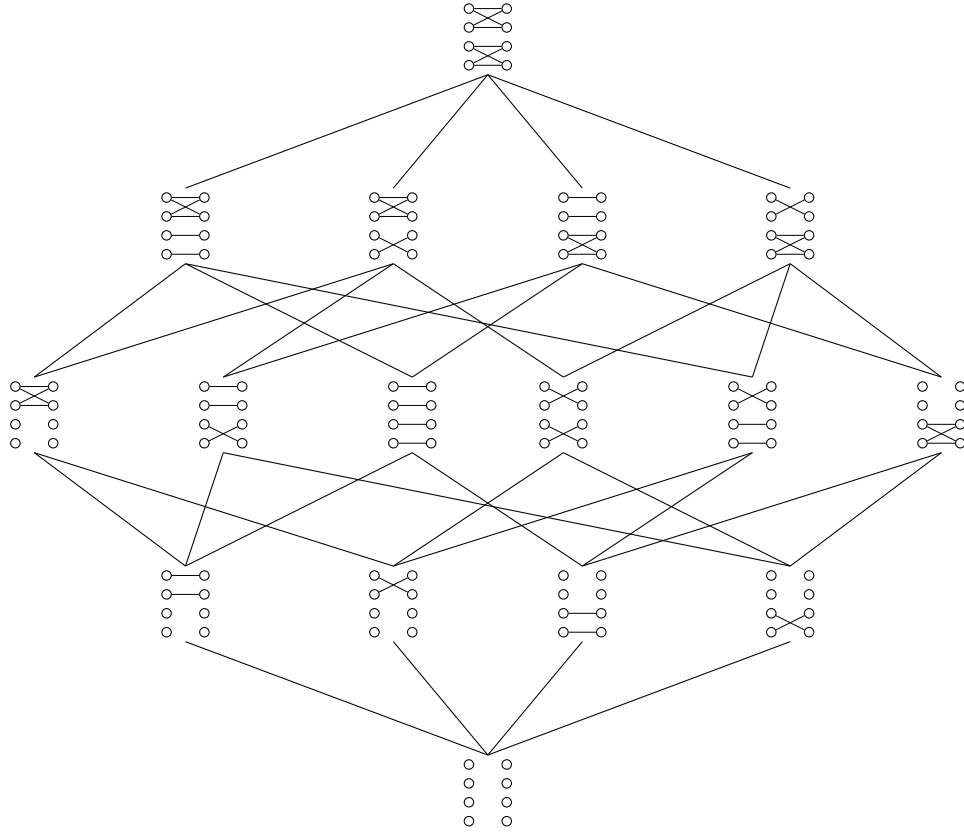
$$\mu = \text{---} \circlearrowleft \text{---}$$

and let \mathfrak{M} be the semialgebra consisting of those relations $R : X \rightarrow X$ of the form

$$R(\mu) = \left\{ \begin{array}{c} \triangle x \\ \text{---} \circlearrowleft \text{---} \\ \text{---} \end{array} \mid \text{for all } \begin{array}{c} \triangle x \\ \text{---} \uparrow \end{array} \right\}$$

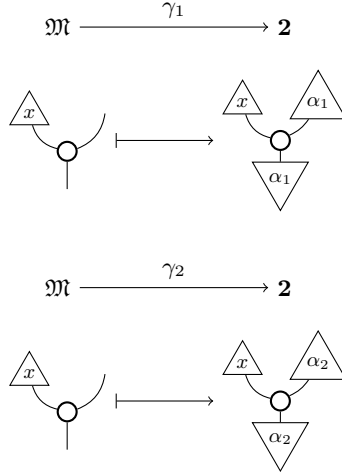
for $x \in X$. Recall, $R(\mu)$ is the commutative semialgebra *generated* by the Frobenius algebra μ_1 , as defined in Definition 3.3.4.

Note that the four Frobenius algebras we defined above, μ_1, μ_2, μ_3 and μ_4 , generate the **2**-semialgebra $\mathfrak{M} = R(\mu_1) = R(\mu_2) = R(\mu_3) = R(\mu_4)$, consisting of the following elements:



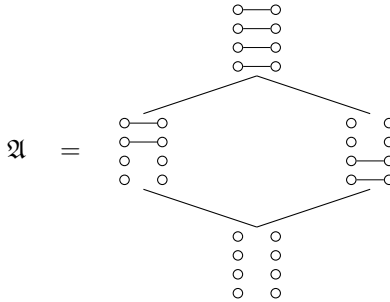
Remark 7.3.2. The fact that these four Frobenius algebras generate the same semialgebra is implicit in our discussion of these groupoids being equal as disjoint unions of torsors, that is, the way they act on the underlying set X is the same; the elements R_x can be viewed as actions on the set X .

By Theorem 3.3.7 the set-like elements of μ naturally form elements of the Gelfand spectrum $\text{Spec}_G(\mathfrak{M})$. In particular, the set-like elements of μ form a subset of the Gelfand spectrum $\text{Spec}_G(\mathfrak{M})$ according to the following correspondence: let α_1 and α_2 be the set-like elements of μ , then we can define the corresponding elements $\gamma_1, \gamma_2 \in \text{Spec}_G(\mathfrak{M})$ as follows



These elements, γ_1 and γ_2 , are all of the elements of $\text{Spec}_G(\mathfrak{M})$. This follows from the fact that homomorphisms into $\mathbf{2}$ are completely determined by how they act on the idempotents in \mathfrak{M} .

In Section 7.2 we used the semialgebra



to represent a measurement context. Notice that $R(\mu) = \mathfrak{M}$ contains the subsemialgebra $i: \mathfrak{Q} \hookrightarrow \mathfrak{M}$, and that these two algebras share the same subunital idempotents. Since elements of the spectrum are completely determined by their values on the subunital idempotents the map

$$\text{Spec}_G(\mathfrak{M}) \xrightarrow{i^*} \text{Spec}_G(\mathfrak{Q})$$

is an isomorphism, and hence the spectrum of \mathfrak{Q} encodes the same information

as the set-like elements of (X, μ) .

$$\{\text{Set-like elements of } \mu\} \cong \text{Spec}_G(\mathfrak{A})$$

We have seen that the Frobenius algebra (X, μ) generates a semialgebra \mathfrak{M} , which is related to the semialgebra \mathfrak{A} that we use in our model of Spekkens' Toy Theory, in the sense that $\mathfrak{A} \hookrightarrow \mathfrak{M}$. The respective spectra are essentially the same in that the corresponding $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{M})$ is an isomorphism, and these spectra are completely characterised by the set-like elements of (X, μ) .

In the following section we will explore the relationship between the phase group of the Frobenius algebra (X, μ) and the generalised phases of the semialgebras \mathfrak{M} and \mathfrak{A} , and we will discuss the interpretation of these group structures in modelling Spekkens' Toy Theory.

Generalised Phase Groups in Spekkens' Toy Theory

In Spekkens' Toy Theory there is a notion of *dynamics* of the underlying state, that is, possible ways in which the state evolves with time. Such dynamics are captured by a function or relation on the underlying ontic states $f : X \rightarrow X$. The *dynamics* of Spekkens' Toy Theory are those transformations compatible with the knowledge balance principle, that is, those which preserve the amount of information we know about the system. Spekkens shows that the transformations which preserve the amount of information are precisely the bijective functions $f : X \rightarrow X$ on the underlying set of ontic states [198, III. B]. In the category **Rel** the unitary maps are precisely the bijective functions. Recall, the unitaries are those isomorphisms $U : X \rightarrow X$ satisfying $UU^\dagger = \text{id}_X$, and we denote the set of unitaries on X by $\mathcal{U}(X) \subseteq \text{Hom}(X, X)$.

Definition 7.3.3. The *dynamics compatible with a measurement* $P_{\{x,y\}} : X \rightarrow X$ are those bijections $f : X \rightarrow X$ such that regardless of the true position of the particle p , measuring $P_{\{x,y\}}$ yields the same outcome as first applying f to the position of p and then measuring $P_{\{x,y\}}$.

For example, regardless of the underlying epistemic state, performing the measurement “is p in a or b ?” will have the same outcome as if one first applies the permutation (a, b) – swapping a and b , leaving c and d unchanged – to the position of p . Permuting (c, d) would similarly have no effect. However, permuting (a, c) could alter the observed outcomes; if for example the epistemic state is $\{a, b\}$ then applying the permutation (a, c) now puts the system in the epistemic state $\{b, c\}$. The measurement “is p in a or b ?” has deterministic outcome for the epistemic state $\{a, b\}$ – the answer is always “yes” – however, for the epistemic state $\{b, c\}$ we expect to only obtain the answer “yes” half of the time, and hence the bijection which swaps a and b is not compatible with the measurement “is p in a or b ?” in the sense of Definition 7.3.3.

We will now show how dynamics in Spekkens' Toy Theory are modelled in the spectral presheaf framework. Recall the phase presheaf, Definition 3.3.9

$$\mathbf{Rel-Alg}(X)^{\text{op}} \xrightarrow{\Phi} \mathbf{Set}$$

defined on objects

$$\Phi(\mathfrak{A}) = \{ U \in \mathcal{U}(X) \mid U \in \mathfrak{A}' \}$$

The following result connects the phase group object with the notion in Spekkens' Toy Theory of dynamics compatible with an observable (in the sense of Definition 7.3.3).

Theorem 7.3.4. *Let \mathfrak{A} be one of the semialgebras corresponding with the families of mutually compatible measurements of Spekkens' Toy Theory – as defined in Section 7.2. The elements of the generalised phase group $\Phi(\mathfrak{A})$ are precisely those dynamics compatible with the corresponding observable.*

Proof. It is straightforward to compute $\Phi(\mathfrak{A})$ directly:

$$\Phi(\mathfrak{A}) = \left\{ \begin{array}{cccc} \circ - \circ & \circ - \circ & \circ - \circ & \circ - \circ \\ \circ - \circ & \circ - \circ & \circ - \circ & \circ - \circ \\ \circ - \circ & \circ - \circ & \circ - \circ & \circ - \circ \\ \circ - \circ & \circ - \circ & \circ - \circ & \circ - \circ \end{array} \right\}$$

This is the group generated by the permutations (a, b) and (c, d) , which are the

bijections compatible with these measurements. □

Remark 7.3.5. Recall, Theorem 2.3.21 asserts the phase groups of internal algebra structures that represent observables in monoidal quantum theory are abelian. Note that the generalised phase group for \mathfrak{A} is isomorphic to the group to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, and is therefore abelian. We might assume that these generalised phase groups associated with commutative semialgebras are likewise abelian groups, however this is not the case. The generalised phase group of \mathfrak{A} is abelian because of the size of the underlying set. We can form a similar toy model in which a particle inhabits one of eight possible boxes but our measuring device can only achieve a resolution of at least four boxes. Let \mathfrak{X} be the commutative semialgebra generated by

$$\mathfrak{X} = \left\{ \begin{array}{cccc} \begin{array}{c} \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \end{array} & \begin{array}{c} \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \quad \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \end{array} \end{array} \right\}$$

The bijections compatible with these measurements are the permutations on the first four elements and the permutations on last set of four elements, or combinations thereof, and the the generalised phase group $\Phi(\mathfrak{X}) \cong S_4 \oplus S_4$, which is not abelian.

By Theorem 3.3.10 we have $\Phi(\mathfrak{M}) = G(\mu)$. Since \mathfrak{M} is a maximal commutative semialgebra, by Lemma 3.3.2 it satisfies $\mathfrak{M}' = \mathfrak{M}$ and hence $\Phi(\mathfrak{M})$ consists of precisely those unitary relations that belong to \mathfrak{M} , that is

$$\Phi(\mathfrak{M}) = \left\{ \begin{array}{cccc} \begin{array}{c} \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \end{array} & \begin{array}{c} \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \\ \circ \text{---} \circ \end{array} \end{array} \right\}$$

which is precisely the set $\Phi(\mathfrak{A})$, that is $\Phi(\mathfrak{M}) = \Phi(\mathfrak{A})$, and therefore the generalised phases of \mathfrak{A} correspond precisely with the phases of the Frobenius algebra (X, μ) .

Hence the Frobenius algebra representation of an observable for Spekkens' Toy Theory (X, μ) encodes all of the same information encoded by our algebra \mathfrak{A} :

the set-like elements of μ correspond precisely with elements of the generalised Gelfand spectrum $\text{Spec}_G(\mathfrak{A})$.

We can summarise the explicit advantages of the spectral presheaf model over the Frobenius algebra model

- The spectral presheaf framework is more expressive, in particular, we can give a finer representation of epistemic and ontic states in terms of operational states. This language allows for the integration of notions of hidden-variable and ontological models.
- There is some redundancy in Frobenius algebra representations in terms of the precise groupoid structure chosen. The passage from Frobenius algebra to semialgebra formally acknowledges this redundancy by equating those Frobenius algebras that are equivalent as observables.
- There is no a priori justification for the dynamics compatible with a particular observable Spekkens' Toy Theory to form an abelian group, and we saw in Remark 7.3.5 that one can easily construct similar models in which such dynamics would form nonabelian groups. A special unital \dagger -Frobenius algebra in **Rel** is commutative if and only if its phase group is abelian, and hence we would not be able to model such a theory using commutative Frobenius algebras. On the other hand, nonabelian generalised phase groups pose no problems for commutative semialgebras.

Chapter 8

Partial Results and Future Work

In this chapter we discuss how our framework can be extended to encompass other aspects of quantum theory, and we give some partial results.

Recall the fundamental aspects of quantum theory we listed in Figure 1.2. In the preceding chapters we have given a thorough account of the first three of these features, but we have yet to discuss the others. In this chapter we give a preliminary account of how our framework can incorporate unitary dynamics, uncertainty relations, and entanglement. We also discuss aspects of classical mechanics, in particular, Hamiltonian mechanics and how these features can be incorporated into our framework.

In Section 8.1 we discuss how one can incorporate Heisenberg's uncertainty relations and dynamics as characterised by unitary operators.

In Section 8.2 we outline one view on how to incorporate spatiotemporal structure into physical theories. We also give a preliminary account of how to describe composite systems and entanglement within the spectral presheaf framework.

In Section 8.3 we discuss quantization of classical systems and in particular

how this relates to POVMs and PVMs. We also discuss aspects of Hamiltonian mechanics which can be incorporated into our framework, and how this can be related to the notion of *quasi-quantization*, introduced by Spekkens [199]. We then propose a toy model – similar in spirit to Spekkens’ Toy Theory – which uses the language of classical mechanics, that is, Hamiltonian vector fields on Poisson manifolds.

8.1 Uncertainty Relations and Unitary Dynamics

In this section we show how to incorporate two of the fundamental aspects of quantum foundations we outlined in Figure 1.2: unitary dynamics; and Heisenberg’s uncertainty relations.

We consider dynamics by generalising unitary dynamics from the usual Hilbert space formalism, where one considers the group of unitaries acting on the state space.

We then attempt to capture something resembling the uncertainty relations of quantum theory, by refining the concept of *contextual entropy*, introduced by Constantin and Doering [55].

Unitary Dynamics: External and Internal Representations

The operational axioms we we gave in Figure 4.2 did not specify any notion of *dynamics*, that is, any description of how the physical system evolves with time. We will now give a brief account of how this can be achieved and interpreted in the spectral presheaf framework.

Typically, in quantum theory *dynamics* are represented by unitary operators $U : H \rightarrow H$ – as we saw in the axioms we presented in Figure 2.2. Recall, a unitary morphism in a \dagger -category \mathcal{A} is an isomorphism $U : X \rightarrow Y$ satisfying $U^{-1} = U^\dagger$. We denote the collection of unitaries $U : X \rightarrow X$ by $\mathcal{U}(X)$. It is easy to see that the set $\mathcal{U}(X)$ forms a group under morphism composition.

If the unitaries determine the dynamics of the physical system represented by the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}_{\text{vN}}(X)^{\text{op}}}$ then we would anticipate a group action of the form

$$\mathcal{U}(X) \times \mathbf{Set}^{\mathcal{A}\text{-Alg}_{\text{vN}}(X)^{\text{op}}} \xrightarrow{\tau} \mathbf{Set}^{\mathcal{A}\text{-Alg}_{\text{vN}}(X)^{\text{op}}}$$

and indeed we show that such an action – known as *conjugation* or the *twisting action* – can be defined in general. However, one can also represent the group $\mathcal{U}(X)$ internally as group object (Definition 3.3.11) in the category $\mathbf{Set}^{\mathcal{A}\text{-Alg}_{\text{vN}}(X)^{\text{op}}}$. We will see that in the case $\mathfrak{A} = \mathbf{Hilb}$ this internal representation can be used to give a nicer description of unitary dynamics.

The following theorem shows that the group action

$$\begin{aligned} \mathcal{U}(X) \times \text{Hom}(X, X) &\longrightarrow \text{Hom}(X, X) \\ (U, A) &\longmapsto UAU^\dagger \end{aligned}$$

lifts to an action on the poset of subsemialgebras.

Theorem 8.1.1. *There is a group action*

$$\begin{aligned} \mathcal{U}(X) \times \mathcal{A}\text{-Alg}_{\text{vN}}(X) &\longrightarrow \mathcal{A}\text{-Alg}_{\text{vN}}(X) \\ (U, \mathfrak{A}) &\longmapsto U\mathfrak{A}^{U^\dagger} = \{ UAU^\dagger \mid A \in \mathfrak{A} \} \end{aligned}$$

which preserves the partial order.

Proof. We need to check two things: firstly, that the set

$$U\mathfrak{A}^{U^\dagger} = \{ UAU^\dagger \mid A \in \mathfrak{A} \}$$

is a von Neumann \mathbb{S}^* -semialgebra; and secondly, that if $\mathfrak{B} \hookrightarrow \mathfrak{A}$ then

$$U\mathfrak{B}^{U^\dagger} \hookrightarrow U\mathfrak{A}^{U^\dagger}.$$

To see that $U\mathfrak{A}^{U^\dagger}$ is a von Neumann semialgebra we need to show that $(U\mathfrak{A}^{U^\dagger})'' = U\mathfrak{A}^{U^\dagger}$. To see this, it is enough to show that $(U\mathfrak{A}^{U^\dagger})' = U(\mathfrak{A}')^{U^\dagger}$.

Then, by applying this equation twice, we obtain $({}^U\mathfrak{A}^{U^\dagger})'' = {}^U({}^U\mathfrak{A}'')^{U^\dagger}$, which, by the assumption that \mathfrak{A} is a von Neumann semialgebra, is equal to ${}^U\mathfrak{A}^{U^\dagger}$.

Let $B \in ({}^U\mathfrak{A}^{U^\dagger})'$ – that is, $BUAU^\dagger = UAU^\dagger B$ for all $A \in \mathfrak{A}$. We need to show that $B \in {}^U({}^U\mathfrak{A}')^{U^\dagger}$, that is, $B = UCU^\dagger$, for some $C \in \mathfrak{A}'$. To see this let $C = U^\dagger BU$, then $B = UCU^\dagger$ and moreover for each $A \in \mathfrak{A}$ we have

$$\begin{aligned} CA &= CAU^\dagger U \\ &= U^\dagger BUAU^\dagger U \\ &= U^\dagger UAU^\dagger BU \\ &= AU^\dagger BU \\ &= AC \end{aligned}$$

and hence $C \in \mathfrak{A}'$, and therefore $B \in {}^U({}^U\mathfrak{A}')^{U^\dagger}$.

Conversely, suppose $B \in {}^U({}^U\mathfrak{A}')^{U^\dagger}$, that is $B = UCU^\dagger$ for $C \in \mathfrak{A}'$. We need to show that $B \in ({}^U\mathfrak{A}^{U^\dagger})'$. We have to show that $BD = DB$ for all $D \in {}^U\mathfrak{A}^{U^\dagger}$. Each $D \in {}^U\mathfrak{A}^{U^\dagger}$ is of the form $D = UAU^\dagger$ for some $A \in \mathfrak{A}$ and hence we have

$$\begin{aligned} BD &= BUAU^\dagger \\ &= UCU^\dagger UAU^\dagger \\ &= UCAU^\dagger \\ &= UACU^\dagger \\ &= UAU^\dagger UCU^\dagger \\ &= UAU^\dagger B \\ &= DB \end{aligned}$$

and hence $B \in ({}^U\mathfrak{A}^{U^\dagger})'$.

The subset inclusion ${}^U\mathfrak{B}^{U^\dagger} \hookrightarrow {}^U\mathfrak{A}^{U^\dagger}$ is trivial. \square

The twisting action is an external representation of the group of unitaries. There is an obvious representation the group of unitaries internally by considering

the group object $C_{\mathcal{U}(X)}$, the constant presheaf on the group $\mathcal{U}(X)$.

The following definition shows that the group action of Theorem 8.1.1 from the previous theorem can be lifted to the level of presheaves, resulting in *twisted presheaves*, which have been studied within the context of topos quantum theory [82, Definition 14.7].

Definition 8.1.2. The group action described in Theorem 8.1.1 lifts to the *twisting action*

$$\mathcal{U}(X) \times \mathbf{Set}^{\mathcal{A}\text{-Alg}_{\mathbb{V}\mathbb{N}}(X)^{\text{op}}} \xrightarrow{\tau} \mathbf{Set}^{\mathcal{A}\text{-Alg}_{\mathbb{V}\mathbb{N}}(X)^{\text{op}}}$$

where for each presheaf \mathcal{F} and unitary $U : X \rightarrow X$ we define the *twisted presheaf*

$$\begin{aligned} \mathcal{A}\text{-Alg}_{\mathbb{V}\mathbb{N}}(X)^{\text{op}} &\xrightarrow{\tau(U, \mathcal{F})} \mathbf{Set} \\ \mathfrak{A} &\longmapsto \mathcal{F}(U\mathfrak{A}U^\dagger) \end{aligned}$$

The fact that the twisting action is an external action is somewhat unsatisfactory; we would like to understand dynamics internally. Looking at what the twisting action does to the presheaf Spec_G we see that rather than acting locally, that is, rather than having at each \mathfrak{A} a group action

$$G \times \text{Spec}_G(\mathfrak{A}) \longrightarrow \text{Spec}_G(\mathfrak{A})$$

natural in \mathfrak{A} – seen as a natural family of locally-acting dynamics at each classical context – the twisting action instead shuffles the contexts, which is group action acting globally. Work has been done by Flori [83], see also [82, Chap. 16], in resolving this picture for the presheaf Spec_G . It might be possible to find a different, or perhaps complementary resolution of this problem for the presheaf Ψ , by *internalising* the action on operational states.

By Theorem 4.2.4 since every global section of Ψ is of the form \tilde{q} for some density operator $q : H \rightarrow H$ we can define the group action

$$\begin{aligned} \mathcal{U}(H) \times \Gamma(\Psi) &\longrightarrow \Gamma(\Psi) \\ (U, \hat{q}) &\longmapsto \widetilde{UqU^\dagger} \end{aligned}$$

Recall Definition 3.3.11, the definition of a group object in a category. One can similarly generalise group actions on sets to group objects acting on objects for an arbitrary category with finite products – see, for example, [155, Chap. V. §6].

Definition 8.1.3. For \mathcal{C} a category with finite products and (G, m, e) a group object in \mathcal{C} . A G -object in \mathcal{C} consists of an object X in \mathcal{C} equipped with an *action morphism*

$$G \times X \xrightarrow{a} X$$

such that the diagrams

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times \text{id}_G} & G \times X \\ \text{id}_G \times a \downarrow & & \downarrow a \\ G \times X & \xrightarrow{a} & X \end{array} \quad \begin{array}{ccc} X & \xrightarrow{\langle e, \text{id}_X \rangle} & G \times X \\ & \searrow \text{id}_X & \downarrow a \\ & & X \end{array}$$

commute.

Rather than having a set of states and a group of dynamics acting on that set, instead we want to find a group object G in $\mathbf{Set}^{\mathcal{A}\text{-Alg}_{\vee\mathbb{N}}(X)^{\text{op}}}$ representing dynamics which acts on the object representing the state space – that is Ψ seen as a G -object in the sense of Definition 8.1.3, that is for each classical context \mathfrak{A} we will have a group action

$$G(\mathfrak{A}) \times \Psi(\mathfrak{A}) \longrightarrow \Psi(\mathfrak{A}) \tag{8.1}$$

The phase presheaf we saw already in Definition 3.3.9 can be seen as an action of the form of (8.1) but where for each \mathfrak{A} the group action $\Psi(\mathfrak{A}) \times \Psi(\mathfrak{A}) \rightarrow \Psi(\mathfrak{A})$ is the trivial action. Hence, one feature of the desired a group object G representing

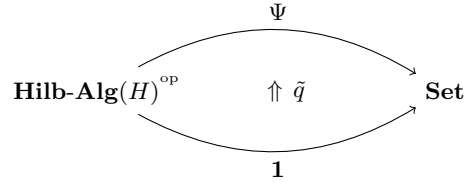
dynamics it that the phase presheaf forms a subobject $\Psi \rightarrow G$ such that Ψ can be characterised as a the *stabiliser object* of the action (8.1), which generalises the stabiliser of a group action on a set.

Understanding the nature of the generalised phases, and how they act on the state space is particularly interesting given the relationship between phase group structure and non-locality [45].

Uncertainty Relations and Entropy

In this section we refine the concept of *contextual entropy* introduced by Constantin and Doering [55]. We use contextual entropy in the pursuit of something resembling Heisenberg’s uncertainty relations.

In this section we consider only finite-dimensional Hilbert spaces. For a finite-dimensional Hilbert space H , every operational state



determines a family of probability distributions

$$\tilde{q}_{\mathfrak{A}} \in \Psi(\mathfrak{A})$$

where, $\tilde{q}_{\mathfrak{A}}$ is a probability distribution on the set $\text{Spec}_G(\mathfrak{A})$. Associated with any finite probability distribution is the *Shannon entropy* of that distribution [193].

Definition 8.1.4. The *Shannon entropy* associated with a finite probability distribution $d : X \rightarrow [0, 1]$ is

$$\mathbb{H}(d) = - \sum_{x \in X} d(x) \ln(d(x))$$

The Shannon entropy is maximised when d is the uniform distribution on X

and is minimised when d is a point-distribution, in which case $\mathbb{H}(d) = 0$.

Probability distributions are a special case of probability measures and hence we can apply the definition of pushforward measure – Definition 4.1.3 – to probability distributions; this gives us the language with which to describe *information loss*.

Definition 8.1.5. Let (X, d) and (Y, e) be a pair of sets equipped with probability distributions. Let $f : X \rightarrow Y$ be a map such that e is the pushforward distribution of d along f , that is, $d(f^{-1}(y)) = e(y)$ for all $y \in Y$. The *information loss* associated with the map of probability distributions $f : (X, d) \rightarrow (Y, e)$ is defined

$$\mathbb{H}(d) - \mathbb{H}(e)$$

We note that Shannon entropy can be completely characterised in terms of *information loss* [15], in particular, for every probability distribution (X, d) there is a map to the distribution which consists of the point distribution on a one-element set $(\{*\}, t)$. This point distribution has minimised Shannon entropy $\mathbb{H}(t) = 0$ and hence the Shannon entropy of (X, d) is equal to the information loss associated with the unique map $f : (X, d) \rightarrow (\{*\}, t)$.

A global section of Ψ consists of a family of probability distributions indexed by \mathfrak{A} , and hence for a given operational state ψ , then to each context we have an associated Shannon entropy for the distribution $\psi_{\mathfrak{A}} \in \Psi(\mathfrak{A})$. Hence we obtain a family of entropy values which vary according to context.

Definition 8.1.6. Consider an operational state

$$\begin{array}{ccc}
 & \Psi & \\
 \text{Hilb-Alg}(H)^{\text{op}} & \xrightarrow{\quad} & \text{Set} \\
 & \uparrow \psi & \\
 & \mathbf{1} &
 \end{array}$$

The *contextual entropies* of ψ are the Shannon entropies $\mathbb{H}(\psi_{\mathfrak{A}})$ of each $\psi_{\mathfrak{A}} \in \Psi(\mathfrak{A})$.

Remark 8.1.7. Definition 8.1.6 is equivalent to the definition of contextual entropy Constantin and Doering [55]. To see this, recall the correspondence between elements of $\text{Spec}_G(\mathfrak{A})$ and the primitive projections in \mathfrak{A} for finite-dimensional C^* -algebras – Lemma 2.2.17.

There is a notion of entropy associated with density operators, namely, *von Neumann entropy*.

Definition 8.1.8. For $q : H \rightarrow H$ a density operator, the *von Neumann entropy* associated with q is

$$\mathbb{S}(q) = -\text{tr}(q \ln(q))$$

where $\ln(q)$ is the matrix logarithm.

The von Neumann entropy is related to Shannon entropy in the following way: consider the diagonalisation $q = \sum \lambda_i |\lambda_i\rangle\langle\lambda_i|$, the von Neumann entropy is computed as $\mathbb{S}(q) = -\sum_i \lambda_i \ln(\lambda_i)$ which is clearly just the Shannon entropy on the probability distribution defined by the eigenvalues λ_i .

Constantin and Doering characterise the relationship between contextual entropy and von Neumann entropy [55].

Theorem 8.1.9. *Let \tilde{q} be the operational state corresponding with the density operator $q : H \rightarrow H$. The von Neumann entropy $\mathbb{S}(q)$ of q is equal to the minimum value $\mathbb{H}(\tilde{q}_{\mathfrak{A}})$ attains across all maximal contexts \mathfrak{A} .*

Theorem 8.1.9 allows us to characterise *pure states* in terms of contextual entropy.

Definition 8.1.10. An operational state ψ is *pure* if the minimum value $\mathbb{H}(\psi_{\mathfrak{A}})$ attains across all maximal contexts is zero.

Given an operational state ψ then for each $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ the corresponding Shannon entropies obey

$$\mathbb{H}(\psi_{\mathfrak{A}}) \leq \mathbb{H}(\psi_{\mathfrak{B}})$$

and because for $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ the probability distribution $\psi_{\mathfrak{B}}$ is the pushforward distribution of $\psi_{\mathfrak{A}}$ along $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$ the value $\mathbb{H}(\psi_{\mathfrak{A}}) - \mathbb{H}(\psi_{\mathfrak{B}})$

characterises the information loss associated with coarse-graining of classical contexts.

Definition 8.1.11. For $i : \mathfrak{B} \hookrightarrow \mathfrak{A}$ we call the information loss associated with the map $i^* : \text{Spec}_G(\mathfrak{A}) \rightarrow \text{Spec}_G(\mathfrak{B})$ the *classical information loss*.

However, even for a pair of contexts \mathfrak{A} and \mathfrak{C} which are not related by inclusion, we want to understand how the probabilities associated with measurement outcomes $\psi_{\mathfrak{A}}$ and $\psi_{\mathfrak{C}}$ are related. This can be achieved by viewing a relation on the respective contextual entropies $\mathbb{H}(\psi_{\mathfrak{A}})$ and $\mathbb{H}(\psi_{\mathfrak{B}})$. For example, suppose \mathfrak{A} and \mathfrak{C} correspond with the bases $\{|0\rangle, |1\rangle\}$ and $\{|+\rangle, |-\rangle\}$ respectively. Heisenberg's uncertainty relations require that perfect knowledge in the first context, means total uncertainty in the second.

We now define the map

$$\mathbf{Hilb-Alg}(H) \times \mathbf{Hilb-Alg}(H) \xrightarrow{\Xi} \mathbb{R}^+$$

Where $\Xi(\mathfrak{A}, \mathfrak{B})$ is defined

$$\Xi(\mathfrak{A}, \mathfrak{B}) = \min\{ \mathbb{H}(\tilde{q}_{\mathfrak{A}}) + \mathbb{H}(\tilde{q}_{\mathfrak{B}}) \mid q : H \rightarrow H \}$$

that is, $\Xi(\mathfrak{A}, \mathfrak{B})$ is the minimum value attained by the sum of Shannon entropies.

We call $\Xi(\mathfrak{A}, \mathfrak{B})$ the *uncertainty relation* between contexts \mathfrak{A} and \mathfrak{B} . The uncertainty relation between \mathfrak{A} and \mathfrak{B} is the least value such that

$$\mathbb{H}(\tilde{q}_{\mathfrak{A}}) + \mathbb{H}(\tilde{q}_{\mathfrak{B}}) \geq \Xi(\mathfrak{A}, \mathfrak{B})$$

for all density operators q . This uncertainty relation bears a resemblance to the *entropic uncertainty* of Maassen and Uffink [153], or the notion of *fidelity* characterised as a minimising condition over POVMs, see for example, Nielsen and Chuang [175, Chap. 9].

If \mathfrak{A} and \mathfrak{B} are maximal contexts their primitive projections correspond with orthonormal bases of H . Now suppose those bases are disjoint, that is, have

no elements in common, then we conclude that $\Xi(\mathfrak{A}, \mathfrak{B}) > 0$. To see this, let q be a density operator such that $\mathbb{H}(\tilde{q}_{\mathfrak{A}}) = 0$. Then q must be a one dimensional projector that belongs to \mathfrak{A} . Then clearly it is not possible for $\mathbb{H}(\tilde{q}_{\mathfrak{B}}) = 0$ as this would imply the one-dimensional projector q also belongs to \mathfrak{B} .

Recall, Lemma 2.2.17 which states that for H a finite dimensional Hilbert space, and \mathfrak{A} a commutative C^* -subalgebra of $\text{Hom}(H, H)$ the elements of $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ are in natural correspondence with the primitive projections in \mathfrak{A} . If \mathfrak{A} is a maximal commutative subalgebra then the primitive projections are one-dimensional, and correspond with an orthogonal basis of H . Future work will show the relationship between our relation Ξ and *mutually unbiased bases*, introduced by Schwinger [190]. In particular we conjecture that the largest possible value for $\Xi(\mathfrak{A}, \mathfrak{B})$ is attained when \mathfrak{A} and \mathfrak{B} correspond with mutually unbiased bases.

Remark 8.1.12. Mutually unbiased bases have been studied within the context of monoidal quantum theory in the form of *complementary observables*, see [42] or [113]. Complementarity or the concept of mutually unbiased bases has a strong information-theoretic flavour, while *strong complementarity* has a strong group theoretic flavour [42, 70].

We will give a complete characterisation of those contexts for which $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$, but first we note the following theorem which suggests that contexts related in a “classical” way satisfy $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$.

Theorem 8.1.13. *If there exists \mathfrak{C} such that $\mathfrak{A} \hookrightarrow \mathfrak{C}$ and $\mathfrak{B} \hookrightarrow \mathfrak{C}$ then $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$. In particular, if $\mathfrak{B} \hookrightarrow \mathfrak{A}$ then $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$.*

Proof. Suppose there exists such a \mathfrak{C} . We can pick a density operator q such that q determines a point-distribution on \mathfrak{C} . For any point distribution on $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ the pushforward distribution on $\text{Spec}_{\mathbb{C}}(\mathfrak{B})$ will also be a point distribution. Hence, the pushforward distribution of $\tilde{q}_{\mathfrak{A}}$ on $\text{Spec}_{\mathbb{C}}(\mathfrak{C})$ is also a point distribution, and

thus $\mathbb{H}(\tilde{q}_{\mathfrak{C}}) = 0$. Hence we have

$$\begin{aligned}\Xi(\mathfrak{A}, \mathfrak{B}) &= \mathbb{H}(\tilde{q}_{\mathfrak{A}}) + \mathbb{H}(\tilde{q}_{\mathfrak{B}}) \\ &= 0\end{aligned}$$

as required. \square

We can refine Theorem 8.1.13 to give a complete characterisation of the condition $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$.

Theorem 8.1.14. *For a pair of contexts $\mathfrak{A}, \mathfrak{B}$ we have $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$ if and only if there exists a one-dimensional projector P such that $P \in \mathfrak{A}'$ and $P \in \mathfrak{B}'$. If \mathfrak{A} and \mathfrak{B} are maximal contexts, then we have $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$ if and only if \mathfrak{A} and \mathfrak{B} have a one-dimensional projection in common;*

We can relate this relation Ξ to Heisenberg's uncertainty relation which we give here in a generalised form as presented by Isham [123, §7.3], which is expressed in terms of *dispersion*.

Definition 8.1.15. Define the *dispersion* $\partial_{\psi}A$ of an operator A for a given $|\psi\rangle \in H$

$$\partial_{\psi}(A) = (\langle\psi|A^2|\psi\rangle - \langle\psi|A|\psi\rangle^2)^{\frac{1}{2}}.$$

Heisenberg's uncertainty relation [123, §7.3] for operators A and B is characterised by the inequality

$$\partial_{\psi}(A) \partial_{\psi}(B) \geq \frac{1}{2} |\langle\psi|AB|\psi\rangle - \langle\psi|BA|\psi\rangle|$$

The following result shows some parallels between Ξ and Heisenberg's uncertainty relation.

Theorem 8.1.16. *The map Ξ is related to dispersion in the following ways:*

1. $\mathbb{H}(\psi_{\mathfrak{A}}) = 0$ if and only if $\partial_{\psi}(A) = 0$ for all $A \in \mathfrak{A}$;

2. for a pair of pure states ψ, ϕ we have

$$\mathbb{H}(\psi_{\mathfrak{A}}) \leq \mathbb{H}(\phi_{\mathfrak{A}})$$

if and only if

$$\partial_{\psi}(A) \leq \partial_{\phi}(A) \text{ for all } A \in \mathfrak{A}$$

3. For \mathfrak{A} and \mathfrak{B} a pair of contexts, we have $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$ if and only if there exists $|\psi\rangle \in H$ such that for all $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$

$$\partial_{\psi}(A) \partial_{\psi}(B) = 0$$

Note that we have only defined Ξ in the way we have because of its resemblance to dispersion. The formula $\mathbb{H}(\psi_{\mathfrak{A}}) + \mathbb{H}(\psi_{\mathfrak{B}})$ is similar to the formula describing *mutual information* of $\mathbb{I}(\mathfrak{A}, \mathfrak{B})$, see for example [215, Chap. 2], computed as

$$\mathbb{I}(\psi_{\mathfrak{A}}, \psi_{\mathfrak{B}}) = \mathbb{H}(\psi_{\mathfrak{A}}) + \mathbb{H}(\psi_{\mathfrak{B}}) - \mathbb{H}(\psi_{\mathfrak{A}}|\psi_{\mathfrak{B}})$$

where $\mathbb{H}(\psi_{\mathfrak{A}}|\psi_{\mathfrak{B}})$ denotes *relative entropy* [215, Chap. 2]. This suggests that it might be possible to import definitions and techniques from the traditional information theory into the context of the spectral presheaf framework.

Future work will give a full characterisation of Heisenberg's uncertainty principle in terms of spectral presheaves and contextual entropy.

We believe that classical information loss in the sense defined in Definition 8.1.11, is – as the name suggests – not a quantum phenomenon, while having non-zero values for the relation $\Xi(\mathfrak{A}, \mathfrak{B})$ reflects a quantum property. We justify this claim by considering operational states in Spekkens' Toy Theory – an essentially classical theory. In Spekkens' Toy Theory one can observe non-trivial information loss coarse-graining contexts, however, $\Xi(\mathfrak{A}, \mathfrak{B}) = 0$ for all measurement contexts.

Theorem 8.1.17. For all $\mathfrak{A}, \mathfrak{B} \in \mathbf{Rel-Alg}_{\mathbf{vN}}(X)$

$$\Xi(\mathfrak{A}, \mathfrak{B}) = 0$$

8.2 Spatiotemporal Structure

In our discussion of Bell’s theorem and ontological models we emphasised the point that we were making no assumptions about spatiotemporal structure in our models, and composite systems made up from two or more subsystems played very little role in our arguments. The traditional argument of Bell’s theorem makes use of spacelike separated measurement sites, invoking principles of special relativity in proving that theorem. Abramsky and Brandenburger have shown [5, Proposition 9.2] that no-signalling is not a property that intrinsically has anything to do with spacelike separation, a result which we improved to include infinite dimensions – Theorem 4.1.29.

Having made a concerted effort to completely ignore all notions of space and time from our model we will now discuss ways in which it could be re-incorporated at the categorical level.

In particular, we show how one can generalise the spectral presheaf approach further for a broader class of categories than the \dagger -symmetric monoidal categories we have so far considered.

There have been other attempts at incorporating spatiotemporal structure into categorical quantum mechanics, for example, using the \dagger -symmetric monoidal category of Hilbert modules [119, 165].

Premonoidal Categories and Spacetime

Note that we use very little of the monoidal structure of the category \mathcal{A} in defining $\mathcal{A}\text{-Alg}(X)$. It is the assumption of \dagger -biproducts that gives each set $\text{Hom}(X, X)$ its additive structure, and the compositional structure is simply morphism composition. The monoidal structure is required for defining the commutative semiring of scalars and the corresponding scalar action. However, in showing this we only need to take the tensor product of the monoidal unit with other objects, using the fact that $I \otimes X \cong X$; we never make use of the fact that we can take tensor products of *arbitrary* objects $X \otimes Y$. We are not making full use of the monoidal structure, and in fact we could apply all of the

same definitions and constructions of Chapter 3 to a broader class of categories, for example, *premonoidal categories* [184].

Definition 8.2.1. A *binoidal category* is a category \mathcal{C} equipped with the following:

1. for each pair of objects X and Y , an object $X \otimes Y$;
2. for each object X a functor $X \rtimes - : \mathcal{C} \rightarrow \mathcal{C}$ where the action on objects is $X \rtimes Y = X \otimes Y$;
3. and for each object X a functor $- \rtimes X : \mathcal{C} \rightarrow \mathcal{C}$ where the action on objects is $Y \rtimes X = Y \otimes X$.

A morphism $f : X \rightarrow Y$ in a binoidal category is said to be *central* if for all $g : W \rightarrow Z$ the diagram

$$\begin{array}{ccc}
 X \otimes W & \xrightarrow{X \rtimes g} & X \otimes Z \\
 \downarrow f \rtimes W & & \downarrow f \rtimes Z \\
 Y \otimes W & \xrightarrow{Y \rtimes g} & Y \otimes Z
 \end{array}$$

commutes.

Definition 8.2.2. A *premonoidal category* consists of a binoidal category equipped with

1. a *unit* object I ;
2. for each triple X, Y, Z of objects a central isomorphism

$$(X \otimes Y) \otimes Z \xrightarrow{\alpha_{X,Y,Z}} X \otimes (Y \otimes Z)$$

such that the pentagon diagram from the definition of a monoidal category is satisfied;

3. for each object X a pair of central isomorphisms

$$I \otimes A \xrightarrow{\lambda_A} A \qquad A \otimes I \xrightarrow{\rho_A} A$$

such that the triangle diagram from the definition of a monoidal category commutes.

Premonoidal categories are known to be related to the concept of control flow graphs in program analysis [126, 12], have been considered in the context of quantum theory with the introduction of *von Neumann categories* [24] for the purpose of formulating *Einstein causality* in a categorical way.

The monoidal categories we considered earlier have biproducts that are distributive with respect to the monoidal structure, Definition 2.3.7. This notion can be generalised for premonoidal categories [184, Definition 3.9].

Definition 8.2.3. A premonoidal category with finite biproducts is said to be *predistributive* if for each object $X \in \mathcal{C}$ the functors $X \times - : \mathcal{C} \rightarrow \mathcal{C}$ and $- \times X : \mathcal{C} \rightarrow \mathcal{C}$ preserve biproducts.

For a premonoidal category with finite predistributive \dagger -biproducts, then since the unit isomorphisms λ and ρ from Definition 8.2.2 are assumed to be central, and therefore set $\mathbb{S} = \text{Hom}(I, I)$ has the structure of a commutative involutive semiring. This is by exactly the same argument for \dagger -symmetric monoidal categories [133, Proposition 6.1], since this result relies not on bifactoriality, but the centrality of the unit isomorphisms. For the same reason, each set $\text{Hom}(X, X)$ has the structure of an \mathbb{S}^* -semialgebra, and hence for any such category \mathcal{C} and a fixed object X , we can define the category $\mathcal{C}\text{-Alg}(X)$.

We want to use the premonoidal structure to encode a sense of spacelike separation, and the notion of one event causally preceding another [177, §2].

Definition 8.2.4. A morphism $h : X \otimes Y \rightarrow X \otimes Y$ in a premonoidal category \mathcal{C} is said to be *locally-acting* if it is of the form $f \times Y$ or $X \times g$.

We think of locally-acting morphisms as processes that only affect one part

of the system. Consider the following diagram:

$$\begin{array}{ccc}
 X \otimes W & \xrightarrow{X \times g} & X \otimes Z \\
 \downarrow f \times W & & \downarrow f \times Z \\
 Y \otimes W & \xrightarrow{Y \times g} & Y \otimes Z
 \end{array}$$

We can think of each path traceable in this diagram as performing the processes f and g in parallel, however, the order in which we perform these parallel processes occur matters; there might be some causal connection between the sites. For this diagram to commute implies that the processes f and g do not affect one another. This idea motivates the following definition.

Definition 8.2.5. For \mathcal{C} a premonoidal category with $f : X \rightarrow Y$ and $g : W \rightarrow Z$ a pair of morphisms in \mathcal{C} , if the diagram

$$\begin{array}{ccc}
 X \otimes W & \xrightarrow{X \times g} & X \otimes Z \\
 \downarrow f \times W & & \downarrow f \times Z \\
 Y \otimes W & \xrightarrow{Y \times g} & Y \otimes Z
 \end{array}$$

commutes then we say that f and g are *separated*.

Theorem 8.2.6. Let $f \times Y : X \otimes Y \rightarrow X \otimes Y$ and $X \times g : X \otimes Y \rightarrow X \otimes Y$ be a pair of locally-acting operators. If there is a classical context $\mathfrak{A} \subseteq \text{Hom}(X \otimes Y, X \otimes Y)$ containing $f \times Y$ and $X \times g$ then f and g are separated.

Proof. Suppose that $f \times Y \in \mathfrak{A}$ and $X \times g \in \mathfrak{A}$ for a pair of locally-acting operators $f : X \rightarrow X$, and $g : Y \rightarrow Y$. Since \mathfrak{A} is a commutative semialgebra we

see that the diagram

$$\begin{array}{ccc}
 X \otimes Y & \xrightarrow{X \times g} & X \otimes Y \\
 f \times Y \downarrow & & \downarrow f \times Y \\
 X \otimes Y & \xrightarrow{X \times g} & X \otimes Y
 \end{array}$$

commutes, and hence f and g are separated. □

That is, if measurements belong to a classical context on a composite system, they must be separated. So while the whole system may have strange causal behaviour, each classical context does not allow such behaviour.

Based on this result we will sketch a definition for a premonoidal category for which we interpret morphisms as quantum mechanical processes, and we interpret the notion of separation (Definition 8.2.5) as *spacelike separation*.

Note that Definition 8.2.7 is not a full definition of a particular premonoidal category, just a set of criteria that we would want such a structure to exhibit. There are numerous ways one could formulate a full definition which we will not go into here. The point we want to make here is that the spectral presheaf framework we have developed could be formulated in terms of premonoidal categories.

Definition 8.2.7. Let \mathcal{C} be a premonoidal \dagger -category with finite distributive \dagger -biproducts in which the underlying objects are the same as **Hilb**. Let M be a space-time [177, Definition 1.1].

A premonoidal space-time for consists of:

1. a forgetful functor $U : \mathcal{C} \rightarrow \mathbf{Hilb}$, that is, underlying any process $f : H \rightarrow K$ in \mathcal{C} there is a bounded linear operator $U(f) : H \rightarrow K$;
2. the assignment of a *causal past*, to every process $f : H \rightarrow K$ in \mathcal{C} ;
3. if two processes $f : X \rightarrow Y$ and $g : W \rightarrow Z$ lie completely outwith one

another's light cones then the diagram

$$\begin{array}{ccc}
 X \otimes W & \xrightarrow{X \times g} & X \otimes Z \\
 f \times W \downarrow & & \downarrow f \times Z \\
 Y \otimes W & \xrightarrow{Y \times g} & Y \otimes Z
 \end{array}$$

commutes, and

$$U((f \times Z) \circ (X \times g)) = f \otimes g$$

that is, the underlying linear operator is the usual tensor product of linear operators in **Hilb**.

By Theorem 8.2.6, such a premonoidal category would be amenable to the techniques to the spectral presheaf framework that we have presented, and could be useful for refining the concepts of spacelike separated measurement sites in, for example, Bell's theorem [19].

Composite Systems

In Chapters 4, 5 and 6 we focussed mainly on isolated systems for which being able to write an operator in a tensor form

$$H \otimes K \xrightarrow{A \otimes B} H \otimes K$$

has no intrinsic physical meaning. Obviously operators of this form are interesting from the perspective of quantum theory.

Note that while for Hilbert spaces H and K there is no canonical linear operator $K \rightarrow H \otimes K$, there is a C^* -algebra homomorphism

$$\begin{array}{ccc}
 \text{Hom}(H, H) & \xrightarrow{\alpha} & \text{Hom}(H \otimes K, H \otimes K) \\
 A & \longmapsto & A \otimes \text{id}_K
 \end{array}$$

and this map α induces a map on the posets of C^* -subalgebras, which we also denote α

$$\begin{aligned} \mathbf{Hilb-Alg}(K) &\xrightarrow{\alpha} \mathbf{Hilb-Alg}(H \otimes K) \\ \mathfrak{A} &\longmapsto \alpha(\mathfrak{A}) = \{ A \otimes \text{id}_K \mid A \in \mathfrak{A} \} \end{aligned}$$

This map of posets induces a functor (in fact, a geometric morphism of toposes)

$$\mathbf{Set}^{\mathbf{Hilb-Alg}(H \otimes K)^{\text{op}}} \xrightarrow{\alpha^*} \mathbf{Set}^{\mathbf{Hilb-Alg}(K)^{\text{op}}}$$

which is defined by restriction. It is easy to check that $\alpha^*(\Psi) = \Psi$ and $\alpha^*(\mathbf{1}) = \mathbf{1}$, and hence α^* maps operational states in the left hand category to operational states in the right hand category. The following theorem gives a characterisation of this transformation of operational states, in particular it essentially reduces to taking the *partial trace*.

Theorem 8.2.8. *For an operational state*

$$\begin{array}{ccc} & \Psi & \\ & \curvearrowright & \\ \mathbf{Hilb-Alg}(H \otimes K)^{\text{op}} & \uparrow \tilde{q} & \mathbf{Set} \\ & \curvearrowleft & \\ & \mathbf{1} & \end{array}$$

corresponding with the density operator

$$H \otimes K \xrightarrow{q} H \otimes K$$

the image of \tilde{q} under the inverse image of the geometric morphism α^ is an operational state and corresponds with the partial trace of q , where H is traced out*

$$K \xrightarrow{\text{tr}(q)_H} K$$

With Theorem 8.2.8 we see the partial trace emerging from the extremely general structure of operational states. In Section 8.1 we saw that we can define *purity* on an abstract level, Theorem 8.2.8 allows us to talk about *purification*.

Note that the functor α^* forms part of a *geometric morphism* of toposes – generally regarded as the right notion of morphism between toposes. This perhaps points to a perspective on composite quantum systems and entanglement in terms of geometric morphisms between the respective toposes. This is perhaps an avenue where we can bring more essentially topos-theoretic constructions to bear on quantum theory. In particular, we would like a more sophisticated notion of no-signalling which incorporates a spatiotemporal structure.

Suppose we have a pair of systems, system A and system B , represented by Hilbert spaces H_A and H_B , and we consider the composite system represented by $H_A \otimes H_B$. Alice can perform measurements on system A , and Bob can perform measurements on the system B .

The pair of functors α^* and β^*

$$\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H_A)^{\text{op}}} \xleftarrow{\alpha^*} \mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H_A \otimes H_B)^{\text{op}}} \xrightarrow{\beta^*} \mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H_B)^{\text{op}}}$$

allow us to reproduce the kind of schematic depicted in Figure 8.1.

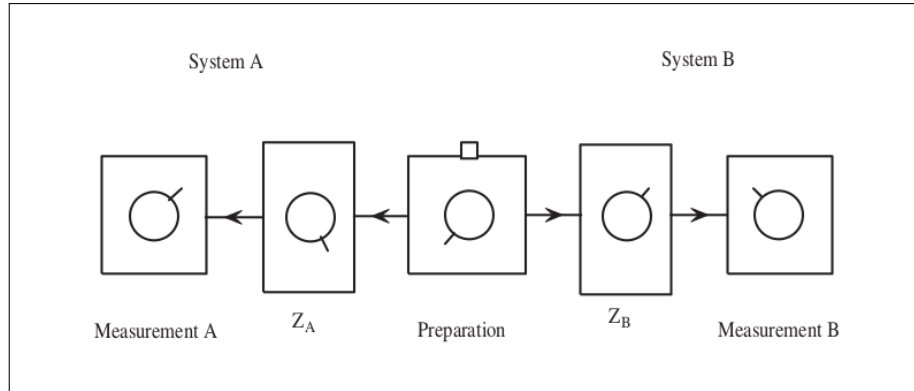


Figure 8.1: A schematic of a bipartite laboratory set-up. Image taken from [102, Fig. 2].

If we prepare an operational state on the composite system $H_A \otimes H_B$, then from Alice’s perspective she has access to an operational state in $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H_A)^{\text{op}}}$, while Bob has access to an operational state in $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H_B)^{\text{op}}}$. Theorem 8.2.8

states that Alice has access to operational state with Bob's part of the system traced out, while Bob has the operational state with Alice's part traced out, which is precisely the notion of no-signalling between measurement sites. Having a full understanding of this more specialised notion of site-specific no-signalling will be needed to better understand Bell's theorem in its traditional presentation.

One approach to this could be to incorporate a theory of composite systems with a premonoidal spatiotemporal structure, like that introduced in the previous section.

8.3 Quantisation and Classical Mechanics

We will now return to classical mechanics which we discussed in Chapter 2. We discuss how quantization can be viewed from the perspective of the spectral presheaf framework by using a well-known relationship between quantization and POVMs.

We will also construct a toy model – similar in principle to Spekkens' Toy Theory – which makes explicit use of mathematical structures central to classical mechanics, namely, differential operators and Hamiltonian vector fields. This can be seen as a generalisation of *quasi-quantization* as formulated by Spekkens [199].

Quantisation and POVMs

Let X be a locally compact Hausdorff topological space, and let $C_0(X)$ be the set of continuous functions $f : X \rightarrow \mathbb{C}$ which vanish at infinity, that is those functions such that for all $\varepsilon > 0$ there exists a compact subset $K \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \in K$. This forms a (non-unital) \mathbb{R} -algebra. If X is compact then this algebra coincides with the algebra of bounded continuous functions $C(X)$.

Definition 8.3.1. Let X be a locally compact Hausdorff topological space. A *quantization* of the algebra $C_0(X)$ consists a Hilbert space H of a positive linear

map of C^* -algebras

$$C_0(X) \xrightarrow{Q} \text{Hom}(H, H)$$

Note that a quantization need not be a homomorphism of C^* -algebras, since it need not preserve multiplication.

Definition 8.3.2. Let X be a set with a σ -algebra Σ of subsets. A *positive operator-valued measure*, or *POVM* on X in a Hilbert space H consists of a map

$$\Sigma \xrightarrow{A} \text{Hom}(H, H)$$

such that $A(\emptyset) = 0$, $A(X) = 1$, and

$$A\left(\bigcup_i Y_i\right) = \sum_i A(Y_i)$$

for any countable collection of disjoint $Y_i \in \Sigma$.

A POVM is called a *projection-valued measure* or *PVM* if

$$A(Y_1 \cap Y_2) = A(Y_1)A(Y_2)$$

for all disjoint Y_1, Y_2 in Σ .

The following is shown in [141, Proposition 4.2.3].

Theorem 8.3.3. *Let X be locally compact Hausdorff topological space, and let Σ be the Borel σ -algebra on X . There is a bijective correspondence between POVMs*

$$\Sigma \xrightarrow{A} \text{Hom}(H, H)$$

and quantizations

$$C_0(X) \xrightarrow{Q} \text{Hom}(H, H)$$

The POVM corresponding with the quantization Q is a PVM if and only if Q is a C^ -algebra homomorphism.*

In particular then we see that every commutative subalgebra $\mathfrak{A} \subseteq \text{Hom}(H, H)$ corresponds with a PVM, defined on the compact topological space $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$.

Let \mathfrak{A} be a commutative C^* -algebra considered as a classical system. Let $\text{Sub}(\mathfrak{A})$ be the poset of subalgebras. Suppose we have a quantization map that is a C^* -algebra homomorphism $Q : \mathfrak{A} \rightarrow \text{Hom}(H, H)$ (in the sense of Definition 8.3.1). Since the image of Q is a commutative C^* -subalgebra of $\text{Hom}(H, H)$, and hence Q extends to a map of posets

$$\text{Sub}(\mathfrak{A}) \xrightarrow{Q} \mathbf{Hilb}\text{-}\mathbf{Alg}(H)$$

Remark 8.3.4. As we noted in Remark 4.2.18, if \mathfrak{A} is a classical mechanical system in the sense of Nestruev, then $\text{Spec}_{\mathbb{C}}(\mathfrak{A})$ is a smooth manifold, then the image of any quantization $Q : \mathfrak{A} \rightarrow \text{Hom}(H, H)$ cannot be a von Neumann algebra, as the spectrum of a von Neumann algebra is a Stone space, and the only compact manifolds that are Stone spaces are the zero-dimensional ones, that is finite discrete topological spaces. Hence, in general there is no equivalent map

$$\text{Sub}(\mathfrak{A}) \xrightarrow{Q} \mathbf{Hilb}\text{-}\mathbf{Alg}_{\text{vN}}(H)$$

into the poset of commutative von Neumann subalgebras. This is further motivation for considering the full poset of commutative C^* -algebras, not just the commutative von Neumann subalgebras.

By [155, Chap. VII. §2. Theorem 2] the map of posets extends to a geometric morphism of toposes

$$\begin{array}{ccc} & Q_* & \\ \text{Set}^{\text{Sub}(\mathfrak{A})^{\text{op}}} & \begin{array}{c} \curvearrowright \\ \top \\ \curvearrowleft \end{array} & \text{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)^{\text{op}}} \\ & Q^* & \end{array}$$

Future work will explore what implications these geometric morphisms have for understanding quantization, and in particular the relationship between a

classical system and its quantised counterpart.

Algebraic Hamiltonian Mechanics

Let M be a compact manifold with dimension n . We think of the points of the manifold as being the possible positions which our system might inhabit. At each point in the manifold there is a space isomorphic to \mathbb{R}^n of vectors which we interpret as representing the collection of possible momenta a particle at this point could have. This vector space at point $x \in M$ we call the tangent space at x and denote $T_x M$.

We combine the tangent spaces to form a new manifold

$$TM = \bigcup_{x \in M} T_x M$$

Where we can identify a point $X \in TM$ with a pair (x, v_x) where $x \in M$ and $v_x \in T_x M$. This manifold, equipped with the projection map

$$\begin{array}{ccc} TM & \xrightarrow{\pi} & M \\ (x, v_x) & \longmapsto & x \end{array}$$

is called the *tangent bundle*.

A *vector field* assigns to each point $x \in M$ a vector in the corresponding tangent space $T_x M$. Equivalently, a vector field is a map

$$M \xrightarrow{F} TM$$

such that $\pi(F(x)) = x$ for all $x \in M$, in other words, a vector field is a *section* of the tangent bundle.

This is the geometric intuition behind a vector field, but we will make the formal definition in the purely algebraic language of Nestruev [172, Chap. 9].

Definition 8.3.5. A map

$$C^\infty(M) \xrightarrow{\xi} \mathbb{R}$$

is called a *tangent vector to M* at the point $x \in M$ if it satisfies the following conditions

1. \mathbb{R} -Linearity: for all $\lambda_j \in \mathbb{R}$ and $f_j \in C^\infty(M)$

$$\xi\left(\sum_{j=1}^k \lambda_j f_j\right) = \sum_{j=1}^k \lambda_j \xi(f_j)$$

2. The local Leibniz rule: for all $f, g \in C^\infty(M)$

$$\xi(fg) = f(z)\xi(g) + g(z)\xi(f)$$

It is easy to check that tangent vectors, as defined here, at a fixed point are closed under point-wise addition and scalar multiplication and so do form a vector space, which we denote $T_x M$.

We can reformulate how we define vector fields. A section of the tangent bundle gives us a family of tangent vectors $\xi_x : C^\infty(M) \rightarrow \mathbb{R}$, indexed by $x \in M$.

With this data we can define a map

$$C^\infty(M) \xrightarrow{F_\xi} C^\infty(M)$$

Where for each $f \in C^\infty(M)$ we define

$$\begin{aligned} M &\xrightarrow{F_\xi(f)} \mathbb{R} \\ x &\longmapsto \xi_x(f) \end{aligned}$$

This map F_ξ satisfies the *Leibniz rule*

$$F_\xi(fg) = F_\xi(f)g + fF_\xi(g)$$

Any map

$$C^\infty(M) \xrightarrow{X} C^\infty(M)$$

satisfying the Leibniz rule defines a section of the tangent bundle, where for each $x \in M$ we define the map

$$\begin{aligned} C^\infty(M) &\xrightarrow{X_x} \mathbb{R} \\ f &\longmapsto X(f)(x) \end{aligned}$$

which is a tangent vector at the point $x \in M$.

Hence vector fields are developed in a purely abstract algebraic way, and can be seen as special cases of abstract differential operators [172, §9.43].

Definition 8.3.6. An *integral curve* for a vector field X on M is a smooth function

$$\mathbb{R} \xrightarrow{c} M$$

such that for all $t \in \mathbb{R}$

$$\frac{d}{dt}c = X_{c(-)}$$

The intuition behind integral curves is as follows: if one pictures a vector field as assigning a vector v_x at a tangent to each point x in M , then an integral curve is a curve on the surface of M such that at each point $c(t)$ the curve c passes through, it passes through parallel to the vector $v_{c(t)}$. If the vector field represented fluid flow, then the integral curves represent the trajectories a body would follow under that flow.

The manifolds used to formulate Hamiltonian mechanics are equipped with additional structure, typically that of a symplectic structure, or more generally *Poisson bracket* [194, 22]. A Poisson bracket consists of a linear map

$$C^\infty(M) \otimes C^\infty(M) \xrightarrow{\{\cdot, \cdot\}} C^\infty(M)$$

satisfying:

1. Anticommutativity:

$$\{f, g\} = -\{g, f\}$$

2. the Leibniz rule:

$$\{fg, h\} = \{f, g\}h + f\{g, h\}$$

3. and the Jacobi identity:

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

It is easy to verify that a map of the form

$$C^\infty(M) \xrightarrow{\{\cdot, H\}} C^\infty(M)$$

for some $H \in C^\infty(M)$ satisfies the properties of being a vector field.

The *Hamiltonian* associated with a physical system consists of an element $H \in C^\infty(M)$. A physical system consists of a symplectic or Poisson manifold equipped with a Hamiltonian $X_H = \{\cdot, H\}$ and one derives properties of the system – for example, the equations of motion – from the properties of the Hamiltonian and the differential calculus associated with the symplectic or Poisson structure.

The following result is shown in, for example, [194, Theorem 18.9].

Theorem 8.3.7. *We have $\{f, H\} = 0$ if and only if f is constant along integral curves of the differential operator $X_H = \{\cdot, H\}$.*

Using the language of Poisson brackets we can generalise the concept of *quasi-quantization* as formulated by Spekkens [199].

Quasi-quantization, as formulated in [199] is a construction on manifolds of the form \mathbb{R}^{2n} , with a restricted class of measurements on this manifold, the so-called *quadrature observables*. Spekkens then groups the quadrature observables into families which are pair-wise commutative with respect to the Poisson bracket.

With spectral presheaves this construction can be generalised for arbitrary Poisson manifolds, which we demonstrate with a toy theory based on Hamiltonian mechanics.

A Hamiltonian-Based Toy Theory

We can consider the poset of subalgebras of $C^\infty(M)$, but we might want to consider a subposet consisting of just those subalgebras satisfying some additional property. Let $\text{Sub}(C^\infty(M))$ be the poset of those subalgebras $\mathfrak{A} \hookrightarrow C^\infty(M)$ such that for all $f, g \in \mathfrak{A}$ we have $\{f, g\} = 0$. These are the algebras which cannot distinguish quantities conserved by some Hamiltonian.

The spectrum defines a functor

$$\text{Sub}(C^\infty(M))^{\text{op}} \xrightarrow{\text{Spec}} \mathbf{Set}$$

and assuming M has the structure of a measure space we can consider the presheaf which assigns some distribution on each spectrum

$$\text{Sub}(C^\infty(M))^{\text{op}} \xrightarrow{\Psi} \mathbf{Set}$$

Suppose we have a particle that can move freely on the surface of a torus T . We can toggle between different settings which impose a field on T . For the sake of this example we will restrict to fields for which the corresponding flow is periodic, that is the flow function $\varphi : \mathbb{R} \times T \rightarrow T$, for each $x \in T$ the corresponding function $\varphi(x) : \mathbb{R} \rightarrow T$ is periodic. The particle p , at initial position $x \in T$ will then move along the trajectory defined by $\varphi(x) : \mathbb{R} \rightarrow T$. Figures 8.2 and Figure 8.3 depict families of possible trajectories for two possible field settings.

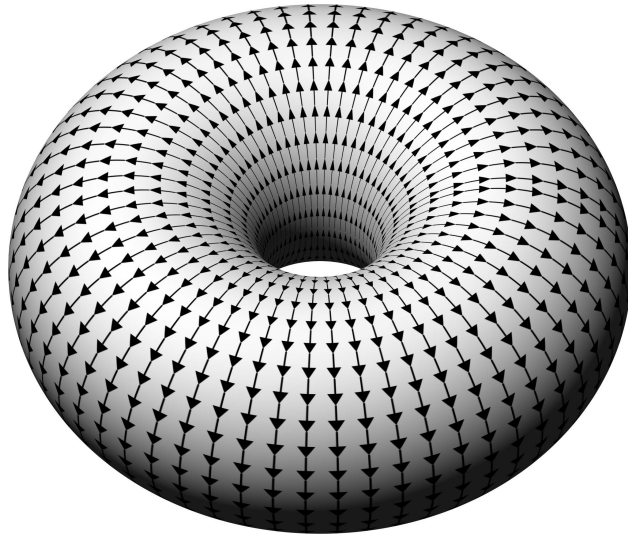


Figure 8.2: *First experimental configuration. Image by RokerHRO, Wikimedia Commons.*

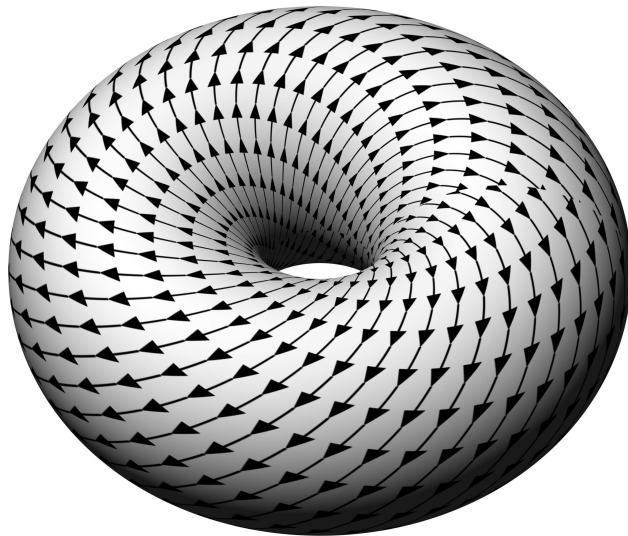


Figure 8.3: *Second experimental configuration. Image by RokerHRO, Wikimedia Commons.*

Suppose the particle is too small to be seen directly, but we are able to detect the path that it traces as it moves under the effect of the field. Now

let H_1, \dots, H_n be a set of Hamiltonians which give periodic flows. Define the collection of subalgebras \mathfrak{A}_i to be those subalgebras which consist of functions integrable with respect to H_i .

At any given moment we do not know the position of the particle, but rather the trajectory on which the particle is moving. If we then switch to a different field the particle will now begin tracing a new trajectory. Analogous to Spekkens' Toy Theory, discussed in Chapter 7, we may be able to deduce something of the position of the particle at the moment the change was made: in particular, the particle must have been at some point where the two trajectories intersect. While we may be able to deduce where the particle was at the moment of measurement, our current knowledge of the position of the particle is restricted.

Note that this toy theory can be quantised in the sense described in the previous sections by considering C^* -algebra homomorphisms of the form

$$C(T) \xrightarrow{Q} \text{Hom}(H, H)$$

for H a Hilbert space, and hence this toy theory can be related to quantum theory much more directly than, say, Spekkens' Toy Theory. Future work will explore the features of this toy theory.

We will end with a general remark about an apparent convergence of ideas for which our techniques could provide a formal link or at least a common language. Nestruiev formulates the language of differential operators and Hamiltonians is developed in a purely algebraic way [172, Chap. 9].

Empirical models of the Abramsky-Brandenburger sheaf formalism can be applied to the study of data structures [4] and computation [8]. It has also been shown that data types admit a differential calculus [1, 2], and also that Noether's theorem – a deep result from classical mechanics relating symmetries of a system to conserved quantities of that system – can be formulated in type theory [14]. Given the highly general and purely algebraic formulation of the differential calculus of Nestruiev, we suspect that the spectral presheaf framework offers a setting in which to further explore these concepts.

This suggests to us that an abstract algebraic formulation of data structures and differential operators on data structures could incorporate the language of contextuality and non-locality.

Chapter 9

Epilogue: A New Metaphysics

In this chapter, we propose a new metaphysics which specifically addresses the conceptual requirements of quantum theory, but which also applies to physics and scientific theories more generally. The position we develop, which we call *representational dualism* is guided and informed by the mathematical constructions we have considered in this work.

We present representational dualism as an alternative to other metaphysical interpretations of physics like realism and instrumentalism, which we discussed in Chapter 1. Realism presents a view that reality, at the most fundamental level, should be conceptualised in an objective way, existing independently from human experience. Instrumentalism, however, asserts that any description of reality must be fundamentally rooted in our subjective experience of reality.

Representational dualism takes incorporates aspects of both the realist and instrumentalist representations of reality, simultaneously treating concepts from each of these seemingly opposite metaphysical perspectives as fundamental. A similar philosophical perspective is proposed by Lawvere [145, p. 16]:

It is my belief that in the next decade and in the next century the

technical advances forged by category theorists will be of value to dialectical philosophy, lending precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc. In turn the explicit attention by mathematicians to such philosophical questions is necessary to achieve the goal of making mathematics (and hence other sciences) more widely learnable and useable. Of course this will require that philosophers learn mathematics and that mathematicians learn philosophy.

Representational dualism is rooted in these ‘ancient philosophical distinctions’, as Lawvere calls them. In particular, we focus on the duality present in classical mechanics between *spaces* of ‘states’, which are objective elements of reality, and algebras of measurements that detect *quantities*, a concept rooted in the subjective, as the notion of ‘observation’ requires the notion of an observer.

One of the most compelling aspects of realism is that it seems to reflect how we conceptualise the world in our day-to-day lives, as Flori puts it [82, p. 9]:

It is worth noting that our own language reflects a realist view of the world: “the tree *is* three meters tall”.

Yet knowing anything about the objective property we call the ‘height’ corresponding with the tree requires the conceptual and physical means of ‘measurement’, a fundamentally subjective phenomenon.

The realist, while believing in the existence of an objective reality, must accept that he can only meaningfully consider those parts of reality accessible to him via subjective measurements and observations; he must, for the practical purposes of reasoning about and navigating the world, think like an instrumentalist.

The instrumentalist, on the other hand, might reject the notion that ‘height’ or carries any objective meaning independent of human experience, yet he certainly reasons about the world in terms of this abstract concept independently

of the notion of measurement; he must, for the practical purposes of reasoning about and navigating the world, think in realist terms.

Recall the pragmatic interpretation of metaphysics characterised Isham as follows [123, pp. 80-81]:

In many respects [the pragmatic interpretation] can be thought of as the ‘safe’, fall-back position: moving towards a full-blown interpretation then resembles looking over a parapet towards the enemy lines whilst reserving the option to duck one’s head at the first signs of fire!

Representational dualism presents a pragmatic, full-blown interpretation of metaphysics which does not merely timidly refrain from ‘picking a side’ in the debate between realism and instrumentalism. We argue that representation dualism is *pragmatic* in the sense of corresponding with how we *actually do* and *conceptualise* science. Representational dualism presents a synthesis of realist and instrumentalist perspectives and we motivate this position by showing that in performing his professional duties a scientist must simultaneously represent reality in both realist and instrumentalist terms, and must pass back and forth between these representations. At the heart of this idea lies the essential structure of the scientific method, which consists of an iterated process:

1. from the result of observations, generate or refine abstract theories of the nature of the world;
2. through reasoning about these abstract theories make predictions of the world;
3. test predictions empirically through the process of observation.

Through the repetition of this process we continuously generate and refine a broader and more detailed theoretical conceptualisation of reality.

The abstract theories we generate are realist representations of reality: abstract representations of elements of reality. In contrast, the process of observation requires notions of measurement, and hence must exist within an

instrumentalist metaphysical paradigm. Hence, the activities of any scientist requires features fundamental to both the realist and the instrumentalist. The only difference between the instrumentalist scientist and the realist scientist is in which aspect they consider to be fundamental and irreducible: objective reality as it exists, or subjective observation as it can be performed.

The fact that the scientific method is premised on both subjective and objective representations of reality is recognised by Worrall – who advocates for realist metaphysics – but the distinction is not well understood or precise [214, p. 166]:

the distinction between theory and observation—and hence the division into theoretical and observational vocabulary—is extraordinarily vague.

One of the central premises of representational dualism lies in articulating this distinction in a precise manner, that is, using mathematical languages. We see an example of this in the algebraic vs. geometric representations of classical mechanics, which we will now discuss.

In his *Principia* Newton modelled a realist interpretation of classical mechanics, asserting that a physical system has objective quantities associated with it, for example, momentum, velocity or position. However, Newton does not give a mathematical formulation of measurements; *momentum* is represented mathematically, but the apparatus and process by which momentum is measured are not. While the theory was developed based on the close analysis of countless experiments, the experimental observations do not appear in the mathematical model itself. While the mathematical machinery has been refined since Newton – now expressed in terms of symplectic and Poisson manifolds – the metaphysics underlying this approach classical mechanics remains largely unchanged. For a detailed, modern account of how to arrive at Newton’s axioms from experimental observations see Spivak [200, 201]. In Newtonian mechanics, ‘measurement’ is just a name given to a function on the state space, a concept we derive from the fundamental notion of ‘state’.

Unlike Newton, Nestruev gives a mathematical model of an instrumentalist approach to classical mechanics, which assigns mathematical objects to the measurements or observations that we can make of a physical system. Nestruev makes these metaphysical assumptions explicit through the so-called *observability principle* [172, p. viii]. In this formulation of mechanics, ‘measurement’ is taken to be the primitive notion, while ‘state’ is merely the name given to the simultaneous assignment of an outcome to every possible measurement.

It is a truly remarkable result that these two models of reality, using fundamentally different metaphysical assumptions, and completely different mathematical techniques turn out to be equivalent – a consequence of the Stone-type duality discussed in Remark 2.2.1. The Stone-type duality which underpins the equivalence of these mathematical representations of this physical theory reflects a fundamental interaction between the ambient metaphysics of those representations; the correspondence at the level of mathematical formalism ‘lifts’ to the level of metaphysical interpretation. This is illustrated in Figure 9.1.

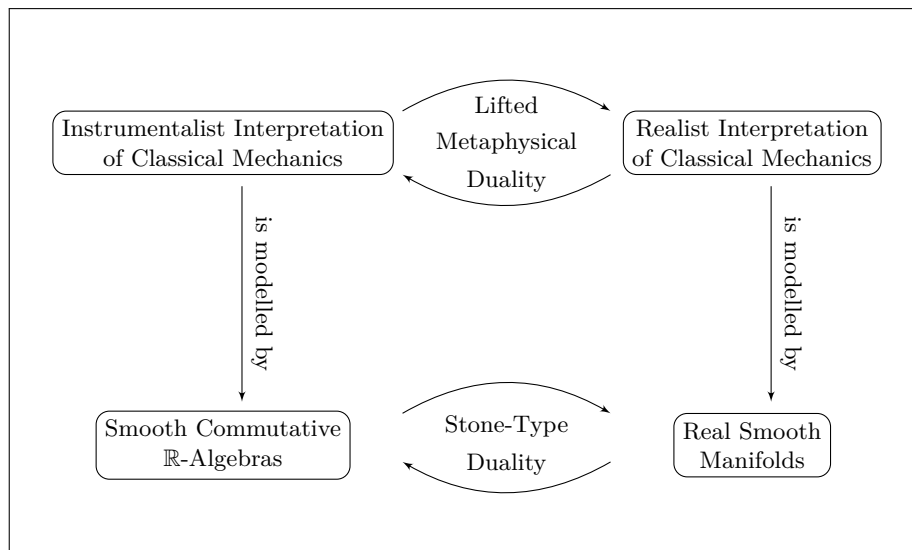


Figure 9.1: *The correspondence between models of classical mechanics and their respective metaphysical interpretations.*

The fundamental interaction between the realist and instrumentalist representations of reality lies at the heart of representational dualism. Representational dualist interpretation of classical mechanics is modelled by smooth algebras, manifolds and the corresponding Stone-type duality between these structures; is modelled by the full picture shown in Figure 9.1. We encode these principles of representational dualism through the precepts shown in Figure 9.2.

1. A physical theory must be represented according to both realist and instrumentalist metaphysical assumptions.
2. Both realist and instrumentalist interpretations of a physical theory should be modelled with an appropriate mathematical language.
3. A formal interaction between these mathematical models should lift to an interaction between metaphysical interpretations of that physical theory.

Figure 9.2: The precepts of representational dualism.

Representational dualism requires placing a subjective characterisation of reality on as fundamental a footing as an objective characterisation of reality. This perspective, perhaps novel from the perspective of physics, exists within other scientific disciplines, for example psychology, as Peterson explains [182, p. 48]:

We tend to view the “environment” as something objective, but one of its most basic features—familiarity, or lack thereof—is something virtually defined by the subjective. This environmental subjectivity is nontrivial.... It appears, indeed, that [a subjective] categorization or characterization of the environment might be regarded as more “fundamental” than any objective characterization—if we make the presumption that what we have adapted to is, by definition, reality.

In order to fully realise the power of the representational dualist perspective on a given physical theory, we must model ‘observation’ and ‘measurement’

mathematically; we must treat mathematics as belonging to the realm of the subjective. This ties into the broader project of Nestruev who states [172, p. 95]:

[We want] to reconsider some paradigms reflecting the relationship of mathematics to the natural sciences and, above all, with physics and mechanics. By including observability in our considerations, we ensure that mathematics may be regarded as a branch of the natural sciences.

This goes against the common intuition that mathematics is a strictly objective exercise, perhaps even the epitome of objectivity; we assign the purest form of ‘objectivity’ and ‘truth’ to statements like ‘ $2+2=4$ ’. However, Nestruev’s project seeks to formulate mathematics as something with foundations in the subjective [172, pp. 211-212]:

It is clear that any rigorous mathematical notion of observability must come from some notion of observer, understood as a kind of mechanism for gathering and processing information. In other words, the notion of observability must be formalised in approximately the same way as Turing machines formalize the notion of algorithm.

The second precept of representational dualism depicted in Figure 9.2 requires that we model an instrumentalist interpretation of a given physical theory according to an ‘appropriate mathematical language’. Nestruev points out that Finding such appropriate mathematical languages presents highly non-trivial conceptual and technical problems [172, p. 208]:

the systematic mathematical formalization of the observability principle requires rethinking many branches of mathematics that seemed established once and for all. The main difficult step that must be taken is to find solutions in the framework of the differential calculus, avoiding the appeal of functional analysis, measure theory, and other purely set-theoretical constructions. In particular, we must refuse

measure theory as integration theory in favor of the purely cohomological approach. One page suffices to write out the main rules of measure theory. The number of pages needed to explain de Rham cohomology is much larger. The conceptual distance between the approaches shows what serious difficulties must be overcome on this road.

This project which Nestruev describes is well underway, see for example, [207, 208, 138]. It is worth noting that Nestruev asserts ‘observable’ mathematics is not equivalent to constructive mathematics [172, pp. 214-215].

In Section 9.1 we give an illustration of representational dualism in action by giving a detailed analysis of the first piece of mathematical physics we learn: the theory of counting finite quantities. Being conceptually simple on the level of physics, this theory provides an appropriate setting in which to consider a representational dualist formulation of a physical theory built from the ground up.

In Section 9.2 we discuss what representational dualism might mean for quantum theory. In particular we will state what we view to be the fundamental problem of quantum metaphysics. We propose a solution to the fundamental problem of quantum metaphysics, which incorporates the projects of Nestruev [172], Lawvere [145], and the project of Bohrification due to Heunen, Spitters and Landsman [114, 116, 115].

In Section 9.3 we discuss representational dualism within the context of the role of metaphysics in the development of scientific theories. We also validate representational dualism as a fully-fledged metaphysical theory by providing an answer to Heidegger’s [107] central question of metaphysics: what is a thing? We then discuss the foundations upon which the assumptions of representational dualism rest: not based on conceptual arguments or justifications, but based on what we know about how the brain represents reality. Based upon these considerations we discuss a possible link between the mathematical and philosophical considerations of Lawvere [145] and Nestruev [172], with the problem

of consciousness.

9.1 The Theory of Finite Quantities and Counting

We will now give an example of a physical theory developed according to the principles of representational dualism: the theory of counting finite quantities.

We will first give a realist formulation of this physical theory, which takes the objective concept of *quantity* associated with a collection of objects to be a fundamental notion. From this notion of quantity we can derive a notion of ‘measurement’ or ‘counting’. We will then give an instrumentalist formulation of the theory of counting finite quantities which takes the notion of counting to be the primitive notion, from which the notion of ‘quantity’ is abstracted, or generated.

It might seem circular to give a notion of counting *before* giving a notion of quantity – after all, how can one describe a means of measuring a specific property without first introducing that property? Although this seems paradoxical, it resembles the way in which we actually learn to count, and indeed resembles the way in which more complicated physical theories are developed. As children we are exposed to many instantiations of ‘one of something’, ‘two of something’ and so on, and we are taught a process how to distinguish such collections from one another using primitive measuring apparatus – our fingers – without being told what ‘one’ or ‘two’ actually *are*. At some point, after achieving sufficient familiarity, we make the abstraction from these concrete instances of ‘*n* of something’ to the intangible concept of ‘*n*’; given a family of collections of *n* objects – e.g. apples, ducks, tractors etc. – we identify the abstract concept of ‘*n*’ as *that which is the same among that family of collections*.

It is not only the notion numbers that we come to know in this way. The modern conceptualisation of ‘time’ only emerged *after* the discovery of devices which can measure time, and as children we learn how to read a clock *before* we

can articulate any notion of time – indeed, most adults, most scientists even, would likely struggle to articulate a coherent notion of time as a concept. Galileo made repeated observations of pendulums, noting that while certain aspects of a pendulum can change – for example, its amplitude – there is something about a pendulum which remains constant: its period. Through the abstraction of *that which is the same among all swings of a given pendulum* we arrive at a concept of time. A concept of time based on the abstraction of motions of heavenly bodies, for example, might be quite different from our modern, linear, notion of time.

We construct a mathematical model of the theory of finite quantities for each of these interpretations, and then show that these models are equivalent at the level of mathematical formalism. This is direct analogy to the realist vs. instrumentalist models of classical mechanics as depicted in Figure 9.1.

In the realist interpretation of quantities, for any collection of objects C , there is assigned an element of reality known as the *quantity* of those objects, for example, associated with the collection of chairs in this room is a number $n \in \mathbb{N}$ which we call the *quantity* of chairs in the room. This notion of quantity associated with C is assumed to exist independently of our ability to observe or conceive of those objects. We give a formal account of the notion of quantity with the axioms shown in Figure 9.3.

1. To each collection of objects C there is an assigned natural number $Q(C) \in \mathbb{N}$ called the *quantity* of those objects.
2. If we have two disjoint collections of objects C and D , with assigned quantities n and m respectively, then the quantity assigned to the collection obtained by combining C and D is $n+m$. That is $Q(C \sqcup D) = Q(C) + Q(D)$.
3. If there exists a bijection between the collections C and D , then $Q(C) = Q(D)$.

Figure 9.3: Axioms for the realist interpretation of the theory of quantities.

Note that while we have an abstract notion of quantity, the axioms in Figure 9.3 tell us nothing of how to actually *compute* the quantity associated with any given collection C . That is, we have a notion of quantity, but no notion of counting. We will now derive a notion of counting that is consistent with how we actually perform and conceptualise the process of counting in day to day life.

Say we wanted to count the collection of cups on a table, then assuming this is a relatively low number we would do something like the following: for each cup on the table we associate a unique finger on our hands. Holding outstretched only those fingers which correspond with a cup we are able recognise the associated quantity. While a child might count by constructing bijections with collections of their fingers, an adult might use more sophisticated collections of arbitrary mouth noises, associating each cup with a utterance like ‘one’, ‘two’, etc. Either approach relies on us recognising that the set

$$\{\text{my left thumb, my left index finger, my left middle finger}\}$$

or

$$\{\text{“one”, “two”, “three”}\}$$

are assigned the quantity 3.

Definition 9.1.1. For each $n \in \mathbb{N}$ let \mathbf{n} be a specified collection of objects such that $Q(\mathbf{n}) = n$.

For example, we might have

$$\mathbf{0} = \emptyset$$

$$\mathbf{1} = \{\text{my left thumb}\}$$

$$\mathbf{2} = \{\text{my left thumb, my left index finger}\}$$

$$\mathbf{3} = \{\text{my left thumb, my left index finger, my left middle finger}\}$$

$$\vdots$$

and so on. Therefore, by axiom 3 from Figure 9.3, to determine the quantity of any collection C , it suffices to find a bijection between C and one of the sets \mathbf{n} . We call this process of explicitly constructing a bijection $C \cong \mathbf{n}$ *counting*.

Each \mathbf{n} is a tangible manifestation of the abstract quantity n . We can use the sets \mathbf{n} as concrete objects against which we can make measurements of arbitrary collections; a kind of yardstick. This is true for all abstract quantities, for example, to measure ‘height’ we need a physical manifestation of that abstract quantity; we need a concrete representation of ‘meter’. This might be physical object, for example, a measuring tape. The height of a given object is determined by comparing that object to the measuring tape. Likewise, the quantity associated with any collection of objects is determined by comparing that collection to the sets \mathbf{n} . This is an inescapable feature of physics, as Born observes [30, pp. 5-6]:

The foundation of every space and time measurement is laid by fixing the unit. The phrase ‘a length of so and so meters’ denotes the ratio of the length to be measured to the length of a meter. The phrase ‘a time of so many seconds’ denotes the ratio of the time to be measured to the duration of a second. Thus we are always dealing with ratios, relative data concerning units which are themselves to a high degree arbitrary, and are chosen for reasons of their being easily reproduced, easily transported, durable and so forth.

Born then goes on to describe that the meter and the second are defined in quite arbitrary terms; for a long time the meter was defined by a rod in Paris, supposed to represent one ten-millionth of a quadrant of the Earth’s circumference, and the second was defined in terms of the duration of a rotation of the Earth. These concrete representations of the abstract quantities of length and time are just as arbitrary as the set

$$\mathbf{3} = \{\text{my thumb, my index finger, my middle finger}\}$$

in representing ‘three’. In this sense each man’s left hand is to the physical

quantity ‘five’, what the rod in Paris is to the physical quantity ‘meter’.

We will now give an instrumentalist formulation of the theory of finite quantities and counting. In this formulation we define and axiomatise a notion of measurement, that is, counting. From this primitive notion we will derive an abstract notion of ‘quantity’.

Counting is a form of measurement, and we perform these measurements with apparatus, for example, we often use our own fingers for this purpose. We will model this process with axioms shown in Figure 9.4.

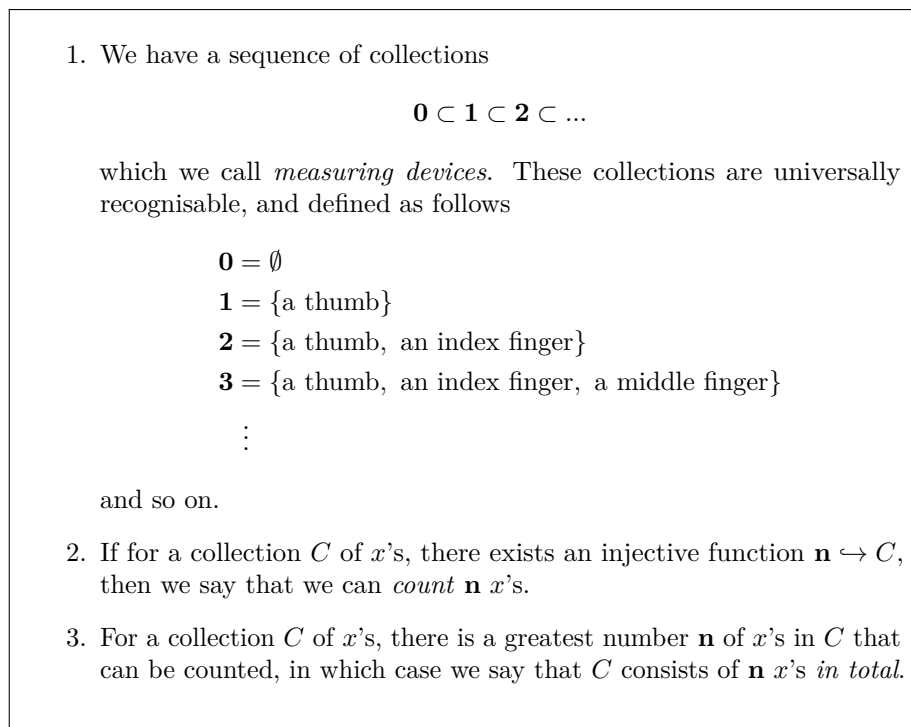


Figure 9.4: *Axioms for the instrumentalist interpretation of the theory of quantities.*

Note that the axioms in Figure 9.4 tell us how to count ‘three of something’ – three chairs, three apples etc. – but not do tell us what ‘three’ *is*. The concept of an abstract quantity is one which we must derive, which we do as follows.

Given lots of examples of ‘three of something’, the abstract concept of ‘three’

is defined to be *that thing which is the same or continuous across all instances of ‘three of something’*. What is the essential thing which ‘three tables’, ‘three rabbits’, and ‘three days’ all have in common? One obvious answer is that all of these collections can be placed in bijective correspondence with one another. We can place an equivalence relation on the class of all collections, where $C \sim D$ if and only if there exists a bijection $C \cong D$ – a perfectly reasonable construction within naive set theory. Denote the equivalence class of a given collection C under this relation by $[C]$. This object $[C]$ is that thing which those different collections of ‘three of something’ have in common; this object $[C]$ is the *quantity* associated with C .

For the collections \mathbf{n} , as axiomatised in Figure 9.4, we will denote the corresponding quantities as follows

$$0 = [\mathbf{0}]$$

$$1 = [\mathbf{1}]$$

$$2 = [\mathbf{2}]$$

$$3 = [\mathbf{3}]$$

$$\vdots$$

and so on. For any finite collection C then, the quantity associated with C will be some n defined in this way.

We will now show that the realist and instrumentalist formulations of the theory of finite quantities and counting are equivalent, in the same way that the realist and instrumentalist formulations of classical mechanics are, as depicted in Figure 9.1.

To show that these two approaches are equivalent we need to show two things: that the notion of ‘counting’ derived from the realist axioms of Figure 9.3 satisfy the instrumentalist axioms of Figure 9.4; and that the notion of ‘quantity’ derived from the instrumentalist axioms satisfies the realist axioms.

To see that the notion of counting derived from the axioms of Figure 9.3

satisfy the axioms of Figure 9.4 is enough to see that the sets \mathbf{n} as defined in Definition 9.1.1 satisfy

$$\mathbf{0} \subset \mathbf{1} \subset \mathbf{2} \subset \dots$$

and that if $Q(C) = n$, then we have $C \cong \mathbf{n}$, that is, if the quantity associated with C is n , then we can count \mathbf{n} elements of C in total.

To see that the notion of quantity derived from the instrumentalist axioms satisfies the realist axioms we must check that the equivalence classes $n = [\mathbf{n}]$ correspond with natural numbers. To see this, it is enough to check that this collection of quantities satisfy the *Peano Postulates*, see for example [196, Chap. 3]. Hence we obtain the picture shown in Figure 9.5.

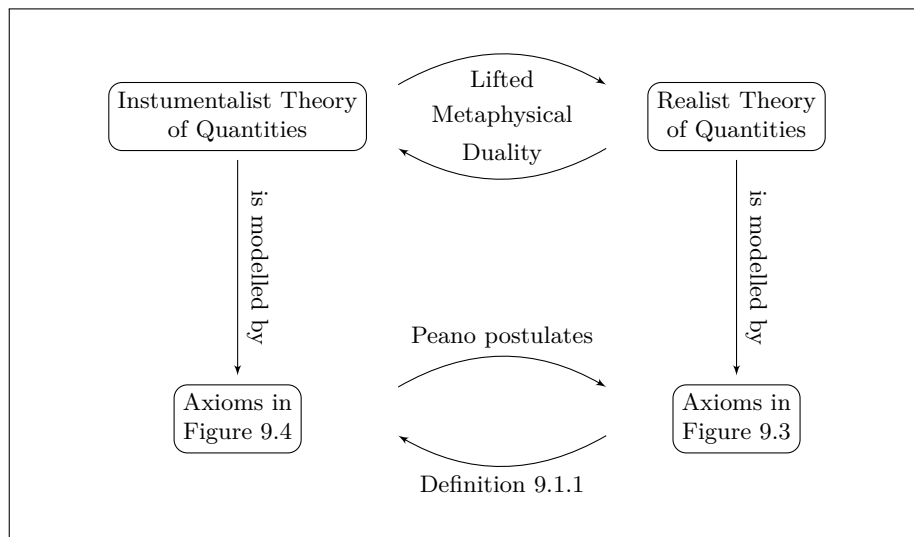


Figure 9.5: The correspondence between models of the theory of quantities and their respective metaphysical interpretations.

The realist treats concept of the natural number ‘ n ’ as fundamental, while the concept of there being ‘ n of something’ is a just the name we give to collections C that happen to satisfy $Q(C) = n$. On the other hand, the instrumentalist takes the concept of being able to detect ‘ n of something’ as fundamental, while the number ‘ n ’ is just the name we give to $[C]$. There is no reason a priori to

assume that these notions coincide, however, the fact that they do coincide is extremely useful for instrumentalists, realists, and everyone else in between.

The realist and the instrumentalist do not disagree on the fact that we come to know numbers through the process of abstraction of concrete instances, they disagree only on the interpretation of this process: the realist asserts that the child has *discovered* objective elements of reality that exist in their own right, while the instrumentalist believes that the child has merely *invented* a useful concept with no intrinsic meaning outside of human experience.

By inventing or discovering abstract quantities we are able to participate in complex interactions with the world and with one another; by ordering the world into abstract concepts we render it more comprehensible.

For some human societies the abstraction of numbers is not a part of their culture or language, with no words in their language corresponding with our ‘one’ or ‘two’. These people are still capable of ‘counting’ in the narrow sense we define in the axioms of Figure 9.4; they can construct bijective correspondences between different collections of objects [95, 85]. However, without being able to articulate the abstract notion of ‘quantity’, the complexity of numerical tasks they can consistently achieve is greatly diminished; tasks which are trivial to anyone in possession of the concept of numbers. For example, if you watch someone place some number of pebbles one-by-one into a bag, and then remove some number of pebbles one-by-one from the bag, answering the question “is this bag empty?” it trivial. However, without the concept of numbers, this task seems to become extremely difficult [95, 85].

The evidence that suggests that ‘number words’, the specific means through which we label the abstract concept of quantities, provide practical means of reasoning about and navigating the world, as suggested by Frank, Everett, Fedorenko and Gibson [85, p. 820]:

number words [should be viewed] as a cognitive technology, a tool for creating mental representations of the exact cardinalities of sets, representations that can be remembered and communicated accurately

across time, space, and changes in modality.

Therefore, a devoted instrumentalist can still accept realist representations of reality as useful ‘cognitive technology’, improving our ability to navigate existence in a complex world, but not corresponding with reality in a fundamental metaphysical sense.

It is important to note that, while in both classical mechanics and theory of quantity and counting the respective mathematical representations for realist and instrumentalist metaphysics turn out to be equivalent, there is no reason to believe a priori that such an equivalence exists. It is precisely this prospect – that there are physical theories for which no reasonable pair of mathematical models, instrumentalist and realist, can be equivalent – that makes representational dualism an interesting avenue to explore. In such a setting the dualist approach offers more than either a realist or instrumentalist approach could individually, and quantum theory offers just such a prospect.

9.2 Dualism and the Problem of Quantum Theory

One of the key insights that our work provides is the following claim: the standard machinery of quantum theory – Hilbert spaces, self-adjoint operators, density matrices – best model aspects of an instrumentalist interpretation of quantum theory. This is one of the central conclusions of this work, developed in Chapter 4.

The task of completing the picture analogous to Figure 9.1 for quantum theory is what we call the *fundamental problem of quantum metaphysics*, that is, the completion of Figure 9.6.

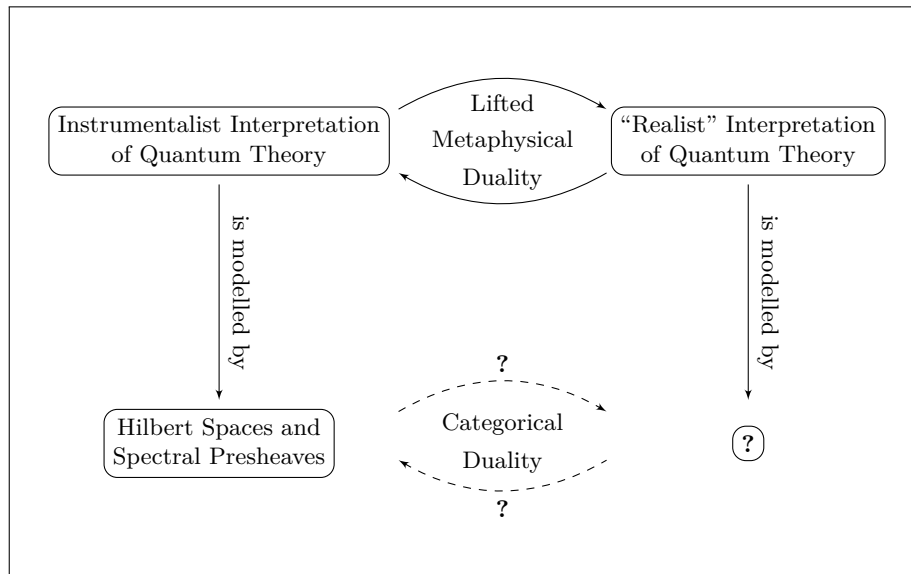


Figure 9.6: *The blueprint for a correspondence between models of quantum theory and their respective metaphysics.*

Resolving the fundamental problem of quantum metaphysics has two components: first, to give a satisfactory mathematical model of a realist interpretation of quantum theory; and second, to relate this mathematical model on a formal level to Hilbert spaces and spectral presheaves.

Notice that we put ‘realist’ in scare-quotes, since it is not clear exactly what a realist interpretation of quantum theory ought to consist of. In particular, in light of the many no-go theorems of quantum theory, Isham’s realist criteria, as shown in Figure 1.6 seem inadmissible. Here we will identify one minimum criterion that we require for a realist interpretation of quantum theory, or of any physical theory for that matter: a realist interpretation of a physical theory represents entities, elements of reality, and in order to be complete there must be some notion of logic facilitating manipulation of these entities at the conceptual level, in a process which we might call ‘thinking’.

For a realist interpretation of a physical theory to be considered complete it must come equipped with a logical structure with which one can make statements

and deductions about the elements of reality represented in the theory. For example, classical mechanics comes naturally endowed with classical Boolean logic, see Isham [123, §4.3].

We understand the term ‘logic’ in quite general terms, following the characterisation of Lawvere [146, p. 43]:

[A logic accompanying a field of study provides a] guide to the complex, but very non-arbitrary constructions of the concepts and their interactions which grow out of the study of [that field].

We certainly mean logic in a sense more general than a syntactic calculus; more general than what Lawvere refers to as *logic in the narrow sense*, which “is related to the inference of statements from statements by means dependent on their form rather than their content” [144, p. 239]. Note that the lack of rules for deduction or predicates makes quantum logic of Birkhoff and von Neumann [23] unsatisfactory as a logic accompanying the field of quantum theory, even in this narrow sense described by Lawvere.

The topos quantum approach of Butterfield, Isham and Doering is an attempt at developing mathematical structures which model a realist interpretation of quantum theory, in particular a neo-realist interpretation. The logical essence of such a realist interpretation is captured by the following assertion of Flori [82, pp. 3-4]:

[In topos quantum theory] propositions can be given truth values without needing to invoke the concepts of ‘measurement’ or ‘observer’.

However, Eva argues that this formulation of topos quantum theory does not provide an adequate representation for the ‘state space’ in a realist interpretation of quantum theory [79, §3].

We will pursue a different mathematical framework within which to model a realist interpretation of quantum theory following a very general categorical construction known as *Isbell duality*, or *Isbell conjugation* [17], or see [147, §7]. Isbell duality describes an adjunction between a category of presheaves and the opposite category of copresheaves.

Definition 9.2.1. For a locally small category \mathcal{C} , *Isbell duality* or *Isbell conjugation* is a contravariant adjunction between the category of presheaves on \mathcal{C} , and the category of copresheaves on \mathcal{C}

$$\begin{array}{ccc}
 & \mathcal{O} & \\
 \text{Set}^{\mathcal{C}^{\text{op}}} & \begin{array}{c} \curvearrowright \\ \perp \\ \curvearrowleft \end{array} & (\text{Set}^{\mathcal{C}})^{\text{op}} \\
 & \Sigma &
 \end{array}$$

where for a presheaf $P : \mathcal{C}^{\text{op}} \rightarrow \mathbf{Set}$ the copresheaf $\mathcal{O}(P)$ is defined

$$\begin{array}{ccc}
 \mathcal{C} & \xrightarrow{\mathcal{O}(P)} & \mathbf{Set} \\
 C & \longmapsto & \text{Nat}(P, \text{Hom}(-, C))
 \end{array}$$

where $\text{Nat}(P, \text{Hom}(-, C))$ denotes the set of natural transformations from the presheaf P to the presheaf $\text{Hom}(-, C)$.

Similarly, for a copresheaf $R : \mathcal{C} \rightarrow \mathbf{Set}$ the presheaf $\Sigma(R)$ is defined

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & \xrightarrow{\Sigma(R)} & \mathbf{Set} \\
 C & \longmapsto & \text{Nat}(\text{Hom}(C, -), R)
 \end{array}$$

According to Lawvere, Isbell duality is a mathematical representation of an archetypal duality which pervades human thought and experience [145, p. 16]:

[which lends] precise form with disputable mathematical models to ancient philosophical distinctions such as general vs. particular, objective vs. subjective, being vs. becoming, space vs. quantity, equality vs. difference, quantitative vs. qualitative etc.

In particular, Lawvere claims that category $\mathbf{Set}^{\mathcal{C}^{\text{op}}}$ can be viewed as the category of generalised spaces which can be ‘probed’ with the help of the objects in \mathcal{C} , while the category $\mathbf{Set}^{\mathcal{C}}$ can be viewed as the category of generalised ‘quantities’ which take values in \mathcal{C} [147, p. 17]. Isbell duality simultaneously

generalises the many Stone-type dualities, which take the general form of a contravariant equivalence between a category of algebraic objects and a category of geometric or topological objects.

In particular, Isbell duality generalises the Stone-type duality between smooth algebras and manifolds used in the representation of classical mechanics, and also generalises Gelfand duality. We will now give a brief account of this.

Let \mathbf{CartSp} be the category whose objects are those \mathbb{R}^n for each $n \in \mathbb{N}$, and whose morphisms consist of smooth maps $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$.

There is an embedding of categories

$$\mathbf{SmothMan} \hookrightarrow \mathbf{Set}^{\mathbf{CartSp}^{\text{op}}}$$

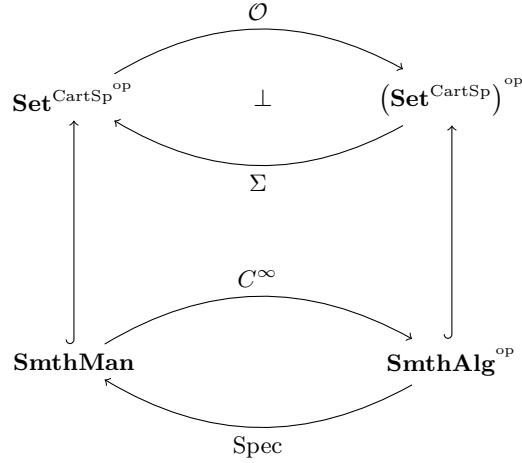
where a manifold M is sent to the presheaf $C^\infty(-, M)$, which sends each \mathbb{R}^n to the set of smooth functions $f : \mathbb{R}^n \rightarrow M$. Manifolds are uniquely determined by the way in which the Euclidean spaces \mathbb{R}^n can be smoothly mapped onto them – usually expressed in terms of *charts* and *atlases*. The category $\mathbf{Set}^{\mathbf{CartSp}^{\text{op}}}$ can be thought of as a category of generalised smooth spaces, see Baez and Hoffnung for a discussion [16].

There is a similar inclusion

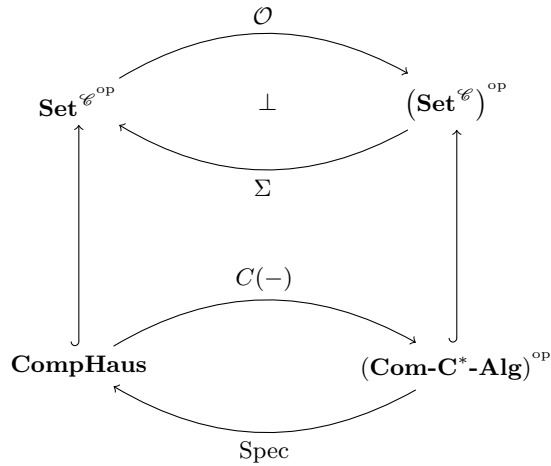
$$\mathbf{SmothAlg} \hookrightarrow \mathbf{Set}^{\mathbf{CartSp}}$$

where for each commutative smooth algebra \mathfrak{A} the presheaf $\text{Hom}(C^\infty(-), \mathfrak{A})$ sends each \mathbb{R}^n to the set of algebra homomorphisms $g : C^\infty(\mathbb{R}^n) \rightarrow \mathfrak{A}$ where $C^\infty(\mathbb{R}^n)$ denotes the commutative algebra of smooth functions $h : \mathbb{R}^n \rightarrow \mathbb{R}$.

Isbell duality between the presheaves and copresheaves on \mathbf{CartSp} restricts to an equivalence of these subcategories, and moreover this equivalence is the Stone-type duality described in Chapter 2.



By a similar construction for compact Hausdorff spaces, Isbell duality also generalises Gelfand duality, that is, for a suitable choice of \mathcal{C} , Isbell duality restricts to Gelfand duality



There have been many efforts to find a non-commutative generalisation of Gelfand duality, see, for example, Heunen, Landsman, Spitters and Wolters [117]. Such a desire led to the development of Connes' non-commutative [54], however, it has been shown that there can exist no straightforward functorial correspondence analogous to Gelfand duality for non-commutative algebras [21].

While there is no ‘Gelfand duality’ for the non-commutative C^* -algebra which represents a given quantum system, we can apply Isbell duality to the category $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)^{\text{op}}}$ which represents that system, that is, we obtain the following adjunction

$$\begin{array}{ccc}
 & \mathcal{O} & \\
 & \curvearrowright & \\
 \mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)^{\text{op}}} & \perp & (\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)})^{\text{op}} \\
 & \curvearrowleft & \\
 & \Sigma &
 \end{array}$$

and hence, Isbell duality naturally presents us with the category of covariant functors $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)}$ – the *Bohr topos* – which is precisely the category considered in the *Bohrification* program, of Heunen, Spitters and Landsman [114, 116, 115]. Bohrification also pursues a new conceptualisation of quantum logic using the language of topos theory, and hence meets our minimal criterion for modelling a realist interpretation of quantum theory.

We conjecture that resolution of the fundamental problem of quantum metaphysics lies in the Isbell duality and the Bohr topos; that the fullest understanding of a quantum mechanical system represented by a Hilbert space H comes from a representational dualist formulation of the quantum system simultaneously represented by the category of spectral presheaves $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)^{\text{op}}}$, and the Bohr topos $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)}$, and through the passage back and forth between these representations via Isbell duality, as depicted in Figure 9.7.

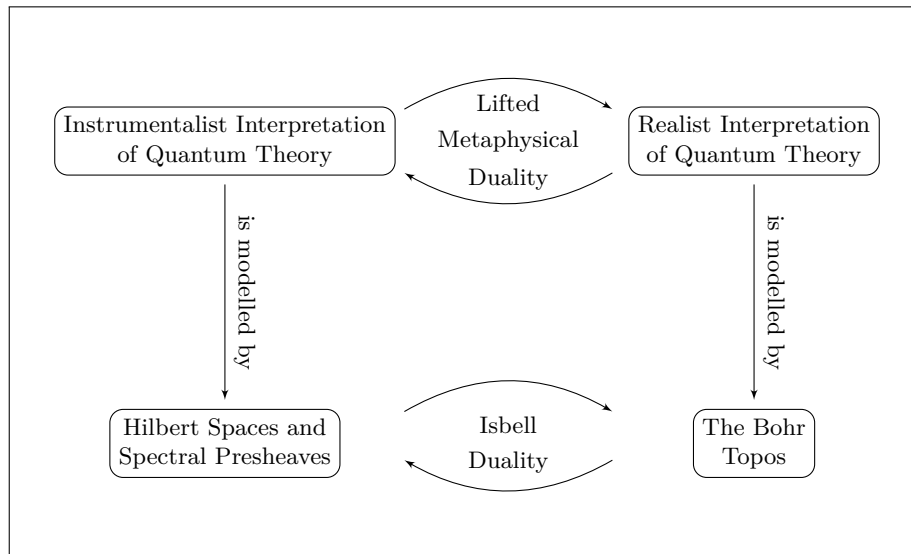


Figure 9.7: Conjecture for the resolution of the fundamental problem of quantum metaphysics according to the metaphysics of representational duality.

There are three main justifications as to why Isbell duality and the Bohr topos provides a good candidate solution for resolving the fundamental problem of quantum metaphysics:

1. Isbell duality directly generalises both Gelfand duality, and the Stone-type duality used in the physical interpretation of classical mechanics due to Nestruev [172]. Moreover, Isbell duality would play the same role in the representational dualist formulation of quantum theory that the Stone-type duality plays for classical mechanics.
2. Isbell duality lands us in the category $\mathbf{Set}^{\mathbf{Hilb}\text{-}\mathbf{Alg}(H)}$, which has already been shown to be a setting in which to formulate quantum logic in the context of Bohrification [114, 116, 115], that is, an environment in which to formulate statements about the nature of reality.
3. More generally, Isbell duality reflects those general dualities between space and quantity, and more broadly, between subjective and objective

perspectives, which lies at the heart of representational dualism; the non-trivial interaction between instrumentalist and realist perspectives.

One feature that marks quantum theory as different from physical theories like classical mechanics of the theory of finite quantities is the prospect that the passage between the instrumentalist and realist representations of reality is not an equivalence; is some weaker, but highly structured correspondence like an adjunction. In this case the representational dualist interpretation offers a genuinely richer physical theory, one which cannot be reduced to one of mere realism or instrumentalism.

9.3 A New Metaphysics

In Chapter 1 we discussed the role of metaphysics as an ambient structure necessary for the development of physical theories. We share the perspective of Lawvere that the proper role of metaphysics is that of a companion to scientific inquiry [145, pp. 14-15]:

The ancient and honorable role of philosophy [was] as a servant to the learning, development and use of scientific knowledge.... In his Lyceum, Aristotle *used* philosophy to lend clarity, directedness, and unity to the investigation and study of particular sciences.

The metaphysical framework we propose, representational dualism, is guided principally by considerations of mathematics and physics. In particular, in addressing the conceptual needs of quantum theory, for which the traditional realist and instrumentalist presentations have proven inadequate. The inadequacy of a purely realist or purely instrumentalist interpretation has led precisely to the discarding of metaphysics altogether by many practising physicists who prefer to simply shut up and calculate; it is not that physicists reject metaphysical doctrines for ideological reasons, but often simply out of exasperation that instead of lending clarity, such metaphysical assumptions seem to render the theory more obscure. However, as Isham points out, such a pragmatic, metaphysical

abstention is insufficient for obtaining a full understanding of quantum theory [123, p. 80].

Representational dualism presents a full metaphysical interpretation that is deeply pragmatic; one based in the actual process by which scientific knowledge is developed. At the heart of representational dualism lies the concept of a non-trivial interaction between two dual representations of reality, which we can broadly characterise as subjective and objective respectively. The case of quantum theory suggests that representational dualism can offer a genuinely richer setting in which to formulate physical theories than either realism or instrumentalism could alone.

Any serious candidate for metaphysics ought to be able to provide an answer to Heidegger's question [107]: what is a thing? The answer to this question provided by representational dualism is as follows: a *thing* is a what emerges through the repeated process of observation and abstraction. For example 'five' is the *thing*, the conceptual entity which emerges from my observation of many examples of collections that have something particular in common with my left hand.

This definition seems sufficiently broad to be a practical notion of 'thing' and encompasses many of the things that ought to be things, for example: 'cup' is a thing, characterised by the abstraction of many observations of containers with a particular shape; this particular cup on my desk is a thing, abstracted from the continuity of its physical form; 'virtue' is a thing, abstracted from all of the instances of good behaviour that we witness in others; 'thing' is a thing, as we have defined 'thing' as the result of taking abstractions, therefore 'thing' is that which emerges by abstracting the process of abstraction; and even 'I' am a thing, abstracted from the continuity of my experiences and memories, as described by John Grey [97, pp. 71-72]:

The notion that our lives are guided by a homunculus – an inner person directing our behaviour – arises from our ability to view ourselves from the outside. We project a self into our actions because

by doing so we can account for the way they seem to hang together.

Our representational dualist conceptualisation of ‘thing’ gives no insight into the realness of things. For example, consider the thing that is the ‘I’, the ‘self’, which is abstracted from a collection of experiences and memories. A Christian, upon identifying this ‘thing’ might determine that he has detected his immortal soul, which exists as an objective element of reality. A Buddhist, on the other hand, recognises the apparent presence of a ‘self’, but concludes that the self abstracted from a continuity of experiences and memories, does not correspond with an element of reality in any meaningful way; that the ‘self’ is an illusion. Regardless of whether one views the self as an illusion, or a real entity, one’s conception of the self arises through the same process of abstraction. Whether this general process of abstraction through which ‘things’ emerge is seen as a process of discovery or a process of creation might vary; some ‘things’ expressible within a theory might correspond with objective elements of reality while other ‘things’ within that theory might not. For example, maybe cups exist but the self does not.

The representational dualist notion of ‘thing’ is defined by an interaction between dual representations of reality, one fundamentally objective, and the other fundamentally subjective; a ‘thing’ emerges and is refined through the repeated process of observation and abstraction. This process seems consistent with the way Peterson believes that the human brain represents reality [182, p. 290]:

the world, as it is experienced... might in fact be considered an emergent property of first-order self-reference; might be regarded as the interaction between the universe as subject and the universe as object.... This idea seems exceedingly foreign to modern sensibility, which is predicated on the historically novel proposition that the objective material in and of itself constitutes the real, and that subjective experience... is merely an epiphenomenal appendage.

The way the brain represents reality is reflected in the physical structure of the brain itself, which is divided into two hemispheres, left and right, each with its own specialized functions.

The right hemisphere, according to Peterson, is responsible for, among other things, pattern recognition and pattern generation, and is specialized for guiding us in exploring the new and unknown; the right hemisphere guides us in situations where we have no suitable representation of what *is* [182, p. 68]. The left hemisphere, seems specialized at linguistic processing, communication and detailed linear thinking; the left hemisphere guides us in situations where we have a representation of what *is* [182, pp. 68-69]. The right brain seems tuned to a reality represented according to instrumentalist metaphysics, while the left hemisphere seems tuned to operating in a reality represented according to realist metaphysics. As Peterson explains [182, p. 72]:

The uniquely specialized capacities of the right hemisphere appear to allow it to derive from repeated observations of behaviour images of action patterns that the verbal left [hemisphere] can arrange, with increasing logic and detail, into *stories*. A story is a map of meaning... and appears generated, in its initial stages, by the capacity for imagery and pattern recognition characteristic of the right hemisphere, which is integrally involved in narrative cognition and in the processes that aid or are analogous to such cognition.... The left-hemisphere “linguistic ” systems “finish” the story, adding logic, proper temporal order, internal consistency, verbal representation, and possibility for rapid abstract explicit communication.

Hence the way we navigate the world – which is always in part known, and in part unknown to us – relies upon the interaction between fundamentally different representations of reality, a process apparently hardwired into the structure of the brain. The scientific method – based on the iterated process of observation and abstraction – appears to be the systematization of a much more fundamental process which underpins all of our interactions with reality.

A metaphysical paradigm that reflects how the human brain represents reality, combined with the prospect of mathematical languages which can formally model these complex, interacting forms of representation could present a paradigm in which consciousness could be analysed, and in particular the *mind-body problem*, which concerns the relationship between consciousness, and the physical substrate which houses consciousness, the brain. Nagel has suggested that the current paradigm of neuroscience is insufficient for the resolution of this problem, and that a new conceptual framework is required [171, §I]:

This is a plea for the project of searching for a solution [for the mind-body problem] that takes conscious points of view as logically irreducible to, but nevertheless necessarily connected with, the physical properties of the organisms whose points of view they are. Consciousness should be recognized as a conceptually irreducible aspect of reality that is necessarily connected with other equally irreducible aspects – as electromagnetic fields are irreducible to but necessarily connected with the behavior of charged particles and gravitational fields with the behavior of masses, and vice versa. But the task of conceiving how a necessary connection might hold between the subjective and the physical cannot be accomplished by applying analogies from within physical science. This is a new ballgame.

Nagel goes on to say [171, §VII]:

The difficulty is that such a viewpoint cannot be constructed by the mere conjunction of the mental and the physical. It has to be something genuinely new, otherwise it will not possess the necessary unity. Truly necessary connections could be revealed only by a new theoretical construction, realist in intention, contextually defined as part of a theory that explained both the familiarly observable phenomenological and the physiological characteristics of these inner events.

Nagel desires a conceptualisation which is not ‘mere conjugation’ of the subjective and objective but requires a connection between these two concepts, each of which must be treated as conceptually irreducible. This position is highly reminiscent of the philosophy of representational duality, while the projects of Lawvere and Nestruev, which seek formal mathematical descriptions of metaphysical concepts as fundamental as the objective and the subjective means that there is a prospect that phenomena like consciousness could be studied formally within the paradigm of representational dualism, as described in the precepts of Figure 9.2.

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