Stability and Stabilisation of Stochastic Delay Systems

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September 2010

This thesis is submitted to the University of Strathclyde for the degree of Doctor of Philosophy in the Faculty of Science.

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Acknowledgements

Many thanks go to Professor Xuerong Mao for his support and guidance as my supervisor throughout my research. Special thanks are due to Dr. Jiazhu Pan (my second supervisor) and Dr. Kan Chen for the helpful conversations. I would also like to thank all my colleagues at the University of Strathclyde for providing me with a stimulating and exciting research environment.

Financial support from the UK ORSAS and the University of Strathclyde is gratefully acknowledged.

I am forever indebted to my family (my parents and my brother) for their support, and, particularly, my wife, Zhujun Deng, for her constant love, patience and kindness. I am also indebted to my son, Zihong Huang. None of these would be possible without the unconditional love from you guys – you are the best!

Abstract

In many real-life dynamical systems, the future evolution of the state variables depends not only on their current values but also on their past values over a finite period of time. Such systems are called time-delay systems or delay systems. For both their theoretical and practical impact, time-delay systems have been an enduring theme in the study of systems and control theory. Stochastic delay systems are those affected by random noise. Although they may be regarded as deterministic when the noise content is negligible, all practical systems are stochastic. Itô stochastic systems are a class of the most important stochastic systems.

In recent years, Itô stochastic delay systems, or simply, stochastic delay systems have been intensively studied since stochastic modelling has come to play an important role in many branches of science and engineering. An area of particular interest has been the stability analysis of this class of dynamical systems and its application to automatic control. This thesis is focused on developing stability criteria and their applications to stabilisation problems of stochastic delay systems.

Due to time spent, e.g., in computation and transfer, control input is usually subject to delays. The presence of input delay may be the cause of the poor performance or even instability of the resulting controlled system if it is not considered in controller design. Problems of stabilisation for deterministic systems with input delay have received a great deal of attention while few works are concerned with those for stochastic systems. This thesis establishes a delay-dependent criterion for exponential stability of stochastic delay systems and, based on the stability result, proposes a state-feedback controller for stabilisation of stochastic systems with input delay. And then this thesis further develops the techniques and obtains the corresponding results for neutral stochastic delay systems.

Sliding mode control (SMC) has various attractive features and has been one of the

most popular control methods. In recent years, there has been a growing interest in extension of SMC to accommodate stochastic systems. However, the existing results employ an assumption on the structure of the control system such that their controller design does not need to deal with the diffusion and then they can use the SMC methods for deterministic systems. This thesis aims to remove such an assumption and propose a practical SMC design method for stochastic (delay) systems.

It is noted that, since Markov jump linear systems were first introduced in the early 1960s, hybrid systems driven by continuous-time Markov chains have been widely employed to model many practical systems where they may experience abrupt changes in system structure and parameters. Consequently, an area of particular interest has been the stability analysis of these hybrid systems. This thesis presents the Razumikhin-type theorems on pth moment asymptotic stability. Since many practical systems are subject to disturbance, the thesis also studies pth moment input-to-state stability of stochastic retarded systems with Markovian switching. Moreover, this thesis investigates almost sure stability of hybrid stochastic systems with mode-dependent delays by proposing a new concept for Markovian jump delay systems and improving an existing result.

Contents

General notation

$a.e.$	:	Almost everywhere.
$a.s.$	:	Almost surely, or with probability 1.
$\emptyset$	:	The empty set.
$A := B$	:	A is defined by B or B is denoted by A.
$\mathbb{I}_A$	:	The indicator function of set A, i.e. $\mathbb{I}_A(x) = 1$ if $x \in A$ or otherwise 0.
A^C	:	The complement of A in Ω, i.e. $A^C = \Omega - A$.
$A \subset B$	:	$A \cap B^C = \emptyset$.
$A \subset B$ $a.s.$	:	$P(A \cap B^C = \emptyset) = 1$.
$\sigma(C)$	:	The σ-algebra generated by C.
$a \vee b$	:	The maximum of a and b.
$a \wedge b$	:	The minimum of a and b.
$f : A \to B$	:	The mapping f from A to B.
$R = R^1$	:	The real line.
R_+	:	The set of all nonnegative real numbers, i.e. $R_+ = [0, \infty)$.
$\mathcal{B} = \mathcal{B}^1$	:	The Borel-σ-algebra on R.
$\mathcal{B}^n$	:	The Borel-σ-algebra on R^n.
$\lvert \cdot \rvert$	:	The Euclidean norm of a vector and its induced norm of a matrix.
$C(D; R^n)$	:	The family of continuous R^n-valued functions defined on D.
$C^m(D; R^n)$	:	The family of continuously m-times differentiable R^n-valued functions defined on D.
$C^{2,1}(D \times R_+; R)$	:	The family of all real-valued functions $V(x,t)$ defined on $D \times R_+$

which are continuously twice differentiable in $x \in D$ and once differentiable in $t \in R_+$.

V_x : $= \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_n} \right)$.

V_{xx} : $= \left(\frac{\partial^2 V}{\partial x_i \partial x_j} \right)_{n \times n}$.

$\|\xi\|_{L^p}$: $= (\mathbb{E}|\xi|^p)^{1/p}$.

$(\Omega, \mathcal{F}, \mathbb{P})$: a complete probability space.

$(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$: a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions.

$L^p(\Omega; R^n)$: The family of R^n-valued random variables X defined on $(\Omega, \mathcal{F}, \mathbb{P})$ with $\mathbb{E}|X|^p < \infty$.

$L^p_{\mathcal{F}_t}(\Omega; R^n)$: The family of R^n-valued $\mathcal{F}_t$-measurable random variables X with $\mathbb{E}|X|^p < \infty$.

$C([-h, 0]; R^n)$: The space of all continuous R^n-valued functions ϕ defined on $[-h, 0]$ with a norm $\|\phi\| = \sup_{-h \leq \theta \leq 0} |\phi(\theta)|$.

$L^p([a, b]; R^n)$: The family of Borel measurable functions $f : [a, b] \to R^n$ such that $\int_a^b |f(t)|^p dt < \infty$.

$\mathcal{L}^p([a, b]; R^n)$: The family of R^n-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a \leq t \leq b}$ such that $\int_a^b |f(t)|^p dt < \infty$ a.s.

$\mathcal{M}^p([a, b]; R^n)$: The family of R^n-valued $\mathcal{F}_t$-adapted processes $\{f(t)\}_{a \leq t \leq b}$ in $\mathcal{L}^p([a, b]; R^n)$ such that $\mathbb{E} \int_a^b |f(t)|^p dt < \infty$.

$\mathcal{K}$: The class of continuous strictly increasing functions $\mu : R_+ \to R_+$ with $\mu(0) = 0$, whose inverse function is denoted by μ^{-1} with domain $[0, \mu(\infty))$.

$\mathcal{K}_\infty$: The class of functions $\mu \in \mathcal{K}$ with $\mu(r) \to \infty$ as $r \to \infty$.

$V\mathcal{K}$: The class of functions $\mu \in \mathcal{K}$ and μ is convex.

$C\mathcal{K}$: The class of functions $\mu \in \mathcal{K}$ and μ is concave.

$\mathcal{KL}$: The class of functions $\beta : R_+ \times R_+ \to R_+$ such that, for each fixed t, the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ while, for each fixed s, $\beta(s,t)$ is decreasing to zero on t as $t \to \infty$.

Other notations will be explained where they first appear.

Chapter 1

Introduction

1.1 Background

The modelling of any physical system is subject to uncertainty. Such uncertainty may be due to nonmeasurable disturbances and unknown or only partly known system parameters. Having realized the necessity of introducing more realistic models of disturbances, we are faced with the problem of finding suitable ways to characterize them. A characteristic feature of practical disturbances is the impossibility of predicting their future values precisely.

Itô stochastic systems, or simply, stochastic systems are widely used to model dynamical processes in many disciplines, ranging from biology to finance (see, e.g., [56] and [94]). The study of stochastic systems is a particularly multidisciplinary subject. Brownian motion for example owes its name to the botanist R. Brown, who observed the incessant random motion of tiny pollen particles in water under a microscope in 1828. There were numerous explanations of such motion of the small pollen grains proposed and disposed of in the more than 70 years until A. Einstein's relatively independent and more famous treatment that the motion of the particles was due to impact with fluid molecules subject to their expected Boltzmann distribution of velocities (see [70] and [56]). The observations of Brownian motion would have significant consequences, ranging from the experimental proof that matter was made from molecules to today's growing understanding of how biological molecules keep cells going such as Brownian ratchets forming the basis of molecular motors [9] and self-avoiding random walks on lattices employed as models of proteins and polymers [107]. In the world of finance, the earliest attempts to mathematically introduce Brownian motion into the financial models can be traced to the work of L. Bachelier in the context of speculation in the stock markets (see, e.g., [16]). However, this result along with mathematical developments closely related to Brownian motion remained obscure until much later when Black and Scholes employed the ideas to investigate how to price derivative securities. Many branches of science, e.g., statistical physics, study stochastic modelling and attempt to describe macroscopic properties in terms of the average behavior of an extremely large number of microscopic degrees of freedom.

For a long time, noise has been a challenging problem that has to be dealt with in engineering (see, e.g., [2], [4], [11], [49] and [125]). This leads to a wide range of fundamental physical discoveries including thermal noise and shot noise (see, e.g., [70] and [56]). Interestingly, in the viewpoint of practice, Brownian motion was also introduced into the early engineering models in the form of white noise. Let us consider the dynamics of a finite dimensional system

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f(x(t), t) + N(t), \qquad \forall\, t \geq 0 \tag{1.1}$$

where $N(t)$ represents the external noise disturbances. The noise process $N(t)$ is, to begin with, not as well-defined as the signal of the state $x(t)$. About all that one can say for the noise process is that, since it accounts for the (non-systematic) instrument error and can often be ascribed to thermodynamic origin, it can be modelled as a wide-sense stationary Gaussian process with a spectral density which is constant over a range of frequencies wider than the frequency range of the state (since the instrument, if well designed, is not supposed to "distort" the signal of the state). In the absence of precise information about the noise bandwidth, which is characteristically the case in practice, it has been customary to translate the concept of "large" bandwidth to infinite bandwidth and refer to it as white noise in the early engineering literature (see, e.g., [4]). Moreover, in a case when $N(t)$ in system (1.1) is a process with large but finite bandwidth, it has been shown that only a correction term need to be introduced if system (1.1) is to be modelled by an Itô equation (see [125]).

Time delay is the property of a dynamical system by which the response to an applied force is delayed in its effect (see [136] and the references therein). Dynamical systems with delays, or say, time-delay systems abound in the world. One of the important reasons is that whenever material, information or energy is physically transmitted from one place to another, there is a delay associated with the transmission. Some delays are short, some are very long. They appear in various systems such as biological, ecological, economic, social, engineering systems etc. For example, in biological systems, delays may well be of a few hundred milliseconds duration (response-time in human being), whereas in signal processing, delays measured in microseconds, or much shorter times, may be very important. Over exposure to radiation increases the risk of cancer, but the onset of cancer typically follows exposure to radiation by many years. In electronic engineering systems, the physical limit to speed of processing in digital computing systems is set by transition times, that is the time for electrons to travel finite distances. The delay between the transmission of electromagnetic waves from an aerial and the reception of its reflection from a distant object though of short duration is the principal feature upon which radar is based. In economics, the central bank in a country often attempts to influence the economy by adjusting interest rates, the effect of which takes months to be translated into an impact on the economy. In politics, politicians need some time to make decisions and they will have to wait for some time before they find out whether the decisions are correct or not. In daily life, the driver has to wait for the steering to take effect when reversing a car around a corner. There are many more examples for real-life systems with time delays, see, e.g., [33], [93], [136] and the references therein. It is observed that the future evolution of a time-delay system depends not only on its present state but also on its history. This particular cause and effect relationship can be most succinctly captured, and has been traditionally modelled, by differential-difference equations, or more generally, by functional differential equations. The presence of time delays makes system analysis and control design much more complex. For both their theoretical and practical impact, time-delay systems have been an enduring theme in the study of systems and control theory, see, e.g., [33], [34], [53], [93] and [136].

Itô stochastic delay systems, or simply, stochastic delay systems are time-delay systems affected by white noise. Consequently, there are many practical examples of stochastic delay systems, see, e.g., [98]. In recent years, stochastic delay systems have received much attention since stochastic modelling has come to play an important role in many branches of science and engineering (see, e.g., [5], [17], [19], [25], [42]-[45], [53], [60], [67],

[72]-[78], [82]-[84], [88], [98], [102], [110], [112]).

Since Markov jump linear systems were first introduced in the early 1960s (see, e.g., [92], [48] and [135]), hybrid systems driven by continuous-time Markov chains have been widely employed to model many real-life systems where they may experience abrupt changes in system structure and parameters such as battle management in command, control and communication (BM/C^3) systems [3], biological systems [41], electric power systems [124], failure prone manufacturing [30], macroeconomic models of national economy [49], population dynamics [68] and solar-powered systems [117]. In his monograph [92], Mariton discussed how such hybrid systems have also emerged as a convenient mathematical framework for the formulation of various design problems in target tracking, fault tolerant control and manufacturing processes. Recently, stochastic delay systems with Markovian switching, also called hybrid stochastic delay systems, have been studied in many works. For example, Mao et al. studied stability and stabilisation of stochastic delay systems with Markovian switching (see, e.g., [79]-[81], [85]-[91], [133]) while Yang et al. proposed a comparison theorem for one-dimensional hybrid stochastic delay systems [132]. Consequently, the stability analysis of these hybrid systems has received a great deal of attention.

1.2 Overview of the study

This thesis focuses on developing stability criteria that are less conservative than the existing results and their applications to stabilisation problems of stochastic delay systems. As is known, hybrid systems driven by continuous-time Markov chains have been widely employed to model many practical systems where they may experience abrupt changes in system structure and parameters. This thesis also studies stability of stochastic delay systems with Markovian switching.

Chapter 2 introduces the basic theory of stochastic analysis. It begins with elementary probability definitions and proceeds to the basic theory of stochastic calculus and stochastic differential equations including the important results that are used in this thesis. It should be pointed out that concepts and theorems in this chapter may be found in many mathematical books on stochastic analysis, see, e.g., [29], [62], [105], [113] and

[123] while Mao's book [88] is the main source of reference for this thesis.

Due to time spent in computation and transfer, control input is usually subject to delays. Problems of deterministic systems with input delay have received considerable attention. However, relatively few works are concerned with problems of stochastic systems with input delay. Chapter 3 studies delayed feedback stabilisation of uncertain stochastic systems. Based on a new delay-dependent stability criterion established in this chapter, a robust delayed-state-feedback controller that exponentially stabilises the uncertain stochastic systems is proposed. Numerical examples are given to verify the effectiveness and less conservativeness of the proposed method.

Extending the techniques proposed in Chapter 3 to neutral-type systems, Chapter 4 studies stability and stabilisation of neutral stochastic delay systems with delayed state feedback control by the linear matrix inequality (LMI) approach. Delay-dependent criteria for exponential stability are presented and a memoryless delayed-state-feedback control law is proposed to exponentially stabilise the neutral stochastic delay systems.

In recent years, there has been a growing interest in extension of sliding mode control (SMC) to accommodate stochastic systems. However, the existing results employ an assumption on the system structure that may be too restrictive in many practical situations (see [13]-[15], [45] and [102]-[104]). Chapter 5 aims to remove such an assumption and propose a sliding mode control design method for stochastic delay systems. Our design method is presented in terms of LMIs, which can be easily implemented.

The Razumikhin method has been developed to cope with the difficulty arisen from the large, fast varying and nondifferentiable time delays. It plays an important role in stability theory of time-delay systems (see [33] and [34]). Chapter 6 studies Razumikhin-type theorems on general asymptotic stability of stochastic retarded systems with Markovian switching, which are a generalization of an existing result on exponential stability of stochastic functional differential equations with Markovian switching (see, e.g., [86]).

Chapter 7 develops a Razumikhin-type theorem on pth moment input-to-state stability of hybrid stochastic retarded systems (also known as stochastic retarded systems with Markovian switching), which is an improvement of the result in Chapter 6. An application to hybrid stochastic delay systems verifies the effectiveness of the improved result.

The classical stochastic analysis theory studies stability not only in moment sense but also in almost sure sense (see, e.g., [36], [84] [97] and [133]). Among the existing results, [133] studied almost sure stability of HSDSs with the techniques proposed in [84] while most of the others dealt with moment stability. However, the results in [133] require the time delays of all subsystems to be equal to a constant. This may be too restrictive to apply to hybrid systems in many practical cases. Chapter 8 extends the results in [133] to hybrid stochastic systems (HSSs) with mode-dependent interval delays, which exploits a relationship between the bounds of time delays and the generator of the continuous Markov chain. A couple of numerical examples are exhibited to show that our result applies to some cases where the existing results do not work.

1.3 Main contributions

The output of the study includes a number of publications. They are main results of Chapters 3-8 in this thesis respectively and listed as follows:

- Lirong Huang and Xuerong Mao, Robust delayed-state-feedback stabilization of uncertain stochastic systems, Automatica, vol.45, 2009, 1332-1339.
- Lirong Huang and Xuerong Mao, Delay-dependent exponential stability of neutral stochastic delay systems, IEEE Transactions on Automatic Control, vol.54, 2009, 147-152.
- Lirong Huang and Xuerong Mao, SMC design for robust H_∞ control of uncertain stochastic delay systems. Automatica, vol.46, 2010, 405-412.
- Lirong Huang, Xuerong Mao and Feiqi Deng, Stability of hybrid stochastic retarded systems, IEEE Transactions on Circuits and Systems I: Regular Papers, vol.55, 2008, 3413-3420.
- Lirong Huang and Xuerong Mao, On input-to-state stability of stochastic retarded systems with Markovian switching, IEEE Transactions on Automatic Control, vol.54, 2009, 1898-1902.
- Lirong Huang and Xuerong Mao, On almost sure stability of hybrid stochastic systems with mode-dependent interval delays, IEEE Transactions on Automatic Control, vol.55, 2010, 1946-1952.

Chapter 2

Basic stochastic analysis

In this chapter, we introduce basic concepts and theory of stochastic analysis that are useful for the development of this thesis. We omit the proofs since they can be found in many textbooks, e.g., [1], [29], [36], [53], [62], [87], [88], [113] and [114].

2.1 Probability theory

Probability theory is the mathematics for trials with uncertainty. The outcome of a trial cannot be precisely predicted but is known to be one of a specified set of possibilities. We call this set the sample space and denote it by Ω. Generally, not every subset of Ω is an observable or interesting event. We denote by $\mathcal{F}$ a family of those observable, interesting and satisfying the following properties

(1) $\emptyset \in \mathcal{F}$, where $\emptyset$ is the empty set.

(2) If $A \in \mathcal{F}$, then the complement of it $A^C \in \mathcal{F}$.

(3) $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$ when $A_i \in \mathcal{F}$ for all $i \geq 1$.

The family $\mathcal{F}$ is called a σ-algebra. The pair $(\Omega, \mathcal{F})$ is called a measurable space; the elements of $\mathcal{F}$ are called measurable sets. If $\mathcal{C}$ is a collection of subsets of Ω, then the smallest σ-algebra $\sigma(\mathcal{C})$ containing $\mathcal{C}$ is called the σ-algebra generated by $\mathcal{C}$. When $\Omega = R^n$ and $\mathcal{C}$ is the collection of all open sets, $\mathcal{B}^n = \sigma(\mathcal{C})$ is called the Borel σ-algebra and the elements of $\mathcal{B}^n$ are called the Borel sets.

Let $(\Omega, \mathcal{F})$ and $(\Omega', \mathcal{F}')$ be measurable spaces. A mapping $X : \Omega \to \Omega'$ is said to be (Ω, Ω')-measurable if

$$\{\omega : X(\omega) \in A'\} \in \mathcal{F}\,, \qquad \forall\ A' \in \mathcal{F}'.$$

Particularly, when $\Omega' = R$ and $\mathcal{F}' = \mathcal{B}$, the function X is called a real-valued ($\mathcal{F}$-measurable) random variable. For example, the indicator of set $A \in \mathcal{F}$

$$\mathbb{I}_A(\omega) = \begin{cases} 1 & \text{for } \omega \in A, \\ 0 & \text{for } \omega \notin A. \end{cases}$$

is an $\mathcal{F}$-measurable random variable.

Let $(\Omega, \mathcal{F})$ be a measurable space. A function $\mu : \mathcal{F} \to R$ is called a measure if

(1) $0 \le \mu(A) \le \infty, \qquad \forall\ A \in \mathcal{F}$,

(2) $\mu(\emptyset) = 0$,

(3) $\mu(\bigcup_{i=1}^{\infty} A_i) = \sum_{i=1}^{\infty} \mu(A_i), \qquad$ if $A_i \in \mathcal{F}$ for all $i \ge 1$ and $A_n \cap A_m = \emptyset$ $(n \ne m)$.

Then the triple $(\Omega, \mathcal{F}, \mu)$ is called a measure space and $\mu(A)$ is called the measure of set A. The measure μ is said to be finite if $\mu(\Omega) < \infty$. A probability measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$ is a finite measure such that $\mathbb{P}(\Omega) = 1$. In this case, the triple $(\Omega, \mathcal{F}, \mathbb{P})$ is called a probability space. Set

$$\overline{\mathcal{F}} = \{A \subset \Omega :\ \exists\, B\,, C \in \mathcal{F} \text{ such that } B \subset A \subset C \text{ and } \mathbb{P}(B) = \mathbb{P}(C)\}$$

is a σ-algebra and is called the completion of $\mathcal{F}$. If $\mathcal{F} = \overline{\mathcal{F}}$, then the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is said to be complete.

In the sequel, let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space.

Suppose $\{A_n\}$ is a sequence of sets in $\mathcal{F}$. The upper limit of $\{A_n\}$ is defined as

$$\limsup_{n\to\infty} A_n = \{\omega : \omega \in A_k \text{ for infinitely many } k\} = \bigcap_{i=1}^{\infty} \bigcup_{n=i}^{\infty} A_n\,.$$

Then we have the following famous lemma.

Lemma 2.1 (Borel-Cantelli lemma)

(1) If $\{A_n\} \subset \mathcal{F}$ and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$, then

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = 0$$

(2) If the sequence $\{A_n\} \subset \mathcal{F}$ is independent and $\sum_{n=1}^{\infty} \mathbb{P}(A_n) = \infty$, then

$$\mathbb{P}(\limsup_{n\to\infty} A_n) = 1.$$

If a real-valued random variable X is absolutely integrable with respect to the probability measure $\mathbb{P}$, then

$$\mathbb{E}X = \int_{\Omega} X(\omega) \mathrm{d}\mathbb{P}(\omega)$$

is called the expectation of X (with respect to $\mathbb{P}$). Similarly, $\mathbb{E}[X^p]$ $(p > 0)$ is called the pth moment of X.

For $p > 0$, let $L^p = L^p(\Omega; R)$. Suppose that $X \in L^2$ and $Y \in L^2$. The covariance of X and Y is given as

$$Cov(X, Y) = \mathbb{E}[(X - \mathbb{E}X)(Y - \mathbb{E}Y)]\,.$$

Random variables X and Y are said to be uncorrelated if $Cov(X, Y) = 0$; otherwise, correlated. Some important inequalities are introduced as follows.

(1) **Hölder's inequality**

$$|E(X^TY)| \le (E|X|^p)^{\frac{1}{p}} (E|Y|^q)^{\frac{1}{q}}, \qquad \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1,\ X \in L^p,\ Y \in L^q.$$

This is also known as the Cauchy-Schwarz inequality when $p = 2$.

(2) **Minkowski's inequality**

$$(E|X + Y|^p)^{\frac{1}{p}} \le (E|X|^p)^{\frac{1}{p}} + (E|Y|^p)^{\frac{1}{p}}, \qquad \text{where } p \ge 1,\ \ X \in L^p,\ Y \in L^p.$$

Sometimes this is called triangle inequality in L^p.

(3) **Chebyshev's inequality**

$$P(|X| \ge c) \le c^{-p} E|X|^p, \qquad \text{where } p > 0,\ c > 0,\ X \in L^p.$$

(4) **Jensen's inequality**

$$\mathbb{E}c(X) \ge c(\mathbb{E}X)\,,$$

where $c : G \to R$ is a convex function on an open subinterval G of R and X is a random variable such that $\mathbb{E}|X| < \infty$, $\mathbb{P}(X \in G) = 1$ and $\mathbb{E}|c(X)| < \infty$.

(5) **Fatou's lemma**

$$\int_\Omega \liminf_{n\to\infty} X_n \,\mathrm{d}\mathbb{P} \le \liminf_{n\to\infty} \int_\Omega X_n \,\mathrm{d}\mathbb{P},$$

where $\{X_n\}$ are a sequence of nonnegative random variables defined on $(\Omega, \mathcal{F}, \mathbb{P})$.

Let $A \in \mathcal{F}$, $B \in \mathcal{F}$ and $\mathbb{P}(B) > 0$. The conditional probability of A under condition B is

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

Suppose that $\mathcal{G}$ is a sub-σ-algebra of $\mathcal{F}$, that is, $\mathcal{G}$ is a σ-algebra such that $\mathcal{G} \subset \mathcal{F}$. Obviously, triplet $(\Omega, \mathcal{G}, \mathbb{P})$ is probability space. Let $X \in L^1$. Although it is $\mathcal{F}$-measurable, X may be not $\mathcal{G}$-measurable. However, by the Radon-Nikodym theorem, there exists an integrable $\mathcal{G}$-measurable random variable Y such that

$$\mathbb{E}(\mathbb{I}_G Y) = \mathbb{E}(\mathbb{I}_G X) \quad \text{or} \quad \int_G Y(\omega)\mathrm{d}\mathbb{P}(\omega) = \int_G X(\omega)\mathrm{d}\mathbb{P}(\omega), \quad \forall\, G \in \mathcal{G}.$$

Moreover, if Y' is another random variable with these properties, then $Y' = Y$ *a.s.*, that is, $\mathbb{P}(Y' = Y) = 1$. Such a random variable Y is called (a version of) the conditional expectation of X given $\mathcal{G}$, which is written as $Y = \mathbb{E}(X|\mathcal{G})$.

2.2 Stochastic Processes

A filtration $\{\mathcal{F}_t\}_{t\ge0}$ is a family of increasing sub-σ-algebras of $\mathcal{F}$, that is, $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \le s < t < \infty$. The filtration is said to be right continuous if $\mathcal{F}_t = \bigcap_{u>t} \mathcal{F}_u$ for all $t \ge 0$. Since $(\Omega, \mathcal{F}, \mathbb{P})$ is a complete probability space, we say the filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfies the usual conditions if it is right continuous and $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets.

In the sequel, let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\ge0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\ge0}$ satisfying the usual conditions.

A stochastic process is a collection of R^n-valued random variables $\{X_t\}_{t\in I}$ defined on $(\Omega, \mathcal{F}, \mathbb{P})$. The parameter set I is usually $\{0, 1, 2, \cdots\}$ (the discrete case), some interval $[t_1, t_2]$ or the halfline $R_+ = [0, \infty)$ of R (the continuous case). It is worth noting that, for each fixed $t \in I$, $X_t : \Omega \to R^n$ is a random variable while, for each fixed $\omega \in \Omega$, $X_t : I \to R^n$ is a function. We also write the stochastic process $X_t(\omega)$ as $X(t, \omega)$, which may be regarded a function of t and ω from $I \times \Omega$ to R^n. When $I = R_+$, the R^n-valued stochastic process $X(t, \omega)$ is said to be measurable if $X : R_+ \times \Omega \to R^n$

is $\mathcal{B}(R_+) \times \mathcal{F}$-measurable, and is said to be progressively measurable or progressive if $X : [0,T] \times \Omega \to R^n$ is $\mathcal{B}([0,T]) \times \mathcal{F}$-measurable for all $T \geq 0$.

An R^n-valued stochastic process $\{X_t\}_{t\geq 0}$ is said to be continuous (resp. left continuous, right continuous) if for almost all $\omega \in \Omega$ the function $X_t(\omega)$ is continuous (resp. left continuous, right continuous) on $t \geq 0$. It is said to be integrable if X_t is an integrable random variable for all $t \geq 0$. It is said to be square integrable if $\mathbb{E}|X_t|^2 < \infty$ for all $t \geq 0$. It is said to be $\mathcal{F}_t$-adapted, or simply, adapted if X_t is $\mathcal{F}_t$-measurable for all $t \geq 0$. It is said to be (strictly) stationary if its finite-dimensional joint probability distribution does not change when shifted in time while it is said to be wide-sense stationary if $X_t \in L^2$, $EX_t = m$ and $Cov(X_t, X_s) = c(t-s)$ for all $s \geq 0$ and $t \geq 0$, where m and c are constants.

An $\mathcal{F}_t$-stopping time, or simply, stopping time is a random variable $\tau : \Omega \to [0,\infty]$ for which $\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t$ for all $t \geq 0$. If $X = \{X_t\}_{t\geq 0}$ is a progressive process and τ is a stopping time, then $X^\tau = \{X_{\tau\wedge t}\}_{t\geq 0}$ is called a stopped process of X.

An R^n-valued $\mathcal{F}_t$-adapted integrable process $\{M_t\}_{t\geq 0}$ is called a martingale with respect to $\mathcal{F}_t$, or simply, a martingale if

$$\mathbb{E}(M_t|\mathcal{F}_s) = M_s \quad a.s. \text{ for all } 0 \leq s < t < \infty.$$

The stochastic process is called a supermartingale or a submartingale when the equality is replaced with $\leq$ or $\geq$ respectively. Two of the well-known Doob's martingale theorems are given as follows.

Theorem 2.1 (Doob's martingale stopping theorem) *Let $\{M_t\}_{t\geq 0}$ be a R^n-valued martingale with respect to $\{\mathcal{F}_t\}$. If τ is a stopping time, then the stopped process $M^\tau = \{M_{\tau\wedge t}\}$ is still a martingale with respect to the same filtration $\{\mathcal{F}_t\}$.*

Theorem 2.2 (Doob's martingale inequality) *Let $M = \{M_t\}_{t\geq 0}$ be a R^n-valued martingale. Let $[a,b]$ be a bounded interval in R_+.*

(1) If $p \geq 1$ and $M_t \in L^p(\Omega; R^n)$, then

$$P\{\omega : \sup_{a\leq t\leq b} |M_t(\omega)| \geq c\} \leq \frac{\mathbb{E}|M_b|^p}{c^p}$$

holds for all $c > 0$

(2) If $p > 1$ and $M_t \in L^p(\Omega; R^n)$, then

$$\mathbb{E}(\sup_{a\le t\le b} |M_t|^p) \le (\frac{p}{p-1})^p \mathbb{E}|M_b|^p.$$

An $\mathcal{F}_t$-adapted process $\{M_t\}_{t\ge 0}$ is called a local martingale if there exists a nondecreasing sequence of stopping times $\{\tau_k\}_{k\ge 1}$ with $\tau_k \uparrow \infty$ a.s. such that $\{M_{\tau_k\wedge t} - M_0\}_{t\ge 0}$ is a martingale for all k. Any martingale is a local martingale while the converse is not true. If $M = \{M_t\}_{t\ge 0}$ is a real-valued continuous (local) martingale, then there exists a unique continuous square-integrable adapted increasing process denoted by $\{\langle M_t, M_t\rangle\}$ such that $\{M_t^2 - \langle M_t, M_t\rangle\}$ is a continuous (local) martingale vanishing at $t = 0$. The process $\{\langle M_t, M_t\rangle\}$ is called the quadratic variation of M, for which we have the following result.

Theorem 2.3 (Strong law of large numbers) *Let $M = \{M_t\}_{t\ge 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then*

$$\limsup_{t\to\infty} \frac{\langle M_t, M_t\rangle}{t} < \infty \quad a.s. \qquad \Rightarrow \qquad \lim_{t\to\infty} \frac{M_t}{t} = 0 \quad a.s.;$$
$$\lim_{t\to\infty} \langle M_t, M_t\rangle = \infty \quad a.s. \qquad \Rightarrow \qquad \lim_{t\to\infty} \frac{M_t}{\langle M_t, M_t\rangle} = 0 \quad a.s..$$

To close this section we state one more useful convergence theorem (see [62] and [88]).

Theorem 2.4 *Let $\{A_t\}_{t\ge 0}$ and $\{U_t\}_{t\ge 0}$ be two continuous adapted increasing process with $A_0 = U_0 = 0$ a.s.. Let $\{M_t\}_{t\ge 0}$ be a real-valued continuous local martingale with $M_0 = 0$ a.s.. Let ξ be a nonnegative $\mathcal{F}_0$-measurable random variable such that $\mathbb{E}\xi < \infty$. Define*

$$X_t = \xi + A_t - U_t + M_t, \quad \forall\, t \ge 0.$$

If X_t is nonnegative, then

$$\{\lim_{t\to\infty} A_t < \infty\} \subset \{\lim_{t\to\infty} X_t < \infty\} \cap \{\lim_{t\to\infty} U_t < \infty\} \quad a.s.$$

where $C \subset D$ a.s. means $\mathbb{P}(C \cap D^C) = 0$. In particular, if $\lim_{t\to\infty} A_t < \infty$ a.s., then for almost all $\omega \in \Omega$

$$\lim_{t\to\infty} X_t < \infty, \quad \lim_{t\to\infty} U_t < \infty \quad \text{and} \quad -\infty < \lim_{t\to\infty} M_t < \infty.$$

That is, the three processes X_t, U_t and M_t almost surely converge to finite random variables.

2.3 Brownian Motion

Botanist R. Brown observed and described the irregular motion of a pollen particle suspended in fluid in 1828. There were numerous explanations of such motion of the small pollen grain, later called Brownian motion, proposed and disposed of in the more than 70 years until A. Einstein, in 1905, argued that the movement is due to bombardment of the particle by the molecules of the fluid. He also obtained the equations of Brownian motion. The mathematical foundation for Brownian motion as a stochastic process was done by N. Wiener in 1931, and this process is also called the Wiener process.

The Brownian motion process $W(t)$ serves as a basic model for the cumulative effect of pure noise (also known as white noise). If $W(t)$ denotes the position of a particle at time t, then the displacement $W(t) - W(0)$ is the effect of the purely random bombardment by the molecules of the fluid, or the effect of noise over time t. Let us now give the mathematical definition of Brownian motion.

Definition 2.1 *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space with a filtration* $\{\mathcal{F}_t\}_{t \geq 0}$. *A (standard) one-dimensional Brownian motion is a real-valued continuous* $\{\mathcal{F}_t\}$*-adapted process* $\{W_t\}_{t \geq 0}$ *with the following properties:*

(i) $W_0 = 0$ *a.s.;*

(ii) for $0 \leq s < t < \infty$*, the increment* $W_t - W_s$ *is normally distributed with mean zero and variance* $t - s$*;*

(iii) for $0 \leq s < t < \infty$*, the increment* $W_t - W_s$ *is independent of* $\mathcal{F}_s$*.*

It is observed that Brownian motion is a wide-sense stationary Gaussian process. Sometimes, we shall speak of a Brownian motion $\{W_t\}_{0 \leq t \leq T}$ for some $T > 0$, and the meaning of this terminology is apparent. The Brownian motion has many important properties, some of which are summarized as follows

(a) $\{-W_t\}$ is a Brownian motion with respect to the same filtration $\{\mathcal{F}_t\}$.

(b) Let $c > 0$. Define

$$X_t = \frac{W_{ct}}{\sqrt{c}}\,, \qquad \forall\, t \geq 0\,.$$

Then $\{X_t\}_{t \geq 0}$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_{ct}\}$.

(c) $\{W_t\}$ is a continuous square-integrable martingale, and its quadratic variation $\langle W_t, W_t \rangle = t$ for all $t \geq 0$.

(d) The strong law of large numbers states that

$$\lim_{t\to\infty} \frac{W_t}{t} = 0 \quad a.s..$$

(e) For almost all $\omega \in \Omega$, the Brownian motion sample path $W(t,\omega)$ is nowhere differentiable with respect to t.

We can easily generalize slightly and consider an m-dimensional Brownian motion. It is not difficult to see that an m-dimensional Brownian motion is an m-dimensional continuous martingale with the joint quadratic variations

$$\langle W_t^i, W_t^j \rangle = \delta_{ij} t\,, \qquad \forall\ 1 \leq i, j \leq m$$

where δ_{ij} is the Dirac delta function, i.e., $\delta_{ij} = \mathbb{I}_{i=j}$.

It turns out that this property characterizes Brownian motion among continuous local martingales. To close this section, we introduce the well-known Lévy theorem.

Theorem 2.5 (Lévy theorem) *Let $\{M_t\} = \{(M_t^1, \cdots, M_t^m)\}$ be an m-dimensional continuous martingale with respect to the filtration $\{\mathcal{F}_t\}$ and $M_0 = 0$ a.s.. If*

$$\langle M_t^i, M_t^j \rangle = \delta_{ij} t\,, \qquad \forall\ 1 \leq i, j \leq m$$

then $\{M_t\}$ is an m-dimensional Brownian motion with respect to $\{\mathcal{F}_t\}$.

2.4 Itô integrals

Let $\{W_t\}_{t\geq 0}$ be a one-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Suppose $T > 0$ and $f(t,\omega)$ is given. We want to define

$$\int_a^b f(s,\omega)\mathrm{d}W(s,\omega)\,, \tag{2.1}$$

where $0 \leq a < b$. In general, it is natural to approximate a given function $f(t,\omega)$ by

$$\sum_{j\geq 0} f(t_j^*, \omega)\mathbb{I}_{t\in[j\cdot 2^{-n},(j+1)\cdot 2^{-n})} \tag{2.2}$$

on $a \le t \le b$, where $t_j^* \in [t_j, t_{j+1}]$ and

$$t_j = t_j^{(n)} = \begin{cases} j \cdot 2^{-n}, & a \le j \cdot 2^{-n} \le b \\ a, & j \cdot 2^{-n} < a \\ b, & j \cdot 2^{-n} > b \end{cases} \tag{2.3}$$

with some natural number n. Then define integral (2.1) as the limit of

$$\sum_{j \ge 0} f(t_j^*, \omega) \left[W(t_{j+1}, \omega) - W(t_j, \omega)\right] \tag{2.4}$$

as $n \to \infty$. However, unlike the Riemann-Stieltjes integral, it is found that the value of the limit of (2.4) depends on what points t_j^* are chosen. The following two choices have turned out to be the most useful ones (see, e.g., [105]): (I) $t_j^* = t_j$ (the left end point), which leads to the Itô integral; and (II) $t_j^* = (t_j + t_{j+1})/2$ (the mid point), which leads to the Stratonovich integral. In this thesis, only Itô integrals are involved.

It is reasonable to start with a definition for a simple class of functions f and then extend by some approximation procedure. Thus, let us first assume that f has the form

$$\phi(t, \omega) = \sum_{j \ge 0} \mathbb{I}_{t \in [j \cdot 2^{-n}, (j+1) \cdot 2^{-n})} \cdot e_j(\omega) \tag{2.5}$$

where, for all $j \ge 0$, e_j is an $\mathcal{F}_{t_j}$-measurable function. A function $\phi \in \mathcal{M}^2([a, b]; R)$ that has the form (2.5) is called elementary. For the family of elementary functions it is reasonable to define

$$\int_a^b \phi(s, \omega) \mathrm{d}W(s, \omega) = \sum_{j \ge 0} e_j(\omega) \left[W(t_{j+1}, \omega) - W(t_j, \omega)\right] . \tag{2.6}$$

Definition 2.2 (The Itô integral) *Let $f \in \mathcal{M}^2([a, b]; R)$. Then the Itô integral of f (from a to b) is defined by*

$$\int_a^b f(t, \omega) \mathrm{d}W(t, \omega) = \lim_{n \to \infty} \int_a^b \phi_n(t, \omega) \mathrm{d}W(t, \omega) \tag{2.7}$$

where $\{\phi_n\}$ is a sequence of elementary functions such that

$$\mathbb{E}\left[\int_a^b [f(t, \omega) - \phi_n(t, \omega)]^2 \mathrm{d}t\right] \to 0 \qquad \text{as } n \to \infty . \tag{2.8}$$

By the definitions of (2.6) and (2.7), it is not difficult to prove

Theorem 2.6 (The Itô isometry)

$$\mathbb{E}\left[\left(\int_a^b f(t,\omega)\mathrm{d}W(t,\omega)\right)^2\right] = \mathbb{E}\left[\int_a^b f^2(t,\omega)\mathrm{d}t\right] \tag{2.9}$$

for all $f \in \mathcal{M}^2([a,b];R)$.

For simplicity, $W(t,\omega)$ is also written as $W(t)$ or W_t. Some properties of the Itô integral are listed as follows.

Theorem 2.7 *Let* $f, g \in \mathcal{M}^2([a,b];R)$, $0 \le a < c < b$ *and let* α, β *be two real numbers. Then*

(1) $\int_a^b f(t)dW_t$ *is* $\mathcal{F}_b$*-measurable;*

(2) $\int_a^b f(t)dW_t = \int_a^c f(t)dW_t + \int_c^b f(t)dW_t$*;*

(3) $\int_a^b [\alpha f(t) + \beta g(t)]dW_t = \alpha \int_a^b f(t)dW_t + \beta \int_a^b g(t)dW_t$.

(4) $\mathbb{E}(\int_a^b f(t)dW_t|\mathcal{F}_a) = 0$*;*

(5) $\mathbb{E}(|\int_a^b f(t)dW_t|^2|\mathcal{F}_a) = \int_a^b \mathbb{E}(|f(t)|^2|\mathcal{F}_a)dt$.

But the basic definition of Itô integrals is not very useful when we try to evaluate a given integral. This is similar to the situation for ordinary Riemann integrals, where we do not use the basic definition but rather the fundamental theorem of calculus plus the chain rule in explicit calculations. However, in the context of Itô integrals, we do not have differentiation theory but only the integration concept. Nevertheless it turns out that it is possible to establish an Itô integral version of the chain rule, called Itô's formula, which is very useful for evaluating Itô integrals.

Let X_t be a scalar continuous $\{\mathcal{F}_t\}$-adapted process on $t \ge 0$ and have the form

$$X_t = X_0 + \int_0^t f(s)\mathrm{d}s + \int_0^t g(s)\mathrm{d}W_s\,, \tag{2.10}$$

where $f \in \mathcal{L}^1(R_+;R)$ and $g \in \mathcal{L}^2(R_+;R)$. This process may also be written in the differential form

$$\mathrm{d}X_t = f(t)\mathrm{d}t + g(t)\mathrm{d}W_t\,. \tag{2.11}$$

Theorem 2.8 (1-dimensional Itô's formula) *Let X_t be a process given by (2.11). Let $y(t,x) \in C^{2,1}(R_+ \times R; R)$. Then $Y_t = y(t, X_t)$ has the form of (2.10) and*

$$\mathrm{d}Y_t = \left[\frac{\partial y}{\partial t}(t,X_t) + \frac{\partial y}{\partial x}(t,X_t)f(t) + \frac{1}{2}g(t)\frac{\partial^2 y}{\partial x^2}(t,X_t)g(t)\right]\mathrm{d}t + \frac{\partial y}{\partial x}(t,X_t)g(t)\mathrm{d}W_t \quad a.s..$$

Let us now turn to the situation in higher dimensions. Let $W_t = (W_t^1, \cdots, W_t^m)^T$ be an m-dimensional Brownian motion. Itô's formula can be easily generalized to the n-dimensional case.

Theorem 2.9 (The general Itô's formula) *Let $v = v(t,x)$ be a function such that $v \in C^{2,1}(R_+ \times R^n; R)$ with partial derivatives denoted by*

$$\frac{\partial}{\partial t}v(t,x) = v_t, \qquad \frac{\partial}{\partial x_i}v(t,x) = v_{x_i}, \qquad \frac{\partial^2}{\partial x_i \partial x_j}v(t,x) = v_{x_i x_j},$$

where $x = [x_1, \cdots, x_n]^T$ and $1 \le i,j \le n$. Moreover, let X_t be an n-dimensional stochastic process defined on $t \ge 0$ by the stochastic differential

$$\mathrm{d}X_t = f(t)\mathrm{d}t + g(t)\mathrm{d}W_t,$$

where $f \in \mathcal{L}^1(R_+; R^n)$ and $g \in \mathcal{L}^2(R_+; R^{n\times m})$. Then $V_t = v(t, X_t)$ is a stochastic process such that

$$\mathrm{d}V_t = [v_t(t,X_t) + v_x(t,X_t)f(t) + \frac{1}{2}trace(g^T(t)v_{xx}(t,X_t)g(t))]\mathrm{d}t + v_x(t,X_t)g(t)\mathrm{d}W_t \quad a.s.$$

where

$$v_x = \left(\frac{\partial v}{\partial x_1}, \cdots, \frac{\partial v}{\partial x_n}\right) \quad \text{and} \quad v_{xx} = \left(\frac{\partial^2 v}{\partial x_i \partial x_j}\right)_{n\times n}.$$

It is easy to verify that the Itô integral

$$X_t = X_0 + \int_0^t g(s)\mathrm{d}W_s\,, \quad t \ge 0$$

is always a martingale with respect to $\{\mathcal{F}_t\}_{t\ge 0}$ when $g \in \mathcal{L}^2([0,t]; R^{n\times m})$ for any $t \ge 0$. But the converse is also true. This result, called the martingale representation theorem, is important for many applications.

Theorem 2.10 (The martingale representation theorem) *Suppose that M_t is an n-dimensional $\{\mathcal{F}_t\}$-martingale and that $M_t \in L^2_{\mathcal{F}_t}(\Omega; R^n)$ for all $t \ge 0$. Then there exists a unique stochastic process $g(s,\omega)$ such that $g \in \mathcal{L}^2([0,t]; R^{n\times m})$ and*

$$M_t(\omega) = \mathbb{E}M_0 + \int_0^t g(s,\omega)\mathrm{d}W_s \quad a.s. \tag{2.12}$$

for all $t \geq 0$*. The process* $g(s,\omega)$ *is said to be unique if any other process* $\bar{g} \in \mathcal{L}^2([0,t]; R^{n\times m})$ *satisfying (2.12) is indistinguishable from* $g(s,\omega)$*, that is*

$$\mathbb{P}\{g(s,\omega) = \bar{g}(s,\omega) \quad \textit{for all } 0 \leq s \leq t\} = 1\,.$$

Finally, we introduce the following important inequalities of the Itô integral that are useful in many applications.

Theorem 2.11 *Let* $p \geq 2$*,* $t \geq 0$ *and* $g \in \mathcal{M}^2([0,t]; R^{n\times m})$*. Then*

$$\mathbb{E}\left|\int_0^t g(s)\mathrm{d}W_s\right|^p \leq \left(\frac{p(p-1)}{2}\right)^{\frac{p}{2}} t^{\frac{p-2}{2}}\mathbb{E}\int_0^t |g(s)|^p\,\mathrm{d}s.$$

In particular, this holds as an equality when $p = 2$*.*

Theorem 2.12 (Burkholder-Davis-Gundy inequality) *Let* $g \in \mathcal{L}^2(R_+; R^{n\times m})$*,*

$$x(t) = \int_0^t g(s)\mathrm{d}W_s \quad \text{and} \quad A(t) = \int_0^t |g(s)|^2\mathrm{d}s$$

for any $t \geq 0$*. Then for every* $p > 0$*, there exist universal positive constants* c_p *and* C_p *such that*

$$c_p\mathbb{E}|A(t)|^{\frac{p}{2}} \leq \mathbb{E}(\sup_{0\leq s\leq t}|x(s)|^p) \leq C_p\mathbb{E}|A(t)|^{\frac{p}{2}}$$

for all $t \geq 0$*. In particular, one may take*

$$\begin{array}{lll} c_p = (p/2)^p, & C_p = (32/p)^{p/2} & \textit{if } 0 < p < 2; \\ c_p = 1, & C_p = 4 & \textit{if } p = 2; \\ c_p = (2p)^{-p/2}, & C_p = [p^{p+1}/2(p-1)^{p-1}]^{p/2} & \textit{if } p > 2. \end{array}$$

2.5 Stochastic differential equations

Let $W(t) = (W_1(t), \cdots, W_m(t))^T$, $t \geq 0$, be an m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Let $f : R^n \times [t_0, T] \to R^n$ and $g : R^n \times [t_0, T] \to R^{n\times m}$ be both Borel measurable with $0 = t_0 < T < \infty$. Let x_0 be an $\mathcal{F}_{t_0}$-measurable R^n-valued random variable such that $\mathbb{E}|x_0|^2 < \infty$. Then the definition of n-dimensional stochastic differential equations (SDEs) of Itô type can be given for the R^n-valued stochastic process $\{X_t\}_{t\in[t_0,T]}$.

Definition 2.3 *An equation of the form*

$$\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}W(t), \quad \forall t \in [t_0, T] \tag{2.13}$$

with initial value $x(t_0) = x_0$*, or, written componentwise as*

$$\mathrm{d}x_i(t) = f_i(x(t),t)\mathrm{d}t + \sum_{j=1}^{m} g_{ij}(x(t),t)\mathrm{d}W_j(t), \tag{2.14}$$

is called a stochastic differential equation (SDE). The random variable x_0 *is called the initial value at time* t_0*.*

Definition 2.4 *An* R^n*-valued stochastic process* $\{x(t)\}_{t_0 \leq t \leq T}$ *is called a (strong) solution of equation (2.13) with initial value* x_0 *if it has the following properties:*

(1) $\{x(t)\}$ *is continuous and* $\mathcal{F}_t$*-adapted;*

(2) $\mathbb{P}\{x(t_0) = x_0\} = 1$*;*

(3) $\{f(x(t),t)\} \in \mathcal{L}^1([t_0,T]; R^n)$ *and* $\{g(x(t),t)\} \in \mathcal{L}^2([t_0,T]; R^{n\times m})$*;*

(4) the integral version of equation (2.13) holds for every $t \in [t_0, T]$ *with probability 1.*

A solution $\{x(t)\}$ *is said to be unique if any other solution* $\{\bar{x}(t)\}$ *is indistinguishable from* $\{x(t)\}$*, that is*

$$\mathbb{P}\{x(t) = \bar{x}(t) \quad \textit{for all } t_0 \leq t \leq T\} = 1\,.$$

The Itô equation (2.13) may not have a unique solution on the whole interval $[t_0, T]$. Let us give the conditions that guarantee the existence and uniqueness of solutions to SDE (2.13).

Theorem 2.13 *Assume that there exist two positive constants* K *and* $\bar{K}$ *such that*

(1) (Lipschitz condition) for all x,y $\in R^n$ *and* $t \in [t_0, T]$*,*

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq K|x-y|^2; \tag{2.15}$$

(2) (linear growth condition) for all (x,t) $\in R^n \times [t_0, T]$*,*

$$|f(x,t)|^2 \vee |g(x,t)|^2 \leq \bar{K}(1+|x|^2); \tag{2.16}$$

Then there exists a unique solution $x(t)$ to equation (2.13) with initial value $x(t_0) = x_0$ and the solution belongs to $\mathcal{M}^2([t_0, T]; R^n)$.

Theorem 2.14 *Assume that the linear growth condition (2.16) holds. But the Lipschitz condition (2.15) is replaced with the following condition*

(3) (local Lipschitz condition) for every integer $k \geq 1$, there exists a positive constant L_k such that, for all $t \in [t_0, T]$ and all $x, y \in R^n$ with $|x| \vee |y| \leq k$,

$$|f(x,t) - f(y,t)|^2 \vee |g(x,t) - g(y,t)|^2 \leq L_k |x - y|^2. \tag{2.17}$$

Then there exists a unique solution $x(t)$ to equation (2.13) with initial value $x(t_0) = x_0$ and the solution belongs to $\mathcal{M}^2([t_0, T]; R^n)$.

The above theory of SDEs can be extended to stochastic functional differential equations (SFDEs) (see, e.g., Sec. 5.2, p149, [88]).

2.6 Stochastic stability theory

Stability of a process (in particular, of a stationary state) is the ability of the process to resist a priori unknown (small) influences. A process is said to be stable if such disturbances do not essentially change it. This property turns out to be of utmost importance. It should be emphasized that an individual predictable process can be physically realized only if it is stable in the corresponding natural sense. It is well known that Lyapunov's second method is an interesting and fruitful technique that has gained increasing significance and has given decisive impetus to the modern development of stability theory of dynamical systems. A manifest advantage of this method is that it does not demand detailed knowledge of solutions and therefore has great power in applications. Lyapunov function serves as a vehicle to transform a given complicated differential system into relatively simpler differential equations and hence it is sufficient to study the properties of solutions of this simpler differential equation.

When we try to carry over the principles of the Lyapunov stability theory for deterministic systems to stochastic systems, we face several problems, the basic one being to give a suitable definition of stochastic stability. In this section, we shall introduce various types of stability for stochastic differential equation (2.13).

(a) Stability in distribution: the solution process $x(t)$ of (2.13) is said to be asymptotically stable in distribution if there exists a probability measure $\pi(\cdot)$ on R^n such that the transition probability $p(t; x; \mathrm{d}y)$ of $x(t)$ converges weakly to $\pi(\mathrm{d}y)$ as $t \to \infty$ for every $x \in R^n$. System (2.13) is said to be asymptotically stable in distribution if $x(t)$ is asymptotically stable in distribution.

(b) Stability in probability: system (2.13) is said to be

(i) stable in probability if $\forall\, \varepsilon > 0$ there is a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\mathbb{P}\{|x(t)| < \gamma(|x_0|)\} \geq 1 - \varepsilon\,, \quad \forall\, t \geq t_0,\, x_0 \in R^n\backslash\{0\};$$

(ii) asymptotically stable in probability if $\forall\, \varepsilon > 0$ there is a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$\mathbb{P}\{|x(t)| < \beta(|x_0|, t)\} \geq 1 - \varepsilon\,, \quad \forall\, t \geq t_0,\, x_0 \in R^n\backslash\{0\}.$$

(c) Stability in pth ($p > 0$) moment sense: system (2.13) is said to be

(i) pth ($p > 0$) moment stable if there is a class $\mathcal{K}$ function $\gamma(\cdot)$ such that

$$\mathbb{E}|x(t)|^p < \gamma(|x_0|)\,, \quad \forall\, t \geq t_0,\, x_0 \in R^n\backslash\{0\};$$

(ii) pth ($p > 0$) moment asymptotically stable if there is a class $\mathcal{KL}$ function $\beta(\cdot,\cdot)$ such that

$$\mathbb{E}|x(t)|^p < \beta(|x_0|, t)\,, \quad \forall\, t \geq t_0,\, x_0 \in R^n\backslash\{0\}.$$

Particularly, when $p = 2$, system (2.13) is said to be mean-square (asymptotically) stable.

(d) Stability in almost sure sense (or say, with probability 1): system (2.13) is said to be almost surely asymptotically stable if

$$\mathbb{P}\{\sup_{t \geq t_0} |x(t; x_0)| < \infty\} = 1 \quad \text{and} \quad \mathbb{P}\{\limsup_{t \to \infty} |x(t; x_0)| = 0\} = 1$$

for all $x_0 \in R^n$.

These concepts of stability have the following relationship, where stability means asymptotic stability.

qth ($q > 0$) moment stability
$\Downarrow$
pth ($0 < p \le q$) moment stability
$\Downarrow$
almost sure stability $\Rightarrow$ stability in probability $\Rightarrow$ stability in distribution

Moreover, we introduce the definitions of exponential stability as follows.

(e) Exponential stability in pth ($p > 0$) moment sense: system (2.13) is said to be pth ($p > 0$) moment exponentially stable if there is a pair of positive constants C and ε such that

$$\mathbb{E}|x(t;x_0)|^p \le C|x_0|^p e^{-\varepsilon t}, \quad \forall\, t \ge t_0,\, x_0 \in R^n$$

which also implies that

$$\limsup_{t\to\infty} \frac{1}{t} \log(\mathbb{E}|x(t;x_0)|^p) \le -\varepsilon, \quad \forall\, x_0 \in R^n.$$

(f) Exponential stability in almost sure sense: system (2.13) is said to be almost surely exponentially stable if there is a positive constant ε such that

$$\limsup_{t\to\infty} \frac{1}{t} \log|x(t;x_0)| \le -\varepsilon \quad a.s., \quad \forall\, x_0 \in R^n.$$

2.7 Continuous-time Markov chains

In this section, we will recall some basic facts about a continuous-time Markov chain (see [1] and [122]). Let $X = \{X_t\}_{t\ge 0}$ be an n-dimensional stochastic process defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$ and taking values in a countable set $\bar{S}$, which is called the state space of the process.

Definition 2.5 *The n-dimensional $\{\mathcal{F}_t\}$-adapted process $X = \{X_t\}_{t\ge 0}$ is called a continuous-time Markov chain if it satisfies the Markov property, that is, for all $0 < t_1 < \cdots < t_k \le t < \infty$, $i_l \in \bar{S}$ $(1 \le l \le k)$ and $j \in \bar{S}$,*

$$\mathbb{P}(X_t = j | X_{t_1} = i_1, \cdots, X_{t_k} = i_k) = \mathbb{P}(X_t = j | X_{t_k} = i_k)$$

for any integer $k > 0$.

Definition 2.6 *A function $p(s,i;t,j) = p_{ij}(s,t)$, defined on $0 \le s \le t < \infty$, $i \in \bar{S}$ and $j \in \bar{S}$, is called the transition probability of the continuous-time Markov chain X and the matrix $P(s,t) = \big(p_{ij}(s,t)\big)_{i,j\in\bar{S}}$ is called the transition matrix of X if the following properties are satisfied:*

(i) $p_{ij}(s,t) = \mathbb{P}(X_t = j | X_s = i)$ *for all* $t \ge s \ge 0$ *and* $i, j \in \bar{S}$;

(ii) $p_{ij}(s,s) = \delta_{ij}$ *for all* $s \ge 0$ *and* $i, j \in \bar{S}$;

(iii) $\sum_{j\in\bar{S}} p_{ij}(s,t) = 1$ *for all* $t \ge s \ge 0$ *and* $i \in \bar{S}$;

(iv) the Chapman-Kolmogorov equation

$$p_{ij}(s,t) = \sum_{k\in\bar{S}} p_{ik}(s,u)p_{kj}(u,t)\,,$$

or in matrix form

$$P(s,t) = P(s,u)P(u,t)$$

holds for all $t \ge u \ge s \ge 0$.

The Markov chain X is said to be stationary if its transition probabilities $p_{ij}(s,t)$, $i, j \in \bar{S}$, are stationary, that is, $p_{ij}(s,t)$ depends only on the difference $t - s$ for all $0 \le s \le t < \infty$ and $i, j \in \bar{S}$, which implies

$$P(s, s+u) = P(u)$$

for all $s \ge 0$ and $u \ge 0$. In this case, the transition probability and the transition matrix of X can be written as $p_{ij}(t)$ and $P(t)$ ($t \ge 0$) respectively. The transition matrix $P(t) = \big(p_{ij}(t)\big)_{i,j\in\bar{S}}$ is said to be standard if $\lim_{t\to 0} p_{ii}(t) = 1$ for all $i \in \bar{S}$.

Theorem 2.15 *Let $P(t)$ be a standard transition matrix, then*

$$\gamma_i = \lim_{t\to 0} \frac{1 - p_{ii}(t)}{t}$$

exists (but may be ∞) for every $i \in \bar{S}$.

A state $i \in \bar{S}$ is said to be stable if $\gamma_i < \infty$.

Theorem 2.16 *Let $P(t)$ be a standard transition matrix and $i \in \bar{S}$ is a stable state. Then*

$$\gamma_{ij} = \lim_{t \to 0} \frac{p_{ij}(t) - p_{ij}(0)}{t}$$

exists and is finite for every $j \in \bar{S}$.

It is observed that $\gamma_{ii} = -\gamma_i$ for all $i \in \bar{S}$. The matrix $\Gamma = \left(\gamma_{ij}\right)_{i,j \in \bar{S}}$ is called the generator of the Markov chain X. The Markov chain X is said to be finite when the number of elements in its state space $\bar{S}$ is finite. In this thesis, we assume that all Markov chains are finite and all their states are stable.

In the sequel, let $\bar{S} = S = \{1, 2, \cdots, N\}$ be the finite state space of continuous-time Markov chain X.

Theorem 2.17 *Let $P(t) = \left(p_{ij}(t)\right)_{N \times N}$ be the transition matrix and $\Gamma = \left(\gamma_{ij}\right)_{N \times N}$ be the generator of the continuous-time Markov chain X. Then*

$$P(t) = e^{t\Gamma}$$

for all $t \geq 0$.

Recall that a continuous-time Markov chain X with generator $\Gamma = \left(\gamma_{ij}\right)_{N \times N}$ can be represented as a stochastic integral with respect to a Poisson random measure (see [114], [30], [6] and [133]). Let $\{\Delta_{ij}\}$ be a sequence of consecutive, left-closed, right-open intervals of length γ_{ij} on the real axis such that

$$\begin{aligned}
\Delta_{12} &= [0,\ \gamma_{12}), \\
\Delta_{13} &= [\gamma_{12},\ \gamma_{12} + \gamma_{13}), \\
&\vdots \\
\Delta_{1N} &= \left[\sum_{j=2}^{N-1} \gamma_{1j},\ \sum_{j=2}^{N} \gamma_{1j}\right), \\
\Delta_{21} &= \left[\sum_{j=2}^{N} \gamma_{1j},\ \sum_{j=2}^{N} \gamma_{1j} + \gamma_{21}\right), \\
\Delta_{23} &= \left[\sum_{j=2}^{N} \gamma_{1j} + \gamma_{21},\ \sum_{j=2}^{N} \gamma_{1j} + \gamma_{21} + \gamma_{23}\right), \\
&\vdots \\
\Delta_{2N} &= \left[\sum_{j=2}^{N} \gamma_{1j} + \sum_{j=1, j \neq 2}^{N-1} \gamma_{2j},\ \sum_{j=2}^{N} \gamma_{1j} + \sum_{j=1, j \neq 2}^{N} \gamma_{2j}\right),
\end{aligned}$$

and so on. Define a function $h : S \times R \to R$ by

$$h(i,y) = \begin{cases} j-i\,, & \text{if } y \in \Delta ij\,, \\ 0\,, & \text{otherwise.} \end{cases} \tag{2.18}$$

Then

$$\mathrm{d}X_t = \int_R h(X_{t-}, y)\nu(\mathrm{d}t, \mathrm{d}y) \tag{2.19}$$

with initial condition $X_0 = i_0 \in S$, where $\nu(\mathrm{d}t, \mathrm{d}y)$ is a Poisson measure with intensity $\mathrm{d}t \times m(\mathrm{d}y)$, in which m is the Lebesgue measure on R.

2.8 Stochastic differential equations with Markovian switching

Let $W(t)$ be an m-dimensional Brownian motion defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}, \mathbb{P})$. Let $r(t), t \geq 0$, be a right-continuous Markov chain, on the same probability space, taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by

$$\mathbb{P}\{r(t+\Delta) = j : r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & if\ i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & if\ i = j, \end{cases} \tag{2.20}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j\neq i}\gamma_{ij}$. Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$. It is known that almost all sample paths of $r(t)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $R_+ := [0, \infty)$.

Consider an n-dimensional stochastic differential equation with Markovian switching

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{2.21}$$

on $t \geq 0$ with initial data $x(0) = x_0 \in R^n$ and $r(0) = r_0 \in S$, where $f : R^n \times R_+ \times S \to R^n$ and $g : R^n \times R_+ \times S \to R^{n\times m}$ are Borel measurable functions. We assume that both f and g are sufficiently smooth so that equation (2.21) has a unique solution (see [79] and [133]).

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x,t,i)$ on $R^n \times R_+ \times S$ that are twice continuously differentiable in x and once in t. If $V \in$

$C^{2,1}(R^n \times R_+ \times S; R_+)$, define an operator associated with (2.21), $\mathcal{L}$, from $R^n \times R_+ \times S$ to R by

$$\begin{aligned}\mathcal{L}V(x,t,i) &= V_t(x,t,i) + V_x(x,t,i)f(x,t,i) \\ &\quad + \frac{1}{2}trace\left[g^T(x,t,i)V_{xx}(x,t,i)g(x,t,i)\right] + \sum_{j=1}^{N}\gamma_{ij}V(x,t,j), \quad (2.22)\end{aligned}$$

where

$$V_t(x,t,i) = \frac{\partial V(x,t,i)}{\partial t}, \quad V_x(x,t,i) = \left(\frac{\partial V(x,t,i)}{\partial x_1}, \cdots, \frac{\partial V(x,t,i)}{\partial x_n}\right)$$

and

$$V_{xx}(x,t,i) = \left(\frac{\partial^2 V(x,t,i)}{\partial x_j \partial x_k}\right)_{n\times n}.$$

The generalized Itô's formula, cited as follows (see p105 [114]), is useful for the development of this thesis.

Lemma 2.2 *If* $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, *then*

$$\begin{aligned}&V(x(t),t,r(t)) \\ &\quad = V(x(0),0,r(0)) + \int_0^t \mathcal{L}V(x(s),s,r(s))\mathrm{d}s \\ &\qquad + \int_0^t V_x(x(s),s,r(s))g(x(s),s,r(s))\mathrm{d}W(s) \\ &\qquad + \int_0^t \int_R \big[V(x(s),s,r(0)+h(r(s),l)) - V(x(s),s,r(s))\big]\mu(\mathrm{d}s,\mathrm{d}l) \quad (2.23)\end{aligned}$$

for all $t \geq 0$, *where function* $h(\cdot,\cdot)$ *is defined as (2.18) and* $\mu(\mathrm{d}s,\mathrm{d}l) = \nu(\mathrm{d}s,\mathrm{d}l) - m(\mathrm{d}l)\mathrm{d}s$ *is a martingale measure (see also [6], [30] and [133]).*

Taking expectation on both sides of (2.23) immediately yields

Lemma 2.3 *Let* $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ *and* ρ_1, ρ_2 *be stopping times such that* $0 \leq \rho_1 \leq \rho_2 < \infty$ *a.s.. If*

$$\sup_{t\in[\rho_1,\rho_2]} |V(x(t),t,r(t))| < \infty \quad \text{and} \quad \sup_{t\in[\rho_1,\rho_2]} |\mathcal{L}V(x(t),t,r(t))| < \infty \quad a.s.,$$

then

$$\mathbb{E}V(x(\rho_2),\rho_2,r(\rho_2)) = \mathbb{E}V(x(\rho_1),\rho_1,r(\rho_1)) + \mathbb{E}\int_{\rho_1}^{\rho_2} \mathcal{L}V(x(s),s,r(s))\mathrm{d}s\,. \quad (2.24)$$

Generally, stopping times ρ_1 and ρ_2 are defined such that $\sup_{t\in[\rho_1,\rho_2]}|x(t)| < \infty$ *a.s.* when this formula (2.24) is applied.

These results can be extended to stochastic functional equations with Markovian switching (see [79], [87] and [133]).

Chapter 3

Robust state-feedback stabilisation of uncertain stochastic systems with input delay

3.1 Introduction

Problems of stability and stabilisation of delay systems have been investigated in many works (see, e.g., [17]-[20], [26]-[28], [31]-[35], [57], [66], [99]-[102], [108], [127]-[135]) over the past decades. Time delays often appear in practical systems and may inhibit the performance of a system. Particularly, due to time spent for example in computation, sensor-to-controller and controller-to-actuator transfer, control input is usually subject to delays. The presence of time delays of input may be the cause of serious deterioration of performance or even instability of the resulting controlled system if it is not considered in a controller design. Problems of stabilisation for deterministic systems with input delay have been intensively studied (see [18], [52], [66], [99], [134] and [135]) over the past few years.

Since stochastic modelling plays an important role in many branches of science and engineering, stochastic systems have received much attention in recent years (see, e.g., [25], [53] and [88]). An area of particular interest has been control of stochastic delay systems, with consequent emphasis on analysis of stability of stochastic models (see [17], [25], [42], [43], [53], [60], [80]-[91], [135]). These works can be classified into two categories according to their dependence on the information about the size of time delays of the system, say, they are either delay-independent results ([25], [42], [53], [60], [75]-[78], [84], [88], [112], [128], [130]) or delay-dependent criteria ([17], [43], [53], [74], [88], [135]). Generally, for the cases of small delays, delay-independent results are more conservative than those dependent on size of delays. Recently, increasing attention has been placed on delay-dependent stability of stochastic delay systems (see [5], [17], [59], [74], [88], [110] and [135]). Particularly, Corollary 6.6, p182, [88] applied the Razumikhin technique to establish a delay-dependent criterion that only requires the time delay $\tau(t)$ to be a bounded function of t, see Example (6.55), p189, [88] and Example 3.1 below. However, these existing methods do not deal with the structure of the diffusion terms but estimate their upper bound, which induces conservativeness in many cases such as Example 3.2. Moreover, stochastic systems with input delay have been studied in [130]. But the results can not be applied to a significant number of cases when the unforced non-delay system is unstable (see, e.g., Example 3.3).

The study in this chapter is concerned with problems of delay-dependent stability and delayed-state-feedback stabilisation of uncertain stochastic systems. By the approach of linear matrix inequalities (LMIs), we present a delay-dependent criterion for exponential stability of uncertain stochastic delay systems, which exploits the advantages of structure of the diffusion term and reduces the conservativeness of the existing methods. Based on this stability result, we further study the problem of stabilisation of systems with input delay and propose a state-feedback controller design that exponentially stabilises the uncertain stochastic system with input delay. Numerical examples are exhibited to show that our results are considerably less conservative than the existing ones.

3.2 Problem statement

Throughout this chapter, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions and $\mathbb{E}[\cdot]$ be the expectation operator with respect to the probability measure. Let $w(t)$ be a scalar Brownian motion defined on the probability space. If A is a vector or matrix, its transpose is denoted by A^T. If P is a square matrix, $P > 0$ ($P < 0$) means that P is a symmetric positive (negative) definite matrix of appropriate dimensions while $P \geq 0$ ($P \leq 0$) is a symmetric positive (negative) semidefinite matrix. I stands for the identity matrix of appropriate dimensions. Denote by $\lambda_M(\cdot)$ and $\lambda_m(\cdot)$ the maximum and minimum eigenvalue of a matrix respectively. Let $|\cdot|$ denote the Euclidean norm of a vector and its induced norm of a matrix. Unless explicitly stated, matrices are assumed to have real entries and compatible dimensions. For $h > 0$ let $C([-h, 0]; R^n)$ denote the family of all continuous R^n-valued functions φ on $[-h, 0]$ with the norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -h \leq \theta \leq 0\}$. Let $\tilde{L}^2_{\mathcal{F}_0}([-h, 0]; R^n)$ be the family of all $\mathcal{F}_0$-measurable $C([-h, 0]; R^n)$-valued random variables φ such that $\mathbb{E}\|\varphi\|^2 < \infty$.

Let us consider an n-dimensional uncertain stochastic delay system

$$\begin{aligned} \mathrm{d}x(t) &= [A_0(t)x(t) + A_1(t)x(t-\tau(t)) + B(t)u(t-\tau(t))]\,\mathrm{d}t \\ &\quad + [H_0(t)x(t) + H_1(t)x(t-\tau(t))]\,\mathrm{d}w(t) \end{aligned} \tag{3.1}$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in \tilde{L}^2_{\mathcal{F}_0}([-h, 0]; R^n)$, where $x(t) \in R^n$ is the state vector; $u(t) \in R^m$ is the control input; the time delay of the system $\tau(t)$, also written as τ in this chapter, is a Borel-measurable function on $t \geq 0$ with $0 \leq \tau(t) \leq h$ for all $t \geq 0$, where h is a positive scalar constant; $A_i(t)$, $B(t)$, and $H_i(t)$, $i = 0, 1$, are matrix functions with time-varying uncertainties described as follows: $A_i(t) = A_i + \Delta A_i(t)$, $B(t) = B + \Delta B(t)$, $H_i(t) = H_i + \Delta H_i(t)$, where A_i, B, and H_i are known constant matrices while uncertainties $\Delta A_i(t)$, $\Delta B(t)$, and $\Delta H_i(t)$ are assumed to be norm bounded, i.e.,

$$\begin{aligned} \begin{bmatrix} \Delta A_i(t) & \Delta B(t) \end{bmatrix} &= L_A F_A(t) \begin{bmatrix} E_{Ai} & E_B \end{bmatrix}, \\ \Delta H_i(t) &= L_H F_H(t) E_{Hi} \end{aligned} \tag{3.2}$$

with known constant matrices L_A, E_{Ai}, E_B, L_H, and E_{Hi}, and unknown matrix functions $F_A(t)$ and $F_H(t)$ having Lebesgue measurable elements and satisfying

$$F_A^T(t)F_A(t) \leq I, \quad F_H^T(t)F_H(t) \leq I, \quad \forall t \geq 0. \tag{3.3}$$

The parameter uncertainties $\Delta A_i(t)$, $\Delta B(t)$, and $\Delta H_i(t)$, $i = 0, 1$, are said to be admissible if both (3.2) and (3.3) hold.

Denote

$$\begin{aligned} f(t) &= f(t, x_t) = A_0(t)x(t) + A_1(t)x(t-\tau), \\ g(t) &= g(t, x_t) = H_0(t)x(t) + H_1(t)x(t-\tau) \end{aligned} \tag{3.4}$$

for all $t \geq 0$. One can observe that

$$|f(t)| + |g(t)| \leq C_L \|x_t\| , \quad \forall t \geq 0 \tag{3.5}$$

where $x_t = \{x(t+\theta) : -h \leq \theta \leq 0\}$ and $C_L = \sum_{i=0}^{1} \left(|A_i| + |L_A| \, |E_{Ai}| + |H_i| + |L_H| \, |E_{Hi}| \right)$. This implies that both $f(\cdot, \cdot)$ and $g(\cdot, \cdot)$ satisfy the local Lipschitz condition and the linear growth condition with respect to the second argument. According to Theorem 2.2, p150, [88], there exists a unique solution denoted by $x(t; \xi)$ to the stochastic functional differential equation

$$\mathrm{d}x(t) = [A_0(t)x(t) + A_1(t)x(t-\tau)] \, \mathrm{d}t + [H_0(t)x(t) + H_1(t)x(t-\tau)] \, \mathrm{d}w(t) \tag{3.6}$$

on $t \geq 0$.

In this chapter, we intend: (i) to establish new delay-dependent sufficient conditions for robust exponential stability of the unforced uncertain stochastic delay system (3.6); and (ii) to design a robust state-feedback controller

$$u(t) = Kx(t) \tag{3.7}$$

which exponentially stabilise system (3.1), where K is a constant gain matrix to be determined. For simplicity only, we take a single delay $\tau = \tau(t)$ in our models. The proposed method can be easily extended to those cases with multiple and distributed delays.

At the end of this section, let us introduce the following definitions and lemmas that are useful for the development of our results.

Definition 3.1 *([88]) The uncertain stochastic delay system (3.6) is said to be robustly exponentially stable in mean square if there is a positive constant ε such that*

$$\limsup_{t \to \infty} \frac{1}{t} \log \mathbb{E}|x(t; \xi)|^2 \leq -\varepsilon \tag{3.8}$$

for all admissible uncertainties (3.2) and (3.3).

Definition 3.2 *([88]) The uncertain stochastic delay system (3.6) is said to be robustly almost surely exponentially stable if there is a positive constant ε such that*

$$\limsup_{t \to \infty} \frac{1}{t} \log |x(t; \xi)| \leq -\varepsilon \quad a.s. \tag{3.9}$$

for all admissible uncertainties (3.2) and (3.3).

Lemma 3.1 *([126]) For any constant matrix $M \in R^{q\times l}$, the inequality*

$$2u^T M v \le r u^T M G M^T u + \frac{1}{r} v^T G^{-1} v\,, \quad u \in R^q\,,\, v \in R^l$$

holds for any symmetric positive definite matrix $G \in R^{l\times l}$ and any positive number $r > 0$.

Lemma 3.2 *([32])For any symmetric positive definite constant matrix $G \in R^{l\times l}$ and any scalar $r > 0$, if there exists a vector function $v : [0, r] \to R^l$ such that integrals $\int_0^r v^T(s)Gv(s)\mathrm{d}s$ and $\int_0^r v(s)\mathrm{d}s$ are well defined, then the following inequality holds*

$$r\int_0^r v^T(s)Gv(s)\mathrm{d}s \ge \left(\int_0^r v(s)\mathrm{d}s\right)^T G\left(\int_0^r v(s)\mathrm{d}s\right).$$

Lemma 3.3 *([100]) Assume that $u \in R^q$, $v \in R^l$ and $M \in R^{q\times l}$. For any constant matrices $X \in R^{q\times q}$, $Y \in R^{q\times l}$ and $Z \in R^{l\times l}$, the inequality*

$$-2u^T M v \le \begin{bmatrix} u \\ v \end{bmatrix}^T \begin{bmatrix} X & Y - M \\ Y^T - M^T & Z \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

holds when $\begin{bmatrix} X & Y \\ Y^T & Z \end{bmatrix} \ge 0\,.$

3.3 Delay-dependent exponential stability

Sufficient conditions for robust exponential stability of the uncertain stochastic delay system (3.6) are proposed as follows.

Theorem 3.1 *The uncertain stochastic delay system (3.6) is robustly mean-square exponentially stable and is also robustly almost surely exponentially stable provided that there exist matrices $P_{11} > 0$, $R > 0$, $S > 0$, $Q \ge 0$, $W \ge 0$, P_{21}, P_{22}, P_{23}, P_{31}, P_{32}, P_{33}, M and scalar numbers $\varepsilon_A > 0$, $\varepsilon_H > 0$ such that*

$$\begin{bmatrix} \Phi_{11} & * & * & * & * & * & * \\ \Phi_{21} & \Phi_{22} & * & * & * & * & * \\ \Phi_{31} & \Phi_{32} & \Phi_{33} & * & * & * & * \\ \Phi_{41} & \Phi_{42} & \Phi_{43} & \Phi_{44} & * & 0 & 0 \\ \Phi_{51} & \Phi_{52} & \Phi_{53} & \Phi_{54} & \Phi_{55} & 0 & 0 \\ L_A^T P_{21} & L_A^T P_{22} & L_A^T P_{23} & 0 & 0 & -\varepsilon_A I & 0 \\ L_H^T P_{31} & L_H^T P_{32} & L_H^T P_{33} & 0 & 0 & 0 & -\varepsilon_H I \end{bmatrix} < 0 \qquad (3.10)$$

and

$$\begin{bmatrix} W & M^T \\ M & Q \end{bmatrix} \geq 0\,, \tag{3.11}$$

where

$$W = \begin{bmatrix} W_{11} & * & * \\ W_{21} & W_{22} & * \\ W_{31} & W_{32} & W_{33} \end{bmatrix}, \qquad M = \begin{bmatrix} M_1 & M_2 & M_3 \end{bmatrix},$$

$$\begin{aligned}
\Phi_{11} = {} & P_{21}^T(A_0 + A_1) + (A_0 + A_1)^T P_{21} + P_{31}^T(H_0 + H_1) + (H_0 + H_1)^T P_{31} + hW_{11} \\
& + \varepsilon_A (E_{A0} + E_{A1})^T (E_{A0} + E_{A1}) + \varepsilon_H (E_{H0} + E_{H1})^T (E_{H0} + E_{H1})\,, \\
\Phi_{21} = {} & P_{22}^T(A_0 + A_1) + P_{32}^T(H_0 + H_1) + P_{11} - P_{21} + hW_{21}\,, \\
\Phi_{31} = {} & P_{23}^T(A_0 + A_1) + P_{33}^T(H_0 + H_1) - P_{31} + hW_{31}\,, \\
\Phi_{41} = {} & h\big(M_1 - A_1^T P_{21} - H_1^T P_{31} - \varepsilon_A E_{A1}^T(E_{A0} + E_{A1}) - \varepsilon_H E_{H1}^T(E_{H0} + E_{H1})\big)\,, \\
\Phi_{51} = {} & -A_1^T P_{21} - H_1^T P_{31} - \varepsilon_A E_{A1}^T(E_{A0} + E_{A1}) - \varepsilon_H E_{H1}^T(E_{H0} + E_{H1})\,, \\
\Phi_{22} = {} & -P_{22}^T - P_{22} + h(W_{22} + R + Q)\,, \quad \Phi_{32} = -P_{23}^T - P_{32} + hW_{32}\,, \\
\Phi_{42} = {} & h\left(M_2 - A_1^T P_{22} - H_1^T P_{32}\right)\,, \quad \Phi_{52} = -A_1^T P_{22} - H_1^T P_{32}\,, \\
\Phi_{33} = {} & -P_{33}^T - P_{33} + P_{11} + h(W_{33} + S)\,, \quad \Phi_{43} = h\left(M_3 - A_1^T P_{23} - H_1^T P_{33}\right)\,, \\
\Phi_{53} = {} & -A_1^T P_{23} - H_1^T P_{33}\,, \quad \Phi_{44} = -hR + \varepsilon_A h^2 E_{A1}^T E_{A1} + \varepsilon_H h^2 E_{H1}^T E_{H1}\,, \\
\Phi_{54} = {} & h\left(\varepsilon_A E_{A1}^T E_{A1} + \varepsilon_H E_{H1}^T E_{H1}\right)\,, \quad \Phi_{55} = -S + \varepsilon_A E_{A1}^T E_{A1} + \varepsilon_H E_{H1}^T E_{H1}\,,
\end{aligned}$$

and entries denoted by $*$ can be readily inferred from symmetry of a matrix.

Proof. By notation (3.4), we can rewrite the unforced system (3.6) for short as

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}w(t) \tag{3.12}$$

on $t \geq 0$ with initial data ξ. So we have

$$x(t_2) - x(t_1) = \int_{t_1}^{t_2} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \tag{3.13}$$

for all $t_2 \geq t_1 \geq 0$.

By (3.4) and (3.13), we can observe that

$$f(t) = \sum_{i=0}^{1} A_i(t)x(t) - A_1(t) \int_{t-\tau}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \tag{3.14}$$

$$g(t) = \sum_{i=0}^{1} H_i(t)x(t) - H_1(t)\int_{t-\tau}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \tag{3.15}$$

for all $t \geq h$.

Choose a Lyapunov-Krasovskii functional candidate for system (3.13) as follows:

$$V(t) = V_1(t) + V_2(t) + V_3(t)\,, \tag{3.16}$$

where

$$\begin{aligned} V_1(t) &= x^T(t)P_{11}x(t)\,, \\ V_2(t) &= \int_{t-h}^{t} (s-t+h)f^T(s)(R+Q)f(s)\mathrm{d}s\,, \\ V_3(t) &= \int_{t-h}^{t} (s-t+h)g^T(s)Sg(s)\mathrm{d}s\,. \end{aligned}$$

for all $t \geq h$. By Itô's formula, we have

$$\mathrm{d}V(t) = \mathcal{L}V(t)\mathrm{d}t + \sigma(t)\mathrm{d}w(t)\,, \tag{3.17}$$

where

$$\begin{aligned} \mathcal{L}V(t) &= 2x^T(t)P_{11}f(t) + g^T(t)P_{11}g(t) + \dot{V}_2(t) + \dot{V}_3(t), \\ \sigma(t) &= 2x^T(t)P_{11}g(t)\,. \end{aligned} \tag{3.18}$$

Denote

$$y(t) = \begin{bmatrix} x(t) \\ f(t) \\ g(t) \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}. \tag{3.19}$$

By equalities (3.14) and (3.15), we have

$$\begin{aligned} 2x^T(t)P_{11}f(t) &= 2\begin{bmatrix} x(t) \\ f(t) \\ g(t) \end{bmatrix}^T \begin{bmatrix} P_{11} & P_{21}^T & P_{31}^T \\ 0 & P_{22}^T & P_{32}^T \\ 0 & P_{23}^T & P_{33}^T \end{bmatrix} \begin{bmatrix} f(t) \\ 0 \\ 0 \end{bmatrix} = 2y^T(t)P^T \begin{bmatrix} f(t) \\ 0 \\ 0 \end{bmatrix} \\ &= 2y^T(t)P^T \left\{ \begin{bmatrix} 0 & I & 0 \\ \sum_{i=0}^{1} A_i(t) & -I & 0 \\ \sum_{i=0}^{1} H_i(t) & 0 & -I \end{bmatrix} y(t) - \begin{bmatrix} 0 \\ A_1(t) \\ H_1(t) \end{bmatrix} \int_{t-\tau}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \right\}. \end{aligned} \tag{3.20}$$

By Lemma 3.3 and equalities (3.11)-(3.13), we see

$$
\begin{aligned}
&-2y^T(t)P^T\begin{bmatrix}0 & A_1^T(t) & H_1^T(t)\end{bmatrix}^T\int_{t-\tau}^t f(s)\mathrm{d}s\\
&= \int_{t-\tau}^t -2y^T(t)P^T\begin{bmatrix}0 & A_1^T(t) & H_1^T(t)\end{bmatrix}^T f(s)\mathrm{d}s\\
&\le \int_{t-\tau}^t \begin{bmatrix}y(t)\\ f(s)\end{bmatrix}^T\begin{bmatrix}W & *\\ M-\begin{bmatrix}0 & A_1(t) & H_1(t)\end{bmatrix}P & Q\end{bmatrix}\begin{bmatrix}y(t)\\ f(s)\end{bmatrix}\mathrm{d}s\\
&\le hy^T(t)Wy(t)+\int_{t-h}^t f^T(s)Qf(s)\mathrm{d}s\\
&\quad + 2y^T(t)\left(M^T-P^T\begin{bmatrix}0 & A_1^T & H_1^T\end{bmatrix}^T\right)\int_{t-\tau}^t f(s)\mathrm{d}s\\
&\quad - 2y^T(t)P^T\begin{bmatrix}0 & \Delta A_1^T(t) & \Delta H_1^T(t)\end{bmatrix}^T\int_{t-\tau}^t f(s)\mathrm{d}s\,. \qquad (3.21)
\end{aligned}
$$

Substitution of (3.21) into (3.20) gives

$$
\begin{aligned}
&2x^T(t)P_{11}f(t)\\
&\le\ y^T(t)\left[P^T\bar{A}+\bar{A}^TP+hW\right]y(t)+2y^T(t)P^T\Delta A_H(t)x(t)\\
&\quad + 2y^T(t)\left(M^T-P^T\begin{bmatrix}0 & A_1^T & H_1^T\end{bmatrix}^T\right)\int_{t-\tau}^t f(s)\mathrm{d}s\\
&\quad - 2y^T(t)P^T\begin{bmatrix}0 & A_1^T & H_1^T\end{bmatrix}^T\int_{t-\tau}^t g(s)\mathrm{d}w(s)+\int_{t-h}^t f^T(s)Qf(s)\mathrm{d}s\\
&\quad - 2y^T(t)P^T\begin{bmatrix}0 & \Delta A_1^T(t) & \Delta H_1^T(t)\end{bmatrix}^T\left(\int_{t-\tau}^t f(s)\mathrm{d}s+\int_{t-\tau}^t g(s)\mathrm{d}w(s)\right), \qquad (3.22)
\end{aligned}
$$

where $\bar{A}=\begin{bmatrix}0 & I & 0\\ \sum_{i=0}^1 A_i & -I & 0\\ \sum_{i=0}^1 H_i & 0 & -I\end{bmatrix}$ and $\Delta A_H(t)=\begin{bmatrix}0\\ \sum_{i=0}^1\Delta A_i(t)\\ \sum_{i=0}^1\Delta H_i(t)\end{bmatrix}$.

Let $\tilde{E}_A=\left[\sum_{i=0}^1 E_{Ai}\ 0\ 0\ -hE_{A1}\ -E_{A1}\right]$, $\tilde{L}_A^T=\left[L_A^TP_{21}\ L_A^TP_{22}\ L_A^TP_{23}\ 0\ 0\right]^T$, $\tilde{E}_H=\left[\sum_{i=0}^1 E_{Hi}\ 0\ 0\ -hE_{H1}\ \ -E_{H1}\right]$, $\tilde{L}_H^T=\left[L_H^TP_{31}\ L_H^TP_{32}\ L_H^TP_{33}\ 0\ 0\right]$, and

$$
\begin{aligned}
z^T(t) &= \left[z_1^T(t)\ z_2^T(t)\ z_3^T(t)\ z_4^T(t)\ z_5^T(t)\right]^T\\
&= \left[x^T(t)\quad f^T(t)\quad g^T(t)\quad \frac{1}{h}\int_{t-\tau}^t f^T(s)\mathrm{d}s\quad \int_{t-\tau}^t g^T(s)\mathrm{d}w(s)\right]^T.
\end{aligned}
$$

Then, by Lemma 3.1, we have

$$
\begin{aligned}
&2y^T(t)P^T\left\{\begin{bmatrix}0 & \sum_{i=0}^1\Delta A_i^T(t) & 0\end{bmatrix}^T x(t)-\begin{bmatrix}0 & \Delta A_1^T(t) & 0\end{bmatrix}^T\right.\\
&\qquad\qquad \left.\cdot\left(\int_{t-\tau}^t f(s)\mathrm{d}s+\int_{t-\tau}^t g(s)\mathrm{d}w(s)\right)\right\}\\
&= 2z^T(t)\tilde{L}_AF_A(t)\tilde{E}_Az(t)\le\varepsilon_A^{-1}z^T(t)\tilde{L}_A\tilde{L}_A^Tz(t)+\varepsilon_Az^T(t)\tilde{E}_A^T\tilde{E}_Az(t) \qquad (3.23)
\end{aligned}
$$

and

$$
\begin{aligned}
&2y^T(t)P^T\left\{\begin{bmatrix}0 & 0 & \sum_{i=0}^{1}\Delta H_i^T(t)\end{bmatrix}^T x(t)-\begin{bmatrix}0 & 0 & \Delta H_1^T(t)\end{bmatrix}^T\right.\\
&\qquad\qquad\left.\cdot\left(\int_{t-\tau}^{t} f(s)\mathrm{d}s+\int_{t-\tau}^{t} g(s)\mathrm{d}w(s)\right)\right\}\\
&\le \varepsilon_H^{-1}z^T(t)\tilde{L}_H\tilde{L}_H^T z(t)+\varepsilon_H z^T(t)\tilde{E}_H^T\tilde{E}_H z(t)\,. \qquad (3.24)
\end{aligned}
$$

Inequalities (3.23) and (3.24) imply

$$
\begin{aligned}
&2y^T(t)P^T\left\{\Delta A_H(t)x(t)-\begin{bmatrix}0 & \Delta A_1^T(t) & \Delta H_1^T(t)\end{bmatrix}^T\left(\int_{t-\tau}^{t} f(s)\mathrm{d}s+\int_{t-\tau}^{t} g(s)\mathrm{d}w(s)\right)\right\}\\
&\le z^T(t)\left[\varepsilon_A^{-1}\tilde{L}_A\tilde{L}_A^T+\varepsilon_A\tilde{E}_A^T\tilde{E}_A+\varepsilon_H^{-1}\tilde{L}_H\tilde{L}_H^T+\varepsilon_H\tilde{E}_H^T\tilde{E}_H\right]z(t)\,. \qquad (3.25)
\end{aligned}
$$

Combination of inequalities (3.20)-(3.25) yields

$$
2x^T(t)P_{11}f(t)+g^T(t)P_{11}g(t)\le z^T(t)\Gamma z(t)+\int_{t-h}^{t} f^T(s)Qf(s)\mathrm{d}s\,, \qquad (3.26)
$$

where Γ is a symmetric matrix, i.e.,

$$
\Gamma=\begin{bmatrix}
\Gamma_{11} & * & * & * & *\\
\Gamma_{21} & \Gamma_{22} & * & * & *\\
\Gamma_{31} & \Gamma_{32} & \Gamma_{33} & * & *\\
\Gamma_{41} & \Gamma_{42} & \Gamma_{43} & \Gamma_{44} & *\\
\Gamma_{51} & \Gamma_{52} & \Gamma_{53} & \Gamma_{54} & \Gamma_{55}
\end{bmatrix}
$$

with

$$
\begin{aligned}
\Gamma_{11} &= \Phi_{11}+\varepsilon_A^{-1}P_{21}^T L_A L_A^T P_{21}+\varepsilon_H^{-1}P_{31}^T L_H L_H^T P_{31}\,,\\
\Gamma_{21} &= \Phi_{21}+\varepsilon_A^{-1}P_{22}^T L_A L_A^T P_{21}+\varepsilon_H^{-1}P_{32}^T L_H L_H^T P_{31}\,,\\
\Gamma_{31} &= \Phi_{31}+\varepsilon_A^{-1}P_{23}^T L_A L_A^T P_{21}+\varepsilon_H^{-1}P_{33}^T L_H L_H^T P_{31}\,,\quad \Gamma_{41}=\Phi_{41}\,,\quad \Gamma_{51}=\Phi_{51}\,,\\
\Gamma_{22} &= \Phi_{22}-h(R+Q)+\varepsilon_A^{-1}P_{22}^T L_A L_A^T P_{22}+\varepsilon_H^{-1}P_{32}^T L_H L_H^T P_{32}\,,\\
\Gamma_{32} &= \Phi_{32}+\varepsilon_A^{-1}P_{23}^T L_A L_A^T P_{22}+\varepsilon_H^{-1}P_{33}^T L_H L_H^T P_{32}\,,\quad \Gamma_{42}=\Phi_{42}\,,\quad \Gamma_{52}=\Phi_{52}\,,\\
\Gamma_{33} &= \Phi_{33}-hS+\varepsilon_A^{-1}P_{23}^T L_A L_A^T P_{23}+\varepsilon_H^{-1}P_{33}^T L_H L_H^T P_{33}\,,\quad \Gamma_{43}=\Phi_{43}\,,\quad \Gamma_{53}=\Phi_{53}\,,\\
\Gamma_{44} &= \Phi_{44}+hR\,,\quad \Gamma_{54}=\Phi_{54}\,,\quad \Gamma_{55}=\Phi_{55}+S\,.
\end{aligned}
$$

Direct computations give

$$
\begin{aligned}
\dot{V}_2(t) &= z_2^T(t)h(R+Q)z_2(t)-z_4^T(t)hRz_4(t)-r_f(t)-\int_{t-h}^{t} f^T(s)Qf(s)\mathrm{d}s, \qquad (3.27)\\
\dot{V}_3(t) &= z_3^T(t)hSz_3(t)-\int_{t-\tau}^{t} g^T(s)Sg(s)\mathrm{d}s-r_g(t), \qquad (3.28)
\end{aligned}
$$

where, according to Lemma 3.2, $r_f(t) = \int_{t-h}^{t} f^T(s)Rf(s)\mathrm{d}s - z_4^T(t)hRz_4(t) \geq 0$ and $r_g(t) = \int_{t-h}^{t-\tau} g^T(s)Sg(s)\mathrm{d}s \geq 0$ for all $t \geq h$.

Substitution of inequalities (3.26)-(3.28) into (3.18) yields

$$\begin{aligned}\mathcal{L}V(t) \;\leq\;& z^T(t)\Gamma z(t) + z_2^T(t)h(R+Q)z_2(t) + z_3^T(t)hSz_3(t)\\ &- z_4^T(t)hRz_4(t) - \int_{t-\tau}^{t} g^T(s)Sg(s)\mathrm{d}s - r_f(t) - r_g(t)\,. \end{aligned} \tag{3.29}$$

By isometry property, we have

$$\mathbb{E}\left[z_5^T(t)Sz_5(t)\right] = \mathbb{E}\left[\int_{t-\tau}^{t} g^T(s)Sg(s)\mathrm{d}s\right].$$

Therefore, taking expectation on both sides of (3.29) yields

$$\mathbb{E}\mathcal{L}V(t) \leq \mathbb{E}\left[z^T(t)\tilde{\Gamma}z(t) - r_f(t) - r_g(t)\right], \tag{3.30}$$

where $\tilde{\Gamma} = \Gamma + diag\left\{0,\quad h(R+Q),\quad hS,\quad -hR,\quad -S\right\}$.

By the Schur complement lemma, inequality (3.10) implies that $\tilde{\Gamma} < 0$. So we have

$$\begin{aligned}\mathbb{E}\mathcal{L}V(t) \;\leq\;& -\lambda_0\mathbb{E}|z(t)|^2 - \mathbb{E}[r_f(t)] - \mathbb{E}[r_g(t)]\\ \leq\;& -\lambda_0\mathbb{E}[|z_1(t)|^2 + |z_4(t)|^2 + |z_5(t)|^2] - \mathbb{E}[r_f(t)] - \mathbb{E}[r_g(t)]\\ \leq\;& -\lambda_0\mathbb{E}|x(t)|^2 - \mathbb{E}[z_4^T(t)(\lambda_0 I - hR)z_4(t)]\\ &- \mathbb{E}\int_{t-h}^{t} f^T(s)Rf(s)\mathrm{d}s - \lambda_g\mathbb{E}\int_{t-h}^{t}|g(s)|^2\mathrm{d}s,\end{aligned}$$

where $\lambda_0 = \lambda_m(-\tilde{\Gamma}) > 0$ and $\lambda_g = \min\{\lambda_0, \lambda_m(S)\} > 0$.

By definition of (3.16), we see

$$\alpha_0|x(t)|^2 \leq V(t) \leq \alpha_1|x(t)|^2 + h\int_{t-h}^{t} f^T(s)(R+Q)f(s)\mathrm{d}s + \alpha_g\int_{t-h}^{t}|g(s)|^2\mathrm{d}s \tag{3.31}$$

for all $t \geq h$, where $\alpha_0 = \lambda_m(P_{11})$, $\alpha_1 = \lambda_M(P_{11})$ and $\alpha_g = h\lambda_M(S)$. Choose $\varepsilon > 0$ such that

$$\lambda_0 \geq \varepsilon\alpha_1, \quad \lambda_g \geq \varepsilon\alpha_g, \quad \lambda_0 \geq \varepsilon\alpha_f \quad \text{and} \quad R_f > 0, \tag{3.32}$$

where $\alpha_f = h^2\lambda_M(R+Q)$ and $R_f = R - \varepsilon h(R+Q)$.

By Itô's formula, we have

$$\mathrm{d}\left[e^{\varepsilon s}V(s)\right] = e^{\varepsilon s}\left[\varepsilon V(s) + \mathcal{L}V(s)\right]\mathrm{d}s + e^{\varepsilon s}\sigma(s)\mathrm{d}w(s)\,. \tag{3.33}$$

Let $t_0 = h$, then, by Lemma 3.2, we have

$$\begin{aligned}
&\mathbb{E}\left[e^{\varepsilon t}V(t)\right] - \mathbb{E}\left[e^{\varepsilon t_0}V(t_0)\right] \\
&\quad = \mathbb{E}\int_{t_0}^{t} e^{\varepsilon s}\left[\varepsilon V(s) + \mathcal{L}V(s)\right]\mathrm{d}s \\
&\quad \le \int_{t_0}^{t} e^{\varepsilon s}\Big\{\mathbb{E}\Big[(\varepsilon\alpha_1 - \lambda_0)|x(s)|^2 - z_4^T(s)(\lambda_0 I - hR)z_4(s) \\
&\qquad\qquad - \int_{s-h}^{s} f^T(v)R_f f(v)\mathrm{d}v + (\varepsilon\alpha_g - \lambda_g)\int_{s-h}^{s}|g(v)|^2\mathrm{d}v\Big]\Big\}\mathrm{d}s \\
&\quad \le -\int_{t_0}^{t} e^{\varepsilon s}\Big\{\mathbb{E}\Big[z_4^T(s)(\lambda_0 I - hR + hR_f)z_4(s)\Big]\Big\}\mathrm{d}s \\
&\quad \le -\int_{t_0}^{t} e^{\varepsilon s}\Big\{\mathbb{E}\Big[z_4^T(s)(\lambda_0 I - \varepsilon h^2(R+Q))z_4(s)\Big]\Big\}\mathrm{d}s \\
&\quad \le -\int_{t_0}^{t} e^{\varepsilon s}\Big\{\mathbb{E}\Big[(\lambda_0 - \varepsilon\alpha_f)|z_4(s)|^2\Big]\Big\}\mathrm{d}s \\
&\quad \le 0
\end{aligned} \tag{3.34}$$

for all $t \ge t_0$. By linear growth condition (3.5) and Theorem 4.1, p160, [88], there are positive constants C_1 and C_2 such that

$$e^{\varepsilon t}\mathbb{E}V(t) \le e^{\varepsilon t_0}\mathbb{E}V(t_0) \le C_1 + C_2\mathbb{E}\|\xi\|^2 \tag{3.35}$$

for all $t \ge t_0$. So we have

$$\mathbb{E}|x(t)|^2 \le \alpha_0^{-1}C_h e^{-\varepsilon t}, \tag{3.36}$$

where $C_h = C_1 + C_2\mathbb{E}\|\xi\|^2 < \infty$. This implies

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}|x(t)|^2 \le -\varepsilon\,. \tag{3.37}$$

The mean-square exponential stability has been proven. Moreover,

$$|f(t)|^2 \le 2\left[|A_0(t)|^2|x(t)|^2 + |A_1(t)|^2|x(t-h)|^2\right].$$

Let $K_f = 2\sum_{i=0}^{1}(|A_i| + |L_A|\cdot|E_{Ai}|)^2$, then we have

$$\begin{aligned}
\mathbb{E}|f(t)|^2 &\le 2\big[|A_0(t)|^2\mathbb{E}|x(t)|^2 + |A_1(t)|^2\mathbb{E}|x(t-h)|^2\big] \\
&\le K_f \sup_{-h\le\theta\le 0}\mathbb{E}|x(t+\theta)|^2\,.
\end{aligned} \tag{3.38}$$

Similarly, letting $K_g = 2\sum_{i=0}^{1}(|H_i| + |L_H|\cdot|E_{Hi}|)^2$, we have

$$\mathbb{E}|g(t)|^2 \le K_g \sup_{-h\le\theta\le 0}\mathbb{E}|x(t+\theta)|^2\,. \tag{3.39}$$

But, by Theorem 6.2, p175, [88], or, Theorem 2.2, [75], inequalities (3.36), (3.38) and (3.39) imply

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \le -\frac{\varepsilon}{2} \quad a.s., \tag{3.40}$$

which completes the proof.

Remark 3.1 *It is easy to derive a corollary from Theorem 3.1 by setting $P_{31} = P_{32} = 0$, which will be used in the study of stabilisation of stochastic delay system (3.1).*

3.4 Robust state-feedback stabilisation

This section is devoted to designing a state-feedback controller of form (3.7) that robustly stabilises uncertain stochastic delay system with input delay (3.1).

Theorem 3.2 *The closed-loop stochastic delay system (3.1) and (3.7) is robustly mean-square exponentially stable and is also robustly almost surely exponentially stable if, for given scalar numbers $\delta_R > 0$, $\delta_S > 0$, $\delta_Q > 0$ and δ_M, there exist matrices $X_1 > 0$, X_2, X_3, Y, Z, $\bar{K}$, $\bar{W}$ and positive numbers $\beta_A > 0$, $\beta_H > 0$ such that*

$$\begin{bmatrix} \Psi_{11} & * & * & 0 & 0 & * & 0 & * & * \\ \Psi_{21} & \Psi_{22} & * & * & * & * & 0 & 0 & 0 \\ \Psi_{31} & \Psi_{32} & \Psi_{33} & * & * & * & * & 0 & 0 \\ 0 & \Psi_{42} & \Psi_{43} & -\delta_R h X_1 & 0 & 0 & 0 & * & * \\ 0 & \Psi_{52} & -X_1 H_1^T & 0 & -\delta_S X_1 & 0 & 0 & * & * \\ \Psi_{61} & \Psi_{62} & \Psi_{63} & 0 & 0 & -X_1 & 0 & 0 & 0 \\ 0 & 0 & \Psi_{73} & 0 & 0 & 0 & -X_1 & 0 & 0 \\ \Psi_{81} & 0 & 0 & \Psi_{84} & \Psi_{85} & 0 & 0 & -\beta_A I & 0 \\ \Psi_{91} & 0 & 0 & -hE_{H1}X_1 & -E_{H1}X_1 & 0 & 0 & 0 & -\beta_H I \end{bmatrix} < 0 \tag{3.41}$$

and

$$\begin{bmatrix} \bar{W} & \bar{M}^T \\ \bar{M} & \delta_Q X_1 \end{bmatrix} \ge 0, \tag{3.42}$$

where

$$\bar{W} = \begin{bmatrix} \bar{W}_{11} & * & * \\ \bar{W}_{21} & \bar{W}_{22} & * \\ \bar{W}_{31} & \bar{W}_{32} & \bar{W}_{33} \end{bmatrix}, \quad \bar{M} = \delta_M \begin{bmatrix} 0 & X_1 A_1^T + \bar{K}^T B^T & X_1 H_1^T \end{bmatrix},$$

$$
\begin{aligned}
\Psi_{11} =\ & -Y^T - Y + h\bar{W}_{11}, \quad \Psi_{21} = (A_0 + A_1)X_1 + B\bar{K} + X_2^T + Y + h\bar{W}_{21}, \\
\Psi_{31} =\ & -Z^T + (H_0 + H_1)X_1 + h\bar{W}_{31}, \quad \Psi_{61} = \sqrt{h(\delta_R + \delta_Q)}\, Y, \\
\Psi_{81} =\ & (E_{A0} + E_{A1})X_1 + E_B\bar{K}, \quad \Psi_{91} = (E_{H0} + E_{H1})X_1, \\
\Psi_{22} =\ & -X_2^T - X_2 + h\bar{W}_{22} + \beta_A L_A L_A^T, \quad \Psi_{32} = Z^T + h\bar{W}_{32}, \\
\Psi_{42} =\ & h(\delta_M - 1)(X_1 A_1^T + \bar{K}^T B^T), \quad \Psi_{52} = -(X_1 A_1^T + \bar{K}^T B^T), \\
\Psi_{62} =\ & -\sqrt{h(\delta_R + \delta_Q)}\, X_2, \quad \Psi_{33} = -X_3^T - X_3 + h\bar{W}_{33} + \beta_H L_H L_H^T, \\
\Psi_{43} =\ & h(\delta_M - 1)X_1 H_1^T, \quad \Psi_{63} = \sqrt{h(\delta_R + \delta_Q)}\, Z, \quad \Psi_{73} = \sqrt{1 + h\delta_S}\, X_3, \\
\Psi_{84} =\ & -h(E_{A1}X_1 + E_B\bar{K}), \quad \Psi_{85} = -(E_{A1}X_1 + E_B\bar{K}).
\end{aligned}
$$

In this case, the gain matrix of (3.7) can be chosen as $K = \bar{K}X_1^{-1}$.

Proof. Substituting (3.7) into (3.1) yields dynamics of the closed-loop system

$$
\mathrm{d}x(t) = [A_0(t)x(t) + (A_1(t) + B(t)K)x(t-\tau)]\,\mathrm{d}t + [H_0(t)x(t) + H_1(t)x(t-\tau)]\,\mathrm{d}w(t) \tag{3.43}
$$

for all $t \geq 0$. From the proof of Theorem 3.1, we observe that system (3.43) is exponentially stable if inequalities (3.11) and $\Theta < 0$ are satisfied, where Θ is derived from $\tilde{\Gamma}$ in (3.30) by replacing A_1 and E_{A1} with $A_1 + BK$ and $E_{A1} + E_B K$ respectively.

In order to obtain a convex optimization problem, we consider the case of $R = \delta_R P_{11}$, $S = \delta_S P_{11}$, $Q = \delta_Q P_{11}$, $M = \delta_M \begin{bmatrix} 0 & (A_1 + BK)^T & H_1^T \end{bmatrix} P$,

$$
P = \begin{bmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & P_{23} \\ 0 & 0 & P_{33} \end{bmatrix}
$$

and

$$
P^{-1} = \begin{bmatrix} P_{11}^{-1} & 0 & 0 \\ -P_{22}^{-1}P_{21}P_{11}^{-1} & P_{22}^{-1} & -P_{22}^{-1}P_{23}P_{33}^{-1} \\ 0 & 0 & P_{33}^{-1} \end{bmatrix}. \tag{3.44}
$$

Define $X_1 = P_{11}^{-1} > 0$, $X_2 = P_{22}^{-1}$, $X_3 = P_{33}^{-1}$, $\beta_A = \varepsilon_A^{-1}$, $\beta_H = \varepsilon_H^{-1}$, $Y = P_{22}^{-1}P_{21}P_{11}^{-1}$, $Z = P_{22}^{-1}P_{23}P_{33}^{-1}$, $\bar{K} = KX_1$, $\bar{W} = (P^{-1})^T W P^{-1}$, and $G = diag\{P^{-1}, P_{11}^{-1}, P_{11}^{-1}\}$. By the Schur complement lemma, inequality (3.41) implies $\tilde{\Theta} < 0$ and hence $\Theta < 0$, where $\tilde{\Theta} = G^T \Theta G$. Moreover, premultiplying by $diag\{(P^{-1})^T, P_{11}^{-1}\}$ and postmultiplying by $diag\{P^{-1}, P_{11}^{-1}\}$ inequality (3.11) leads to LMI (3.42), which implies that (3.11) and

(3.42) are equivalent when $Q = \delta_Q P_{11}$ and $M = \delta_M \begin{bmatrix} 0 & (A_1 + BK)^T & H_1^T \end{bmatrix} P$. The proof is complete.

3.5 Comments on "Delay-dependent robust stability for stochastic time-delay systems with polytopic uncertainties"

Recently, [59] proposes in terms of LMIs a delay-dependent criterion for stability of stochastic delay systems with polytopic uncertainties

$$\begin{aligned} \mathrm{d}x(t) &= [A_\alpha x(t) + B_\alpha x(t-h)]\mathrm{d}t + [C_\alpha x(t) + D_\alpha x(t-h)]\mathrm{d}w(t) \\ x(\theta) &= \phi(\theta), \quad \theta \in [-h, 0] \end{aligned} \tag{3.45}$$

where matrices A_α, B_α, C_α and D_α are subject to uncertainties satisfying real convex polytopic model

$$\begin{bmatrix} A_\alpha & B_\alpha & C_\alpha & D_\alpha \end{bmatrix} \in \Omega$$

$$\Omega = \left\{ \begin{bmatrix} A_\alpha & B_\alpha & C_\alpha & D_\alpha \end{bmatrix} = \sum_{i=1}^{N} \alpha_i \begin{bmatrix} A_i & B_i & C_i & D_i \end{bmatrix}, \sum_{i=1}^{N} \alpha_i = 1, \alpha_i \geq 0 \right\} \tag{3.46}$$

where $\alpha = [\,\alpha_1 \quad \alpha_2 \quad \cdots \quad \alpha_N\,]$ denotes an uncertain vector satisfying $\alpha_i \geq 0$ and $\sum_{i=1}^{N} \alpha_i = 1$, and A_α, B_α, C_α and D_α are constant matrices (see [59]).

However, there appears to be a technical error in the proof of Theorem 1 in [59], which may lead to the unjustified result. Let us look at matrices Ψ_α and Ξ that are given in [59] as follows

$$\Psi_\alpha = \begin{bmatrix} \Phi_\alpha & P_\alpha B_\alpha - M_\alpha & M_\alpha & C_\alpha^T P_\alpha & \mu C_\alpha^T P_\alpha & \mu A_\alpha^T P_\alpha \\ * & -S_\alpha & 0 & D_\alpha^T P_\alpha & \mu D_\alpha^T P_\alpha & \mu B_\alpha^T P_\alpha \\ * & * & -R_\alpha & 0 & 0 & 0 \\ * & * & * & -P_\alpha & 0 & 0 \\ * & * & * & * & -\mu R_\alpha & 0 \\ * & * & * & * & * & -\mu Q_\alpha \end{bmatrix}$$

with $\Phi_\alpha = P_\alpha A_\alpha + A_\alpha^T P_\alpha + \mu W_\alpha + M_\alpha + M_\alpha^T + S_\alpha$, and

$$\Xi = \begin{bmatrix} \Xi_{11} & \Xi_{12} \\ * & \Xi_{22} \end{bmatrix}$$

with

$$\begin{aligned} \Xi_{11} &= P_\alpha A_\alpha + A_\alpha^T P_\alpha + \mu W_\alpha + M_\alpha + M_\alpha^T + S_\alpha + M_\alpha R_\alpha^{-1} M_\alpha^T + C_\alpha^T P_\alpha C_\alpha + \mu C_\alpha^T R_\alpha C_\alpha \\ &\quad + \mu A_\alpha^T Q_\alpha A_\alpha, \\ \Xi_{12} &= P_\alpha B_\alpha - M_\alpha + C_\alpha^T P_\alpha D_\alpha + \mu C_\alpha^T R_\alpha D_\alpha + \mu A_\alpha^T Q_\alpha B_\alpha, \\ \Xi_{22} &= -S_\alpha + D_\alpha^T P_\alpha D_\alpha + \mu D_\alpha^T R_\alpha D_\alpha + \mu B_\alpha^T Q_\alpha B_\alpha. \end{aligned}$$

In the proof of Theorem 1 in [59], the authors claimed that

$$\Psi_\alpha < 0 \;\Rightarrow\; \Xi < 0\,. \tag{3.47}$$

But, unless $P_\alpha = R_\alpha = Q_\alpha$, this may not be true. Instead, by the Schur complement lemma, one has

$$\Pi_\alpha < 0 \;\Rightarrow\; \Xi < 0\,, \tag{3.48}$$

where

$$\Pi_\alpha = \begin{bmatrix} \Phi_\alpha & P_\alpha B_\alpha - M_\alpha & M_\alpha & C_\alpha^T P_\alpha & \mu C_\alpha^T R_\alpha & \mu A_\alpha^T Q_\alpha \\ * & -S_\alpha & 0 & D_\alpha^T P_\alpha & \mu D_\alpha^T R_\alpha & \mu B_\alpha^T Q_\alpha \\ * & * & -R_\alpha & 0 & 0 & 0 \\ * & * & * & -P_\alpha & 0 & 0 \\ * & * & * & * & -\mu R_\alpha & 0 \\ * & * & * & * & * & -\mu Q_\alpha \end{bmatrix}.$$

Consequently, a modified version of Theorem 1 in [59] may be given as follows

Theorem 3.3 *Given a constant $\mu > 0$, uncertain stochastic delay system (3.45) is robustly stochastically stable for any h satisfying $0 \le h \le \mu$, if there exist matrices $P_j > 0$, $Q_j > 0$, $S_j > 0$, $R_j > 0$, W_j and M_j such that*

$$\Pi_{ii} < 0\,, \quad i = 1, 2, \cdots, N \tag{3.49}$$

$$\Pi_{ik} + \Pi_{ki} < 0\,, \quad 1 \le i < k \le N \tag{3.50}$$

and

$$\begin{bmatrix} W_i & M_i \\ * & Q_i \end{bmatrix} \ge 0\,, \quad i = 1, 2, \cdots, N \tag{3.51}$$

where

$$\Pi_{ij} = \begin{bmatrix} \Omega_{ij} & P_jB_i - M_j & M_j & C_i^TP_j & \mu C_i^TR_j & \mu A_i^TQ_j \\ * & -S_j & 0 & D_i^TP_j & \mu D_i^TR_j & \mu B_i^TQ_j \\ * & * & -R_j & 0 & 0 & 0 \\ * & * & * & -P_j & 0 & 0 \\ * & * & * & * & -\mu R_j & 0 \\ * & * & * & * & * & -\mu Q_j \end{bmatrix}.$$

with $\Omega_{ij} = P_jA_i + A_i^TP_j + \mu W_j + M_j + M_j^T + S_j$ *for* $i = 1, 2, \cdots, N$ *and* $j = 1, 2, \cdots, N$.

3.6 Examples

In this section, examples are given to verify the effectiveness of the proposed method. To compare with the existing results, we consider the case of constant delay $\tau \equiv h$ in some of the following examples. It should be pointed out that our results apply to the systems with time-varying delay $\tau(t) \in [0, h]$.

Example 3.1 Let us consider a stochastic delay system

$$\mathrm{d}x(t) \;=\; A_1x(t-\tau)\mathrm{d}t + H_1x(t-\tau)\mathrm{d}w(t), \tag{3.52}$$

where $A_1 = \begin{bmatrix} -c & 0 \\ 0.5 & -1 \end{bmatrix}$ with $c > 0$ and $H_1 = \begin{bmatrix} 0.5 & 1 \\ -0.5 & 0.5 \end{bmatrix}$.

It is noted that delay-independent results can not be applied in this case. Moreover, many delay-dependent results (see, e.g., [17], [59] and [135]) do not work when the time-varying delay $\tau(t)$ is not differentiable or there is $\theta > 0$ such that $\dot{\tau}(\theta) \geq 1$. Criterion (6.54), p189, [88] gives the upper bound of time delay for exponential stability

$$h_{max} < \frac{1}{2\,a_1^2}\left(\sqrt{h_1^4 + \frac{1}{2}(\lambda_1 - h_1^2)^2} - h_1^2\right) \tag{3.53}$$

if $\lambda_1 > h_1^2$, where $\lambda_1 = \lambda_m(-A_1 - A_1^T)$, $a_1 = |A_1|$ and $h_1 = |H_1|$. When $c = 1.5$ and $\lambda_1 = 1.7929 > h_1^2 = 1.3257$, (3.53) yields $h_{max} < 0.0076$; by Theorem 3.1, it is found that the upper bound of time delay $h_{max} = 0.1978$. When $c = 0.8$ and $\lambda_1 = 1.2615 < h_1^2 = 1.3257$, criterion (6.54), p189, [88] does not work; but Theorem 3.1 gives $h_{max} = 0.1523$. This example shows that the result of the proposed method is an improvement.

Example 3.2 Let us look at a scalar stochastic delay system

$$\mathrm{d}x(t) = -bx(t-\tau)\mathrm{d}t + [c_0 x(t) + c_1 x(t-\tau)]\mathrm{d}w(t) \tag{3.54}$$

with $2b > c_1^2 > 0$.

Example (6.55), p189, [88] has studied system (3.54) with $c_0 = 0$. It is found that Theorem 3.1 yields $h_{max} = 0.1339$ for exponential stability of system (3.54) with $b = -c_1 = 1$ and $c_0 = 0$, which is better than $h_{max} < \sqrt{3/8} - 1/2 = 0.1124$ given by criterion (6.54), p187, [88]. To compare with other results, we assume $\tau(t) \equiv h$ for all $t \geq 0$. Results by different methods for stability of constant delay case with $b = -c_1 = 1$ and various values of c_0 are listed in Table 3.1, where Thm=Theorem and na=not applicable.

Table 3.1: h_{max} by different methods

c_0	[5]	[17]	Thm 3.3	Thm 3.1
0.0	0.4999	0.1715	0.1339	0.1339
0.2	0.2799	0.1454	0.1760	0.1856
0.4	0.0199	0.0818	0.1894	0.2300
0.6	na	0.0159	0.1818	0.2627
0.8	na	na	0.1612	0.2927
1.0	na	na	0.1339	0.3200
1.2	na	na	0.1046	0.3345
1.4	na	na	0.0763	0.2233

Example 3.3 Consider the following stochastic system that describes a scalar linear stochastic oscillator (see Example 3.7, p125, [88])

$$\mathrm{d}x(t) = (A_0 x(t) + Bu(t-\tau))\mathrm{d}t + H_0 x(t)\mathrm{d}w(t)\,, \tag{3.55}$$

where $A_0 = \begin{bmatrix} 0 & 1 \\ 0 & 2 \end{bmatrix}$, $H_0 = \begin{bmatrix} 0 & 0 \\ 2 & 0.5 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

By Theorem 3.5, p123, [88] with $V(x,t) = |x(t)|^2$, it is easy to verify that system (3.55) is almost surely exponentially unstable when $u(t-\tau) = 0$ for all $t \geq 0$. Since there is no matrix $X > 0$ such that $A_0 X + X A_0^T < 0$, Theorems in [130] do not work when $h > 0$. By Theorem 3.2 with $\delta_M = 1$, $\delta_R = 10^{-4}$, $\delta_S = 27$ and $\delta_Q = 3$, the sufficient condition for stabilisability of system (3.55) is $0 \le h \le 0.1256$. When $h = 0.1256$, solving LMIs (3.41)-(3.42) gives $K = [-10.4787 \ \ -5.4418]$, which implies system (3.55) is exponentially stabilised by delayed-state-feedback controller $u(t-\tau) = [-10.4787 \ \ -5.4418]\, x(t-\tau)$.

Example 3.4 Deterministic systems may be regarded a special class of stochastic systems, e.g., the following deterministic delay system is exactly system (3.6) with $H_0(t) = H_1(t) = 0$ and $\tau(t) = h$ for all $t \ge 0$ (see [66])

$$\dot{x}(t) = A_0(t)x(t) + A_1(t)x(t-h) + W(t)u(t-h)\,, \tag{3.56}$$

where $A_0 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$, $A_1 = \begin{bmatrix} -2 & 0.1 \\ -1 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1.05 \end{bmatrix}$, $L_A = 0.1I$, $E_{A0} = E_{A1} = I$ and $E_B = \begin{bmatrix} 0 \\ 0.5 \end{bmatrix}$.

The result of [66] guarantees that system (3.56) is asymptotically stabilised by the delayed-state-feedback control $u(t-h) = [-0.0499 \ \ -1.005]\, x(t-h)$ if $0 \le h < 0.1512$. However, by Theorem 3.2 with $\delta_M = 1$, $\delta_R = 0.1$, $\delta_S = 600$ and $\delta_Q = 3$, it is found that the closed-loop system (3.56) and (3.7) with $K = [-0.2237 \ \ -0.6356]$ is exponentially stable when $0 \le h \le 0.4182$, which has a larger upper bound of time delay but a smaller feedback gain. By Theorem 3.1, closed-loop system (3.56) with $u(t-h) = [-0.0499 \ \ -1.005]\, x(t-h)$ is exponentially stable for $0 \le h \le 0.4468$ while that with $u(t-h) = [-0.2237 \ \ -0.6356]\, x(t-h)$ is exponentially stable for $0 \le h \le 0.4676$. Moreover, when $h = 0.4468$, the estimate of Lyapunov exponent of the former closed-loop system is $-\varepsilon = -0.0690$ while that of the latter one is $-\varepsilon = -0.0778$.

3.7 Summary

This chapter has presented a delay-dependent criterion for exponential stability of uncertain stochastic delay systems in terms of LMIs. It should be pointed out that introduction of (3.14) and (3.15), as well as techniques such as Lemmas 1-3, helps exploit the structure of diffusion of the system and deal with the crossing terms. This leads to a less conservative result. Based on the newly established stability criterion, a robust state-feedback controller (3.7) has been proposed to exponentially stabilise the uncertain stochastic system with input delay (3.1). This design method involves four tuning parameters. It is observed that δ_R in LMI (3.41) may be chosen as a small positive number when $\delta_M = 1$, which is also suggested in the examples of last section. The above numerical examples have verified the effectiveness of the proposed method.

Chapter 4

State-feedback stabilisation of neutral stochastic delay systems with input delay

4.1 Introduction

Many dynamical systems are described with neutral functional differential equations that include neutral delay differential equations. And hence these systems are called neutral-type systems, or, neutral systems. Motivated by chemical engineering systems as well as theory of aero elasticity, studies on deterministic neutral systems have been of research interest over the past decades (see, e.g., [20], [26], [27], [34], [35], [61], [101] and the references therein). As stochastic modelling has come to play an important role in many branches of science and industry, neutral stochastic delay systems, which are described with neutral stochastic functional equations and neutral stochastic delay equations, have been intensively studied over recent years (see [44], [53], [67], [73], [76], [80], [129], [130]). Mao ([73], [76] and [80]) initiated the study of exponential stability of neutral stochastic functional equations, developed Razumikhin-type theorems further for exponential stability of neutral stochastic functional equations, and studied asymptotic properties of neutral stochastic delay differential equations. More recently, Luo et al. ([67]) proposed new criteria for exponential stability of neutral stochastic delay differential equations while Xu et al. ([129] and [130]) investigated exponential dynamic output feedback control of neutral stochastic systems with distributed delays and robust H_∞ control of neutral stochastic systems with single time delay. Generally, for systems with small delays, these delay-independent criteria are likely to be conservative. However, few existing works study delay-dependent stability of neutral stochastic delay systems. Neutral stochastic delay systems with input delay have been considered in [130]. But the result can not be applied to a significant number of cases when the non-delay system matrix is unstable (see Example 4.3). This chapter further develops the techniques proposed in the previous chapter to cope with problems of stability and stabilisation of linear neutral stochastic delay systems. Delay-dependent exponential stability criteria are established by linear matrix inequality (LMI) approach. Based on these stability results, a memoryless delayed-state-feedback controller is designed to exponentially stabilise the neutral stochastic delay systems. Moreover, our results are developed to remove an assumption that a norm of the delayed difference operator is less than one, which is employed by the existing results. Numerical examples are conducted to verify the effectiveness of our proposed method.

4.2 Problem statement

Throughout this chapter, unless otherwise specified, we will employ the same notation as Chapter 3.

Let us consider an n-dimensional neutral stochastic delay system with delayed feedback control

$$\begin{aligned} \mathrm{d}\big[x(t) - Cx(t-h_1)\big] \;=\;& [A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2) + Bu(t-h_2)]\,\mathrm{d}t \\ &+ [H_0x(t) + H_1x(t-h_1) + H_2x(t-h_2)]\,\mathrm{d}w(t) \end{aligned} \tag{4.1}$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in \tilde{L}^2_{\mathcal{F}_0}([-h, 0]; R^n)$, where $x(t) \in R^n$ is the state vector; $u(t-h_2) \in R^m$ is the control input; positive scalar constants h_1, h_2 are time delays of the system and $h = \max\{h_1, h_2\}$; C, A_i, B and H_i, $i = 0, 1, 2$, are known matrices.

Denote

$$\begin{aligned} f(t) =\;& A_0x(t) + A_1x(t-h_1) + A_2x(t-h_2)\,, \\ g(t) =\;& H_0x(t) + H_1x(t-h_1) + H_2x(t-h_2) \end{aligned} \tag{4.2}$$

for all $t \geq 0$. One can observe that

$$|f(t)|^2 \leq K_f \|x_t\|^2 , \quad |g(t)|^2 \leq K_g \|x_t\|^2 \tag{4.3}$$

for all $t \geq 0$, where $x_t = \{x(t+\theta) : -h \leq \theta \leq 0\}$, $K_f = 3\sum_{i=0}^{2} |A_i|^2$ and $K_g = 3\sum_{i=0}^{2} |H_i|^2$. This implies that both $f(\varphi, t)$ and $g(\varphi, t)$ satisfy the global Lipschitz condition and the linear growth condition. It is easy to verify, by the way of induction proposed in the proof of Theorem 3.1, p210, [88], that there exists a unique continuous solution denoted by $x(t;\xi)$ to neutral stochastic delay differential equation

$$\begin{aligned} \mathrm{d}\big[x(t) - Cx(t-h_1)\big] &= [A_0 x(t) + A_1 x(t-h_1) + A_2 x(t-h_2)]\,\mathrm{d}t \\ &\quad + [H_0 x(t) + H_1 x(t-h_1) + H_2(t) x(t-h_2)]\,\mathrm{d}w(t) \end{aligned} \tag{4.4}$$

for all $t \geq 0$.

The objectives of this chapter are: (i) to establish sufficient conditions for exponential stability of system (4.1) with $u(t) = 0$, that is, the unforced uncertain neutral stochastic delay system (4.4); and (ii) to design a state-feedback controller

$$u(t) = Kx(t) \tag{4.5}$$

to exponentially stabilise system (4.1) with input delay, where K is a constant gain matrix to be determined. The proposed method can be easily extended to those cases with norm-bounded uncertainties in parameters A_i, B and H_i. The method can also be applied to systems with multiple, distributed and time-varying delays. It should be pointed out that, for simplicity only, we consider a relatively simple model (see also, e.g., [26] and [27]).

4.3 Delay-dependent exponential stability

Delay-dependent stability of neutral deterministic delay systems has been intensively studied over recent years (see, e.g., [20], [26], [27], [35], [61], [101]). However, so far little is known about delay-dependent criteria for stability of neutral stochastic delay systems. Denote $\bar{A}_0 = A_0$, $\bar{A}_1 = A_0 C + A_1$, $\bar{A}_2 = A_2$, $\bar{H}_0 = H_0$, $\bar{H}_1 = H_0 C + H_1$, $\bar{H}_2 = H_2$, $\bar{A} = \sum_{i=0}^{2} \bar{A}_i$ and $\bar{H} = \sum_{i=0}^{2} \bar{H}_i$. Sufficient conditions for delay-dependent exponential stability of system (4.4) are proposed as follows.

Theorem 4.1 *The neutral stochastic delay system (4.4) is mean-square exponentially stable and is also almost surely exponentially stable provided that there exist matrices* $P_{11} > 0$, $Q_k > 0$, $R_k > 0$, $S > 0$, $T_k > 0$, P_{21}, P_{22}, P_{23}, P_{31}, P_{32}, P_{33} *and* $k = 1, 2$ *such that*

$$\Gamma = \begin{bmatrix} \Gamma_{11} & * & * & * & * & * & * & * & * & * \\ \Gamma_{21} & \Gamma_{22} & * & * & * & * & * & 0 & * & * \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & * & * & * & * & 0 & * & * \\ h_1 L_{11}^T & h_1 L_{21}^T & h_1 L_{31}^T & -h_1 Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_2 L_{12}^T & h_2 L_{22}^T & h_2 L_{32}^T & 0 & -h_2 Q_2 & 0 & 0 & 0 & 0 & 0 \\ L_{11}^T & L_{21}^T & L_{31}^T & 0 & 0 & -R_1 & 0 & 0 & 0 & 0 \\ L_{12}^T & L_{22}^T & L_{32}^T & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 \\ \Gamma_{81} & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{88} & 0 & 0 \\ C^T L_{11}^T & C^T L_{21}^T & C^T L_{31}^T & 0 & 0 & 0 & 0 & 0 & -T_1 & 0 \\ C^T L_{12}^T & C^T L_{22}^T & C^T L_{32}^T & 0 & 0 & 0 & 0 & 0 & 0 & -T_2 \end{bmatrix} < 0, \tag{4.6}$$

where

$$\begin{aligned}
\Gamma_{11} &= P_{21}^T \bar{A} + \bar{A}^T P_{21} + P_{31}^T \bar{H} + \bar{H}^T P_{31} + S + T_1 + T_2, \\
\Gamma_{21} &= P_{22}^T \bar{A} + P_{32}^T \bar{H} + P_{11} - P_{21}, \quad \Gamma_{31} = P_{23}^T \bar{A} + P_{33}^T \bar{H} - P_{31}, \\
\Gamma_{81} &= C^T (S + T_1 + T_2), \quad \Gamma_{22} = -P_{22}^T - P_{22} + h_1 Q_1 + h_2 Q_2, \quad \Gamma_{32} = -P_{23}^T - P_{32}, \\
\Gamma_{33} &= -P_{33}^T - P_{33} + P_{11} + h_1 R_1 + h_2 R_2\,, \quad \Gamma_{88} = -S + C^T (S + T_1 + T_2) C, \\
L_{11} &= P_{21}^T \bar{A}_1 + P_{31}^T \bar{H}_1\,, \quad L_{21} = P_{22}^T \bar{A}_1 + P_{32}^T \bar{H}_1\,, \quad L_{31} = P_{23}^T \bar{A}_1 + P_{33}^T \bar{H}_1\,, \\
L_{12} &= P_{21}^T \bar{A}_2 + P_{31}^T \bar{H}_2\,, \quad L_{22} = P_{22}^T \bar{A}_2 + P_{32}^T \bar{H}_2\,, \quad L_{32} = P_{23}^T \bar{A}_2 + P_{33}^T \bar{H}_2\,,
\end{aligned}$$

and entries denoted by $*$ *can be readily inferred from symmetry of the matrix.*

Proof. To simplify the expression, we define

$$\eta(t) = x(t) - Cx(t - h_1) \tag{4.7}$$

for all $t \geq 0$. With notation (4.2) and (4.7), we can rewrite the unforced system (4.4) as

$$\mathrm{d}\eta(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}w(t) \tag{4.8}$$

on $t \geq 0$ with initial data ξ.

So we have

$$\eta(t_2) - \eta(t_1) = \int_{t_1}^{t_2} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) \tag{4.9}$$

for all $t_2 \geq t_1 \geq 0$.

By (4.2) and (4.9), we can observe that

$$\begin{aligned} f(t) &= \sum_{i=0}^{2} \bar{A}_i \eta(t) - \sum_{i=1}^{2} \bar{A}_i \big[\eta(t) - \eta(t-h_i)\big] + \sum_{i=1}^{2} \bar{A}_i C x(t-h_1-h_i) \\ &= \bar{A}\,\eta(t) - \sum_{i=1}^{2} \bar{A}_i \int_{t-h_i}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) + \sum_{i=1}^{2} \bar{A}_i C x(t-h_1-h_i)\,, \end{aligned} \tag{4.10}$$

$$g(t) = \bar{H}\eta(t) - \sum_{i=1}^{2} \bar{H}_i \int_{t-h_i}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) + \sum_{i=1}^{2} \bar{H}_i C x(t-h_1-h_i) \tag{4.11}$$

for all $t \geq h$. Choose a Lyapunov-Krasovskii functional candidate for system (4.8) as follows:

$$V(t) = \sum_{j=1}^{5} V_j(t)\,, \tag{4.12}$$

where

$$\begin{aligned} V_1(t) &= \eta^T(t)P_{11}\eta(t)\,, \quad V_2(t) = \sum_{i=1}^{2} \int_{t-h_i}^{t} (s-t+h_i) f^T(s) Q_i f(s)\mathrm{d}s\,, \\ V_3(t) &= \sum_{i=1}^{2} \int_{t-h_i}^{t} (s-t+h_i) g^T(s) R_i g(s)\mathrm{d}s\,, \quad V_4(t) = \int_{t-h_1}^{t} x^T(s) S x(s)\mathrm{d}s\,, \\ V_5(t) &= \sum_{i=1}^{2} \int_{t-h_1-h_i}^{t} x^T(s) T_i x(s)\mathrm{d}s\,. \end{aligned}$$

By Itô's formula, we have

$$\mathrm{d}V(t) = \mathcal{L}V(t)\mathrm{d}t + \sigma(t)\mathrm{d}w(t)\,, \tag{4.13}$$

where

$$\begin{aligned} \mathcal{L}V(t) &= \sum_{j=1}^{5} \mathcal{L}V_j(t) = 2\eta^T(t)P_{11}f(t) + g^T(t)P_{11}g(t) + \sum_{j=2}^{5} \dot{V}_j(t)\,, \\ \sigma(t) &= 2\eta^T(t)P_{11}g(t)\,. \end{aligned} \tag{4.14}$$

Denote

$$y(t) = \begin{bmatrix} \eta(t) \\ f(t) \\ g(t) \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} P_{11} & 0 & 0 \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{bmatrix}. \tag{4.15}$$

By equalities (4.10) and (4.11), we have

$$\begin{aligned}2\eta^T(t)P_{11}f(t) &= 2\begin{bmatrix}\eta(t)\\ f(t)\\ g(t)\end{bmatrix}^T\begin{bmatrix}P_{11} & P_{21}^T & P_{31}^T\\ 0 & P_{22}^T & P_{32}^T\\ 0 & P_{23}^T & P_{33}^T\end{bmatrix}\begin{bmatrix}f(t)\\ 0\\ 0\end{bmatrix} = 2y^T(t)P^T\begin{bmatrix}f(t)\\ 0\\ 0\end{bmatrix} = 2y^T(t)P^T\\ &\cdot\left\{\begin{bmatrix}0 & I & 0\\ \bar{A} & -I & 0\\ \bar{H} & 0 & -I\end{bmatrix}y(t) - \sum_{i=1}^{2}\begin{bmatrix}0\\ \bar{A}_i\\ \bar{H}_i\end{bmatrix}\left(\int_{t-h_i}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s)\right)\right.\\ &\left.+\sum_{i=1}^{2}\begin{bmatrix}0\\ \bar{A}_i\\ \bar{H}_i\end{bmatrix}Cx(t-h_1-h_i)\right\}.\end{aligned}$$

or

$$\begin{aligned}2\eta^T(t)P_{11}f(t) &= y^T(t)(P^TA+A^TP)y(t) - 2y^T(t)\sum_{i=1}^{2}P^T\begin{bmatrix}0 & \bar{A}_i^T(t) & \bar{H}_i^T(t)\end{bmatrix}^T\\ &\cdot\left(\int_{t-h_i}^{t} f(s)\mathrm{d}s + g(s)\mathrm{d}w(s) - Cx(t-h_1-h_i)\right), \qquad (4.16)\end{aligned}$$

where

$$A = \begin{bmatrix}0 & I & 0\\ \bar{A} & -I & 0\\ \bar{H} & 0 & -I\end{bmatrix},\ P^TA + A^TP = \begin{bmatrix}P_{A1} & P_{A2} & P_{A3}\\ P_{A2}^T & -P_{22}^T - P_{22} & -P_{32}^T - P_{23}\\ P_{A3}^T & -P_{32} - P_{23}^T & -P_{33}^T - P_{33}\end{bmatrix}$$

with $P_{A1} = P_{21}^T\bar{A} + \bar{A}^TP_{21} + P_{31}^T\bar{H} + \bar{H}^TP_{31}$, $P_{A2} = \bar{A}^TP_{22} + \bar{H}^TP_{32} + P_{11} - P_{21}^T$, $P_{A3} = \bar{A}^TP_{23} + \bar{H}^TP_{33} - P_{31}^T$ and $P^T\begin{bmatrix}0 & \bar{A}_i^T & \bar{H}_i^T\end{bmatrix}^T = \begin{bmatrix}L_{1i}^T & L_{2i}^T & L_{3i}^T\end{bmatrix}^T$ for $i = 1, 2$.

Direct computations with Lemma 3.2 and equation (4.7) give

$$\dot{V}_2(t) \leq \sum_{i=1}^{2}\left[f^T(t)h_iQ_if(t) - \int_{t-h_i}^{t}\frac{1}{h_i}f^T(s)\mathrm{d}s(h_iQ_i)\int_{t-h_i}^{t}\frac{1}{h_i}f(s)\mathrm{d}s\right], \qquad (4.17)$$

$$\dot{V}_3(t) = \sum_{i=1}^{2}\left[g^T(t)h_iR_ig(t) - \int_{t-h_i}^{t}g^T(s)R_ig(s)\mathrm{d}s\right], \qquad (4.18)$$

$$\dot{V}_4(t) = \begin{bmatrix}\eta(t)\\ x(t-h_1)\end{bmatrix}^T\begin{bmatrix}S & SC\\ C^TS & -S + C^TSC\end{bmatrix}\begin{bmatrix}\eta(t)\\ x(t-h_1)\end{bmatrix}, \qquad (4.19)$$

$$\dot{V}_5(t) = \sum_{i=1}^{2}\begin{bmatrix}\eta(t)\\ x(t-h_1)\\ x(t-h_1-h_i)\end{bmatrix}^T\begin{bmatrix}T_i & T_iC & 0\\ C^TT_i & C^TT_iC & 0\\ 0 & 0 & -T_i\end{bmatrix}\begin{bmatrix}\eta(t)\\ x(t-h_1)\\ x(t-h_1-h_i)\end{bmatrix}. \qquad (4.20)$$

By isometry property, for $i = 1, 2$, we have

$$\begin{aligned}\mathbb{E}\left[\int_{t-h_i}^{t} g^T(s)R_i g(s)\mathrm{d}s\right] &= \int_{t-h_i}^{t} \mathbb{E}\left[g^T(s)R_i g(s)\right]\mathrm{d}s \\ &= \mathbb{E}\left[\int_{t-h_i}^{t} g^T(s)\mathrm{d}w(s)\, R_i \int_{t-h_i}^{t} g(s)\mathrm{d}w(s)\right].\end{aligned}$$

Therefore, substituting inequalities (4.16)-(4.20) into (4.14) and taking expectation on the both sides yield

$$\mathbb{E}\mathcal{L}V(t) \le \mathbb{E}\left[z^T(t)\Gamma z(t)\right] , \tag{4.21}$$

where $z^T(t) = \big[\eta^T(t)\ \ f^T(t)\ \ g^T(t)\ \ -\frac{1}{h_1}\int_{t-h_1}^{t} f^T(s)\mathrm{d}s\ \ -\frac{1}{h_2}\int_{t-h_2}^{t} f^T(s)\mathrm{d}s\ \ -\int_{t-h_1}^{t} g^T(s)\mathrm{d}w(s)$

$-\int_{t-h_2}^{t} g^T(s)\mathrm{d}w(s)\ \ x^T(t-h_1)\ \ x^T(t-2h_1)\ \ x^T(t-h_1-h_2)\big]^T$.

By LMI (4.6), we have

$$\mathbb{E}\mathcal{L}V(t) \le -\lambda_\Gamma \mathbb{E}|z(t)|^2 \le -\lambda_\Gamma \mathbb{E}\left[|\eta(t)|^2 + |x(t-h_1)|^2\right] \tag{4.22}$$

with $\lambda_\Gamma = \lambda_m(-\Gamma)$ and

$$C^T SC - S < 0\,. \tag{4.23}$$

For any $\kappa \in (0, 1)$, equation (4.7), inequalities (4.22)-(4.23) and Lemma (3.1) give

$$\begin{aligned}\mathbb{E}\mathcal{L}V(t) &\le -(1-\kappa)\lambda_\Gamma \mathbb{E}|\eta(t)|^2 - \kappa\lambda_\Gamma(\mathbb{E}|\eta(t)|^2 + \frac{1}{\kappa}\mathbb{E}|x(t-h_1)|^2) \\ &\le -(1-\kappa)\lambda_\Gamma \mathbb{E}|\eta(t)|^2 - \frac{\kappa\lambda_\Gamma}{\lambda_M(S)}\mathbb{E}\Big[\big(x(t)-Cx(t-h_1)\big)^T S \\ &\qquad \cdot \big(x(t)-Cx(t-h_1)\big) + \frac{1}{\kappa}x^T(t-h_1)Sx(t-h_1)\Big] \\ &\le -(1-\kappa)\lambda_\Gamma \mathbb{E}|\eta(t)|^2 - \frac{\kappa\lambda_\Gamma}{\lambda_M(S)}\mathbb{E}\big[x^T(t)Sx(t) \\ &\qquad - 2x^T(t)SCx(t-h_1) + \frac{1+\kappa}{\kappa}x^T(t-h_1)C^T SCx(t-h_1)\big] \\ &\le -(1-\kappa)\lambda_\Gamma \mathbb{E}|\eta(t)|^2 - \frac{\kappa\lambda_\Gamma\lambda_m(S)}{(1+\kappa)\lambda_M(S)}\mathbb{E}|x(t)|^2\,. \\ &\le -\lambda_0 \mathbb{E}\left[|\eta(t)|^2 + |x(t)|^2\right],\end{aligned}$$

where $\lambda_0 = \min\big\{(1-\kappa)\lambda_\Gamma,\ \kappa\lambda_\Gamma\lambda_m(S)[(1+\kappa)\lambda_M(S)]^{-1}\big\} > 0\,.$

It is obvious from the definition of $V(t)$ that

$$\alpha_0|\eta(t)|^2 \le V(t) \le \alpha_1|\eta(t)|^2 + \alpha_2\int_{t-2h}^{t}|x(s)|^2\mathrm{d}s\,, \tag{4.24}$$

where $\alpha_0 = \lambda_m(P_{11})$, $\alpha_1 = \lambda_M(P_{11})$, $\alpha_2 = \sum_{i=1}^{2} h_i[\lambda_M(Q_i)K_f + \lambda_M(R_i)K_g] + \lambda_M(S) + \sum_{i=1}^{2} \lambda_M(T_i)$. Choose $\varepsilon > 0$ such that

$$\max\{\varepsilon\alpha_1, 2h\varepsilon\alpha_2 e^{2h\varepsilon}\} \le \lambda_0 \quad \text{and} \quad e^{2h\varepsilon}C^T SC - S < 0. \tag{4.25}$$

By Itô's lemma, we have

$$\mathrm{d}\big[e^{\varepsilon s}V(s)\big] = e^{\varepsilon s}\big[\varepsilon V(s) + \mathcal{L}V(s)\big]\mathrm{d}s + e^{\varepsilon s}\sigma(s)\mathrm{d}w(s), \quad \forall\, s \ge 0. \tag{4.26}$$

Let $t_0 = h$, then integrating from t_0 to t and taking expectation on (4.26) give

$$\begin{aligned}
&e^{\varepsilon t}\mathbb{E}V(t) - e^{\varepsilon t_0}\mathbb{E}V(t_0) \\
&\quad = \mathbb{E}\int_{t_0}^{t} e^{\varepsilon s}\big[\varepsilon V(s) + \mathcal{L}V(s)\big]\mathrm{d}s \\
&\quad \le \mathbb{E}\int_{t_0}^{t} e^{\varepsilon s}\Big[\varepsilon\alpha_1|\eta(s)|^2 + \varepsilon\alpha_2\int_{s-2h}^{s}|x(v)|^2\mathrm{d}v - \lambda_0(|\eta(s)|^2 + |x(s)|^2)\Big]\mathrm{d}s \\
&\quad \le \mathbb{E}\int_{t_0}^{t} e^{\varepsilon s}\Big[\varepsilon\alpha_2\int_{s-2h}^{s}|x(v)|^2\mathrm{d}v - \lambda_0|x(s)|^2\Big]\mathrm{d}s\,.
\end{aligned} \tag{4.27}$$

Since

$$\begin{aligned}
\int_{t_0}^{t} e^{\varepsilon s}\mathrm{d}s\int_{s-2h}^{s}|x(v)|^2\mathrm{d}v &\le \int_{t_0-2h}^{t}|x(v)|^2\mathrm{d}v\int_{v}^{v+2h}e^{\varepsilon s}\mathrm{d}s \le 2he^{2h\varepsilon}\int_{t_0-2h}^{t}|x(s)|^2e^{\varepsilon s}\mathrm{d}s \\
&\le 2he^{2h\varepsilon}\int_{t_0}^{t}|x(s)|^2e^{\varepsilon s}\mathrm{d}s + 2he^{2h\varepsilon}\int_{t_0-2h}^{t_0}|x(s)|^2\mathrm{d}s\,,
\end{aligned}$$

it follows

$$\alpha_0 e^{\varepsilon t}\mathbb{E}|\eta(t)|^2 \le e^{\varepsilon t}\mathbb{E}V(t) \le \alpha_0 C_h \quad \text{or} \quad \mathbb{E}|\eta(t)|^2 \le C_h e^{-\varepsilon t}\,, \tag{4.28}$$

where $C_h = \alpha_h \sup_{-h\le\theta\le h}\mathbb{E}|x(\theta)|^2$ with $\alpha_h = \alpha_0^{-1}e^{\varepsilon h}[\alpha_1 + 2h\alpha_2(1 + 2h\varepsilon e^{2h\varepsilon})] \ge 1$. Since neutral stochastic delay differential equation (4.4) has a unique continuous solution, C_h is a nonnegative finite number for any $0 \le h < \infty$.

Since $e^{2\varepsilon h}C^T SC < S$, there exists a number $\mu \in (0,1)$ such that

$$e^{2\varepsilon h}C^T SC < \mu S < S\,. \tag{4.29}$$

Note that $\eta^T(t)S\eta(t) = x^T(t)Sx(t) - 2x^T(t)SCx(t-h_1) + x^T(t-h_1)C^T SCx(t-h_1)$ for all $t \ge 0$. By Lemma 3.1, we have

$$e^{\varepsilon t}x^T(t)Sx(t) \le \frac{e^{\varepsilon t}}{1-\mu}\eta^T(t)S\eta(t) + \frac{e^{\varepsilon t}}{\mu}x^T(t-h_1)C^T SCx(t-h_1). \tag{4.30}$$

Let ρ be any nonnegative real number. For all $0 \le t \le \rho$, we have

$$\begin{aligned} e^{\varepsilon t}\mathbb{E}\big[x^T(t)Sx(t)\big] &\le \frac{1}{1-\mu}\sup_{0\le t\le\rho}\mathbb{E}\big[e^{\varepsilon t}\eta^T(t)S\eta(t)\big] + \frac{1}{\mu}\sup_{0\le t\le\rho}\mathbb{E}\big[e^{\varepsilon t}x^T(t-h_1)C^TSCx(t-h_1)\big] \\ &\le \frac{1}{1-\mu}\lambda_M(S)\sup_{0\le t\le\rho}\mathbb{E}[e^{\varepsilon t}|\eta(t)|^2] + \frac{e^{\varepsilon h_1}}{\mu}\sup_{-h_1\le t\le\rho}\mathbb{E}\big[e^{\varepsilon t}x^T(t)C^TSCx(t)\big] \\ &\le \frac{1}{1-\mu}\lambda_M(S)C_h + e^{-\varepsilon h}\sup_{-h\le t\le\rho}\Big\{e^{\varepsilon t}\mathbb{E}\big[x^T(t)Sx(t)\big]\Big\}. \end{aligned}$$

But this holds for all $-h \le t \le \rho$. So

$$\sup_{-h\le t\le\rho}\Big\{e^{\varepsilon t}\mathbb{E}\big[x^T(t)Sx(t)\big]\Big\} \le \frac{\lambda_M(S)C_h}{(1-e^{-\varepsilon h})(1-\mu)}. \tag{4.31}$$

Since ρ is an arbitary nonnegative number, we have

$$\mathbb{E}|x(t)|^2 \le \frac{\lambda_M(S)C_h e^{-\varepsilon t}}{(1-e^{-\varepsilon h})(1-\mu)\lambda_m(S)}, \quad \forall\, t \ge -h. \tag{4.32}$$

The mean-square exponential stability has been proven.

Now let us proceed to discuss the almost sure exponential stability. Let $\gamma \in (0, \varepsilon)$ be arbitrary. We claim that there are a finite positive number t_h such that for all $t \ge t_h$

$$|\eta(t)|^2 \le e^{-(\varepsilon-\gamma)t} \quad a.s. \tag{4.33}$$

Therefore, for all $t \ge t_h$, inequality (4.30) implies

$$e^{(\varepsilon-\gamma)t}x^T(t)Sx(t) \le \frac{\lambda_M(S)e^{(\varepsilon-\gamma)t}}{1-\mu} + \frac{e^{(\varepsilon-\gamma)t}}{\mu}x^T(t-h_1)C^TSCx(t-h_1) \quad a.s.$$

Using similar reasoning to above and letting $\gamma \to 0$, we have $|x(t)|^2 \le \lambda_M(S)e^{-\varepsilon t}[(1 - e^{-\varepsilon h})(1-\mu)\lambda_m(S)]^{-1}$ $a.s.$ for all $t \ge t_h - h$. This implies immediately

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)| \le -\frac{\varepsilon}{2} \quad a.s.$$

We complete the proof by showing that inequality (4.33) is true. Note that

$$\mathbb{E}|f(t)|^2 \le K_f \sup_{-h\le\theta\le 0}\mathbb{E}|x(t+\theta)|^2 \quad \text{and} \quad \mathbb{E}|g(t)|^2 \le K_g \sup_{-h\le\theta\le 0}\mathbb{E}|x(t+\theta)|^2$$

for all $t \ge 0$. For any integer $k \ge 1$, by Hölder's inequality and the Burkholder-Davis-Gundy inequality, one can derive that

$$\begin{aligned} \mathbb{E}\left[\sup_{0\le\theta\le h}|\eta(kh+\theta)|^2\right] &\le 3\Bigg[\mathbb{E}|\eta(kh)|^2 + h\int_{kh}^{(k+1)h}\mathbb{E}|f(s)|^2\mathrm{d}s \\ &\quad + \mathbb{E}\left(\sup_{0\le\theta\le h}\left|\int_{kh}^{kh+\theta}g(s)\mathrm{d}w(s)\right|^2\right)\Bigg] \le \beta_h e^{-kh\varepsilon}, \end{aligned} \tag{4.34}$$

where $\beta_h = 3C_h(1 + K_f h^2 e^{h\varepsilon} + 4K_g h e^{h\varepsilon})$. But, by Chebyshev's inequality, this implies

$$\mathbb{P}\left\{\omega : \sup_{0\le\theta\le h} |\eta(kh+\theta)|^2 > e^{-(\varepsilon-\gamma)kh}\right\} \le \beta_h e^{-\gamma kh}.$$

By the Borel-Cantelli lemma, there is a finite random variable k_0 such that

$$\sup_{0\le\theta\le h} |\eta(kh+\theta)|^2 \le e^{-(\varepsilon-\gamma)kh} \quad a.s.$$

for all $k \ge k_0$. Therefore, inequality (4.33) holds with $t_h \ge k_0 h$.

From the proof of Theorem 4.1, we observe that

$$z^T(t)\Gamma z(t) \;=\; \bar{z}^T(t)\bar{\Gamma}\bar{z}(t) + \sum_{i=1}^{2} x^T(t-h_1-h_i)\left(-T_i + C^T W_i C\right) x(t-h_1-h_i),$$

where $\bar{z}^T(t) = \big[\eta^T(t)\;\; f^T(t)\;\; g^T(t)\;\; -\frac{1}{h_1}\int_{t-h_1}^{t} f^T(s)\mathrm{d}s \;\; -\frac{1}{h_2}\int_{t-h_2}^{t} f^T(s)\mathrm{d}s \;\; -\int_{t-h_1}^{t} g^T(s)\mathrm{d}w(s)$

$-\int_{t-h_2}^{t} g^T(s)\mathrm{d}w(s) \quad x^T(t-h_1)\;\; x^T(t-2h_1)C^T \;\; x^T(t-h_1-h_2)C^T\big]$. By inequalities (4.21) and (4.37), this implies

$$\mathbb{E}\mathcal{L}V(t) \le \mathbb{E}\left[\bar{z}^T(t)\bar{\Gamma}\bar{z}(t)\right]. \tag{4.35}$$

Also inequality (4.36) implies (4.23). By repeating the same reasoning as that in the proof of Theorem 4.1, one can prove

Theorem 4.2 *The neutral stochastic delay system (4.4) is mean-square exponentially stable and is also almost surely exponentially stable provided that there exist matrices* $P_{11} > 0$, $Q_i > 0$, $R_i > 0$, $S > 0$, $T_i > 0$, $W_i > 0$, P_{21}, P_{22}, P_{23}, P_{31}, P_{32}, P_{33} *and* $i = 1, 2$ *such that*

$$\bar{\Gamma} = \begin{bmatrix} \Gamma_{11} & * & * & * & * & * & * & * & * & * \\ \Gamma_{21} & \Gamma_{22} & * & * & * & * & * & 0 & * & * \\ \Gamma_{31} & \Gamma_{32} & \Gamma_{33} & * & * & * & * & 0 & * & * \\ h_1 L_{11}^T & h_1 L_{21}^T & h_1 L_{31}^T & -h_1 Q_1 & 0 & 0 & 0 & 0 & 0 & 0 \\ h_2 L_{12}^T & h_2 L_{22}^T & h_2 L_{32}^T & 0 & -h_2 Q_2 & 0 & 0 & 0 & 0 & 0 \\ L_{11}^T & L_{21}^T & L_{31}^T & 0 & 0 & -R_1 & 0 & 0 & 0 & 0 \\ L_{12}^T & L_{22}^T & L_{32}^T & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 \\ \Gamma_{81} & 0 & 0 & 0 & 0 & 0 & 0 & \Gamma_{88} & 0 & 0 \\ L_{11}^T & L_{21}^T & L_{31}^T & 0 & 0 & 0 & 0 & 0 & -W_1 & 0 \\ L_{12}^T & L_{22}^T & L_{32}^T & 0 & 0 & 0 & 0 & 0 & 0 & -W_2 \end{bmatrix} < 0 \tag{4.36}$$

and

$$\begin{bmatrix} -T_i & C^T W_i \\ W_i C & -W_i \end{bmatrix} \leq 0, \quad i = 1, 2 \tag{4.37}$$

where entries $\Gamma_{..}$ *and* $L_{..}$ *are given in (4.6).*

4.4 State-feedback stabilisation

The presence of time delay in control input (4.5), if not considered in a controller design, may be the cause of serious deterioration of performance or even instability of the resulting controlled system. Relatively few works are concerned with problem of neutral systems with input delay. A state-feedback control for neutral deterministic delay systems with input delay was proposed in Corollary 3.3, [27] while stabilisation of neutral stochastic delay systems with input delay was studied in [130]. However, the result in [130] is not applicable to a significant number of cases when the non-delay system matrix is unstable (see, e.g., Examples 4.3). In this section, the stability result obtained in the previous section is applied to design a memoryless delayed state feedback controller (4.5), which exponentially stabilises the neutral stochastic delay system (4.1). Let us consider the case of $P_{31} = P_{32} = 0$ in (4.15), which implies

$$P^{-1} = \begin{bmatrix} P_{11}^{-1} & 0 & 0 \\ -P_{22}^{-1} P_{21} P_{11}^{-1} & P_{22}^{-1} & -P_{22}^{-1} P_{23} P_{33}^{-1} \\ 0 & 0 & P_{33}^{-1} \end{bmatrix}. \tag{4.38}$$

In this case, by the Schur complement lemma, condition (4.36) will be satisfied if linear matrix inequality $\Lambda < 0$ holds, where symmetric matrix Λ is given by (4.39) on the next page with $\Lambda_{11} = P_{21}^T \bar{A} + \bar{A}^T P_{21}$, $\Lambda_{21} = P_{22}^T \bar{A} + P_{11} - P_{21}$, $\Lambda_{31} = P_{23}^T \bar{A} + P_{33}^T \bar{H}$, $\Lambda_{22} = -P_{22}^T - P_{22}$, and $\Lambda_{33} = -P_{33}^T - P_{33}$. The delayed-state-feedback controller (4.5) that exponentially stabilises neutral stochastic delay system (4.1) can be designed as follows.

$$
\Lambda = \begin{bmatrix}
\Lambda_{11} & * & * & * & * & * & * & 0 & * & * & 0 & 0 & 0 & 0 & * & * & * & 0 \\
\Lambda_{21} & \Lambda_{22} & * & * & * & * & * & 0 & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
\Lambda_{31} & -P_{23}^T & \Lambda_{33} & * & * & * & * & 0 & * & * & 0 & 0 & * & * & 0 & 0 & 0 & * \\
h_1\bar{A}_1^T P_{21} & h_1\bar{A}_1^T P_{22} & h_1(\bar{A}_1^T P_{23} + \bar{H}_1^T P_{33}) & -h_1 Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
h_2\bar{A}_2^T P_{21} & h_2\bar{A}_2^T P_{22} & h_2(\bar{A}_2^T P_{23} + \bar{H}_2^T P_{33}) & 0 & -h_2 Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{A}_1^T P_{21} & \bar{A}_1^T P_{22} & \bar{A}_1^T P_{23} + \bar{H}_1^T P_{33} & 0 & 0 & -R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{A}_2^T P_{21} & \bar{A}_2^T P_{22} & \bar{A}_2^T P_{23} + \bar{H}_2^T P_{33} & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -S & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
\bar{A}_1^T P_{21} & \bar{A}_2^T P_{22} & \bar{A}_1^T P_{23} + \bar{H}_1^T P_{33} & 0 & 0 & 0 & 0 & 0 & -W_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\bar{A}_2^T P_{21} & \bar{A}_2^T P_{22} & \bar{A}_2^T P_{23} + \bar{H}_2^T P_{33} & 0 & 0 & 0 & 0 & 0 & 0 & -W_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{h_1} Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Q_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \sqrt{h_2} Q_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -Q_2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{h_1} R_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{h_2} R_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -R_2 & 0 & 0 & 0 & 0 \\
S & 0 & 0 & 0 & 0 & 0 & 0 & SC & 0 & 0 & 0 & 0 & 0 & 0 & -S & 0 & 0 & 0 \\
T_1 & 0 & 0 & 0 & 0 & 0 & 0 & T_1 C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_1 & 0 & 0 \\
T_2 & 0 & 0 & 0 & 0 & 0 & 0 & T_2 C & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -T_2 & 0 \\
0 & 0 & P_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -P_{11}
\end{bmatrix} \tag{4.39}
$$

Theorem 4.3 *The closed-loop neutral stochastic delay system (4.1) and (4.5) is mean-square exponentially stable and is also almost surely exponentially stable if, for given positive scalar numbers* δ_Q*,* δ_R *and* δ_W*, there exist matrices* $X_1 > 0$*,* $\bar{Q} > 0$*,* $\bar{R} > 0$*,* $\bar{S} > 0$*,* $\bar{W} > 0$*,* $U_1 > 0$*,* $U_2 > 0$*,* X_2*,* X_3*,* Y*,* Z *and* $\bar{K}$ *such that*

$$\Phi < 0, \tag{4.40}$$

$$\begin{bmatrix} -U_1 & U_1 C^T \\ CU_1 & -\bar{W} \end{bmatrix} \leq 0 \quad \text{and} \quad \begin{bmatrix} -U_2 & U_2 C^T \\ CU_2 & -\delta_W X_1 \end{bmatrix} \leq 0\,, \tag{4.41}$$

where symmetric matrix Φ *is given by (4.43) on the next page. In this case, the gain matrix of (4.5) can be chosen as* $K = \bar{K} X_1^{-1}$*.*

Proof. Substitution of (4.5) into (4.1) yields dynamics of the closed-loop system

$$\begin{aligned} \mathrm{d}\big[x(t) - Cx(t-h_1)\big] \;&=\; [A_0 x(t) + A_1 x(t-h_1) + (A_2 + BK)x(t-h_2)]\,\mathrm{d}t \\ &\quad + [H_0 x(t) + H_1 x(t-h_1) + H_2 x(t-h_2)]\,\mathrm{d}w(t) \end{aligned} \tag{4.42}$$

for all $t \geq 0$. In order to construct an LMI problem, we consider the case of (4.38) with $Q_2 = \delta_Q^{-1} P_{11}$, $R_2 = \delta_R^{-1} P_{11}$ and $W_2 = \delta_W^{-1} P_{11}$. In this case, by Theorem 4.2 and conditions (4.36)-(4.37), it is noted that system (4.42) is exponentially stable if LMIs $M < 0$ and (4.37) are satisfied, where symmetric matrix M is derived from Λ defined in (4.39) by replacing $\bar{A}$ and $\bar{A}_2$ with $\bar{A} + BK$ and $\bar{A}_2 + BK$ respectively.

Let $X_1 = P_{11}^{-1} > 0$, $\bar{Q} = Q_1^{-1}$, $\bar{R} = R_1^{-1}$, $\bar{S} = S^{-1}$, $\bar{W} = W_1^{-1}$, $U_1 = T_1^{-1}$, $U_2 = T_2^{-1}$, $X_2 = P_{22}^{-1}$, $X_3 = P_{33}^{-1}$, $Y = P_{22}^{-1} P_{21} P_{11}^{-1}$, $Z = P_{22}^{-1} P_{23} P_{33}^{-1}$, and $\bar{K} = K P_{11}^{-1}$, then $Q_2^{-1} = \delta_Q X_1$, $R_2^{-1} = \delta_R X_1$ and $W_2^{-1} = \delta_W X_1$. Define

$$G = diag\{P^{-1}, Q_1^{-1}, Q_2^{-1}, R_1^{-1}, R_2^{-1}, S^{-1}, W_1^{-1}, W_2^{-1}, Q_1^{-1}, Q_2^{-1}, R_1^{-1}, R_2^{-1}, S^{-1}, T_1^{-1}, T_2^{-1}, P_{11}^{-1}\}.$$

It is not difficult to verify that $\Phi = G^T M G$. Therefore LMI (4.40) implies $M < 0$. Moreover, pre-multiplying and post-multiplying LMIs (4.37) by $diag\{T_i^{-1}, W_i^{-1}\}$, $i = 1, 2$, leads to (4.41), which implies that (4.37) and (4.41) are equivalent. The proof is complete.

$$\Phi = \begin{bmatrix}
-Y^T - Y & * & * & 0 & 0 & 0 & 0 & 0 & 0 & 0 & * & * & 0 & 0 & * & * & * & 0 \\
\bar{A}X_1 + B\bar{K} + X_2^T + Y & -X_2^T - X_2 & * & * & * & * & * & 0 & * & * & * & * & 0 & 0 & 0 & 0 & 0 & 0 \\
-Z^T + \bar{H}_1 X_1 & Z & -X_3^T - X_3 & * & * & * & * & 0 & * & * & * & * & * & * & 0 & 0 & 0 & * \\
0 & h_1\bar{Q}\bar{A}_1^T & h_1\bar{Q}\bar{H}_1^T & -h_1\bar{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_Q h_2(X_1\bar{A}_2^T + \bar{K}^T B^T) & \delta_Q h_2 X_1 \bar{H}_2^T & 0 & -\delta_Q h_2 X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \bar{R}\bar{A}_1^T & \bar{R}\bar{H}_1^T & 0 & 0 & -\bar{R} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_R(X_1\bar{A}_2^T + \bar{K}^T B^T) & \delta_R X_1 \bar{H}_2^T & 0 & 0 & 0 & -\delta_R X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{S} & 0 & 0 & 0 & 0 & 0 & 0 & * & * & * & 0 \\
0 & \bar{W}\bar{A}_1^T & \bar{W}\bar{H}_1^T & 0 & 0 & 0 & 0 & 0 & -\bar{W} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & \delta_W(X_1\bar{A}_2^T + \bar{K}^T B^T) & \delta_W X_1 \bar{H}_2^T & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_W X_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{h_1}Y & \sqrt{h_1}X_2 & -\sqrt{h_1}Z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{Q} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-\sqrt{h_2}Y & \sqrt{h_2}X_2 & -\sqrt{h_2}Z & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_Q X_1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{h_1}X_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{R} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \sqrt{h_2}X_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\delta_R X_1 & 0 & 0 & 0 & 0 \\
X_1 & 0 & 0 & 0 & 0 & 0 & 0 & C\bar{S} & 0 & 0 & 0 & 0 & 0 & 0 & -\bar{S} & 0 & 0 & 0 \\
X_1 & 0 & 0 & 0 & 0 & 0 & 0 & C\bar{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -U_1 & 0 & 0 \\
X_1 & 0 & 0 & 0 & 0 & 0 & 0 & C\bar{S} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -U_2 & 0 \\
0 & 0 & X_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -X_1
\end{bmatrix} \tag{4.43}$$

4.5 Examples

In this section, a number of numerical examples are conducted to verify the effectiveness of our proposed methods.

Example 4.1 Let us look at the neutral stochastic delay system (4.4) with parameters

$$C = \begin{bmatrix} -0.2 & 0 \\ 1 & 0.2 \end{bmatrix}, A_0 = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.3 \end{bmatrix}, A_1 = \begin{bmatrix} -1 & 0 \\ -1 & -1 \end{bmatrix}, H_0 = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, H_1 = \begin{bmatrix} 0.3 & 0 \\ 0 & 0.3 \end{bmatrix}. \tag{4.44}$$

It is easy to verify that the existing results (see [19], [53], [67], [73], [76] and [80]) do not work. But, by Theorem 4.1 or 4.2, the upper bounds of time delay for exponential stability of system (4.44) is $h_{max} = 0.3585$.

Example 4.2 Deterministic systems may be regarded a special class of stochastic systems, e.g., the following deterministic neutral system is exactly system (4.4) with $A_0 = A$, $A_1 = B$ and $A_2 = H_0 = H_1 = H_2 = 0$, i.e.,

$$\dot{x}(t) - C\dot{x}(t-h) = Ax(t) + Bx(t-h) \tag{4.45}$$

for all $t \geq 0$, where $A = \begin{bmatrix} -0.9 & 0.2 \\ 0.1 & -0.9 \end{bmatrix}$, $B = -\begin{bmatrix} 1.1 & 0.2 \\ 0.1 & 1.1 \end{bmatrix}$, $C = \begin{bmatrix} -0.2 & \gamma \\ 0.2 & -0.1 \end{bmatrix}$ and γ is a constant real number.

The case of $\gamma = 0$ has been studied in many works (see, e.g., [26], [35] and [61]). However, results of [26], [61] and [101] are not applicable when $|\gamma| \geq 1$. For $\gamma \geq 2$, the criterion in [27] does not work, but the upper bounds h_{max} for exponential stability of (4.45) by other methods are listed in Table 4.1, which shows that the results obtained by the methods proposed in this chapter are less conservative in these cases.

Table 4.1: h_{max} by different methods

	$\gamma = 2.0$	$\gamma = 2.2$	$\gamma = 2.4$
[20]	0.2954	0.2552	0.2163
[35]	0.3934	0.3189	0.2526
Theorem 4.1 or 4.2	0.4602	0.3957	0.3050

The following example is devoted to applying Theorem 4.3 and designing delayed-state-feedback controller (4.5) to exponentially stabilise the neutral system (4.1).

Example 4.3 Consider a neutral stochastic delay system of the form (4.1) with $A_2 = H_2 = 0$, $h_1 = 0.15$ and other parameters as follows

$$\begin{aligned} C &= \begin{bmatrix} 0.2 & 0 \\ 0.1 & -0.1 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0.2 \\ 0.1 & -0.3 \end{bmatrix}, \quad A_1 = \begin{bmatrix} -0.3 & 0.2 \\ 0.1 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} 1.5 & 2.6 \\ -2.2 & 1.3 \end{bmatrix}, \quad H_0 = \begin{bmatrix} -0.3 & 0 \\ 0.2 & 0.1 \end{bmatrix}, \quad H_1 = \begin{bmatrix} 0.3 & 0.1 \\ 0.2 & -0.1 \end{bmatrix}. \end{aligned} \tag{4.46}$$

Problem of non-delay dynamic output-feedback stabilisation of system (4.46) with distributed delays was studied in [129]. It is noted that existing results ([129] and [130]) do not work when $h_2 > 0$ since there is no matrix $X > 0$ such that $A_0X + XA_0^T < 0$. For convenience, in this example, we choose $\delta_Q = \delta_R = \delta_W = 1$. By Theorem 4.3, the sufficient condition for stabilisability of system (4.46) is $0 \le h_2 \le 0.7873$. For the case of $h_2 = 0.7873$, solving LMIs (4.40)-(4.41) gives $K = \begin{bmatrix} -0.0527 & 0.0180 \\ -0.1001 & -0.0722 \end{bmatrix}$, which implies system (4.46) is exponentially stabilised by delayed-state-feedback controller $u(t - h_2) = \begin{bmatrix} -0.0456 & 0.0368 \\ -0.1001 & -0.0718 \end{bmatrix} x(t - h_2)$. According to Theorem 4.1 or 4.2, this resulting closed-loop system is exponentially stable when $0 \le h_2 \le 1.3015$.

4.6 Summary

In this chapter, delay-dependent criteria for stability of neutral stochastic delay systems have been presented by approach of LMIs. Based on these newly-established stability results, a state-feedback controller design has been proposed to exponentially stabilise neutral stochastic delay system with input delay (4.1). Numerical examples have been given to verify the effectiveness of the methods proposed in this chapter. Example 2 shows that our results developed for stochastic systems are competitive even when they are specialized to the deterministic cases.

Chapter 5

SMC design for robust H_∞ control of uncertain stochastic delay systems

5.1 Introduction

Sliding mode control (SMC) has various attractive features such as fast response, good transient performance, order reduction and, particularly, robust with matched uncertainties, and is well known to be an effective way to handle many challenging problems of robust stabilization. Over the past decades, SMC has been one of the most popular control methods among the control community and found wide applications to automotive systems, observers design, chemical processes, electrical motor control, aero engineering and so on (see, e.g., [21], [31], [41], [40], [47], [58], [106], [119], [120] and the references therein). Generally speaking, SMC uses a discontinuous control law (relays) to force and restrict the state trajectories to a predefined sliding surface on which the system has some desired properties such as stability, disturbance rejection capability and tracking (see [31], [58] and [119]).

In recent years, there has been a growing interest in extension of SMC to accommodate stochastic systems (see, e.g., [13], [14, 15], [45], [102], [103, 104]) since stochastic modeling has come to play an important role in many branches of science and engineering (see, e.g., [25], [53] and [88]). For example, [102] studied integral SMC for stochastic delay system

$$\begin{aligned} \mathrm{d}x(t) &= [(A+\Delta A(t))x(t) + (A_d+\Delta A_d(t))x(t-\tau(t)) + B(u(t)+f(x(t),t))]dt \\ &\quad + D[(C+\Delta C(t))x(t) + (C_d+\Delta C_d(t))x(t-\tau(t))]\mathrm{d}w(t)\,, \end{aligned} \tag{5.1}$$

where it is assumed there is matrix $G \in R^{m\times n}$ such that

$$\det(GB) \neq 0 \quad \text{and} \quad GD = 0 \tag{5.2}$$

with $\det(\cdot)$ denoting the determinant of a matrix. However, these existing results employ assumptions such as (5.2) on structure of the control system such that their controller design does not need to deal with stochastic perturbation and hence they can use the SMC design method for deterministic systems (see Remark 1 and 4 in [45]). These existing results may be considered as studies of SMC with stochastic perturbation in sliding mode. But such an assumption may be too restrictive for stochastic systems in many practical situations.

The main purpose of this chapter is to remove this assumption and propose a practical SMC design method for stochastic systems. Moreover, in some cases, our design method provides a control scheme for finite-time stabilisation of stochastic delay systems (see Remark 5.2 and the Example). Problems of finite-time stabilization of stochastic systems ([131]) are relatively seldom studied while those of deterministic systems have received much attention (see [38], [46] and the references therein). Our proposed design method is presented in terms of LMIs (see [12]), which can be easily implemented.

5.2 Problem statement

Throughout the chapter, unless otherwise specified, we will employ the notation as before. Let $W(t) = (W_1(t), \cdots, W_{r_w}(t))^T$ be an r_w-dimensional Brownian motion defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let $|\cdot|$ and $|\cdot|_1$ denote the Euclidean norm and 1-norm of a vector and their induced norms of a matrix respectively.

Let us consider an n-dimensional uncertain stochastic system with state delay

$$\begin{aligned} \mathrm{d}x(t) &= \big[A_0(t)x(t) + A_1(t)x(t-h) + B(u(t)+\phi(t,x_t)) + B_v v(t)\big]\mathrm{d}t \\ &\quad + g(t,x(t))\mathrm{d}W(t)\,, \end{aligned} \tag{5.3}$$

$$z(t) = Cx(t) + Dv(t) \tag{5.4}$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in \tilde{L}^2_{\mathcal{F}_0}([-h, 0]; R^n)$, where $x(t) \in R^n$ is the state vector; $u(t) \in R^m$ is the control input;$z(t) \in R^p$ is the controlled output; $v(t) \in R^q$ is the exogenous disturbance input belonging to $L_2[0, \infty)$; $h > 0$, time delay of the system, is a known number; B, B_v, C, D are constant matrices and B is of full column rank; $A_i(t)$, $i = 0, 1$, are matrix functions with time-varying uncertainties described as $A_i(t) = A_i + \Delta A_i(t)$, where A_i, $i = 0, 1$, are known constant matrices while uncertainties $\Delta A_i(t)$ are assumed to be norm bounded, i.e.,

$$\Delta A_i(t) = L_i F_i(t) E_i, \quad i = 0, 1 \tag{5.5}$$

with known constant matrices L_i, E_i, and unknown matrix functions $F_i(t)$ having Lebesgue measurable elements and satisfying $F_i^T(t)F_i(t) \leq I$ for all $t \geq 0$; matched uncertainty $\phi(t, x_t)$ satisfies

$$|\phi(t, x_t)| \leq k_\phi(|x(t)| + |x(t-h)|), \quad \forall\, t \geq 0 \tag{5.6}$$

where k_ϕ is a nonnegative number; $g(t, x(t))$ may be not exactly known but there is a constant matrix G such that

$$\text{trace}[g^T(t, x(t))g(t, x(t))] \leq |Gx(t)|^2 \tag{5.7}$$

for all $t \geq 0$ (see, e.g., [17] and [135]). It is also assumed that pair (A_0, B) is controllable, that is, there exists matrix $K_0 \in R^{m \times n}$ such that matrix $A_0 + BK_0$ is stable.

It is easy to verify that equation (5.3) with $u(t) = 0$ and $v(t) = 0$ has a unique solution (see, e.g., [84] and [88]). In this chapter, we intend to design a sliding surface and a switching control law such that the state trajectories are drawn in finite time to the sliding surface with probability 1, on which system (5.3)-(5.4) is robustly mean-square exponentially stable with some prescribed disturbance attenuation $\gamma(> 0)$ (see Definition 5.1 below). It should be noted that, for simplicity only, we take a relatively simple model. The proposed method can be easily extended to many systems such as those of large scale, with Markovian switching and time-varying and multiple delays (see [13], [14, 15], [17], [45], [102], [103, 104]).

At the end of this section, let us introduce the following definitions.

Definition 5.1 *Uncertain stochastic delay system (5.3)-(5.4) is said to be robustly mean-square exponentially stable with disturbance attenuation* γ (> 0) *if system (5.3) with*

$v(t) = 0$ is robustly mean-square exponentially stable and moreover, under zero initial condition,

$$\mathbb{E}\int_0^\infty |z(t)|^2 \mathrm{d}t \le \gamma^2 \int_0^\infty |v(t)|^2 \mathrm{d}t \tag{5.8}$$

for all nonzero $v \in L_2[0,\infty)$ and admissible uncertainties (5.5).

For definitions of mean-square stability with a given disturbance attenuation γ, please see, e.g., [7], [103] and [128]. Moreover, let us present the definition of finite-time stability of stochastic systems, which is consistent with that of deterministic systems (see, e.g., [8], [38] and [46])

Definition 5.2 *The equilibrium $x = 0$ of uncertain stochastic delay system (5.3) with $u(t) = 0$ and $v(t) = 0$ is said to be pth ($p > 0$) moment finite-time stable if system (5.3) with $u(t) = 0$ and $v(t) = 0$ is pth moment stable and if for every $\xi \in \tilde{L}^p_{\mathcal{F}_0}([-h,0];R^n)$, there exists (a settling time) $T = T(\xi) > 0$ such that $0 < \mathbb{E}|x(t;\xi)|^p < \infty$ for all $0 \le t < T$, $\lim_{t\to T}\mathbb{E}|x(t;\xi)|^p = 0$ and $\mathbb{E}|x(t;\xi)|^p = 0$ for all $t > T$.*

For definition of pth moment stability, please see, e.g., [44]. We also cite the following well-known result that is useful for the development of this chapter (see, e.g., p44 [69]).

Lemma 5.1 *For a pair of constant matrices $G \in R^{p\times p}$ and $M \in R^{p\times q}$, if $G \ge 0$, then*

$$\mathrm{trace}(M^T G M) \le \lambda_M(G)\mathrm{trace}(M^T M)\,.$$

5.3 Switching surface and control scheme design

This section is devoted to designing the sliding surface and the switching control law such that the task of this chapter is fulfilled. We present the design method as follows.

Given constant $\gamma > 0$, assume that there exist matrices $X > 0$, $R > 0$, Y_0, Y_1 and positive numbers β_0, β_1, λ_g, ζ_g such that

$$\Theta = \begin{bmatrix} \Theta_1 & * & * & * & 0 & * & * \\ XA_1^T + Y_1^T B^T & -R & 0 & 0 & * & 0 & 0 \\ GX & 0 & -\lambda_g I & 0 & 0 & 0 & 0 \\ E_0 X & 0 & 0 & -\beta_0 I & 0 & 0 & 0 \\ 0 & E_1 X & 0 & 0 & -\beta_1 I & 0 & 0 \\ B_v^T + D^T CX & 0 & 0 & 0 & 0 & \Theta_6 & 0 \\ CX & 0 & 0 & 0 & 0 & 0 & -I \end{bmatrix} < 0, \tag{5.9}$$

$$\lambda_g I \leq X \,, \tag{5.10}$$

and

$$\begin{bmatrix} -BB^T & XG^T \\ GX & -\zeta_g I \end{bmatrix} \leq 0 \,, \tag{5.11}$$

where $\Theta_1 = A_0 X + XA_0^T + BY_0 + Y_0^T B^T + R + \beta_0 L_0 L_0^T + \beta_1 L_1 L_1^T$, $\Theta_6 = -\gamma^2 I + D^T D$ and entries denoted by $*$ can be readily inferred from symmetry of the matrix. Let $P = X^{-1}$. It is easy to find $\zeta_b > 0$ such that

$$PBB^T P \leq \zeta_b I \,. \tag{5.12}$$

And then let

$$\beta = \frac{1}{2}\zeta_b \zeta_g \,. \tag{5.13}$$

In this work, we choose the switching surface as a linear function of the current states

$$s(t) = s(t, x(t)) = B^T P x(t) = 0 \tag{5.14}$$

for all $t \geq 0$. Note that matrix B is of full column rank and matrix $P > 0$. It is easy to see that

$$B^T P B > 0 \,. \tag{5.15}$$

Moreover, function sgn : $R^m \to R^m$ is defined by

$$\text{sgn}(u) = \begin{bmatrix} \text{sgn}(u_1) & \text{sgn}(u_2) & \cdots & \text{sgn}(u_m) \end{bmatrix}^T , \tag{5.16}$$

where

$$\text{sgn}(u_i) = \begin{cases} 1\,, & u_i > 0 \\ 0\,, & u_i = 0 \\ -1\,, & u_i < 0 \end{cases}$$

for $i = 1, 2, \cdots, m$. Since time delay $h > 0$ is known, the past state $x(t-h)$ can be used in the control law (see, e.g., [31], [58], [102] and [106]). In this case, we design the switching control law as follows

$$u(t) = -(B^T PB)^{-1}\big[\beta s(t) + u_1(t) + u_2(t)\big] \tag{5.17}$$

for all $t \geq 0$, where $u_1(t) = B^T P(A_0 x(t) + A_1 x(t-h))$ and $u_2(t) = [\alpha + \rho(t)]\text{sgn}(s(t))$ with $\alpha > 0$ and

$$\begin{aligned} \rho(t) &= |B^T PL_0|\,|E_0 x(t)| + |B^T PL_1|\,|E_1 x(t-h)| \\ &\quad + k_\phi\,|B^T PB|\,(|x(t)| + |x(t-h)|) + |B^T P|\,|B_v v(t)|\,. \end{aligned} \tag{5.18}$$

5.4 Reachability analysis

In this section, we consider reachability of the sliding surface (5.14).

Theorem 5.1 *The state trajectories of system (5.3) synthesized with switching control (5.17) are drawn to sliding surface (5.14) in finite time almost surely, or say, with probability 1.*

Proof. Without loss of generality, assume $|s(0)| > 0$. Define a stopping time

$$\tau_s = \inf\{t \geq 0 : s(t) = 0\}\,. \tag{5.19}$$

We need to prove that there exists $0 < t_r < \infty$ such that $\tau_s \leq t_r$ *a.s.*, or say, $\mathbb{P}\{\tau_s \leq t_r\} = 1$.

Let us consider function $U(t) = s^T(t)s(t)$ for all $t \geq 0$. By Itô's formula, we have

$$\mathrm{d}U(t) = \mathcal{L}U(t)\mathrm{d}t + 2s^T(t)B^T Pg(t, x(t))\mathrm{d}W(t)\,, \tag{5.20}$$

where

$$\begin{aligned} \mathcal{L}U(t) &= 2s^T(t)B^T P\big[A_0(t)x(t) + A_1(t)x(t-h) + B(u(t) + \phi(t, x_t)) + B_v v(t)\big] \\ &\quad + \text{trace}\big[g^T(t, x(t))PBB^T Pg(t, x(t))\big]\,. \end{aligned} \tag{5.21}$$

Substitution of (5.17) into (5.21) yields

$$\begin{aligned}\mathcal{L}U(t) &= 2s^T(t)B^TP\big[\Delta A_0(t)x(t) + \Delta A_1(t)x(t-h) + B\phi(t,x_t) + B_v v(t)\big] \\ &\quad - 2\rho(t)s^T(t)\,\mathrm{sgn}(s(t)) - 2\alpha s^T(t)\,\mathrm{sgn}(s(t)) - 2\beta s^T(t)s(t) \\ &\quad + \mathrm{trace}\big[g^T(t,x(t))PBB^TPg(t,x(t))\big] \\ &\le -2\alpha|s(t)| - 2\beta s^T(t)s(t) + \mathrm{trace}\big[g^T(t,x(t))PBB^TPg(t,x(t))\big]\,. \qquad (5.22)\end{aligned}$$

Inequality $|s(t)|_1 \ge |s(t)|$ is used in the last step of inequality (5.22).
But LMI (5.11) implies

$$G^TG \le \zeta_g PBB^TP\,. \qquad (5.23)$$

Combination of Lemma 5.1 and inequalities (5.7), (5.12) and (5.23) gives

$$\begin{aligned}\mathrm{trace}\big[g^T(t,x(t))PBB^TPg(t,x(t))\big] &\le \lambda_M(PBB^TP)\mathrm{trace}\big[g^T(t,x(t))g(t,x(t))\big] \\ &\le \zeta_b\mathrm{trace}\big[g^T(t,x(t))g(t,x(t))\big] \\ &\le x^T(t)(\zeta_b G^TG)x(t) \\ &\le x^T(t)(\zeta_b\zeta_g PBB^TP)x(t) \\ &= 2\beta s^T(t)s(t)\,. \qquad (5.24)\end{aligned}$$

Inequalities (5.22) and (5.24) imply

$$\mathcal{L}U(t) \le -2\alpha\sqrt{U(t)}\,, \quad \forall\, t \ge 0. \qquad (5.25)$$

But, by Itô's formula, this yields

$$\mathcal{L}|s(t)| = \mathcal{L}\sqrt{U(t)} \le -\alpha\,, \qquad (5.26)$$

and hence

$$\mathbb{E}|s(t)| \le \mathbb{E}|s(0)| - \alpha\, t\,, \qquad (5.27)$$

which implies $\mathbb{E}|s(t)|$ converges to zero in finite time. Specifically, there is $t_r = r_0/\alpha$ such that $\mathbb{E}|s(t)| = 0$ for all $t \ge t_r$, where $r_0 = \mathbb{E}|s(0)| < \infty$. This implies $\mathbb{E}|s(t)| = 0$ and hence $|s(t)| = 0$ *a.s.*, or say, $\mathbb{P}\{|s(t)| = 0\} = 1$ for all $t \ge t_r$. For any $\varepsilon_r > 0$, suppose that $\mathbb{P}\{\tau_s > t_r\} \ge \varepsilon_r$. Then $\mathbb{P}\{|s(t_r)| > 0\} \ge \varepsilon_r$, which leads to a contradiction. Therefore we have $\tau_s \le t_r$ almost surely. The proof is complete.

Remark 5.1 *It is observed that we may choose $\beta = 0$ in a case when the assumption $B^TPg(t,x(t)) = 0$ (see [13], [14, 15], [102], [103, 104] and [45]) holds.*

Remark 5.2 *In the case when $m = n$, the design method (5.17) proposes a control scheme for 1st moment finite-time stabilisation of stochastic delay system (5.3)(see Definition 5.2).*

5.5 Stability of sliding mode

Since it has been shown that the state trajectories of closed-loop system (5.3) and (5.17) are drawn to sliding surface (5.14) in finite time, we proceed to discuss stability of the sliding mode. Let us rewrite system (5.3) in the following form

$$\begin{aligned} \mathrm{d}x(t) &= \big[\bar{A}_0(t)x(t) + \bar{A}_1(t)x(t-h) + B(u(t) + \bar{\phi}(t, x_t)) + B_v v(t)\big]\mathrm{d}t \\ &\quad + g(t, x(t))\mathrm{d}W(t)\,, \end{aligned} \tag{5.28}$$

where $\bar{A}_0(t) = \bar{A}_0 + \Delta A_0(t) = (A_0 + BK_0) + \Delta A_0(t)$, $\bar{A}_1(t) = \bar{A}_1 + \Delta A_1(t) = (A_1 + BK_1) + \Delta A_1(t)$, $\bar{\phi}(t, x_t) = \phi(t, x_t) - K_0 x(t) - K_1 x(t-h)$ and matrices K_i, $i = 0, 1$, are to be determined.

Remark 5.3 *System (5.3) may also be rewritten in the form of (see, e.g., [58])*

$$\begin{aligned} \mathrm{d}x(t) &= \big[\bar{A}_0(t)x(t) + \bar{A}_1(t)x(t-h) + B(\bar{u}(t) + \phi(t, x_t)) \\ &\quad + B_v v(t)\big]\mathrm{d}t + g(t, x(t))\mathrm{d}W(t)\,, \end{aligned} \tag{5.29}$$

where $\bar{u}(t) = u(t) - K_0 x(t) - K_1 x(t-h)$. But (5.29) may be somewhat misleading that control law (5.17) is changed. In fact, control commands are always input as the scheme (5.17). Note that control scheme (5.17) is different from that in [58] even in the case when $\beta = 0$ (see Remark 5.1). At this point, system (5.28) is clear to show that part of the system dynamics is treated as perturbation (but not counteracted by control input). This may also help highlight the advantage of SMC that sliding mode dynamics is insensitive to matched uncertainties.

Remark 5.4 *It should be stressed that, unlike many cases in references, matrices K_0 and K_1 are not feedback gain matrices. As a matter of fact, there is neither K_0 nor K_1 in control scheme (5.17). Matrices BK_0 and BK_1 are introduced into the stability analysis of sliding mode because the sliding surface and the Lyapunov-Krasovskii functional are* *chosen as (5.14) and (5.30) respectively, by which we take advantage of their relationship on the sliding surface* $s(t) = B^T P x(t) = 0$.

In this section, we consider stability of dynamics of the sliding mode, that is, system (5.28) restricted on sliding surface (5.14).

Theorem 5.2 *Given constant* $\gamma > 0$*, sliding mode dynamics of system (5.28) and (5.4) on sliding surface (5.14) is robustly mean-square exponentially stable with disturbance attenuation* γ *provided that LMIs (5.9)-(5.10) are satisfied.*

Proof. First, let us consider stability of system (5.28) with $v(t) = 0$ restricted on sliding manifold (5.14). Choose a Lyapunov-Krasovskii functional candidate as

$$V(t) = x^T(t)Px(t) + \int_{t-h}^{t} x^T(\tau)Qx(\tau)\mathrm{d}\tau \tag{5.30}$$

for all $t \geq t_0 = t_r + h$, where $P = X^{-1}$ and $Q = PRP$ while matrices $X > 0$ and $R > 0$ are determined by LMIs (5.9) and (5.10). By Itô's formula, we have

$$\mathrm{d}V(t) = \mathcal{L}V(t)\mathrm{d}t + 2x^T(t)Pg(t,x(t))\mathrm{d}W(t)\,, \tag{5.31}$$

where

$$\begin{aligned} \mathcal{L}V(t) &= 2x^T(t)P\big[\bar{A}_0(t)x(t) + \bar{A}_1(t)x(t-h)\big] + 2x^T(t)PB\big[u(t) + \bar{\phi}(t,x_t)\big] \\ &\quad + \mathrm{trace}\big[g^T(t,x(t))Pg(t,x(t))\big] + x^T(t)Qx(t) - x^T(t-h)Qx(t-h) \\ &= 2x^T(t)P\big[\bar{A}_0(t)x(t) + \bar{A}_1(t)x(t-h)\big] + \mathrm{trace}\big[g^T(t,x(t))Pg(t,x(t))\big] \\ &\quad + x^T(t)Qx(t) - x^T(t-h)Qx(t-h)\,, \end{aligned} \tag{5.32}$$

since system (5.28) is restricted on sliding surface (5.14). By Lemma 5.1, we obtain

$$\begin{aligned} \mathcal{L}V(t) &\leq 2x^T(t)P\big[\bar{A}_0(t)x(t) + \bar{A}_1(t)x(t-h)\big] + \lambda_M(P)\mathrm{trace}\big[g^T(t,x(t))g(t,x(t))\big] \\ &\quad + x^T(t)Qx(t) - x^T(t-h)Qx(t-h)\,. \end{aligned} \tag{5.33}$$

Moreover, LMI (5.10) implies

$$P \leq \lambda_g^{-1} I. \tag{5.34}$$

Substitution of (5.7) and (5.34) into (5.33) yields

$$\begin{aligned}\mathcal{L}V(t) &\leq 2x^T(t)P\big[\bar{A}_0(t)x(t)+\bar{A}_1(t)x(t-h)\big]+x^T(t)(\lambda_g^{-1}G^TG)x(t)\\ &\quad +x^T(t)Qx(t)-x^T(t-h)Qx(t-h)\\ &= x^T(t)\big(P\bar{A}_0+\bar{A}_0^TP+\lambda_g^{-1}G^TG+Q\big)x(t)\\ &\quad +2x^T(t)P\bar{A}_1x(t-h)-x^T(t-h)Qx(t-h)\\ &\quad +2x^T(t)P\Delta A_0(t)x(t)+2x^T(t)P\Delta A_1(t)x(t-h)\,. \end{aligned} \tag{5.35}$$

But, by Lemma 3.1, we see

$$2x^T(t)P\Delta A_0(t)x(t) \leq x^T(t)\beta_0PL_0L_0^TPx(t)+x^T(t)\beta_0^{-1}E_0^TE_0x(t)\,, \tag{5.36}$$

$$\begin{aligned}2x^T(t)P\Delta A_1(t)x(t-h) &\leq x^T(t)\beta_1PL_1L_1^TPx(t)\\ &\quad +x^T(t-h)\beta_1^{-1}E_1^TE_1x(t-h)\,. \end{aligned} \tag{5.37}$$

This implies

$$\begin{aligned}\mathcal{L}V(t) &\leq x^T(t)\big(P\bar{A}_0+\bar{A}_0^TP+\lambda_g^{-1}G^TG+Q\\ &\quad +\beta_0PL_0L_0^TP+\beta_0^{-1}E_0^TE_0+\beta_1PL_1L_1^TP\big)x(t)\\ &\quad +2x^T(t)P\bar{A}_1x(t-h)\\ &\quad +x^T(t-h)\big(-Q+\beta_1^{-1}E_1^TE_1\big)x(t-h)\\ &= \Big[x^T(t) \quad x^T(t-h)\Big]\,\Omega\,\Big[x^T(t) \quad x^T(t-h)\Big]^T\,, \end{aligned} \tag{5.38}$$

where

$$\Omega=\begin{bmatrix}\Omega_1 & P(A_1+BK_1)\\ (K_1^TB^T+A_1^T)P & -Q+\beta_1^{-1}E_1^TE_1\end{bmatrix} \tag{5.39}$$

with $\Omega_1 = P(A_0+BK_0)+(A_0+BK_0)^TP+\lambda_g^{-1}G^TG+Q+\beta_0PL_0L_0^TP+\beta_0^{-1}E_0^TE_0+\beta_1PL_1L_1^TP$.

Let us look at matrix Γ given as follows

$$\Gamma=\begin{bmatrix}\Gamma_1 & * & * & * & 0\\ XA_1^T+Y_1^TB^T & -R & 0 & 0 & *\\ GX & 0 & -\lambda_gI & 0 & 0\\ E_0X & 0 & 0 & -\beta_0I & 0\\ 0 & E_1X & 0 & 0 & -\beta_1I\end{bmatrix}, \tag{5.40}$$

where $\Gamma_1 = A_0X + XA_0^T + BY_0 + Y_0^TB^T + R + \beta_0L_0L_0^T + \beta_1L_1L_1^T$. Observe that Γ is a principal submatrix of matrix Θ given in (5.9). By Schur complement lemma, LMI (5.9) implies $\Gamma < 0$. But, also by Schur complement lemma, this implies $\tilde{\Omega} < 0$, where $\tilde{\Omega}$ is given as

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_1 & A_1X + BY_1 \\ Y_1^TB^T + XA_1^T & -R + \beta_1^{-1}XE_1^TE_1X \end{bmatrix} \tag{5.41}$$

with $\tilde{\Omega}_1 = A_0X + XA_0^T + BY_0 + Y_0^TB^TP + \lambda_g^{-1}XG^TGX + R + \beta_0L_0L_0^T + \beta_0^{-1}XE_0^TE_0X + \beta_1L_1L_1^T$. Let $K_i = Y_iP$, $i = 0, 1$, then it is observed $\Omega = D_p\tilde{\Omega}D_p$, where $D_p = \mathrm{diag}\{P,\ P\}$. This implies $\Omega < 0$ and hence

$$\mathcal{L}V(t) \le -\lambda_0|x(t)|^2\,, \tag{5.42}$$

where $\lambda_0 = \lambda_m(-\Omega) > 0$.

According to (5.30), we have

$$\alpha_0|x(t)|^2 \le V(t) \le \alpha_1|x(t)|^2 + \alpha_2\int_{t-h}^{t}|x(\tau)|^2\mathrm{d}\tau \tag{5.43}$$

for all $t \ge t_0$, where $\alpha_0 = \lambda_m(P)$, $\alpha_1 = \lambda_M(P)$ and $\alpha_2 = \lambda_M(Q)$. Choose $\varepsilon_0 > 0$ such that

$$\varepsilon_0(\alpha_1 + \alpha_2he^{h\varepsilon_0}) \le \lambda_0\,. \tag{5.44}$$

By Itô's formula, we have

$$\mathrm{d}\left[e^{\varepsilon_0\tau}V(\tau)\right] = e^{\varepsilon_0\tau}\left[\varepsilon_0V(\tau) + \mathcal{L}V(\tau)\right]\mathrm{d}s + 2e^{\varepsilon_0\tau}x^T(t)Pg(\tau, x(\tau))\mathrm{d}W(\tau)\,. \tag{5.45}$$

Integrating from t_0 to t and taking expectation on both sides of (5.45) yield

$$\begin{aligned} &\mathbb{E}\left[e^{\varepsilon_0t}V(t)\right] - \mathbb{E}\left[e^{\varepsilon_0t_0}V(t_0)\right] \\ &\quad = \mathbb{E}\int_{t_0}^{t}e^{\varepsilon_0\tau}\left[\varepsilon_0V(\tau) + \mathcal{L}V(\tau)\right]\mathrm{d}\tau \\ &\quad \le \int_{t_0}^{t}e^{\varepsilon_0\tau}\Big\{\mathbb{E}\Big[\varepsilon_0\alpha_1|x(\tau)|^2 + \varepsilon_0\alpha_2\int_{\tau-h}^{\tau}|x(v)|^2\mathrm{d}v\Big] - \lambda_0\mathbb{E}|x(\tau)|^2\Big\}\mathrm{d}\tau\,. \end{aligned} \tag{5.46}$$

Since

$$\begin{aligned} \int_{t_0}^{t}e^{\varepsilon_0\tau}\mathrm{d}\tau\int_{\tau-h}^{\tau}|x(v)|^2\mathrm{d}v &\le \int_{t_0-h}^{t}|x(v)|^2\mathrm{d}v\int_{v}^{v+h}e^{\varepsilon_0\tau}\mathrm{d}\tau \le he^{h\varepsilon_0}\int_{t_0-h}^{t}|x(\tau)|^2e^{\varepsilon_0\tau}\mathrm{d}\tau \\ &\le he^{h\varepsilon_0}\int_{t_0}^{t}|x(\tau)|^2e^{\varepsilon_0\tau}\mathrm{d}\tau + he^{h\varepsilon_0}\int_{t_0-h}^{t_0}|x(\tau)|^2\mathrm{d}\tau\,, \end{aligned}$$

it follows

$$\begin{aligned} e^{\varepsilon_0 t}\mathbb{E}V(t) &\leq e^{\varepsilon_0 t_0}\mathbb{E}V(t_0) + \int_{t_0}^{t} e^{\varepsilon_0 \tau}\left[\varepsilon_0(\alpha_1 + \alpha_2 h e^{h\varepsilon_0}) - \lambda_0\right]\mathbb{E}|x(\tau)|^2 \mathrm{d}\tau \\ &\quad + h\varepsilon_0\alpha_2 e^{h\varepsilon_0}\int_{t_0-h}^{t_0}\mathbb{E}|x(\tau)|^2\mathrm{d}\tau \\ &\leq C_{t_0}\,, \end{aligned} \tag{5.47}$$

where $C_{t_0} = [\alpha_1 e^{\varepsilon_0 t_0} + \alpha_2 h(e^{\varepsilon_0 t_0} + \varepsilon_0 h e^{h\varepsilon_0})]\sup_{t_r \leq \theta \leq t_0}\mathbb{E}|x(\theta)|^2$. So we have

$$\alpha_0|x(t)|^2 \leq \mathbb{E}V(t) \leq C_{t_0}e^{-\varepsilon_0 t}, \quad \forall\, t \geq 0 \tag{5.48}$$

or

$$\limsup_{t\to\infty}\frac{1}{t}\log\mathbb{E}|x(t;\xi)|^2 \leq -\varepsilon_0\,. \tag{5.49}$$

The mean-square exponential stability of the sliding mode dynamics has been proved. In fact, by Theorem 6.2, p175, [88] or Theorem 2.2, [75], (5.49) also implies almost sure exponential stability. We proceed to show

$$\mathbb{E}\int_{t_0}^{\infty}|z(t)|^2\mathrm{d}t \leq \gamma^2\int_{t_0}^{\infty}|v(t)|^2\mathrm{d}t \tag{5.50}$$

for all nonzero $v \in L_2[t_0,\infty)$ and admissible uncertainties (5.5) under zero initial condition $x(\theta) = 0$ for all $\theta \in [t_0 - h, t_0]$.

For prescribed constant $\gamma > 0$, define the performance index function

$$J(t) = \int_{t_0}^{t}[z^T(\tau)z(\tau) - \gamma^2 v^T(\tau)v(\tau)]\mathrm{d}\tau \tag{5.51}$$

for all $t > t_0$. Let

$$Y(t) = J(t) + V(t)\,, \quad \bar{J}(t) = \mathbb{E}J(t), \quad \bar{Y}(t) = \mathbb{E}Y(t)\,. \tag{5.52}$$

Obviously, $Y(t) \geq J(t)$ and $\bar{Y}(t) \geq \bar{J}(t)$ for all $t > t_0$. Since $x(\theta) = 0$ for all $\theta \in [t_0 - h, t_0]$, by Dynkin's formula, we have

$$\mathbb{E}V(t) = \mathbb{E}\int_{t_0}^{t}\mathcal{L}V(\tau)\mathrm{d}\tau, \quad \forall\, t > t_0 \tag{5.53}$$

and therefore

$$\begin{aligned} \bar{Y}(t) &= \mathbb{E}\int_{t_0}^{t}[z^T(\tau)z(\tau) - \gamma^2 v^T(\tau)v(\tau) + \mathcal{L}V(\tau)]\mathrm{d}\tau \\ &= \mathbb{E}\int_{t_0}^{t}\left[x^T(\tau) \quad x^T(t-h) \quad v^T(\tau)\right]\Omega_v\left[x^T(\tau) \quad x^T(t-h) \quad v^T(\tau)\right]^T\mathrm{d}\tau\,, \end{aligned} \tag{5.54}$$

where

$$\Omega_v = \begin{bmatrix} \Omega_{v1} & P(A_1 + BK_1) & PB_v + C^T D \\ (K_1^T B^T + A_1^T)P & -Q + \beta_1^{-1} E_1^T E_1 & 0 \\ B_v^T P + D^T C & 0 & -\gamma^2 I + D^T D \end{bmatrix}$$

with $\Omega_{v1} = P(A_0 + BK_0) + (A_0 + BK_0)^T P + \lambda_g^{-1} G^T G + Q + \beta_0 P L_0 L_0^T P + \beta_0^{-1} E_0^T E_0 + \beta_1 P L_1 L_1^T P + C^T C$.

Using similar techniques as above, we find

$$\Theta < 0 \;\Rightarrow\; \Omega_v < 0\,. \tag{5.55}$$

But this implies

$$\bar{J}(t) \le \bar{Y}(t) \le -\lambda_v \int_{t_0}^{t} |v(\tau)|^2 \mathrm{d}\tau \quad \forall\, t \ge t_0 \tag{5.56}$$

with $\lambda_v = \lambda_m(-\Omega_v) > 0$, which completes the proof.

5.6 Example

Let us consider a water-quality dynamic model subject to environmental noise (see Example 4.2, p157, [71])

$$\begin{aligned} \mathrm{d}x(t) &= \big[A_0(t)x(t) + A_1(t)x(t-h) + Bu(t) + B_v v(t)\big]\mathrm{d}t \\ &\quad + g(t, x(t))\mathrm{d}W(t)\,, \end{aligned} \tag{5.57}$$

$$z(t) = Cx(t) \tag{5.58}$$

with

$$A_0 = \begin{bmatrix} -1 & 1 \\ -2 & -3 \end{bmatrix}, L_0 = \begin{bmatrix} 0.6 & -0.2 \\ 0 & 0.8 \end{bmatrix}, E_0 = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.2 \end{bmatrix},$$

$$A_1 = \begin{bmatrix} 0 & -0.1 \\ 0.5 & 1 \end{bmatrix}, L_1 = \begin{bmatrix} 0.25 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.2 \end{bmatrix},$$

$$E_1 = \begin{bmatrix} 0 & 0.4 \\ 0.2 & 0.2 \end{bmatrix}, F_0(t) = \begin{bmatrix} 0.7\sin(t) & 0 \\ 0 & 0.3\sin(3t) \end{bmatrix},$$

$$F_1(t) = \begin{bmatrix} 0.5\sin(2t) & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0.4\sin(t) \end{bmatrix}^T,$$

and

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 0.5 \end{bmatrix}, B_v = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \end{bmatrix}, G = \begin{bmatrix} 3 & 0 \\ 3 & 0 \end{bmatrix}, C = \begin{bmatrix} 0.5 & 0 \\ 0 & 0.5 \end{bmatrix}.$$

Obviously, in this case, we have $m = n = 2$ (see Remark 5.2). For $\gamma = 1$, solving inequalities (5.9)-(5.13) yields

$$P = \begin{bmatrix} 2.2138 & 0.0002 \\ 0.0002 & 0.3918 \end{bmatrix}, \quad \beta = 15.5091 \,.$$

By Theorem 5.1, closed-loop system (5.57) and (5.17) designed with the parameters above converges to sliding surface

$$s(t) = B^T P x(t) = \begin{bmatrix} 2.2138 & 0.0002 \\ 0.0001 & 0.1959 \end{bmatrix} x(t) = 0 \tag{5.59}$$

in finite time. Since $\text{rank}(B^T P) = \text{rank}(B) = m = n = 2$, $s(t) = 0 \Rightarrow x(t) = 0$. That is, system (5.57) and (5.17) arrives at the equilibrium in finite time and stays there afterwards. By Definition 5.2 and Remark 5.2, we see that our method provides a control scheme for 1st moment finite-time stabilization of system (5.57). This control strategy is more desired than the control method in [71] in a case when the states of the water quality are required to reach the equilibrium in finite time and stay at the point.

In the following, let us consider system (5.57)-(5.58) with single input, to which SMC strategy may be applied. To illustrate the effectiveness of the result proposed in this chapter, we consider the case of

$$B = \begin{bmatrix} 2 & 2 \end{bmatrix}^T, \tag{5.60}$$

where it is easy to see that the assumption (5.2) in the existing results (see [13], [14, 15], [102], [103, 104] and [45]) is not satisfied and hence those results can not be applied in this case. For prescribed $\gamma = 1$, inequalities (5.9)-(5.13) give

$$P = \begin{bmatrix} 0.0729 & -0.0726 \\ -0.0726 & 0.0726 \end{bmatrix}, \quad \beta = 31.6466 \,.$$

Hence, sliding surface (5.14) and control scheme (5.17) can be designed with these given parameters. In this case, the state trajectories of system (5.57)-(5.58) converge to sliding manifold

$$s(t) = B^T P x(t) = 10^{-3} \times \begin{bmatrix} 0.4970 & 0 \end{bmatrix} x(t) = 0 \tag{5.61}$$

in finite time, on which the sliding mode is robustly mean-square exponentially stable with disturbance attenuation $\gamma = 1$. The curves given in Figure 5.1-5.5 are the result of a simulation with diffusion $g(t, x(t)) = Gx(t)$, design parameter $\alpha = 10^{-4}$, initial condition $x(\theta) = [10\ 10]^T$, $\theta \in [-h, 0]$, and time delay $h = 10^3\text{Dt}$, where step sizes $\text{Dt} = R\text{dt} = 10^{-4}$ and $\text{dt} = 0.5 \times 10^{-4}$ (see [37]). The curve of mean square of 1000 samples are given in Figure 5.6.

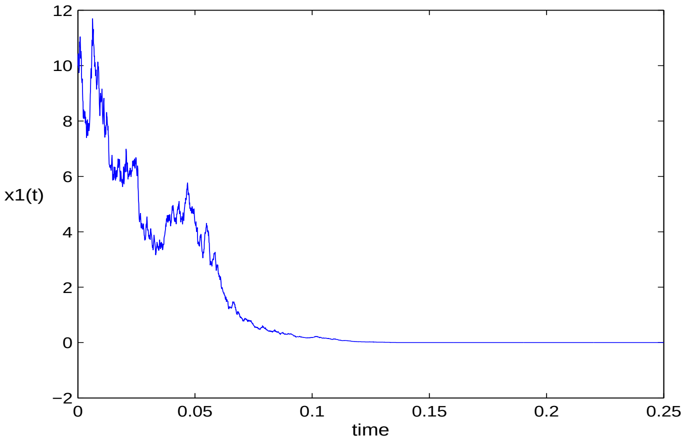

Figure 5.1: The curve of $x_1(t)$.

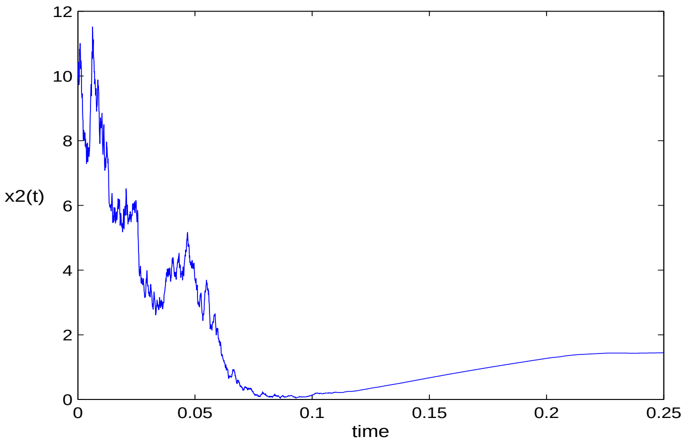

Figure 5.2: The curve of $x_2(t)$ before entering sliding mode.

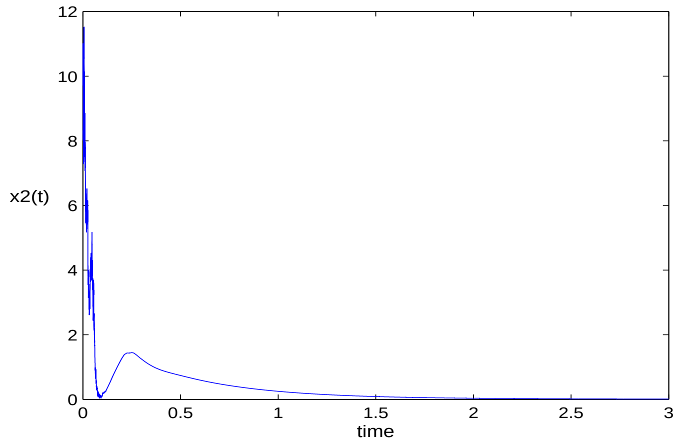

Figure 5.3: The curve of $x_2(t)$.

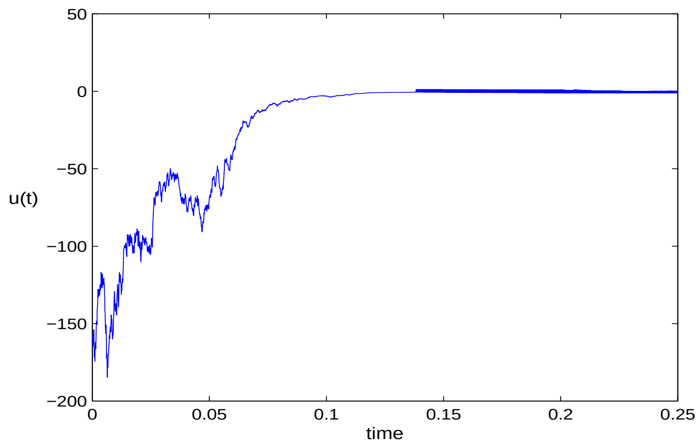

Figure 5.4: The curve of $u(t)$ before entering sliding mode.

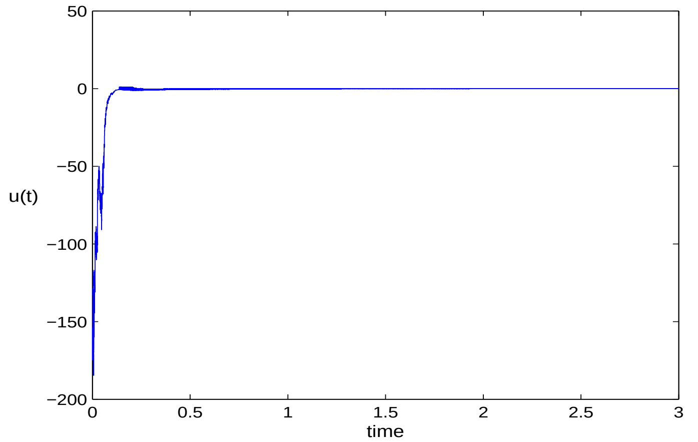

Figure 5.5: The curve of $u(t)$.

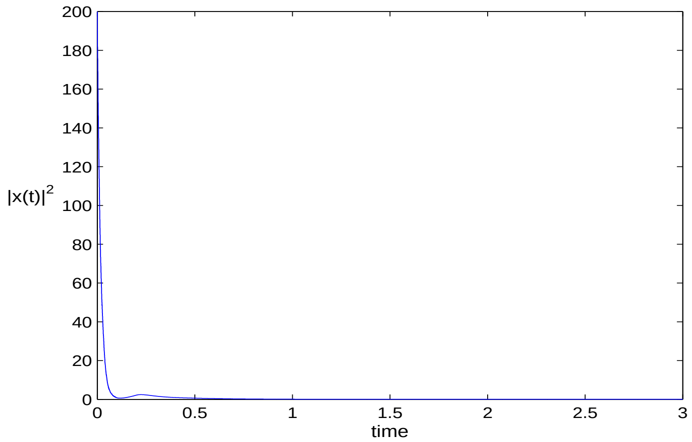

Figure 5.6: The curve of mean square of 1000 paths.

5.7 Summary

This chapter presents a SMC design for robust H_∞ control for uncertain stochastic delay systems. The proposed method removes a restriction in the existing results. The idea in this chapter may also be applied in an alternative way to linear stochastic delay systems with $m < n$. Since pair (A_0, B) is controllable and matrix B is of full column rank, system (5.3) can be transformed to a variant of canonical controller-type form (see Remark 4 [45]). This may be considered as a decomposition into two interconnected subsystems, one of which, denoted by subsystem $y_2(t) \in R^m$, includes control input $u(t)$ and the other denoted by subsystem $y_1(t) \in R^{n-m}$ is free of input. If the sliding mode is chosen as $s(t) = Sy_2(t) = 0$, where S is a nonsingular matrix, then the condition for reachability of sliding mode can be figured out from the subsystem $y_2(t)$ while the condition for stability of sliding mode is indeed that for stability of subsystem $y_1(t)$ with $y_2(t) = 0$.

Chapter 6

Razumikhin-type theorems on stability of stochastic retarded systems with Markovian switching

6.1 Introduction

Since Markov jump linear systems were first introduced in early 1960s (see, e.g., [92] and [135]), hybrid systems driven by continuous-time Markov chains have been widely employed to model many practical systems that may experience abrupt changes in system structure and parameters such as BM/C^3 (battle management in command, control and communication) systems, electric power systems, failure prone manufacturing, macroeconomic models of national economy, population dynamics and solar-powered systems (see [3], [30], [68], [84], [92], [117], [132] and the references therein). An area of particular interest has been the stability analysis of this class of hybrid systems and its applications to automatic control (see, e.g., [10], [48] and [92]).

When time delays and environmental noise, which are often encounterd in real systems and may be the cause of poor performance and instability, are taken into account, the hybrid systems are described by stochastic functional differential equations with Markovian switching and called hybrid stochastic retarded systems (HSRSs). Some of the most important HSRSs that frequently appear in engineering are those called hybrid stochastic delay systems (HSDSs), which are also known as stochastic delay systems with

Markovian switching (SDSwMS) and described by stochastic differential delay equations with Markovian switching (see, e.g., [84], [86] and [133]). Recently, HSRSs and HSDSs have been widely used since stochastic modelling plays an important role in many branches of science and engineering (see, e.g., [85], [132], [135] and the references therein). Consequently, the stability analysis of HSRSs including HSDSs has been studied by many works, see, e.g., [79], [80], [81], [85], [87], [133] and [135]. Mao et al. ([81], [82] and [85]) established a number of exponential stability criteria for stochastic differential delay equations with Markovian switching while Yue et al. ([135]) considered delay-dependent exponential stability of a class of stochastic systems with time delay, nonlinearity and Markovian switching. However, these results require the time delay of the systems to be a constant or a differentiable function that varies slowly, or say, the derivative of which is upper bounded by a constant number less than one. To remove this restriction in the result of [85] and allow the time delay to be a bounded variable only, Mao et al. ([79] and [86]) proposed the Razumikhin-type theorem on exponential stability of HSRSs and its application to linear uncertain HSDSs.

The Razumikhin method is developed to cope with the difficulty arisen from the large, fast varying and non-differentiable time delays. However, the importance of general asymptotic stability has not been considered. In many cases, the exponential stability of the equilibrium of the system is not necessary and to stabilize the system exponentially fast is economically, and sometimes practically, unfeasible. In fact, the criteria for exponential stability of HSRSs implicitly require the diffusion operator associated with the underlying HSRSs of the Lyapunov function along a solution of the system to be negative and have the same order as that of the function itself at some instants, which is not satisfied in many nonlinear systems. In these cases, the existing results (see [79]-[81], [85], [86], [87] and [135]) can not be applied. For example, consider the following scalar stochastic delay system driven by a right-continuous Markov chain $r(t)$ that is independent of the one-dimensional standard Brownian motion $W_1(t)$ and takes values in $S = \{1, 2\}$ with generator

$$\Gamma = \begin{pmatrix} -\gamma_1 & \gamma_1 \\ \gamma_2 & -\gamma_2 \end{pmatrix}, \quad \gamma_1 > 0, \quad \gamma_2 > 0.$$

The HSDS is described by the following nonlinear stochastic delay equation with Markovian switching

$$\mathrm{d}x(t) \;=\; -[\frac{1}{2}x(t)+\zeta(x(t),r(t))]\mathrm{d}t+\sigma(x(t),x(t-\tau(t)),r(t))\mathrm{d}W_1(t) \qquad (6.1)$$

on $t \geq 0$, where $\tau : R_+ \to (0,h]$ is Borel measurable and the nonlinear term $\zeta(x(t),r(t))$ and the diffusion term $\sigma(x(t),x(t-\tau(t)),i)$ are given as follow

$$\zeta(x(t),i) = \begin{cases} \frac{1}{6}x^3(t), & i=1 \\ \frac{1}{10}x(t)\sqrt{|x(t)|}, & i=2 \end{cases}$$

and

$$\sigma(x(t),x(t-\tau(t)),i) = \begin{cases} \frac{\sqrt{2}}{4}x^2(t)+\frac{\sqrt{2}}{2}x(t-\tau(t)), & i=1 \\ x(t-\tau(t)), & i=2 \end{cases}$$

for all $t \geq 0$. We encounter a problem when we attempt to apply the existing results to analyze the stability of the solution to equation (6.1). To see this problem, let us set $V(x(t),t,r(t)) = x^2(t)$ and calculate

$$\mathcal{L}V(x_t,t,i) \leq \begin{cases} -x^2(t)-\frac{1}{12}x^4(t)+x^2(t-\tau(t)), & i=1 \\ -x^2(t)-\frac{1}{5}x^2(t)\sqrt{|x(t)|}+x^2(t-\tau(t)), & i=2 \end{cases} \qquad (6.2)$$

on $t \geq 0$, where operator $\mathcal{L}$ is defined in (6.5) or (6.29) (see, e.g., [79]). It is easy to verify that the Razumikin-type theorem on exponential stability (see Theorem 4.2 [79], Theorem 2.1 [86] or Theorem 8.9, p311, [87]) is not applicable to this case. However, the solution to equation (6.1) can be asymptotically stable in mean-square sense though it might be not exponentially stable (see [54] and [126]). This chapter studies the general asymptotic stability of HSRSs with Razumikhin-type arguments, which is a generalization of the result on exponential stability (see [79], [86] and [87]).

6.2 Notation

Throughout the chapter, unless otherwise specified, we will employ the following notation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is right continuous and $\mathcal{F}_0$ contains all P-null sets)and $\mathbb{E}[\cdot]$ be the expectation operator with respect to the probability measure. Let $W(t) = (W_1(t), \cdots, W_m(t))^T$ be an m-dimensional Brownian motion defined on the probability space. If x, y are real numbers, then $x \vee y$ denotes the maximum of x and y, and $x \wedge y$ stands for the minimum of x and y. Let $|\cdot|$ denote the Euclidean norm in R^n. Let $h \geq 0$ and $C([-h, 0]; R^n)$ denote the family of all continuous R^n-valued functions φ on $[-h, 0]$ with the norm $\|\varphi\| = \sup\{|\varphi(\theta)| : -h \leq \theta \leq 0\}$. Let $C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ be the family of all $\mathcal{F}_0$-measurable bounded $C([-h, 0]; R^n)$-valued random variables $\xi = \{\xi(\theta) : -h \leq \theta \leq 0\}$. For $p > 0$ and $t \geq 0$, denote by $L^p_{\mathcal{F}_t}([-h, 0]; R^n)$ the family of all $\mathcal{F}_t$-measurable $C([-h, 0]; R^n)$-valued random processes $\phi = \{\phi(\theta) : -h \leq \theta \leq 0\}$ such that $\sup_{-h \leq \theta \leq 0} \mathbb{E}|\phi(\theta)|^p < \infty$. We let $\mathcal{K}$ denote the class of continuous strictly increasing functions μ from R_+ to R_+ with $\mu(0) = 0$. Let $\mathcal{K}_\infty$ denote the class of functions $\mu \in \mathcal{K}$ with $\mu(r) \to \infty$ as $r \to \infty$. Functions in $\mathcal{K}$ and $\mathcal{K}_\infty$ are called class $\mathcal{K}$ and $\mathcal{K}_\infty$ functions, respectively.

Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$P\{r(t+\Delta) = j : r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta) & if\ i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta) & if\ i = j, \end{cases} \tag{6.3}$$

where $\Delta > 0$ and $\gamma_{ij} \geq 0$ is the transition rate from i to j if $i \neq j$ while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$.

Assume that the Markov chain $r(\cdot)$ is independent of the Brownian motion $W(\cdot)$. It is known that almost all sample paths of $r(t)$ are right-continuous step functions with a finite number of simple jumps in any finite subinterval of $R_+ := [0, \infty)$.

Let us consider an n-dimensional HSRS

$$\mathrm{d}x(t) = f(x_t, t, r(t))\mathrm{d}t + g(x_t, t, r(t))\mathrm{d}W(t) \tag{6.4}$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ and $r(0) = r_0 \in S$. Moreover,

$$f : C([-h, 0]; R^n) \times R_+ \times S \to R^n$$

and

$$g : C([-h, 0]; R^n) \times R_+ \times S \to R^{n \times m}$$

are measurable functions with $f(0, t, i) \equiv 0$ and $g(0, t, i) \equiv 0$ for all $t \geq 0$. So equation (6.4) admits a trivial solution $x(t; 0) \equiv 0$. Here, $x_t = \{x(t+\theta) : -h \leq \theta \leq 0\}$ is regarded as a $C([-h, 0]; R^n)$-valued stochastic process. We assume that f and g are sufficiently smooth so that equation (6.4) has a unique solution on $t \geq 0$ (see, e.g., [63], [84], [112], [133] and Appendix A), which is denoted by $x(t; x_0)$ or $x(t; \xi)$ in this chapter.

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times R_+ \times S$ that are twice continuously differentiable in x and once in t. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, define an operator $\mathcal{L}$ associated with system (6.4) from $C([-h, 0]; R^n) \times R_+ \times S$ to R by

$$\begin{aligned}\mathcal{L}V(x_t, t, i) &= V_t(x, t, i) + V_x(x, t, i)f(x_t, t, i) + \frac{1}{2} trace \left[g^T(x_t, t, i)V_{xx}(x, t, i)g(x_t, t, i)\right] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(x, t, j), \end{aligned} \tag{6.5}$$

where

$$\begin{aligned} V_t(x, t, i) &= \frac{\partial V(x, t, i)}{\partial t}, \\ V_x(x, t, i) &= \left(\frac{\partial V(x, t, i)}{\partial x_1}, \cdots, \frac{\partial V(x, t, i)}{\partial x_n} \right), \\ V_{xx}(x, t, i) &= \left(\frac{\partial^2 V(x, t, i)}{\partial x_j \partial x_k} \right)_{n \times n}. \end{aligned}$$

The purpose of this chapter is to further develop Razumihkin-type theorems on stability of HSRSs initiated by [79]. Let us begin with the following definition (see, e.g., [25]) and lemma.

Definition 6.1 *The solution of equation (6.4), or simply, equation (6.4) is said to be*

1. *pth ($p > 0$) moment stable if for every $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that*

$$\mathbb{E}|x(t; \xi)|^p \leq \varepsilon, \quad \forall t \geq 0$$

whenever $\mathbb{E}\|\xi\|^p < \delta_0$.

2. *pth moment asymptotically stable if it is p-th moment stable and, moreover, for every $\varepsilon > 0$, there exist $\delta_0 = \delta_0(\varepsilon)$ and $T = T(\varepsilon)$ such that*

$$\mathbb{E}|x(t; \xi)|^p \leq \varepsilon, \quad \forall t \geq T$$

whenever $\mathbb{E}\|\xi\|^p < \delta_0$.

3. *globally pth moment asymptotically stable if it is pth moment stable and, moreover, for all $\xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$,*

$$\lim_{t \to \infty} \mathbb{E}|x(t; \xi)|^p = 0.$$

6.3 Asymptotic Stability of HSRSs

As the main results of this chapter, we present Razumikhin-type theorems on general stability of HSRSs (6.4) as follows.

Theorem 6.1 *Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in C\mathcal{K}_\infty$ and $w : R \times R_+ \times S \to R$ be a nonnegative continuous function with $w(y,t,i) > 0$ if $y > 0$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that*

$$u(|x|^p) \le V(x,t,i) \le v(|x|^p), \quad \forall (x,t,i) \in R^n \times [-h,\infty) \times S \tag{6.6}$$

and, moreover,

$$\mathbb{E}\mathcal{L}V(\phi,t,i) \le -w(\mathbb{E}|\phi(0)|^p, t, i) \tag{6.7}$$

for all $(t,i) \in R_+ \times S$ and those $\phi \in L^p_{\mathcal{F}_t}([-h,0]; R^n)$ satisfying

$$\min_{k\in S} \mathbb{E}V(\phi(\theta), t+\theta, k) < q(\max_{k\in S} \mathbb{E}V(\phi(0), t, k), i) \tag{6.8}$$

on $-h \le \theta \le 0$, where $q : R \times S \to R$ is a continuous nondecreasing function with respect to $s \in R$ for all $s \ge 0$ and $i \in S$. Moreover $q(s,i) > s$ for all $s > 0$ and $i \in S$. Then the trivial solution of HSRS (6.4) is globally pth moment asymptotically stable.

Let us first present the following lemma that is will be used to prove the main results.

Lemma 6.1 *Let $V(t) = V(x(t), t, r(t))$ for $t \ge 0$, then $\mathbb{E}V(t)$ is continuous on $t \ge 0$.*

Proof. For any initial data $\xi \in C^b_{\mathcal{F}_0}([-h,0]; R^n)$, write $x(t) = x(t;\xi)$ and extend $r(t)$ to $[-h,0)$ by setting $r(t) = r(0) = r_0$ for all $t \in [-h,0)$. The generalized Itô's formula (2.23) (see p105, [114]) can be easily extended to HSRSs (6.4) (see [82] and [133])

$$\begin{aligned} &V(x(t), t, r(t)) \\ &\quad = V(x(0), 0, r(0)) + \int_0^t \mathcal{L}V(x_s, s, r(s))\mathrm{d}s + \int_0^t V_x(x(s), s, r(s)) g(x_s, s, r(s))\mathrm{d}W(s) \\ &\qquad + \int_0^t \int_R \big[V(x(s), s, r(0) + h(r(s), l)) - V(x(s), s, r(s))\big]\mu(\mathrm{d}s, \mathrm{d}l) \end{aligned} \tag{6.9}$$

for all $t \ge 0$, where function $h(\cdot,\cdot)$ and martingale measure $\mu(\cdot,\cdot)$ are defined as, e.g., (2.18) and (2.23) (see also [30], [6] and [133]).

Since $\xi \in C^b_{\mathcal{F}_0}([-h,0];R^n)$, we can find an integer k_0 such that $\|\xi\| < k_0$ *a.s.*. For any integer $k > k_0$, define the stopping time

$$\rho_k = \inf\{t \geq 0 : |x(t)| \geq k\}, \tag{6.10}$$

where we set $\inf \emptyset = \infty$ as usual. Note that $x(t)$ is continuous and so are $|x(t)|$ and $v(|x(t)|)$ on $t \geq -h$. Clearly, $\rho_k \to \infty$ almost surely as $k \to \infty$. Moreover, since $x_0 = \xi \in C^b_{\mathcal{F}_0}([-h,0];R^n)$, $\mathbb{E}V(x(0),0,r(0)) \leq \mathbb{E}v(|\xi(0)|) \leq v(k_0)$. It then follows from (6.9) that

$$\mathbb{E}V(x(t_k),t_k,r(t_k)) = \mathbb{E}V(x(0),0,r(0)) + \mathbb{E}\int_0^{t_k} \mathcal{L}V(x_s,s,r(s))\mathrm{d}s \tag{6.11}$$

where $t_k = t \wedge \rho_k$. So, letting $k \to \infty$, by Fubini's theorem, we have

$$\mathbb{E}V(t) = \mathbb{E}V(0) + \mathbb{E}\int_0^t \mathcal{L}V(x_s,s,r(s))\mathrm{d}s = \mathbb{E}V(0) + \int_0^t \mathbb{E}\mathcal{L}V(x_s,s,r(s))\mathrm{d}s \tag{6.12}$$

for all $t \geq 0$. This implies $\mathbb{E}V(t)$ is continuous on $t \geq 0$.

Now we proceed to prove Theorem 6.1.

Proof. By Lemma 6.1, we see that $\mathbb{E}V(x(t),t,r(t))$ is continuous on $t \geq -h$. Define

$$U(t) = \sup_{-h\leq\theta\leq 0} \mathbb{E}V(x(t+\theta),t+\theta,r(t+\theta)) \qquad \forall t \geq 0. \tag{6.13}$$

We claim that

$$D_+U(t) := \limsup_{s\to 0+} \frac{U(t+s)-U(t)}{s} \leq 0 \qquad \forall\, t \geq 0. \tag{6.14}$$

To show inequality (6.14), for each $t \geq 0$ (fix t for the moment), we define

$$\bar{\theta} = \max\{\theta \in [-h,0] : \mathbb{E}V(x(t+\theta),t+\theta,r(t+\theta)) = U(t)\}. \tag{6.15}$$

Obviously, $\bar{\theta}$ is either less than 0 or equal to 0.

If $\bar{\theta} < 0$, then

$$\mathbb{E}V(x(t+\theta),t+\theta,r(t+\theta)) < \mathbb{E}V(x(t+\bar{\theta}),t+\bar{\theta},r(t+\bar{\theta})) = U(t), \quad \forall \theta \in (\bar{\theta},0]. \tag{6.16}$$

It follows from the continuity of $\mathbb{E}V(x(t),t,r(t))$ that for every sufficiently small $s > 0$

$$\mathbb{E}V(x(t+s),t+s,r(t+s)) \leq U(t),$$

hence

$$U(t+s) \le U(t) \quad \text{and} \quad D_+U(t) \le 0.$$

If $\bar{\theta} = 0$, then

$$\mathbb{E}V(x(t+\theta), t+\theta, r(t+\theta)) \le \mathbb{E}V(x(t), t, r(t)) = U(t), \quad \forall \theta \in [-h, 0]. \tag{6.17}$$

Note that either $\mathbb{E}V(x(t), t, r(t)) = 0$ or $\mathbb{E}V(x(t), t, r(t)) > 0$. In the former case, i.e., $\mathbb{E}V(x(t), t, r(t)) = 0$, inequalities (6.17) and (6.6) yield that $x(t+\theta) = 0$ a.s. for all $-h \le \theta \le 0$. Recalling that $f(0, t, i) = 0$ and $g(0, t, i) = 0$, we see $x(t) = 0$ *a.s.* for all $t > 0$ hence $D_+U(t) = 0$. In the other case when $\mathbb{E}V(x(t), t, r(t)) > 0$, inequality (6.17) implies

$$\begin{aligned} &\mathbb{E}V(x(t+\theta), t+\theta, r(t+\theta)) \\ &\quad \le \mathbb{E}V(x(t), t, r(t)) < q(\mathbb{E}V(x(t), t, r(t)), r(t)), \quad \forall \theta \in [-h, 0]. \end{aligned} \tag{6.18}$$

Consequently inequality (6.8) holds, that is,

$$\min_{k \in S} \mathbb{E}V(x(t+\theta), t+\theta, k) < q(\max_{k \in S} \mathbb{E}V(x(t), t, k), r(t))$$

on all $-\tau \le \theta \le 0$. Moreover, by condition (6.6) and Jensen's inequality, $\mathbb{E}V(x(t), t, r(t)) > 0$ yields $\mathbb{E}|x(t)|^p > 0$.Thus, by condition (6.7), we have

$$\mathbb{E}\mathcal{L}V(x_t, t, i) < 0 \tag{6.19}$$

for all $i \in S$. By the right continuity of the processes concerned, we see that for all $h_s > 0$ sufficiently small

$$\mathbb{E}\mathcal{L}V(x_s, s, i) \le 0, \quad \forall t \le s \le t + h_s, i \in S.$$

By formula (6.12), we observe

$$\begin{aligned} &\mathbb{E}(x(t+h_s), t+h_s, r(t+h_s)) \\ &\quad = \mathbb{E}V(x(t), t, r(t)) + \int_t^{t+h_s} \mathbb{E}\mathcal{L}V(x_s, s, r(s)) \mathrm{d}s \le \mathbb{E}V(x(t), t, r(t)). \end{aligned}$$

Hence we have

$$U(t+h_s) = U(t) = \mathbb{E}V(x(t), t, r(t)) \quad \text{and} \quad D_+U(t) = 0.$$

Inequality (6.14) has been proved. It follows immediately that

$$U(t) \le U(0), \quad \forall t \ge 0. \tag{6.20}$$

Together with the definition of $U(t)$, condition (6.6) and Jensen's inequality, inequality (6.20) yields

$$\mathbb{E}|x(t)|^p \le u^{-1}(v(\mathbb{E}\|\xi\|^p)), \qquad \forall t \ge 0. \tag{6.21}$$

So, for any $\epsilon > 0$, we can find $\delta(\epsilon) = v^{-1}(u(\epsilon))$ such that

$$\mathbb{E}|x(t)|^p \le \epsilon, \qquad \forall t \ge 0$$

whenever $\mathbb{E}\|\xi\|^p < \delta(\epsilon)$. The pth moment stability is proved.

Now we proceed to show the convergence of $\mathbb{E}|x(t)|^p \to 0$ as $t \to \infty$. For any initial data $\xi \in C^b_{\mathcal{F}_0}([-h,0];R^n)$, let $\delta > 0$ and $\varepsilon > 0$ be such that $\mathbb{E}\|\xi\|^p < \delta$ and $U(0) < v(\delta) = u(\varepsilon)$. So, by inequalities (6.20) and (6.21), we have $\mathbb{E}V(x(t),t,r(t)) < v(\delta)$ and $\mathbb{E}|x(t)|^p < \varepsilon$ for all $t \ge 0$. Suppose $0 < \beta \le \varepsilon$ is arbitrary. We need to show there is a number $T = T(\beta,\delta)$ such that $\mathbb{E}|x(t)|^p \le \beta$ for all $t \ge T$. This will be true by condition (6.6) and Jensen's inequality if we show that $\mathbb{E}V(x(t),t,r(t)) \le u(\beta)$ for all $t \ge T$.

According to the property of function $q(\cdot,\cdot)$, there is a positive real number $a > 0$ such that $q(s,i) - s > a$ for all $u(\beta) \le s \le v(\delta)$ and $i \in S$. Let J be the minimal nonnegative integer such that $u(\beta) + Ja \ge v(\delta)$, and $\gamma = \inf\{w(\mathbb{E}|x(t)|^p,t,i) : \beta \le \mathbb{E}|x(t)|^p \le \varepsilon, t \ge 0, i \in S\}$. So $\gamma > 0$, since $w(y,t,i) > 0$ for all $y > 0$, $t \ge 0$ and $i \in S$. Let $\tilde{\tau} = h \vee \frac{v(\delta)}{\gamma}$ and $T_j = j\tilde{\tau}$ with $j = 0,1,\cdots,J$.

We claim that $\mathbb{E}V(x(t),t,r(t)) \le u(\beta)$ for all $t \ge T_J$. First we show that $\mathbb{E}V(x(t),t,r(t)) \le u(\beta) + (J-1)a$ for all $t \ge T_1$. Let $t_1 = \inf\{t \ge T_0 : \mathbb{E}V(x(t),t,r(t)) \le u(\beta) + (J-1)a\}$. If $t_1 > T_1$, then, $\forall T_0 \le t \le T_1$, we have

$$\begin{aligned} & q(\max_{k\in S} \mathbb{E}V(x(t),t,k), r(t)) \\ & \quad \ge q(\mathbb{E}V(x(t),t,r(t)),r(t)) > \mathbb{E}V(x(t),t,r(t)) + a \ge u(\beta) + Ja \ge v(\delta) \\ & \quad > \mathbb{E}V(x(t+\theta),t+\theta,r(t+\theta)) \ge \min_{k\in S} \mathbb{E}V(x(t+\theta),t+\theta,k), \quad \forall \theta \in [-h,0]. \end{aligned}$$

This, by condition (6.7), implies

$$\mathbb{E}\mathcal{L}V(x_t,t,r(t)) \le -w(\mathbb{E}|x(t)|^p,t,r(t)) \le -\gamma, \quad \forall T_0 \le t \le T_1.$$

Consequently, by formula (6.12), we see

$$\mathbb{E}V(x(T_1),T_1,r(T_1)) \le \mathbb{E}V(x(T_0),T_0,r(T_0)) - \gamma(\bar{T}_1 - \bar{T}_0) < v(\delta) - \gamma\tilde{\tau} \le 0,$$

which contradicts the nonnegative property of $\mathbb{E}V(x(t), t, r(t))$. So we must have

$$t_1 \le T_1 \quad \text{and} \quad \mathbb{E}\mathcal{L}V(x(t_1), t_1, r(t_1)) \le -\gamma.$$

In fact, $\forall t_{11} \in \{t \ge T_0 : \mathbb{E}V(x(t), t, r(t)) = u(\beta) + (J-1)a\}$, we have

$$\mathbb{E}\mathcal{L}V(x_{t_{11}}, t_{11}, r(t_{11})) \le -\gamma$$

because

$$\begin{aligned} q(\max_{k\in S} \mathbb{E}V(x(t_{11}), t_{11}, k), r(t)) &\ge q(\mathbb{E}V(x(t_{11}), t_{11}, r(t_{11})), r(t)) > u(\beta) + Ja \ge v(\delta) \\ &> \mathbb{E}V(x(t_{11}+\theta), t_{11}+\theta, r(t_{11}+\theta)) \ge \min_{k\in S} \mathbb{E}V(x(t_{11}+\theta), t_{11}+\theta, k), \ \forall \theta \in [-h, 0]. \end{aligned}$$

So we have $\mathbb{E}V(x(t), t, r(t)) \le u(\beta) + (J-1)a$ for all $t \ge T_1$.

Define $t_j = \inf\{t \ge T_{j-1} : \mathbb{E}V(x(t), t, r(t)) \le u(\beta) + (J-j)a\}$ for $j = 2, 3, \cdots, J$. By the same type of reasoning as above, we have

$$\mathbb{E}V(x(t), t, r(t)) \le u(\beta) + (J-j)a$$

for all $t \ge T_j$ and $j = 2, 3, \cdots, J$. In particular, $\mathbb{E}V(x(t), t, r(t)) \le u(\beta)$ for all $t \ge T_J$. This completes the proof.

Theorem 6.2 *Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in C\mathcal{K}_\infty$ and $w : R \times R_+ \times S \to R_+$ be a nonnegative continuous function with $w(y, t, i) > 0$ if $y > 0$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that*

$$u(|x|^p) \le V(x, t, i) \le v(|x|^p), \quad \forall (x, t, i) \in R^n \times [-h, \infty) \times S \tag{6.22}$$

and, moreover,

$$\mathbb{E}\mathcal{L}V(\phi, t, i) \le -w(\mathbb{E}|\phi(0)|^p, t, i) \tag{6.23}$$

for all $(t, i) \in R_+ \times S$ and those $\phi \in L^p_{\mathcal{F}_t}([-h, 0]; R^n)$ satisfying

$$\min_{k\in S} \mathbb{E}V(\phi(\theta), t+\theta, k) < \max_{k\in S} \mathbb{E}\bar{q}(V(\phi(0), t, k), i) \tag{6.24}$$

on $-h \le \theta \le 0$, where $\bar{q} : R \times S \to R$ is a continuous nondecreasing function with respect to $s \in R$ for all $s \ge 0$ and $i \in S$. Moreover, $\bar{q}(s, i) > s$ for all $(s, i) \in R_+ \times S$ and $\bar{q}(s, i)/s > 1$ as $s \to \infty$ for all $i \in S$. Then the trivial solution of HSRS (6.4) is globally pth moment asymptotically stable.

Proof. As above, the proof is composed of two parts. The first part to show the pth moment stability of equation (6.4) is similar to that of Theorem 6.1. One only needs to note that the properties of function $\bar{q}(\cdot,\cdot)$ yield the following inequality

$$\begin{aligned} &\mathbb{E}\bar{q}(V(x(t),t,r(t)),r(t)) \\ &\quad\geq \int_{0<V<\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}\mathbb{P} + \int_{V\to\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}\mathbb{P} \\ &\quad> \int_{0<V<\infty} V(x(t),t,r(t)))\mathrm{d}\mathbb{P} + \int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}\mathbb{P} \\ &\quad= \mathbb{E}V(x(t),t,r(t)) \end{aligned} \tag{6.25}$$

for all $t \geq 0$. Inequalities (6.17) and (6.25) imply that condition (6.24) is satisfied. Moreover, $\mathbb{E}V(x(t),t,r(t)) > 0$ implies $\mathbb{E}|x(t)|^p > 0$. Thus, by condition (6.23) and the property of function $w(\cdot,\cdot,\cdot)$, we are led to (6.19) in the case when $\mathbb{E}V(x(t),t,r(t)) > 0$.

The other part to show the convergence of $\mathbb{E}|x(t)|^p \to 0$ as $t \to \infty$ is slightly different and given as follows.

Numbers δ, ε , γ and $\tilde{\tau}$ are defined as above while the positive real number $\bar{a} = a_1 \wedge a_2$, where $a_1 > 0$ and $a_2 > 0$ are such that

$$\bar{q}(s,i) - s > a_1 \quad \forall u(\beta) \leq s < \infty$$

and

$$\frac{\bar{q}(s,i) - s}{s} > a_2 \quad as\ s \to \infty$$

for all $i \in S$. Let us now consider the expectation of function $V(x(t),t,r(t))$

$$\begin{aligned} \mathbb{E}V(x(t),t,r(t)) \;=\; &\int_{V<u(\beta)} V(x(t),t,r(t))\mathrm{d}\mathbb{P} \\ &+ \int_{u(\beta)\leq V<\infty} V(x(t),t,r(t))\mathrm{d}\mathbb{P} + \int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}\mathbb{P}. \end{aligned}$$

for any $t \geq 0$. Obviously there is a positive number $0 < \bar{p} < 1$ such that

$$\alpha_1 \vee \alpha_2 \geq \bar{p} \tag{6.26}$$

for any $t \geq 0$ whenever $\mathbb{E}V(x(t),t,r(t)) \geq u(\beta)$, where

$$\alpha_1 = \mathbb{P}\left\{u(\beta) \leq V(x(t),t,r(t)) < \infty\right\} \quad \text{and} \quad \alpha_2 = \int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}\mathbb{P}.$$

Let $\bar{J}$ be the minimal nonnegative integer such that $u(\beta) + \bar{J}\bar{p}\bar{a} \geq v(\delta)$, and $\bar{T}_j = j\tilde{\tau}$ with $j = 0, 1, \cdots, \bar{J}$. To prove that $\mathbb{E}V(x(t),t,r(t)) \leq u(\beta)$ for all $t \geq \bar{T}_{\bar{J}}$, we first show that

$\mathbb{E}V(x(t),t,r(t)) \le u(\beta)+(\bar{J}-1)\bar{p}\bar{a}$ for all $t \ge \bar{T}_1$. Let $\bar{t}_1 = \inf\{t \ge \bar{T}_0 : \mathbb{E}V(x(t),t,r(t)) \le u(\beta)+(\bar{J}-1)\bar{p}\bar{a}\}$. If $\bar{t}_1 > \bar{T}_1$, then, $\forall \bar{T}_0 \le t \le \bar{T}_1$, we have

$$\begin{aligned}
&\max_{k\in S} \mathbb{E}\bar{q}(V(x(t),t,k),r(t)) \ge \mathbb{E}\bar{q}(V(x(t),t,r(t)),r(t)) \\
&\quad= \int_{V<u(\beta)} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}\mathbb{P} + \int_{u(\beta)\le V<\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}\mathbb{P} \\
&\qquad+ \int_{V\to\infty} \bar{q}(V(x(t),t,r(t)),r(t))\mathrm{d}\mathbb{P} \\
&\quad> \int_{V<u(\beta)} V(x(t),t,r(t))\mathrm{d}\mathbb{P} + \int_{u(\beta)\le V<\infty} \left[V(x(t),t,r(t))+\bar{a}\right]\mathrm{d}\mathbb{P} \\
&\qquad+ (1+\bar{a})\int_{V\to\infty} V(x(t),t,r(t))\mathrm{d}\mathbb{P} \\
&\quad\ge \mathbb{E}V(x(t),t,r(t)) + \bar{p}\bar{a} \ge u(\beta)+\bar{J}\bar{p}\bar{a} \ge v(\delta) > \mathbb{E}V(x(t+\theta),t+\theta,r(t+\theta)) \\
&\quad\ge \min_{k\in S} \mathbb{E}V(x(t+\theta),t+\theta,k) \qquad (6.27)
\end{aligned}$$

for all $\theta \in [-h,0]$. This, by condition (6.23), implies

$$\mathbb{E}\mathcal{L}V(x_t,t,r(t)) \le -w(\mathbb{E}|x(t)|^p,t,r(t)) \le -\gamma, \quad \forall \bar{T}_0 \le t \le \bar{T}_1.$$

Consequently, we see

$$\mathbb{E}V(x(\bar{T}_1),\bar{T}_1,r(\bar{T}_1)) \le \mathbb{E}V(x(\bar{T}_0),\bar{T}_0,r(\bar{T}_0)) - \gamma(\bar{T}_1-\bar{T}_0) < v(\delta) - \gamma\tilde{\tau} \le 0,$$

which contradicts the nonnegative property of $\mathbb{E}V(x(t),t,r(t))$. So we have

$$\bar{t}_1 \le \bar{T}_1 \quad \text{and} \quad \mathbb{E}\mathcal{L}V(x(\bar{t}_1),\bar{t}_1,r(\bar{t}_1)) \le -\gamma.$$

Moreover, $\forall \bar{t}_{11} \in \{t \ge \bar{T}_0 : \mathbb{E}V(x(t),t,r(t)) = u(\beta)+(\bar{J}-1)\bar{p}\bar{a}\}$, we have

$$\mathbb{E}\mathcal{L}V(x_{\bar{t}_{11}},\bar{t}_{11},r(\bar{t}_{11})) \le -\gamma$$

because inequality (6.24), or say, (6.27) holds on $t = \bar{t}_{11}$. So we have $\mathbb{E}V(x(t),t) \le u(\beta)+(\bar{J}-1)\bar{p}\bar{a}$ for all $t \ge \bar{T}_1$.

Define $\bar{t}_j = \inf\{t \ge \bar{T}_{j-1} : \mathbb{E}V(x(t),t,r(t)) \le u(\beta)+(\bar{J}-j)\bar{p}\bar{a}\}$ for $j = 2,3,\cdots,\bar{J}$. By the same type of reasoning, we have

$$\mathbb{E}V(x(t),t,r(t)) \le u(\beta)+(\bar{J}-j)\bar{p}\bar{a}$$

for all $t \ge \bar{T}_j$ and $j = 2,3,\cdots,\bar{J}$. Therefore, $\mathbb{E}V(x(t),t,r(t)) \le u(\beta)$ for all $t \ge \bar{T}_J$. The proof is complete.

Remark 6.1 *By Fatou's lemma, we note that conditions (6.8) and (6.24) are less conservative than the corresponding ones in the existing results (see, e.g., inequality (2.5), Theorem 2.1 ,[87]) and are convenient for application.*

6.4 Application to HSDSs

Hybrid stochastic delay systems (HSDSs) described with stochastic differential delay equations with Markovian switching are an important class of HSRSs that are frequently used in engineering. As an illustrative example of applications of our new results, we will apply Theorem 6.2 to establish a criterion for stability of HSDEs in this section.

Let us consider the HSDSs of the form

$$\mathrm{d}x(t) = F(x(t), x(t-\tau(t)), t, r(t))\mathrm{d}t + G(x(t), x(t-\tau(t)), t, r(t))\mathrm{d}W(t) \tag{6.28}$$

on $t \geq 0$ with initial data $x_0 = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$, where $\tau : R_+ \to [0, h]$ is Borel measurable while

$$F : R^n \times R^n \times R_+ \times S \to R^n$$

and

$$G : R^n \times R^n \times R_+ \times S \to R^{n\times m}$$

are measurable functions with $F(0, 0, t, i) \equiv 0$ and $g(0, 0, t, i) \equiv 0$ for all $t \geq 0$ and $i \in S$. It is easy to see that this is a special case of equation (6.4) with

$$f(\phi, t, i) = F(\phi(0), \phi(-\tau(t)), t, i) \quad \text{and} \quad g(\phi, t, i) = G(\phi(0), \phi(-\tau(t)), t, i)$$

for $(\phi, t, i) \in C([-h, 0]; R^n) \times R_+ \times S$. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, for the special case of (6.28) the operator $\mathcal{L}$ defined in (6.5) becomes from $R^n \times R^n \times R_+ \times S$ to R as

$$\begin{aligned} \mathcal{L}V(x, y, t, i) &= V_t(x, t, i) + V_x(x, t, i)F(x, y, t, i) \\ &\quad + \frac{1}{2}trace\left[G^T(x, y, t, i)V_{xx}(x, t, i)G(x, y, t, i)\right] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij}V(x, t, j). \end{aligned} \tag{6.29}$$

To give our new result for the HSDSs (6.28), let us introduce one more notation that $L^p_{\mathcal{F}_t}([\Omega; R^n)$ are the collection of all $\mathcal{F}_t$-measurable $C([-h, 0]; R^n)$-valued random variables X such that $\mathbb{E}|X|^p < \infty$ and state the corresponding version of Theorem 6.2 for equation (6.28) as follows

Theorem 6.3 *Let $p > 0$, $c_2 \geq c_1 > 0$ and $w : R \times R_+ \times S \to R_+$ be a nonnegative continuous function with $w(y,t,i) > 0$ for $y > 0$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that*

$$c_1|X|^p \leq V(X,t,i) \leq c_2|X|^p, \quad \forall (x,t,i) \in R^n \times [-h,\infty) \times S \tag{6.30}$$

and, moreover,

$$\mathbb{E}\mathcal{L}V(X,Y,t,i) \leq -w(\mathbb{E}|X|^p,t,i) \tag{6.31}$$

for all $(t,i) \in R_+ \times S$ and those $X, Y \in L^p_{\mathcal{F}_t}(\Omega; R^n)$ satisfying

$$\min_{k \in S} \mathbb{E}V(Y, t-\tau(t), k) < \max_{k \in S} \mathbb{E}\bar{q}(V(X,t,k), i), \tag{6.32}$$

where $\bar{q} : R \times S \to R$ is a continuous nondecreasing function with respect to $s \in R$ for all $s \geq 0$ and $i \in S$. Moreover, $\bar{q}(s,i) > s$ for all $(s,i) \in R_+ \times S$ and $\bar{q}(s,i)/s > 1$ as $s \to \infty$ for all $i \in S$. Then the trivial solution of HSDS (6.28) is globally pth moment asymptotically stable.

This is a corollary from Theorem 6.2 and will be used to establish the following useful result.

Theorem 6.4 *Let $p > 0$, $c_2 \geq c_1 > 0$, $\lambda_{0i} \geq \lambda_{1i} \geq 0$ and $\lambda : R \times S \to R$ be a continuous nondecreasing convex function with respect to $s \in R$ for all $s \geq 0$ and $i \in S$. Moreover $\lambda(s,i) > s$ for all $(s,i) \in R_+ \times S$ and $\lambda(s,i)/s > 0$ as $s \to \infty$ for all $i \in S$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that inequality (6.30) is satisfied and, moreover,*

$$\mathcal{L}V(X,Y,t,i) \leq -\lambda_{0i} \max_{k \in S} V(X,t,k) + \lambda_{1i} \min_{k \in S} V(Y, t-h(t), k) - \lambda(\max_{k \in S} V(X,t,k), i) \tag{6.33}$$

for all $X, Y \in R^n$, $t \geq 0$ and $i \in S$. Then the trivial solution of HSDS (6.28) is globally pth moment asymptotically stable.

Proof. In condition (6.32), let

$$\bar{q}(s,i) = s + \frac{1}{2(1+\lambda_{1i})}\lambda(s,i). \tag{6.34}$$

For all $(t,i) \in R_+ \times S$ and $X, Y \in L^p_{\mathcal{F}_t}(\Omega; R^n)$ satisfying condition (6.32) with function (6.34), i.e.,

$$\min_{k \in S} \mathbb{E}V(Y, t-\tau(t), k) < \max_{k \in S} \mathbb{E}V(X,t,k) + \frac{1}{2(1+\lambda_{1i})}\mathbb{E}\lambda(\max_{k \in S} V(X,t,k), i),$$

by inequality (6.33), Fatou's lemma and condition (6.30) we have

$$
\begin{aligned}
&\mathbb{E}\mathcal{L}V(X,Y,t,i)\\
&\quad\le -\lambda_{0i}\max_{k\in S}\mathbb{E}V(X,t,k)+\lambda_{1i}\min_{k\in S}\mathbb{E}V(Y,t-\tau(t),k)-\mathbb{E}\lambda(\max_{k\in S}V(X,t,k),i)\\
&\quad\le -(\lambda_{0i}-\lambda_{1i})\max_{k\in S}\mathbb{E}V(X,t,k)-\frac{1}{2}\mathbb{E}\lambda(\max_{k\in S}V(X,t,k),i)\\
&\quad\le -\frac{1}{2}\mathbb{E}\lambda(c_1|X|^p,i).
\end{aligned}
$$

According to the properties of function $\lambda(\cdot,\cdot)$, it is easy to verify that $\mathbb{E}\lambda(c_1|X|^p,i)>0$ if $\mathbb{E}|X|^p>0$. Let $w(\mathbb{E}|X|^p,t,i)=-\frac{1}{2}\lambda(c_1\mathbb{E}|X|^p,i)\ge-\frac{1}{2}\mathbb{E}\lambda(c_1|X|^p,i)$ in condition (6.31), then, by Theorem 6.3, it follows the conclusion.

Remark 6.2 *In many cases, this useful criterion may be applied with* $\lambda(s,i)=\tilde{\lambda}_i s^{k_i}$, $k_i\ge 1$ *and* $\tilde{\lambda}_i>0$ *for* $i\in S$. *In a special case when* $\lambda(s,i)=\tilde{\lambda}_0 s$ *and* $\tilde{\lambda}_0>0$ *for all* $i\in S$, *the above result is exactly the Theorem 4.2 in [79]. However, our result works for the particular cases when* $\lambda_{0i}-\lambda_{1i}=0$ *for some* $i\in S$, *to which the existing results (see [79]-[81], [86], [87] and [135]) do not apply.*

Using the above skills, Theorem 6.4 can be developed to cope with systems with multiple delays of the form

$$
\begin{aligned}
\mathrm{d}x(t) &= F(x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_L(t)),t,r(t))\mathrm{d}t\\
&\quad+G(x(t),x(t-\tau_1(t)),\cdots,x(t-\tau_L(t)),t,r(t))\mathrm{d}W(t)
\end{aligned}
\tag{6.35}
$$

on $t\ge 0$, where $\tau_l:R_+\to[0,h]$ is Borel measurable, $l=1,2,\cdots,L$.

Let us state the following generalized result, which can be proven in the same way as the proof of Theorem 6.4.

Theorem 6.5 *Let* $p>0$, $c_2\ge c_1>0$, *and* $\lambda_{0i}\ge 0,\lambda_{1i}\ge 0,\cdots\lambda_{Li}\ge 0$ *such that* $\lambda_{0i}\ge\sum_{l=1}^{L}\lambda_{li}$ *for all* $1\le i\le N$. *Let* $\lambda:R\times S\to R$ *be a continuous nondecreasing convex function with respect to* $s\in R$ *for all* $s\ge 0$ *and* $i\in S$. *Moreover* $\lambda(s,i)>s$ *for all* $(s,i)\in R_+\times S$ *and* $\lambda(s,i)/s>0$ *as* $s\to\infty$ *for all* $i\in S$. *Assume that there exists a function* $V\in C^{2,1}(R^n\times R_+\times S;R_+)$ *such that inequality (6.30) is satisfied and,* *moreover,*

$$\begin{aligned}
&\mathcal{L}V(X, Y_1, \cdots, Y_L, t, i)\\
&\quad \leq -\lambda_{0i} \max_{k\in S} V(X,t,k) + \lambda_{1i} \min_{k\in S} V(Y_1, t-\tau_1(t), k) + \cdots\\
&\qquad + \lambda_{Li} \min_{k\in S} V(Y_L, t-\tau_L(t), k) - \lambda(\max_{k\in S} V(X,t,k), i) \qquad (6.36)
\end{aligned}$$

for all $X, Y_1, \cdots, Y_L \in R^n$*,* $t \geq 0$ *and* $i \in S$*. Then the trivial solution of HSDS (6.35) is globally pth moment asymptotically stable.*

6.5 Example

Example 6.1 Let us now return to the scalar HSDS (6.1). For the previous calculation (6.2), let

$$\lambda_{01} = \lambda_{11} = 1, \quad \lambda(s,1) = \frac{1}{12}s^2, \quad \lambda_{02} = \lambda_{12} = 1, \quad \lambda(s,2) = \frac{1}{5}s^{5/4}$$

in condition (6.33). It immediately follows from Theorem 6.4 that the trivial solution of system (6.1) is mean-square asymptotically stable. Clearly, this is in fact an application of Theorem 6.2. Alternatively, we can use Theorem 6.1 and have the same conclusion. Let

$$q(s,i) = \begin{cases} s + \frac{1}{24}s^2, & i = 1 \\ s + \frac{1}{10}s^{5/4}, & i = 2 \end{cases}$$

in condition (6.8), then previous calculation (6.2) yields

$$\mathbb{E}\mathcal{L}V(x_t, t, i) \leq \begin{cases} -\frac{1}{24}\mathbb{E}x^4(t), & i = 1 \\ -\frac{1}{10}\mathbb{E}\left[x^2(t)\sqrt{|x(t)|}\right], & i = 2 \end{cases}$$

when condition (6.8) is satisfied. Let

$$w(\mathbb{E}x^2(t), t, i) = \begin{cases} \frac{1}{24}(\mathbb{E}x^2(t))^2, & i = 1 \\ \frac{1}{10}(\mathbb{E}x^2(t))^{5/4}, & i = 2 \end{cases}$$

in inequality (6.7), then the inequality holds. According to Theorem 6.1, this implies that the trivial solution of system (6.1) is mean-square asymptotically stable.

6.6 Summary

In this chapter, the general pth moment asymptotic stability of HSRSs (6.4) is studied with Razumikhim-type arguments. Theorems on asymptotic stability are established. Their applications to HSDSs (6.28) and (6.35) are also proposed. The Razumikhin-type theorems work for many HSRSs including some complicated cases to which the existing results do not apply. In a special case of the above results when $w(\mathbb{E}|x(t)|^p, t, r(t)) = \alpha(t)\mathbb{E}V(x(t), t, r(t))$ for all $t \geq 0$ with $\alpha(t) > 0$, using the techniques similar to the proof of Theorem 4.2 [79] (see also Theorem 8.9, p311, [87]), a Razumikhin-type theorem on generalized exponential stability of HSRSs (6.4) may be obtained.

Chapter 7

Input-to-state stability of stochastic retarded systems with Markovian switching

7.1 Introduction

Recently, hybrid stochastic retarded systems (HSRSs) have been widely used since stochastic modelling plays an important role in many branches of science and engineering. Consequently, stability analysis of HSRSs and HSDSs has been studied by many works, see, e.g., [81], [84], [86], [133] and [135]. Among the key results, [86] and Chapter 6 proposed the Razumikhin-type theorems on stability of hybrid stochastic retarded systems and their applications to hybrid stochastic delay systems. The Razumikhin method is developed to cope with the difficulty arisen from the large, fast varying and nondifferentiable time delays (see, e.g., [84] and [86]). Since the results for non-switched systems cannot be simply extended to systems with jumps and switching (see, e.g., [48] and [92]), Razumikhin-type Theorems for HSRSs and their applications are developed in [86] and Chapter 6. However, some conditions of results in [86] and Chapter 6 may be too conservative. Moreover, practical systems are often subject to disturbance input. This chapter is to improve the Razumikhin-type theorem proposed in Chapter 6 and make it more applicable.

7.2 Problem formulation

Throughout this chapter, unless otherwise specified, we shall employ the same notation as Chapter 6. Moreover, a function $\beta : R_+ \times R_+ \to R_+$ is said to be of class $\mathcal{KL}$ if for each fixed t the mapping $\beta(\cdot, t)$ is of class $\mathcal{K}$ and for each fixed s $\beta(s,t)$ is decreasing to zero on t as $t \to \infty$. We also let $\mathcal{L}^l_\infty$ denote the class of essentially bounded functions $u : R_+ \to R^l$ with $\|u\|_\infty = \operatorname{ess\,sup}_{t\geq 0} |u(t)| < \infty$. Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N\times N}$ given by (6.3).

Let us consider an n-dimensional HSRS

$$\mathrm{d}x(t) = f(x_t, t, r(t), u_d(t))\mathrm{d}t + g(x_t, t, r(t), u_d(t))\mathrm{d}W(t) \tag{7.1}$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ and $r(0) = r_0 \in S$, where $x_t = \{x(t+\theta) : -h \leq \theta \leq 0\}$ is regarded as a $C([-h,0]; R^n)$-valued random variable and $u_d \in \mathcal{L}^l_\infty$ the disturbance input. Moreover, $f : C([-h, 0]; R^n) \times R_+ \times S \times R^l \to R^n$ and $g : C([-h, 0]; R^n) \times R_+ \times S \times R^l \to R^{n\times m}$ are measurable functions with $f(0, t, i, 0) \equiv 0$ and $g(0, t, i, 0) \equiv 0$ for all $t \geq 0$. So equation (7.1) admits a trivial solution $x(t; 0) \equiv 0$. We assume that f and g are sufficiently smooth so that equation (7.1) has a unique solution on $t \geq -h$ (see, e.g., [63], [84], [133] and Appendix A), which is denoted by $x(t; x_0, r(0))$ or $x(t; \xi, r_0)$ in this chapter. It should be noted that equation (7.1) is a very general type of equation and includes stochastic differential equations, stochastic delay differential equations, integro-differential equations and those with Markovian switching. Much more equations are also included in equation (7.1) (see, e.g., [34] and [127]).

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times R_+ \times S$ that are twice continuously differentiable in x and once in t. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, define an operator $\mathcal{L}$ associated with system (7.1) from $C([-h, 0]; R^n) \times R_+ \times S$ to R by

$$\begin{aligned} \mathcal{L}V(x_t, t, i) &= V_t(x, t, i) + V_x(x, t, i)f(x_t, t, i, u_d) \\ &\quad + \frac{1}{2} trace \left[g^T(x_t, t, i, u_d)V_{xx}(x, t, i)g(x_t, t, i, u_d)\right] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(x, t, j), \end{aligned} \tag{7.2}$$

where $V_t(x,t,i)$, $V_x(x,t,i)$ and $V_{xx}(x,t,i)$ are partial derivatives defined by (6.5).

The purpose of this chapter is to develop the Razumihkin-type theorem on pth moment input-to-state stability (ISS) of HSRSs and its applications. For definitions of pth moment stability of stochastic systems and input-to-state stability of deterministic systems, readers are referred to, e.g., [25], [51], [64], [109], [115], [116] and [118]. Let us introduce the definition of pth moment ISS of HSRSs, which is consistent with the definition of ISS for deterministic systems (see, e.g., [51], [109], [115], [116] and [118]).

Definition 7.1 *The system (7.1) is said to be pth ($p > 0$) moment input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$ and $\gamma \in \mathcal{K}$ such that the solution $x(t) = x(t;\xi, r_0)$ satisfies*

$$\mathbb{E}|x(t)|^p \leq \beta(\mathbb{E}\|\xi\|^p, t) + \gamma(\|u_d\|_\infty) \quad \forall t \geq 0 \tag{7.3}$$

for any essentially bounded input $u_d \in \mathcal{L}^l_\infty$ and any initial data $\xi \in C^b_{\mathcal{F}_0}([-h,0];R^n)$, $r_0 \in S$.

Remark 7.1 *It is observed that, if $\|u_d\|_\infty = 0$, pth moment ISS of system (7.1) implies the pth moment asymptotic stability of the system (see Chapter 6).*

7.3 Razumikhin-type theorem on ISS of HSRSs

As the main result of this chapter, we present a Razumikhin-type theorem on pth moment ISS of HSRSs (7.1) as follows.

Theorem 7.1 *Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in \mathcal{K}_\infty$ and $\lambda \in \mathcal{K}$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that*

$$u(|x|^p) \leq V(x,t,i) \leq v(|x|^p), \quad \forall (x,t,i) \in R^n \times [-h,\infty) \times S \tag{7.4}$$

and, moreover,

$$\mathbb{E}\mathcal{L}V(\phi,t,i) \leq \lambda(|u_d(t)|) - \mathbb{E}w(\phi(0),i) \tag{7.5}$$

for all $(t,i) \in R_+ \times S$ and those $\phi \in L^p_{\mathcal{F}_t}([-h,0];R^n)$ satisfying

$$\min_{k\in S} \mathbb{E}V(\phi(\theta), t+\theta, k) < \mathbb{E}q(\phi(0),t,i) \tag{7.6}$$

on $-h \le \theta \le 0$, *where* $w : R^n \times S \to R_+$ *is a nonnegative function such that there is* $\bar{w} \in \mathcal{K}_\infty$ *with* $w(x,i) \ge \bar{w}(|x|)$ *and* $\lim_{|x|\to\infty} \frac{\bar{w}(|x|)}{v(|x|^p)} > 0$ *for all* $i \in S$*;* $q : R^n \times R_+ \times S \to R$ *is a function such that* $q(x,t,i) - V(x,t,i) \ge \zeta(|x|)$ *for all* $(x,t,i) \in R^n \times [-\tau,\infty) \times S$ *with* $\zeta \in \mathcal{K}_\infty$ *and* $\lim_{|x|\to\infty} \frac{\zeta(|x|)}{v(|x|^p)} > 0$*. Then system (7.1) is pth moment ISS.*

In order to prove this theorem, let us present the following useful lemmas

Lemma 7.1 *Let* $V(t) = V(x(t),t,r(t))$ *for* $t \ge 0$*, then* $\mathbb{E}V(t)$ *is continuous on* $t \ge 0$*.*

The proof is the same as that of Lemma 6.1 and hence omitted.

Lemma 7.2 *For any* $t \ge 0$*, there is* $a_w > 0$ *such that* $\mathbb{E}w(x,i) \ge a_w$ *for all* $i \in S$ *whenever* $\mathbb{E}V(x,t,i) \ge a_v > 0$*.*

Proof . It immediately follows the desired conclusion if we show there is $\mu_w \in \mathcal{K}_\infty$ such that

$$\mathbb{E}\bar{w}(|x(t)|) \ge \mu_w(a_v) \tag{7.7}$$

whenever $\mathbb{E}v(|x|^p) \ge \mathbb{E}V(x,t,i) \ge a_v > 0$.
Fix t for the moment. We define a nondecreasing function $b : R_+ \to R_+$ as

$$b(y) = \inf_{|x|^p \ge v^{-1}(y/2)} \frac{\bar{w}(|x|)}{v(|x|^p)}, \quad y \ge 0. \tag{7.8}$$

By property of function $\bar{w}(\cdot)$, $b(y) > 0$ when $y > 0$. So, for any $a_v > 0$, we have

$$\begin{aligned} \mathbb{E}\bar{w}(|x|) &\ge \int_{|x|^p \ge v^{-1}(\frac{a_v}{2})} \bar{w}(|x|)\mathrm{d}\mathbb{P} \\ &\ge b(a_v) \int_{v(|x|^p) \ge \frac{a_v}{2}} v(|x|^p)\mathrm{d}\mathbb{P} \ge \frac{a_v b(a_v)}{2} \end{aligned}$$

whenever $\mathbb{E}v(|x|^p) \ge \mathbb{E}V(x,t,i) \ge a_v$. Inequality (7.7) holds with $\mu_w(a_v) = \frac{1}{2}a_v b(a_v)$.

Lemma 7.3 *For any* $t \ge 0$*, there is* $a_q > 0$ *such that* $\mathbb{E}q(x,t,i) \ge a_q + \mathbb{E}V(x,t,i)$ *for all* $i \in S$ *whenever* $\mathbb{E}V(x,t,i) \ge a_v > 0$*.*

Proof It is noted that $\mathbb{E}q(x,t,i) - \mathbb{E}V(x,t,i) \ge \mathbb{E}\zeta(|x|)$ for all $t \ge 0$. According to the property of function $\zeta(|x|)$, the rest of the proof is similar to that of Lemma 7.2 and hence omitted.

We can now begin to prove Theorem 7.1.

Proof Denote $\alpha_\lambda = \lambda(\|u_d\|_\infty)$ and $\bar{V}_0 = u(\mathbb{E}\|\xi\|^p)$. Without loss of generality, assume $0 < \mu_w^{-1}(2\alpha_\lambda) < u(\sup_{-h\le\theta\le 0}\mathbb{E}|\xi(\theta)|^p) \le \bar{V}_0$. For any $t \ge 0$, by Lemma 7.2, $\mathbb{E}w(x(t),i) \ge 2\alpha_\lambda$ whenever $\mathbb{E}V(x,t,i) \ge \mu_w^{-1}(2\alpha_\lambda)$ for all $i \in S$. By Lemma 7.3, there is $a > 0$ such that $\mathbb{E}q(x,t,i) - \mathbb{E}V(x,t,i) \ge a$, $i \in S$, whenever $\mathbb{E}V(x,t,i) \ge \mu_w^{-1}(2\alpha_\lambda)$. Let J be the minimal nonnegative integer such that $M_0 = \mu_w^{-1}(2\alpha_\lambda) + Ja > \bar{V}_0$. Moreover, let $\tilde{\tau} = h \vee \frac{M_0}{\alpha_\lambda}$ and $t_j = j\tilde{\tau}$ for $j = 0,1,2,\cdots,J$. We claim that

$$\mathbb{E}V(x(t),t,r(t)) \le \bar{V}_0 \wedge M_j \tag{7.9}$$

for all $t \ge t_j$, where $M_j = \mu_w^{-1}(2\alpha_\lambda) + (J-j)a$ and $j = 0,1,2,\cdots,J$.

First we show that

$$\mathbb{E}V(x(t),t,r(t)) \le \bar{V}_0, \quad \forall\, t \ge t_0\,. \tag{7.10}$$

Suppose that $t_a = \inf\{t > t_0 : \mathbb{E}V(x(t),t,r(t)) > \bar{V}_0\} < \infty$. Since $\mathbb{E}V(x(t),t,r(t))$ is continuous on $t \ge 0$, there exist a pair of constants t_b and t_c such that $t_0 \le t_b \le t_a < t_c$ and

$$\begin{cases} \mathbb{E}V(x(t),t,r(t)) = \bar{V}_0, & t = t_b; \\ \bar{V}_0 < \mathbb{E}V(x(t),t,r(t)) < \bar{V}_0 + a, & t_b < t \le t_c. \end{cases} \tag{7.11}$$

However, by equation (6.12) and condition (7.5), we have

$$\begin{aligned} \mathbb{E}V(x(t),t,r(t)) &= \mathbb{E}V(x(t_b),t_b,r(t_b)) + \int_{t_b}^{t} \mathbb{E}\mathcal{L}V(x_s,s,r(s))\mathrm{d}s \\ &\le \bar{V}_0 - \alpha_\lambda(t-t_b) < \bar{V}_0 \end{aligned}$$

for every $t \in (t_b, t_c]$, which contradicts (7.11). So inequality (7.10) must be true.

We further show that $\mathbb{E}V(x(t),t,r(t)) \le M_1$ for all $t \ge t_1$. Let $\tau_1 = \inf\{t \ge t_0 : \mathbb{E}V(x(t),t,r(t)) \le M_1\}$. If $\tau_1 > t_1$, then, $\forall t_0 \le t \le t_1$, we have

$$\begin{aligned} \mathbb{E}q(x(t),t,r(t)) &\ge \mathbb{E}V(x(t),t,r(t)) + a > M_1 + a > \bar{V}_0 \\ &\ge \mathbb{E}V(x(t+\theta),t+\theta,r(t+\theta)) \\ &\ge \min_{k\in S}\mathbb{E}V(\phi(\theta),t+\theta,k), \quad \forall\,\theta \in [-h,0]. \end{aligned}$$

This, by condition (7.5), implies $\mathbb{E}\mathcal{L}V(x_t,t,r(t)) \le -\alpha_\lambda$ a.e. on $[t_0,t_1]$. Consequently, by (6.12), we have $\mathbb{E}V(x(t_1),t_1,r(t_1)) \le \bar{V}_0 - \alpha_\lambda\tilde{\tau} < 0$, which contradicts the nonnegative property of $\mathbb{E}V(x(t), t, r(t)) \geq 0$ for all $t \geq 0$. So we must have $\tau_1 \leq t_1$. Let $t_{1a} = \inf\{t > \tau_1 : \mathbb{E}V(x(t), t, r(t)) > M_1\}$. If $t_{1a} < \infty$, then there are constants t_{1b} and t_{1c} such that $t_1 \leq t_{1b} \leq t_{1a} < t_{1c}$ and

$$\begin{cases} \mathbb{E}V(x(t), t, r(t)) = M_1, & t = t_{1b}; \\ M_1 < \mathbb{E}V(x(t), t, r(t)) < M_1 + a, & t_{1b} < t \leq t_{1c}. \end{cases} \tag{7.12}$$

Similarly, by (6.12) and (7.5), we find a contradiction and hence have (7.9) for $j = 1$.

Define $\tau_j = \inf\{t \geq t_{j-1} : \mathbb{E}V(x(t), t, r(t)) \leq M_j\}$ for $j = 2, 3, \cdots, J$. By the same type of reasoning, we have $\mathbb{E}V(x(t), t, r(t)) \leq M_j$ for all $t \geq t_j$ and $j = 2, 3, \cdots, J$. Particularly, $\mathbb{E}V(x(t), t, r(t)) \leq M_J = \mu_w^{-1}(2\alpha_\lambda)$ for all $t \geq t_J$. By Jensen's inequality, we have

$$\mathbb{E}|x(t)|^p \leq \gamma(\|u_d\|_\infty), \quad \forall\, t \geq t_J \tag{7.13}$$

where $\gamma(\cdot) = u^{-1}(\mu_w^{-1}(2\lambda(\cdot)))$.

Let $k = \frac{\bar{V}_0}{t_J}$. Choose $\tilde{\beta} \in \mathcal{KL}$ such that $\tilde{\beta}(\bar{V}_0, t) \geq 2\bar{V}_0 - kt$ for all $0 \leq t \leq t_J$. So we have $\mathbb{E}V(x(t), t, r(t)) \leq \tilde{\beta}(\bar{V}_0, t)$ for all $0 \leq t \leq t_J$, which implies

$$\mathbb{E}|x(t)|^p \leq u^{-1}(\tilde{\beta}(\bar{V}_0, t)) = \beta(\mathbb{E}\|\xi\|^p, t), \quad \forall\, 0 \leq t \leq t_J \tag{7.14}$$

where $\beta(\cdot, \cdot) = u^{-1}(\tilde{\beta}(u(\cdot), \cdot))$ is a $\mathcal{KL}$ function. This completes the proof.

Remark 7.2 *Obviously, inequality (7.3) implies that system (7.1) with $u_d(t) \equiv 0$ is globally pth moment asymptotically stable (see Remark 7.1). Moreover, it is not difficult to show that if $|u(t)| \to 0$ as $t \to \infty$, so does $\mathbb{E}|x(t)|^p$ (see, e.g., Exercise 4.58, [51]). Therefore, by Theorem 7.1, it is easy to find that the HSDS, considered in Example 2.1 [133] but with mode-dependent and time-varying delay $\tau : R_+ \times S \to [0, h]$, is mean-quare asymptotically stable while the results in [133] do not work.*

Remark 7.3 *It is noted that, compared with Theorem 6.2, Theorem 7.1 has an additional term with respect to the disturbance input in the condition (7.5) but, in the inequality (7.6), removes the maximum operator on the right-hand side of corresponding conditions in the existing results (see Theorem 2.1, [86] and Theorem 6.2, Chapter 6), which makes Theorem 7.1 less conservative but more applicable (see Example 7.1).*

7.4 Application and Example

Hybrid stochastic delay systems (HSDSs) described with stochastic differential delay equations with Markovian switching are an important class of HSRSs that are frequently used in engineering. As an illustrative example of applications of our new result, we consider the following HSDE

$$\begin{aligned} \mathrm{d}x(t) &= F(x(t), x(t-\tau(t,r(t))), t, r(t), u_d(t))\mathrm{d}t \\ &\quad + G(x(t), x(t-\tau(t,r(t))), t, r(t), u_d(t))\mathrm{d}W(t) \end{aligned} \tag{7.15}$$

on $t \geq 0$, where $\tau : R_+ \times S \to [0,h]$ is Borel measurable while $F : R^n \times R^n \times R_+ \times S \times R^l \to R^n$ and $G : R^n \times R^n \times R_+ \times S \times R^l \to R^{n\times m}$ are measurable functions with $F(0,0,t,i,0) \equiv 0$ and $g(0,0,t,i,0) \equiv 0$ for all $t \geq 0$ and $i \in S$. Actually, this is a special case of equation (7.1) when $f(\phi,t,i,u_d) = F(\phi(0), \phi(-\tau(t,i)), t, i, u_d)$ and $g(\phi,t,i,u_d) = G(\phi(0), \phi(-\tau(t,i)), t, i, u_d)$ for $(\phi,t,i) \in C([-h,0];R^n) \times R_+ \times S \times R^l$ while the operator $\mathcal{L}$ defined in (7.2) becomes from $R^n \times R^n \times R_+ \times S$ to R as

$$\begin{aligned} \mathcal{L}V(x,y,t,i) &= V_t(x,t,i) + V_x(x,t,i)F(x,y,t,i,u_d) \\ &\quad + \frac{1}{2}trace\left[G^T(x,y,t,i,u_d)V_{xx}(x,t,i)G(x,y,t,i,u_d)\right] \\ &\quad + \sum_{j=1}^{N} \gamma_{ij} V(x,t,j). \end{aligned} \tag{7.16}$$

Assume that both $F(x,y,t,i)$ and $G(x,y,t,i)$ satisfy the local Lipschitz condition. That is, for each $c > 0$, there is a $K_c > 0$ such that

$$|F(x_1,y_1,t,i) - F(x_2,y_2,t,i)| + |G(x_1,y_1,t,i) - G(x_2,y_2,t,i)| \leq K_c(|x_1-x_2| + |y_1-y_2|) \tag{7.17}$$

for all $(t,i) \in R_+ \times S$ and $x_1, y_1, x_2, y_2 \in R^n$ with $\max\{|x_1|, |y_1|, |x_2|, |y_2|\} \leq c$. Let us use Theorem 7.1 to establish a useful criterion for system (7.15).

Theorem 7.2 *Let $p > 0$, $u \in V\mathcal{K}_\infty$, $v \in \mathcal{K}_\infty$, $\lambda \in \mathcal{K}$ and $\kappa_{0i} \geq \kappa_{1i} \geq 0$, $i \in S$. Assume that there exists a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ such that inequality (7.4) holds and, moreover,*

$$\mathcal{L}V(x,y,t,i) \leq \lambda(|u_d(t)|) - \hat{\zeta}(x,i) - \kappa_{0i}V(x,t,i) + \kappa_{1i}\min_{k\in S} V(y, t-\tau(t,i), k) \tag{7.18}$$

for all $(x,y,t,i) \in R^n \times R^n \times R_+ \times S$, where $\hat{\zeta} : R^n \times S \to R$ is a function such that there is $\hat{w} \in \mathcal{K}$ with $\hat{\zeta}(x,i) \geq \hat{w}(|x|)$ for all $i \in S$ and $\lim_{|x|\to\infty} \hat{w}(|x|)/v(|x|^p) > 0$. Then system (7.15) is pth moment ISS.

Proof. According to Theorem A.1 (see Appendix A), equation (7.15) has a unique solution. For any $i \in S$, let

$$w(x,i) = \frac{1}{1+\kappa_{0i}}\hat{\zeta}(x,i) \text{ and } q(x,t,i) = V(x,t,i) + w(x,i) \tag{7.19}$$

in inequalities (7.5) and (7.6). By inequality (7.18) and Fatou's lemma, we have

$$\begin{aligned}
&\mathbb{E}\mathcal{L}V(x,y,t,i) \\
&\quad \leq \lambda(|u_d(t)|) - \mathbb{E}\hat{\zeta}(x,i) - \kappa_{0i}\mathbb{E}V(x,t,i) + \kappa_{1i}\,\mathbb{E}\left[\min_{k\in S} V(y,t-\tau(t,i),k)\right] \\
&\quad \leq \lambda(|u_d(t)|) - \kappa_{0i}\big[\mathbb{E}V(x,t,i) + \mathbb{E}w(x,i)\big] + \kappa_{1i}\min_{k\in S}\mathbb{E}V(y,t-\delta(t,i),k) - \mathbb{E}w(x,i) \\
&\quad \leq \lambda(|u_d(t)|) - \mathbb{E}w(x,i) - (\kappa_{0i}-\kappa_{1i})\big[\mathbb{E}V(x,t,i) + \mathbb{E}w(x,i)\big] \\
&\quad \leq \lambda(|u_d(t)|) - \mathbb{E}w(x,i)
\end{aligned}$$

for all $t \geq 0$, $i \in S$ and $x_t \in L^p_{\mathcal{F}_t}([-h,0];R^n)$ satisfying condition (7.6) with function $q(x,t,i)$ defined in (7.19), i.e., $\min_{k\in S}\mathbb{E}V(y,t-\delta(t,i),k) < \mathbb{E}V(x,t,i)+\mathbb{E}w(x,i)$. Moreover, $\bar{w}(\cdot) = \zeta(\cdot) = \frac{1}{1+\kappa}\hat{w}(\cdot)$ satisfy the properties required in (7.5) and (7.6). By Theorem 7.1, inequality (7.3) holds for system (7.15).

Remark 7.4 *As in Chapter 6, Theorem 7.2 can be easily generalised to cope with the case with multiple delays.*

To compare with the Theorem 6.4 in Chapter 6, let us consider the following example.

Example 7.1 Let $W(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markovian chain independent of $W(t)$ and taking values in $S = \{1,2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ 2 & -2 \end{pmatrix}.$$

Consider a scalar uncertain stochastic delay system with Markovian switching of the form

$$\mathrm{d}x(t) = f(x(t),t,r(t))\mathrm{d}t + g(x(t-\tau(t,r(t))),t,r(t))\mathrm{d}W(t) \tag{7.20}$$

on $t \geq 0$, where $\tau : R_+ \times S \to [-h, 0]$ is a continuous but non-differentiable function with respect to t and

$$f(x,t,1) = \frac{1}{4}x - \frac{1}{8}|x|\sqrt[3]{x}, \quad f(x,t,2) = -bx - \frac{1}{10}x^3,$$
$$g(y,t,1) = \frac{1}{4}y\cos t, \quad g(y,t,2) = \sqrt{2}y\sin t.$$

with $x = x(t)$, $y = x(t - \tau(t, r(t)))$ and positive constant b.

It is noted that the existing results ([84], [86], [133], [135]) can not be applied to system (7.20), which has mode-dependent and time-varying delay $\tau(t, r(t))$. Observe that

$$\begin{cases} 2xf(x,t,1) \leq \frac{1}{2}x^2 - \frac{1}{4}|x|^{\frac{7}{3}}, \\ g^2(y,t,1) \leq \frac{1}{16}y^2; \end{cases}$$

and

$$\begin{cases} 2xf(x,t,2) \leq -2bx^2 - \frac{1}{5}x^4, \\ g^2(y,t,2) \leq 2y^2. \end{cases}$$

To examine the stability of system (7.20), we construct a Lyapunov function candidate $V : R \times S \to R_+$ as $V(x,i) = \alpha_i x^2$ with $\alpha_2 = 1$ and $\alpha_1 > 0$ to be determined. By computation, we have

$$\mathcal{L}V(x,y,t,1) \leq -\frac{\alpha_1}{4}|x|^{\frac{7}{3}} - (\frac{\alpha_1}{2} - 1)x^2 + \frac{\alpha_1}{16}y^2, \tag{7.21}$$
$$\mathcal{L}V(x,y,t,2) \leq -\frac{1}{5}x^4 - (2 + 2b - 2\alpha_1)x^2 + 2y^2. \tag{7.22}$$

According to Theorem 6.4, inequalities (7.21) and (7.22) give

$$\lambda_{01} = \frac{1}{2} - \frac{1}{\alpha_1}, \quad \lambda_{11} = \frac{\alpha_1}{16}, \quad \lambda(s,1) = \frac{1}{4\sqrt[7]{\alpha_1}}s^{\frac{7}{6}};$$
$$\lambda_{02} = \frac{2(1+b)}{\alpha_1} - 2, \quad \lambda_{12} = 2, \quad \lambda(s,2) = \frac{1}{5\alpha_1^2}s^2.$$

Inequalities $\lambda_{01} \geq \lambda_{11}$ and $\lambda_{02} \geq \lambda_{12}$ yield $\alpha_1 = 4$ and $b \geq 7$. Then, by Theorem 6.4, system (7.20) is mean-square asymptotically stable if $b \geq 7$. However, for inequalities (7.21) and (7.22), we have

$$\kappa_{01} = \frac{1}{2} - \frac{1}{\alpha_1}, \quad \kappa_{11} = \frac{\alpha_1}{16}, \quad \hat{\zeta}(x,1) = \frac{\alpha_1}{4}|x|^{\frac{7}{3}};$$
$$\kappa_{02} = 2(1 + b - \alpha_1), \quad \kappa_{12} = 2, \quad \hat{\zeta}(x,2) = \frac{1}{5}x^4.$$

Inequalities $\kappa_{01} \geq \kappa_{11}$ and $\kappa_{02} \geq \kappa_{12}$ imply $\alpha_1 = 4$ and $b \geq 4$. By Theorem 7.2, the sufficient condition for mean-square asymptotical stability of system (7.20) is $b \geq 4$.

Note that, when $4 \leq b < 7$, Theorem 6.4 does not work while Theorem 7.2 is still applicable to system (7.20). This shows Theorem 7.2 is more applicable.

7.5 Summary

This chapter improves the existing result in Chapter 6 and develops a Razumikhin-type theorem on input-to-state stability of HSRSs in pth ($p > 0$) moment sense. It is seen that this improved result is less conservative but more applicable (see Remark 7.1, Remark 7.2 and Example 7.1).

Chapter 8

Almost sure stability of hybrid stochastic systems with mode-dependent interval delays

8.1 Introduction

Recently, hybrid stochastic delay systems (HSDSs) have received considerable attention (see, e.g., [90], [111] and [132]). The presence of the Markovian switching is quite involved in stability analysis of the hybrid systems (see, e.g., [6], [30], [48], [92] and [89]). Even if all the subsystems are stable, the hybrid system may not be stable; on the other hand, the hybrid system may be stable even if all the subsystems are unstable (see, e.g., [6], [30] and [92]).

The classical stochastic analysis theory studies stability not only in moment sense but also in almost sure sense (see, e.g., [36], [84], [133], [135] and the references therein). Among the existing results, [133] studied almost sure stability of HSDSs with the techniques proposed in [84] while most of the others dealt with moment stability. However, the results in [133] require the time delays of all subsystems to be equal to a constant. This may be too restrictive to apply to hybrid systems in many situations. This chapter studies almost sure stability of hybrid stochastic systems (HSSs) with mode-dependent interval time delays by extending the results in [133] to hybrid stochastic systems (HSSs) with mode-dependent interval delays, which reveals an important role that the Markovian jumps play in the stability analysis of hybrid stochastic delay systems. It is found that the upper bounds of derivatives of time delays of some subsystems may be equal to or larger than one, which are required to be less than one in the case without jumps (see Example 8.2). This shows that the presence of Markovian switching is quite involved in stability analysis of the delay systems.

8.2 System description

Throughout this chapter, unless otherwise specified, we shall employ the same notation as Chapter 6. If A is a subset of R^n and $x \in R^n$, denote by $d(x, A) = \inf_{a \in A} |x - a|$ the distance from x to A. We also denote by $L^1(R_+; R_+)$ the family of functions $\lambda : R_+ \to R_+$ such that $\int_0^\infty \lambda(t)\mathrm{d}t < \infty$. Let $r(t), t \geq 0$, be a right-continuous Markov chain on the probability space taking values in a finite state space $S = \{1, 2, \cdots, N\}$ with generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by (6.3).

Let us consider an n-dimensional HSSs with mode-dependent interval time delays

$$\mathrm{d}x(t) = f(x(t), x(t - \tau(t, r(t))), t, r(t))\mathrm{d}t + g(x(t), x(t - \tau(t, r(t))), t, r(t))\mathrm{d}W(t) \quad (8.1)$$

on $t \geq 0$ with initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ and $r(0) = r_0 \in S$, where $f : R^n \times R^n \times R_+ \times S \to R^n$ and $g : R^n \times R^n \times R_+ \times S \to R^{n \times m}$ satisfy the local Lipschitz condition in (x, y), that is, for any $K > 0$, there is $L_K > 0$ such that

$$|f(x, y, t, i) - f(\bar{x}, \bar{y}, t, i)| \vee |g(x, y, t, i) - g(\bar{x}, \bar{y}, t, i)| \leq L_K(|x - \bar{x}| + |y - \bar{y}|) \quad (8.2)$$

for all $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq K$, $t \geq 0$ and $i \in S$, and moreover, $\sup_{t \geq 0, i \in S}\{|f(0, 0, t, i)| \vee |g(0, 0, t, i)| : t \geq 0, i \in S\} \leq K_0$ with some nonnegative number K_0 (see [133]); time delay of the system $\tau : R_+ \times S \to R_+$, also written as τ or $\tau(t)$ where there is no ambiguity, is differentiable in t for all $i \in S$ and there are a pair of nonnegative numbers l and h such that $l \leq \tau(t, i) \leq h$ for all $t \geq 0$ and $i \in S$.

Let $C^{2,1}(R^n \times R_+ \times S; R_+)$ denote the family of all nonnegative functions $V(x, t, i)$ on $R^n \times R_+ \times S$ that are twice continuously differentiable in x and once in t. If $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, define an operator $\mathcal{L}$ associated with (8.1) from $R^n \times R^n \times R_+ \times S$

to R by

$$\mathcal{L}V(x,y,t,i) = V_t(x,t,i) + V_x(x,t,i)f(x,y,t,i) + \frac{1}{2}trace\left[g^T(x,y,t,i)V_{xx}(x,t,i)g(x,y,t,i)\right] + \sum_{j=1}^{N}\gamma_{ij}V(x,t,j), \quad (8.3)$$

where where $V_t(x,t,i)$, $V_x(x,t,i)$ and $V_{xx}(x,t,i)$ are partial derivatives defined by (6.5).

The purpose of this chapter is to propose a criterion for almost sure stability of the HSS with mode-dependent delays (8.1). For definition of almost sure stability, please refer to Section 2.6 in Chapter 2.

8.3 Almost sure stability of HSSs with mode-dependent interval delays

As the main result of this chapter, we present a criterion for almost sure stability of HSDSs (8.1) as follows.

Theorem 8.1 *Suppose that there are nonnegative numbers l_i, h_i, δ_i and $\bar{\delta}$ such that*

$$l_i \le \tau(t,i) \le h_i, \quad \tau_t(t,i) = \frac{\partial \tau(t,i)}{\partial t} \le \delta_i, \quad \bar{\delta}_i = \delta_i + \gamma_{ii}l_i + \sum_{j\neq i}\gamma_{ij}h_j \le \bar{\delta} < 1 \quad (8.4)$$

for all $t \ge 0$ and $i \in S$ with $l = \min_{i\in S} l_i$ and $h = \max_{i\in S} h_i$. Assume that there exist nonnegative functions $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, $\lambda \in L^1(R_+; R_+)$, and $w_1, w_2 \in C(R^n; R_+)$ such that

$$\mathcal{L}V(x,y,t,i) \le \lambda(t) - k_1 w_1(x) + k_2 w_2(y), \quad \forall\, (x,y,t,i) \in R^n \times R^n \times R_+ \times S \quad (8.5)$$

$$w_1(x) > w_2(x), \quad \forall\, x \neq 0 \quad (8.6)$$

and

$$\lim_{|x|\to\infty} \inf_{t\ge 0, i\in S} V(x,t,i) = \infty\,, \quad (8.7)$$

where k_1 and k_2 are positive numbers such that $k_1 \ge k_2/(1-\bar{\delta})$. Then the solution of HSDS (8.1)

$$\lim_{t\to\infty} x(t;\xi,r_0) = 0 \quad a.s..$$

In order to prove this theorem, let us present the following useful lemmas

Lemma 8.1 *(p105 [114]) For $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, the generalized Itô's formula is given as*

$$\mathrm{d}V(x(t),t,r(t)) = \mathcal{L}V(x(t),x(t-\tau),t,r(t))\mathrm{d}t + V_x(x(t),t,r(t))g(x(t),x(t-\tau),r(t))\mathrm{d}W(t)$$
$$+ \int_R [V(x(t),t,r(t)+h(r(t),\alpha)) - V(x(t),t,r(t))]\mu(\mathrm{d}t,\mathrm{d}\alpha)$$

for all $t \geq 0$, where function $h(\cdot,\cdot)$ and martingale measure $\mu(\cdot,\cdot)$ are defined as, e.g., (2.18) and (2.23) (see also [30], [6] and [133]).

Lemma 8.2 *The following inequality holds for $t \geq 0$*

$$(1-\bar{\delta})\int_0^t w_2(x(s-\tau(s,r(s))))\mathrm{d}s \leq \int_{-h}^t w_2(x(s))\mathrm{d}s$$
$$+ \int_0^t \int_R w_2(x(s-\tau(s,r(s))))[\tau(s,r_0+h(r(s),\alpha)) - \tau(s,r(s))]\mu(\mathrm{d}s,\mathrm{d}\alpha)\,.$$

Proof. Let $s = u + \tau(s,r(s))$ and $r(s) = i \in S$. Then

$$\mathrm{d}s = \mathrm{d}u + \mathrm{d}\tau(s,i)\,. \tag{8.8}$$

By Lemma 8.1 and inequalities (8.4), we have

$$\begin{aligned}
\mathrm{d}\tau(s,i) &= \Big[\tau_s(s,i) + \sum_{i=1}^N \gamma_{ij}\tau(s,j)\Big]\mathrm{d}s + \int_R [\tau(s,i+h(i,\alpha)) - \tau(s,i)]\mu(\mathrm{d}s,\mathrm{d}\alpha) \\
&\leq \Big[\delta_i + \gamma_{ii}l_i + \sum_{j\neq i}\gamma_{ij}h_j\Big]\mathrm{d}s + \int_R [\tau(s,i+h(i,\alpha)) - \tau(s,i)]\mu(\mathrm{d}s,\mathrm{d}\alpha) \\
&= \bar{\delta}_i\mathrm{d}s + \int_R [\tau(s,i+h(i,\alpha)) - \tau(s,i)]\mu(\mathrm{d}s,\mathrm{d}\alpha) \\
&\leq \bar{\delta}\mathrm{d}s + \int_R [\tau(s,i+h(i,\alpha)) - \tau(s,i)]\mu(\mathrm{d}s,\mathrm{d}\alpha)\,.
\end{aligned} \tag{8.9}$$

Substitution of (8.9) into (8.8) yields

$$(1-\bar{\delta})\mathrm{d}s \leq \mathrm{d}u + \int_R [\tau(s,i+h(i,\alpha)) - \tau(s,i)]\mu(\mathrm{d}s,\mathrm{d}\alpha)\,. \tag{8.10}$$

This implies

$$\begin{aligned}
&(1-\bar{\delta})\int_0^t w_2(x(s-\tau(s,r(s))))\mathrm{d}s \\
&\quad\leq \int_{-\tau(0,r_0)}^{t-\tau(t,r(t))} w_2(x(u))\mathrm{d}u \\
&\qquad + \int_0^t w_2(x(s-\tau(s,r(s)))) \int_R [\tau(s,r_0+h(r(s),\alpha)) - \tau(s,r(s))]\mu(\mathrm{d}s,\mathrm{d}\alpha) \\
&\quad\leq \int_{-h}^t w_2(x(s))\mathrm{d}s \\
&\qquad + \int_0^t \int_R w_2(x(s-\tau(s,r(s))))[\tau(s,r_0+h(r(s),\alpha)) - \tau(s,r(s))]\mu(\mathrm{d}s,\mathrm{d}\alpha)
\end{aligned}$$

for all $t \geq 0$.

Lemma 8.3 *For any initial data $x_0 = \{x(\theta) : -h \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ and $r(0) = r_0 \in S$, equation (8.1) has a unique global solution.*

This lemma is proved with the standard truncated technique in Appendix B, which is similar to the proof of Lemma 2.1 in [133].

We can now begin to prove Theorem 8.1.

Proof. Let $\eta(x) = w_1(x) - w_2(x)$ for all $x \in R^n$. Inequality (8.6) implies $\eta(x) > 0$ whenever $x \neq 0$. We decompose the sample space into three mutually exclusive events as follows

$$\begin{aligned}
E_1 &= \left\{\omega : \limsup_{t\to\infty} \eta(x(t)) \geq \liminf_{t\to\infty} \eta(x(t)) > 0\right\}, \\
E_2 &= \left\{\omega : \limsup_{t\to\infty} \eta(x(t)) > 0 \text{ and } \liminf_{t\to\infty} \eta(x(t)) = 0\right\}, \\
E_3 &= \left\{\omega : \lim_{t\to\infty} \eta(x(t)) = 0\right\}.
\end{aligned}$$

Obviously, it follows the desired result $\mathbb{P}(E_3) = 1$ if we show $\mathbb{P}(E_1) = \mathbb{P}(E_2) = 0$. By inequality (8.5), Lemma 8.1 and Lemma 8.2, we have

$$\begin{aligned}
&V(x(t), t, r(t)) \\
&\quad = V(x(0), 0, r_0) + \int_0^t \mathcal{L}V(x(s), x(t - \tau(s, r(s))), s, r(s))\mathrm{d}s \\
&\qquad + \int_0^t V_x(x(s), s, r(s))g(x(s), x(s - \tau(s, r(s))), s, r(s))\mathrm{d}W(s) \\
&\qquad + \int_0^t \int_R \big[V(x(s), s, r_0 + h(r(s), \alpha)) - V(x(s), s, r(s))\big]\mu(\mathrm{d}s, \mathrm{d}\alpha) \\
&\quad \leq V(x(0), 0, r_0) + \int_0^t \lambda(s)\mathrm{d}s - k_1 \int_0^t w_1(x(s))\mathrm{d}s + k_2 \int_0^t w_2(x(s - \tau(s, r(s))))\mathrm{d}s \\
&\qquad + \int_0^t V_x(x(s), s, r(s))g(x(s), x(s - \tau(s, r(s))), s, r(s))\mathrm{d}W(s) \\
&\qquad + \int_0^t \int_R \big[V(x(s), s, r_0 + h(r(s), \alpha)) - V(x(s), s, r(s))\big]\mu(\mathrm{d}s, \mathrm{d}\alpha) \\
&\quad \leq V(x(0), 0, r_0) + \int_0^t \lambda(s)\mathrm{d}s + k_1 \int_{-h}^0 w_2(x(s))\mathrm{d}s - k_1 \int_0^t \eta(x(s))\mathrm{d}s \\
&\qquad + \int_0^t V_x(x(s), s, r(s))g(x(s), x(s - \tau(s, r(s))), s, r(s))\mathrm{d}W(s) \\
&\qquad + \int_0^t \int_R \Big\{V(x(s), s, r_0 + h(r(s), \alpha)) - V(x(s), s, r(s)) + k_1 w_2(x(s - \tau(s, r(s)))) \\
&\qquad\qquad \cdot \big[\tau(s, r_0 + h(r(s), \alpha)) - \tau(s, r(s))\big]\Big\}\mu(\mathrm{d}s, \mathrm{d}\alpha). \qquad (8.11)
\end{aligned}$$

But according to Lemma 2.4, this implies

$$\lim_{t\to\infty}\int_0^t \eta(x(s))\mathrm{d}s = \int_0^\infty \eta(x(s))\mathrm{d}s < \infty \quad \text{and} \quad \limsup_{t\to\infty} V(x(t),t,r(t)) < \infty \tag{8.12}$$

hold almost surely. It immediately follows that $\mathbb{P}(E_1) = 0$ and

$$\sup_{-h\le t<\infty} V(x(t),t,r(t)) < \infty \quad a.s..$$

Define $\beta : R_+ \to R_+$ by

$$\beta(r) = \inf_{|x|\ge r, t\ge 0, i\in S} V(x,t,i). \tag{8.13}$$

Then inequality

$$\sup_{0\le t<\infty} \beta(|x(t)|) \le \sup_{0\le t<\infty} V(x(t),t,r(t)) < \infty \quad a.s.$$

and condition (8.7) imply that

$$\sup_{0\le t<\infty} |x(t)| < \infty \quad a.s..$$

Since initial data $\xi \in C^b_{\mathcal{F}_0}([-h,0];R^n)$, we can find an integer k_0 such that $\|\xi\| < k_0$ $a.s.$. For any integer $k \ge k_0$, define the stopping time

$$\rho_k = \inf\{t \ge 0 : |x(t)| \ge k\}\,, \tag{8.14}$$

where we set $\inf \emptyset = \infty$ as usual. Clearly, $\rho_k \to \infty$ almost surely as $k \to \infty$. Moreover, for any given $\varepsilon > 0$, there is $k_\varepsilon \ge k_0$ such that $\mathbb{P}\{\rho_k < \infty\} \le \varepsilon$ for any $k \ge k_\varepsilon$.

Now we proceed to prove $\mathbb{P}(E_2) = 0$ by contradiction. Suppose that $\mathbb{P}(E_2) > 0$. There exist $\varepsilon_0 > 0$ and $\varepsilon_1 > 0$ such that

$$\mathbb{P}\{\omega : \text{there are infinitely many } j \text{ such that } \sigma_j < \infty\} = \mathbb{P}(\sigma_j < \infty : j \in Z) \ge \varepsilon_0\,, \tag{8.15}$$

where $\{\sigma_j\}_{j\ge 1}$ are a sequence of stopping times defined as

$$\begin{aligned}
\sigma_1 &= \inf\{t \ge 0 : \eta(x(t)) \ge 2\varepsilon_1\}\,,\\
\sigma_{2j} &= \inf\{t \ge \sigma_{2j-1} : \eta(x(t)) \le \varepsilon_1\}\,,\\
\sigma_{2j+1} &= \inf\{t \ge \sigma_{2j} : \eta(x(t)) \ge 2\varepsilon_1\}\,, \quad j = 1,2,3,\cdots;
\end{aligned}$$

and Z is a set of natural numbers that includes infinitely many elements. Since $x(t)$ and hence $\eta(x(t))$ are continuous on $t \ge 0$, we see that $\sigma_j \to \infty$ a.s. as $j \to \infty$. By local Lipschitz condition (8.2), for any given $k > 0$, there exists $K_k > 0$ such that

$$|f(x,y,t,i)| \vee |g(x,y,t,i)| \le K_k \tag{8.16}$$

for all $|x| \vee |y| \le k$, $t \ge 0$ and $i \in S$. Let $\tau_k^j(t) = (\sigma_j + t) \wedge \rho_k$ for $t \ge 0$ and $\mathbb{I}_A$ be the indicator of set A. For any $j \in Z$, by Hölder's inequality and Doob's martingale inequality, we compute

$$\begin{aligned}
&\mathbb{E}\left\{\mathbb{I}_{\{\sigma_j<\rho_k\}} \sup_{0\le t\le T} |x(\tau_k^j(t)) - x(\sigma_j)|^2\right\} \\
&\quad = \mathbb{E}\left\{\mathbb{I}_{\{\sigma_j<\rho_k\}} \sup_{0\le t\le T} \left| \int_{\sigma_j}^{\tau_k^j(t)} f(x(s), x(s-\tau), s, r(s))\mathrm{d}s \right.\right. \\
&\qquad\qquad\qquad \left.\left. + \int_{\sigma_j}^{\tau_k^j(t)} g(x(s), x(s-\tau), s, r(s))\mathrm{d}W(s) \right|^2 \right\} \\
&\quad \le 2\mathbb{E}\left\{\mathbb{I}_{\{\sigma_j<\rho_k\}} \sup_{0\le t\le T} \left| \int_{\sigma_j}^{\tau_k^j(t)} f(x(s), x(s-\tau), s, r(s))\mathrm{d}s \right|^2 \right\} \\
&\qquad\quad + 2\mathbb{E}\left\{\mathbb{I}_{\{\sigma_j<\rho_k\}} \sup_{0\le t\le T} \left| \int_{\sigma_j}^{\tau_k^j(t)} g(x(s), x(s-\tau), s, r(s))\mathrm{d}W(s) \right|^2 \right\} \\
&\qquad \le 2\mathbb{E}\left\{\mathbb{I}_{\{\sigma_j<\rho_k\}} \int_{\sigma_j}^{\tau_k^j(T)} |f(x(s), x(s-\tau), s, r(s))|^2 \,\mathrm{d}s\right\} \\
&\qquad\qquad + 8\mathbb{E}\left\{\mathbb{I}_{\{\sigma_j<\rho_k\}} \int_{\sigma_j}^{\tau_k^j(T)} |g(x(s), x(s-\tau), s, r(s))|^2 \,\mathrm{d}s\right\} \\
&\qquad \le 2K_k^2 T(T+4)\,,
\end{aligned} \tag{8.17}$$

where T is some positive constant. Since $\eta(\cdot)$ is continuous in R^n, it must be uniformly continuous in the closed ball $\bar{S}_k = \{x \in R^n : |x| \le k\}$. For any given $b > 0$, we can choose $c_b > 0$ such that $|\eta(x) - \eta(y)| < b$ whenever $x, y \in \bar{S}_k$ and $|x - y| < c_b$. Let us choose

$$\varepsilon = \frac{\varepsilon_0}{3}, \quad k \ge k_\varepsilon \quad \text{and} \quad b = \varepsilon_1.$$

By inequality (8.17) and Chebyshev's inequality, we have

$$\begin{aligned}
&\mathbb{P}\Big\{\omega : \sigma_j < \rho_k \text{ and } \sup_{0\le t\le T} |\eta(x(\sigma_j + t)) - \eta(x(\sigma_j))| \ge \varepsilon_1\Big\} + \mathbb{P}\Big\{\omega : \rho_k \le \sigma_j\Big\} \\
&\quad \le \mathbb{P}\Big\{\omega : \sigma_j + T < \rho_k \text{ and } \sup_{0\le t\le T} |\eta(x(\sigma_j + t)) - \eta(x(\sigma_j))| \ge \varepsilon_1\Big\} \\
&\qquad + \mathbb{P}\Big\{\omega : \sigma_j < \rho_k \le \sigma_j + T\Big\} + \mathbb{P}\Big\{\omega : \rho_k \le \sigma_j\Big\} \\
&\quad \le \mathbb{P}\Big\{\omega : \sigma_j + T < \rho_k \text{ and } \sup_{0\le t\le T} |x(\sigma_j + t) - x(\sigma_j)| \ge c_{\varepsilon_1}\Big\} + \mathbb{P}\Big\{\omega : \rho_k \le \sigma_j + T\Big\} \\
&\quad \le \frac{2K_k^2 T(T+4)}{c_{\varepsilon_1}^2} + (1 - 2\varepsilon)\,.
\end{aligned} \tag{8.18}$$

We furthermore choose $T = T(\varepsilon, \varepsilon_1, k) > 0$ sufficiently small for

$$\frac{2K_k^2 T(T+4)}{c_{\varepsilon_1}^2} \le \varepsilon\,. \tag{8.19}$$

Inequalities (8.18) and (8.19) yield

$$\mathbb{P}\Big\{\omega : \sigma_j < \rho_k \text{ and } \sup_{0\le t\le T} |\eta(x(\sigma_j + t)) - \eta(x(\sigma_j))| < \varepsilon_1\Big\} \ge \varepsilon\,. \tag{8.20}$$

According to (8.15), $j-1 \in Z$ whenever $j \in Z$ and $j \ge 2$, which implies there are infinitely many even numbers in Z. By inequalities (8.11), (8.12) and (8.20), we have

$$\begin{aligned}
\infty &> \mathbb{E}\int_0^\infty \eta(x(t))\mathrm{d}t \ge \sum_{2j\in Z} \mathbb{E}\Big[\mathbb{I}_{\{\sigma_{2j-1}<\rho_k\}} \int_{\sigma_{2j-1}}^{\sigma_{2j}} \eta(x(t))\mathrm{d}t\Big] \\
&\ge \sum_{2j\in Z} \varepsilon_1 \mathbb{E}\Big[\mathbb{I}_{\{\sigma_{2j-1}<\rho_k\}}(\sigma_{2j} - \sigma_{2j-1})\Big] \\
&\ge \sum_{2j\in Z} T\varepsilon_1\, \mathbb{P}\Big\{\omega : \sigma_{2j-1} < \rho_k \text{ and } \sup_{0\le t\le T} |\eta(x(\sigma_{2j-1}+t)) - \eta(x(\sigma_{2j-1}))| < \varepsilon_1\Big\} \\
&\ge \sum_{2j\in Z} T\varepsilon_1\varepsilon = \frac{1}{3}\sum_{2j\in Z} T\varepsilon_0\varepsilon_1 = \infty\,,
\end{aligned} \tag{8.21}$$

which is a contradiction. So we must have $\mathbb{P}(E_2) = 0$ and hence $\mathbb{P}(E_3) = 1$. This implies there is an $\Omega_0 \subset \Omega$ with $\mathbb{P}(\Omega_0) = 1$ such that

$$\lim_{t\to\infty} \eta(x(t,\omega)) = 0 \quad \text{and} \quad \sup_{0\le t<\infty} |x(t,\omega)| < \infty\,, \quad \forall\, \omega \in \Omega_0\,.$$

Fix any $\omega \in \Omega_0$. Then $\{x(t,\omega)\}_{t\ge 0}$ is bounded in R^n. By Bolzano-Weierstrass theorem, there is an increasing sequence $\{t_i\}_{i\ge 1}$ such that $\{x(t,\omega)\}_{i\ge 1}$ converges to some $y \in R^n$ with $|y| < \infty$. Since $\eta(x) > 0$ whenever $x \ne 0$, we must have $\eta(x) = 0$ if and only if $x = 0$. Then $\mathbb{P}(E_3) = 1$ implies the solution

$$\lim_{t\to\infty} x(t;\xi, r_0) = 0 \quad a.s.\,.$$

This completes the proof.

Similarly, Theorem 2.2 in [133] can be generalized to system (8.1) as a LaSalle-type theorem (see [84]) for hybrid stochastic systems with mode-dependent interval delays while other results in [133] can be extended to system (8.1) as well. The LaSalle-type theorem for HSDS (8.1) is given as follows.

Theorem 8.2 *Suppose inequalities (8.4) hold. Assume that there exist nonnegative functions* $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$, $\lambda \in L_1(R_+; R_+)$, *and* $w_1, w_2 \in C(R^n; R_+)$ *such that*

$$\mathcal{L}V(x, y, t, i) \le \lambda(t) - k_1 w_1(x) + k_2 w_2(y), \quad \forall\, (x, y, t, i) \in R^n \times R^n \times R_+ \times S \tag{8.22}$$

$$w_1(x) \ge w_2(x), \quad \forall\, x \in R^n \tag{8.23}$$

and

$$\lim_{|x|\to\infty} \inf_{t\ge 0, i\in S} V(x, t, i) = \infty\,, \tag{8.24}$$

where k_1 *and* k_2 *are positive numbers such that* $k_1 \ge k_2/(1-\bar{\delta})$. *Then* $Ker(w_1 - w_2) \neq \emptyset$ *and*

$$\lim_{t\to\infty} d(x(t;\xi, r_0), Ker(w_1 - w_2)) = 0 \quad a.s.$$

for all $\xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ *and* $r_0 \in S$.

Remark 8.1 *Theorem 8.1 is a generalization of Theorem 2.1 in [133]. In a very special case when* $\tau(t, i) = h$ *for all* $t \ge 0$ *and* $i \in S$, *it is easy to see that* $l_i = h_i = l = h$, $\delta_i = 0$, $\bar{\delta}_i = 0$, $\bar{\delta} = 0$ *for all* $i \in S$, *and Theorem 8.1 is exactly Theorem 2.1 in [133].*

Remark 8.2 *Theorem 8.1 can be specialized to the case of* $N = 1$, *which is a modified version of Corollary 3.1 in [84] for stochastic delay systems (8.1) with* $N = 1$ *and* $\delta_N \le \bar{\delta} < 1$.

Remark 8.3 *Unlike the existing results that assume* $\delta_i \le \bar{\delta} < 1$ *for all* $i \in S$ *(see [84], [111], [135] and the references therein), we propose an alternative assumption (8.4) on the time delays, which reveals an important role the Markovian jumps play in the stability analysis of delay systems. That is, we do not need to require the derivative of time delay* $\delta_i \le \bar{\delta} < 1$ *for all* $i \in S$. *Instead, we assume the upper bound of the average variation rate of time delay* $\bar{\delta}_i \le \bar{\delta} < 1$ *for each subsystem* $i \in S$.

Remark 8.4 *It is noted that, compared with* δ_i *for* $i \in S$, $\bar{\delta}_i$ *in assumption (8.4) is calculated by making use of much more information of the jump delay system (8.1) including the generator of the Markovian jumps and the bounds of the mode-dependent time delays. Let us look at inequality (8.9). The drift of the differential of time delay* $\mathrm{d}\tau(s, i)$ *is composed of two parts, that is,* $\tau_s(s, i)$ *contributed by the change of time delay* $\tau(s, i)$ *in mode* i *and* $\sum_{j=1}^{N} \gamma_{ij}\tau(s, j)$ *caused by the Markovian jumps. In fact,* $\bar{\delta}_i$ *is the estimate* *of the upper bound of the drift of* $\mathrm{d}\tau(s,i)$ *in mode* i *while* δ_i *is the upper bound of* $\tau_s(s,i)$ *that is only part of the drift. Taking expectation on both sides of inequality (8.9), we see*

$$\mathbb{E}\mathrm{d}\tau(s,i) \le \bar{\bar{\delta}}_i \mathrm{d}s \quad \Rightarrow \quad \mathbb{E}\frac{\mathrm{d}\tau(s,i)}{\mathrm{d}s} \le \bar{\bar{\delta}}_i$$

for all $i \in S$*, which is exactly the upper bound of the average variation rate of time delay in mode* i*. This shows that assumption (8.4) may be more sensible for the Markovian jump delay systems than the assumption* $\delta_i \le \bar{\delta} < 1$ *in the existing literatures.*

Remark 8.5 *The techniques in Lemma 8.2 can be adjusted to deal with nonlinear systems with stochastically varying delays of the sawtooth form recently presented in [121], particularly, those with the same minimum delay and the slope of the sawtooth less than one for all subsystems and hence inequality (8.9) still satisfied. Obviously, the existing results ([84], [111], [133], [135] and the references therein) are not applicable to system (1) in [121] even when the slope of the sawtooth is less than one.*

8.4 Examples

In this section, two numerical examples are given to verify the effectiveness of the improved result.

Example 8.1 As a practical example, let us consider Example 3.2 in [133]. The charge $Q(t)$ at time t in an electrical circuit satisfies the second-order differential equation

$$H\ddot{Q}(t) + (R+q)\dot{Q}(t) + \frac{1}{C}Q(t) = F(t)\,, \tag{8.25}$$

where H is the inductance, R and q the resistance, C the capacitance, and $F(t)$ the potential source (see, e.g., p52, [88]). In practice, if the voltage across q is applied to an amplifier and the output is provided with a special phase-shifting network, it will introduce a constant time delay between the input and the output. In this case, we have

$$H\ddot{Q}(t) + R\dot{Q}(t) + q\dot{Q}(t-\tau) + \frac{1}{C}Q(t) = F(t)\,. \tag{8.26}$$

Suppose that the potential source is subject to the environmental noise and is described by $F(t) = G(t) + \alpha(t)\dot{W}(t)$, where $\dot{W}(t)$ is a scalar white noise, that is, $W(t)$ is a Brownian motion, and $\alpha(t)$ is the intensity of the noise. Then equation (8.26) becomes

$$H\ddot{Q}(t) + R\dot{Q}(t) + q\dot{Q}(t-\tau) + \frac{1}{C}Q(t) = G(t) + \alpha(t)\dot{W}(t)\,. \tag{8.27}$$

Assume that the electric circuit experiences abrupt changes in their structure in the sense that the parameters will switch from one to the other as described by

$$
\begin{aligned}
&H = H(r(t))\,, \quad R = R(r(t))\,, \quad q = q(r(t))\,, \quad \tau = \tau(r(t)) \\
&C = C(r(t))\,, \quad G(t) = G(t, r(t))\,, \quad \alpha = \alpha(t, r(t))\,,
\end{aligned}
$$

where $r(t)$ is a Markov chain taking values in $S = \{1, 2\}$ with generator

$$
\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix}.
$$

Let $x_1(t) = Q(t)$ and $x_2(t) = \dot{Q}(t)$, then equation (8.27) can be rewritten as an Itô equation with Markovian switching

$$
\mathrm{d}x_1(t) = x_2(t)\mathrm{d}t \tag{8.28}
$$

$$
\begin{aligned}
\mathrm{d}x_2(t) &= \frac{1}{H(r(t))}\left[-R(r(t))x_2(t) - q(r(t))x_2(t-\tau(r(t))) - \frac{1}{C(r(t))}x_1(t) + G(t, r(t))\right]\mathrm{d}t \\
&\quad + \frac{\alpha(t, r(t))}{H(r(t))}\mathrm{d}W(t)\,.
\end{aligned} \tag{8.29}
$$

To stabilise the fluctuation of the current, a state-feedback controller is introduced and the controlled system is described by

$$
\mathrm{d}x_1(t) = [x_2(t) + u(r(t))x_1(t)]\,\mathrm{d}t \tag{8.30}
$$

$$
\begin{aligned}
\mathrm{d}x_2(t) &= \frac{1}{H(r(t))}\left[-R(r(t))x_2(t) - q(r(t))x_2(t-\tau(r(t))) - \frac{1}{C(r(t))}x_1(t) + G(t, r(t))\right]\mathrm{d}t \\
&\quad + \frac{\alpha(t, r(t))}{H(r(t))}\mathrm{d}W(t)\,.
\end{aligned} \tag{8.31}
$$

For simplicity, we also write $H(i) = H_i$, $\alpha(t, i) = \alpha_i(t)$ etc. for $i \in S$. Example 3.2 in [133] has considered the case when the parameters are given as

$$
H_1 = H_2 = 1,\ C_1 = C_2 = 1,\ R_1 = 6,\ R_2 = \frac{11}{2},\ q_1 = 3,\ q_2 = \frac{4}{3},\ \tau_1 = \tau_2 = h,
$$

Assume that $\bar{\beta} = 4/3$, $\bar{G}(t) = [\bar{\beta}G_1^2(t) + 2\alpha_1^2(t)] \vee [G_2^2(t) + 2\alpha_2^2(t)]$ and $\int_0^\infty \bar{G}(t)\mathrm{d}t < \infty$. By Theorem 2.1 in [133], the closed-loop system (8.30)-(8.31) with $u_1 = -2$ and $u_2 = -3$ is almost surely stable for any $h > 0$.

To compare with the result in [133], we do not change any other condition but assume that the time delays of the subsystems $\tau_1 = h_1$ and $\tau_2 = h_2$ may be two different positive numbers, which is more reasonable in practice. We employ the Lyapunov function candidate

$$
V(x, 1) = \bar{\beta}(x_1^2 + x_2^2) \quad \text{and} \quad V(x, 2) = x_1^2 + x_2^2 \tag{8.32}
$$

and the control strategy $u_1 = -2$ and $u_2 = -3$, which are proposed in Example 3.2 of [133]. It is easy to show that

$$\begin{aligned} \mathcal{L}V(x,y,t,1) &\le (2\bar{\beta}u_1 + 1 - \bar{\beta})x_1^2 - (9\bar{\beta} - 1)x_2^2 + 3\bar{\beta}y_2^2 + \bar{\beta}G_1^2(t) + 2\alpha_1^2(t), \quad (8.33) \\ \mathcal{L}V(x,y,t,2) &\le (2u_2 - 1 + \bar{\beta})x_1^2 - (\frac{29}{3} - \bar{\beta})x_2^2 + \frac{4}{3}y_2^2 + G_2^2(t) + 2\alpha_2^2(t). \quad (8.34) \end{aligned}$$

Inequalities (8.33) and (8.34) imply

$$\mathcal{L}V(x,y,t,i) \le \tilde{G}(t) - \frac{17}{3}|x|^2 + 4y_2^2 \le \tilde{G}(t) - \frac{17}{3}|x|^2 + 4|y|^2 \tag{8.35}$$

for $t \ge 0$ and $i = 1, 2$. Theorem 2.1 in [133] works in the special case when $h_1 = h_2$ but fails when $h_1 \ne h_2$. However, by Theorem 8.1, the closed-loop system is almost surely stable if $|h_1 - h_2| < 5/17$, which shows our result is an improvement.

Example 8.2 Let $W(t)$ be a scalar Brownian motion. Let $r(t)$ be a right-continuous Markovian chain independent of $W(t)$ and taking values in $S = \{1, 2\}$ with generator

$$\Gamma = (\gamma_{ij})_{2\times 2} = \begin{pmatrix} -0.5 & 0.5 \\ 0.1 & -0.1 \end{pmatrix}.$$

Consider a scalar nonlinear stochastic delay system with Markovian switching of the form

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t - \tau(t, r(t))), t, r(t))\mathrm{d}W(t) \tag{8.36}$$

on $t \ge 0$, where

$$\begin{aligned} f(x,t,1) &= -\frac{5}{2}\sqrt[3]{x}, \quad g(y,t,1) = 2\sqrt[3]{y^2}, \quad \tau(t,1) = \tau_1(t), \quad (8.37) \\ f(x,t,2) &= \frac{2}{\sqrt{1+t}} - \frac{21}{8}\sqrt[3]{x}, \quad g(y,t,2) = \frac{9}{5}\sqrt[3]{y^2}\sin t, \quad \tau(t,2) = h_2; \quad (8.38) \end{aligned}$$

$h_2 > 0$ is a constant and $\tau_1 : R_+ \to [0, h]$ is continuously differentiable function.

It is observed that the existing results on stability in moment sense (see [86], [135], Chapter 7 and the references therein) do not apply to system (8.36). To examine the stability of system (8.36), we consider a Lyapunov function candidate $V : R \times S \to R_+$ as $V(x, i) = x^2$ for $i = 1, 2$. By computation, we have

$$\begin{aligned} \mathcal{L}V(x,y,t,1) &= -5x^{\frac{4}{3}} + 4y^{\frac{4}{3}}, \quad (8.39) \\ \mathcal{L}V(x,y,t,2) &\le \frac{4x}{\sqrt{1+t}} - \frac{21}{4}x^{\frac{4}{3}} + \frac{81}{25}y^{\frac{4}{3}}. \quad (8.40) \end{aligned}$$

By the elementary inequality

$$\alpha^c\beta^{1-c} \le c\alpha + (1-c)\beta, \qquad \forall\ \alpha \ge 0,\ \beta \ge 0,\ 0 \le c \le 1$$

we see that inequality

$$\frac{4x}{\sqrt{1+t}} \le \kappa x^{\frac{4}{3}} + \frac{\kappa_1}{(1+t)^2} \tag{8.41}$$

holds for any $\kappa > 0$, where $\kappa_1 = (\kappa/3)^{-3}$.

From inequalities (8.39), (8.40) and (8.41), we have

$$\mathcal{L}V(x, y, t, i) \le \frac{\kappa_1}{(1+t)^2} - (5-\kappa)x^{\frac{4}{3}} + 4y^{\frac{4}{3}} \tag{8.42}$$

for all $t \ge 0$ and $i \in S$. In a special case when $\tau_1(t) \equiv h_2$ for all $t \ge 0$, by Theorem 2.1 in [133] (see also Remark 8.1), inequality (8.42) with $0 < \kappa < 1$ implies that system (8.36) is almost surely asymptotically stable for any $h_2 > 0$. But Theorem 2.1 in [133] does not apply to the case with time-varying delay $\tau_1(t)$. Let us turn to Theorem 8.1 above. For any $\bar{\delta} < 1/5$, we choose constant κ such that $0 < \kappa < (1-5\bar{\delta})/(1-\bar{\delta})$ and hence condition (8.5) is satisfied. For $h_2 = 1$, various bounds of interval time delay $\tau_1(t)$ for almost sure asymptotic stability are listed in Table 8.1, where it should be pointed out that $\delta_1 = 0$ refers to the case of constant delay $\tau_1(t) \equiv \tau_1$, in which system (8.36) is almost surely asymptotically stable if $0.6001 = l_1 \le \tau_1 \le h_1 = 2.9999$. It is also noted that δ_1 may be equal to or larger than 1, which is required to be less than 1/5 in the case of subsystem (8.37) without jumps (see Remark 8.2).

Table 8.1: bounds of $\tau_1(t)$ for stability

δ_1	0.0	0.2	0.4	0.6	0.8	1.0
l_1	0.6001	1.0001	1.4001	1.8001	2.2001	2.6001
h_1	2.9999	2.9999	2.9999	2.9999	2.9999	2.9999

8.5 Summary

This chapter extends the results in [133] to hybrid stochastic systems with mode-dependent interval time delays (8.1) by exploiting the information of the jump delay systems including the generator of the Makovian jumps and the bounds of the mode-dependent time delays. The proposed techniques may not only be applied to generalize the results in [91] to neutral hybrid stochastic systems with mode-dependent interval time delays but also be extended to Lyapunov-Krasovskii functional method, particularly, for delay-range-dependent stability and stabilization of hybrid stochastic delay systems (see, e.g., [111] and [135]).

Chapter 9

Conclusion and future work

In this thesis, we have developed stability criteria and their applications to stabilisation problems of stochastic delay systems.

In Chapters 3 and 4, we have studied state-feedback stabilisation of linear stochastic systems with input delay by using the newly established delay-dependent stability criteria. It is noted that few results are concerned with this important issue for nonlinear systems (see [90] and [95]). Clearly, the development of delay-dependent stability criteria for nonlinear delay systems plays an important role in the study of such issue (see, e.g., [22] and [95]). However, due to the difficulties in system analysis, there are few results even for nonlinear deterministic delay systems (see [22]). The topic of delay-dependent stability criteria and their applications such as delayed-feedback stabilisation is an important issue in the study of stochastic delay systems.

Chapter 5 has presented a SMC design for robust H_∞ control for uncertain stochastic delay systems, which has removed a restriction in the existing results. But it is noted that the case when state delay appears in diffusion has not been considered. In that case, inequality (5.22) would involve a positive definite function with respect to the delay states such that control law (5.17) could not guarantee that the state trajectories would be drawn onto sliding surface (5.14) in finite time. This is one of the problems of SMC for stochastic delay systems that are to be studied.

Razumikhin-type theorems on stability of stochastic retarded systems with Markovian switching have been proposed in Chapters 6 and 7. It is noted that these results all deal with pth moment stability of stochastic retarded systems with Markovian switching.

The classical stochastic analysis theory studies stability not only in moment sense but also in almost sure sense. Although almost sure stability has been studied in Chapter 8, those improved results can apply to systems with differentiable delays only. The Razumikhin method is developed to cope with the difficulty arisen from the large, fast varying and nondifferentiable time delays and plays an important role in stability theory of delay systems. It is very desirable to have a Razumikhin-type theorem on almost sure stability of stochastic retarded systems with Markovian switching that is applicable to some cases when the pth moment versions do not work. This may be a challenging problem.

Let us cite the well-known quotation (see [105]) as the end of this thesis

> *"We have not succeeded in answering all of our problems. The answers we have found only serve to raise a whole set of new questions. In some ways we feel we are as confused as ever, but we believe we are confused on a higher level and about more important things."*

Appendix A

Existence and uniqueness of solutions to HSDSs

In this appendix, we shall establish a useful criterion for the existence and uniqueness of the solution to equation (7.15). Let us introduce one of the well-known Gronwall-type inequalities that have been widely applied in the theory of ordinary differential equations and stochastic differential equations to prove the results on existence, uniqueness and boundedness of solutions.

Lemma A.1 (Gronwall inequality) *Let $\alpha : R_+ \to R_+$ be a continuous function, $u : R_+ \to R_+$ be a Borel measurable bounded function and $\kappa : R_+ \to R_+$ be a integrable function. If*

$$u(t) \le \alpha(t) + \int_0^t \kappa(s)u(s)\mathrm{d}s$$

for all $t \ge 0$, then

$$u(t) \le \alpha(t) + \int_0^t \alpha(s)\kappa(s)e^{\int_s^t \kappa(r)\mathrm{d}r}\mathrm{d}s$$

for all $t \ge 0$.

Theorem A.1 *Assume that there are a function $V \in C^{2,1}(R^n \times R_+ \times S; R_+)$ and a constant $K > 0$ such that*

$$\mathcal{L}V(x,y,t,i) \le K\big[1 + V(x,t,i) + V(y,t-\tau(t,i),i)\big] \tag{A.1}$$

and

$$\lim_{|x|\to\infty} \inf_{0\le t\le\infty, i\in S} V(x,t,i) = \infty \tag{A.2}$$

for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$. *Then equation (7.15) has a unique (global) solution for any initial data* $x_0 = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ *and* $r(0) = r_0 \in S$.

Proof. Since both $F(x, y, t, i)$ and $G(x, y, t, i)$ satisfy the local Lipschitz condition (7.17), equation (7.15) has a unique maximal solution $x(t)$ on $[-h, \sigma_\infty)$ for any initial data $\xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ and $r_0 \in S$, where σ_∞ is the explosion time (see [72] and [84]). Therefore, we only need to show that $\sigma_\infty = \infty$ *a.s.* . Let k_0 be an integer such that $\|\xi\| \le k_0$ *a.s.* . For any integer $k \ge k_0$, define the stopping time

$$\rho_k = \sigma_\infty \wedge \inf\{t \in [0, \sigma_\infty) : |x(t)| \ge k\}, \tag{A.3}$$

where, as usual, we set $\inf \emptyset = \infty$. Clearly, $\{\rho_k\}_{k \ge k_0}$ are an increasing sequence and they have the limit $\rho_\infty = \lim_{k\to\infty} \rho_k$. Obviously, $\rho_\infty \le \sigma_\infty$ *a.s.* . For any $k \ge k_0$ and $t \ge 0$, formula (6.11) gives

$$\begin{aligned}
&\mathbb{E}V(x(t \wedge \rho_k), t \wedge \rho_k, r(t \wedge \rho_k)) \\
&\quad = \mathbb{E}V(x(0), 0, r(0)) + \mathbb{E}\int_0^{t\wedge\rho_k} \mathcal{L}V(x(s), x(s - \tau(s, r(s))), s, r(s))\mathrm{d}s.
\end{aligned}$$

Let $t_k = t \wedge \rho_k$ for any $k \ge k_0$ and $t \ge 0$. By condition (A.1), we have

$$\begin{aligned}
&\mathbb{E}V(x(t_k), t_k, r(t_k)) \\
&\quad \le \mathbb{E}V(x(0), 0, r(0)) + \mathbb{E}\int_0^{t_k} K\big[1 + V(x(s), s, r(s)) \\
&\qquad\qquad + V(x(s - \tau(s, r(s))), s - \tau(s, r(s)), r(s))\big]\mathrm{d}s \\
&\quad \le \mathbb{E}V(x(0), 0, r(0)) + Kt + \mathbb{E}\int_0^{t} K\big[V(x(s_k), s_k, r(s_k)) \\
&\qquad\qquad + V(x((s - \tau(s, r(s)))_k), (s - \tau(s, r(s)))_k, r(s_k))\big]\mathrm{d}s \\
&\quad = \mathbb{E}V(x(0), 0, r(0)) + Kt + K\int_0^{t} \big[\mathbb{E}V(x(s_k), s_k, r(s_k)) + \\
&\qquad\qquad + \mathbb{E}V(x((s - \tau(s, r(s)))_k), (s - \tau(s, r(s)))_k, r(s_k))\big]\mathrm{d}s \\
&\quad = C + Kt + 2K\int_0^{t} \sup_{0\le\delta\le s, i\in S} \mathbb{E}V(x(\delta_k), \delta_k, i)\mathrm{d}s\,,
\end{aligned} \tag{A.4}$$

where $C = \mathbb{E}V(x(0), 0, r(0)) + K\int_0^h \mathbb{E}V(x(s - \tau(s, r(s))), s - \tau(s, r(s)), r(s))\mathrm{d}s < \infty$. Since inequality (A.4) holds for $r(t_k) = i$ for all $i \in S$ and the right-hand side of (A.4) is increasing in t, we must have

$$\sup_{0\le\delta\le t, i\in S} \mathbb{E}V(x(\delta_k), \delta_k, i) \le C + Kt + \int_0^t 2K\left[\sup_{0\le\delta\le s, i\in S} \mathbb{E}V(x(\delta_k), \delta_k, i)\right]\mathrm{d}s, \quad \forall\, t \ge 0. \tag{A.5}$$

But, by Lemma A.1, this yields

$$\sup_{0 \le \delta \le t, i \in S} \mathbb{E}V(x(\delta_k), \delta_k, i) \le C + Kt + \int_0^t 2K(C + Ks)e^{2K(t-s)} \mathrm{d}s, \quad \forall\, t \ge 0. \tag{A.6}$$

This implies there are positive constants C_1 and C_2 such that

$$\mathbb{E}V(x(t_k), t_k, r(t_k)) \le C_1 + C_2 e^{2Kt}, \quad \forall\, t \ge 0 \tag{A.7}$$

where $C_1 = \frac{1}{2}$ and $C_2 = C + \frac{1}{2}$.

On the other hand, define $\mu : R_+ \to R_+$ by

$$\mu(r) = \inf_{|x| \ge r, t \in R_+, i \in S} V(x, t, i)\,.$$

Obviously, $\mu(|x(t)|) \le V(x(t), t, r(t))$ for all $t \ge 0$ and, by condition (A.2),

$$\lim_{r \to \infty} \mu(r) = \infty.$$

But inequality (A.7) implies

$$C_1 + C_2 e^{2Kt} \ge \mathbb{E}\mu(|x(t_k)|) \ge \mu(k)\mathbb{P}(\rho_k \ge t)\,.$$

Letting $k \to \infty$ and then $t \to \infty$, we obtain

$$\mathbb{P}(\rho_\infty < \infty) = 0\,.$$

That is, $\rho_\infty = \infty$ $a.s.$, which implies $\sigma_\infty = \infty$ $a.s.$. This completes the proof.

Appendix B

Existence and uniqueness of solutions to HSSs with mode-dependent delays

Proof of Lemma 8.3: We use the standard truncated technique (see, e.g., [67], [88] and [133]) and therefore only outline the proof. Suppose that k_0 is an integer such that $\|\xi\| < k_0$ *a.s.*. For any integer $k \geq k_0$, define

$$f_k(x, y, t, i) = f\left(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, t, i\right) \tag{B.1}$$

and

$$g_k(x, y, t, i) = g\left(\frac{|x| \wedge k}{|x|}x, \frac{|y| \wedge k}{|y|}y, t, i\right) \tag{B.2}$$

for all $(x, y, t, i) \in R^n \times R^n \times R_+ \times S$. It is easy to see that f_k and g_k satisfy the global Lipschitz condition and the linear growth condition. Let us consider the following equation

$$\mathrm{d}x_k(t) = f_k(x_k(t), x_k(t - \tau(t, r(t))), t, r(t))\mathrm{d}t + g(x_k(t), x_k(t - \tau(t, r(t))), t, r(t))\mathrm{d}W(t) \tag{B.3}$$

on $t \geq 0$ with initial data $x_{k0} = \{x_k(\theta) : -h \leq \theta \leq 0\} = \xi \in C^b_{\mathcal{F}_0}([-h, 0]; R^n)$ and $r(0) = r_0 \in S$. By Theorem 7.10, p277, [87], equation (B.3) has a unique global solution on $t \geq -h$. Define the stopping time

$$\bar{\rho}_k = \inf\{t \geq 0 : |x_k(t)| \geq k\}, \tag{B.4}$$

where we set $\inf \emptyset = \infty$ as usual. It is straightforward to show that

$$x_k(t) = x_{k+1}(t) \quad \text{if } -h \leq t \leq \bar{\rho}_k \text{ and } k \geq k_0\,. \tag{B.5}$$

This implies that $\{\bar{\rho}_k\}_{k\geq k_0}$ are an increasing sequence. Let $\bar{\rho}_\infty = \lim_{k\to\infty}\bar{\rho}_k$. In view of (B.5), we can define $x(t)$ for $t \in [-h, \bar{\rho}_\infty)$ uniquely by

$$x(t) = x_k(t) \quad \text{if } -h \leq t \leq \bar{\rho}_k \tag{B.6}$$

for $k \geq k_0$.

Now, for any stopping time $\bar{\rho} < \bar{\rho}_\infty$, we set $\hat{\rho}_k = \bar{\rho} \wedge \bar{\rho}_k$ and derive that

$$\begin{aligned} x(\hat{\rho}_k) - x(0) &= x_k(\hat{\rho}_k) - x_k(0) \\ &= \int_0^{\hat{\rho}_k} f_k(x_k(s), x_k(s-\tau(s,r(s))), s, r(s))\mathrm{d}s \\ &\quad + \int_0^{\hat{\rho}_k} g_k(x_k(s), x_k(s-\tau(s,r(s))), s, r(s))\mathrm{d}W(s) \\ &= \int_0^{\hat{\rho}_k} f(x(s), x(s-\tau(s,r(s))), s, r(s))\mathrm{d}s \\ &\quad + \int_0^{\hat{\rho}_k} g(x(s), x(s-\tau(s,r(s))), s, r(s))\mathrm{d}W(s)\,. \end{aligned} \tag{B.7}$$

Letting $k \to \infty$ in (B.7) gives

$$\begin{aligned} x(\rho) - x(0) &= \int_0^{\rho} f(x(s), x(s-\tau(s,r(s))), s, r(s))\mathrm{d}s \\ &\quad + \int_0^{\rho} g(x(s), x(s-\tau(s,r(s))), s, r(s))\mathrm{d}W(s)\,. \end{aligned} \tag{B.8}$$

This means that $x(t)$ is a unique solution to equation (8.1) on $t \in [-h, \bar{\rho}_\infty)$. To complete the proof, we need to show that $\bar{\rho}_\infty = \infty$ *a.s.* . Let $\bar{t}_k = t \wedge \bar{\rho}_k$ for any $t \geq 0$ and $k \geq k_0$. By inequality (8.5), Lemma 8.1 and Lemma 8.2, we have

$$\begin{aligned} &\mathbb{E}V(x(\bar{t}_k), \bar{t}_k, r(\bar{t}_k)) \\ &= \mathbb{E}V(x(0), 0, r(0)) + \mathbb{E}\int_0^{\bar{t}_k} \mathcal{L}V(x(s), x(t-\tau(s,r(s))), s, r(s))\mathrm{d}s \\ &\leq \mathbb{E}V(x(0), 0, r(0)) + \int_0^{\bar{t}_k} \lambda(s)\mathrm{d}s \\ &\qquad - k_1\mathbb{E}\int_0^{\bar{t}_k} w_1(x(s))\mathrm{d}s + k_2\mathbb{E}\int_0^{\bar{t}_k} w_2(x(s-\tau(s,r(s))))\mathrm{d}s \\ &\leq \mathbb{E}V(x(0), 0, r(0)) + \int_0^{\bar{t}_k} \lambda(s)\mathrm{d}s + k_1\mathbb{E}\int_{-h}^{0} w_2(x(s))\mathrm{d}s - k_1\mathbb{E}\int_0^{\bar{t}_k} \eta(x(s))\mathrm{d}s \\ &\leq \mathbb{E}V(x(0), 0, r(0)) + \int_0^{\bar{t}_k} \lambda(s)\mathrm{d}s + k_1\mathbb{E}\int_{-h}^{0} w_2(x(s))\mathrm{d}s := C_0. \end{aligned} \tag{B.9}$$

Clearly, C_0 is a positive constant independent of t and k. This yields

$$\mathbb{P}\{\bar{\rho}_k \leq t\} \leq \frac{C_0}{\beta(k)}, \tag{B.10}$$

where function $\beta(\cdot)$ is defined by (8.13).

Letting $k \to \infty$ in (B.10), we see

$$\mathbb{P}\{\bar{\rho}_\infty \leq t\} = 0.$$

Since $t \geq 0$ is arbitrary, we must have $\bar{\rho}_\infty = \infty$ *a.s.*, which completes the proof.

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