On Word-Representability of Apollonian Graphs and 1-11-Representability of Graphs

PhD Thesis

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Abstract

A graph $G = (V, E)$ is *word-representable* if there exists a word w over the alphabet V such that two letters x and y alternate in w if and only if $\{x, y\} \in E$. The notion was introduced in the early 2000s by Kitaev and is now the subject of an extensive theory connecting graph theory, combinatorics on words, and order theory. A central result of Halldórsson, Kitaev, and Pyatkin states that a graph is word-representable if and only if it admits a *semi-transitive orientation*. While the class of word-representable graphs is rich containing all 3-colourable graphs, all comparability graphs, and all circle graphs, not every graph is word-representable, and characterising word-representable planar graphs remains a long-standing open problem.

This thesis is divided into two parts. In the first part, we study the word-representability of *Apollonian graphs*, a classical family of maximal planar graphs constructed by recursively subdividing triangular faces. We focus on a natural subfamily encoded by sequences of the symbols m, ℓ, and r, which record whether each subdivision step is performed in the middle, left, or right triangle relative to the most recently created vertex. Working through the cases of zero, one, two, and three changes of direction, we obtain a complete classification of word-representability in terms of parity conditions on the runs of m-steps. In the representable cases, we exhibit explicit semi-transitive orientations arising from carefully chosen 4-colourings; in the non-representable cases, we identify induced subgraphs whose neighbourhoods are not comparability graphs, or which lie in a known family of forbidden subgraphs of Hauser. We then refine these results to show that within this subfamily, word-representability is in fact equivalent to being a *circle graph*, by constructing 2-uniform word-representations whenever the parity conditions hold. The proof is built around an *extension invariant* which tracks a local structure preserved by each subdivision step.

In the second part, we turn to the more general notion of k-11-*representability*, introduced by Remmel and first developed by Cheon, Kim, Kim, Kitaev, and Pyatkin. A graph is k-11-representable if it admits a word in which two letters violate strict alternation at most k times exactly when the corresponding vertices are adjacent. Word-representable graphs are precisely the 0-11-representable graphs, and Cheon et al. proved that every graph is 2-11-representable, leaving open the case $k = 1$. Our main contribution to this problem, joint with Kitaev, Tang, Tao, and Zhang, is a proof that every graph on at most eight vertices is 1-11-representable, extending the previously known bound of seven vertices. We also introduce and study the *multi*-1-11-*representation number* of a graph and show that every graph on at most 24 vertices has multi-1-11-representation number at most 2.

Subsequent to the publication of these results, Hefty, Horn, and Muir resolved the existence question completely by showing that *every* graph is 1-11-representable, in fact, permutationally so, and obtained near-optimal bounds on representation length. We summarise this work and discuss the open questions it raises, which now form the natural sequel to the line of research begun in this thesis.

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Chapter 1

Introduction

1.1 Encoding graphs by words

A central problem at the interface of combinatorics, graph theory, and computer science is to find efficient and informative ways of encoding graphs. The classical approaches, adjacency matrices, adjacency lists, and incidence matrices, have been studied extensively, and each carries its own strengths and weaknesses; an excellent survey of efficient graph representations can be found in Spinrad's monograph [38]. Among the more structurally rich and combinatorially illuminating approaches is the idea of representing a graph by a *word* (i.e., a finite string of symbols) whose alphabet is the vertex set, where the adjacency relation is encoded by some property of the patterns formed by pairs of letters.

The most studied representation of this kind is the notion of a *word-representable graph*, introduced by Kitaev around 2004 and put on systematic foundations in the seminal paper of Kitaev and Pyatkin [31], motivated by earlier work of Kitaev and Seif on the Perkins semigroup [36]. Two distinct letters x and y are said to *alternate* in a word w if, after deleting all occurrences of letters other than x and y from w, one obtains a string of the form $xyxy\cdots$ or $yxyx\cdots$. A graph $G = (V, E)$ is word-representable if there exists a word w over the alphabet V such that two distinct letters $x, y \in V$ alternate in w if and only if $\{x, y\} \in E$.

The class of word-representable graphs has proved to be remarkably rich. It contains all 3-colourable graphs, all circle graphs, and all comparability graphs [30]. A landmark result of Halldórsson, Kitaev, and Pyatkin [20] established that a graph is word-representable if and only if it admits a *semi-transitive orientation*, providing a structural characterisation that has become indispensable in the area. The book by Kitaev and Lozin [30], together with the survey [32] and the introductory chapter [27], give a comprehensive account of the theory as it stands.

Despite this progress, the class of word-representable graphs is not closed under all natural operations and, importantly, not every graph is word-representable. The smallest non-word-representable graph is the wheel graph W_5 on six vertices, and there are exactly twenty-five non-word-representable graphs on seven vertices and 929 on eight vertices [30, 37]. Determining which graphs in a given family are word-representable is therefore a non-trivial structural question. Computational investigations, including the use of SAT solvers and integer-sequence methods, have also played a role in understanding the structure of word-representable graphs; see, for example, Akgün, Gent, Kitaev, and Zantema [1]. Particular attention has been paid to families of *planar* graphs, where a complete characterisation remains famously open even in restricted settings: triangulations of convex polyominoes are word-representable if and only if they are 3-colourable [2], and analogous structural conditions have been obtained for several near-triangulation classes by Glen [18], but no general characterisation of word-representable planar graphs is known.

1.2 Two themes of this thesis

This thesis contributes to two complementary lines of research within the broad area of word-representations.

Apollonian graphs. The first part of the thesis concerns the word-representability of *Apollonian graphs*, a classical family of maximal planar graphs constructed by starting from a triangle and recursively subdividing triangular faces, each new vertex being joined to the three vertices of the face it subdivides [4, 12]. Apollonian graphs are uniquely 4-colourable and have served as extremal examples in chromatic and geometric problems for nearly a century. They are, however, no longer triangle-free, so the sufficient condition of 3-colourability for word-representability does not apply.

We focus on a natural subfamily of Apollonian graphs in which subdivisions are constrained to take place around the most recently created vertex. Each construction can then be encoded by a word over the three-letter alphabet $\{m, \ell, r\}$, where m records that the new vertex is placed in the "middle" triangle and ℓ, r record placements in the "left" or "right" triangle relative to the most recently created vertex. Following the convention introduced for this thesis (and used throughout Chapters 3 and 4), the initial outer triangle is labelled with vertex 3 at the top and vertices 2 and 1 in clockwise order, so that the second-created vertex (i.e. the first interior vertex) is labelled 4. The resulting Apollonian graph corresponding to the encoding word w is denoted $A(w)$.

Within this family, every graph admits a sequence of *changes of direction*, positions where consecutive symbols of the encoding differ. Our central question is:

> *Given the change-of-direction structure of an Apollonian graph $A(w)$, can one decide whether $A(w)$ is word-representable, and if so, exhibit an explicit representation?*

We give a complete answer for graphs with up to three changes of direction, in terms of parity conditions on the lengths of m-runs surrounding each direction change. The representable cases admit semi-transitive orientations derived from particular 4-colourings; the non-representable cases contain induced subgraphs that obstruct word-representability, either because they are not comparability graphs in the neighbourhood of some vertex, or because they belong to a forbidden family.

After the bulk of this work was completed, Hauser's PhD thesis [21] appeared, in which the word-representability of *all* Apollonian graphs (not only those in our restricted subfamily) is characterised in terms of two structural conditions: being free of a certain forbidden family $\mathcal{F}_1$, and being *locally comparability*, meaning that the neighbourhood of every vertex is a comparability graph. Although Hauser's characterisation is more general in scope, the work of Section 3.1 retains independent value within our subfamily on three fronts: it gives an *easily checkable parity* on the encoding (Hauser's conditions require induced-subgraph inspection); it produces *explicit* *semi-transitive orientations* in the representable cases (Hauser's existence proof is non-constructive); and on the non-representability side, the proofs of obstruction extend the human-verifiable proof format of [37] for the first time, as far as we are aware, from individual graphs to infinite parametric families, by combining a base-case derivation with a subdivision-induction step.

We then go further, in Section 3.2, and prove a result that does not follow from Hauser's work: within this subfamily, word-representability is *equivalent* to being a circle graph, by constructing 2-uniform word-representations in every representable case. This is the new contribution of the thesis to the Apollonian theory.

k-11-representations. The second part of the thesis is concerned with a generalisation of word-representability inspired by work on permutation patterns. The notion of k-11-*representable graphs* was introduced by Jeff Remmel and developed by Cheon, Kim, Kim, Kitaev, and Pyatkin in [8]. A graph $G = (V, E)$ is k-11-representable if there exists a word w over V such that for any pair of distinct letters x, y, the subword $w|_{\{x,y\}}$ contains at most k occurrences of consecutive equal letters (i.e. factors xx or yy) if and only if $\{x, y\} \in E$. Setting $k = 0$ recovers exactly word-representability.

Cheon et al. [8] proved that every graph is 2-11-representable. The case $k = 1$ was left open, and was a notoriously difficult question: at the time of writing the published paper presented in Chapter 4, only the cases of graphs on at most seven vertices had been settled [8, 13]. Our contribution, joint with Kitaev, Tang, Tao, and Zhang [3], is a proof that every graph on at most eight vertices is 1-11-representable. The proof considers all such graphs by chromatic number, building a semi-transitive orientation after applying a sequence of star and matching operations to remove or add carefully chosen edges. As an application, we show that every graph on at most 24 vertices has *multi*-1-11-*representation number* at most 2, where the multi-version of the notion was inspired by Kenkireth and Malhotra [25].

After our paper was published, Hefty, Horn, and Muir [22] settled the existence question completely: *every* graph is 1-11-representable, in fact *permutationally* so. Their construction yields representations using only a linear number of permutations, which is tight up to a logarithmic factor. Their proof is independent of ours and uses an entirely different method. The published 8-vertex result is thereby subsumed as a special case, but the work of Chapter 4 retains independent interest: the proof technique of the 8-vertex result remains an instructive worked example of chromatic-number case analysis combined with the type-1, type-2, and type-3 lemmas; the formalism of multi-1-11-representation (Definition 4.2.2), including the strict-versus-non-strict distinction, is introduced for the first time in our paper and remains a valid framework for finer quantitative questions; and the partition technique of Theorem 4.2.4 (decomposing the graph into 8-vertex blocks with a 3-colourable cross-block graph) is a constructive approach that is independent of the existential argument of [22]. We summarise the Hefty–Horn–Muir results and the open questions they raise at the end of Chapter 4.

1.3 Related work and historical context

Word-representable graphs have grown into a substantial body of literature in the two decades since their introduction. We give here a brief summary of the developments that bear most directly on the work of this thesis.

Foundational results. The combinatorial framework was developed in [31, 36]. The fundamental tool that has been used throughout this area, and is used heavily in this thesis, is the equivalence between word-representability and the existence of a semi-transitive orientation [20]. The class is hereditary [30], contains all 3-colourable graphs and all comparability graphs, and is properly contained in the class of 4-colourable graphs (since the wheel W_5 is 4-chromatic but not word-representable). Bounds on the number of n-vertex word-representable graphs and on the chromatic number of certain non-representable families were obtained by Collins, Kitaev, and Lozin [11]. The hierarchy of k-uniform representations is also of fundamental importance: the class of 2-uniform word-representable graphs coincides with the class of *circle graphs* [30], which links our 2-word constructions in Chapter 3 to a classical and well-studied graph class. The *representation number* of a graph, the minimum k such that the graph admits a k-uniform word-representation, has been studied in its own right: Kitaev [26] characterised graphs with representation number 3 and showed that the representation number can be arbitrarily large, while Glen, Kitaev, and Pyatkin [19] determined the representation number of crown graphs. More recently, Hefty, Horn, Muir, and Owens [23] introduced poset-theoretic, Ramsey-theoretic, and probabilistic tools to establish new bounds on the existence and length of word-representations, and Srinivasan and Hariharasubramanian [39] studied the minimum length of word-representants. Connections to other types of graph encoding have also been explored: Jones, Kitaev, Pyatkin, and Remmel [24] introduced a framework for representing graphs via pattern-avoiding words, while Kitaev [28] established the existence of u-representations of graphs, a generalisation in which the forbidden pattern is allowed to vary. Operations that preserve semi-transitive orientability, both edge operations and graph products, were systematically studied by Choi, Kim, and Kim [9].

Word-representability of planar and near-triangulated graphs. Word-representability of planar graphs is one of the most actively studied open problems in the area. Glen [18] characterises word-representable triangle-free near-triangulations via 3-colourability. Akrobotu, Kitaev, and Masárová [2] establish the analogous result for triangulations of convex polyominoes, while Chen, Kitaev, and Sun [7] extended this line to face subdivisions of triangular grid graphs; further refinements have also been pointed out to us in private communication [5]. Kitaev and Pyatkin's introduction [27] surveys what was known up to 2017. The Hauser thesis [21] of 2025 gives a complete characterisation of word-representable Apollonian networks, and more generally, of word-representable chordal near-triangulations with outer cycle of length 3 or 5, in terms of forbidden induced subgraphs. The Hauser thesis is methodologically very different from our approach in Chapter 3: whereas Hauser's results take the form of an inductive structural argument over all Apollonian networks, our results give a direct, constructive recipe (with parity tests on the encoding) for the subfamily of greatest interest in this thesis.

Word-representability of structured graph families. Beyond the planar setting, the word-representability of several important graph classes has been investigated. Collins, Kitaev, and Lozin [11] obtained a number of structural results on word-representable graphs, including the characterisation of word-representable line graphs (further developed by Kitaev, Salimov, Severs, and Úlfarsson [35]) and several operations preserving word-representability. Split graphs, a class appearing naturally in algorithmic graph theory, have received sustained attention: Kitaev, Long, Ma, and Wu [29] gave a structural characterisation of word-representable split graphs, and Chen, Kitaev, and Saito [6] developed an alternative approach to their representation. Semi-transitive orientability has also been investigated for several structured non-planar classes, including triangle-free graphs [33] and Kneser graphs and their complements [34]. Gaetz and Ji [17] studied the enumeration of word-representants and introduced extensions of the word-representation framework to count the number of representing words with particular properties.

Tools for 1-11-representations. Several toolkits have been developed for 1-11-representability. The original paper [8] introduced what we call the *type-1, type-2, and type-3 lemmas* of Section 2.3.1, which describe operations preserving 1-11-representability. These include the disjoint union, gluing along a vertex, joining by an edge, the *star operation*, and the *matching operation*. Futorny, Kitaev, and Pyatkin [13] developed additional tools, including a powerful colouring lemma, that we use in Chapter 4. Kenkireth and Malhotra [25] studied the multi-word-representation number, which inspired our multi-1-11-representation number.

Subsequent developments. After the publication of our paper [3], the work of Hefty, Horn, and Muir [22] appeared, settling the central existence question for 1-11-representations. They prove that every graph is permutationally 1-11-representable (Theorem 4.3.1 in Chapter 4); that there exist graphs requiring $\Omega(n^2/\log n)$ characters in any 1-11-representation (so the Cheon et al. bound is tight up to a logarithmic factor); that every graph admits a permutational 1-11-representation using only $O(n)$ permutations (Chapter 4); and that the crown graph $H_{n,n}$, despite having a large word-representation number, has bounded permutational 1-11-representation number. They also show, in a complementary direction, that almost no graphs admit *even–odd* representations, in which the parity of the number of 11-patterns determines adjacency. We discuss these results in detail at the end of Chapter 4, together with the open questions that now drive the area.

1.4 Contributions of this thesis

This section gives an honest accounting of the contributions of the thesis, including how each result is positioned with respect to subsequent developments (Hauser's PhD thesis [21] and the work of Hefty, Horn, and Muir [22]) that came to light between the work being completed and the thesis being submitted.

- **(Chapter 3, Section 3.2.)** The headline new contribution of this thesis to the Apollonian theory: within the change-of-direction subfamily, word-representability is equivalent to being a circle graph (i.e. to being 2-uniformly word-representable). The proof is constructive: in each representable case, we exhibit an explicit 2-uniform word-representation built by induction using a notion we introduce, the *extension invariant*. In each non-representable case, an obstruction in the forbidden families $\mathcal{F}_1$ and $\mathcal{F}_2$ is identified. This refines and goes strictly beyond Hauser's characterisation, which establishes word-representability for all Apollonian networks but does not address whether the representable cases are circle graphs, and does not produce explicit words.

- **(Chapter 3, Section 3.1.)** A self-contained route to word-representability on the change-of-direction subfamily, via semi-transitive orientations and explicit 4-colourings. The classification is given by easily-checked parity conditions on the runs of m-steps surrounding each direction change. This work was completed independently of, and is methodologically different from, the more general characterisation in Hauser's thesis [21]; it has independent value as a decidable, constructive route to the question on this subfamily.

  The section also contains a methodological contribution: the proofs of non-word-representability in Lemmas 3.1.11, 3.1.16, and 3.1.17 use the human-verifiable proof format of Kitaev, Salimov, and Pyatkin [37] (which had previously been applied to specific small obstructions, one graph at a time), and extend it for the first time, as far as we are aware, to prove non-word-representability for *infinite families* of graphs, by combining a base-case derivation with an induction step that subdivides a parametric edge.

- **(Chapter 4, Section 4.1.)** The main result of the published paper [3], joint with Kitaev, Tang, Tao, and Zhang: every graph on at most eight vertices is 1-11-representable, extending the previously known bound of seven vertices. The proof works by chromatic-number case analysis, building a semi-transitive orientation after applying star and matching operations to add carefully chosen edges. The result was published in *Utilitas Mathematica* (2025), and at the time of publication addressed a notoriously difficult open question. Subsequently, Hefty, Horn, and Muir [22] resolved the existence question completely, by proving that every graph is 1-11-representable. Their work subsumes our 8-vertex result, but uses an entirely different (and independent) method.

- **(Chapter 4, Section 4.2.)** The introduction and first study of the *multi-1-11-representation number* of a graph (Definition 4.2.2), distinguishing strict and non-strict variants. The principal result is that every graph on at most 24 vertices has strict multi-1-11-representation number at most 2, via a partition technique that decomposes the graph into three 8-vertex blocks and uses the 3-colourability of the cross-block graph. Like the 8-vertex result, this contribution has been subsumed by Hefty–Horn–Muir [22] (since their result implies every graph has multi-1-11-representation number 1), but the partition technique and the strict/non-strict formalism are independent of their approach, and the finer quantitative questions they raise can still be revisited in this framework.

- **(Chapter 4, Section 4.3.)** A summary of the recent landmark work of Hefty, Horn, and Muir [22], which resolves the existence question for 1-11-representability of all graphs. We summarise their main results (universal 1-11-representability, length bounds, the crown-graph phenomenon, even–odd representations) and present the open questions raised in their paper as the natural sequel to the line of research this thesis contributes to.

Sole and joint contributions.

The work in Chapter 4 (specifically Sections 4.1 and 4.2) is joint work with Kitaev, Tang, Tao, and Zhang, and reproduces the published paper [3] with adjustments for notation, additional commentary, and the new Section 4.3 which is a summary of subsequent work and is not part of the original publication. The work in Chapter 3 (both Sections 3.1 and 3.2) is sole-authored by the candidate and has not been published at the time of thesis submission. The material in Chapter 2 is largely expository, presenting standard definitions, known theorems, and the toolkits from the literature used in the later chapters. Section 2.4 contains a small number of technical lemmas (in particular, Lemmas 2.4.5 and 2.4.7) that are needed to set up Chapter 3; these are original to this thesis and sole-authored by the candidate.

1.5 Organisation of the thesis

The thesis is organised as follows.

Chapter 2 provides the necessary preliminaries. Section 2.1 reviews basic graph-theoretic notation and standard results. Section 2.2 develops the theory of word-representable graphs in the form needed throughout the thesis: words, alternation, the representation number, semi-transitive orientations, comparability graphs, and the fundamental Halldórsson–Kitaev–Pyatkin theorem. Section 2.3 introduces 1-11-representations, defines the relevant notation and terminology, and presents the toolkits of Cheon et al. and Futorny et al. that we use in Chapter 4. Section 2.4 introduces Apollonian graphs and the change-of-direction encoding used throughout the rest of the thesis.

Chapter 3 contains the work on Apollonian graphs. The chapter is organised into two main sections, told as a single narrative reflecting the chronological development of the work. Section 3.1 presents the classification via semi-transitive orientations, working through the cases of zero, one, two, and three changes of direction; it includes a methodological contribution on the non-representability side, generalising the human-verifiable proof format of [37] from individual graphs to infinite families. After this work was completed, the Hauser thesis [21] came to light, which gave a complete characterisation theorem for word-representable Apollonian networks via the conditions of $\mathcal{F}_1$-freeness and local comparability; the section retains independent value as a decidable, constructive route on the subfamily, and is positioned with explicit reference to Hauser's work. Section 3.2 contains the new contribution to the Apollonian theory of this thesis: we show that, within our subfamily, every word-representable graph is in fact a circle graph, by constructing 2-uniform representations directly via an extension invariant. The two sections use slightly different vertex labellings; the conventions are made explicit at the start of each section, and the labelling of Section 3.2 is the convention used throughout the rest of the thesis.

Chapter 4 reproduces the published paper [3] on 1-11-representations. The proof that all graphs on at most eight vertices are 1-11-representable is given in Section 4.1, the multi-1-11-representation results in Section 4.2. The chapter closes with Section 4.3, where we summarise the subsequent work of Hefty, Horn, and Muir [22] that resolves the open question motivating our paper, and we present the open questions raised in their paper as the natural sequel to this line of work.

Chapter 5 concludes with a summary of the contributions and a discussion of open problems that span both halves of the thesis.

The bibliography lists all references cited in the thesis.

Chapter 2

Preliminaries

This chapter sets out the definitions, notation, and background results that the rest of the thesis builds upon. Section 2.1 reviews the graph-theoretic notation and standard concepts needed throughout. Section 2.2 develops the theory of word-representable graphs at the level of detail required in Chapters 3 and 4. Section 2.3 introduces 1-11-representable graphs and the various toolkits we use in Chapter 4. Finally, Section 2.4 introduces Apollonian graphs and the change-of-direction encoding that is central to Chapter 3.

The treatment here is mostly drawn from standard sources, in particular the book of Kitaev and Lozin [30], the foundational paper of Kitaev and Pyatkin [31], and the survey [32], with additional definitions adapted from Hauser's thesis [21] where these are needed in Chapter 3.

2.1 Graph-theoretic background

A *graph* $G = (V, E)$ is a pair consisting of a finite set V of *vertices* and a set E of two-element subsets of V, called *edges*. We write $V(G)$ and $E(G)$ when the graph G requires emphasis. We write $\{u, v\} \in E(G)$ to indicate that u and v are adjacent in G, and shall often abbreviate this to $uv \in E(G)$. The *order* of G is $|V(G)|$ and the *size* of G is $|E(G)|$. All graphs in this thesis are simple, finite, and undirected unless otherwise stated.

The *neighbourhood* of a vertex $v \in V(G)$ is $N(v) = N_G(v) = \{u \in V(G) : uv \in E(G)\}$. The *closed neighbourhood* is $N[v] = N(v) \cup \{v\}$. The *degree* of v is $\deg_G(v) = |N_G(v)|$.

A graph H is a *subgraph* of G if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. The subgraph H is *induced* if for all $u, v \in V(H)$, we have $uv \in E(H)$ if and only if $uv \in E(G)$. We write $G[X]$ for the subgraph induced by a vertex subset $X \subseteq V(G)$. We write $G - v$ for the subgraph induced by $V(G) \setminus \{v\}$. A class $\mathcal{C}$ of graphs is *hereditary* if for every $G \in \mathcal{C}$ and every induced subgraph H of G, $H \in \mathcal{C}$.

A *path* of length k is a sequence of distinct vertices $v_0 v_1 \cdots v_k$ such that consecutive vertices are adjacent. The *path on n vertices*, denoted P_n, is the graph with vertices $v_1, \ldots, v_n$ and edges $v_i v_{i+1}$ for $1 \leq i \leq n-1$; thus P_3 is the path on three vertices, having two edges. A *cycle* of length $k \geq 3$ is similar but with $v_0 = v_k$. The *cycle on n vertices* is denoted C_n.

A *(proper) k-colouring* of G is a map $c : V(G) \to \{1, 2, \ldots, k\}$ such that $c(u) \neq c(v)$ whenever $uv \in E(G)$. The smallest k for which G admits a k-colouring is the *chromatic number* of G, denoted $\chi(G)$.

We adopt the following notational convention used in Chapter 4. An $(a_1, a_2, \ldots, a_k)$-*colouring* of a graph G is a proper k-colouring with colour classes $V_1, \ldots, V_k$ of sizes $|V_i| = a_i$, where $a_1 \geq a_2 \geq \cdots \geq a_k$, and where $k = \chi(G)$. We stress that the number of colours is required to equal the chromatic number, so that an $(a_1, \ldots, a_k)$-colouring is in particular a *minimal* colouring: no proper colouring of G uses fewer than k colours. This minimality is used repeatedly in Chapter 4, where recolouring arguments derive a contradiction from the existence of a $(k-1)$-colouring. The number of edges between V_i and V_j is denoted $e(V_i, V_j)$.

Orientations

An *orientation* of G is a directed graph $\Omega(G)$ obtained by replacing each edge $uv \in E(G)$ by exactly one of the arcs $u \to v$ or $v \to u$. An orientation is *acyclic* if it contains no directed cycle.

An orientation of G is *transitive* if, whenever $u \to v$ and $v \to w$ are arcs, the arc

$u \to w$ is also present. A graph G is a *comparability graph* if it admits a transitive orientation. Comparability graphs, and the transitive orientations that certify them, are treated in detail by Golumbic [16]; a classical theorem of Gallai [14] gives the complete list of the minimal graphs that are *not* comparability graphs, that is, the graphs that are not comparability graphs but all of whose proper induced subgraphs are. Comparability graphs are precisely the graphs whose edges can be interpreted as the comparable pairs of some partially ordered set.

Comparability graphs have a deep connection to the theory of partially ordered sets. Recall that a (strict) *partial order* on a set P is a relation $\prec$ that is irreflexive, anti-symmetric, and transitive. A *linear extension* of a partial order is a total ordering of P that is consistent with $\prec$. A *realiser* of $(P, \prec)$ is a collection of linear extensions $\pi_1, \ldots, \pi_t$ such that $a \prec b$ in $(P, \prec)$ if and only if a precedes b in every π_i. The *dimension* of a partially ordered set is the smallest cardinality of a realiser.

Example 2.1.1. Let $P = \{a, b, c\}$ with the single relation $a \prec b$ (so c is incomparable to both a and b). The two linear extensions

$$\pi_1 = c\,a\,b, \qquad \pi_2 = a\,b\,c$$

form a realiser of $(P, \prec)$. Indeed, a precedes b in both π_1 and π_2, which records $a \prec b$; whereas c precedes a in π_1 but follows a in π_2, so a and c are incomparable, and similarly for b and c. No single linear extension can be a realiser here, since a linear order makes every pair comparable. Hence the dimension of $(P, \prec)$ is 2.

Some classes of graphs

We will encounter the following classes throughout; three of them are illustrated in Figure 2.1.

- The *complete graph* K_n has vertex set $\{v_1, \ldots, v_n\}$ and every pair adjacent.
- A *clique* of G is a set of pairwise adjacent vertices of G; equivalently, a subset $X \subseteq V(G)$ such that $G[X]$ is a complete graph. An *independent set* (or *stable*

set) of G is a set of pairwise non-adjacent vertices of G; equivalently, a subset $X \subseteq V(G)$ such that $G[X]$ has no edges.

- A graph G is *bipartite* if its vertex set admits a partition $V(G) = A \cup B$ with $A \cap B = \emptyset$ such that every edge of G has one endpoint in A and the other in B; equivalently, there is no edge with both endpoints in A, and no edge with both endpoints in B. Such a partition (A, B) is called a *bipartition* of G.

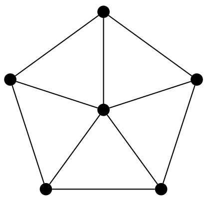
- The *complete bipartite graph* $K_{m,n}$ is the bipartite graph with bipartition (A, B) where $|A| = m$ and $|B| = n$, in which every $a \in A$ is adjacent to every $b \in B$, and there are no edges within A and no edges within B. The case $K_{3,3}$ is shown in Figure 2.1(b).

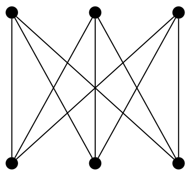
- The *wheel* W_n is obtained from a cycle C_n on n vertices by adding one new vertex (the *hub*) adjacent to every vertex of the cycle. The wheel W_5, shown in Figure 2.1(a), is the unique non-word-representable graph with six vertices [30].

- A *matching* in a graph G is a set of edges no two of which share an endpoint; it is a *perfect matching* if in addition every vertex of G is an endpoint of exactly one edge of the set. *The crown graph* $H_{n,n}$ is obtained from $K_{n,n}$ by deleting a perfect matching; concretely, if the bipartition classes are $\{a_1, \dots, a_n\}$ and $\{b_1, \dots, b_n\}$, then the edges $a_i b_i$ for $1 \leq i \leq n$ form a perfect matching of $K_{n,n}$, and $H_{n,n}$ is obtained by removing them, so that a_i is adjacent to b_j precisely when $i \neq j$. The case $n = 4$ is depicted in Figure 2.1(c).

- A graph is *planar* if it can be drawn in the plane with no two edges crossing. A planar graph is *maximal planar* (or a *triangulation*) if no edge can be added without violating planarity.

- A graph is *chordal* if every cycle of length at least four has a chord (i.e. an edge between two non-consecutive vertices of the cycle).


(a) The wheel W_5.


(b) The complete bipartite graph $K_{3,3}$.

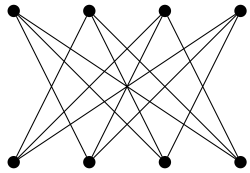

(c) The crown graph $H_{4,4}$.

Figure 2.1: Three standard graphs used throughout this thesis: the wheel W_5 (the smallest non-word-representable graph), the complete bipartite graph $K_{3,3}$, and the crown graph $H_{4,4}$ (obtained from $K_{4,4}$ by deleting a perfect matching).

2.2 Word-representable graphs

We now turn to the central notions that drive this thesis.

2.2.1 Words and alternation of letters

A *word* w over an alphabet V is a finite sequence $w_1w_2\cdots w_n$ of letters from V. The *length* of w is the number n of letters appearing (counting repetitions); we denote this by $|w|$. A *factor* of w is a contiguous subword $w_iw_{i+1}\cdots w_j$, where $1 \le i \le j \le n$. For a subset $S \subseteq V$, the *restriction of w to S*, denoted $w|_S$, is the word obtained from w by deleting all letters not in S. If $S = \{x, y\}$, we write $w|_{\{x,y\}}$ or simply $w|_{xy}$.

For a word w we let $\pi(w)$ denote the *initial permutation* of w, that is, the permutation of the letters of w obtained by recording the leftmost occurrence of each distinct letter when reading w from left to right. We let $r(w)$ denote the *reverse* of w, namely w written backwards. For example, if $w = abacbc$ then reading w from left to right the leftmost occurrences of the distinct letters appear in the order a, b, c, so $\pi(w) = abc$; while for $w = cbabca$ the leftmost occurrences appear in the order c, b, a, so $\pi(w) = cba$.

Two distinct letters x and y are said to *alternate* in w if $w|_{\{x,y\}}$ is one of the four words $xyxy\cdots$, $yxyx\cdots$ (each of which may be of even or odd length). Equivalently, no factor of $w|_{\{x,y\}}$ is of the form xx or yy.

A word w over an alphabet V is *k-uniform* if every letter of V occurs in w exactly k times. In particular, no letter of the alphabet may be absent: a k-uniform word over V has length exactly $k|V|$. (When a word-representant of a graph G is under discussion the alphabet is $V(G)$, so this says that every vertex of G occurs exactly k times.) In particular, 1-uniform words are precisely the permutations of an alphabet. A word is *uniform* if it is k-uniform for some $k \geq 1$.

2.2.2 Definition and basic properties of word-representable graphs

Definition 2.2.1. A graph $G = (V, E)$ is *word-representable* if there exists a word w over alphabet V such that, for every pair of distinct vertices $x, y \in V$,

$$xy \in E(G) \iff x \text{ and } y \text{ alternate in } w.$$

The word w is called a *word-representant* of G.

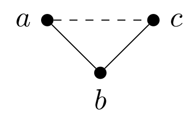
Example 2.2.2. The complete graph K_n is word-representable, since any permutation of its vertices is a word-representant. The empty graph $\overline{K_n}$ is word-representable by $v_1v_1v_2v_2 \cdots v_nv_n$ (each pair fails to alternate). The path P_3 on vertices a, b, c with edges ab, bc is word-representable by the word $abacbc$, as illustrated in Figure 2.2: the restrictions $w|_{ab}$ and $w|_{bc}$ alternate, while $w|_{ac} = aacc$ contains the factors aa and cc and hence a, c do not alternate.

$w =$ a b a c b c

$w|_{ab} =$ a b a b $\boxed{\text{alternates}}$ $\Rightarrow$ edge ab

$w|_{bc} =$ b c b c $\boxed{\text{alternates}}$ $\Rightarrow$ edge bc

$w|_{ac} =$ a a c c $\boxed{\text{contains } aa \text{ and } cc}$ $\Rightarrow$ no edge ac



Path P_3 (dashed: non-edge)

Figure 2.2: The word $w = abacbc$ represents the path P_3 on vertices a, b, c with edges ab and bc: the restrictions $w|_{ab}$ and $w|_{bc}$ alternate, while $w|_{ac} = aacc$ contains the factors aa and cc, so a and c do not alternate.

The class of word-representable graphs has the following fundamental properties.

Theorem 2.2.3 ([31, 30])**.**

(i) Every word-representable graph admits a k-uniform representant for some $k \geq 1$.

(ii) Every 3-colourable graph is word-representable.

(iii) Every comparability graph is word-representable.

(iv) The class of word-representable graphs is hereditary.

The first item motivates the following definition.

Definition 2.2.4. The *representation number* of a word-representable graph G, denoted $\rho(G)$, is the smallest k for which G admits a k-uniform word-representant. A word-representable graph G has *representation number* 1 if and only if it is a complete graph; this is straightforward to see since every 1-uniform word is a permutation, and any two letters in a permutation alternate.

Definition 2.2.5. A *circle graph* is a graph that can be realised as the intersection graph of a family of chords of a circle: that is, a graph G for which there exist a circle and a family of chords $\{c_v : v \in V(G)\}$ of that circle, indexed by the vertices of G, such that two distinct vertices u and v are adjacent in G if and only if the chords c_u and c_v cross. Circle graphs have been studied extensively in their own right; a characterisation by forbidden induced subgraphs was obtained by Bouchet [15], and recognition algorithms are surveyed by Spinrad [38].

Theorem 2.2.6 ([30])**.** *The class of graphs with representation number at most* 2 *coincides with the class of circle graphs.*

This theorem will play a substantive role in Chapter 3: we shall ultimately show that within our Apollonian subfamily, word-representability is equivalent to representation number ≤ 2, and hence equivalent to being a circle graph.

2.2.3 Neighbourhood structure of word-representable graphs

The following classical result constrains the local structure of word-representable graphs and provides one of the principal tools by which non-word-representability is established in this thesis. The wheel W_5 of Figure 2.1(a) is the smallest example: the neighbourhood of its hub is the cycle C_5, which is not a comparability graph.

Theorem 2.2.7 ([31])**.** *If a graph G is word-representable, then for every vertex $v \in V(G)$, the induced subgraph on the neighbourhood $N(v)$ is a comparability graph.*

If some neighbourhood fails to be a comparability graph, the graph cannot be word-representable. Following Hauser [21], we call a graph G *locally comparability* if the neighbourhood $N(v)$ induces a comparability graph for every $v \in V(G)$.

2.2.4 Minimum word-representants

A word w representing G is a *minimum word-representant* if no shorter word represents G. The structure of minimum word-representants is generally subtle: even for the complete bipartite graph $K_{n,n}$, exhibiting a minimum word-representant is non-trivial. The minimum-uniform-representation problem (find the smallest k for which there is a k-uniform word-representant) is also non-trivial; indeed, computing $\rho(G)$ is open even for some natural graph families. For the crown graph $H_{n,n}$ (illustrated in the case $n = 4$ in Figure 2.1(c)), it is known [19] that $\rho(H_{n,n}) = \lceil n/2 \rceil$, while in Chapter 4 we shall see that the permutational 1-11-representation number of $H_{n,n}$ is bounded by 4 independently of n.

2.2.5 Semi-transitive orientations

The most important structural result on word-representable graphs is Theorem 2.2.9 below.

Definition 2.2.8. An orientation $\Omega(G)$ of a graph G is *semi-transitive* if it is acyclic and *shortcut-free*, where a *shortcut* is an induced subgraph on at least four vertices $\{v_0, v_1, \ldots, v_k\}$ that is acyclic, non-transitive, and contains both the directed path $v_0 \to v_1 \to \cdots \to v_k$ and the arc $v_0 \to v_k$ (called the *shortcutting edge*).

Figure 2.3 illustrates the forbidden configuration: in the shortcut on the left, the arc $v_1 \to v_3$ is missing, so the directed path $v_0 \to v_1 \to v_2 \to v_3$ is short-circuited by $v_0 \to v_3$ without the intermediate transitivity arcs being present. Adding all transitivity arcs, as on the right, eliminates the shortcut.

Equivalently, an orientation is semi-transitive if it is acyclic and, for every directed path $v_0 \to v_1 \to \cdots \to v_k$, either there is no arc from v_0 to v_k, or every pair v_i, v_j with $i < j$ has the arc $v_i \to v_j$.

dashed: $v_1 \to v_3$ is absent

(a) A shortcut.

transitive closure: no shortcut

(b) A semi-transitive configuration.

Figure 2.3: A shortcut (left) is an induced subgraph on at least four vertices that is acyclic, non-transitive, and contains both a directed path $v_0 \to v_1 \to \cdots \to v_k$ and the shortcutting arc $v_0 \to v_k$. Adding all transitivity arcs (right) eliminates the shortcut.

Theorem 2.2.9 ([20])**.** *A graph G is word-representable if and only if it admits a semi-transitive orientation.*

This characterisation is used pervasively throughout Chapter 3, both to construct word-representations (by exhibiting a semi-transitive orientation) and to rule them out (by showing that no orientation can avoid both directed cycles and shortcuts).

The following two results are also used heavily.

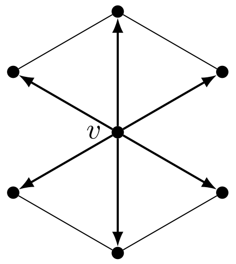
Theorem 2.2.10 ([37])**.** *If G is word-representable, then for every vertex $v \in V(G)$, G admits a semi-transitive orientation in which v is either a source (all edges incident to v point away from v) or a sink (all edges incident to v point towards v).*

Figure 2.4 illustrates the two configurations guaranteed by Theorem 2.2.10; the orientations of edges not incident to v are suppressed.

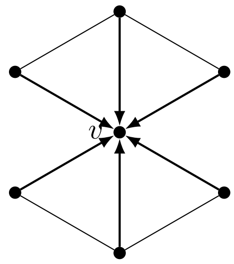
Lemma 2.2.11 ([31])**.** *Suppose that G has a cycle $C = x_1x_2\cdots x_mx_1$, with $m \geq 4$, such that the vertices $\{x_1, \ldots, x_m\}$ do not induce a clique. If G is oriented semi-transitively and $m-2$ edges of C are oriented in the same direction (i.e. from x_i to x_{i+1}, with indices mod m, or vice versa), then the remaining two edges of C are oriented in the opposite direction.*

2.2.6 Format for non-representability proofs

In Chapter 3 we use a compact notation, introduced in [37], to record proofs of non-word-representability via case analysis on potential semi-transitive orientations. We



(a) v as a source: all edges incident to v point away.



(b) v as a sink: all edges incident to v point towards v.

Figure 2.4: Theorem 2.2.10 guarantees that any chosen vertex v of a word-representable graph can be made either a source or a sink in some semi-transitive orientation. Orientations of the remaining edges are suppressed.

follow [37] closely; the reader is referred to that paper for further details.

By a *partially oriented* graph we mean a graph G together with a choice of orientation for some (possibly empty) subset $F \subseteq E(G)$ of its edges, the edges of $E(G) \setminus F$ being left unoriented; the case analysis proceeds by repeatedly orienting further edges, so that a fully oriented graph is reached when $F = E(G)$.

By a *line* of a proof we mean a sequence of instructions for orienting a partially oriented graph, ending with the detection of a shortcut or another contradiction that closes off the orientation branch under consideration. Each proof begins by orienting an edge $A \to B$ (the choice of which is determined by an algorithm); the symmetric branch $B \to A$ may be omitted by symmetry. The four types of instructions used are:

- "**B**" followed by "$X \to Y$ (Copy Z)": branch on edge XY, orient as $X \to Y$, and create a copy of the graph (called Copy Z) in which XY is oriented in the opposite direction. We continue with $X \to Y$, returning to Copy Z later.

- "**MC**" followed by a number X: move to Copy X of the graph that was created previously.

- One "**O**" followed by "$X \to Y$, (C X-Y-Z)": orient the edge XY as $X \to Y$, because otherwise the triangle XYZ would form a directed cycle (or, in the case of a longer cycle, an application of Lemma 2.2.11 forces the orientation).

- Two "**O**"s followed by "$X \to Y$, (C X-Y-Z-$\cdots$)": identify the cycle to which

Lemma 2.2.11 can be applied and the resulting forced orientations.

Each line ends with "**S**: X-Y-$\cdots$-Z" to indicate that a shortcut (in the sense of Figure 2.3) with shortcutting edge $X \to Z$ has been obtained.

A graph is non-word-representable if every branch of the case analysis terminates in a shortcut or a directed cycle, since this shows that no semi-transitive orientation exists, contradicting Theorem 2.2.9.

2.3 1-11-representable graphs

The notion of word-representable graphs admits a natural generalisation, in which strict alternation is relaxed by allowing a controlled number of violations.

Definition 2.3.1. Let $k \geq 0$. A graph $G = (V, E)$ is k-11-*representable* if there exists a word w over the alphabet V such that, for every pair of distinct vertices $x, y \in V$,

$$xy \in E(G) \iff w|_{\{x,y\}} \text{ contains at most } k \text{ factors of the form } xx \text{ or } yy \text{ in total.}$$

A word with this property is called a k-11-*representant* of G. A graph is *permutationally* k-11-*representable* if it admits a k-11-representant that is the concatenation of permutations of V.

The terminology comes from regarding the factors xx and yy as occurrences of the consecutive pattern 11 in the subword $w|_{\{x,y\}}$. A pair of vertices is connected by an edge precisely when this consecutive pattern occurs at most k times.

When $k = 0$, we recover word-representable graphs: factors xx and yy are forbidden in $w|_{\{x,y\}}$ exactly when x and y alternate in w. Thus every 0-11-representable graph is word-representable, and conversely.

Example 2.3.2. The wheel W_5 (Figure 2.1(a)) is not 0-11-representable (i.e. not word-representable) but is 1-11-representable [8]. More dramatically, the following theorem of Cheon et al. shows the contrast between $k = 2$ and the boundary case $k = 1$.

Theorem 2.3.3 ([8])**.** *Every graph is permutationally* 2-11-*representable.*

The case $k = 1$ remained open at the time of writing the published paper presented in Chapter 4, and was the focus of considerable effort. We discuss the state of the art in detail in Chapter 4.

2.3.1 Tools for studying 1-11-representations

We collect here the principal tools used in Chapter 4. The first lemma collects the operations preserving 1-11-representability that were introduced in [8]; we refer to them collectively as the *type-1 lemmas*.

Lemma 2.3.4 ([8], type-1)**.**

(a) *If G_1 and G_2 are* 1-11*-representable graphs, then their disjoint union, gluing them at a single vertex, or connecting them by a single edge, all yield* 1-11*-representable graphs. Here* gluing G_1 and G_2 at a single vertex *means the following: choose a vertex $u \in V(G_1)$ and a vertex $v \in V(G_2)$, take the disjoint union of G_1 and G_2, and then identify u with v into a single vertex (which is therefore adjacent to $N_{G_1}(u) \cup N_{G_2}(v)$). The resulting graph has $|V(G_1)| + |V(G_2)| - 1$ vertices and $|E(G_1)| + |E(G_2)|$ edges, and G_1 and G_2 meet in exactly the identified vertex.*

(b) *If G is* 1-11*-representable, then for any edge $xy \in E(G)$, the graph obtained by adding a new vertex adjacent to x and y only is* 1-11*-representable.*

We pause to fix the two notations used in the next two lemmas, since in each case the symbol $\setminus$ does *not* denote a plain set-theoretic difference between a graph and a set.

In Lemma 2.3.5 the symbol is applied to a set of edges. For a set of edges $E' \subseteq E(G)$, we write

$$G \setminus E' := \big(V(G),\, E(G) \setminus E'\big),$$

that is, the graph obtained from G by deleting the edges of E' while keeping *all* vertices of G (so no vertex is removed, even if it becomes isolated). In particular, for the specific edge sets appearing in Lemma 2.3.5, such as $\{uv \in E(G) : u \in N_A(v)\}$, only those edges are deleted and every vertex of G is retained.

In Lemma 2.3.6 the symbol is applied to a single vertex. For a vertex $v \in V(G)$, we write

$$G \setminus v := G\big[V(G) \setminus \{v\}\big],$$

that is, the graph obtained from G by deleting the vertex v together with all edges incident to it; this is the induced subgraph on the remaining vertices, and it agrees with the notation $G - v$ introduced earlier. More generally, for a set of vertices $V' \subseteq V(G)$ the same convention gives $G \setminus V' := G[V(G) \setminus V']$, the induced subgraph on $V(G) \setminus V'$.

For the next lemma, given $A \subseteq V(G)$ and $v \in V(G)$, let $N_A(v) = \{u \in A : uv \in E(G)\}$.

Lemma 2.3.5 ([8], type-2)**.** *Let G be a word-representable graph. Then for every $A \subseteq V(G)$ and $v \in V(G)$:*

(a) $G \setminus \{xy \in E(G) : x, y \in A\}$ *is* 1-11-*representable.*

(b) $G \setminus \{uv \in E(G) : u \in N_A(v)\}$ *is* 1-11-*representable.*

The operation in part (b) is colloquially referred to as "adding a star" or the *star operation*. The terminology deserves a word of explanation, since the operation as stated *removes* the edges $\{uv : u \in N_A(v)\}$, which form a star centred at v. The name reflects the direction in which the lemma is applied rather than the direction in which it is stated. In practice one starts with the graph H that is to be shown 1-11-representable, and one seeks a word-representable graph G from which H arises by the operation; equivalently, one *adds* to H a star centred at v so as to obtain a word-representable graph $G = H \cup \{uv : u \in N_A(v)\}$. Thus in the intended workflow a star is added to the graph under consideration, and the lemma then certifies that the original graph is 1-11-representable. This is illustrated on the left of Figure 2.5.

The next lemma combines several results from [8] and [13].

Lemma 2.3.6 ([8], type-3)**.** *Let G be a graph with a vertex $v \in V(G)$. Then G is* 1-11-*representable if at least one of the following holds:*

(a) $G \setminus v$ *is a comparability graph;*

(b) *$G \setminus v$ is a circle graph (equivalently, by Theorem 2.2.6, $\rho(G \setminus v) \leq 2$).*

The next tool is the *matching operation*, due to Futorny, Kitaev, and Pyatkin [13].

Lemma 2.3.7 ([13])**.** *Let H be a graph and let G be obtained from H by adding a matching, that is, by adding a (possibly empty) set M of new edges no two of which share a vertex, so that $G = (V(H), E(H) \cup M)$ and $M \cap E(H) = \emptyset$. If G is word-representable, then H is* 1-11*-representable.*

The direction of this statement is worth emphasising, since it is the graph H with the *fewer* edges that is concluded to be 1-11-representable, on the hypothesis that the *larger* graph G is word-representable. In applications one therefore starts with the graph H under consideration, adds a matching so as to reach a word-representable graph G, and concludes that H is 1-11-representable. In this sense the lemma complements Lemma 2.3.5(b): there one removes a star from a word-representable graph, here one adds a matching to the graph of interest in order to reach a word-representable one. The contrast between the two operations is depicted in Figure 2.5.

In practice, these operations are typically applied to add or remove a small number of edges, often a single edge or a pair of edges forming a partial matching. In Chapter 4, where the case analysis for the main theorem repeatedly invokes both operations, the goal is usually to complete a small bipartite block to a complete bipartite graph, which requires only one or two additional edges at each step.

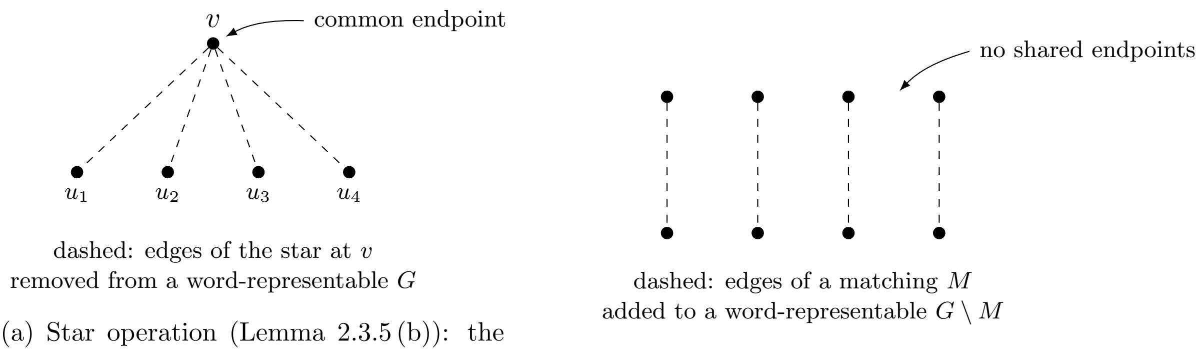
A useful structural observation, also from [13], is the following.

Lemma 2.3.8 ([13])**.** *Every split graph (a graph whose vertex set partitions into a clique and an independent set) is permutationally* 1-11*-representable.*

This is used in the proof of one auxiliary result in Chapter 4.

We will also need the following easy observation, which we record now for use in Chapter 4.

Remark 2.3.9. In an $(a_1, a_2, \ldots, a_k)$-colouring of G with $\chi(G) = k$, the vertices in colour classes V_t with $a_t = 1$ are adjacent to at least one vertex in every other colour class. (Otherwise, recolouring such a vertex with the missing colour would yield a $(k-1)$-colouring.)



(a) Star operation (Lemma 2.3.5 (b)): the dashed edges $\{u_iv : u_i \in N_A(v)\}$ all share the common endpoint v and are *removed* from G.

(b) Matching operation (Lemma 2.3.7): the dashed edges of M pairwise share no endpoint and are *added* to $G \setminus M$ to obtain G.

Figure 2.5: The star operation removes a set of edges all sharing a common endpoint from a word-representable graph, while the matching operation adds a set of edges no two of which share an endpoint. Both operations produce 1-11-representable graphs.

2.4 Apollonian graphs and direction encoding

We now introduce the family of graphs that is the focus of Chapter 3.

2.4.1 Apollonian graphs

Definition 2.4.1 ([21])**.** An *Apollonian graph* (or *Apollonian network*) is constructed recursively as follows:

- A triangle is an Apollonian graph.
- If G is an Apollonian graph and F is a triangular face of some planar embedding of G, then the graph obtained from G by inserting a new vertex v inside F and connecting v to each of the three vertices of F is again an Apollonian graph.

Example 2.4.2. The smallest Apollonian graph is the triangle K_3 itself, obtained by applying no subdivision at all. Applying a single subdivision to the unique face of K_3 inserts one vertex joined to all three existing vertices, and produces the complete graph K_4. Applying a second subdivision, to any one of the three interior triangular faces of K_4, produces the 5-vertex Apollonian graph sometimes drawn as a triangle with two nested vertices; it has 5 vertices and 9 edges, and is the graph denoted $A(m^2)$ in the encoding introduced below. Continuing in this way, every Apollonian graph on n

vertices has exactly $3n - 6$ edges, the maximum possible for a planar graph, and is obtained from K_3 by exactly $n - 3$ subdivisions. The graph $A(m\ell r)$ on six vertices is illustrated in Figure 2.7.

Apollonian graphs are maximal planar (every face other than the outer face is a triangle, and no edge can be added without creating a crossing or a multi-edge) and uniquely 4-colourable [4, 12]. The family is hereditary in a structural sense (a vertex of degree 3 may always be removed without leaving the class), but in the wider sense induced subgraphs of Apollonian graphs need not be Apollonian.

Throughout this thesis, we focus on a natural subclass of Apollonian graphs in which subdivisions are constrained to take place around the most recently created vertex. We now describe this construction precisely.

2.4.2 Direction encoding

We adopt the following labelling convention, used in Chapters 3 (Section 3.2) and 4.

Notation 2.4.3 (**Vertex labelling convention**)**.** The initial outer triangle of an Apollonian graph in our subfamily is labelled with vertex 3 at the top, vertex 2 at the bottom-right, and vertex 1 at the bottom-left, so that reading the triangle clockwise from the top gives the sequence $3, 2, 1$. The fourth vertex (which subdivides the initial triangle) is labelled 4, and subsequent vertices are labelled $5, 6, 7, \dots$ in the order they are added.

Note. In Section 3.1 of Chapter 3, which uses a slightly different labelling, we use $v_1, v_2, v_3, \dots$ in place of $1, 2, 3, \dots$. The geometry is identical: v_3 is at the top, v_2 at the bottom-right, v_1 at the bottom-left, v_4 is the first interior vertex, and so on. This local convention is restated at the start of Section 3.1.

The construction

Starting from the initial triangle on vertices $\{1, 2, 3\}$, the first interior vertex (vertex 4) subdivides this triangle. Moving from vertex 3 to vertex 4 inside the triangle, we identify the three new triangles thus created (see Figure 2.6):

- the *middle* triangle, namely $\{4, 1, 2\}$ (which does not contain vertex 3);
- the *left* triangle, namely $\{3, 2, 4\}$ (lying to the left of the segment 3-4);
- the *right* triangle, namely $\{3, 4, 1\}$ (lying to the right of the segment 3-4).

At each subsequent step, we choose one of the three triangles surrounding the most recently created vertex, middle, left, or right, and subdivide it, introducing a new vertex adjacent to the three vertices of the chosen triangle. After this subdivision, the most recently created vertex changes, and so do the local notions of "middle", "left", and "right". Since this updating rule is used constantly in Chapter 3, we state it precisely.

Suppose that at some step the most recently created vertex is u, that u was inserted into the triangle $\{p, q, r\}$, and that the vertex created immediately before u is p. (At the first step $u = 4$, $\{p, q, r\} = \{3, 2, 1\}$ and $p = 3$.) Inserting u into $\{p, q, r\}$ creates exactly three new triangles, namely $\{u, q, r\}$, $\{u, p, q\}$ and $\{u, p, r\}$, and these are the three triangles surrounding u. They are named as follows:

- the *middle* triangle is $\{u, q, r\}$, the unique one of the three that does *not* contain the previously most recently created vertex p;
- the *left* and *right* triangles are the two containing p, namely $\{u, p, q\}$ and $\{u, p, r\}$, distinguished by which side of the segment p-u they lie on in the planar drawing: reading the boundary of the old triangle clockwise starting at p, the first of q, r encountered gives the left triangle and the second gives the right one.

Now suppose the next symbol of the encoding causes us to subdivide one of these three triangles, inserting a new vertex u'. Then for the following step the roles shift: u' becomes the most recently created vertex, u takes over the role that p had (it is now the previously most recently created vertex), and the triangle just subdivided takes over the role of $\{p, q, r\}$. The three triangles surrounding u' are then named by exactly the same rule, with (u', u) in place of (u, p). In particular the labels "middle", "left" and "right" are always relative to the current pair consisting of the most recently created vertex and its immediate predecessor, and never refer to a fixed position in the plane; this is why the same symbol ℓ can subdivide geometrically quite different triangles at different steps. Figure 2.7 traces this through for the encoding $m\ell r$, where the symbol r subdivides the triangle that is to the right *relative to vertex* 5, not relative to the original drawing.

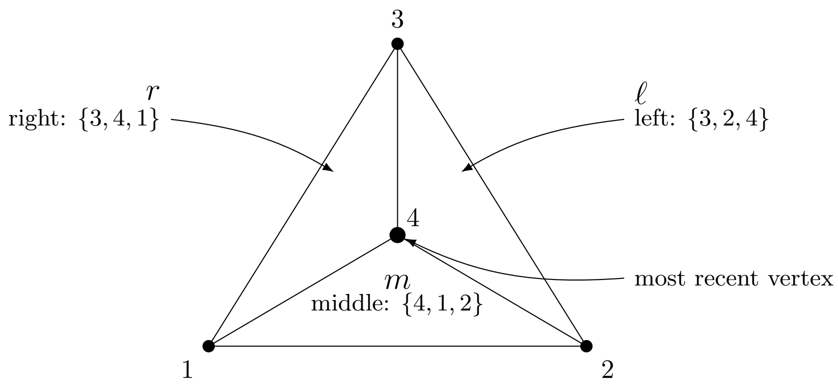


Figure 2.6: After the first m-subdivision, the three triangles surrounding the most recently created vertex (vertex 4). A subsequent m, ℓ, or r symbol subdivides the corresponding triangle. The text is read with vertex 3 at the top, vertex 2 at the bottom-right, and vertex 1 at the bottom-left.

The construction is encoded by a word over the alphabet $\{m, \ell, r\}$, with m denoting a middle subdivision, ℓ a left subdivision, and r a right subdivision. We write $A(w)$ for the Apollonian graph obtained from the encoding w. Following the conventions used in Chapter 3, we write w in the form

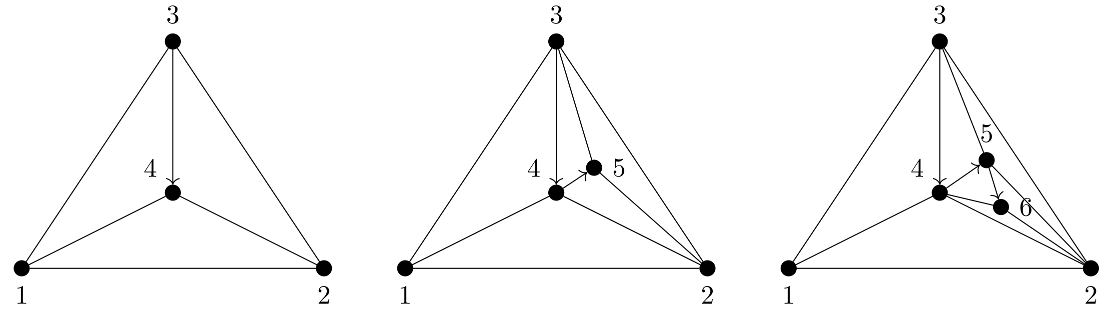
$$m^{s_0} x_1 m^{s_1} x_2 m^{s_2} \cdots x_p m^{s_p}, \qquad x_i \in \{\ell, r\}, \quad s_0 \geq 1, \quad s_i \geq 0 \text{ for } i \geq 1, \quad p \geq 0,$$

so that s_0 counts the initial run of m-subdivisions and s_i counts the run between the i-th and $(i+1)$-st direction changes. The number p is the number of ℓ/r-symbols, which we call the number of *direction symbols* in the encoding. The number of *changes of direction* is the number of indices i for which $x_i \neq x_{i+1}$; this counts the positions where the direction symbol differs from its predecessor.

When parameters are unimportant, we abbreviate Apollonian graphs as $A_1, A_2, \ldots$.

Example 2.4.4. Figure 2.7 illustrates the construction of $A(m\ell r)$. After the first subdivision m (placing vertex 4, with surrounding triangles as labelled in Figure 2.6), the second symbol ℓ subdivides the left triangle (placing vertex 5), and the third symbol

r subdivides what is now the right triangle relative to vertex 5 (placing vertex 6). The resulting Apollonian graph has 6 vertices.



(a) After m. (b) After $m\ell$. (c) After $m\ell r$.

Figure 2.7: The construction of the Apollonian graph $A(m\ell r)$. The arrows indicate the order in which vertices are created.

Some basic properties

We record two simple but important results that will be used throughout Chapter 3.

Lemma 2.4.5 (Direction reduction). *Let $x_0, x_1, \ldots, x_p \in \{\ell, r\}$, $p \geq 0$, and $s_1, \ldots, s_p \geq 0$. Then the Apollonian graphs*

$$A_1 = A(m\, x_0\, m^{s_1}\, x_1\, m^{s_2} \cdots x_p\, m^{s_p}) \qquad \textit{and} \qquad A_2 = A\big(m^{s_1+2}\, x_1\, m^{s_2} \cdots x_p\, m^{s_p}\big)$$

are isomorphic.

Proof. We give an explicit graph isomorphism $\varphi : V(A_1) \to V(A_2)$.

Recall the labelling convention (Notation 2.4.3): in any encoded graph $A(w)$, the outer triangle carries vertex 3 at the top, vertex 2 at the bottom-right, and vertex 1 at the bottom-left; the interior vertices are numbered $4, 5, 6, \ldots$ in the order they are created. We treat the case $x_0 = r$; the case $x_0 = \ell$ is symmetric and is discussed at the end of the proof. Figure 2.8 illustrates the base case.

Step 1: identify the two five-vertex starting subgraphs.

The prefix $m\, r$ of A_1 produces the five-vertex graph $A(mr)$:

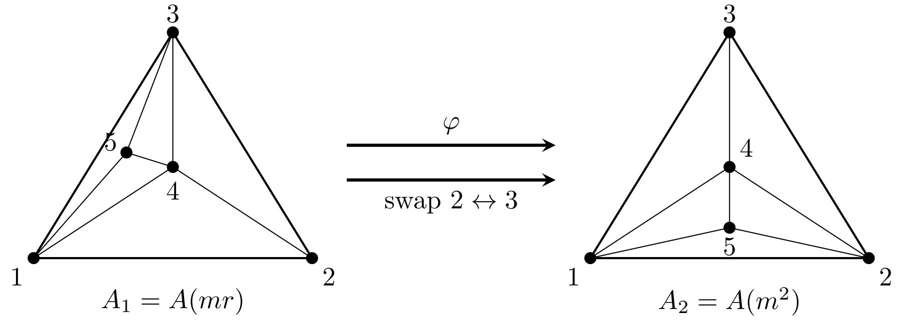


Figure 2.8: Illustration of Lemma 2.4.5 in the base case $p = 0$, $x_0 = r$: the graphs $A(mr)$ and $A(m^2)$ are both K_5 minus a single edge, and the isomorphism φ swaps the outer-triangle labels 2 and 3 while fixing 1, 4, and 5. The non-edge $\{2, 5\}$ of $A(mr)$ is sent to the non-edge $\{3, 5\}$ of $A(m^2)$. The same swap, extended by $\varphi(k) = k$ for $k \geq 6$, gives the isomorphism for any $p \geq 0$.

- the outer triangle has edges $12, 13, 23$;
- the m-step creates vertex 4 inside $\{1, 2, 3\}$, adding edges $14, 24, 34$;
- the r-step creates vertex 5 inside the right triangle $\{3, 4, 1\}$, adding edges $35, 45, 15$.

Thus $E(A(mr)) = \{12, 13, 23, 14, 24, 34, 35, 45, 15\}$, which is K_5 minus the edge $\{2, 5\}$.

The prefix m^2 of A_2 produces the five-vertex graph $A(m^2)$:

- outer triangle edges $12, 13, 23$;
- the first m-step creates vertex 4, adding $14, 24, 34$;
- the second m-step creates vertex 5 inside the middle triangle $\{4, 1, 2\}$, adding $15, 25, 45$.

Thus $E(A(m^2)) = \{12, 13, 23, 14, 24, 34, 15, 25, 45\}$, which is K_5 minus the edge $\{3, 5\}$.

Step 2: the base isomorphism. Define φ on $\{1, 2, 3, 4, 5\}$ by swapping vertices 2 and 3 and fixing $1, 4, 5$:

$$\varphi(1) = 1, \quad \varphi(2) = 3, \quad \varphi(3) = 2, \quad \varphi(4) = 4, \quad \varphi(5) = 5.$$

Applying φ to each edge of $A(mr)$:

$$\begin{aligned}\{12,13,23,14,24,34,35,45,15\} \xmapsto{\varphi} &\{13,12,32,14,34,24,25,45,15\}\\ &= \{12,13,23,14,24,34,15,25,45\}\\ &= E(A(m^2)).\end{aligned}$$

So φ is an isomorphism on the first five vertices. The non-edge $\{2,5\}$ of $A(mr)$ is sent to the non-edge $\{3,5\}$ of $A(m^2)$, as required.

Step 3: φ identifies the local picture for subsequent subdivisions.

After the prefix, both A_1 and A_2 have the same most recently created vertex, namely vertex 5. The three triangles surrounding vertex 5 in each graph are identified by which one is the "middle" (the triangle not containing the previously newest vertex 4):

- In $A_1 = A(mr\cdots)$, the middle triangle around vertex 5 is $\{5,1,3\}$; the left and right triangles are $\{5,3,4\}$ and $\{5,4,1\}$, respectively.

- In $A_2 = A(m^2\cdots)$, the middle triangle around vertex 5 is $\{5,1,2\}$; the left and right triangles are $\{5,2,4\}$ and $\{5,4,1\}$, respectively.

Under φ (which swaps 2 and 3 and fixes $1,4,5$):

- the middle triangle $\{5,1,3\}$ of A_1 is sent to $\{5,1,2\}$ of A_2;

- the left triangle $\{5,3,4\}$ of A_1 is sent to $\{5,2,4\}$ of A_2;

- the right triangle $\{5,4,1\}$ of A_1 is sent to itself.

So the local labels "middle", "left", "right" are identified by φ, and the next symbol of the encoding $m^{s_1}x_1m^{s_2}\cdots$ creates a vertex inside corresponding triangles in A_1 and A_2.

Step 4: extending φ to all of $V(A_1)$.

For each $k \geq 6$, set $\varphi(k) = k$. By induction on k, the vertex created at step k in A_1 subdivides a triangle $\{u,v,w\}$ which, by Step 3 applied recursively, is sent by φ to the triangle $\{\varphi(u),\varphi(v),\varphi(w)\}$ in A_2 in which the corresponding step creates vertex k.

Hence the three new edges added at step k in A_1 are sent by φ to the three new edges added at step k in A_2.

Combining Steps 1–4, φ is an edge-preserving bijection $V(A_1) \to V(A_2)$, hence a graph isomorphism.

The case $x_0 = \ell$. By the left/right symmetry of the construction (which swaps the bottom-left and bottom-right outer vertices), the analogous isomorphism is the swap $\varphi(1) = 2,\ \varphi(2) = 1$, fixing $3, 4, 5$, with $\varphi(k) = k$ for $k \geq 6$. The verification is identical with the roles of vertices 1 and 2 interchanged. $\square$

Example 2.4.6. The smallest non-trivial instance is $p = 0$, $x_0 = r$: the lemma asserts $A(mr) \cong A(m^2)$. Both graphs are K_5 minus an edge, and the isomorphism is the swap $2 \leftrightarrow 3$ shown in Figure 2.8.

A larger instance: with $p = 1$, $x_0 = r$, $x_1 = \ell$, $s_1 = 1$, $s_2 = 0$, the lemma asserts $A(mrm\ell) \cong A(m^3\ell)$, both on six vertices. The isomorphism is $\varphi(2) = 3,\ \varphi(3) = 2,\ \varphi(k) = k$ for $k \in \{1, 4, 5, 6\}$. One checks that the edges added by the trailing $m\ell$ in A_1 and by the trailing $m^2\ell$ in A_2 correspond under φ.

Lemma 2.4.7 (Hereditary property)**.** *Let $s_0 \geq 1$, $p \geq 0$, $s_1, \ldots, s_p \geq 0$, and $x_1, \ldots, x_p \in \{\ell, r\}$. Then for every $k \geq 0$, the Apollonian graph*

$$A_1 \;=\; A(m^{s_0}\, x_1\, m^{s_1} \,\cdots\, x_p\, m^{s_p})$$

is an induced subgraph of

$$A_2 \;=\; A\Big(m^{s_0+k}\, x_1\, m^{s_1} \,\cdots\, x_p\, m^{s_p}\Big)\,.$$

Consequently, if A_1 is non-word-representable then so is A_2, and if A_2 is word-representable then so is A_1.

Proof. The case $k = 0$ is trivial: $A_1 = A_2$ and the inclusion is an equality. For $k \geq 1$ we proceed by induction on k, the inductive step being the following claim. Figure 2.9 illustrates the smallest non-trivial case.

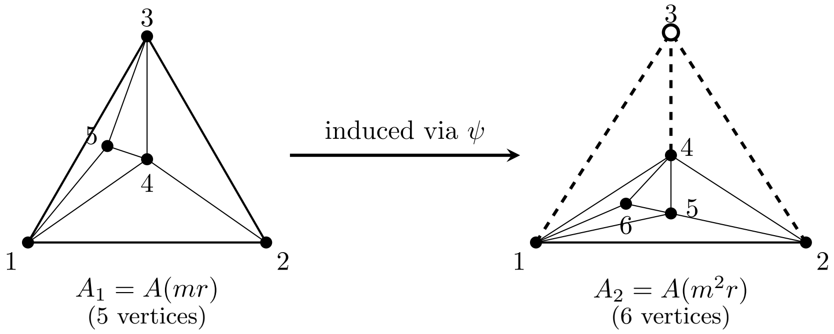


Figure 2.9: Illustration of Lemma 2.4.7 in the smallest non-trivial case $s_0 = 1$, $k = 1$, $p = 1$, $x_1 = r$, $s_1 = 0$. The graph $A_1 = A(mr)$ (left, 5 vertices) is realised as an induced subgraph of $A_2 = A(m^2r)$ (right, 6 vertices) on the vertex subset $\{1, 2, 4, 5, 6\}$ of A_2, via the embedding $\psi(1) = 1,\ \psi(2) = 2,\ \psi(3) = 4,\ \psi(4) = 5,\ \psi(5) = 6$. Vertex 3 of A_2 (drawn as a white circle) is not in the image of ψ, and its three incident edges (dashed) are the only edges of A_2 that lie outside the embedded copy of A_1. The key structural point is that no construction step of A_2 after the very first m-step ever subdivides a triangle that contains vertex 3, so $\deg_{A_2}(3) = 3$ and $N_{A_2}(3) = \{1, 2, 4\}$.

Claim. For every $s \geq 1$, the graph $A(m^s\, x_1\, m^{s_1}\, \cdots\, x_p\, m^{s_p})$ embeds as an induced subgraph of $A\big(m^{s+1}\, x_1\, m^{s_1}\, \cdots\, x_p\, m^{s_p}\big)$ via deletion of the top outer vertex 3.

The claim, iterated k times, gives a chain of induced inclusions

$$A_1 \;\hookrightarrow\; A\big(m^{s_0+1}\, x_1\, \cdots\big) \;\hookrightarrow\; \cdots \;\hookrightarrow\; A\Big(m^{s_0+k}\, x_1\, \cdots\Big) \;=\; A_2,$$

where at each step the smaller graph embeds into the larger by relabelling the outer triangle (the top apex of the smaller graph becomes an interior vertex of the larger). Composing the embeddings yields the desired induced inclusion $A_1 \subseteq A_2$, and the word-representability conclusions follow from the heredity of the class of word-representable graphs [30]: if A_1 is non-word-representable then so is its supergraph A_2, and if A_2 is word-representable then so is its induced subgraph A_1.

Proof of the claim:

Write $B = A\big(m^{s+1}\, x_1\, m^{s_1}\, \cdots\, x_p\, m^{s_p}\big)$ and $C = A(m^s\, x_1\, m^{s_1}\, \cdots\, x_p\, m^{s_p})$. The first symbol of B's encoding is m, which creates vertex 4 inside the outer triangle $\{1, 2, 3\}$, with new edges 14, 24, 34. After this first m-step, the most recently created vertex is 4, and the three triangles surrounding it are

$$\text{middle: } \{4,1,2\}, \qquad \text{left: } \{3,2,4\}, \qquad \text{right: } \{3,4,1\}$$

Key observation. Every symbol of B's encoding after the very first one subdivides a triangle that lies strictly inside the middle triangle $\{4,1,2\}$.

To see this, recall that in our change-of-direction subfamily each symbol subdivides one of the three triangles around the *most recently created* vertex. The next symbol of B is m (the second m in the m^{s+1} prefix), which subdivides the middle triangle around vertex 4, namely $\{4,1,2\}$. This creates vertex 5 inside $\{4,1,2\}$, with the three triangles around vertex 5 all contained in $\{4,1,2\}$. By induction on the position in the encoding, every subsequent step occurs strictly inside $\{4,1,2\}$.

In particular, the left triangle $\{3,2,4\}$ and the right triangle $\{3,4,1\}$ are *never* subdivided in the construction of B. Hence the only edges of B incident to vertex 3 are 13, 23, 34, and

$$\deg_B(3) = 3, \qquad N_B(3) = \{1,2,4\}.$$

The embedding. Define $\psi : V(C) \to V(B)$ by

$$\psi(1) = 1, \qquad \psi(2) = 2, \qquad \psi(3) = 4, \qquad \psi(i) = i+1 \quad \text{for } i \geq 4.$$

Thus ψ fixes the bottom outer vertices, sends the top outer vertex of C to vertex 4 of B, and shifts all interior-vertex labels of C up by one. The image of ψ is $V(B) \setminus \{3\}$, and we claim that ψ is an isomorphism between C and the induced subgraph $B[V(B) \setminus \{3\}]$.

We verify this step by step on the construction of C:

- *Outer triangle.* C has outer-triangle edges 12, 13, 23; ψ sends these to 12, 14, 24, which are precisely the edges of the triangle $\{1,2,4\}$ in B.

- *Each subdivision step.* The first m-step in C's encoding creates vertex 4 of C inside the outer triangle $\{1,2,3\}$, with edges to $1,2,3$; under ψ this becomes vertex 5 of B adjacent to $\{1,2,4\}$, which matches the second m-step in B's encoding (creating vertex 5 of B inside the middle triangle $\{4,1,2\}$). Subsequent steps correspond similarly: the j-th subdivision step of C matches the $(j+1)$-th step of B, because the encodings are identical apart from the extra leading m, and the surrounding triangles around the newest vertex coincide under ψ.

Hence ψ is a graph isomorphism $C \to B[V(B) \setminus \{3\}]$, so C is an induced subgraph of B. This proves the claim.

This completes the induction and the proof of the lemma. $\square$

Remark 2.4.8. The proof above strips the extra m-vertices from A_2 in *outside-in* order: at each step the vertex removed is the top of the current outer triangle (vertex 3), not the most recently created vertex. Attempting to remove the extra m-vertices in the order they were created ("last in, first out") does not work for the lemma as stated, because the extra m-steps in A_2's encoding occur at the *start* of the construction, so the vertices they create lie at the innermost reaches of the graph, not on its boundary. For example, in $A_2 = A(m^2r)$ the vertex created by the second m-step is vertex 5, with neighbours $\{1, 2, 4\}$; deleting vertex 5 leaves the r-vertex (vertex 6, originally adjacent to $\{4, 5, 1\}$) with only two neighbours, so the result is not even an Apollonian graph, let alone $A_1 = A(mr)$.

In Chapter 3 we will also use additional structural notions specific to the change-of-direction encoding (such as the *base* of a triangle being subdivided, and the *extension invariant* of Section 3.2). These are introduced where they are needed.

Chapter 3

Word-Representability of Apollonian Graphs

Overview

This chapter is the first of two main research chapters of the thesis. It tells the story of our investigation into word-representability of Apollonian graphs, a story that unfolded in two phases. In the first phase (Section 3.1), we characterised word-representability of Apollonian graphs in our change-of-direction subfamily by constructing explicit semi-transitive orientations and identifying induced obstructions case by case. This work is independent of Hauser's; it gives a decidable parity check on the encoding string, produces explicit semi-transitive orientations in the representable cases, and, on the non-representability side, generalises the human-verifiable proof format of [37] from individual graphs to infinite families, by combining a base-case derivation with an induction step that subdivides a parametric edge. After this work was completed, we discovered the PhD thesis of Hauser [21], which gave a complete characterisation of word-representable Apollonian networks, in fact for arbitrary Apollonian networks, not only the change-of-direction subfamily, in terms of two structural conditions: $\mathcal{F}_1$-freeness and local comparability. Hauser's result is more general but is non-constructive: the work of Section 3.1 retains independent value as a decidable, constructive route on the subfamily.

The discovery of Hauser's characterisation prompted us to re-examine our work and ask a sharper question: *when an Apollonian graph in our subfamily is word-representable, is it in fact a circle graph?* The class of 2-uniformly word-representable graphs is precisely the class of circle graphs (Theorem 2.2.6), so this question is equivalent to asking whether every word-representable graph in our subfamily admits a 2-uniform representation. The answer, which forms the substance of Section 3.2, is yes: word-representability and 2-word-representability coincide on this family. The proof is constructive: for each representable case, we exhibit a 2-uniform word-representation, built by induction using a notion we call the *extension invariant*.

A note on notation

In Section 3.1 we use the vertex labels $v_1, v_2, v_3, \ldots$, while in Section 3.2 we use the bare labels $1, 2, 3, \ldots$. The two notations refer to the same vertices: in both sections, v_3 (or equivalently 3) is the vertex at the top of the outer triangle, v_2 (2) is the vertex at the bottom-right, and v_1 (1) is the vertex at the bottom-left, so that reading the triangle clockwise from the top gives $3, 2, 1$. The fourth vertex is in both cases labelled v_4 (4), and so on. Thus the two sections differ only typographically, not geometrically.

3.1 A semi-transitive orientation approach

Throughout this section, we use the labelling $v_1, v_2, v_3, \ldots$ explained at the start of the chapter: vertex v_3 is at the top of the outer triangle, v_2 at the bottom-right, v_1 at the bottom-left, and v_4 is the first interior vertex. We follow the construction and conventions of Section 2.4.

The classification we develop in this section is independent of Hauser's general characterisation (Theorem 3.1.30, discussed at the end of the section), and offers three specific contributions within our subfamily.

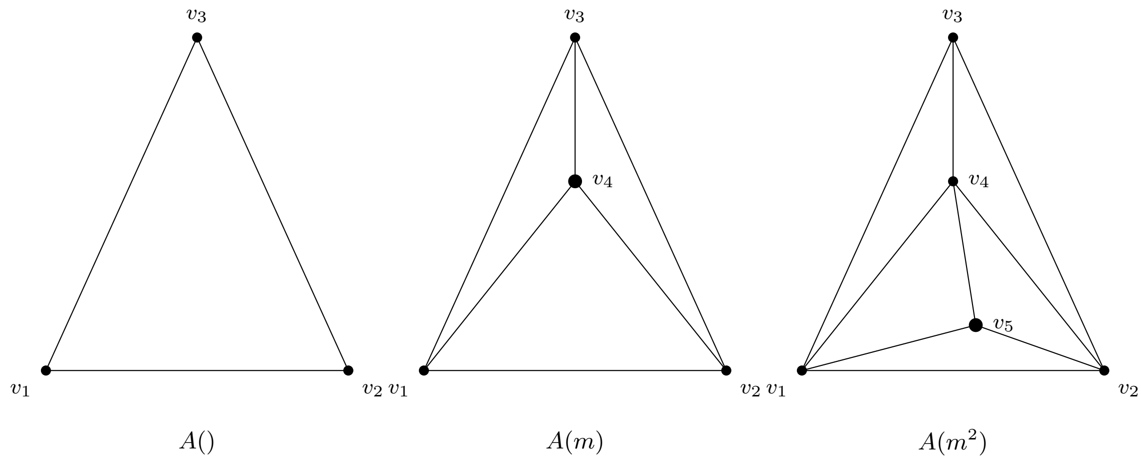
First, it provides a *decidable parity check* on the encoding string: word-representability is determined by reading the parities of the integers $s_0, s_1, \ldots, s_p$ and the equality pattern of the directions $x_1, \ldots, x_p$, with no graph-isomorphism or subgraph-search step required. By contrast, Hauser's characterisation in terms of $\mathcal{F}_1$-freeness and local comparability requires, in principle, the inspection of induced subgraphs of unbounded size.

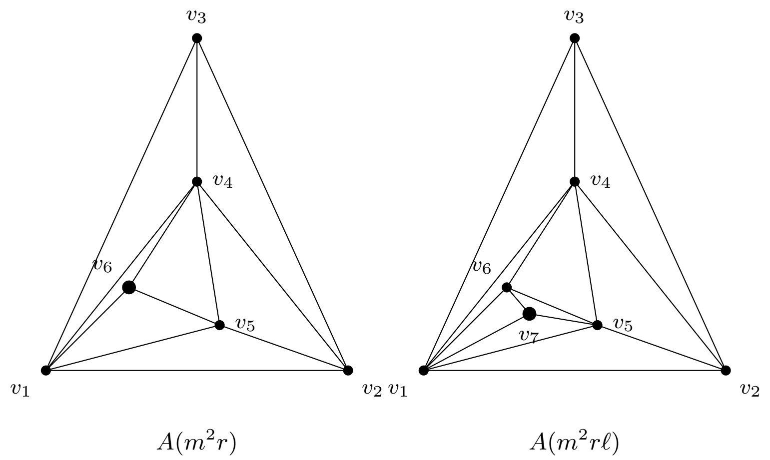
Second, in the representable cases the proof produces an explicit semi-transitive orientation by colour-class, which Section 3.2 later strengthens to a 2-uniform word-representation. By contrast, Hauser's argument is a non-constructive existence proof: even in cases where the result establishes word-representability, the proof yields no explicit word.

Third, on the non-representability side, the proofs of Lemmas 3.1.11, 3.1.16, and 3.1.17 use the human-verifiable proof format introduced in [37] to establish that no semi-transitive orientation exists, but generalised here, for the first time as far as we are aware, from individual graphs to infinite families. The previous applications of the format in the literature (e.g. to specific small obstructions for word-representability) treat one graph at a time. The technique we use here, of carrying out the human-verifiable case analysis on a base graph and then extending the resulting derivation through subdivision of a parametric edge, producing a single uniform proof valid for all values of the parameter, is, we believe, a genuinely new methodological contribution. It is what allows the parity classification to encompass arbitrary values of s_1, s_2, s_3, rather than just the smallest non-representable instances.

Figure 3.1 illustrates the construction for the small case $A(m^2r\ell)$, showing how the graph grows one vertex at a time as the encoding string is read.

We treat the cases of zero, one, two, and three changes of direction in turn. The strategy is uniform: in the representable cases we exhibit a 4-colouring of the graph and prove that orienting edges from smaller to larger colours yields a semi-transitive orientation; in the non-representable cases we exhibit an induced subgraph whose neighbourhood (in the larger graph) fails to be a comparability graph, so that the graph cannot be word-representable by Theorem 2.2.7. To organise the colouring arguments, we first record a small abstract verification lemma.





vertex	neighbours
v_4	v_1, v_2, v_3
v_5	v_1, v_2, v_4
v_6	v_1, v_4, v_5
v_7	v_1, v_5, v_6

Adjacencies in $A(m^2r\ell)$

Figure 3.1: Step-by-step construction of the Apollonian graph $A(m^2r\ell)$ on seven vertices. Each new vertex (drawn slightly larger) subdivides a triangle: an m-step subdivides the triangle whose base is unchanged from the previous step, while a turn (r or ℓ) subdivides a different adjacent triangle, changing the base. The right turn moves the base from v_1v_2 to v_1v_5, so v_6 is created adjacent to $\{v_1, v_4, v_5\}$ rather than to v_2. In the final panel the new vertex v_7 is adjacent to $\{v_1, v_5, v_6\}$; since this triangle still contains v_5, the vertex created immediately before v_6, the step is a turn rather than an m-step, and the encoding of the seven-vertex graph is $A(m^2r\ell)$.

3.1.1 A shortcut verification lemma

We frequently use the following abstract criterion to verify that a 4-colouring of a graph yields a semi-transitive orientation. Once stated, every representability proof in this section reduces to: exhibit a 4-colouring, enumerate the maximal directed colour-paths under the induced orientation, and check that each carries the two supporting transitivity edges.

Lemma 3.1.1 (Shortcut Verification Lemma)**.** *Let G be a graph and let $c : V(G) \to \{1, 2, 3, 4\}$ be a proper 4-colouring. Orient each edge of G from its smaller-colour endpoint to its larger-colour endpoint, obtaining an acyclic orientation σ. Suppose that for every directed path of colour pattern $1 \to 2 \to 3 \to 4$ in σ, say $u_1 \to u_2 \to u_3 \to u_4$ with $c(u_i) = i$, both of the supporting transitivity edges u_1u_3 and u_2u_4 are present in G.*

Then σ is semi-transitive; in particular, G is word-representable.

Proof. Recall (Definition 2.2.8) that an acyclic orientation is semi-transitive if and only if it admits no shortcut, where a shortcut is an induced acyclic non-transitive subgraph on at least four vertices having a unique source and a unique sink such that the source and sink are joined by an edge.

Since c takes only four values, any shortcut in σ must arise from a directed path of length at least three whose colour sequence is strictly increasing and includes all four colours. So the offending path has colour pattern $1 \to 2 \to 3 \to 4$, with a shortcutting edge from its colour-1 source to its colour-4 sink.

For a shortcut to exist, one of the two transitivity edges u_1u_3 or u_2u_4 must be absent from G (if both are present, the four vertices induce a transitive tournament, so they form no shortcut). Our hypothesis is precisely that both supporting edges are present for every $1 \to 2 \to 3 \to 4$ path; hence no shortcut exists, and σ is semi-transitive. $\square$

Remark 3.1.2. All the colouring proofs in this section, Theorems 3.1.3, 3.1.6, 3.1.10, 3.1.12, 3.1.18, 3.1.19, 3.1.20, 3.1.21, 3.1.22, and 3.1.24, are applications of Lemma 3.1.1. In each case we specify a 4-colouring of the encoded Apollonian graph (or of an auxiliary supergraph in the even-s_0 subcases, after which the orientation is restricted), enumerate the directed paths of colour pattern $1 \to 2 \to 3 \to 4$, and observe that for every such path the supporting edges are present in G. To keep the case analyses self-contained, the explicit enumerations are retained in each proof; the reader is invited to verify them against Lemma 3.1.1 once and then read the remaining proofs as a sequence of colouring choices.

3.1.2 One change of direction

We begin with the cases of zero or one change of direction. We treat them together because the result is the same.

Theorem 3.1.3. *Any Apollonian graph of the form $A(m^{s_0}xm^{s_1})$, where $x \in \{\ell, r\}$ and $s_0 \geq 1$, $s_1 \geq 0$, is word-representable. The same conclusion holds when $s_0 = 0$; see Remark 3.1.4 below.*

Proof. We exhibit a semi-transitive orientation by applying Lemma 3.1.1. Suppose first that s_0 is even. Consider the following 4-colouring of the graph using the vertex labelling above. Assign colours $1, 2, 3, 4$ to v_4, v_3, v_2, v_1 respectively. By symmetry, assume without loss of generality that $x = r$. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 1 and 2, starting and ending with colour 2. The vertices $v_{s_0+4}, v_{s_0+5}, \ldots, v_{s_0+s_1+4}$ alternate between colours 2 and 3, starting with colour 3 and ending with colour 2 or 3 depending on the parity of s_1.

We claim that orienting edges from smaller to larger colours yields a semi-transitive orientation. By Lemma 3.1.1, it suffices to check that for every directed path of colour pattern $1 \to 2 \to 3 \to 4$, both supporting transitivity edges $1 \to 3$ and $2 \to 4$ are present.

Note that all vertices coloured 2 are connected to the vertex v_1, which is the only vertex coloured 4. So the supporting edge $2 \to 4$ is always present, and the only possible obstruction would be a missing edge $1 \to 3$. This does not involve the vertex v_2, which is coloured 3, as v_2 is connected to all vertices coloured 1. However, the other vertices coloured 3 are connected to only one vertex coloured 1, namely v_{s_0+2}, and any potential shortcut starting at v_{s_0+2} would follow one of the following paths:

- $v_{s_0+2} \to v_{s_0+1} \to v_2 \to v_1$;
- $v_{s_0+2} \to v_{s_0+3} \to v_2 \to v_1$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+4} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_1}{2}$ if s_1 is even or $k = 0, 1, 2, \ldots, \frac{s_1-1}{2}$ if s_1 is odd;

- $v_{s_0+2} \to v_{s_0+2k+5} \to v_{s_0+2k+4} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_1}{2}$ if s_1 is even or $k = 0, 1, 2, \ldots, \frac{s_1-1}{2}$ if s_1 is odd.

For all such paths, the supporting edge $1 \to 3$ is in fact present in G. Hence by Lemma 3.1.1, the orientation is semi-transitive.

When s_0 is odd, add a vertex v_0 connected to v_1, v_2, and v_3. Then make the triangle $v_0v_2v_1$ the initial triangle, assign colour 1 to v_0, and apply exactly the same colouring and orientation procedure as described above to obtain a semi-transitive orientation of the augmented graph. Restricting this orientation to the original graph (i.e. deleting v_0) gives a semi-transitive orientation of $A(m^{s_0}xm^{s_1})$. $\square$

Remark 3.1.4. The hypothesis $s_0 \geq 1$ is assumed in the statement because the colouring constructed in the proof refers to the vertices $v_5, \ldots, v_{s_0+3}$ produced by the initial run of m-steps, and to the vertex v_{s_0+2} at which the change of direction occurs; these are meaningful only when at least one m-step precedes the turn. The excluded case $s_0 = 0$ is nevertheless covered, for the following reason. An encoding beginning with a turn describes the same construction as one beginning with an m-step, because at the very first subdivision the vertex v_4 is inserted into the initial triangle $\{v_1, v_2, v_3\}$, and the three triangles surrounding v_4 are interchangeable up to relabelling of the outer triangle: there is no earlier vertex relative to which "middle", "left" and "right" are distinguished. Hence $A(xm^{s_1}) \cong A(m\,m^{s_1-1}x')$ for a suitable x' when $s_1 \geq 1$, and $A(x) \cong A(m) \cong K_5$ minus nothing, i.e. the 5-vertex Apollonian graph, when $s_1 = 0$. In every case the graph is word-representable, either by the theorem applied with $s_0 \geq 1$ or, for the two smallest graphs $A()$ and $A(x)$, because K_4 and the 5-vertex Apollonian graph are 3-colourable and hence word-representable by Theorem 2.2.3(ii).

Figure 3.2 illustrates the construction concretely for the smallest non-trivial case $A(m^2r)$, with $s_0 = 2$ and $s_1 = 0$. The same colouring strategy underlies all of the representability proofs that follow in this section.

Corollary 3.1.5. *Every Apollonian graph of the form $A(m^{s_0})$, $s_0 \geq 0$, is word-representable.*

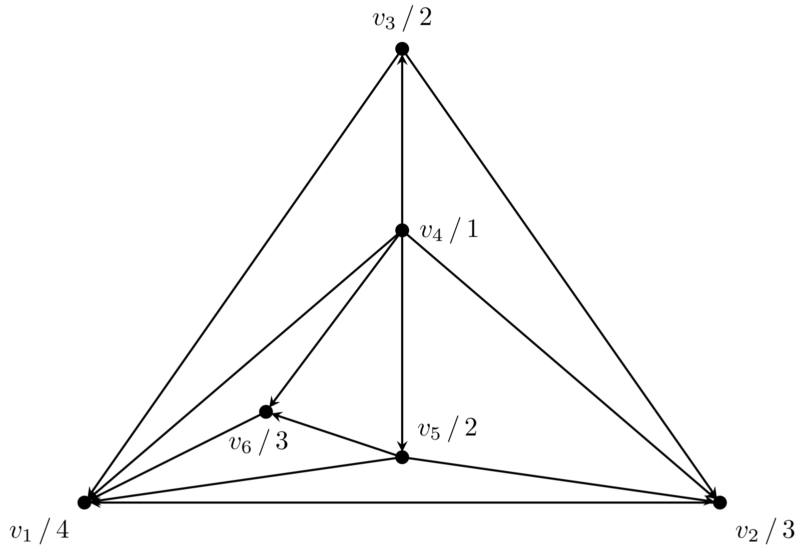


Figure 3.2: A semi-transitive orientation of the Apollonian graph $A(m^2r)$, produced by the construction in the proof of Theorem 3.1.3. Each vertex carries the label *vertex / colour*; edges are oriented from smaller to larger colour. No shortcut exists, since for every directed path of colours $1 \to 2 \to 3 \to 4$, the supporting transitivity edges (colour $1 \to 3$ and colour $2 \to 4$) are present in the graph.

Proof. $A(m^{s_0})$ is an induced subgraph of $A(m^{s_0}xm^0)$, which is word-representable by Theorem 3.1.3. □

3.1.3 Two changes of direction

We now consider Apollonian graphs of the form $A(m^{s_0}x_1m^{s_1}x_2m^{s_2})$, $x_i \in \{\ell, r\}$. We first dispose of the case $s_0 = 2$.

Theorem 3.1.6. *Every Apollonian graph of the form $A(m^2x_1m^{s_1}x_2m^{s_2})$, $x_i \in \{\ell, r\}$, $s_i \geq 0$, is word-representable.*

Proof. We claim that this graph admits a semi-transitive orientation. We consider four sub-cases according to whether $x_1 = x_2$ and the parity of s_1, using the vertex labelling above. By symmetry, we may assume throughout that $x_1 = r$.

Case 1.1. $x_1 = x_2$ and s_1 is odd. Consider the following 4-colouring of the graph. Assign colours $1, 2, 3, 4$ to v_2, v_3, v_1, and v_4 respectively. Then v_5 is assigned colour 2 and v_6 receives colour 1. The vertices $v_7, v_8, \ldots, v_{s_1+6}$ alternate between colours 1 and 2 starting with colour 2; since s_1 is odd, the chain ends with v_{s_1+6} on colour 2. The remaining vertices $v_{s_1+7}, v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ are coloured alternately with colours 2 and 3 starting with colour 2 and ending with colour 2 or 3 depending on the parity of s_2.

We verify the hypothesis of Lemma 3.1.1. All vertices coloured 2 are connected to v_4, the only vertex coloured 4, so the supporting edge $2 \to 4$ is always present. For the supporting edge $1 \to 3$: v_1, which is coloured 3, is connected to all vertices coloured 1, so paths through v_1 are fine. The other vertices coloured 3 are connected to only one vertex coloured 1, namely v_{s_1+5}, and any potential shortcut starting at v_{s_1+5} follows one of the paths:

- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7} \to v_4$ for $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd or $k = 0, 1, 2, \ldots, \frac{s_2}{2} - 1$ when s_2 is even;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7} \to v_4$ for $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd or $k = 0, 1, 2, \ldots, \frac{s_2}{2} - 1$ when s_2 is even.

For all such paths, the supporting edge $1 \to 3$ is present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 1.2. $x_1 = x_2$ and s_1 is even. Consider the following 4-colouring of the graph. Assign colours $1, 2, 3, 4$ to v_3, v_2, v_1, and v_4 respectively. Then v_5 is assigned colour 1 and v_6 receives colour 2. The vertices $v_7, v_8, \ldots, v_{s_1+6}$ alternate between colours 1 and 2 starting with colour 1; since s_1 is even, the chain ends with v_{s_1+6} on colour 2. The remaining vertices $v_{s_1+7}, v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ are coloured alternately with colours 2 and 3 starting with colour 2 and ending with colour 2 or 3 depending on the parity of s_2.

We verify the hypothesis of Lemma 3.1.1. All vertices coloured 2 are connected to v_4, the only vertex coloured 4, so the supporting edge $2 \to 4$ is always present. For the supporting edge $1 \to 3$: v_1, which is coloured 3, is connected to all vertices coloured 1, so paths through v_1 are fine. The other vertices coloured 3 are connected to only one vertex coloured 1, namely v_{s_1+5} (the last colour-1 vertex before the second turn), and any potential shortcut starting at v_{s_1+5} follows one of the paths:

- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7} \to v_4$ for $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd or $k = 0, 1, 2, \ldots, \frac{s_2}{2} - 1$ when s_2 is even;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7} \to v_4$ for $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd or $k = 0, 1, 2, \ldots, \frac{s_2}{2} - 1$ when s_2 is even.

The index ranges match those of Case 1.1 because the post-turn colour pattern is the same and only depends on s_2. For all such paths, the supporting edge $1 \to 3$ is present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2.1. $x_1 \neq x_2$ and s_1 is odd. Consider the following 4-colouring of the graph. Assign colours $1, 2, 3, 4$ to v_2, v_3, v_4, and v_1 respectively. Then v_5 is assigned colour 2 and v_6 receives colour 1. The vertices $v_7, v_8, \ldots, v_{s_1+6}$ alternate between colours 1 and 2 starting with colour 2 and ending with colour 2 (since s_1 is odd). The vertices $v_{s_1+7}, v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ are coloured alternately with colours 2 and 3 starting with colour 3 and ending with colour 2 or 3 depending on the parity of s_2.

We verify the hypothesis of Lemma 3.1.1. All vertices coloured 2 are connected to v_1, the only vertex coloured 4, so the supporting edge $2 \to 4$ is always present. For the supporting edge $1 \to 3$: v_4, coloured 3, is connected to all vertices coloured 1, so paths through v_4 are fine. The other vertices coloured 3 are v_{s_1+2k+7} for $k = 0, 1, 2, \ldots, \frac{s_2}{2}$ if s_2 is even or $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ if s_2 is odd; these are connected to only one vertex coloured 1, namely v_{s_1+5}. Any potential shortcut starting at v_{s_1+5} follows one of the paths:

- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_2}{2}$ when s_2 is even or $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_2}{2} - 1$ when s_2 is even or $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd.

For all such paths, the supporting edge $1 \to 3$ is present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2.2. $x_1 \neq x_2$ and s_1 is even. Consider the following 4-colouring of the graph. Assign colours $1, 2, 3, 4$ to v_3, v_2, v_4, and v_1 respectively. Then v_5 is assigned colour

1 and v_6 receives colour 2. The vertices $v_7, v_8, \ldots, v_{s_1+6}$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 2 (since s_1 is even). The vertices $v_{s_1+7}, v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ are coloured alternately with colours 2 and 3 starting with colour 3 and ending with colour 2 or 3 depending on the parity of s_2.

We verify the hypothesis of Lemma 3.1.1. All vertices coloured 2 are connected to v_1, the only vertex coloured 4, so the supporting edge $2 \to 4$ is always present. For the supporting edge $1 \to 3$: v_4, coloured 3, is connected to all vertices coloured 1, so paths through v_4 are fine. The other vertices coloured 3 are v_{s_1+2k+7} for $k = 0, 1, 2, \ldots, \frac{s_2}{2}$ if s_2 is even or $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ if s_2 is odd; these are connected to only one vertex coloured 1, namely v_{s_1+5}. Any potential shortcut starting at v_{s_1+5} follows one of the paths:

- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_2}{2}$ when s_2 is even or $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_2}{2} - 1$ when s_2 is even or $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ when s_2 is odd.

The index ranges match those of Case 2.1 because the post-turn colour pattern is the same and only depends on s_2. For all such paths, the supporting edge $1 \to 3$ is present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Since we have provided a semi-transitive orientation in all four sub-cases, the graph $A(m^2 x_1 m^{s_1} x_2 m^{s_2})$ is word-representable. □

Corollary 3.1.7. *Every Apollonian graph of the form* $A(m x_1 m^{s_1} x_2 m^{s_2})$, $x_i \in \{\ell, r\}$, $s_i \geq 0$, *is word-representable.*

Proof. $A(m x_1 m^{s_1} x_2 m^{s_2})$ is an induced subgraph of $A(m^2 x_1 m^{s_1} x_2 m^{s_2})$ by Lemma 2.4.7. □

We now turn to the cases $s_0 \geq 3$, where parity matters.

The non-comparability obstruction

We need a structural lemma that identifies a forbidden induced neighbourhood. Consider the graph $H_{n,k,i}$ depicted in Figure 3.3, defined for $n \geq 5$, $k \geq 1$, and $3 \leq i \leq n-2$. The vertex set of $H_{n,k,i}$ is $\{a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_{2k}\}$, with edges:

- $a_j a_{j+1}$ for $j = 1, \ldots, n-1$ (the "top" path of a-vertices);
- $b_j b_{j+1}$ for $j = 1, \ldots, 2k-1$ (the "bottom" path of b-vertices);
- $a_i b_j$ for all $j = 1, \ldots, 2k$ (vertex a_i is adjacent to every b-vertex);
- $a_j b_1$ for all $j = 1, \ldots, i$ (the vertex b_1 is adjacent to $a_1, a_2, \ldots, a_i$);
- $a_j b_{2k}$ for all $j = i, \ldots, n$ (the vertex b_{2k} is adjacent to $a_i, a_{i+1}, \ldots, a_n$).

Note that the two "end" vertices b_1 and b_{2k} are each joined to a whole run of consecutive a-vertices (up to a_i from the left, and from a_i onwards to the right, respectively), while the interior vertices $b_2, \ldots, b_{2k-1}$ are joined among the a-vertices only to a_i. In particular the edges $a_i b_1$ and $a_i b_{2k}$ are already included in the third family, and the earlier "end" edges $a_1 b_1$ and $a_n b_{2k}$ are the extreme cases $j = 1$ and $j = n$ of the two new families.

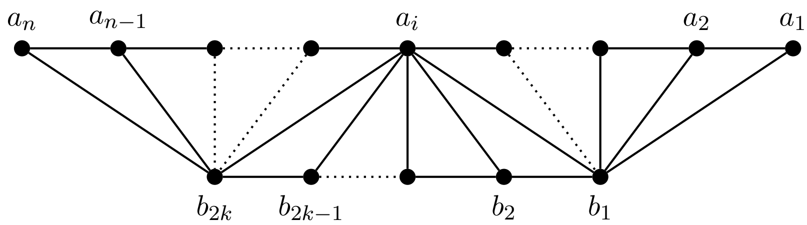


Figure 3.3: The graph $H_{n,k,i}$ used in Lemma 3.1.8. The vertex b_1 is joined to every a-vertex from a_1 up to a_i, and b_{2k} to every a-vertex from a_i up to a_n; the interior vertices $b_2, \ldots, b_{2k-1}$ are joined among the a-vertices only to a_i. Dotted edges indicate that further a-vertices or b-vertices may appear, with the obvious adjacencies.

Lemma 3.1.8. *For all $n \geq 5$, $k \geq 1$, and $3 \leq i \leq n-2$, the graph $H_{n,k,i}$ in Figure 3.3 is not a comparability graph.*

Proof. We claim that the graph admits no transitive orientation. Suppose for contradiction that it does. Throughout we use the defining property of a transitive orientation:

whenever $x \to y$ and $y \to z$ are arcs, the edge xz must be present and oriented $x \to z$; equivalently, a directed path of length two may never join a non-adjacent pair. We record two facts about the graph that will be used repeatedly. First, a_i is adjacent to every b-vertex, whereas each interior vertex $b_2, b_3, \ldots, b_{2k-1}$ is adjacent among the a-vertices to a_i only. Second, a_ia_1 and a_ia_n are non-edges, because $i \geq 3$ and $i \leq n-2$ make a_i non-consecutive with each of a_1 and a_n on the top path.

We first orient the top and bottom paths. Since a_1a_3 is not an edge, the two arcs at a_2 cannot form a directed path, so a_2 is either a source or a sink of the top path; the same holds at every internal vertex a_j, because $a_{j-1}a_{j+1}$ is not an edge. Hence the orientation alternates along the top path $a_1a_2 \cdots a_n$, and without loss of generality we assume it begins $a_1 \to a_2$ (otherwise reverse every arc of the whole orientation, which preserves transitivity). Then a_i is a sink of the top path when i is even and a source when i is odd. Likewise, since a_i is adjacent to every b-vertex, each interior b_j lies on the non-edge $b_{j-1}b_{j+1}$, so by the same reasoning the orientation alternates along the bottom path $b_1b_2 \cdots b_{2k}$.

Case 1: i is even, so a_i is a sink of the top path. Then every edge at a_i is oriented into a_i; in particular $b_j \to a_i$ for all j, and $a_{i-1} \to a_i$, $a_{i+1} \to a_i$.

We propagate along the bottom path from the right end. Since $b_{2k} \to a_i$ and a_ia_n is a non-edge, orienting $a_n \to b_{2k}$ would give $a_n \to b_{2k} \to a_i$ and force the absent edge a_na_i; hence

$$b_{2k} \to a_n.$$

Now consider the edge $b_{2k-1}b_{2k}$. The arc $b_{2k} \to a_n$ together with the non-edge a_nb_{2k-1} (the vertex b_{2k-1} is adjacent among the a-vertices to a_i only, and $a_n \neq a_i$) forces $b_{2k} \to b_{2k-1}$: otherwise $b_{2k-1} \to b_{2k} \to a_n$ would demand the absent edge $b_{2k-1}a_n$. By the alternation of the bottom path established above, this fixes its orientation from the right: $b_{2\ell} \to b_{2\ell-1}$ and $b_{2\ell} \to b_{2\ell+1}$ for every ℓ, so the even-indexed b-vertices are sources.

We also propagate from the left end. Since $b_1 \to a_i$ and a_ia_1 is a non-edge, orienting $a_1 \to b_1$ would give $a_1 \to b_1 \to a_i$ and force the absent edge a_1a_i; hence $b_1 \to a_1$. Then a_1b_2 is a non-edge, since the a-vertices adjacent to b_2 are a_i (when $k \geq 2$, as b_2 is then interior) or $a_i, a_{i+1}, \ldots, a_n$ (when $k = 1$, as then $b_2 = b_{2k}$), and in either case a_1 is excluded because $1 < i$. Hence, orienting $b_2 \to b_1$ would give $b_2 \to b_1 \to a_1$ and force the absent edge b_2a_1; so $b_1 \to b_2$. By the alternation of the bottom path, this fixes its orientation from the left: the odd-indexed b-vertices are sources.

These two determinations are incompatible on the middle edge b_kb_{k+1}. Propagating from the right makes the even-indexed endpoints sources, while propagating from the left makes the odd-indexed endpoints sources; since b_k and b_{k+1} have opposite parities, one determination orients $b_k \to b_{k+1}$ and the other $b_{k+1} \to b_k$. As the bottom path has an *even* number $2k$ of vertices, the two propagations meet at this single central edge and disagree there, which is the required contradiction. (This is precisely where the evenness of the number of b-vertices is used.)

Case 2: i is odd, so a_i is a source of the top path. This case is obtained from Case 1 by reversing every arc, and we spell it out for completeness. Now every edge at a_i is oriented away from a_i; in particular $a_i \to b_j$ for all j, and $a_i \to a_{i-1}$, $a_i \to a_{i+1}$.

From the right end: since $a_i \to b_{2k}$ and a_ia_n is a non-edge, orienting $b_{2k} \to a_n$ would give $a_i \to b_{2k} \to a_n$ and force the absent edge a_ia_n; hence $a_n \to b_{2k}$. Then, since $a_n \to b_{2k}$ and a_nb_{2k-1} is a non-edge, orienting $b_{2k} \to b_{2k-1}$ would give $a_n \to b_{2k} \to b_{2k-1}$ and force the absent edge a_nb_{2k-1}; hence $b_{2k-1} \to b_{2k}$. By alternation this fixes the bottom path from the right, with the even-indexed b-vertices sinks.

From the left end: since $a_i \to b_1$ and a_ia_1 is a non-edge, orienting $b_1 \to a_1$ would give $a_i \to b_1 \to a_1$ and force the absent edge a_ia_1; hence $a_1 \to b_1$. Then a_1b_2 is a non-edge (as noted in Case 1, a_1 is adjacent to b_2 neither when $k \geq 2$ nor when $k = 1$), so orienting $b_1 \to b_2$ would give $a_1 \to b_1 \to b_2$ and force the absent edge a_1b_2; hence $b_2 \to b_1$. By alternation this fixes the bottom path from the left, with the odd-indexed b-vertices sinks. Exactly as in Case 1, the two determinations disagree on the central edge b_kb_{k+1}, since $2k$ is even; this contradiction completes Case 2.

In both cases, no transitive orientation exists, so the graph is not a comparability graph. □

Remark 3.1.9. It is natural to ask how the family $H_{n,k,i}$ relates to Gallai's classification [14] of the *minimal* non-comparability graphs, that is, those graphs that are not comparability graphs but all of whose proper induced subgraphs are. The graphs $H_{n,k,i}$ are in general *not* minimal in this sense: for large n and k the top path of a-vertices and the bottom path of b-vertices can be shortened while retaining a non-comparability obstruction, so a proper induced subgraph of $H_{n,k,i}$ may already fail to be a comparability graph. What Lemma 3.1.8 provides is not a new minimal obstruction but a single uniform parametric family, realised explicitly inside the neighbourhoods arising in our Apollonian constructions, and this uniformity is what makes it usable in the induction. Concretely, each $H_{n,k,i}$ contains an induced member of Gallai's list, but the point of the lemma is that the containment holds simultaneously for all admissible n, k and i, with the obstruction located in a prescribed position relative to the encoding word. For the general theory of transitive orientations and the accompanying decomposition, we refer the reader to Golumbic [16].

The two-change classification

Theorem 3.1.10. *Let $G = A(m^{s_0}x_1m^{s_1}x_2m^{s_2})$, with $x_i \in \{\ell, r\}$, $x_1 \neq x_2$, and $s_0 \geq 3$. Then G is word-representable if and only if s_1 is odd.*

Proof. Suppose s_1 is even. We claim that the neighbourhood of v_1, denoted $N(v_1)$, is isomorphic to the graph $H_{n,k,i}$ of Figure 3.3 for suitable n, k and i. We describe the isomorphism explicitly.

Recall from the construction of $A(m^{s_0}x_1m^{s_1}x_2m^{s_2})$ that v_1 is a vertex of the initial outer triangle, and that it remains on the boundary of every triangle subdivided during the runs m^{s_0} and m^{s_1}, so that every vertex created during those runs is joined to v_1. Consequently $N(v_1)$ consists of the following vertices: the two other vertices of the initial triangle together with the vertices created during the run m^{s_0}, which we list in order of creation as $a_1, a_2, \ldots, a_n$; and the vertices created during the run m^{s_1} after the change of direction x_1, which we list in order of creation as $b_1, b_2, \ldots, b_{2k}$. Since consecutive vertices in a run of m-steps are joined (each new vertex is placed in the middle triangle and is therefore adjacent to its immediate predecessor), the a-vertices induce the path $a_1a_2\cdots a_n$ and the b-vertices induce the path $b_1b_2\cdots b_{2k}$; these are the top and bottom paths in Figure 3.3.

It remains to identify the edges between the two paths. The change of direction x_1 places the first vertex of the m^{s_1}-run inside a triangle one of whose vertices is the vertex a_i created at the position of the turn; as the run m^{s_1} proceeds, every vertex it creates is placed in a triangle having a_i on its boundary, and hence a_i is adjacent to every one of $b_1, \ldots, b_{2k}$. This gives the edges $a_i b_j$ for $1 \leq j \leq 2k$. Finally, the first vertex b_1 of the run is also adjacent to the vertex a_1 bounding the triangle in which it was placed, and the last vertex b_{2k} is adjacent to a_n, giving the two end edges $a_1 b_1$ and $a_n b_{2k}$. No further edges are present, since a vertex created in the m^{s_1}-run is joined only to the three vertices of the triangle it subdivides. The index i is determined by the position of the change of direction within the m^{s_0}-run, and satisfies $3 \leq i \leq n-2$ because $s_0 \geq 3$.

The hypothesis that s_1 is even is exactly what guarantees that the number of b-vertices is even, i.e. of the form $2k$ with $k \geq 1$, which is required for $H_{n,k,i}$ to be defined. Hence $N(v_1) \cong H_{n,k,i}$, and by Lemma 3.1.8, $N(v_1)$ is not a comparability graph. Hence by Theorem 2.2.7, G is not word-representable.

Conversely, suppose s_1 is odd. We construct a semi-transitive orientation. By symmetry, assume without loss of generality that $x_1 = r$. We treat two subcases according to the parity of s_0.

Case 1. s_0 is odd. Consider the following 4-colouring of the graph. Assign colours $1, 2, 3, 4$ to v_3, v_4, v_2, and v_1 respectively. Then v_5 receives colour 1, and vertices $v_6, v_7, \ldots, v_{s_0+3}$ alternate between colours 1 and 2 starting with colour 2 and ending with colour 2. Consequently, the vertices $v_{s_0+4}, v_{s_0+5}, \ldots, v_{s_0+s_1+4}$ alternate between colours 2 and 3 starting with colour 3 and ending with colour 2. Vertices $v_{s_0+s_1+5}, v_{s_0+s_1+6}, \ldots$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 1 or 2 depending on the parity of s_2.

We verify the hypothesis of Lemma 3.1.1. All vertices coloured 2 are adjacent to v_1, the only vertex coloured 4, so the supporting edge $2 \to 4$ is present along every candidate path. For the supporting edge $1 \to 3$, the candidate paths are of the following forms:

- $v_3 \to v_4 \to v_2 \to v_1$;

- $v_{2k+5} \to v_{2k+4} \to v_2 \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_0-3}{2}$;
- $v_{2k+5} \to v_{2k+6} \to v_2 \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_0-3}{2}$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+4} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+2} \to v_{s_0+2k+5} \to v_{s_0+2k+4} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+s_1+2k+5} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+3} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_2-1}{2}$ if s_2 is odd, or $k = 0, 1, 2, \ldots, \frac{s_2}{2}$ if s_2 is even;
- $v_{s_0+s_1+2k+5} \to v_{s_0+s_1+2k+6} \to v_{s_0+s_1+3} \to v_1$ for $k = 0, 1, 2, \ldots, \frac{s_2-3}{2}$ if s_2 is odd, or $k = 0, 1, 2, \ldots, \frac{s_2-2}{2}$ if s_2 is even.

For all such paths, the supporting edge $1 \to 3$ is present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2. s_0 is even. In this case, we add a new vertex v_0 coloured 2, and connect it to vertices v_1, v_2, and v_3. We then consider the triangle $v_0v_2v_1$ as the initial triangle and follow the same process as in Case 1, adding one extra path $v_3 \to v_0 \to v_2 \to v_1$ to the enumeration above. Restricting the resulting semi-transitive orientation to the original graph (i.e. deleting v_0) gives a semi-transitive orientation of G. □

The same-direction case requires a different obstruction, identified through a small case analysis using the human-verifiable proof format of Section 2.2.6. The minimal example is the 9-vertex graph $A(m^3rmr)$ depicted in Figure 3.4, which the reader is invited to consult while following the branching proof below.

Lemma 3.1.11. *For $x \in \{\ell, r\}$ and s_1 odd, the Apollonian graph $A(m^3xm^{s_1}xm^{s_2})$ is not word-representable.*

Proof. We first prove the case $s_1 = 1$, $s_2 = 0$: namely, that $A(m^3xmx)$ is non-word-representable. By symmetry assume $x = r$, and by Theorem 2.2.10 let v_5 be a source. The graph has nine vertices $v_1, \ldots, v_9$, where v_3 is at the top, v_2 at the bottom-right, and v_1 at the bottom-left of the outer triangle, and $v_4, v_5, v_6, v_7, v_8, v_9$ are the interior vertices in order of creation (Figure 3.4).

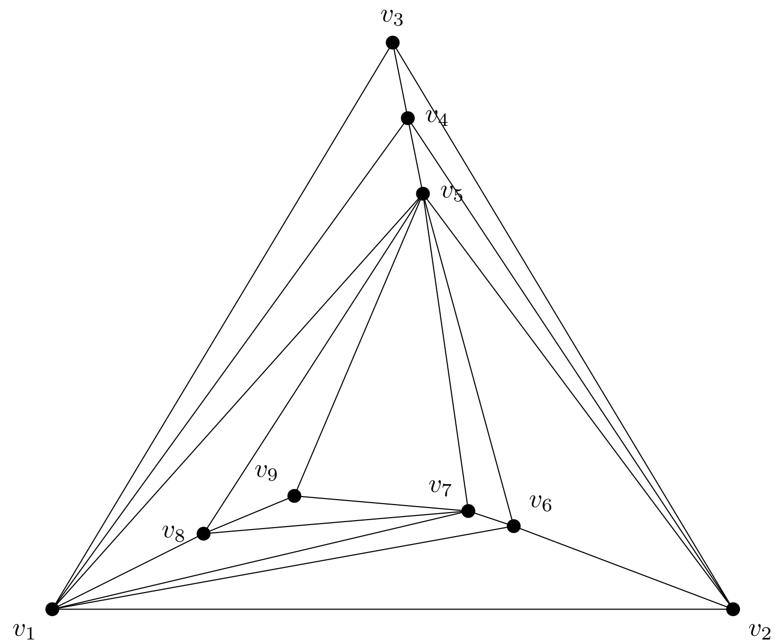


Figure 3.4: The Apollonian graph $A(m^3rmr)$ on nine vertices, drawn as a schematic. Lemma 3.1.11 shows that this graph is non-word-representable: every choice of semi-transitive orientation forces a shortcut. The vertices v_4, v_5, v_6 are produced by the three m-steps with base v_1v_2; the right turn produces v_7 inside triangle $v_1v_5v_6$ with new base v_1v_5; the m-step gives v_8; and the final right turn gives v_9 inside triangle $v_5v_7v_8$.

We use the human-verifiable proof format of Section 2.2.6. The labels **B**, **MC**, **O**, **C**, **S** stand for *branch*, *move to copy*, *orient*, *by cycle*, and *shortcut*, respectively (with **C** α-β-γ-δ meaning the orientation is forced by Lemma 2.2.11 applied to the 4-cycle on those vertices). We branch on the orientation of the edge v_8v_9.

Branch 1. **B** $v_8 \to v_9$ (Copy 2) **O** $v_8 \to v_1$ (**C** v_1-v_8-v_9-v_5) **O** $v_2 \to v_1$ (**C** v_2-v_5-v_8-v_1) **O** $v_7 \to v_1$ (**C** v_2-v_5-v_7-v_1) **O** $v_4 \to v_1$ (**C** v_1-v_7-v_5-v_4) **O** $v_6 \to v_1$ (**C** v_1-v_6-v_5-v_4) **O** $v_7 \to v_9$ (**C** v_1-v_7-v_9-v_5) **O** $v_7 \to v_6$ (**C** v_5-v_9-v_7-v_6) **O** $v_2 \to v_6$ (**C** v_2-v_6-v_7-v_1) **O** $v_3 \to v_1$ **O** $v_2 \to v_3$ (**C** v_3-v_1-v_6-v_2) **S**: v_5-v_2-v_3-v_1.

Branch 2. **MC** 2, taking $v_9 \to v_8$: **O** $v_1 \to v_8$ (**C** v_1-v_8-v_9-v_5) **O** $v_1 \to v_2$ (**C** v_2-v_5-v_8-v_1) **O** $v_1 \to v_7$ (**C** v_2-v_5-v_7-v_1) **O** $v_1 \to v_4$ (**C** v_1-v_7-v_5-v_4) **O** $v_1 \to v_6$ (**C** v_1-v_6-v_5-v_4) **O** $v_9 \to v_7$ (**C** v_1-v_7-v_9-v_5) **O** $v_6 \to v_7$ (**C** v_5-v_9-v_7-v_6) **O** $v_6 \to v_2$ (**C** v_2-v_6-v_7-v_1) **O** $v_1 \to v_3$ **O** $v_3 \to v_2$ (**C** v_3-v_1-v_6-v_2) **S**: v_5-v_1-v_3-v_2.

In each branch a shortcut is found, so no semi-transitive orientation of $A(m^3rmr)$

exists.

Hence $A(m^3xmx)$ is non-word-representable. Since this graph is an induced subgraph of $A(m^{s_0}xmxm^{s_2})$ for any $s_0 \geq 3$ and $s_2 \geq 0$ by Lemma 2.4.7, the latter is also non-word-representable.

For odd $s_1 = 2k + 1$ with $k \geq 1$, write $A_4 = A(m^3xm^{2k+1}x)$. The graph A_4 is obtained from $A_3 = A(m^3xmx)$ by subdividing the edge v_6v_7, introducing $2k$ new vertices $u_1, u_2, \dots, u_{2k}$ in order from v_6 to v_7, each adjacent to both v_1 and v_5 in A_4 (the subdivision converts the single edge v_6v_7 into the path $v_6, u_1, u_2, \dots, u_{2k}, v_7$, with each new vertex inheriting the common neighbour set $\{v_1, v_5\}$ from the original edge endpoints).

Case $k = 1$. The new graph has two extra vertices u_1, u_2. We modify Branch 1 of the $s_1 = 1$ proof above. The instructions through **O** $v_4 \to v_1$ remain valid because none of the cited cycles use the edge v_6v_7 (they involve $v_1, v_4, v_5, v_6, v_7, v_8, v_9$ only in 4-cycles passing through the existing K_4-subgraphs around those vertices, which are unchanged by the subdivision). The instruction **O** $v_7 \to v_6$ (**C** v_5-v_9-v_7-v_6), however, cites a 4-cycle that *does* use the edge v_6v_7, which is now subdivided; that cycle no longer exists in A_4.

We replace this single instruction with:

O $u_1 \to v_1$ (**C** v_5-v_7-v_1-u_1)
O $u_2 \to v_1$ (**C** v_5-v_7-v_1-u_2)
O $v_7 \to u_2$ (**C** v_5-v_9-v_7-u_2)
O $u_1 \to u_2$ (**C** v_5-v_7-u_2-u_1)
O $u_1 \to v_6$ (**C** v_6-v_5-u_2-u_1).

Each cited 4-cycle lies entirely within the local neighbourhood of v_5 and is present in A_4. The remaining instructions of Branch 1 then proceed unchanged, since they do not reference any of the subdivision vertices, and the cited cycles among $\{v_1, v_2, v_3, v_4, v_5, v_6, v_8, v_9\}$ are unaffected. The branch closes with the same shortcut v_5-v_2-v_3-v_1.

General $k \geq 1$. For $k \geq 2$, the same idea extends. We replace the single instruction **O** $v_7 \to v_6$ by the block:

> $\big[$**O** $u_i \to v_1$ (**C** v_5-v_7-v_1-u_i) for $i = 1, 2, \ldots, 2k\big]$ **O** $v_7 \to u_{2k}$ (**C** v_5-v_9-v_7-u_{2k}) **O** $u_{2k-1} \to u_{2k}$ (**C** v_5-v_7-u_{2k}-u_{2k-1}) $\big[$**O** $u_{i+1} \to u_i$ (**C** v_5-u_i-u_{i+1}-u_{i+2}), **O** $u_{i-1} \to u_i$ (**C** v_5-u_{i-1}-u_i-u_{i+1}) for $i = 2k-2, 2k-3, \ldots, 2\big]$ **O** $u_1 \to v_6$ (**C** v_6-v_5-u_2-u_1).

Each cited 4-cycle is present in A_4: v_5 is adjacent to every u_i (because each u_i was inserted adjacent to the common neighbour set $\{v_1, v_5\}$); consecutive u_i, u_{i+1} are adjacent (as the subdivided path); and the side vertices v_1, v_6, v_7 are adjacent to v_5 and to the chain endpoints. The induction on k goes through: the $k = 1$ case is verified above, and each step from k to $k + 1$ adds two more applications of Lemma 2.2.11 to 4-cycles strictly inside the chain.

Branch 2 is modified analogously by reversing every orientation in the inserted block; the cycle citations remain valid (Lemma 2.2.11 is direction-symmetric). Both branches close with shortcuts, so A_4 is non-word-representable.

Since A_4 is an induced subgraph of $A(m^{s_0}xm^{s_1}xm^{s_2})$ for any $s_0 \geq 3$, odd $s_1 \geq 1$, and $s_2 \geq 0$ by Lemma 2.4.7, the lemma follows. □

Theorem 3.1.12. *Let $G = A(m^{s_0}xm^{s_1}xm^{s_2})$, with $x \in \{\ell, r\}$ and $s_0 \geq 3$. Then G is word-representable if and only if s_1 is even.*

Proof. The forward direction follows from Lemma 3.1.11: if s_1 is odd, then $A(m^3xm^{s_1}x)$ is an induced subgraph of G by Lemma 2.4.7, so G is not word-representable.

For the backward direction, suppose s_1 is even. We construct a semi-transitive orientation through a 4-colouring. By symmetry, assume without loss of generality that $x = r$. We consider two cases based on the parity of s_0.

Case 1. s_0 is odd. Consider the following 4-colouring of the graph. Assign colours $1, 2, 3, 4$ to the vertices v_3, v_4, v_2, and v_1 respectively. Then the vertices $v_5, \ldots, v_{s_0+3}$ receive colours 1 and 2 in alternating order starting and ending with colour 2. Vertices $v_{s_0+4}, v_{s_0+5}, v_{s_0+6}, \ldots, v_{s_0+s_1+4}$ are coloured alternately with 2 and 3, beginning and ending with colour 3. The vertices $v_{s_0+s_1+5}, v_{s_0+s_1+6}$ and so on are coloured alternately with 4 and 3, starting with colour 4 and ending at the last vertex which is coloured 4 or 3 depending on the parity of s_2.

We verify the hypothesis of Lemma 3.1.1. For any candidate path $1 \to 2 \to 3 \to 4$, there are two possibilities:

- Vertex v_1 is on the path. In every directed path with colour sequence $1 \to 2 \to 3 \to 4$ involving v_1, both $1 \to 3$ and $2 \to 4$ are present.
- Vertex v_1 is not on the path. Then some vertex such as $v_{s_0+s_1+5}, v_{s_0+s_1+7}, \dots$, which is coloured 4, takes the role of u_4. Again the supporting edges $1 \to 3$ and $2 \to 4$ exist (with the vertex coloured 1 being v_{s_0+2} in this case).

By Lemma 3.1.1, the orientation is semi-transitive.

Case 2. s_0 is even. In this case, we add a new vertex v_0 coloured 2, and connect it to vertices v_1, v_2, and v_3. We then consider the triangle $v_0v_2v_1$ as the initial triangle and follow the same process as in Case 1. Restricting the resulting semi-transitive orientation to the original graph (i.e. deleting v_0) gives a semi-transitive orientation of G. □

Remark 3.1.13. Theorems 3.1.6, 3.1.10, and 3.1.12 together provide a complete classification of word-representable Apollonian graphs with exactly two changes of direction.

3.1.4 Three changes of direction

We now turn to graphs of the form $A(m^{s_0}x_1m^{s_1}x_2m^{s_2}x_3m^{s_3})$, $x_i \in \{\ell, r\}$. The classification depends on the pattern of equal and unequal direction changes.

We first record two "inheritance" theorems that follow from the two-change classification via Lemma 2.4.5.

Theorem 3.1.14. *Let $G = A(mx_1m^{s_1}x_2m^{s_2}x_3m^{s_3})$ with $x_i \in \{\ell, r\}$, $x_2 \neq x_3$, $s_1 \geq 1$. Then G is word-representable if and only if s_2 is odd.*

Proof. By Lemma 2.4.5, G is isomorphic to $A(m^{s_1+2}x_2m^{s_2}x_3m^{s_3})$. The result now follows from Theorem 3.1.10. □

Theorem 3.1.15. *Let $G = A(mx_1m^{s_1}x_2m^{s_2}x_2m^{s_3})$ with $x_1, x_2 \in \{\ell, r\}$, $s_1 \geq 1$. Then G is word-representable if and only if s_2 is even.*

Proof. By Lemma 2.4.5, G is isomorphic to $A(m^{s_1+2}x_2m^{s_2}x_2m^{s_3})$. The result now follows from Theorem 3.1.12. □

The case $s_0 = 2$

We require two non-representability lemmas before stating the main results for $s_0 = 2$. The proofs follow the same human-verifiable format as Lemma 3.1.11.

Lemma 3.1.16. *The Apollonian graph $A(m^{s_0}xm^{s_1}xm^{s_2}xm^{s_3})$, where $x \in \{\ell, r\}$, $s_0 \geq 2$, $s_1, s_2, s_3 \geq 0$, and s_2 is odd, is non-word-representable.*

Proof. We prove that $A(m^2xm^{s_1}xm^{s_2}x)$ is non-word-representable for s_2 odd; the lemma then follows from the hereditary property (Lemma 2.4.7). We may therefore assume $s_0 = 2$ and $s_3 = 0$. By symmetry assume $x = r$, and by Theorem 2.2.10 let v_4 be a source. Write $s_2 = 2k + 1$ with $k \geq 0$.

Case 1. $s_1 \geq 0$, $k = 0$. The graph is $A_2 = A(m^2rm^{s_1}rmr)$. When $s_1 = 0$, the graph $A_3 = A_2$ has nine vertices $v_1, \dots, v_9$. When $s_1 \geq 1$, the graph A_2 is obtained from A_3 by subdividing the edge v_2v_5, adding s_1 new vertices, but A_3 remains as an induced subgraph by Lemma 2.4.7. Therefore it suffices to prove that A_3 is non-word-representable. We branch on the orientation of v_7v_8.

Branch 1. **B** $v_7 \to v_8$ (Copy 2) **O** $v_7 \to v_6$ (**C** v_4-v_8-v_7-v_6) **O** $v_1 \to v_6$ (**C** v_1-v_6-v_7-v_4) **O** $v_1 \to v_3$ (**C** v_3-v_4-v_6-v_1) **O** $v_1 \to v_5$ (**C** v_3-v_4-v_5-v_1) **O** $v_7 \to v_5$ (**C** v_1-v_5-v_7-v_4) **O** $v_8 \to v_5$ (**C** v_1-v_5-v_8-v_4) **O** $v_6 \to v_5$ (**C** v_4-v_8-v_5-v_6) **O** $v_9 \to v_5$ **O** $v_7 \to v_9$ (**C** v_5-v_9-v_7-v_6) **S**: v_4-v_7-v_9-v_5.

Branch 2. **MC** 2, taking $v_8 \to v_7$: **O** $v_6 \to v_7$ (**C** v_4-v_8-v_7-v_6) **O** $v_6 \to v_1$ (**C** v_1-v_6-v_7-v_4) **O** $v_3 \to v_1$ (**C** v_3-v_4-v_6-v_1) **O** $v_5 \to v_1$ (**C** v_3-v_4-v_5-v_1) **O** $v_5 \to v_7$ (**C** v_1-v_5-v_7-v_4) **O** $v_5 \to v_8$ (**C** v_1-v_5-v_8-v_4) **O** $v_5 \to v_6$ (**C** v_4-v_8-v_5-v_6) **O** $v_5 \to v_9$ **O** $v_9 \to v_7$ (**C** v_5-v_9-v_7-v_6) **S**: v_4-v_5-v_9-v_7.

Each branch closes with a shortcut, so A_3 is non-word-representable. Hence the lemma holds for $k = 0$ and any $s_1 \geq 0$.

Case 2. $s_1 \geq 0$, $k \geq 1$. The graph $A_4 = A(m^2rm^{s_1}rm^{2k+1}r)$ is constructed from A_2 by subdividing the edge v_6v_7 and inserting $2k$ vertices $w_1, w_2, \dots, w_{2k}$ from v_6 to v_7, each adjacent to both v_4 and v_5. We argue on the induced subgraph A_5 obtained from A_3 by the same subdivision; this is an induced subgraph of A_4 by Lemma 2.4.7.

The argument follows the same pattern as the $k \geq 1$ extension in the proof of Lemma 3.1.11. We modify the proof of Branch 1 above by inserting the following block of instructions immediately after **O** $v_7 \to v_6$:

> **O** $v_7 \to w_{2k}$ (**C** v_4-v_8-v_7-w_{2k}) **O** $w_{2k-1} \to w_{2k}$ (**C** v_4-v_7-w_{2k}-w_{2k-1}) $\big[$**O** $w_{j-1} \to w_{j-2}$ (**C** v_4-w_j-w_{j-1}-w_{j-2}), **O** $w_{j-3} \to w_{j-2}$ (**C** v_4-w_{j-1}-w_{j-2}-w_{j-3}) for $j = 2k, 2k-2, \ldots, 6, 4\big]$ **O** $w_1 \to v_6$ (**C** v_4-w_2-w_1-v_6).

For $k = 1$ explicitly: the block reduces to a single **O** $v_7 \to w_2$ and analogous chain through w_1, with each cited 4-cycle present in the subdivided graph because w_1 and w_2 inherit the common neighbour set $\{v_4, v_5\}$ from the original edge v_6v_7. The induction on k proceeds as in Lemma 3.1.11.

After this insertion, the remaining instructions of Branch 1 proceed unchanged, with the additional forced orientations

$$\big[\textbf{O}\ w_j \to v_5\ (\textbf{C}\ v_1\text{-}v_5\text{-}w_j\text{-}v_4) \text{ for } j = 1, 2, \ldots, 2k\big]$$

inserted before the final shortcut closure. Branch 2 is modified by reversing the directions in the inserted block. Each branch terminates in the shortcut v_4-v_7-v_9-v_5 (or its symmetric counterpart v_4-v_5-v_9-v_7), so A_5 is non-word-representable. Since $A_5 \subseteq A_4 \subseteq A_1$ as induced subgraphs, this completes the proof. □

Lemma 3.1.17. *The Apollonian graph $A(m^{s_0}x_1m^{s_1}x_2m^{s_2}x_2m^{s_3})$, where $x_1, x_2 \in \{\ell, r\}$, $x_1 \neq x_2$, $s_0 \geq 2$, $s_1, s_2, s_3 \geq 0$, and s_2 is odd, is non-word-representable.*

Proof. We prove that $A(m^2x_1m^{s_1}x_2m^{s_2}x_2)$ is non-word-representable for s_2 odd; the lemma then follows from Lemma 2.4.7. We may assume $s_0 = 2$ and $s_3 = 0$. By symmetry assume $x_1 = r$ (so $x_2 = \ell$), and by Theorem 2.2.10 let v_1 (the top vertex) be a source. Write $s_2 = 2k + 1$ with $k \geq 0$.

Case 1. $s_1 \geq 0$, $k = 0$. The graph is $A_2 = A(m^2rm^{s_1}\ell m\ell)$. When $s_1 = 0$, set $A_3 = A_2$, a graph on nine vertices $v_1, \ldots, v_9$. When $s_1 \geq 1$, A_2 is obtained from A_3 by subdividing the edge v_2v_5 (with new vertices $u_1, \ldots, u_{s_1}$ each adjacent to v_1 and v_4), but A_3 is still an induced subgraph of A_2 by Lemma 2.4.7. We show A_3 is non-word-representable. Branch on the orientation of v_7v_8.

Branch 1. **B** $v_7 \to v_8$ (Copy 2) **O** $v_7 \to v_6$ (**C** v_1-v_8-v_7-v_6) **O** $v_4 \to v_6$ (**C** v_1-v_7-v_6-v_4) **O** $v_4 \to v_3$ (**C** v_3-v_4-v_6-v_1) **O** $v_4 \to v_5$ (**C** v_3-v_4-v_5-v_1) **O** $v_7 \to v_5$ (**C** v_1-v_7-v_5-v_4) **O** $v_8 \to v_5$ (**C** v_1-v_8-v_5-v_4) **O** $v_6 \to v_5$ (**C** v_1-v_8-v_5-v_6) **O** $v_9 \to v_5$ **O** $v_7 \to v_9$ (**C** v_5-v_9-v_7-v_6) **S**: v_1-v_7-v_9-v_5.

Branch 2. **MC** 2, taking $v_8 \to v_7$: **O** $v_6 \to v_7$ (**C** v_1-v_8-v_7-v_6) **O** $v_6 \to v_4$ (**C** v_1-v_7-v_6-v_4) **O** $v_3 \to v_4$ (**C** v_3-v_4-v_6-v_1) **O** $v_5 \to v_4$ (**C** v_3-v_4-v_5-v_1) **O** $v_5 \to v_7$ (**C** v_1-v_7-v_5-v_4) **O** $v_5 \to v_8$ (**C** v_1-v_8-v_5-v_4) **O** $v_5 \to v_6$ (**C** v_1-v_8-v_5-v_6) **O** $v_5 \to v_9$ **O** $v_9 \to v_7$ (**C** v_5-v_9-v_7-v_6) **S**: v_1-v_5-v_9-v_7.

Both branches terminate in a shortcut, so A_3 is non-word-representable. Hence the lemma holds for $k = 0$ and $s_1 \geq 0$.

Case 2. $s_1 \geq 0$, $k \geq 1$. Define $A_4 = A(m^2rm^{s_1}\ell m^{2k+1}\ell)$, obtained from A_2 by subdividing the edge v_6v_7 with $2k$ new vertices $w_1, \ldots, w_{2k}$ from v_6 to v_7, each adjacent to both endpoints' common neighbours in the construction. We argue on the induced subgraph A_5 obtained from A_3 by the same subdivision; this is induced in A_4 by Lemma 2.4.7.

The argument follows the same pattern as in Case 2 of Lemma 3.1.16 and the $k \geq 1$ extension in Lemma 3.1.11. We modify Branch 1 by inserting, immediately after **O** $v_7 \to v_6$, the following block of forced orientations:

> **O** $v_7 \to w_{2k}$ (**C** v_1-v_8-v_7-w_{2k}) **O** $w_{2k-1} \to w_{2k}$ (**C** v_1-v_7-w_{2k}-w_{2k-1}) $\big[$**O** $w_{j-1} \to w_{j-2}$ (**C** v_1-w_j-w_{j-1}-w_{j-2}), **O** $w_{j-3} \to w_{j-2}$ (**C** v_1-w_{j-1}-w_{j-2}-w_{j-3}) for $j = 2k, 2k-2, \ldots, 7, 5\big]$ **O** $w_1 \to w_2$ (**C** v_1-w_3-w_2-w_1) **O** $w_1 \to v_6$ (**C** v_1-w_2-w_1-v_6).

For $k = 1$ explicitly, the block reduces to verifying the orientations of the 4-cycles on $\{v_1, v_8, v_7, w_2\}$, $\{v_1, v_7, w_2, w_1\}$, and $\{v_1, w_2, w_1, v_6\}$, each present in the subdivided graph for the same reasons as in Lemma 3.1.11. The induction on k proceeds analogously.

The remaining instructions of Branch 1 proceed as before, augmented by

$$\big[\mathbf{O}\ w_j \to v_5\ (\mathbf{C}\ v_1\text{-}w_j\text{-}v_5\text{-}v_4) \text{ for } j = 1, 2, \ldots, 2k\big],$$

and conclude with the shortcut v_1-v_7-v_9-v_5. Branch 2 is modified analogously by reversing the directions in the inserted block, terminating in the shortcut v_1-v_5-v_9-v_7. Hence A_5 is non-word-representable, and so is $A_4 \supseteq A_5$ and $A_1 \supseteq A_4$. □

Theorem 3.1.18. *The Apollonian graph $A(m^2 x_1 m^{s_1} x_2 m^{s_2} x_2 m^{s_3})$, with $x_1, x_2 \in \{\ell, r\}$, is word-representable if and only if s_2 is even.*

Proof. If s_2 is odd, then by Lemma 3.1.16 (when $x_1 = x_2$) or Lemma 3.1.17 (when $x_1 \neq x_2$), the graph is non-word-representable.

Conversely, suppose s_2 is even. We construct a semi-transitive orientation through a 4-colouring and verify the hypothesis of Lemma 3.1.1 in each of the four sub-cases (by the parity of s_1 and whether $x_1 = x_2$); by symmetry, $x_1 = r$.

Case 1.1. $x_1 = x_2$, s_1 even. Assign colours $1, 2, 3, 4$ to v_4, v_3, v_2, v_1 respectively. Then v_5 receives colour 2, the vertices $v_6, v_7, \ldots, v_{s_1+6}$ alternate between colours 3 and 2 starting and ending with colour 3, vertex v_{s_1+7} is coloured 4, the vertices $v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ alternate between colours 3 and 4 starting with 3 and ending with 4, and vertices from $v_{s_1+s_2+8}$ onward alternate between colours 1 and 4 starting with 1.

The directed paths $1 \to 2 \to 3 \to 4$ are enumerated below; in each case the supporting edges $1 \to 3$ and $2 \to 4$ are present in G, so by Lemma 3.1.1 the orientation is semi-transitive.

- $v_4 \to v_3 \to v_2 \to v_1$, and $v_4 \to v_5 \to v_2 \to v_1$;
- $v_4 \to v_{2k+5} \to v_{2k+4} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1}{2}$;
- $v_4 \to v_{2k+5} \to v_{2k+6} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7}$ for $k = 0, 1, \ldots, \frac{s_2}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+5}$ for $k = 1, 2, \ldots, \frac{s_2}{2}$;

- $v_{s_1+s_2+2k+8} \to v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+7}$ for $k = 0, 1, \ldots, \lfloor s_3/2 \rfloor$;
- $v_{s_1+s_2+2k+8} \to v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+9}$ for $k = 1, 2, \ldots, \lceil s_3/2 \rceil$.

Case 1.2. $x_1 = x_2$, s_1 odd. Assign colours $1, 2, 3, 4$ to v_4, v_2, v_3, v_1 respectively. Vertex v_5 receives colour 3, vertices $v_6, \ldots, v_{s_1+6}$ alternate between colours 2 and 3 starting with 2 and ending with 3, vertex v_{s_1+7} is coloured 4, vertices $v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ alternate between colours 3 and 4 starting with 3 and ending with 4, and the rest alternate between 1 and 4.

The directed paths $1 \to 2 \to 3 \to 4$ are:

- $v_4 \to v_2 \to v_3 \to v_1$ and $v_4 \to v_2 \to v_5 \to v_1$;
- $v_4 \to v_{2k+4} \to v_{2k+3} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1+1}{2}$;
- $v_4 \to v_{2k+4} \to v_{2k+5} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1+1}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7}$ for $k = 0, 1, \ldots, \frac{s_2}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+5}$ for $k = 1, 2, \ldots, \frac{s_2}{2}$;
- $v_{s_1+s_2+2k+8} \to v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+7}$ for $k = 0, 1, \ldots, \lfloor s_3/2 \rfloor$;
- $v_{s_1+s_2+2k+8} \to v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+9}$ for $k = 1, 2, \ldots, \lceil s_3/2 \rceil$.

In every such path the supporting edges $1 \to 3$ and $2 \to 4$ are present.

Case 2.1. $x_1 \neq x_2$, s_1 even. Assign colours $1, 2, 3, 4$ to v_3, v_2, v_4, v_1 respectively. Vertex v_5 receives colour 1, vertices $v_6, \ldots, v_{s_1+6}$ alternate between colours 1 and 2 starting and ending with colour 2, vertex v_{s_1+7} is coloured 3, vertices $v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ alternate between colours 2 and 3 starting with 2 and ending with 3, and the rest alternate between 3 and 4 starting with 4.

The directed paths $1 \to 2 \to 3 \to 4$ are:

- $v_3 \to v_2 \to v_4 \to v_1$ and $v_5 \to v_2 \to v_4 \to v_1$;
- $v_{2k+5} \to v_{2k+4} \to v_4 \to v_1$ for $k = 0, 1, \ldots, \frac{s_1}{2}$;

- $v_{2k+5} \to v_{2k+6} \to v_4 \to v_1$ for $k = 0, 1, \ldots, \frac{s_1}{2} - 1$;
- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+7} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2}{2}$;
- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2}{2} - 1$;
- $v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+8} \to v_1$ for $k = 0, 1, \ldots, \lfloor s_3/2 \rfloor$;
- $v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+7}$ for $k = 1, 2, \ldots, \lceil s_3/2 \rceil$.

Case 2.2. $x_1 \neq x_2$, s_1 odd. Assign colours $1, 2, 3, 4$ to v_2, v_3, v_4, v_1 respectively. Vertex v_5 receives colour 2, vertices $v_6, \ldots, v_{s_1+6}$ alternate between colours 1 and 2 starting with 1 and ending with 2, vertex v_{s_1+7} is coloured 3, vertices $v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ alternate between colours 2 and 3 starting with 2 and ending with 3, and the rest alternate between 3 and 4 starting with 4.

The directed paths $1 \to 2 \to 3 \to 4$ are:

- $v_2 \to v_3 \to v_4 \to v_1$ and $v_2 \to v_5 \to v_4 \to v_1$;
- $v_{2k+5} \to v_{2k+6} \to v_4 \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_{2k+5} \to v_{2k+4} \to v_4 \to v_1$ for $k = 1, 2, \ldots, \frac{s_1-1}{2}$;
- $v_{s_1+5} \to v_{s_1+6} \to v_{s_1+7} \to v_1$;
- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2}{2} - 1$;
- $v_{s_1+5} \to v_{s_1+2k+6} \to v_{s_1+2k+5} \to v_1$ for $k = 1, 2, \ldots, \frac{s_2}{2}$;
- $v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+7}$ for $k = 0, 1, \ldots, \lfloor s_3/2 \rfloor$;
- $v_{s_1+5} \to v_{s_1+s_2+6} \to v_{s_1+s_2+2k+8} \to v_1$ for $k = 0, 1, \ldots, \lceil s_3/2 \rceil - 1$.

In every path the supporting edges are present, so by Lemma 3.1.1 the orientation is semi-transitive in all four sub-cases, and G is word-representable. $\square$

Theorem 3.1.19. *The Apollonian graph $G = A(m^2 x_1 m^{s_1} x_2 m^{s_2} x_3 m^{s_3})$, with $x_2 \neq x_3$ and $x_i \in \{\ell, r\}$ for $i = 1, 2, 3$, is word-representable if and only if s_2 is even.*

Proof. Forward direction. Suppose s_2 is odd. If $x_1 = x_2$, then by Lemma 3.1.16 the graph is non-word-representable. If $x_1 \neq x_2$, then by Lemma 3.1.17 the graph is non-word-representable.

Backward direction. Suppose s_2 is even. We use the vertex labelling of Section 2.4 and, by symmetry, assume $x_2 = r$. We consider four sub-cases according to whether $x_1 = x_2$ and the parity of s_1, and in each case apply Lemma 3.1.1.

Case 1.1. s_1 is odd and $x_1 = x_2$. Assign colours $1, 2, 3, 4$ to v_4, v_2, v_3, and v_1 respectively. Then v_5 must be coloured 3, and the vertices $v_6, v_7, \ldots, v_{s_1+6}$ alternate between colours 2 and 3 starting with colour 2 and ending with colour 3. Vertex v_{s_1+7} is coloured 4, and the vertices $v_{s_1+8}, v_{s_1+9}, \ldots, v_{s_1+s_2+7}$ alternate between colours 3 and 4 starting and ending with colour 3. The vertices from $v_{s_1+s_2+8}$ onward alternate between colours 2 and 3 starting with colour 2 and ending with colour 2 or 3 depending on the parity of s_3.

We verify the hypothesis of Lemma 3.1.1. The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ under the induced orientation are:

- $v_4 \to v_2 \to v_3 \to v_1$, and $v_4 \to v_2 \to v_5 \to v_1$;
- $v_4 \to v_{2k+6} \to v_{2k+5} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_4 \to v_{2k+6} \to v_{2k+7} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+6} \to v_1$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7}$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+9}$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_4 \to v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+7} \to v_{s_1+s_2+6}$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even or $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_4 \to v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+9} \to v_{s_1+s_2+6}$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even or $k = 1, 2, \ldots, \frac{s_3-1}{2}$ if s_3 is odd.

For each such path, both supporting transitivity edges (colour $1 \to 3$ and colour $2 \to 4$) are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 1.2. s_1 is even and $x_1 = x_2$. Assign colours $1, 2, 3, 4$ to v_4, v_3, v_2, and v_1 respectively. Then v_5 must be coloured 2, and the vertices $v_6, v_7, \ldots, v_{s_1+6}$ alternate between colours 2 and 3 starting and ending with colour 3. Vertex v_{s_1+7} is coloured 4, and the vertices $v_{s_1+8}, v_{s_1+9}, \ldots, v_{s_1+s_2+7}$ alternate between colours 3 and 4 starting and ending with colour 3. The remaining vertices alternate between colours 3 and 2 starting with colour 2 and ending with colour 2 or 3 depending on the parity of s_3.

We verify the hypothesis of Lemma 3.1.1. The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_4 \to v_3 \to v_2 \to v_1$, and $v_4 \to v_5 \to v_2 \to v_1$;
- $v_4 \to v_{2k+5} \to v_{2k+6} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1}{2}$;
- $v_4 \to v_{2k+5} \to v_{2k+4} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+6} \to v_1$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7}$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_4 \to v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+9}$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_4 \to v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+7} \to v_{s_1+s_2+6}$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even or $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_4 \to v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+9} \to v_{s_1+s_2+6}$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even or $k = 1, 2, \ldots, \frac{s_3-1}{2}$ if s_3 is odd.

For each such path, both supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2.1. s_1 is odd and $x_1 \neq x_2$. Assign colours $1, 2, 3, 4$ to v_2, v_3, v_4, and v_1 respectively. Then v_5 must be coloured 2, and the vertices $v_6, v_7, \ldots, v_{s_1+6}$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 2. Vertex v_{s_1+7} is coloured 3, and the vertices $v_{s_1+8}, \ldots, v_{s_1+s_2+7}$ alternate between colours 2 and 3 starting and ending with colour 2. The remaining vertices alternate between colours 1 and 2 starting with colour 1 and ending with colour 1 or 2 depending on the parity of s_3.

We verify the hypothesis of Lemma 3.1.1. The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_2 \to v_3 \to v_4 \to v_1$, and $v_2 \to v_5 \to v_4 \to v_1$;
- $v_{2k+6} \to v_{2k+7} \to v_4 \to v_1$ for $k = 0, 1, \dots, \frac{s_1-1}{2}$;
- $v_{2k+6} \to v_{2k+5} \to v_4 \to v_1$ for $k = 1, 2, \dots, \frac{s_1-1}{2}$;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+9} \to v_1$ for $k = 0, 1, \dots, \frac{s_2-1}{2}$;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, \dots, \frac{s_2-1}{2}$;
- $v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+7} \to v_{s_1+s_2+6} \to v_1$ for $k = 0, 1, \dots, \frac{s_3}{2}$ if s_3 is even or $k = 0, 1, \dots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+9} \to v_{s_1+s_2+6} \to v_1$ for $k = 1, 2, \dots, \frac{s_3}{2}$ if s_3 is even or $k = 1, 2, \dots, \frac{s_3-1}{2}$ if s_3 is odd.

For each such path, both supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2.2. s_1 is even and $x_1 \neq x_2$. Assign colours $1, 2, 3, 4$ to v_3, v_2, v_4, and v_1 respectively. Then v_5 is coloured 1, and the vertices $v_6, v_7, \dots, v_{s_1+6}$ alternate between colours 1 and 2 starting and ending with colour 2. Vertex v_{s_1+7} is coloured 3, and the vertices $v_{s_1+8}, \dots, v_{s_1+s_2+7}$ alternate between colours 2 and 3 starting and ending with colour 2. The remaining vertices alternate between colours 1 and 2 starting with colour 1 and ending with colour 1 or 2 depending on the parity of s_3.

We verify the hypothesis of Lemma 3.1.1. The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_3 \to v_2 \to v_4 \to v_1$, and $v_5 \to v_2 \to v_4 \to v_1$;
- $v_{2k+5} \to v_{2k+6} \to v_4 \to v_1$ for $k = 0, 1, \dots, \frac{s_1}{2}$;

- $v_{2k+5} \to v_{2k+4} \to v_4 \to v_1$ for $k = 1, 2, \ldots, \frac{s_1}{2}$;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+9} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_{s_1+5} \to v_{s_1+2k+8} \to v_{s_1+2k+7} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+7} \to v_{s_1+s_2+6} \to v_1$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even or $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_1+s_2+2k+8} \to v_{s_1+s_2+2k+9} \to v_{s_1+s_2+6} \to v_1$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even or $k = 1, 2, \ldots, \frac{s_3-1}{2}$ if s_3 is odd.

For each such path, both supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Since we have provided a semi-transitive orientation in all four sub-cases, the graph $G = A(m^2 x_1 m^{s_1} x_2 m^{s_2} x_2 m^{s_3})$ is word-representable. $\square$

The cases $s_0 \geq 3$: a complete classification by direction pattern

When $s_0 \geq 3$, the longer initial m-prefix introduces enough additional structure for further obstructions to appear. The classification depends on the full pattern of equalities and inequalities among x_1, x_2, x_3. We give the four results, with full proofs in each case according to the parity of s_0.

Theorem 3.1.20. *The Apollonian graph $A(m^{s_0} x m^{s_1} x m^{s_2} x m^{s_3})$, with $x \in \{\ell, r\}$ and $s_0 \geq 3$, is word-representable if and only if both s_1 and s_2 are even.*

Proof. Forward direction. If s_1 is odd, then $A(m^3 x m^{s_1} x m^{s_2})$ is an induced subgraph of G by Lemma 2.4.7, and by Lemma 3.1.11 this is non-word-representable. If s_2 is odd, then by Lemma 3.1.16 the graph $A(m^{s_0} x m^{s_1} x m^{s_2} x)$ is non-word-representable, hence so is G.

Backward direction. Suppose s_1 and s_2 are both even. By symmetry, $x = r$. We consider two cases according to the parity of s_0, applying Lemma 3.1.1 in each.

Case 1. s_0 is even. Assign colours $1, 2, 3, 4$ to v_4, v_3, v_2, and v_1 respectively. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 1 and 2 starting and ending with colour 2.

Vertices $v_{s_0+4}, \ldots, v_{s_0+s_1+4}$ alternate between colours 2 and 3 starting and ending with colour 3. Vertex $v_{s_0+s_1+5}$ is coloured 4. Vertices $v_{s_0+s_1+6}, \ldots, v_{s_0+s_1+s_2+4}$ alternate between colours 3 and 4 starting with colour 3 and ending with colour 4. Vertices from $v_{s_0+s_1+s_2+5}$ onward alternate between colours 1 and 4 starting with colour 1 and ending with colour 1 or 4 depending on the parity of s_3.

The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ under this orientation are:

- $v_4 \to v_3 \to v_2 \to v_1$;
- $v_{2k+4} \to v_{2k+5} \to v_2 \to v_1$ for $k = 0, 1, \ldots, \frac{s_0}{2}$;
- $v_{2k+4} \to v_{2k+3} \to v_2 \to v_1$ for $k = 1, 2, \ldots, \frac{s_0}{2}$;
- $v_{s_0+2} \to v_{s_0+3} \to v_2 \to v_1$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+4} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1}{2}$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+2} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1}{2}$;
- $v_{s_0+2} \to v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+5}$ for $k = 0, 1, \ldots, \frac{s_2}{2}$;
- $v_{s_0+2} \to v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+3}$ for $k = 1, 2, \ldots, \frac{s_2}{2}$;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+5}$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+7}$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 1, 2, \ldots, \frac{s_3+1}{2}$ if s_3 is odd.

For each such path the supporting edges $1 \to 3$ and $2 \to 4$ are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2. s_0 is odd. Assign colours $1, 2, 3, 4$ to v_3, v_4, v_2, and v_1 respectively. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 1 and 2 starting and ending with colour 2, and the vertices $v_{s_0+4}, \ldots, v_{s_0+s_1+4}$ alternate between colours 2 and 3 starting and ending with colour 3. Vertex $v_{s_0+s_1+5}$ is coloured 4, and the vertices $v_{s_0+s_1+6}, \ldots, v_{s_0+s_1+s_2+4}$ alternate between colours 3 and 4 starting with colour 3 and ending with colour 4.

The vertices from $v_{s_0+s_1+s_2+5}$ onward alternate between colours 1 and 4 starting with colour 1 and ending with colour 1 or 4 depending on the parity of s_3.

The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_3 \to v_4 \to v_2 \to v_1$;
- $v_{2k+5} \to v_{2k+6} \to v_2 \to v_1$ for $k = 1, 2, \ldots, \frac{s_0-1}{2}$;
- $v_{2k+5} \to v_{2k+4} \to v_2 \to v_1$ for $k = 0, 1, \ldots, \frac{s_0-1}{2}$;
- $v_{s_0+2} \to v_{s_0+3} \to v_2 \to v_1$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+4} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1}{2}$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+2} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1}{2}$;
- $v_{s_0+2} \to v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+5}$ for $k = 0, 1, \ldots, \frac{s_2}{2}$;
- $v_{s_0+2} \to v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+3}$ for $k = 1, 2, \ldots, \frac{s_2}{2}$;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+5}$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+7}$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 1, 2, \ldots, \frac{s_3+1}{2}$ if s_3 is odd.

For each such path the supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive. $\square$

Theorem 3.1.21. *The Apollonian graph $A(m^{s_0}x_1m^{s_1}x_1m^{s_2}x_2m^{s_3})$, with $x_1, x_2 \in \{\ell, r\}$, $x_1 \neq x_2$, and $s_0 \geq 3$, is word-representable if and only if s_1 is even and s_2 is odd.*

Proof. Forward direction. If s_1 is odd, then $A(m^3x_1m^{s_1}x_1m^{s_2})$ is an induced subgraph by Lemma 2.4.7 and is non-word-representable by Lemma 3.1.11. If s_2 is even, the neighbourhood of an appropriate vertex contains a copy of the graph $H_{n,k,i}$ from Lemma 3.1.8 (see Remark 3.1.23), so G is non-word-representable by Theorem 2.2.7.

Backward direction. Suppose s_1 is even and s_2 is odd. By symmetry, $x_1 = r$. We consider two cases according to the parity of s_0, applying Lemma 3.1.1 in each.

Case 1. s_0 is odd. Assign colours $1, 2, 3, 4$ to v_3, v_4, v_2, and v_1 respectively. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 2, and the vertices $v_{s_0+4}, \ldots, v_{s_0+s_1+4}$ alternate between colours 2 and 3 starting and ending with colour 3. Vertex $v_{s_0+s_1+5}$ is coloured 4, and the vertices $v_{s_0+s_1+6}, \ldots, v_{s_0+s_1+s_2+5}$ alternate between colours 3 and 4 starting and ending with colour 3. The vertices from $v_{s_0+s_1+s_2+6}$ onward alternate between colours 2 and 3 starting with colour 2 and ending with colour 2 or 3 depending on the parity of s_3.

The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_3 \to v_4 \to v_2 \to v_1$, and $v_{s_0+2} \to v_{s_0+3} \to v_2 \to v_1$;
- $v_{2k+5} \to v_{2k+4} \to v_2 \to v_1$ for $k = 0, 1, \ldots, \frac{s_0-3}{2}$;
- $v_{2k+5} \to v_{2k+6} \to v_2 \to v_1$ for $k = 0, 1, \ldots, \frac{s_0-3}{2}$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+4} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1}{2}$;
- $v_{s_0+2} \to v_{s_0+2k+3} \to v_{s_0+2k+2} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1}{2}$;
- $v_{s_0+2} \to v_{s_0+s_1+3} \to v_{s_0+s_1+4} \to v_1$;
- $v_{s_0+2} \to v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+5}$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_{s_0+2} \to v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+3}$ for $k = 1, 2, \ldots, \frac{s_2-1}{2}$;
- $v_{s_0+2} \to v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+s_2+2k+5} \to v_{s_0+s_1+s_2+4}$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_0+2} \to v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+s_2+2k+7} \to v_{s_0+s_1+s_2+4}$ for $k = 1, 2, \ldots, \frac{s_3-2}{2}$ if s_3 is even, and $k = 1, 2, \ldots, \frac{s_3-1}{2}$ if s_3 is odd.

For each such path the supporting edges $1 \to 3$ and $2 \to 4$ are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2. s_0 is even. Add a virtual vertex v_0 adjacent to v_1, v_2, v_3, take the triangle $v_0v_2v_1$ as the new initial triangle, and assign v_0 colour 2. The remaining colour assignments follow Case 1 verbatim. The directed-path enumeration of Case 1 remains valid;

one additional path arises through v_0:

$$v_3 \to v_0 \to v_2 \to v_1,$$

with the supporting edges $1 \to 3$ and $2 \to 4$ present in the augmented graph. Hence the orientation restricted to G (i.e. deleting v_0) is semi-transitive. □

Theorem 3.1.22. *The Apollonian graph $A(m^{s_0}x_1m^{s_1}x_2m^{s_2}x_2m^{s_3})$, with $x_1, x_2 \in \{\ell, r\}$, $x_1 \neq x_2$, and $s_0 \geq 3$, is word-representable if and only if s_1 is odd and s_2 is even.*

Proof. Forward direction. If s_1 is even, the neighbourhood $N(v_1)$ contains an induced copy of $H_{n,k,i}$ from Lemma 3.1.8 (with apex v_4; see Remark 3.1.23 for the identification), so G is non-word-representable by Theorem 2.2.7. If s_2 is odd, then $A(m^{s_0}x_1m^{s_1}x_2m^{s_2}x_2)$ is non-word-representable by Lemma 3.1.17, hence so is G.

Backward direction. Suppose s_1 is odd and s_2 is even. By symmetry, $x_1 = r$. We consider two cases according to the parity of s_0, applying Lemma 3.1.1 in each.

Case 1. s_0 is odd. Assign colours $1, 2, 3, 4$ to v_2, v_4, v_3, and v_1 respectively. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 2 and 3 starting with colour 3 and ending with colour 2, and the vertices $v_{s_0+4}, \ldots, v_{s_0+s_1+4}$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 2. Vertex $v_{s_0+s_1+5}$ is coloured 3. Vertices $v_{s_0+s_1+6}, \ldots, v_{s_0+s_1+s_2+5}$ alternate between colours 2 and 3 starting with colour 2 and ending with colour 3. The vertices from $v_{s_0+s_1+s_2+6}$ onward alternate between colours 3 and 4 starting with colour 4 and ending with colour 3 or 4 depending on the parity of s_3.

The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_2 \to v_4 \to v_3 \to v_1$, and $v_2 \to v_{s_1+3} \to v_{s_1+2} \to v_1$;
- $v_2 \to v_{2k+4} \to v_{2k+5} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1-1}{2}$;
- $v_2 \to v_{2k+4} \to v_{2k+3} \to v_1$ for $k = 1, 2, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+2k+4} \to v_{s_0+2k+3} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;

- $v_{s_0+2k+4} \to v_{s_0+2k+5} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+5} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+3} \to v_1$ for $k = 1, 2, \ldots, \frac{s_2}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+5} \to v_{s_0+s_1+s_2+2k+6}$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+7} \to v_{s_0+s_1+s_2+2k+6}$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 1, 2, \ldots, \frac{s_3+1}{2}$ if s_3 is odd.

For each such path the supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2. s_0 is even. Assign colours $1, 2, 3, 4$ to v_2, v_3, v_4, and v_1 respectively. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 2 and 3 starting and ending with colour 2, and the vertices $v_{s_0+4}, \ldots, v_{s_0+s_1+4}$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 2. Vertex $v_{s_0+s_1+5}$ is coloured 3. Vertices $v_{s_0+s_1+6}, \ldots, v_{s_0+s_1+s_2+5}$ alternate between colours 2 and 3 starting with colour 2 and ending with colour 3. The vertices from $v_{s_0+s_1+s_2+6}$ onward alternate between colours 3 and 4 starting with colour 4 and ending with colour 3 or 4 depending on the parity of s_3.

The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_2 \to v_3 \to v_4 \to v_1$, and $v_2 \to v_{s_0+3} \to v_{s_0+2} \to v_1$;
- $v_2 \to v_{2k+5} \to v_{2k+6} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-3}{2}$;
- $v_2 \to v_{2k+5} \to v_{2k+4} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+2k+4} \to v_{s_0+2k+3} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+2k+4} \to v_{s_0+2k+5} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+5} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+3} \to v_1$ for $k = 1, 2, \ldots, \frac{s_2}{2}$;

- $v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+5} \to v_{s_0+s_1+s_2+2k+6}$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+s_2+4} \to v_{s_0+s_1+s_2+2k+7} \to v_{s_0+s_1+s_2+2k+6}$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 1, 2, \ldots, \frac{s_3+1}{2}$ if s_3 is odd.

For each such path the supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive. □

Remark 3.1.23. The same kind of explicit $H_{n,k,i}$-realisation is the obstruction in the forward direction of Theorems 3.1.21 and 3.1.24. In each instance, the path of vertices created during the run of s_i violating the parity rule plays the role of the inner path, and the most recently created vertex before that run (which has not yet been "cut off" by a later turn) plays the role of the apex a_i. The neighbourhood of the appropriate boundary vertex, v_1 or v_4 depending on the direction pattern, contains the realised $H_{n,k,i}$ as an induced subgraph, and the rest follows from Lemma 3.1.8 and Theorem 2.2.7. We omit the routine identification of the specific vertices in each case.

Theorem 3.1.24. *The Apollonian graph $A(m^{s_0}x_1m^{s_1}x_2m^{s_2}x_1m^{s_3})$, with $x_1, x_2 \in \{\ell, r\}$, $x_1 \neq x_2$, and $s_0 \geq 3$, is word-representable if and only if both s_1 and s_2 are odd.*

Proof. Forward direction. If s_1 is even, the graph $H_{n,k,i}$ of Lemma 3.1.8 appears as an induced subgraph of $N(v_1)$ (see Remark 3.1.23), so by Theorem 2.2.7, G is non-word-representable. If s_2 is even, an analogous obstruction appears in another neighbourhood, again by Remark 3.1.23.

Backward direction. Suppose both s_1 and s_2 are odd. By symmetry, $x_1 = r$. We consider two cases according to the parity of s_0, applying Lemma 3.1.1 in each.

Case 1. s_0 is odd. Assign colours $1, 2, 3, 4$ to v_2, v_4, v_3, and v_1 respectively. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 2 and 3 starting with colour 3 and ending with colour 2, and the vertices $v_{s_0+4}, v_{s_0+5}, \ldots, v_{s_0+s_1+5}$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 2. Vertex $v_{s_0+s_1+6}$ is coloured 3, and the vertices $v_{s_0+s_1+7}, \ldots, v_{s_0+s_1+s_2+5}$ alternate between colours 2 and 3 starting and ending with colour 2. The vertices from $v_{s_0+s_1+s_2+6}$ onward alternate between colours 1 and 2 starting with colour 1 and ending with colour 1 or 2 depending on the parity of s_3.

The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_2 \to v_4 \to v_3 \to v_1$, and $v_2 \to v_{s_0+3} \to v_{s_0+2} \to v_1$;
- $v_2 \to v_{2k+4} \to v_{2k+5} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-3}{2}$;
- $v_2 \to v_{2k+4} \to v_{2k+3} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-3}{2}$;
- $v_{s_0+2k+4} \to v_{s_0+2k+3} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+2k+4} \to v_{s_0+2k+5} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \ldots, \frac{s_1-1}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+5} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2-1}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+6} \to v_{s_0+s_1+2k+5} \to v_1$ for $k = 0, 1, \ldots, \frac{s_2-3}{2}$;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+s_2+2k+5} \to v_{s_0+s_1+s_2+4} \to v_1$ for $k = 0, 1, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 0, 1, \ldots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+s_2+2k+7} \to v_{s_0+s_1+s_2+4} \to v_1$ for $k = 1, 2, \ldots, \frac{s_3}{2}$ if s_3 is even, and $k = 1, 2, \ldots, \frac{s_3-1}{2}$ if s_3 is odd.

For each such path the supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive.

Case 2. s_0 is even. Assign colours $1, 2, 3, 4$ to v_2, v_3, v_4, and v_1 respectively. Then $v_5, v_6, \ldots, v_{s_0+3}$ alternate between colours 2 and 3 starting and ending with colour 2, and the vertices $v_{s_0+4}, v_{s_0+5}, \ldots, v_{s_0+s_1+5}$ alternate between colours 1 and 2 starting with colour 1 and ending with colour 2. Vertex $v_{s_0+s_1+6}$ is coloured 3, and the vertices $v_{s_0+s_1+7}, \ldots, v_{s_0+s_1+s_2+5}$ alternate between colours 2 and 3 starting and ending with colour 2. The vertices from $v_{s_0+s_1+s_2+6}$ onward alternate between colours 1 and 2 starting with colour 1 and ending with colour 1 or 2 depending on the parity of s_3.

The directed paths of colour pattern $1 \to 2 \to 3 \to 4$ are:

- $v_2 \to v_3 \to v_4 \to v_1$, and $v_2 \to v_{s_0+3} \to v_{s_0+2} \to v_1$;

- $v_2 \to v_{2k+4} \to v_{2k+5} \to v_1$ for $k = 0, 1, \dots, \frac{s_1-3}{2}$;
- $v_2 \to v_{2k+4} \to v_{2k+3} \to v_1$ for $k = 0, 1, \dots, \frac{s_1-3}{2}$;
- $v_{s_0+2k+4} \to v_{s_0+2k+3} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \dots, \frac{s_1-1}{2}$;
- $v_{s_0+2k+4} \to v_{s_0+2k+5} \to v_{s_0+2} \to v_1$ for $k = 0, 1, \dots, \frac{s_1-1}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+4} \to v_{s_0+s_1+2k+5} \to v_1$ for $k = 0, 1, \dots, \frac{s_2-1}{2}$;
- $v_{s_0+s_1+3} \to v_{s_0+s_1+2k+6} \to v_{s_0+s_1+2k+5} \to v_1$ for $k = 0, 1, \dots, \frac{s_2-3}{2}$;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+s_2+2k+5} \to v_{s_0+s_1+s_2+4} \to v_1$ for $k = 0, 1, \dots, \frac{s_3}{2}$ if s_3 is even, and $k = 0, 1, \dots, \frac{s_3-1}{2}$ if s_3 is odd;
- $v_{s_0+s_1+s_2+2k+6} \to v_{s_0+s_1+s_2+2k+7} \to v_{s_0+s_1+s_2+4} \to v_1$ for $k = 1, 2, \dots, \frac{s_3}{2}$ if s_3 is even, and $k = 1, 2, \dots, \frac{s_3-1}{2}$ if s_3 is odd.

For each such path the supporting edges are present in G. By Lemma 3.1.1, the orientation is semi-transitive. □

3.1.5 Synthesis

Combining Theorems 3.1.3–3.1.24 (and Lemma 2.4.5, which reduces the case of an initial direction symbol x_0 to the run-of-m case), we obtain the following.

Theorem 3.1.25 (Three-change classification, closed form)**.** *Let $G = A(m^{s_0}x_1m^{s_1}x_2m^{s_2}x_3m^{s_3})$ be an Apollonian graph in our subfamily, with $s_0 \geq 1$, $s_1, s_2, s_3 \geq 0$, and $x_1, x_2, x_3 \in \{\ell, r\}$. Set $\delta_i = 0$ if $x_i = x_{i+1}$ and $\delta_i = 1$ if $x_i \neq x_{i+1}$, for $i = 1, 2$. Define the set of* constrained indices

$$I = \begin{cases} \{1,2\}, & \text{if } s_0 \geq 3, \\ \{2\}, & \text{if } s_0 \leq 2. \end{cases}$$

Then G is word-representable if and only if

$$s_i \equiv \delta_i \pmod 2 \qquad \text{for every } i \in I.$$

In words: the parameter s_2 is always constrained, and must have the same parity as δ_2; the parameter s_1 is constrained in the same way precisely when the initial run satisfies $s_0 \geq 3$, and is unconstrained when $s_0 \leq 2$; and s_0 and s_3 are never constrained.

Proof. This is the combined statement of Theorems 3.1.3, 3.1.6, 3.1.10, 3.1.12, 3.1.18, 3.1.19, 3.1.20, 3.1.21, 3.1.22, and 3.1.24. The exemption of $s_0 \leq 2$ from the parity condition on s_1 reflects the special role of the initial m-block: a short initial run does not yet provide enough structure for the relevant obstructions to appear, as is also reflected in Theorem 3.1.6 ($s_0 = 2$) and Theorems 3.1.14, 3.1.15 ($s_0 = 1$). □

Theorem 3.1.26 (General classification)**.** *Let G be an Apollonian graph in our subfamily with p changes of direction, $p \in \{0, 1, 2, 3\}$. Then G is word-representable if and only if $s_i \equiv \delta_i \pmod 2$ holds for every constrained index i, where the constrained indices are those i with $1 \leq i \leq p-1$, excluding $i = 1$ in the case $s_0 \leq 2$. In particular, when $p \leq 1$ there are no constrained indices and every such graph is word-representable; and when $p = 2$ and $s_0 \leq 2$ the conditions are likewise vacuous.*

3.1.6 From semi-transitive orientations to a general principle

The classification above suggests a uniform pattern: *s_i is even when $x_i = x_{i+1}$, and odd when $x_i \neq x_{i+1}$.* This pattern is exactly the one stated in the four theorems for $s_0 \geq 3$: in Theorem 3.1.20, all changes are equal, so all parity conditions are "even"; in Theorems 3.1.21 and 3.1.22, one of the parity conditions flips because one of the changes is unequal; and in Theorem 3.1.24, both parity conditions are "odd" because both changes are unequal.

This uniform pattern strongly suggests that the classification extends naturally to graphs with more direction changes. Indeed, this turns out to be the case. However, rather than continuing to grind out individual classifications by case analysis on potential shortcuts, we adopt in Section 3.2 a quite different approach. There, we work with 2-uniform word-representations directly, build them inductively by maintaining a precise local invariant, and show that the same parity pattern characterises 2-word-representability for the entire family. As an immediate consequence, this also gives the desired classification of word-representability.

The motivation for this shift in approach was the discovery, after the work of this section was complete, of the work described in the next subsection.

3.1.7 The Hauser characterisation theorem

While completing the classifications above, we became aware of the recently completed PhD thesis of Hauser [21]. Hauser considers the larger class of *chordal near-triangulations*, planar graphs whose every internal face is a triangle, and which are chordal, and characterises the word-representable members of this class in terms of forbidden induced subgraphs. As a corollary, Hauser obtains a complete characterisation of word-representable Apollonian networks (in the sense of Definition 2.4.1, not only the change-of-direction subfamily considered here).

The relevant structural conditions are as follows.

Definition 3.1.27 (Family $\mathcal{F}_1$, cf. [21])**.** The family $\mathcal{F}_1$ consists of certain Apollonian-like ladder graphs that arise as forbidden induced subgraphs for word-representability. A representative member $F_1^{(m)}$ of $\mathcal{F}_1$, on $|V| = 2m$ vertices for $m \geq 4$, is depicted in Figure 3.5: there are two distinguished "apex" vertices, labelled 4 and 1 in Figure 3.5, and an inner path connecting them (the vertices labelled $5, 6, 7, 8$ in the figure) whose vertices alternate in adjacency between 4 and 1. The full family $\mathcal{F}_1$ is generated by varying the path length and certain optional cross-edges around the two ends of the path, as catalogued in detail in Hauser [21]; we shall not need the exact catalogue, only the fact that membership in $\mathcal{F}_1$ implies non-word-representability.

The smallest non-word-representable graphs in our change-of-direction subfamily, $A(m^3rmr)$, $A(m^2rmrmr)$, and so on, are members of $\mathcal{F}_1$ in Hauser's sense, as established by the human-verifiable proofs of Lemmas 3.1.11, 3.1.16, and 3.1.17.

Lemma 3.1.28 ([21])**.** *Every graph $G \in \mathcal{F}_1$ is non-word-representable.*

The second condition in Hauser's characterisation is the local one we have already encountered.

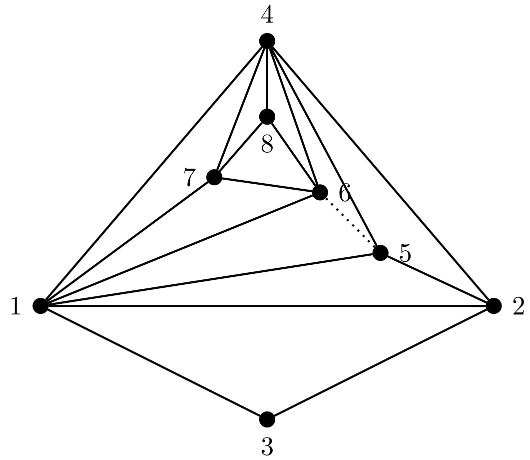


Figure 3.5: The structure of a typical member of $\mathcal{F}_1$ (cf. [21]). Vertices 1 and 4 are the two distinguished "apex" vertices, 2 and 3 are the lower outer-triangle vertices, and $5, 6, 7, 8$ form an inner path whose vertices alternate in adjacency between 1 and 4. Dotted edges indicate that more vertices, alternately attached to 4 and 1, may be inserted along the inner path; the family is parametrised by the order $|V| = 2m$, $m \geq 4$.

Definition 3.1.29. A graph G is *locally comparability* if for every vertex $v \in V(G)$, the induced subgraph on $N(v)$ is a comparability graph.

By Theorem 2.2.7, every word-representable graph is locally comparability. Hauser's main result is that, restricted to Apollonian networks, this necessary condition together with $\mathcal{F}_1$-freeness is also *sufficient*.

Theorem 3.1.30 ([21])**.** *An Apollonian network is word-representable if and only if it is $\mathcal{F}_1$-free and locally comparability.*

Definition 3.1.31. Let $\mathcal{F}_2$ be the family of graphs G with $|V(G)| = 2m - 1$, $m \geq 4$, depicted in Figure 3.6.

Lemma 3.1.32. *Every graph in $\mathcal{F}_2$ fails to be a comparability graph.*

In light of Theorem 3.1.30, our classification of Section 3.1 can be viewed as exhibiting concrete realisations of the Hauser obstructions in the change-of-direction subfamily. The graphs in $\mathcal{F}_2$ are exactly the non-comparability subgraphs that appear in our induced neighbourhoods in the proofs of Theorems 3.1.10, 3.1.22, 3.1.24, and so on when the parity condition fails. Specifically:

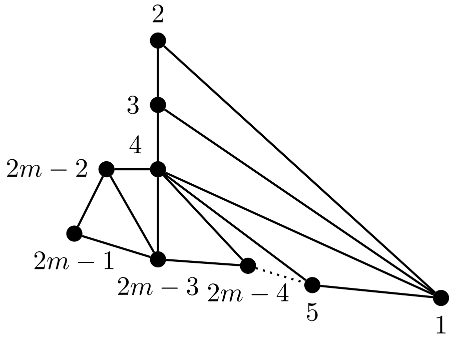


Figure 3.6: The structure of graphs in the family $\mathcal{F}_2$. When the vertex set $\{5, 6, \ldots, 2m-4\}$ is empty, vertex $2m-3$ is adjacent directly to vertex 1.

- The non-comparability graph $H_{n,k,i}$ identified in Lemma 3.1.8 is the canonical realisation of $\mathcal{F}_2$ that appears throughout our analysis.
- Every non-comparability obstruction we identified (in Theorems 3.1.10, 3.1.21, 3.1.22, 3.1.24, and 3.1.19) is a copy of $H_{n,k,i}$, hence a member of $\mathcal{F}_2$.
- The minimal non-word-representable graphs we identified in Lemmas 3.1.11, 3.1.16, and 3.1.17 are members of $\mathcal{F}_1$.

So a unified perspective is now available: Hauser's $\mathcal{F}_1$ and $\mathcal{F}_2$ are the two families of obstructions, and the parity conditions of Section 3.1 are exactly the conditions under which these obstructions appear or are avoided in the change-of-direction subfamily.

This perspective leads naturally to the second phase of our work, presented in the next section, where we go further: we show that within the change-of-direction subfamily, when the obstructions $\mathcal{F}_1$ and $\mathcal{F}_2$ are absent, the resulting graph admits not just any word-representation but a 2-uniform one, i.e. it is a circle graph.

3.2 From semi-transitive orientations to circle graphs: 2-word-representability

The vertex labelling in this section is the bare-numbers convention announced at the start of the chapter: vertex 3 at the top of the outer triangle, vertex 2 at the bottom-right, vertex 1 at the bottom-left, and the first interior vertex is 4.

3.2.1 Aim and outline

The classification of Section 3.1 establishes, for each Apollonian graph in the change-of-direction subfamily *with at most three changes of direction*, exactly when it is word-representable. (The results of Section 3.1 are stated and proved for $p \leq 3$ changes of direction; graphs in the subfamily with $p \geq 4$ are not covered by that classification.) In each representable case, the proof gives an explicit semi-transitive orientation; that is, a 0-uniform property of the graph (the existence of an orientation), but not directly a word-representant.

Theorem 2.2.6 states that the class of 2-uniformly word-representable graphs (graphs admitting a representation in which each letter appears exactly twice) coincides with the class of *circle graphs*: intersection graphs of chords of a circle. So the question

> *Is every word-representable Apollonian graph in our subfamily* 2*-uniformly representable?*

is equivalent to asking whether they are circle graphs. This question is non-trivial: in general, the representation number $\rho(G)$ can be much larger than 2, and there are word-representable graphs that are not circle graphs (e.g. K_4 with a pendant, see [30, Example 5.4.4]). On the other hand, all the small representable Apollonian graphs we encountered in Section 3.1 can be checked by hand to be circle graphs.

In this section, we answer the question affirmatively. The main result is the following.

Theorem 3.2.1 (Main result)**.** *An Apollonian graph G in the change-of-direction subfamily, encoded as $A(m^{s_0}x_1m^{s_1}x_2m^{s_2}\cdots x_pm^{s_p})$ with $s_0 \geq 1$, $p \geq 0$, $s_i \geq 0$, is word-representable if and only if it is a circle graph.*

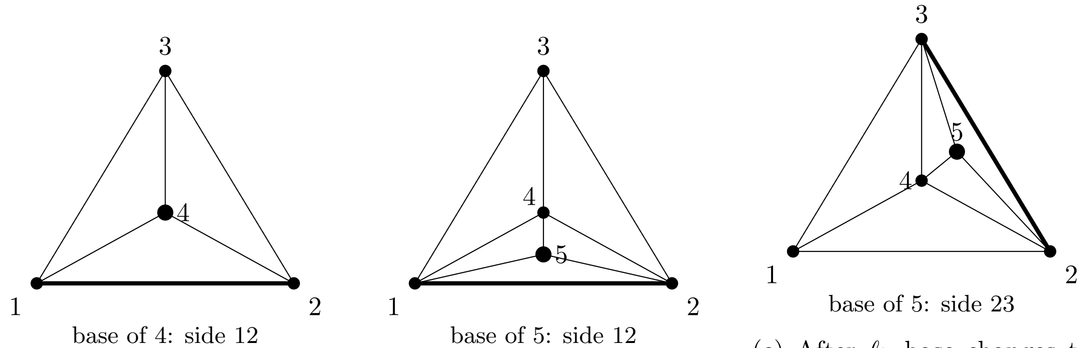
The proof of Theorem 3.2.1 is constructive in both directions. The forward implication, representable $\Rightarrow$ circle graph, is established by exhibiting an explicit 2-uniform word-representation of G. The reverse implication, circle graph $\Rightarrow$ representable, is immediate, since circle graphs are word-representable [30].

3.2.2 The base of a subdivision and the extension invariant

To carry out the construction, we introduce a small piece of structural language.

Definition 3.2.2 (Base of a subdivision)**.** The *base* for the subdivision when vertex i is created is the side of the subdivided triangle opposite to vertex $i-1$. (Recall that, by the construction in Section 2.4, each new vertex is created in a triangle one of whose vertices is the most recently created vertex; so the "opposite side" is well-defined.) For example, in the graph in Figure 2.7 of Section 2.4, the base of vertex 4 is 12, and the base of vertex 5 is 23. Note that during m-steps, the base remains unchanged from the last change of direction in $\{r, \ell\}$.

Figure 3.7 illustrates the rule: the base is fixed at creation, preserved by m-steps, and changes after each turn.



(a) Vertex 4 subdivides $\triangle 123$. (b) After m: base unchanged. (c) After ℓ: base changes to 23.

Figure 3.7: The base of a subdivision (Definition 3.2.2). The base is the side of the subdivided triangle opposite to the previously created vertex (shown in bold). During an m-step (centre), the base is inherited from the previous step; during a turn (right), the base changes accordingly.

The key device of the construction is the following.

Definition 3.2.3 (Extension invariant)**.** We say that a 2-uniform word w representing an Apollonian graph satisfies the *extension invariant* with base ab and newest vertex n if w contains a factor of one of the two forms

$$n(n{-}1)ab\,n \quad \text{(type I)} \qquad \text{or} \qquad n\,ab(n{-}1)n \quad \text{(type II)},$$

where ab is the base of the triangle subdivided in the step that created n, and $n-1$ is the vertex created immediately before n.

The structure of the two factor types is shown in Figure 3.8.

Example 3.2.4. Consider the smallest non-trivial Apollonian graph $A(m) = K_4$, on the vertices $1, 2, 3$ of the initial triangle together with the vertex 4 inserted into it. The 2-uniform word

$$w = 3\,1\,2\,4\,3\,1\,2\,4$$

represents K_4: each of the four letters occurs exactly twice, and every pair of letters alternates, which is what is required since K_4 is complete. This word satisfies the extension invariant with base 12 and newest vertex 4: reading the underlined factor

$$w = 3\,\underline{4\,3\,1\,2\,4}\,\ldots$$

contains $4\,3\,1\,2\,4$, which is of type I, namely $n(n{-}1)ab\,n$ with $n = 4$, $n-1 = 3$ (the vertex created immediately before 4), and base $ab = 12$ (the side of the triangle $\{1, 2, 3\}$ opposite the previous vertex 3, in the sense of Definition 3.2.2).

Applying an m-step to this word, as in Lemma 3.2.5(a), rewrites the factor $4\,3\,1\,2\,4 \mapsto 4\,3\,5\,1\,2\,4\,5$ and yields

$$w' = 3\,1\,2\,4\,3\,5\,1\,2\,4\,5,$$

a 2-uniform word representing $A(m^2)$ (in which the new vertex 5 is adjacent to $1, 2, 4$ but not to 3). The word w' contains the factor $5\,1\,2\,4\,5$, which is of type II with base 12 and newest vertex 5, illustrating that an m-step reverses the invariant type while preserving the base.

Lemma 3.2.5 (Step lemma)**.** *Let w be a 2-uniform word representing an Apollonian graph G and satisfying the extension invariant with base ab and newest vertex n.*

(a) m***-step.*** *If the next subdivision is an m-step, then the resulting graph admits a 2-uniform word w' satisfying the extension invariant with the same base ab and newest vertex $n{+}1$. The type of w' is the opposite of that of w (an m-step* reverses *the type).*

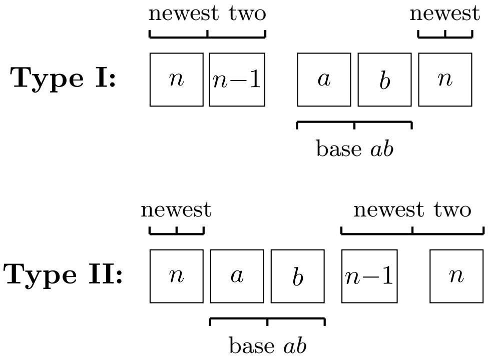


Figure 3.8: The two types of factor admitted by the *extension invariant* (Definition 3.2.3). In both cases, n is the newest vertex, $n-1$ the vertex created immediately before, and ab is the base of the triangle subdivided to create n. The two occurrences of n bracket the base in the type I factor and the previous newest vertex in the type II factor.

(b) ***Turn step.*** *Suppose the next subdivision is a turn. Of the two turn directions, exactly one is* admissible*, in the following sense: the construction requires the letters of the new base and the previously added vertex to appear consecutively in w, and the type of w determines which of r, ℓ satisfies this. Specifically, a right turn is admissible when w has type I and a left turn is admissible when w has type II.*

If the turn taken is the admissible one, then the base changes from ab to $a(n-1)$ (right turn, type I) or to $(n-1)b$ (left turn, type II), and the resulting graph admits a 2-uniform word w' satisfying the extension invariant with the new base and newest vertex $n+1$; the type of w' is the same as that of w, so a turn preserves *the type.*

If the turn taken is the inadmissible one, the rewriting rule does not apply and the construction does not directly yield a word for the resulting graph; such a step must instead be reached by first applying an m-step, which reverses the type by part (a) and thereby makes the desired turn admissible.

Proof. We verify both parts by examining the replacement rules at the level of factors of the word.

For an m-step, suppose w has type I, so it contains the factor $n(n-1)ab\,n$. An

m-step inserts a new vertex $n+1$ into the triangle formed by $n-1$, a, and b. We replace the factor by

$$n(n-1)ab\,n \longmapsto n(n-1)(n+1)ab\,n(n+1).$$

The result contains the factor $(n+1)ab\,n(n+1)$, which is a type II invariant with base ab and newest vertex $n+1$. One checks directly that $n+1$ alternates with each of its three neighbours $n-1$, a, and b (the two occurrences of each letter pair are interleaved), and does not alternate with any non-neighbour.

If w has type II, the factor $n\,ab(n-1)n$ is replaced by

$$n\,ab(n-1)n \longmapsto (n+1)n\,ab(n+1)(n-1)n,$$

producing a type I invariant $(n+1)n\,ab(n+1)$. The verification is analogous.

For a turn step, suppose w has type I, so it contains the factor $n(n-1)ab\,n$. A right turn creates a new vertex $n+1$ in the right triangle, changing the base from ab to $a(n-1)$. We replace

$$n(n-1)a \longmapsto (n+1)n(n-1)a(n+1),$$

producing a word w' containing the factor $(n+1)n(n-1)a(n+1)$, which is a type I invariant with base $a(n-1)$ and newest vertex $n+1$. The replacement is valid because the letters $(n-1)$ and a already appear consecutively in w (within the factor $n(n-1)ab\,n$). A left turn, however, would require $(n-1)$ and b to be consecutive in w; since they are separated by a, a left turn is not admissible from a type I invariant.

If w has type II, the factor is $n\,ab(n-1)n$, and now a left turn is admissible (since b and $n-1$ are consecutive) while a right turn is not (since a and $n-1$ are separated by b). The replacement for a left turn is analogous and produces a type II invariant with the new base.

Global alternation check. We must also verify that the rewritten word w' continues to alternate correctly for every pair of letters not involving the new letter $n+1$. In every rewrite rule above, the two new occurrences of $n+1$ are inserted at positions in w' that are either adjacent or flank a small constant-length factor of w. Consider any pair of letters $\{p, q\} \subseteq V(w)$ with $p, q \neq n+1$. Each of p and q occurs exactly twice in w, hence exactly twice in w', and the insertion of $n+1$ adds two new positions to the word but does not delete or move existing positions. So the relative order of the four occurrences of p and q in w' is the same as in w, meaning that p and q alternate in w' if and only if they alternate in w. The adjacency of p and q is therefore preserved.

The remaining case is $p = n+1$ paired with some $q \neq n+1$: this is exactly the local verification performed above for each of the three neighbours $n-1, a, b$ of $n+1$ and the non-neighbour check for letters outside the local factor.

This establishes that w' is a 2-uniform representation of the full graph after the subdivision step, completing the proof. □

The four replacement rules are summarised in Figure 3.9. Reading them off the figure makes plain a fact that the proof above states only in passing, but which drives the main classification theorem of this section: *m-steps reverse the invariant type, while turn steps preserve it.* This is the structural reason why the parity conditions of Section 3.1 appear so naturally.

3.2.3 2-word-representability for graphs with at most one direction change

Theorem 3.2.6. *Every Apollonian graph $G = A(m^{s_0} x m^{s_1})$ with $x \in \{\ell, r\}$, $s_0 \geq 1$, $s_1 \geq 0$ is 2-word-representable (equivalently, a circle graph).*

Proof. We construct a 2-uniform representation by maintaining the extension invariant.

The graph $A(m)$ is isomorphic to the complete graph K_4 on vertices $\{1, 2, 3, 4\}$. We represent it by the word

$$w_1 = 31243124,$$

which has base 12 and newest vertex 4. The factor 43124 is of type I, namely $n(n-1)ab\,n$ with $n = 4$, $a = 1$, $b = 2$. So w_1 satisfies the extension invariant of type I with base 12.

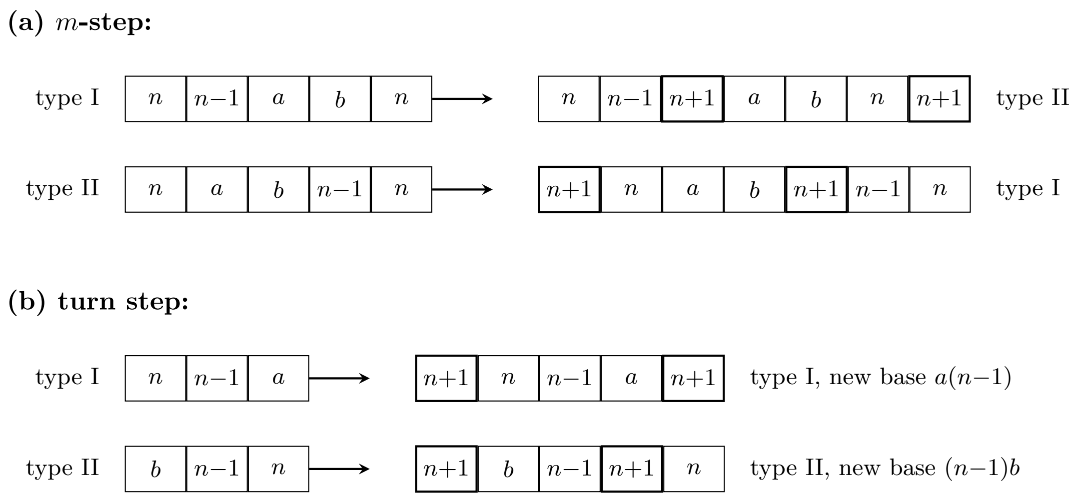


Figure 3.9: The replacement rules of Lemma 3.2.5 (Step Lemma) for the extension invariant. New letter occurrences inserted by each step are drawn in bold. Part (a): an m-step inserts the new vertex $n+1$ at two positions flanking either the base ab (from type I) or the pair $n(n-1)$ (from type II); in both cases the resulting factor has the *opposite* type. Part (b): a turn step inserts $n+1$ flanking the part of the factor adjacent to the new base; the resulting factor has the *same* type, and only one of the two turn directions is admissible from a given type. Only the locally rewritten factor of the word is shown; the rest of the word is unchanged.

By Lemma 2.4.5, $A(mr) \cong A(m\ell) \cong A(m^2)$. Applying the m-step replacement of Lemma 3.2.5(a) to w_1, we obtain

$$w_2 = 31243\,\mathbf{5}1224\,\mathbf{5},$$

which represents $A(m^2)$, with the invariant now of type II, base 12, newest vertex 5.

By induction, repeated m-steps produce a 2-uniform word w_{s_0} representing $A(m^{s_0})$, satisfying the invariant with base 12, newest vertex s_0+3, and type alternating with each step (type I for s_0 odd, type II for s_0 even).

Without loss of generality assume s_0 is odd (the even case is analogous, with the roles of 1 and 2 swapped) and $x = r$ (by symmetry). Then the invariant has type I, and by Lemma 3.2.5(b), a right turn is admissible, producing a word w_{s_0+1} satisfying the invariant with new base $(s_0+2)\,1$ and newest vertex s_0+4, still of type I.

The remaining s_1 subdivisions are m-steps with base $(s_0+2)\,1$. By repeated application of Lemma 3.2.5(a), each step maintains the invariant, switching its type, and increments the newest vertex. After all s_1 steps, we obtain a 2-uniform word $w_{s_0+s_1+1}$ representing $A(m^{s_0}rm^{s_1})$.

The word satisfies the invariant with base $(s_0+2)\,1$, type I if s_1 is even (i.e. even total number of m-steps after the turn), and type II if s_1 is odd. This completes the construction. $\square$

Corollary 3.2.7. *Every Apollonian graph $G = A(m^{s_0})$, $s_0 \geq 1$, is 2-word-representable.*

Proof. Run the construction in the proof of Theorem 3.2.6 and stop immediately before the direction change is applied, that is, apply only the s_0 initial m-steps. The construction is well defined at that stage and produces a 2-uniform word representing $A(m^{s_0})$, together with its extension invariant, which is all that is required here. $\square$

Figure 3.10 traces the construction for the small case $A(m^4)$, showing how the 2-uniform word grows step by step under repeated m-steps, and how the extension invariant alternates between type I and type II at each step. Table 3.1 collects the resulting representations for the Apollonian graphs in our subfamily on up to eight vertices.

$A(m)$	: $w_1 = 3\,1\,2\,4\,3\,1\,2\,4$	*(type I, base* 12, *newest* 4*)*
$A(m^2)$	: $w_2 = 3\,1\,2\,4\,3\,\mathbf{5}\,1\,2\,4\,\mathbf{5}$	*(type II, base* 12, *newest* 5*)*
$A(m^3)$	: $w_3 = 3\,1\,2\,4\,3\,\mathbf{6}\,5\,1\,2\,\mathbf{6}\,4\,5$	*(type I, base* 12, *newest* 6*)*
$A(m^4)$	: $w_4 = 3\,1\,2\,4\,3\,6\,5\,\mathbf{7}\,1\,2\,6\,\mathbf{7}\,4\,5$	*(type II, base* 12, *newest* 7*)*

Figure 3.10: Step-by-step construction of a 2-uniform word representing $A(m^4)$ on seven vertices, following the proof of Theorem 3.2.6. At each step, the newly inserted pair of letters is shown in bold. The base 12 is preserved by every m-step; the invariant alternates between types I and II at each step.

graph	$\lvert V\rvert$	a 2-uniform representation
$A()$	3	1 2 3 1 2 3
$A(m)$	4	3 1 2 4 3 1 2 4
$A(m^2)$	5	3 1 2 4 3 5 1 2 4 5
$A(m^3) \cong A(m^2r)$	6	3 1 2 4 3 6 5 1 2 6 4 5
$A(m^4) \cong A(m^2rm)$	7	3 1 2 4 3 6 5 7 1 2 6 7 4 5
$A(m^5) \cong A(m^2rm^2)$	8	3 1 2 4 3 6 5 8 7 1 2 8 6 7 4 5

Table 3.1: 2-uniform word-representations of the small Apollonian graphs in the change-of-direction subfamily, on up to eight vertices. Each row is obtained from the previous by a single application of Lemma 3.2.5(a) (m-step). Where the graph admits two encodings in our subfamily, both are shown; each isomorphism is verified by exhibiting an explicit vertex bijection between the two constructions (see Observation 3.2.8 for the structural picture). The two-turn graphs $A(m^2rm^{s_1}x_2m^{s_2})$ start at nine vertices and are handled separately in Theorem 3.2.9.

Observation 3.2.8 (Isomorphism classes for small vertex counts)**.** Let $\mathcal{A}_n$ denote the set of Apollonian graphs in the change-of-direction subfamily on n vertices, taken up to isomorphism. Then:

(i) For $n \in \{3, 4, 5, 6\}$, the set $\mathcal{A}_n$ has exactly one element. Every encoding of length $n - 3$ produces the same graph up to isomorphism.

(ii) For $n = 7$, the set $\mathcal{A}_7$ has exactly two elements, distinguished by the last symbol of the encoding:

- the nine encodings of length 4 ending in m form one isomorphism class, with degree sequence $(3, 3, 4, 4, 4, 6, 6)$;
- the eighteen encodings of length 4 ending in ℓ or r form the other, with degree sequence $(3, 3, 4, 4, 5, 5, 6)$.

For $n \geq 8$ the subfamily produces strictly more than two isomorphism classes; numerical computation gives $\lvert\mathcal{A}_8\rvert = 4$ and $\lvert\mathcal{A}_9\rvert = 10$.

Proof. Part (i). For $n = 3$ the encoding is empty and the graph is K_3, the outer triangle. For $n = 4$ the only encoding is m, giving K_4.

For $n = 5$, the three encodings $mm, m\ell, mr$ all give isomorphic graphs by Lemma 2.4.5: $A(mr) \cong A(m\ell) \cong A(m^2)$. So $\lvert\mathcal{A}_5\rvert = 1$.

For $n = 6$, there are nine length-3 encodings. By Lemma 2.4.5 applied to each encoding whose second symbol is a turn:

- $A(m\ell m), A(mrm) \cong A(m^3)$;
- $A(m\ell\ell), A(mr\ell) \cong A(m^2\ell)$;
- $A(m\ell r), A(mrr) \cong A(m^2 r)$.

The remaining three encodings mmm, $mm\ell$, mmr are not reducible by the lemma. So Lemma 2.4.5 alone leaves three potential reductions: $A(m^3)$, $A(m^2\ell)$, $A(m^2r)$. The left-right reflection of the outer triangle (swap $v_1 \leftrightarrow v_2$, $\ell \leftrightarrow r$) is a graph isomorphism, so $A(m^2\ell) \cong A(m^2r)$. It remains to show $A(m^3) \cong A(m^2r)$, which is not covered by Lemma 2.4.5 since the encoding m^2r does not start with $m \cdot x_0 \cdots$ for $x_0 \in \{\ell, r\}$.

We exhibit the isomorphism explicitly. The edge sets are

$$
\begin{aligned}
E(A(m^3)) &= \{12, 13, 23, 14, 24, 34, 15, 25, 45, 16, 26, 56\}, \\
E(A(m^2r)) &= \{12, 13, 23, 14, 24, 34, 15, 25, 45, 16, 46, 56\},
\end{aligned}
$$

differing only in $\{2, 6\}$ versus $\{4, 6\}$. The transposition $\varphi : 2 \leftrightarrow 4$ (fixing $1, 3, 5, 6$) sends every edge of $A(m^2r)$ to an edge of $A(m^3)$, as one verifies on the twelve edges. Hence $A(m^3) \cong A(m^2r)$, and $|\mathcal{A}_6| = 1$.

Part (ii). For $n = 7$, the 27 length-4 encodings partition into Group M (last symbol m, nine encodings) and Group T (last symbol a turn, eighteen encodings). A vertex-by-vertex degree computation shows:

- For every encoding in M, the resulting graph has degree sequence $(3, 3, 4, 4, 4, 6, 6)$. The bottom vertices v_1 and v_2 are both adjacent to every interior vertex created during the entire run, since the base v_1v_2 is preserved by the final m-step regardless of the preceding turns; hence both have degree 6. The final m-vertex has degree 3.
- For every encoding in T, the resulting graph has degree sequence $(3, 3, 4, 4, 5, 5, 6)$. The final turn moves the seventh vertex off the base v_1v_2, so exactly one of v_1, v_2

loses an adjacency (dropping from degree 6 to 5), and a new vertex with the previously-newest neighbour configuration appears with degree 5 instead of 6.

Since these degree sequences differ, no graph in Group M is isomorphic to any graph in Group T. So $|\mathcal{A}_7| \geq 2$.

For equality, we show that within each group the graphs are pairwise isomorphic. The argument has three parts: a Structural Fact about the early K_4, a direct enumeration of the parent triangles in each step, and case analyses for Group M and Group T.

Structural Fact. In every graph $A(w)$ built from an encoding w in our subfamily (of any length ≥ 1), the first four vertices $\{v_1, v_2, v_3, v_4\}$ induce a copy of K_4.

Proof of the Structural Fact. The outer triangle gives the three edges v_1v_2, v_1v_3, v_2v_3. The first step of the encoding is always m (by convention $s_0 \geq 1$), creating v_4 inside the outer triangle $\{v_1, v_2, v_3\}$, and hence adjacent to all three. So $\{v_1, v_2, v_3, v_4\}$ has all six edges and forms a K_4. $\square$

For each length-4 encoding $w = x_1x_2x_3x_4$, the construction rules of Section 2.4 determine the parent triangle subdivided at each step, and thus the adjacencies of each new vertex. We catalogue this data below for each group. Recall (Definition 3.2.2) that the *base* b_i of step i is the side of the parent triangle opposite to the previously-newest vertex; the new vertex v_{i+3} created at step i is adjacent to both endpoints of b_i and to the previously-newest vertex.

Within Group M. Let $w = m\,x_2\,x_3\,m$ be a Group M encoding with $x_2, x_3 \in \{m, \ell, r\}$. Applying the construction rules to each of the nine encodings yields the following bases and adjacencies for the late vertices:

w	b_2	b_3	nbrs of v_5	nbrs of v_6	nbrs of v_7
$mmmm$	$\{v_1, v_2\}$	$\{v_1, v_2\}$	$\{v_1, v_2, v_4\}$	$\{v_1, v_2, v_5\}$	$\{v_1, v_2, v_6\}$
$mm\ell m$	$\{v_1, v_2\}$	$\{v_2, v_4\}$	$\{v_1, v_2, v_4\}$	$\{v_2, v_4, v_5\}$	$\{v_2, v_4, v_6\}$
$mmrm$	$\{v_1, v_2\}$	$\{v_1, v_4\}$	$\{v_1, v_2, v_4\}$	$\{v_1, v_4, v_5\}$	$\{v_1, v_4, v_6\}$
$m\ell mm$	$\{v_2, v_3\}$	$\{v_2, v_3\}$	$\{v_2, v_3, v_4\}$	$\{v_2, v_3, v_5\}$	$\{v_2, v_3, v_6\}$
$m\ell\ell m$	$\{v_2, v_3\}$	$\{v_3, v_4\}$	$\{v_2, v_3, v_4\}$	$\{v_3, v_4, v_5\}$	$\{v_3, v_4, v_6\}$
$m\ell rm$	$\{v_2, v_3\}$	$\{v_2, v_4\}$	$\{v_2, v_3, v_4\}$	$\{v_2, v_4, v_5\}$	$\{v_2, v_4, v_6\}$
$mrmm$	$\{v_1, v_3\}$	$\{v_1, v_3\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_5\}$	$\{v_1, v_3, v_6\}$
$mr\ell m$	$\{v_1, v_3\}$	$\{v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_3, v_4, v_5\}$	$\{v_3, v_4, v_6\}$
$mrrm$	$\{v_1, v_3\}$	$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_4, v_5\}$	$\{v_1, v_4, v_6\}$

From the table we read off the structure of $A(w)$:

- v_5 is adjacent to three early vertices, which form a triangle in the K_4. The fourth early vertex, the *cut-off* c, has no neighbour in $\{v_5, v_6, v_7\}$. The cut-off depends on x_2: $c = v_3$ if $x_2 = m$, $c = v_1$ if $x_2 = \ell$, $c = v_2$ if $x_2 = r$.
- v_6 is adjacent to two of the three active early vertices (namely both endpoints of b_2), plus v_5.
- v_7 is adjacent to two of the three active early vertices (namely both endpoints of b_3), plus v_6.

Both b_2 and b_3 are 2-subsets of the active triple $\{v_1, v_2, v_3, v_4\} \setminus \{c\}$. The encoding determines the triple (c, b_2, b_3), but the abstract structure is the same: a K_4 on the early vertices with a $v_5v_6v_7$ path attached so that v_5 hits three early vertices (the active triple), v_6 hits a 2-subset b_2 of those three, and v_7 hits another 2-subset b_3.

Let w and w' be two Group M encodings with data (c, b_2, b_3) and (c', b_2', b_3'). Choose a bijection $\varphi : \{v_1, v_2, v_3, v_4\} \to \{v_1, v_2, v_3, v_4\}$ with $\varphi(c) = c'$, $\varphi(b_2) = b_2'$ (as a set), and $\varphi(b_3) = b_3'$ (as a set). The active triple maps to the active triple, and since b_2, b_3 are 2-subsets of a 3-element set, any pair of pairs can be matched to any other pair of pairs by a permutation. Extending φ by the identity on $\{v_5, v_6, v_7\}$ gives a graph isomorphism: the early K_4 is preserved by any permutation of its vertices; $v_5v_6 \in E$

and $v_6v_7 \in E$ are fixed; the cross-edges are preserved by φ's definition. So all nine Group M graphs are pairwise isomorphic.

Within Group T. Let $w = m\, x_2\, x_3\, x$ with $x \in \{\ell, r\}$. The eighteen Group T encodings yield the following bases and adjacencies:

w	b_2	b_3	nbrs of v_5	nbrs of v_6	nbrs of v_7
$mmm\ell$	$\{v_1, v_2\}$	$\{v_1, v_2\}$	$\{v_1, v_2, v_4\}$	$\{v_1, v_2, v_5\}$	$\{v_2, v_5, v_6\}$
$mmmr$	$\{v_1, v_2\}$	$\{v_1, v_2\}$	$\{v_1, v_2, v_4\}$	$\{v_1, v_2, v_5\}$	$\{v_1, v_5, v_6\}$
$mm\ell\ell$	$\{v_1, v_2\}$	$\{v_2, v_4\}$	$\{v_1, v_2, v_4\}$	$\{v_2, v_4, v_5\}$	$\{v_4, v_5, v_6\}$
$mm\ell r$	$\{v_1, v_2\}$	$\{v_2, v_4\}$	$\{v_1, v_2, v_4\}$	$\{v_2, v_4, v_5\}$	$\{v_2, v_5, v_6\}$
$mmr\ell$	$\{v_1, v_2\}$	$\{v_1, v_4\}$	$\{v_1, v_2, v_4\}$	$\{v_1, v_4, v_5\}$	$\{v_4, v_5, v_6\}$
$mmrr$	$\{v_1, v_2\}$	$\{v_1, v_4\}$	$\{v_1, v_2, v_4\}$	$\{v_1, v_4, v_5\}$	$\{v_1, v_5, v_6\}$
$m\ell m\ell$	$\{v_2, v_3\}$	$\{v_2, v_3\}$	$\{v_2, v_3, v_4\}$	$\{v_2, v_3, v_5\}$	$\{v_3, v_5, v_6\}$
$m\ell mr$	$\{v_2, v_3\}$	$\{v_2, v_3\}$	$\{v_2, v_3, v_4\}$	$\{v_2, v_3, v_5\}$	$\{v_2, v_5, v_6\}$
$m\ell\ell\ell$	$\{v_2, v_3\}$	$\{v_3, v_4\}$	$\{v_2, v_3, v_4\}$	$\{v_3, v_4, v_5\}$	$\{v_4, v_5, v_6\}$
$m\ell\ell r$	$\{v_2, v_3\}$	$\{v_3, v_4\}$	$\{v_2, v_3, v_4\}$	$\{v_3, v_4, v_5\}$	$\{v_3, v_5, v_6\}$
$m\ell r\ell$	$\{v_2, v_3\}$	$\{v_2, v_4\}$	$\{v_2, v_3, v_4\}$	$\{v_2, v_4, v_5\}$	$\{v_4, v_5, v_6\}$
$m\ell rr$	$\{v_2, v_3\}$	$\{v_2, v_4\}$	$\{v_2, v_3, v_4\}$	$\{v_2, v_4, v_5\}$	$\{v_2, v_5, v_6\}$
$mrm\ell$	$\{v_1, v_3\}$	$\{v_1, v_3\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_5\}$	$\{v_3, v_5, v_6\}$
$mrmr$	$\{v_1, v_3\}$	$\{v_1, v_3\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_3, v_5\}$	$\{v_1, v_5, v_6\}$
$mr\ell\ell$	$\{v_1, v_3\}$	$\{v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_3, v_4, v_5\}$	$\{v_4, v_5, v_6\}$
$mr\ell r$	$\{v_1, v_3\}$	$\{v_3, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_3, v_4, v_5\}$	$\{v_3, v_5, v_6\}$
$mrr\ell$	$\{v_1, v_3\}$	$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_4, v_5\}$	$\{v_4, v_5, v_6\}$
$mrrr$	$\{v_1, v_3\}$	$\{v_1, v_4\}$	$\{v_1, v_3, v_4\}$	$\{v_1, v_4, v_5\}$	$\{v_1, v_5, v_6\}$

From the table we read off the structure of $A(w)$:

- As in Group M, v_5 is adjacent to three early vertices (the active triple $\{v_1, v_2, v_3, v_4\} \setminus \{c\}$), determined by x_2.
- v_6 is adjacent to two active early vertices (the endpoints of b_2), plus v_5.

- v_7 is adjacent to one active early vertex (call it r, the attachment), plus v_5, v_6. The attachment r is one of the two endpoints of b_3, the specific choice determined by x_4.

The encoding determines the four-tuple (c, b_2, b_3, r); the table shows all eighteen instances. As in Group M, b_2, b_3 are 2-subsets of the active triple, and $r \in b_3$. Any two four-tuples can be related by a permutation φ of $\{v_1, v_2, v_3, v_4\}$ that sends $c \mapsto c'$, $b_2 \mapsto b_2'$, $b_3 \mapsto b_3'$, and $r \mapsto r'$. Extending φ by the identity on $\{v_5, v_6, v_7\}$ gives a graph isomorphism by the same verification as in Group M. So all eighteen Group T graphs are pairwise isomorphic.

This completes the proof that $|\mathcal{A}_7| = 2$. □

3.2.4 Two direction changes with $s_0 = 2$

The case $s_0 = 2$ is special and requires a slightly different construction, because the prefix $m^{s_0} = m^2$ does not yet provide enough room to apply Lemma 3.2.10 or its analogues used later for non-representability.

Theorem 3.2.9. *Every Apollonian graph $A(m^2 x_1 m^{s_1} x_2 m^{s_2})$, with $s_1, s_2 \geq 0$ and $x_1, x_2 \in \{\ell, r\}$, is 2-word-representable (equivalently, a circle graph).*

Proof. By symmetry assume $x_1 = r$. One can verify directly that the words

$$w_3 = 321435261456 \qquad \text{and} \qquad w_3' = 324135264156$$

both represent the graph $A(m^2 r)$, with base 14 for the last (sixth) step.

We use w_3 or w_3' as the starting point depending on the direction x_2 of the second turn, ensuring that the extension invariant is satisfied from this point onward. We give the four resulting cases.

Case 1. s_1 even and $x_2 = r$, or s_1 odd and $x_2 = \ell$. Starting from w_3, the m-step construction of Lemma 3.2.5(a) produces a 2-uniform word w_{s_1+3} representing $A(m^2 r m^{s_1})$ that contains the factor:

- $4(s_1+5)(s_1+6)$ if s_1 is even. Then w_{s_1+4}, representing $A(m^2rm^{s_1}r)$, is obtained from w_{s_1+3} by replacing this factor with $(s_1+7)4(s_1+5)(s_1+6)(s_1+7)$, creating the new base $4(s_1+5)$.

- $(s_1+6)(s_1+5)1$ if s_1 is odd. Then w_{s_1+4}, representing $A(m^2rm^{s_1}\ell)$, is obtained by replacing this factor with $(s_1+7)(s_1+6)(s_1+5)1(s_1+7)$, creating the new base $1(s_1+5)$.

In either subcase, subsequent m-steps apply Lemma 3.2.5(a) iteratively to extend the word to a 2-uniform representant of $A(m^2rm^{s_1}x_2m^{s_2})$.

Case 2. s_1 even and $x_2=\ell$, or s_1 odd and $x_2=r$. Starting from w_3', the construction proceeds analogously. The relevant factors are $1(s_1+5)(s_1+6)$ if s_1 is even (giving base $1(s_1+5)$ after the turn), and $(s_1+6)(s_1+5)4$ if s_1 is odd (giving base $4(s_1+5)$).

In all four subcases, repeated application of Lemma 3.2.5(a) extends to the final word representing $A(m^2rm^{s_1}x_2m^{s_2})$. □

3.2.5 Obstructions: the families $\mathcal{F}_1$ and $\mathcal{F}_2$

For graphs with at least two direction changes and $s_0 \geq 3$, the parity conditions become essential: when they fail, the graph contains an induced obstruction. We use the families $\mathcal{F}_1$ and $\mathcal{F}_2$ from Definitions 3.1.27 and 3.1.31, recalled here for convenience.

We need two "localisation lemmas" that pinpoint where these obstructions occur within the change-of-direction subfamily.

Lemma 3.2.10. *Let $G = A(m\,X\,x\,m^{2k+1}\,x\cdots x_pm^{s_p})$, where $k \geq 0$, $p \geq 3$, $s_i \geq 0$, X is any word over $\{r,\ell,m\}$ of length at least 2, and $x \in \{r,\ell\}$, with $|V(G)| \geq 9$. Then G contains an induced subgraph in $\mathcal{F}_1$, hence is not word-representable.*

Proof. By symmetry assume $x=r$.

Case $k=0$. Consider the consecutive substring $y_1y_2r_1mr_2$ of the encoding of G, where $y_1,y_2 \in \{m,r,\ell\}$ and r_1,r_2 are consecutive r-subdivisions. Let $n,n-1,n-2,n-3,n-4$ denote the vertices created by r_2,m,r_1,y_2,y_1 respectively. The vertices $\{n-3,n-2,n-1,n\}$ all lie inside the triangle determined by $n-4$ and two further boundary vertices

i, j, giving a 7-vertex induced subgraph $\{i, j, n-4, n-3, n-2, n-1, n\}$. We add an eighth vertex according to whether y_1 is an m-step (in which case the eighth vertex is $n-6$, the vertex created two steps before y_2) or a turn (in which case the eighth vertex is the third vertex of the triangle in which $n-5$ was created, distinct from $n-6$). In either case, the resulting eight vertices induce a graph in $\mathcal{F}_1$ (cf. Figure 3.5).

Case $k \geq 1$. The graph G has the encoding $\ldots xm^{2k+1}x\ldots$, with $2k$ additional m-steps inserted between the two x's of the relevant substring. These create $2k$ new vertices $z_1, z_2, \ldots, z_{2k}$ in creation order, each adjacent to the previous newest vertex and to the two "base" vertices of the triangle being subdivided. Every z_l is adjacent to both of the two distinguished vertices $n-6$ and n identified in the $k=0$ case (because these vertices remain on the boundary of the inner region throughout the subdivisions), and consecutive z_l are adjacent.

The new $\mathcal{F}_1$-subgraph in G has:

- the two distinguished vertices $n-6$ and n (the two "apex" positions in $\mathcal{F}_1$);
- the inner path: the original path $n-4, n-3, n-2, n-1$ of length 3 in the $k=0$ case is now extended to a path of length $3+2k$ by interleaving the z_l in the appropriate positions. Specifically, the z_l are inserted between $n-3$ and $n-2$ in the path order;
- the additional "outer triangle" vertices i and j.

Adjacencies in the new inner path: every z_l is adjacent to both distinguished vertices (as noted), to its predecessor and successor in the chain (consecutive subdivisions), and to no other inner-path vertex (the construction places each z_l inside its parent triangle, with only local adjacency). The outer adjacencies ij, $i(n-6)$, jn, $j(n-2)$ from the $k=0$ structure remain unchanged.

The result is a member of $\mathcal{F}_1$ on $2m'$ vertices with $m' = m+k$ (i.e. inner path of length $2m'-2 = 2m-2+2k$), where m was the parameter of the $k=0$ instance. Since $m \geq 4$ in the $k=0$ instance and $k \geq 1$, we have $m' \geq 5 \geq 4$. Hence G contains a member of $\mathcal{F}_1$ as an induced subgraph.

By Lemma 3.1.28, no graph in $\mathcal{F}_1$ is word-representable, so G is not word-representable.

□

Lemma 3.2.11. *Let* $G = A(m\,X\,x_1\,m^{2k}\,x_2 \cdots x_p m^{s_p})$, *where* X *is any word over* $\{r, \ell, m\}$ *of length at least* 2, $x_1, x_2 \in \{r, \ell\}$, $x_1 \neq x_2$, $k, s_i \geq 0$, $p \geq 3$, *with* $|V(G)| \geq 8$. *Then* G *contains a vertex* s *whose neighbourhood* $N(s)$ *contains an induced subgraph in* $\mathcal{F}_2$. *Hence by Theorem 2.2.7 (and Lemma 3.1.32),* G *is not word-representable.*

Proof. By symmetry, assume $x_1 = r$.

Case $k = 0$. Consider the consecutive substring $y_1 y_2 r \ell$ of the encoding, with $y_1, y_2 \in \{m, r, \ell\}$. Let $n, n-1, n-2, n-3$ be the vertices created by ℓ, r, y_2, y_1 respectively. These four vertices, together with a boundary vertex j, lie within a triangle formed by $j, n-4$, and another vertex which we call s; the vertex s is connected to all the vertices $\{j, n-4, n-3, n-2, n-1, n\}$. The Apollonian construction further guarantees a vertex i adjacent to $n-4$ and j but not adjacent to any of the inner vertices. The induced subgraph on $\{i, j, n-4, n-3, n-2, n-1, n\}$ is a member of $\mathcal{F}_2$ (cf. Figure 3.6), and is contained in $N(s)$.

Case $k \geq 1$. The graph G has the encoding $\ldots x_1 m^{2k} x_2 \ldots$ with $2k$ additional m-steps between the two turns. These create $2k$ new vertices $z_1, \ldots, z_{2k}$ analogous to those in the proof of Lemma 3.2.10. Each z_l is adjacent to s (the apex of the $\mathcal{F}_2$ structure, since s is constructed at a step before the z_l-block and remains on the boundary of the inner region) and to the chain endpoints $n-3$ and $n-2$ as well as to consecutive z_l's.

The new $\mathcal{F}_2$-subgraph in $N(s)$ has:

- the outer-triangle vertices $\{i, j, n-4\}$ (unchanged);
- the chain $n-3, z_1, z_2, \ldots, z_{2k}, n-2, n-1, n$ extended by $2k$ vertices in the middle;
- the apex s remains apex, now adjacent to all $2k+5$ chain vertices and to all three outer-triangle vertices.

The total vertex count is $2m' - 1$ with $m' = m + k$, again ≥ 4 since $m \geq 4$.

By Lemma 3.1.32, this graph fails to be a comparability graph. Hence $N(s)$ contains a non-comparability induced subgraph, and by Theorem 2.2.7, G is not word-representable. □

3.2.6 The general classification theorem

We can now state and prove the main classification result for graphs with $s_0 \geq 3$ and at least two direction changes. The condition stated below is exactly the parity rule observed in Section 3.1: s_i is even when $x_i = x_{i+1}$, and odd when $x_i \neq x_{i+1}$.

Theorem 3.2.12. *Let $G = A(m^{s_0} x_1 m^{s_1} x_2 m^{s_2} \cdots x_p m^{s_p})$, where $s_0 \geq 3$, $p \geq 2$, $s_i \geq 0$ for $i \geq 1$, and $x_i \in \{\ell, r\}$. Then G is 2-word-representable (equivalently, a circle graph) if and only if for every $1 \leq i \leq p-1$,*

$$s_i \text{ is even when } x_i = x_{i+1}, \qquad s_i \text{ is odd when } x_i \neq x_{i+1}.$$

Proof. Suppose the parity condition fails for some index $1 \leq i \leq p-1$. Then either:

- $x_i = x_{i+1}$ and s_i is odd, or
- $x_i \neq x_{i+1}$ and s_i is even.

In the first case, the encoding contains the substring $xm^{2k+1}x$ (with $k = (s_i - 1)/2$) within it, and since $s_0 \geq 3$ and $i \geq 1$, the prefix $m^{s_0}x_1$ provides a word X of length at least 2 before the violating substring. So Lemma 3.2.10 applies, giving an induced $\mathcal{F}_1$-subgraph in G.

In the second case, the encoding contains the substring $x_1 m^{2k} x_2$ with $x_1 \neq x_2$ and $k = s_i/2$, and the same prefix gives the required X. So Lemma 3.2.11 applies, giving a vertex whose neighbourhood contains an induced $\mathcal{F}_2$-subgraph.

In either case, G is not word-representable, hence not 2-word-representable.

Conversely, suppose the parity condition holds for every $1 \leq i \leq p-1$. We construct a 2-uniform representation of G by induction, using the extension invariant (Figure 3.8) and the step rules of Lemma 3.2.5 (Figure 3.9).

By Theorem 3.2.6, the subgraph $A(m^{s_0}x_1m^{s_1})$ admits a 2-uniform representation w satisfying the invariant with newest vertex $n = s_0 + s_1 + 4$, base $a'b'$ (some specific edge), and a known type:

- If s_1 is even, the type after the s_1 m-steps is the same as immediately after the turn x_1, which is type I (assuming s_0 is odd; the other case is symmetric).
- If s_1 is odd, the type has been reversed, becoming type II.

Suppose we have constructed a 2-uniform word w representing the subgraph built from the first $s_0 + s_1 + \cdots + s_{j-1} + j$ subdivision steps, with w satisfying the invariant with some base $a''b''$, newest vertex n, and known type.

The next operation is the turn x_j, followed by s_j middle steps (and then potentially the next turn x_{j+1}).

By Lemma 3.2.5(b), exactly one of the two turn directions is admissible from the current invariant, and a turn preserves the type. We must verify that the admissible direction is in fact the required direction x_j.

The parity condition ensures that the admissible turn direction always matches x_j. To see this, observe:

1. By Lemma 3.2.5(a), each m-step reverses the invariant type. So after s_{j-1} consecutive m-steps following the turn x_{j-1}, the type has been reversed exactly s_{j-1} times.
2. By Lemma 3.2.5(b), the type immediately after a turn determines which direction was taken. If the invariant after a right turn has type I, then the next admissible turn from a type I invariant is also a right turn, while from a type II invariant the admissible turn is a left turn.
3. Combining: after s_{j-1} middle steps following the turn x_{j-1}, the admissible turn direction is the *same* as x_{j-1} if s_{j-1} is even, and the *opposite* of x_{j-1} if s_{j-1} is odd.

By the parity condition, s_{j-1} is even when $x_{j-1} = x_j$ and odd when $x_{j-1} \neq x_j$. In both cases, the admissible direction is exactly x_j, as required. This proves the claim.

By Lemma 3.2.5(a), we perform s_j consecutive m-steps. Each maintains the invariant with the (new) base fixed and the type alternating; the resulting word is a valid 2-uniform representation of the graph through the end of the j-th block of m-steps, with the invariant ready for the next turn.

By induction on j from 1 to $p-1$, the construction successfully processes every turn and every block of m-steps. The final block of s_p m-steps after the last turn x_p is handled by Lemma 3.2.5(a) alone, with no admissibility constraint to verify. The resulting word is a 2-uniform representation of G, so G is 2-word-representable. □

A parallel result holds with $s_0 = 2$ instead of $s_0 \geq 3$.

Theorem 3.2.13. *Let $G = A(m^2 x_1 m^{s_1} x_2 m^{s_2} \cdots x_p m^{s_p})$, where $p \geq 3$, $s_i \geq 0$ for $i \geq 1$, and $x_i \in \{\ell, r\}$. Then G is 2-word-representable (equivalently, a circle graph) if and only if for every $2 \leq i \leq p-1$, s_i is even when $x_i = x_{i+1}$ and odd when $x_i \neq x_{i+1}$.*

Proof. The forward implication uses Lemmas 3.2.10 and 3.2.11 as in Theorem 3.2.12. The relevant prefix $m^2 x_1 m^{s_1} x_2$ provides the required X of length at least 2 when applying these lemmas, so the obstructions appear whenever the parity condition fails for $i \geq 2$. (No parity condition is imposed on s_1, consistent with Theorem 3.2.9.)

For the backward implication, Theorem 3.2.9 initialises the construction of the graph: $A(m^2 x_1 m^{s_1} x_2 m^{s_2})$ admits a 2-uniform representation regardless of s_1, s_2, and the proof of that theorem ensures that the result satisfies the extension invariant after the second direction change. The inductive step from the proof of Theorem 3.2.12 now applies for all subsequent turns $x_3, x_4, \ldots, x_p$. □

3.2.7 The main characterisation

Combining everything, we obtain the following sharp characterisation.

Theorem 3.2.14. *An Apollonian graph $G = A(m^{s_0} x_1 m^{s_1} x_2 m^{s_2} \cdots x_p m^{s_p})$, where $s_i \geq 0$ for $i \geq 0$ and $x_i \in \{\ell, r\}$, is word-representable if and only if it is a circle graph.*

Proof. Combine Corollary 3.2.7, Theorems 3.2.6, 3.2.9, 3.2.12, and 3.2.13: in every case, when G is word-representable, the construction produces an explicit 2-uniform representation, so G is a circle graph; the converse direction is the well-known fact that circle graphs are word-representable. $\square$

3.3 Concluding remarks for Chapter 3

Theorem 3.2.14 provides a complete characterisation of 2-word-representability (equivalently, of circle-graph membership) for Apollonian graphs in our change-of-direction subfamily. Together with Hauser's general characterisation (Theorem 3.1.30), the picture for Apollonian networks is now reasonably complete:

- For *arbitrary* Apollonian networks, word-representability is captured by the conditions of $\mathcal{F}_1$-freeness and local comparability [21].
- For the change-of-direction subfamily, word-representability is equivalent to 2-word-representability, equivalently to being a circle graph, equivalently to a parity condition on the encoding.

A natural direction for further work is to extend this analysis beyond Apollonian networks. Hauser's thesis [21] characterises word-representable chordal near-triangulations with outer cycle of length 3 or 5, and the broader programme of characterising word-representable planar graphs remains open. Within the Apollonian setting, we note first that the question of which graphs in the subfamily have representation number 1 has an immediate answer: by Definition 2.2.4, representation number 1 characterises complete graphs, and the only complete Apollonian graphs are K_3 and K_4 (any Apollonian graph on $n \geq 5$ vertices has $3n - 6 < \binom{n}{2}$ edges). Every other word-representable graph in the subfamily therefore has representation number exactly 2, by Theorem 3.2.14. More substantial questions include the following:

- Apollonian graphs not in our subfamily can have arbitrarily many simultaneous direction choices at each step. Can the parity-based classification of Theorem 3.2.14 be extended (perhaps in a more elaborate form) to such broader classes?

- For circle-graph membership, beyond representation number, are there structural parameters (such as the chord length distribution in a chord diagram) that distinguish Apollonian circle graphs from arbitrary circle graphs?

We turn now to the very different setting of 1-11-representations.

Chapter 4

1-11-Representability of Graphs

Overview

This chapter addresses the second main theme of the thesis: k-11-*representations* of graphs. Although the underlying combinatorics is closely related to that of word-representations, the methods are quite different from those of Chapter 3, here, the question is global rather than local, and the techniques rely on chromatic-number case analysis combined with the toolkit assembled in Section 2.3.1.

The chapter is divided into three parts. Section 4.1 reproduces the main result of the published paper [3], joint with Kitaev, Tang, Tao, and Zhang: every graph on at most eight vertices is 1-11-representable, extending the previously known bound of seven vertices. Section 4.2 introduces and studies the *multi*-1-11-*representation number*, with the main result that every graph on at most 24 vertices has multi-1-11-representation number at most 2. Finally, Section 4.3 summarises the subsequent work of Hefty, Horn, and Muir [22], which has dramatically resolved the central open question motivating our paper, and presents the open problems they raise as the natural sequel to this line of research.

Figure 4.1 illustrates the basic definition concretely on a small graph that is not word-representable but is 1-11-representable.

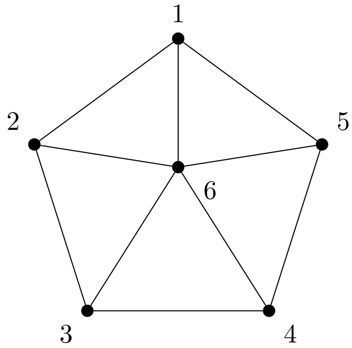


Figure 4.1: The 5-wheel W_5 on 6 vertices: a hub (vertex 6) adjacent to every vertex of an outer 5-cycle on vertices 1 through 5. This graph is known to be non-word-representable, but is 1-11-representable. The non-edges of W_5 are precisely the five diagonals of the pentagon: $\{1,3\}$, $\{2,4\}$, $\{3,5\}$, $\{4,1\}$, $\{5,2\}$. Every other pair of vertices is connected by an edge.

4.1 All graphs on at most 8 vertices are 1-11-representable

This section reproduces the main result of the published paper [3]. We use the notation $\chi(G)$ for the chromatic number of G, and the $(a_1, a_2, \ldots, a_k)$-colouring terminology of Section 2.1: a graph is $(a_1, a_2, \ldots, a_k)$-*colourable* with $a_1 \geq a_2 \geq \cdots \geq a_k$ if it has chromatic number k and admits a colouring with class sizes a_i. We write the colour classes as $V_i = \{v_i, v_i', v_i'', \ldots\}$ in size a_i. Recall from Remark 2.3.9 that the union $\bigcup_{a_i=1} V_i$ induces a clique.

For sets $X, Y \subseteq V(G)$, $e(X, Y)$ denotes the number of edges with one endpoint in X and the other in Y; for a singleton $\{x\}$, we write $e(x, Y)$. For disjoint independent sets $V_1, \ldots, V_m$, we write $G[V_1, \ldots, V_m]$ for the multipartite induced subgraph on $\bigcup V_i$.

4.1.1 A shortcut classification

The following terminology, used throughout the proofs below, follows [3].

Definition 4.1.1. A $(b_1 b_2 \ldots b_m)$-*shortcut* is a shortcut with directed path $w_{b_1} \to w_{b_2} \to \cdots \to w_{b_m}$, where $w_{b_i} \in V_{b_i}$ for $i = 1, \ldots, m$. A $(b_1 b_2 \ldots b_s-)$-*shortcut* is any $(c_1 c_2 \ldots c_t)$-shortcut with $b_i = c_i$ for $i \in \{1, \ldots, s\}$ and $t \geq s$.

For example, in a (12345)-shortcut, the directed path passes through colour classes

V_1, V_2, V_3, V_4, V_5 in that order, and there is a shortcutting edge from V_1 to V_5. Figure 4.2 depicts the schematic shape of a (1234)-shortcut.

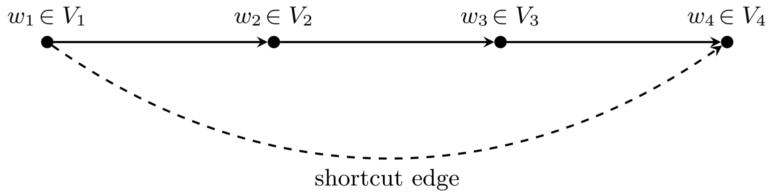


Figure 4.2: Schematic of a (1234)-shortcut in the sense of Definition 4.1.1. The solid arrows form a directed path $w_1 \to w_2 \to w_3 \to w_4$ visiting the colour classes V_1, V_2, V_3, V_4 in order; the dashed arrow is the shortcutting edge from V_1 directly to V_4. The path is a *shortcut* (rather than just a directed walk) precisely when at least one of the intermediate edges w_1w_3 or w_2w_4 is missing.

4.1.2 Preparatory lemmas

We first record two preparatory results.

Lemma 4.1.2. *If a graph G admits a minimal $(k, 1, 1, \ldots, 1)$-colouring (that is, a colouring with these class sizes using exactly $\chi(G)$ colours), then G is 1-11-representable.*

Proof. G is a split graph: its vertex set partitions into the clique $\bigcup_{i\geq 2} V_i$ (which has size $\chi(G) - 1$) and the independent set V_1 (of size k). By Lemma 2.3.8 (i.e. Theorem 6 of [13]), every split graph is permutationally 1-11-representable, and in particular 1-11-representable. □

Lemma 4.1.3. *Let G be a graph admitting a minimal $(a_1, a_2, \ldots, a_k)$-colouring, so that $k = \chi(G)$, with $a_1 \geq a_2 \geq \cdots \geq a_i > a_{i+1} = a_{i+2} = \cdots = a_k = 1$. (Minimality is essential: the proof derives a contradiction from the construction of a $(k-1)$-colouring.) If $e(V_s, V_t) = 1$ for some $s \leq i < t$, then the unique vertex in V_t together with its unique neighbour in V_s is adjacent to all vertices in V_j for every $j > i$.*

Proof. Write $V_t = \{v_t\}$, and let $v_s \in V_s$ be its unique neighbour in V_s. Since $a_t = 1$, by Remark 2.3.9, v_t is adjacent to every vertex in V_j for $j > i$. Suppose for contradiction that v_s is not adjacent to some vertex $v_\ell \in V_\ell$ for some $\ell > i$. Then we can recolour $V_s \setminus$

$\{v_s\}$ with colour t and v_s with colour ℓ, yielding a $(k-1)$-colouring of G, contradicting $\chi(G) = k$. Hence v_s is adjacent to all vertices in V_j for $j > i$. $\square$

Lemma 4.1.4 (Shortcut elimination). *Let G be a graph admitting a minimal $(a_1, a_2, \ldots, a_k)$-colouring, so that $k = \chi(G)$, with $k \geq 4$.*[1] *Consider the orientation σ of G that directs every edge uv from the endpoint with the smaller colour* index *to the endpoint with the larger colour index; that is, if $u \in V_i$ and $v \in V_j$ with $i < j$, the edge is oriented $u \to v$. (The comparison is between the indices i and j labelling the classes, not between the cardinalities a_i and a_j.) Fix a shortcut type $(b_1 b_2 \ldots b_m)$ with $b_1 < b_2 < \cdots < b_m$ and $m \geq 4$. If for every pair (i, j) with $1 \leq i < j \leq m$ and $(i, j) \neq (1, m)$ and $(j - i) \neq 1$ (i.e. every transitivity-edge index pair within the shortcut), the bipartite induced subgraph $G[V_{b_i}, V_{b_j}]$ is complete bipartite, then no $(b_1 b_2 \ldots b_m)$-shortcut exists in σ.*

In the case $m = 4$, the condition reduces to: $G[V_{b_1}, V_{b_3}]$ and $G[V_{b_2}, V_{b_4}]$ are both complete bipartite.

Proof. A $(b_1 b_2 \ldots b_m)$-shortcut consists of an induced acyclic subgraph on m vertices $\{w_{b_1}, \ldots, w_{b_m}\}$ with $w_{b_i} \in V_{b_i}$, with w_{b_1} a unique source, w_{b_m} a unique sink, the source–sink edge $w_{b_1} w_{b_m}$ present, and the subgraph not transitive. For the subgraph to be non-transitive, at least one transitivity-supporting edge $w_{b_i} w_{b_j}$ with $j - i \geq 2$ and $(i, j) \neq (1, m)$ must be missing from G.

Under our hypothesis, every such pair (b_i, b_j) corresponds to a complete bipartite $G[V_{b_i}, V_{b_j}]$, so the edge $w_{b_i} w_{b_j}$ is present. Hence no transitivity edge can be missing, and no shortcut of type $(b_1 b_2 \ldots b_m)$ exists.

For $m = 4$, the transitivity edges within the shortcut are $w_{b_1} w_{b_3}$ and $w_{b_2} w_{b_4}$, so the condition reduces to the two stated pairs. $\square$

4.1.3 The main theorem

Theorem 4.1.5 ([3]). *All graphs on at most* 8 *vertices are* 1-11-*representable.*

[1] Minimality is assumed here for uniformity with Lemmas 4.1.2 and 4.1.3; unlike those two, the present lemma is in fact purely structural, and its proof uses only that G is equipped with a proper k-colouring into classes $V_1, \ldots, V_k$ with $k \geq 4$, not that this colouring is minimal.

The proof considers all graphs on 8 vertices by chromatic number; smaller graphs follow because the class of 1-11-representable graphs is hereditary [8]. The proof of this theorem could in principle be reduced to checking the 929 non-word-representable graphs on 8 vertices [30, p. 47], since every word-representable graph is 1-11-representable. However, our proof works directly with the chromatic-number structure, which is more uniform and which lends itself to extension.

Proof. We proceed by chromatic number. The cases $\chi(G) \leq 3$, $\chi(G) = 8$, and $\chi(G) = 7$ are short. For $\chi(G) \in \{4, 5, 6\}$, the strategy is uniform:

> Choose an $(a_1, \ldots, a_k)$-colouring with $a_1 \geq \cdots \geq a_k$. Orient edges uv from smaller to larger colour. Then apply a star or matching operation (Lemma 2.3.5 or 2.3.7) to add carefully chosen edges to yield a graph that is word-representable.

The added edges are oriented from smaller to larger colour. We then verify that the resulting orientation is semi-transitive (which by Theorem 2.2.9 establishes word-representability of the augmented graph), and conclude 1-11-representability of the original graph by Lemma 2.3.5(b) or Lemma 2.3.7.

Table 4.1 summarises the cases handled below, including for each colourability type the pairs of colour classes that may contain missing edges and the operation used by the proof to bring the relevant pairs to complete bipartite.

Case 1: $\chi(G) \leq 3$. Every 3-colourable graph is word-representable (Theorem 2.2.3), and hence 1-11-representable.

Case 2: $\chi(G) = 8$. On 8 vertices, $\chi(G) = 8$ forces $G = K_8$, which is word-representable.

Case 3: $\chi(G) = 7$. Then G is $(2, 1, 1, 1, 1, 1, 1)$-colourable, so by Lemma 4.1.2, G is 1-11-representable.

Case 4: $\chi(G) = 6$. Then G is either $(3, 1, 1, 1, 1, 1)$-colourable or $(2, 2, 1, 1, 1, 1)$-colourable. By Lemma 4.1.2, the first case is settled. So we may assume G is $(2, 2, 1, 1, 1, 1)$-colourable, with vertex labels as in Figure 4.3. The clique structure forces $C = \{v_3, v_4, v_5, v_6\}$ to induce a clique.

Case	Colourability	Pairs that may be missing	Operation
$\chi = 8$	K_8	none (already K_8)	word-representable
$\chi = 7$	$(2,1,1,1,1,1,1)$	none	split graph
$\chi = 6$	$(3,1,1,1,1,1)$	none	split graph
$\chi = 6$	$(2,2,1,1,1,1)$	$G[V_1, C], G[V_2, C]$	star at v_1'
$\chi = 5$	$(4,1,1,1,1)$	none	split graph
$\chi = 5$	$(3,2,1,1,1)$	$G[V_1, V_j],\ G[V_i, V_5]$	star or matching
$\chi = 5$	$(2,1,1,2,2)$	$G[V_a, V_b]$ for $a, b \in \{1,4,5\}$,	matching
		and $G[V_i, V_j]$ for $i \in \{2,3\}, j \in \{4,5\}$	
$\chi = 4$	$(5,1,1,1)$	none	split graph
$\chi = 4$	$(4,2,1,1)$	$G[V_1, V_3],\ G[V_2, V_4]$	star at v_3
$\chi = 4$	$(3,3,1,1)$	$G[V_1, V_3],\ G[V_2, V_4]$	star or matching
$\chi = 4$	$(3,2,1,2)$	$G[V_1, V_3],\ G[V_2, V_4]$	matching
$\chi = 4$	$(2,2,2,2)$	$G[V_1, V_3],\ G[V_2, V_4]$	matching

Table 4.1: Overview of the case analysis in the proof of Theorem 4.1.5. The cases marked "split graph" are handled directly by Lemma 4.1.2; the case $\chi = 8$ on eight vertices forces $G = K_8$, which is itself word-representable. In the $(2,2,1,1,1,1)$ row, $C = V_3 \cup V_4 \cup V_5 \cup V_6$ denotes the union of the four singleton classes. In the $(3,2,1,1,1)$ row, the ranges are $j \in \{3,4\}$ and $i \in \{1,2,3\}$, and V_5 refers to the re-indexed three-element class. The third column lists the pairs of colour classes that may contain missing edges; the fourth column indicates which lemma is used to bring the relevant pairs to complete bipartite. In the $(3,2,1,1,1)$- and $(2,1,1,2,2)$-colourable sub-cases of $\chi(G) = 5$, the exact operation depends on a sub-case parameter κ counting how many of the singleton classes have only one neighbour in a fixed two-element class.

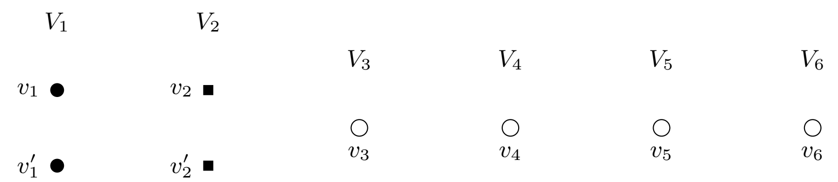


Figure 4.3: Vertex labelling for Case 4 of the proof of Theorem 4.1.5 ($\chi(G) = 6$, $(2,2,1,1,1,1)$-colourable). The two two-element colour classes V_1, V_2 are drawn with filled circles and filled squares respectively; the four singleton classes $V_3, \ldots, V_6$ are drawn with hollow circles, and induce a clique among themselves (edges omitted for clarity).

Suppose v_1 is not adjacent to some vertex in C; without loss of generality, $v_1v_3 \notin E(G)$. Since $\chi(G) = 6$, the vertex v_3 cannot be recoloured with colour 1, so $v_1'v_3 \in$

$E(G)$. By Lemma 4.1.3, v_1' is adjacent to every vertex in C. The same argument applies to v_2, v_2'. Renaming vertices if necessary, we may assume that both v_1 and v_2 are adjacent to every vertex in C.

Add (using a star operation) any missing edges between v_1' and the vertices of C to obtain G'. Then rename the vertices of C, if necessary, so that the neighbours of v_2' in C form the set $C' = \{v_i, v_{i+1}, \ldots, v_6\}$ for some $3 \le i \le 7$ (where $i = 7$ means C' is empty). Now orient: $v_a \to v_b$ and $v_a' \to v_b$ for $1 \le a < b \le 6$, and $v_1 \to v_2'$ and $v_1' \to v_2'$ if these edges exist. Direct inspection shows the orientation is transitive, hence semi-transitive, so G' is word-representable. By Lemma 2.3.5(b), G is 1-11-representable.

Case 5: $\chi(G) = 5$. The possible shortcuts are of types (12345), (1234), (1235), (1245), (1345), (2345). Possible missing edges in G (relative to a fixed $(a_1, \ldots, a_5)$-colouring) appear only in $G[V_1, V_3]$, $G[V_1, V_4]$, $G[V_2, V_4]$, $G[V_2, V_5]$, $G[V_3, V_5]$.

The graph G is $(4,1,1,1,1)$-, $(3,2,1,1,1)$-, or $(2,2,2,1,1)$-colourable. By Lemma 4.1.2, the first case is settled.

$(3,2,1,1,1)$-*colourable:* We adopt the equivalent labelling $(2,1,1,1,3)$-colourable, with the colour classes shown in Figure 4.4, with V_5 of size 3 at the top of the colour list (we re-index for convenience). The clique condition forces $G[\{v_2, v_3, v_4\}]$ to be a triangle. Each vertex in $\{v_2, v_3, v_4\}$ has at least one neighbour in V_1 and at least one in V_5 (otherwise we could recolour to obtain a 4-colouring), and $e(V_1, V_5) \ge 2$ (otherwise we could recolour into $(4,1,1,1,1)$).

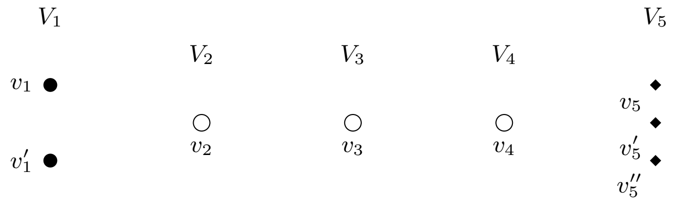


Figure 4.4: Vertex labelling for the $(3,2,1,1,1)$-colourable sub-case of Case 5 ($\chi(G) = 5$) in the proof of Theorem 4.1.5, re-indexed so that V_5 has size 3. Filled circles mark the two-element class V_1; diamonds mark the three-element class V_5; hollow circles mark the singleton classes V_2, V_3, V_4, which induce a triangle among themselves.

We first prove a useful claim: *if at least two vertices among* $\{v_2, v_3, v_4\}$ *have only one neighbour in* V_5*, then* G *is* 1-11-*representable.* Without loss of generality, $e(v_3, V_5) =$

$e(v_4, V_5) = 1$ and $v_3v_5 \in E(G)$. By Lemma 4.1.3, v_5 is adjacent to all of v_2, v_3, v_4. Add all edges in $G[V_1, v_3]$ and $G[V_1, v_4]$, which adds at most two edges and can be realised either by a star operation or by a matching operation. By Lemma 2.3.5 or Lemma 2.3.7, the resulting graph (after this edge-augmentation) is word-representable, and G is 1-11-representable.

The shortcut analysis verifying this conclusion proceeds as follows. After the augmentation, $G[V_1, V_3]$ and $G[V_1, V_4]$ are complete bipartite (by the explicit edge-addition), and $G[V_2, V_4]$ is complete bipartite to begin with (both classes are singletons and $v_2v_4 \in E$ by the singleton-clique structure). We must rule out all shortcut types listed at the start of Case 5.

- Shortcut types (1234) and (12345) contain $G[V_1, V_3]$ and $G[V_2, V_4]$ among their transitivity-edge pairs; since both are complete bipartite, Lemma 4.1.4 rules these out.

- Shortcut types (1235) and (2345) require the pair $G[V_3, V_5]$ to support a transitivity edge. By hypothesis $e(v_3, V_5) = 1$ with $v_3v_5 \in E$; for any other $w_5 \in V_5 \setminus \{v_5\}$, the edge v_3w_5 is missing from G. But then the path edge $w_3w_5 = v_3w_5$ of the shortcut is itself missing, so no such shortcut exists.

- Shortcut types (1245) and (1345) require the pair $G[V_4, V_5]$ to support a transitivity edge. By hypothesis $e(v_4, V_5) = 1$ with $v_4v_5 \in E$; for $w_5 \neq v_5$, the path edge $w_4w_5 = v_4w_5$ is missing, so no such shortcut exists.

This rules out all shortcut types in the list, so the augmented graph has a semi-transitive orientation by Theorem 2.2.9.

So we may assume that at most one of $\{v_2, v_3, v_4\}$ has only one neighbour in V_5.

Let κ denote the number of $i \in \{2, 3, 4\}$ such that $e(V_1, v_i) = 1$. Consider the four cases.

(i) $\kappa = 0$. So $e(V_1, v_i) = 2$ for all $i \in \{2, 3, 4\}$. We may also assume $e(v_2, V_5) \geq 2$ and $e(v_3, V_5) \geq 2$ (by the claim above, at least two of v_2, v_3, v_4 have ≥ 2 neighbours in V_5). By adding at most two edges, we can make $G[V_2, V_5]$ and $G[V_3, V_5]$ complete bipartite.

Since $G[V_1, V_3]$, $G[V_1, V_4]$, $G[V_2, V_4]$ are already complete bipartite (no missing edges in these sets), the orientation has no shortcut, so G is 1-11-representable.

(ii) $\kappa = 1$. By recolouring if necessary, $e(V_1, v_2) = 1$ and $v_1v_2 \in E(G)$. By symmetry, $e(v_2, V_5) \geq e(v_1, V_5)$. Apply a star or matching operation to add the edges v_2w for each $w \in V_5$ such that $v_1w \in E(G)$, and add all edges in $\{v_3w : w \in V_5\}$. At most two edges are added (since $e(v_2, V_5) \geq e(v_1, V_5)$ and $e(v_3, V_5) \geq 2$). Now $N_{V_5}(v_1) \subseteq N_{V_5}(v_2)$, and $G[V_1, V_3]$, $G[V_1, V_4]$, $G[V_2, V_4]$, $G[V_3, V_5]$ are all complete bipartite. Hence the orientation has no shortcut. Figure 4.5 illustrates the structure of this sub-case.

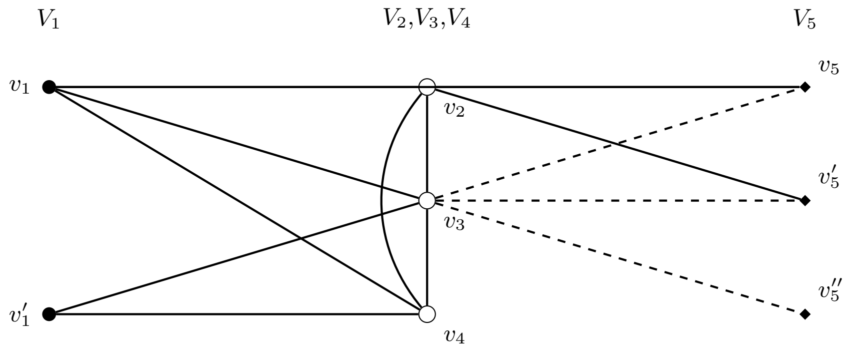


Figure 4.5: Schematic of sub-case (ii) of the $(3, 2, 1, 1, 1)$-colourable analysis in the proof of Theorem 4.1.5. Solid edges are present in G before the augmentation (a sample of the configuration consistent with $\kappa = 1$, $e(V_1, v_2) = 1$, $e(v_2, V_5) \geq e(v_1, V_5)$); dashed edges are added by the star/matching operation so that $N_{V_5}(v_1) \subseteq N_{V_5}(v_2)$ and $G[V_3, V_5]$ becomes complete bipartite. After the augmentation, $G[V_1, V_3]$, $G[V_1, V_4]$, $G[V_2, V_4]$, and $G[V_3, V_5]$ are all complete bipartite, ruling out every shortcut type listed at the start of Case 5.

(iii) $\kappa = 2$. By recolouring, $e(V_1, v_2) = 1, e(V_1, v_3) = 1, e(V_1, v_4) = 2$. By the earlier claim, we may assume $e(v_3, V_5) \geq 2$. Assume $v_1v_2 \in E(G)$, so $v_1v_3, v_1v_4 \in E(G)$. By symmetry $e(v_2, V_5) \geq e(v_1, V_5)$. Add at most one edge to ensure $N_{V_5}(v_1) \subseteq N_{V_5}(v_2)$ and at most one further edge so that $G[V_3, V_5]$ becomes complete bipartite. The result has no shortcut.

(iv) $\kappa = 3$. So $e(V_1, v_i) = 1$ for $i = 2, 3, 4$. Then $v_1v_2, v_1v_3, v_1v_4 \in E(G)$. By symmetry $e(v_2, V_5) \geq e(v_1, V_5)$. As in case (iii), we obtain a semi-transitive orientation by adding at most two edges.

$(2, 1, 1, 2, 2)$-*colourable:* We may assume that $G[V_1, V_4]$, $G[V_1, V_5]$, $G[V_4, V_5]$ all have perfect matchings (otherwise we recolour to a $(3,2,1,1,1)$-colouring and apply the earlier case). The vertex labelling for this sub-case is shown in Figure 4.6.

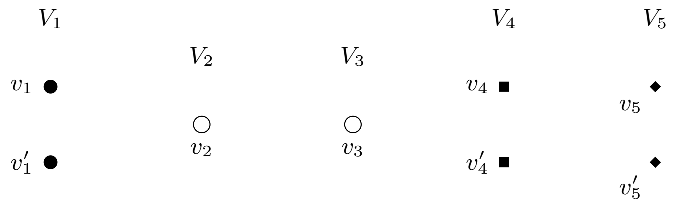


Figure 4.6: Vertex labelling for the $(2,1,1,2,2)$-colourable sub-case of Case 5 in the proof of Theorem 4.1.5. Filled circles, squares, and diamonds mark the two-element classes V_1, V_4, V_5 respectively; hollow circles mark the singleton classes V_2, V_3, which together with the singleton structure form the clique $G[\{v_2, v_3\}]$.

For $i \in \{2,3\}$, define $f(i) = (e(V_1, v_i), e(v_i, V_4), e(v_i, V_5))$, so $f(i) \in \{1,2\}^3$. Let k_i be the number of 2's in $f(i)$, and let $\kappa = \max\{k_2, k_3\}$. Note $\kappa \geq 1$, since otherwise we can find $s, t \in \{1,4,5\}$ with $e(V_s, v_i) = e(V_t, v_i) = 1$ for $i = 2, 3$, leading to a sub-case analysed below.

We discuss two preliminary sub-cases.

Special case: there exist $s, t \in \{1,4,5\}$ with $e(V_s, v_2) = e(V_t, v_2) = e(V_s, v_3) = e(V_t, v_3) = 1$. By symmetry $s = 1, t = 4$. By Lemma 4.1.3, $v_1v_2, v_1v_3, v_2v_4, v_3v_4 \in E(G)$. Then $v_1v_4 \in E(G)$, since otherwise we can swap colours to obtain a 4-colouring. By a matching operation we can make $G[V_2, V_5]$ and $G[V_3, V_5]$ complete bipartite. The shortcut analysis then verifies 1-11-representability.

Now suppose the special case does not apply. The four sub-cases below treat the values of κ.

(i) $\kappa = 3$ and $f(i) \neq (1,1,1)$ for $i = 2,3$. By symmetry $e(v_2, V_4) = e(v_2, V_5) = e(V_1, v_3) = 2$. Add a matching to make $G[V_1, V_4]$ and $G[V_3, V_5]$ complete bipartite. Since $G[V_1, V_3]$, $G[V_2, V_4]$, $G[V_2, V_5]$ are already complete bipartite, no shortcut exists.

(ii) $\kappa = 2$. By symmetry, $e(V_1, v_2) = 1$ and $e(v_2, V_4) = e(v_3, V_5) = 2$. By Lemma 4.1.3, $v_1v_2, v_1v_3 \in E(G)$. After adding all edges in $G[V_1, V_4]$ via a matching, the only possible shortcuts are (123–)-shortcuts, which start with v_1v_2 and v_2v_3. But $v_1v_3 \in E(G)$, so no such shortcut exists.

(iii) $\kappa = 1$. By the absence of the special case and by symmetry, $e(V_1, v_2) =$

$e(v_2, V_4) = e(v_3, V_4) = e(v_3, V_5) = 1$ and $e(v_2, V_5) = e(V_1, v_3) = 2$. Lemma 4.1.3 gives $v_1v_2, v_2v_4, v_3v_4, v_3v_5 \in E(G)$. By adding at most a matching, we can make $G[V_1, V_4]$ and $G[V_3, V_5]$ complete bipartite.

We verify each shortcut type explicitly. The only edges that may be missing in G (relative to the $(2, 1, 1, 2, 2)$-colouring) are in the pairs $G[V_1, V_4]$, $G[V_1, V_5]$, $G[V_4, V_5]$, $G[V_2, V_4]$, $G[V_2, V_5]$, $G[V_3, V_4]$, $G[V_3, V_5]$. After the augmentation, $G[V_1, V_4]$ and $G[V_3, V_5]$ are complete bipartite. The shortcut types from Case 5 that could appear are:

- (12–)-shortcuts: these begin with the path edge v_1v_2 from a vertex of V_1 to $v_2 \in V_2$; the next path edge goes from v_2 to a vertex of some V_j with $j \geq 3$. For any such shortcut to be a shortcut, some transitivity edge must be missing. The first transitivity edge from the source $w_1 \in V_1$ is $w_1 \to v_3 \in V_3$ (since the source has colour 1 and the third path vertex has colour at least 3); but $G[V_1, V_3]$ is complete bipartite by the singleton-clique structure of V_3, so this edge is present. The remaining transitivity edges in a (12–)-shortcut land in pairs whose edges are guaranteed present by either the augmentation, the singleton-clique structure (V_2, V_3 are singletons), or the explicit hypotheses ($v_1v_2, v_2v_4, v_3v_4 \in E$).

- (1345)-shortcut: needs $G[V_1, V_4]$ and $G[V_3, V_5]$ to support transitivity, both complete bipartite by augmentation.

- (2345)-shortcut: needs $G[V_2, V_4]$ and $G[V_3, V_5]$ to support transitivity. V_2 is a singleton with $v_2v_4 \in E$, and $G[V_3, V_5]$ is complete bipartite.

So Lemma 4.1.4 rules out every shortcut type, and the augmented graph has a semi-transitive orientation.

(iv) $\kappa = 3$ but some $f(i) = (1, 1, 1)$. By symmetry, $e(v_2, V_i) = 1$ and $e(v_3, V_i) = 2$ for $i \in \{1, 4, 5\}$, and $v_1v_2, v_2v_4, v_2v_5 \in E(G)$. We claim $e(v_1, V_4) = e(v_1, V_5) = e(v_4, V_1) = e(v_4, V_5) = e(v_5, V_1) = e(v_5, V_4) = 1$ (otherwise we recolour to obtain a $(2, 1, 1, 2, 2)$-colouring with f-triples in case (i)–(iii) above). Further, $G[\{v_1, v_4, v_5\}]$ is not edgeless (else we recolour into $(3, 2, 1, 1, 1)$). Without loss, $v_1v_5 \in E(G)$. Then $v_1v_5', v_1'v_5 \notin E(G)$. Now we recolour v_1, v_5 to obtain a new $(2, 1, 1, 2, 2)$-colouring of G, this time

with the swapped pair playing different roles: the new colour class structure has v_5 in the role formerly occupied by v_1 and vice versa.

We must verify that the new f-triples $\widetilde{f}(2), \widetilde{f}(3)$ are not both $(1,1,1)$ again, otherwise the recursion would not terminate. Suppose for contradiction $\widetilde{f}(2) = \widetilde{f}(3) = (1,1,1)$ under the new colouring. By the same argument applied to the new colouring, we would have $e(v_2, V_i) = 1$ and $e(v_3, V_i) = 2$ for the new classes $i \in \{1,4,5\}$. But under the swap $v_1 \leftrightarrow v_5$, the old triple $f(3) = (2,2,2)$ (which we have just established) becomes the new $\widetilde{f}(3) = (2,2,2)$ as well, since v_3's degree to each of the (renamed) classes V_1, V_4, V_5 is unchanged in terms of total count of neighbours, only the identities of the classes are shuffled. So $\widetilde{f}(3) \neq (1,1,1)$, and we land in case (i), (ii), or (iii), which has already been handled. This completes case (iv).

Case 6: $\chi(G) = 4$**.** The only possible shortcuts are (1234)-shortcuts, and missing edges appear only in $G[V_1, V_3]$ and $G[V_2, V_4]$.

If G is $(5,1,1,1)$-colourable, then by Lemma 4.1.2, G is 1-11-representable.

$(4,2,1,1)$-*colourable:* By symmetry $e(V_2, v_3) \leq e(V_2, v_4)$.

If $e(V_2, v_4) = 2$, add (via a star operation) all edges in $G[V_1, v_3]$, making $G[V_1, V_3]$ and $G[V_2, V_4]$ complete bipartite. No shortcut is possible.

If $e(V_2, v_3) = e(V_2, v_4) = 1$, by Lemma 4.1.3 $v_2v_3, v_2v_4 \in E(G)$. Adding all edges in $G[V_1, v_3]$ via a star, the resulting graph has no (1234)-shortcut, since such a shortcut would end with v_2v_3 and v_3v_4, but $G[V_1, V_3]$ and $G[v_2, V_4]$ are now complete bipartite.

$(3,3,1,1)$-*colourable:* Then $e(V_i, V_j) \geq 1$ for $i \in \{1,2\}$ and $j \in \{3,4\}$. If $e(V_2, v_3) = 1$, by Lemma 4.1.3, $v_2v_3, v_2v_4 \in E(G)$, and adding all edges in $G[V_1, v_3]$ via a star gives no shortcut. Otherwise $e(V_i, V_j) \geq 2$ for all such i, j, and by adding at most two edges (using a matching), we make $G[V_1, V_3]$ and $G[V_2, V_4]$ complete bipartite, eliminating all shortcuts.

$(3,2,1,2)$-*colourable:* We may assume G is not $(3,3,1,1)$-colourable. So we can add a matching to $G[V_2, V_4]$ to make it complete bipartite. If $e(V_1, v_3) \geq 2$, we add another matching to make $G[V_1, V_3]$ complete bipartite, eliminating all shortcuts.

If $e(V_1, v_3) = 1$, by swapping V_2 and V_3 we may assume $E(V_1, V_2) = \{v_1v_2\}$. Note $e(v_2, V_3) \geq 1$, so by recolouring v_1 with colour 3 and v_2 with colour 1, we may assume

$e(v_1, V_3) \geq 1$. Then a matching makes $G[v_1, V_3]$ and $G[V_2, V_4]$ complete bipartite, and no (1234)-shortcut survives.

$(2, 2, 2, 2)$-*colourable:* We may assume G is not $(3, 2, 2, 1)$-colourable. Then for any $1 \leq i < j \leq 4$, we can add a matching to make $G[V_i, V_j]$ complete bipartite, since otherwise some vertex $v \in V_i \cup V_j$ has no edge in $G[V_i, V_j]$, allowing us to recolour and obtain a $(3, 2, 2, 1)$-colouring. Adding the matching to make $G[V_1, V_3]$ and $G[V_2, V_4]$ complete bipartite removes all shortcuts.

This exhausts all cases with $\chi(G) \in \{4, 5, 6\}$, completing the proof. $\square$

Comments on the proof structure

The proof of Theorem 4.1.5 fits into a coherent overall scheme: orient edges from smaller to larger colour, augment by carefully chosen extra edges (a star or a matching), and verify that the resulting orientation is semi-transitive. The case analysis is unavoidably intricate when $\chi(G) = 5$ because there are many possible non-edges to track and several types of shortcut to rule out. As noted in Section 4.3 below, the general 1-11-representability question, already a long-standing open problem at the time, has subsequently been resolved by an entirely different and remarkably efficient method.

4.2 The multi-1-11-representation number

Remark 4.2.1 (Status of this section in light of Hefty–Horn–Muir [22])*.* The content of this section was published in [3] as a step towards the then-open question of whether every graph is 1-11-representable. Theorem 4.2.4 establishes that every graph on at most 24 vertices has strict multi-1-11-representation number at most 2, building on the 8-vertex result of Theorem 4.1.5. At the time of publication, this was a substantive partial result: no general bound on the multi-1-11-representation number was known, and the construction used here (partitioning into three 8-vertex blocks and exploiting the 3-colourability of the cross-block graph) is a constructive approach to the more general question.

Subsequent to the publication of [3], Hefty, Horn, and Muir [22] proved the much stronger result that every graph is 1-11-representable as a single word (Theorem 4.3.1). This subsumes the multi-representation question in the form posed here: it follows that every graph has multi-1-11-representation number 1. The construction of [22] is independent of ours and proceeds by a different route.

We retain this section as a record of the published paper [3], including the formalism of Definition 4.2.2 (the distinction between the strict and non-strict multi-representation numbers, introduced for the first time in [3]). The open questions stated at the end of this section, several of which remain interesting in their finer quantitative forms, are revisited in Section 4.3 in light of [22].

In this section we generalise the notion of representation by a single word to representation by a tuple of words sharing the same vertex set, motivated by the multi-word-representation number introduced by Kenkireth and Malhotra [25]. The development follows [3] closely.

Definition 4.2.2 (Multi-k-11-representation number)**.** Let G be a graph and let $k \geq 0$. A *multi-k-11-representation* of G is a family $G_1, \ldots, G_m$ of graphs such that

- $V(G_i) = V(G)$ for every i;
- $E(G) = \bigcup_{i=1}^{m} E(G_i)$;
- each G_i is k-11-representable.

The *multi-k-11-representation number* of G is the least m for which a multi-k-11-representation of G by m graphs exists.

Definition 4.2.3 (Strict multi-k-11-representation number)**.** Let G be a graph and let $k \geq 0$. A *strict multi-k-11-representation* of G is a multi-k-11-representation $G_1, \ldots, G_m$ of G in the sense of Definition 4.2.2 which in addition satisfies

$$E(G_i) \cap E(G_j) = \emptyset \qquad \text{for all } 1 \leq i < j \leq m,$$

so that the edge sets $E(G_1), \ldots, E(G_m)$ partition $E(G)$. The *strict multi-k-11-representation number* of G is the least m for which a strict multi-k-11-representation of G by m graphs exists.

Separating the two notions in this way makes clear that the disjointness requirement is a condition on the family $G_1, \ldots, G_m$, and not a condition on the represented graph.

In this terminology, the multi-word-representation number of [25] is the multi-0-11-representation number. The strict version is introduced in [3] for the first time.

Definition 4.2.2 is interesting only in the cases $k \in \{0, 1\}$: if $k \geq 2$, every graph is k-11-representable by a single word (Theorem 2.3.3), so the multi-k-11-representation number is always 1.

The (non-strict) multi-1-11-representation number is at most equal to the strict multi-1-11-representation number, since a strict multi-representation is also a (non-strict) multi-representation.

Theorem 4.2.4 ([3])**.** *All graphs on at most* 24 *vertices have a strict multi-*1-11-*representation number of at most* 2*.*

The construction underlying the proof is depicted schematically in Figure 4.7.

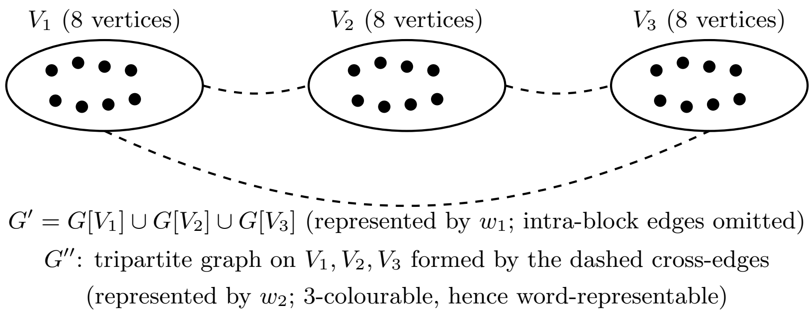


Figure 4.7: Schematic of the construction in the proof of Theorem 4.2.4. The 24 vertices of G are partitioned into three blocks V_1, V_2, V_3 of 8 vertices each. The first word w_1 represents the disjoint union $G' = G[V_1] \cup G[V_2] \cup G[V_3]$ of the three induced 8-vertex subgraphs (intra-block edges are omitted for clarity). The second word w_2 represents the cross-block graph G'' (dashed edges), which is 3-colourable with V_1, V_2, V_3 as colour classes and hence word-representable.

Proof. Suppose G is a graph on 24 vertices (for smaller graphs, the conclusion follows from the hereditary nature of 1-11-representation). Partition $V(G)$ into three disjoint subsets V_1, V_2, V_3, each containing 8 vertices. By Theorem 4.1.5, each induced subgraph $G[V_i]$, $i \in \{1, 2, 3\}$, is 1-11-representable. By Lemma 2.3.4(a), the disjoint union $G' := G[V_1] \cup G[V_2] \cup G[V_3]$ is 1-11-representable; let w_1 be a 1-11-representant. Here the *disjoint union* of graphs $H_1, \ldots, H_r$ on pairwise disjoint vertex sets is the graph with vertex set $\bigcup_i V(H_i)$ and edge set $\bigcup_i E(H_i)$; that is, the graphs are placed side by side with no edges added between them. In the present case V_1, V_2, V_3 partition $V(G)$, so $V(G') = V(G)$, while $E(G')$ consists exactly of the edges of G having both endpoints inside a single part, the edges of G running between different parts being omitted.

Now, removing all edges within $G[V_1], G[V_2], G[V_3]$ from G yields a graph G'' that is 3-colourable (with V_1, V_2, V_3 as colour classes), hence word-representable, hence 1-11-representable [8]. Let w_2 be a 1-11-representant of G''.

Since $V(G) = V(G') = V(G'')$, $E(G) = E(G') \cup E(G'')$, and $E(G') \cap E(G'') = \emptyset$, the strict multi-1-11-representation number of G is at most 2. □

Open questions on multi-1-11-representation

The published paper [3] closes with several open questions, which we record here for completeness:

- Are all 4-colourable graphs 1-11-representable? If so, the bound of 24 in Theorem 4.2.4 can be improved to 32 (partition into four sets of size 8, and use the 4-colourable G'').

- If proving or disproving 1-11-representability for all 4-colourable graphs is too challenging, one could ask the question for planar graphs, a subclass.

- Assuming that proving or disproving 1-11-representability for all graphs is infeasible with existing tools, one could focus on showing that the (strict) multi-1-11-representation number is at most 2 for all graphs.

- The notions of strict and non-strict multi-k-11-representation number are equivalent for $k \geq 2$. What can be said about $k \in \{0, 1\}$? Are there counter-examples?

- Definition 4.2.2 can be refined to the *ℓ-multi-k-11-representation number*, where any edge can belong to at most ℓ subgraphs.

- Numerical experiments by Herman Chen suggest that the 1-11-representability of every graph on 8 vertices can be established by adding *at most one new edge*;

indeed, each of the 929 non-word-representable 8-vertex graphs can be made word-representable by adding a single edge. Our proof of Theorem 4.1.5, by contrast, often requires adding more than one edge. Can a non-computational proof be given that adding one edge suffices? Such a proof might extend to graphs on 9 vertices.

These questions were posed before the resolution of the general existence problem, and their status must now be read in the light of Theorem 4.3.1 below, which establishes that *every* graph is (permutationally) 1-11-representable. In particular, the first three questions are answered affirmatively and are no longer open: all 4-colourable graphs, all planar graphs, and indeed all graphs are 1-11-representable, and consequently every graph has (strict) multi-1-11-representation number 1. The fourth question, on the relationship between the strict and non-strict numbers for $k \in \{0, 1\}$, is likewise resolved for $k = 1$ by the same theorem, but remains meaningful for $k = 0$, that is, for multi-word-representation. The fifth question, on the refined ℓ-multi-k-11-representation number, and the sixth, on whether a single added edge suffices for graphs on eight vertices, are not settled by Theorem 4.3.1 and remain of interest; the sixth in particular concerns the *method* of proof rather than the existence statement. We discuss the questions that replace these in Section 4.3.

4.3 Subsequent work: every graph is 1-11-representable

Subsequent to the publication of the paper that forms the basis of this chapter, Hefty, Horn, and Muir [22] resolved the central open question of 1-11-representability completely. Their main results, summarised below, transform the landscape of the area.

4.3.1 The main theorems of Hefty–Horn–Muir

Theorem 4.3.1 ([22]). *Every graph is permutationally* 1-11*-representable.*

This not only settles the existence question for 1-11-representability that motivated [3], but does so in the strongest possible form: every graph admits a 1-11-representant that is the concatenation of permutations of its vertex set. Their proof is constructive and can be adapted to produce k-11-representations for any $k \geq 1$.

Theorem 4.3.2 ([22])**.** *Every graph G on n vertices admits a* 1-11*-representation consisting of the concatenation of $O(n)$ permutations.*

This length bound is essentially tight in the worst case. Hefty, Horn, and Muir prove a matching lower bound of $\Omega(n/\log n)$ permutations on certain random-like graphs, so the upper bound of Theorem 4.3.2 is tight up to a logarithmic factor. Here we use the standard asymptotic notation: for functions f, g of n, we write $f(n) = \Omega(g(n))$ to mean that there exist a constant $c > 0$ and an integer n_0 such that $f(n) \geq c\, g(n)$ for all $n \geq n_0$; that is, g is an asymptotic lower bound for f up to a constant factor. Correspondingly $f(n) = O(g(n))$ means $f(n) \leq C\, g(n)$ for some constant $C > 0$ and all sufficiently large n.

Theorem 4.3.3 ([22])**.** *There exist graphs on n vertices requiring $\Omega(n^2/\log n)$ characters in any* 1-11*-representation, whether permutational or not.*

This is a strong character-count lower bound, distinct from the permutation-count lower bound mentioned above. The previous construction from [8] of 2-11-representations for arbitrary graphs uses $\Omega(n^2)$ permutations, so the result of Hefty, Horn, and Muir is also a substantial improvement on the constructive side.

4.3.2 The crown graph phenomenon

Hefty, Horn, and Muir highlight a striking contrast with classical word-representability. Recall that the *crown graph* $H_{n,n}$ is obtained from the complete bipartite graph $K_{n,n}$ by deleting a perfect matching. The crown graph is well known to require $\rho(H_{n,n}) = \lceil n/2 \rceil$ for word-representability [19], where $\rho(\cdot)$ denotes the representation number introduced in Definition 2.2.4, namely the least k for which the graph admits a k-uniform word-representant. Figure 4.8 depicts $H_{4,4}$. By contrast:

Theorem 4.3.4 ([22])**.** *The crown graph $H_{n,n}$ is permutationally* 1-11*-representable using only* 4 *permutations, independent of n.*

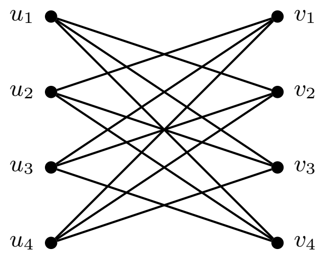


Figure 4.8: The crown graph $H_{4,4}$: two columns of 4 vertices with every cross-edge except the perfect matching $\{u_iv_i : i = 1, 2, 3, 4\}$. The word-representation number satisfies $\rho(H_{n,n}) = \lceil n/2 \rceil$ (so $\rho(H_{4,4}) = 2$), and grows linearly in n. Theorem 4.3.4 shows that, by contrast, the permutational 1-11-representation number is bounded by 4 for every n.

This shows that the relationship between word-representability and 1-11-representability is not straightforwardly monotone: some graphs that are difficult to word-represent are easy to 1-11-represent. Intuitively, the additional flexibility of allowing one 11-pattern in a non-edge subword can dramatically simplify the encoding of certain graphs.

4.3.3 Even–odd representations

Hefty, Horn, and Muir also study a parity-based variant of 11-representations. A word w is an *even–odd-representation* of G if, for every pair of letters x, y, the parity of the number of 11-occurrences in $w|_{\{x,y\}}$ determines whether $xy \in E(G)$. They prove the following:

Theorem 4.3.5 ([22]). *Almost no graph admits an even–odd-representation; the graphs that are permutationally even–odd-representable have been completely characterised.*

This answers a question of Wanless from 2018 (raised at a conference talk on the existence of 2-11-representations).

4.3.4 Open questions raised by the new resolution

With Theorem 4.3.1 settling the central existence question, the natural new questions concern the *quantitative* structure of 1-11-representations. These are phrased in terms of two parameters, which we introduce now.

Notation 4.3.6. For a graph G, let $R(G)$ denote the least number of permutations of $V(G)$ whose concatenation is a 1-11-representant of G, taken over all 1-11-representants of G that are concatenations of permutations; and let $R_\pi(G)$ denote the corresponding quantity when the representation is additionally required to be permutational in the sense of Definition 2.3.1, that is, when every permutation used is a permutation of the full vertex set $V(G)$. Both parameters are finite for every graph by Theorem 4.3.1, and Theorem 4.3.2 states that $R_\pi(G) = O(n)$ for every graph on n vertices.

Following [22], we record the principal open problems:

- **Identifying graphs with high $R_\pi(G)$.** Theorem 4.3.2 provides a lower bound of $\Omega(n/\log n)$ on the number of permutations needed in a 1-11-representation of some graphs, but the bound is ineffective: it is known to be achieved by "almost all" graphs in a probabilistic sense, but no specific construction is exhibited. Is there an easily-stated structural property of a graph that implies that $R(G)$ or $R_\pi(G)$ is large? Can one prove, for an explicitly given graph, that $R_\pi(G)$ is large?

  A natural place to look would be quasi-random graphs, particularly the *Paley graphs*, which are known to be quasi-random in the sense of Chung–Graham–Wilson [10]. Hefty, Horn, and Muir suggest that Paley graphs may have large $R_\pi(G)$.

- **Improving the upper bound for restricted classes.** The upper bound of $O(n)$ permutations in Theorem 4.3.2 is tight only up to a logarithmic factor, and the lower bound proof relies on counting-based arguments. For many natural families of graphs, bipartite graphs, graphs with a fixed chromatic number, split graphs, the family contains 2^{cn^2} graphs of order n, and counting-based arguments give matching $O(\log n)$-factor lower bounds.

  The family of *planar graphs* is particularly interesting. There are only $\exp(O(n))$ planar graphs on n vertices, so counting alone gives essentially no non-trivial lower bound. Can one find planar graphs with large $R_\pi(G)$, or prove an $o(n)$ or even $O(n^{1-\varepsilon})$ upper bound for $R_\pi(G)$ on planar graphs?

- **The crown-graph phenomenon.** Theorem 4.3.4 shows that the word-representation number can be much larger than the permutational 1-11-representation number. This is despite the intuition that encoding non-edges (which is what the 11-patterns do) requires additional "effort". It is interesting to study when this phenomenon occurs: *what causes a graph to be easier to* k-11-*represent than to* $(k-1)$-11-*represent?*

These three families of open problems, structural characterisations of graphs with large $R_\pi(G)$, sharper bounds for restricted graph classes (especially planar graphs), and a deeper understanding of the crown-graph phenomenon, now form the natural sequel to the line of research initiated in [3]. The questions of Section 4.2 can be revisited in this new light: with Theorem 4.3.1 now in hand, the multi-1-11-representation number of every graph is at most 1 (since every graph is 1-11-representable as a single word), so the multi-version of the question is no longer of interest in the form posed in [3]. However, finer quantitative questions about the lengths and structures of multi-representations remain.

Chapter 5

Conclusion

This thesis has explored two complementary themes within the broad area of graph representations by words: word-representability of Apollonian graphs, and 1-11-representability of all graphs. Although these two themes use different tools and address different questions, they are unified by the underlying combinatorial framework and by the central role played by semi-transitive orientations.

5.1 Summary of contributions

Word-representability of Apollonian graphs

In Chapter 3, we studied the natural change-of-direction subfamily of Apollonian graphs, those constructed by recursive subdivision around the most recently created vertex, encoded by a word over $\{m, \ell, r\}$. The chapter contains two main results; we present them here in the order of their position within the thesis as contributions, following the framing of the Introduction.

- The headline new contribution of the thesis to the Apollonian theory, in Section 3.2, is the result that within the change-of-direction subfamily, word-representability is in fact equivalent to being a circle graph. The proof uses an *extension invariant* that tracks a particular local structure preserved by each subdivision step, and it is constructive: in every representable case, we exhibit an explicit 2-uniform word-representation. This goes strictly beyond Hauser's characterisation [21]: while Hauser settles the word-representability question for all Apollonian networks (in terms of $\mathcal{F}_1$-freeness and local comparability), Hauser's result does not address whether the representable cases are also circle graphs, and does not produce explicit words.

- In Section 3.1, we developed a self-contained route to word-representability on the change-of-direction subfamily via semi-transitive orientations and explicit 4-colourings. This work was completed independently of, and is methodologically different from, Hauser's. It yields a complete classification of word-representability for graphs in this subfamily having at most three changes of direction, with the classification given by decidable parity conditions on the runs of m-steps surrounding each direction change. The section also contains a methodological contribution: the proofs of non-word-representability extend the human-verifiable proof format of Kitaev, Salimov, and Pyatkin [37] for the first time, as far as we are aware, from individual graphs to *infinite parametric families*, by combining a base-case derivation with a subdivision-induction step. Within our subfamily, the section retains independent value as a decidable, constructive route, complementing rather than being replaced by Hauser's more general but non-constructive existence proof.

The two main theorems of the chapter (Theorems 3.1.26 and 3.2.14) thus give:

- For arbitrary graphs in the subfamily, the equivalence between circle-graph and word-representability membership (Section 3.2).
- For graphs with at most three direction changes, a complete classification by parity conditions on the m-runs (Section 3.1).

1-11-representability

In Chapter 4, we addressed the 1-11-representability question, which at the time of the published paper [3] (joint with Kitaev, Tang, Tao, and Zhang) was a notoriously difficult open problem in the area.

- Section 4.1 extended the previously known existence result, due to Cheon et al. [8] and Futorny et al. [13], from 7 vertices to 8 vertices. The proof considers all graphs by chromatic number, applying star or matching operations to add a small number of carefully chosen edges and verifying that the resulting orientation is semi-transitive. At the time of publication this addressed a notoriously difficult open question; the result has subsequently been subsumed by Hefty, Horn, and Muir [22], whose construction is independent of ours and uses an entirely different method.

- Section 4.2 introduced the *multi*-1-11-*representation number* of a graph, with the distinction between strict and non-strict variants introduced for the first time, generalising the multi-word-representation number of Kenkireth and Malhotra [25]. As an application of the 8-vertex result, every graph on at most 24 vertices was shown to have strict multi-1-11-representation number at most 2 (Theorem 4.2.4), via a partition technique that decomposes the graph into three 8-vertex blocks and exploits the 3-colourability of the cross-block graph. Like the 8-vertex result, this contribution has been subsumed by [22], but the partition technique and the strict/non-strict formalism are independent of their approach.

- Section 4.3 summarised the recent breakthrough work of Hefty, Horn, and Muir [22], which resolves the 1-11-representability question definitively: every graph is permutationally 1-11-representable. We also discussed the new open questions raised by [22], which now form the natural successor to this line of research.

5.2 Cross-cutting observations and open problems

A few remarks span both halves of the thesis.

The role of semi-transitive orientations

In both Chapters 3 and 4, semi-transitive orientations play a fundamental role, although the way they enter the arguments differs. In Chapter 3, semi-transitive orientations are constructed directly, often from a clever 4-colouring, and the verification that no shortcut exists is itself the substance of the proof. In Chapter 4, semi-transitive orientations enter indirectly: we orient a colouring, augment by edges, and verify semi-transitivity to conclude that the augmented graph is word-representable, which then implies 1-11-representability of the original via the type-2 lemma or matching lemma. In both settings, the equivalence between word-representability and the existence of a semi-transitive orientation (Theorem 2.2.9) is the bridge between the combinatorial encoding and the structural property.

Constructive versus existential proofs

A second cross-cutting observation is the value of *constructive* representations. In Chapter 3, the move from semi-transitive orientations (Section 3.1) to explicit 2-uniform representations (Section 3.2) gave us much more than the bare existence of word-representations: it placed the graphs in the well-studied class of circle graphs, and it produced word-representants that one could write down. Similarly, in Chapter 4, the work of Hefty, Horn, and Muir [22] not only resolves the existence question for 1-11-representability but also provides quantitative bounds on representation length and even a polynomial-time construction. Future work in this area is likely to focus on the quantitative side, representation lengths, permutation numbers, and structural complexity, rather than on existence.

Some open problems

Beyond the open questions listed at the ends of Chapters 3 and 4, we record a few problems that span both halves.

- **Word-representability of Apollonian graphs as circle graphs.** Hauser [21] characterises word-representable Apollonian networks via $\mathcal{F}_1$-freeness and local comparability. In Chapter 3, we showed that within the change-of-direction subfamily, word-representable Apollonian graphs are circle graphs. *Is every word-representable Apollonian network a circle graph?* If so, this would give a sharp characterisation of word-representable Apollonian networks within a classical, well-studied class.

- **Extending the parity classification.** The parity classification of Section 3.1 covers all encodings with at most three changes of direction. *Does the parity pattern extend to encodings with arbitrarily many changes of direction?* The proof techniques (the human-verifiable proof format and the subdivision-induction step) appear to scale, but the case analysis grows with the number of direction changes, and a uniform statement of the parity condition for the general case is not yet known. A positive answer would give a decidable characterisation of word-representability for the entire change-of-direction subfamily, complementing the the existence of Hauser's result on this subfamily.

- **Combining the two themes: 1-11-representability of Apollonian graphs.** The work of Hefty, Horn, and Muir [22] shows that all graphs are 1-11-representable, but says little about the representation length or structural complexity for specific families. *What is the permutational 1-11-representation number of an Apollonian network?* For the change-of-direction subfamily, where we have an explicit 2-uniform word-representation when the graph is word-representable, can the same construction be adapted to give a small (i.e., $O(1)$ or $O(\log n)$) permutational 1-11-representation? This question fits with the open problem of Hefty–Horn–Muir on planar graphs (which include Apollonian networks).

- **Other recursive families.** The change-of-direction encoding used throughout Chapter 3 captures a natural recursive structure of Apollonian graphs, but it is not the only one: for instance, one might consider Apollonian graphs constructed by simultaneously subdividing multiple non-adjacent triangular faces, or the *generalised Apollonian networks* obtained from higher-dimensional analogues. The interaction between recursion and word-representability for such families remains largely unexplored.

- **Constructive characterisations of circle graphs.** Within the change-of-direction subfamily, we characterised circle-graph membership via an explicit constructive recipe (Theorem 3.2.14). *Is there a similarly constructive characterisation of circle-graph membership for arbitrary planar graphs, or for chordal* *near-triangulations more generally?* This would be a substantial step towards the much harder problem of characterising word-representable planar graphs.

- **Quantitative bounds for representation numbers.** For the change of direction subfamily of Apollonian graphs, we showed that the representation number is at most 2 when the graph is word-representable. *Can finer quantitative results be obtained, for example, the minimum length of a* 2*-uniform word-representant in terms of the encoding?* This question connects to the broader study of minimum-length word-representations, which remains open even for many small graph classes.

We close by remarking on the changing landscape of the area. At the start of this doctoral research, the question "is every graph 1-11-representable?" was a notable open problem; within the span of this work, that question has been definitively answered by Hefty, Horn, and Muir. The result has not closed the area but redirected it towards more refined questions of length, structure, and concrete constructibility. The journey of the area from existence questions towards quantitative and constructive ones mirrors the structure of this thesis itself: from semi-transitive orientations to explicit 2-uniform representations on the Apollonian side, and from 8-vertex existence results towards the present landscape on the 1-11 side. We hope that the techniques and perspectives developed here will be of use to those who pursue the next steps.

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