The Discrete Coagulation–Fragmentation System

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Declaration

‘This thesis is the result of the author’s original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

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Abstract

In this thesis, we examine an infinite system of ordinary differential equations that models the evolution of fragmenting and coalescing discrete-sized particle clusters. We express this discrete coagulation–fragmentation system as an abstract Cauchy problem posed in an appropriate Banach space of sequences. The theory of operator semigroups is then used to establish the existence and uniqueness of solutions. We also investigate properties of the solutions, such as positivity, mass-conservation and asymptotic behaviour. A main aim of this thesis is to relax the assumptions that have previously been required, when using a semigroup approach, to obtain the existence and uniqueness of physically relevant solutions. Moreover, we consider the case when the infinite system is non-autonomous due to time-dependent coagulation and fragmentation coefficients.

Contents

Chapter 1

Introduction

In this thesis, we consider a system consisting of clusters of particles that can break apart (i.e. fragment) to produce smaller clusters and can merge together (i.e. coagulate) to produce larger clusters. For example, in industry, the coagulation and fragmentation of clusters occur in polymer science, [1, 78, 79], in the formation of aerosols, [29], and in the powder production industry, [69, 73]. Moreover, these processes also appear in nature, including in blood aggregation, [62], in the formation of preplanetesimals, [36], and in animal groupings, [28].

The evolution of coalescing and fragmenting clusters can be described by coagulation–fragmentation equations, and various methods have been developed to analyse these equations; see, for example, [18, 19, 23] . We concentrate on the approach discussed in [18], which is based upon the theory of strongly continuous semigroups (C_0-semigroups). In particular, semigroup perturbation theory plays a pivotal role in the theory and results presented in this thesis.

Suppose that we wish to predict the density evolution in a system of fragmenting and coalescing clusters, where density is the number per unit volume. If we assume that each cluster is made up of identical units, then cluster size is a discrete variable and a cluster of size $n \in \mathbb{N}$, an n-mer, is made up of n identical units. We refer to these identical units as monomers and we scale the mass such that a monomer has unit mass. It follows that the size variable, n, also represents the mass of a cluster. In this case, the density evolution of clusters may be described by the infinite system of ordinary differential equations, [18, (2.2.1)],

$$
\begin{aligned}
u_n'(t) &= -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \\
&\qquad + \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j} u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j} u_n(t) u_j(t), \quad t > 0; \\
u_n(0) &= \mathring{u}_n, \quad n = 1, 2, \ldots,
\end{aligned}
\tag{1.1.1}
$$

where we define $\sum_{j=1}^{0}$ here, and henceforth, to be the empty sum. In (1.1.1), $u_n(t)$ denotes the density of clusters of mass $n \in \mathbb{N}$ at time $t \geq 0$. For $n \in \mathbb{N}$, the coefficient $a_n \geq 0$ is the rate at which clusters of mass n fragment and, for $n, j \in \mathbb{N}$, $j > n$, $b_{n,j}$ is the average number, per unit volume, of clusters of mass n that are created when a cluster of mass j fragments. Since monomers cannot fragment, the case $a_1 > 0$ represents a system in which monomers are being removed from the system at rate a_1. We interpret the coagulation kernel $k_{n,j} = k_{j,n}$ as the rate at which clusters of mass n merge with those of mass j, when $j \neq n$, and as double the rate at which clusters of mass n merge with those of the same size when $n = j$.

Hence, the first term in (1.1.1) represents the loss of clusters of mass n as they fragment into smaller clusters and the second term represents the gain in clusters of mass n due to the fragmentation of larger clusters. The third term in (1.1.1) represents the gain in clusters of mass n due to smaller clusters merging together. Note that, since $k_{n,j} = k_{j,n}$ for all $n, j \in \mathbb{N}$, the coefficient of $\frac{1}{2}$ prevents the double counting of n-mers produced when clusters of masses j and $n-j$ merge together. Finally, the fourth term represents the loss of n-mers as they merge with other clusters. It is assumed in this model that the population of clusters is dilute so that only binary collisions, and so binary coagulation, occurs, i.e. each coagulation event occurs between exactly two clusters. It is also assumed that fragmentation events do not arise due to interactions between clusters. In addition, (1.1.1) is an infinite system of equations because we assume that there is no upper bound on the size of a cluster. We note here that certain coagulation and fragmentation coefficients can be shown to give rise to a phenomenon known as gelation, where clusters of infinite size are created. However we do not study this phenomenon in this thesis. We refer to (1.1.1) as the coagulation–fragmentation system (or the C–F system). If the coagulation and fragmentation rates are all independent of time, then the associated C–F model is said to be autonomous. On the other hand, if one or more of the rates depend on time, the model is non-autonomous.

We can write any solution of the system (1.1.1) as a sequence $u(t) = (u_n(t))_{n=1}^{\infty}$ and, if $u(t)$ is a solution of (1.1.1), then the total mass of clusters in the system at time $t \geq 0$ is given by

$$M_1(u(t)) = \sum_{n=1}^{\infty} n u_n(t). \tag{1.1.2}$$

The total mass of daughter clusters that are produced when a j-mer fragments is given by

$$\sum_{n=1}^{j-1} n b_{n,j}.$$

To compare the mass before and after a fragmentation event we set

$$\sum_{n=1}^{j-1} n b_{n,j} = (1-\lambda_j) j, \qquad \lambda_j \in \mathbb{R},\ j = 2, 3, \ldots. \tag{1.1.3}$$

If $\lambda_j = 0$, then mass is conserved during the break up of a j-mer. On the other hand, the case $\lambda_j > 0$ (resp. $\lambda_j < 0$) corresponds to mass being lost (resp. gained) during the fragmentation of a j-mer. Using (1.1.1), (1.1.2) and (1.1.3) we find that, at least formally, for $t > 0$,

$$\begin{aligned}
\frac{d}{dt} M_1(u(t)) = \sum_{n=1}^{\infty} n u_n'(t) &= \sum_{n=1}^{\infty} n \Big(- a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \\
&\qquad + \frac{1}{2} \sum_{j=1}^{n-1} k_{n-j,j} u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j} u_n(t) u_j(t) \Big) \\
&= - \sum_{n=1}^{\infty} n a_n u_n(t) + \sum_{j=2}^{\infty} \left(\sum_{n=1}^{j-1} n b_{n,j} \right) a_j u_j(t) \\
&\quad + \frac{1}{2} \sum_{j=1}^{\infty} \left(\sum_{n=j+1}^{\infty} n k_{n-j,j} u_{n-j}(t) u_j(t) \right) - \sum_{n=1}^{\infty} \left(\sum_{j=1}^{\infty} n k_{n,j} u_n(t) u_j(t) \right) \\
&= - \sum_{n=1}^{\infty} n a_n u_n(t) + \sum_{j=2}^{\infty} \left(1 - \lambda_j \right) j a_j u_j(t)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}(l+j)k_{l,j}u_l(t)u_j(t)-\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}nk_{n,j}u_n(t)u_j(t)\\
=&-a_1u_1(t)-\sum_{j=2}^{\infty}\lambda_j j a_j u_j(t).
\end{aligned}
$$

Hence, the change of mass in the system is given by

$$\frac{d}{dt}M_1(u(t))=-a_1u(t)-\sum_{j=2}^{\infty}\lambda_j j a_j u_j(t). \tag{1.1.4}$$

Motivated by (1.1.2), most previous investigations into (1.1.1), for example [46], have concentrated on writing the system as an abstract Cauchy problem (ACP) in the space $X_{[1]}$, where

$$X_{[1]} := \left\{ f=(f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} n|f_n| < \infty \right\} \tag{1.1.5}$$

is equipped with the norm

$$\|f\|_{[1]} = \sum_{n=1}^{\infty} n|f_n|. \tag{1.1.6}$$

The space $X_{[1]}$ is of physical relevance to the system since, from (1.1.2) and (1.1.6), the norm of a non-negative solution (i.e. a solution in which $u_n(t) \geq 0$ for all $n \in \mathbb{N}$, $t \geq 0$) gives the total mass of clusters in the system.

However, for reasons that we explain later in this section, we formulate (1.1.1) as an ACP in a more general weighted ℓ^1 space. Let $w=(w_n)_{n=1}^{\infty}$ be such that $w_n > 0$ for all $n \in \mathbb{N}$. Then we write (1.1.1) as an ACP in ℓ^1_w, where

$$\ell^1_w := \left\{ f=(f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} w_n|f_n| < \infty \right\} \tag{1.1.7}$$

is equipped with the norm

$$\|f\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n|f_n|. \tag{1.1.8}$$

We refer to ℓ^1_w as the weighted ℓ^1 space with weight $w=(w_n)_{n=1}^{\infty}$.

As well as the space $X_{[1]}$, $X_{[p]}$ spaces, for $p > 1$, have also attracted attention in previous examinations of (1.1.1) in, for example, [9, 15]. For $p \geq 1$, the space $X_{[p]}$ coincides with ℓ_w^1 when the weight is of the form $w_n = n^p$ for all $n \in \mathbb{N}$.

In this thesis, we show that it is possible to obtain results regarding the well-posedness of the ACP associated with (1.1.1) when working with a more general weight, $w = (w_n)_{n=1}^{\infty}$, which reduce to those obtained in previous investigations on setting $w_n = n^p$ for $p \geq 1$, $n \in \mathbb{N}$. There are several advantages to be gained by working with more general weights, including the ability to obtain results regarding analytic semigroups that do not necessarily hold if the weight is of the form $w_n = n^p$ for $p \geq 1$. Also, even in the case when $w_n = n^p$ for $p \geq 1$, we prove stronger results than those deduced in prior examinations of (1.1.1). We note that more general weighted spaces have been previously used when examining (1.1.1) by Laurençot, [40, Theorem 2.5], who exploits an alternative method to the semigroup approach investigated in this thesis.

We refer to a solution of (1.1.1) (or the corresponding ACP) as being physically relevant if it is non-negative and displays the expected change in mass. An explicit definition of what we regard as a "classical" solution of a linear ACP is given in Chapter 3 and definitions are given in Chapter 4 of "mild" and "classical" solutions to semi-linear ACPs. In Chapter 5, we begin our investigation into (1.1.1) by examining the pure fragmentation system where we set $k_{n,j} = 0$ for all $n, j \in \mathbb{N}$. We write this fragmentation system as a linear ACP in ℓ_w^1 involving the sum of two operators, $A^{(w)}$ and $B^{(w)}$, which correspond, respectively, to the first and second terms on the right-hand side of (1.1.1). In Theorem 5.2.7 we use a novel argument, based on a perturbation result from [68], to show that $\overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$. The approach that we use differs from that used in previous $X_{[p]}$-based investigations and this result allows us to establish that, for $t \geq 0$, $S^{(w)}(t)\mathring{u}$ is the unique physically relevant classical solution of an ACP associated with the pure fragmentation system, where $\mathring{u} = (\mathring{u}_n)_{n=1}^{\infty}$ is the initial condition. Moreover, we obtain an additional invariance result that can be used to prove that for $t \geq 0$, $S^{(w)}(t)\mathring{u}$ provides a solution of the pure fragmentation system (1.1.1) for all $\mathring{u} \in \ell_w^1$; see Section 5.3.

The main motivation for working with more general weighted ℓ^1 spaces is the results that can be obtained regarding analytic semigroups. For example, in Theorem 5.4.5 we show that for any given fragmentation coefficients we are able to find a corresponding weight, w, such that the semigroup $(S^{(w)}(t))_{t\geq 0}$ on ℓ_w^1 is analytic and is generated by $A^{(w)} + B^{(w)}$. This leads to the existence and uniqueness of physically relevant classical solutions to the fragmentation ACP for all initial conditions in ℓ_w^1; see Theorem 5.4.9.

Furthermore, to prove the existence and uniqueness of solutions, we require the existence of some $\delta \in (0,1]$ such that the weight, $w = (w_n)_{n=1}^{\infty}$, satisfies

$$\sum_{n=1}^{j-1} w_n b_{n,j} \leq \delta w_j \qquad \text{for all } j = 2, 3, \ldots. \tag{1.1.9}$$

If we take the weight w to be such that $w_n = n$ for all $n \in \mathbb{N}$, and (1.1.9) holds, then

$$\sum_{n=1}^{j-1} n b_{n,j} \leq \delta j.$$

Since $\delta \in (0,1]$, it follows that (1.1.3) cannot hold with $\lambda_j < 0$ in this case, i.e. we cannot consider mass gain. By working with a more general weight than $w_n = n$ for all $n \in \mathbb{N}$, we allow the possibility that mass is gained during fragmentation.

We also study the asymptotic behaviour of solutions and show that the unique classical solution, $S^{(w)}(t)\mathring{u}$, of the fragmentation ACP decays to zero as $t \to \infty$ if and only if $a_n > 0$ for all $n \in \mathbb{N}$; see Theorem 5.5.1(i). In Theorem 5.5.4, we study the case where mass is conserved in the system and we show in part (i) that the classical solution converges to a steady state consisting entirely of monomers if and only if $a_n > 0$ for all $n \geq 2$. In Theorem 5.5.1(ii) (resp. Theorem 5.5.4(ii)), we investigate the case where $A^{(w)} + B^{(w)}$ generates an analytic semigroup, and we demonstrate that the solution decays to zero (resp. the solution decays to the monomer state) at an exponential rate, which we quantify, if and only if $\inf_{n\in\mathbb{N}} a_n > 0$ (resp. $\inf_{n\in\mathbb{N}:n\geq 2} a_n > 0$).

We then move on to investigate the full coagulation–fragmentation system (1.1.1), which we write as a semi-linear ACP in a weighted ℓ^1 space. The system (1.1.1) originates from work by Smoluchowski, [64, 65], who derived an equation to describe pure coagulation. Smoluchowski's derivation of this equation leads initially to a system involving time-dependent rates which is subsequently reduced, via a simplifying assumption, to the more tractable case of constant rate coefficients. Unlike previous analytical investigations into the discrete C–F system, we allow the coagulation rates in (1.1.1) to be time-dependent. Under particular assumptions on the weight and coagulation rates, we establish the existence and uniqueness of physically relevant mild and classical solutions of the semi-linear ACP; see Propositions 6.3.4 and 6.3.7. Our results on the analyticity of the fragmentation semigroup in the autonomous (time-independent) case can be used in conjunction with the theory of interpolation spaces to further relax the conditions required on the coagulation rates to obtain the existence and uniqueness of physically relevant mild and classical solutions; see Theorems 6.5.9, 6.5.10 and 6.5.11. The analyticity of the fragmentation semigroup in the autonomous case is also used in Theorem 7.2.4 to prove the existence of a unique classical solution of the non-autonomous pure fragmentation ACP.

It should be noted that there is also a continuous version of (1.1.1), where the size of a cluster is allowed to take any real positive value. The infinite system of equations (1.1.1) is then replaced by an integro-differential equation of the form

$$\begin{aligned}\frac{\partial}{\partial t}u(x,t) &= -a(x)u(x,t) + \int_x^\infty a(y)b(x,y)u(y,t)\,\mathrm{d}y \\ &\quad + \frac{1}{2}\int_0^x k(x-y,y)u(x-y,t)u(y,t)\,\mathrm{d}y \\ &\quad - u(x,t)\int_0^\infty k(x,y)u(y,t)\,\mathrm{d}y,\ \ t>0; \\ u(x,0) &= \mathring{u}(x), \quad x>0,\end{aligned} \tag{1.1.10}$$

where $u(x,t)$ is the density of clusters of size $x>0$ at time $t \geq 0$. The coefficients and terms in (1.1.10) are defined in an analogous way to those in (1.1.1). For some function $w : [0,\infty) \to [0,\infty)$ such that $w(x) > 0$ for a.e. $x > 0$, the continuous system can be posed as an ACP in L_w^1, where

$$L_w^1 = \left\{ f : [0,\infty) \to \mathbb{R} : \int_0^\infty w(x)|f(x)|\,\mathrm{d}x < \infty \right\} \tag{1.1.11}$$

is equipped with the norm

$$\|f\|_{L^1_w} = \int_0^\infty w(x)|f(x)|\,\mathrm{d}x. \tag{1.1.12}$$

At time $t \geq 0$, the total number of clusters and the total mass of clusters in the system are given, respectively, by

$$\hat{N}(u(x,t)) = \int_0^\infty u(x,t)\,\mathrm{d}x \qquad \text{and} \qquad \hat{M}(u(x,t)) = \int_0^\infty xu(x,t)\,\mathrm{d}x.$$

The case where $w(x) = x$ is of physical relevance to the system since the norm of a non-negative solution will then give the total mass in the system. However, if we need to control both the number of clusters and the mass of clusters, then it is clear that the weight $w(x) = 1 + x$ is most appropriate. In this thesis we concentrate on the discrete system (1.1.1), with the continuous system, (1.1.10), being mentioned only briefly in relation to prior investigations into coagulation–fragmentation.

We now provide an outline of the thesis. In the next chapter we discuss coagulation–fragmentation models and previous results that have been obtained using the semigroup approach that we adopt. In Chapter 3 we provide abstract results regarding Banach lattices and linear operators. We introduce operator semigroups, perturbation results relating to these semigroups and we discuss the connection between operator semigroups and solutions of linear ACPs. These results are useful later when we examine pure fragmentation. We then move on to examine non-linear operators in Chapter 4. In particular, we define what it means for an operator to satisfy a Lipschitz condition and to be Fréchet differentiable. These concepts are important when obtaining existence and uniqueness results for the C–F system as we require the operator used to describe coagulation to possess these properties. Moreover, in Chapter 4 we provide existence and uniqueness results for semi-linear ACPs that we use when examining the ACP formulation of (1.1.1). Chapters 3 and 4 contain mostly known results, and any that are new, or that we have adapted or extended, will be highlighted as they arise. In Chapter 5, we examine pure, autonomous fragmentation. We write this system as a linear ACP in an ℓ_w^1 space and use semigroup perturbation theory to obtain the existence and uniqueness of classical solutions. In Chapter 5 we also investigate properties of this solution including positivity, mass conservation and asymptotic behaviour. The results in Chapter 6 relate to the full coagulation–fragmentation system, (1.1.1), where the coagulation rates are allowed to be time-dependent. We write this system as a semi-linear ACP and show that, under certain assumptions on the coagulation rates, the operator that is used to describe coagulation is Lipschitz and Fréchet differentiable. This then leads to the existence and uniqueness of physically relevant mild and classical solutions to the semi-linear ACP. In Chapter 7 we return to the pure fragmentation system, where the fragmentation coefficients are now time dependent. Under certain assumptions we establish the existence and uniqueness of physically relevant classical solutions for the ACP related to this system. Finally, in Chapter 8, we discuss the conclusions of the work presented in this thesis and possible future steps that can be taken.

We note that part of the material in Chapter 3 and much of Chapter 5 has been accepted for publication in the Journal of Evolution Equations.

Chapter 2

Coagulation–Fragmentation Systems

In this chapter we provide a brief historical account of coagulation–fragmentation (C–F) models and previous investigations into (1.1.1).

2.1 Coagulation–Fragmentation Models

We first turn our attention to C–F models. As mentioned in Chapter 1, the system (1.1.1) originates from work by Smoluchowski, [64, 65], who in 1916 introduced an infinite system of ordinary differential equations, known as the Smoluchowski equation, to model pure coagulation. The coagulation in this case occurs as a result of Brownian motion, where two clusters merge into a larger cluster whenever they become sufficiently close together. A continuous size version of the Smoluchowski equation was later introduced by Müller in 1928, [58]..

C–F equations with binary fragmentation have been extensively studied. In binary fragmentation, each fragmentation event results in the creation of exactly two daughter clusters. This means that we have

$$\sum_{n=1}^{j-1} b_{n,j} = 2 \qquad \text{for } j \geq 2, \tag{2.1.1}$$

since the left-hand side of (2.1.1) gives the total number of daughter clusters produced during the fragmentation of a j-mer. In the case of binary fragmentation, (1.1.1) can be written as

$$\begin{aligned} u_n'(t) = \frac{1}{2}\sum_{j=1}^{n-1}\Big(k_{n-j,j}u_{n-j}(t)u_j(t) - F_{n-j,j}u_n(t)\Big) \\ - \sum_{j=1}^{\infty}\Big(k_{n,j}u_n(t)u_j(t) - F_{n,j}u_{n+j}(t)\Big), \quad t > 0; \\ u_n(0) = \mathring{u}_n, \quad n = 1, 2, \dots, \end{aligned} \tag{2.1.2}$$

where $F_{n,j} = a_{n+j}b_{n,n+j}$ is the rate at which clusters of mass $n + j$ fragment to produce a cluster of mass n and a cluster of mass j; see [18, Equations (2.2.6) and (2.2.7)].

The first discrete model to include both coagulation and fragmentation was due to Becker and Döring in 1935, [20], and models a specific form of binary fragmentation. The Becker–Döring cluster equations examine the particular case where clusters can only merge with one single monomer at a time and each fragmentation of an n-mer results in the production of a monomer and an $(n-1)$-mer. In this case, the coefficients $(b_{n,j})_{n,j\in\mathbb{N}:j>n}$ are given by

$$\begin{aligned} b_{1,2} = 2; \qquad b_{1,j} = b_{j-1,j} = 1, \qquad j \geq 3; \\ b_{n,j} = 0, \qquad 2 \leq n \leq j-2, \end{aligned} \tag{2.1.3}$$

and the coefficients $(k_{n,j})_{n,j\in\mathbb{N}}$ satisfy

$$k_{n,j} = 0, \qquad n, j > 1. \tag{2.1.4}$$

It is assumed that monomers cannot fragment so that $a_1 = 0$. In this case, as in [6, Equation 1.1], we can write the C–F system as

$$\begin{aligned} u_1'(t) &= -J_1(t) - \sum_{n=1}^{\infty} J_n(t); \\ u_n'(t) &= J_{n-1}(t) - J_n(t), \qquad n \geq 2; \\ u_n(0) &= \mathring{u}_n, \qquad n = 1, 2, \dots, \end{aligned} \tag{2.1.5}$$

where

$$J_n(t) = k_{n,1}u_n(t)u_1(t) - F_{n,1}u_{n+1}(t) = k_{n,1}u_n(t)u_1(t) - a_{n+1}b_{n,n+1}u_{n+1}(t)$$

for all $n \in \mathbb{N}$. We note here that in [20], Becker and Döring consider the case in which the concentration of monomers is constant. Hence the equation for u_1 in (2.1.5) does not appear in [20] and was introduced by Burton in 1977, [24].

The first time that a model appeared featuring both coagulation and multiple fragmentation was in 1957, when Melzak formulated a continuous C–F model of the form

$$\begin{aligned} \frac{\partial}{\partial t}u(x,t) = -\frac{u(x,t)}{x}\int_0^x y\varphi(x,y)\,\mathrm{d}y + \int_x^\infty \varphi(y,x)u(y,t)\,\mathrm{d}y \\ + \frac{1}{2}\int_0^x k(x-y,y)u(x-y,t)u(y,t)\,\mathrm{d}y \\ - u(x,t)\int_0^\infty k(x,y)u(y,t)\,\mathrm{d}y, \qquad t>0; \\ u(x,0) = \mathring{u}(x), \quad x > 0, \end{aligned} \tag{2.1.6}$$

see [52]. Note that by setting $\varphi(y,x) = a(y)b(x,y)$ we can obtain (1.1.10) from (2.1.6). In 1957, Melzak also considered a more general form of (2.1.6), where the coefficients $\varphi(x,y)$ and $k(x,y)$ are allowed to be time-dependent, [53], resulting in (2.1.6) being a non-autonomous model.

It is worthwhile noting that the system (1.1.1) is not the only model for discrete C–F processes. For example, equations have been studied in which the fragmentation terms are non-linear; see, for example, [38, 66]. The case of collision-induced fragmentation is considered in [38]. Here, fragmentation occurs due to two clusters colliding, resulting in one of them breaking into two smaller clusters. Another case of collisional fragmentation is studied in [41]. In [41], the authors consider the possibility that a collision between two clusters can produce clusters that are larger than the colliding clusters, e.g. a j-mer and k-mer colliding may produce a $(j+k-1)$-mer and a monomer. Moreover, stochastic models for C–F have also been developed; see, for example, [2, 23, 44, 45].

2.2 Semigroup Approach

There are various approaches that have been used to examine (1.1.1) and we concentrate on a method that is based upon the theory of operator semigroups. The idea is to first consider the case where only fragmentation occurs. The corresponding discrete fragmentation system is given by (1.1.1) with $k_{n,j} = 0$ for all $n, j \in \mathbb{N}$. Similarly, the continuous fragmentation equation is given by (1.1.10), with $k(x, y) = 0$ for all $x, y > 0$. The associated initial-value problem describing the fragmentation process is first written as an abstract Cauchy problem (ACP) posed in some Banach space using two operators, A and B. As mentioned in Chapter 1, in the discrete case, the most physically relevant Banach space to work in is the space $X_{[1]}$, defined by (1.1.5). We also note that the operators A and B will reflect, respectively, the first and second terms on the right-hand side of (1.1.1). A unique classical solution of the fragmentation ACP is then shown to exist and to be defined in terms of a substochastic C_0-semigroup. Finally, the full C–F system is written as a semi-linear ACP and, by treating the operator arising from the coagulation terms as a perturbation of the fragmentation operator, existence and uniqueness results are obtained. One of the major benefits of using the semigroup approach is that the existence and uniqueness of solutions can be obtained simultaneously, while these must be proven separately using other methods. However, the semigroup approach does have restrictions. In particular, the semigroup approach relies on the ability to treat coagulation as a perturbation of fragmentation and so it cannot be used to deal with the case of pure coagulation. We now provide some background on the history of this method in dealing with C–F equations.

Operator semigroups were used for the first time to analyse C–F equations in 1979 when Aizenman and Bak, [1], examined the binary fragmentation version of the continuous C–F equation (1.1.10). In [1], the coagulation and fragmentation coefficients are assumed to be constant and mass is assumed to be conserved during each fragmentation event. The authors begin by considering the linear fragmentation integro-differential equation which they write as an ACP in L^1_w, where $w(x) = x$ for all $x \geq 0$. The strategy used is to examine a sequence of "truncated" ACPs that involve functions which are identically equal to zero on $[n, \infty)$,

$n = 1, 2, \ldots$. Each of these truncated problems is expressed in terms of an appropriately truncated fragmentation operator, which, being bounded, generates a uniformly continuous semigroup. It is then shown that these semigroups converge (as $n \to \infty$) in an appropriate sense to a contraction semigroup, which, in turn, provides a unique, non-negative, mass-conserving classical solution of the original (untruncated) fragmentation ACP for suitably restricted initial conditions. Finally, the full C–F equation is dealt with by expressing it as a semi-linear ACP and treating the coagulation operator as a perturbation of the fragmentation operator.

McLaughlin, Lamb and McBride build upon the work of Aizenman and Bak in [47, 48], where they also examine the continuous equation (1.1.10). However, unlike [1], they allow more than two daughter clusters to be created during a single fragmentation event, and non-constant coagulation and fragmentation coefficients are permitted. The methods used in [47, 48] are similar to those in [1]. In [47], the fragmentation ACP is considered and a classical solution is obtained by considering first a sequence of truncated, and more tractable, ACPs, and then using a limit argument. The full C–F equation is then examined in [48], where the coagulation terms are treated as a perturbation of the fragmentation terms.

The first time that results associated with substochastic semigroups were used to deal with pure fragmentation was in 2001 by Banasiak, [7], who examined continuous fragmentation with specific fragmentation coefficients. As in [1], the fragmentation equation is written as an ACP in L^1_w, with $w(x) = x$ for all $x \geq 0$, using two operators, A and B. Banasiak then shows that these operators satisfy the conditions of the Kato–Voigt Perturbation Theorem, [71, Proposition 1.4], and concludes that there exists an extension, G, of $A + B$ that generates a substochastic C_0-semigroup. It is then shown that $G = \overline{A + B}$ is the generator using a method based on Arlotti extensions; see [12, §6.3].

In 2010, the Kato–Voigt Perturbation Theorem was applied to the discrete fragmentation system by McBride, Smith and Lamb, [46]. The authors work with general fragmentation coefficients which satisfy the natural assumptions that monomers cannot fragment and that mass is conserved during each fragmentation event. Mathematically, this leads to the assumptions

$$a_1 = 0 \qquad \text{and} \qquad \sum_{n=1}^{j-1} n b_{n,j} = j. \tag{2.2.1}$$

The discrete fragmentation system is posed as a linear ACP in $X_{[1]}$ using two operators A and B. The Kato–Voigt Perturbation Theorem and an Arlotti extension argument is then used to show that $G = \overline{A+B}$ is the generator of a stochastic C_0-semigroup, $(S(t))_{t\geq 0}$. In the particular case where $(a_n)_{n=1}^{\infty}$ is monotone increasing, the authors show that the domain of A, $\mathcal{D}(A) = \mathcal{D}(A+B)$, is invariant under $(S(t))_{t\geq 0}$ and it is concluded that $(S(t))_{t\geq 0}$ provides a unique, non-negative, mass-conserving classical solution to the fragmentation ACP for any initial condition in $\mathcal{D}(A)$. The authors then move on to examine the full system (1.1.1), where the coagulation rates are uniformly bounded, i.e. there exists some $k > 0$ such that

$$k_{n,j} \leq k \qquad \text{for all } n, j \in \mathbb{N}. \tag{2.2.2}$$

The full C–F system is written as a semi-linear ACP in $X_{[1]}$ and the non-linear operator, corresponding to the coagulation terms, is shown to satisfy a Lipschitz condition and to be Fréchet differentiable. This yields the local existence of unique mild and classical solutions of the semi-linear ACP. These solutions are shown to be positive for positive initial conditions and, by proving that there is no finite-time blow up, it is concluded that the solutions also exist globally.

The discrete fragmentation system has since attracted a lot of interest and a number of subsequent investigations have been carried out. In 2012, these equations were examined by Smith, Lamb, Langer and McBride, [63], who, rather than assuming (2.2.1), allow the possibility that mass is lost during a fragmentation event and that monomers may be removed from the system. The fragmentation system is written as a linear ACP in the space $X_{[1]}$, as in [46]. The same methods as used in [46] for the mass-conserving case then show that $G = \overline{A+B}$ is the generator of a substochastic semigroup, $(S(t))_{t\geq 0}$, and, when $(a_n)_{n=1}^{\infty}$ is increasing, $(S(t))_{t\geq 0}$ provides a unique classical solution of the fragmentation ACP for any initial condition in the domain $\mathcal{D}(A)$.

The authors of [63] proceed to examine the particular example of "random bond annihilation”. Here, it is assumed that clusters are long strings of particles that are linked by bonds. A fragmentation event occurs when a bond is broken. The mass is taken to be contained within the bonds between particles and so each fragmentation event results in mass being lost from the system. It is assumed that only binary fragmentation occurs and that bonds are annihilated randomly and with equal probability. This results in

$$a_n = n \qquad \text{and} \qquad b_{n,j} = \frac{2}{j} \qquad \text{for all } n, j \in \mathbb{N} \text{ such that } j > n. \tag{2.2.3}$$

In this particular case, the authors use an inductive argument to obtain an explicit expression for the semigroup $(S(t))_{t\geq 0}$ and then for the resolvent of its infinitesimal generator, G. These expressions enable the analyticity of $(S(t))_{t\geq 0}$ to be established. An explicit description of the domain $\mathcal{D}(G)$ is also deduced and a unique, non-negative classical solution of the fragmentation ACP is obtained for all non-negative initial conditions in $\mathcal{D}(G)$. Analytic semigroups have very desirable properties, as we discuss later. In particular, in [63], the analyticity of the semigroup $(S(t))_{t\geq 0}$ is exploited to enable an extrapolation space to be constructed and used to explain an apparent non-uniqueness of solutions emanating from a zero initial condition.

While the majority of past investigations into (1.1.1) have concentrated on writing the system as an ACP in $X_{[1]}$, the system has also been considered in the moment spaces $X_{[p]}$, for $p \geq 1$. We recall that $X_{[p]} = \ell_w^1$ with $w_n = n^p$ for all $n \in \mathbb{N}$. When we work in $X_{[p]}$, we denote the fragmentation operators as $A_{[p]}$ and $B_{[p]}$. In 2012, Banasiak considered the mass-conserving fragmentation system (i.e. (2.2.1) is satisfied) in $X_{[p]}$ spaces, [9]. The same methods as in [46] show that $G_{[p]} = \overline{A_{[p]} + B_{[p]}}$ is the generator of a substochastic C_0-semigroup and additional assumptions are provided under which we can conclude that $G_{[p]} = A_{[p]} + B_{[p]}$. Moreover, for $p > 1$, conditions are given under which the semigroup generated by $G_{[p]} = A_{[p]} + B_{[p]}$ is analytic and, in this case, the theory of interpolation spaces is used to examine the full C–F system.

We note, however, that simple examples of fragmentation coefficients are given in an appendix to [9] for which $G_{[p]} \neq A_{[p]} + B_{[p]}$ for any $p \geq 1$, and the semigroup generated is not analytic. It is noted that this is true, in particular, if the coefficients $(b_{n,j})_{n,j\in\mathbb{N}:j>n}$ are defined as in (2.1.3) and we take $a_1 = 0$, $a_n = n$ for $n \geq 2$.

After the existence and uniqueness of solutions has been established, another area of interest is the long-term behaviour of solutions. The asymptotic behaviour of solutions to the pure fragmentation system was examined in 2011 by Banasiak, [8], where he considers the discrete, mass-conserving fragmentation system in $X_{[1]}$. He shows that the solution converges to a state consisting entirely of monomers if and only if $a_n > 0$ for all $n \geq 2$. Additional conditions are also provided in [8] under which the solution can be shown to converge to the monomer state at an exponential rate. In 2012, this work was extended by Banasiak and Lamb in [15], where the results in [8] are shown to also hold in the moment spaces, $X_{[p]}$, for $p \geq 1$.

As well as the long-term behaviours mentioned above, there are other asymptotic behaviours that have been shown to arise for certain coagulation and fragmentation coefficients. Since the asymptotic behaviour of solutions is of great interest and importance, we include some references in the following paragraph to results regarding asymptotic behaviour that have been obtained by alternative methods to the semigroup approach that is adopted in this thesis.

As mentioned in the introduction, a phenomenon known as gelation is possible for certain coagulation and fragmentation coefficients, where clusters of infinite size are created at some finite gel time $T_{gel} > 0$. Gelation was shown to occur for the first time in [42], for coagulation coefficents that grow “fast enough” as cluster size increases. It was then shown in [39] that gelation can also occur when fragmentation, satisfying a certain boundedness assumption, is added to the system in [42]. There has been a lot interest in the topic of gelation and related work can be found, for example, in [26, 31, 77]. Moreover, the approach of solutions to similarity solutions has also been studied extensively in the case of pure fragmentation, pure coagulation and the full coagulation–fragmentation system; see, for example, [51, 55, 54, 56, 60, 79]

It is worthwhile noting that equations based on (1.1.1) but incorporating other factors have also been considered. For example, [13] examines the system (1.1.1), where additional growth and decay terms are included. These additional terms are important in applications such as animal groupings, where groups of animals can split apart, merge together, lose individuals through death and gain individuals through birth.

If one or more of the coagulation or fragmentation coefficients are time dependent, then we are dealing with a non-autonomous model and this case is also of interest. Indeed, as already mentioned, the original derivation of Smoluchowski's coagulation equation contained time-dependent rate coefficients until simplifying assumptions were made to obtain an autonomous system; see [18, §2.2.1] for a summary of the derivation. Non-autonomous coagulation systems play an important role in applications relating to the powder production industry, where industrial spray drying causes the coagulation of small particles and produces powders that possess desired physical properties. A model is investigated in [73] for coagulation and droplet transport in a spray dryer's hollow conical spray. A "slurry" of small droplets is created at the top of the tower and, as the droplets fall to the bottom of the tower, they dry and merge together to form a powder; see [73, §2.1.1] for a detailed account of the physical configuration of the dryer tower examined. The volume of a droplet and its distance from the top of the tower are denoted, respectively, by x and z and the state of a droplet is described by (x, z). An equation that describes these coalescing clusters is derived and it is shown that if we assume the existence of a steady-state solution, then the resulting equation can be interpreted as a continuous non-autonomous Smoluchowski equation, with z being regarded as a "time-like" variable; see [73, (3.9)].

Moreover, in the pioneering work by Melzak into the continuous C–F equation, [53], the possibility of time-dependent coefficients is included. The continuous non-autonomous fragmentation equation has also been investigated by McLaughlin, Lamb and McBride in 1997, [49], using the theory of evolution families. In [49], mass is assumed to be conserved during each fragmentation event and the authors prove that, under a certain Lipschitz assumption on the fragmentation rates, there exists a non-negative, mass-conserving solution of the continuous fragmentation equation for all compactly supported initial conditions. Moreover, this is shown to be unique in the sense that it is the only solution that is also a classical solution of the corresponding non-autonomous fragmentation ACP. Under the additional assumption that the Lipschitz condition is uniform in time, the authors are able to demonstrate that there exists at least one mass-conserving solution of the non-autonomous equation for an arbitrary initial condition.

The same equation was examined again in 2010 by Arlotti and Banasiak, [4], but this time using the theory of evolution semigroups. The assumptions required in [4] are weaker than those in [49] but a solution is only obtained for an integrated version of the continuous non-autonomous fragmentation equation. This solution is shown to be given by an evolution family and to be non-negative and mass conserving for non-negative initial conditions. However, neither the uniqueness of the solution nor the existence of a solution to the corresponding non-autonomous ACP is mentioned.

As well as the semigroup approach, weak-compactness methods have also been used to deal with (1.1.1). In this approach finite-dimensional truncations of the full C–F system are considered and a sequence of solutions to these truncated equations is obtained. Weak compactness arguments can then be used to show that this sequence has a subsequence that converges to a solution of an integrated version of the full C–F system. Additional assumptions are required to prove the uniqueness of the solution. This method was first introduced by Ball, Carr and Penrose in 1986 to deal with the Becker–Döring cluster equations, [6]. The weak compactness approach has since been used to deal with the C–F system with binary fragmentation, see [5, 27], and was first used to deal with the C–F system with multiple fragmentation by Laurençot, [40]. In [40], the author works with fragmentation coefficients satisfying (2.2.1) and coagulation coefficients satisfying

$$0 \leq k_{n,j} \leq K(n+j), \qquad n, j \geq 1,$$

for some $K > 0$. Of particular relevance to the work in this thesis is [40, Theorem 2.5]. In this result, the existence and uniqueness of solutions to (1.1.1) in ℓ_w^1 is obtained, where $w = (w_n)_{n=1}^{\infty}$ may not take the "usual" form of $w_n = n^p$ for some $p \geq 1$ and all $n \in \mathbb{N}$. In fact the existence and uniqueness of solutions in ℓ_w^1 is obtained for any $w = (w_n)_{n=1}^{\infty}$ satisfying particular properties described in [40]. The majority of the work presented in this thesis also works with weights that are more general than the usual choice of $w_n = n^p$, for some $p \geq 1$ and all $n \in \mathbb{N}$. However, we restrict our attention to methods involving the theory of operator semigroups rather than using a weak-compactness approach.

Chapter 3

Banach Lattices and Linear Operators

When we examine the discrete C–F system, (1.1.1), we begin by considering the case when only fragmentation occurs (i.e. when $k_{n,j} = 0$ for all $n, j \in \mathbb{N}$). We write the resulting fragmentation system as an abstract Cauchy problem (ACP) in an appropriate Banach space. Since the pure fragmentation equation arising from (1.1.1) is linear, so is the resulting fragmentation ACP. Linear operators are therefore vital in our investigation into the pure fragmentation system and we dedicate the bulk of this chapter to the examination of these operators. We begin in Section 3.1 and 3.2 by providing some prerequisite results and concepts on Banach lattices and linear operators. In Section 3.3 we examine operator semigroups, which are crucial in the methods that we use to examine the discrete C–F system. Section 3.4 concentrates on linear abstract Cauchy problems and their solutions and gives results that are closely linked to the semigroups introduced in Section 3.3; these will help us to obtain solutions to the pure fragmentation system. While many of the results in this chapter are known, there are some that extend or generalise those that have previously been obtained. Moreover, we also present several results that we believe to be new and we point these out at the start of each section, and again before they are stated.

3.1 Banach Lattices

In this section we provide some preliminary results on Banach lattices that we require later when we examine the discrete C–F system. Much of this subsection is based on material in [12, Section 2.2] and, unless otherwise stated, we assume X to be a real vector space. We begin by introducing a partial order. We note that, while a partial order can be defined on any set, we are only interested here in a partial order on a vector space X.

Definition 3.1.1. Let X be a vector space. A *partial order* on X is a binary relation, denoted by $\leq$, which satisfies the following conditions for all $f, g, h \in X$:

(i) $f \leq f$;

(ii) if $f \leq g$ and $g \leq f$ then $f = g$;

(iii) if $f \leq g$ and $g \leq h$ then $f \leq h$.

Note that if $f, g \in X$, then, by definition, $g \geq f$ is equivalent to $f \leq g$ and in this case we say that g is greater than or equal to f or, equivalently, that f is less than or equal to g. We refer to an element $f \in X$ as being *positive* if $f \geq 0$, where 0 is the zero element in X.

Example 3.1.2. Assume that X is a vector space of real-valued functions defined on a set S. Then we define a partial order on X in a pointwise sense. Suppose $f, g \in X$. Then we say that $f \leq g$ if $f(x) \leq g(x)$ for all $x \in S$.

Remark 3.1.3. We refer to $\leq$ as a partial order on X, rather than a (total) order, as it is not always possible to compare two elements in X. For example, if $f, g \in X$ and X is as in Example 3.1.2 then we may have $f(x) > g(x)$ for some $x \in S$, while $f(x) < g(x)$ for other $x \in S$.

Using a partial order we can define what it means for a vector space X to be an ordered vector space.

Definition 3.1.4. An *ordered vector space* is a real vector space X equipped with a partial ordering that is compatible with the vector structure of X, in the sense that

(i) if $f \leq g$ in X then $f + h \leq g + h$ for all $h \in X$;

(ii) if $f \leq g$ in X then $\alpha f \leq \alpha g$ for all $\alpha \geq 0$.

Note that the space X in Example 3.1.2 is an ordered vector space. We are now able to state what is meant by the positive cone in the vector space X that is associated with the partial order $\leq$.

Definition 3.1.5. Let X be an ordered vector space and let U be a subset of X. The set $X_+ \coloneqq \{f \in X : f \geq 0\}$ is called the *positive cone* of X and we define $U_+ \coloneqq \{f \in U : f \geq 0\}$.

Note that for f, $g \in X_+$ and $\alpha \geq 0$,

$$f + g \in X_+ \qquad \text{and} \qquad \alpha f \in X_+.$$

In many cases, the positive cone X_+ has the additional property of being a generating cone in the following sense.

Definition 3.1.6. We say that X_+ is a *generating cone* for X if every $f \in X$ has a decomposition $f = g - h$ where $g, h \in X_+$.

It is not always the case that X_+ is a generating cone, as the next example shows.

Example 3.1.7. Let X be a vector space consisting of more than one element and let $f, g \in X$. We define $\leq$ by

$$f \leq g \text{ if and only if } f = g.$$

Then $\leq$ defines a partial order on X. In this case the positive cone, X_+, is given by $X_+ = \{f \in X : f = 0\}$. Take a non-zero $f \in X$. Then there do not exist $g, h \in X_+$ such that $f = g - h$. Thus X_+ is not a generating cone.

The partial order on X can be used to define what is meant by the supremum and infimum of a set $\mathcal{S} \subseteq X$.

Definition 3.1.8. Let X be an arbitrary ordered vector space and let $\mathcal{S} \subseteq X$.

(i) An element $f \in X$ is a *supremum* of $\mathcal{S}$ if

   (a) $f \geq g$ for all $g \in \mathcal{S}$;

   (b) whenever $h \in X$ is such that $h \geq g$ for all $g \in \mathcal{S}$, then $h \geq f$.

(ii) An element $f \in X$ is an *infimum* of $\mathcal{S}$ if

   (a) $f \leq g$ for all $g \in \mathcal{S}$;

   (b) whenever $h \in X$ is such that $h \leq g$ for all $g \in \mathcal{S}$, then $h \leq f$.

A supremum of a set does not always exist. However, if a supremum does exist then it is unique. Analogous statements hold for infima.

Example 3.1.9. Suppose that X and S are as in Example 3.1.2 and let $f, g \in X$. If the functions $f \vee g$ and $f \wedge g$, given by

$$(f \vee g)(x) = \max\{f(x), g(x)\}, \quad x \in S, \tag{3.1.1}$$

and

$$(f \wedge g)(x) = \min\{f(x), g(x)\}, \quad x \in S, \tag{3.1.2}$$

exist as elements in X then

(i) $\sup\{f, g\} = f \vee g$;

(ii) $\inf\{f, g\} = f \wedge g$.

As mentioned above, and as we now show, the supremum or infimum of two elements need not exist. Let X be the space of continuously differentiable functions on $\mathbb{R}$ with the natural pointwise order. Set $f(x) = x$ and $g(x) = -x$ so that $f, g \in X$. Then, if $\sup\{f, g\}$ exists, it must be of the form

$$\sup\{f, g\}(x) = |x|.$$

Similarly, if $\inf\{f, g\}$ exists, then it is of the form

$$\inf\{f, g\}(x) = -|x|.$$

Both $|x|$ and $-|x|$ have a corner and so are not elements in X.

The concept of a vector lattice will be important in later chapters.

Definition 3.1.10. A *vector lattice*, or a *Riesz space*, is an ordered vector space such that for any two elements f, g, there exist $\sup\{f, g\}$ and $\inf\{f, g\}$.

Example 3.1.11. Define elements $f \vee g$ and $f \wedge g$ as in (3.1.1) and (3.1.2). Let $X = C(\mathbb{R})$ and let $f, g \in X$. Then $f \vee g$ and $f \wedge g \in X$ and it follows that $\sup\{f, g\} = f \vee g$ and $\inf\{f, g\} = f \wedge g$ exist. Hence $C(\mathbb{R})$ is a vector lattice.

Lemma 3.1.12. *Let $w = (w_n)_{n=1}^{\infty}$ be such that $w_n > 0$ for all $n \in \mathbb{N}$ and let*

$$\ell_w^1 = \left\{ f = (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} w_n |f_n| < \infty \right\}. \tag{3.1.3}$$

Then ℓ_w^1 is a vector lattice.

Proof. Suppose $f, g \in \ell_w^1$. Then the elements $f \vee g$ and $f \wedge g$ are given by

$$(f \vee g)_n = \max\{f_n, g_n\} \quad \text{and} \quad (f \wedge g)_n = \min\{f_n, g_n\}, \quad n = 1, 2, \ldots.$$

Clearly, $(f \vee g)_n$ and $(f \wedge g)_n \in \mathbb{R}$ for all $n \in \mathbb{N}$. Also,

$$\sum_{n=1}^{\infty} w_n |(f \vee g)_n| = \sum_{n=1}^{\infty} w_n |\max\{f_n, g_n\}| \leq \sum_{n=1}^{\infty} w_n |f_n| + \sum_{n=1}^{\infty} w_n |g_n| < \infty.$$

Similarly,

$$\sum_{n=1}^{\infty} w_n |(f \wedge g)_n| = \sum_{n=1}^{\infty} w_n |\min\{f_n, g_n\}| \leq \sum_{n=1}^{\infty} w_n |f_n| + \sum_{n=1}^{\infty} w_n |g_n| < \infty.$$

Hence, $f \vee g$ and $f \wedge g \in \ell_w^1$ and so $\sup\{f, g\} = f \vee g$ and $\inf\{f, g\} = f \wedge g$ exist. Therefore ℓ_w^1 is a vector lattice. $\square$

Definition 3.1.13. Let X be a vector lattice. For an element $f \in X$ we define the *positive part*, *negative part* and *modulus* of f, respectively, by

$$f_+ \coloneqq \sup\{f, 0\}; \quad f_- \coloneqq \sup\{-f, 0\}; \quad |f| \coloneqq \sup\{f, -f\}.$$

From Definition 3.1.13, we have that $f_+ \in X_+$ and $f_- \in X_+$ for all $f \in X$. The following result can be found in [12, Proposition 2.46].

Proposition 3.1.14. *If X is a vector lattice and $f \in X$, then*

$$f = f_+ - f_- \tag{3.1.4}$$

and

$$|f| = f_+ + f_-. \tag{3.1.5}$$

Thus, in particular, if X is a vector lattice, then X_+ is a generating cone for X. By (3.1.5)*, we have that $|f| \in X_+$ for all $f \in X$.*

A vector lattice then leads on to the idea of a lattice norm.

Definition 3.1.15. Let X be a vector lattice and suppose that $\|\cdot\|$ is a norm on X.

(a) We say that $\|\cdot\|$ is a *lattice norm* if

$$|f| \leq |g| \Rightarrow \|f\| \leq \|g\|.$$

(b) If X is also complete under this lattice norm, then we say that X is a *Banach lattice.*

Note that if X is a vector lattice with norm $\|\cdot\|$, then the positive cone, X_+, is closed.

For a normed lattice, X, it will be useful for us to know when a dense subspace, U, of X is such that U_+ is dense in X_+. We provide the following lemma which gives a sufficient condition for this to be true.

Lemma 3.1.16. *Let X be a normed lattice and suppose that U is a dense subspace of X, such that for all $f \in U$ we have $f_+, f_- \in U_+$. Then U_+ is dense in X_+.*

Proof. Suppose that $f \in X_+$. Since U is dense in X, there exists a sequence $(g^{(n)})_{n=1}^{\infty}$, with $g^{(n)} \in U$ for all $n \in \mathbb{N}$, such that $g^{(n)} \to f$ as $n \to \infty$. We have, for each $n \in \mathbb{N}$, $g^{(n)} = (g^{(n)})_+ - (g^{(n)})_-$. Since X is a normed lattice, we can deduce from [12, Proposition 2.55], that the mappings $g^{(n)} \mapsto (g^{(n)})_+$ and $g^{(n)} \mapsto (g^{(n)})_-$ are continuous, and it follows that $(g^{(n)})_- \to f_- = 0$. Hence $(g^{(n)})_+ \to f_+ = f$ as $n \to \infty$. Since $(g^{(n)})_+ \in U_+$ for all $n \in \mathbb{N}$, it follows that U_+ is dense in X_+. □

We can now state what it means for a space to be an AL-space.

Definition 3.1.17. Let X be a Banach lattice and let $\|\cdot\|$ be a norm on X.

(i) The norm $\|\cdot\|$ is *additive on the positive cone*, X_+, if

$$\|f + g\| = \|f\| + \|g\| \qquad \text{for all } f, g \in X_+. \tag{3.1.6}$$

(ii) If $\|\cdot\|$ is a lattice norm and (3.1.6) is satisfied, then X is an *AL-space*, see [12, Definition 2.56].

Lemma 3.1.18. *Let X be an AL-space with norm $\|\cdot\|$. Then there exists a unique, bounded, linear functional, ϕ, on X, which coincides with $\|\cdot\|$ on X_+.*

Proof. For all $f \in X$, let

$$\phi(f) = \|f_+\| - \|f_-\|. \tag{3.1.7}$$

For $f \in X_+$, we have $f_- = 0$. Hence

$$\phi(f) = \|f_+\| - \|0\| = \|f_+\| = \|f_+ - 0\| = \|f\| \geq 0$$

for all $f \in X_+$. Also, for f, $g \in X$,

$$\phi(f + g) = \|f_+ + g_+\| - \|f_- + g_-\| = \|f_+\| + \|g_+\| - \|f_-\| - \|g_-\| = \phi(f) + \phi(g)$$

where the second equality follows since X is an AL-space. Now let $\alpha \geq 0$ and $f \in X$. We have

$$\phi(\alpha f) = \|\alpha f_+\| - \|\alpha f_-\| = \alpha(\|f_+\| - \|f_-\|) = \alpha\phi(f).$$

Moreover, for $f \in X$, $-f = f_- - f_+$ and so $(-f)_+ = f_-$ and $(-f)_- = f_+$. It follows that

$$\phi(-f) = \|f_-\| - \|f_+\| = -(\|f_+\| - \|f\|_-) = -\phi(f).$$

It is therefore clear that the functional ϕ is a positive, linear functional that coincides with $\|\cdot\|$ on X_+. To show uniqueness, let γ be another positive, linear operator that coincides with $\|\cdot\|$ on X_+. Then, for $f \in X$,

$$\gamma(f) = \gamma(f_+ - f_-) = \gamma(f_+) - \gamma(f_-) = \|f_+\| - \|f_-\| = \phi(f),$$

where the linearity of γ is used to obtain the first inequality. This proves uniqueness.

Finally, the functional ϕ is bounded since

$$|\phi(f)| = |\phi(f_+) - \phi(f_-)| = |\|f_+\| - \|f_-\|| \leq \|f_+\| + \|f_-\| = \|f_+ + f_-\| = \||f|\|$$

for all $f \in X$. Note that the first equality follows from the linearity of ϕ and the third equality follows from the fact that X is an AL-space. □

Example 3.1.19. Consider the space ℓ^1_w in Example 3.1.11(ii) equipped with the norm

$$\|f\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n |f_n|. \tag{3.1.8}$$

For each $f \in \ell^1_w$, $(f_+)_n = \max\{f_n, 0\}$ and $(f_-)_n = \max\{-f_n, 0\}$. Then clearly $f_+, f_- \in (\ell^1_w)_+$ and $f = f_+ - f_-$. Using (3.1.5), the modulus $|\cdot|$ on ℓ^1_w is $|f| = (|f_n|)_{n=1}^{\infty}$.

Also, as we now show, the norm on ℓ^1_w is a lattice norm. Suppose $f, g \in \ell^1_w$ are such that $|f| \leq |g|$. By considering sequences $\tilde{f}, \tilde{g} \in \ell^1_w$ as functions of n then, using the definition of $\leq$ given in Example 3.1.2, we have that $\tilde{f} \leq \tilde{g}$ if and only if $\tilde{f}_n \leq \tilde{g}_n$ for all $n \in \mathbb{N}$. Consequently, $|f| \leq |g|$ implies that $|f_n| \leq |g_n|$ for all $n \in \mathbb{N}$, and therefore

$$\|f\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n |f_n| \leq \sum_{n=1}^{\infty} w_n |g_n| = \|g\|_{\ell^1_w}.$$

So the norm on ℓ^1_w is a lattice norm and, since ℓ^1_w is complete with respect to this norm, ℓ^1_w is a Banach lattice. Let $f, g \in (\ell^1_w)_+$. Then

$$\|f+g\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n|(f+g)_n| = \sum_{n=1}^{\infty} w_n|f_n| + \sum_{n=1}^{\infty} w_n|g_n| = \|f\|_{\ell^1_w} + \|g\|_{\ell^1_w}.$$

Thus, ℓ^1_w is an AL-space and the unique, bounded, linear extension, $\phi_{\ell^1_w}$, of $\|\cdot\|_{\ell^1_w}$ (see Lemma 3.1.18) is given, for all $f \in \ell^1_w$, by

$$\phi_{\ell^1_w}(f) = \sum_{n=1}^{\infty} w_n f_n. \tag{3.1.9}$$

Remark 3.1.20. At the start of this chapter we assumed X be to a real vector space. However, there are occasions when it is necessary to work in a complex space. We can move from a real space to a complex space using a process called *complexification*; see [12, Section 2.2.5]. As in [12, Definition 2.85], for a real vector lattice, X, we define X_c by

$$X_c = \{(x,y) : x, y \in X\}.$$

We write $(x,y) \in X_c$ as $x + iy$. Let $x_0, y_0, x_1, y_1 \in X$. As in [12, Definition 2.85], the vector operations are defined by

$$\begin{aligned} x_0 + iy_0 + x_1 + iy_1 &= x_0 + x_1 + i(y_0 + y_1), \\ (\alpha + i\beta)(x_0 + iy_0) &= \alpha x_0 - \beta y_0 + i(\beta x_0 + \alpha y_0). \end{aligned}$$

Moreover, the partial order on X_c is given by

$$x_0 + iy_0 \leq x_1 + iy_1 \text{ if and only if } x_0 \leq x_1 \text{ and } y_0 = y_1.$$

As in [12, Section 2.2.5], the norm, $\|\cdot\|_c$, on X_c is defined by

$$\|z\|_c = \||x+iy|\|,$$

for all $z = x + iy \in X_c$, where

$$|x + iy| = \sup_{\theta \in [0,2\pi]} \Big(x\cos\theta + y\sin\theta \Big).$$

The concepts introduced in this section for real vector spaces can be defined analogously in complex vector spaces.

3.2 Linear Operators

Throughout this thesis, if U is a linear or nonlinear operator in a vector space, X, then we denote the domain of U by $\mathcal{D}(U)$. In this section we devote our attention to linear operators.

Definition 3.2.1. Let X be an ordered vector space and $U : \mathcal{D}(U) \to X$ be a linear operator, where $\mathcal{D}(U) \subseteq X$. Then we say that U is a *positive operator* if $Uf \in X_+$ for all $f \in \mathcal{D}(U)_+$. In this case we write $U \geq 0$.

Note that the above definition of positivity also holds for non-linear operators. We now introduce a partial order on $\mathcal{B}(X)$, the space of bounded, linear operators on an ordered vector space X, as follows.

Definition 3.2.2. Let X be an ordered vector space and let $U, V \in \mathcal{B}(X)$. Then we write $U \leq V$ if and only if $V - U \geq 0$.

Remark 3.2.3. The partial order above can be introduced on any space of operators that map an ordered vector space X into itself.

Lemma 3.2.4. *Let U be a positive, linear operator in a vector lattice, X, with lattice norm $\|\cdot\|$. Then $\|U|f|\| \geq \|Uf\|$ for all $f \in X$.*

Proof. Let $f \in X$. It is clear that $|f| \geq f$ and so

$$U|f| - Uf = U(|f| - f) \geq 0$$

since U is a positive operator. It follows that $U|f| \geq Uf$. Similarly, $|f| \geq -f$. Hence, $U|f| \geq U(-f) = -Uf$. Thus, $|Uf| = \sup\{Uf, -Uf\} \leq U|f| = \Big|U|f|\Big|$.

Since $\|\cdot\|$ is a lattice norm, it follows that

$$\|Uf\| \leq \|U|f|\|. \qquad \square$$

We now define what we mean by an extension of an operator. Note that this definition holds for both linear and nonlinear operators.

Definition 3.2.5. Let U, V be operators in a vector space X. Then V is said to be an *extension* of U if $\mathcal{D}(U) \subseteq \mathcal{D}(V)$ and $Uf = Vf$, for all $f \in \mathcal{D}(U)$. This is equivalent to the graph of U being a subset of the graph of V, i.e.

$$\{(f, Uf) : f \in \mathcal{D}(U)\} \subseteq \{(g, Vg) : g \in \mathcal{D}(V)\}.$$

If V is an extension of U, then we write $U \subseteq V$.

A concept that we also require is that of one space being continuously embedded in another space.

Definition 3.2.6. Let $(X, \|\cdot\|_X), (Y, \|\cdot\|_Y)$ be normed vector spaces such that $Y \subseteq X$. We say that $(Y, \|\cdot\|_Y)$ is *continuously embedded* in $(X, \|\cdot\|_X)$ (written as $Y \hookrightarrow X$) if there exists $C \geq 0$ such that

$$\|f\|_X \leq C\|f\|_Y,$$

for all $f \in Y$.

From the definition above it is clear that the embedding operator is bounded. We note that if, in addition, the linear embedding operator

$$\iota : Y \to X, \ f \mapsto f \qquad \text{for all } f \in Y,$$

is compact, then we say that $(Y, \|\cdot\|_Y)$ is *compactly embedded* in $(X, \|\cdot\|_X)$.

We now state and prove a proposition that is used later when we examine the discrete C–F system in weighted ℓ^1 spaces.

Proposition 3.2.7. *Let $(w_n)_{n=1}^{\infty}$ and $(v_n)_{n=1}^{\infty}$ be sequences such that w_n, $v_n > 0$, for all $n \in \mathbb{N}$. Consider the spaces $(\ell^1_w, \|\cdot\|_{\ell^1_w})$ and $(\ell^1_v, \|\cdot\|_{\ell^1_v})$, where ℓ^1_w and $\|\cdot\|_{\ell^1_w}$*

are as defined in (3.1.3) *and* (3.1.8) *respectively, and* ℓ_v^1, $\|\cdot\|_{\ell_v^1}$ *are defined in an analogous manner. Let* $\left(\frac{w_n}{v_n}\right)_{n=1}^{\infty}$ *be bounded. Then* $(\ell_v^1, \|\cdot\|_{\ell_v^1})$ *is continuously embedded in* $(\ell_w^1, \|\cdot\|_{\ell_w^1})$.

Also, under the stronger condition,

$$\frac{w_n}{v_n} \to 0 \qquad \text{as } n \to \infty,$$

$(\ell_v^1, \|\cdot\|_{\ell_v^1})$ *is compactly embedded in* $(\ell_w^1, \|\cdot\|_{\ell_w^1})$.

Proof. Let $\left(\frac{w_n}{v_n}\right)_{n=1}^{\infty}$ be bounded. Then there exists a constant $C > 0$ such that $\frac{w_n}{v_n} \leq C$ for all $n \in \mathbb{N}$ and therefore, for all $f \in \ell_v^1$,

$$\|f\|_{\ell_w^1} = \sum_{n=1}^{\infty} w_n|f_n| = \sum_{n=1}^{\infty} \frac{w_n}{v_n} v_n|f_n| \leq C \sum_{n=1}^{\infty} v_n|f_n| = C\|f\|_{\ell_v^1}.$$

It follows that $(\ell_v^1, \|\cdot\|_{\ell_v^1})$ is continuously embedded in $(\ell_w^1, \|\cdot\|_{\ell_w^1})$.

Now suppose that we also have $\frac{w_n}{v_n} \to 0$ as $n \to \infty$, and consider the embedding operator

$$\iota : \ell_v^1 \to \ell_w^1, \ f \mapsto f, \qquad \text{for all } f \in \ell_v^1.$$

For each $k \in \mathbb{N}$, let $\iota^{(k)} : \ell_v^1 \to \ell_w^1$ be the operator defined by $\iota^{(k)} f = f^{(k)}$, where,

$$f_n^{(k)} = \begin{cases} f_n & \text{when } n \leq k, \\ 0 & \text{when } n > k. \end{cases} \tag{3.2.1}$$

Since $\iota^{(k)}$ is a finite rank operator, it is compact for each $k \in \mathbb{N}$. Choose $\varepsilon > 0$. There exists $N \in \mathbb{N}$ such that $\frac{w_n}{v_n} < \varepsilon$ if $n \geq N$. Hence, for $f \in \ell_v^1$ and $k \geq N$, we obtain

$$\begin{aligned} \|(\iota^{(k)} - \iota)f\|_{\ell_w^1} &= \sum_{n=1}^{k} w_n|f_n - f_n| + \sum_{n=k+1}^{\infty} w_n|f_n| = \sum_{n=k+1}^{\infty} w_n|f_n| \\ &= \sum_{n=k+1}^{\infty} \frac{w_n}{v_n} v_n|f_n| \leq \varepsilon \sum_{n=k+1}^{\infty} v_n|f_n| \leq \varepsilon\|f\|_{\ell_v^1}. \end{aligned}$$

It follows that $\|\iota^{(k)} - \iota\| \leq \varepsilon$ if $k \geq N$. Thus $\iota^{(k)} \to \iota$ as $k \to \infty$ and so ι is compact. Thus $(\ell^1_v, \|\cdot\|_{\ell^1_v})$ is compactly embedded in $(\ell^1_w, \|\cdot\|_{\ell^1_w})$. $\square$

We now give an example where the above holds.

Example 3.2.8. Suppose $w_n = n^p$ for some $p \geq 0$. Then we obtain the Banach space $\ell^1_w := X_{[p]}$ defined by

$$X_{[p]} = \left\{ (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} n^p |f_n| < \infty \right\}$$

equipped with the norm

$$\|f\|_{[p]} = \sum_{n=1}^{\infty} n^p |f_n|.$$

Let $q > p \geq 0$. Then

$$\frac{n^p}{n^q} = \frac{1}{n^{q-p}} \to 0 \qquad \text{as } n \to \infty.$$

Hence, by Proposition 3.2.7, $X_{[q]}$ is compactly embedded in $X_{[p]}$.

3.3 Operator Semigroups

In this section we introduce operator semigroups and results relating to these families of linear operators which will be essential later. There is extensive theory relating to operator semigroups and many books have been written on the subject matter; see, for example, [21, 22, 30, 32, 34, 35, 43, 61]. Section 3.3.1 examines strongly continuous semigroups, often referred to as C_0-semigroups. In connection with the C–F problem that we study in later chapters, substochastic and stochastic semigroups are of particular importance. We examine these semigroups in Section 3.3.2. The results in these sections are primarily known results, which are mostly taken from [30]. However, Proposition 3.3.9 provides a different proof and a more general formulation of a result given in [75, Theorem 2.3.4 (i)]. Moreover, it is believed that Proposition 3.3.16 is a new result. In Section 3.3.3 we discuss analytic semigroups, which will be crucial when we investigate the C–F system in general weighted ℓ^1 spaces. In particular, the existence of certain analytic semigroups allows us to weaken the required assumptions on the time-dependent coagulation rates. The definitions and results given here are again based on material in [30] but Propositions 3.3.26 and 3.3.37 are new. Finally, Section 3.3.4 consists of perturbation theorems, the majority of which are known. In Corollary 3.3.30 and Theorem 3.3.35, however, we formulate corollaries of published results that can be applied in an elegant way to the pure fragmentation system, posed in a weighted ℓ^1 space.

3.3.1 Strongly Continuous Semigroups

We begin by defining a C_0-semigroup. The following definition is taken from [30, Definition I.5.1].

Definition 3.3.1. A family, $(S(t))_{t\geq 0}$, of bounded linear operators on a Banach space X is called a *strongly continuous semigroup* (or a C_0*-semigroup*) if

(i) $S(t+s) = S(t)S(s)$ for all $t, s \geq 0$;

(ii) $S(0) = I$;

(iii) the mappings $t \mapsto S(t)f$ are continuous from $[0, \infty)$ into X for every $f \in X$.

We make use of the following proposition in many of the results that follow.

Proposition 3.3.2. [30, Proposition I.5.5] *For every C_0-semigroup, $(S(t))_{t\geq 0}$, there exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that, for all $t \geq 0$,*

$$\|S(t)\| \leq Me^{\omega t}. \tag{3.3.1}$$

Definition 3.3.3. If $(S(t))_{t\geq 0}$ is a C_0-semigroup and (3.3.1) holds with $M = 1$ and $\omega = 0$, then $(S(t))_{t\geq 0}$ is called a *contraction semigroup*.

We can now define the growth bound of a C_0-semigroup.

Definition 3.3.4. The *growth bound* of a C_0-semigroup, ω_0, is given by

$$\omega_0 := \omega_0(S) := \inf \Big\{\omega \in \mathbb{R} : \text{ there exists } M_\omega \geq 1 \text{ such that } \|S(t)\| \leq M_\omega e^{\omega t} \text{ for all } t \geq 0\Big\}.$$

With each semigroup we associate a unique operator which we refer to as the generator (or infinitesimal generator) of the semigroup. The following definition is taken from [30, Definition II.1.2].

Definition 3.3.5. Let X be a Banach space and $(S(t))_{t\geq 0}$ be a C_0-semigroup on X. The *generator* of $(S(t))_{t\geq 0}$ is the operator A defined by

$$Af := \lim_{h\downarrow 0} \frac{1}{h}\Big(S(h)f - f\Big), \tag{3.3.2}$$

for all $f \in \mathcal{D}(A)$, where

$$\mathcal{D}(A) = \left\{ f \in X : \lim_{h\downarrow 0} \frac{1}{h}\Big(S(h)f - f\Big) \text{ exists} \right\}. \tag{3.3.3}$$

Note that if A is the generator of a C_0-semigroup, then we sometimes denote by $(e^{tA})_{t\geq 0}$ the semigroup generated by A, where the definition of the operator exponential, e^{tA}, is given below in (3.3.5). The following properties of a generator are taken from [30, Lemma II.1.3 and Theorem II.1.4].

Lemma 3.3.6. *Let A be the generator of a C_0-semigroup, $(S(t))_{t\geq 0}$. Then A is a closed and densely defined linear operator. Moreover, $(S(t))_{t\geq 0}$ leaves $\mathcal{D}(A)$ invariant and*

$$\frac{d}{dt}S(t)f = S(t)Af = AS(t)f$$

for all $t \geq 0$ and $f \in \mathcal{D}(A)$.

For a closed operator A in a Banach space X, the resolvent set of A, $\rho(A)$, is given by

$$\rho(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ is bijective from } \mathcal{D}(A) \text{ onto } X\}.$$

Let $\lambda \in \rho(A)$. Then we define the *resolvent operator*, $R(\lambda, A)$, of A by

$$R(\lambda, A) = (\lambda I - A)^{-1}. \tag{3.3.4}$$

We note that, by the Closed Graph Theorem, $R(\lambda, A)$ is a bounded operator for all $\lambda \in \rho(A)$.

The following theorem is known as the Feller–Miyadera–Phillips theorem and is taken from [30, Theorems II.1.10 and II.3.8 and Corollary III.5.5].

Theorem 3.3.7. *Let $\omega \in \mathbb{R}$, $M \geq 1$. An operator A is the generator of a C_0-semigroup, $(S(t))_{t\geq 0}$, satisfying (3.3.1), on a Banach space X if and only if*

(i) *A is closed and densely defined,*

(ii) *$(\omega, \infty) \subseteq \rho(A)$,*

(iii) *$\| [(\lambda - \omega)R(\lambda, A)]^n \| \leq M$ for all $\lambda > \omega$.*

Moreover, if these conditions are satisfied, then

$$S(t)f = e^{tA}f = \lim_{k\to\infty}\left(I - \frac{t}{k}A\right)^{-k} f \qquad \text{for all } f \in X, \tag{3.3.5}$$

uniformly for t on compact intervals, and the resolvent operator of A can be expressed as

$$R(\lambda, A)f = \int_0^\infty e^{-\lambda s} S(s) f \mathrm{d}s, \tag{3.3.6}$$

for all $f \in X$ and $\lambda : \mathrm{Re}(\lambda) > \omega_0(S)$, where $\omega_0(S)$ is as in Definition 3.3.4.

We note that we interpret the integral in (3.3.6) as

$$\int_0^\infty e^{-\lambda s} S(s) f \mathrm{d}s := \lim_{t\to\infty} \int_0^t e^{-\lambda s} S(s) f \mathrm{d}s, \tag{3.3.7}$$

where the integral on the right-hand side of (3.3.7) is the Riemann integral.

The following theorem is the particular case of Theorem 3.3.7 when $M = 1$ and $\omega = 0$. It is called the Hille–Yosida theorem and is taken from [30, Theorem II.3.5, Corollary III.5.5 and Theorem II.1.10]. The Hille–Yosida theorem is used to prove the more general Feller–Miyadera–Phillips theorem given above.

Theorem 3.3.8. *An operator A is the generator of a contraction semigroup, $(S(t))_{t\geq 0}$, on a Banach space X if and only if*

(i) *A is closed and densely defined,*

(ii) $(0, \infty) \subseteq \rho(A)$,

(iii) $\|\lambda R(\lambda, A)\| \leq 1$ *for all* $\lambda > 0$.

Moreover, if these conditions hold, then (3.3.5) *and* (3.3.6) *hold where* $\omega_0(S) \leq 0$.

Note that, in general, Theorem 3.3.8 is much easier to apply than Theorem 3.3.7 because in the former one has to estimate only the norm of the resolvent of A whereas in the latter one has to estimate the norm of all powers of the resolvent.

The following result is given for the case in which the closure of an operator A generates a substochastic C_0-semigroup (see Definition 3.3.15 below for the definition of a substochastic semigroup) in [18, Theorem 4.10.28] and [74, Theorem 2.3.4 (i)]. We give a more general case here.

Proposition 3.3.9. *Let A be an operator in a real Banach space X. If $G = \overline{A}$ is the generator of a C_0-semigroup, then no other extension of A is the generator of a C_0-semigroup.*

Proof. Suppose $G = \overline{A}$ is the generator of a C_0-semigroup. Then, from Theorem 3.3.7, there exists $\omega_1 \in \mathbb{R}$ such that $(\omega_1, \infty) \subseteq \rho(G)$. Assume that $H \supseteq A$ (and so $H \supseteq \overline{A} = G$ since H is closed) is also the generator of a C_0-semigroup. Then there exists $\omega_2 \in \mathbb{R}$ such that $(\omega_2, \infty) \subseteq \rho(H)$.

Let $\lambda > \max\{\omega_1, \omega_2\}$. Then since $\lambda \in \rho(G)$, we have $\lambda I - G : \mathcal{D}(G) \to X$ is a bijective mapping. We also have $\lambda \in \rho(H)$ and so $\lambda I - H : \mathcal{D}(H) \to X$ is bijective.

However, if $\lambda I - G : \mathcal{D}(G) \to X$ is a bijection and $\mathcal{D}(H) \supsetneq \mathcal{D}(G)$, then $\lambda I - H$ cannot be injective. This is a contradiction. Hence we can conclude that $G = \overline{A}$ is the only generator of a C_0-semigroup which is an extension of A. $\square$

Another concept that is important to us is that of an operator, and in particular the generator of a C_0-semigroup, being resolvent positive.

Definition 3.3.10. Let X be an ordered Banach space and let $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subseteq X$. We say that A is *resolvent positive* if there exists $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subseteq \rho(A)$ and $R(\lambda, A) \geq 0$ for all $\lambda > \omega$.

We now define what it means for a semigroup to be positive.

Definition 3.3.11. A semigroup, $(S(t))_{t\geq 0}$, on an ordered Banach space, X, is a *positive semigroup* if $S(t) \geq 0$ for all $t \geq 0$.

We also have the following useful result, which is mentioned in [18, p. 128].

Lemma 3.3.12. *Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on an ordered Banach space, X, with generator A. Then $(S(t))_{t\geq 0}$ is positive if and only if A is resolvent positive.*

Proof. Let $(S(t))_{t\geq 0}$ be positive. Then $(\omega_0(S), \infty) \subseteq \rho(A)$ and, from (3.3.6), it is clear that $R(\lambda, A) \geq 0$ for all $\lambda > \omega_0(S)$. Conversely, suppose that $R(\lambda, A)$ is defined and $R(\lambda, A) \geq 0$ for all $\lambda > \lambda_0$, where $\lambda_0 \in \mathbb{R}$. From (3.3.5), for each $t > 0$, $f \in X_+$, we have

$$S(t)f = \lim_{k\to\infty} \left(I - \frac{t}{k}A\right)^{-k} f = \lim_{k\to\infty} \left(\frac{k}{t}\left(\frac{k}{t}I - A\right)^{-1}\right)^{k} f \geq 0$$

since, for each $t > 0$,

$$\left(\frac{k}{t}I - A\right)^{-1} = R\left(\frac{k}{t}, A\right) \geq 0$$

for all k sufficiently large. The result follows. $\square$

We find it useful on numerous occasions to be able to split an element from a subspace, U, of a vector lattice, X, into the difference of two elements in U_+. In particular, suppose that A is the generator of a positive C_0-semigroup, $(S(t))_{t\geq 0}$, on a Banach lattice X and we want to write $f \in \mathcal{D}(A)$ as $f = g - h$, where $g, h \in \mathcal{D}(A)_+$. The following lemma, taken from [12, Remark 6.6], allows us to do this.

Lemma 3.3.13. [12, Remark 6.6] *Let $U \subseteq X$ be a subspace of a vector lattice, X, such that $U = RX$, where R is a positive linear operator on X. Let $f \in U$. Then $f = g - h$ for some $g, h \in U_+$.*

Proof. For a fixed $f \in U$, we have $f = Ru$ for some $u \in X$. Then, since $u = u_+ - u_-$ and R is positive and linear, we have that $Ru = Ru_+ - Ru_-$, where Ru_+, $Ru_- \in U_+$. Hence, $f = g - h = Ru_+ - Ru_-$, where $g, h \in U_+$. $\square$

Remark 3.3.14. Suppose that A is the generator of a positive C_0-semigroup, $(S(t))_{t\geq 0}$, and $U = \mathcal{D}(A)$. Then, for any $\lambda > \omega_0(S)$, the assumptions of Lemma 3.3.13 hold with $R = R(\lambda, A)$.

3.3.2 Substochastic and Stochastic Semigroups

We now define particular semigroups that are of interest to us in connection with the C–F system.

Definition 3.3.15. Suppose that $(S(t))_{t\geq 0}$ is a C_0-semigroup on an ordered Banach space X. Then

(i) $(S(t))_{t\geq 0}$ is said to be *substochastic* if $S(t) \geq 0$ and $\|S(t)f\| \leq \|f\|$ for all $t \geq 0$, $f \in X_+$;

(ii) $(S(t))_{t\geq 0}$ is said to be *stochastic* if $S(t) \geq 0$ and $\|S(t)f\| = \|f\|$ for all $t \geq 0$, $f \in X_+$.

Note that if $(S(t))_{t\geq 0}$ is substochastic, then the relation $\|S(t)f\| \leq \|f\|$, $t \geq 0$, extends to all $f \in X$ by [12, Remark 2.68].

In later chapters, stochastic semigroups play an important role in obtaining mass-conserving solutions to the C–F system. The next proposition is applied to the pure fragmentation system later. Part (i) of the proposition is, as far as we are aware, a new result. Part (ii) and (iii) are discussed briefly in [68, Remark 2.1(a)].

Proposition 3.3.16. *Let $(S(t))_{t\geq 0}$ be a positive C_0-semigroup on an AL-space, X, with generator G, and let ϕ be the unique bounded linear extension of the norm $\|\cdot\|$ from X_+ to X.*

(i) *The semigroup $(S(t))_{t\geq 0}$ is stochastic if and only if*

$$\phi\Big(S(t)f\Big) = \phi(f) \qquad \text{for all } f \in X. \tag{3.3.8}$$

(ii) *If $\phi(Gf) = 0$ for all $f \in \mathcal{D}(G)_+$, then* (3.3.8) *holds and hence the semigroup $(S(t))_{t\geq 0}$ is stochastic.*

(iii) *Let G_0 be an operator such that $G = \overline{G_0}$. If $\phi(G_0 f) = 0$ for all $f \in \mathcal{D}(G_0)_+$ and each $f \in \mathcal{D}(G_0)$ can be written as $f = g - h$, where $g, h \in \mathcal{D}(G_0)_+$, then* (3.3.8) *holds and hence $(S(t))_{t\geq 0}$ is stochastic.*

Proof. (i) Assume that $(S(t))_{t\geq 0}$ is stochastic and let $f \in X$ and $t \geq 0$. Then $f = f_+ - f_-$, where $f_+, f_- \in X_+$, and therefore

$$\begin{aligned}\phi\big(S(t)f\big) &= \phi\big(S(t)f_+\big) - \phi\big(S(t)f_-\big) = \|S(t)f_+\| - \|S(t)f_-\| = \|f_+\| - \|f_-\| \\ &= \phi(f_+) - \phi(f_-) = \phi(f).\end{aligned}$$

Conversely, when (3.3.8) holds, we have $\|S(t)f\| = \phi(S(t)f) = \phi(f) = \|f\|$ for $f \in X_+$ and $t \geq 0$.

(ii) Let $f \in \mathcal{D}(G)$. From Lemma 3.3.13 and Remark 3.3.14, we have that there exist $g, h \in \mathcal{D}(G)_+$ such that $f = g - h$. Then, since ϕ is continuous,

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\big(\phi(S(t)f)\big) &= \phi\bigg(\frac{\mathrm{d}}{\mathrm{d}t}\big(S(t)f\big)\bigg) = \phi\big(GS(t)f\big) \\ &= \phi\big(GS(t)g\big) - \phi\big(GS(t)h\big) = 0\end{aligned}$$

since $S(t)g$, $S(t)h \in \mathcal{D}(G)_+$. Thus $\phi(S(t)f) = \phi(f)$ for all $f \in \mathcal{D}(G)$.

Now let $f \in X$. There exists a sequence $(f^{(n)})_{n=1}^{\infty}$ such that $f^{(n)} \in \mathcal{D}(G)$ for all $n \in \mathbb{N}$ and $f^{(n)} \to f$ as $n \to \infty$. Moreover, for fixed $t \geq 0$,

$$\|S(t)f^{(n)} - S(t)f\| \to 0 \qquad \text{as } n \to \infty.$$

We have $\phi(S(t)f^{(n)}) = \phi(f^{(n)})$ and so, taking the limit as $n \to \infty$ and using the fact that ϕ is continuous, (3.3.8) follows and $(S(t))_{t\geq 0}$ is stochastic.

(iii) Let $f \in \mathcal{D}(G_0)$. Then $f = g - h$ for some $g, h \in \mathcal{D}(G_0)_+$ by assumption, and

$$\phi(G_0 f) = \phi\big(G_0(g - h)\big) = \phi(G_0 g) - \phi(G_0 h) = 0.$$

Thus $\phi(G_0 f) = 0$ for all $f \in \mathcal{D}(G_0)$. Now let $f \in \mathcal{D}(G)$. Then there exist $f^{(n)} \in \mathcal{D}(G_0)$, $n \in \mathbb{N}$, such that $f^{(n)} \to f$ and $G_0 f^{(n)} \to Gf$ as $n \to \infty$. Therefore

$$\phi(Gf) = \phi\Big(\lim_{n\to\infty} G_0 f^{(n)}\Big) = \lim_{n\to\infty} \phi(G_0 f^{(n)}) = 0,$$

and the result follows from part (ii). □

In subsequent sections we will obtain results that tell us when an extension of a given operator is the generator of a substochastic C_0-semigroup, $(S(t))_{t\geq 0}$. We now define what it means for $(S(t))_{t\geq 0}$ to be the smallest such semigroup.

Definition 3.3.17. Suppose that $(S(t))_{t\geq 0}$ is a substochastic C_0-semigroup on an ordered Banach space X that is generated by an extension, V, of an operator U. Then $(S(t))_{t\geq 0}$ is referred to as the *smallest* such semigroup if, given any other substochastic semigroup $(T(t))_{t\geq 0}$ on X that is generated by an extension of U, we have $S(t) \leq T(t)$, for all $t \geq 0$.

3.3.3 Analytic Semigroups

In later chapters we also require the concept of an analytic semigroup. Before we define analytic semigroups, however, we define a sectorial operator. The following definitions are from [30, Definition II.4.1].

Definition 3.3.18. Let $\alpha \in (0, \pi]$. We define the *sector*, Σ_α, to be

$$\Sigma_\alpha := \{\lambda \in \mathbb{C}\backslash\{0\} : |\arg \lambda| < \alpha\}. \tag{3.3.9}$$

Definition 3.3.19. Let $A : \mathcal{D}(A) \to X$ be a closed, linear, densely defined operator in a complex Banach space, X. Then A is a *sectorial operator* (of angle δ) if there exists $\delta \in (0, \frac{\pi}{2}]$ such that the sector $\Sigma_{\frac{\pi}{2}+\delta}$ is contained in the resolvent set $\rho(A)$, and if, for each $\varepsilon \in (0, \delta)$, there exists $M_\varepsilon \geq 1$ such that

$$\|R(\lambda, A)\| \leq \frac{M_\varepsilon}{|\lambda|}$$

for all $\lambda \in \overline{\Sigma}_{\frac{\pi}{2}+\delta-\varepsilon} \setminus \{0\}$.

We now define what it means for a family of operators to be an analytic semigroup. This definition is taken from [30, Definition II.4.5].

Definition 3.3.20. Let $\delta \in (0, \frac{\pi}{2}]$ and let $(S(t))_{t\in\Sigma_\delta\cup\{0\}}$ be a family of bounded, linear operators on a complex Banach space X. Then $(S(t))_{t\in\Sigma_\delta\cup\{0\}}$ is an *analytic semigroup* (of angle δ) if

(i) $S(0) = I$ and $S(t_1 + t_2) = S(t_1)S(t_2)$ for all $t_1, t_2 \in \Sigma_\delta$,

(ii) the map $t \mapsto S(t)$ is analytic in Σ_δ,

(iii) $\lim\limits_{t \in \Sigma_{\delta'} : t \to 0} S(t)x = x$ for all $x \in X$ and $0 < \delta' < \delta$.

If, in addition,

(iv) $\|S(t)\|$ is bounded in $\Sigma_{\delta'}$ for every $0 < \delta' < \delta$,

then $(S(t))_{t \in \Sigma_\delta \cup \{0\}}$ is a *bounded, analytic semigroup.*

If $(S(t))_{t \in \Sigma_{\delta \cup \{0\}}}$ is an analytic semigroup, then the restriction $(S(t))_{t \geq 0}$ is a C_0-semigroup. The generator of the latter is also referred to as the generator of the analytic semigroup $(S(t))_{t \in \Sigma_{\delta \cup \{0\}}}$. The connection between analytic semigroups and sectorial operators is given in the next theorem.

Theorem 3.3.21. [30, Theorem II.4.6] *Let X be a Banach space. For an operator $A : \mathcal{D}(A) \to X$, where $\mathcal{D}(A) \subseteq X$, the following assertions are equivalent.*

(i) *The operator A generates a bounded, analytic semigroup on X.*

(ii) *The operator A generates a bounded C_0-semigroup, $(S(t))_{t \geq 0}$, on X, and there exists a constant $C > 0$ such that*

$$\|R(r + is, A)\| \leq \frac{C}{|s|}$$

for all $r > 0$ and $0 \neq s \in \mathbb{R}$.

(iii) *The operator A is sectorial of angle* $\arctan\left(\frac{1}{C}\right)$.

The next proposition describes some of the desirable properties that analytic semigroups possess.

Proposition 3.3.22. [43, Proposition 2.1.1] *Let X be a Banach space and let $A : \mathcal{D}(A) \to X$ be the generator of an analytic semigroup, $(S(t))_{t \in \Sigma_{\delta \cup \{0\}}}$, on X, where $\mathcal{D}(A) \subseteq X$. Then for $t > 0$ and $f \in X$, $S(t)f \in \mathcal{D}(A^n)$ for all $n \in \mathbb{N}$ and $t \mapsto S(t)f$ is in $C^\infty((0, \infty), X)$, with*

$$\frac{d^k}{dt^k} S(t)f = A^k S(t)f \qquad \textit{for all } k \in \mathbb{N}.$$

Interpolation Spaces

When we examine the C–F system we make certain assumptions on the coagulation rates to obtain the properties that are required to apply existence and uniqueness results. For a certain class of initial conditions, we can weaken the assumptions required on the coagulation rates using the theory of interpolation spaces. We now introduce interpolation spaces and some useful results relating to them. The theory presented in this subsection is based on [43, §2.2.1]. Throughout this subsection, X denotes a complex Banach space and A is the generator of an analytic C_0-semigroup, $(S(t))_{t\in\Sigma_{\delta\cup\{0\}}}$, on X. For the definition of fractional powers, see (3.3.13) below, we need A to be an invertible operator and so, for simplicity, we assume throughout this discussion that $0 \in \rho(A)$ so that A is invertible.

Definition 3.3.23. Let $\mathcal{D}(A)$ be equipped with the graph norm (or equivalently with the norm $\|A\cdot\|$). A Banach space, Y, such that

$$\mathcal{D}(A) \hookrightarrow Y \hookrightarrow X,$$

where $\hookrightarrow$ denotes continuous embedding, is said to be an *intermediate space* between X and $\mathcal{D}(A)$. If, in addition, for all $T \in \mathcal{B}(X)$ with $T|_{\mathcal{D}(A)} \in \mathcal{B}(\mathcal{D}(A))$, we have that $T|_Y \in \mathcal{B}(Y)$, then Y in an *interpolation space* between X and $\mathcal{D}(A)$.

As in [43, §2.2.1], for $\alpha \in (0,1)$ and $p \in [1,\infty]$, we define a class of intermediate spaces, $D_A(\alpha,p)$, between X and $\mathcal{D}(A)$ by

$$D_A(\alpha,p) = \{f \in X : t \mapsto v(t) = \|t^{1-\alpha-\frac{1}{p}}AS(t)f\| \in L^p(0,1)\}, \tag{3.3.10}$$

$$\|f\|_{D_A(\alpha,p)} = \|f\| + \|v\|_{L^p(0,1)}, \tag{3.3.11}$$

and for all $p \in [1,\infty]$, we set

$$D_A(0,p) = X; \qquad \|\cdot\|_{D_A(0,p)} = \|\cdot\|.$$

The spaces $D_A(\alpha,p)$ are *real interpolation spaces* (see [43, Chapter 1]). We note that, from [43, p. 253], the part of A in $D_A(\alpha,p)$ is the generator of an analytic C_0-semigroup on $D_A(\alpha,p)$.

Proposition 3.3.24. [43, Propositions 1.2.3 and 2.2.2 and Corollary 2.2.3(ii)] *Let* $1 \leq p_1 \leq p_2 \leq \infty$ *and* $\alpha \in (0,1)$. *Then*

$$\mathcal{D}(A) \hookrightarrow D_A(\alpha, p_1) \hookrightarrow D_A(\alpha, p_2) \hookrightarrow \overline{\mathcal{D}(A)} = X.$$

Moreover, for $0 < \alpha_1 < \alpha_2 < 1$ *we have*

$$D_A(\alpha_2, \infty) \hookrightarrow D_A(\alpha_1, 1).$$

The following result will also be useful.

Corollary 3.3.25. [43, Corollary 2.2.3(i)] *Let* $\alpha \in (0,1)$ *and* $p \in [1,\infty]$. *Let* $A : \mathcal{D}(A) \to X$ *be the generator of an analytic semigroup such that* A *is invertible, where* $\mathcal{D}(A) \subseteq X$. *In addition, let* $C : \mathcal{D}(C) = \mathcal{D}(A) \to X$ *also be the generator of an analytic semigroup such that* C *is invertible and, for all* $f \in \mathcal{D}(A)$ *and some* $c \geq 1$,

$$c^{-1}\|Af\| \leq \|Cf\| \leq c\|Af\|. \tag{3.3.12}$$

Then $D_C(\alpha, p) = D_A(\alpha, p)$, *with equivalence of the respective norms.*

We now formulate a condition under which (3.3.12) holds.

Proposition 3.3.26. *Let* A *and* B *be linear operators in a Banach space,* X, *with* $\mathcal{D}(A) \subseteq \mathcal{D}(B) \subseteq X$. *Moreover, assume that there exists* $\kappa \in (0,1)$ *satisfying*

$$\|Bf\| \leq \kappa\|Af\| \qquad \textit{for all } f \in \mathcal{D}(A).$$

Then (3.3.12) *holds with* $C = A + B$ *and* $c = \max\left\{1+\kappa, \frac{1}{1-\kappa}\right\}$.

Proof. For all $f \in \mathcal{D}(A)$, we have

$$\|Cf\| \leq \|Af\| + \|Bf\| \leq \|Af\| + \kappa\|Af\| = (1+\kappa)\|Af\|.$$

Also, for $f \in \mathcal{D}(A)$,

$$\|Af\| = \|Af + Bf - Bf\| \leq \|(A+B)f\| + \|Bf\| \leq \|Cf\| + \kappa\|Af\|$$

and so

$$(1-\kappa)\|Af\| \leq \|Cf\|.$$

Hence (3.3.12) holds with $c = \max\left\{1+\kappa, \frac{1}{1-\kappa}\right\}$. □

We also use the concept of a fractional power of $-A$ in connection with intermediate spaces. The following definition can be found in [43, §2.2.2].

Definition 3.3.27. For every $\alpha > 0$ we define

$$(-A)^{-\alpha} = \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} S(t)\,\mathrm{d}t, \tag{3.3.13}$$

where

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1}\,\mathrm{d}t. \tag{3.3.14}$$

We set $\mathcal{D}((-A)^\alpha) := \mathrm{range}((-A)^{-\alpha})$ and

$$(-A)^\alpha := ((-A)^{-\alpha})^{-1}. \tag{3.3.15}$$

Moreover, we set $A^0 = I$.

Another result from [43] that we require is the following.

Proposition 3.3.28. [43, Proposition 2.2.15] *For $\alpha \in (0,1)$ we have*

$$D_A(\alpha, 1) \hookrightarrow \mathcal{D}((-A)^\alpha) \hookrightarrow D_A(\alpha, \infty). \tag{3.3.16}$$

3.3.4 Semigroup Perturbation Theory

Of particular importance to the results in this thesis is perturbation theory for strongly continuous semigroups. We, therefore, begin this subsection by providing a brief history of the main results from this field that we utilise and build upon later. We note that there are many perturbation theorems relating to operator semigroups, but here we comment only on those that are significantly related to the results in this thesis.

There is much interest in identifying whether an operator is the generator of a C_0-semigroup. A seminal breakthrough in relation to this topic appeared in 1948, [34, 76], in the form of the Hille–Yosida Theorem; see Theorem 3.3.8. This theorem was named after Einar Hille and Kōsaku Yosida, who simultaneously, but independently, obtained the result. The theorem gives necessary and sufficient conditions for an operator to be the generator of a contraction C_0-semigroup.

Now suppose that an operator A is the generator of a C_0-semigroup. A question that has attracted a great deal of interest is: if we perturb A by some operator B, is $A+B$ also the generator of a C_0-semigroup? An influential result, prompted by this question, which proves to be very useful when investigating fragmentation models, is the Kato–Voigt Perturbation Theorem. This theorem, first introduced in 1954 by Kato, [37, Theorem 1], and further developed by Voigt in 1987, [71, Proposition 1.4], provides sufficient conditions for A and B, under which there exists an extension, G, of $A+B$ that generates a substochastic C_0-semigroup. In [71, Remark 1.5] it is noted that if G is the closure of $A+B$ and a certain condition holds, then the semigroup generated by G is stochastic. Following on from this, in 2006 Thieme and Voigt proved a theorem, [68, Theorem 2.7], where they re-examined the Kato–Voigt Perturbation Theorem and provided conditions under which it can be deduced that the generator G in the Kato–Voigt Perturbation Theorem is indeed the closure of $A+B$.

Analytic semigroups also have particularly desirable properties that we exploit in this thesis. A perturbation theorem proved by Arendt and Rhandi, [3, Theorem 1.1] in 1991, considers the unperturbed operator A to be the generator of an analytic semigroup. Conditions are then provided under which we can deduce that a perturbation, $A+B$, of A is the generator of an analytic semigroup.

Perturbation results are invaluable in our investigation into the C–F system and this subsection collates those that we make use of. The following theorem is important in our examination of the pure fragmentation system.

Theorem 3.3.29. [68, Theorem 2.7] *Let $(X, \|\cdot\|)$ be an ordered real Banach lattice with generating cone X_+ such that $\|\cdot\|$ is additive on X_+. Let $X_{(1)}$ be a subspace of X such that*

(i) *$(X_{(1)}, \|\cdot\|_{(1)})$ is a Banach lattice for some norm $\|\cdot\|_{(1)}$;*

(ii) $(X_{(1)}, \|\cdot\|_{(1)})$ *is continuously embedded in* $(X, \|\cdot\|)$*;*

(iii) $(X_{(1)})_+$ *is dense in* X_+*;*

(iv) $(X_{(1)})_+$ *is a generating cone for* $X_{(1)}$ *and* $\|\cdot\|_{(1)}$ *is additive on* $(X_{(1)})_+$*.*

Also, let ϕ *and* $\phi_{(1)}$ *be the linear extensions, see Lemma 3.1.18, of* $\|\cdot\|$ *from* X_+ *to* X *and of* $\|\cdot\|_{(1)}$ *from* $(X_{(1)})_+$ *to* $X_{(1)}$ *respectively. Let* $A : \mathcal{D}(A) \to X$, $B : \mathcal{D}(B) \to X$ *be operators in* X *such that* $\mathcal{D}(A) \subseteq \mathcal{D}(B) \subseteq X$*. Now consider the following conditions:*

(a) $-A$ *is positive;*

(b) A *generates a positive* C_0*-semigroup,* $(T(t))_{t\geq 0}$*, on* X*;*

(c) *the restriction of* $(T(t))_{t\geq 0}$ *to* $X_{(1)}$ *is a* C_0*-semigroup on* $X_{(1)}$*, denoted by* $(\tilde{T}(t))_{t\geq 0}$*, with generator* $\tilde{A}$*, defined by*

$$\tilde{A}f = Af \qquad \text{for all } f \in \mathcal{D}(\tilde{A}) = \{f \in \mathcal{D}(A) \cap X_{(1)} : Af \in X_{(1)}\};$$

(d) $B|_{\mathcal{D}(A)}$ *is a positive linear operator;*

(e) $\phi((A+B)f) \leq 0$ *for all* $f \in \mathcal{D}(A)_+$*;*

(f) $B(D(\tilde{A})) \subseteq X_{(1)}$*;*

(g) *there exist* $c, \varepsilon > 0$ *such that*

$$\phi_{(1)}((A+B)f) \leq c\|f\|_{(1)} - \varepsilon\|Af\|$$

for all $f \in D(\tilde{A})_+$*.*

If these conditions hold, then there exists a substochastic C_0*-semigroup on* X *that is generated by the closure of* $A + B$*. Moreover, the semigroup,* $(S(t))_{t\geq 0}$*, generated by* G *leaves* $X_{(1)}$ *invariant.*

We now formulate a corollary of Theorem 3.3.29 that we believe to be a new result, and which can be applied in an elegant way to the pure fragmentation problem.

Corollary 3.3.30. *Let* $(X, \|\cdot\|)$ *and* $(X_{(1)}, \|\cdot\|_{(1)})$ *be AL-spaces such that*

(i') $X_{(1)}$ *is dense in* X,

(ii') $(X_{(1)}, \|\cdot\|_{(1)})$ *is continuously embedded in* $(X, \|\cdot\|)$.

Let $A : \mathcal{D}(A) \to X, B : \mathcal{D}(B) \to X$ *be operators in* X *where* $\mathcal{D}(A) \subseteq \mathcal{D}(B) \subseteq X$. *Also, let* ϕ *and* $\phi_{(1)}$ *be the linear extensions, see Lemma 3.1.18, of* $\|\cdot\|$ *from* X_+ *to* X *and of* $\|\cdot\|_{(1)}$ *from* $(X_{(1)})_+$ *to* $X_{(1)}$ *respectively. Let conditions* (a)–(e) *of Theorem 3.3.29 hold and assume that*

(f') $(A+B)f \in X_{(1)}$ *and* $\phi_{(1)}((A+B)f) \leq 0$ *for all* $f \in \mathcal{D}(\tilde{A})_+$;

(g') $\|Af\| \leq \|f\|_{(1)}$ *for all* $f \in \mathcal{D}(\tilde{A})_+$.

Then there exists a unique substochastic C_0*-semigroup on* X *that is generated by an extension,* G*, of* $A+B$*. The operator* G *is the closure of* $A+B$*. Moreover, the semigroup,* $(S(t))_{t\geq 0}$*, generated by* G *leaves* $X_{(1)}$ *invariant. If* $\phi((A+B)f) = 0$ *for all* $f \in \mathcal{D}(A)_+$*, then* $(S(t))_{t\geq 0}$ *is stochastic.*

Proof. We first show that conditions (i)–(iv) in Theorem 3.3.29 are satisfied. Since $(X_{(1)}, \|\cdot\|)_{(1)}$ is an AL-space, it is clearly a Banach space. It is also clear that (ii) is satisfied. Since $X_{(1)}$ is an AL-space then (3.1.4) holds for any $f \in X_{(1)}$. Thus, from condition (i') and Lemma 3.1.16, we have that $(X_{(1)})_+$ is dense in X_+. Moreover, (iv) in Theorem 3.3.29 holds since $X_{(1)}$ is an AL-space.

We require to prove that conditions (f) and (g) in Theorem 3.3.29 follow from (f') and (g'). Let $f \in D(\tilde{A})_+$. Then, from (f'), $\phi_{(1)}((A+B)f) \leq 0$ and, from (g'), $\|f\|_{(1)} - \|Af\| \geq 0$. Hence $\phi_{(1)}((A+B)f) \leq \|f\|_{(1)} - \|Af\|$ and so, choosing $c = \varepsilon = 1$, we confirm that (g) in Theorem 3.3.29 holds.

Finally, for (f) of Theorem 3.3.29, we have that $(A+B)f \in X_{(1)}$ and $Af \in X_{(1)}$ for all $f \in \mathcal{D}(\tilde{A})_+$. It follows that $Bf \in X_{(1)}$ for all $f \in \mathcal{D}(\tilde{A})_+$. Now suppose that $f \in D(\tilde{A})$. Then, from Lemma 3.3.13 and Remark 3.3.14, $f = g - h$ for some $g, h \in D(\tilde{A})_+$. Thus $Bf = Bg - Bh \in X_{(1)}$, since $X_{(1)}$ is a subspace. Hence $B(D(\tilde{A})) \subseteq X_{(1)}$ as required.

Finally, the uniqueness of the semigroup follows from Proposition 3.3.9. Since A generates a substochastic C_0-semigroup then, using Lemma 3.3.13 and Remark 3.3.14, we can write any $f \in \mathcal{D}(A)$ as $f = g - h$, where $g, h \in \mathcal{D}(A)_+$. The stochasticity result then follows from Proposition 3.3.16(iii). $\square$

We now introduce some concepts that we use in results that follow.

Definition 3.3.31. Let X be a Banach space and let $A : \mathcal{D}(A) \to X$ and $B : \mathcal{D}(B) \to X$ be operators with $\mathcal{D}(A) \subseteq \mathcal{D}(B) \subseteq X$. Then B is A-*bounded* if there exist $a, b \geq 0$ such that

$$\|Bf\| \leq a\|Af\| + b\|f\| \quad \text{for all } f \in \mathcal{D}(A). \tag{3.3.17}$$

If B is A-bounded, then the A-*bound*, or the *relative bound*, is

$$a_0 := \inf \{a \geq 0 : \text{ there exists } b \geq 0 \text{ such that (3.3.17) holds}\}. \tag{3.3.18}$$

The following lemma will be useful, together with Corollary 3.3.30, when the pure fragmentation system is examined in general weighted ℓ^1 spaces.

Lemma 3.3.32. [30, Lemma III.2.4] *Let X be a Banach space. Moreover, let $A : \mathcal{D}(A) \to X$, $B : \mathcal{D}(B) \to X$ be operators with $\mathcal{D}(A) \subseteq \mathcal{D}(B) \subseteq X$. If A is a closed operator and B is A-bounded with relative bound $a_0 < 1$, then $(A + B, \mathcal{D}(A))$ is a closed operator.*

We also require the definition of a Miyadera perturbation.

Definition 3.3.33. Let X be a Banach space and let $A : \mathcal{D}(A) \to X$ and $B : \mathcal{D}(B) \to X$, where $\mathcal{D}(A) \subseteq \mathcal{D}(B) \subseteq X$. Let A be the generator of a C_0-semigroup, $(T(t))_{t\geq 0}$, on X. Then B is a *Miyadera perturbation* of A if B is A-bounded and there exist numbers α and γ, with $0 < \alpha < \infty$, $0 \leq \gamma < 1$ such that

$$\int_0^\alpha \|BT(t)f\| \,\mathrm{d}t \leq \gamma\|f\| \quad \text{for all } f \in \mathcal{D}(A). \tag{3.3.19}$$

The following theorem, which is the Miyadera–Voigt Perturbation Theorem, originates from work in [57, 70].

Theorem 3.3.34. [12, Theorem 4.16] *Let X be a Banach space and let A be the generator of a C_0-semigroup on X. Moreover, let B be a Miyadera Perturbation of A. Then $(A + B, \mathcal{D}(A))$ is the generator of a C_0-semigroup.*

Proof. We give a brief outline of the proof used in [12, Theorem 4.16]. From [12, Lemma 4.15], we have that B is a Miyadera perturbation of A if and only if B is a Miyadera perturbation of $A - \lambda I$, for any $\lambda \in \mathbb{R}$. Moreover, $A + B$ is the generator of a semigroup, $(S(t))_{t\geq 0}$, if and only if $A - \lambda I + B$ is the generator of a semigroup, $(e^{-\lambda t}S(t))_{t\geq 0}$. Hence, without loss of generality, we can assume that the semigroup, $(T(t))_{t\geq 0}$, generated by A has negative growth bound.

Now set

$$S_1(t)f = \int_0^t T(t-s)BT(s)f\,\mathrm{d}s, \qquad f \in \mathcal{D}(A),\ t \geq 0.$$

As in [12, Theorem 4.16], for each $t \geq 0$, $S_1(t)$ can be extended in a unique way to a bounded linear operator on X. We now define $S_j(t)$, $j \in \mathbb{N}$ such that $j \geq 2$, recursively by

$$S_j(t)f = \int_0^t S_{j-1}(t-s)BT(s)f\,\mathrm{d}s, \qquad f \in \mathcal{D}(A),\ t \geq 0,$$

where we can again extend $S_j(t)$ in a unique way to a bounded linear operator on X. For $t \geq 0$, set $S_0(t) = T(t)$. It can then be shown that the family of operators defined via

$$S(t) = \sum_{j=0}^{\infty} S_j(t), \qquad t \geq 0 \tag{3.3.20}$$

satisfies the Duhamel formula

$$S(t)f = T(t)f + \int_0^t S(t-s)BT(t)f\,\mathrm{d}s, \qquad f \in \mathcal{D}(A),\ t \geq 0, \tag{3.3.21}$$

and that $(S(t))_{t\geq 0}$ is a C_0-semigroup on X which is generated by $A + B$. □

The next theorem, which we believe is an original result, deals with the special case of an AL-space where the condition (3.3.19) can be replaced by the simpler assumption that B is A-bounded, with A-bound less than one. It is convenient to apply this result to the discrete fragmentation system examined in certain weighted ℓ^1 spaces. The proof is based on ideas in [71].

Theorem 3.3.35. *Let the operator A be the generator of a positive C_0-semigroup on an AL-space, X, such that $-A$ is positive. Moreover, let B be an A-bounded linear operator, with A-bound less than 1. Then $A+B$, with $\mathcal{D}(A+B) = \mathcal{D}(A)$, is the generator of a C_0-semigroup on X. If, in addition, B is a positive operator, then the semigroup generated by $A+B$ is positive.*

Proof. We show that B is a Miyadera perturbation of A and then apply Theorem 3.3.34. Since B is A-bounded, we need only show that there exist α and γ such that $0 < \alpha < \infty$, $0 \le \gamma < 1$ and (3.3.19) holds. Let ϕ be the unique, linear extension of $\|\cdot\|$ from X_+ to X. Let $(T(t))_{t\ge 0}$ be the semigroup generated by A. For $\alpha > 0$ and $f \in \mathcal{D}(A)_+$, we have

$$\begin{aligned}\int_0^\alpha \|AT(t)f\|\,\mathrm{d}t &= \int_0^\alpha \phi(-AT(t)f)\,\mathrm{d}t = \phi\Big(-\int_0^\alpha AT(t)f\,\mathrm{d}t\Big)\\ &= \phi\Big(-\int_0^\alpha \frac{d}{dt}\big(T(t)f\big)\,\mathrm{d}t\Big) = \phi\big(f - T(\alpha)f\big)\\ &= \|f\| - \|T(\alpha)f\| \le \|f\|.\end{aligned}$$

Thus,

$$\int_0^\alpha \|AT(t)f\|\mathrm{d}t \le \|f\| \quad \text{for all } f \in \mathcal{D}(A)_+. \tag{3.3.22}$$

Now, for $\delta > 0$, define $T_\delta := \delta^{-1}\int_0^\delta T(t)\,\mathrm{d}t$. Let $f \in \mathcal{D}(A)$. Then $|f| \in X_+$ and $T_\delta|f| \in \mathcal{D}(A)_+$, from [30, Lemma II.1.3 (iii)]. Since, for all $t \ge 0$, $-A$, $T(t)$ and T_δ are positive operators, it follows from Lemma 3.2.4 that

$$\|-AT(t)T_\delta f\| \le \|-AT(t)T_\delta|f|\|.$$

Hence,

$$\begin{aligned}\int_0^\alpha \|AT(t)T_\delta f\|\,\mathrm{d}t &= \int_0^\alpha \|-AT(t)T_\delta f\|\,\mathrm{d}t \le \int_0^\alpha \|-AT(t)T_\delta|f|\|\,\mathrm{d}t\\ &= \int_0^\alpha \|AT(t)T_\delta|f|\|\,\mathrm{d}t \le \|T_\delta|f|\|,\end{aligned}$$

where (3.3.22) is used to obtain the final inequality. So we have

$$\int_0^\alpha \|AT(t)T_\delta f\| \,\mathrm{d}t \leq \|T_\delta |f|\| \qquad \text{for all } f \in \mathcal{D}(A). \tag{3.3.23}$$

Also,

$$\begin{aligned}\|(T_\delta - I)f\| &= \left\| \delta^{-1} \int_0^\delta T(t)f \,\mathrm{d}t - f \right\| = \left\| \delta^{-1} \int_0^\delta (T(t)f - f) \,\mathrm{d}t \right\| \\ &\leq \max_{t\in[0,\delta]} \|T(t)f - f\| \to 0 \quad \text{as } \delta \to 0.\end{aligned}$$

It follows that $T_\delta f \to f$ as $\delta \to 0$. Similarly, $T_\delta |f| \to |f|$ as $\delta \to 0$ and so $\|T_\delta |f|\| \to \|f\|$ as $\delta \to 0$. Furthermore, for $f \in \mathcal{D}(A)$, using [30, Lemma II.1.3 (iv)] to obtain the last equality, we have

$$\begin{aligned}AT(t)T_\delta f &= \frac{1}{\delta} AT(t) \int_0^\delta T(s)f \,\mathrm{d}s = \frac{1}{\delta} A \int_0^\delta T(t)T(s)f \,\mathrm{d}s \\ &= \frac{1}{\delta} A \int_0^\delta T(s)T(t)f \,\mathrm{d}s = \frac{1}{\delta} \int_0^\delta T(s)AT(t)f \,\mathrm{d}s.\end{aligned}$$

Hence for $f \in \mathcal{D}(A)$, using [30, Lemma II.1.3 (ii)] to obtain the fourth equality,

$$\begin{aligned}AT(t)T_\delta f - AT(t)f &= \frac{1}{\delta} \int_0^\delta T(s)AT(t)f \,\mathrm{d}s - AT(t)f \\ &= \frac{1}{\delta} \int_0^\delta \Big(T(s)AT(t)f - AT(t)f \Big) \,\mathrm{d}s \\ &= \frac{1}{\delta} \int_0^\delta (T(s) - I)AT(t)f \,\mathrm{d}s \\ &= \frac{1}{\delta} \int_0^\delta (T(s) - I)T(t)Af \,\mathrm{d}s \\ &= \frac{1}{\delta} \int_0^\delta T(t)(T(s) - I)Af \,\mathrm{d}s\end{aligned}$$

$$= T(t)\frac{1}{\delta}\int_0^\delta (T(s)-I)Af\,\mathrm{d}s.$$

It follows that, for $f \in \mathcal{D}(A)$,

$$\begin{aligned}\|AT(t)T_\delta f - AT(t)f\| &\le \|T(t)\|\frac{1}{\delta}\int_0^\delta \|(T(s)-I)Af\|\,\mathrm{d}s \\ &\le \|T(t)\| \sup_{s\in[0,\delta]} \|(T(s)-I)Af\| \to 0 \text{ as } \delta \to 0\end{aligned}$$

uniformly in t on $[0,\alpha]$. Hence, for $f \in \mathcal{D}(A)$, $AT(t)T_\delta f \to AT(t)f$ as $\delta \to 0$ and so

$$\int_0^\alpha \|AT(t)T_\delta f\|\,\mathrm{d}t \to \int_0^\alpha \|AT(t)f\|\,\mathrm{d}t \qquad \text{as } \delta \to 0.$$

Taking the limit as $\delta \to 0$ in (3.3.23) we have

$$\int_0^\alpha \|AT(t)f\|\,\mathrm{d}t \le \|f\| \text{ for all } f \in \mathcal{D}(A).$$

Now, $(T(t))_{t\ge0}$ is a C_0-semigroup and so there exist $M \ge 1$, $\omega \ge 0$ such that $\|T(t)\| \le Me^{\omega t}$ for all $t \ge 0$. Moreover, B is A-bounded with A-bound $a_0 < 1$. Hence, there exist $a, b \ge 0$ such that $a < 1$ and (3.3.17) holds, which implies that

$$\begin{aligned}\int_0^\alpha \|BT(t)f\|\,\mathrm{d}t &\le a\int_0^\alpha \|AT(t)f\|\,\mathrm{d}t + b\int_0^\alpha \|T(t)f\|\,\mathrm{d}t \\ &\le a\|f\| + bM\int_0^\alpha e^{\omega t}\|f\|\,\mathrm{d}t \\ &\le a\|f\| + bM\alpha e^{\omega\alpha}\|f\| \\ &= (a + bM\alpha e^{\omega\alpha})\|f\|\end{aligned}$$

for all $f \in \mathcal{D}(A)$. We know that $a < 1$ and so we choose $\alpha > 0$ such that $\alpha e^{\omega\alpha} < \frac{1-a}{Mb}$. With this choice of α we have that $a + bM\alpha e^{\omega\alpha} < 1$ and so (3.3.19) holds, with $\gamma = a + bM\alpha e^{\alpha\omega}$. It follows from Theorem 3.3.34 that $A + B$ is the generator of a C_0-semigroup, $(S(t))_{t\ge0}$.

Finally, we prove the positivity. Let ω be the growth bound of $(T(t))_{t\geq 0}$ and choose $\lambda > \omega$. Then $A - \lambda I$ generates the semigroup $(e^{-\lambda t}T(t))_{t\geq 0}$ and, moreover, $(e^{-\lambda t}T(t))_{t\geq 0}$ has negative growth bound. We also have that $A - \lambda I + B$ is the generator of the semigroup $(e^{-\lambda t}S(t))_{t\geq 0}$. Now, from the discussion in the proof of Theorem 3.3.34, we have that B is a Miyadera Perturbation of $A - \lambda I$ and, from (3.3.20), the semigroup $(e^{-\lambda t}S(t))_{t\geq 0}$, satisfies

$$e^{-\lambda t}S(t) = \sum_{j=0}^{\infty} S_j(t), \qquad t \geq 0, \tag{3.3.24}$$

where

$$\begin{aligned}
S_0(t)f &= e^{-\lambda t}T(t)f, && f \in X;\\
S_1(t)f &= \int_0^t e^{-\lambda(t-s)}T(t-s)Be^{-\lambda s}T(s)f \,\mathrm{d}s, && f \in \mathcal{D}(A);\\
S_j(t)f &= \int_0^t S_{j-1}(t-s)Be^{-\lambda s}T(s)f \,\mathrm{d}s, && j = 2, 3, \ldots, \;\; f \in \mathcal{D}(A).
\end{aligned}$$

Since $(T(t))_{t\geq 0}$ is a positive semigroup, it follows that if B is a positive operator, then $(e^{-\lambda t}S(t))$ is positive on $\mathcal{D}(A)$. Hence the semigroup $(S(t))_{t\geq 0}$ is positive on $\mathcal{D}(A)$. We have that $\mathcal{D}(A)$ is dense in X and so, since the positive cone is closed, $(S(t))_{t\geq 0}$ is positive on X. □

The following perturbation result, from [3, Theorem 1.1], provides sufficient conditions under which a perturbation of the generator of an analytic semigroup also generates an analytic semigroup. We use this result when we examine the pure fragmentation system in certain weighted ℓ^1 spaces.

Theorem 3.3.36. *Let X be a complex Banach lattice (see Remark 3.1.20) and let $A : \mathcal{D}(A) \to X$ be the generator of a positive, analytic semigroup, where $\mathcal{D}(A) \subseteq X$. Moreover, let $B : \mathcal{D}(A) \to X$ be a positive, linear operator. If $A + B$ is resolvent positive, then $A + W$ generates an analytic C_0-semigroup for all linear mappings $W : \mathcal{D}(A) \to X$ satisfying*

$$|Wu| \leq Bu, \quad u \in \mathcal{D}(A)_+.$$

In particular, the result holds for $W = B$.

Proposition 3.3.37. *Let A be the generator of an analytic, positive C_0-semigroup on an AL-space, X, such that A is invertible and $-A$ is positive. Moreover, let B be a positive, linear operator such that there exists $\kappa \in (0,1)$ satisfying*

$$\|Bf\| \leq \kappa \|Af\|, \quad \text{for all } f \in \mathcal{D}(A).$$

Then $C = A + B$ is the generator of a positive, analytic semigroup on X and (3.3.12) holds for $c = \max\left\{1+\kappa, \frac{1}{1-\kappa}\right\}$.

Proof. From Theorem 3.3.35 we have that $A + B$ is the generator of a positive C_0-semigroup and so $A + B$ is resolvent positive. It follows from Theorem 3.3.36 that $A + B$ generates an analytic semigroup on X. The bounds (3.3.12) follow from Proposition 3.3.26. □

3.4 Linear Abstract Cauchy Problems (ACPs)

The C–F system, (1.1.1), is an infinite system of ODEs. Our strategy for dealing with this infinite system is to write it as an abstract Cauchy problem (ACP) in a weighted ℓ^1 space. We begin our investigation of (1.1.1) by examining the pure fragmentation system, which we pose as a linear ACP. We therefore examine linear ACPs in this section. We define what it means for a function to be a solution of an ACP and then devote the remainder of the section to results regarding the existence and uniqueness of solutions to ACPs. These results are applied later to the pure fragmentation system. Basic concepts and results relating to linear ACPs are provided in Section 3.4.1. In Section 3.4.2 we examine linear ACPs using the theory of Sobolev towers. While the theory used to construct these towers is taken from [30], more detail is provided in Section 3.4.2 than in [30]. Moreover, Corollary 3.4.7 has been generalised. The existence and uniqueness results, Theorems 3.4.11, 3.4.13 and Proposition 3.4.12, are believed to be new.

3.4.1 Operator Semigroups and Linear ACPs

The following definition can be found in [22, Definition 2.38].

Definition 3.4.1. Let X be a Banach space and let $A : \mathcal{D}(A) \to X$ be a linear operator, where $\mathcal{D}(A) \subseteq X$. The homogeneous *ACP* associated with A is

$$u'(t) = Au(t), \quad t > 0; \quad u(0) = \mathring{u}, \tag{3.4.1}$$

where $\mathring{u} \in X$ is given.

We interpret the derivative in Definition 3.4.1 as

$$u'(0) = \lim_{h \to 0^+} \frac{u(t+h) - u(t)}{h} \qquad \text{and} \qquad u'(t) = \lim_{h \to 0} \frac{u(t+h) - u(t)}{h}, \ t > 0.$$

As in [12, Definition 3.1] we define what it means for a function to be a classical solution of (3.4.1) in the following way.

Definition 3.4.2. We call a function $u : [0, \infty) \to X$ a *classical solution* of the ACP (3.4.1) if

(i) $u \in C([0, \infty), X)$;

(ii) $u \in C^1((0, \infty), X)$;

(iii) $u(t) \in \mathcal{D}(A)$ for all $t > 0$;

(iv) u satisfies (3.4.1).

The following existence and uniqueness result will be of extreme importance when we look for solutions of the pure fragmentation system and can be found in [22, Theorems 2.40 and 2.41].

Theorem 3.4.3. *Let X be a Banach space and A be the generator of a C_0-semigroup, $(S(t))_{t \geq 0}$, on X. Then, for $\mathring{u} \in \mathcal{D}(A)$, the ACP* (3.4.1) *has a unique classical solution, $u(t) = S(t)\mathring{u}$, for all $t \geq 0$.*

If the semigroup $(S(t))_{t \geq 0}$ in Theorem 3.4.3 is analytic, then it provides a solution for a larger class of initial conditions.

Proposition 3.4.4. *If the semigroup $(S(t))_{t \geq 0}$ in Theorem 3.4.3 is analytic, then $u(t) = S(t)\mathring{u}$, $t \in [0, \infty)$, is the unique classical solution of* (3.4.1) *for all $\mathring{u} \in X$.*

Remark 3.4.5. In physical problems we often also require solutions to be non-negative. For example in the case of coagulation–fragmentation, u will be an infinite sequence from a weighted ℓ^1 space such that each component in the sequence represents a density. In addition, at times we assume that the total mass of clusters remains constant and this leads to the condition that the solution be mass conserving.

3.4.2 Sobolev Towers

When we examine the pure fragmentation system on weighted ℓ^1 spaces we are interested in obtaining a solution to the system for as large a class of initial conditions as possible. Sobolev towers prove themselves to be useful with respect to this. In this section, we follow the Sobolev tower construction found in [30, §II.5]. The existence and uniqueness results that appear in this subsection are believed to be new results.

CASE 1: Semigroups with Negative Growth Bound

Let H be the generator of a C_0-semigroup, $(T(t))_{t\geq 0}$, on a Banach space, X. We are interested in solutions of the ACP

$$u'(t) = Hu(t), \quad t > 0; \quad u(0) = \mathring{u}. \tag{3.4.2}$$

From Theorem 3.4.3 we know that $u(t) = T(t)\mathring{u}$ is the unique solution of (3.4.2) for all $t \geq 0$, $\mathring{u} \in \mathcal{D}(H)$. We aim to use a Sobolev tower construction to widen the class of initial conditions, $\mathring{u}$, for which we can obtain a solution. The following Sobolev tower construction can be found in [30, §II.5].

Assume that the growth bound of $(T(t))_{t\geq 0}$ is negative. If the growth bound is not negative then we can rescale to obtain a negative growth bound—we deal with this case later. It follows that $0 \in \rho(H)$ and so the operator $H^{-1} : X \to \mathcal{D}(H)$ exists, i.e. $H^{-1} \in \mathcal{B}(X)$.

Let $n \in \mathbb{N}$ and consider $X_n = (\mathcal{D}(H^n), \|\cdot\|_n)$, where

$$\|f\|_n \coloneqq \|H^n f\| \qquad \text{for all } f \in \mathcal{D}(H^n).$$

The operator H has a bounded inverse on X and so the norm on X_n is equivalent to the graph norm on $\mathcal{D}(H^n)$. It follows that X_n is a Banach space for each $n \in \mathbb{N}$. Moreover, each X_n, $n \in \mathbb{N}$, is continuously embedded in X since there exists $C > 0$ such that

$$\|f\| = \|H^{-n}H^n f\| \le C^n \|H^n f\| = C^n \|f\|_n \qquad \text{for all } f \in \mathcal{D}(H^n).$$

The space X_n is known as the Sobolev space of order n associated with $(T(t))_{t\ge 0}$.

Let $n \in \mathbb{N}$. We denote by $(T_n(t))_{t\ge 0}$, the restriction of $(T(t))_{t\ge 0}$ to X_n. By [30, Proposition II.5.2(ii), (iii)], $(T_n(t))_{t\ge 0}$ is a strongly continuous semigroup on X_n and its generator, H_n, is the part of H in X_n, i.e.

$$H_n f = Hf, \qquad \mathcal{D}(H_n) = \{f \in X_n : Hf \in X_n\}.$$

The restriction H_n is a closed linear operator in X_n which maps $\mathcal{D}(H_n)$ onto X_n. However, we can also view H_n as a bounded operator from X_{n+1} onto X_n and in this case it becomes an isometry in $\mathcal{B}(X_{n+1}, X_n)$. To distinguish between these two different interpretations of H_n, we denote the isometry version by $\mathcal{H}_n$.

For the construction of Sobolev spaces of negative order, it is convenient to set $X_0 := X$, $T_0(t) := T(t)$, $\|\cdot\|_0 := \|\cdot\|$. We also let $H_0 = H$ denote the, potentially unbounded, operator $H_0 : X_0 \supseteq \mathcal{D}(H) \to X_0$, and let $\mathcal{H}_0 = H$ denote the isometry $\mathcal{H}_0 : X_1 \to X_0$.

As noted in [30, p. 125], for $n = 0, 1, 2, \ldots$, we can equivalently define X_n as the completion of X_{n+1} with regard to the norm

$$\|f\|_n := \|H_{n+1}^{-1} f\|_{n+1}.$$

From this equivalent definition, we are led naturally to Sobolev spaces of negative order that are associated with $(T(t))_{t\ge 0}$, and which are defined recursively in the following manner. First, we define $X_{-1} = (X, \|\cdot\|_{-1})^{\sim}$, where,

$$\|f\|_{-1} = \|H_0^{-1} f\|_0 \qquad \text{for all } f \in X_0$$

and $\tilde{}$ denotes the completion. This is the Sobolev space of order -1 associated with $(T(t))_{t\geq 0}$. For $f \in X_1$,

$$\|\mathcal{H}_0 f\|_{-1} = \|H_0^{-1}\mathcal{H}_0 f\|_0 = \|H_0^{-1}H_0 f\|_0 = \|f\|_0.$$

Moreover, X_1 is dense in X_0 from Theorem 3.3.8(i). It follows that we can define $\mathcal{H}_{-1} : X_0 \to X_{-1}$ to be the unique, bounded, continuous, linear extension of $\mathcal{H}_0 : X_1 \to X_0$. We now define $H_{-1} : X_{-1} \supseteq \mathcal{D}(H_{-1}) \to X_{-1}$ as

$$H_{-1} = \mathcal{H}_{-1}, \quad \mathcal{D}(H_{-1}) = \{f \in X_{-1} : \mathcal{H}_{-1} f \in X_{-1}\}.$$

Once again, $\mathcal{H}_{-1}$ is a bounded operator from $X_0 \to X_{-1}$ while H_{-1} is potentially unbounded.

The growth bound of $(T_0(t))_{t\geq 0}$ is negative and so there exists $\tilde{c} \geq 1$ such that, for all $f \in X_0$, we have that

$$\|T_0(t)f\|_0 \leq \tilde{c}\|f\|_0.$$

Let $f \in X_0$. There exists $g \in \mathcal{D}(H_0)$ such that $f = H_0 g$. From Lemma 3.3.6 we have

$$T_0(t)H_0 g = H_0 T_0(t) g$$

and so

$$T_0(t)f = H_0 T_0(t) g.$$

Applying H_0^{-1} to both sides we obtain

$$H_0^{-1}T_0(t)f = T_0(t)g = T_0(t)H_0^{-1}f.$$

Hence, for all $f \in X_0$,

$$\|T_0(t)f\|_{-1} = \|H_0^{-1}T_0(t)f\|_0 = \|T_0(t)H_0^{-1}f\|_0 \leq \tilde{c}\|H_0^{-1}f\|_0 = \tilde{c}\|f\|_{-1}.$$

Since X_0 is dense in X_{-1}, it follows that, for $t \geq 0$, $T_0(t)$ extends to a bounded operator, $T_{-1}(t)$, on X_{-1}.

We now use X_{-1} to construct the Sobolev space of order -2 associated with

$(T(t))_{t\geq 0}$. Let $X_{-2} = (X_{-1}, \|\cdot\|_{-2})^{\sim}$, where

$$\|f\|_{-2} = \|H_{-1}^{-1} f\|_{-1} \qquad \text{for all } f \in X_{-1}.$$

Iteratively, for all $n \in \mathbb{N}$, we can argue as before to deduce that the bounded operator $\mathcal{H}_{-n+1} : X_{-n+2} \to X_{-n+1}$ can be extended to a unique, bounded, continuous, linear operator, $\mathcal{H}_{-n} : X_{-n+1} \to X_{-n}$. In a similar manner as before, we denote by $H_{-n} : X_{-n} \supseteq \mathcal{D}(H_{-n}) \to X_{-n}$ the, potentially unbounded, operator given by

$$H_{-n} = \mathcal{H}_{-n}, \quad \mathcal{D}(H_{-n}) = \{f \in X_{-n} : \mathcal{H}_{-n} f \in X_{-n}\}, \tag{3.4.3}$$

We can therefore recursively define

$$X_{-n} = (X_{-n+1}, \|\cdot\|_{-n})^{\sim}, \tag{3.4.4}$$

for all $n \in \mathbb{N}$, where,

$$\|f\|_{-n} = \|H_{-n+1}^{-1} f\|_{-n+1} \qquad \text{for all } f \in X_{-n+1}. \tag{3.4.5}$$

We thus obtain a "tower" of spaces where $\mathcal{H}_n : X_{n+1} \to X_n$ and $\mathcal{H}_n^{-1} : X_n \to X_{n+1}$ are bounded operators for all $n \in \mathbb{Z}$.

Using a similar argument as before, for all $n \in \mathbb{N}$, we can recursively show that $T_{-n+1}(t)$ can be continuously extended to a bounded operator, $T_{-n}(t)$, on the extrapolated space X_{-n}. We now obtain the following theorem.

Theorem 3.4.6. [30, Theorem II.5.5] *For all $m \geq n \in \mathbb{Z}$ the following holds.*

(i) *Each X_n is a Banach space containing X_m as a dense subspace.*

(ii) *The operators $T_n(t)$ form a C_0-semigroup, $(T_n(t))_{t\geq 0}$, on X_n.*

(iii) *The generator of $(T_n(t))_{t\geq 0}$ is $H_n : X_n \supseteq \mathcal{D}(H_n) \to X_n$ given, as above, by*

$$H_n = \mathcal{H}_n, \quad \mathcal{D}(H_n) = \{f \in X_n : \mathcal{H}_n f \in X_n\} = X_{n+1}. \tag{3.4.6}$$

This generator is the unique continuous linear extension of the operator $H_m : X_{m+1} \to X_m$ to an isometry from X_{n+1} to X_n, but considered as an *unbounded operator in* X_n.

Moreover, we have the following similarity result. This corollary is stated, without proof, in [30, Corollary II.5.3] for $n \in \mathbb{N}$. Here, we prove a more general result for $n \in \mathbb{Z}$.

Corollary 3.4.7. *All of the semigroups,* $(T_n(t))_{t\geq 0}$*, are similar. More precisely,*

$$T_{n+1}(t) = \mathcal{H}_n^{-1} T_n(t) \mathcal{H}_n = T_n(t)|_{X_{n+1}} \qquad \text{for all } n \in \mathbb{Z}. \tag{3.4.7}$$

Proof. We know that $\mathcal{H}_n : X_{n+1} \to X_n$ is an isomorphism. From [30, Sections I.5.10 and II.2.1], we have that $(S(t))_{t\geq 0}$, defined by

$$S(t) = \mathcal{H}_n^{-1} T_n(t) \mathcal{H}_n, \ t \geq 0,$$

is a C_0-semigroup on X_{n+1}, with generator, A, defined by

$$A = \mathcal{H}_n^{-1} H_n \mathcal{H}_n = H_n^{-1} H_n H_n = H_n|_{\mathcal{D}(A)} \text{ (since } H_n \text{ and } \mathcal{H}_n \text{ coincide on } \mathcal{D}(H_n))$$

$$\begin{aligned} \mathcal{D}(A) &= \{f \in X_{n+1} : H_n f \in \mathcal{D}(H_n)\} \\ &= \{f \in X_{n+1} : H_n f \in X_{n+1}\} \\ &= X_{n+2}. \end{aligned}$$

We justify the last equality as follows. Since $H_n|_{X_{n+2}} = H_{n+1}|_{X_{n+2}} = \mathcal{H}_{n+1}$ is an isomorphism onto X_{n+1}, and $X_{n+2} \subseteq X_{n+1}$, it follows that $X_{n+2} \subseteq \mathcal{D}(A)$. Now suppose $g \in \mathcal{D}(A) \backslash X_{n+2}$. Then $H_n g = h$ for some $h \in X_{n+1}$. Similarly, $\mathcal{H}_{n+1} f = H_{n+1} f = H_n f = h$ for some $f \in X_{n+2}$. Since H_n is injective, $f = g$. This contradicts $g \notin X_{n+2}$ and so $\mathcal{D}(A) = X_{n+2}$.

It follows that $A = H_n|_{X_{n+2}} = H_{n+1}$ and so $(S(t))_{t\geq 0} = (T_{n+1}(t))_{t\geq 0}$. Thus, (3.4.7) holds. □

Remark 3.4.8. From [30, Section II.2.1], the spectra of $(T_n(t))_{t\geq 0}$ must coincide, and $R(\lambda, H_{n+1}) = H_n^{-1} R(\lambda, H_n) H_n$ for all $n \in \mathbb{Z}$. Moreover, as mentioned immediately after [30, Corollary II.5.3], the spectral bound and growth bound of $(T_n(t))_{t\geq 0}$, coincide for all $n \in \mathbb{Z}$.

From Corollary 3.4.7 we now have

$$T_{n+1}(t) = \mathcal{H}_n^{-1} T_n(t) \mathcal{H}_n \qquad \text{for all } t \geq 0, n \in \mathbb{Z}$$

and so

$$T_n(t) = \mathcal{H}_n T_{n+1}(t) \mathcal{H}_n^{-1} \qquad \text{for all } t \geq 0, n \in \mathbb{Z}.$$

We know that $\mathcal{H}_n$, $\mathcal{H}_n^{-1}$ are continuous, and it follows that, if $(T_0(t))_{t\geq 0}$ is analytic, then $(T_n(t))_{t\geq 0}$ is an analytic C_0-semigroup on X_n, for all $n \in \mathbb{Z}$. Hence, in this case, $T_n(t)\mathring{u} \in \mathcal{D}(H_n^p)$ for all $p \in \mathbb{N}$, $\mathring{u} \in X_n$, $t > 0$, and so, in particular, $T_n(t)\mathring{u} \in \mathcal{D}(H_n)$ for all $\mathring{u} \in X_n$, $t > 0$. We use this to prove the following new result.

Theorem 3.4.9. *Let $(T(t))_{t\geq 0}$ be analytic. Fix $n \in \mathbb{Z}$ and let $\mathring{u} \in X_n$. Then $T_n(t)\mathring{u} \in X_m$ for all $m \geq n$, $t > 0$.*

Proof. We have, for all $t > 0$, that $T_n(t)\mathring{u} \in X_n = X_{n+0}$. Now fix $k \in \mathbb{N} \cup \{0\}$ and assume that, for all $t > 0$, $T_n(t)\mathring{u} \in X_{n+k}$. Then, for $t_0 \in (0, t)$,

$$\begin{aligned} T_n(t)\mathring{u} &= T_n(t - t_0)T_n(t_0)\mathring{u} \\ &= T_{n+k}(t - t_0)T_n(t_0)\mathring{u} \in \mathcal{D}(H_{n+k}) = \mathcal{D}(\mathcal{H}_{n+k}) = X_{n+k+1}, \end{aligned}$$

since $T_{n+k}(t)$ and $T_n(t)$ coincide on X_{n+k} and, for all $f \in X_{n+k}$, $t > 0$, we have $T_{n+k}(t)f \in \mathcal{D}(H_{n+k})$. Hence $T_n(t)\mathring{u} \in X_m$ for all $m \geq n$ and $t > 0$ by induction. □

The following example is [30, Example II.5.7] in the particular case when $(X_0, \|\cdot\|_0) = (\ell_w^1, \|\cdot\|_{\ell_w^1})$, where ℓ_w^1 and $\|\cdot\|_{\ell_w^1}$ are as defined in (3.1.3) and (3.1.8) respectively. These weighted l^1 spaces are of particular relevance later since we choose to examine the coagulation–fragmentation system as an ACP in such spaces.

Example 3.4.10. Suppose $(X_0, \|\cdot\|_0) = (\ell_w^1, \|\cdot\|_{\ell_w^1})$ and $(h_k)_{k=1}^{\infty}$ is such that $h_k \in \mathbb{R}$ and $h_k < 0$ for all $k \in \mathbb{N}$. The corresponding multiplication operator $H : \mathcal{D}(H) \to X_0$, given by

$$[Hf]_k = h_k f_k, \qquad \text{for all } f \in \mathcal{D}(H),\ k \in \mathbb{N},$$

is the generator of the semigroup, $(T_0(t))_{t\geq 0}$, where

$$[T_0(t)f]_k = e^{h_k t} f_k, \qquad \text{for all } f \in X_0,\ t \geq 0,\ k \in \mathbb{N}.$$

For all $n \in \mathbb{Z}$, the Sobolev spaces, X_n, are then

$$X_n = \{g = (g_k)_{k=1}^{\infty} : g_k \in \mathbb{R} \text{ for all } k \in \mathbb{N} \text{ and } H^n g \in X_0\},$$

where $[H^n f]_k = h_k^n f_k, \qquad$ for all $k \in \mathbb{N}$, $f \in X_0$.

The following new theorem allows us to obtain a solution of the ACP (3.4.2) for any $\mathring{u} \in X_{-n}$, where $n \in \mathbb{N}$.

Theorem 3.4.11. *Let $n \in \mathbb{N}$ and let the semigroup $(T(t))_{t\geq 0}$ be analytic. Then (3.4.2) has a unique solution $u \in C^1((0,\infty), X) \cap C([0,\infty), X_{-n})$ for all $\mathring{u} \in X_{-n}$. This solution is given by $u(t) = T_{-n}(t)\mathring{u}$, $t \geq 0$.*

Proof. Let $n \in \mathbb{N}$ and take $\mathring{u} \in X_{-n}$. From Theorem 3.4.9, $T_{-n}(t)\mathring{u} \in X_1 = \mathcal{D}(\mathcal{H})$ for all $t > 0$. Hence, $T_{-n}(t)\mathring{u} \in \mathcal{D}(H)$ for all $t > 0$. Since $T_{-n}(t)\mathring{u} \in \mathcal{D}(H_{-n})$ for all $t > 0$, we have $u(t) = T_{-n}(t)\mathring{u} \in C^1((0,\infty), X_{-n}) \cap C([0,\infty), X_{-n})$ is the unique classical solution of

$$u'(t) = H_{-n}u(t), \quad t > 0; \quad u(0) = \mathring{u}. \tag{3.4.8}$$

For fixed $t > 0$, let t_0 be such that $0 < t_0 < t$. Then

$$T_{-n}(t)\mathring{u} = T_{-n}(t-t_0)T_{-n}(t_0)\mathring{u} = T(t-t_0)T_{-n}(t_0)\mathring{u}$$

since $T_{-n}(t_0)\mathring{u} \in \mathcal{D}(H)$. It follows that, for all $t > 0$, $T_{-n}(t)\mathring{u}$ is differentiable with respect to the norm, $\|\cdot\|$, on X, with

$$\frac{d}{dt}(T_{-n}(t)\mathring{u}) = HT(t-t_0)T_{-n}(t_0)\mathring{u} = HT_{-n}(t)\mathring{u}.$$

It follows that $u(t) = T_{-n}(t)\mathring{u}$ satisfies (3.4.2) for $t > 0$, $\mathring{u} \in X_{-n}$.

Now, if $u(t)$ is differentiable with respect to the norm in X, then it is differentiable with respect to the norm in X_{-n} and the derivatives coincide. Also,

H and H_{-n} coincide on $\mathcal{D}(H)$. It follows that if $v(t)$ satisfies (3.4.2), then $v(t)$ satisfies (3.4.8). Since $T_{-n}(t)\mathring{u} \in \mathcal{D}(H)$ is the unique solution of (3.4.8), we have that $u(t) = T_{-n}(t)\mathring{u}$ is the unique solution of (3.4.2). $\square$

CASE 2: Semigroups with General Growth Bound

When we examine the pure fragmentation system in general weighted ℓ^1 spaces we obtain a semigroup with growth bound zero and so we cannot use this semigroup directly to construct a Sobolev tower as in CASE 1. We deal with general growth bounds here. Let X be a Banach space, equipped with the norm $\|\cdot\|$. We now consider the case where G is the generator of a C_0-semigroup, $(S(t))_{t\geq 0}$, on X, with growth bound $\omega_0 \in \mathbb{R}$. As in CASE 1, we are interested in solutions of the ACP

$$u'(t) = Gu(t), \quad t > 0; \quad u(0) = \mathring{u}. \tag{3.4.9}$$

The way we deal with this more general problem is to rescale the semigroup $(S(t))_{t\geq 0}$ to obtain a semigroup with a negative growth bound. We can then apply the Sobolev tower construction described in CASE 1, which allows us to obtain existence and uniqueness results to a rescaled ACP. Finally, we scale back to obtain results relating to (3.4.9).

Choose $\mu > \omega_0$ and define the rescaled semigroup, $(T(t))_{t\geq 0}$, by

$$T(t) := e^{-\mu t} S(t) \quad \text{for all } t \geq 0.$$

The generator of this rescaled semigroup is $H = G - \mu I$ and we have, for some $M \geq 1$,

$$\|T(t)\| = \|e^{-\mu t} S(t)\| \leq e^{-\mu t} e^{\omega_0 t} \cdot M = e^{(\omega_0 - \mu)t} \cdot M \qquad \text{for all } t \geq 0.$$

Since $\omega_0 - \mu < 0$, the growth bound of $(T(t))_{t\geq 0}$ is negative and so $0 \in \rho(H)$. Note that, if the semigroup $(S(t))_{t\geq 0}$ is analytic, then the rescaled semigroup $(T(t))_{t\geq 0}$ is also analytic.

As in CASE 1, we can now use $(T(t))_{t\geq 0}$ and H to construct a Sobolev tower where, for $n \in \mathbb{Z}$, X_n is the Sobolev space of order n associate with $(T(t))_{t\geq 0}$. We adopt the same notation as used in CASE 1.

For $n \in \mathbb{N}$, the unique, continuous extension of $T(t)$ from X to X_{-n} is given by $T_{-n}(t) = e^{-\mu t} S_{-n}(t)$, where $S_{-n}(t)$ is the unique extension of $S(t)$ from X to X_{-n}.

We have formulated the following proposition which gives the relationship between a solution of a rescaled ACP and a solution of the ACP (3.4.9).

Proposition 3.4.12. *Suppose that for some $n \in \mathbb{N}$ and $\mathring{u} \in X_{-n}$, $u(t) = T_{-n}(t)\mathring{u}$ satisfies*

$$u'(t) = Hu(t) = (G - \mu I)u(t), \quad t > 0; \quad u(0) = \mathring{u}. \tag{3.4.10}$$

Then $u(t) = S_{-n}(t)\mathring{u}$ satisfies (3.4.9). *If, in addition, $u(t) = T_{-n}(t)\mathring{u}$ is the unique function that satisfies* (3.4.10)*, then $u(t) = S_{-n}(t)\mathring{u}$ is the unique function that satisfies* (3.4.9)*.*

Proof. Let $u(t)$ satisfy (3.4.10). It follows that

$$\frac{d}{dt}(e^{\mu t}u(t)) = \mu e^{\mu t}u(t) + e^{\mu t}\frac{d}{dt}(u(t)) = \mu e^{\mu t}u(t) + e^{\mu t}(G - \mu I)u(t) = Ge^{\mu t}u(t).$$

Also, $e^{0t}u(0) = e^0\mathring{u} = \mathring{u}$ and so $e^{\mu t}u(t)$ satisfies (3.4.9). Thus, if $u(t) = T_{-n}(t)\mathring{u}$ satisfies (3.4.10), then $u(t) = e^{\mu t}T_{-n}(t)\mathring{u} = S_{-n}(t)\mathring{u}$ satisfies (3.4.9).

We can similarly show that if $u(t)$ satisfies (3.4.9), then $e^{-\mu t}u(t)$ satisfies (3.4.10). Hence, if $u(t) = T_{-n}(t)\mathring{u}$ is the unique function that satisfies (3.4.10), then $u(t) = S_{-n}(t)\mathring{u}$ is the unique function that satisfies (3.4.9). $\square$

We now give a new result that is analogous to Theorem 3.4.11. This result is useful later, when we examine the pure fragmentation system in weighted ℓ^1 spaces.

Theorem 3.4.13. *Let $n \in \mathbb{N}$ and let $(S(t))_{t \geq 0}$ be an analytic C_0-semigroup. For all $\mathring{u} \in X_{-n}$,* (3.4.9) *has a unique solution $u \in C^1((0,\infty), X) \cap C([0,\infty), X_{-n})$. This solution is given by $u(t) = S_{-n}(t)\mathring{u}$.*

Proof. Since $(S(t))_{t\geq 0}$ is an analytic semigroup, the rescaled semigroup $(T(t))_{t\geq 0}$ is also analytic. From Theorem 3.4.11 we know that (3.4.10) has a unique solution $u \in C^1((0,\infty), X) \cap C(([0,\infty), X_{-n})$, given by $u(t) = T_{-n}(t)\mathring{u} = e^{-\mu t}S_{-n}(t)\mathring{u}$, for all $\mathring{u} \in X_{-n}$. The result then follows from Proposition 3.4.12. $\square$

3.5 Evolution Families and Non-Autonomous Abstract Cauchy Problems

In this section we briefly discuss evolution families and the role that they play in solving non-autonomous ACPs. This is of importance later when non-autonomous fragmentation is examined. We begin by providing a precise definition of an evolution family that is based on [30, Definition VI.9.2].

Definition 3.5.1. Let X be a Banach space and let I be an interval such that $I \subseteq \mathbb{R}$. A family of bounded linear operators, $(U(t,s))_{t,s\in I:t\geq s}$, on X is called an *evolution family* (or a *strongly continuous evolution family*) on X if

(i) $U(t,r) = U(t,s)U(s,r)$ and $U(r,r) = I$ for all $t,s,r \in I$ such that $r \leq s \leq t$;

(ii) the mapping $(t,s) \mapsto U(t,s)$ is strongly continuous on $\{(t,s) \in I^2 : t \geq s\}$.

We now turn our attention to non-autonomous ACPs. Let X be a Banach space and let $s, T \in \mathbb{R}$ be such that $s < T \leq \infty$. Let $I = [s,T]$ if $T < \infty$ and $I = [s,\infty)$ if $T = \infty$. We examine equations of the form

$$u'(t) = G(t)u(t), \quad t \in I,\ t > s; \qquad u(s) = \mathring{u}, \tag{3.5.1}$$

where $(G(t))_{t\geq 0}$ is a family of linear operators with domains $\mathcal{D}(G(t))$, $t \geq 0$. In the following definition, from [59, Definition 2.1], we define solutions of (3.5.1).

Definition 3.5.2. Consider a continuous function $u : I \to X$. Then u is a

(i) *classical solution* of (3.5.1) if $u \in C^1(I\backslash\{s\}, X)$, $u(t) \in \mathcal{D}(G(t))$ for all $t \in I$, $t > s$ and (3.5.1) is satisfied;

(ii) *strict solution* of (3.5.1) if $u \in C^1(I, X)$, $u(t) \in \mathcal{D}(G(t))$ for all $t \geq s$ and (3.5.1) is satisfied,

where

$$C^1(I,X) = \left\{ f \in C^1\Big(I\backslash\{s\}, X\Big) : f'(s) := \lim_{x\to s^+} f'(x) \text{ exists} \right\}.$$

The following result from [61, Theorem 5.6.8] regarding the existence and uniqueness of solutions is crucial in Chapter 7, where we examine non-autonomous pure fragmentation. The condition (P2) given here differs slightly from that in [61]. This is due to the fact that [61] considers the case where $-G(t)$ is a generator, while here we take $G(t)$ to be the generator.

Theorem 3.5.3. *Let X be a Banach space and $T > 0$. For each $t \in [0, T]$, let $G(t)$ be the generator of an analytic C_0-semigroup, $(S_t(s))_{s\geq 0}$, on X. Assume that the following conditions are satisfied.*

(P1) The domain $\mathcal{D}(G(t)) := \mathcal{D}$ of $G(t)$ is dense in X and independent of $t \in [0, T]$.

(P2) For $t \in [0, T]$, the resolvent $R(\lambda, G(t))$ exists for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda \geq 0$ and there is a constant $\mathscr{M}$ such that

$$\|R(\lambda, G(t))\| \leq \frac{\mathscr{M}}{|\lambda| + 1} \qquad \text{for all } t \in [0, T],\ \lambda \in \mathbb{C} : \operatorname{Re}\lambda \geq 0.$$

(P3) There exist constants L and $\sigma \in (0, 1]$ such that

$$\left\|\Big(G(t) - G(s)\Big)G(\tau)^{-1}\right\| \leq L|t - s|^{\sigma} \qquad \text{for } s, t, \tau \in [0, T].$$

Then, for every $0 \leq s < T$ and $\mathring{u} \in X$ the initial value problem

$$u'(t) = G(t)u(t), \qquad s < t \leq T; \qquad u(s) = \mathring{u} \tag{3.5.2}$$

has a unique classical solution. This solution is given by $u(t) = W(t, s)\mathring{u}$, where $(W(t, s))_{(t,s)\in[0,T]:t\geq s}$ is an evolution family.

Chapter 4

Nonlinear Operators

Our approach for dealing with the full system (1.1.1) is to pose it as a semi-linear ACP in an appropriate Banach space. The non-linearity in this ACP arises from the introduction of a non-linear coagulation operator to represent the coagulation terms in (1.1.1). In this chapter we provide some abstract results relating to non-linear operators that we require when we examine the C–F system. In Section 4.1, we provide definitions and results regarding the Lipschitz continuity and Fréchet differentiability of an operator. These properties are very important in Section 4.2, where abstract existence and uniqueness results for semi-linear ACPs are provided. We again highlight new results at the start of each section and when they are stated.

4.1 Lipschitz Conditions and Fréchet Differentiability

To apply existence and uniqueness results to the C–F system, we require the non-linear coagulation operator to be Lipschitz continuous. To obtain classical solutions, the coagulation operator must also be Fréchet differentiable. Results are provided in [46, Theorems 4.3 and 4.4] regarding the Lipschitz continuity and Fréchet differentiability of the coagulation operator, when the coagulation rates are time-independent and uniformly bounded. We provide abstract results in Lemma 4.1.6, Proposition 4.1.8 and Corollary 4.1.9 to prove the Lipschitz continuity and Fréchet differentiability of general operators that satisfy certain conditions. When applied to the coagulation operator, these results are stronger than those in [46] since they allow the possibility that the coagulation operator is time-dependent and not uniformly bounded.

We require the following definitions and results.

Definition 4.1.1. Let X be a normed vector space with norm $\|\cdot\|$. We define the *open ball*, centred at $h \in X$ with radius $r > 0$, to be the set

$$B_X(h, r) \coloneqq \{g \in X : \|h - g\| < r\}. \tag{4.1.1}$$

Similarly, the *closed ball*, centred at $h \in X$ with radius $r > 0$, is the set

$$\overline{B}_X(h, r) \coloneqq \{g \in X : \|h - g\| \leq r\}. \tag{4.1.2}$$

The following definition is based on [22, Definition 3.6 (ii)].

Definition 4.1.2. Let t_0, $T \in \mathbb{R}$ such that $0 \leq t_0 < T < \infty$ and let I be an interval of the form $[t_0, T)$, $[t_0, T]$ or $[t_0, \infty)$. Let $(Y, \|\cdot\|_Y)$, $(X, \|\cdot\|_X)$ be normed vector spaces. Then an operator $F : I \times Y \to X$ satisfies a *local Lipschitz condition in the second argument, uniformly in the first argument on compact intervals* if, for every $t' \in I$ and $h \in Y$, there exist constants, $r > 0$, $L = L(t', h, r) > 0$, such that for all $f, g \in \overline{B}_Y(h, r)$ and $t \in [t_0, t']$,

$$\|F(t, f) - F(t, g)\|_X \leq L\|f - g\|_Y. \tag{4.1.3}$$

Note that when $I = [t_0, T]$, we can take $t' = T$ so that (4.1.3) holds with the same L for all $t \in [t_0, T]$.

The following definition is taken from [61, p. 185].

Definition 4.1.3. Let t_0, $T \in \mathbb{R}$ such that $0 \leq t_0 < T < \infty$ and let I be an interval of the form $[t_0, T)$, $[t_0, T]$ or $[t_0, \infty)$. Let $(Y, \|\cdot\|_Y)$, $(X, \|\cdot\|_X)$ be normed vector spaces. Then an operator $F : I \times Y \to X$ satisfies a *Lipschitz condition in the second argument on bounded sets, uniformly in the first argument* *on compact intervals* if, for every $t' \in I$ and constant $r > 0$, there exists a constant $l = l(t', r) > 0$ such that

$$\|F(t, f) - F(t, g)\|_X \leq l\|f - g\|_Y, \tag{4.1.4}$$

for all $f, g \in Y$, with $\|f\|_Y \leq r$, $\|g\|_Y \leq r$ and $t \in [t_0, t']$.

Note that when $I = [t_0, T]$, we can take $t' = T$ so that (4.1.4) holds with the same l for all $t \in [t_0, T]$.

Remark 4.1.4. Let F satisfy a Lipschitz condition in the second argument on bounded sets, uniformly in the first argument on compact intervals as in Definition 4.1.3. Then F also satisfies a local Lipschitz condition in the second argument, uniformly in the first argument on compact intervals, as in Definition 4.1.2. We can show this as follows. Let $h \in Y$, $t' \in I$. Choose an arbitrary $\tilde{r} > 0$ and set $r = \tilde{r} + \|h\|_Y$. For all $f, g \in \overline{B}_Y(h, \tilde{r})$, we have

$$\begin{aligned}\|f\|_Y &\leq \|f - h\|_Y + \|h\|_Y \leq \tilde{r} + \|h\|_Y = r,\\ \|g\|_Y &\leq \|g - h\|_Y + \|h\|_Y \leq \tilde{r} + \|h\|_Y = r,\end{aligned}$$

and so

$$\|F(t, f) - F(t, g)\|_X \leq L(t', h, \tilde{r})\|f - g\|_Y \text{ for all } t \in [t_0, t'],$$

where $L(t', h, \tilde{r}) = l(t', r) = l(t', \tilde{r} + \|h\|_Y)$. Thus there exists a constant $L(t', h, r)$ such that (4.1.3) holds.

We also need the concept of Fréchet differentiability. The following definition is taken from [22, Definition 3.27].

Definition 4.1.5. Let $(Y, \|\cdot\|_Y)$ and $(X, \|\cdot\|_X)$ be Banach spaces and let $F : \mathcal{D}(F) \to X$ where $\mathcal{D}(F) \subseteq Y$. Let $f \in \mathcal{D}(F)$. The operator F is said to be *Fréchet differentiable* at f if $B_Y(f, r) \subseteq \mathcal{D}(F)$ for some $r > 0$ and there exists a linear operator $DF(f) \in \mathcal{B}(Y, X)$ and a mapping $R(f, \cdot) : B_Y(0, r) \to X$ such that, for all $\delta \in B_Y(0, r)\backslash\{0\}$,

$$F(f + \delta) = F(f) + DF(f)\delta + R(f, \delta), \tag{4.1.5}$$

where R satisfies
$$\lim_{\|\delta\|_Y \to 0} \frac{\|R(f,\delta)\|_X}{\|\delta\|_Y} = 0.$$
The operator $DF(f)$ is the *Fréchet derivative* of F at f. The operator F is *Fréchet differentiable on* an open subset $D_0 \subseteq \mathcal{D}(F)$ if it is Fréchet differentiable at every $f \in D_0$.

In the following results we provide sufficient conditions under which an operator is locally Lipschitz and Fréchet differentiable. These properties will be of importance later when we apply existence and uniqueness results to the full coagulation–fragmentation system. The next lemma has been formulated by McBride, Lamb and Smith in [46, Theorems 4.3 and 4.4] for the specific case where the operator F describes time-independent coagulation. We prove the result for a more general operator, $F : I \times Y \to X$, where $I \subseteq [0,\infty)$ and Y, X are Banach spaces. We note that the operator F in this lemma is basically quadratic in the space variable, in the sense that

$$F(t, \alpha f) = \alpha^2 F(t,f) \qquad \text{for all } (t,f) \in I \times Y \text{ and } \alpha \in \mathbb{R}.$$

Lemma 4.1.6. *Let t_0, $T \in \mathbb{R}$ be such that $0 \leq t_0 < T < \infty$ and let I be an interval of the form $[t_0,T)$, $[t_0,T]$ or $[t_0,\infty)$. Let $(Y, \|\cdot\|_Y)$, $(X, \|\cdot\|_X)$ be Banach spaces and let $\tilde{F} : I \times Y \times Y \to X$ be such that*

(a) *$\tilde{F}$ is linear in the second and third arguments;*

(b) *for each $t' \in I$ there exists a $c(t') > 0$ such that*
$$\|\tilde{F}[t,f,g]\|_X \leq c(t')\|f\|_Y \|g\|_Y, \tag{4.1.6}$$
for all $f, g \in Y$ and $t \in [t_0, t']$.

Define $F : I \times Y \to X$ by $F(t,f) := \tilde{F}[t,f,f]$ for $(t,f) \in I \times Y$.

(i) *Then F is Lipschitz in the second argument on bounded sets, uniformly in the first argument on compact intervals.*

(ii) *If, in addition, $t \mapsto \tilde{F}[t,f,g]$ is continuous on I for every fixed f, $g \in Y$, then $\tilde{F} : I \times Y \times Y \to X$ is continuous and hence $F : I \times Y \to X$ is* *continuous.*

Proof. Let $t' \in I$, $r > 0$ and let f, $g \in Y$ be such that $\|f\|_Y \leq r$ and $\|g\|_Y \leq r$. Then, for all $t \in [t_0, t']$, we have

$$\begin{aligned}
\|F(t,f) - F(t,g)\|_X &= \|\tilde{F}[t,f,f] - \tilde{F}[t,g,g]\|_X \\
&= \|\tilde{F}[t,f,f] - \tilde{F}[t,g,f] + \tilde{F}[t,g,f] - \tilde{F}[t,g,g]\|_X \\
&\leq \|\tilde{F}[t,f-g,f]\|_X + \|\tilde{F}[t,g,f-g]\|_X \\
&\leq c(t')\|f-g\|_Y\|f\|_Y + c(t')\|g\|_Y\|f-g\|_Y \\
&\leq 2rc(t')\|f-g\|_Y \\
&= L\|f-g\|_Y
\end{aligned}$$

where $L = L(t', r) = 2rc(t')$. Thus, F is Lipschitz in Y on bounded sets, uniformly in t on compact intervals and (i) holds.

Now suppose that, in addition, $t \mapsto \tilde{F}[t,f,g]$ is continuous on I for every fixed f, $g \in Y$. Fix $(s_0, f_0, g_0) \in I \times Y \times Y$, and choose $r > 0$ such that $\|f_0\|_Y < r$ and $\|g_0\|_Y < r$. Moreover, choose $t' \in I$ such that $t' > s_0$ (if $I = [t_0, T]$ and $s_0 = T$, we take $t' = s_0 = T$). Then, for $(t, f, g) \in I \times Y \times Y$, such that $s \in [t_0, t']$ and $\|f\|_Y \leq r$, $\|g\|_Y \leq r$,

$$\begin{aligned}
&\|\tilde{F}(s,f,g) - \tilde{F}(s_0,f_0,g_0)\|_X \\
&= \|\tilde{F}(s,f,g) - \tilde{F}(s,f_0,g) + \tilde{F}(s,f_0,g) - \tilde{F}(s,f_0,g_0) \\
&\qquad\qquad\qquad + \tilde{F}(s,f_0,g_0) - \tilde{F}(s_0,f_0,g_0)\|_X \\
&= \|\tilde{F}(s,f-f_0,g) + \tilde{F}(s,f_0,g-g_0) + \tilde{F}(s,f_0,g_0) - \tilde{F}(s_0,f_0,g_0)\|_X \\
&\leq \|\tilde{F}(s,f-f_0,g)\|_X + \|\tilde{F}(s,f_0,g-g_0)\|_X + \|\tilde{F}(s,f_0,g_0) - \tilde{F}(s_0,f_0,g_0)\|_X \\
&\leq c(t')\|f-f_0\|_Y\|g\|_Y + c(t')\|f_0\|_Y\|g-g_0\|_Y + \|\tilde{F}(s,f_0,g_0) - \tilde{F}(s_0,f_0,g_0)\|_X \\
&\leq c(t')r\|f-f_0\|_Y + c(t')r\|g-g_0\|_Y + \|\tilde{F}(s,f_0,g_0) - \tilde{F}(s_0,f_0,g_0)\|_X \\
&\to 0 \qquad \text{as } (s,f,g) \to (s_0,f_0,g_0).
\end{aligned}$$

It follows that $\tilde{F} : I \times Y \times Y$ is continuous, and hence so is $F : I \times Y$, i.e. (ii) holds. □

Remark 4.1.7. Note that, from Remark 4.1.4, if the conditions of Lemma 4.1.6 hold, then it follows that, for $h \in Y$, F is Lipschitz on $\overline{B}_Y(h, r)$ for any $r > 0$.

To obtain the existence and uniqueness of classical solutions of the C–F system, we require the operator that we use to describe coagulation to be Fréchet differentiable. Since we allow the coagulation rates to be time-dependent, this operator is also time-dependent and so the following, specific case of Definition 4.1.5 is of particular interest to us. Consider the case where $(Y, \|\cdot\|_Y)$, $(X, \|\cdot\|_X)$ are Banach spaces. Let the norm, $\|\cdot\|_{\mathbb{R}\times Y}$, on the space $\mathbb{R} \times Y$ be given by

$$\|(t, f)\|_{\mathbb{R}\times Y} = |t| + \|f\|_Y \qquad \text{for all } (t, f) \in \mathbb{R} \times Y. \tag{4.1.7}$$

Let t_0, $T \in \mathbb{R}$ be such that $0 \leq t_0 < T < \infty$ and let I be an interval of the form $[t_0, T)$ or $[t_0, \infty)$. Let $F : I \times Y \to X$. From Definition 4.1.5, F is Fréchet differentiable at $(t, f) \in I \backslash \{t_0\} \times Y$ if there exists a linear operator $DF(t, f) \in \mathcal{B}(\mathbb{R} \times Y, X)$ and a mapping $R((t, f), \cdot) : B_{\mathbb{R}\times Y}(0, r) \to X$, for some $r > 0$, such that, for all $(\delta_t, \delta_f) \in (\mathbb{R} \times Y)\backslash\{(0, 0)\}$ satisfying $t + \delta_t \in I\backslash\{t_0\}$,

$$F\Big((t, f) + (\delta_t, \delta_f)\Big) = F(t, f) + DF(t, f)(\delta_t, \delta_f) + R\Big((t, f), (\delta_t, \delta_f)\Big)$$

where

$$\lim_{\|(\delta_t,\delta_f)\|_{\mathbb{R}\times Y}\to 0} \frac{\|R((t, f), (\delta_t, \delta_f))\|_X}{\|(\delta_t, \delta_f)\|_{\mathbb{R}\times Y}} = 0.$$

The next proposition is probably new in this form and provides conditions that are sufficient to obtain Fréchet differentiability of F. Moreover, we show that under these conditions the Fréchet derivative can be expressed as the sum of the derivative of F with respect to the first argument and the Fréchet derivative of F with respect to the second argument.

Proposition 4.1.8. *Let $(Y, \|\cdot\|_Y)$, $(X, \|\cdot\|_X)$ be Banach spaces and let t_0, $T \in \mathbb{R}$ such that $0 \leq t_0 < T < \infty$. Let I be an interval of the form $[t_0, T)$ or $[t_0, \infty)$ and consider the space $(\mathbb{R} \times Y, \|\cdot\|_{\mathbb{R}\times Y})$, where $\|\cdot\|_{\mathbb{R}\times Y}$ is defined by* (4.1.7). *Let $F : I \times Y \to X$ and assume that*

(a) *$(t, f) \mapsto F(t, f)$ is Fréchet differentiable with respect to f, with a uniform remainder term on bounded time intervals in the sense that, for all $t \in I$*

and $f, \delta_f \in Y$ *with* $\delta_f \neq 0$*, we have*

$$F(t, f + \delta_f) = F(t, f) + D_Y F(t, f)\delta_f + R_Y(t, f, \delta_f), \tag{4.1.8}$$

where for fixed f*,*

$$\frac{\|R_Y(t, f, \delta_f)\|_X}{\|\delta_f\|_Y} \to 0 \text{ as } \|\delta_f\|_Y \to 0, \tag{4.1.9}$$

uniformly in t *on compact subintervals of* I*, and for fixed* $(t, f) \in I \times Y$*,* $D_Y F(t, f) \in \mathcal{B}(Y, X)$*;*

(b) $t \mapsto F(t, f)$ *is strongly differentiable for fixed* $f \in Y$*;*

(c) $(t, f) \mapsto D_Y F(t, f)$ *and* $(t, f) \mapsto \frac{\partial}{\partial t} F(t, f)$ *are continuous;*

(d) $t \mapsto D_Y F(t, f)$ *is continuously differentiable for fixed* $f \in Y$*.*

Then F *is Fréchet differentiable on* $I \backslash \{t_0\} \times Y$ *and the Fréchet derivative of* F *at* $(t, f) \in I \backslash \{t_0\} \times Y$*,* $DF(t, f)$*, is given by*

$$DF(t, f)(s, g) = \frac{\partial}{\partial t} F(t, f)s + D_Y F(t, f)g, \tag{4.1.10}$$

for all $(s, g) \in \mathbb{R} \times Y$*. Moreover,* $DF(t, f)$ *is continuous with respect to* (t, f)*.*

Proof. Assumption (b) implies that for all $t \in I \backslash \{t_0\}$, $\delta_t \in \mathbb{R} \backslash \{0\}$ such that $t + \delta_t \in I \backslash \{t_0\}$ and $f \in Y$,

$$F(t + \delta_t, f) = F(t, f) + \frac{\partial}{\partial t} F(t, f)\delta_t + R_t(t, f, \delta_t) \tag{4.1.11}$$

where

$$\frac{\|R_t(t, f, \delta_t)\|_X}{|\delta_t|} \to 0 \qquad \text{as } |\delta_t| \to 0. \tag{4.1.12}$$

Let $t \in I \backslash \{t_0\}$, $\delta_t \in \mathbb{R} \backslash \{0\}$ such that $t + \delta_t \in I \backslash \{t_0\}$, $f \in Y$ and $\delta_f \in Y \backslash \{0\}$. Then from (4.1.8), (4.1.11) and assumption (d) we obtain that

$$\begin{aligned} &F(t + \delta_t, f + \delta_f) - F(t, f) \\ &= F(t + \delta_t, f + \delta_f) - F(t + \delta_t, f) + F(t + \delta_t, f) - F(t, f) \end{aligned}$$

$$= D_Y F(t+\delta_t, f)\delta_f + R_Y(t+\delta_t, f, \delta_f) + \frac{\partial}{\partial t}F(t,f)\delta_t + R_t(t,f,\delta_t)$$

$$\begin{aligned}
&= D_Y F(t,f)\delta_f + \int\limits_t^{t+\delta_t} \frac{\partial}{\partial \tau}\Big(D_Y F(\tau, f)\Big)\,\mathrm{d}\tau\delta_f + R_Y(t+\delta_t, f, \delta_f) \\
&\qquad + \frac{\partial}{\partial t}F(t,f)\delta_t + R_t(t,f,\delta_t).
\end{aligned}$$

Note that $\frac{\partial}{\partial t}(D_Y F(t,f))$ is continuous and therefore integrable. We now fix $(t,f) \in I\backslash\{t_0\} \times Y$. Then $D_Y F(t,f)\delta_f + \frac{\partial}{\partial t}F(t,f)\delta_t$ is linear in (δ_t, δ_f), since $D_Y F(t,f)$ is linear, and

$$\begin{aligned}
\left\| D_Y F(t,f)\delta_f + \frac{\partial}{\partial t}F(t,f)\delta_t \right\|_X &\le \|D_Y F(t,f)\| \|\delta_f\|_Y + \left\| \frac{\partial}{\partial t}F(t,f) \right\| |\delta_t| \\
&\le C(t,f)\|(\delta_f, \delta_t)\|_{\mathbb{R}\times Y},
\end{aligned}$$

where $C(t,f) = \max\left\{ \|D_Y F(t,f)\|, \left\| \frac{\partial}{\partial t}F(t,f) \right\| \right\} \ge 0$. Also,

$$\begin{aligned}
\frac{1}{\|(\delta_t, \delta_f)\|_{\mathbb{R}\times Y}} \left\| \int\limits_t^{t+\delta_t} \frac{\partial}{\partial \tau}\Big(D_Y F(\tau, f)\Big)\,\mathrm{d}\tau\delta_f \right\|_X &\le \frac{\|\delta_f\|_Y}{\|\delta_f\|_Y} \left\| \int\limits_t^{t+\delta_t} \frac{\partial}{\partial \tau}\Big(D_Y F(\tau, f)\Big)\,\mathrm{d}\tau \right\| \\
&\le \int\limits_t^{t+\delta_t} \left\| \frac{\partial}{\partial t}\Big(D_Y F(\tau, f)\Big) \right\| \mathrm{d}\tau \to 0
\end{aligned}$$

as $\|(\delta_t, \delta_f)\|_{\mathbb{R}\times Y} \to 0$.

As $\|(\delta_t, \delta_f)\|_{\mathbb{R}\times Y} \to 0$ we have $\|\delta_f\|_Y \to 0$ and $t + \delta_t \to t$. Thus, since (4.1.9) holds uniformly in t on compact subintervals of I, we have

$$\frac{\|R_Y(t+\delta_t, f, \delta_f)\|_X}{\|(\delta_t, \delta_f)\|_{\mathbb{R}\times Y}} \le \frac{\|R_Y(t+\delta_t, f, \delta_f)\|_X}{\|\delta_f\|_Y} \to 0 \qquad \text{as } \|(\delta_t, \delta_f)\|_{\mathbb{R}\times Y} \to 0$$

and similarly, by (4.1.12),

$$\frac{\|R_t(t,f,\delta_t)\|_X}{\|(\delta_t, \delta_f)\|_{(\mathbb{R}\times Y)}} \le \frac{\|R_t(t,f,\delta_t)\|_X}{|\delta_t|} \to 0 \qquad \text{as } \|(\delta_t, \delta_f)\|_{\mathbb{R}\times Y} \to 0.$$

It follows that F is Fréchet differentiable on $I\backslash\{t_0\}\times Y$ and the Fréchet derivative of F at $(t,f)\in I\backslash\{t_0\}\times Y$ is given by (4.1.10).

Since $D_Y F(t,f)$ and $\frac{\partial}{\partial t}F(t,f)$ are continuous with respect to (t,f), it follows from (4.1.10) that $DF(t,f)$ is also continuous with respect to (t,f). □

We now return to the situation as in Lemma 4.1.6. The conditions that are used to show Fréchet differentiability in the next result are more convenient to check for the coagulation operator used later than those in Proposition 4.1.8.

Corollary 4.1.9. *Let the conditions of Lemma 4.1.6(ii) be satisfied with $I=[t_0,T)$ or $I=[t_0,\infty)$. Moreover, assume that*

(a) *for fixed $f,g\in Y$, $t\mapsto\tilde{F}[t,f,g]$ is continuously differentiable;*

(b) *for each $t'\in I$, there exists $\tilde{c}(t')>0$ such that*

$$\Big\|\frac{\partial}{\partial t}\tilde{F}[t,f,g]\Big\|_X\le\tilde{c}(t')\|f\|_Y\|g\|_Y$$

for all $f,\ g\in Y$ and $t\in(t_0,t']$.

Then F is Fréchet differentiable with respect to (t,f) on $I\backslash\{t_0\}\times Y$ and the Fréchet derivative at $(t,f)\in I\backslash\{t_0\}\times Y$ is given by

$$DF(t,f)(s,g)=\frac{\partial}{\partial t}\tilde{F}[t,f,f]s+\tilde{F}[t,f,g]+\tilde{F}[t,g,f],\tag{4.1.13}$$

for all $(s,g)\in\mathbb{R}\times Y$. Moreover, $DF(t,f)$ is continuous with respect to (t,f).

Proof. Let $t\in I$, f, $\delta_f\in Y$ such that $\delta_f\neq 0$. Then

$$\begin{aligned}F(t,f+\delta_f)&=\tilde{F}[t,f+\delta_f,f+\delta_f]\\&=\tilde{F}[t,f,f]+\tilde{F}[t,f,\delta_f]+\tilde{F}[t,\delta_f,f]+\tilde{F}[t,\delta_f,\delta_f]\\&=F(t,f)+\tilde{F}[t,f,\delta_f]+\tilde{F}[t,\delta_f,f]+F(t,\delta_f).\end{aligned}$$

So equation (4.1.8) is satisfied where

$$D_Y F(t,f)\delta_f=\tilde{F}[t,f,\delta_f]+\tilde{F}[t,\delta_f,f]\tag{4.1.14}$$

and

$$R_Y(t, f, \delta_f) = F(t, \delta_f).$$

Since $\|\tilde{F}[t, f, g]\|_X \leq c(t)\|f\|_Y\|g\|_Y$, it follows that

$$\|\tilde{F}[t, f, \delta_f] + \tilde{F}[t, \delta_f, f]\|_X \leq 2c(t)\|f\|_Y\|\delta_f\|_Y.$$

Thus, for each fixed $(t, f) \in I \times Y$, we have that $D_Y F(t, f)$ is a bounded linear operator on Y.

Take $t' \in I$ and fix $f \in Y$. Then, for $\delta_f \in Y \setminus \{0\}$ and $t \in [t_0, t']$, we have

$$\frac{\|R_Y(t, f, \delta_f)\|_X}{\|\delta_f\|_Y} = \frac{\|F(t, \delta_f)\|_X}{\|\delta_f\|_Y} = \frac{\|\tilde{F}[t, \delta_f, \delta_f]\|_X}{\|\delta_f\|_Y} \leq c(t')\|\delta_f\|_Y \to 0$$

as $\|\delta_f\|_Y \to 0$, and the convergence is uniform in $t \in [t_0, t']$. Thus F is Fréchet differentiable with respect to the second argument and the Fréchet derivative at $f \in Y$ is given by (4.1.14) for all $\delta_f \in Y$. Hence assumption (a) of Proposition 4.1.8 holds.

Fix $(t, f) \in I \backslash \{t_0\} \times Y$. From assumption (a) we have, for $\delta_t \in \mathbb{R} \backslash \{0\}$ such that $t + \delta_t \in I \backslash \{t_0\}$,

$$\begin{aligned} F(t + \delta_t, f) = \tilde{F}[t + \delta_t, f, f] &= \tilde{F}[t, f, f] + \frac{\partial}{\partial t}\tilde{F}[t, f, f]\delta_t + R_t(t, f, f, \delta_t) \\ &= F(t, f) + \frac{\partial}{\partial t}\tilde{F}[t, f, f]\delta_t + R_t(t, f, f, \delta_t), \end{aligned}$$

where

$$\frac{\|R_t(t, f, f, \delta_t)\|_X}{|\delta_t|} \to 0 \qquad \text{as } |\delta_t| \to 0.$$

Thus F is strongly differentiable with respect to the first argument and so assumption (b) in Proposition 4.1.8 holds.

From part (ii) of Lemma 4.1.6 we have that $\tilde{F}$ is continuous. It follows from (4.1.14) that $D_Y F(t, f)$ is also continuous with respect to (t, f). Using assumption (b) and the continuity of the derivative from (a) we can prove in the same way as in the second half of the proof of Lemma 4.1.6 that $(t, f, g) \mapsto \frac{\partial}{\partial t}\tilde{F}[t, f, g]$ is continuous, and hence $(t, f) \mapsto \frac{\partial}{\partial t}F(t, f)$ is continuous. It follows from (4.1.14) and the continuity of $\tilde{F}$ that $D_Y F(t, f)$ is continuously differentiable with respect to t. Thus assumptions (c) and (d) in Proposition 4.1.8 hold and the statements of the corollary follow. □

4.2 Semi-Linear ACPs

The concepts and results that we introduce in this section are useful when we examine the full coagulation–fragmentation system, posed as a semi-linear ACP in a weighted ℓ^1 space. Of particular importance is Theorem 4.2.5, which provides conditions under which a semi-linear ACP has a unique "mild" solution; see Definition 4.2.2 below. The proof of Theorem 4.2.5 is based on those of [61, Theorem 6.1.4] and [43, Theorem 7.1.2], and uses Lemma 4.2.8(i), where a contraction mapping argument shows that a certain integral equation has a unique fixed point. Unlike in [61, Theorem 6.1.4] and [43, Theorem 7.1.2], however, we allow the non-linear operator in the semi-linear ACP to map from one arbitrary Banach space, Y, into another Banach space, X, with Y continuously embedded in X. From this theorem we are then able to obtain Propositions 4.2.12 and 4.2.15, which correspond, respectively, to the cases considered in [61, Theorem 6.1.4] and [43, Theorem 7.1.2].

Moreover, conditions that guarantee the positivity of the unique mild solution are given in Theorem 4.2.6, the proof of which uses Lemma 4.2.8(ii). Positivity of the unique mild solution was not included in [61, Theorem 6.1.4] and [43, Theorem 7.1.2], and we believe Theorem 4.2.6 and Lemma 4.2.8(ii) to be new results.

Let X, Y be Banach spaces such that Y is continuously embedded in X. We first state what we mean by a solution of an ACP of the form

$$u'(t) = Gu(t) + F(t, u(t)), \qquad t \in (t_0, T), \tag{4.2.1}$$

$$u(t_0) = \mathring{u}, \tag{4.2.2}$$

where $0 \le t_0 < T \le \infty$, G is the generator of a C_0-semigroup, $(S(t))_{t\ge 0}$, on X, $F : [t_0, T) \times Y \to X$ and $\mathring{u} \in Y$. We define a classical solution of (4.2.1), (4.2.2) in an analogous way as for linear ACPs.

Definition 4.2.1. We say that u is a *classical solution* of (4.2.1), (4.2.2) on $[t_0, T)$ if $u : [t_0, T) \to Y$ is continuous in Y, $u : (t_0, T) \to Y$ is continuously differentiable in X, $u(t) \in \mathcal{D}(G)$ for $t \in (t_0, T)$ and (4.2.1) and (4.2.2) are satisfied.

Note that in some books, see for example [21, Definition 2.38], a classical solution, as defined here, is referred to as a strong solution. We also require the concept of a mild solution.

Definition 4.2.2. A continuous solution $u : [t_0, T) \to Y$ of the integral equation

$$u(t) = S(t - t_0)\mathring{u} + \int_{t_0}^{t} S(t - s)F(s, u(s))\mathrm{d}s, \quad t \in [t_0, T), \tag{4.2.3}$$

is said to be a *mild solution* of (4.2.1), (4.2.2).

Note that the continuity in Definition (4.2.2) is with respect to the norm in Y and we take the equality (4.2.3) to hold in the space X.

The following argument from [61, p. 105] proves that any classical solution of (4.2.1), (4.2.2) is also a mild solution. Let u be a classical solution of (4.2.1), (4.2.2). Then, for $t_0 < s < t < T$, we have $u(s) \in \mathcal{D}(G)$ and the function $g(s) = S(t - s)u(s)$ is differentiable, with

$$\begin{aligned}
\frac{dg}{ds}(s) &= -GS(t - s)u(s) + S(t - s)u'(s) \\
&= -GS(t - s)u(s) + S(t - s)\big(Gu(s) + F(s, u(s))\big) \\
&= -GS(t - s)u(s) + GS(t - s)u(s) + S(t - s)F(s, u(s)) \\
&= S(t - s)F(s, u(s)).
\end{aligned}$$

Integrating between t_0 and t we obtain

$$u(t) - S(t - t_0)u(t_0) = \int_{t_0}^{t} S(t - s)F(s, u(s))\mathrm{d}s$$

and so

$$u(t) = S(t - t_0)\mathring{u} + \int_{t_0}^{t} S(t - s)F(s, u(s))\,\mathrm{d}s.$$

Definition 4.2.3. A mild solution (resp. classical solution), u, of (4.2.1), (4.2.2) on $[t_0, t_{max})$ is called *maximal* if there does not exist a $\tilde{t} > t_{max}$ and an extension, $\tilde{u}$, of u such that $\tilde{u}$ is a mild solution (resp. classical solution) on $[t_0, \tilde{t})$.

The following lemma will be useful in the proofs of later results.

Lemma 4.2.4. *Let X and Y be Banach spaces, with Y continuously embedded in X, and let $0 < T \leq \infty$. Let $(S(t))_{t\geq 0}$ be a C_0-semigroup on X and let $F : [0,T) \times Y \to X$. Let $0 \leq t_0 < T$ and $\mathring{u} \in Y$. Let $u : [t_0, T) \to Y$ satisfy*

$$u(t) = S(t-t_0)\mathring{u} + \int_{t_0}^{t} S(t-s)F(s,u(s))\,\mathrm{d}s, \tag{4.2.4}$$

for $t \in [\tau_0, \tau]$, where $t_0 \leq \tau_0 \leq \tau < T$. Then $u(t)$ also satisfies

$$u(t) = S(t-\tau_0)u(\tau_0) + \int_{\tau_0}^{t} S(t-s)F(s,u(s))\,\mathrm{d}s, \tag{4.2.5}$$

for $t \in [\tau_0, \tau]$.

Proof. Let $t_0 \leq \tau_0 \leq \tau < T$ and suppose that $u(t)$ satisfies (4.2.4) for $t \in [\tau_0, \tau]$. Further, let $t \in [\tau_0, \tau]$ and set $\hat{t} := t - \tau_0 \geq 0$. We have

$$\begin{aligned}
u(t) &= u(\tau_0 + \hat{t}) \\
&= S(\tau_0 + \hat{t} - t_0)\mathring{u} + \int_{t_0}^{\tau_0+\hat{t}} S(\tau_0 + \hat{t} - s)F(s,u(s))\,\mathrm{d}s \\
&= S(\tau_0 + \hat{t} - t_0)\mathring{u} + \int_{t_0}^{\tau_0} S(\tau_0 + \hat{t} - s)F(s,u(s))\,\mathrm{d}s \\
&\qquad + \int_{\tau_0}^{\tau_0+\hat{t}} S(\tau_0 + \hat{t} - s)F(s,u(s))\,\mathrm{d}s \\
&= S(\hat{t})\left(S(\tau_0 - t_0)\mathring{u} + \int_{t_0}^{\tau_0} S(\tau_0 - s)F(s,u(s))\,\mathrm{d}s\right)
\end{aligned}$$

$$
\begin{aligned}
&\quad + \int_{\tau_0}^{\tau_0+\hat{t}} S(\tau_0+\hat{t}-s)F(s,u(s))\,\mathrm{d}s \\
&= S(\hat{t})u(\tau_0) + \int_{\tau_0}^{\tau_0+\hat{t}} S(\tau_0+\hat{t}-s)F(s,u(s))\,\mathrm{d}s \\
&= S(t-\tau_0)u(\tau_0) + \int_{\tau_0}^{t} S(t-s)F(s,u(s))\,\mathrm{d}s.
\end{aligned}
$$

This proves the result. □

We now provide some existence and uniqueness results that we apply to the coagulation–fragmentation system later.

Theorem 4.2.5. *Let $(X,\|\cdot\|_X)$, $(Y,\|\cdot\|_Y)$ be Banach spaces such that Y is continuously embedded in X and let $0<T\le\infty$. Assume that*

(a) *$F:[0,T)\times Y\to X$ is such that $t\mapsto F(t,v)$ is continuous for $v\in Y$ and F is Lipschitz in the second argument on bounded sets, uniformly in the first argument on compact intervals;*

(b) *G is the generator of a C_0-semigroup, $(S(t))_{t\ge 0}$, on X such that $S(t)$ leaves Y invariant, for $t\in[0,T)$, and $t\mapsto S(t)y$ is continuous from $[0,T)$ into Y for every $y\in Y$;*

(c) *for $s_0,s_1\in[0,T)$, such that $s_0\le s_1$,*

$$\int_{s_0}^{s_1} S(s_1-s)\varphi(s)\,\mathrm{d}s \in Y \qquad \textit{for every } \varphi\in C([s_0,s_1],X);$$

(d) *there exists an increasing function $\eta:[0,\infty)\to[0,\infty)$ such that $\eta(\delta)\to 0$ as $\delta\to 0^+$ and*

$$\left\|\int_{s_0}^{s_1} S(s_1-s)\varphi(s)\,\mathrm{d}s\right\|_Y \le \eta(s_1-s_0)\|\varphi\|_{C([s_0,s_1],X)}$$

for every $\varphi\in C([s_0,s_1],X)$.

Then the following statements hold.

(i) *For every* $\mathring{u} \in Y$*, there exists a* t_{max} *satisfying* $0 < t_{max} \leq T$*, such that* (4.2.1)*,* (4.2.2) *(with* $t_0 = 0$*) has a unique maximal mild solution,* $u \in C\Big([0, t_{max}), Y\Big)$*, i.e. there exists a unique solution,* u*, of* (4.2.3) *(with* $t_0 = 0$*) on* $[0, t_{max})$*.*

(ii) *If* $t_{max} < T$*, then* $\|u(t)\|_Y \to \infty$ *as* $t \to t_{max}^-$*.*

(iii) *Let* $(\mathring{u}^{(m)})_{m=1}^{\infty}$ *be such that* $\mathring{u}^{(m)} \in Y$ *for each* $m \in \mathbb{N}$ *and* $\mathring{u}^{(m)} \to \mathring{u}$ *in* Y *as* $m \to \infty$*. Let* $u^{(m)}$ *and* u *be the unique, maximal mild solutions of* (4.2.1)*,* (4.2.2) *(with* $t_0 = 0$*), corresponding to the initial conditions* $\mathring{u}^{(m)}$ *and* $\mathring{u}$ *respectively. Then there exists* $0 < t'_{max} \leq T$ *such that, for all* $m \in \mathbb{N}$*,* $u^{(m)}$ *and* u *exist in* Y *for* $t \in [0, t'_{max})$*. Moreover,* $u^{(m)} \to u$*, in* Y*, as* $m \to \infty$ *uniformly on* $[0, T_0]$*, for some* $0 < T_0 < t'_{max}$*.*

(iv) *Let the assumptions in* (iii) *hold and for every* $\tau \in [0, t'_{max})$ *let there exists a* $C_\tau > 0$ *such that, for all* $m \in \mathbb{N}$*,* $\|u(t)\|_Y$*,* $\|u^{(m)}(t)\|_Y \leq C_\tau$ *for all* $t \in [0, \tau]$*. Then* $u^{(m)} \to u$*, in* Y*, as* $m \to \infty$ *uniformly in* t *on compact subintervals of* $[0, t'_{max})$*.*

We also provide the following result regarding the positivity of the mild solution in Theorem 4.2.5(i).

Theorem 4.2.6. *Let assumptions (a)–(d) of Theorem 4.2.5 hold and let* u *be the maximal mild solution on* $[0, t_{max})$*, for some* $0 < t_{max} \leq T$*. In addition, let* X *be an ordered Banach space,* $\mathcal{D}(G) \subseteq Y$ *be dense in* Y*, and* $\mathring{u} \in Y_+$*. Assume that*

(a) *there exists a bounded, linear operator* $H : Y \to X$ *such that, for all* $\gamma \geq 0$*,* $G - \gamma H$ *generates the semigroup* $(S_\gamma(t))_{t \geq 0}$ *on* X*, satisfying assumptions* (b)–(d) *of Theorem 4.2.5;*

(b) *for each* $\gamma \geq 0$ *the semigroup* $(S_\gamma(t))_{t \geq 0}$ *is positive and satisfies*

$$\|S_\gamma(t)|_Y\|_{\mathcal{B}(Y)} \leq \|S(t)\Big|_Y\|_{\mathcal{B}(Y)} \qquad \text{for all } t \in [0, T).$$

Moreover, let $\tau \in (0, t_{max})$ *and assume that*

(c) *for every* $r > 0$*, there exists* $\gamma \geq 0$ *such that*

$$F(t, v) + \gamma H v \geq 0 \qquad \text{for all } v \in \overline{B}_Y(0, r)_+,\ t \in [0, \tau]. \tag{4.2.6}$$

Then $u(t) \geq 0$ *for all* $t \in [0, \tau]$.

The following lemmas will be useful in the proofs of Theorems 4.2.5 and 4.2.6.

Lemma 4.2.7. *Let assumptions (b)–(d) of Theorem 4.2.5 hold. Then*

(i) $S(t)\big|_Y$ *is a bounded operator on* Y *for every* $t \in [0, T)$*, and*

$$\left\{ \left\| S(t)\big|_Y \right\|_{\mathcal{B}(Y)} : t \in [0, t_1] \right\} \tag{4.2.7}$$

is bounded for every $t_1 \in [0, T)$.

(ii) *For* $0 \leq s_0 \leq s_1 < T$*, the function*

$$t \mapsto \int_{s_0}^{t} S(t - s)\varphi(s)\,\mathrm{d}s$$

is continuous from $[s_0, s_1]$ *to* Y *for every* $\varphi \in C([s_0, s_1], X)$.

Proof. Let $t \in [0, T)$. Since $S(t)$ is closed in X and Y is continuously embedded in X, $S(t)\big|_Y$ is closed in Y. It follows from the Closed Graph Theorem that $S(t)\big|_Y$ is bounded. Also, by assumption, for $y \in Y$ and every $t_1 \in [0, T)$, $t \mapsto S(t)y$ is continuous from $[0, t_1]$ into Y. It follows that $\{\|S(t)y\| : t \in [0, t_1]\}$ is bounded and, by the Uniform Boundedness Principle, the set in (4.2.7) is bounded. This proves part (i).

To prove part (ii), let $0 < s_0 \leq s_1 < T$ and let $\varphi \in C([s_0, s_1], X)$. Moreover, let $s_2, s_3 \in [s_0, s_1]$ such that $s_2 < s_3$ and set $h := s_3 - s_2$. Then

$$\begin{aligned}
&\left\| \int_{s_0}^{s_3} S(s_3 - s)\varphi(s)\,\mathrm{d}s - \int_{s_0}^{s_2} S(s_2 - s)\varphi(s)\,\mathrm{d}s \right\|_Y \\
&= \left\| \int_{s_0}^{s_3} S(s_3 - s)\varphi(s)\,\mathrm{d}s - \int_{s_0}^{s_3 - h} S(s_3 - h - s)\varphi(s)\,\mathrm{d}s \right\|_Y
\end{aligned}$$

$$
\begin{aligned}
&= \left\| \int_{s_0}^{s_3} S(s_3 - s)\varphi(s)\,\mathrm{d}s - \int_{s_0+h}^{s_3} S(s_3 - s)\varphi(s-h)\,\mathrm{d}s \right\|_Y \\
&\leq \left\| \int_{s_0}^{s_0+h} S(s_3 - s)\varphi(s)\,\mathrm{d}s \right\|_Y + \left\| \int_{s_0+h}^{s_3} S(s_3 - s)\big[\varphi(s) - \varphi(s-h)\big]\,\mathrm{d}s \right\|_Y \\
&= \left\| S(s_3 - s_0 - h) \int_{s_0}^{s_0+h} S(s_0 + h - s)\varphi(s)\,\mathrm{d}s \right\|_Y \\
&\qquad + \left\| \int_{s_0+h}^{s_3} S(s_3 - s)\big[\varphi(s) - \varphi(s-h)\big]\,\mathrm{d}s \right\|_Y \\
&\leq \big\| S(s_3 - s_0 - h) \big\|_{\mathcal{B}(Y)} \left\| \int_{s_0}^{s_0+h} S(s_0 + h - s)\varphi(s)\,\mathrm{d}s \right\|_Y \\
&\qquad + \eta(s_3 - s_0 - h) \big\| \varphi(\cdot) - \varphi(\cdot - h) \big\|_{C([s_0+h, s_1], X)} \\
&\leq \sup_{t \in [0, s_1]} \|S(t)|_Y\|_{\mathcal{B}(Y)} \eta(h) \|\varphi\|_{C([s_0,s_1],X)} + \eta(s_1 - s_0) \big\| \varphi(\cdot) - \varphi(\cdot - h) \big\|_{C([s_0+h, s_1], X)}
\end{aligned}
$$

The first term clearly converges to 0 as $h \to 0$. For the second term note that φ is uniformly continuous on $[s_0, s_1]$ as it is continuous on a compact interval. Hence $\|\varphi(\cdot) - \varphi(\cdot - h)\big\|_{C([s_0+h,s_1],X)} \to 0$ as $h \to 0$. □

Before proving Theorem 4.2.5 and Theorem 4.2.6, we provide some notation and a useful lemma. Let the assumptions (a)–(d) of Theorem 4.2.5 hold. For all $t \in (0, T)$ let

$$
\widehat{M}(t) = \sup \left\{ \|S(s)\big|_Y\|_{\mathcal{B}(Y)} : s \in [0, t] \right\}
$$

and let $\widehat{M}_0 = \limsup_{t \to 0^+} \widehat{M}(t)$. From Lemma 4.2.7(i), $\widehat{M}(t)$ is well defined for each $t \in (0, T)$ and, since $S(0)\big|_Y = I\big|_Y$, we have $\widehat{M}(t) \geq 1$ for all $t \in (0, T)$ and $\widehat{M}_0 \geq 1$. Moreover, for $t \in (0, T)$, let

$$
N(t) \coloneqq \max\Big\{ \|F(s, 0)\|_X : s \in [0, t] \Big\}.
$$

Then N and $\widehat{M}$ are both monotone increasing functions. Let $C > 0$ be given and let $\tau \in (0, T)$. We now fix $r > 2C\widehat{M}_0$, and define

$$\delta(C, r, \tau) \coloneqq \frac{1}{2} \sup\left\{ d \in (0, \tau) : l(\tau, r)\eta(d) \leq \frac{1}{2}\,,\ C\widehat{M}(d) + N(\tau)\eta(d) \leq \frac{r}{2} \right\}. \tag{4.2.8}$$

Note that l is the Lipschitz constant associated with F on $[0, \tau] \times \overline{B}_Y(r)$, which is chosen such that it is increasing in the first argument. The set on the right-hand side is non-empty because

$$\lim_{d \to 0^+} l(\tau, r)\eta(d) = 0 \qquad \text{and} \qquad \limsup_{d \to 0^+} \Big(C\widehat{M}(d) + N(\tau)\eta(d) \Big) = C\widehat{M}_0 < \frac{r}{2}\,,$$

and hence $\delta = \delta(C, r, \tau) > 0$. Moreover, the factor of $\frac{1}{2}$ before the supremum and the monotonicity of η and $\widehat{M}$ imply that δ is contained in the set on the right hand side of (4.2.8). Hence, we have

$$l(\tau, r)\eta(\delta) \leq \frac{1}{2} \qquad \text{and} \qquad C\widehat{M}(\delta) + N(\tau)\eta(\delta) \leq \frac{r}{2}. \tag{4.2.9}$$

Let $t_0, t_1 \in (0, T)$ be such that $t_0 < t_1$. We now consider the Banach space $Z \coloneqq C\big([t_0, t_1], Y\big)$ equipped with the norm $\|v\|_Z = \max\limits_{t \in [t_0, t_1]} \|v(t)\|_Y$, for $v \in Z$. Let

$$\Sigma \coloneqq \overline{B}_Z(0, r) = \{ v \in Z : \|v\|_Z \leq r \} \tag{4.2.10}$$

and

$$(Qv)(t) \coloneqq S(t - t_0)\mathring{u} + \int_{t_0}^{t} S(t - s)F(s, v(s))\, \mathrm{d}s, \qquad v \in \Sigma,\ t_0 \leq t \leq t_1. \tag{4.2.11}$$

We adopt this notation in the following lemma. The proof of part (i) of this lemma is based on the proof of [61, Theorem 6.1.4]. However, [61, Theorem 6.1.4] deals with the particular case where $Y = X$ and so the result presented here is more general. Moreover, we provide a positivity result that is absent in [61, Theorem 6.1.4].

Lemma 4.2.8. *Let assumptions (a)–(d) of Theorem 4.2.5 hold and let $\mathring{u} \in Y$.*

Choose $C > 0$ such that $\|\mathring{u}\|_Y \leq C$ and choose $r > 2C\widehat{M}_0$. Let Σ and Q be defined by (4.2.10) *and* (4.2.11) *respectively. Let t_0, $\tau \in [0, T)$ be such that $t_0 < \tau$ and choose*

$$t_1 = \min\Big\{\tau, t_0 + \delta(C, r, \tau)\Big\}, \tag{4.2.12}$$

where $\delta(C, r, \tau)$ is as in (4.2.8).

(i) *Then the operator Q is a contraction, with contraction constant $\frac{1}{2}$, that maps Σ into itself, and hence has a unique fixed point, u, in Σ.*

(ii) *Now suppose that, in addition, X is an ordered Banach space, $(S(t))_{t\geq 0}$ is a positive semigroup and $\mathring{u} \in Y_+$. Assume that*

$$F(t, v) \geq 0 \qquad \text{for all } v \in \overline{B}_Y(0, r)_+,\ t \in [t_0, t_1].$$

Let u be the unique fixed point of Q from (i). Then $u \in \Sigma_+$.

Proof. Since $C < r$, we have that $\mathring{u} \in \Sigma$, where $\mathring{u}$ is considered as a constant function in t. Also, we have $Qv \in C([t_0, t_1], Y) = Z$ since $t \mapsto S(t)\mathring{u}$ is continuous from $[t_0, t_1]$ into Y, by assumption (b) of Theorem 4.2.5, and, taking $\phi(s) = F(s, v(s))$ for $v \in \Sigma$ in Lemma 4.2.7(ii), we have that

$$t \mapsto \int_{t_0}^{t} S(t - s)F(s, v(s))\,\mathrm{d}s$$

is continuous from $[t_0, t_1]$ into Y. Let $v, w \in \Sigma$. Then, for $t \in [t_0, t_1]$,

$$(Qv)(t) - (Qw)(t) = \int_{t_0}^{t} S(t - s)\Big(F(s, v(s)) - F(s, w(s))\Big)\,\mathrm{d}s.$$

Now,

$$\begin{aligned}
\|F(\cdot, v(\cdot)) - F(\cdot, w(\cdot))\|_{C([t_0,t_1],X)} &= \max_{s\in[t_0,t_1]} \|F(s, v(s)) - F(s, w(s))\|_X \\
&\leq l\,(\tau, r) \max_{s\in[t_0,t_1]} \|v(s) - w(s)\|_Y \\
&= l\,(\tau, r)\,\|v - w\|_Z.
\end{aligned}$$

Hence

$$\begin{aligned}\|Qv - Qw\|_Z &= \left\| \int_{t_0}^{\cdot} S(\cdot - s)\big(F(s, v(s)) - F(s, w(s))\big)\,\mathrm{d}s \right\|_Z \\ &\le \eta(t_1 - t_0) l\,(\tau, r)\,\|v - w\|_Z \\ &\le \eta(\delta) l\,(\tau, r)\,\|v - w\|_Z \\ &\le \frac{1}{2}\|v - w\|_Z,\end{aligned}$$

where (4.2.9) is used to obtain the last inequality. We therefore have

$$\|Qv - Qw\|_Z \le \frac{1}{2}\|v - w\|_Z. \tag{4.2.13}$$

Using (4.2.13), we can show that Q maps Σ into itself since, for $v \in \Sigma$,

$$\begin{aligned}\|Qv\|_Z &\le \|Qv - Q0\|_Z + \|Q0\|_Z \\ &\le \frac{1}{2}\|v - 0\|_Z + \|S(\cdot - t_0)\mathring{u}\|_Z + \left\| \int_{t_0}^{\cdot} S(\cdot - s) F(s, 0)\,\mathrm{d}s \right\|_Z \\ &\le \frac{r}{2} + \widehat{M}(t_1 - t_0)\|\mathring{u}\|_Y + \eta(t_1 - t_0) N(t_1) \\ &\le \frac{r}{2} + C\widehat{M}(\delta) + \eta(\delta) N(\tau) \le r,\end{aligned}$$

where the last inequality is obtained using (4.2.9). Thus, from (4.2.13), Q is a contraction with contraction constant $\frac{1}{2}$. It follows from the Contraction Mapping Theorem that Q has a unique fixed point, u, in Σ. Note that this means that $u(t) \in \overline{B}_Y(0, r)$ for $t \in [t_0, t_1]$. This proves part (i).

We now turn our attention to the proof of part (ii). We have shown that Q is a contraction that maps Σ into Σ. Moreover, $F(t, f) \ge 0$ for $t \in [t_0, t_1]$, $f \in \Sigma_+$. Since $\mathring{u} \in Y_+$ and $(S(t))_{t \ge 0}$ is a positive semigroup, it follows that Q maps Σ_+ into Σ_+. Also, $\mathring{u} \in \Sigma_+$ and, by the Contraction Mapping Theorem, we have that $u = \lim_{n \to \infty} Q^n \mathring{u}$. Since Q maps Σ_+ into Σ_+, we can deduce that $Q^n \mathring{u} \in \Sigma_+$ for all $n \in \mathbb{N}$ and so, since the positive cone is closed, $u \in \Sigma_+$. $\square$

We can now use Lemma 4.2.8 to prove Theorem 4.2.5. We adopt the notation introduced in the discussion preceding Lemma 4.2.8.

Proof of Theorem 4.2.5. In Lemma 4.2.8(i) we have shown the existence of a unique $u \in \Sigma$ satisfying

$$u(t) = S(t-t_0)\mathring{u} + \int_{t_0}^{t} S(t-s)F(s,v(s))\,\mathrm{d}s. \tag{4.2.14}$$

We now prove that this is the only solution of (4.2.14) in Y.

Suppose that u_1 and u_2 are two solutions of (4.2.14) on $[t_0,t_1]$. Since $\mathring{u} \in \Sigma$, with $\|\mathring{u}\|_Y < r$, and there is a unique, continuous solution in Σ, we have that u_1 and u_2 must coincide on some interval $[t_0,t']$. Define

$$\tau_0 = \sup\left\{t \in [t_0,t_1] : u_1(s) = u_2(s) \text{ for all } s \in [t_0,t]\right\}.$$

We set $y := u_1(\tau_0) = u_2(\tau_0)$. Note that $u_1(\tau_0) = u_2(\tau_0)$ since u_1 and u_2 are continuous.

Let $\tau_0 < t_1$. Then u_1, u_2 are distinct solutions of (4.2.14) on $(\tau_0,t_1]$. Moreover, by Lemma 4.2.4, u_1 and u_2 both satisfy

$$u(t) = S(t-\tau_0)y + \int_{\tau_0}^{t} S(t-s)F(s,u(s))\,\mathrm{d}s \qquad \text{on } [\tau_0,t_1]. \tag{4.2.15}$$

Using Lemma 4.2.8, with t_0 replaced by τ_0 and $\mathring{u}$ replaced by y, we can show that there exists a unique, solution of (4.2.15) on $[\tau_0,\tau_0+\varepsilon]$ for some $\varepsilon > 0$. This is a contradiction and so $\tau_0 = t_1$, i.e. the solution is unique on $[t_0,t_1]$. Hence, for each $t_0 \in [0,T)$, there exists a $t_1 \in (t_0,T)$ such that (4.2.1), (4.2.2) has a unique, maximal mild solution on $[t_0,t_1]$. Taking $t_0 = 0$ we obtain the existence of a unique mild solution on $[0,t_1]$ and so there must exist a unique maximal mild solution. Hence (i) holds.

We now prove (ii). Assume that $t_{max} < T$ and that $\|u(t)\|_Y \nrightarrow \infty$ as $t \to t_{max}^-$. Then there exist $s_n \in [0,t_{max})$ such that $(s_n)_{n=1}^{\infty}$ is strictly increasing, $s_n \to t_{max}$ as $n \to \infty$ and $\|u(s_n)\|_Y \le C$ for some $C > 0$. Choose $\tau \in (t_{max},T)$, $r > 2C\widehat{M}_0$ and set $\delta := \delta(C,r,\tau)$. For some $n \in \mathbb{N}$, we have $s_n > t_{max} - \delta$. Choose t_1 as in

(4.2.12) with $t_0 = s_n$. Then $t_1 > t_{max}$. From Lemma 4.2.8(i), it follows that the solution can be extended beyond t_{max}. This contradicts the definition of t_{max}.

We now prove (iii). Since $\mathring{u}^{(m)} \to \mathring{u}$ as $m \to \infty$, we can choose $C > 0$ large enough such that $\|\mathring{u}^{(m)}\|_Y$, $\|\mathring{u}\|_Y \le C$ for all $m \in \mathbb{N}$. Let $\tau \in (0, T)$, $r > 2C\widehat{M}_0$ and let $\hat{t}_1 = \min\{\tau, \delta(C, r, \tau)\}$. From Lemma 4.2.8(i), u and $u^{(m)}$, $m \in \mathbb{N}$, all exist on $[0, \hat{t}_1]$ and so there exists a $t'_{max} > 0$ such that, for all $m \in \mathbb{N}$, u and $u^{(m)}$ exist for $t \in [0, t'_{max})$.

We now consider an arbitrary, fixed, $t_0 \in [0, t'_{max})$, satisfying, for all $m \in \mathbb{N}$, $\|u^{(m)}(t_0)\|_Y$, $\|u(t_0)\|_Y \le C$. From Lemma 4.2.4 we have that u satisfies

$$u(t) = S(t - t_0)u(t_0) + \int_{t_0}^{t} S(t - s)F(s, u(s))\, \mathrm{d}s, \tag{4.2.16}$$

for $t \in [t_0, t'_{max})$. Let $t_0 < \tau < T$ and let t_1 be as in (4.2.12). From Lemma 4.2.8(i) we can deduce that (4.2.16) has a unique solution, $u \in \Sigma$. Similarly, $u^{(m)}$ satisfies

$$u^{(m)}(t) = S(t - t_0)u^{(m)}(t_0) + \int_{t_0}^{t} S(t - s)F(s, u^{(m)}(s))\, \mathrm{d}s,$$

for $t \in [t_0, t'_{max})$ and $u^{(m)} \in \Sigma$ for each $m \in \mathbb{N}$. From Lemma 4.2.8(i), Q is a contraction on Σ, with contraction constant $\frac{1}{2}$. Suppose that $u^{(m)}(t_0) \to u(t_0)$ as $m \to \infty$. Then, for $t \in [t_0, t_1]$,

$$\begin{aligned}
\|u - u^{(m)}\|_Z \le{}& \|S(\cdot - t_0)(u(t_0) - u^{(m)}(t_0))\|_Z \\
&+ \left\| \int_{t_0}^{\cdot} S(\cdot - s)(F(s, u(s)) - F(s, u^{(m)}(s))\, \mathrm{d}s \right\|_Z \\
\le{}& \widehat{M}(\delta)\|u(t_0) - u^{(m)}(t_0)\|_Y + \|Qu - Qu^{(m)}\|_Z \\
\le{}& \widehat{M}(\delta)\|u(t_0) - u^{(m)}(t_0)\|_Y + \frac{1}{2}\|u - u^{(m)}\|_Z.
\end{aligned}$$

Thus,

$$\|u - u^{(m)}\|_Z \le 2\widehat{M}(\delta)\|u(t_0) - u^{(m)}(t_0)\|_Y.$$

It follows that $u^{(m)}(t) \to u(t)$ as $m \to \infty$ for $t \in [t_0, t_1]$. This holds, in particular, with $t_0 = 0$, $u(0) = \mathring{u}$ and $u^{(m)}(0) = \mathring{u}^{(m)}$. Hence $u^{(m)} \to u$ as $m \to \infty$ on $[0, \hat{t}_1]$, i.e. (iii) holds.

Finally, we prove (iv). Let $\tau \in (0, t'_{max})$ and choose $r > 2C_\tau \widehat{M}_0$. From part (iii), we know that $u^{(m)} \to u$ as $m \to \infty$ uniformly in t on $[0, \hat{t}_1]$, where $\hat{t}_1 = \min\{\tau, \delta(C_\tau, r, \tau)\}$. If $\hat{t}_1 < \tau$, i.e. $\hat{t}_1 = \delta(C_\tau, r, \tau)$, then, since $u^{(m)}(\hat{t}_1) \to u(\hat{t}_1)$ as $m \to \infty$ and $\|u^{(m)}(\hat{t}_1)\|_Y, \|u^{(m)}(\hat{t}_1)\|_Y \le C_\tau$ for all $m \in \mathbb{N}$, we can use the argument from the proof of part (iii) to extend the result to $[0, \min\{\tau, 2\delta(C_\tau, r, \tau)\}]$. Repeating this argument a finite number of times we can deduce that $u^{(m)} \to u$ as $m \to \infty$, uniformly in t on $[0, \tau]$. $\square$

To prove Theorem 4.2.6, we require the following two lemmas, the first of which is based on [30, Corollary III.1.7].

Lemma 4.2.9. *Let assumptions (a) and (b) of Theorem 4.2.6 hold. Then*

$$S(t)g = S_\gamma(t)g + \int_0^t S_\gamma(t-s)(\gamma H)S(s)g \, \mathrm{d}s \tag{4.2.17}$$

for every $t \ge 0$ and $g \in Y$.

Proof. Let $\gamma \ge 0$ and take $g \in \mathcal{D}(G)$, $t \in [0, T)$. Consider the functions

$$s \mapsto \xi(s) \coloneqq S_\gamma(t-s)S(s)g \in X, \quad s \in [0, t].$$

Since $\mathcal{D}(G) = \mathcal{D}(G - \gamma H)$ is invariant under $(S(t))_{t \ge 0}$ and $(S_\gamma(t))_{t \ge 0}$, we can use [30, Lemma B.16] to deduce that $\xi(\cdot)$ is differentiable and

$$\begin{aligned} \frac{d}{ds} S_\gamma(t-s)S(s)g &= -S_\gamma(t-s)(G - \gamma H)S(s)g + S_\gamma(t-s)GS(s)g \\ &= S_\gamma(t-s)(\gamma H)S(s)g. \end{aligned}$$

Hence

$$S(t)g - S_\gamma(t)g = \int_0^t S_\gamma(t-s)(\gamma H)S(s)g \, \mathrm{d}s.$$

If $g \in Y$, then there exists $(g^{(n)})_{n=1}^\infty$ such that $g^{(n)} \in \mathcal{D}(G)$ for all $n \in \mathbb{N}$ and

$g^{(n)} \to g$ as $n \to \infty$ in Y. Also

$$S(t)g^{(n)} - S_\gamma(t)g^{(n)} = \int_0^t S_\gamma(t-s)(\gamma H)S(s)g^{(n)}\,\mathrm{d}s. \tag{4.2.18}$$

Now, $S(\cdot)g^{(n)} \to S(\cdot)g$ as $n \to \infty$ in $C([0,t],Y)$ because, from Lemma 4.2.7(i), the set $\{\|S(t)|_Y\|_{\mathcal{B}(Y)} : s \in [0,t]\}$ is bounded. Moreover, $H \in \mathcal{B}(Y,X)$ implies that $HS(\cdot)g^{(n)} \to HS(\cdot)g$ in $C([0,t],X)$ as $n \to \infty$. Hence, from assumption (d) in Theorem 4.2.5 (applied to $(S_\gamma(t))_{t\geq 0}$) we have

$$\begin{aligned}
&\Big\| \int_0^t S_\gamma(t-s)(\gamma H)S(s)g^{(n)}\,\mathrm{d}s - \int_0^t S_\gamma(t-s)(\gamma H)S(s)g\,\mathrm{d}s \Big\|_Y \\
&= \Big\| \int_0^t S_\gamma(t-s)\Big((\gamma H)S(s)g^{(n)} - (\gamma H)S(s)g\Big)\,\mathrm{d}s \Big\|_Y \\
&\leq \gamma\eta(t)\|HS(\cdot)g^{(n)} - HS(\cdot)g\|_{C([0,t],X)} \to 0 \text{ as } n \to \infty.
\end{aligned}$$

Taking limits in (4.2.18), we therefore obtain (4.2.17). □

We now use Lemma 4.2.9 to prove the following result.

Lemma 4.2.10. *Let assumptions (a) and (b) of Theorem 4.2.6 hold. Moreover, let $f \in C([t_0,\tau],X)$, for some $0 \leq t_0 < \tau < T$, and suppose that for $\mathring{u} \in Y$, $u \in C([t_0,\tau],Y)$ satisfies*

$$u(t) = S(t-t_0)\mathring{u} + \int_{t_0}^t S(t-s)f(s)\,\mathrm{d}s, \qquad t \in [t_0,\tau]. \tag{4.2.19}$$

Then u also satisfies

$$u(t) = S_\gamma(t-t_0)\mathring{u} + \int_{t_0}^t S_\gamma(t-s)\Big(f(s) + \gamma H u(s)\Big)\,\mathrm{d}s, \qquad [t_0,\tau]. \tag{4.2.20}$$

Proof. We first consider the case where $f \in C([t_0,\tau],Y)$. From Lemma 4.2.9 we have, for $t \in [t_0, \tau]$,

$$
\begin{aligned}
u(t) &= S(t-t_0)\mathring{u} + \int_{t_0}^{t} S(t-s)f(s)\,\mathrm{d}s \\
&= S_\gamma(t-t_0)\mathring{u} + \int_0^{t-t_0} S_\gamma(t-t_0-s)(\gamma H)S(s)\mathring{u}\,\mathrm{d}s \\
&\qquad + \int_{t_0}^{t} \Big(S_\gamma(t-s)f(s) + \int_0^{t-s} S_\gamma(t-s-r)(\gamma H)S(r)f(s)\,\mathrm{d}r\Big)\mathrm{d}s \\
&= S_\gamma(t-t_0)\mathring{u} + \int_{t_0}^{t} S_\gamma(t-s)f(s)\,\mathrm{d}s \\
&\quad + \int_0^{t-t_0} S_\gamma(t-t_0-s)(\gamma H)S(s)\mathring{u}\,\mathrm{d}s + \int_{t_0}^{t}\int_0^{t-s} S_\gamma(t-s-r)(\gamma H)S(r)f(s)\,\mathrm{d}r\mathrm{d}s.
\end{aligned}
$$

Now,

$$
\begin{aligned}
I &:= \int_0^{t-t_0} S_\gamma(t-t_0-s)(\gamma H)S(s)\mathring{u}\,\mathrm{d}s + \int_{t_0}^{t}\int_0^{t-s} S_\gamma(t-s-r)(\gamma H)S(r)f(s)\,\mathrm{d}r\mathrm{d}s \\
&= \int_{t_0}^{t} S_\gamma(t-v)(\gamma H)S(v-t_0)\mathring{u}\,\mathrm{d}v + \int_{t_0}^{t}\int_{s}^{t} S_\gamma(t-v)(\gamma H)S(v-s)f(s)\,\mathrm{d}v\mathrm{d}s \\
&= \int_{t_0}^{t} S_\gamma(t-v)(\gamma H)S(v-t_0)\mathring{u}\,\mathrm{d}v + \int_{t_0}^{t}\int_{t_0}^{v} S_\gamma(t-v)(\gamma H)S(v-s)f(s)\,\mathrm{d}s\mathrm{d}v \\
&= \int_{t_0}^{t} S_\gamma(t-v)(\gamma H)S(v-t_0)\mathring{u}\,\mathrm{d}v + \int_{t_0}^{t} S_\gamma(t-v)(\gamma H)\Big(\int_{t_0}^{v} S(v-s)f(s)\,\mathrm{d}s\Big)\mathrm{d}v \\
&= \int_{t_0}^{t} S_\gamma(t-v)(\gamma H)\Big(S(v-t_0)\mathring{u} + \int_{t_0}^{v} S(v-s)f(s)\,\mathrm{d}s\Big)\mathrm{d}v \\
&= \int_{t_0}^{t} S_\gamma(t-v)(\gamma H)u(v)\,\mathrm{d}v.
\end{aligned}
$$

It follows that

$$\begin{aligned} u(t) &= S_\gamma(t-t_0)\mathring{u} + \int_{t_0}^{t} S_\gamma(t-s)f(s)\,\mathrm{d}s + I \\ &= S_\gamma(t-t_0)\mathring{u} + \int_{t_0}^{t} S_\gamma(t-s)(f(s)+\gamma Hu(s))\,\mathrm{d}s. \end{aligned}$$

Hence (4.2.20) holds if $f \in C([t_0,\tau],Y)$.

We now show that $C([t_0,\tau],Y)$ is dense in $C([t_0,\tau],X)$. Take $f \in C([t_0,\tau],X)$. Since f is continuous on the compact interval $[t_0,\tau]$, it is uniformly continuous. Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that, for $t,s \in [t_0,\tau]$, $\|f(t)-f(s)\|_X \le \varepsilon$ whenever $|t-s| < \delta$. Choose $m \in \mathbb{N}$ such that $\frac{\tau-t_0}{m} < \delta$ and set $t_k = t_0 + \frac{k}{m}(\tau - t_0)$, $k = 0,\ldots,m$. Since $\mathcal{D}(G) \subseteq Y$ is dense in X, there exist $y_k \in Y$ such that $\|y_k - f(t_k)\| < \varepsilon$. We now construct $g \in C([t_0,\tau],Y)$ as

$$g(t) = \frac{m}{\tau - t_0}[(t-t_{k-1})y_k - (t-t_k)y_{k-1}] \qquad \text{when } t \in [t_{k-1},t_k].$$

Then for $t \in [t_{k-1},t_k]$,

$$\begin{aligned} \|y_k - g(t)\|_X &= \|g(t_k) - g(t)\|_X \\ &= \frac{m}{\tau-t_0}\|(t_k-t_{k-1})y_k - (t-t_{k-1})y_k + (t-t_k)y_{k-1}\|_X \\ &= \frac{m}{\tau-t_0}\|(t_k-t)y_k - (t_k-t)y_{k-1}\|_X \\ &= \frac{m}{\tau-t_0}\|(t_k-t)(y_k-y_{k-1})\|_X \\ &\le \frac{m}{\tau-t_0}(t_k-t_{k-1})\|y_k-y_{k-1}\|_X \\ &= \|y_k - y_{k-1}\|_X \\ &\le \|y_k - f(t_k)\|_X + \|f(t_k)-f(t_{k-1})\|_X + \|f(t_{k-1})-y_{k-1}\|_X \\ &< 3\varepsilon. \end{aligned}$$

Hence, for $t \in [t_{k-1},t_k]$,

$$\|f(t)-g(t)\|_X \le \|f(t)-f(t_k)\|_X + \|f(t_k)-y_k\|_X + \|y_k-g(t)\|_X < 5\varepsilon.$$

It follows that $C([t_0,\tau],Y)$ is dense in $C([t_0,\tau],X)$.

Let $f \in C([t_0,\tau)],X)$. Then there exists $(f_n)_{n=1}^{\infty}$ such that $f_n \in C([t_0,\tau],Y)$ for each $n \in \mathbb{N}$ and $f_n \to f$ as $n \to \infty$ in $C([t_0,\tau],X)$. For $t \in [t_0,\tau]$, define

$$u_n(t) := S(t-t_0)\mathring{u} + \int_{t_0}^{t} S(t-s)f_n(s)\,\mathrm{d}s. \tag{4.2.21}$$

Then, by the first part of the proof, for $t \in [t_0,\tau]$ we have

$$u_n(t) = S_\gamma(t-t_0)\mathring{u} + \int_{t_0}^{t} S_\gamma(t-s)(f_n(s) + \gamma H u_n(s))\,\mathrm{d}s. \tag{4.2.22}$$

Now, from assumption (d) in Theorem 4.2.5,

$$\int_{t_0}^{t} S(t-s)f_n(s)\,\mathrm{d}s \to \int_{t_0}^{t} S(t-s)f(s)\,\mathrm{d}s \qquad \text{as } n \to \infty$$

in Y for all $t \in [t_0,\tau]$, and so, taking limits in (4.2.21), we have from (4.2.19) that $u_n \to u$ in $C([t_0,\tau],Y)$.

Moreover, since $H \in \mathcal{B}(Y,X)$, we know that $f_n + \gamma H u_n \to f + \gamma H u$ as $n \to \infty$ in $C([t_0,\tau],X)$. Taking the limit in (4.2.22), and again using assumption (d) in Theorem 4.2.5 to obtain convergence of the integral on the right-hand side in $C([t_0,\tau],Y)$, we have

$$u(t) = S_\gamma(t-t_0)\mathring{u} + \int_{t_0}^{t} S_\gamma(t-s)(f(s) + \gamma H u(s))\,\mathrm{d}s.$$

Hence (4.2.20) holds. □

Theorem 4.2.6 can now be obtained from Lemma 4.2.8(ii).

Proof of Theorem 4.2.6. Let $\gamma \geq 0$. From Theorem 4.2.5(i), we have that $u \in C([0,t_{max}),Y)$ is the unique mild solution of (4.2.1) and (4.2.2). Hence, by

Lemma 4.2.10, u is also the unique mild solution of

$$u'(t) = (G - \gamma H)u(t) + F(t, u(t)) + \gamma H u(t), \qquad t \in (0, t_{max}), \tag{4.2.23}$$

with $u(0) = \mathring{u}$.

We know that $G - \gamma H$ satisfies assumptions (b)–(d) of Theorem 4.2.5 and that the semigroup $(S_\gamma(t))_{t\geq 0}$ is positive on X. Also, $\|H\|_{\mathcal{B}(Y,X)} \leq \hat{h}$ for some $\hat{h} \geq 0$. We now show that the operator $F_\gamma : [0, T) \times Y \to X$, given by

$$F_\gamma(t, v) = \gamma H v + F(t, v),$$

satisfies assumption (a) of Theorem 4.2.5.

Take $t' \in [0, T)$, $r' > 0$. For all $t \in [0, t']$, $u, v \in Y$ such that $\|u\|_Y, \|v\|_Y \leq r'$, we have

$$\begin{aligned}
\|F_\gamma(t, u) - F_\gamma(t, v)\|_X &\leq \|F(t, u) - F(t, v)\|_X + \gamma \|Hu - Hv\|_X \\
&\leq l(t', r')\|u - v\|_Y + \gamma \hat{h}\|u - v\|_Y \\
&= (l(t', r') + \gamma \hat{h})\|u - v\|_Y \\
&= l_\gamma(t', r')\|u - v\|_Y,
\end{aligned}$$

where $l_\gamma(t', r') = l(t', r') + \gamma \hat{h}$ is increasing in the first argument (since $l(t', r')$ is increasing in the first argument). It follows that F_γ is Lipschitz in the second argument on bounded sets, uniformly in the first argument on compact intervals. Moreover, since, for $v \in Y$, $t \mapsto F(t, v)$ is continuous in X, it is clear that $t \mapsto F_\gamma(t, v)$ is also continuous. Hence F_γ satisfies assumption (a) in Theorem 4.2.5.

Let $C > 0$ be such that $\|u(t)\|_Y \leq C$ for all $t \in [0, \tau]$ and choose $r > 2C\widehat{M}_0$. Choose $\gamma \geq 0$ such that (4.2.6) holds. Then $F_\gamma(t, f) \geq 0$ for $t \in [0, \tau]$ and $f \in \Sigma$. Taking $t_0 = 0$ in Lemma 4.2.8, the positivity of u on $[0, t_1]$ follows from Lemma 4.2.8(ii). If $\tau > t_1$, i.e. $t_1 = \delta(C, r, \tau)$, then since $\|u(t_1)\|_Y \leq C$ and $\delta(C, r, \tau)$ is a constant that is independent of $t_0 \in [0, \tau]$, we can apply Lemma 4.2.8(i) and (ii) again and obtain the existence of a unique, positive mild solution for $t \in [0, 2\delta(C, r, \tau)]$ (or on $[0, \tau]$ if $\tau < 2\delta(C, r, \tau)$). We can repeat this argument a finite number of times to obtain $u(t) \geq 0$ for all $t \in [0, \tau]$. $\square$

The following lemma gives conditions under which we can conclude that a non-negative solution is norm conserving. This is important when we pose the coagulation–fragmentation system as an ACP in the most physically relevant weighted ℓ^1 space, where the weight $w = (w_n)_{n=1}^{\infty}$ satisfies $w_n = n$ for all $n \in \mathbb{N}$. In this space the norm of a non–negative solution is equal to the total mass of clusters in the system and so a non-negative, norm-conserving solution is also a mass-conserving solution.

Lemma 4.2.11. *Let* $(X, \|\cdot\|_X)$ *and* $(Y, \|\cdot\|_Y)$ *be AL-spaces such that* Y *is continuously embedded in* X. *Moreover, let* ϕ_X *be the unique bounded linear extension of* $\|\cdot\|_X$ *from* X_+ *to* X, *as in Lemma 3.1.18. Let* $\mathring{u} \in Y_+$ *and let* $u \in C\Big([0,s), Y_+\Big) \subseteq C\Big([0,s), X_+\Big)$ *be a mild solution of* (4.2.1), (4.2.2) *(with* $t_0 = 0$*), for some* $0 < s \le T$. *In addition, let* $(S(t))_{t\ge 0}$ *be stochastic and assume that*

$$\phi_X(F(t,f)) = 0 \qquad \textit{for all } f \in Y_+, t \in [0,s).$$

Then

$$\|u(t)\|_X = \|\mathring{u}\|_X \qquad \textit{for all } t \in [0,s).$$

Proof. The function u satisfies

$$u(t) = S(t)\mathring{u} + \int_0^t S(t-s)F(s,u(s))\,\mathrm{d}s,$$

for $t \in [0,s)$. Let $t \in [0,s)$. Then $u(t) \in X_+$ and so

$$\begin{aligned}
\|u(t)\|_X = \phi_X\Big(u(t)\Big) &= \phi_X\left(S(t)\mathring{u} + \int_0^t S(t-s)F(s,u(s))\,\mathrm{d}s\right) \\
&= \phi_X\Big(S(t)\mathring{u}\Big) + \phi_X\left(\int_0^t S(t-s)F(s,u(s))\,\mathrm{d}s\right) \\
&= \|S(t)\mathring{u}\|_X + \int_0^t \phi_X\Big(S(t-s)F(s,u(s))\Big)\,\mathrm{d}s \\
&= \|\mathring{u}\|_X + \int_0^t \phi_X\Big(S(t-s)F(s,u(s))\Big)\,\mathrm{d}s.
\end{aligned}$$

From Proposition 3.3.16(i), we have, since $(S(t))_{t\geq 0}$ is stochastic, that $\phi_X(S(t)f) = \phi_X(f)$ for all $t \geq 0$, $f \in X$. Thus, for $t \in [0, s)$,

$$\|u(t)\|_X = \|\mathring{u}\|_X + \int\limits_0^t \phi_X\Big(F(s, u(s)\Big)\,\mathrm{d}s = \|\mathring{u}\|_X,$$

since $u(t) \in Y_+$ for all $t \in [0, s)$ and so $\phi(F(t, u(t)) = 0$ for all $t \in [0, s)$. It follows that $\|u(t)\|_X$ is independent of t for $t \in [0, s)$. $\square$

We devote the remainder of this section to examining two particular cases where the assumptions of Theorem 4.2.5 hold. We first examine the case where $Y = X$.

Proposition 4.2.12. *Let X be a Banach space with norm $\|\cdot\|$ and let $0 < T \leq \infty$. Let $F : [0, T) \times X \to X$ be such that $t \mapsto F(t, v)$ is continuous for $v \in X$, and Lipschitz in the second argument on bounded sets, uniformly in the first argument on compact intervals. Moreover, let G be the generator of a C_0-semigroup on X. Then assumptions (a)–(d) of Theorem 4.2.5 are satisfied with $Y = X$.*

Proof. Clearly (a) and (b) in Theorem 4.2.5 are satisfied. Also, there exists $M \geq 1$, $\omega \geq 0$ such that

$$\|S(t)v\| \leq Me^{\omega t}\|v\| \qquad \text{for all } v \in X,\ t \geq 0.$$

Let s_0, s_1 be such that $0 \leq s_0 < s_1 < T$. Then, for all $\varphi \in C([s_0, s_1], X)$,

$$\int\limits_{s_0}^{s_1} S(s_1 - s)\varphi(s)\,\mathrm{d}s \in X$$

and

$$\begin{aligned}\left\|\int\limits_{s_0}^{s_1} S(s_1 - s)\varphi(s)\,ds\right\| &\leq \int\limits_{s_0}^{s_1} \|S(s_1 - s)\varphi(s)\|\,\mathrm{d}s \leq M\int\limits_{s_0}^{s_1} e^{\omega(s_1 - s)}\|\varphi(s)\|\,\mathrm{d}s \\ &\leq Me^{\omega(s_1 - s_0)}(s_1 - s_0)\|\varphi\|_{C([s_0, s_1], X)},\end{aligned}$$

where $\eta(\delta) = Me^{\omega\delta}\delta : [0, \infty) \to [0, \infty)$ is an increasing function satisfying

$Me^{\omega\delta}\delta \to 0$ as $\delta \to 0^+$. It follows that assumptions (c) and (d) of Theorem 4.2.5 are satisfied. □

From Proposition 4.2.12 and Theorem 4.2.5, we have that if the assumptions of Lemma 4.1.6(ii) hold, with $Y = X$, then there exists a unique, mild solution of (4.2.1), (4.2.2) on $[0, t_{max})$, for some $t_{max} > 0$. The following theorem gives an additional assumption under which the mild solution is also a classical solution.

Theorem 4.2.13. *Let the assumptions of Proposition 4.2.12 hold. If the operator* $F : [0,T) \times X \to X$ *is continuously Fréchet differentiable from* $(0,T) \times X$ *into* X*, then the mild solution of* (4.2.1), (4.2.2)*, with* $u(0) = \mathring{u} \in \mathcal{D}(G)$*, is a classical solution of the initial value problem.*

Proof. See [61, Theorem 6.1.5]. □

We note that, since every classical solution is also a mild solution, and the mild solution mentioned before Theorem 4.2.13 is unique, it follows that this classical solution must also be unique.

We now consider the case where the space Y in Theorem 4.2.5 is an interpolation space. Let X be a Banach space. For each $\gamma \in (0,1)$, $I \subseteq \mathbb{R}$, let $C_b(I, X)$ be the set of bounded, continuous functions from I into X and define the spaces of Hölder continuous functions,

$$C^\gamma(I,X) = \left\{ f \in C_b(I,X) : [f]_{C^\gamma(I,X)} = \sup_{t,s\in I:\, s<t} \frac{\|f(t) - f(s)\|}{(t-s)^\gamma} < \infty \right\}.$$

We equip $C^\gamma(I,X)$ with the norm $\|\cdot\|_{C^\gamma(I,X)}$, given by

$$\|f\|_{C^\gamma(I,X)} = \|f\|_{C(I,X)} + [f]_{C^\gamma(I,X)}.$$

The following lemma, which is based on [43, Lemma 7.1.1], is useful to us.

Lemma 4.2.14. *Let* G *be the generator of an analytic semigroup,* $(S(t))_{t\geq 0}$*, on a Banach space* X*, such that* G *is invertible. Let* $\alpha \in (0,1)$*,* $1 \leq p \leq \infty$ *and let* $Y = D_G(\alpha, p)$*,* $\|\cdot\|_{D_G(\alpha,p)}$ *be as in Section 3.3.3. Moreover, let* s_0*,* $s_1 \in \mathbb{R}$ *be such* *that* $0 \leq s_0 < s_1$ *and let* $\varphi \in C([s_0, s_1], X)$. *Then*

$$\int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \in C([s_0, s_1], D_G(\alpha, p))$$

and

$$\left\| \int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \right\|_{C([s_0,s_1],D_G(\alpha,p))} \leq \eta(s_1 - s_0)\|\varphi\|_{C([s_0,s_1],X)},$$

where $\eta(\delta) = C\delta^{1-\alpha}$ *for some* $C > 0$.

Proof. Let $\gamma = 1 - \alpha$ and let $f \in C^\gamma([s_0, s_1], Y)$ with $f(s_0) = 0$. Then

$$\begin{aligned}
[f]_{C^\gamma([s_0,s_1],Y)} &= \sup_{t,\tau\in[s_0,s_1]:\tau<t} \frac{\|f(t) - f(\tau)\|_{D_G(\alpha,p)}}{(t-\tau)^\gamma} \geq \sup_{t\in(s_0,s_1]} \frac{\|f(t) - f(s_0)\|_{D_G(\alpha,p)}}{(t - s_0)^\gamma} \\
&= \sup_{t\in(s_0,s_1]} \frac{\|f(t)\|_{D_G(\alpha,p)}}{(t-s_0)^\gamma},
\end{aligned}$$

which implies that

$$\|f(t)\|_{D_G(\alpha,p)} \leq [f]_{C^\gamma([s_0,s_1],Y)}(t - s_0)^\gamma, \qquad t \in [s_0, s_1]. \tag{4.2.24}$$

We now set $\varphi(s) = 0$ for $s \in [0, s_0)$ so that $\varphi \in L^\infty([0, s_1], X)$. From [43, Lemma 7.1.1], we can deduce that for $\alpha \in (0, 1)$,

$$\int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s = \int_{0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \in C^\gamma([s_0, s_1], D_G(\alpha, p))$$

and

$$\begin{aligned}
\left\| \int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \right\|_{C^\gamma([s_0,s_1],D_G(\alpha,p))} &= \left\| \int_{0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \right\|_{C^\gamma([s_0,s_1],D_G(\alpha,p))} \\
&\leq C\|\varphi\|_{L^\infty([0,s_1],X)} \\
&= C\|\varphi\|_{C([s_0,s_1],X)}.
\end{aligned}$$

Moreover,

$$\int_{s_0}^{s_0} S(s_0 - s_0)\varphi(s)\,\mathrm{d}s = 0,$$

and so, using (4.2.24), we have

$$\begin{aligned}\left\| \int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \right\|_{C([s_0,s_1],D_G(\alpha,p))} &\leq \left[\int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \right]_{C^\gamma([s_0,s_1],Y)} (s_1 - s_0)^\gamma \\ &\leq C(s_1 - s_0)^\gamma \|\varphi\|_{C([s_0,s_1],X)}.\end{aligned}$$

□

Using this lemma, we obtain the following proposition.

Proposition 4.2.15. *Let $(X, \|\cdot\|)$ be a Banach space and let $(S(t))_{t\geq 0}$ be an analytic C_0-semigroup on X, generated by an invertible operator, G. Let $\alpha \in [0,1)$, $1 \leq p \leq \infty$ and $0 < T \leq \infty$. Moreover, let $F : [0,T) \times D_G(\alpha,p) \to X$ be such that $t \mapsto F(t,v)$ is continuous for $v \in D_G(\alpha,p)$, and Lipschitz in the second argument on bounded sets, uniformly in the first argument on compact intervals. Then assumptions (a)–(d) of Theorem 4.2.5 are satisfied, with $Y = D_G(\alpha,p)$.*

Proof. From Proposition 3.3.24, $D_G(\alpha,p)$ is continuously embedded in X. If $\alpha = 0$, then $D_G(\alpha,p) = X$ and the result follows from Proposition 4.2.12. Now suppose that $\alpha \in (0,1)$. As discussed in [43, p. 253], we have that [43, (7.0.2)] is satisfied if we take $X_\alpha = D_G(\alpha,p)$. In particular, the part of G in $D_G(\alpha,p)$ is sectorial in $D_G(\alpha,p)$ and so generates the C_0-semigroup, $(S(t)|_{D_G(\alpha,p)})_{t\geq 0}$, on $D_G(\alpha,p)$. It follows that $S(t)$ leaves $D_G(\alpha,p)$ invariant for $t \in [0,T)$. Moreover, $t \mapsto S(t)f$ is continuous from $[0,T)$ into $D_G(\alpha,p)$ for every $f \in D_G(\alpha,p)$. Finally, we have from Lemma 4.2.14 that

$$\int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \in C([s_0,s_1],D_G(\alpha,p))$$

for each $\varphi \in C([s_0, s_1], X)$ and

$$\left\| \int_{s_0}^{\cdot} S(\cdot - s)\varphi(s)\,\mathrm{d}s \right\|_{C([s_0,s_1],D_G(\alpha,p))} \leq \eta(s_1 - s_0)\|\varphi\|_{C([s_0,s_1],X)},$$

where $\eta : [0, \infty) \to [0, \infty)$ is a continuous, increasing function, satisfying $\eta(\delta) \to 0$ as $\delta \to 0^+$. The result then follows. $\square$

Chapter 5

Pure Autonomous Fragmentation

We begin our examination of the coagulation–fragmentation system by considering the case where each fragmentation event is irreversible, i.e. no coagulation occurs. This results in a pure fragmentation system, which we write as a linear ACP in a weighted ℓ^1 space with weight $w = (w_n)_{n=1}^{\infty}$. We then use the theory of operator semigroups to obtain results regarding solutions of the system. Previous work on this system, see for example [9, 46], have examined the system as an ACP in a weighted ℓ^1 space, where the weight is of the form $w_n = n^p$ for $p \geq 1$. However, we find it beneficial to examine the system as an ACP in more general weighted spaces. The main benefit of working in other weighted spaces is the ability to prove results regarding analytic semigroups that do not necessarily hold when the weight is of the form $w_n = n^p$.

5.1 Setting up the Problem

For the convenience of the reader, we begin by recalling, from Chapter 1, that we are considering a system consisting of clusters of particles where we assume that each cluster of size $n \in \mathbb{N}$, an n-mer, is made up of n identical units. We refer to these individual units as monomers and we scale the mass such that a monomer has unit mass. The mass of any cluster is then a positive integer, i.e. an n-mer has mass n for any $n \in \mathbb{N}$, and the size variable n also represents the mass of a cluster. In this chapter, we assume that no coagulation is occurring, and therefore clusters of sizes $n \geq 2$ can only fragment to form smaller clusters. We shall refer to this case as a pure fragmentation process, which is modelled by the pure fragmentation linear system of ordinary differential equations

$$u_n'(t) = -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t), \quad t > 0, \quad n = 1, 2, 3, \ldots; \tag{5.1.1}$$

$$u_n(0) = \mathring{u}_n, \qquad n = 1, 2, 3, \ldots. \tag{5.1.2}$$

The coefficients in (5.1.1) are defined in the same way as in (1.1.1). Consequently, the first term in (5.1.1) describes the loss of n-mers as they fragment and the second term describes the gain in n-mers due to larger clusters fragmenting. The solution of the system (5.1.1), (5.1.2) can be represented by a sequence, $u(t) = (u_n(t))_{n=1}^{\infty}$.

Throughout this thesis we make the following assumption on the fragmentation rates.

Assumption 5.1.1. For all $n, j \in \mathbb{N}$, assume that $a_n \geq 0$ and $b_{n,j} \geq 0$. Moreover, let $b_{n,j} = 0$ for all $j \leq n$.

If $u(t)$ is a solution of (5.1.1), (5.1.2), then the total mass, per unit volume, of the system at time t is given by

$$M_1(u(t)) = \sum_{n=1}^{\infty} n u_n(t). \tag{5.1.3}$$

As explained in Chapter 1 (see (1.1.3)), to compare the mass before and after a fragmentation event we set

$$\sum_{n=1}^{j-1} n b_{n,j} = (1 - \lambda_j) j, \qquad \lambda_j \in \mathbb{R},\ j = 2, 3, \ldots. \tag{5.1.4}$$

The left-hand side of (5.1.4) gives the total mass of daughter clusters that are produced when a cluster of mass j fragments. Clearly, for each $j = 2, 3, \ldots$, we must have $\lambda_j \leq 1$. Note that previous investigations into (5.1.1), (5.1.2) have concentrated on the case where $\lambda_j \in [0, 1]$ for each $j = 2, 3, \ldots$. For most of the results in this thesis, we do not require this assumption and so allow mass even to be gained during a fragmentation event. The case when $\lambda_j = 0$ for all $j = 2, 3, \ldots$, corresponds to the case where mass is conserved during each fragmentation event. If $\lambda_j > 0$, then mass is lost during the break up of a j-mer. Moreover, since it is assumed that cluster mass (or size) is a discrete variable, the smallest possible mass of a cluster is one. It follows that clusters of mass one cannot fragment into smaller clusters. If $a_1 > 0$, we therefore assume that monomers are removed from the system through some external mechanism and the rate of removal is given by a_1. This removal of monomers will again result in mass being lost from the system.

If (5.1.4) holds and u is a solution of the pure fragmentation system, then, as shown in Chapter 1, a formal calculation leads to

$$\frac{d}{dt} M_1(u(t)) = -a_1 u_1(t) - \sum_{j=2}^{\infty} \lambda_j j a_j u_j(t).$$

Note that $\frac{d}{dt} M_1(u(t)) \leq 0$ if $\lambda_j \geq 0$ for all $j \in \mathbb{N}$. This shows that in this case, at least formally, mass will be either conserved or lost in the system. Of particular importance is the case where

$$a_1 = 0 \qquad \text{and} \qquad \sum_{n=1}^{j-1} n b_{n,j} = j \qquad \text{for } j = 2, 3, \ldots. \tag{5.1.5}$$

If we assume that (5.1.5) holds, then $\frac{d}{dt} M_1(u(t)) = 0$. Hence, as we intuitively expect since, from (5.1.5), mass is conserved during each fragmentation event and no monomers are being removed from the system, the total mass in the system is conserved. Since we are dealing with real sequences $u = (u_n)_{n=1}^{\infty}$, we work in a real sequence space, although in Section 5.4 we shall also consider complex spaces. As in (3.1.3) and (3.1.8), for a non-negative sequence $(w_n)_{n=1}^{\infty}$ we define the weighted ℓ^1 space, ℓ^1_w, and its norm by

$$\ell^1_w = \left\{ f = (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} w_n |f_n| < \infty \right\},$$

$$\|f\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n |f_n| \qquad \text{for all } f \in \ell^1_w.$$

As in Example 3.1.19, for any $f, g \in \ell^1_w$, we have that $f \leq g$ if and only if $f_n \leq g_n$ for all $n \in \mathbb{N}$. Moreover, since the density of clusters of size $n \in \mathbb{N}$ will be non-negative, we are mostly interested in sequences $f \in \ell^1_w$ such that $f_n \geq 0$ for all $n \in \mathbb{N}$. Such elements form the positive cone, $(\ell^1_w)_+$, as in Definition 3.1.5. Moreover, from Example 3.1.19, ℓ^1_w is an AL-space and, for all $f \in \ell^1_w$, the unique bounded, linear functional, $\phi_{\ell^1_w}$, which extends the norm on $(\ell^1_w)_+$ to ℓ^1_w is given by $\phi_{\ell^1_w}(f) = \sum_{n=1}^{\infty} w_n f_n$.

We now pose the discrete fragmentation system as an ACP in ℓ^1_w. Motivated by the terms in (5.1.1), we introduce the formal expressions

$$\mathcal{A} : (f_n)_{n=1}^{\infty} \mapsto (-a_n f_n)_{n=1}^{\infty} \qquad \text{and} \qquad \mathcal{B} : (f_n)_{n=1}^{\infty} \mapsto \left(\sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \right)_{n=1}^{\infty}.$$

Operator realisations, $A^{(w)}$ and $B^{(w)}$, of $\mathcal{A}$ and $\mathcal{B}$ respectively, are defined in ℓ^1_w by

$$A^{(w)} f = \mathcal{A} f, \qquad \mathcal{D}(A^{(w)}) = \{ f \in \ell^1_w : \mathcal{A} f \in \ell^1_w \}; \tag{5.1.6}$$

$$B^{(w)} f = \mathcal{B} f, \qquad \mathcal{D}(B^{(w)}) = \left\{ f \in \ell^1_w : \mathcal{B} f \in \ell^1_w \right\}. \tag{5.1.7}$$

An abstract Cauchy problem (ACP) corresponding to (5.1.1) in ℓ^1_w then takes the form

$$u'(t) = A^{(w)} u(t) + B^{(w)} u(t), \quad t > 0; \qquad u(0) = \mathring{u}. \tag{5.1.8}$$

Here, $u(t) = (u_n(t))_{n=1}^{\infty}$, where $u_n(t)$ is the density of clusters of size n at time t, and $A^{(w)}$ and $B^{(w)}$ are as defined above. For u to be a classical solution of (5.1.8) we require $u(t) \in \mathcal{D}(A^{(w)}) \cap \mathcal{D}(B^{(w)})$ for all $t > 0$, u to be continuous on $[0, \infty)$ and strongly differentiable on $(0, \infty)$, i.e. $u \in C([0, \infty), \ell^1_w) \cap C^1((0, \infty), \ell^1_w)$, and we require u to satisfy (5.1.8). Since the solution that we seek, $u(t) = (u_n(t))_{n=1}^{\infty}$, is a sequence of densities, we also desire $u(t) \in (\ell^1_w)_+$.

The most physically relevant choice of weight is $w_n = n$, $n = 1, 2, \ldots$. In this case we denote $X_{[1]} \coloneqq \ell^1_w$ and $\|\cdot\|_{[1]} \coloneqq \|\cdot\|_{\ell^1_w}$, i.e.

$$X_{[1]} = \left\{ (f_n)_{n=1}^{\infty} : f_n \in \mathbb{R} \text{ for all } n \in \mathbb{N} \text{ and } \sum_{n=1}^{\infty} n|f_n| < \infty \right\} \tag{5.1.9}$$

and

$$\|f\|_{[1]} = \sum_{n=1}^{\infty} n|f_n| \qquad \text{for } f = (f_n)_{n=1}^{\infty} \in X_{[1]}. \tag{5.1.10}$$

To distinguish the physical space $X_{[1]}$ from other weighted spaces, we drop the w in the notation and set $A \coloneqq A^{(w)}$ and $B \coloneqq B^{(w)}$ when we work in $X_{[1]}$. We are then interested in solutions of the ACP

$$u'(t) = Au(t) + Bu(t), \quad t > 0; \qquad u(0) = \mathring{u}. \tag{5.1.11}$$

If $u(t)$ is a non-negative solution of (5.1.11), then $\|u(t)\|_{[1]}$ gives the total mass in the system (5.1.3).

When we assume that (5.1.5) holds, i.e. that mass is conserved during each fragmentation event, we want to obtain a non-negative, mass-conserving solution. For this to be the case we require a solution $u(t)$, such that $u(t) \in (X_{[1]})_+$ and $\|u(t)\|_{[1]} = \|\mathring{u}\|_{[1]}$ for all $t \geq 0$, i.e. we require a non-negative solution that is norm-conserving in $X_{[1]}$.

5.2 The Fragmentation ACP

In this section we aim to establish the existence and uniqueness of physically relevant classical solutions of (5.1.8).

5.2.1 The Fragmentation Semigroup

We begin by concentrating on the multiplication operator $A^{(w)}$ defined by (5.1.6). We use the same strategy here as used previously when $w_n = n^p$ for some $p \geq 1$; see for example [9, 15, 46, 63]. We first show that the operator $A^{(w)}$ is the generator of a substochastic C_0-semigroup and we then treat the operator $B^{(w)}$ as a perturbation of the operator $A^{(w)}$.

In the following lemma we use the Hille–Yosida Theorem to show that the operator $A^{(w)}$ is the generator of a substochastic C_0-semigroup.

Lemma 5.2.1. *The operator $A^{(w)}$ generates $(T^{(w)}(t))_{t\geq 0}$, a substochastic C_0-semigroup on ℓ^1_w. Moreover, this semigroup is given by the infinite diagonal matrix diag$(v_1(t), v_2(t), \ldots)$, where $v_n = e^{-a_n t}$ for all $n \in \mathbb{N}$, $t \geq 0$.*

Proof. Consider the multiplication operator $A^{(w)}$. This operator can be written as an infinite-dimensional diagonal matrix with $-a_n$ as its n^{th} diagonal element.

We show that $A^{(w)}$ satisfies the conditions of the Hille-Yosida theorem (Theorem 3.3.8). We can deduce that $A^{(w)}$ is closed as follows. Let $(f^{(k)})_{k=1}^{\infty}$ be such that $f^{(k)} \in \mathcal{D}(A^{(w)})$ for all $k \in \mathbb{N}$ and $f^{(k)} \to f$, $A^{(w)} f^{(k)} \to g$ in ℓ^1_w as $k \to \infty$. Then

$$\|g - A^{(w)} f^{(k)}\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n |g_n + a_n f_n^{(k)}| \to 0 \qquad \text{as } k \to \infty$$

and so, for each $n \in \mathbb{N}$,

$$-a_n f_n^{(k)} \to g_n \qquad \text{as } k \to \infty.$$

Since $f^{(k)} \to f$, we similarly have that $f_n^{(k)} \to f_n$ as $k \to \infty$ and so, for each $n \in \mathbb{N}$,

$$-a_n f_n^{(k)} \to -a_n f_n \qquad \text{as } k \to \infty.$$

It follows that $g_n = -a_n f_n$ for $n \in \mathbb{N}$. Hence

$$\|A^{(w)} f\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n |a_n f_n| = \sum_{n=1}^{\infty} w_n |g_n| < \infty,$$

and so $f \in \mathcal{D}(A^{(w)})$ and $g = A^{(w)} f$, establishing that $A^{(w)}$ is closed.

Clearly $A^{(w)}$ is also densely defined. For example, given any $f \in \ell^1_w$, if we define $(f^{(k)})_{k=1}^{\infty}$ by

$$(f^{(k)})_n = \begin{cases} f_n & \text{if } n \leq k \\ 0 & \text{if } n > k, \end{cases}$$

then $f^{(k)} \to f$ as $k \to \infty$ and $f^{(k)} \in \mathcal{D}(A^{(w)})$ for all $k \in \mathbb{N}$. Since $a_n \geq 0$ for all $n \in \mathbb{N}$, it is straightforward to show that the resolvent operator $R(\lambda, A^{(w)})$ can be written as an infinite diagonal matrix, with n^{th} diagonal element $(\lambda + a_n)^{-1}$ for each $\lambda > 0$. Also, for $\lambda > 0$,

$$\|\lambda R(\lambda, A^{(w)})f\|_{\ell^1_w} = \sum_{n=1}^{\infty} w_n \frac{\lambda}{\lambda + a_n}|f_n| \leq \sum_{n=1}^{\infty} w_n |f_n| = \|f\|_{\ell^1_w}.$$

Thus $\|\lambda R(\lambda, A^{(w)})\| \leq 1$ for all $\lambda > 0$.

It follows from Theorem 3.3.8 that $A^{(w)}$ generates a C_0-semigroup of contractions, $(T^{(w)}(t))_{t\geq 0}$. Since $T^{(w)}(t) = \lim_{k\to\infty} \left(I - \frac{t}{k}A^{(w)}\right)^{-k}$ and, for $k \in \mathbb{N}$, $t \geq 0$, $\left(I - \frac{t}{k}A^{(w)}\right)^{-1}$ is an infinite diagonal matrix, $T^{(w)}(t)$ is also an infinite diagonal matrix. In addition, for $g \in \ell^1_w$, $t > 0$, setting $\mu = \frac{k}{t}$, we have

$$\begin{aligned}\left(I - \frac{t}{k}A^{(w)}\right)^{-k} g = \left(I - \frac{1}{\mu}A^{(w)}\right)^{-k} g &= \left[\left(\frac{1}{\mu}\left(\mu I - A^{(w)}\right)\right)^{-1}\right]^{k} g \\ &= \left[\mu\left(\mu I - A^{(w)}\right)^{-1}\right]^{k} g.\end{aligned}$$

Hence, for $n \in \mathbb{N}$,

$$\left(\left(I - \frac{t}{k}A^{(w)}\right)^{-k} g\right)_n = \left(\frac{\mu}{\mu + a_n}\right)^{k} g_n = \frac{1}{(1 + \frac{a_n}{\mu})^k} g_n.$$

It follows that for each $t \geq 0$, $g \in \ell^1_w$ and $n \in \mathbb{N}$,

$$(T^{(w)}(t)g)_n = \lim_{k\to\infty} \frac{1}{(1 + \frac{a_n t}{k})^k} g_n = e^{-a_n t} g_n.$$

We now show that the semigroup generated by $A^{(w)}$ is substochastic. We already know that $\|T^{(w)}(t)\|_{\ell^1_w} \leq 1$ for all $t \geq 0$ and so we need only show that $T^{(w)}(t) \geq 0$. Take $f \in (\ell^1_w)_+$. Then $[T^{(w)}(t)f]_n = e^{-a_n t} f_n \geq 0$ for all $n \in \mathbb{N}$, $t \geq 0$, and so $T^{(w)}(t)f \geq 0$, i.e. $T^{(w)}(t) \geq 0$ for all $t \geq 0$. This shows that $A^{(w)}$ does indeed generate a substochastic C_0-semigroup, given for all $t \geq 0$, $n \in \mathbb{N}$ and $f \in \ell^1_w$ by $(T^{(w)}(t)f)_n = e^{-a_n t} f_n$. □

We now treat $B^{(w)}$ as a perturbation of the operator $A^{(w)}$. To make progress we need to make some assumptions on the weight $w = (w_n)_{n=1}^{\infty}$.

Assumption 5.2.2.

(i) Assume that there exists $\delta \in (0, 1]$ such that

$$\sum_{n=1}^{j-1} w_n b_{n,j} \leq \delta w_j \qquad \text{for all } j = 2, 3, \ldots. \tag{5.2.1}$$

(ii) Assume that $w_n \geq n$ for all $n \in \mathbb{N}$.

Remark 5.2.3. We require Assumption 5.2.2(i) to obtain the existence and uniqueness of solutions. Assumption 5.2.2(ii) is mostly required when we want to show that a solution conserves mass.

We note that if Assumption 5.2.2(ii) holds, then $\frac{n}{w_n} \leq 1$ for all $n \in \mathbb{N}$ and so, by Proposition 3.2.7, ℓ_w^1 is continuously embedded in $X_{[1]}$. We also note that if (5.2.1) holds for $w_n = n$, then (5.1.4) automatically holds with $\lambda_j \in [0, 1)$ for $j = 2, 3, \ldots$.

As we show in the following proposition, if (5.1.5) holds and $w_1 \geq 1$, then Assumption 5.2.2(i) immediately implies Assumption 5.2.2(ii).

Proposition 5.2.4. *Let* (5.1.5) *and Assumption 5.2.2(i) hold. Moreover, let* $w_1 \geq 1$. *Then* $w_n \geq n$ *for all* $n \in \mathbb{N}$, *i.e. Assumption 5.2.2(ii) holds. Moreover,* ℓ_w^1 *is continuously embedded in* $X_{[1]}$.

Proof. We know $w_1 \geq 1$. Suppose that $w_k \geq k$ for all $k < j$, where $j \in \mathbb{N}$ such that $j \geq 2$. Then

$$j = \sum_{n=1}^{j-1} n b_{n,j} \leq \sum_{n=1}^{j-1} w_n b_{n,j} \leq \delta w_j \leq w_j.$$

Hence, by induction, $w_n \geq n$ for all $n \in \mathbb{N}$. The continuous embedding follows from Proposition 3.2.7. $\square$

The following lemma allows us to deduce simple examples where Assumption 5.2.2 holds.

Lemma 5.2.5. *Let* (5.1.4) *hold with* $\lambda_j \in [0, 1]$ *for all* $j = 2, 3, \ldots$. *If* $(w_n)_{n=1}^{\infty}$ *is such that* $w_n \geq n$ *for all* $n \in \mathbb{N}$ *and* $\left(\frac{w_n}{n}\right)_{n=1}^{\infty}$ *is increasing, then* (5.2.1) *holds with* $\delta = 1$. *In particular, if* $w_n = n^p$ *for some* $p \geq 1$, *then* (5.2.1) *holds with* $\delta = 1$.

Proof. In this case, for $j \in \mathbb{N}$ such that $j \geq 2$, we have

$$\sum_{n=1}^{j-1} w_n b_{n,j} = \sum_{n=1}^{j-1} \frac{w_n}{n} n b_{n,j} \leq \frac{w_j}{j} \sum_{n=1}^{j-1} n b_{n,j} \leq \frac{w_j}{j} j = w_j.$$

Hence (5.2.1) holds with $\delta = 1$. □

It follows from Lemma 5.2.5 that, if (5.1.4) holds with $\lambda_j \in [0,1]$ for $j = 2, 3, \ldots$, then Assumption 5.2.2 is satisfied when we take $w_n = n$ for all $n \in \mathbb{N}$, i.e. when we work in the space $X_{[1]}$. We now drop Assumption 5.2.2(ii) and show that, under Assumption 5.2.2(i), $\mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(B^{(w)})$.

Lemma 5.2.6. *Let Assumption 5.2.2(i) hold. Then* $\mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(B^{(w)})$ *and*

$$\|B^{(w)} f\|_{\ell^1_w} \leq \delta \|A^{(w)} f\|_{\ell^1_w} \qquad \textit{for all } f \in \mathcal{D}(A^{(w)}). \tag{5.2.2}$$

Proof. Let $f \in \mathcal{D}(A^{(w)})$. Then we have that $\sum\limits_{n=1}^{\infty} w_n a_n |f_n| < \infty$. Also

$$\begin{aligned}
\|B^{(w)} f\|_{\ell^1_w} &= \sum_{n=1}^{\infty} w_n \left| \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \right| \leq \sum_{n=1}^{\infty} w_n \sum_{j=n+1}^{\infty} a_j b_{n,j} |f_j| \\
&= \sum_{j=2}^{\infty} \sum_{n=1}^{j-1} w_n a_j b_{n,j} |f_j| = \sum_{j=2}^{\infty} \left(\sum_{n=1}^{j-1} w_n b_{n,j} \right) a_j |f_j| \\
&\leq \delta \sum_{j=2}^{\infty} w_j a_j |f_j| \leq \delta \|A^{(w)} f\|_{\ell^1_w} < \infty \quad \text{since } f \in \mathcal{D}(A^{(w)}).
\end{aligned}$$

The change in the order of summation in the calculation above is justified since each term is positive. From this calculation, for all $f \in \mathcal{D}(A^{(w)})$, we can conclude that $f \in \mathcal{D}(B^{(w)})$ and (5.2.2) holds. □

The proof of the next proposition requires the existence of a monotone increasing sequence, $(c_n)_{n=1}^{\infty}$, that dominates $(a_n)_{n=1}^{\infty}$, i.e. we need a sequence $(c_n)_{n=1}^{\infty}$ such that

$$c_{n+1} \geq c_n \qquad \text{and} \qquad c_n \geq a_n \qquad \text{for all } n \in \mathbb{N}. \tag{5.2.3}$$

It is always possible to find such a sequence, e.g. we could take $c_1 = a_1$ and

$c_n = \max\{a_n, c_{n-1}\}$ for all $n \in \mathbb{N}$. This is equivalent to choosing

$$c_n = \max\{a_1, \ldots, a_n\}. \tag{5.2.4}$$

Then $(c_n)_{n=1}^{\infty}$ is a monotone increasing sequence that dominates $(a_n)_{n=1}^{\infty}$.

Now take $(c_n)_{n=1}^{\infty}$ to be any sequence satisfying (5.2.3) and, for all $n \in \mathbb{N}$, define the operator $C^{(w)}$ by

$$[C^{(w)}f]_n = -c_n f_n, \qquad \mathcal{D}(C^{(w)}) = \left\{ f \in \ell_w^1 : \sum_{n=1}^{\infty} w_n c_n |f_n| < \infty \right\}. \tag{5.2.5}$$

Note that, since $(c_n)_{n=1}^{\infty}$ dominates $(a_n)_{n=1}^{\infty}$ and $a_n \geq 0$ for all $n \in \mathbb{N}$, we have that $c_n \geq 0$ for all $n \in \mathbb{N}$.

We also take $\|\cdot\|_{C^{(w)}}$ to be the graph norm of $C^{(w)}$, i.e.

$$\|f\|_{C^{(w)}} = \|f\|_{\ell_w^1} + \|C^{(w)}f\|_{\ell_w^1} \tag{5.2.6}$$

for all $f \in \mathcal{D}(C^{(w)})$, and let $(X_{(1)}, \|\cdot\|_{(1)}) = (\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}})$. Note that $(\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}})$ is a weighted ℓ^1 space and the unique linear extension of $\|\cdot\|_{C^{(w)}}$ from $(\mathcal{D}(C^{(w)}))_+$ to $\mathcal{D}(C^{(w)})$, as in Example 3.1.19, is $\phi_{C^{(w)}}(f) = \sum_{n=1}^{\infty}(w_n + w_n c_n) f_n$ for all $f \in \mathcal{D}(C^{(w)})$.

We now apply Corollary 3.3.30 to the operators $A^{(w)}$ and $B^{(w)}$.

Theorem 5.2.7. *Let Assumption 5.2.2(i) hold. Then $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. Moreover, $\mathcal{D}(C^{(w)})$ is invariant under the semigroup $(S^{(w)}(t))_{t\geq 0}$.*

Proof. We show that the operators $A^{(w)}$ and $B^{(w)}$ satisfy the conditions of Corollary 3.3.30 with $(X_{(1)}, \|\cdot\|_{(1)}) = (\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}})$, where $C^{(w)}$ and $\|\cdot\|_{C^{(w)}}$ are defined by (5.2.5) and (5.2.6) respectively. The result then follows immediately.

From Lemma 5.2.6 we know that $\mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(B^{(w)})$ and

$$\|B^{(w)}f\|_{\ell_w^1} \leq \delta \|A^{(w)}f\|_{\ell_w^1} \leq \|A^{(w)}f\|_{\ell_w^1} \tag{5.2.7}$$

for all $f \in \mathcal{D}(A^{(w)})$.

Since $C^{(w)}$ is a multiplication operator we can argue as in Lemma 5.2.1 to deduce that $C^{(w)}$ generates a positive (in fact, substochastic) C_0-semigroup of contractions on ℓ_w^1. Thus $\mathcal{D}(C^{(w)})$ is dense in ℓ_w^1 and so assumption (i') in Corollary 3.3.30 holds.

From Example 3.1.19, we have that $(\ell_w^1, \|\cdot\|_{\ell_w^1})$ and $(\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}})$ are AL-spaces. Moreover, $\|f\|_{\ell_w^1} \leq \|f\|_{\ell_w^1} + \|C^{(w)}f\|_{\ell_w^1} = \|f\|_{C^{(w)}}$ for all $f \in \mathcal{D}(C^{(w)})$ and so $(\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}})$ is continuously embedded in $(\ell_w^1, \|\cdot\|_{\ell_w^1})$. Hence assumption (ii') in Corollary 3.3.30 holds. Consequently, $(\mathcal{D}(C^{(w)}), \|\cdot\|_{C^{(w)}})$ satisfies the conditions of $(X_{(1)}, \|\cdot\|_{(1)})$ in Corollary 3.3.30.

We now need to check that conditions (a)–(e) of Theorem 3.3.29, and (f')–(g') of Corollary 3.3.30 also hold. Firstly, it is clear that $-A^{(w)}$ is positive. From Lemma 5.2.1, we have that $A^{(w)}$ generates a substochastic C_0-semigroup, $(T^{(w)}(t))_{t\geq 0}$, on ℓ_w^1, given by $(T^{(w)}(t)f)_n = e^{-a_n t} f_n$ for all $t \geq 0$, $n \in \mathbb{N}$ and $f \in \ell_w^1$.

Let

$$\tilde{A}^{(w)} f = A^{(w)} f,$$

where

$$\mathcal{D}(\tilde{A}^{(w)}) = \{f \in \mathcal{D}(C^{(w)}) : A^{(w)} f \in \mathcal{D}(C^{(w)})\}$$

and

$$\tilde{B}^{(w)} f = B^{(w)} f,$$

where

$$\mathcal{D}(\tilde{B}^{(w)}) = \{f \in \mathcal{D}(C^{(w)}) : B^{(w)} f \in \mathcal{D}(C^{(w)})\}.$$

Since $\mathcal{D}(C^{(w)})$ is a weighted ℓ_w^1 space, we can apply Lemma 5.2.1 and obtain that $\tilde{A}^{(w)}$ generates a C_0-semigroup of contractions on $\mathcal{D}(C^{(w)})$. Moreover, this semigroup is the restriction of $(T^{(w)}(t))_{t\geq 0}$ to $\mathcal{D}(C^{(w)})$.

It is clear that $B^{(w)}$ is positive on $\mathcal{D}(B^{(w)})$, and hence on $\mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(B^{(w)})$. Let $f \in \mathcal{D}(A^{(w)})_+$. Then, from (5.2.7),

$$\begin{aligned} \phi_{\ell_w^1}((A^{(w)} + B^{(w)})f) &= \sum_{n=1}^{\infty} -w_n a_n f_n + \sum_{n=1}^{\infty} w_n \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \\ &= -\|A^{(w)} f\|_{\ell_w^1} + \|B^{(w)} f\|_{\ell_w^1} \leq 0. \end{aligned}$$

Since $(c_n)_{n=1}^{\infty}$ is monotone increasing, we deduce that

$$\sum_{n=1}^{j-1}(w_n + w_n c_n)b_{n,j} \leq (1+c_j)\sum_{n=1}^{j-1} w_n b_{n,j} \leq \delta(1+c_j)w_j = \delta(w_j + w_j c_j),$$

for $j \geq 2$. Hence, from Lemma 5.2.6, we can conclude that $\mathcal{D}(\tilde{A}^{(w)}) \subseteq \mathcal{D}(\tilde{B}^{(w)})$ and $\|B^{(w)}f\|_{C^{(w)}} = \|\tilde{B}^{(w)}f\|_{C^{(w)}} \leq \delta\|\tilde{A}^{(w)}f\|_{C^{(w)}} = \delta\|A^{(w)}f\|_{C^{(w)}} \leq \|A^{(w)}f\|_{C^{(w)}}$ for all $f \in \mathcal{D}(\tilde{A}^{(w)})$. Take $f \in \mathcal{D}(\tilde{A}^{(w)})_+ \subseteq \mathcal{D}(\tilde{B}^{(w)})_+$. Then $f \in \mathcal{D}(C^{(w)})$, $A^{(w)}f \in \mathcal{D}(C^{(w)})$ and $B^{(w)}f \in \mathcal{D}(C^{(w)})$. So we have $(A^{(w)} + B^{(w)})f \in \mathcal{D}(C^{(w)})$ for all $f \in \mathcal{D}(\tilde{A}^{(w)})_+$. Moreover, for $f \in \mathcal{D}(\tilde{A}^{(w)})_+$, since $-A^{(w)}f \geq 0$ and $B^{(w)}f \geq 0$, we have

$$\begin{aligned}
\phi_{C^{(w)}}((A^{(w)} + B^{(w)})f) &= \phi_{C^{(w)}}(A^{(w)}f) + \phi_{C^{(w)}}(B^{(w)}f) \\
&= -\|A^{(w)}f\|_{C^{(w)}} + \|B^{(w)}f\|_{C^{(w)}} \\
&\leq 0.
\end{aligned}$$

Finally,

$$\|A^{(w)}f\|_{\ell_w^1} = \sum_{n=1}^{\infty} w_n a_n f_n \leq \sum_{n=1}^{\infty} w_n c_n f_n \leq \sum_{n=1}^{\infty}(w_n + w_n c_n)f_n = \|f\|_{C^{(w)}}$$

for all $f \in D(\tilde{A}^{(w)})_+$. Hence all the conditions of Corollary 3.3.30 hold. □

5.2.2 Classical Solutions of the Fragmentation ACP

If Assumption 5.2.2(i) holds, then, from Theorem 5.2.7, $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of

$$u'(t) = G^{(w)}u(t) \tag{5.2.8}$$

$$u(0) = \mathring{u} \tag{5.2.9}$$

in ℓ_w^1, for all $\mathring{u} \in \mathcal{D}(G^{(w)})$ and $t \geq 0$. Moreover, if $\mathring{u} \in \mathcal{D}(G^{(w)})_+$, then, since $(S^{(w)}(t))_{t\geq 0}$ is substochastic, $u(t)$ is non-negative.

We now want to show that if (5.1.5) holds, then the solution of (5.2.8), (5.2.9) is mass conserving. We first consider the case where $\ell_w^1 = X_{[1]}$, i.e. when $w_n = n$

for all $n \in \mathbb{N}$. As previously mentioned, to distinguish the space $X_{[1]}$ we drop the w in the notation, i.e. we denote $A \coloneqq A^{(w)}$, $B \coloneqq B^{(w)}$, $G \coloneqq G^{(w)}$, $C \coloneqq C^{(w)}$, $(S(t))_{t\geq 0} \coloneqq (S^{(w)}(t))_{t\geq 0}$, etc, when we work in $X_{[1]}$. We also recall that, from (5.1.3), $\phi_{\ell^1_w} = M_1$ when we work in the space $X_{[1]}$.

Proposition 5.2.8. *Let* (5.1.5) *hold. Then* $G = \overline{A+B}$ *generates a stochastic* C_0*-semigroup,* $(S(t))_{t\geq 0}$*, on* $X_{[1]}$*. Moreover, for all* $\mathring{u} \in \mathcal{D}(G)_+$*,* $u(t) = S(t)\mathring{u}$ *is the unique, non-negative, mass-conserving classical solution of*

$$u'(t) = Gu(t), \qquad t > 0, \tag{5.2.10}$$

$$u(0) = \mathring{u}. \tag{5.2.11}$$

Proof. We have that Assumption 5.2.2(i) holds with $w_n = n$ for $n \in \mathbb{N}$ and $\delta = 1$. It follows from Theorem 5.2.7 that $G = \overline{A+B}$ generates a substochastic C_0-semigroup, $(S(t))_{t\geq 0}$, on $X_{[1]}$. To show that $(S(t))_{t\geq 0}$ is stochastic we show that $M_1((A+B)f) = 0$ for all $f \in \mathcal{D}(A)_+$ and apply the last statement in Corollary 3.3.30. For $f \in \mathcal{D}(A)_+$,

$$\begin{aligned} M_1(Af+Bf) &= \sum_{n=1}^{\infty} n(Af+Bf)_n = \sum_{n=1}^{\infty} -na_nf_n + \sum_{n=1}^{\infty} n \sum_{j=n+1}^{\infty} a_jb_{n,j}f_j \\ &= -\sum_{n=1}^{\infty} na_nf_n + \sum_{j=2}^{\infty}\sum_{n=1}^{j-1} na_jb_{n,j}f_j = -\sum_{n=1}^{\infty} na_nf_n + \sum_{j=1}^{\infty} ja_jf_j \\ &= 0. \end{aligned} \tag{5.2.12}$$

It follows from the last statement in Corollary 3.3.30 that $(S(t))_{t\geq 0}$ is stochastic.

We know that $u(t) = S(t)\mathring{u}$ is the unique, non-negative classical solution of (5.2.10), (5.2.11) for all $\mathring{u} \in \mathcal{D}(G)_+$. Moreover,

$$\|u(t)\|_{[1]} = \|S(t)\mathring{u}\|_{[1]} = \|\mathring{u}\|_{[1]},$$

for all $\mathring{u} \in \mathcal{D}(G)_+$, i.e. for $\mathring{u} \in \mathcal{D}(G)_+$ the solution is norm conserving in $X_{[1]}$. Since the norm of a non-negative solution coincides with the mass in the system, it follows that for $\mathring{u} \in \mathcal{D}(G)_+$, $u(t) = S(t)\mathring{u}$ is the unique, non-negative, mass-conserving classical solution of (5.2.10), (5.2.11). $\square$

We now use Proposition 5.2.8 to obtain the following result regarding the mass conservation of the solution to (5.2.8), (5.2.9) for a general weight $w = (w_n)_{n=1}^{\infty}$ satisfying Assumption 5.2.2(i) and (ii).

Lemma 5.2.9. *Let* (5.1.5) *hold and let Assumptions 5.2.2*(i) *and* (ii) *hold. Then* $u(t) = S^{(w)}(t)\mathring{u}$ *is the unique, non-negative, mass-conserving classical solution of* (5.2.8), (5.2.9) *for all* $\mathring{u} \in \mathcal{D}(G^{(w)})_+$.

Proof. From Remark 5.2.3 we have that ℓ_w^1 is continuously embedded in $X_{[1]}$. It follows that $\ell_w^1 \subseteq X_{[1]}$ and $\|f\|_{[1]} \leq c\|f\|_{\ell_w^1}$ for all $f \in \ell_w^1$ and some $c > 0$.

Hence $A+B$ is an extension of $A^{(w)}+B^{(w)}$. Let $f \in D(G^{(w)}) = D(\overline{A^{(w)} + B^{(w)}})$. There exists $f^{(k)}$, $k \in \mathbb{N}$ such that $f^{(k)} \in \mathcal{D}(A^{(w)} + B^{(w)})$ for all $k \in \mathbb{N}$, $\|f^{(k)} - f\|_{\ell_w^1} \to 0$ and $\|(A^{(w)} + B^{(w)})f^{(k)} - G^{(w)}f\|_{\ell_w^1} \to 0$ as $k \to \infty$. Since $A + B$ is an extension of $A^{(w)} + B^{(w)}$, $f^{(k)} \in \mathcal{D}(A + B)$ for all $k \in \mathbb{N}$. Moreover, since ℓ_w^1 is continuously embedded in $X_{[1]}$, we have $\|f^{(k)} - f\|_{[1]} \to 0$ and

$$\|(A^{(w)} + B^{(w)})f^{(k)} - G^{(w)}f\|_{[1]} = \|(A + B)f^{(k)} - G^{(w)}f\|_{[1]} \to 0 \text{ as } k \to \infty.$$

We have that $G = \overline{A + B}$ and so, by the definition of the closure, $f \in \mathcal{D}(G)$ and $Gf = G^{(w)}f$. Thus G and $G^{(w)}$ coincide on $\mathcal{D}(G^{(w)})$ and so G is an extension of $G^{(w)}$.

Take $\mathring{u} \in \mathcal{D}(G^{(w)})_+$. Then $\mathring{u} \in \mathcal{D}(G)_+$. We know that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique, non-negative classical solution of the ACP (5.2.8), (5.2.9) and it follows that $u(t) \in \mathcal{D}(G^{(w)})_+ \subseteq \mathcal{D}(G)_+$ for all $t \geq 0$. Moreover u is strongly differentiable in ℓ_w^1 and it follows that u is strongly differentiable in $X_{[1]}$ and the derivatives coincide. Since G is an extension of $G^{(w)}$, $u(t) = S^{(w)}(t)\mathring{u}$ is also a classical solution of the ACP (5.2.10), (5.2.11). However, we know from Proposition 5.2.8 that $\tilde{u}(t) = S(t)\mathring{u}$ is the unique, non-negative, mass-conserving classical solution of (5.2.10), (5.2.11). Hence $u(t)$ and $\tilde{u}(t)$ must coincide for $\mathring{u} \in \mathcal{D}(G^{(w)})_+$ and so $u(t)$ is mass conserving. $\square$

We now return to the fragmentation ACP (5.1.8). Under the assumptions of Theorem 5.2.7, we have that $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ generates a substochastic semigroup, $(S^{(w)}(t))_{t\geq 0}$. We know that $G^{(w)}$ and $A^{(w)} + B^{(w)}$ coincide on $\mathcal{D}(A^{(w)})$. However, if $\mathring{u} \in \mathcal{D}(A^{(w)})$, then we cannot guarantee that $S^{(w)}(t)\mathring{u} \in \mathcal{D}(A^{(w)})$ and so we cannot say in general that the classical solution of (5.2.8), (5.2.9) is also a classical solution of (5.1.8). However, as we show in the next theorem, the invariance result in Theorem 5.2.7 allows us to obtain a solution of (5.1.8) for a certain class of initial conditions. We again define $C^{(w)}$ and $\mathcal{D}(C^{(w)})$ by (5.2.5), where $(c_n)_{n=1}^{\infty}$ is a monotone increasing sequence that dominates the sequence $(a_n)_{n=1}^{\infty}$.

Theorem 5.2.10. *Let Assumption 5.2.2(i) hold. Then $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of* (5.1.8) *for all $\mathring{u} \in \mathcal{D}(C^{(w)})$. If $\mathring{u} \in \mathcal{D}(C^{(w)})_+$, then this solution is non-negative.*

Moreover, if (5.1.5) *and Assumption 5.2.2(ii) holds, then $u(t)$ is a mass-conserving solution.*

Proof. Let $\mathring{u} \in \mathcal{D}(C^{(w)})$. Since the sequence $(c_n)_{n=1}^{\infty}$ dominates the sequence $(a_n)_{n=1}^{\infty}$ we have $\mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(A^{(w)})$. Also, from Theorem 5.2.7, $\mathcal{D}(C^{(w)})$ is invariant under $(S^{(w)}(t))_{t\geq 0}$. Hence $S^{(w)}(t)\mathring{u} \in \mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(A^{(w)})$. Moreover, $G^{(w)}$ and $A^{(w)} + B^{(w)}$ coincide on $\mathcal{D}(A^{(w)})$, and we know that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.2.8), (5.2.9). It follows that if $\mathring{u} \in \mathcal{D}(C^{(w)})$, then $u(t) = S^{(w)}(t)\mathring{u} \in \mathcal{D}(A^{(w)})$ is the unique classical solution of (5.1.8). The non-negativity result follows from the positivity of the semigroup $(S^{(w)}(t))_{t\geq 0}$. The mass conservation result follows from Lemma 5.2.9. □

In Theorem 5.2.10, under Assumption 5.2.2(i), we have obtained a unique solution of (5.1.8) for any $\mathring{u} \in \mathcal{D}(C^{(w)})$ and, moreover, this solution is non-negative whenever $\mathring{u} \in \mathcal{D}(C^{(w)})_+$. As we now remark, we can "optimise" $\mathcal{D}(C^{(w)})$ through our choice of $(c_n)_{n=1}^{\infty}$.

Remark 5.2.11. Let the assumptions of Theorem 5.2.7 hold and let $(c_n)_{n=1}^{\infty}$ be as in (5.2.4). We note that this choice is the "minimal" choice of $(c_n)_{n=1}^{\infty}$ such that (5.2.3) holds. By this we mean that if $(\tilde{c}_n)_{n=1}^{\infty}$ is any other sequence satisfying (5.2.3), then $c_n \leq \tilde{c}_n$ for all $n \in \mathbb{N}$. It follows that $\mathcal{D}(\tilde{C}^{(w)}) \subseteq \mathcal{D}(C^{(w)})$, where

$$[\tilde{C}^{(w)}f]_n = -\tilde{c}_n f_n, \qquad \mathcal{D}(\tilde{C}^{(w)}) = \left\{ f \in \ell_w^1 : \sum_{n=1}^{\infty} w_n \tilde{c}_n |f_n| < \infty \right\}.$$

Hence, for a given sequence $(a_n)_{n=1}^{\infty}$, choosing $(c_n)_{n=1}^{\infty}$ as in (5.2.4) gives rise to the existence of a unique solution for the largest possible class of initial conditions, which can be obtained using Theorem 5.2.10.

On the other hand, let the assumptions of Theorem 5.2.7 hold and suppose that $\mathring{u} \in \mathcal{D}(C^{(w)})$, where $(c_n)_{n=1}^{\infty}$ is given by (5.2.4). Since the choice of monotone increasing sequence that dominates $(a_n)_{n=1}^{\infty}$ is arbitrary, we can choose the sequence $(\tilde{c}_n)_{n=1}^{\infty}$, defined as above, to be as large as possible such that $\mathring{u} \in \mathcal{D}(\tilde{C}^{(w)})$. From Theorem 5.2.7 we know that $u(t) = S^{(w)}\mathring{u} \in \mathcal{D}(\tilde{C}^{(w)})$. Hence, by choosing $(\tilde{c}_n)_{n=1}^{\infty}$ to be as large as possible, we restrict $\mathcal{D}(\tilde{C}^{(w)})$ and so obtain a clearer idea of where the unique classical solution lies.

For a given $(a_n)_{n=1}^{\infty}$, choose $(c_n)_{n=1}^{\infty}$ as in (5.2.4). If $\mathring{u} \in \mathcal{D}(A^{(w)})\backslash\mathcal{D}(C^{(w)})$, for this choice of $(c_n)_{n=1}^{\infty}$, then we are unable to use Theorem 5.2.10 to obtain a solution of (5.1.8). In the next example we examine the set $\mathcal{D}(A^{(w)})\backslash\mathcal{D}(C^{(w)})$ in more detail and show that there are situations where it is non-empty. First we note that, due to our choice of $(c_n)_{n=1}^{\infty}$, we have

$$\mathcal{D}(A^{(w)})\backslash\mathcal{D}(C^{(w)}) = \left\{ f \in \ell_w^1 \,:\, A^{(w)}f \in \ell_w^1 \text{ and } \sum_{n=1}^{\infty} w_n A_n |f_n| \text{ diverges} \right\},$$

where $A_n = \max\{a_1, a_2, \ldots, a_n\}$.

Example 5.2.12. For a given $(a_n)_{n=1}^{\infty}$, let $(c_n)_{n=1}^{\infty}$ be as in (5.2.4). Then we have that $\mathcal{D}(C^{(w)})$ is a strict subset of $\mathcal{D}(A^{(w)})$ if there exists at least one $f = (f_n)_{n=1}^{\infty}$ satisfying

$$\sum_{n=1}^{\infty} w_n |f_n| < \infty \qquad \text{and} \qquad \sum_{n=1}^{\infty} w_n a_n |f_n| < \infty$$

but

$$\sum_{n=1}^{\infty} w_n A_n |f_n| = \infty.$$

By setting $g_n = w_n|f_n|$, we see that this is equivalent to establishing the existence of a sequence $(g_n)_{n=1}^{\infty} \in \ell_+^1$ such that

$$\sum_{n=1}^{\infty} a_n g_n < \infty \qquad \text{but} \qquad \sum_{n=1}^{\infty} A_n g_n = \infty.$$

Let $(a_n)_{n=1}^{\infty}$ be defined by

$$a_n = \begin{cases} 0 & \text{when } n = 1 \\ 1 & \text{if } n \text{ is odd and } n \geq 3 \\ n^p & \text{if } n \text{ is even,} \end{cases}$$

where $p \geq 2$. Now take $h \in \ell_+^1$ and define $(g_n)_{n=1}^{\infty}$ by

$$g_n = \begin{cases} \frac{1}{n^2} & \text{if } n \text{ is odd} \\ \frac{1}{n^p} h_n & \text{if } n \text{ is even.} \end{cases}$$

Then $(g_n)_{n=1}^{\infty} \in \ell_+^1$ and

$$\begin{aligned} \sum_{n=1}^{\infty} a_n g_n = \sum_{n=1}^{\infty} a_{2n-1} g_{2n-1} + \sum_{n=1}^{\infty} a_{2n} g_{2n} &= \sum_{n=2}^{\infty} \frac{1}{(2n-1)^2} + \sum_{n=1}^{\infty} h_{2n} \\ &\leq \sum_{n=2}^{\infty} \frac{1}{n^2} + \sum_{n=1}^{\infty} h_n < \infty. \end{aligned}$$

However,

$$\sum_{n=1}^{\infty} A_n g_n = \sum_{n=1}^{\infty} A_{2n-1} g_{2n-1} + \sum_{n=1}^{\infty} A_{2n} g_{2n} = \sum_{n=2}^{\infty} \frac{(2n-2)^p}{(2n-1)^2} + \sum_{n=1}^{\infty} h_{2n}$$

and

$$\frac{(2n-2)^p}{(2n-1)^2} \geq \left(\frac{2n-2}{2n-1} \right)^2 = \left(1 - \frac{1}{2n-1} \right)^2 \to 1 \text{ as } n \to \infty.$$

Hence $\frac{(2n-2)^p}{(2n-1)^2}$ does not tend to zero as $n \to \infty$ and so $\sum_{n=1}^{\infty} A_n g_n$ diverges.

Let Assumption 5.2.2(i) hold. From Theorem 5.2.10, $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.1.8) for $\mathring{u} \in \mathcal{D}(C^{(w)})$. Hence, if we choose $(c_n)_{n=1}^{\infty}$ such that (5.2.3) holds and $\mathcal{D}(C^{(w)}) = \mathcal{D}(A^{(w)})$, then we obtain a unique classical solution to the fragmentation ACP, (5.1.8), for all $\mathring{u} \in \mathcal{D}(A^{(w)})$. Unfortunately it will not always be possible to choose $(c_n)_{n=1}^{\infty}$ such that $\mathcal{D}(A^{(w)}) = \mathcal{D}(C^{(w)})$; see Example 5.2.12. However, if we impose some further restrictions on the sequence $(a_n)_{n=1}^{\infty}$, we find that there are situations in which it is possible to choose $(c_n)_{n=1}^{\infty}$

such that $\mathcal{D}(C^{(w)}) = \mathcal{D}(A^{(w)})$.

Proposition 5.2.13. *Let Assumption 5.2.2(i) hold and let $(a_n)_{n=1}^{\infty}$ be an unbounded sequence. Define the sequence $(c_n)_{n=1}^{\infty}$ by (5.2.4). Then $\mathcal{D}(C^{(w)}) = \mathcal{D}(A^{(w)})$ if and only if*

$$\liminf_{n\to\infty} \frac{a_n}{c_n} > 0. \tag{5.2.13}$$

Proof. We first note that, since $(a_n)_{n=1}^{\infty}$ is unbounded, then $(c_n)_{n=1}^{\infty}$ is a monotone increasing sequence such that $c_n \to \infty$ as $n \to \infty$. Hence the quotient on the left-hand side of (5.2.13) is well defined for all n large enough. Since $c_n \geq a_n$ for all $n \in \mathbb{N}$, we have $\mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(A^{(w)})$. If (5.2.13) holds, then there exist $\gamma \in (0,1]$, $N \in \mathbb{N}$ such that $a_n \geq \gamma c_n$ for all $n \geq N$. Let $f \in \mathcal{D}(A^{(w)})$. Then

$$\|C^{(w)}f\|_{\ell_w^1} \leq \sum_{n=1}^{N-1} w_n c_n |f_n| + \frac{1}{\gamma}\sum_{n=N}^{\infty} w_n a_n |f_n| \leq \sum_{n=1}^{N-1} w_n c_n |f_n| + \frac{1}{\gamma}\|A^{(w)}f\|_{\ell_w^1} < \infty.$$

Hence $\mathcal{D}(A^{(w)}) = \mathcal{D}(C^{(w)})$.

Now suppose that $\liminf_{n\to\infty} \frac{a_n}{c_n} = 0$. Then there exists a subsequence, $\left(\frac{a_{n_k}}{c_{n_k}}\right)_{k=1}^{\infty}$, such that

$$\frac{a_{n_k}}{c_{n_k}} \leq \frac{1}{k} \qquad \text{and} \qquad \frac{1}{c_{n_k}} \leq \frac{1}{k} \qquad \text{for all } k \in \mathbb{N}.$$

Consider f such that

$$f_j = \begin{cases} \frac{1}{c_{n_k} w_{n_k} k} & \text{when } j = n_k, \\ 0 & \text{otherwise.} \end{cases} \tag{5.2.14}$$

Then

$$\sum_{n=1}^{\infty} w_n |f_n| = \sum_{k=1}^{\infty} \frac{1}{c_{n_k} k} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

and

$$\sum_{n=1}^{\infty} a_n w_n |f_n| = \sum_{k=1}^{\infty} a_{n_k} w_{n_k} \frac{1}{c_{n_k} w_{n_k} k} \leq \sum_{k=1}^{\infty} \frac{1}{k^2} < \infty$$

but

$$\sum_{n=1}^{\infty} c_n w_n |f_n| = \sum_{k=1}^{\infty} c_{n_k} w_{n_k} \frac{1}{c_{n_k} w_{n_k} k} = \sum_{k=1}^{\infty} \frac{1}{k} = \infty.$$

Hence $f \in \mathcal{D}(A^{(w)}) \backslash \mathcal{D}(C^{(w)})$, i.e. $\mathcal{D}(A^{(w)}) \neq \mathcal{D}(C^{(w)})$, and the result follows. □

We can apply this result to eventually monotone increasing $(a_n)_{n=1}^{\infty}$, as we show in the following remark.

Remark 5.2.14. Let Assumption 5.2.2(i) hold and let $(a_n)_{n=1}^{\infty}$ be an unbounded, eventually monotone increasing sequence. Then there exists an $M \in \mathbb{N}$ such that $(a_n)_{n=M}^{\infty}$ is monotone increasing and $a_n \leq a_M$ for all $n < M$. Hence we can take $(c_n)_{n=1}^{\infty}$ such that $c_n = \max\{a_1, \ldots, a_{M-1}\}$ for $n = 1, \ldots, M-1$ and $a_n = c_n$ for all $n \geq M$. Then $(c_n)_{n=1}^{\infty}$ is monotone increasing, $a_n \leq c_n$ for all $n \in \mathbb{N}$ and $\gamma c_n \leq a_n$, with $\gamma = 1$, for all $n \geq M$. Hence (5.2.13) is satisfied and we can conclude that $\mathcal{D}(C^{(w)}) = \mathcal{D}(A^{(w)})$. It follows from Theorem 5.2.10 that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.1.8) for $\mathring{u} \in \mathcal{D}(A^{(w)})$. This solution is non-negative if $\mathring{u} \in \mathcal{D}(A^{(w)})_+$.

We will now give a result regarding the existence of moments of solutions. These results are slightly more general than the moment result in [16, Theorem 4.1], where the authors use an approach involving Fréchet spaces, and follow almost immediately from the invariance of $\mathcal{D}(C^{(w)})$. In particular, we avoid the need to work in Fréchet spaces.

Corollary 5.2.15. *Let Assumption 5.2.2(i) and (ii) hold. Consider the following conditions:*

(a) $\mathring{u} \in \ell_w^1$ *is such that* $\sum_{n=1}^{\infty} w_n^p |\mathring{u}_n| < \infty$ *and there exists* $M > 0$ *such that* $a_n \leq M n^{p-1}$ *for some* $p \geq 1$*;*

(b) $\mathring{u} \in \mathcal{D}(C^{(w)})$*, where* $(c_n)_{n=1}^{\infty}$ *is given by* (5.2.4)*, is such that* $\sum_{n=1}^{\infty} w_n^p |\mathring{u}_n| < \infty$ *for some* $p \geq 1$*, and* $(w_n)_{n=1}^{\infty}$ *is monotone increasing.*

If either (a) or (b) hold, then $u(t) = S^{(w)}(t)\mathring{u}$ *is the unique classical solution of* (5.1.8)*. Moreover,*

$$\sum_{n=1}^{\infty} n^q |u_n(t)| < \infty \qquad \text{for all } q \in [0,p],\ t \geq 0. \tag{5.2.15}$$

Proof. We first prove the result under the conditions in (a). We choose the sequence $(c_n)_{n=1}^{\infty}$, from Theorem 5.2.7, to be $c_n = M n^{p-1}$ for all $n \in \mathbb{N}$. Then $(c_n)_{n=1}^{\infty}$ is a monotone increasing sequence that dominates the sequence $(a_n)_{n=1}^{\infty}$

and

$$[C^{(w)}f]_n = -c_n f_n = -Mn^{p-1}f_n; \quad \mathcal{D}(C^{(w)}) = \left\{ f \in \ell_w^1 : \sum_{n=1}^{\infty} w_n n^{p-1}|f_n| < \infty \right\}.$$

We have

$$\sum_{n=1}^{\infty} w_n n^{p-1}|\mathring{u}_n| \le \sum_{n=1}^{\infty} w_n^p |\mathring{u}_n| < \infty,$$

since $w_n \ge n$ for all $n \in \mathbb{N}$. Consequently, $\mathring{u} \in \mathcal{D}(C^{(w)})$. From Theorem 5.2.10, we know that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.1.8). Also, $(S^{(w)}(t))_{t\ge 0}$ leaves $\mathcal{D}(C^{(w)})$ invariant and so $u(t) \in \mathcal{D}(C^{(w)})$ for all $t \ge 0$. Thus, we have

$$\sum_{n=1}^{\infty} n^p |u_n(t)| \le \sum_{n=1}^{\infty} w_n n^{p-1}|u_n(t)| < \infty,$$

since $w_n \ge n$ for all $n \in \mathbb{N}$. If follows that

$$\sum_{n=1}^{\infty} n^q |u_n(t)| \le \sum_{n=1}^{\infty} n^p |u_n(t)| < \infty \qquad \text{for all } q \in [0,p] \text{ and } t \ge 0.$$

We now prove the result under the conditions in (b). That $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.1.8) follows immediately from Theorem 5.2.10. Consider $(\tilde{c}_n)_{n=1}^{\infty}$, where $\tilde{c}_n = \max\{c_n, w_n^{p-1}\}$. Then $(\tilde{c}_n)_{n=1}^{\infty}$ is a monotone increasing sequence that dominates the sequence $(a_n)_{n=1}^{\infty}$. From Theorem 5.2.7, $\mathcal{D}(\tilde{C})$ is invariant under $(S^{(w)}(t))_{t\ge 0}$, where

$$[\tilde{C}f]_n = -\tilde{c}_n f_n, \qquad \mathcal{D}(\tilde{C}) = \left\{ f \in \ell_w^1 : \sum_{n=1}^{\infty} w_n \tilde{c}_n |f_n| < \infty \right\}.$$

We have

$$\sum_{n=1}^{\infty} w_n \tilde{c}_n |\mathring{u}_n| \le \sum_{n=1}^{\infty} w_n c_n |\mathring{u}_n| + \sum_{n=1}^{\infty} w_n^p |\mathring{u}_n| < \infty$$

and so $\mathring{u} \in \mathcal{D}(\tilde{C})$. Hence $u(t) \in \mathcal{D}(\tilde{C})$ and so

$$\sum_{n=1}^{\infty} n^p |u_n(t)| \le \sum_{n=1}^{\infty} w_n^p |u_n(t)| = \sum_{n=1}^{\infty} w_n w_n^{p-1} |u_n(t)| \le \sum_{n=1}^{\infty} w_n \tilde{c}_n |u_n(t)| < \infty.$$

The result then follows immediately. $\square$

5.3 The Pointwise Fragmentation System

In this section we examine the pointwise fragmentation system (5.1.1), (5.1.2). Throughout this subsection we let Assumption 5.2.2(i) hold so that, from Theorem 5.2.7, $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$. We note that if (5.1.4) holds with $\lambda_j \in [0, 1)$, $j \geq 2$, then Assumption 5.2.2(i) is satisfied for $w_n = n$, $n \in \mathbb{N}$, i.e. if (5.1.4) holds with $\lambda_j \in [0, 1)$, $j \geq 2$, then Assumption 5.2.2(i) holds when $\ell^1_w = X_{[1]}$. It follows that the results in this section hold if (5.1.4) holds with $\lambda_j \in [0, 1)$,$j \geq 2$ and we work in $X_{[1]}$.

As mentioned at the start of Section 5.2.2, if Assumption 5.2.2(i) holds, then $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.2.8), (5.2.9).

As we now show, this provides us with a solution to the pointwise problem

$$u_n'(t) = [G^{(w)}u(t)]_n, \quad t > 0, \quad n = 1, 2, 3, \ldots \tag{5.3.1}$$

$$u_n(0) = \mathring{u}_n, \qquad n = 1, 2, 3, \ldots \tag{5.3.2}$$

for all $\mathring{u} \in \mathcal{D}(G^{(w)})$.

Let Assumption 5.2.2(i) hold and let $\mathring{u} \in \mathcal{D}(G^{(w)})$. We have that $S^{(w)}(t)\mathring{u}$ is strongly differentiable with respect to t on ℓ^1_w, with

$$\frac{d}{dt}S^{(w)}(t)\mathring{u} = G^{(w)}S^{(w)}(t)\mathring{u}.$$

This means that

$$\left\| \frac{S^{(w)}(t+h)\mathring{u} - S^{(w)}(t)\mathring{u}}{h} - G^{(w)}S^{(w)}(t)\mathring{u} \right\|_{\ell^1_w} \to 0 \qquad \text{as } h \to 0.$$

From this we can deduce that, for each $n \in \mathbb{N}$,

$$w_n \left| \frac{(S^{(w)}(t+h)\mathring{u})_n - (S^{(w)}(t)\mathring{u})_n}{h} - (G^{(w)}S^{(w)}(t)\mathring{u})_n \right|$$

$$\leq \sum_{k=1}^{\infty} w_k \left| \frac{(S^{(w)}(t+h)\mathring{u})_k - (S^{(w)}(t)\mathring{u})_k}{h} - (G^{(w)}S^{(w)}(t)\mathring{u})_k \right|$$
$$= \left\| \frac{S^{(w)}(t+h)\mathring{u} - S^{(w)}(t)\mathring{u}}{h} - G^{(w)}S^{(w)}(t)\mathring{u} \right\|_{\ell_w^1} \to 0 \qquad \text{as } h \to 0.$$

Thus, if $S^{(w)}(t)\mathring{u}$ is strongly differentiable with respect to $t > 0$ in ℓ_w^1, then for all $n \in \mathbb{N}$, $(S^{(w)}(t)\mathring{u})_n$ is differentiable with respect to $t > 0$ and

$$\frac{d}{dt}(S^{(w)}(t)\mathring{u})_n = (G^{(w)}S^{(w)}(t)\mathring{u})_n.$$

Hence $u_n(t) = (S^{(w)}(t)\mathring{u})_n$ is a solution of the pointwise problem (5.3.1), (5.3.2), for all $n \in \mathbb{N}$, $t \geq 0$, $\mathring{u} \in \mathcal{D}(G^{(w)})$. Moreover, $u(t) = S^{(w)}(t)\mathring{u} \geq 0$ for all $\mathring{u} \in \mathcal{D}(G^{(w)})_+$, $n \in \mathbb{N}$, $t \geq 0$ and so $u_n(t) = (S^{(w)}(t)\mathring{u})_n \geq 0$ for all $\mathring{u} \in \mathcal{D}(G^{(w)})_+$.

However, since we do not have an explicit expression for $G^{(w)}$, we do not know in what sense, if any, the function $u(t) = S^{(w)}(t)\mathring{u}$ satisfies the original problem (5.1.1) for $\mathring{u} \in \mathcal{D}(G^{(w)})$ (or more generally for $\mathring{u} \in \ell_w^1$). We deal with this in the next theorem. The proof of this theorem requires the concept of absolute continuity.

Definition 5.3.1. Let f be a real-valued function on a compact interval $[a, b]$. Then f is *absolutely continuous* on $[a, b]$ if there exists a Lebesgue integrable function g on $[a, b]$ such that

$$f(x) = f(a) + \int_a^x g(t)\mathrm{d}t, \tag{5.3.3}$$

for all x in $[a, b]$.

The following lemma, based on [21, p. 25], gives an equivalent definition of absolute continuity which we use in this section.

Lemma 5.3.2. *Let f be a real-valued function on a compact interval $[a, b]$. Then the following statements are equivalent:*

(i) *f is absolutely continuous;*

(ii) *f has a derivative f' almost everywhere on $[a,b]$ and the derivative is Lebesgue integrable on $[a,b]$.*

If f is absolutely continuous and (5.3.3) holds, then $g = f'$ almost everywhere on $[a,b]$.

We now obtain the following theorem, regarding a solution of the pointwise fragmentation system (5.1.1). (5.1.2).

Theorem 5.3.3. *Let $\mathring{u} \in \ell_w^1$. Then $u_n(t) = [S^{(w)}(t)\mathring{u}]_n$ satisfies the pointwise fragmentation system,* (5.1.1) *and* (5.1.2), *for almost all $t \geq 0$.*

Proof. As previously, we define a sequence $(c_n)_{n=1}^{\infty}$ by

$$\begin{aligned} c_1 &= a_1 \\ c_n &= \max\{c_{n-1}, a_n\} = \max\{a_1, \dots, a_n\} \qquad \text{for } n \geq 2 \end{aligned}$$

and let

$$(C^{(w)}f)_{n=1}^{\infty} = (-c_n f_n)_{n=1}^{\infty}$$

with

$$\mathcal{D}(C^{(w)}) = \left\{ f \in \ell_w^1 : \sum_{n=1}^{\infty} w_n c_n |f_n| < \infty \right\}.$$

Then $a_n \leq c_n$ for all $n \in \mathbb{N}$ and it follows that $\mathcal{D}(C^{(w)}) \subseteq \mathcal{D}(A^{(w)})$.

Taking $\mathring{u} \in \mathcal{D}(C^{(w)})$ we have, from Theorem 5.2.10, that $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.1.8) for each $t \geq 0$. By a similar argument as used at the start of this section, it follows that for $t \geq 0$, $u_n(t)$ is a solution of the pointwise problem

$$u_n'(t) = -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t), \quad n = 1, 2, \dots$$

and so is also a solution of

$$u_n(t) - \mathring{u}_n = -a_n \int_0^t u_n(s)\,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(s)\,\mathrm{d}s. \qquad (5.3.4)$$

Fix $t \geq 0$. Take $\mathring{u} \in (\ell_w^1)_+$ and let $u(t) = S^{(w)}(t)\mathring{u}$. Then there exists a sequence $(\mathring{u}^{(k)})_{k=1}^{\infty}$, with $\mathring{u}^{(k)} \in \mathcal{D}(C^{(w)})_+$ for all $k \in \mathbb{N}$, such that $\mathring{u}^{(k)} \to \mathring{u}$ as $k \to \infty$. For example, take $\mathring{u}^{(k)}$ such that

$$(\mathring{u}^{(k)})_n = \begin{cases} \mathring{u}_n & \text{if } n \leq k, \\ 0 & \text{if } n > k. \end{cases} \tag{5.3.5}$$

Then $\mathring{u}^{(k)} \in \mathcal{D}(C^{(w)})_+$ for each $k \in \mathbb{N}$ and $(\mathring{u}^{(k)})_{k=1}^{\infty}$ is monotone increasing.

Let $k_1 \leq k_2$. Then $\mathring{u}^{(k_1)} \leq \mathring{u}^{(k_2)}$. Thus we have

$$S^{(w)}(t)\mathring{u}^{(k_2)} - S^{(w)}(t)\mathring{u}^{(k_1)} = S^{(w)}(t)(\mathring{u}^{(k_2)} - \mathring{u}^{(k_1)}) \geq 0$$

since $(S^{(w)}(t))_{t\geq 0}$ is substochastic and $\mathring{u}^{(k_2)} - \mathring{u}^{(k_1)} \geq 0$. Similarly, for all $k \in \mathbb{N}$,

$$S^{(w)}(t)\mathring{u} - S^{(w)}(t)\mathring{u}^{(k)} = S^{(w)}(t)(\mathring{u} - \mathring{u}^{(k)}) \geq 0.$$

It is therefore clear that $(S^{(w)}(t)\mathring{u}^{(k)})_{k=1}^{\infty}$ is monotone increasing and bounded above by $S^{(w)}(t)\mathring{u}$. Hence, for fixed $n \in \mathbb{N}$, we also have that $((S^{(w)}(t)\mathring{u}^{(k)})_n)_{k=1}^{\infty}$ is monotone increasing and bounded above by $(S^{(w)}(t)\mathring{u})_n$. We can also deduce that $S^{(w)}(t)\mathring{u}^{(k)} \to S^{(w)}(t)\mathring{u}$ as $k \to \infty$ since we have

$$\begin{aligned} \|S^{(w)}(t)\mathring{u} - S^{(w)}(t)\mathring{u}^{(k)}\|_{\ell_w^1} &= \|S^{(w)}(t)(\mathring{u} - \mathring{u}^{(k)})\|_{\ell_w^1} \leq \|\mathring{u} - \mathring{u}^{(k)}\|_{\ell_w^1} \\ &= \sum_{n=1}^{\infty} w_n(\mathring{u}_n - \mathring{u}_n^{(k)}) = \sum_{n=k+1}^{\infty} w_n\mathring{u}_n \to 0 \qquad \text{as } k \to \infty. \end{aligned}$$

It follows that $(S^{(w)}(t)\mathring{u}^{(k)})_n \to (S^{(w)}(t)\mathring{u})_n$ as $k \to \infty$.

Since $\mathring{u}^{(k)} \in \mathcal{D}(C^{(w)})_+$ for all $k \in \mathbb{N}$, we know that $S^{(w)}(t)\mathring{u}^{(k)}$ satisfies

$$(S^{(w)}(t)\mathring{u}^{(k)})_n = (\mathring{u}^{(k)})_n - \int_0^t a_n(S^{(w)}(s)\mathring{u}^{(k)})_n \,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j \,\mathrm{d}s.$$

Taking limits on both sides we have

$$(S^{(w)}(t)\mathring{u})_n = \mathring{u}_n + \lim_{k\to\infty}\left(\int_0^t -a_n(S^{(w)}(s)\mathring{u}^{(k)})_n \,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j \,\mathrm{d}s\right). \tag{5.3.6}$$

By the monotone convergence theorem we have

$$\lim_{k\to\infty}\int_0^t (S^{(w)}(s)\mathring{u}^{(k)})_n \,\mathrm{d}s = \int_0^t \lim_{k\to\infty}(S^{(w)}(s)\mathring{u}^{(k)})_n \,\mathrm{d}s = \int_0^t (S^{(w)}(s)\mathring{u})_n \,\mathrm{d}s.$$

Since all the other limits in (5.3.6) exist we must have that

$$\lim_{k\to\infty}\int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j \,\mathrm{d}s$$

exists. Thus $\sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j$, and so also $a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j$, must be finite except on sets of measure zero.

Again the sequence $\left(a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j\right)_{k=1}^{\infty}$ is increasing for fixed n, j and so $\left(\sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j\right)_{k=1}^{\infty}$ is increasing. It follows, by the monotone convergence theorem, that

$$\begin{aligned}\lim_{k\to\infty}\int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j \,\mathrm{d}s &= \int_0^t \lim_{k\to\infty}\sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}^{(k)})_j \,\mathrm{d}s \\ &= \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u})_j \,\mathrm{d}s.\end{aligned}$$

Thus,

$$\begin{aligned}(S^{(w)}(t)\mathring{u})_n &= \mathring{u}_n - \int_0^t a_n(S^{(w)}(s)\mathring{u})_n \,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u})_j \,\mathrm{d}s \\ &= \mathring{u}_n + \int_0^t \left(-a_n(S^{(w)}(s)\mathring{u})_n + \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u})_j\right) \mathrm{d}s.\end{aligned}$$

It follows that $(S^{(w)}(t)\mathring{u})_n$ is absolutely continuous with respect to t for each $n = 1, 2, \ldots$. From Lemma 5.3.2 we can then deduce that

$$\frac{d}{dt}(S^{(w)}(t)\mathring{u})_n = -a_n(S^{(w)}(t)\mathring{u})_n + \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(t)\mathring{u})_j \tag{5.3.7}$$

for all $\mathring{u} \in (\ell_w^1)_+$ and almost every $t \geq 0$, i.e. for all $\mathring{u} \in (\ell_w^1)_+$ and almost every $t \geq 0$, $u_n(t) = (S^{(w)}(t)\mathring{u})_n$ satisfies the original pointwise discrete fragmentation equation (5.1.1), (5.1.2). Moreover, for $\mathring{u} \in (\ell_w^1)_+$ we have

$$(S^{(w)}(t)\mathring{u})_n = \mathring{u}_n - \int_0^t a_n(S^{(w)}(s)\mathring{u})_n \,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u})_j \,\mathrm{d}s. \tag{5.3.8}$$

Take $\mathring{u} \in \ell_w^1$. Then $\mathring{u} = \mathring{u}_+ - \mathring{u}_-$ where $\mathring{u}_+,\ \mathring{u}_- \in (\ell_w^1)_+$. We have

$$\begin{aligned}
(S^{(w)}(t)\mathring{u})_n &= (S^{(w)}(t)(\mathring{u}_+ - \mathring{u}_-))_n \\
&= (S^{(w)}(t)\mathring{u}_+)_n - (S^{(w)}(t)\mathring{u}_-)_n \\
&= (\mathring{u}_+)_n - \int_0^t a_n(S^{(w)}(s)\mathring{u}_+)_n \,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}_+)_j \,\mathrm{d}s \\
&\qquad - (\mathring{u}_-)_n + \int_0^t a_n(S^{(w)}(s)\mathring{u}_-)_n \,\mathrm{d}s - \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u}_-)_j \,\mathrm{d}s \\
&= ((\mathring{u}_+)_n - (\mathring{u}_-)_n) - \int_0^t a_n[(S^{(w)}(s)\mathring{u}_+)_n - (S^{(w)}(s)\mathring{u}_-)_n] \,\mathrm{d}s \\
&\qquad + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}[(S^{(w)}(s)\mathring{u}_+)_j - (S^{(w)}(s)\mathring{u}_-)_j] \,\mathrm{d}s \\
&= \mathring{u}_n - \int_0^t a_n(S^{(w)}(s)\mathring{u})_n \,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j}(S^{(w)}(s)\mathring{u})_j \,\mathrm{d}s.
\end{aligned}$$

Hence, for any initial condition $\mathring{u} \in \ell_w^1$, we can use an identical absolute continuity argument as above to deduce that $(S^{(w)}(t)\mathring{u})_n$ satisfies the original pointwise fragmentation system, (5.1.1), (5.1.2), for almost all $t \geq 0$. $\square$

Theorem 5.3.3 now enables us to obtain an explicit expression for the operator

$G^{(w)} = \overline{A^{(w)} + B^{(w)}}$.

Theorem 5.3.4. *For all $g \in \mathcal{D}(G^{(w)})$, we have*

$$[G^{(w)} g]_n = -a_n g_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} g_j. \tag{5.3.9}$$

Proof. Let $g \in \mathcal{D}(G^{(w)})$. Then there exists $f \in \ell_w^1$ such that $g = R(1, G^{(w)})f$. Moreover, $f = f_+ - f_-$, where $f_+, f_- \in (\ell_w^1)_+$. Using (5.3.8) from the proof of Theorem 5.3.3 to obtain the second equality, we have

$$\begin{aligned}
&[R(1, G^{(w)}) f_+]_n \\
&= \int_0^\infty e^{-t} [S^{(w)}(t) f_+]_n \,\mathrm{d}t \\
&= \int_0^\infty e^{-t} \left[(f_+)_n - \int_0^t a_n (S^{(w)}(s) f_+)_n \,\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} (S^{(w)}(s) f_+)_j \,\mathrm{d}s \right] \mathrm{d}t.
\end{aligned}$$

Clearly,

$$\int_0^\infty e^{-t} (f_+)_n \,\mathrm{d}t = (f_+)_n$$

exists. Moreover, we have $\|S^{(w)}(s) f_+\|_{\ell_w^1} \leq \|f_+\|_{\ell_w^1}$, for all $s \geq 0$. Hence, for fixed $n \in \mathbb{N}$,

$$\begin{aligned}
\int_0^\infty \int_0^t |e^{-t} a_n (S^{(w)}(s) f_+)_n| \,\mathrm{d}s \,\mathrm{d}t &= \int_0^\infty \int_0^t e^{-t} a_n (S^{(w)}(s) f_+)_n \,\mathrm{d}s \,\mathrm{d}t \\
&\leq a_n \int_0^\infty \int_0^t e^{-t} \|S^{(w)}(s) f_+\|_{\ell_w^1} \,\mathrm{d}s \,\mathrm{d}t \\
&\leq a_n \int_0^\infty t e^{-t} \|f_+\|_{\ell_w^1} \,\mathrm{d}t \\
&= a_n \left([-te^{-t}]_0^\infty + \int_0^\infty e^{-t} \,\mathrm{d}t \right) \|f_+\|_{\ell_w^1} \\
&= a_n [-e^{-t}]_0^\infty \|f_+\|_{\ell_w^1} = a_n \|f_+\|_{\ell_w^1} < \infty.
\end{aligned}$$

It follows that

$$\int_0^\infty e^{-t} \int_0^t \sum_{j=n+1}^\infty a_j b_{n,j} (S^{(w)}(s) f_+)_j \, \mathrm{d}s \, \mathrm{d}t$$

also exists and so we have

$$\begin{aligned}[R(1, G^{(w)}) f_+]_n = (f_+)_n - \int_0^\infty \int_0^t e^{-t} a_n (S^{(w)}(s)(f_+))_n \, \mathrm{d}s \, \mathrm{d}t \\ + \int_0^\infty \int_0^t \sum_{j=n+1}^\infty e^{-t} a_j b_{n,j} (S^{(w)}(s)(f_+))_j \, \mathrm{d}s \, \mathrm{d}t.\end{aligned}$$

Now, $\int_0^\infty \int_0^t |e^{-t} a_n (S^{(w)}(s) f_+)_n| \, \mathrm{d}s \, \mathrm{d}t < \infty$ and so, by the Fubini–Tonelli Theorem, we have

$$\begin{aligned}\int_0^\infty \int_0^t e^{-t} a_n (S^{(w)}(s) f_+)_n \, \mathrm{d}s \, \mathrm{d}t &= \int_0^\infty \int_s^\infty e^{-t} a_n (S^{(w)}(s) f_+)_n \, \mathrm{d}t \, \mathrm{d}s \\ &= a_n \int_0^\infty e^{-s} (S^{(w)}(s) f_+)_n \, \mathrm{d}s \\ &= a_n [R(1, G^{(w)}) f_+]_n.\end{aligned}$$

Similarly, since $\int_0^\infty e^{-t} \int_0^t \sum_{j=n+1}^\infty a_j b_{n,j} (S^{(w)}(s) f_+)_j \, \mathrm{d}s \, \mathrm{d}t$ exists, we can use the Fubini–Tonelli Theorem three times to obtain

$$\begin{aligned}\int_0^\infty e^{-t} \int_0^t \sum_{j=n+1}^\infty a_j b_{n,j} (S^{(w)}(s) f_+)_j \, \mathrm{d}s \, \mathrm{d}t &= \int_0^\infty \int_s^\infty \sum_{j=n+1}^\infty e^{-t} a_j b_{n,j} (S^{(w)}(s) f_+)_j \, \mathrm{d}t \, \mathrm{d}s \\ &= \int_0^\infty \sum_{j=n+1}^\infty \int_s^\infty e^{-t} a_j b_{n,j} (S^{(w)}(s) f_+)_j \, \mathrm{d}t \, \mathrm{d}s \\ &= \sum_{j=n+1}^\infty \int_0^\infty \int_s^\infty e^{-t} a_j b_{n,j} (S^{(w)}(s) f_+)_j \, \mathrm{d}t \, \mathrm{d}s \\ &= \sum_{j=n+1}^\infty a_j b_{n,j} \int_0^\infty e^{-s} (S^{(w)}(s) f_+)_j \, \mathrm{d}s\end{aligned}$$

$$= \sum_{j=n+1}^{\infty} a_j b_{n,j} [R(1, G^{(w)}) f_+]_j.$$

So we have

$$[R(1, G^{(w)}) f_+]_n = (f_+)_n - a_n [R(1, G^{(w)}) f_+]_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} [R(1, G^{(w)}) f_+]_j.$$

Similarly, we have

$$[R(1, G^{(w)}) f_-]_n = (f_-)_n - a_n [R(1, G^{(w)}) f_-]_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} [R(1, G^{(w)}) f_-]_j.$$

Thus

$$\begin{aligned}
g_n &= [R(1, G^{(w)}) f]_n \\
&= [R(1, G^{(w)}) f_+]_n - [R(1, G^{(w)}) f_-]_n \\
&= (f_+)_n - (f_-)_n - \Big(a_n [R(1, G^{(w)}) f_+]_n - a_n [R(1, G^{(w)}) f_-]_n \Big) \\
&\qquad\qquad + \left(\sum_{j=n+1}^{\infty} a_j b_{n,j} [R(1, G^{(w)}) f_+]_j - \sum_{j=n+1}^{\infty} a_j b_{n,j} [R(1, G^{(w)}) f_-]_j \right) \\
&= f_n - a_n [R(1, G^{(w)}) f]_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} [R(1, G^{(w)}) f]_j \\
&= [(I - G^{(w)}) g]_n - a_n [R(1, G^{(w)}) f]_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} [R(1, G^{(w)}) f]_j.
\end{aligned}$$

Hence (5.3.9) holds. $\square$

We note that while in Theorem 5.3.3 we found that $(S^{(w)}(t))_{t\geq 0}$ provides a solution to the pointwise system (5.1.1), (5.1.2) *for almost all* $t \geq 0$, we know that for $\mathring{u} \in \mathcal{D}(G^{(w)})$ and $n \in \mathbb{N}$, $u_n(t) = (S^{(w)}(t)\mathring{u})_n$ solves (5.3.1), (5.3.2) for all $t \geq 0$. Hence, Theorem 5.3.4 enables us to deduce that when $\mathring{u} \in \mathcal{D}(G^{(w)})$, $u_n(t) = (S^{(w)}(t)\mathring{u})_n$ provides a solution to (5.1.1), (5.1.2) *for all* $t \geq 0$.

5.4 Analytic Fragmentation Semigroups

The results obtained in this section provide the main motivation for studying (5.1.1), (5.1.2) in general weighted ℓ^1 spaces. In particular, when working in these general weighted spaces we can obtain results relating to the analyticity of the fragmentation semigroup, $(S^{(w)}(t))_{t\geq 0}$, that do not necessarily hold when $w_n = n$ for all $n \in \mathbb{N}$.

We once again recall, from Theorem 5.2.7, that if Assumption 5.2.2(i) holds, then $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup on ℓ^1_w. We now use Theorem 3.3.35 to show that $G^{(w)} = A^{(w)} + B^{(w)}$ under certain conditions. Moreover, we show that under these conditions the semigroup $(S^{(w)}(t))_{t\geq 0}$ is analytic. We note that, unlike in (5.2.1), we exclude the case $\delta = 1$ in assumption (5.4.1) below.

Theorem 5.4.1. *Suppose that there exists $\delta \in (0,1)$ such that*

$$\sum_{n=1}^{j-1} w_n b_{n,j} \leq \delta w_j \qquad \text{for all } j = 2, 3, \ldots. \tag{5.4.1}$$

Then $G^{(w)} = A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ^1_w.

Proof. First note that, under the conditions of this theorem, Assumption 5.2.2(i) holds. To show that $A^{(w)} + B^{(w)}$ is a generator we check that, under the assumptions of this theorem, the conditions of Theorem 3.3.35 hold. It is clear that $-A^{(w)}$ is a positive operator and, from Lemma 5.2.1, $A^{(w)}$ is the generator of a substochastic C_0-semigroup on ℓ^1_w.

From Lemma 5.2.6, $\mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(B^{(w)})$ and $\|B^{(w)}f\|_{\ell^1_w} \leq \delta \|A^{(w)}f\|_{\ell^1_w}$ for all $f \in \mathcal{D}(A^{(w)})$. Hence, by Theorem 3.3.35, $G^{(w)} = A^{(w)} + B^{(w)}$ is the generator of a C_0-semigroup on ℓ^1_w. Moreover, since $B^{(w)}$ is a positive operator, it follows from Theorem 3.3.35 that the semigroup generated by $A^{(w)} + B^{(w)}$ is positive. From Theorem 5.2.7, we have that $\overline{A^{(w)} + B^{(w)}}$ is also the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$. However, since $A^{(w)} + B^{(w)}$ is a generator, it is a closed operator. It follows that $A^{(w)} + B^{(w)} = \overline{A^{(w)} + B^{(w)}}$ and so $A^{(w)} + B^{(w)}$ generates the substochastic semigroup $(S^{(w)}(t))_{t\geq 0}$.

We now use Theorem 3.3.36 to prove that the semigroup, $(S^{(w)}(t))_{t\geq 0}$, is analytic. We know from Lemma 5.2.1 that $A^{(w)}$ is the generator of a substochastic C_0-semigroup, $(T^{(w)}(t))_{t\geq 0}$, on ℓ^1_w. Also, for $\lambda = r + is$ with $r > 0$, $s \in \mathbb{R}\backslash\{0\}$,

$$\begin{aligned}\|R(\lambda, A^{(w)})f\|_{\ell^1_w} &= \sum_{n=1}^{\infty} w_n \frac{1}{|\lambda + a_n|}|f_n| = \sum_{n=1}^{\infty} w_n \frac{1}{|(r + a_n) + is|}|f_n| \\ &\leq \frac{1}{|s|}\sum_{n=1}^{\infty} w_n |f_n| = \frac{1}{|s|}\|f\|_{\ell^1_w}.\end{aligned}$$

It follows that

$$\|R(\lambda, A^{(w)})\| \leq \frac{1}{|s|}.$$

Hence, by Theorem 3.3.21, $(T^{(w)}(t))_{t\geq 0}$ is an analytic semigroup.

Also, since the semigroup $(S^{(w)}(t))_{t\geq 0}$ is positive, it follows from Lemma 3.3.12 that $A^{(w)} + B^{(w)}$ is resolvent positive. The analyticity of $(S^{(w)}(t))_{t\geq 0}$ then follows immediately from Theorem 3.3.36. □

Remark 5.4.2. Now consider the case where $w_n = n$ for all $n \in \mathbb{N}$. Suppose that (5.1.4) holds and that there exists $\lambda_0 \in (0, 1)$ such that $\lambda_j > \lambda_0$, $j = 2, 3, \ldots$. Then (5.4.1) holds with $w_n = n$ for all $n \in \mathbb{N}$ and $\delta = (1 - \lambda_0) \in (0, 1)$.

On the other hand, suppose that for some $j = 2, 3, \ldots$, (5.1.4) holds with $\lambda_j \leq 0$. Then

$$\sum_{n=1}^{j-1} n b_{n,j} = (1 - \lambda_j)j \geq j$$

and so in this case (5.4.1) does not hold for $w_n = n$ and $\delta \in (0, 1)$. Thus, we cannot use Theorem 5.4.1 when we work in the space $X_{[1]}$ and we consider a mass-conserving system (i.e. a system where (5.1.5) holds) or a mass-gain system.

Under the assumptions of Theorem 5.4.1, we also have the following, alternative, proof that $G^{(w)} = A^{(w)} + B^{(w)}$ is the generator of a substochastic, C_0-semigroup on ℓ^1_w.

Remark 5.4.3. Let the assumptions of Theorem 5.4.1 hold. The first step in the proof of Theorem 5.4.1 uses the Miyadera–Voigt Perturbation Theorem to show that $A^{(w)} + B^{(w)}$ is a generator. Using Theorem 5.2.7 we can shorten this part of the proof. Under the conditions of Theorem 5.4.1, we have that Assumption 5.2.2(i) holds. Hence, by Theorem 5.2.7, $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup. Also, $A^{(w)}$ is a closed operator and, from Lemma 5.2.6, we know that $B^{(w)}$ is $A^{(w)}$-bounded, with $A^{(w)}$-bound $\delta < 1$. It follows from Lemma 3.3.32 that $A^{(w)}+B^{(w)}$ is closed, and so $G^{(w)} = A^{(w)}+B^{(w)}$ is the generator of a substochastic, C_0-semigroup on ℓ^1_w.

We now give an important remark regarding Theorem 5.4.1. This remark explains that we can always find a space, ℓ^1_w, such that Theorem 5.4.1 can be applied.

Remark 5.4.4. It is always possible to choose a weight $(w_n)_{n=1}^{\infty}$ such that the conditions of Theorem 5.4.1 hold. For given fragmentation coefficients $(b_{n,j})_{n,j\in\mathbb{N}:n<j}$, we can take, for example, $w_1 = 1$ and then choose each w_n, $n \geq 2$, iteratively so that (5.4.1) is satisfied. Thus, for any fragmentation coefficients, we can always find a weighted ℓ^1 space such that $A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup.

Theorem 5.4.5. *For any fragmentation coefficients, $(b_{n,j})_{n,j\in\mathbb{N}:n<j}$, we can always find a weight, $(w_n)_{n=1}^{\infty}$, such that the conditions of Theorem 5.4.1 hold.*

Proof. As explained in Remark 5.4.4, we can choose w_n iteratively such that (5.4.1) is satisfied. $\square$

We now give a sufficient condition for the assumptions of Theorem 5.4.1 to hold.

Lemma 5.4.6. *Let* (5.1.4) *hold with $\lambda_j \in [0,1]$ for $j = 2, 3, \ldots$. Moreover, let $(w_n)_{n=1}^{\infty}$ be such that there exists $\delta \in (0,1)$ such that*

$$\delta \frac{w_{n+1}}{n+1} \geq \frac{w_n}{n} \qquad \textit{for all } n \in \mathbb{N}.$$

Then the assumptions of Theorem 5.4.1 hold.

Proof. We have

$$\frac{w_1}{1} \leq \delta \frac{w_2}{2} \leq \delta^2 \frac{w_3}{3} \leq \ldots .$$

Hence $\frac{w_n}{n} \leq \delta^{j-n}\frac{w_j}{j} \leq \delta\frac{w_j}{j}$ for all $j \geq 2$, $n = 1, \ldots, j-1$, since $\delta \in (0,1)$. It follows that

$$\sum_{n=1}^{j-1} w_n b_{n,j} = \sum_{n=1}^{j-1} \frac{w_n}{n} n b_{n,j} \leq \delta \frac{w_j}{j} \sum_{n=1}^{j-1} n b_{n,j} \leq \delta w_j$$

for $j = 2, 3, \ldots$, where we use (5.1.4) to obtain the last inequality. Since $\delta \in (0,1)$, the result then follows. □

Using Lemma 5.4.6 we can show that, in the case where mass is either lost or conserved, we can always choose the weight $(w_n)_{n=1}^{\infty}$, as in Theorem 5.4.5, to be at most exponentially growing.

Proposition 5.4.7. *Let* (5.1.4) *hold with* $\lambda_j \in [0,1]$ *for all* $j = 2, 3, \ldots$ *and let* $w_n = r^n$ *for some* $r > 2$. *Then the assumptions of Theorem 5.4.1 hold.*

Proof. Taking $\delta > 0$ we have, for all $n \in \mathbb{N}$,

$$\delta \frac{r^{n+1}}{n+1} - \frac{r^n}{n} = \frac{\delta n r^{n+1} - (n+1)r^n}{n(n+1)} = \frac{r^n((\delta r - 1)n - 1)}{n(n+1)} \geq 0$$

if $\delta \geq \frac{1}{r}\left(\frac{1}{n} + 1\right)$. Since $r > 2$ and $n \geq 1$, $\frac{1}{r}\left(\frac{1}{n} + 1\right) \leq \frac{2}{r} < 1$. Choosing $\delta = \frac{2}{r}$, the result then follows from Lemma 5.4.6. □

In [9], the case is considered where (5.1.5) holds. A particular weight of the form $w_n = n^p$, for $p \geq 1$ and all $n \in \mathbb{N}$, is examined and, under an equivalent assumption as (5.4.1), it is shown in [9, Theorem 2.1.3] that $A^{(w)} + B^{(w)}$ is a generator. However, the analyticity of the semigroup $(S^{(w)}(t))_{t\geq 0}$ under this condition is not mentioned. Moreover, simple examples of fragmentation coefficients are given in [9] for which this condition does not hold when $w_n = n^p$ for any $p \geq 1$. For example, in the appendix of [9] a fragmentation process is considered in which a cluster of mass n splits into two clusters with masses 1 and $n-1$. In this case we have

$$\begin{aligned} b_{1,2} = 2; \qquad b_{1,j} = b_{j-1,j} &= 1, \qquad j \geq 3; \\ b_{n,j} &= 0, \qquad 2 \leq n \leq j-2, \end{aligned} \tag{5.4.2}$$

It is mentioned in [9] that, in this case, the condition equivalent to (5.4.1) is not satisfied when $w_n = n^p$, for any $p \geq 1$. Moreover, in [9, Proposition A3] it is shown that if $w_n = n^p$ for $p \geq 1$ and $a_n = n$ for $n \geq 2$, then $A^{(w)} + B^{(w)}$ is not the generator of a C_0-semigroup and the semigroup that is generated by $\overline{A^{(w)} + B^{(w)}}$ is not analytic.

We now show that it is possible to construct a weight $(w_n)_{n=1}^{\infty}$ satisfying (5.4.1) when the fragmentation coefficients are given by (5.4.2).

Example 5.4.8. Consider the case where the fragmentation coefficients $b_{n,j}$ satisfy (5.4.2). We want to choose $(w_n)_{n=1}^{\infty}$ such that (5.4.1) is satisfied for some $\delta \in (0,1)$. For $n = 2$,

$$\sum_{k=1}^{n-1} w_k b_{k,n} = 2w_1,$$

and, for $n = 3, 4, \ldots$, we have

$$\sum_{k=1}^{n-1} w_k b_{k,n} = w_1 + w_{n-1}.$$

Thus we require

$$2w_1 \leq \delta w_2 \qquad \text{and} \qquad w_1 + w_{n-1} \leq \delta w_n \qquad \text{for } n = 3, 4, \ldots,$$

for some $\delta \in (0,1)$.

Take $w_1 = 1$ and $\delta = \frac{5}{8}$. Then we can choose $w_2 = 4 = 2^2$. We also want

$$\frac{8}{5}(1 + w_{n-1}) \leq w_n \qquad \text{for } n = 3, 4, \ldots. \tag{5.4.3}$$

If we take $w_3 = 8 = 2^3$ then (5.4.3) is satisfied for $n = 3$. Now assume that (5.4.3) holds, with $w_k = 2^k$, $k = 2, \ldots, n$, for some $n \geq 3$. Then

$$\frac{8}{5}(1 + 2^n) = \frac{8}{5}(1 + 2 \cdot 2^{n-1}) \leq 2\left(\frac{8}{5}(1 + 2^{n-1})\right) \leq 2 \cdot 2^n = 2^{n+1}.$$

Hence, by induction, (5.4.3) holds if we take $w_n = 2^n$ for $n \geq 2$. Thus if we consider ℓ_w^1 where w_n is given by

$$w_1 = 1, \quad w_n = 2^n \qquad \text{for } n = 2, 3, \ldots,$$

then (5.4.1) is satisfied. It follows, from Theorem 5.4.1, that $A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup on ℓ^1_w.

We can now obtain the following existence and uniqueness result.

Theorem 5.4.9. *Let the assumptions of Theorem 5.4.1 hold.*

(i) Then for all $\mathring{u} \in \ell^1_w$, $u(t) = S^{(w)}(t)\mathring{u}$ is the unique classical solution of (5.1.8)*. If $\mathring{u} \in (\ell^1_w)_+$, then this solution is non-negative.*

(ii) Moreover, if (5.1.5) *and Assumption 5.2.2(ii) hold, then the solution is mass conserving for $\mathring{u} \in (\ell^1_w)_+$.*

Proof. (i) This follows from the analyticity of $(S^{(w)}(t))_{t\geq 0}$, Proposition 3.4.4 and the fact that $(S^{(w)}(t))_{t\geq 0}$ is substochastic.

(ii) In this case ℓ^1_w is continuously embedded in $X_{[1]}$ and, from Proposition 5.2.8, $G = \overline{A+B}$ is the generator of the stochastic C_0-semigroup, $(S(t))_{t\geq 0}$, on $X_{[1]}$. Moreover, $A^{(w)} + B^{(w)} \subseteq A + B \subseteq G$. Hence, $(S(t))_{t\geq 0}$ and $(S^{(w)}(t))_{t\geq 0}$ must coincide on ℓ^1_w. The mass conservation then follows from the stochasticity of $(S(t))_{t\geq 0}$ □

We now use the Sobolev Tower construction as introduced in Section 3.4.2. Let $(S^{(w)}(t))_{t\geq 0}$ be an analytic semigroup, generated by $G^{(w)}$, on a fixed space $Y = \ell^1_w$ and let $\omega_0 \in \mathbb{R}$ be the growth bound of $(S^{(w)}(t))_{t\geq 0}$. Choose $\mu > \omega_0$ and consider the rescaled semigroup $(T(t))_{t\geq 0} = (e^{-\mu t} S^{(w)}(t))_{t\geq 0}$, with generator $H = G^{(w)} - \mu I$. As explained in Section 3.4.2, $(T(t))_{t\geq 0}$ has a negative growth bound. We use Y_n to denote the Sobolev space of order $n \in \mathbb{Z}$, associated with $(T(t))_{t\geq 0}$ and, as in Section 3.4.2, we denote by $(T_n(t))_{t\geq 0}$ and H_n, the corresponding analytic semigroup and its generator, respectively, on Y_n. Recall that, for $n \in \mathbb{N}$, $T_{-n}(t) = e^{-\mu t} S^{(w)}_{-n}(t)$, where $S^{(w)}_{-n}(t)$ is the unique, continuous extension of $S^{(w)}(t)$ from Y to Y_{-n}.

Theorem 5.4.10. *Let the conditions of Theorem 5.4.1 hold, and let $(S^{(w)}(t))_{t\geq 0}$ be the analytic, substochastic semigroup, generated by $G^{(w)} = A^{(w)} + B^{(w)}$, on $Y_0 = \ell^1_w$. Fix $n \in \mathbb{N}$. Then, for all $\mathring{u} \in Y_{-n}$,* (5.1.8) *has a unique solution $u \in C^1((0,\infty), \ell^1_w) \cap C([0,\infty), Y_{-n})$, given by $u(t) = S^{(w)}_{-n}(t)\mathring{u}$, $t \geq 0$. Moreover, if $\mathring{u} \in (Y_{-n})_+$, then this solution is non-negative.*

Proof. The function $u(t) = S^{(w)}_{-n}(t)\mathring{u} \in C^1((0,\infty), \ell^1_w) \cap C([0,\infty), Y_{-n})$ is the unique solution of (5.1.8) for all $\mathring{u} \in Y_{-n}$, from Theorem 3.4.13. The positivity result follows since $(S^{(w)}(t))_{t\geq 0}$, and so also $(S^{(w)}_{-n}(t))_{t\geq 0}$, is a substochastic semigroup. □

Remark 5.4.11. We make the following observations regarding Theorem 5.4.10.

(i) For any given fragmentation coefficients we can recursively choose a weight $w = (w_n)_{n=1}^{\infty}$ that satisfies the conditions of Theorem 5.4.1. Hence, we can always choose a weight such that Theorem 5.4.10 can be applied.

(ii) Suppose that, in addition to the assumptions of Theorem 5.4.10, we also have $X_{[1]} \subseteq Y_{-n}$ for some $n \in \mathbb{N}$. Then, in particular, we obtain a unique, non-negative solution of (5.1.8) for all $\mathring{u} \in \left(X_{[1]}\right)_+$. Moreover, as we now show, this solution is mass-conserving if (5.1.5) and Assumption 5.2.2(ii) both hold.

From Proposition 5.2.8 we have, in this case, that $G = \overline{A+B}$ is the generator of a stochastic C_0-semigroup, $(S(t))_{t\geq 0}$, on $X_{[1]}$. In addition, since $w_n \geq n$ for all $n \in \mathbb{N}$, we have that ℓ^1_w is continuously embedded in $X_{[1]}$. Also, H_{-n-1} is the unique extension of $H = G^{(w)} - \mu I$ to Y_{-n} and $G^{(w)} = A^{(w)} + B^{(w)} \subseteq A + B \subseteq \overline{A+B} = G$. This then implies that $H = G^{(w)} - \mu I \subseteq G - \mu I$, i.e. $G - \mu I$ is also an extension of H.

Since $\mathcal{D}(G - \mu I) \subseteq X_{[1]} \subseteq Y_{-n} = \mathcal{D}(H_{-n-1})$, we have $G - \mu I \subseteq H_{-n-1}$. Hence, for all $t \geq 0$, $e^{-\mu t}S(t)$ and $T_{-n-1}(t) = e^{-\mu t}S^{(w)}_{-n-1}(t)$ coincide on $X_{[1]}$, i.e. $S(t)$ and $S^{(w)}_{-n-1}(t)$ coincide on $X_{[1]}$.

Thus, by the stochasticity of $(S(t))_{t\geq 0}$, we obtain that, for all $\mathring{u} \in \left(X_{[1]}\right)_+$,

$$u(t) = S^{(w)}_{-n}(t)\mathring{u} = S^{(w)}_{-n-1}(t)\mathring{u} = S(t)\mathring{u}$$

is the unique, non-negative, mass-conserving solution of (5.1.8).

Let the Assumptions of Theorem 5.4.10 hold. We know that, for all $n \in \mathbb{N}$, $Y_n = (\mathcal{D}\big((A^{(w)} + B^{(w)} - \mu I)^n\big), \|\cdot\|_n)$, where

$$\|f\|_n = \|(A^{(w)} + B^{(w)} - \mu I)^n f\|_{\ell^1_w}.$$

It follows that

$$\begin{aligned} Y_1 &= \Big\{ f \in \mathcal{D}(A^{(w)} + B^{(w)} - \mu I) : \|(A^{(w)} + B^{(w)} - \mu I) f\|_{\ell^1_w} < \infty \Big\} \\ &= \mathcal{D}(A^{(w)} + B^{(w)} - \mu I) = \mathcal{D}(A^{(w)}). \end{aligned}$$

Similarly, for Y_2, we have

$$\begin{aligned} Y_2 &= \mathcal{D}\Big((A^{(w)} + B^{(w)} - \mu I)^2\Big) \\ &= \Big\{ f \in \mathcal{D}\Big(A^{(w)}\Big) : (A^{(w)} + B^{(w)} - \mu I) f \in \mathcal{D}\Big(A^{(w)}\Big) \Big\} \\ &= \Big\{ f \in \mathcal{D}\Big(A^{(w)}\Big) : \Big(A^{(w)} + B^{(w)}\Big) f \in \mathcal{D}\Big(A^{(w)}\Big) \Big\} \\ &= \mathcal{D}\Big(\Big(A^{(w)} + B^{(w)}\Big)^2\Big). \end{aligned}$$

Suppose $Y_{n-1} = \mathcal{D}\Big(\Big(A^{(w)} + B^{(w)}\Big)^{n-1}\Big)$ for some $n \geq 3$. Then

$$\begin{aligned} Y_n &= \mathcal{D}\Big((A^{(w)} + B^{(w)} - \mu I)^n\Big) \\ &= \Big\{ f \in \mathcal{D}\Big((A^{(w)} + B^{(w)} - \mu I)^{n-1}\Big) : \\ &\qquad\qquad (A^{(w)} + B^{(w)} - \mu I) f \in \mathcal{D}\Big((A^{(w)} + B^{(w)} - \mu I)^{n-1}\Big) \Big\} \\ &= \Big\{ f \in Y_{n-1} : \Big(A^{(w)} + B^{(w)}\Big) f \in Y_{n-1} \Big\} \\ &= \Big\{ f \in \mathcal{D}\Big(\Big(A^{(w)} + B^{(w)}\Big)^{n-1}\Big) : \Big(A^{(w)} + B^{(w)}\Big) f \in \mathcal{D}\Big(\Big(A^{(w)} + B^{(w)}\Big)^{n-1}\Big) \Big\} \\ &= \mathcal{D}\Big(\Big(A^{(w)} + B^{(w)}\Big)^n\Big). \end{aligned}$$

It follows, by induction, that $Y_n = \mathcal{D}\Big((A^{(w)} + B^{(w)})^n\Big)$ for all $n \in \mathbb{N}$.

Fix $t > 0$, $n \in \mathbb{N}$ and take $\mathring{u} \in Y_{-n}$. From Theorem 3.4.9, we have that $T_{-n}(t)\mathring{u} \in Y_m$ for all $m \geq -n$. It follows that $S_{-n}(t)\mathring{u} \in Y_m$ for all $m \geq -n$. Hence, from the characterisation of Y_m, for $m \in \mathbb{N}$, we have that the solution in Theorem 5.4.10 must be in $Y_m = \mathcal{D}\Big((A^{(w)} + B^{(w)})^m\Big)$ for all $m \in \mathbb{N}$.

We have shown that for any given fragmentation coefficients we can find a sequence $(w_n)_{n=1}^{\infty}$ such that $A^{(w)}+B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup on ℓ^1_w. Now suppose we are in the situation where we have a fixed space, ℓ^1_w, for which we know that $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a C_0-semigroup. Can we deduce whether the semigroup that is generated on this space is analytic? This is what we now examine.

We now consider the case where (5.1.5) holds, i.e. we consider the case where we have a mass-conserving system. Let Assumption 5.2.2(i) hold. Then, from Theorem 5.2.7, $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ generates a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. As in [9] we can write $R(\lambda, G^{(w)})$, where $\lambda > 0$, as an infinite dimensional matrix $(r_{k,n}(\lambda))_{1\leq k,n<\infty}$, where $r_{n,n}(\lambda) = \frac{1}{\lambda+a_n}$ for $n \geq 1$ and $r_{k,n}(\lambda) = 0$ for $k > n$. For any $k \in \mathbb{N}$ and $n \geq k+1$ we have, from [9, Lemma 2.2],

$$r_{k,n}(\lambda) = \frac{a_n}{\lambda + a_n} \sum_{j=k}^{n-1} b_{j,n} r_{k,j}(\lambda). \tag{5.4.4}$$

We note that, as remarked in [9], $r_{k,n}(\lambda)$ can be extended to an analytic function on $\{\lambda \in \mathbb{C} : \mathrm{Re}(\lambda) > 0\}$. The following lemma is a generalisation of [9, Lemma 3.1]. It will be useful in determining conditions under which the semigroup $(S^{(w)}(t))_{t\geq 0}$ is analytic.

Lemma 5.4.12. *Let* (5.1.4) *hold with* $\lambda_j \in [0, 1)$ *for* $j = 2, 3, \ldots$, *let Assumption 5.2.2(i) hold and let* $(S^{(w)}(t))_{t\geq 0}$ *be the substochastic* C_0*-semigroup on* ℓ_w^1, *generated by* $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$. *Moreover, assume that there is a positive integer* $n_0 \geq 2$, *a constant* $c > 0$ *and a sequence,* $(\psi_k)_{k=1}^{\infty}$, *with* $\psi_k > 0$ *for all* $k \in \mathbb{N}$, *such that*

$$\psi_k b_{k,n} \leq c \sum_{j=1}^{k} j b_{j,n}, \quad \text{for all } n, k \in \mathbb{N} \text{ such that } n \geq n_0 \text{ and } n > k. \tag{5.4.5}$$

Then, for all $n, k \in \mathbb{N}$ *such that* $n > k$,

$$|r_{k,n}(\lambda)| \leq \frac{cn}{|\lambda + a_k|\psi_k}. \tag{5.4.6}$$

Proof. We first use an inductive argument with respect to n to show that (5.4.6) holds when $n_0 = 2$. Fix $k \in \mathbb{N}$ and let $\lambda \in \mathbb{C}$ be such that $\mathrm{Re}(\lambda) > 0$. Then $\lambda \in \rho(G^{(w)})$. We have $|r_{k,k}(\lambda)| = \frac{1}{|\lambda+a_k|}$ and $a_n \leq |\lambda + a_n|$. Thus we have

$$|r_{k,k+1}(\lambda)| = \left| \frac{a_{k+1}}{\lambda + a_{k+1}} \sum_{j=k}^{k} b_{j,k+1} r_{k,j}(\lambda) \right| = \frac{a_{k+1}}{|\lambda + a_{k+1}|} b_{k,k+1} |r_{k,k}(\lambda)|$$

$$
\begin{aligned}
&\leq b_{k,k+1}|r_{k,k}(\lambda)| = \frac{1}{|\lambda+a_k|}b_{k,k+1} \\
&\leq \frac{1}{|\lambda+a_k|}\frac{c}{\psi_k}\sum_{j=1}^{k} jb_{j,k+1} \quad \text{(from (5.4.5) since } k+1 \geq 2) \\
&\leq \frac{c(k+1)}{|\lambda+a_k|\psi_k}.
\end{aligned}
$$

Hence (5.4.6) is satisfied for $n = k+1$.

Now fix $n \in \mathbb{N}$ such that $n > k$ and assume

$$
|r_{k,j}(\lambda)| \leq \frac{cj}{|\lambda+a_k|\psi_k} \quad \text{for all } j \in \{k+1,\ldots,n-1\}.
$$

We want to show that this inequality also holds when $j = n$. We have

$$
\begin{aligned}
|r_{k,n}(\lambda)| &= \left|\frac{a_n}{\lambda+a_n}\sum_{j=k}^{n-1} b_{j,n}r_{k,j}(\lambda)\right| \\
&\leq \frac{a_n}{|\lambda+a_n|}b_{k,n}|r_{k,k}(\lambda)| + \frac{a_n}{|\lambda+a_n|}\sum_{j=k+1}^{n-1} b_{j,n}|r_{k,j}(\lambda)| \\
&\leq b_{k,n}|r_{k,k}(\lambda)| + \sum_{j=k+1}^{n-1} b_{j,n}|r_{k,j}(\lambda)| \\
&\leq b_{k,n}\frac{1}{|\lambda+a_k|} + \sum_{j=k+1}^{n-1} b_{j,n}\frac{cj}{|\lambda+a_k|\psi_k} \\
&= \frac{1}{|\lambda+a_k|\psi_k}\left(b_{k,n}\psi_k + c\sum_{j=k+1}^{n-1} jb_{j,n}\right) \\
&\leq \frac{1}{|\lambda+a_k|\psi_k}\left(c\sum_{j=1}^{k} jb_{j,n} + c\sum_{j=k+1}^{n-1} jb_{j,n}\right) \\
&= \frac{1}{|\lambda+a_k|\psi_k}\left(c\sum_{j=1}^{n-1} jb_{n,j}\right) \\
&\leq \frac{cn}{|\lambda+a_k|\psi_k},
\end{aligned}
$$

where the second last inequality holds from (5.4.5) since $n \geq k+1 \geq 2 = n_0$. Hence, by induction, we have that (5.4.6) holds in the case $n_0 = 2$.

Now suppose (5.4.5) holds for some $n_0 > 2$. We show that, in this case, there exists $\tilde{c} > 0$ such that, for all $n \geq 2$,

$$\psi_k b_{k,n} \leq \tilde{c} \sum_{j=1}^{k} j b_{j,n}, \quad 1 \leq k \leq n-1.$$

Since $b_{j,n} \geq 0$ for all $j, n \in \mathbb{N}$, the right-hand side of (5.4.5) is equal to zero for some $n \in \mathbb{N}$ if and only if $b_{j,n} = 0$ for all $1 \leq j \leq k$. In this case (5.4.5) is satisfied for all $c > 0$ and $n, k \in \mathbb{N}$ such that $n \geq n_0$ and $k < n$. Otherwise, for $2 \leq n \leq n_0 - 1$ and $k < n$ we have

$$\psi_k b_{k,n} = \frac{\psi_k b_{k,n}}{\sum\limits_{j=1}^{k} j b_{j,n}} \sum_{j=1}^{k} j b_{j,n} \leq \max_{(k,n)\in\Omega} \left\{ \frac{\psi_k b_{k,n}}{\sum\limits_{j=1}^{k} j b_{j,n}} \right\} \sum_{j=1}^{k} j b_{j,n} = d \sum_{j=1}^{k} j b_{j,n},$$

where

$$\Omega = \{(k,n) : 1 \leq k \leq n-1, 2 \leq n \leq n_0 - 1, \sum_{j=1}^{k} j b_{j,n} \neq 0\}$$

and

$$d = \max_{(k,n)\in\Omega} \left\{ \frac{\psi_k b_{k,n}}{\sum\limits_{j=1}^{k} j b_{j,n}} \right\}.$$

Choose $\tilde{c} = \max\{c, d\}$. Then, for all $n \geq 2$,

$$\psi_k b_{k,n} \leq \tilde{c} \sum_{j=1}^{k} j b_{j,n}, \qquad 1 \leq k \leq n-1.$$

The result then follows from the case $n_0 = 2$. $\square$

The following theorem is a routine extension of [9, Theorem 3.1] and gives conditions under which it can be concluded that the semigroup, $(S^{(w)}(t))_{t\geq 0}$, is analytic.

Theorem 5.4.13. *Let (5.1.4) hold with $\lambda_j \in [0,1)$ for $j = 2, 3, \ldots$, let Assumption 5.2.2(i) hold, and let $(S^{(w)}(t))_{t\geq 0}$ be the substochastic C_0-semigroup on ℓ^1_w, generated by $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$. Moreover, assume that there is a positive in-*

teger $n_0 \geq 2$, a constant $c > 0$ and a sequence, $(\psi_k)_{k=1}^{\infty}$, with $\psi_k > 0$ for all $k \in \mathbb{N}$, such that (5.4.5) holds for all $n, k \in \mathbb{N}$ such that $n \geq n_0$ and $k < n$. In addition, let $(\psi_k)_{k=1}^{\infty}$ be such that, for all $n \in \mathbb{N}$,

$$\sum_{k=1}^{n-1} \frac{w_k}{\psi_k} \leq \tilde{C}\frac{w_n}{n} \tag{5.4.7}$$

for some constant $\tilde{C} > 0$. Then the semigroup $(S^{(w)}(t))_{t\geq 0}$ is analytic.

Proof. Let $\mathrm{Re}(\lambda) > 0$. Then, for $f \in \ell_w^1$,

$$\begin{aligned}
&\|R(\lambda, S^{(w)})f\|_{\ell_w^1} \\
&= \sum_{k=1}^{\infty} w_k \left|R(\lambda, S^{(w)})f\right|_k \leq \sum_{k=1}^{\infty} w_k \sum_{n=k}^{\infty} |r_{k,n}(\lambda)||f_n| \\
&= \sum_{k=1}^{\infty} w_k |r_{k,k}(\lambda)||f_k| + \sum_{k=1}^{\infty} w_k \sum_{n=k+1}^{\infty} |r_{k,n}(\lambda)||f_n| \\
&\leq \frac{1}{|\lambda|}\|f\|_{\ell_w^1} + \sum_{n=2}^{\infty}\sum_{k=1}^{n-1} w_k |r_{k,n}(\lambda)||f_n| \\
&= \frac{1}{|\lambda|}\|f\|_{\ell_w^1} + \sum_{n=2}^{\infty} w_n|f_n| \left(\frac{1}{w_n}\sum_{k=1}^{n-1} w_k|r_{k,n}(\lambda)|\right) \\
&\leq \frac{1}{|\lambda|}\|f\|_{\ell_w^1} + \sum_{n=2}^{\infty} w_n|f_n| \left(\frac{1}{w_n}\sum_{k=1}^{n-1} w_k \frac{cn}{|\lambda + a_k|\psi_k}\right) \\
&\leq \frac{1}{|\lambda|}\|f\|_{\ell_w^1} + \sqrt{2}c\sum_{n=2}^{\infty} w_n|f_n| \left(n\frac{1}{w_n}\sum_{k=1}^{n-1} w_k \frac{1}{(|\lambda| + a_k)\psi_k}\right) \\
&\leq \frac{1}{|\lambda|}\|f\|_{\ell_w^1} + \frac{\sqrt{2}c}{|\lambda|}\sum_{n=2}^{\infty} w_n|f_n| \left(n\frac{1}{w_n}\sum_{k=1}^{n-1} \frac{w_k}{\psi_k}\right),
\end{aligned}$$

since $|\lambda + a_k| \geq \frac{|\lambda| + a_k}{\sqrt{2}}$ for all λ with $\mathrm{Re}(\lambda) > 0$. If (5.4.7) holds, then

$$\|R(\lambda, S^{(w)})f\|_{\ell_w^1} \leq \frac{1}{|\lambda|}\|f\|_{\ell_w^1} + \frac{\sqrt{2}c\tilde{C}}{|\lambda|}\sum_{n=2}^{\infty} w_n|f_n| \leq \frac{M}{|\lambda|}\|f\|_{\ell_w^1},$$

where $M = 1 + \sqrt{2}c\tilde{C}$. Thus from Theorem 3.3.21 we can conclude that the semigroup $(S^{(w)}(t))_{t\geq 0}$ is analytic. $\square$

5.5 Asymptotic Behaviour of Solutions

In this section we examine the asymptotic behaviour of the solutions obtained in Section 5.2.2 and Section 5.4. In the case where mass is conserved, i.e. when (5.1.5) holds, the asymptotic behaviour of solutions of (5.1.8) is examined in [8, 15] when the weight is of the form $w_n = n^p$ for $p \geq 1$, $n \in \mathbb{N}$. In particular, the solution in this case is shown to converge to a system that consists entirely of monomers if and only if $a_n > 0$ for all $n \geq 2$. Moreover, in [15], for $w_n = n^p$ for $p > 1$, a result is given in which it is shown that this decay occurs at an exponential rate. However, the rate of this decay is not quantified in [15].

In this section we work with a more general weight and obtain results regarding the asymptotic behaviour of solutions both in the mass-loss case and in the mass-conserving case. Moreover, we use the results regarding analytic semigroups, given in Section 5.4, to obtain results regarding the exponential decay of solutions. In addition, we quantify the rate of decay.

Let Assumption 5.2.2(i) hold so that, from Theorem 5.2.7, $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$ on ℓ_w^1. We begin by obtaining a matrix representation of $(S^{(w)}(t))_{t\geq 0}$. For $n \in \mathbb{N}$, let $e_n \in \ell_w^1$ be given by

$$(e_n)_k = \begin{cases} 1 & \text{if } n = k, \\ 0 & \text{otherwise.} \end{cases} \tag{5.5.1}$$

Note that $(e_n)_{n=1}^{\infty}$ is a Schauder basis for ℓ_w^1. We now define an infinite matrix, $\mathbb{S}(t) = (s_{m,n}(t))_{m,n\in\mathbb{N}}$, by

$$s_{m,n}(t) = (S^{(w)}(t)e_n)_m$$

for all $m, n \in \mathbb{N}$. Note that, since $(S^{(w)}(t))_{t\geq 0}$ is positive, $s_{m,n}(t) \geq 0$ for all $m, n \in \mathbb{N}$.

Now, for $f \in \ell_w^1$, we have $f = \sum\limits_{n=1}^{\infty} f_n e_n$. Hence, for each $t \geq 0$ and $m \in \mathbb{N}$, using the linearity and continuity of $S^{(w)}(t)$, we have,

$$(S^{(w)}(t)f)_m = \left(\sum_{n=1}^{\infty} f_n S^{(w)}(t)e_n\right)_m = \sum_{n=1}^{\infty} f_n s_{m,n}(t) = (\mathbb{S}(t)f)_m,$$

i.e. $(S^{(w)}(t))_{t\geq 0}$ can be represented by the matrix $\mathbb{S}(t) = (s_{m,n}(t))_{m,n\in\mathbb{N}}$.

From Theorem 5.2.10, for fixed $n \in \mathbb{N}$, $u(t) = S^{(w)}(t)e_n$ is the unique, classical solution of (5.1.8) with $\mathring{u} = e_n$. Now, for fixed $n \in \mathbb{N}$, let $(\tilde{s}_{1,n}(t), \ldots, \tilde{s}_{n,n}(t))$ be the unique solution of the n-dimensional system

$$\tilde{s}'_{m,n}(t) = -a_m\tilde{s}_{m,n}(t) + \sum_{j=m+1}^{n} a_j b_{m,j}\tilde{s}_{j,n}(t), \quad t > 0, \qquad m = 1, 2, \ldots, n, \tag{5.5.2}$$

with

$$\tilde{s}_{n,n}(0) = 1 \qquad \text{and} \qquad \tilde{s}_{m,n}(0) = 0 \qquad \text{for } m < n,$$

where $\sum\limits_{j=m+1}^{n} a_j b_{m,j}\tilde{s}_{j,n}(t) = 0$ for $m = n$. Note that the n^{th} equation is given by

$$\tilde{s}'_{n,n}(t) = -a_n\tilde{s}_{n,n}(t), \quad t > 0, \qquad \tilde{s}_{n,n}(0) = 1, \tag{5.5.3}$$

and so $\tilde{s}_{n,n}(t) = e^{-a_n t}$. It is clear that $(\tilde{s}_{1,n}(t), \ldots, e^{-a_n t}, 0, \ldots)$ is a solution of

$$\tilde{s}'_{m,n}(t) = -a_m\tilde{s}_{m,n}(t) + \sum_{j=m+1}^{\infty} a_j b_{m,j}\tilde{s}_{j,n}(t), \quad t > 0, \qquad m = 1, 2, \ldots, \tag{5.5.4}$$

with

$$\tilde{s}_{n,n}(0) = 1 \qquad \text{and} \qquad \tilde{s}_{m,n}(0) = 0 \qquad \text{for } m \neq n.$$

Moreover, $(\tilde{s}_{1,1}(t), \ldots, e^{-a_n t}, 0, \ldots) \in \mathcal{D}(A^{(w)}) \subseteq \mathcal{D}(G^{(w)})$ and it follows that $(\tilde{s}_{1,n}(t), \ldots, e^{-a_n t}, 0, \ldots)$ is a solution of (5.1.8), with $\mathring{u} = e_n$. Hence, by the uniqueness of solutions to (5.1.8),

$$u(t) = S^{(w)}e_n = (s_{1,n}(t), \ldots, s_{n,n}(t), s_{n+1,n}(t), \ldots) = (\tilde{s}_{1,n}(t), \ldots, e^{-a_n t}, 0, \ldots).$$

Since n was arbitrary, we have for all $t \geq 0$,

$$S^{(w)}(t) = \left[\begin{array}{c|ccc} e^{-a_1 t} & s_{1,2}(t) & s_{1,3}(t) & \cdots \\ \hline 0 & e^{-a_2 t} & s_{2,3}(t) & \cdots \\ 0 & 0 & e^{-a_3 t} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{array}\right] = \begin{bmatrix} e^{-a_1 t} & S^{(w)}_{(12)}(t) \\ \mathbf{0} & S^{(w)}_{(22)}(t) \end{bmatrix}, \tag{5.5.5}$$

where $\mathbf{0}$ is an infinite column vector consisting entirely of zeros, $S^{(w)}_{(12)}(t)$ is a non-negative infinite row vector and $S^{(w)}_{(22)}(t)$ is an infinite-dimensional, non-negative, upper triangular matrix.

In particular, we note that the entries $(s_{m,n}(t))_{m,n\in\mathbb{N}}$ are independent of the weight $w = (w_n)_{n=1}^{\infty}$ and so, for two weights w and v satisfying Assumption 5.2.2(i), the semigroups $(S^{(w)}(t))_{t\geq 0}$ and $(S^{(v)}(t))_{t\geq 0}$ coincide on the intersection of their domains. Moreover, Banasiak obtains the infinite matrix representation for the semigroup $(S(t))_{t\geq 0}$ in [8, Equation (10) and Lemma 1], where the mass conservation case is considered in the space $X_{[1]}$. In [8], a recursive formula is found for $s_{m,n}(t)$, $m < n$. We omit this here since it is not required in the results that follow.

We now consider the case where $a_n > 0$ for all $n \in \mathbb{N}$. Note that, since $a_1 > 0$, this indicates that we have a fragmentation system where mass is being lost. We obtain the following theorem which tells us that if $a_n > 0$ for all $n \in \mathbb{N}$, then the solution of the fragmentation system converges to zero as $t \to \infty$. Moreover, if $(a_n)_{n=1}^{\infty}$ is bounded below by a positive constant and $\delta < 1$ in (5.2.1), then we can obtain an exponential rate of decay for the solutions. For all $N \in \mathbb{N}$, we define $P_N : \ell^1_w \to \ell^1_w$, to be

$$P_N f = (f_1, f_2, \ldots, f_N, 0, \ldots).$$

Theorem 5.5.1. *Let Assumption 5.2.2(i) hold.*

(i) *We have*

$$\lim_{t\to\infty} \|S^{(w)}(t)\mathring{u}\|_{\ell^1_w} = 0 \tag{5.5.6}$$

for all $\mathring{u} \in \ell^1_w$ if and only if $a_n > 0$ for all $n \in \mathbb{N}$.

(ii) *Choose the weight w such that Assumption 5.2.2(i) holds for $\delta \in (0,1)$ and let $a_0 = \inf_{n\in\mathbb{N}} a_n$. We have*

$$\|S^{(w)}(t)\| \leq e^{-(1-\delta)a_0 t}, \tag{5.5.7}$$

and hence, if $a_0 > 0$ and $\alpha \in [0, (1-\delta)a_0)$,

$$\lim_{t\to\infty} e^{\alpha t}\|S^{(w)}(t)\mathring{u}\|_{\ell^1_w} = 0 \qquad \text{for every } \mathring{u} \in \ell^1_w. \tag{5.5.8}$$

If $\alpha > a_0$, then (5.5.8) *does not hold. In particular, if $a_0 = 0$, then* (5.5.8) *does not hold for any $\alpha > 0$.*

Proof. (i) First assume that $a_n > 0$ for all $n \in \mathbb{N}$. Let $\mathring{u} \in \ell_w^1$. From the discussion leading to (5.5.5), we have that for each $n \in \mathbb{N}$, $(S^{(w)}(t)e_n)_m = s_{m,n}(t) = 0$ for $m > n$. Moreover, we have that $(S^{(w)}(t)e_n)_m = s_{m,n}(t) \geq 0$ solves (5.5.2) for $m = 1, \ldots, n$. Since $a_k > 0$ for all $k \in \mathbb{N}$, the matrix associated with the finite-dimensional system, (5.5.2), has purely negative eigenvalues. It follows that $(S^{(w)}(t)e_n)_m = s_{m,n}(t) \to 0$ as $t \to \infty$ for $m = 1, \ldots, n$.

For all $N \in \mathbb{N}$, we have $P_N\mathring{u} = \sum_{n=1}^{N} \mathring{u}_n e_n$ and so

$$
\begin{aligned}
\|S^{(w)}(t)P_N\mathring{u}\|_{\ell_w^1} &= \left\| \sum_{n=1}^{N} \mathring{u}_n S^{(w)}(t)e_n \right\|_{\ell_w^1} \leq \sum_{n=1}^{N} |\mathring{u}_n| \|S^{(w)}(t)e_n\|_{\ell_w^1} \\
&= \sum_{n=1}^{N} |\mathring{u}_n| \sum_{m=1}^{n} w_m s_{m,n}(t) \leq \sum_{n=1}^{N} |\mathring{u}_n| \sum_{m=1}^{N} w_m s_{m,n}(t) \\
&\to 0 \qquad \text{as } t \to \infty.
\end{aligned}
$$

Let $\varepsilon > 0$ and choose $N \in \mathbb{N}$ such that

$$
\|\mathring{u} - P_N\mathring{u}\|_{\ell_w^1} < \frac{\varepsilon}{2}.
$$

We also take $t_0 > 0$ such that

$$
\|S^{(w)}(t)P_N\mathring{u}\|_{\ell_w^1} < \frac{\varepsilon}{2} \qquad \text{for all } t \geq t_0.
$$

Then

$$
\begin{aligned}
\|S^{(w)}(t)\mathring{u}\|_{\ell_w^1} &= \|S^{(w)}(t)(\mathring{u} - P_N\mathring{u}) + S^{(w)}(t)P_N\mathring{u}\|_{\ell_w^1} \\
&\leq \|S^{(w)}(t)(\mathring{u} - P_N\mathring{u})\|_{\ell_w^1} + \|S^{(w)}(t)P_N\mathring{u}\|_{\ell_w^1} \\
&\leq \|\mathring{u} - P_N\mathring{u}\|_{\ell_w^1} + \|S^{(w)}(t)P_N\mathring{u}\|_{\ell_w^1} \\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \qquad \text{for all } t \geq t_0.
\end{aligned}
$$

Since we can choose any $\varepsilon > 0$, it follows that (5.5.6) holds.

On the other hand, let $a_N = 0$ for some $N \in \mathbb{N}$. Then for $\mathring{u} = e_N$, we have that the unique solution of (5.1.8) is given by $u(t) = S^{(w)}(t)e_N = (s_{m,N}(t))_{m=1}^{\infty}$. Since $s_{N,N}(t) = e^{-a_N t} = 1$, it is clear that $u(t) \nrightarrow 0$ as $t \to \infty$.

(ii) Now let $\mathring{u} \in (\ell_w^1)_+$. Since $\delta \in (0,1)$ in (5.2.1), we know, from Theorem 5.4.1, that $G^{(w)} = A^{(w)} + B^{(w)}$ generates the substochastic semigroup $(S^{(w)}(t))_{t\geq 0}$ and, moreover, $(S^{(w)}(t))_{t\geq 0}$ is analytic. Hence $u(t) = S^{(w)}(t)\mathring{u}$ is the unique, non-negative solution of (5.1.8). Let $t > 0$. From Lemma 5.2.6, $\|B^{(w)}f\|_{\ell_w^1} \leq \delta\|A^{(w)}f\|_{\ell_w^1}$ for all $f \in \mathcal{D}(A^{(w)})$. Consequently, since the operators $-A^{(w)}$ and $B^{(w)}$ are positive, we have

$$\phi_{\ell_w^1}(B^{(w)}u(t)) \leq -\delta\phi_{\ell_w^1}(A^{(w)}u(t)).$$

Now

$$\begin{aligned}
\frac{d}{dt}\phi_{\ell_w^1}(u(t)) = \phi_{\ell_w^1}(u'(t)) &= \phi_{\ell_w^1}\left(A^{(w)}u(t) + B^{(w)}u(t)\right) \\
&= \phi_{\ell_w^1}\left(A^{(w)}u(t)\right) + \phi_{\ell_w^1}\left(B^{(w)}u(t)\right) \\
&\leq \phi_{\ell_w^1}\left(A^{(w)}u(t)\right) - \delta\phi_{\ell_w^1}\left(A^{(w)}u(t)\right) \\
&= -(1-\delta)\sum_{n=1}^{\infty} w_n a_n u_n(t) \\
&\leq -(1-\delta)a_0\sum_{n=1}^{\infty} w_n u_n(t) \\
&= -(1-\delta)a_0\phi_{\ell_w^1}(u(t)).
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\frac{\frac{d}{dt}\phi_{\ell_w^1}(u(t))}{\phi_{\ell_w^1}(u(t))} \leq -(1-\delta)a_0 \\
&\implies \ln\phi_{\ell_w^1}(u(t)) - \ln\phi_{\ell_w^1}(u(0)) \leq -(1-\delta)a_0 t \\
&\implies \phi_{\ell_w^1}(u(t)) \leq \phi_{\ell_w^1}(u(0))e^{-(1-\delta)a_0 t} \\
&\implies \|S^{(w)}(t)\mathring{u}\|_{\ell_w^1} \leq e^{-(1-\delta)a_0 t}\|\mathring{u}\|_{\ell_w^1}.
\end{aligned}$$

Hence (5.5.7) follows from the positivity of $(S^{(w)}(t))_{t\geq 0}$ and [12, Proposition 2.67].

On the other hand, if we choose $\alpha > a_0$, then there exists $N \in \mathbb{N}$ such that $a_N < \alpha$. Take $\mathring{u} = e_N$ so that $(S^{(w)}(t)e_N)_N = e^{-a_N t} > e^{-\alpha t}$. It follows that

$$e^{\alpha t}\|S^{(w)}(t)e_N\|_{\ell^1_w} \geq e^{\alpha t}w_N|(S^{(w)}(t)e_N)_N| > e^{\alpha t}w_N e^{-\alpha t} = w_N.$$

Hence (5.5.8) cannot hold for any $\alpha > a_0$. □

Remark 5.5.2. Let the assumptions of Theorem 5.5.1 hold. It is clear that $u(t) = 0$ is an equilibrium solution of (5.1.8). Moreover, if $a_n > 0$ for all $n \in \mathbb{N}$, then from (5.5.6) we can deduce that this is the only equilibrium solution of the system and, moreover, 0 is a global attractor.

On the other hand, if $a_n = 0$ for at least one $n \in \mathbb{N}$, then we have, from Theorem 5.5.1, that the equilibrium point $u(t) = 0$ is not a global attractor.

We now consider the case where (5.1.5) holds, i.e. we consider the case where we have a mass-conserving fragmentation system. We aim to show that in this case, as $t \to \infty$, the fragmentation system in ℓ^1_w converges to a system consisting entirely of monomers. We note here that in this mass-conserving case, from Proposition 5.2.8, the semigroup $(S(t))_{t\geq 0}$ on the space $X_{[1]}$ is stochastic. As in previous sections, the norm on the space $X_{[1]}$ is denoted by $\|\cdot\|_{[1]}$ and the linear extension of this norm from $\left(X_{[1]}\right)_+$ to $X_{[1]}$ is denoted by M_1. We also define the space $Y^{(w)}$ and its norm $\|\cdot\|_{Y^{(w)}}$ by

$$Y^{(w)} = \left\{\tilde{f} = (f_n)_{n=2}^{\infty} : f = (f_n)_{n=1}^{\infty} \in \ell^1_w\right\} \qquad \text{and} \qquad \|f\|_{Y^{(w)}} = \sum_{n=2}^{\infty} w_n|f_n|,$$

respectively. Note that

$$Y^{(w)} = \ell^1_{\tilde{w}}, \qquad \text{with } \tilde{w}_n = w_{n+1} \qquad \text{for } n \in \mathbb{N}.$$

Moreover, we define the operator $J : Y^{(w)} \to \ell^1_w$ by

$$Jf = (0, f_2, f_3, \ldots) \qquad \text{for all } f \in \ell^1_w.$$

Lemma 5.5.3. *Let $\alpha \geq 0$ and $f \in \ell_w^1$ be fixed. Let (5.1.5) hold. Moreover, let Assumption 5.2.2(i) hold with $w_1 \geq 1$. Define $\tilde{f} := (f_n)_{n=2}^{\infty}$. Then*

$$\|S_{(22)}^{(w)}(t)\tilde{f}\|_{Y^{(w)}} \leq \|S^{(w)}(t)f - M_1(f)e_1\|_{\ell_w^1} \leq (w_1+1)\|S_{(22)}^{(w)}(t)\tilde{f}\|_{Y^{(w)}}, \quad (5.5.9)$$

where $(S_{(22)}^{(w)}(t))_{t\geq 0}$ is as in (5.5.5).

Proof. From (5.5.5) we have that

$$S^{(w)}(t)f = \left(f_1 + S_{(12)}^{(w)}(t)\tilde{f}\right)e_1 + JS_{(22)}^{(w)}(t)\tilde{f}.$$

From this we deduce that

$$\|S^{(w)}(t)f - M_1(f)e_1\|_{\ell_w^1} = w_1\left|f_1 + S_{(12)}^{(w)}(t)\tilde{f} - M_1(f)\right| + \left\|S_{(22)}^{(w)}(t)\tilde{f}\right\|_{Y^{(w)}} \quad (5.5.10)$$

and so

$$\|S^{(w)}(t)f - M_1(f)e_1\|_{\ell_w^1} \geq \|S_{(22)}^{(w)}(t)\tilde{f}\|_{Y^{(w)}}.$$

On the other hand, from Proposition 5.2.8 we have that the semigroup $(S(t))_{t\geq 0}$ is stochastic. Note also that, from Proposition 5.2.4, $w_n \geq n$ for all $n \in \mathbb{N}$ and so ℓ_w^1 is continuously embedded in $X_{[1]}$. Hence, using Proposition 3.3.16(i) and the fact that $S^{(w)}(t)f$ and $S(t)f$ coincide for $t \geq 0$, $f \in \ell_w^1 \cap X_{[1]} = \ell_w^1$, we have that $M_1(S^{(w)}(t)f) = M_1(S(t)f) = M_1(f)$ and so

$$\begin{aligned}
\left|f_1 + S_{(12)}^{(w)}(t)\tilde{f} - M_1(f)\right| &= \left|M_1(f) - M_1\left(\left(f_1 + S_{(12)}^{(w)}(t)\tilde{f}\right)e_1\right)\right| \\
&= \left|M_1\left(S^{(w)}(t)f\right) - M_1\left(\left(f_1 + S_{(12)}^{(w)}(t)\tilde{f}\right)e_1\right)\right| \\
&= \left|M_1\left(S^{(w)}(t)f - \left(f_1 + S_{(12)}^{(w)}(t)\tilde{f}\right)e_1\right)\right| \\
&\leq M_1\left(\left|S^{(w)}(t)f - \left(f_1 + S_{(12)}^{(w)}(t)\tilde{f}\right)e_1\right|\right) \\
&\leq \phi_{\ell_w^1}\left(\left|S^{(w)}(t)f - \left(f_1 + S_{(12)}^{(w)}(t)\tilde{f}\right)e_1\right|\right)
\end{aligned}$$

$$
\begin{aligned}
&= \phi_{\ell^1_w}\left(\left|JS^{(w)}_{(22)}(t)\tilde{f}\right|\right) \\
&= \|S^{(w)}_{(22)}(t)\tilde{f}\|_{Y^{(w)}}.
\end{aligned}
$$

The relation (5.5.9) then follows from (5.5.10). □

We can now prove the following theorem. The first part of this theorem tells us that if (5.1.5) holds and the fragmentation rates a_n are positive for all $n \geq 2$, then the solution of the fragmentation system converges to a state in which we have only monomers. Moreover, as we intuitively expect, since the mass of a monomer is one and we now assume that mass is conserved during fragmentation, we show that the number of monomers as $t \to \infty$ converges to the value of the total mass in the system.

In the second part of this theorem we choose w such that (5.4.1) holds. This allows us to obtain a result which tells us that the solution converges exponentially to the monomer state. In [15, Section 4], the case where $w_n = n^p$ for $p > 1$ is examined and a result is obtained in which the fragmentation semigroup is shown to have the "asynchronous exponential growth property" (AEG), i.e. there exists $\alpha > 0$ such that for any $\mathring{u} \in \ell^1_w$,

$$\|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_{\ell^1_w} \leq Ke^{-\alpha t}$$

for some $K > 0$. However, to obtain this result, assumptions are required in [15] that ensure that the resolvent operator, $R(\lambda, G^{(w)})$, is compact for $\lambda > 0$. These assumptions are difficult to check and, moreover, no specific value for α is given. Here, under the assumption that $(a_n)_{n=2}^{\infty}$ is bounded below by a positive constant, we obtain an AEG result and we quantify α. We note that "asynchronous exponential growth" is also sometimes referred to as "balanced exponential growth"; see, for example, [67, 72].

Theorem 5.5.4. *Let* (5.1.5) *and Assumption 5.2.2(i) hold with* $w_1 \geq 1$.

(i) *We have*

$$\lim_{t\to\infty} \|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_{\ell^1_w} = 0 \tag{5.5.11}$$

for all $\mathring{u} \in \ell^1_w$ *if and only if* $a_n > 0$ *for all* $n \geq 2$.

(ii) *Choose the weight w such that Assumption 5.2.2(i) holds with $\delta \in (0,1)$. Let $\hat{a}_0 = \inf_{n\in\mathbb{N}:n\geq 2} a_n$. Then*

$$\|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_{\ell^1_w} \leq (w_1+1)e^{-(1-\delta)\hat{a}_0 t}\|\mathring{u}\|_{\ell^1_w} \qquad \text{for every } \mathring{u} \in \ell^1_w, \tag{5.5.12}$$

and so, if $\hat{a}_0 > 0$ and $\alpha \in [0,(1-\delta)\hat{a}_0)$, then

$$\lim_{t\to\infty} e^{\alpha t}\|S^{(w)}(t)\mathring{u} - M_1(\mathring{u})e_1\|_{\ell^1_w} = 0 \qquad \text{for any } \mathring{u} \in \ell^1_w. \tag{5.5.13}$$

Equation (5.5.13) *does not hold for any $\alpha > \hat{a}_0$. In particular, if $\hat{a}_0 = 0$, then* (5.5.13) *does not hold for any $\alpha > 0$.*

Proof. Consider the fragmentation system with the equation for $n = 1$ removed. Then we obtain a new fragmentation system that we can write as an ACP in $Y^{(w)} = \ell^1_{\tilde{w}}$, where $\tilde{w}_n = w_{n+1}$ for all $n \in \mathbb{N}$. The new fragmentation coefficients, $(\tilde{a}_n)_{n=1}^{\infty}$ and $(\tilde{b}_{n,j})_{n,j\in\mathbb{N}:n<j}$, that are associated with this system in $Y^{(w)}$ are given by $\tilde{a}_n = a_{n+1}$ and $\tilde{b}_{n,j} = b_{n+1,j+1}$. We first check that $\tilde{w}$, $(\tilde{a}_n)_{n=1}^{\infty}$, $(\tilde{b}_{n,j})_{n,j\in\mathbb{N}:n<j}$ satisfy Assumption 5.1.1 and Assumption 5.2.2(i). Clearly $\tilde{a}_n = a_{n+1} \geq 0$ and $\tilde{b}_{n,j} = b_{n+1,j+1} \geq 0$ for all n, $j \in \mathbb{N}$. Moreover, $\tilde{b}_{n,j} = b_{n+1,j+1} = 0$ for $j < n$. Also, for $j = 2, 3, \ldots$,

$$\sum_{n=1}^{j-1} \tilde{w}_n \tilde{b}_{n,j} = \sum_{n=1}^{j-1} w_{n+1} b_{n+1,j+1} = \sum_{k=2}^{j} w_k b_{k,j+1} \leq \sum_{k=1}^{j} w_k b_{k,j+1} \leq \delta w_{j+1} = \delta \tilde{w}_j.$$

Hence Assumption 5.2.2(i) is satisfied and so, from Theorem 5.2.7, there is a semigroup that is generated by $\overline{\mathbb{A}^{(w)} + \mathbb{B}^{(w)}}$, where $\mathbb{A}^{(w)}$, $\mathbb{B}^{(w)}$ are the fragmentation operators in $Y^{(w)}$ associated to the new system. Moreover, this semigroup takes the form

$$\begin{bmatrix} e^{-\tilde{a}_1 t} & \hat{s}_{1,2}(t) & \hat{s}_{1,3}(t) & \cdots \\ 0 & e^{-\tilde{a}_2 t} & \hat{s}_{2,3}(t) & \cdots \\ 0 & 0 & e^{-\tilde{a}_3 t} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix} = \begin{bmatrix} e^{-a_2 t} & \hat{s}_{1,2}(t) & \hat{s}_{1,3}(t) & \cdots \\ 0 & e^{-a_3 t} & \hat{s}_{2,3}(t) & \cdots \\ 0 & 0 & e^{-a_4 t} & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}, \tag{5.5.14}$$

where, for all $n \in \mathbb{N}$, $m = 1, \ldots, n-1$, $t \geq 0$, $\hat{s}_{m,n}(t)$ is the unique solution of

$$\begin{aligned}
\hat{s}'_{m,n}(t) &= -\tilde{a}_m \hat{s}_{m,n}(t) + \sum_{j=m+1}^{n} \tilde{a}_j \tilde{b}_{m,j} \hat{s}_{j,n}(t) \\
&= -a_{m+1}\hat{s}_{m,n}(t) + \sum_{j=m+1}^{n} a_{j+1} b_{m+1,j+1} \hat{s}_{j,n}(t) \\
&= -a_{m+1}\hat{s}_{m,n}(t) + \sum_{k=m+2}^{n+1} a_k b_{m+1,k} \hat{s}_{k-1,n}(t).
\end{aligned}$$

It follows from (5.5.2) that $\hat{s}_{m,n}(t) = \tilde{s}_{m+1,n+1}(t)$ for all $n \in \mathbb{N}$, $m = 1, \ldots, n-1$, $t \geq 0$, i.e. the semigroup on $Y^{(w)}$ is given by $(S^{(w)}_{(22)}(t))_{t \geq 0}$, where $(S^{(w)}_{(22)}(t))_{t \geq 0}$ is as in (5.5.5).

(i) Let $\mathring{u} \in \ell^1_w$ and denote $\tilde{\mathring{u}} = (\mathring{u}_2, \mathring{u}_3, \ldots)$. We can use Theorem 5.5.1 to deduce that

$$\lim_{t \to \infty} \left\| S^{(w)}_{(22)}(t) \tilde{\mathring{u}} \right\|_{Y^{(w)}} = \lim_{t \to \infty} \left\| S^{(w)}_{(22)}(t) \tilde{\mathring{u}} \right\|_{\ell^1_{\tilde{w}}} = 0,$$

if and only if $a_n > 0$ for all $n \geq 2$. Thus, from Lemma 5.5.3, (5.5.11) holds if and only if $a_n > 0$ for all $n \geq 2$.

(ii) If Assumption 5.2.2(i) is satisfied with $\delta < 1$, then, from Theorem 5.5.1,

$$\| S^{(w)}_{(22)}(t) \| \leq e^{-(1-\delta)\hat{a}_0 t}.$$

Thus, (5.5.12) follows from Lemma 5.5.3 and, if $\hat{a}_0 > 0$, $\alpha \in [0, (1-\delta)\hat{a}_0)$, (5.5.13) follows immediately.

If $\alpha > \hat{a}_0$, then, from Theorem 5.5.1, it is untrue that for all $\mathring{u} \in \ell^1_w$,

$$\lim_{t \to \infty} e^{\alpha t} \left\| S^{(w)}_{(22)}(t) \tilde{\mathring{u}} \right\|_{Y^{(w)}} = \lim_{t \to \infty} e^{\alpha t} \left\| S^{(w)}_{(22)}(t) \tilde{\mathring{u}} \right\|_{\ell^1_{\tilde{w}}} = 0.$$

Hence, from Lemma 5.5.3, (5.5.13) does not hold if $\alpha > \hat{a}_0$. □

Remark 5.5.5. Let the assumptions of Theorem 5.5.4 hold and let $\mathring{u} = Me_1$ for some $M \in \mathbb{R}$. Then, from (5.5.5) with $a_1 = 0$, $u(t) = S^{(w)}(t)\mathring{u} = Me_1$ for all $t \geq 0$. Hence $u(t) = Me_1$ is an equilibrium solution for all $M \in \mathbb{R}$. Assume that $a_n > 0$ for all $n \geq 2$. Then, from Theorem 5.5.4, we can deduce that Me_1 is an attractor for any solution whose initial condition $\mathring{u}$ satisfies $M_1(\mathring{u}) = M$. If, on the other hand, $a_N = 0$ for some $N \geq 2$, then from (5.5.5) Me_N is also an equilibrium point for every $M \in \mathbb{R}$.

Chapter 6

Full C–F System with Time-Dependent Coagulation

In this chapter we examine the full discrete coagulation–fragmentation system. Although investigations such as [50, 53] have been carried out on continuous C–F equations in which the coagulation and fragmentation kernels depend on time, this does not appear to be the case for the discrete C–F system with time dependent coagulation and fragmentation rate coefficients. In this chapter, our aim is to address this deficiency, at least partially, by considering discrete C–F systems in which time-dependent coagulation coefficients are permitted. Clearly, the results we obtain will also hold for the purely autonomous case, when all the coefficients are independent of time. Since we work in general weighted ℓ^1 spaces, these "autonomous" results extend those established in [15, 46] where the weight is restricted to $w_n = n^p$ for some $p \geq 1$.

6.1 Setting up the C–F Problem

We now consider the full coagulation–fragmentation system

$$\begin{aligned} u_n'(t) &= -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \\ &\qquad + \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t) u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j}(t) u_n(t) u_j(t), \quad t > 0; \\ u_n(0) &= \mathring{u}_n, \qquad n = 1, 2, \ldots \end{aligned} \tag{6.1.1}$$

on weighted ℓ^1 spaces, with the coefficients and terms in (6.1.1) being interpreted in the same way as in (1.1.1). We require the following assumption to hold throughout this chapter.

Assumption 6.1.1. Let $a_n \geq 0$, $b_{n,j} \geq 0$ and $k_{n,j}(t) \geq 0$ for all $n, j \in \mathbb{N}$ and $t \geq 0$. Moreover, we assume that $b_{n,j} = 0$ for all $j \leq n$ and that $k_{n,j}(t) = k_{j,n}(t)$ for all $n, j \in \mathbb{N}$, $t \geq 0$. In addition, let $(w_n)_{n=1}^{\infty}$ be monotone increasing and let Assumption 5.2.2(i) hold.

Note that, in contrast to previous semigroup-based investigations, we allow the possibility that the coagulation rates are time-dependent. Recall from Proposition 5.2.4 that if $w_1 \geq 1$ and (5.1.5) holds, then under Assumption 6.1.1, $w_n \geq n$ for all $n \in \mathbb{N}$ and so ℓ_w^1 is continuously embedded in $X_{[1]}$. Moreover, we note that if (5.1.4) holds with $\lambda_j \in [0, 1]$ for $j = 2, 3, \ldots$, then, from Lemma 5.2.5, it is clear that any weight of the form $w_n = n^p$, for $p \geq 1$, satisfies Assumption 6.1.1.

Let $(v_n)_{n=1}^{\infty}$ be such that $v_n > 0$ for all $n \in \mathbb{N}$ and $\ell_v^1 \subseteq \ell_w^1$. We note that $(v_n)_{n=1}^{\infty}$ will be chosen later as the weight of a weighted ℓ^1 space that plays the role of Y in Chapter 4. We now list various assumptions on the coagulation rates, $k_{n,j}(t)$, that will be needed to obtain results later. Let $0 < T \leq \infty$.

Assumption 6.1.2.

(A1) For every $t' \in [0, T)$, there exists a constant $C(t') > 0$ such that

$$\frac{w_{n+j}}{v_n v_j} k_{n,j}(t) \leq C(t') \tag{6.1.2}$$

for all $n, j \in \mathbb{N}$ and $t \in [0, t']$.

(A2) The family of functions $\left(\frac{w_{n+j}}{v_n v_j} k_{n,j}(\cdot)\right)_{n,j\in\mathbb{N}}$ is equicontinuous on $[0, T)$.

(A3) The coagulation coefficients, $k_{n,j}$, are differentiable with respect to t for all $n, j \in \mathbb{N}$ in the sense that for fixed $t \in (0, T)$ and $\delta_t \in \mathbb{R}$ such that $t + \delta_t \in (0, T)$ we have

$$k_{n,j}(t + \delta_t) = k_{n,j}(t) + \frac{d}{dt} k_{n,j}(t)\delta_t + R_{n,j}(t, \delta_t) \tag{6.1.3}$$

where, for fixed $t' \in (0, T)$, there exists a $\tilde{C}(t') > 0$ satisfying, for all $n, j \in \mathbb{N}$,

$$\left|\frac{w_{n+j}}{v_n v_j} \frac{d}{dt} k_{n,j}(t)\right| \leq \tilde{C}(t') \qquad \text{for all } t \in (0, t']$$

and, for fixed $t \in (0, T)$,

$$\frac{w_{n+j}|R_{n,j}(t, \delta_t)|}{v_n v_j |\delta_t|} \to 0 \qquad \text{as } |\delta_t| \to 0$$

uniformly in $n, j \in \mathbb{N}$. Moreover,

$$t \mapsto \left(\frac{w_{n+j}}{v_n v_j} \frac{d}{dt} k_{n,j}(t)\right)_{n,j\in\mathbb{N}}$$

is equicontinuous on $(0, T)$.

Remark 6.1.3. Suppose that the first and second derivatives of $t \mapsto k_{n,j}(t)$ exist and are continuous. For all $t' \in (0, T)$ assume that there exists $\tilde{c}(t') > 0$ such that, for all $n, j \in \mathbb{N}$, $t \in (0, t']$,

$$\left|\frac{w_{n+j}}{v_n v_j} \frac{d}{dt} k_{n,j}(t)\right| \leq \tilde{c}(t') \qquad \text{and} \qquad \left|\frac{w_{n+j}}{v_n v_j} \frac{d^2}{dt^2} k_{n,j}(t)\right| \leq \tilde{c}(t'). \tag{6.1.4}$$

Let $t' \in (0, T)$. By the Mean Value Theorem, for $t \in (0, t']$, $h \in \mathbb{R}$ satisfying $t + h \in (0, t']$, there exists ξ between t and $t + h$ such that

$$\left|\frac{w_{n+j}}{v_n v_j} \frac{d}{dt} k_{n,j}(t + h) - \frac{w_{n+j}}{v_n v_j} \frac{d}{dt} k_{n,j}(t)\right| = \frac{w_{n+j}}{v_n v_j}\left|\frac{d^2}{dt^2} k_{n,j}(\xi)\right||h|$$

$$\leq \tilde{c}(t')|h| \to 0 \text{ as } h \to 0.$$

It follows that $t \mapsto \left(\frac{w_{n+j}}{v_n v_j}\frac{d}{dt}k_{n,j}(t)\right)_{n,j\in\mathbb{N}}$ is equicontinuous on $(0,T)$.

Now let $t \in (0,T)$ and let $\delta_t \in \mathbb{R}$ be such that $t+\delta_t \in (0,T)$. Then, from Taylor's Theorem,

$$k_{n,j}(t+\delta_t) = k_{n,j}(t) + \frac{d}{dt}k_{n,j}(t)\delta_t + R_{n,j}(t,\delta_t),$$

where

$$R_{n,j}(t,\delta_t) = \frac{1}{2}\frac{d^2}{dt^2}k_{n,j}(\varepsilon)\delta_t^2$$

for some ε between t and $t+\delta_t$. Let $t' \in (0,T)$ such that $t,\ t+\delta_t \in (0,t']$. We have

$$\frac{w_{n+j}|R_{n,j}(t,\delta_t)|}{v_n v_j|\delta_t|} = \frac{w_{n+j}}{v_n v_j}\frac{1}{2}\left|\frac{d^2}{dt^2}k_{n,j}(\varepsilon)\right||\delta_t| \leq \frac{\tilde{c}(t')}{2}|\delta_t| \to 0 \qquad \text{as } |\delta_t| \to 0,$$

uniformly in $n,\ j \in \mathbb{N}$. It follows that (A3) holds.

The following corollary deals with the case where $k_{n,j}(t) \coloneqq k_{n,j}$ for all $t \geq 0$, i.e. the coagulation rates are time-independent.

Remark 6.1.4. Let $k_{n,j}$ be time-independent and let $C > 0$ be such that

$$\frac{w_{n+j}}{v_n v_j}k_{n,j} \leq C$$

for all $n,j \in \mathbb{N}$. Since $k_{n,j}$ is time independent for all $n,j \in \mathbb{N}$, we can take $C(t') \coloneqq C$ in (6.1.2). Also, it is clear that (A2) holds and that $k_{n,j}$ is differentiable with respect to t, with $\frac{d}{dt}k_{n,j} = 0$. Hence (A1)–(A3) hold with $C(t) \coloneqq C$.

The following proposition is the general mean inequality and is useful in calculations that follow. For $x \neq y$, this proposition is a specific example of [33, Theorem 16.b]. When we take $x = y$, the proposition follows easily with equality in (6.1.5).

Proposition 6.1.5. *Let x, $y \geq 0$ and let $0 < p \leq q$. Then*

$$\left(\frac{x^p + y^p}{2}\right)^{\frac{1}{p}} \leq \left(\frac{x^q + y^q}{2}\right)^{\frac{1}{q}}. \tag{6.1.5}$$

We use this proposition in the following remark, which gives sufficient conditions for the weights, $(w_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$, and coagulation rates, $k_{n,j}(t)$, under which (A1) holds. This is of particular interest when the weights satisfy $w_n = v_n = n^p$ for $p \geq 1$.

Remark 6.1.6. Suppose that there exists an $\alpha > 0$ such that, for all $n, j \in \mathbb{N}$,

$$w_{n+j} \leq \alpha(v_n + v_j) \tag{6.1.6}$$

and, for all $t' \in [0, T)$, there exists a $\tilde{C}(t') > 0$ such that

$$k_{n,j}(t) \leq \tilde{C}(t') \min\{v_n, v_j\} \tag{6.1.7}$$

for all $t \in [0, t']$. We now show that (A1) holds under these assumptions. We have, for all $t \in [0, t']$,

$$\begin{aligned} w_{n+j} k_{n,j}(t) &\leq \alpha(v_n + v_j)\tilde{C}(t') \min\{v_n, v_j\} \\ &= \alpha\tilde{C}(t')\Big(v_n \min\{v_n, v_j\} + v_j \min\{v_n, v_j\}\Big) \\ &\leq 2\alpha\tilde{C}(t') v_n v_j \\ &= C(t') v_n v_j, \end{aligned}$$

where $C(t') = 2\alpha\tilde{C}(t')$.

For example, if we take $v_n = w_n = n^p$ for all $n \in \mathbb{N}$ and some $p \geq 1$ then, from Proposition 6.1.5, we have

$$w_{n+j} = (n + j)^p \leq 2^{p-1}(n^p + j^p) = 2^{p-1}(w_n + w_j).$$

Hence (A1) holds if $v_n = w_n = n^p$, where $p \geq 1$, and $k_{n,j}$ satisfies (6.1.7).

We now give specific examples where the coagulation rates, $k_{n,j}$, do indeed satisfy (A1).

Example 6.1.7.

(i) Fix $N \in \mathbb{N}$ and consider the case where a cluster of size n can only merge with another cluster if $n \leq N$. Then we have

$$k_{n,j}(t) = \begin{cases} c_{n,j}(t) & \text{if } 1 \leq n, j \leq N, \\ 0 & \text{otherwise,} \end{cases} \tag{6.1.8}$$

where we assume $c_{n,j}(t) = c_{j,n}(t)$. Suppose that for each $t' \in [0, T)$ we also have $0 \leq c_{n,j}(t) < \frac{C(t')v_n v_j}{w_{n+j}}$ for all $t \in [0, t']$. Then it follows that

$$\frac{w_{n+j}}{v_n v_j} k_{n,j}(t) \leq C(t')$$

for all $t \in [0, t']$.

(ii) Suppose that the coagulation kernel $k_{n,j}(t)$ satisfies

$$k_{n,j}(t) \leq c(t) \frac{v_n v_j}{v_n + v_j}, \quad n, j \in \mathbb{N},$$

where for every $t' > 0$ there exists a $C(t') > 0$ such that $c(t) \leq C(t')$ for all $t \in [0, t']$. Then, since $\frac{v_j}{v_n + v_j} < 1$ and $\frac{v_n}{v_n + v_j} < 1$, it follows that $k_{n,j}(t) \leq C(t') \min\{v_n, v_j\}$ for all $t \in [0, t']$. It follows that if, in addition, (6.1.6) holds, then (A1) holds.

(iii) Suppose that for all $n \in \mathbb{N}$, $v_n = \beta_n w_n$ for some $\beta_n \geq 1$ and that for every $t' \in [0, T)$ there exists a constant $C(t') > 0$ such that

$$k_{n,j}(t) \leq C(t') \frac{w_n w_j}{w_{n+j}} \qquad \text{for all } t \in [0, t'],\ n, j \in \mathbb{N}. \tag{6.1.9}$$

Then

$$\frac{w_{n+j}}{v_n v_j} k_{n,j}(t) = \frac{w_{n+j}}{\beta_n w_n \beta_j w_j} k_{n,j}(t) \leq \frac{w_{n+j}}{w_n w_j} k_{n,j}(t) \leq C(t'),$$

for all $t \in [0, t']$, i.e. (A1) holds. Note that the case where $v_n = \beta_n w_n$ will be of interest later, specifically for $\beta_n = (1 + a_n)^\alpha$ for some $\alpha \in [0, 1)$.

If Assumption 6.1.1 holds, then from Theorem 5.2.7, $G^{(w)} = \overline{A^{(w)} + B^{(w)}}$ is the generator of a substochastic C_0-semgroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ^1_w. Moreover, from Theorem 5.3.4, $G^{(w)}$ takes the form

$$[G^{(w)}f]_n = -a_n f_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j, \qquad f \in \mathcal{D}(G^{(w)}). \tag{6.1.10}$$

Assume that (A1) and (A2) holds. The ACP corresponding to (6.1.1) on ℓ^1_w takes the form

$$u'(t) = G^{(w)}u(t) + K^{(v,w)}(t, u(t)), \quad t \in [0, T), \tag{6.1.11}$$

$$u(0) = \mathring{u}, \tag{6.1.12}$$

where, for some $0 < T \leq \infty$, we define $K^{(v,w)} : [0, T) \times \ell^1_v \mapsto \ell^1_w$ by

$$[K^{(v,w)}(t, f)]_n = \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t) f_{n-j} f_j - \sum_{j=1}^{\infty} k_{n,j}(t) f_n f_j,$$

for all $t \in [0, T)$, $f \in \ell^1_v$, $n = 1, 2, \ldots$. The fact that $K^{(v,w)}$ maps $[0, T) \times \ell^1_v$ into ℓ^1_w will be shown in Lemma 6.2.1. Note that u is a function with values in ℓ^1_v and so (6.1.11), (6.1.12) is an ACP of the form (4.2.1), (4.2.2), with $X = \ell^1_w$ and $Y = \ell^1_v$.

The following lemma is required in various results that follow.

Lemma 6.1.8. *Let $(v_n)_{n=1}^{\infty}$, $(w_n)_{n=1}^{\infty}$ be such that v_n, $w_n > 0$ for all $n \in \mathbb{N}$ and $(w_n)_{n=1}^{\infty}$ is monotone increasing. Consider the Banach spaces $(\ell^1_v, \|\cdot\|_{\ell^1_v})$, $(\ell^1_w, \|\cdot\|_{\ell^1_w})$. Let $\varphi_{n,j} \in \mathbb{R}$ for all n, $j \in \mathbb{N}$.*

(i) *Then*

$$\left\| \left(\frac{1}{2}\sum_{j=1}^{n-1} \varphi_{n-j,j} f_{n-j} g_j - \sum_{j=1}^{\infty} \varphi_{n,j} f_n g_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w} \leq \frac{3}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} w_{n+j} |\varphi_{n,j}| |f_n| |g_j|$$

for f, $g \in \ell^1_v$.

(ii) *Moreover, suppose that there exists $C > 0$ such that*

$$\frac{w_{n+j}}{v_n v_j}|\varphi_{n,j}| \leq C \qquad \text{for all } n, j \in \mathbb{N}. \tag{6.1.13}$$

Then

$$\left\| \left(\frac{1}{2} \sum_{j=1}^{n-1} \varphi_{n-j,j} f_{n-j} g_j - \sum_{j=1}^{\infty} \varphi_{n,j} f_n g_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w} \leq \frac{3C}{2} \|f\|_{\ell^1_v} \|g\|_{\ell^1_v}$$

for $f,\, g \in \ell^1_v$.

Proof. Let $f,\, g \in \ell^1_v$. Then we have

$$\begin{aligned} &\left\| \left(\frac{1}{2} \sum_{j=1}^{n-1} \varphi_{n-j,j} f_{n-j} g_j - \sum_{j=1}^{\infty} \varphi_{n,j} f_n g_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w} \\ &\leq \left\| \left(\frac{1}{2} \sum_{j=1}^{n-1} \varphi_{n-j,j} f_{n-j} g_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w} + \left\| \left(\sum_{j=1}^{\infty} \varphi_{n,j} f_n g_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w}. \end{aligned}$$

Now,

$$\begin{aligned} \left\| \left(\frac{1}{2} \sum_{j=1}^{n-1} \varphi_{n-j,j} f_{n-j} g_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w} &= \sum_{n=1}^{\infty} w_n \left| \frac{1}{2} \sum_{j=1}^{n-1} \varphi_{n-j,j} f_{n-j} g_j \right| \\ &\leq \frac{1}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} w_n |\varphi_{n-j,j}||f_{n-j}||g_j| \\ &= \frac{1}{2} \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} w_n |\varphi_{n-j,j}||f_{n-j}||g_j| \\ &= \frac{1}{2} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} w_{l+j} |\varphi_{l,j}||f_l||g_j|, \end{aligned}$$

and using the monotonicity of $(w_n)_{n=1}^{\infty}$ we obtain

$$\begin{aligned} \left\| \left(\sum_{j=1}^{\infty} \varphi_{n,j} f_n g_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w} &\leq \sum_{n=1}^{\infty} w_n \left| \sum_{j=1}^{\infty} \varphi_{n,j} f_n g_j \right| \leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} w_n |\varphi_{n,j}||f_n||g_j| \\ &\leq \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} w_{n+j} |\varphi_{n,j}||f_n||g_j|. \end{aligned}$$

Hence

$$\left\|\left(\frac{1}{2}\sum_{j=1}^{n-1}\varphi_{n-j,j}f_{n-j}g_j-\sum_{j=1}^{\infty}\varphi_{n,j}f_ng_j\right)_{n\in\mathbb{N}}\right\|_{\ell_w^1}\leq\frac{3}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}w_{n+j}|\varphi_{n,j}||f_n||g_j|.$$

If, in addition, (6.1.13) holds, then

$$\begin{aligned}\left\|\left(\frac{1}{2}\sum_{j=1}^{n-1}\varphi_{n-j,j}f_{n-j}g_j-\sum_{j=1}^{\infty}\varphi_{n,j}f_ng_j\right)_{n\in\mathbb{N}}\right\|_{\ell_w^1}&\leq\frac{3C}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{\infty}v_nv_j|f_n||f_j|\\&=\frac{3C}{2}\|f\|_{\ell_v^1}\|g\|_{\ell_v^1}.\end{aligned}$$

□

6.2 Lipschitz Continuity and Fréchet Differentiability of the Coagulation Operator

In this section we show that under (A1) and (A2), the coagulation operator $K^{(v,w)}$ is locally Lipschitz. Moreover, we show that if, in addition, (A3) holds, then $K^{(v,w)}$ is Fréchet differentiable with respect to (t,f) and the Fréchet derivative is continuous. These properties are important when we apply the existence and uniqueness results from Section 4.2 to the coagulation–fragmentation system.

We now introduce the mapping $\tilde{K}^{(v,w)}:[0,T)\times\ell_v^1\times\ell_v^1\mapsto\ell_w^1$ defined by

$$(\tilde{K}^{(v,w)}[t,f,g])_n:=\frac{1}{2}\sum_{j=1}^{n-1}k_{n-j,j}(t)f_{n-j}g_j-\sum_{j=1}^{\infty}k_{n,j}(t)f_ng_j,$$

for $f,g\in\ell_v^1$, $t\in[0,T)$ and $n\in\mathbb{N}$. Set

$$(\tilde{K_1}^{(v,w)}[t,f,g])_n=\frac{1}{2}\sum_{j=1}^{n-1}k_{n-j,j}(t)f_{n-j}g_j$$

and

$$(\tilde{K_2}^{(v,w)}[t,f,g])_n=\sum_{j=1}^{\infty}k_{n,j}(t)f_ng_j.$$

Then

$$(\tilde{K}^{(v,w)}[t,f,g])_n=(\tilde{K_1}^{(v,w)}[t,f,g])_n-(\tilde{K_2}^{(v,w)}[t,f,g])_n$$

for all $f, g \in \ell_v^1$, $t \in [0,T)$ and $n \in \mathbb{N}$. We also set $K_1^{(v,w)}(t,f) = \tilde{K}_1^{(v,w)}[t,f,f]$ and $K_2^{(v,w)}(t,f) = \tilde{K}_2^{(v,w)}[t,f,f]$ so that

$$K^{(v,w)}(t,f) = \tilde{K}^{(v,w)}[t,f,f] = K_1^{(v,w)}(t,f) - K_2^{(v,w)}(t,f).$$

We now show that under assumptions (A1) and (A2), $K^{(v,w)}$ is indeed a continuous mapping defined on $[0,T) \times \ell_v^1$ and $K^{(v,w)}$ satisfies a Lipschitz condition.

Lemma 6.2.1. *Let Assumption 6.1.1,* (A1) *and* (A2) *hold. Then the operator $K^{(v,w)}$ is a continuous mapping from $[0,T) \times \ell_v^1$ into ℓ_w^1. Moreover, $K^{(v,w)}$ is Lipschitz on bounded sets in the second argument, uniformly in the first argument on compact intervals.*

Proof. We show that the operator $\tilde{K}^{(v,w)}$ satisfies the assumptions of Lemma 4.1.6. By definition, it is clear that $\tilde{K}^{(v,w)}$ is linear in the second and third arguments. Take $t' \in (0,T)$. From (A1) and Lemma 6.1.8(ii), we have that $\tilde{K}^{(v,w)}$ maps $[0,T) \times \ell_v^1 \times \ell_v^1$ into ℓ_w^1 and

$$\|\tilde{K}^{(v,w)}[t,f,g]\|_{\ell_w^1} \leq \frac{3}{2} C(t') \|f\|_{\ell_v^1} \|g\|_{\ell_v^1} \tag{6.2.1}$$

for all $f, g \in \ell_v^1$, $t \in [0,t']$. Hence $\tilde{K}^{(v,w)}$ is bounded, and so continuous, in the second and third arguments separately. It follows from Lemma 4.1.6 that $K^{(v,w)}$ is Lipschitz on bounded sets in the second argument, uniformly in the first argument on compact intervals.

We now show that the mapping $t \mapsto \tilde{K}(t,f,g)$ is continuous on $[0,T)$ for fixed $f,\, g \in \ell_v^1$. Let $t,\, t_0 \in [0,T)$ and $f,\, g \in \ell_v^1$. Then, from Lemma 6.1.8(i),

$$\begin{aligned}
&\|\tilde{K}^{(v,w)}[t,f,g] - \tilde{K}^{(v,w)}[t_0,f,g]\|_{\ell_w^1} \\
&= \left\| \frac{1}{2} \sum_{j=1}^{n-1} \Big(k_{n-j,j}(t) - k_{n-j,j}(t_0)\Big) f_{n-j} g_j - \sum_{j=1}^{\infty} \Big(k_{n,j}(t) - k_{n,j}(t_0)\Big) f_n g_j \right\|_{\ell_w^1} \\
&\leq \frac{3}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} w_{n+j} \Big| k_{n,j}(t) - k_{n,j}(t_0) \Big| |f_n| |g_j| \\
&= \frac{3}{2} \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} \left| \frac{w_{n+j} k_{n,j}(t)}{v_n v_j} - \frac{w_{n+j} k_{n,j}(t_0)}{v_n v_j} \right| v_n |f_n| v_j |g_j|
\end{aligned}$$

$$\to 0 \qquad \text{as } t \to t_0 \text{ by } (A2).$$

Hence $t \mapsto \tilde{K}^{(v,w)}[t, f, g]$ is continuous on $[0, T)$ for fixed f, $g \in \ell_v^1$. From Lemma 4.1.6(ii) we then deduce that $K^{(v,w)} : [0, T) \times \ell_v^1 \mapsto \ell_w^1$ is continuous. □

Remark 6.2.2. Combining the previous result with Remark 4.1.7, we have that for $f \in \ell_v^1$, $K^{(v,w)}$ is Lipschitz on $\overline{B}_{\ell_v^1}(f, r)$ for any $r > 0$.

The following theorem is proved using Corollary 4.1.9. This result tells us that if assumptions (A1)–(A3) hold, then $K^{(v,w)}$ is Fréchet differentiable with respect to (t, f) and the Fréchet derivative is continuous.

Theorem 6.2.3. *Let Assumption 6.1.1,* (A1), (A2) *and* (A3) *hold. Then the operator $K^{(v,w)}$ is Fréchet differentiable with respect to (t, f), with the Fréchet derivative at fixed $(t, f) \in (0, T) \times \ell_v^1$ given, for all $(s, g) \in \mathbb{R} \times \ell_v^1$, by*

$$DK^{(v,w)}(t, f)(s, g) = \frac{\partial}{\partial t}\tilde{K}^{(v,w)}(t, f, f)s + \tilde{K}^{(v,w)}[t, f, g] + \tilde{K}^{(v,w)}[t, g, f].$$

Moreover, this derivative is continuous with respect to (t, f).

Proof. We show that $\tilde{K}^{(v,w)}$ satisfies the conditions of Corollary 4.1.9. From Lemma 6.2.1 we know that $\tilde{K}^{(v,w)}$ satisfies the conditions of Lemma 4.1.6. Then, from (6.1.3), we have for $f, g \in \ell_v^1$, $t \in (0, T)$ and $\delta_t \in \mathbb{R}$ such that $t + \delta_t \in (0, T)$,

$$\begin{aligned}
&(\tilde{K}^{(v,w)}[t + \delta_t, f, g])_n \\
&= \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t + \delta_t) f_{n-j} g_j - \sum_{j=1}^{\infty} k_{n,j}(t + \delta_t) f_n g_j \\
&= \frac{1}{2}\sum_{j=1}^{n-1} \Big(k_{n-j,j}(t) + \frac{d}{dt}k_{n-j,j}(t)\delta_t + R_{n-j,j}(t, \delta_t)\Big) f_{n-j} g_j \\
&\qquad - \sum_{j=1}^{\infty} \Big(k_{n,j}(t) + \frac{d}{dt}k_{n,j}(t)\delta_t + R_{n,j}(t, \delta_t)\Big) f_n g_j \\
&= \left(\frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t) f_{n-j} g_j - \sum_{j=1}^{\infty} k_{n,j}(t) f_n g_j\right) \\
&\qquad + \left(\frac{1}{2}\sum_{j=1}^{n-1} \frac{d}{dt}k_{n-j,j}(t)\delta_t f_{n-j} g_j - \sum_{j=1}^{\infty} \frac{d}{dt}k_{n,j}(t)\delta_t f_n g_j\right)
\end{aligned}$$

$$
\begin{aligned}
&+\left(\frac{1}{2}\sum_{j=1}^{n-1} R_{n-j,j}(t,\delta_t) f_{n-j} g_j - \sum_{j=1}^{\infty} R_{n,j}(t,\delta_t) f_n g_j\right) \\
&= (\tilde{K}^{(v,w)}[t,f,g])_n + \left(\frac{\partial}{\partial t}\tilde{K}^{(v,w)}[t,f,g]\right)_n \delta_t + (R(t,f,g,\delta_t))_n
\end{aligned}
$$

where

$$
\left(\frac{\partial}{\partial t}\tilde{K}^{(v,w)}[t,f,g]\right)_n = \frac{1}{2}\sum_{j=1}^{n-1}\frac{d}{dt}k_{n-j,j}(t) f_{n-j} g_j - \sum_{j=1}^{\infty}\frac{d}{dt}k_{n,j}(t) f_n g_j
$$

and

$$
(R(t,f,g,\delta_t))_n = \frac{1}{2}\sum_{j=1}^{n-1} R_{n-j,j}(t,\delta_t) f_{n-j} g_j - \sum_{j=1}^{\infty} R_{n,j}(t,\delta_t) f_n g_j.
$$

Let $t' \in (0,T)$. Then using (A3) and Lemma 6.1.8(ii) we have, for fixed $(t,f,g) \in (0,T) \times \ell_v^1 \times \ell_v^1$ such that $t \in [0,t']$,

$$
\begin{aligned}
\left\|\frac{\partial}{\partial t}\tilde{K}^{(v,w)}[t,f,g]\right\|_{\ell_w^1} &= \left\|\frac{1}{2}\sum_{j=1}^{n-1}\frac{d}{dt}k_{n-j,j}(t) f_{n-j} g_j - \sum_{j=1}^{\infty}\frac{d}{dt}k_{n,j}(t) f_n g_j\right\|_{\ell_w^1} \\
&\leq \frac{3\tilde{C}(t')}{2}\|f\|_{\ell_v^1}\|g\|_{\ell_v^1} < \infty.
\end{aligned}
$$

Also, for fixed $(t,f,g) \in (0,T) \times \ell_v^1 \times \ell_v^1$, we have, from Lemma 6.1.8(i),

$$
\begin{aligned}
\frac{\|R(t,f,g,\delta_t)\|_{\ell_w^1}}{|\delta_t|} &= \frac{1}{|\delta_t|}\left\|\frac{1}{2}\sum_{j=1}^{n-1} R_{n-j,j}(t,\delta_t) f_{n-j} g_j + \sum_{j=1}^{\infty} R_{n,j}(t,\delta_t) f_n g_j\right\|_{\ell_w^1} \\
&\leq \frac{3}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} w_{l+j}\frac{|(R_{l,j}(t,\delta_t))|}{|\delta_t|}|f_l||g_j| \\
&= \frac{3}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\frac{w_{l+j}}{v_l v_j}\frac{|(R_{l,j}(t,\delta_t))|}{|\delta_t|} v_l|f_l| v_j|g_j| \\
&\to 0 \qquad \text{as } |\delta_t| \to 0 \text{ by (A3).}
\end{aligned}
$$

The continuity of the derivative with respect to t follows from the equicontinuity of $t \mapsto \left(\frac{w_{n+j}}{v_n v_j}\frac{d}{dt}k_{n,j}(t)\right)_{n,j\in\mathbb{N}}$ and Lemma 6.1.8. Thus assumption (a) in Corollary 4.1.9 holds. Let $t' \in (0,T)$ and $f,\ g \in \ell_v^1$. Then, for all $t \in (0,t']$, we can deduce from Lemma 6.1.8 that

$$\Big\|\frac{\partial}{\partial t}\tilde{K}^{(v,w)}[t,f,g]\Big\|_{\ell^1_w} = \Big\|\frac{1}{2}\sum_{j=1}^{n-1}\frac{d}{dt}k_{n-j,j}(t)f_{n-j}g_j - \sum_{j=1}^{\infty}\frac{d}{dt}k_{n,j}(t)f_n g_j\Big\|_{\ell^1_w}$$
$$\leq \frac{3}{2}\tilde{C}(t')\|f\|_{\ell^1_v}\|g\|_{\ell^1_v},$$

which shows that assumption (b) in Corollary 4.1.9 holds. The result then follows from Corollary 4.1.9. □

6.3 Solutions of the C–F System

In this section we consider the specific case where $\ell^1_v = \ell^1_w$. For convenience, we explicitly state the assumptions (A1)–(A3) in this case. For notational convenience, we set

$$W_{n,j} = \frac{w_{n+j}}{w_n w_j}.$$

Assumption 6.3.1. When $\ell^1_v = \ell^1_w$ assumptions (A1)–(A3) become, respectively,

(A*1) for every $t' \in [0,T)$, there exists a constant $C(t') > 0$ such that

$$W_{n,j}k_{n,j}(t) \leq C(t') \tag{6.3.1}$$

for all $n, j \in \mathbb{N}$ and $t \in [0,t']$;

(A*2) the family of functions $\big(W_{n,j}k_{n,j}(\cdot)\big)_{n,j\in\mathbb{N}}$ is equicontinuous on $[0,T)$;

(A*3) the coagulation coefficients, $k_{n,j}$, are differentiable with respect to t for all $n, j \in \mathbb{N}$ in the sense that for fixed $t \in (0,T)$ and $\delta_t \in \mathbb{R}$ such that $t+\delta_t \in (0,T)$ we have

$$k_{n,j}(t+\delta_t) = k_{n,j}(t) + \frac{d}{dt}k_{n,j}(t)\delta_t + R_{n,j}(t,\delta_t)$$

where, for fixed $t' \in (0,T)$, there exists a $\tilde{C}(t') > 0$ satisfying, for all $n, j \in \mathbb{N}$,

$$\left|W_{n,j}\frac{d}{dt}k_{n,j}(t)\right| \leq \tilde{C}(t') \qquad \text{for all } t \in (0,t']$$

and, for fixed $t \in (0, T)$,

$$\frac{W_{n,j}|R_{n,j}(t, \delta_t)|}{|\delta_t|} \to 0 \qquad \text{as } |\delta_t| \to 0$$

uniformly in $n, j \in \mathbb{N}$. Moreover,

$$t \mapsto \Big(W_{n,j} \frac{d}{dt} k_{n,j}(t)\Big)_{n,j \in \mathbb{N}}$$

is equicontinuous on $(0, T)$.

Theorem 6.3.2. *Let Assumption 6.1.1 hold and let $\mathring{u} \in \ell^1_w$.*

(i) *Under assumptions (A*1) and (A*2) there exists a unique maximal mild solution $u \in C\Big([0, t_{max}), \ell^1_w\Big)$, of* (6.1.11), (6.1.12), *where $t_{max} > 0$.*

(ii) *If, in addition, (A*3) holds and $\mathring{u} \in \mathcal{D}(G^{(w)})$, then the mild solution in (i) is also the unique classical solution.*

Proof. Under assumptions (A*1) and (A*2) we have, from Lemma 6.2.1, that $K^{(w,w)}$ is continuous from $[0, T) \times \ell^1_w$ into ℓ^1_w and is Lipschitz on bounded sets in the second argument, uniformly in the first argument on compact intervals. Moreover, we know that $G^{(w)}$ is the generator of a C_0-semigroup on ℓ^1_w. Part (i) then follows from Proposition 4.2.12 and Theorem 4.2.5.

If we also have that (A*3) holds then, from Theorem 6.2.3, $K^{(w,w)}$ is Fréchet differentiable on $[0, T)$ and the Fréchet derivative of $K^{(w,w)}$ is continuous. Part (ii) then follows from Theorem 4.2.13, $\square$

We now aim to show that the unique solution will be non-negative if the initial condition is non-negative. Let $t' \in (0, T)$, $r > 0$. The following lemma shows that if we choose $\gamma > 0$ to be suitably large, then $K^{(w,w,\gamma)}(t, f) \in (\ell^1_w)_+$ for all $t \in [0, t']$, $f \in \overline{B}_{\ell^1_w}(0, r)_+$, where the operator $K^{(w,w,\gamma)}$ is defined by

$$K^{(w,w,\gamma)}(t, f) = K^{(w,w)}(t, f) + \gamma f.$$

This result and proof is based on [46, Lemma 4.5].

Lemma 6.3.3. *Let $\ell_v^1 = \ell_w^1$ and let Assumptions 6.1.1,* (A*1) *and* (A*2) *hold. Let $r > 0$, $t' \in [0, T)$ and choose $\gamma > C(t')r$. Then $K^{(w,w,\gamma)}(t, f) \in (\ell_w^1)_+$ for all $f \in \overline{B}_{\ell_w^1}(0, r)_+$, $t \in [0, t']$.*

Proof. We have for $f \in (\ell_w^1)_+$, $t \in [0, T)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
(K^{(w,w,\gamma)}(t, f))_n &= \gamma f_n + (\tilde{K}^{(w,w)}[t, f, f])_n \\
&= \gamma f_n + (\tilde{K_1}^{(w,w)}[t, f, f])_n - (\tilde{K_2}^{(w,w)}[t, f, f])_n \\
&= \gamma f_n + (K_1^{(w,w)}(t, f))_n - (K_2^{(w,w)}(t, f))_n,
\end{aligned}$$

where

$$(K_1^{(w,w)}(t, f))_n = \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t) f_{n-j} f_j \geq 0 \qquad \text{for all } f \in \overline{B}_{\ell_w^1}(0, r)_+.$$

Letting $t' \in [0, T)$, we have

$$\begin{aligned}
(K_2^{(w,w)}(t, f))_n &= \sum_{j=1}^{\infty} k_{n,j}(t) f_n f_j \leq \sum_{j=1}^{\infty} \frac{C(t')}{W_{n,j}} f_n f_j \\
&= C(t') \sum_{j=1}^{\infty} \frac{w_n w_j}{w_{n+j}} f_j f_n \leq C(t') \sum_{j=1}^{\infty} w_j f_j f_n, \quad (\text{since } w_n \leq w_{n+j}) \\
&= C(t') \|f\|_{\ell_w^1} f_n
\end{aligned}$$

for all $t \in [0, t']$.

If $f \in \overline{B}_{\ell_w^1}(0, r)_+$, then for $t \in [0, t']$,

$$(K_2^{(w,w)}(t, f))_n \leq C(t') r f_n.$$

Hence for $t \in [0, t']$, $\gamma f_n - (K_2^{(w,w)}(t, f))_n \geq \gamma f_n - C(t') r f_n \geq 0$ if $\gamma \geq C(t')r$. Consequently, if $\gamma \geq C(t')r$, we have

$$(K^{(w,w,\gamma)}(t, f))_n = \gamma f_n + (K_1^{(w,w)}(t, f))_n - (K_2^{(w,w)}(t, f))_n \geq 0,$$

i.e. $K^{(w,w,\gamma)}(t, f) \in (\ell_w^1)_+$, for all $f \in \overline{B}_{\ell_w^1}(0, r)_+$ and $t \in [0, t']$. $\square$

This leads immediately to the following result.

Proposition 6.3.4. *Let $\ell_v^1 = \ell_w^1$ and let Assumption 6.1.1,* (A*1) *and* (A*2) *hold. Moreover, let $\mathring{u} \in (\ell_w^1)_+$.*

(i) *Then, for some t_{max} satisfying $0 < t_{max} \leq T$, there exists a unique, non-negative mild solution $u \in C\Big([0, t_{max}), (\ell_w^1)_+\Big)$ of* (6.1.11), (6.1.12).

(ii) *If, in addition, (A*3) holds and $\mathring{u} \in \mathcal{D}(G^{(w)})_+$, then this solution is also the unique, non-negative classical solution.*

Proof. The existence and uniqueness of a mild solution, u, on $[0, t_{max})$, for some $0 < t_{max} \leq T$, follow from Theorem 6.3.2(i), where we showed that the assumptions of Proposition 4.2.12 are satisfied.

Clearly, $\mathcal{D}(G^{(w)})$ is dense in ℓ_w^1. For each $\gamma \geq 0$, we have that $\gamma I : \ell_w^1 \mapsto \ell_w^1$ is a bounded operator and $G^{(w)} - \gamma I$ is the generator of the positive C_0-semigroup, $(S_\gamma^{(w)}(t))_{t\geq 0} = (e^{-\gamma t} S^{(w)}(t))_{t\geq 0}$ on ℓ_w^1. From Proposition 4.2.12 we can deduce that $(S_\gamma^{(w)}(t))_{t\geq 0}$ satisfies assumptions (b)–(d) of Theorem 4.2.5. It follows that (a) of Theorem 4.2.6 holds.

Moreover, it is clear that, for $t \in [0, T)$, $S_\gamma^{(w)}(t) = e^{-\gamma t} S^{(w)}(t)$ is positive and satisfies $\|S_\gamma^{(w)}(t)\|_{\mathcal{B}(\ell_w^1)} \leq \|S^{(w)}(t)\|_{\mathcal{B}(\ell_w^1)}$ and so assumption (b) in Theorem 4.2.6 is satisfied for any $\gamma \geq 0$.

Moreover, from Lemma 6.3.3, for every $r > 0$ and $t' \in [0, t_{max})$ there exists $\gamma > 0$ such that $K^{(w,w,\gamma)}(t, f) \in (\ell_w^1)_+$ for all $\overline{B}_{\ell_w^1}(0, r)_+$, $t \in [0, t']$. It follows from Theorem 4.2.6 that $u \geq 0$ on $[0, t']$. Since $t' \in [0, t_{max})$ was arbitrary, it follows that u is positive on $[0, t_{max})$, which proves part (i). Part (ii) follows from Theorem 6.3.2 (ii). $\square$

We recall that we use $X_{[1]}$ to denote ℓ_w^1 in the case where $w_n = n$ for all $n \in \mathbb{N}$ and we use $\|\cdot\|_{[1]}$ and M_1 to denote the norm and the corresponding bounded linear functional on $X_{[1]}$. As before, to distinguish between this physical space and other weighted ℓ^1 spaces, we drop the w in the notation when we work in $X_{[1]}$ and so $A \coloneqq A^{(w)}$, $B \coloneqq B^{(w)}$, $G \coloneqq G^{(w)}$, $S \coloneqq S^{(w)}$, $K \coloneqq K^{(v,w)}$, $K_1 \coloneqq K_1^{(v,w)}$, $K_2 \coloneqq K_2^{(v,w)}$, etc, when $w_n = n$. We also recall that a non-negative solution that is norm conserving in $X_{[1]}$ is also a mass-conserving solution. We now aim to obtain results regarding mass conservation.

Proposition 6.3.5. *Let Assumptions 6.1.1 and 5.2.2(ii) hold and let* (5.1.5) *hold. Let* $0 < T \leq \infty$, $\mathring{u} \in (\ell^1_w)_+$ *and let* $u \in C\Big([0, t_{max}), (\ell^1_w)_+\Big)$ *be a mild solution of* (6.1.11), (6.1.12) *for* $t \in [0, t_{max})$, *where* $0 < t_{max} \leq T$. *Then* u *is a mass-conserving solution, i.e.* $\|u(t)\|_{[1]} = \|\mathring{u}\|_{[1]}$ *for all* $t \in [0, t_{max})$.

Proof. For $t \in [0, t_{max})$, we have that u satisfies

$$u(t) = S^{(w)}(t)\mathring{u} + \int_0^t S^{(w)}(t-s)K^{(w,w)}(s, u(s))\,\mathrm{d}s.$$

Since $\ell^1_w \subseteq X_{[1]}$, it follows from (5.5.5), that $S^{(w)}(t)f$ and $S(t)f$ coincide for $f \in \ell^1_w$, $t \geq 0$. It is also clear that $K^{(w,w)}(t,f)$ and $K(t,f)$ coincide for $t \geq 0$, $f \in \ell^1_w$. Hence u must also satisfy

$$u(t) = S(t)\mathring{u} + \int_0^t S(t-s)K(s, u(s))\,\mathrm{d}s,$$

for $t \in [0, t_{max})$, i.e. u is a mild solution of (6.1.11), (6.1.12) posed in $X_{[1]}$, for $t \in [0, t_{max})$. Moreover, from Proposition 5.2.8, $(S(t))_{t\geq 0}$ is a stochastic C_0-semigroup. We now show that $M_1(K(t,f)) = 0$ for all $f \in \Big(X_{[1]}\Big)_+$ and then apply Lemma 4.2.11. Let $f \in (X_{[1]})_+$, $t \in [0, t_{max})$. Then we have

$$\begin{aligned}
M_1(K(t,f)) &= \sum_{n=1}^{\infty} n[K(t,f)]_n = \sum_{n=1}^{\infty} n\Big([K_1(t,f)]_n - [K_2(t,f)]_n\Big) \\
&= \sum_{n=1}^{\infty} n[K_1(t,f)]_n - \sum_{n=1}^{\infty} n[K_2(t,f)]_n.
\end{aligned}$$

Now,

$$\begin{aligned}
\sum_{n=1}^{\infty} n[K_1(t,f)]_n &= \frac{1}{2}\sum_{n=1}^{\infty} n \sum_{j=1}^{n-1} k_{n-j,j}(t) f_{n-j} f_j = \frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty} n k_{n-j,j}(t) f_{n-j} f_j \\
&= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} (l+j) k_{l,j}(t) f_l f_j \\
&= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} l k_{l,j}(t) f_j f_l + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} j k_{l,j}(t) f_l f_j
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} l k_{l,j}(t) f_j f_l + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} j k_{j,l}(t) f_l f_j \\
&= \sum_{j=1}^{\infty}\sum_{l=1}^{\infty} j k_{l,j}(t) f_j f_l \\
&= \sum_{j=1}^{\infty} j [K_2(t,f)]_j,
\end{aligned}
$$

where the changes in the order of summation are justified since each term is positive. It follows that $M_1(K(t,f)) = 0$ for all $f \in (X_{[1]})_+$, $t \in [0, t_{max})$ and so, from Lemma 4.2.11, we have that

$$
\|u(t)\|_{[1]} = \|\mathring{u}\|_{[1]}, \tag{6.3.2}
$$

for all $t \in [0, t_{max})$. Hence $u(t)$ is a mass-conserving solution. □

Remark 6.3.6. Let the assumptions of Proposition 6.3.5 hold. Then $w_n \geq n$ for all $n \in \mathbb{N}$ and so $\|f\|_{[1]} \leq \|f\|_{\ell^1_w}$ for all $f \in \ell^1_w$. Hence, while the solution in Proposition 6.3.5 is bounded in $X_{[1]}$, this does not imply that the solution is also bounded in ℓ^1_w. It follows that we may have $\lim\limits_{t \to \hat{T}^-} \|u(t)\|_{\ell^1_w} = \infty$ for some $\hat{T} < T$. In this case $t_{max} = \hat{T} \neq T$.

Using the previous result we can now draw some conclusions regarding the existence of a mass-conserving solution of (6.1.11) and (6.1.12).

Proposition 6.3.7. *Let* (5.1.5)*, the assumptions of Proposition 6.3.4(i) and Assumption 5.2.2(ii) hold. Then*

(i) *for some $t_{max} \in (0, T)$, there exists a unique, non-negative, mass-conserving mild solution $u \in C\Big([0, t_{max}), (\ell^1_w)_+\Big)$ of* (6.1.11), (6.1.12)*;*

(ii) *if $w_n = n$, then there exists a unique, non-negative, mass-conserving mild solution $u \in C\Big([0, T), (X_{[1]})_+\Big)$ of* (6.1.11), (6.1.12)*;*

(iii) *if, in addition, Assumption (A*3) holds and $\mathring{u} \in \mathcal{D}(G^{(w)})_+$, then the solutions in (i) and (ii) are unique, non-negative, mass-conserving classical solutions.*

Proof. We obtain part (i) immediately from Proposition 6.3.4(i) and Proposition 6.3.5. If $w_n = n$, then the system (6.1.11), (6.1.12) is posed in the space $X_{[1]}$. The existence of a unique, non-negative, mass-conserving mild solution follows as in part (i). Moreover, since the solution is bounded in $X_{[1]}$ by Proposition 6.3.5, it follows from Theorem 4.2.5(ii) that $t_{max} = T$. Hence part (ii) of this result holds. Finally, part (iii) follows from Proposition 6.3.4(ii). $\square$

The last proposition allows us to obtain the following result, which tells us that, when $w_n = n$ for all $n \in \mathbb{N}$, the solution in Proposition 6.3.5 depends continuously on the initial condition.

Corollary 6.3.8. *Let $\mathring{u} \in (X_{[1]})_+$. Suppose that the assumptions of Proposition 6.3.7(ii) hold and let u be the unique, non-negative, global mild solution of* (6.1.11), (6.1.12)*. Now suppose that $(\mathring{u}^{(m)})_{m=1}^{\infty}$ is such that $\mathring{u}^{(m)} \in (X_{[1]})_+$ for all $m \in \mathbb{N}$ and $\|\mathring{u}^{(m)} - \mathring{u}\|_{[1]} \to 0$ as $m \to \infty$. Then*

$$\|u^{(m)}(t) - u(t)\|_{[1]} \to 0 \qquad \textit{as } m \to \infty$$

uniformly in t on compact subintervals of $[0, T)$, where $u^{(m)}$ is the unique, non-negative, global mild solution of (6.1.11), (6.1.12) *corresponding to initial condition $\mathring{u}^{(m)}$.*

Proof. Since $\mathring{u}^{(m)} \to \mathring{u}$ as $m \to \infty$ in $X_{[1]}$, we have that there exists $C_0 > 0$, such that $\|\mathring{u}\|_{[1]} \leq C_0$ and $\|\mathring{u}^{(m)}\|_{[1]} \leq C_0$ for all $m \in \mathbb{N}$. Hence, by Proposition 6.3.5, $\|u(t)\|_{[1]}$, $\|u^{(m)}(t)\|_{[1]} \leq C_0$ for all $t \in [0, T)$ and $m \in \mathbb{N}$. The result then follows from Theorem 4.2.5(iv). $\square$

In Proposition 6.3.7(ii), we obtain a global solution in the space $X_{[1]}$. We now provide an additional assumption on the weight and coagulation rates, under which the local mild solution in Proposition 6.3.7(i) is a global mild solution in ℓ_w^1. In general, when $w_n = n^p$ for $p \geq 1$, we denote $X_{[p]} := \ell_w^1$, $\|\cdot\|_{[p]} := \|\cdot\|_{\ell_w^1}$, $\phi_{[p]} := \phi_{\ell_w^1}$ $G_{[p]} := G^{(w)}$, $A_{[p]} := A^{(w)}$, $B_{[p]} := B^{(w)}$ and $K_{[p]} := K^{(w,w)}$. We recall that a mass-conserving solution is a solution that satisfies $\|u(t)\|_{[1]} = \|\mathring{u}\|_{[1]}$. The following lemma, proved in [25, Lemma 2.3], is useful in the next proposition.

Lemma 6.3.9. *Let $p > 1$. Then*

$$(x+y)[(x+y)^p - x^p - y^p] \le c_p(x^p y + x y^p), \qquad x, y \in (0,\infty), \tag{6.3.3}$$

where $c_p = p$ if $p \in (1,2]$ and $c_p = 2^p - 2$ if $p > 2$.

Proposition 6.3.10. *For some $p > 1$, let Assumptions 6.1.1, 5.2.2(ii),* (A*1) *and* (A*2) *hold for $w_n = n^p$, $n \in \mathbb{N}$, and let* (5.1.5) *hold. Moreover, assume that for each $t' \in (0,T)$, there exists $h(t') > 0$ such that*

$$k_{n,j}(t) \le h(t')(n+j) \qquad \text{for all } n, j \in \mathbb{N} \text{ and } t \in [0,t']. \tag{6.3.4}$$

Fix $\mathring{u} \in (X_{[p]})_+$ and let u be the unique, non-negative, mass-conserving, maximal mild solution of (6.1.11), (6.1.12)*, as in Proposition 6.3.7(i). Then u is a global mild solution. In particular, for each $t' \in (0,T)$, there exists $\gamma > 0$ such that,*

$$\|u(t)\|_{[p]} \le \|\mathring{u}\|_{[p]} e^{t'(c_p h(t')\|\mathring{u}\|_{[1]} + \gamma)} \qquad \text{for all } t \in [0,t'], \tag{6.3.5}$$

where $c_p = p$ if $p \in (1,2]$ and $c_p = 2^p - 2$ if $p > 2$.

Proof. We first note that $\mathring{u} \in (X_{[p]})_+ \subseteq (X_{[1]})_+$. Let $[0, t_{max})$ be the maximal interval of existence of u. Since $G_{[p]} = \overline{A_{[p]} + B_{[p]}}$ is the generator of the substochastic C_0-semigroup, $(S_{[p]}(t))_{t\ge0}$, on $X_{[p]}$, for each $\gamma > 0$, $G_{[p]} - \gamma I$ is the generator of the substochastic C_0-semigroup, $(S_{[p],\gamma}(t))_{t\ge0} := (e^{-\gamma t} S_{[p]}(t))_{t\ge0}$. Moreover, from Proposition 4.2.12, we have that $(S_{[p,\gamma]}(t))_{t\ge0}$ satisfies (b)–(d) of Theorem 4.2.5. Hence assumptions (a) and (b) of Theorem 4.2.6 hold and, by Lemma 4.2.10, u satisfies

$$u(t) = S_{[p],\gamma}(t)\mathring{u} + \int_0^t S_{[p],\gamma}(t-s) K_{[p],\gamma}(s, u(s))\,\mathrm{d}s, \qquad t \in [0, t_{max}),$$

where we define $K_{[p],\gamma}(t,f) := K_{[p]}(t,f) + \gamma f$ for all $(t,f) \in [0,T) \times X_{[p]}$.

Let $t' \in [0, t_{max})$ and choose $r > 0$ such that $u(t) \in \overline{B}_{X_{[p]}}(0,r)_+$ for $t \in [0,t']$. Let $\gamma > C(t')r$. From Lemma 6.3.3, $K_{[p],\gamma}(t, u(t)) \in (X_{[p]})_+$ for $t \in [0,t']$.

Hence, using Lemma 6.3.9 to obtain the first inequality, for $t \in [0, t']$, we have

$$
\begin{aligned}
&\|K_{[p],\gamma}(t,u(t))\|_{[p]} \\
&= \phi_{[p]}\Big(K_{[p],\gamma}(t,u(t))\Big) \\
&= \sum_{n=1}^{\infty} n^p \left(\frac{1}{2} \sum_{j=1}^{n-1} k_{n-j,j}(t) u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j}(t) u_n(t) u_j(t) \right) + \gamma \|u(t)\|_{[p]} \\
&= \frac{1}{2} \sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} n^p k_{n-j,j}(t) u_{n-j}(t) u_j(t) - \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} n^p k_{n,j}(t) u_n(t) u_j(t) + \gamma \|u(t)\|_{[p]} \\
&= \frac{1}{2} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} (l+j)^p k_{l,j}(t) u_l(t) u_j(t) - \frac{1}{2} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} j^p k_{l,j}(t) u_l(t) u_j(t) \\
&\qquad\qquad - \frac{1}{2} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} l^p k_{l,j}(t) u_l(t) u_j(t) + \gamma \|u(t)\|_{[p]} \\
&= \frac{1}{2} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} [(l+j)^p - j^p - l^p] k_{l,j}(t) u_l(t) u_j(t) + \gamma \|u(t)\|_{[p]} \\
&\le \frac{c_p}{2} \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} \frac{l^p j + l j^p}{l+j} k_{l,j}(t) u_l(t) u_j(t) + \gamma \|u(t)\|_{[p]} \\
&\le \frac{c_p}{2} h(t') \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} (l^p j + l j^p) u_l(t) u_j(t) + \gamma \|u(t)\|_{[p]} \\
&= \frac{c_p}{2} h(t') \Big(\|u(t)\|_{[1]} \|u(t)\|_{[p]} + \|u(t)\|_{[1]} \|u(t)\|_{[p]} \Big) + \gamma \|u(t)\|_{[p]} \\
&= c_p h(t') \|\mathring{u}\|_{[1]} \|u(t)\|_{[p]} + \gamma \|u(t)\|_{[p]} \\
&= (c_p h(t') \|\mathring{u}\|_{[1]} + \gamma) \|u(t)\|_{[p]},
\end{aligned}
$$

where $c_p = p$ if $p \in (1, 2]$ and $c_p = 2^p - 2$ if $p > 2$.

Now, for $t \in [0, t']$,

$$
\begin{aligned}
\|u(t)\|_{[p]} &= \left\| S_{[p],\gamma}(t)\mathring{u} + \int_0^t S_{[p],\gamma}(t-s) K_{[p],\gamma}(s, u(s)) \,\mathrm{d}s \right\|_{[p]} \\
&\le \left\| S_{[p],\gamma}(t)\mathring{u} \right\|_{[p]} + \int_0^t \left\| S_{[p],\gamma}(t-s) K_{[p],\gamma}(s, u(s)) \right\|_{[p]} \,\mathrm{d}s \\
&= \left\| e^{-\gamma t} S_{[p]}(t)\mathring{u} \right\|_{[p]} + \int_0^t \left\| e^{-\gamma t} S_{[p]}(t-s) K_{[p],\gamma}(s, u(s)) \right\|_{[p]} \,\mathrm{d}s
\end{aligned}
$$

$$
\begin{aligned}
&\leq \|\mathring{u}\|_{[p]} + \int_0^t \left\| K_{[p],\gamma}(s, u(s)) \right\|_{[p]} \mathrm{d}s \\
&\leq \|\mathring{u}\|_{[p]} + \int_0^t \Big(c_p h(t') \|\mathring{u}\|_{[1]} + \gamma \Big) \|u(s)\|_{[p]} \, \mathrm{d}s.
\end{aligned}
$$

It follows from Grönwall's inequality that

$$
\|u(t)\|_{[p]} \leq \|\mathring{u}\|_{[p]} e^{\int_0^t c_p h(t') \|\mathring{u}\|_{[1]} + \gamma \, \mathrm{d}s} \leq \|\mathring{u}\|_{[p]} e^{t' (c_p h(t') \|\mathring{u}\|_{[1]} + \gamma)}
$$

for all $t \in [0, t']$. Since $t' \in [0, t_{max})$ was chosen arbitrarily, it follows that $\|u(t)\|_{[p]}$ does not blow up in finite time and so u is a global solution, i.e. $t_{max} = T$. □

Remark 6.3.11. We make the following observations.

(i) We note that in Proposition 6.3.7(ii), we obtain a global solution of (6.1.11), (6.1.12), in $X_{[1]}$, without imposing (6.3.4).

(ii) We obtain a global mild solution for all $\mathring{u} \in (X_{[p]})_+$, for some $p \in \mathbb{N}$, in Proposition 6.3.10. However, for the case where $w_n = n$ for all $n \in \mathbb{N}$, we obtain a global solution for a larger class of initial conditions. In particular, in Proposition 6.3.7(ii) and (iii), we obtain a global mild solution for all $\mathring{u} \in (X_{[1]})_+$ and a global classical solution for all $\mathring{u} \in \mathcal{D}(G)_+$.

(iii) In a similar way as in Corollary 6.3.8, we can show that the global solution obtained in Proposition 6.3.10 depends continuously on the initial condition $\mathring{u} \in (X_{[1]})_+$.

6.4 The Pointwise C–F system

In this section we consider the case where $w_n = n$. We assume that Assumption 6.1.1, (A*1), (A*2) and (A*3) hold for $w_n = n$. In addition, we let Assumption 5.2.2(ii) and (5.1.5) hold. We now also assume that as $j \to \infty$,

$$
a_j b_{n,j} = \mathcal{O}(j) \qquad \text{for every fixed } n \in \mathbb{N}.
$$

In a similar way as for the pure fragmentation system, we can show that, for $\mathring{u} \in \mathcal{D}(G)_+$, the solution u of (6.1.11), (6.1.12) is such that $u_n(t) = (u(t))_n$ satisfies (6.1.1). In this section we try to deduce in what way $u_n(t)$ satisfies (6.1.1) if $\mathring{u} \in (X_{[1]})_+$.

From Proposition 6.3.7 we have that, for each $\mathring{u} \in (X_{[1]})_+$, the ACP (6.1.11), (6.1.12) has a unique, non-negative, mass-conserving mild solution, which exists globally in time. Moreover, if $\mathring{u} \in \mathcal{D}(G)_+$, then this is the unique classical solution. Let $\mathring{u} \in (X_{[1]})_+$ and, for $m \in \mathbb{N}$, define $\mathring{u}^{(m)}$ by

$$\mathring{u}_n^{(m)} = \begin{cases} \mathring{u}_n & \text{if } n \leq m, \\ 0 & \text{if } n > m. \end{cases}$$

Since $\mathring{u} \in (X_{[1]})_+$ then $(\mathring{u}^{(m)})_{m=1}^{\infty}$ is such that $\mathring{u}^{(m)} \in \mathcal{D}(A)_+ \subseteq \mathcal{D}(G)_+$ for all $m \in \mathbb{N}$.

Let $u^{(m)}$ be the classical (and so also the mild) solution of (6.1.11) with $u(0) = \mathring{u}^{(m)}$ and let u be the mild solution for $u(0) = \mathring{u}$. Since u and $u^{(m)}$ are mass conserving solutions and $\|\mathring{u}^{(m)}\|_{[1]} \leq \|\mathring{u}\|_{[1]}$ for all $m \in \mathbb{N}$, we have that, $\|u(t)\|_{[1]}, \|u^{(m)}(t)\|_{[1]} \leq \|\mathring{u}\|_{[1]}$ for all $m \in \mathbb{N}$, $t \in [0, T)$ and so, from Theorem 4.2.5(iv), $u^{(m)} \to u$ uniformly as $m \to \infty$ on compact intervals $[0, \tau]$.

We now examine in what way u satisfies (6.1.1). Since $\mathring{u}^{(m)} \in \mathcal{D}(G)_+$, we have that $u_n^{(m)}$ satisfies (6.1.1) and hence also satisfies the integral equation

$$\begin{aligned} u_n^{(m)}(t) = \mathring{u}_n^{(m)} + \int_0^t (-a_n u_n^{(m)}(s))\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(m)}(s)\mathrm{d}s \\ + \frac{1}{2}\int_0^t \sum_{j=1}^{n-1} k_{n-j,j}(s) u_{n-j}^{(m)}(s) u_j^{(m)}(s)\mathrm{d}s \\ - \int_0^t \sum_{j=1}^{\infty} k_{n,j}(s) u_n^{(m)}(s) u_j^{(m)}(s)\,\mathrm{d}s \end{aligned} \tag{6.4.1}$$

for all $t \in [0, T)$. Let $\tau \in [0, T)$. Then $u^{(m)}(t) \to u(t)$ as $m \to \infty$ uniformly for all $t \in [0, \tau]$. Taking limits in (6.4.1) we obtain, for $t \in [0, \tau]$,

$$
\begin{aligned}
u_n(t) = \mathring{u}_n + \lim_{m\to\infty} \Bigg\{ & \int_0^t (-a_n u_n^{(m)}(s))\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(m)}(s)\mathrm{d}s \\
& + \frac{1}{2}\int_0^t \sum_{j=1}^{n-1} k_{n-j,j}(s) u_{n-j}^{(m)}(s) u_j^{(m)}(s)\mathrm{d}s \\
& - \int_0^t \sum_{j=1}^{\infty} k_{n,j}(s) u_n^{(m)}(s) u_j^{(m)}(s)\mathrm{d}t \Bigg\}. \quad (6.4.2)
\end{aligned}
$$

It follows that

$$
\lim_{m\to\infty} \int_0^t (-a_n u_n^{(m)}(s))\,\mathrm{d}s = \int_0^t (-a_n u_n(s))\,\mathrm{d}s.
$$

Choose $r > 0$ such that $u^{(m)}(t),\ u(t) \in \overline{B}_{X_{[1]}}(0,r)$ for $t \in [0,\tau]$. Similarly to Lemma 6.2.1, we can show that there exists an $L_1(\tau,0,r) > 0$ such that for $t \in [0,\tau]$,

$$
\|K_1(t,u^{(m)}(t)) - K_1(t,u(t))\|_{[1]} \le L_1(\tau,0,r)\|u^{(m)}(t) - u(t)\|_{[1]}.
$$

It follows that $K_1(t,u^{(m)}(t)) \to K_1(t,u(t))$ as $m \to \infty$, uniformly in t on $[0,\tau]$. Similarly, $K_2(t,u^{(m)}(t)) \to K_2(t,u(t))$ as $m \to \infty$, uniformly in t on $[0,\tau]$. Thus, we also have that

$$
\begin{aligned}
\lim_{m\to\infty} \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t) u_{n-j}^{(m)}(t) u_j^{(m)}(t) &= \lim_{m\to\infty} \Big(K_1(t,u^{(m)}(t))\Big)_n = \Big(K_1(t,u(t))\Big)_n \\
&= \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t) u_{n-j}(t) u_j(t)
\end{aligned}
$$

uniformly in t, for all $t \in [0,\tau]$. Hence for $t \in [0,\tau]$,

$$
\lim_{m\to\infty} \int_0^t \sum_{j=1}^{n-1} k_{n-j,j}(s) u_{n-j}^{(m)}(s) u_j^{(m)}(s)\,\mathrm{d}s = \int_0^t \sum_{j=1}^{n-1} k_{n-j,j}(s) u_{n-j}(s) u_j(s)\,\mathrm{d}s.
$$

Next, we have

$$\begin{aligned}\lim_{m\to\infty}\sum_{j=1}^{\infty} k_{n,j}(t)u_n^{(m)}(t)u_j^{(m)}(t) &= \lim_{m\to\infty}\Big(K_2(t,u^{(m)}(t))\Big)_n = \Big(K_2(t,u(t))\Big)_n \\ &= \sum_{j=1}^{\infty} k_{n,j}(t)u_n(t)u_j(t)\end{aligned}$$

uniformly in t, for all $t \in [0,\tau]$, and so

$$\lim_{m\to\infty}\int_0^t \sum_{j=1}^{\infty} k_{n,j}(s)u_n^{(m)}(s)u_j^{(m)}(s)\,\mathrm{d}s = \int_0^t \sum_{j=1}^{\infty} k_{n,j}(s)u_n(s)u_j(s)\,\mathrm{d}s.$$

Consider

$$\begin{aligned}&\left|\sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(m)}(t) - \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t)\right| \\ &\le \sum_{j=n+1}^{\infty} a_j b_{n,j}|u_j^{(m)}(t)-u_j(t)| = \sum_{j=n+1}^{\infty}\frac{a_j b_{n,j}}{j} j|u_j^{(m)}(t)-u_j(t)| \\ &\le \sum_{j=n+1}^{\infty} Lj|u_j^{(m)}(t)-u_j(t)| \qquad (\text{for some } L>0) \\ &\to 0 \qquad \text{as } m\to\infty \text{ uniformly in } t \text{ on } [0,\tau].\end{aligned}$$

Thus,

$$\lim_{m\to\infty}\sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(m)}(t) = \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t),$$

uniformly in t on $[0,\tau]$. It follows that, for $t \in [0,\tau]$,

$$\lim_{m\to\infty}\int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j^{(m)}(s)\,\mathrm{d}s = \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(s)\,\mathrm{d}s.$$

Hence, from (6.4.2), for $t \in [0,\tau]$ and $\mathring{u} \in X_+$ we have that u satisfies

$$u_n(t) = \mathring{u}_n + \int_0^t (-a_n u_n(s))\mathrm{d}s + \int_0^t \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(s)\mathrm{d}s$$
$$+ \frac{1}{2}\int_0^t \sum_{j=1}^{n-1} k_{n-j,j}(s) u_{n-j}(s) u_j(s)\mathrm{d}s$$
$$- \int_0^t \sum_{j=1}^{\infty} k_{n,j}(s) u_n(s) u_j(s)\mathrm{d}s. \quad (6.4.3)$$

Since $\tau \in (0, T)$ was arbitrary, (6.4.3) holds for all $t \in [0, T)$ and we have that $u_n(t)$ is absolutely continuous for each $n \in \mathbb{N}$. So, from the definition of absolute continuity, we have that $u_n(t)$ satisfies the original system (6.1.1) for almost every $t \in [0, T)$.

Remark 6.4.1. It is worth noting again that, from Remark 6.1.4, the arguments presented in this section all hold for the special case of a system where both the fragmentation coefficients and the coagulation coefficients are independent of time for all $n, j \in \mathbb{N}$.

6.5 Interpolation Spaces and the C–F System

In this section we use the theory in Section 3.3.3 to relax the assumptions required on the coagulation rates and obtain the existence and uniqueness of a non-negative solution to (6.1.11), (6.1.12) for a certain class of initial conditions. We note that the theory of interpolation spaces has been used in [10], to examine discrete time-independent coagulation–fragmentation, and in [11, 14, 17] to deal with continuous time-independent coagulation–fragmentation. We make the following assumption throughout this subsection.

Assumption 6.5.1. Let $a_n \geq 0$, $b_{n,j} \geq 0$, $k_{n,j}(t) \geq 0$ for all n, $j \in \mathbb{N}$ and $t \geq 0$. In addition, let $b_{n,j} = 0$ for $j \leq n$ and let $k_{n,j}(t) = k_{j,n}(t)$ for all n, $j \in \mathbb{N}$ and $t \geq 0$. Moreover, let $(w_n)_{n=1}^{\infty}$ be a monotone increasing sequence such that Assumption 5.2.2(i) holds for some $\delta \in (0, 1)$.

From Theorem 5.4.1 we have that $G^{(w)} = A^{(w)} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup, $(S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. Since $(S^{(w)}(t))_{t\geq 0}$ is substochastic, to obtain a semigroup with a negative growth bound we write the coagulation–fragmentation system as

$$\begin{aligned} u_n'(t) &= -a_n u_n(t) - u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \\ &\quad + u_n(t) + \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}(t) u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j}(t) u_n(t) u_j(t), \quad t > 0 \\ u_n(0) &= \mathring{u}. \end{aligned} \tag{6.5.1}$$

Fix $0 < T \leq \infty$. For $(w_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$ as in Section 6.1 we can write (6.5.1) as an ACP in ℓ_w^1 and obtain

$$\begin{aligned} u'(t) &= G_{w,1} u(t) + u(t) + K^{(v,w)}(t, u(t)), \quad t \in [0, T); \\ u(0) &= \mathring{u}, \end{aligned} \tag{6.5.2}$$

where $G_{w,1} \coloneqq A_{w,1} + B^{(w)}$, with $A_{w,1} = A^{(w)} - I$, and $K^{(v,w)} : [0,T) \times \ell_v^1 \to \ell_w^1$. The operator $G_{w,1}$ is invertible and generates the analytic, substochastic semigroup given by $(S_{w,1}(t))_{t\geq 0} = (e^{-t} S^{(w)}(t))_{t\geq 0}$.

Lemma 6.5.2. *Let Assumption 6.5.1 hold and let $G_{w,1}$ and $A_{w,1}$ be as defined above. Then, for all $f \in \mathcal{D}(A_{w,1})$,*

$$c^{-1} \|A_{w,1} f\|_{\ell_w^1} \leq \|G_{w,1} f\|_{\ell_w^1} \leq c \|A_{w,1} f\|_{\ell_w^1}. \tag{6.5.3}$$

where $c = \max\{1 + \delta, \frac{1}{1-\delta}\}$.

Proof. From Lemma 5.2.6 we have, for all $f \in \mathcal{D}(A_{w,1}) = \mathcal{D}(A^{(w)})$, that

$$\|B^{(w)} f\|_{\ell_w^1} \leq \delta \|A^{(w)} f\|_{\ell_w^1} \leq \delta \|A_{w,1} f\|_{\ell_w^1}.$$

Since $\delta \in (0, 1)$ the result follows from Proposition 3.3.26. □

Let $D_A(\alpha, p)$ and $\|\cdot\|_{D_A(\alpha,p)}$ be as in (3.3.10) and (3.3.11) respectively. Our aim is to apply Proposition 4.2.15, with $p = 1$, to (6.5.2). We therefore first want to characterise $D_{G_{w,1}}(\alpha, 1)$. As the next proposition shows, this is equivalent to characterising $D_{A_{w,1}}(\alpha, 1)$.

Proposition 6.5.3. *Let Assumption 6.5.1 hold. For* $\alpha \in (0, 1)$ *and* $p \in [1, \infty]$, *we have*

$$D_{G_{w,1}}(\alpha, p) = D_{A_{w,1}}(\alpha, p), \tag{6.5.4}$$

with equivalence of their respective norms.

Proof. We have shown in the proof of Theorem 5.4.1 that the substochastic semigroup, $(T^{(w)}(t))_{t\geq 0}$, generated by $A^{(w)}$ is analytic. Hence, $(e^{-t}T^{(w)}(t))_{t\geq 0}$, the C_0-semigroup generated by $A_{w,1}$, is also analytic. The growth bound of the semigroup $(e^{-t}T^{(w)}(t))_{t\geq 0}$ is negative and so $A_{w,1}$ is invertible. The result then follows immediately from Lemma 6.5.2 and Corollary 3.3.25. □

For $\alpha > 0$, we define the fractional power $(-A_{w,1})^\alpha$ as in (3.3.15). To characterise $D_{G_{w,1}}(\alpha, 1)$, the following lemma will be useful.

Lemma 6.5.4. *Let Assumption 6.5.1 hold. For* $\alpha > 0$,

$$\mathcal{D}((-A_{w,1})^\alpha) = X_{w,\alpha} := \left\{ g \in \ell^1_w : \sum_{n=1}^{\infty} w_n(a_n+1)^\alpha |g_n| < \infty \right\}. \tag{6.5.5}$$

Proof. Let $f \in \ell^1_w$. Then, from (3.3.13),

$$\begin{aligned}
&[(-A_{w,1})^{-\alpha} f]_n \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty t^{\alpha-1} e^{-(a_n+1)t}\,\mathrm{d}t\, f_n \\
&= \frac{1}{\Gamma(\alpha)} \int_0^\infty \left(\frac{u}{a_n+1}\right)^{\alpha-1} e^{-u}\,\frac{\mathrm{d}u}{a_n+1}\, f_n \qquad \text{(taking } u = (a_n+1)t\text{)} \\
&= \frac{1}{\Gamma(\alpha)} (a_n+1)^{-\alpha} \int_0^\infty u^{\alpha-1} e^{-u}\,\mathrm{d}u\, f_n \\
&= (a_n+1)^{-\alpha} f_n.
\end{aligned}$$

It follows that

$$\begin{aligned}\operatorname{range}((-A_{w,1})^{-\alpha}) &= \{g \in \ell_w^1 : g_n = (a_n+1)^{-\alpha} f_n \text{ for some } f \in \ell_w^1\} \\ &= \left\{ g \in \ell_w^1 : \sum_{n=1}^{\infty} w_n (a_n+1)^{\alpha} |g_n| < \infty \right\}.\end{aligned}$$

We have that $\mathcal{D}((-A_{w,1})^{\alpha}) = \operatorname{range}((-A_{w,1})^{-\alpha})$ and so (6.5.5) holds. $\square$

The following result gives a characterisation of $D_{G_{w,1}}(\alpha, 1)$.

Proposition 6.5.5. *Let Assumption 6.5.1 hold and let* $\alpha \in (0,1)$. *Then*

$$D_{G_{w,1}}(\alpha,1) = X_{w,\alpha} = \left\{ f \in \ell_w^1 : \sum_{n=1}^{\infty} w_n (a_n+1)^{\alpha} |f_n| < \infty \right\}. \tag{6.5.6}$$

Proof. We have

$$\begin{aligned}D_{G_{w,1}}(\alpha,1) &= D_{A_{w,1}}(\alpha,1) \\ &= \{ f \in \ell_w^1 : t \mapsto v(t) := \|t^{-\alpha} A_{w,1} e^{A_{w,1} t} f\|_{\ell_w^1} \in L^1(0,1)\}.\end{aligned}$$

Let $f \in X_{w,\alpha}$. Then, using the Fubini–Tonelli Theorem, we have

$$\begin{aligned}&\|v(t)\|_{L^1(0,1)} \\ &= \int_0^1 \sum_{n=1}^{\infty} w_n t^{-\alpha} (a_n+1) e^{-(a_n+1)t} |f_n| \,\mathrm{d}t \\ &= \sum_{n=1}^{\infty} w_n (a_n+1)|f_n| \int_0^{a_n+1} \left(\frac{u}{a_n+1} \right)^{-\alpha} e^{-u} \frac{\mathrm{d}u}{a_n+1} \qquad \text{(taking } u = (a_n+1)t\text{)} \\ &= \sum_{n=1}^{\infty} w_n (a_n+1)^{\alpha} |f_n| \int_0^{a_n+1} u^{-\alpha} e^{-u} \,\mathrm{d}u \\ &\le \sum_{n=1}^{\infty} w_n (a_n+1)^{\alpha} |f_n| \int_0^{\infty} u^{-\alpha} e^{-u} \,\mathrm{d}u \\ &= \sum_{n=1}^{\infty} w_n (a_n+1)^{\alpha} |f_n| \int_0^{\infty} u^{(1-\alpha)-1} e^{-u} \,\mathrm{d}u\end{aligned}$$

$$= \Gamma(1-\alpha)\sum_{n=1}^{\infty} w_n(a_n+1)^{\alpha}|f_n| < \infty.$$

Hence $X_{w,\alpha} \subseteq D_{G_{w,1}}(\alpha, 1)$.

Also, from (3.3.16), we have that

$$D_{G_{w,1}}(\alpha,1) = D_{A_{w,1}}(\alpha,1) \subseteq \mathcal{D}((-A_{w,1})^{\alpha})$$

and, from (6.5.5), $\mathcal{D}((-A_{w,1})^{\alpha}) = X_{w,\alpha}$. Thus $D_{G_{w,1}}(\alpha,1) \subseteq X_{w,\alpha}$ and so (6.5.6) holds. □

Let $\alpha \in (0,1)$ and let $0 < T \leq \infty$. We now consider the case where $K^{(v,w)} : [0,T) \times D_{G_{w,1}}(\alpha,1) \mapsto \ell_w^1$. We recall that, for all $n, j \in \mathbb{N}$,

$$W_{n,j} = \frac{w_{n+j}}{w_n w_j}.$$

Assumption 6.5.6. In this case, i.e. when $\ell_v^1 = D_{G_{w,1}}(\alpha,1)$, $0 < \alpha < 1$, we can write assumptions (A1) and (A2), respectively, as

(A1) for each $t' \in [0,T)$, there exists a constant $C(t') > 0$ such that for all $n, j \in \mathbb{N}$, $t \in [0,t']$

$$\frac{W_{n,j}}{(a_n+1)^{\alpha}(a_j+1)^{\alpha}} k_{n,j}(t) \leq C(t'); \tag{6.5.7}$$

(A2) the family of functions $\left(\frac{W_{n,j}}{(a_n+1)^{\alpha}(a_j+1)^{\alpha}} k_{n,j}(\cdot)\right)_{n,j\in\mathbb{N}}$ is equicontinuous on the interval $[0,T)$.

We have that (A1) and (A2) are (A1) and (A2) with $v_n = w_n(a_n+1)^{\alpha}$ for all $n \in \mathbb{N}$, respectively. Note that (A1) is weaker than Assumption (A*1) used to obtain the existence and uniqueness results in Section 6.3.

Proposition 6.5.7.

(i) *Let $w_n = r^n$ for some $r > 0$ and all $n \in \mathbb{N}$. Then (A1) is satisfied if for each $t' \in [0,T)$, there exists a constant $\tilde{C}(t') > 0$ such that for all $t \in [0,t']$*

$$k_{n,j}(t) \leq \tilde{C}(t')(a_n+1)^{\alpha}(a_j+1)^{\alpha} \tag{6.5.8}$$

for all $n, j \in \mathbb{N}$.

(ii) *Let* $w_n = n^p$ *for all* $n \in \mathbb{N}$ *and some* $p \geq 1$. *Then* (A1) *is satisfied if for each* $t' \in [0, T)$, *there exists a constant* $\hat{C}(t') > 0$ *such that for all* $t \in [0, t']$

$$k_{n,j}(t) \leq \hat{C}(t')(\min\{n, j\})^p (a_n + 1)^\alpha (a_j + 1)^\alpha \tag{6.5.9}$$

for all $n, j \in \mathbb{N}$.

Proof. Suppose that the weight w is of the form $w_n = r^n$ for some $r > 0$ and all $n \in \mathbb{N}$. Then

$$\frac{w_n w_j}{w_{n+j}} = 1$$

and so, for $t' \in [0, T)$ and $t \in [0, t']$,

$$\frac{W_{n,j}}{(a_n + 1)^\alpha (a_j + 1)^\alpha} k_{n,j}(t) = \frac{1}{(a_n + 1)^\alpha (a_j + 1)^\alpha} k_{n,j}(t) \leq \tilde{C}(t')$$

if (6.5.8) is satisfied. Hence (i) holds.

To prove (ii) we now consider the case when $w_n = n^p$ for some $p \geq 1$ and all $n \in \mathbb{N}$. In this case

$$\frac{w_n w_j}{w_{n+j}} = \frac{n^p j^p}{(n+j)^p} = \left(\frac{nj}{n+j}\right)^p \geq \left[\frac{\min\{n,j\} \max\{n,j\}}{2 \max\{n,j\}}\right]^p = \left(\frac{\min\{n,j\}}{2}\right)^p.$$

Hence, for $t' \in [0, T)$ and all $t \in [0, t']$,

$$\begin{aligned} \frac{W_{n,j}}{(a_n + 1)^\alpha (a_j + 1)^\alpha} k_{n,j}(t) &\leq \frac{2^p}{(\min\{n,j\})^p (a_n + 1)^\alpha (a_j + 1)^\alpha} k_{n,j}(t) \\ &\leq 2^p \hat{C}(t') \end{aligned}$$

if (6.5.9) is satisfied. It follows that (ii) holds. □

Remark 6.5.8. The theory of interpolation spaces is used by Banasiak in [10] to examine (6.1.1), where the fragmentation coefficients $(b_{n,j})_{n,j \in \mathbb{N}: n \leq j}$ are such that Assumption 6.1.1 holds for $w_n = n^p$ for some $p \geq 1$ and all $n \in \mathbb{N}$. Moreover, [10] considers the case when $k_{n,j}(t) \coloneqq k_{n,j}$ is independent of time and satisfies

$$k_{n,j} \leq C\Big((a_n+1)^\alpha + (a_j+1)^\alpha\Big), \tag{6.5.10}$$

for some $C > 0$. We now compare our assumption (A1) with (6.5.10).

We first note that

$$\begin{aligned}(a_n+1)^\alpha(a_j+1)^\alpha &= \frac{1}{2}\Big[(a_n+1)^\alpha(a_j+1)^\alpha + (a_n+1)^\alpha(a_j+1)^\alpha\Big] \\ &\geq \frac{1}{2}\Big[(a_n+1)^\alpha + (a_j+1)^\alpha\Big].\end{aligned}$$

It follows that

$$(a_n+1)^\alpha + (a_j+1)^\alpha \leq 2(\min\{n,j\})^p(a_n+1)^\alpha(a_j+1)^\alpha \tag{6.5.11}$$

for all $p \geq 1$. Hence (6.5.9), considered in Proposition 6.5.7(ii) in the case when $w_n = n^p$ for some $p \geq 1$ and all $n \in \mathbb{N}$, is less severe than (6.5.10) and, moreover, allows for time-dependent coagulation coefficients. When the fragmentation coefficients are such that we can choose $w_n = n^p$ for all $n \in \mathbb{N}$ and some $p \geq 1$, we have shown in Proposition 6.5.7(ii) that (6.5.9) implies that (A1) holds. It follows from (6.5.11) that $k_{n,j}(t)$, for n, $j \geq 1$, can be larger under (A1) than under the condition (6.5.10).

Also, we have shown in Proposition 5.4.7 and Theorem 5.4.1 that, when there is no mass gain in the C–F system, i.e. (5.1.4) holds with $\lambda_j \in [0,1]$ for $j = 2,3,\ldots$, we are always able to choose the weight w such that Assumption 6.1.1 holds and $w_n = r^n$ for some $r > 0$ and all $n \in \mathbb{N}$. It follows that for *any* fragmentation coefficients $(b_{n,j})_{n,j\in\mathbb{N}:n\leq j}$, we are always able to find a space ℓ_w^1 such that the results in this section hold for coagulation coefficients satisfying (6.5.8).

Let assumptions (A1) and (A2) hold. We have that the operator $G_{w,1}$ generates a positive, analytic C_0-semigroup. From Lemma 6.2.1 and Remark 6.2.2, we have that $K^{(v,w)}$ is a continuous mapping from $[0,T) \times D_{G_{w,1}}(\alpha,1)$ into ℓ_w^1 and, for $r > 0$, $t' \in [0,T)$, there exists a constant $L(t',r) > 0$ such that for all

$f, g \in \overline{B}_{D_{G_{w,1}}(\alpha,1)}(0, r)$ and $t \in [0, t']$,

$$\|K^{(v,w)}(t, f) - K^{(v,w)}(t, g)\|_{\ell^1_w} \le L(t', r)\|f - g\|_{D_{G_{w,1}}(\alpha,1)}.$$

It follows that, for all $t \in [0, t']$, $r > 0$, $f, g \in \overline{B}_{D_{G_{w,1}}(\alpha,1)}(0, r)$,

$$\begin{aligned}
&\|f + K^{(v,w)}(t, f) - (g + K^{(v,w)}(t, g))\|_{\ell^1_w} \\
&\le \|f - g\|_{\ell^1_w} + L(t', r)\|f - g\|_{D_{G_{w,1}}(\alpha,1)} \\
&\le \|f - g\|_{D_{G_{w,1}}(\alpha,1)} + L(t', r)\|f - g\|_{D_{G_{w,1}}(\alpha,1)} \\
&= (1 + L(t', r))\|f - g\|_{D_{G_{w,1}}(\alpha,1)},
\end{aligned}$$

i.e.

$$\|f + K^{(v,w)}(t, f) - (g + K^{(v,w)}(t, g))\|_w \le (1 + L(t', r))\|f - g\|_{D_{G_{w,1}}(\alpha,1)}, \quad (6.5.12)$$

for all $t \in [0, t']$.

We can now state the following existence and uniqueness result. We set

$$D_{G_{w,1}}(0, 1) = \ell^1_w; \quad \|\cdot\|_{D_{G_{w,1}}(0,1)} = \|\cdot\|_{\ell^1_w}.$$

Theorem 6.5.9. *Let $\alpha \in [0, 1)$. Let Assumption 6.5.1, (A1) and (A2) hold and let $0 < T \le \infty$. Take $\mathring{u} \in D_{G_{w,1}}(\alpha, 1) = X_{w,\alpha}$.*

(i) *Then for some t_{max} such that $0 < t_{max} \le T$, the ACP (6.5.2) has a unique mild solution, $u \in C\Big([0, t_{max}), D_{G_{w,1}}(\alpha, 1)\Big)$.*

(ii) *Let $(\mathring{u}^{(m)})_{m=1}^{\infty}$ be such that $\mathring{u}^{(m)} \to \mathring{u}$ as $m \to \infty$ and let $u^{(m)}$ be the unique, maximal, mild solution corresponding to $\mathring{u}^{(m)}$ for $m \in \mathbb{N}$. Then there exists some $t'_{max} > 0$ such that, for all $m \in \mathbb{N}$, the solutions u, $u^{(m)}$ exist on $[0, t'_{max})$. Moreover, there exists a constant $\delta_0 > 0$ such that $u^{(m)}(t) \to u(t)$ as $m \to \infty$ uniformly for $t \in [0, \delta_0]$.*

Proof. The assertions follow from Proposition 4.2.15 and Theorem 4.2.5, since (6.5.12) holds, $t \mapsto f + K^{(v,w)}(t, f)$ is continuous on $[0, T)$ for $f \in D_{G_{w,1}}(\alpha, 1)$, and $G_{w,1}$ is an invertible operator that generates an analytic, substochastic, C_0-

semigroup. □

The next result shows that the mild solution from Theorem 6.5.9 is non-negative if the initial condition is non-negative. The technique used to prove this theorem is based on the proof of [14, Theorem 2.2].

Theorem 6.5.10. *Let Assumption 6.5.1,* (A1) *and* (A2) *hold. Let* $\alpha \in [0,1)$*,* $0 < T \leq \infty$ *and* $\mathring{u} \in D_{G_{w,1}}(\alpha,1)_+$*. Then the mild solution,* u*, from Theorem 6.5.9 is non-negative.*

Proof. We first note that, from [43, Corollary 2.2.3(iv)], $\mathcal{D}(A^{(w)}) = \mathcal{D}(G_{w,1})$ is dense in $D_{G_{w,1}}(\alpha,1)$. Let $\gamma \geq 0$. We have that $G_{w,1} = A_{w,1} + B^{(w)}$ is the generator of an analytic, substochastic C_0-semigroup, $(e^{-t}S^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. Also, $D((-A_{w,1})^\alpha) = X_{w,\alpha} = D_{G_{w,1}}(\alpha,1)$ and it follows from [43, Proposition 2.4.1 (i)] that the operator $G_{w,1} - \gamma(-A_{w,1})^\alpha$ is the generator of an analytic C_0-semigroup, $(S_\gamma^{(w)}(t))_{t\geq 0}$, on ℓ_w^1. Moreover the operator $\gamma(-A_{w,1})^\alpha$ satisfies

$$\|\gamma(-A_{w,1})^\alpha f\|_{\ell_w^1} = \gamma\|f\|_{D_{G_{w,1}}(\alpha,1)} \qquad \text{for all } f \in D_{G_{w,1}}(\alpha,1)$$

and so is a bounded operator from $D_{G_{w,1}}(\alpha,1)$ into ℓ_w^1.

Now, $-\gamma(-A_{w,1})^\alpha$ generates the substochastic C_0-semigroup, $(\tilde{S}(t))_{t\geq 0}$, given by $[\tilde{S}(t)f]_n = e^{-\gamma(a_n+1)^\alpha t} f_n$, for all $n \in \mathbb{N}$, $t \geq 0$ and $f \in \ell_w^1$. Hence

$$\left\| \left[e^{-\frac{t}{n}} S^{(w)}\left(\frac{t}{n}\right) \tilde{S}\left(\frac{t}{n}\right) \right]^n \right\|_{\mathcal{B}(\ell_w^1)} \leq e^{-t} \leq 1 \qquad \text{for all } t \geq 0 \text{ and } n \in \mathbb{N}, \tag{6.5.13}$$

and so, from [30, Corollary III.5.8], the semigroup $(S_\gamma^{(w)}(t))_{t\geq 0}$, is given by

$$\lim_{n\to\infty} \left[e^{-\frac{t}{n}} S^{(w)}\left(\frac{t}{n}\right) \tilde{S}\left(\frac{t}{n}\right) \right]^n$$

and so is substochastic (in particular, positive). In addition, from (6.5.13), we have that $G_{w,1} - \gamma(-A_{w,1})^\alpha$ is invertible.

From Proposition 4.2.15, we can conclude that $G_{w,1} - \gamma(-A_{w,1})^\alpha$ satisfies assumptions (b)–(d) of Theorem 4.2.5. Moreover,

$$
\begin{aligned}
&\left\| \left(\left[e^{-\frac{t}{n}} S^{(w)} \left(\frac{t}{n} \right) \tilde{S} \left(\frac{t}{n} \right) \right]^n \right) \Big|_{D_{G_{w,1}}(\alpha,1)} \right\|_{\mathcal{B}(D_{G_{w,1}}(\alpha,1))} \\
&\leq \left\| \left(\left[S^{(w)} \left(\frac{t}{n} \right) \right]^n \right) \Big|_{D_{G_{w,1}}(\alpha,1)} \right\|_{\mathcal{B}(D_{G_{w,1}}(\alpha,1))} \\
&= \left\| S^{(w)}(t)|_{D_{G_{w,1}}(\alpha,1)} \right\|_{\mathcal{B}(D_{G_{w,1}}(\alpha,1))}
\end{aligned}
$$

and so

$$
\|S_\gamma^{(w)}(t)\Big|_{D_{G_{w,1}}(\alpha,1)}\|_{\mathcal{B}(D_{G_{w,1}}(\alpha,1))} \leq \left\| S^{(w)}(t)|_{D_{G_{w,1}}(\alpha,1)} \right\|_{\mathcal{B}(D_{G_{w,1}}(\alpha,1))}.
$$

It follows that (a) and (b) of Theorem 4.2.6 hold for any $\gamma \geq 0$.

Fix $\tau \in [0, t_{max})$, let $r > 0$ and choose $\gamma \geq C(\tau) r$. For $f \in \overline{B}_{D_{G_{w,1}}(\alpha,1)}(0, r)_+$, $t \in [0, \tau]$ and $n \in \mathbb{N}$,

$$
(K_1^{(v,w)}(t, f))_n = \frac{1}{2} \sum_{j=1}^{n-1} k_{n-j,j}(t) f_{n-j} f_j \geq 0
$$

and, since $(w_n)_{n=1}^\infty$ is monotone increasing,

$$
\begin{aligned}
(K_2^{(v,w)}(t, f))_n = \sum_{j=1}^\infty k_{n,j}(t) f_n f_j &\leq \sum_{j=1}^\infty \frac{C(\tau)(a_n + 1)^\alpha (a_j + 1)^\alpha}{W_{n,j}} f_n f_j \\
&= C(\tau) \sum_{j=1}^\infty \frac{w_n w_j}{w_{n+j}} (a_n + 1)^\alpha (a_j + 1)^\alpha f_n f_j \\
&\leq C(\tau)(a_n + 1)^\alpha \sum_{j=1}^\infty w_j (a_j + 1)^\alpha f_j f_n \\
&= C(\tau)(a_n + 1)^\alpha \|f\|_{D_{G_{w,1}}(\alpha,1)} f_n \\
&\leq C(\tau)(a_n + 1)^\alpha r f_n.
\end{aligned}
$$

Hence, for $t \in [0, \tau]$ and $f \in \overline{B}_{D_{G_{w,1}}(\alpha,1)}(0, r)_+$,

$$\begin{aligned}\Big(\gamma(-A_{w,1})^\alpha f - K_2^{(v,w)}(t,f)\Big)_n &\geq \gamma(a_n+1)^\alpha f_n - C(\tau)(a_n+1)^\alpha r f_n \\ &= (\gamma - C(\tau)r)(a_n+1)^\alpha f_n \\ &\geq 0 \qquad \text{since } \gamma \geq C(\tau)r.\end{aligned}$$

It follows from Theorem 4.2.6 that $u(t) \geq 0$ on $[0, \tau]$. Since $\tau \in [0, t_{max})$ was arbitrary, we can conclude that $u(t) \geq 0$ for $t \in [0, t_{max})$. □

We can now give a result regarding mass conservation.

Theorem 6.5.11. *Let the assumptions of Theorem 6.5.10,* (5.1.5) *and Assumption 5.2.2(ii) hold. Then, for all $\mathring{u} \in \mathcal{D}_{G_{w,1}}(\alpha, 1)_+$, there exists a unique, non-negative, mass-conserving mild solution of* (6.1.11), (6.1.12) *on* $[0, t_{max})$*, for some* $t_{max} > 0$.

Proof. Since (6.5.2) is equivalent to (6.1.11), (6.1.12), the existence of a unique mild solution, $u(t)$, follows from Theorem 6.5.9. In addition, from Theorem 6.5.10 and Assumption 5.2.2(ii), we have $u(t) \in (\ell_w^1)_+ \subseteq (X_{[1]})_+$ for all $t \in [0, t_{max})$. It follows from Proposition 6.3.5 that this solution is mass conserving. □

The following proposition provides an additional assumption, under which we can conclude that the mild solution in Theorem 6.5.9 is a classical solution.

Proposition 6.5.12. *Let the assumptions of Theorem 6.5.9 hold. Moreover, assume that $\mathring{u} \in D_{G_{w,1}}(\alpha, 1)$ and that there exist $\Omega > 0$, $\theta \in (0, 1)$ such that, for all $n, j \in \mathbb{N}$, $0 \leq s \leq t < T$,*

$$\frac{W_{n,j}}{(a_n+1)^\alpha (a_j+1)^\alpha} \Big| k_{n,j}(t) - k_{n,j}(s) \Big| \leq \Omega (t-s)^\theta. \tag{6.5.14}$$

Then the mild solution in Theorem 6.5.9 is the unique classical solution.

Proof. Let $f \in D_{G_{w,1}}(\alpha, 1)$. For $0 \leq s \leq t < T$, from Lemma 6.1.8(ii), we have

$$\|K^{(v,w)}(t,f) - K^{(v,w)}(s,f)\|_{\ell_w^1}$$

$$= \left\| \left(\frac{1}{2} \sum_{j=1}^{n-1} \Big(k_{n-j,j}(t) - k_{n-j,j}(s) \Big) f_{n-j} f_j - \sum_{j=1}^{\infty} \Big(k_{n,j}(t) - k_{n,j}(s) \Big) f_n f_j \right)_{n\in\mathbb{N}} \right\|_{\ell^1_w}$$
$$\leq \frac{3}{2} \Omega (t-s)^{\theta} \|f\|^2_{D_{G_{w,1}}(\alpha,1)}.$$

Now fix $g \in D_{G_{w,1}}(\alpha, 1)$ and choose $r > 0$. Moreover, let f be such that $\|g - f\|_{D_{G_{w,1}}(\alpha,1)} \leq r$. Then $\|f\|_{D_{G_{w,1}}(\alpha,1)} \leq \|g\|_{D_{G_{w,1}}(\alpha,1)} + r$. Hence

$$\|K^{(v,w)}(t, f) - K^{(v,w)}(s, f)\|_{\ell^1_w} \leq \frac{3\Omega}{2} \Big(\|g\|_{D_{G_{w,1}}(\alpha,1)} + r \Big)^2 (t-s)^{\theta} = \sigma(t-s)^{\theta},$$

where $\sigma = \frac{3\Omega}{2} \Big(\|g\|_{D_{G_{w,1}}(\alpha,1)} + r \Big)^2 > 0$. The result then follows from part (i) of [43, Proposition 7.1.10]. □

The existence of a classical solution for $t \in [0, t_{max})$, $t_{max} \leq T$, has been established under the assumptions of Proposition 6.5.12. The fact that we now have a classical solution enables us to obtain the following result regarding when $t_{max} = T$, i.e. when the classical solution is a global solution in $D_{G_{w,1}}(\alpha, 1)$, $0 < \alpha < 1$. In this theorem, for each $t' \in [0, T)$, we require the existence of a sequence $(q_l(t'))_{l=1}^{\infty}$ satisfying certain properties. An example of the construction of such a sequence will be given in Example 6.5.15.

Theorem 6.5.13. *Fix $\alpha \in [0, 1)$ and choose $\mathring{u} \in D_{G_{w,1}}(\alpha, 1)_+$. Moreover, let the assumptions of Proposition 6.5.12 hold and let $u \in C\Big([0, t_{max}), D_{G_{w,1}}(\alpha, 1)_+\Big)$ be a maximal, non-negative, mass-conserving classical solution of* (6.1.11), (6.1.12). *Assume that there exists $\delta_0 \in (0, 1)$ such that*

$$\sum_{n=1}^{j-1} w_n (a_n + 1)^{\alpha} b_{n,j} \leq \delta_0 w_j (a_j + 1)^{\alpha} \qquad \textit{for all } j = 2, 3, \ldots. \tag{6.5.15}$$

Moreover, assume that for each $t' \in [0, T)$, there exists a sequence $(q_l(t'))_{l=1}^{\infty}$ such that $q_l(t') > 0$ for all $l \in \mathbb{N}$ and either

(a) $q_l(t') \to 0$ *as* $l \to \infty$

or

(b) $(q_l(t'))_{l=1}^{\infty}$ *is bounded,* $\limsup_{l\to\infty} q_l(t') > 0$ *and* $\|\mathring{u}\|_{[1]} < \frac{1-\delta_0}{\limsup_{l\to\infty} q_l(t')}$.

Suppose that, in addition, for $t \in [0, t']$,

$$k_{l,j}(t) \le \frac{q_l(t')(a_l+1)^{\alpha+1}w_l j + q_j(t')(a_j+1)^{\alpha+1}w_j l}{(a_{l+j}+1)^{\alpha}w_{l+j}}. \tag{6.5.16}$$

Then u *is a global solution, i.e.* $t_{max} = T$.

Proof. We aim to show that $\|u(t)\|_{D_{G_{w,1}}(\alpha,1)} \nrightarrow \infty$ as $t \to \hat{T}^-$ for any $\hat{T} < T$. In particular, we will show that for $t \in [0, t_{max})$,

$$\|u(t)\|_{D_{G_{w,1}}(\alpha,1)} \le \|\mathring{u}\|_{D_{G_{w,1}}(\alpha,1)} e^{Ct}. \tag{6.5.17}$$

It then follows that $t_{max} = T$.

Let $v_n = w_n(a_n+1)^{\alpha}$ for all $n \in \mathbb{N}$. For $t \in [0, t_{max})$, we have

$$\begin{aligned}
\frac{d}{dt}\|u(t)\|_{D_{G_{w,1}}(\alpha,1)} = \frac{d}{dt}\|u(t)\|_{\ell_v^1} &= \frac{d}{dt}\Big(\phi_{\ell_v^1}\big(u(t)\big)\Big) \\
&= \phi_{\ell_v^1}\big(u'(t)\big) \\
&= \phi_{\ell_v^1}\big(G^{(w)}u(t)\big) + \phi_{\ell_v^1}\big(K^{(v,w)}(t, u(t))\big).
\end{aligned}$$

Now, using (6.5.15),

$$\begin{aligned}
\phi_{\ell_v^1}\big(G^{(w)}u(t)\big) &= \sum_{n=1}^{\infty} v_n\left(-a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t)\right) \\
&= -\sum_{n=1}^{\infty} v_n a_n u_n(t) + \sum_{j=2}^{\infty}\left(\sum_{n=1}^{j-1} v_n b_{n,j}\right) a_j u_j(t) \\
&\le -\sum_{n=1}^{\infty} v_n a_n u_n(t) + \sum_{j=1}^{\infty} \delta_0 v_j a_j u_j(t) \\
&= -(1-\delta_0)\sum_{n=1}^{\infty} a_n v_n u_n(t).
\end{aligned}$$

Also, for $t \in [0, t_{max})$,

$$\phi_{\ell_v^1}\big(K^{(v,w)}(t, u(t))\big)$$

$$
\begin{aligned}
&= \frac{1}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{n-1} v_n k_{n-j,j}(t)u_{n-j}(t)u_j(t) - \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} v_n k_{n,j}(t)u_n(t)u_j(t) \\
&= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty} v_n k_{n-j,j}(t)u_{n-j}(t)u_j(t) - \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} v_n k_{n,j}(t)u_n(t)u_j(t) \\
&= \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} v_{l+j} k_{l,j}(t)u_l(t)u_j(t) - \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} v_n k_{n,j}(t)u_n(t)u_j(t) \\
&\leq \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} v_{l+j} k_{l,j}(t)u_l(t)u_j(t).
\end{aligned}
$$

Suppose that $t_{max} < T$ and take $t' > t_{max}$. Then, for $t \in [0, t_{max})$,

$$
\begin{aligned}
&\frac{d}{dt}\|u(t)\|_{D_{G_{w,1}}(\alpha,1)} \\
&\leq -(1-\delta_0)\sum_{n=1}^{\infty} a_n v_n u_n(t) + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} v_{l+j}k_{l,j}(t)u_l(t)u_j(t) \\
&\leq -(1-\delta_0)\sum_{n=1}^{\infty} a_n v_n u_n(t) \\
&\qquad + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\Big(q_l(t')(a_l+1)^{\alpha+1}w_l j + q_j(t')(a_j+1)^{\alpha+1}w_j l\Big)u_l(t)u_j(t) \\
&= -(1-\delta_0)\sum_{n=1}^{\infty} a_n v_n u_n(t) + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\Big(q_l(t')(a_l+1)^{\alpha+1}w_l j\Big)u_l(t)u_j(t) \\
&\qquad + \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\Big(q_j(t')(a_j+1)^{\alpha+1}w_j l\Big)u_l(t)u_j(t) \\
&= -(1-\delta_0)\sum_{n=1}^{\infty} a_n v_n u_n(t) + \sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\Big(q_l(t')(a_l+1)^{\alpha+1}w_l j\Big)u_l(t)u_j(t) \\
&= -(1-\delta_0)\sum_{n=1}^{\infty} a_n v_n u_n(t) + \sum_{j=1}^{\infty} j u_j(u)\sum_{l=1}^{\infty}\Big(q_l(t')(a_l+1)v_l\Big)u_l(t) \\
&= -(1-\delta_0)\sum_{n=1}^{\infty} a_n v_n u_n(t) + \|u(t)\|_{[1]}\sum_{l=1}^{\infty} q_l(t')(a_l+1)v_l u_l(t) \\
&= \sum_{n=1}^{\infty}\Big[a_n\Big(-(1-\delta_0) + \|\mathring{u}\|_{[1]}q_n(t')\Big) + \|\mathring{u}\|_{[1]}q_n(t')\Big] v_n u_n(t).
\end{aligned}
$$

If (a) holds, then there exists $N \in \mathbb{N}$ such that

$$
-(1-\delta_0) + \|\mathring{u}\|_{[1]}q_n(t') < 0 \qquad \text{and} \qquad q_n(t') \leq q_N(t') \qquad \text{for all } n \geq N.
$$

In this case,

$$\begin{aligned}
&\frac{d}{dt}\phi_{D_{G_{w,1}}(\alpha,1)}(u(t)) \\
&\leq \sum_{n=1}^{N-1}\Big[a_n\Big(-(1-\delta_0)+\|\mathring{u}\|_{[1]}q_n(t')\Big)+\|\mathring{u}\|_{[1]}q_n(t')\Big]\,v_nu_n(t) \\
&\qquad\qquad + \sum_{n=N}^{\infty}\|\mathring{u}\|_{[1]}q_N(t')v_nu_n(t) \\
&\leq C\sum_{n=1}^{\infty}v_nu_n(t) = C\phi_{\ell^1_v}(u(t)) = C\phi_{D_{G_{w,1}}(\alpha,1)}(u(t)),
\end{aligned}$$

where

$$C = \max\left\{\|\mathring{u}\|_{[1]}q_N(t'),\max_{1\leq n\leq N-1}\Big[a_n\Big(-(1-\delta_0)+\|\mathring{u}\|_{[1]}q_n(t')\Big)+\|\mathring{u}\|_{[1]}q_n(t')\Big]\right\} > 0.$$

If (b) holds, then there exists $\varepsilon > 0$ such that

$$\|\mathring{u}\|_{[1]} < \frac{1-\delta_0}{\limsup_{l\to\infty} q_l(t')+\varepsilon}.$$

Moreover, there exists $N_0 \in \mathbb{N}$ such that $q_n(t') \leq \limsup_{l\to\infty} q_l(t') + \varepsilon$ for all $n \geq N_0$. Then

$$\begin{aligned}
&\frac{d}{dt}\phi_{D_{G_{w,1}}(\alpha,1)}(u(t)) \\
&\leq \sum_{n=1}^{\infty}\left[a_n\left(-(1-\delta_0)+\frac{(1-\delta_0)}{\limsup_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right)+\frac{(1-\delta_0)}{\limsup_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right]v_nu_n(t) \\
&\leq \sum_{n=1}^{N_0-1}\left[a_n\left(-(1-\delta_0)+\frac{(1-\delta_0)}{\limsup_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right)+\frac{(1-\delta_0)}{\limsup_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right]v_nu_n(t) \\
&\quad + \sum_{n=N_0}^{\infty}\Big[a_n\Big(-(1-\delta_0)+(1-\delta_0)\Big)+(1-\delta_0)\Big]\,v_nu_n(t)
\end{aligned}$$

$$
\begin{aligned}
&\leq \sum_{n=1}^{N_0-1}\left[a_n\left(-(1-\delta_0)+\frac{(1-\delta_0)}{\limsup\limits_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right)+\frac{(1-\delta_0)}{\limsup\limits_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right]v_n u_n(t)\\
&\qquad+\sum_{n=N_0}^{\infty}(1-\delta_0)v_n u_n(t)\\
&\leq C\sum_{n=1}^{\infty}v_n u_n(t)=C\phi_{\ell_v^1}(u(t))=C\phi_{D_{G_{w,1}}(\alpha,1)}(u(t)),
\end{aligned}
$$

where

$$
\begin{aligned}
C=\max\{&(1-\delta_0),\\
&\max_{1\leq n\leq N_0-1}\left[a_n\left(-(1-\delta_0)+\frac{(1-\delta_0)}{\limsup\limits_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right)+\frac{(1-\delta_0)}{\limsup\limits_{l\to\infty} q_l(t')+\varepsilon}q_n(t')\right]\Bigg\}.
\end{aligned}
$$

In either case, (a) or (b), we have for $t\in[0,t_{max})$,

$$
\frac{d}{dt}\phi_{D_{G_{w,1}}(\alpha,1)}(u(t))\leq C\phi_{D_{G_{w,1}}(\alpha,1)}(u(t)),
$$

i.e.

$$
\|u(t)\|_{D_{G_{w,1}}(\alpha,1)}\leq\|\mathring{u}\|_{D_{G_{w,1}}(\alpha,1)}e^{Ct}\nrightarrow\infty \text{ as } t\to t_{max}. \tag{6.5.18}
$$

Hence, we must have that $t_{max}=T$. □

Remark 6.5.14. We give the following remarks.

(i) We can use Theorem 4.2.5(iv) and (6.5.18) to show that this classical solution in Theorem 6.5.13 depends continuously on the initial condition $\mathring{u}$.

(ii) For given fragmentation coefficients, we can always choose $(w_n)_{n=1}^{\infty}$ iteratively such that both Assumption 6.5.1 and (6.5.15) holds.

(iii) A sufficient condition for (6.5.15) to hold is that $(a_n)_{n=1}^{\infty}$ is monotone increasing since, from Assumption 6.5.1, we then have

$$
\sum_{n=1}^{j-1}w_n(a_n+1)^{\alpha}b_{n,j}\leq(a_j+1)^{\alpha}\sum_{n=1}^{j-1}w_n b_{n,j}\leq\delta w_j(a_j+1)^{\alpha}.
$$

We now provide an example of a situation in which we can apply Theorem 6.5.13.

Example 6.5.15. Let $(a_n)_{n=1}^{\infty}$ be monotone increasing. Let there exist $N \in \mathbb{N}$, $c > 1$, $s > 1$, $p \geq 1$ satisfying

$$\frac{1}{c} n^s \leq a_n \leq c n^s \qquad \text{and} \qquad \frac{1}{c} n^p \leq w_n \leq c n^p$$

for all $n \geq N$. Assume that for each $t' \in [0, T)$, there exists $\hat{C}(t') > 0$ such that

$$k_{l,j}(t) \leq \hat{C}(t') l j \qquad \text{for all } l, j \in \mathbb{N}, t \in [0, t'].$$

From the previous remark, we know that (6.5.15) holds.

Now, for l, $j \in \mathbb{N}$, we have

$$\begin{aligned}
\frac{q_l(a_l+1)^{\alpha+1} w_l j + q_j (a_j+1)^{\alpha+1} w_j l}{(a_{l+j}+1)^{\alpha} w_{l+j}} &\geq \frac{q_l(\frac{1}{c} l^s + 1)^{\alpha+1} \frac{1}{c} l^p j + q_j (\frac{1}{c} j^s + 1)^{\alpha+1} \frac{1}{c} j^p l}{(c(l+j)^s + 1)^{\alpha} c (l+j)^p} \\
&\geq \frac{q_l(\frac{1}{c} l^s)^{\alpha+1} \frac{1}{c} l^p j + q_j (\frac{1}{c} j^s)^{\alpha+1} \frac{1}{c} j^p l}{((c+1)(l+j)^s)^{\alpha} c (l+j)^p} \\
&= \frac{\frac{1}{c^{\alpha+2}} \left[q_l l^{s+s\alpha+p} j + q_j j^{s+s\alpha+p} l \right]}{c(c+1)^{\alpha} (l+j)^{s\alpha+p}} \\
&= \frac{1}{c^{\alpha+3}(c+1)^{\alpha}} \left(\frac{q_l l^{s+s\alpha+p} j + q_j j^{s+s\alpha+p} l}{(l+j)^{s\alpha+p}} \right).
\end{aligned}$$

From Proposition 6.1.5 with $p = 1$, $q = s\alpha + p$, we have

$$\frac{l^{s\alpha+p} + j^{s\alpha+p}}{(l+j)^{s\alpha+p}} \geq 2^{1-s\alpha-p}.$$

This motivates to choose the sequence $(q_l)_{l=1}^{\infty}$ such that

$$q_l l^{s+s\alpha+p} j + q_j j^{s+s\alpha+p} l = l j (l^{s\alpha+p} + j^{s\alpha+p}). \tag{6.5.19}$$

Let $q_l = l^{1-s}$ for all $l \in \mathbb{N}$. Then (6.5.19) holds and, since $s > 1$, $q_l \to 0$ as $l \to \infty$.

Hence, using 6.5.19, we have

$$\begin{aligned}
&\frac{q_l(a_l+1)^{\alpha+1}w_l j + q_j(a_j+1)^{\alpha+1}w_j l}{(a_{l+j}+1)^{\alpha}w_{l+j}} \\
&\geq \frac{1}{c^{\alpha+3}(c+1)^{\alpha}}\left(\frac{lj(l^{s\alpha+p}+j^{s\alpha+p})}{(l+j)^{s\alpha+p}}\right) \\
&\geq \frac{2^{1-s\alpha-p}}{c^{\alpha+3}(c+1)^{\alpha}}lj \\
&\geq \frac{2^{1-s\alpha-p}}{c^{\alpha+3}(c+1)^{\alpha}\hat{C}(t')}k_{l,j}(t) \qquad \text{for all } t \in [0,t'].
\end{aligned}$$

It follows that (6.5.16) is satisfied.

We conclude this chapter with a remark relating to Example 6.5.15.

Remark 6.5.16. In the absence of fragmentation, i.e. when we have a pure coagulation system, it has been shown that a phenomena known as gelation occurs when the coagulation rates are of the form $k_{l,j} = lj$ for all l, $j \in \mathbb{N}$. Gelation occurs when the rate of coagulation increases rapidly as particle size increases, resulting in the creation of clusters of "infinite" size. Since the C–F equations only describe the behaviour of clusters of finite size, this results in solutions that can be observed to lose mass. An explicit global solution was obtained for the case of pure coagulation, when $k_{l,j} = lj$ for all l, $j \in \mathbb{N}$, in [42], where the solution was shown to conserve mass for $t \in [0,1]$ until gelation occurs at $t = 1$, after which the solution loses mass. A discussion of this case is provided in [18, Example 2.3.8(iii)]. However, Example 6.5.15, Theorem 6.5.13 and Proposition 6.3.5 show that, when (5.1.5) holds, we can add fragmentation to coagulation of the form $k_{l,j} = lj$ for all l, $j \in \mathbb{N}$ and obtain a global mass-conserving solution in $X_{[1]}$. Since gelation manifests itself as a loss in mass, this confirms that no gelation occurs in the space $X_{[1]}$ in this case.

Chapter 7

Non-autonomous Fragmentation System

In this chapter we return to the pure fragmentation system (5.1.1), (5.1.2), with the important difference that the fragmentation coefficients may now be time-dependent. As mentioned in § 2.2, the case of continuous non-autonomous fragmentation has been examined in [4, 49] but there appears to be no corresponding treatment of non-autonomous discrete fragmentation. Here we use the theory of evolution families to investigate these time-dependent equations. In particular, the results obtained in § 5.4 regarding analytic semigroups are used to obtain a unique, non-negative, mass-conserving classical solution to the non-autonomous fragmentation ACP.

7.1 Setting up the Problem

As in Chapter 5 we consider a discrete pure fragmentation system. We now, however, allow the fragmentation coefficients to be time-dependent. As before, $u_n(t)$ denotes the density of clusters of size n at time $t \geq 0$. Then, for fixed $s \geq 0$, the discrete non-autonomous fragmentation system that we now examine is of the form

$$\begin{aligned} u_n'(t) &= -a_n(t)u_n(t) + \sum_{j=n+1}^{\infty} a_j(t)b_{n,j}(t)u_j(t), \qquad t > s; \\ u_n(s) &= \mathring{u}_n, \qquad n = 1, 2, \ldots. \end{aligned} \tag{7.1.1}$$

The coefficient $a_n(t)$ denotes the rate, at time t, at which clusters of size n are lost due to fragmentation. The coefficient $b_{n,j}(t)$ denotes the average number, at time t, of clusters of size n that are produced when a larger cluster of size j fragments. For a positive sequence, $w = (w_n)_{n=1}^{\infty}$, we again define the weighted ℓ^1 space, ℓ_w^1, and its norm, $\|\cdot\|_{\ell_w^1}$, as in the previous chapters. Fix $\Theta > 0$ and consider $\mathscr{I} = [0, \Theta]$. Let

$$D_{\mathscr{I}} = \{(t,s) : t, s \in \mathscr{I} \text{ and } s \leq t\}.$$

Throughout this chapter, we make the following assumption on the fragmentation coefficients.

Assumption 7.1.1. For all $n, j \in \mathbb{N}$ and $t \in \mathscr{I}$, let $a_n(t) \geq 0$ and $b_{n,j}(t) \geq 0$, with $b_{n,j}(t) = 0$ if $j \leq n$. Moreover, for all $n \in \mathbb{N}$, let $t \mapsto a_n(t) \in L^1(0, \Theta)$.

Note that, in an analogous way as in § 5.1, if mass is conserved during each fragmentation event, then we have

$$a_1(t) = 0 \qquad \text{and} \qquad \sum_{n=1}^{j-1} n b_{n,j}(t) = j \qquad \text{for all } t \in \mathscr{I}. \tag{7.1.2}$$

We also make the following assumption on the weight w throughout this chapter.

Assumption 7.1.2. For all $j \in \mathbb{N}$, let $w_j \geq j$ and assume that there exists some $\delta \in (0, 1)$ such that for all $t \in [0, \Theta]$,

$$\sum_{n=1}^{j-1} w_n b_{n,j}(t) \leq \delta w_j.$$

Motivated by the terms in (7.1.1), for $t \in \mathscr{I}$, we introduce the formal expressions

$$\mathcal{A}(t) : (f_n)_{n=1}^{\infty} \mapsto (-a_n(t)f_n)_{n=1}^{\infty}$$

and

$$\mathcal{B}(t) : (f_n)_{n=1}^{\infty} \mapsto \left(\sum_{j=n+1}^{\infty} a_j(t)b_{n,j}(t)f_j\right)_{n=1}^{\infty}.$$

For each $t \in \mathscr{I}$, operator realisations, $A^{(w)}(t)$ and $B^{(w)}(t)$, of $\mathcal{A}(t)$ and $\mathcal{B}(t)$ respectively, are defined in ℓ_w^1 by

$$A^{(w)}(t)f = \mathcal{A}(t)f, \qquad \mathcal{D}(A^{(w)}(t)) = \left\{f \in \ell_w^1 : \mathcal{A}(t)f \in \ell_w^1\right\} \tag{7.1.3}$$

and

$$B^{(w)}(t)f = \mathcal{B}(t)f, \qquad \mathcal{D}(B^{(w)}(t)) = \left\{f \in \ell_w^1 : \mathcal{B}(t)f \in \ell_w^1\right\}. \tag{7.1.4}$$

Note that, from Lemma 5.2.6, for each $t \in \mathscr{I}$, $\mathcal{D}(A^{(w)}(t)) \subseteq \mathcal{D}(B^{(w)}(t))$ and

$$\|B^{(w)}(t)f\|_{\ell_w^1} \leq \delta \|A^{(w)}(t)f\|_{\ell_w^1} \qquad \text{for all } f \in \mathcal{D}(A^{(w)}(t)). \tag{7.1.5}$$

For fixed $s \in \mathscr{I}$, we now write (7.1.1) as the non-autonomous ACP

$$\begin{aligned} u'(t) &= A^{(w)}(t)u(t) + B^{(w)}(t)u(t), \qquad t \in (s, \Theta]; \\ u(s) &= \mathring{u}. \end{aligned} \tag{7.1.6}$$

7.2 Solutions of the Non-Autonomous Fragmentation System

Fix $t \in \mathscr{I}$. Then $(a_n(t))_{n=1}^{\infty}$, $(b_{n,j}(t))_{n,j\in\mathbb{N}}$ and $(w_n)_{n=1}^{\infty}$ satisfy the conditions of Theorem 5.4.1 and so $G^{(w)}(t) = A^{(w)}(t) + B^{(w)}(t)$ is the generator of an analytic, substochastic C_0-semigroup, $(S_t^{(w)}(\tau))_{\tau\geq 0}$, on ℓ_w^1. We now rescale this semigroup. For each $t \geq 0$, $H^{(w)}(t) := A^{(w)}(t) + B^{(w)}(t) - I$ is the generator of the analytic C_0-semigroup, $(e^{-\tau}S_t^{(w)}(\tau))_{\tau\geq 0}$, which has a growth bound less than or equal to -1. The following proposition shall prove to be useful.

Proposition 7.2.1. *For all $t \in \mathscr{I}$ and $\lambda \in \mathbb{C}$ such that* $\operatorname{Re}\lambda > -1$*, we have*

$$\|R(\lambda, H^{(w)}(t))\| = \left\| \left(A^{(w)}(t) + B^{(w)}(t) - (1+\lambda)I \right)^{-1} \right\| \le \frac{\tilde{C}}{|1+\lambda|}, \tag{7.2.1}$$

where $\tilde{C} = \frac{1}{1-\delta} > 1$.

Proof. Let $t \in \mathscr{I}$ and let $\lambda \in \mathbb{C}$ be such that $\operatorname{Re}\lambda > -1$. For $f \in \ell_w^1$, we have from (7.1.5), that

$$\begin{aligned}
\left\| B^{(w)}(t) \left((1+\lambda)I - A^{(w)}(t) \right)^{-1} f \right\|_{\ell_w^1} &\le \delta \left\| A^{(w)}(t) \left((1+\lambda)I - A^{(w)}(t) \right)^{-1} f \right\|_{\ell_w^1} \\
&= \delta \sum_{n=1}^{\infty} w_n \left| \frac{a_n(t)}{1+\lambda+a_n(t)} f_n \right| \\
&\le \delta \|f\|_{\ell_w^1},
\end{aligned}$$

since $\operatorname{Re}\lambda > -1$. Now,

$$\begin{aligned}
&\|R(\lambda, H^{(w)}(t))\| \\
&= \|(A^{(w)}(t) + B^{(w)}(t) - (1+\lambda)I)^{-1}\| \\
&= \left\| \left[\left(B^{(w)}(t) \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} + I \right) (A^{(w)}(t) - (1+\lambda)I) \right]^{-1} \right\| \\
&= \left\| \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} \left(B^{(w)}(t) \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} + I \right)^{-1} \right\| \\
&\le \left\| \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} \right\| \left\| \left(B^{(w)}(t) \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} + I \right)^{-1} \right\| \\
&= \left\| \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} \right\| \left\| \left(I - B^{(w)}(t) \left((1+\lambda)I - A^{(w)}(t) \right)^{-1} \right)^{-1} \right\| \\
&\le \left\| \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} \right\| \sum_{n=0}^{\infty} \left\| B^{(w)}(t) \left((1+\lambda)I - A^{(w)}(t) \right)^{-1} \right\|^n \\
&\le \left\| \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} \right\| \sum_{n=0}^{\infty} \delta^n \\
&= \frac{1}{1-\delta} \left\| \left(A^{(w)}(t) - (1+\lambda)I \right)^{-1} \right\| \\
&\le \frac{1}{1-\delta} \cdot \frac{1}{|1+\lambda|},
\end{aligned}$$

where the last inequality follows from the fact that $1+\lambda > 0$ and $-a_n(t) \le 0$ for all $n \in \mathbb{N}$, $t \in \mathscr{I}$. The result then follows immediately. $\square$

For the remainder of this section we impose the following assumption on the fragmentation rates.

Assumption 7.2.2. Let

(i) $\mathcal{D} := \mathcal{D}(A^{(w)}(t))$ be independent of $t \in [0, \Theta]$;

(ii) there exist $C_1 \ge 0$ and $\sigma \in (0,1]$ such that

$$|a_n(t) - a_n(s)| \le C_1 |t-s|^{\sigma} \qquad \text{for all } n \in \mathbb{N},\ s,t \in \mathscr{I};$$

(iii) there exist some $C_2 \ge 0$ such that

$$\sum_{n=1}^{j-1} w_n \left| a_j(t) b_{n,j}(t) - a_j(s) b_{n,j}(s) \right| \le C_2 w_j \left| a_j(t) - a_j(s) \right|$$

for all $j = 2, 3, \dots$ and $t, s \in \mathscr{I}$.

Lemma 7.2.3. *Let Assumption 7.2.2 hold. Then there exists an evolution family, $(\mathscr{V}^{(w)}(t,s))_{(t,s)\in D_{\mathscr{I}}}$, such that $u(t) = \mathscr{V}^{(w)}(t,s)\mathring{u}$ is the unique classical solution of the non-autonomous ACP*

$$\begin{aligned} u'(t) &= H^{(w)}(t)u(t) = (A^{(w)}(t) + B^{(w)}(t) - I)u(t), \qquad t > s; \\ u(s) &= \mathring{u} \in \ell^1_w. \end{aligned} \tag{7.2.2}$$

Moreover, if $\mathring{u} \in (\ell^1_w)_+$, then the solution is non-negative.

Proof. Fix $t \in \mathscr{I}$. We begin by showing that $H^{(w)}(t)$ satisfies the assumptions of Theorem 3.5.3. Since $\mathcal{D} = \mathcal{D}(A^{(w)}(t)) = \mathcal{D}(H^{(w)}(t))$ for all $t \in [0,\Theta]$, it is clear that (P1) in Theorem 3.5.3 is satisfied. Also, $(e^{-\tau} S_t^{(w)}(\tau))_{\tau \ge 0}$ has growth bound less than or equal to -1. Thus, $R(\lambda, H^{(w)}(t))$ exists for all $t \in \mathscr{I}$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \ge 0$. Let $\lambda = \hat{\alpha} + i\hat{\beta}$ for some $\hat{\alpha}$, $\hat{\beta} \in \mathbb{R}$ such that $\hat{\alpha} \ge 0$. From Proposition 6.1.5, with $p = 1$, $q = 2$, $x = \hat{\alpha} + 1$, $y = |\hat{\beta}|$, we have

$$\sqrt{(\hat{\alpha}+1)^2 + \hat{\beta}^2} \ge \frac{\hat{\alpha} + 1 + |\hat{\beta}|}{\sqrt{2}}.$$

Now,

$$|\lambda+1| = \sqrt{(\hat{\alpha}+1)^2+\hat{\beta}^2} \geq \frac{\hat{\alpha}+1+|\hat{\beta}|}{\sqrt{2}} = \frac{\sqrt{(\hat{\alpha}+|\hat{\beta}|)^2}+1}{\sqrt{2}} \geq \frac{\sqrt{\hat{\alpha}^2+\hat{\beta}^2}+1}{\sqrt{2}} = \frac{|\lambda|+1}{\sqrt{2}},$$

where $\hat{\alpha} \geq 0$ is used to obtain the second inequality. It follows that

$$\frac{\sqrt{2}}{|\lambda|+1} \geq \frac{1}{|\lambda+1|}.$$

Using (7.2.1), for $t \in \mathscr{I}$ and $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda \geq 0$,

$$\|R(\lambda, H^{(w)}(t))\| \leq \frac{\tilde{C}}{|1+\lambda|} \leq \frac{\sqrt{2}\tilde{C}}{|\lambda|+1}.$$

We now need only show that (P3) in Theorem 3.5.3 holds. Let $f \in \mathcal{D}$. Then, for $\tau, t \in \mathscr{I}$,

$$\begin{aligned}
&\|(H^{(w)}(t) - H^{(w)}(\tau))f\|_{\ell^1_w} \\
&\leq \sum_{n=1}^{\infty} w_n |a_n(t) - a_n(\tau)||f_n| + \sum_{n=1}^{\infty} w_n \sum_{j=n+1}^{\infty} |a_j(t)b_{n,j}(t) - a_j(\tau)b_{n,j}(\tau)||f_j| \\
&= \sum_{n=1}^{\infty} w_n |a_n(t) - a_n(\tau)||f_n| \\
&\qquad + \sum_{j=2}^{\infty} |a_j(t) - a_j(\tau)| \sum_{n=1}^{j-1} w_n \left| \frac{a_j(t)b_{n,j}(t) - a_j(\tau)b_{n,j}(\tau)}{a_j(t) - a_j(\tau)} \right| |f_j| \\
&\leq C_1|t-\tau|^{\sigma} \sum_{n=1}^{\infty} w_n|f_n| + C_1|t-\tau|^{\sigma} C_2 \sum_{j=2}^{\infty} w_j|f_j| \\
&\leq (1+C_2)C_1|t-\tau|^{\sigma}\|f\|_{\ell^1_w} \\
&= C_0|t-\tau|^{\sigma}\|f\|_{\ell^1_w} \qquad \text{where } C_0 = (1+C_2)C_1 \geq 0.
\end{aligned}$$

Take $f \in \ell^1_w$. Then, using (7.2.1) with $\lambda = 0$, we have

$$\|(H^{(w)}(t) - H^{(w)}(\tau))(H^{(w)}(u))^{-1}f\|_{\ell^1_w} \leq C_0|t-\tau|^{\sigma}\|(H^{(w)}(u))^{-1}f\|_{\ell^1_w}$$

$$\leq \tilde{C}C_0|t-\tau|^\sigma \|f\|_{\ell_w^1}$$

for all $t, \tau, u \in \mathscr{I}$. It follows from Theorem 3.5.3 that there exists an evolution family, $(\mathscr{V}^{(w)}(t,\tau))_{(t,\tau)\in D_{\mathscr{I}}}$, such that $u(t) = \mathscr{V}^{(w)}(t,s)\mathring{u}$ is the unique classical solution of (7.2.2).

We now show that this solution is non-negative for non-negative initial conditions. To this end, following the procedure in § 5.5, we find a matrix representation of $(\mathscr{V}^{(w)}(t,\tau))_{(t,\tau)D_{\mathscr{I}}}$. Let $e_n \in \ell_w^1$ be defined as in (5.5.1) and, for $(t,s) \in D_{\mathscr{I}}$, define a matrix $\mathbb{V}(t,s) = (v_{m,n}(t,s))_{m,n\in\mathbb{N}}$ by $v_{m,n}(t,s) = (\mathscr{V}^{(w)}(t,s)e_n)_m$. We have $f = \sum\limits_{n=1}^{\infty} f_n e_n$ for all $f \in \ell_w^1$. Hence, for all $m \in \mathbb{N}$,

$$(\mathscr{V}^{(w)}(t,s)f)_m = \left(\sum_{n=1}^{\infty} f_n \mathscr{V}^{(w)}(t,s)e_n\right)_m = \sum_{n=1}^{\infty} f_n v_{m,n}(t,s) = (\mathbb{V}(t,s)f)_m.$$

Thus, for $(t,s) \in D_{\mathscr{I}}$, we can represent $\mathscr{V}^{(w)}(t,s)$ by the matrix $\mathbb{V}(t,s)$. Let $n \in \mathbb{N}$. We know that $u(t) = \mathscr{V}^{(w)}(t,s)e_n$ is the unique classical solution of (7.2.2) with $\mathring{u} = e_n$. Now, for fixed $n \in \mathbb{N}$, let $(v_{1,n}(t,s), \ldots, v_{n,n}(t,s))$ be the unique solution of the n-dimensional system

$$\frac{\partial}{\partial t} v_{m,n}(t,s) = -(1 + a_m(t))v_{m,n}(t,s) + \sum_{j=m+1}^{n} a_j(t)b_{m,j}(t)v_{j,n}(t,s),\ (t,s) \in D_{\mathscr{I}}, \tag{7.2.3}$$

for $m = 1, 2, \ldots, n$, with $v_{n,n}(s,s) = 1$ and $v_{m,n}(s,s) = 0$ for $m < n$, where $\sum\limits_{j=n+1}^{n} a_j(t)b_{m,j}(t)v_{j,n}(t,s) = 0$. We note that, since this is a linear, finite dimensional system, a unique solution is given in the form of an exponential defined via a power series. The n^{th} equation in this finite system is given by

$$\frac{\partial}{\partial t} v_{n,n}(t,s) = -(1 + a_n(t))v_{n,n}(t,s)), \qquad v_{n,n}(s,s) = 1, \qquad (t,s) \in D_{\mathscr{I}};$$

and so $v_{n,n}(t,s) = \exp\left(-\int\limits_s^t (1 + a_n(\tau))\,\mathrm{d}\tau\right)$.

It is clear that $(v_{1,n}(t,s),\ldots,v_{n,n}(t,s),0,0,\ldots)$ is a solution of

$$\frac{\partial}{\partial t}v_{m,n}(t,s)=-(1+a_m(t))v_{m,n}(t,s)+\sum_{j=m+1}^{\infty}a_j(t)b_{m,j}(t)v_{j,n}(t,s),\ (t,s)\in D_{\mathscr{I}},$$

for $m=1,2,\ldots$, with $v_{n,n}(s,s)=1$ and $v_{m,n}(s,s)=0$ for $n\in\mathbb{N}$, $m\neq n$. Moreover,

$$(v_{1,n}(t,s),\ldots,v_{n,n}(t,s),0,0,\ldots)\in\mathcal{D}=\mathcal{D}(A^{(w)}(t))\qquad\text{for all }(t,s)\in D_{\mathscr{I}}.$$

It follows that $(v_{1,n}(t,s),\ldots,v_{n,n}(t,s),0,0,\ldots)$ is a classical solution of (7.2.2) with $\mathring{u}=e_n$. Hence, by the uniqueness of solutions of (7.2.2),

$$u(t)=\mathscr{V}^{(w)}(t,s)e_n=(v_{1,n}(t,s),\ldots,v_{n,n}(t,s),0,0,\ldots).$$

Since $n\in\mathbb{N}$ was arbitrary, we have for all $(t,s)\in D_{\mathscr{I}}$,

$$\mathscr{V}^{(w)}(t,s)=\begin{bmatrix}v_{1,1}(t,s)&v_{1,2}(t,s)&v_{1,3}(t,s)&\ldots\\0&v_{2,2}(t,s)&v_{2,3}(t,s)&\ldots\\0&0&v_{3,3}(t,s)&\ldots\\\vdots&\vdots&\vdots&\ddots\end{bmatrix}\tag{7.2.4}$$

where $v_{n,n}(t,s)=\exp\left(-\int\limits_s^t\Big(1+a_n(\tau)\Big)\,\mathrm{d}\tau\right)$ for all $n\in\mathbb{N}$.

Let $\mathring{u}\in(\ell_w^1)_+$. If we can show that $v_{m,n}(t,s)\geq 0$ for all m, $n\in\mathbb{N}$, $(t,s)\in D_{\mathscr{I}}$, then it is immediate from (7.2.4) that $u(t)=\mathscr{V}^{(w)}(t,s)\mathring{u}$ is a non-negative solution.

Choose $n\in\mathbb{N}$ and fix $s\in\mathscr{I}$. It is clear that

$$v_{n,n}(t,s)=\exp\left(-\int\limits_s^t(1+a_n(\tau))\,\mathrm{d}\tau\right)\geq 0,\qquad\text{for all }t\in[s,\Theta].$$

If $n > 1$, then the $(n-1)^{th}$ equation in (7.2.3) is given by

$$\frac{\partial}{\partial t} v_{n-1,n}(t,s) = -(1 + a_{n-1}(t))v_{n-1,n}(t,s) + a_n(t)b_{n-1,n}(t)v_{n,n}(t,s), \qquad t \in [s,\Theta]. \tag{7.2.5}$$

Now suppose that $v_{n-1,n}(t,s) < 0$ for t in some maximal interval $(\varepsilon_{n-1}, \hat{\varepsilon}_{n-1})$, where $s < \varepsilon_{n-1} \leq \hat{\varepsilon}_{n-1} \leq \Theta$. Then, using the fact that $v_{n,n}(t,s) \geq 0$ for $t \in [s,\Theta]$, we have that the right-hand side of (7.2.5) is positive on $(\varepsilon_{n-1}, \hat{\varepsilon}_{n-1})$. On the other hand, since $v_{n-1,n}(s,s) = 0$, it follows from the Mean Value Theorem that there exists some $\varepsilon \in (\varepsilon_{n-1}, \hat{\varepsilon}_{n-1})$ such that $\frac{\partial}{\partial t} v_{n-1,n}(\varepsilon, s) < 0$. This is a contradiction and so $v_{n-1,n}(t,s) \geq 0$ for all $t \in [s,\Theta]$.

If $n > 2$, we can then use a similar argument to deduce that $v_{n-2,n}(t,s) \geq 0$ for $t \in [s,\Theta]$ and continuing in this way leads to $v_{m,n}(t,s) \geq 0$ for all $t \in [s,\Theta]$ and $m \leq n$. Moreover, from (7.2.4), $v_{m,n}(t,s) = 0$ for all $m > n$.

Since $n \in \mathbb{N}$ and $s \in \mathscr{I}$ were chosen arbitrarily, it follows that $v_{m,n}(t,s) \geq 0$ for all m, $n \in \mathbb{N}$ and $(t,s) \in D_{\mathscr{I}}$. Hence, if $\mathring{u} \in (\ell^1_w)_+$, then $u(t) = \mathscr{V}^{(w)}(t,s)\mathring{u}$ is the unique, non-negative classical solution of (7.2.2). $\square$

Lemma 7.2.3 then leads us to a unique, non-negative solution of (7.1.6).

Theorem 7.2.4. *Let Assumption 7.2.2 hold and let $\mathring{u} \in \ell^1_w$. Then there exists an evolution family, $(V^{(w)}(t,s))_{(t,s)\in D_{\mathscr{I}}}$, such that $u(t) = V^{(w)}(t,s)\mathring{u}$ is the unique classical solution of* (7.1.6)*. If $\mathring{u} \in (\ell^1_w)_+$, then the solution is non-negative.*

Proof. From Lemma 7.2.3 we can deduce that there exists an evolution family, $(\mathscr{V}^{(w)}(t,s))_{(t,s)\in D_{\mathscr{I}}}$, such that $u(t) = \mathscr{V}^{(w)}(t,s)\mathring{u}$ is the unique solution of (7.2.2). For $(t,s) \in D_{\mathscr{I}}$, we have

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(e^{t-s}\mathscr{V}^{(w)}(t,s)\mathring{u}\right) \\
&= e^{t-s}\left(\mathscr{V}^{(w)}(t,s)\mathring{u}\right) + e^{t-s}\frac{\partial}{\partial t}\left(\mathscr{V}^{(w)}(t,s)\mathring{u}\right) \\
&= e^{t-s}\mathscr{V}^{(w)}(t,s)\mathring{u} + e^{t-s}(A^{(w)} + B^{(w)} - I)\mathscr{V}^{(w)}(t,s)\mathring{u} \\
&= (A^{(w)} + B^{(w)})e^{t-s}\mathscr{V}^{(w)}(t,s)\mathring{u}
\end{aligned}$$

and $e^{s-s}\mathscr{V}^{(w)}(s,s)\mathring{u} = \mathring{u}$.

Hence, we can conclude that $u(t) = V^{(w)}(t,s)\mathring{u} := e^{t-s}\mathscr{V}^{(w)}(t,s)\mathring{u}$ is a classical solution of (7.1.6). We now note that $(V^{(w)}(t,s))_{(t,s)\in D_{\mathscr{I}}}$ is an evolution family since,

- for $0 \le r \le s \le t < \Theta$,

$$\begin{aligned} V^{(w)}(t,s)V^{(w)}(s,r) &= e^{t-s}\mathscr{V}^{(w)}(t,s)e^{s-r}\mathscr{V}^{(w)}(s,r) \\ &= e^{t-r}\mathscr{V}^{(w)}(t,r) \\ &= V^{(w)}(t,r); \end{aligned}$$

- $V^{(w)}(s,s) = e^{s-s}\mathscr{V}^{(w)}(s,s) = I$;
- the mapping $(t,s) \mapsto \mathscr{V}^{(w)}(t,s)$ is strongly continuous and so the mapping $(t,s) \mapsto e^{t-s}\mathscr{V}^{(w)}(t,s) = V^{(w)}(t,s)$ is strongly continuous.

It then follows that $(V^{(w)}(t,s))_{(t,s)\in D_{\mathscr{I}}}$ is an evolution family that solves (7.1.6). Fix $s \in \mathscr{I}$ and let $v(t)$ be any solution of (7.1.6) for $t \in [s,\Theta]$. Then, by differentiating $\tilde{u}(t) = e^{-(t-s)}v(t)$, we can show that $\tilde{u}(t)$ solves (7.2.2). By the uniqueness of solutions to (7.2.2), it follows that $\mathscr{V}^{(w)}(t,s)\mathring{u} = \tilde{u}(t) = e^{-(t-s)}v(t)$ for all $t \in [s,\Theta]$. Hence, for $t \in [s,\Theta]$,

$$v(t) = e^{t-s}\mathscr{V}^{(w)}(t,s)\mathring{u} = V^{(w)}(t,s)\mathring{u},$$

i.e. the classical solution of (7.1.6) is unique.

Finally, from Lemma 7.2.3, if $\mathring{u} \in (\ell_w^1)_+$, then $\mathscr{V}^{(w)}(t,s)\mathring{u} \in (\ell_w^1)_+$ and so $V^{(w)}(t,s)\mathring{u} = e^{t-s}\mathscr{V}^{(w)}(t,s)\mathring{u} \in (\ell_w^1)_+$. $\square$

We are now able to obtain the following result regarding mass conservation. We once again set $X_{[1]} := \ell_w^1$, $\|\cdot\|_{[1]} := \|\cdot\|_{\ell_w^1}$ and $M_1(\cdot) := \phi_{\ell_w^1}$ when $w_n = n$ for all $n \in \mathbb{N}$.

Corollary 7.2.5. *Let $\mathring{u} \in (\ell_w^1)_+$ and let* (7.1.2) *and Assumption 7.2.2 hold. Then there exists an evolution family, $(V^{(w)}(t,s))_{(t,s)\in D_{\mathscr{I}}}$, such that $u(t) = V^{(w)}(t,s)\mathring{u}$ is the unique, non-negative, mass-conserving classical solution of* (7.1.6), *in the sense that*

$$\|u(t)\|_{[1]} = \|\mathring{u}\|_{[1]} \qquad \textit{for all } (t,s) \in D_{\mathscr{I}}.$$

Proof. Since $w_n \geq n$ for all $n \in \mathbb{N}$, ℓ_w^1 is continuously embedded in $X_{[1]}$. Let $s \in \mathscr{I}$. From Theorem 7.2.4 we know that $u(t) = V^{(w)}(t,s)\mathring{u} \in (\ell_w^1)_+ \subseteq (X_{[1]})_+$ is the unique, non-negative solution of (7.1.6). Hence, for $t \in [s, \Theta]$,

$$\frac{d}{dt}\|u(t)\|_{[1]} = \frac{d}{dt}M_1\Big(u(t)\Big) = M_1\Big(u'(t)\Big) = M_1\Big(A^{(w)}(t)u(t) + B^{(w)}(t)u(t)\Big).$$

Now, for $t \in [s, \Theta]$,

$$\begin{aligned}
&M_1\Big(A^{(w)}(t)u(t) + B^{(w)}(t)u(t)\Big) \\
&= \sum_{n=1}^{\infty} n \left(-a_n(t)u_n(t) + \sum_{j=n+1}^{\infty} a_j(t)b_{n,j}(t)u_j(t)\right) \\
&= -\sum_{n=1}^{\infty} na_n(t)u_n(t) + \sum_{j=2}^{\infty}\left(\sum_{n=1}^{j-1} nb_{n,j}(t)\right) a_j(t)u_j(t) \\
&= -\sum_{n=1}^{\infty} na_n(t)u_n(t) + \sum_{j=1}^{\infty} ja_j(t)u_j(t) = 0.
\end{aligned}$$

Hence,

$$\frac{d}{dt}\|u(t)\|_{[1]} = 0 \qquad \text{and so} \qquad \|u(t)\|_{[1]} = \|\mathring{u}\|_{[1]}.$$

$\square$

Chapter 8

Conclusion and Future Work

To summarise, in this thesis we have examined the autonomous fragmentation system, (5.1.1), the C–F system with time-dependent coagulation, (6.1.1), and the non-autonomous fragmentation system, (7.1.1), using the theory of operator semigroups and evolution families. By writing (5.1.1) as an ACP in a general weighted ℓ^1 space, we were able to obtain various results regarding the existence and uniqueness of solutions. Properties of these solutions such as positivity, mass conservation and asymptotic behaviour were also established. In particular, when examining (5.1.1), we obtained results regarding analytic semigroups that do not necessarily hold when the "usual" weights, $w_n = n^p$, for $p \geq 1$ and $n \in \mathbb{N}$, are considered.

We then examined (6.1.1), where the coagulation was allowed to be time-dependent. This system was written as a semi-linear ACP in a general weighted ℓ^1 space and the existence and uniqueness of positive solutions were established under certain conditions on the coagulation coefficients. Under additional assumptions on the fragmentation rates, mass-conservation results were also provided. The analytic semigroups obtained when examining (5.1.1) were then used with the theory of interpolation spaces to relax the conditions required on the coagulation rates to obtain the existence and uniqueness of solutions.

Finally, we further exploited the results regarding analytic semigroups to examine non-autonomous fragmentation. Here we established that under certain assumptions on the time-dependent fragmentation rates, a unique, non-negative solution of (7.1.1) is given by an evolution family. Mass-conservation can again be shown to hold under additional assumptions on the fragmentation rates.

A natural extension of the work in this thesis is to examine the full C–F system where both the coagulation and the fragmentation are time-dependent. While results have previously been obtained for continuous C–F equations in which the coagulation and fragmentation coefficients are both permitted to be time-dependent, we are unaware of similar investigations into the discrete case. To use the semigroup perturbation approach adopted in this thesis with regard to a fully time-dependent system will require existence and uniqueness results for semi-linear ACPs, where both the associated linear and the associated non-linear operator are time-dependent. We are unaware of any such results but believe that it may be possible to extend Theorem 4.2.5 to find "mild" solutions of fully non-autonomous semi-linear equations; see [61, (7.2)]. However, there are also open questions regarding the full system where only the coagulation is time-dependent, (6.1.1). For example, we examined the asymptotic behaviour of solutions of the pure autonomous fragmentation system, (5.1.1), and similar topics could be investigated for the solutions of (6.1.1). Moreover, the non-autonomous fragmentation system was only briefly discussed in Chapter 7, and it may be possible to obtain the existence of solutions under more relaxed conditions than those provided in this thesis.

Furthermore, future investigations into non-autonomous C–F systems may consider using the theory of evolution semigroups, which was used to examine continuous non-autonomous fragmentation in [4]. Recent work in [13] has also considered discrete fragmentation models which incorporate additional birth and death terms. The analysis in [13] is carried out in an ℓ_w^1 space with a weight of the form $w_n = n^p$, for some $p \geq 1$ and all $n \in \mathbb{N}$, and future investigations could be carried out regarding whether the results obtained in [13] can be improved upon by working with more general weights, as in this thesis. A final extension of the work in this thesis would be to examine whether similar weighted L^1 theory as used in this thesis could be applied to continuous C–F.

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