THE PARADOXES OF ZENO

by

Raymond A. Thomson

THESIS SUBMITTED FOR THE DEGREE

of

DOCTOR OF PHILOSOPHY

THE UNIVERSITY OF STRATHCLYDE

GLASGOW

DEPARTMENT OF ENGLISH STUDIES

1985

Contents

Page

Ecclesiastes, IX, 10

Acknowledgements

I wish to acknowledge the patience and understanding afforded me throughout this research by my wife, Anne. Philosophy is a taskmaster from whom despondency is never far distant, and her support has been a constant encouragement. My greatest debt, however, is to my supervisor, Hywel Thomas, who has stinted neither hospitality, wit nor erudition is his attempts to clarify the obscurities in my mind. He has not always succeeded, but this is my fault and the responsibility for this work's imperfections is mine.

Abstract

In the Introduction it is shown that there are scientific theories which are real world examples of Zeno's Paradoxes. The Paradoxes are therefore not to be dismissed as frivolous. An analysis of what "paradox" is sets out the subsequent strategy of the thesis. Part One is an historical examination of Pythagoras, Heraclitus and Parmenides which places Zeno and the Paradoxes in context, showing why he was led to formulate them and how they work. Part Two is a grouping and analysis of refutations based on mathematics, logic and science. These refutations which utilize the theory of geometric progressions are shown to fail because the limit of an infinite series is not part of that series. Refutations based on the Cantor-Russell analysis of the continuum are seen to refute the Paradoxes of Plurality but to be harmless against the Paradoxes of Motion. It is shown that the Achilles Paradox can, through the use of elementary Relativity Theory, be reduced to the Stadium Paradox. Part Three deals with attempts to refute Zeno, (a) by using circumlocution of the problems as a strategy, and (b) through an analysis of the terms "motion" and "infinite". This section describes a convincing refutation of the first version of the Stadium Paradox. Part Four is concerned with refutations based on metaphysical theories of the working of the intellect with regard to motion and perception, and intellectual analysis of the space-time continuum. These are shown to provide a refutation of the Paradox of the Flying Arrow.

Part Five contains an explanation of the problems in mathematics uncovered by Zeno through his Paradoxes and a refutation, based on non-standard analysis of the infinitesimal, of the second version of the Stadium Paradox and the Paradox of the Moving Rows. An Appendix contains an hypothetical reconstruction of Zeno's lost Paradoxes of Time.

Introduction

"There are optical illusions in time as well as in space."
(Marcel Proust)

This research has as its subject the Paradoxes of Zeno the Eleatic, paradoxes which have confused, perplexed and fascinated the minds of philosophers for over two thousand years. Their fascination derives from an extraordinary subtlety, with ever another avenue to investigate and ever another confusion to tease out, but, like the knot of Gordias, there seems never a point where one can say that the paradox is unravelled and that the task is finished.

More particularly, this research will concern itself with the refutations offered up by later philosophers in their dealings with Zeno. Philosophical opinion on the gravity, to say nothing of the validity of Zeno's arguments, is curiously divided: many seem not so concerned to challenge their soundness as to impugn their philosophical importance. Certainly, despite the venerable tradition of controversy surrounding the paradoxes, there is reason to believe that by calling the Achilles a "very old and, I think, very silly problem,"[1] Richard Taylor expresses the feelings of many. There are those in every age who regard with suspicion any effort to establish philosophically important truths with a technical apparatus depending more on logical manipulation than on enlightened empirical judgements, especially when these "truths" conflict with common sense. This approach to Zeno borders on the facetious, considering him to be no more than an ingenious riddler whose aim was to confuse those less clever than he, a propounder of trivia whose cogitations are

---

1. Richard Taylor, "Mr. Wisdom on Temporal Paradoxes" in Analysis, 13, (1952), p.17.

unworthy of direct engagements; he is to be circumvented by the caustic comment.[2] This approach is not very helpful as it labours, not only under an entirely groundless disrespect for our philosophical heritage, but also under a misapprehension of what Zeno had set out to do. It fails utterly to realize the subtlety of what he has to say and their accusations of levity and philosophical irrelevance betray nothing more than impatience. However, there are other writers who have struggled admirably and honestly to find solutions to what Zeno has offered and, although they often emerge from the fray defeated and more confused than when they started, they are worthy of the greatest consideration. At the very least, their work is a tacit acknowledgement that Zeno has something serious to say: indeed, he had a very serious purpose.

Zeno was a disciple of Parmenides and his arguments must never by construed as witty bagatelles; every word in them is designed to support the position of the Master. Parmenides, as shall later be seen, combated pluralism and declared change and motion to be illusory, and "Toute l'argumentation de Zénon est dirigée contre le mouvement; car le mouvement supprimé, il emporte nécessairement avec lui la géneration et la mort, l'accroiseement et la diminution, le changement, en un mot, tous les phénonèmes de la nature elle-même."[3] His task

---

2. Vide Bertrand Russell, Wisdom of the West, (London, 1970) and Abner Shimony, "Resolution of the Paradox" in Zeno's Paradoxes, (New York, 1970), ed. Wesley C. Salmon.

3. "Zénon d'Elee", Dictionaire de Science Philosophique, (Paris, 1852), p. 1020.

clearly is colossal and it is because he seems to be flying in the face of the facts that there is an air of the ridiculous in what he is trying to do. For Parmenides and Zeno to be correct, then our perceptual apparatus cannot be conveying to us what is "out there". Cleverly, Zeno does not concern himself with the ability or inability of the senses to provide us with knowledge, but shifts the ground radically by saying that what we perceive is logically impossible. He argues for Parmenides' position by showing not only that the hypothesis of the One is correct, but also by showing that the Pythagorean hypothesis of the Many has as a consequence the impossibility of motion and change.

The charge is often levelled that situations such as Zeno suggests, cannot occur in the real world. If this were the case, would they still be a subject worthy of study? The answer must be affirmative. We may well strip away all of the layers and provide a complete resolution of all of the difficulties that arise out of the paradoxes, and find that we are left with nothing, that there are no fundamental truths about the nature of reality to be discovered in solving the paradoxes. Philosophy, however, has truth as only one of its objectives, and, were the above to happen, that would not be to say that nothing of value had been done. The analysis itself would nevertheless be serendipitous; because it deals in detail with fundamental problems of space, time, infinity, motion and continuity, the process will be richly rewarding.

Against the change of unreality, however, it must be noted that examples of Zeno-like situations, which are readily accepted, do occur in the world. An example can be found in science, in the field of the physics of Relativity theory. This example, which is discussed by Pieirls,[4] is double-edged in that two forms of infinity feature in it, the infinitely small and the infinitely large. Physics states that it is impossible for any material object to exceed the speed of light, or even, in fact, to attain it. This is because of the phenomenon that, as the speed of an object increases towards that of light, it becomes increasingly more difficult to accelerate that object. In other words, the "chunks" of energy required, if the rate of acceleration is to be maintained, must become larger and larger. This is because the mass of a particle moving at speed has a different value for different observers: if we position an observer on the particle (in thought at least), he will find a standard value for the mass, the "rest mass", which will remain identical throughout the acceleration. But an observer seeing the particle whizz past him will find that its mass has increased. This makes no difference at ordinary everyday speeds: a motor car moving at fifty miles per hour will increase its mass to a stationary observer by 0.0000000000003%.[5] In an atomic accelerator or

---

4. R.E. Pieirls, "Relativity" in Problems of Space and Time, (New York, 1964), ed. J.C.C. Smart, p.267.

5. E.M. Rogers, Physics for the Enquiring Mind, (New Jersey, 1960), p.487.

cyclotron, however, accelerated particles increase their mass significantly. Therefore, as their speed rises nearer to the speed of light, it becomes increasingly harder to accelerate them, for their mass sweeps up towards infinite mass at the speed of light. Experiments using linear accelerators, which are devices for shooting electrons straight ahead, show that, at high energies, the electrons approach the speed of light but never exceed it. The electrons gain more energy at each successive push, and therefore more mass, but hardly move any faster. This mass growing towards infinity at the speed of light means also that resistance to acceleration also grows towards infinity. Relativity theory, therefore, predicts that no piece of matter can move faster than light since, in attempting to accelerate it to that speed, we should encounter more and more mass and therefore obtain less and less response to our accelerating force. Therefore, this prediction that to reach the velocity of light will require an infinitely strong force gives us two diagrams which will become increasingly familiar as this enquiry progresses.

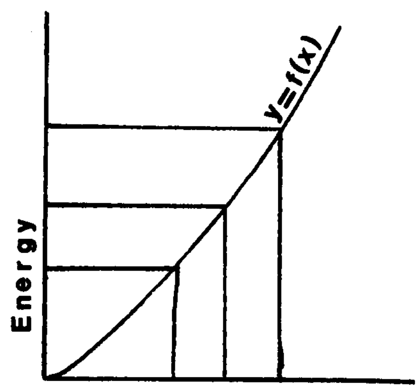


FIG. 1

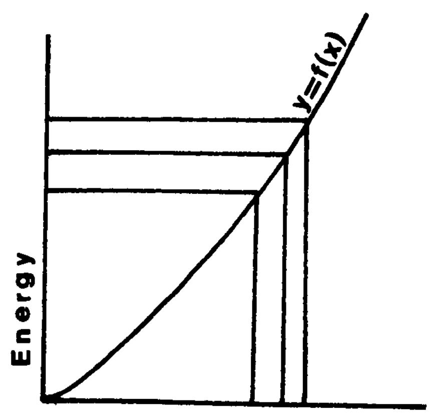


FIG. 2

Figure 1 indicates that to maintain a regular acceleration, shown on the horizontal axis, more and more energy, shown on the vertical axis, is required, and that this requirement tends to infinity. The "chunks" of energy needed become infinitely large. Figure 2 indicates that the further application of regular quanta of energy will produce less and less accelerative effect, and that the increase in speed tends to the infinitesimal.

These two diagrams represent a sophisticated example of the Stadium Paradox: there will never be a point where an application of energy produces no effect at all, but never an amount of energy available, adequate to cause the speed of light to be attained. We always get nearer to the speed of light but never attain it.

I wish to consider one further example from science. Thermodynamicists have deduced that nothing can ever attain the temperature absolute zero. Adkins states that "It is impossible to reduce the temperature of any system or part of a system to the absolute zero in a finite number of operations"[6] and Pippard has it that "By no finite series of processes is the absolute zero attainable."[7] Measuring with gas thermometers, and by using gas liquifiers, very low temperatures have been attained. Liquid helium, for example, has been lowered in temperature to 0.8°K, i.e. approximately 0.8° Centigrade above absolute zero. This is done by pumping away the vapour in the vessel which

---

6. C.J. Adkins, Equilibrium Thermodynamics, (London, 1969) p.241

7. A.B. Pippard, The Elements of Classical Thermodynamics, (Cambridge, 1966) p.51.

contains the helium, causing the liquid to boil at very low temperatures. It is not feasible to go much lower than this with helium, however, since the vapour pressure of helium falls very rapidly towards zero below the temperature of 0.8°K, and no pump will maintain a low enough pressure against the large volume of gas which evaporates from the liquid under these conditions.[8] This seems to be a flaw in the pumping process and in the efficiency of pumps, and, by using other methods, temperatures as low as 0.001°K have been attained. The experiments in this field of low temperature thermodynamics suggest that however low the temperature may be brought, there may be some limitation to all cooling methods which prevents the absolute zero, 0.0°K, from ever being attained.

The equations lying behind the third law of thermodynamics are too complex for a layman such as I am to understand, but what I take them to mean is something like this: in order to be at absolute zero, the loss of pressure on the system must be such that the molecules within the system are no longer in motion. Amontons deduced in 1703 that at the temperature - 237°C, the pressure would become zero; the most peculiar things would happen to the atomic structure of any system.

Also, to cool something, the means of cooling must be colder than that which is to be cooled. A moment's reflection will show that this is a guarantee that absolute zero cannot be reached. The similarity with the Stadium Paradox is very exciting; in order to attempt to reach the finishing tape, the tape must <u>always</u>

---

8. ibid, p.48.

be ahead of the Runner.

Observe that the impossibility of attaining the speed of light or the temperature absolute zero has been deduced by logical inference. We accept as fact, situations which appear to be paradoxical, simply because we are inferring from logic to reality.

Zeno's paradoxes are extremely damaging to the foundations of the calculus. Consider, for example, this early attempt by Leibniz[9] to base the integral calculus on the infinitesimal:

> It is useful to consider quantities so infinitely small that when there ratio is looked for, they may not be considered zero, but which are to be rejected if they occur with quantities incomparably greater. Thus if we have x + dx, dx is rejected ... Similarly we cannot stand xdx and dxdx together. Hence if we differentiate xy we are to write:
>
> $(x + dx)(y + dy) - xy = xdy + ydx + dxdy$
>
> But dxdy is here to be rejected as far less than xdy + ydx. Thus in any particular case the error is less than any finite quantity.

Clearly we do not have to wait for a Berkeley to come along to show that something is wrong with this. Zeno would surely contend that, no matter how small the thing is that is neglected, we can no longer claim exactness, only approximation: in rebus mathematicis errores quam minimi non sunt contemnendi. This strategy by Leibniz would be dismissed as unintelligible because of what is meant by these ratios so small that they need not be considered. Are they something or nothing?[10] Whichever

---

9. Gerhardt, (ed.), Leibnitzens Mathematische Schriften, IV.63, in Leibnitzens Gesammelte Werke, Pertz ed., (Halle, 1859).

10. Berkeley's caustic conclusion was that they were the "ghosts of departed quantities."

Leibniz chooses he will confirm Zeno's paradoxes of Plurality which assert that any interval is simultaneously without size and of infinite dimension.[11]

Weierstrasse disposed of the infinitesimal by resting the foundation of the calculus on the limit.

This has its genesis in Antiphon's "Method of Exhaustion" which he used to attempt to calculate the area of a circle. In a fragment of Eudoxus[12] we have the following account:

> Antiphon, having drawn a circle, inscribed in it ... a square. He then bisected each side of this square, and through the points of the section drew straight lines at right angles to them, producing them to meet the circumference ... He then joined the new points of section to the ends of the sides of the square, so that four triangles were formed, and the whole inscribed figure became an octagon, ... he concluded that in this manner a polygon would be inscribed in the circle, the sides of which, on account of their minuteness, would coincide with the circumference of the circle.

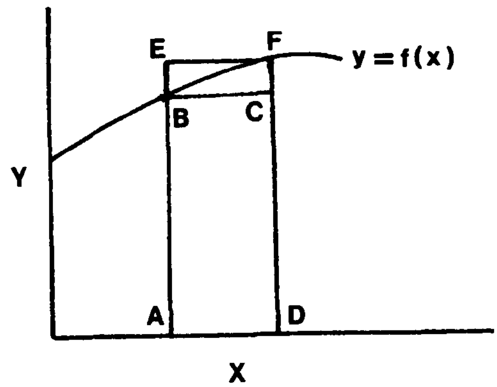
Zeno, however, will say that if this "method of exhaustion" constitutes the basis of the calculus then it is on very soft foundations indeed. Irrespective of the "minuteness" of the sides, there will always be radians left using this method, and so the area of the circle cannot accurately be deduced.

---

11. For an entertaining account of the controversy caused in Scottish Universities by fluxions and infinitesimals, see G.E. Davie, The Democratic Intellect, (Edinburgh, 1961).

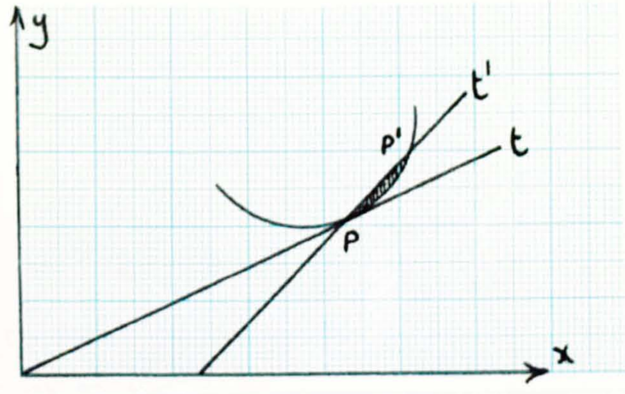
12. Quoted in D.E. Smith, History of Mathematics, Vol. II, (New York, 1958), pp. 677-678.

This method was expanded into Riemann's notion of the limit. Let us attempt to calculate the area under this curve:



Rectangles are drawn from the curve to the X-axis. There is a finite number of non-overlapping small rectangles, e.g. ABCD, and a finite number of non-overlapping large rectangles, e.g. AEFD. The area under the curve lies between the aggregate of the areas of the large rectangles and the aggregate of the areas of the small rectangles. If we make the rectangles narrower, so packing more in under the curve, the two areas converge. The aggregate area of the small rectangles becomes larger while the aggregate area of the large rectangles becomes smaller, and their limit is the area under the curve. However, Zeno will again object, saying that their limit occurs only when the rectangles have no width. As soon as this occurs we are no longer using the method of limits but something altogether different and impossible. As long as the rectangles have width we cannot know the area under the curve. For the rectangles to become so narrow that they turn into straight lines (with length and no breadth) would be like the Runner in the Stadium reaching the tape.

This problem of basing the calculus on limits is best seen by examining the following diagram:



The problem is to calculate the slope of the curve at point P. PP^1 is the secant t^1 which approximates to the tangent at P when P^1 is near P. The tangent is the limit of the slope of the secant. The slope of the tangent equals the slope of the secant when

$$\lim \frac{f(x^1) - f(x)}{x^1 - x} = \lim \frac{\Delta y}{\Delta x},$$

where the limits are evaluated as $x^1 \longrightarrow x$, i.e. as $\Delta x = x^1 - x \rightarrow 0$.

That is, the slope of the tangent t to the curve is the limit of the difference quotient $\Delta y/\Delta x$ as $\Delta x = x^1 - x$ approaches zero.

This, of course, is less metaphysical than the method of infinitesimals, but Zeno still has something pertinent to say. The Paradox of Achilles and the Tortoise and the Paradox of

the Stadium show that a movement from P^1 to P is impossible, that is, that P^1 will not attain its limit P. His basic strategy, which will be examined in more detail later, is to invoke a geometric progression which shows that P^1 always has some interval to cross on its journey to P.
The charge of unreality should therefore be dropped as it should by now be apparent that Zeno is a highly important thinker and that what his paradoxes offer must, somehow, be shown to be false; if, that is, we wish to maintain the edifice of calculus and geometry and not allow them to tumble into anarchy.

I wish now to turn to the approach which I intend to adopt and explain the method I propose to use. It has been realized before that the paradoxes work concomitantly. Refutations, however, fail to reflect this: there are those writers who consider that they have refuted one of the paradoxes, but when that refutation is examined in the light of what the other paradoxes have to say, it can be seen that their confidence is misplaced. Taken together as a unity the paradoxes form a defensive bloc of enormous potency. This research, then, consists mainly of an examination of the attempts by philosophers, mathematicians and scientists to disprove Zeno's Paradoxes. Even a cursory survey of the vast literature on the topic will show that there is a wide variety of interpretations concerning the difficulties raised by Zeno and I propose to investigate closely the three main forms of criticism used against him:

the thesis that an infinite number of acts can be done in a finite time, the thesis that circumlocution will solve the problem and the thesis that Zeno is operating with a most peculiar view of the spatio-temporal continuum. These forms I term the Mathematico-Scientific, Periphrastic, and Metaphysical refutations respectively.

These headings under which I propose to categorise and study the refutations of the paradoxes are not arbitrary, but derive from the unique nature of Zeno's Paradoxes themselves.

Quine[13] calls the paradoxes "falsidical" in that the propositions which they purport to prove are not only absurd but also patently false: despite their having an argument in support, the falsidical paradox is characterized by there being always a fallacy in that argument.[14] What makes the paradoxes of Zeno falsidical, he claims, is that the propositions which Zeno seeks to establish are absurd, false, and based on the fallacy of thinking that any finite succession of intervals of time has to add up to all eternity. All is clarified when we consider the nature of convergent series.[15]

Quine, however, is correct only so far: the paradoxes of Zeno cannot be accepted as veridical; but they cannot be

---

13. W.V. Quine, The Ways of Paradox and other essays, (New York 1966) p.5: this is contrasted with "veridical" paradoxes such as being 21 years old but having had only five birthdays. Being born on Leap Year's Day dissolves the paradox.

14. ibid., p.5

15. ibid., p.6

accepted as falsidical either, because I will show that there is no fallacy in Zeno's reasoning. The paradoxes are therefore to be seen as antinomies. A veridical paradox seems surprising at first, but this surprise disappears when an explanation is offered: a falsidical paradox can be equally surprising, but this surprise disappears when we solve the underlying fallacy. An antinomy, however, packs a surprise "that can be accomodated by nothing less than a repudiation of our conceptual heritage."[16] This is precisely what will happen.

In many paradoxes there is a clear inconsistency between what propositions say about the world and the way in which the world corresponds to these propositions. It is not just that the propositions fail to square with the facts: we simply call these "false propositions". What gives Antinomy its unique qualities is that the propositions have an extremely powerful internal consistency. This is clearly the case with the paradoxes of motion where Zeno's description of Achilles and the Tortoise and of the Stadium is intellectually acceptable. These propositions cannot be accepted, but the problem arises as to what should constitute good grounds for their rejection. What gives Antinomy its paradoxicality is, I think, that it implies one of three things. It could be that there is something wrong with Logic; it could be that the language we have evolved is simply inadequate to describe phenomena, such

16. ibid., p.11.

as plurality and motion, which are to be found in the world; or, finally, and more remotely, it could be that there is something wrong with the world itself, that its reality is very different from its appearance. Corresponding to these three possibilities are the types of categorisation I will employ. The Mathematico-Scientific refutations are concerned with Logic, the Periphrastic refutations are concerned with the way we describe the Stadium, the Achilles and the Moving Rows Paradoxes, and the Metaphysical refutations are concerned with the true nature of the spatio-temporal continuum.

The refutations analysed herein are chosen because they exemplify certain aspects of the literature on Zeno which I consider to be either important or interesting or both. Hence there were refutations which I encountered but did not include because a better, or more interesting argument had occurred elsewhere. The material of the articles and learned works studied has been broken down into its component parts so that its organisational structure can be understood. Different parts of the material have been identified as important, and there has been an analysis of the relationship between important parts, together with a recognition of the organisational principles involved. It will be found that, with only a few exceptions, the conviction of later writers that they have solved the paradoxes is mistaken, for my approach to these solutions is that which Zeno himself would have used: *reductio ad absurdum*. This may make my remarks seem uniformly hostile but has the advantage that the refutations encountered will not survive long against someone who plays at being Zeno.

Each age, from Aristotle onwards, seems to find in the paradoxes "difficulties that are roughly commensurate with the mathematical, logical, and philosophical resources then available,"[17] but, armed with the knowledge of these previous errors, I have constructed a general theory which is, I think, capable of overturning all of the paradoxes simultaneously. In this way there can be no mutual defensive support among the paradoxes. Because of this approach, the objections which can be raised against those who have attempted to gainsay Zeno in the past cannot be applied to any original refutation offered here. I have said that the refutations offered so far against Zeno can be classified into the following categories: Mathematico-Scientific, Periphrastic, Metaphysical. One would therefore expect that an ultimate refutation, if it can be found, will be of one of two types: (a) a co-ordination or integration of the special contributions derived from each of the above categories, or (b) some completely new form of refutation.

My conclusion is that the paradoxes are logical antinomies and simply do not admit of explanations of the traditional kind. It is not that I am not up to the task of refuting them in this way. This research, however, has not failed to generate a theory which is adequate to finally overthrow Zeno and firmly and legitimately put him to the rear of the philosophical mind. By use of non-standard mathematical analysis, I show how the

---

17. W. Salmon, op. cit., pp. 43-44.

paradoxes succeed in what they set out to do, but how they collapse in the face of recent (and highly controversial) mathematics. However, many side issues arise in the process of this enquiry which I hope that subsequent investigations will pursue: that is, I intend that a solid foundation be laid for further studies in this field.

A brief account of the intellectual background to the paradoxes will finally dispel any residual unease which may still be felt as to the significance of the paradoxes and why Zeno should have been led to undertake such a bizarre course in his work: to this I propose now to turn.

Part One

The Intellectual Background to the Paradoxes

Chapter One

Pythagoras and Heraclitus

"Here is the world, sound as a nut, not the smallest piece of chaos left, never a stitch nor an end, not a mark of haste, or botching, or second thought: but the theory of the world is a thing of shreds and patches."

(Ralph Waldo Emerson)

It is one of history's more striking curiosities that the ancient world's most revolutionary thinkers emerged almost simultaneously. Confucius in China, Gautama Buddha in India, Zoroaster in Persia, Pythagoras, Heraclitus and Parmenides[1] in Ionian Greece were all near contemporaries. This revolutionary thought divided into two distinct streams: the scientific-philosophical and the mystical-religious. Greece shows particularly good examples of this.

The scientific movement in Greece was the earlier, was based at Miletus, and is particularly associated with Thales, Anaximander and Anaximenes. The world view of this group was elemental, finding its principles by using observation and speculation combined, and not merely by guesswork and recourse to myth. It was Nature, not Man, that was the object of intellectual investigation and this group represents a turning away from mythical cosmology and a movement towards natural philosophy. The question that exercised them ran thus: What is it that all things have in common? What is the one unchanging, ageless and deathless being underlying all these many shifting,

---

1. Parmenides' ideas could be considered to be scientifically retograde; Aristotle called him an antinaturalist.

   Nevertheless, the daring character of his thought must class him as a revolutionary thinker, as he is responsible for the course taken by natural philosophy in the fifth century: no advance could be made without breaking through the network of his logic.

decaying and perishing things? Thales, as we know, suggested that the being out of which all things come, is Water. The fact that this observation seems to us to be fantastic is neither here nor there. What mattered was that Thales and his successors abandoned once and for all the old method of explaining the universe through myth. Instead of invoking the gods, they substituted the beginnings of scientific-philosophic thought, systematic, ordered, self-controlled, in which generalization, supplied by intuition, stimulates and directs observation, while observation in turn provokes and controls generalization.

This gave way between 550 and 500 B.C., a time of great strife and social and political instability, to a more mystically oriented movement, a movement to which we assign thinkers such as Pythagoras, Heraclitus and Parmenides.

The reaction against the elemental school of Thales, detectable in these thinkers, their swing away from natural science to mystical absolutism may have been influenced by contemporary political events.[2] Many commentators believe that there may have been some psychological motivation to discover an intangible, stable pattern behind the appearance of the unstable world in which they lived.[3]

It was not a clear cut movement; rational and irrational elements frequently mingled together. Pythagoras (c582-500B.C.),

---

2. See B. Russell, History of Western Philosophy, (London, 1961), p. 44.

3. See V. Ehrenberg, From Solon to Socrates, (London, 1968), p. 102 passim.

for example, united in his teaching the two opposing trends of his time, the scientific and the mystical. Contemporary opinions of Pythagoras varied. He is one of those historical figures who become legend almost before they are dead and it is difficult to assert with any degree of certainty that for which he was responsible and that which was conceived by his disciples. Plutarch maintained that he wrote nothing but that his disciples kept this fact to themselves.[4] We know that he came from Samos and founded at Croton in Southern Italy a school of scientific philosophy which was simultaneously a religious and mystical fraternity.

The religious and ascetic ideas and practices of the Pythagoreans border on the fantastic. The School, for example, abstained from eating beans, refused to walk in a main street, took care to spit on the trimmings of their hair and nails, and so on. However trivial these mystical elements in Pythagoreanism seem, there was one other aspect of the School's teaching which is of importance in understanding Zeno: their doctrine of the soul. There is confusion about the Pythagorean view of the soul: Aristotle[5] gives four separate accounts of it, but one thing is clear: the soul could transmigrate. Copleston[6] speaks

---

4. Porphyrius, Vita Pythagorea 19, in Kirk and Raven, The PreSocratic Philosophers, (Cambridge, 1962), p.221

5. Aristotle, De Anima A2, 404a16, De Anima A3, 407b20 De Anima A4, 407b27, Metaphysics A5, 985b29.

6. F. Copleston, History of Philosophy, Vol. 1 (New York, 1962), p.48.

of the Pythagorean soul as being the "real" man, although this is not very helpful, but "... the soul wanders from point of being to point of being, or, the same thing, from life to life."[7] (my translation). Porphyrius[8] tells us that Pythagoras "maintains that the soul is immortal: next that it changes into other living things." This accounts for the story told by Diogenes Laertius that Pythagoras prevented a dog being whipped as he claimed to hear the cries of a friend in the howling of the animal.

How is this of significance in understanding Zeno? The soul could be trained and tended to, and the two most valuable aids to this were the study of music and mathematics. Cramer[9] says that "The Pythagoreans, according to Aristonexus, practiced the purification ... of the soul by music."

As well as probably finding the solution to some geometrical propositions, the Pythagoreans made the great discovery of the numerical ratios which determine the concordant intervals of the musical scale. The sounds which strings produce are dependent on their length, and musical intervals such as the octave, perfect fifth and perfect fourth can be explained in terms of the ratio of the lengths of the sounding strings. This ratio is always constant irrespective of the note being considered. The octave can always be expressed as a ratio between strings

---

7. Julius Stanzel, quoted in Copleston, ibid., p.48, "... die Seele wandert von Ichzustand zu Ichzustand, oder, was dasselbe ist, von Leib zu Leib."

8. Porphyrius, loc. cit., p.223.

9. Quoted in Kirk and Raven, ibid., p.229.

of 2:1, the fifth by 3:2, the fourth by 4:3. Sound can therefore be explained in terms of number and proportion. They were so fired by this discovery that they were led to extend it beyond music to the cosmos. Aristotle[10] says that they saw "... the whole heaven to be a musical scale and a number."

They also linked geometry with numbers showing that, not only is the world of sound governed by numbers, so also is the world of vision. Pythagoras' famous geometrical theorem, that the square on the hypotenuse of a right-angled triangle equals the sum of the squares on the other two sides, established a fundamental characterisation of the space in which we move. It is a characterisation that is translatable into numbers. The exact fit of the numbers in the theorem describes the exact laws that bind the universe: the horizon line is horizontal, the force of gravity is vertical and cuts the horizon at right-angles. The unchanging physis was not an elemental substance such as the water of Thales, but something altogether different. It was the thing which Pythagoras had discovered to be underlying and unifying so many variables without itself being either one of these variables or resembling any of them: number or proportion.

As Aristotle[11] has it, from Pythagorean insights into the construction of musical sound, the Pythagoreans "thought its

---

10. Aristotle, Metaphysics, A5, 985b31-986a3

11. Aristotle, Metaphysics, A5, 985b23-6.

(mathematical) principles were the principles of all things," and that "numbers seemed to be the first things in the whole of nature."[12]

Not content with stressing the importance of number in the universe, they went on to maintain that things are numbers. "Evidently, then, these thinkers also consider that number is the principle both as matter for things ..."[13] We know that they believed in one kind of number," ... the mathematical, only they say it is not separate but sensible substances are formed out of it. For they construct the whole universe out of numbers ... they suppose the numbers to have magnitude."[14]

This is very difficult to understand but we must remember that "The early Pythagoreans, having no simple form of numerical notation, chose to express numbers in the form of patterns similar to those now found on dominoes or dice."[15] A large part of Pythagorean arithmetic, then, consisted of a study of the various series resulting from arranging units in geometrical patterns. According to the Pythagorean view, geometry was simply an application of arithmetic, and the point only differs from the arithmetical unit insofar as it has position,[16, 17]

---

12. Aristotle, Metaphysics A5, 985b31-986a3
13. Aristotle, Metaphysics,A5, 986a15
14. Aristotle, Metaphysics M6, 1080b16
15. Kirk and Raven, op. cit., p.243.
16. J. Burnet, Greek Philosophy, (London, 1962), p.83.
17. F. Cornford, Plato and Parmenides, (London, 1958), p.11: "The two sciences were not yet distinguished."

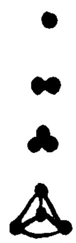
and lines, surfaces and solids were built up of adjacent points. Speusippus tells us that

1 is the point

2 is the line

3 is the plane

4 is the solid



This diagrammatic representation of numbers led them to think of numbers as spatially extended and confusing the points of geometry with the units of arithmetic, imagining both to possess magnitude.[18] By confusing the unit of arithmetic and the geometrical point, Speusippus shows that the application of the second unit has generated the line, i.e. one dimension, the application of the third unit has generated the surface, i.e. two dimensions, the application of the fourth unit has generated the solid, i.e. three dimensions. The number 4, being composed of four unit-points, is equated with the simplest geometrical solid, the tetrahedron, and so also is the geometrical solid, being composed of four unit-points, equated with the physical body.

To say that things are numbers means that all bodies consist of unit-points in space, which, when aggregated, constitute a number.[19] The cosmos is therefore a vast mathematical pattern in which things are not simply numerable: space and time are made up of indivisible minima which are material points.

Heraclitus (c535-475B.C.) came from Ephesus near Miletus. The key to understanding this gnomic writer lies in the phrase

---

18. Kirk and Raven, op. cit., p.246-247.

19. F. Copleston, op. cit., p.50.

"Know thyself" which was inscribed on the temple wall at Delphi and which was "a household phrase in those days."[20] Heraclitus says "I have sought for myself"[21] and "You cannot find the boundaries of the soul: it is too deep to be measured."[22] Words such as these heralded a new era in which man became the object of investigation, and human reason the decisive factor over and against divine guidance,[23] and, indeed, his personality is very important in the shaping of his philosophy. His writing style is vivid, aphoristic and so cryptic that he earned the soubriquet "the Obscure". It is interesting that he justifies his oracular style by reference to the God at Delphi who "neither utters nor hides his meaning, but signifies it,"[24] for he was an aristocrat by descent and by conviction. In Fragment 16 he maintains that the learning of Xenophanes, Hesiod, Pythagoras and Hecataios did not teach them to think and that wisdom is not about many things; Wisdom is about one thing, the Word of Heraclitus which is true evermore, even though the common man of Ephesus and the Sages of the past cannot understand him.[25]

---

20. J. Burnet, op. cit., p.59
21. Heraclitus, Fragment 101.
22. Heraclitus, Fragment 45.
23. V. Ehrenberg, op. cit., p.325.
24. Heraclitus, Fragment 11.
25. Heraclitus, Fragment 2.

He was guided by intuition as well as observation and believed in a governing logos, the universal formula according to which all natural events occur and which men should be able to isolate and understand. The most important part of this single unifying principle, according to Heraclitus, was the underlying connection between opposites. For example, he says that "Sea is the most pure and most polluted water; for fishes it is drinkable and salutary, but for men it is undrinkable and deleterious,"[26] and that "The path up and down is one and the same."[27] Fragment 88 declares "And as the same thing there exists in us living and dead and the waking and the sleeping and young and old." The sort of opposites he exemplifies are: the same thing producing different effects, (the sea water), the same thing justifying different descriptions, (the path), different characteristics occurring in a single process, (life). Each pair of opposites applies to the one subject and so the world is not an arbitrary, indeterminate collocation of distinct parts, but a coherent and therefore discoverable system.[28] He says in Fragment 10 that "Things taken together are whole and not whole, ... out of all things there comes a unity, and out of a unity all things." This unity depends upon a balanced reaction between opposites. He uses (Fr. 51) a metaphor of a bow and a lyre in which the tension of the strings has an equivalence in the bow itself

---

26. Heraclitus, Fragment 61.

27. Heraclitus, Fragment 60.

28. "Heraclitus" in Encyclopedia Brittanica (1964 Edition).

or the arms of the lyre. The equivalence produces a coherent, unified and stable complex: if there were no equivalence the system would collapse.[29] If this strife, the action and reaction between opposites were to cease, then one or other of the pairs of opposites would dominate and the world would be destroyed. There must, therefore, be unending strife in the universe if it is to remain balanced. Aristotle[30] has it that "Out of the Different cometh Harmony in her fairest form."

This is usually taken to mean that the cosmos must always be in a state of flux. I am, however, not so sure that this is the case.[31] Heraclitus never gives a reason for supposing the world to move out of equilibrium. He seems to be saying that if a change takes place, unless equilibrium is restored by a corresponding change elsewhere, the world will be destroyed. However, Plato and Aristotle certainly considered that Heraclitus believed in a state of cosmic flux, and it will be argued that Parmenides and Zeno also had this opinion of Heraclitus.

The Fragment which I feel causes the problem is his twelfth, which illustrates the unity that depends on balance in change: "Upon those that step into the same river different and different waters flow ... It scatters and ... gathers ... it comes together and flows away ... approaches and departs." Plato's transformation

---

29. Kirk and Raven, op. cit., pp. 194-195.

30. Aristotle, Nichomachean Ethics, Book 8, ch.1.

31. See Kirk and Raven, op. cit., p.167 passim.

of this fragment occurs in the Cratylus[32] where it has evolved into "Heraclitus somewhere says that all things are in process and nothing stays still, and likening existing things to the stream of a river he says that you would not step twice into the same river." This is further transformed by Aristotle[33] into "And some say not that some existing things are moving, and not others, but that all things are in motion all the time, but that this escapes our perception," and "All things are in motion, nothing steadfastly is."[34]

According to this view, objects partake of opposites simultaneously; they are constantly in a state of being something and not being that something. They are in a constant state of becoming. Everything, even though this may be invisible, is in motion.

These, then, are the two major world-views against which the words of Parmenides and Zeno must be seen. The Pythagorean school asserted that the world is made up from indivisible minima, little points which they also took to be numbers. The followers of Heraclitus, even if the precise meanings of Heraclitus are obscure, asserted that the cosmos was in a constant state of flux: everything moves and nothing stays still.

---

32. Plato, Cratylus, 402a

33. Aristotle, Physics 03,253b9

34. Aristotle, De Caelo, 298b30 (III,i).

Chapter Two

The Parmenidean Sphere

"I have made such wonderful discoveries
that I am myself lost in astonishment:
Out of nothing I have created a new
and another world."

(John Bolyai)

Parmenides of Elea is reckoned by Cornford in From Religion to Philosophy to be the "discoverer of logic." Certainly his application of logical analysis to metaphysical problems was a profoundly original contribution to philosophy. Little is known about the date of his birth, but in Plato's Parmenides he is said to be about sixty-five years of age. This gives him a birth date of approximately 515 B.C. He began his philosophical life as a Pythagorean: Strabo[1] has it that "Elea ... whence Parmenides and Zeno came, both Pythagoreans." Diogenes Laertius[2] records that Parmenides "associated also, as Sotion recorded, with the Pythagorean Ameinias ... It was by Ameinias ... that he was converted to the contemplative life."

His tenets are embodied in a short poem called "On Nature" of which about 160 lines have been preserved in the writings of Sextus Empiricus and others. For us, the most important part of the poem is the section entitled "The Way of Truth." Here Parmenides reacted against the Heraclitean theory of flux, declaring that Being was one, finite, motionless, unchanging, and indivisible, with neither beginning nor end. For him, Being is unique and necessary,[3] and is deduced from the proposition "It is, and not-being is impossible." The proposition implies that change is a mere illusion because it involves something coming into being

---

1. Quoted in Kirk and Raven, op. cit., p.264.

2. Quoted in Kirk and Raven, ibid., p. 264.

3. J.B. Chethimattam, Consciousness and Reality (London, 1971), p.3

out of nothing or passing out of being into nothing. His ideas formed a major contribution to early science[4] and they became one axiom which regulated the thinking of his immediate successors.[5] Parmenides' ideas can even be found lurking at the base of Aquinas' Third Way of God's Existence.[6]

Many later philosophers have been confused by Parmenides, often misunderstanding what he said.[7]

His thoughts on the nature of the universe are contained in a poem addressed to Zeno, of which nineteen fragments are considered to be authentic, with six other fragments either spurious or doubtful. This means that the sources for a reconstruction of what Parmenides meant are not very promising, but the early part of the poem, the "Way of Truth", was preserved for us by Simplicius who writes: "The lines of Parmenides on the One Being are not many, and I should like to append them to this commentary both as a confirmation of what I say and because of the rarity of the book."[8]

The poem, which is written in hexameter verse and has three sections, the "Prologue", the "Way of Truth" and the "Way of

---

4. One thinks today of the laws of the conservation of energy.

5. See Ch.1, note 1.

6. "Si igitur omnia sunt possibilia non esse aliquando nihil fuit in rebus. Sed si hoc est verum etiam nunc nihil esset, quia quod non est non incipit esse nisi per aliquid quod est. Si igitur nihil fuet ens, impossibile fuit quod aliquid inciperet esse, et sic modo nihil esset, quod patet esse falsum."

7. For example, B. Russell, op. cit., p.67.

8. Simplicius, Physics, 115.26, quoted in Guthrie, History of Greek Philosophy, Vol. 2. (Cambridge, 1979), p.3.

Opinion", begins in grand style. It is reminiscent of Homer in that Parmenides declares that mares carried him to a goddess after a magical journey through the gates of Day and Night. The words spoken by the goddess to Parmenides are very significant:

> Young man, companion of immortal charioteers, who comest by the help of the steeds which bring thee to our dwelling; welcome! no evil fate has despatched thee on thy journey by this road (for truly it is far from the path trodden by mankind); no, it is divine command and Right. Thou shalt enquire into everything: both the motionless heart of well-rounded Truth, and also the opinions of mortals, in which there is no true reliability. [9]

Here, already, we have the notion of motionlessness, but the more important features of the Prologue are these: it is not open to anyone to make this journey for it must be by divine command, which means that Parmenides is privileged above all other men. Right and divine consent have permitted his insight into Truth.[10] He puts his philosophy into the mouth of the goddess and the subsequent fragments are to be seen as divine revelations because Parmenides is allowed only to accept what the goddess says: " ... and you must accept my word when you have heard it."[11]

The goddess then indicates the ways of enquiry open to Parmenides: "the one that it is, and it is impossible for it not to be, is the way of persuasion, for it follows Truth; the other, that it is not, and that it must necessarily not be: this I

---

9. Parmenides, Fragment 1, (my underlining).

10. For a discussion of how much Parmenides thought that this was divine revelation and how much was simply convention, see Guthrie, op. cit., p.11, footnotes.

11. Parmenides, Fragment 2, lines 1-2.

declare to be a wholly indiscernible tract, for you could neither recognise that which is not, nor express it. For it is the same thing that can be thought and can be."[12]

There is no suggestion that anyone ever takes this second path although there may be an imputation in the "you" that the goddess could recognise and express the path of what is-not. Parmenides, however, dismisses this second path as inconceivable; the statement "It is not" is impossible, for the subject of such a statement could never enter our minds because it could neither be recognised as the subject nor be shown to anyone else. Fragment 6, line 1, amplifies this, where the goddess says that "What can be spoken and thought of must be." This explains also the rather obscure Fragment 4 where the goddess claims that "... things absent are securely present to the mind." The object of thought does not have to be before us: it could be "... scattered everywhere utterly throughout the Universe"[13] but as long as the object of thought is, it can be thought, recognised and spoken of.

We are also warned in Fragment 6 from considering another path, the path on which "mortals, knowing nothing, wander in two minds, for perplexity in their bosoms steers their intelligence astray. They are carried along as deaf as they are blind, with

---

12. Parmenides, Fragments 2 and 3.

13. Parmenides, Fragment 4, lines 3-4.

no judgement, who believe that to be and not to be are the same and not the same, and for whom in everything there is a way of opposing stress."[14]

The goddess has therefore mentioned three paths of enquiry, one correct and two full of error: firstly the assertion that what is, is; secondly, the assertion that there is nothing; thirdly, a confusion between "is" and "is not". This third way results from the use of the senses, (why else the references to blindness and deafness?), and in Fragments 7 and 8 it becomes clear that the goddess means to deny the belief in any change, motion, becoming or perishing of what is, a belief which results naturally from the use of eyes, ears and other sense organs.[15] Indeed, Fragment 7 commands that we must not "let ordinary experience in its variety force you along this way"; we must not be ruled by the senses, but place our faith in Reason.

The poem now embarks on the central theme of the doctrine expounded to Parmenides by the goddess, the deductions to be drawn from the simple statement "It is", and it is here that the position which the disciple Zeno is defending becomes apparent.

---

14. See Heraclitus, Fragment 8: "That which is in opposition is in concert ..." and Fragment 55: "Those things of which there is sight, hearing, knowledge; these are what I honour most." I feel that the Parmenides fragment is clearly directed at the Heracliteans.

15. Guthrie, op. cit., p.23.

The first deduction made by the goddess is that "What is" is eternal, neither coming into being nor perishing. She says that Being "was not in the past, nor yet shall it be, since it now is, all together, one and continuous."[16] The basis of this part of the theory is an axiom of Greek thought, an axiom which is clearly expressed by Aristotle[17]: "Generation from the non-existent is impossible: in this opinion all the natural philosophers concur." It follows from this that there can have been no point in time when there was no Being. Also, if Being is to be in the future, there is an implication that, at the moment, it is-not. If that is the case, then Being cannot come into being in the future because it would come from nothing, and generation from the non-existent is impossible. It follows, therefore, that past and future are both meaningless, the only time is an eternal omnipresent time, and Being "must of necessity be both uncreated and imperishable."[18] Guthrie[19] has it that "Being can only be thought of in the present."

Line 3 of Fragment 8 asserts that Being is imperishable, and, although the argument for this is obscure, some sense can be made of it. To suppose that what is can perish is to suppose that at some future time it will be possible to say that it is not. This, however, we are excluded from saying for two reasons: firstly,

---

16. Parmenides, Fragment 8, line 5.

17. Aristotle, Physics, 187a34.

18. Kirk and Raven, op. cit., p.274.

19. Guthrie, op. cit., p.29.

that the notion of "future" has been shown to be meaningless, and, secondly, that the notion of "what is-not" is itself also meaningless. Being is therefore continuously present with no possibility of any other temporal state of affairs, i.e. it is timeless. Guthrie[20] points out that this is a truly intellectual achievement in that it draws out the distinction between "eternal" and "everlasting". To conceive of something as "everlasting" is tacitly to set it in time, but to conceive of something as "eternal" is to say that "was" and "will be" are meaningless and cannot be applied to it, thus rendering the concept of a temporal continuum redundant.

From what has been so far said it follows that Being is unique. There are only two possible alternative states, Being and Non-being, and the latter is not allowed by the logic of the goddess. Therefore, everything that is, is, and there can only be Being alone.

Fragment 8, lines 22-25, goes on to indicate the next characteristic of Being, that it is continuous and indivisible: "Nor is it divisible, since it all equally is. It does not exist more fully in one direction, which would prevent it from holding together, nor more weakly in another, but all is full of what is. Therefore it is all continuous, for what is is close to what is." This is to say that there is no degree of Being: "... it all equally is." We have seen that the only choice is between "It is" and "It is not". Guthrie[21] claims that

---

20. ibid., p.29.
21. ibid., p.33.

what Parmenides means is that "the lack of homogeneity which would result from "what is" existing more or less might cause it to fall apart and be divided". If we consider what has so far been deduced, there must be a denial of the void, the "what is-not". There would have to be the "what is-not" to act as that which separated the parts of Being. As there can be no such thing as the "what is-not", Being must be indivisible and continuous.

Parmenides now goes on to make his next deduction, that Being is immobile: "... unmoved, in the grip of mighty bonds ... remaining the same in the same place it rests by itself and so remains firmly where it is".[22] Little argument is advanced here but we can reconstruct the process which led Parmenides to this conclusion. In order to move, Being must move to some place where it is not. But the existence of the "is-not" has been shown to be impossible. That is to say that Being can only move where there already is Being, i.e. move internally. This, too, is impossible because movements within Being imply empty space into which blocks of Being could move, which has been seen to be impossible. As well as this, the very notion of "blocks" of Being has been shown to be illegitimate. Therefore there can be no locomotion and so the goddess condemns all human experience as illusory. This condemnation of human experience is further amplified in Fragment 8, line 38, where Parmenides says that "... all things that mortals have established, believing in their truth, are just a name: coming into being and perishing, being

---

22. Parmenides, Fragment 8, lines 26-33.

and not being, change of place and alterations of bright colour." Names are therefore mere sounds standing for nothing real. Fragment 19 has it that "Thus according to appearance were these things created ... and men have laid down a distinguishing name for each of them." By this he means not that names are arbitrary sounds which could easily be attached to something else, but that there is no real object for names to be attached to at all. What he seems not to have noticed is that there can be neither sounds nor anyone to do the naming. This, together with the fact that the senses must be an illusion, renders the concept of "a name" redundant.

This is very difficult, but the same sort of point is discussed by Descartes in the Third Meditation where he talks of a situation in which "ideas represent nothing as though it were something."[23] Descartes asks whether cold is a privation of heat, or heat a privation of cold, or whether both are real qualities. Now, when I experience the feeling of being cold, it seems that I am not simply experiencing the absence of heat, but something which is altogether more profound and positive. Descartes, however, says that "If it is correct that cold is merely a privation of heat, the idea which represents it to me as something real and positive will not be improperly termed false."[24] For Parmenides, the senses are always mistaken because the things which they represent to us as positive cannot possibly be positive

---

23. Descartes, Philosophical Works, Vol. 1, (Cambridge 1970), trans. Haldane and Ross, p.164.

24. ibid., p.164.

because they are, in fact, absent. From this it follows that the word "is" can never be used copulatively because there are no qualities to be attached to objects nor objects to which objects can be attached. To say that a "tree is green" is to utter two names which have neither sense nor reference. As well as this, change can only take place through time, and, as Being is timeless, change is impossible.

The section of the poem on Truth concludes in Fragment 8, lines 42-49, where the goddess claims that Being is spherical: "But since there is a furthest limit, it is complete on every side, like the mass of a well-rounded ball, equal every way from the centre; for it may not be at all greater or smaller in this direction or in that", and later, "For in all directions equal to itself, it reaches its limits uniformly."

The problem for scholars finally begins to surface in this section and the problem is this: does Parmenides conceive of his one true Being as purely conceptual or physically occupying space? The fact that it cannot be grasped by the senses but only through intellectual insight, the fact that it is immutable and timeless with neither change through time nor movement through space, the fact that it is unique, all of a sort and indivisible, leads many scholars to the conclusion that the Parmenidean sphere is not a body physically fitting space in the same way as Saturn or a tennis ball, but that it is an idea. They maintain that Zeno is not defending a position in which Being is substantially real in the elemental sense of the Milesians, but is defending a position in which Being is something like an abstract governing principle. What seems to lead them to this conclusion is the difficulty which arises if we ask what lies outside the boundaries of the sphere, of this spatially finite Being? Where it stops, there must be either something else or empty space, both of which seem to put Parmenides in a difficult position. Non-Euclidean geometries such as those of Lobachevsky and Riemann seem to overcome this problem. However, if Being were not finite, that is, if it had no spatial limits, then in what sense could it be said to exist completely? The answer is, that, for Parmenides, it could never exist completely, and therefore, as it <u>does</u> exist completely (there are no degrees of Being) it must be finite. Fragment 8, lines 42-43, has it: "Since there is a furthest limit, it is complete on every side, like a round ball." Those who see Parmenides as an idealist may persist in asking what lies beyond the sphere. Clearly, Parmenides himself would not have asked this sort of question, but if I can speak for him with the logic of the way of Truth: there cannot be empty space beyond the sphere, because empty space is nothing and nothing neither exists nor can be imagined. Similarly, there cannot be something beyond the sphere because that thing would have to <u>be</u> in order to be beyond the sphere and would therefore be part of Being and so part <u>of</u> the sphere.

This is not all that strange given the general tenor of Hellenic thought. Plato declares in the Timeus (33b.c) that the cosmos is spherical because the sphere is "the most perfect of shapes." Elsewhere he uses the idea of roundness to indicate status in the kingdom of animals and that the fish is the lowest of creatures by virtue of having a flat head. More significantly, both Plato and Aristotle agree that there is nothing outside of the spherical cosmos, neither matter nor empty space. Aristotle says that "There is neither place nor void nor time outside the heaven."[25]

I therefore consider that the Parmenidean sphere is a substantial object although much august opinion stands against me. Brehier[26] maintains that, for Parmenides, Being is neither an abstract notion nor a sensible image: he (Brehier) calls it a sort of geometrical image. This is rather dark. Possibly he is trying to show that we cannot see the sphere, but that there _is_ a sphere, the notion of geometry coming in when we try to describe it, perhaps as "the locus of all points round a given point". Guthrie[27] declares that Parmenides is arguing against the Pythagoreans who maintain, as shown earlier, that the line springs from the point, the surface or plane from the line, the solid figure from the plane, and perceptible bodies from the solid. Parmenides "pounces" on the leap from geometrical solid to physical nature by asserting that it is illegitimate to move from the intelligible geometrical figure to the moving and perceptible world: "His reality is the spherical solid of the geometer, now for the first time separated from its physical manifestations, an object of thought, not sense".[28]

---

25. Aristotle, _Physics_, 207a7.
26. E. Brehier, _The Hellenic Age_, Vol.1. (Chicago, 1965), p.56.
27. Guthrie, op. cit., p.49.
28. ibid., p.49

This is absolutely unacceptable even though it concurs with Brehier's aphoristic statement. Who or what is doing the thinking? Certainly, if Guthrie's point of view is correct, then Zeno's position becomes untenable and Zeno's purpose in the Paradoxes will have to be completely rethought. However, I think it can be shown that Guthrie and Brehier are in error when they say that the Parmenidean sphere is incorporeal, "an object of thought". We have seen that the Parmenidean sphere is the only thing that is, therefore it must be self-subsistent thought, keeping itself in existence through contemplation of itself. What else could it think of?

This is to argue metaphysically: we can also argue historically for the materialist viewpoint. Philosophers at this time had not really distinguished between the incorporeal and the non-existent, thinking that "Being is just as much as is sensible."[29] Surely if Parmenides had represented his One as self-subsistent thought in an intellectual climate like this, then Plato (and, certainly, Aristotle) would not have failed to record this startling fact! Clearly the sphere is not an object apparent to the senses for it is apprehensible only in thought. But does it follow from this that it is Idea and that the sphere does not occupy space, but is "no more and no less (in space) than the figures to which Euclid supplied definitions at the beginning of the various books of the Elements."[30]? Stace[31]

---

29. Kirk and Raven, op cit., p.216.
30. Guthrie, op. cit., p.49.
31. Stace, A Critical History of Greek Philosophy, (London, 1920) pp. 47-48.

concedes that Parmenides and the Eleatics generally regard Being as material, but he still wants to say that the absolute reality, of which the world is a manifestation, consists in thought, in concepts.

There seems to be a fundamental confusion in this school of thought between "being comprehended through thinking and reason" and "Being Thought or Idea itself." Copleston[32] gives an example which makes this clearer: Thales said that the universe was water; clearly, this is not obvious to the senses and can only be comprehended through thinking. Is this to say that, because the notion of water being the only thing-that-is is comprehended through thought, the water is Idea and not in space? If water is not in space, and everything is composed of water, then nothing is in space. Thus we arrive at the opposite, and unlikely, conclusion to Thales. The sphere is a material object, all that there is.

We can now turn to a final point about the sphere, a point which emerges if we maintain that Being is a physical object: can we say anything about Being other than that it is? Aristotle[33] couches the problem thus: "... if there is to be a being-itself and a unity-itself, there is much difficulty in seeing how there will be anything else besides these ... For what is different from being does not exist." In the ultimate part of the poem, the Way of Opinion, the goddess describes a physical pluralised

---

32. Copleston, op. cit., p.68.

33. Aristotle, Metaphysics, 1001a(11)

universe, the world as it seems to be, but she states in Fragment 19: "Thus according to appearance these things have arisen and now are, and as they have grown will end in time to come; and men have laid down a distinguishing name for each of them." There is no real object, as we have earlier deduced, for these names to be attached to. From this we can cogently maintain that there can be no predicative nor copulative use of "is" for Parmenides: the only "is" he will allow is the "is" of existence. Guthrie's conclusion is that the phenomenal world is like a hallucination or dream, and so it "becomes clear that the attributes we use in the real world are hallucinatory, signifying nothing."[34] We are, in fact, driven to a position remarkably akin to that adopted by Descartes in his First Meditation: "I shall consider that the heavens, the earth ... and all other external things are nought but ... illusions and dreams."[35]

Nothing, therefore, can be predicated of Being. It may be objected that Parmenides himself gives Being attributes: it is motionless, spherical, continuous, indivisible and eternal. However, the proposition "Being is motionless, spherical, etc." cannot be empirical. This is because Being is the only thing-that-is. If the concept of "proposition" makes any sense at all in Parmenides' schema (which I doubt), any proposition which can be made must be made by Being itself, and, if nothing exists

---

34. Guthrie, op. cit., p.75.

35. Descartes, op. cit., p.148: Descartes was at least left with a problematical self, but Parmenides does not even have that.

apart from Being, must therefore be analytic. The notions of continuity, immobility, etc. are implicitly contained in the concept of Being in the same way as the suckling of infants or giving birth to live young is implicit in the concept of "mammal". Neither "immobility" nor "giving suck" amplify the notions of Being or mammal, but are actually contained within these notions.

Parmenides' monism is a "fight against common sense",[36] but Zeno's position has become clear. The Eleatics deny the reality of plurality and motion, and, although they do not deny that we experience or sense plurality and motion, they declare that what we sense is illusion. Reason has shown that there can be no plurality, no movement and no change. All that is is the sphere: Being. The notion of perpetual flux of the Heracliteans, the notion of unit-points of the Pythagoreans, and the common sense view of a world with occurrences have been overturned. It is in a bizarre defence of this world-view that Zeno puts forward his paradoxes. They are of a complexity and elegance such that they are still marvelled at today and, ever since they were constructed, have exercised a hypnotic and corrosive power on philosophical minds.

---

36. Ehrenberg, op. cit., p.108.

Chapter Three

Zeno the Eleatic

"Confusion now hath made his masterpiece!"

(Macbeth)

We do not know precisely when Zeno was born. Plato[1] says that "According to Pythodoros, Zeno and Parmenides once came to Athens for the Great Panathanea. Parmenides was already well on in years; but although sixty-five and most white, he was a fine looking old man. Zeno was about forty at the time. He was tall and graceful and was said to be a particular favourite of Parmenides." This would lead us to give him a birth year of around 490 B.C. He was the son of Teleutagoras and, like Parmenides, was a citizen of Elea. According to Strabo,[2] "through their agency the city was well governed" and it is presumed that Zeno spent most of his life there. Plato[3] says that "... I might cite Pythodoros, the son of Isolochus, and Callias, the son of Calliades, who have grown wiser in the society of Zeno, for which privilege they have each of them paid him the sum of a hundred minae to the increase of their wisdom and fame." Plutarch[4] says that "Pericles also studied under Zeno the Eleatic at the period when, like Parmenides, he was lecturing on natural philosophy." Clearly, even if he spent most of his life in Elea, Zeno was no stranger to Athens. Many anecdotes attest to his great physical courage and a most extraordinary death.

---

1. Plato, Parmenides, (127)
2. Quoted in Kirk and Raven, op. cit., p.264.
3. Plato, Alcibiades, 1.119a.
4. Plutarch, "The Life of Pericles" in The Rise and Fall of Athens, (London, 1969), pp. 172-173.

Traditionally, he is supposed to have been subjected to torture for conspiring against the tyrant Nearchus of Elea. Refusing to name his fellow conspirators, he died at his torturer's hands. Some anecdotes claim that he showed his cleverness by revealing the names of Nearchus' friends as his fellow conspirators; others that, rather than betray his friends, he bit off his tongue and spat it at the tyrant. Lee[5] suggests 430 B.C. as the date of his murder.

He is famous for his paradoxes in which, to recommend the Parmenidean doctrine of the "One", i.e. an indivisible reality, he sought to gainsay the common sense belief in the existence of the "Many", i.e. plurality, and things capable of motion. His method is described by Plutarch[6]: "Zeno had perfected a technique of cross-examination which enabled him to corner his opponent by the method of question and answer, and Timon of Phlius has described his as "Zeno, assailer of all things, whose tongue like a double-edged weapon argued on either side with an irresistable fury."" Thus did this single minded and enthusiastic disciple of Parmenides bring his remarkable intellectual powers to bear on the defence of the _logos_ of Parmenides.[7] All his arguments are aimed at making us accept the strange truth that reality is one, indivisible

---

5. H.D.P. Lee, _Zeno of Elea_, (Amsterdam, 1967).
6. Plutarch, op. cit., pp. 172-173.
7. The claim (_Metaphysics_ B4, 1001b14) that "His speculations are crude" protests a total misunderstanding on Aristotle's part.

and motionless, by using the method of taking the hypothesis contrary to his own and showing how it leads to a logical conclusion in complete absurdity: "(My treatise) pays them in their own coin with a vengeance; and is meant to show that their hypothesis, ("Reality is Many"), when closely examined, involves yet greater absurdities than our assumption of the One."[8] Aristotle[9] calls him the inventor of dialectic on account of this method, in which he takes his opponents' position and shows that it leads to a pair of contradictory conclusions. Of the paradoxes, over forty of which are said to have been created, only eight survive.

The paradoxes can be divided into two basic groups directed respectively against the ideas of plurality and motion. It seems, therefore, obvious that Zeno was a faithful disciple of Parmenides and followed the Master in the belief that Reality is one, indivisible and motionless. However, some scholars have put forward the unlikely view that Zeno had no wish to deny the possibility of motion, only that it was incompatible with a belief in plurality.[10] They say something like this: "If there is a plurality there can be no motion, but as there is motion it follows that there can be no plurality." This is extraordinary, taking insufficient account of Zeno's Eleatic background and his relationship to Parmenides, a relationship clearly attested to

---

8. Plato, Parmenides (128)

9. Aristotle, Physics Z

10. Paul Tannery was the leader of this school of thought.

by Plato[11]: "I see, Parmenides, that Zeno's intention is to associate himself with you by means of his treatise no less intimately than by his personal attachment." Zeno was the chief disciple of Parmenides; indeed, Parmenides' poem is dedicated to him, and it seems clear from the paradoxes themselves that the statements "there is plurality" and "there is motion", both of which were asserted by everyone who was not an Eleatic, are the objects of his attack. It also seems highly unlikely that Zeno is saying that motion is possible and perceivable through the senses, while at the same time, through the Paradox of the Millet Seed, denying that we can have confidence in the senses.

The paradoxes are stated* as follows:

1. First he takes the assumption that things are a Many, that there is a plurality of things in the world:

   (a) "If things are Many, there must be as many as there are and neither more nor less than this. But if they are as many as they are, they must be finite in number."

   (b) "If things are Many, they must be infinite in number. For there are always other things between those that are, and again others between those. And thus they are infinite in number."

A way to escape this is to say that there need be no intervention of a third thing because these ultimate "things"

---

11. Plato, Parmenides (128)

* My couchings of the paradoxes are a conflation of various translations and utilise largely those contained in Kirk and Raven, op. cit., and Freeman, op.cit.

have no magnitude. This, however, is precluded by:

2. "If things are a Many, they must be a number of units. These units may be either with magnitude or without magnitude.

   (a) "If they are without magnitude (size, thickness or bulk) then such a unit, if added to any other thing will not make it larger. For nothing can gain in magnitude by the addition of that which has no magnitude. And so it follows at once that that which was added was nothing ...[12] So that if the object is not decreased by the subtraction of the unit, it is clear that the thing added was nothing, and that the thing subtracted was nothing. That is, everything is infinitely small, so small as to have no magnitude."

   (b) "If the many things are units with magnitude, that is, if the unit is something having size and thickness, then it must have a definite size and thickness and each part of it must be a definite distance from each other part. And if you take one such part, the same argument applies: it will have a definite size, and therefore parts, and each part will be a certain distance from each other part. The same reasoning, in fact, applies always. There can never be a subdivision so small that it cannot be redivided, that is, so small that it will not have a "one part" and "another part", whose relations to each other can be stated in terms of the distance between them.

---

12. Here probably followed a similar argument about subtraction.

> Thus you get an infinite number of things each having magnitude, and this infinite number of magnitudes added together make up infinite sizes. Thus things are infinitely great."
>
> Therefore if things are Many, they must be both small and great; so small as to have no size, so large as to be infinite."

Some commentators[13] are puzzled as to whether it is the One or the Many that Zeno is attacking in this argument because it can be applied also to Parmenides' sphere. Freeman[14] suggests that the One which he is attacking is not the Sphere, but "the One regarded as something from which the Many could be derived, and the concepts of the One necessary to make this possible." She clearly feels that it is the Pythagorean unit which is under attack. If, like the Pythagoreans, we take as the unit some ultimate indivisible, then we are starting with something that is nothing; something that has no size, extension or bulk. We cannot get size by adding together things which have no size, no matter how many we add. If, on the other hand, we take as our fundamental building brick, a something which has size, then it must be infinitely divisible and therefore, from paradoxes 1(b) and 2(b), of infinite size already. Nothing, therefore, can have

---

13. See N.B. Booth, "Were Zeno's arguments a reply to attacks upon Parmenides?" in Phronesis, 2, (1957), pp. 1-9.
14. K. Freeman, The PreSocratic Philosophers, (Oxford, 1946), pp. 155-156.

definite size; we are always left with zero magnitude or infinite magnitude.

The attack reduces to absurdity the Pythagorean view that by the addition or multiplication of units we can get the Many of the cosmos. Parmenides' Sphere, however, also has size and extension: Parmenides claims that it is finite yet Zeno has just shown that it is either of zero magnitude or of infinite magnitude. That this is a problem is acknowledged by Seneca who says somewhere that "If I am to accept Parmenides, there is nothing except the One; if Zeno, that there is no One, even." Zeno, then, appears to be a nihilist. Freeman[15] says that "whether he was not aware of this development and remained always in his own view a disciple of Parmenides, is difficult to determine." Booth[16] however, reckons that "Zeno probably did not notice that his arguments applied equally well to Parmenides."

Leaving this problem aside, Aristotle and others saw that this paradox was a clear attack on the Pythagorean unit-point. If we designate "Reality is composed of points" as P, and "Reality has magnitude" as R, the structure of his argument is clear:

$$(P \longrightarrow -R) \ \& \ (P \longrightarrow R)$$

This is unacceptable and must lead to the following:

$$(P \longrightarrow -R) \ \& \ (P \longrightarrow R) \longrightarrow (-P)$$

---

15. K. Freeman, ibid., p.154.

16. Booth, loc. cit., p.

If the attack carries away the Parmenidean Sphere as well as the Pythagorean point, tough luck on Parmenides: "... it is open to doubt whether, even if he was aware of this fact, he would have allowed it to deter him."[17] It does not seem likely, however, given what we know about Zeno, that it was a deliberate intention on his part.

3. The Paradox on Space is:

> "Everything is in Space. By this we must mean that it is in something. But if Space is something, then Space itself is in something, and so on, *ad infinitum*."

There is no opposite half to this paradox, that "Space is nothing", probably because he felt that it was absurd to assert the proposition "Everything is in nothing." This paradox, according to Aristotle[18], can easily be resolved because Zeno does not seem to have realised that the proposition "Space is something" is equally absurd. Zeno is using the term "Space" in the sense of "Place", and place is not a thing in itself; nor is it nothing. Space is the outer *boundary* of an object.

While this may shed some light on the problem exposed by Zeno, Aristotle has lost his way in what this argument is doing. In order to support Parmenides, Zeno is saying that there is no such thing as space. He is not saying that space is a thing-that-is-not, but that the very concept of space is incoherent; what Parmenides would call a "name".

---

17. Kirk and Raven, op. cit., p.303.

18. Aristotle, *Physics* Δ 3, 210b22.

To argue against Zeno by saying that Space is not a thing is to make Zeno's point for him! Kant[19], agreeing with Zeno, contends that space is not a a posteriori intuition, but that things are intuited in space. Space, for Kant, is the framework in the intellect ("the subjective constitution of our mind"), an a priori intuition conditioning whatever is apprehended through the senses.

4. The Paradox of the Millet Seed runs:

> "A grain of millet seed falling makes no sound. Each part of the grain of millet falling makes no sound; how, therefore, can a bushel falling make a sound?"

I first thought that this argument set out to show that a collection of nothings could become a something, a collection of silences could become a sound. However, I was mistaken: this argument is against sense preception and means that Zeno has changed his strategy. Parmenides and he have shown that the deductions of logic are incompatible with experience and so the world of sense-experience must be rejected. We can now say that it is no longer to be rejected because logic is superior to the senses, but also because of the inability of the senses to convey accurate information.

Aristotle tells us that there were four arguments on Motion. Theories of motion are dependent on theories of the nature of space and time; Zeno is arguing against the

---

19. I.Kant, Critique of Pure Reason, (Macmillan 1970), pp.67-74.

two views prevalent in his time. The first view was that space and time were infinitely divisible in which case motion is continuous and smooth-flowing: the second view is that space and time are made up of indivisible minima. Zeno's arguments are directed against both views because "Motionlessness" is his ultimate destination. It is surprising, then, to encounter so many philosophers who have said that Zeno can be overturned if we think of motion as continuous, or, if we think of space as being composed of minimal particles. Zeno's attack on motion through the four paradoxes, then, is two-pronged. The paradoxes can be divided into two pairs, the first pair against the "continuity" theory, the second against the "indivisible minima" theory. There is a lot of confusion about the precise nomenclature to be used for these paradoxes, but I name them as follows:

5. The Stadium Paradox, which has two versions:

(a) "Before a body in motion can reach a given point, it must first traverse the half of the distance; before it can traverse the half, it must traverse the quarter, and so on. Hence for a body to be set in motion it must perform an infinite number of operations and so can never get started."

(b) "In order to reach a given point a body in motion must traverse half the distance; thereafter, it must traverse half the remainder, then half of that remainder, and so on. Hence, the body will never reach its destination because it must first traverse an infinite number of divisions."

Which of these two versions, progression or regression, is the correct one? Aristotle is rather confusing. His Physics, Z2, 233a21, declares that it should be the regression, but he says in the same section that "the argument assumes falsely that it is impossible to traverse, or come in contact with, each one of an infinite number of things in a finite time." Vlastos[20] opts for the progression. I propose to consider both versions as many later philosophers have been rather arbitrary in deciding exactly what Zeno did say.

6. Achilles and the tortoise:
"Achilles and the tortoise are having a race and the tortoise gets a start. Achilles can never overtake the tortoise because, by the time he reaches the point from which the tortoise started, it will have moved on to another point; by the time he reaches that second point it will have moved on again, and so on."

7. The Flying Arrow, which is the first of the pair directed against indivisible minima, survives in a fragment attributed[21] to Zeno: "That which moves, moves neither

---

20. G. Glastos, "Zeno of Elea" in Encyclopedia of Philosophy, (New York, 1972), ed. P. Edwards.

21. Zeno, Fragment 4, in K. Freeman, Ancilla to the PreSocratic Philosophers, (Oxford, 1948), p.47.

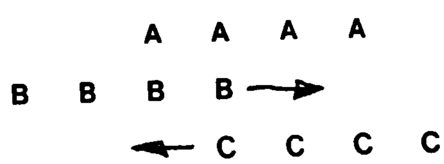
in the place in which it is, nor in that which it is not."

This is more often reconstructed as:

"An object is at rest when it occupies a space equal to its own dimensions. An arrow in flight occupies, at any given moment, a space equal to its own dimensions. Therefore an arrow in flight is at rest."

8. The Moving Rows, which is by far the most complex of the surviving paradoxes:

"Consider three rows of solid objects, As, Bs, and Cs. These objects are the same size as the indivisible minima from which reality is created. The As are stationary while the Bs move to the right and the Cs move to the left at the same speed as the Bs. The Bs and Cs move such that they pass an A in an indivisible minimum



of time. But, while each B has passed two As, (which, by the information given, means in two indivisible minima of time), each C has passed four Bs, (which, by the information given, must have taken four indivisible minima). Therefore, the indivisible minima are divisible after all."

These, then are Zeno's arguments[22] and it will help our understanding of them if we clarify precisely what their objective is and who they were directed against.

Many feel that the arguments are directed very firmly at the school of Pythagoras: "And upon the unfortunate Pythagoreans, who had hitherto confused the indivisible units of arithmetic with the points in infinitely divisible geometrical magnitudes, this last argument (the Moving Rows) must finally have impressed the urgent need for revision of their suppositions."[23] This is not universally accepted but it is generally agreed that opponents of the anti-Pythagorean view have not made their case.[24]

However, it must be noted that in Zeno's arguments the target is the whole idea of plurality, the whole idea of motion, the whole idea of place. In the arguments about plurality there is no real evidence that Zeno's hypothesis really means "If there is a plurality of Pythagorean units." It should be remembered that the Pythagoreans were not at all sure about what their

---

22. See Appendix One for an attempted construction of Paradox of Time which may help to clarify what is going on in the Paradox of the Moving Rows.

23. Kirk and Raven, op. cit., p.297.

24. See Booth, "Were Zeno's arguments directed against the Pythagoreans?" in Phronesis, 2, (1957), for a very full discussion of this topic.

points really were and also that Zeno's hypothesis is stated in such a way that it is difficult to see how it can be taken as anything other than an attack on the general assumption of plurality, the assumption that "existing things are many."

If Zeno's arguments are anti-Pythagorean, then they are so only because anti-Pythagoreanism is an element in them. They were directed against the Pythagoreans insofar as that school believed in plurality and motion, but they were "also directed against all others who subscribed to such beliefs"[25] and presumably included the Ionians and the common man.

Are the paradoxes directed at all against Heraclitus? Scholars such as Diels thought that Heraclitus was the main target for Parmenides' poem, and presumably, then, Zeno's defence of Parmenides would also constitute an attack on Heraclitus. There is still much controversy about the relationship between Heraclitus and Parmenides, whether they were aware of the other's work and some have thought[26] that Parmenides' poem is directed only against the common man.

How does this controversy relate to Zeno's arguments? Firstly, we can say that the fact, if it is a fact, that Parmenides and Heraclitus did not know of each other, does not constitute sufficient a reason for saying that Zeno's arguments were not

---

25. ibid., p.92.

26. Based on Parmenides, Fragment 6: "... mortals, knowing nothing wander in two minds ... They are carried along as deaf as they are blind..."

directed against Heraclitus. Zeno, after all, was a younger man than Parmenides and could possibly have become aware that Heraclitus and Parmenides maintained beliefs that were incompatible. Zeno, were this the case, would be quite capable of going to his master's defence. It could therefore be that Zeno's paradoxes have an element of anti-Heracliteanism in them. Secondly, we can ask if it really is the case that there is no reference to Heraclitus in Parmenides' poem? Note 14 of the previous chapter indicated what I took to be a reference to Heraclitus, and Fragment 6, lines 8-9, seem to constitute a further attack on him, saying: "... hordes, devoid of judgement, who are persuaded that to be and to be - not are the same, yet not the same, and for whom the path of all things is backward-turning." Burnet[27] thinks it reasonable to suppose that this is a reference to Heraclitus because it is one of the most prominent features of Heraclitus' system that it involved the putting together of opposites.

Booth[28] points out that it is probable that Heraclitus had not heard of Parmenides, and this seems <u>prima facie</u> reasonable because Parmenides would surely represent a prime target for Heraclitus' spleen. After all, he attacks Hesiod, Pythagoras, Xenophanes and Hecateus, but never mentions Parmenides by name. There is no definite evidence, however, that Parmenides had ever

---

27. Burnet, op. cit., p.57: he even uses this reference as a means of dating Heraclitus.

28. Booth, loc. cit., p.94.

heard of Heraclitus and "in view of the content of the poem it seems prima facie possible that he had heard of him."[29] Zeno's defence of Parmenides, then, is almost certainly directed against Heraclitus as well as the Pythagoreans.

Booth[30] also tries to show that Parmenides' poem can be seen as a development of Pythagoreanism and is not an attack on Pythagoras at all. The evidence he presents for this most peculiar interpretation is that Parmenides' "spiritual affinities" were the same as the Pythagoreans. The Pythagorean universe was peaceful and based on harmony. Into this Heraclitus brought the war of opposites and the theory of everlasting Strife. The Strife was based on motion and plurality and, because Parmenides denies motion and plurality, he must have been fundamentally on the side of the Pythagoreans. To bolster this case, Booth points out that Parmenides began his philosophy with Ameinias, a Pythagorean.

This is a queer way to use the evidence, because it would seem to imply that Zeno was a supporter of Pythagoras. Booth's conclusions are highly conjectural and he eventually says that Zeno seems to have been concerned to attack <u>everyone</u>: "... the whole tone of Zeno's arguments seems to me to be that of a man who is out to attack anyone and everyone who believed in motion, plurality, and the like."[31]

---

29. ibid., p.94.

30. ibid., p.98.

31. Booth, ibid., p.103.

I feel that the least conjectural line to take with regard to the paradoxes is this: Zeno's arguments were directed against all comers,[32] Pythagoreans, Heraclitus and the common man included. The paradoxes concerned with a plurality of units and points seem almost certainly directed at the Pythagoreans, and the paradoxes of motion seem to be directed at Heraclitus as well as the Pythagoreans. To boot, the whole antipathy towards the testimony of the senses seems directed at anyone who is prepared to take up the challenge. The paradoxes, then, are directed against the commonly accepted views of space, motion, objects and the senses.

The effect which his paradoxes had on subsequent thinking is interesting. Hasse and Scholz have claimed[33] that Zeno was "the man of destiny of ancient mathematics in the hour of its gravest crisis." There were two main consequences of Zeno's attacks. Firstly, arithmetic became separated from geometry. The arithmetical unit was no longer the same thing as the geometrical point, and, from this, the geometrical solid was no longer confused with the sensible body. The geometrical point was no longer the same thing as the atom. Aristotle[34] shows this

---

32. See Plato, Phaedrus, 261d, where Socrates cites Zeno as a rhetorician who uses his art "... as a method of winning men's souls by means of words ... (in) ... public assemblies." Here is a clear implication that Zeno would speak to any who would listen.

33. Quoted in Vlastos, loc. cit., p.377.

34. Aristotle, Metaphysics, (997b35)

distinction when he says that "Perceptible lines have not the properties of the lines of which geometers speak." Secondly, and equally important, Leucippus and Democritus saw that if physical bodies need not have the properties of geometrical solids, Zeno's paradoxes could be resolved. They maintained that their atoms[35] were not like geometrical points, but compact bodies: "...they are so small as to elude our senses, but they have all sorts of forms and shapes and differences in size. So he (Democritus) is already enabled from them, as from elements, to create by aggregation bulks that are perceptible to sight and the other senses."[36] Zeno's paradoxes, then, can be seen to lead to early atomic theory. As well as these two major consequences, Zeno's influence on Aristotle is certain: in his attempt to solve the Stadium Paradox, Aristotle was led to present his theory of the distinction between the actual and potential infinite.

However, I feel that the most important contribution of Zeno is this: until his time, philosophy proceeded by assertion rather than argument. Argument began with Parmenides but true argumentative prose begins with Zeno. The philosophical paradox and <u>reductio ad absurdum</u> are his invention and no better device has ever been found for the stimulation of philosophical

---

35. It is not at all clear to me just how the claims of Leucippus and Democritus overturn the Flying Arrow and the Moving Rows.
36. Aristotle, "On Democritus", quoted in Kirk and Raven, op. cit., p.407.

discussion. The reductio form breeds discussion and dispute rather than disciples who faithfully accept and promulgate the master's teaching.[37] If this is true, and it seems a reasonable point of view, then Zeno's work is the foundation of one of the greatest cultural achievements of all time: the creation of a heritage of discussion rather than the blind acceptance of truths laid down by authority.

When confronted by contradictions the human mind cannot remain passive: it is set in motion with the aim of resolving the contradiction. This will surely be Zeno's most lasting legacy: the inability to dismiss from our mind these haunting paradoxes, which will mean that, for as long as philosophy is allowed by society, Zeno will be there to puzzle and confuse.

For the two and a half millenia since the paradoxes were formulated, thinkers have been motivated to show that they do not work. The time has now come to examine these refutations.

---

37. Howard DeLong, A Profile of Mathematical Logic, (Addison-Wesley Publishing Company, n.p., 1970), p.9.

Part Two

Mathematico-Scientific Refutations

Coleridge wrote of mathematics in 1791 that "Though Reason is feasted, Imagination is starved; whilst Reason is luxuriating in its proper Paradise, Imagination is wearily travelling on a dreary desert." Often, the same holds true of philosophical discourse, but in this section our imagination will be stimulated by many strange and wonderful suggestions. We will be asked to use our imagination for baking cakes, bouncing balls and constructing a glittering array of inordinately powerful machinery.

The strategy is simple: let us construct a model of the Achilles or Stadium Paradox, show how the model can succeed, and, if the model is well constructed, ipso facto, how the paradoxes can be resolved. This unwillingness to face the paradoxes square-on, and to seek to construct models, however, has an interesting motivation. The theoretical models to be discussed carry surplus meaning because they are richer than the explanandum, imparting concepts and conceptual relations not immediately apparent in Zeno's descriptions in his paradoxes. The models therefore convey associations and implications which will be of assistance in investigating the paradoxes. But there seems to me to be yet another motivation behind the device of constructing models: it is an attempt to make mathematics manifest.

Zeno has confronted us with problems which are soundly based in logic, and, to talk of runners and tortoises when discussing the paradoxes, is tacitly to place faith in something as frail as human nature. To construct a model, and especially a machine model, however, is to imbue one's approach with a solid logic derived from mathematics and hence to confront Zeno on a more equal footing.

As Wittgenstein[1] has it, "The logical picture can depict the world."

The physical models which we will encounter, such as bouncing balls and cutting cakes, are not particularly useful: If Zeno proposes to cast doubt on our perception of runners traversing an interval in space, he is hardly likely to accede to our perception of balls bouncing to rest.

The most significant models, therefore, are the mechanical models. They are mechanical by definition in that they are machines. However, they are also imaginary. There is no way in which these machine models could ever be constructed. If they were manifested into physical machines they would simply be more complex versions of the bouncing ball. That the models are both mechanical and imaginary is what gives them their fascination.

I do not think that they are to be seen as analogues of the Zeno paradoxes. The properties of the paradoxes are not represented by different but similar properties in the models. For any property in the paradox, the self-same property can be found in the model so that the relationship between these models and the explanandum is iconic. Because these models are icons then the same arguments which one applies to the Stadium or Achilles Paradoxes can be applied to them, but, because they are imaginary and have surplus meaning, they have implications which mean that they must be treated on their own merits. The reason for this will become clearer as we proceed.

---

1. Wittgenstein, Tractatus Logico-Philosophicus, (London, 1961), 2.19

Chapter Four

Conceptual Technology and Geometric Progressions

"O, for an engine to keep back all clocks!"

(Ben Jonson)

The backbone of the Stadium and Achilles Paradoxes is Zeno's contention that all movement requires an infinite number of subsidiary movements to take place and, therefore, as movements take place through time, an infinite time to elapse before a movement can be completed. If, however, it can be shown that an infinite number of movements can be completed in a finite time, then Zeno will have been dealt an exceedingly robust and mortal blow. This becomes more intelligible if the Paradoxes of the Stadium and of Achilles and the tortoise are restated in a clearer argument form.

(a) On a journey from A to Z we must first go from A to B, which is some point between A and Z, and then to C, some point between B and Z, and then to D, and so on. (Note that the Achilles Paradox can be accomodated to this argument form by not specifying the mid-point of AZ, etc.)

(b) It is absurd that someone should have completed all of an infinite number of actions or tasks.

(c) Therefore it is absurd to suppose that anyone is now completing, has ever completed, or will ever complete, any journey.

This, while not an exact copy of the Achilles paradox, because it does not allow for any movement of Z, is a reasonable prototype for both the Achilles and the Stadium paradoxes. The argument appears to be valid; for most of us, however, the conclusion is ridiculous and, this being so, we must deny one of the premises. Here the problem arises: which is to be accepted and which denied, for has not each the ring of truth? Those who try to show that an infinite number of acts can be performed in a finite time accept the truth of premise (a), the falsity of the conclusion (c), and therefore infer the falsity of the second premise (b). They say, in effect, not only that we must complete an infinite number of acts in a finite time, but also that we can complete an infinite number of acts in a finite time.

In addressing myself to the problem of whether an infinite number of actions can be performed in a finite time, I propose to take as the seminal work in my investigation, Chapter Three of Ryle's Dilemmas.[1] The reasons for this are as follows. Firstly, it shows the sort of errors which can easily be made by the unwary, for Ryle represents an excellent example of a solution which is totally misplaced because it has been based on a fundamental confusion between the paradoxes. Secondly, it shows the difficulties incurred in the construction and use of models of the paradoxes. Thirdly, Ryle raises points which are pertinent to the next three chapters, because Ryle forms the genesis of a discussion on whether a geometric progression shows that an infinite number of acts can be performed in a finite time. The ideas which he presents are amplified by other writers who will also be considered.

In Chapter Three of his Dilemmas, Ryle proposes to offer a solution to Zeno's paradox of Achilles and the tortoise. He

---

1. G. Ryle, Dilemmas, (Cambridge, 1954).

purports to offer not one, but two solutions, adding the rider that even if he fails, he may have betrayed some factor which has succeeded in tricking him. He begins by announcing the paradox:

> Achilles is in pursuit of the tortoise and before he catches him he has to reach the tortoise's starting-line, by which time the tortoise has advanced a little way ahead of this line. So Achilles has to make up this new reduced lead and does so; but by the time he has done this, the tortoise has got a little bit further ahead. Ahead of each lead that Achilles makes up, there always remains a further, though always diminished lead for him still to make up. There is no number of such leads at the end of which no lead remains to be made up. So Achilles never catches the tortoise. He whittles down the distance, but never whittles it down to nothing. [2]

This is an excellently lucid exposition of Zeno's argument and gives little hint of the blunder that is soon to occur in Ryle's thinking.

In his first attempt to solve the paradox Ryle asks us to consider a cake, a very special sort of cake, which is to be cut by a family of children in such a way that each child is to cut off a bit and only a bit of what is left on the cake plate. No child, it can be seen, is to take the whole of what he sees on the plate before him. This is intriguing, but Ryle breezily assures us that this scenario presents no problem, for, as he says, the cake is "really the sum of the consumed parts <u>plus the unconsumed part</u>."[3] The pieces already eaten, plus the fragment which is left,

---

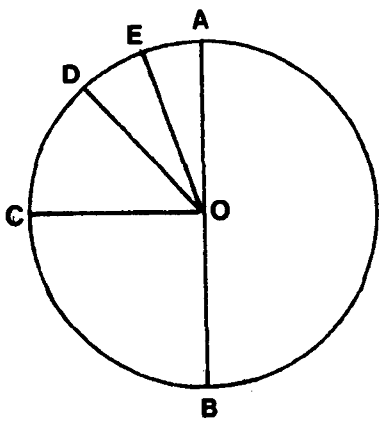
2. ibid., p.36.

3. ibid., p.39.

constitute the whole cake.

Ryle now makes this more precise, and in so doing exposes the germ of his fatal error. The children's mother passes the plate round with the instructions that the eldest shall have half the cake, the second eldest the half of the remainder, the third the half of that remainder, and so on.

Let us imagine this diagram as the top view of Ryle's cake.



The eldest child will get segment AOB, the second eldest will get segment BOC, the third will get segment COD, etc. Ryle now states: "The size of each slice, if the bisection is exact, is a measurable and calculable fraction of the size of the original whole cake."[4] Therefore, if the second child, "playing the Zeno", were to say that because what we consume never amounts to the whole cake, he will not believe that there ever was a whole cake capable of consumption, Ryle can ask him what his own first slice

---

4. ibid., p.40.

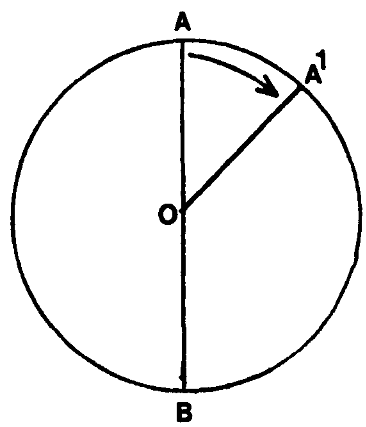
was a quarter of. There must have been a whole cake for him to get a quarter of it, and it must have been a finite cake, because his quarter of it was finite. No-one would dispute this, but Zeno's Paradox of Achilles and the tortoise is not like this at all. The analogy is false.

Ryle has clearly made an error somewhere and this can easily be seen if we take a closer look at the Paradox of the Stadium. An athlete is running across the stadium AZ:

He will first of all have to run halfway across the stadium to point B. He then runs across half the remainder BC, and then across half the new remainder CD. Clearly he will never reach Z, but more of this later; we will know where Z is, because the distance from A to Z is the distance covered to the last run plus the distances covered in the last run. That is, if the athlete is at C, the distance from A to Z is as follows: AB + BC + BC. If the athlete is at E, then the distance from A to Z is AB + BC + CD + DE + DE. If our athlete, like the child, says that because he can never reach Z he does not believe that there actually is a Z, Ryle can say that there must be a Z if he can run halfway towards it. Besides, I have just worked out where point Z lies and how it can be calculated.

Ryle's account of the cake and my account of Zeno's Paradox of the Stadium can be seen to be about exactly the same thing if we look at the arithmetic of both versions. Consider for a moment the circumference of the cake. If the mother were to tie a string round the cake and mark it where each child should make his cut, on straightening the string out we would get a copy of the diagram drawn for the Stadium paradox. This can be verified empirically.

Any relationship between Ryle's model and the Achilles paradox is purely fortuitous.[5] This can be shown if we push Ryle to a logical conclusion. Let us take the diagram of his cake to also represent Zeno's racetrack, on which both Achilles and the tortoise start at A, but the tortoise has one lap of a start.



The first cut in the cake was AOB, and this is equivalent to Achilles having run round half a lap, but before the first cut AOB is made, the line OA will have revolved a little to position OA^1, the point which the tortoise has reached. This is where an objector could ask if there is such a thing as a growing cake, but, if Ryle's analogy is to hold good, then culinary experiences must go by the board and we must admit that the cake can grow.

---

5. Although the connection between them will become clearer in Chapters Seven and Eight.

It does not seem so strange if we use my racetrack model and say that the tortoise has moved, but it still presents a problem: B is still halfway round the racetrack, but is not halfway towards our tortoise. Point B is, in fact, still $\frac{AA^1}{2}$ short of being halfway to the tortoise. Ryle seems to have found an answer, because his tortoise is stationary, whereas Zeno's moves. Ryle's method is inapplicable because of the motion of the tortoise, and it is now obvious, for his model to be a proper parellel, that his cake must grow after all.

Ryle, ignorant of the fact that his model proves nothing, says, "I now want to show you that the race between Achilles and the tortoise exemplifies just what is exemplified by the mother's division of the cake by the second method."[6] This should be interesting. He now discards the cake model and begins to plant flags.

We have, he says, two options. In a race we can plant flags at regular intervals, every furlong or chain, etc., and if we do this there is no problem. The tortoise crawls at one mile per hour and Achilles saunters at two miles per hour. If the tortoise has one mile of a start then the race will be over after one hour or after sixteen furlong flags have been planted. We could also plant the flags in a converging geometric progression, (although Ryle does not call it this), i.e. we can stick a flag into the ground where Achilles started, a second at the halfway

6. G. Ryle, ibid., pp. 40-41.

point of his total course, a third at the halfway point of the second half of the course, etc.

A very interesting point crops up here. This "planting of flags" is not one uninterrupted motion which could subsequently be analyzed into an infinite number of sub-motions, as Bergson will later be seen to claim, but is literally a series of actions separated, as it were, by pauses of rest. These pauses of rest occur when the runner is _en route_ to the next point. The point which this raises concerns _dynamics_. If we assume for the moment that in each act of planting a flag, the runner's hands and arms describe the same length of path each time he raises the flag and drives it into the ground, and also that the amount of energy he expends in each "driving" action is the same, then, (a) his hands and arms will literally have travelled an _infinite_ distance, even though the _run_ itself may be finite. This is because the total distance travelled by his hands and arms is the product of a finite quantity (the distance travelled in raising and driving one flag) multiplied by an infinite number (the number of times he raises and drives the flag), (b) the total energy or work expended by the runner in raising and driving the flag is infinite. This, again, is because the total energy expended is the product of the energy expended in one raising and driving of the flag multiplied by the infinite number of times the flag is raised and driven. He must therefore have an infinite store of energy when he sets out on his run. These specifically dynamic difficulties will be analyzed in more detail later[7], but we can see that any flag-planting or other marking process which would require discontinuous change in any component of the velocity of the runner's limbs will positively ensure that the runner will not reach his destination.

To return to more immediate problems: "Clearly", says Ryle, "for every flag we plant, there is always another flag to put in halfway between it and the place where Achilles caught the tortoise." (Note the as yet unproven assumption that Achilles does catch the tortoise). He then goes on to say: "At each stage the total distance run by Achilles does consist of all the distances between the flags plus the distance between the last flag planted and the point where the race ended."[8] At one stage three quarters of the course has been flagged and one quarter of the course still remains ahead of the last flag planted, and three quarters plus one quarter equals one. Let us put all this talk about planting flags into diagrammatic form and see what happens:

This seems similar to the earlier diagram of the Stadium paradox. It is, in fact, exactly similar. This new model therefore suffers from exactly the same drawbacks as the magic cake model, by taking no account of the fact that the tortoise moves. Ryle speaks glibly of "the total course" but that seems

7. See Chapter Six, "The Dilemma of Dynamics".

8. G. Ryle, ibid., p.41.

to me to be begging the question somewhat. B is the mid-point of AZ and, according to Ryle, to be able to calculate where B is implies that we know where Z is. This is an obvious truth, but in Zeno's argument about Achilles and the tortoise we do not know where Z is. Therefore we cannot know the total course, i.e. where Achilles will catch the tortoise, and therefore cannot find the halfway point. Point Z is continually moving and how can one calculate the mid-point of the expanding line AZ? This may seem easy to answer by saying that the mid-point B moves at half the rate of point Z, but we do not know if the line is expanding at a constant speed, i.e. whether the tortoise is moving always at the same speed. Clearly we either have to stop the motion of the line or set out rules for its expansion in order to do our calculations. Ryle's talk of calculating the position B implying that Z can be calculated is also misplaced. Point B in the paradox is the tortoise's starting point which must remain stationary!

Any attempt to stop the tortoise as a means of solving the paradox is clearly illegitimate, because it changes what Zeno says about Achilles and the tortoise into what Zeno says about running across a Stadium. This is not allowed: therefore his refutation will not work.

Ryle has completely lost his way in this problem and we must now consider why it is that his refutations prove nothing. This, as will be seen in the penultimate chapter, is not the sort of problem that can be solved by baking cakes, sticking flags into the ground, or any other such strategy. There is, I think, no endlessness to Achilles' pursuit of the tortoise, but to attempt to work out the solution using Ryle's models and simple arithmetic is doomed to failure.

What, then, of positive merit can we take from Ryle and his "Dilemmas"? Firstly, it should immediately put us on our guard, because it shows how easy it is for a thinker as clever as Professor Ryle to become confused. Secondly, it should now be apparent how difficult and dangerous it is to try to create models for the Achilles paradox. Having seen the imbroglio into which Ryle led himself, simply through misreading the situation, his fate should be a warning to all. Thirdly, and more positively,it introduces us to the idea of a geometric progression as the key to understanding Zeno.

With all the above considerations in mind I turn now to David Bostock's article, "Aristotle, Zeno, and the Potential Infinite",[9] an article which, in an aside, sets out to give a restatement of Zeno's Paradox of Achilles and the tortoise in such a way that a solution is said to become clear. The restatement he chooses is a model based on a rubber ball released from a height on to a firm and unyielding surface so that it bounces.

> The ball is elastic and so resists deformation, with the result that the downward motion of the ball is eventually halted and the deforming process comes to an end. Thereupon, the reverse process begins: the ball reverts to its natural shape, exerting pressure on the surface

---

9. David Bostock, "Aristotle, Zeno, and the Potential Infinite", in <u>Proceedings of the Aristotelian Society</u>, 1972-73, pp.37-51.

> and thereby raising itself once more, and the upward momentum which it thus acquires carries it away from the surface and so brings about a further bounce. [10]

Thereafter follows a passage of elementary mechanics which Bostock elaborates in order to show that the model is an entirely plausible one, and does not involve anything unintelligible.

The main portent of this passage is to show that the rubber ball has a "rest shape", the shape it will assume when at rest on a surface (subject to normal atmospheric pressure), and that the position it is in at this point is its "rest position". Whenever it is below its rest position and compressed beyond its rest shape, its resistance to deformation (being rubber) will cause it to try to resume its rest position. It will acquire momentum in doing so and be carried beyond its rest position where it becomes subject to the force of gravity which will tend to carry it down to its rest position. It acquires momentum in doing so and the ball is therefore compressed beyond its rest shape, thereafter to spring into the air again, etc. According to this theory, Bostock points out that, "once a ball has started bouncing there is no bounce that is its last bounce,"[11] and so it represents a model of the Achilles paradox.

Bostock now attempts to show that it is not the case that the ball will bounce for ever and, ipso facto, that Achilles will catch the tortoise. He declares that it is "an obvious fact of experience" that bouncing balls tend to bounce to a lesser height

---

10. ibid., p.43.

11. ibid., p.44.

on each successive bounce and "a moment's reflection" is all that is needed to show that the theory of the bouncing ball can cope with a situation where each bounce takes only half as long as the previous bounce, so that the times of the successive bounces are in the proportion:

"1, 1/2, 1/4, 1/8, 1/16, ...

and all the bounces are completed in a finite time."[12]

This is nothing but a disguised geometric progression. Bostock, however, has further information to reveal to us. He says that it is equally consistent with the theory to suppose that "the times taken by successive bounces are in some different proportions, for instance:

1, 1/2, 1/3, 1/4, 1/5 ....

so that the bouncing goes on for ever."[13]

This gives us two series:

(a) 1, 1/2, 1/4, 1/8 ......

(b) 1, 1/2, 1/3, 1/4 ......

(a), according to Bostock, will be completed in a finite time, while (b) bounces on "for ever". This is based on the assumption that series (a) is summable and that his assumed summability

---

12. ibid., p.45.

13. ibid., p.45. This is called a Harmonic Progression, and presumably argues some peculiar kind of elasticity: the elasticity must alter after each bounce in order to give these strange proportions.

ensures its completability. Thereafter, by spurious logical reasoning (denying the antecedent), we infer that the non-summability of (b) ensures its non-completability.

This is mysterious. Why cannot they both be completed in a finite time, or, more pertinently, why cannot they both bounce for ever? We are given no assistance on this point, because of assistance there can be none. Every fraction which appears in series (a) will, sooner or later, appear in series (b), so that if series (a) can be completed, (the proof of which is that it is an "ordinary fact of experience"), then so can series (b); it will simply take a while longer. Bostock, with his talk of experience, has too readily overlooked the warning of Parmenides and Zeno. One hesitates to invoke the Paradox of the Millet Seed, but suffice it to say that placing faith in the senses for what is a logical or metaphysical problem seems to be a bizarre strategy. What causes the ball to stop bouncing is not that a geometric series is summable, but that the ball loses excess energy on each bounce through air resistance, etc.

Bostock now abandons empiricism in favour of logical investigation. He admits that this infinite series has no last member, and therefore that "it is indeed impossible to come to the last member of the series."[14] He continues, however, that it does not follow from this that it "is impossible to finish the series, i.e. to come to a state in which no member remains outstanding."[15] I find an inconsistency in these remarks: I

---

14. ibid., p.46.

15. ibid., p.46.

would like to think that if I cannot come to the last member of the series, I cannot perform the last member of the series. Bostock, however, says that "from the fact that there is no last member it does not follow that we cannot perform every member."[16]

This is an equivocation; it is an ambiguous sentence and I am sure that Bostock has failed to see the ambiguity. An example will clarify this. If there is an infinite number of pieces of cake to be eaten, this does not mean that there are pieces of cake which cannot be eaten. I can eat any one because each is capable of being eaten, but I could not eat every one, because, as Bostock says, it is impossible to come to the last member of the series. He says that "it does not follow that we cannot perform every member," when he means, or should mean, "every member is capable of being performed." These are clearly not the same thing. He has neglected the fact that a geometric series is a series which cannot be broken into arbitrarily: we cannot, for example, do the last bit first. (We could make Achilles' progress very erratic if Bostock were correct.)

Bostock now attempts to defend his theory against possible criticism. There are those who argue that a consistent account cannot be given of the state which results when every member of an infinite series of tasks has been performed.[17] In order to confront these people, the bouncing ball is now made to change

---

16. ibid., p.46.

17. c.f. J.F. Thomson, "Tasks and Supertasks" in Analysis, 15, 1954, reprinted in The Philosophy of Time, ed. R.M. Gale, (New York, 1967).

colour at each bounce; more specifically, when it passes through its rest position on the way up. The ball is white to start with. After the first bounce it changes to black, and back to white again on the next bounce, etc. The problem, says Bostock is to discover the colour of the ball when it finally comes to rest. (Note the assumption, based on eyesight, that it does come to rest.) Bostock maintains that if we agree that the original account of the ball was intelligible, then this more specific account must also be intelligible. Critics hold that since the ball can only be black or white, it must be either black or white when the bouncing has stopped. If it is white then there were an even number of bounces in all, and if it is black there were an odd number of bounces in all, and yet "neither of these is possible since only a finite number can be odd or even, and the number of bounces was supposed not to be a finite number."[18]

This is the argument of the critic and Bostock contends that this will not stand as a criticism of his theory. We have been given enough information to determine the colour of the ball at any moment in its bouncing, but not enough to determine its colour when it is finally at rest. We can predict the colour after a finite number of bounces, but not after infinitely many bounces. We can therefore maintain, without contradiction, either that it is white or that it is black.

---

18. D. Bostock, ibid., p.48.

This is mathematically correct[19], but Bostock's proof of this shows that he does not know where he is being led by the paradox. At its first moment of rest, the ball is stopped and the ball does not change colour, for it only changes colour when it is passing through the rest position on the way up. If the ball does not change colour we should be able to say that it is now the colour it was at the previous moment. The trouble here, says Bostock, is that there is no such thing as "the previous moment", for "quite generally no two moments are next moments (since between any two moments there are others)."[20] I intend to consider this notion of time forming a Cantorean continuum of point-events in Chapter Seven, but let me say for the moment that Zeno can lead Bostock to the following consequences:

(a) Any time interval must be infinite in duration, because between any two moments must lie a third, with others between them, etc.

(b) Motion is impossible, because motion takes place through time but if one cannot get from moment 1 to moment 2 because of the infinity of intervening moments, then motion between moment 1 and moment 2 is impossible.[21]

Bostock's ultimate point in this section is that all methods of deducing the colour of the ball must fail, and, consequently,

---

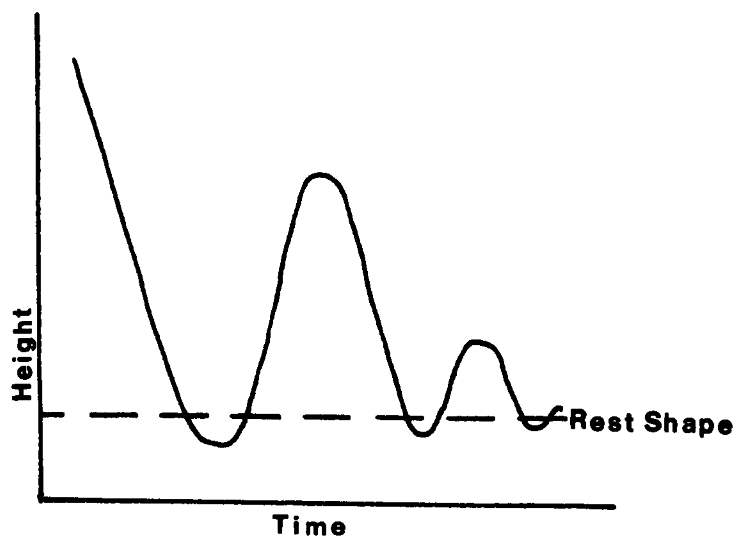
19. See the discussion of Bolzano later in this chapter.

20. D. Bostock, ibid., p.48.

21. This will be examined more closely in Chapter Seven, "Infinity and Kinematics."

we "are at liberty to make what assumptions we like about it."[22] His conclusion is that there is not in general any logical impossibility in the idea of completing an infinite series of tasks: "this is actually achieved by Achilles when he catches his tortoise, and there is no difficulty in supposing it is achieved by the bouncing ball."[23] Comment on this is surely superfluous.

He then gives a graphical representation of what his ball achieves, and it is here that he shows that he has made exactly the same mistake as Ryle, because his model is not a model of the Achilles paradox. The graph of the motion of the ball has, he says, the following general shape:



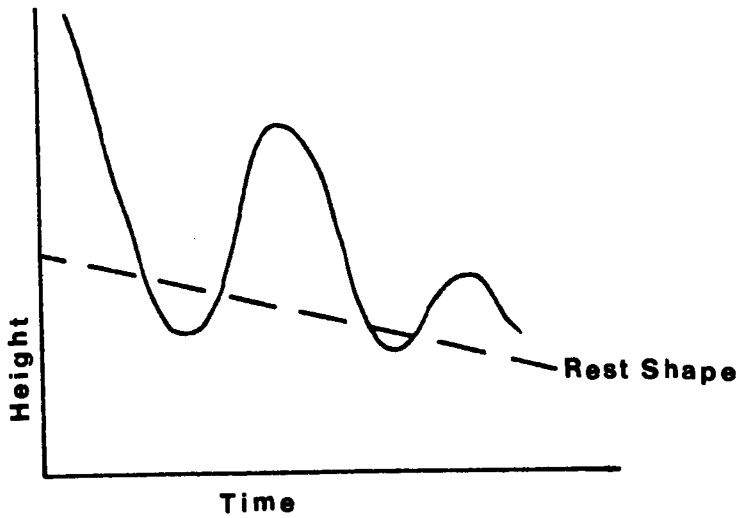
The thing to notice in the graph is that the line which runs parallel to the time axis, the "rest position", is always horizontal. Now, if we relate this to the Achilles paradox, we

---

22. D. Bostock, ibid., p.48.

23. ibid., p.49.

can see that this "rest position" in the model takes the place of the tortoise, with "being at rest" the equivalent of "catching the tortoise". The movement of the tortoise, which is so important to Zeno's paradox, is clearly not indicated on the graph, and in order to accomodate this variable the graph would have to look like this:



Bostock has clearly confused the Achilles paradox with the Stadium paradox. Not only that, he has also failed to provide an answer to either paradox, for he has not shown how his theory can account for a ball bouncing to rest after an infinitude of bounces. All he has shown is that the ball's coming to rest is not incompatible with his theory. I have tried to show that to attempt to base a theory on the observation that the ball stops bouncing (or Achilles catches the tortoise) is redundant. The consequences of his theory have also been shown to play into Zeno's hands by ultimately denying the possibility of motion.

Bostock has not refuted this paradox, but note again the assumption that we can reach the end of a geometric series, an assumption that he has failed to substantiate. Can it be shown that a geometric progression reaches its limit?

In his article, "The Sum of an Infinite Series"[24], J. Watling declares not only that the notion of completing an infinity of tasks is logically consistent, but also that an infinity of tasks can be physically completed.

He opens by giving us his definition for the expression "an infinite number of things". He says (page 39) that "there is an infinite number of things in a collection if, and only if, there is at least one more than any finite number." What this seems to mean is that when I add an extra member to my collection and no longer have a finite number, then I have reach infinity. As I always end up with finite numbers through addition, it follows that I cannot reach infinity by any process of addition: if we take any finite number, then we must know the next number, which must also be a finite number. We cannot suddenly arrive at some magical infinitieth number. Generally there seems to be a confusion about the two following statements, of which I give a translation into symbolic logic in order to clearly show the error:

---

24. J. Watling, "The Sum of an Infinite Series", in Analysis, 13, 1952-1953, pp. 39-46.

(a) "There is a number which is greater than any given number." or "There is an X such that, whatever Y may be, if Y is finite then X is greater than it." That is:

$$(\exists x)(y)\left[Fy \rightarrow (x>y)\right]$$

(b) "Whatever number you take, as long as it is finite, there is some number greater than it" or "Whatever Y may be, if Y is finite, then there is some X such that X is greater than Y." That is:

$$(y)\left[Fy \rightarrow (\exists x)(x>y)\right]$$

By using this definition, Watling can translate "I have performed an infinite number of acts" into "I have performed one more act, than any finite number." This shows, says Watling, that there is nothing contradictory in the notion of completing an infinite series of acts. The requirement that most people have thought of with regard to the Stadium or Achilles paradoxes is that, no matter how many acts have been done, more must remain: this has been thought to entail that more acts must always remain to be done. Using his new definition, however, Watling is able to say that this requirement should be "whatever finite number has been done, more must remain," and "it is clear that this does not entail that after an infinite number have been done more must remain."[25] This is all very fine, but it does not show how we can

25. ibid., p.41.

do it. Watling is being hypnotised by his own logic into forgetting the problem. Redefining "infinity" will not cause motion to take place.

Watling now applies this to the Achilles paradox and his first suggestion is that we simplify the problem by considering the tortoise as standing still. As we have already seen, this is to turn the Achilles paradox into the Stadium paradox, but let us leave that criticism aside. We now, as in Ryle, have to go planting flags, halfway, threequarterway, etc., up the course. This gives, as usual, a converging geometric series and we are told (page 43) that the summing of this series "requires no more than that every term of the series shall be added to the first by ordinary addition." In other words, we can add the terms of the sequence (1/2, 1/4, 1/8, ...) in literally the same way as we would add together a finite number of numbers. We add 1/4 to 1/2, 1/8 to the result, 1/16 to the result, and so on, until an infinite number of additions have been completed. This last sentence must be agreed with because that is what we have to do, but I find it difficult to suppose that someone could physically make an infinite number of additions. The limit to this operation would be the number 1 and "Nothing else is required to give the sum than making every one of the additions,"[26] because "each term brings the sum nearer to the limit than the sum of finite numbers preceding that term."[27] Watling now mentions briefly how this

26. ibid., p.44.

27. ibid., p.45.

supposed addition could take place. I will examine this strategy more closely in the next chapter, but suffice to say that he utilizes Aristotle's[28] suggestion of dividing the time taken for each part of the operation, i.e. we add the first two terms in one minute, the next term to the result in half a minute, and so on. In this way, he says that every one of the additions is made. But have we arrived at the limit, the number 1? I do not think that we can say what number we will arrive at, simply because there is never any last addition. There is, in fact, a confusion here: to say that we have only to add the numbers in the sequence, ("Nothing else is required to give the sum than making every one of the additions"), and that we can compute the number 1, is to utter a flat contradiction.

Let us take a closer look at this notion of "limit" which Watling is using. He says[29] that if we start to add together the terms of the sequence to be summed, every term that is added brings the sum nearer to the limit of the sequence of partial sums. Therefore, when we have added together all the terms, there is no difference between the sum arrived at and the limit. This, however, is only to say that the sequence of partial sums converges to a limit and therefore that the sum of all the terms is the limit: "Therefore the sum of an infinite series, whose terms all have the same sign (i.e. positive or negative), is equal

---

28. Aristotle, Physics, 233a.

29. J. Watling, ibid., p.45.

to the limit of the sequence of sums of its terms."[30] But this, however, is only how we <u>define</u> the sum of an infinite series, and Thomson[31] has pointed out that those who rely tacitly on this definition are guilty of circularity in their proof. Watling gives no other method whereby the sum of an infinite sequence might be specified, relying on our <u>definition</u> of a limit operation instead. That is, he begs the question.

Watling concludes that Achilles can catch the tortoise because "an infinite series has a finite sum." What he really should say is that we stipulate that it has a finite sum. The "sum" in the Stadium paradox, which is the paradox he actually addresses himself to, is equivalent to the end of the race track, i.e. the destination. But <u>we already know</u> that this is where the runner has to run to, that that is the limit of his operation. Watling has failed to show <u>how</u> he can get there, which is surely the most important aspect of the paradox. His example of adding number to number to number is simply the equivalent of saying to the runner, "You want to get to the far end of the Stadium? Well, just keep running until you get there." This is hardly satisfactory.

Watling's solution of turning the paradox into a limit-operation is, I think, useful, but to suppose that a limit-operation is completed by simple addition is totally wrong. Even if we were to accept that his notion of how limit-operations work is correct,

---

30. ibid., p.45.

31. J.F. Thomson, loc. cit., p.416.

which I cannot, Watling has failed to show how it can physically apply to the paradox.

It seems to me that the notion that an infinite number of acts being done in a finite time is fundamentally unsound. Naturally, there is a certain hypnotic fascination in its possible truth because it completely discredits Zeno in a robustly straightforward manner. Unfortunately, as has been shown, the desire to have their point of view hold the day has caused those whose refutations to the paradoxes lie along this path to think in a woolly manner. One cannot stop en route to a logical conclusion at a point where the new theory seems to be working.

I have shown how the suggestions advanced in the foregoing articles hold some sort of error, but I have yet to put forward what I consider to be convincing reasons why refutations of this sort will always be in error.

Some philosophers have directed attention to the possible falsity of the first premise of the argument (see page 73), by affirming the self-contradictory nature of the second premise, viz., that the concept of "a completed infinite number of tasks" is self-contradictory, that there is something absurd in the notion of "a supertask". These philosophers accept (rightly) that there is nothing absurd in the concept of "infinity" and nothing absurd in the concept of "a completed sequence of tasks". However, when these two concepts are brought together into a supposedly meaningful juxtaposition, then something results which is self-contradictory. This is the approach taken by Professor Max Black in his "Achilles and the Tortoise."[32]

Black is therefore setting out to show that Zeno's second premise is correct, and this intention has my clear and unequivocal support. I must confess that I am not happy with his arguments. They portray yet another case of being led astray by models which are not quite accurate. However, an examination of Black's arguments (and later, Thomson's) will help to clear the ground for a proper affirmation of the self-contradictory nature of Zeno's second premise.

Black begins from the argument that the Achilles paradox is a geometric progression: "Suppose Achilles runs ten times as fast as the tortoise and gives him a hundred yards start."[33] There are those, he maintains, who claim that the total number of yards which Achilles must travel to catch the tortoise is 100 + 10 + 1 + 1/10 + ... yards. This is a convergent geometrical series in space giving us a total of 111.1111 repeater yards, or exactly 111 and 1/9 yards. If Achilles can run one hundred yards in ten seconds, the time taken to do this is 10 + 1 + 1/10 + ... seconds, or, exactly 11 and 1/9 seconds. Black points out that those who hold this viewpoint only show where Achilles and the tortoise will meet *if* they meet, yet it does not show how Zeno is wrong in claiming that they cannot meet.

---

32. M. Black, "Achilles and the Tortoise" in *Analysis*, (11), (1950-51), pp. 91 ff.

33. ibid., p.91.

To say that the sum of the infinite series 100+10 + 1 + ... is 111 and 1/9 does not mean, despite what Watling tried to say, that mathematicians have succeeded in adding together an infinite number of terms. Frege[34] says that this would require infinite supplies of paper, ink and time. By addition alone, we could never prove that an infinite series had a finite sum. Black says (page 100) that if I were commanded to count a collection and to keep counting until there was nothing left to count, I would assume that there will be a point where my task will be completed, i.e. I have nothing left to count. Were I commanded to count an infinite collection and keep counting until there was nothing left to count, the commander "would be practising a deception." There can be no last term in an infinite collection. He then goes on, "to say that the sum of the series is 111 and 1/9 is to say that if enough terms of the series are taken, the difference between the sum of that finite number of terms and the number 111 and 1/9 becomes, and stays, as small as we please."[35] That is to say, we perform a limit operation. If we apply this to the case of Achilles and the tortoise it seems the series of distances crossed by Achilles is convergent: the first step is 100 yards (?), the second step is 10 yards, the third step is 1 yard, etc. The distance still to go becomes "as small as we please". The difficulty, as Professor Black points out, is that

---

34. G. Frege, Translations from the Writings of Gottlob Frege, (1st ed.), ed. Geach and Black, (Oxford, 1952), p.219.

35. M. Black, ibid., p.93

after each step there are still infinitely many steps to be taken: "The difficulty is to understand how Achilles could arrive anywhere at all without first having performed an infinite series of acts."[36] This, of course, is Zeno's problem.

To bring out the self-contradictory nature of completing an infinite series of tasks, and thus to affirm Zeno's second premise, he introduces us to the Alpha Marble Counting Machine.

This truly formidable piece of apparatus is capable of counting infinitely fast. The reason for this theoretical ability is that some philosophers, such as Watling, have maintained that the difficulty of counting an infinite collection is simply a lack of time.[37] We are asked to believe that the machine moves one marble from the left hand tray to the right hand tray in the first minute of operation. It then rests for a minute and moves another marble in the next half minute. It then rests for half a minute, after which it transmits another marble in the next quarter minute, etc. At the end of exactly four minutes, he says the machine will come to a halt, having transferred all the marbles from left to right.

---

36. ibid., p.94.

37. Couturat, De l'Infini Mathematique, (Paris, 1896), p.462: "Quand vous dites qu'un collection infinie ne pouvra jamais être numerotée tout entière, il ne s'agit pas là d'une impossibilité intrinsique et logique, mais d'une impossibilité pratique et materielle: c'est tout simplement une question de temps."

If we think about this we can see that the arithmetic of his claim is at fault. It implies that the last operation of the machine, resting, is an action. Surely the last action would be a transmitting action, after which the machine would cease operating. If this more specific examination of the machine's functions is correct, then an observer could legitimately ask Black when it was that the machine completed the transmitting. He would be hard pushed to conjure up a reply. This objection may smack of pedantry, but it is extremely important. If rest is <u>not</u> to be included then we cannot know when it stops. Could we, for instance, ask how long the last <u>transmitting</u> action took? It will assume a significance later on where Black's misunderstanding of his machine causes total confusion. More fundamental a criticism for now is that in the Alpha Machine model there is an infinitieth number, the last marble to be transmitted from left to right. Black has said (page 100) that if I were commanded to count an infinite collection the commander "would be practising a deception", because there can be no last member in an infinite collection, but here he says that there <u>is</u> a last member. He therefore is guilty of utilizing a self-inconsistent notion of an infinite collection, "that it has and has not a last member".[38] Were there less than an infinite number of marbles then the Alpha machine would complete its operation in less than four minutes, but Taylor supports my contention that if the collection is

---

38. Taylor, "Mr. Black on Tempral Paradoxes", in <u>Analysis</u>, 12, 1951-1952, p.39.

infinite, there is never a last marble. He (Taylor) claims that were the machine to have an equal supply of red and black marbles, and choose to transmit only the red, "there would be no danger of the available marbles becoming exhausted; yet there would remain uncounted all the black ones."[39] (This notion of "infinity" will be examined later.)

Also, if we wish to use this machine as a model of the Achilles paradox, we must show the motion of the tortoise. Black's machine has a stationary tortoise: to indicate its motion the number of marbles in the left hand tray would have to be topped up.

Black eventually concludes that the Alpha machine is technologically inadequate and invents the Beta and Gamma machines. Here a single marble is transferred from tray to tray. As soon as the Beta machine transfers the marble from the left hand tray it is returned by the Gamma machine. Beta takes one minute to transfer the marble to the right hand tray for the first time; thereafter it takes its rest for one minute while Gamma returns it. While Gamma takes its rest, the Beta machine transfers the marble again to the right hand tray, but this time in half a minute. Black claims that "the successive working periods and pauses of Gamma are then equal in length to those of Beta, except Gamma is working while Beta is resting and vice versa."[40]

---

39. ibid., p.39.

40. Black, loc. cit., p.98.

Black's model of the Beta and Gamma machines is meant to show that "if the result of the whole four minute's operation by the first machine is to transfer the marble from left to right, the result of the whole four minute's operation by the second machine must be to transfer the marble from right to left. But there is only one marble, and it must end somewhere!"[41]

Let us enquire into what lies behind Black's assertion that there is something wrong in a process where the marble ends up somewhere. He is assuming[42] that an infinite number of moves is completed only when the last, or infinitieth, has been performed. This shows that he thinks that an infinitude has not only a first member but also a last member. This would mean that it was not an infinitude at all, for we can always, given any number, add on another number. For Black to say that something is contradictory in the notion that the marble is somewhere at the end of the operation seems to be strange and requires a bit more argument than he gives it. Is he saying that when a marble has been transferred between two locations an infinite number of times (were this possible), then the marble should be either nowhere or in both locations simultaneously? No answer is given although one will shortly be attempted by J.F. Thomson.

Black then modifies his model so that the machines, now Delta and Epsilon, operate with two marbles, passing them back and forth and so eliminating the pauses, taking a minute for the first

---

41. ibid., p.98.

42. Taylor, loc. cit., p.40.

exchange, a half minute for the second exchange, and so on. He declares that if the Delta machine succeeds in performing its infinite number of actions then both marbles will end up in the right hand tray: if Epsilon succeeds, then both marbles will end up in the left hand tray. Hence neither of them can succeed and so the notion of completing an infinite number of tasks is a contradiction. This is strange: any time a machine has a marble in front of it, it transfers the marble to the other machine, and so on. When it has performed an infinite number of transfers it will still have a marble in front of it. From this we are supposed to deduce that the notion of completing an infinite number of actions is self-contradictory. But Black is obviously using as the kingpin of the "proof" that there ought to be a last transposition of the marbles, but clearly Zeno, even if he sympathises with Black's aims, will not allow this. Taylor points out that as the motions of the machines are to correspond with time intervals of the ratio 1, 1/2, 1/4, etc., there cannot be a last movement: "At any stage of the process, further movements are discernible."[43]

Black has failed to sustain the charge of self-contradiction in the notion "completing an infinite series of acts", because he bases his argument on a self-contradictory notion of infinity, that it has and has not a last act. However, there is a more subtle error than this in his reasoning. I will consider it shortly.

---

43. ibid., p.41.

This same path, which has as its destination the proof that the concept of completing an infinite number of tasks is self-contradictory, is taken, and more coherently explored, by J.F. Thomson, in his remarkable and stimulating paper "Tasks and Supertasks."[44] Like Black and Bostock, he is intrigued by the idea of creating models of the paradoxes. Thomson refers to tasks such as those put forward by Zeno as "supertasks" and he proposes to argue that there is something absurd about the accomplishment of a supertask.

His first argument, concerning a lamp, is short, imaginative and compelling. It follows the same sort of line as the coloured bouncing ball of Bostock but draws a remarkably different conclusion. It appears to demonstrate that completing an infinite series of tasks is a self-contradictory concept:

> There are certain reading-lamps which have a button in the base. If the lamp is off and you press the button the lamp goes on, and if the lamp is on and you press the button, the lamp goes off. So if the lamp was originally off and you pressed the button an odd number of times, the lamp is on, and if you pressed the button an even number of times, the lamp is off. Suppose now that the lamp is off, and I succeed in pressing the button an infinite number of times, perhaps making one jab in one minute, another jab in the next half minute, and so on ... After I have completed the whole infinite sequence of jobs, i.e. at the end of two minutes, is the lamp on or off? It seems impossible to answer this question. It cannot be on, because I did not ever turn it on without at once turning it off. It cannot be off, because I did in the first place turn it on, and thereafter I never turned it off without at once turning it on. But the lamp must be either on or off. This is a contradiction.[45]

---

44. J.F. Thomson, "Tasks and Supertasks" in <u>Analysis</u>, 15, (1954), pp. 1-10.

45. ibid., p.5.

This possesses everything that an argument should have: it has neatness, brevity and credibility, but unfortunately, it lacks validity. We can visualize two people, Mary and John, jabbing their lamps, (starting at time t_o), and at the end of two minutes (t_1) (this will be examined more fully in the next chapter) the first lamp is on while the second lamp is off. As far as I can see, Thomson's argument shows neither of these situations to be self-contradictory.

According to Thomson, Mary's lamp cannot be on at (t_1) because she turned it off after each time she turned it on. But this is true only of instants before (t_1). From this it follows that there is no time between (t_0) and (t_1) at which the lamp was on and which was not followed by a time, also before (t_1), at which it was off. Ditto for John's lamp. But nothing is contained in these instructions for the lamps at time (t_1) or later. Thomson's instructions do not cover the state of the lamp at (t_1), although they cover every moment between and including (t_0) up to, but not including, (t_1). This is Bostock's point about the coloured balls being either black or white without contradiction. As Thomson says, the lamp must be on or off at (t_1) but nothing we are told implies which it is to be. The arguments to the effect that it cannot be neither on nor off are redundant because they have no bearings on the case. To suppose that they do is to suppose that a physical description of the lamp at (t_1) is a logical consequence of a description of its state prior to time (t_1). In order to make that argument valid there would have to be an extra premise to that effect. Thomson suggests that "It is impossible to answer this question." It _is_ impossible on the information given. A contradiction, which is what Thomson "finds", can be shown to arise "only by assuming that his instructions are complete in the sense that the statement that they have been followed entails either that the lamp is on at (t_1) or that it is off at (t_1). They are not."[46]

Thomson then goes on to analyse why the "contradiction" arises in the case of the lamp. If his explanation is correct then it means that _I_ have made an error in _my_ argument that a contradiction does _not_ arise. This must therefore be investigated closely.

> Now what exactly do these arguments come to? Say that the reading lamp has either of two light values, 0 ("off") and 1 ("on"). To switch the lamp on is then to add 1 to its value and to switch it off is to subtract 1 from its value. Then the question whether the lamp is on or off after an infinite number of switchings have been performed is a question about the value of the lamp after an infinite number of alternating additions and subtractions of 1 to and from its value, i.e. is the question: What is the sum of the infinite divergent sequence +1 -1 -1 ...? Now mathematicians do say that this sequence has a sum; they say that its sum is 1/2. And this answer does not help us, since we attach no sense here to saying that the lamp is half-on.[47]

This seems to be convincing. Has Thomson discovered a mathematical analogue of the lamp model? It appears that he has, that it is perfectly self-consistent, and that the "contradiction" in the case of the physical example carries into its mathematical

---

46. P. Benacerraf, "Tasks, Supertasks and the Modern Eleatics" in _Journal of Philosophy_, 59, (1962), p.769.

47. Thomson, loc. cit., p.8.

counterpart. It makes no sense to speak of the lamp as being "half-on." If the initial value of the lamp was 0, then the value of the lamp after the first X switchings is the sum of the first X terms of the series. Consequently the value of the lamp after all the switchings is the sum of all the terms, or 1/2. But, if the value of the lamp at (t_1) is always the value of the lamp at (t_0) plus the sum of the values corresponding to the sequence of switchings, then there are only two possible values for the lamp, 0 or 1. Hence the contradiction is derived.

Thomson has maintained that mathematicians "say that this sequence (1 -1 +1 -1 +1 -1 ...) has a sum; they say that its sum is 1/2". This is an argument from spurious authority because it is simply wrong. Bolzano considered this series in 1851 and discovered that if we wish to find a value for S where S = 1 -1 +1 -1 +1 ..., we can group the terms in several ways:

(a) $S = (1 -1) + (1 -1) + (1 -1) + \ldots$

$= 0 +0 +0 +0 +0 +0 +0 + \ldots$

$= 0$

(b) $S = 1 - (1-1) - (1-1) - (1-1) - \ldots$

$= 1 - 0 -0 -0 -0 -0 - \ldots$

$= 1$

(c) $S = 1 -(1 -1 +1 -1 +1 -1 + \ldots)$

$= 1 -S$

Therefore, $2S = 1$, and

$S = 1/2$.

This then, is an infinite series whose limit can have one of three values: 0, 1, or 1/2. Mathematicians, in fact, call this an oscillating series and now agree that it has no fixed sum. Notice, however, that 0 and 1 are possible limits; the switch, it seems can be on or off at the end of the operation!

Thomson then says that this shows that there is no established method for deciding what is done when a super-task is done, but the argument does not do this at all. To return to my first objection to Thomson, there is no reason to expect that the sum of the infinite series (+1 -1 +1 -1 ...) will "represent" the value of the lamp after the hypothesised infinite series of lamp switchings. The fact that 1/2 cannot be said to represent the lamp value does not show that there cannot be an infinite series of lamp switchings. Thomson thinks that he is describing a super-task. However, by reaching the destination after the completion of an infinite number of switchings, a "super-duper-task"[48] has been performed, to wit, completing the series plus reaching the limit. This is to say that the limit does not form part of the convergent series, a point which is fatal to the whole geometric progression strategy and which I propose now to develop.

Thomson now relates his example to the Stadium Paradox where Z is to be the set of points (0, 1/2, 3/4, ...) from left to right along the racecourse. His argument runs: "suppose someone could have occupied every Z-point without having occupied any point external to Z. Where would he be? Not at any Z-point, for there

---

48. P. Benacerraf, loc. cit., p.772, makes this point.

would be an unoccupied Z-point to the right. Not, for the same reason, between Z-points. And, ex hypothesi, not at any point external to Z. But these possibilities are exhaustive."[49] We must occupy every Z-point, and this also means occupying 1. Thomson continues, "The absurdity of having occupied all the Z-points without having occupied any point external to Z is exactly like the absurdity of having pressed the lamp switch an infinite number of times."[50]

I find this passage rather dark, but it indicates the following: Thomson believes that, just as we cannot go through all the Z-points without reaching a point outside of Z (a situation apparently precluded by the above), the description of the lamp super-task is self-contradictory because it fails to provide an answer to the question about the state of the lamp at the moment after the infinitieth act. This seems to show that Zeno's description of Achilles and the Stadium Runner are based on a contradiction, but, Zeno's description can be seen to be perfectly valid by the introduction of a magical leprechaun. Thomson bases his "discovery" of self-contradiction on the belief that one cannot obey the instructions concerning the set of points, Z, without occupying some point, namely 1, outside the series, and this is precluded by the instructions. The leprechaun will show how this need not be the case. Let me introduce this strange creature and mention some of his characteristics. This little man has the amazing ability of being able to change his size at

---

49. Thomson, loc. cit., p.10.

50. ibid., p.10.

will and is even able to disappear entirely.[51] The leprechaun can show how it is not the case that we have to be somewhere after the completion of an infinite number of actions, (if that were, in the first instance, possible.) The leprechaun is a very shy creature, particularly with regard to the number 1. This causes him to reduce in size as he approaches this number; he is half his normal size at the halfway mark, quarter his normal size at the threequarter mark, etc. This means that he vanishes at the destination, the number 1! He occupied every point to the left of 1, i.e. every Z-point, but does not occupy 1 because there is nothing left of him at 1. Indeed, it is evidence of his reaching 1 without occupying it that he has disappeared! The leprechaun, doing nothing which Thomson will not allow,[52] has therefore been able to do what Thomson claims to be impossible, viz. occupying every Z-point without occupying any point external to Z.

---

51. This is not as daft as it sounds: let the runner accelerate throughout his run and imagine, if you will, that the rate of acceleration is such that he will be travelling at half the speed of light at the halfway point, threequarter the speed of light at the threequarter mark, etc. But, $l = l_0\sqrt{1-\beta^2}$ where ($\beta^2 = \frac{v^2}{c^2}$) where ($\beta$) signifies length, (v) is velocity and (c) signifies the speed of light. As $v^2 = c^2$ then $l_0 = 0$. That is to say as an object approaches the speed of light, its length approaches zero.

52. He must allow this because he allows the completion of an infinite number of acts, i.e. the completion of the Z-run.

Thomson, despite his remarkable arguments, has failed to establish the logical impossibility of super-tasks. What misleads him in each argument is the belief that to complete a super-task means to occupy the limit point. He does not show that to occupy all the points in an infinite convergent series of points logically entails occupying the limit point; the leprechaun has seen to that. What makes us feel that we must occupy point 1 is an existential compulsion, not a logical compulsion. Let me give an analogy to indicate my point more clearly. We can, without too much difficulty, imagine our life as the Stadium Paradox. The limit point in this case would be death. The objective of the analogy is graphically illustrated by the Wittgensteinian epigram, that "Dying is not the last thing we do." This, I think, illustrated perfectly the mistake that Thomson, Black, Ryle, Watling et al. have made.

What Black and Thomson have attempted to do is to establish that logical contradiction results if the completion of an infinite sequence of tasks were followed by a determinate state. They have not shown what they set out to show because there is an extremely subtle error in their reasoning. Like Bostock, all that they have succeeded in showing is that either of two determinate states (on-off, left-right, black-white) would be perfectly consistent with the premises. There is therefore a fatal disanalogy between these models and Zeno's paradoxes; they do not, in fact, speak in his defence.

The models offered by Black and Thomson are *finite discrete state* systems.[53] A finite state system is one where at every instant of its history, it is in one of two non-instantaneous states, (on-off, left-right, black-white). In a discrete state system, the occurence of each state is immediately followed and immediately preceded by the unique occurence of the other state. Consider now the Stadium:

The progress of a runner towards Z by way of all the sub-runs, AB, BC, etc. is not a finite discrete state system. The state of being at Z does not bear the *same* relation to the state represented by some sub-run which any occurence of being on (left, black) bears to being off (right, white). In the latter case we can always count on an immediate predecessor for any given state, but not so in the Achilles or Stadium paradox. There are infinitely many sub-runs between being at Z and any sub-run we care to isolate. In Black and Thomson's models the final state (black, left, off) has an immediate predecessor (white, right, on): in the case of being at Z, there is *no* immediate predecessor. There is therefore no analogy.

Despite the fact that neither Black nor Thomson has succeeded in showing that the notion of completing an infinite sequence of acts is self-contradictory, it can nevertheless be shown that

---

53. G. Vlastos, "Zeno's Race Course" in *Journal of the History of Philosophy*, (4), 1966, pp. 99-101.

writers such as Ryle, Bostock, and their supporters who place faith in geometric progressions will always be wrong. They have failed to show that motion is in fact possible, taking motion as something given by Nature. Clearly, it would be tilting at windmills to argue for the Paradox of the Millet Seed, and I do not wish to make anything of this. The concept of "infinity" which they use is fundamentally flawed and will be discussed in the chapter, "Infinity and Kinematics". The most potent criticism which can be levelled at Ryle and the others is to consider the whole notion of a geometric solution.

Every refutation encountered so far has taken its cue from the theory of geometric progressions. These refutations have tried to show that we will get an answer to the question of when and where Achilles will catch the tortoise (or when the Runner will breast the tape) by the application of the formula for summing geometric series.

As I have tried to show, the geometric solution is far from convincing. How is it that the mathematical device of equating an infinite series to a single number, (1/2 + 1/4 + ... tends to 1), makes the series any less infinite? The solutions have said that when we consider a convergent geometric series, what is meant is that, by going along the sequence, we can approach as close as we wish to a certain specific number. To add cogency to this explanation, Watling and others have tried to show that the geometric series, whose terms represent the intervals of space, converges step-by-step with another geometric series whose terms represent intervals in time.[54] This is because of the following: that the sequence of rational numbers 1/2, 3/4, 7/8, ... converges to 1, does not mean that we shall eventually get to 1, if only we keep going for long enough. (Although we will come upon a philosopher who actually maintains this!)

But even this device is not enough. If Achilles or the tortoise do not move regularly, and there is no reason that they should (other than to make the sums more plausible), we can conceive of a random sequence of steps, the partial sums of which, in the numerical representation of the sequence, would not converge to any particular number. For example, to what number does the sequence 1/12 + 1/8 + 1/5 + 1/3 ... converge? These, however, could easily be isolated as points on the course.

These geometric refutations have all been seen to need a general rule enabling us to say "and so on", which shows clearly that the convergence of a numerical infinite series representing problems like Achilles and the tortoise is always presumed. Let us grant this presumption for the moment: it still does not follow that Achilles catches the tortoise, or that the Runner succeeds in reaching the finishing tape. Let me recouch the Stadium Paradox in a way which Zeno could have done.[55]

---

54. This tactic will be examined in great detail in the next chapter.

55. Had Zeno done this, a very boisterous philosophical child would have been stillborn.

"Before crossing a stadium, a Runner has to get to a point A, one third of the way across; after that he must get to a point B one ninth of the total distance of the course beyond point A: and then to a point one twentyseventh of the total distance of the course beyond point B, and so on."[56] This, as can be verified, is a converging geometric series. Unfortunately, to grant this is to grant nothing, for while it converges, it does not converge to 1, the end of the Stadium, but rather to some other number. So the mere convergence of a geometric series is not enough in itself to prove that we can ever get to where we want to go, because it can converge to the wrong number and the number it converges to does not form part of the geometric series.

I have tried to show in this chapter that the notion of completing the enumeration of an infinitely converging geometric series in a finite time is ill-founded. This is because the limit of a geometric series is not a member of the series and because it can imply that "infinity" is a number in the number series just like any other except that it is rather farther away than the rest. This is nonsense as will be clearly shown in Chapter Seven. The two major articles which supported Zeno in trying to show that the notion of completing an infinite series of actions is self-contradictory have also been seen to be wrong. This is because, by using models which were inappropriate, they tried to do too much. Zeno's description of Achilles and the tortoise and the Runner in the Stadium is of one object X approaching another

---

56. With a little ingenuity the same treatment can be given to the Achilles Paradox.

object Y asymptotically, a word which means "approaching, but in such a way that the object approached is never reached", in the same way as Lobachevskian parallel lines. From this it follows that to say "X approached Y asymptotically and X reached Y" is to say "X approached Y in such a way that he neither would nor could reach Y and X reached Y." This is a flat self-contradiction.

It was observed en passant during this chapter that the notion of a time asymptote would solve the problem of an infinite number of actions requiring an infinite passage of time. This took the form of saying that "half the distance could be covered in half the time, threequarter the distance in threequarter the time, etc." The next chapter will see if this is of any assistance in showing that the limit could be reached.

Chapter Five

Clockwatching and Geometric Progressions

"You are not born for fame if you do not know the value of time."

(Vauvenargues)

Aristotle noted that if we have a finite whole and take away equal parts of it sufficiently often we will eventually exhaust the whole. If, however, as Zeno wants us to do, we take away successive parts diminishing in a constant ratio, we will never exhaust the whole. Our subtraction of parts of this kind is potentially infinite. This notion of "potentially infinite" is significant, because Artistotle was the first to note that Zeno was guilty of confusing two senses of "infinite": "... there are two senses in which a distance or a period of time (or indeed any continuum) may be regarded as illimitable, viz. in respect to its divisibility or in respect to its extension."[1]

What Zeno has shown is that we can think of any finite bounded interval as composed of an infinite set of subintervals. We assume, perfectly reasonably, that we are unable to pass over an infinite extent in a finite time, i.e. that humans are in principle unable to traverse an infinite extent. Zeno concludes, however, that neither can we pass over a finite extent decomposed into an infinite set of subintervals. Aristotle has it that "it is not possible to come in contact with quantitatively illimitable things in a limited time, but it is possible to traverse what is illimitable in its divisibility."[2] Aristotle has, so far, given no reason to show how we can pass over an infinitely subdivided finite interval, because, even if the boundaries of the interval are a finite distance part, the subdivisions which have to be

---

1. Aristotle, Physics, 233a.

2. Aristotle, Physics, 233a.

crossed are still infinite in number. We have to come "in contact with quantitatively illimitable things in a limited time."

This last piece of reasoning is considered by Aristotle to be fallacious "for in this respect time itself is also illimitable."[3] i.e. time, like space, is infinitely divisible:

> ... for divisions and subdivisions of the given time and the given magnitude can always be made to keep pace in number and in ratio without limit ... for we do not hesitate to say that half the time suffices to cover half the distance, or generally the lesser time the lesser distance; for the divisions of the distance can always be made in the same ratio as the divisions of the time.[4]

Aristotle's idea occurs with great frequency throughout the history of attacks on Zeno's paradoxes, but there is not, in this approach, the tremendous variety and wonderful ingenuity which characterized the geometric progression school of thought. Each subsequent employment of the theory of dividing time infinitely is more or less a regurgitation of what Aristotle has to say. These repetitions, however, help to develop what Aristotle says and to enable us to deal more fully with his arguments.

One of the best modern utilizations of this strategy is by R.M. Blake.[5] Blake opens his paper by stating (correctly) that there is no last fractional part in the series 1/2, 1/4, 1/8, etc., so that there seems to be an implausibility in the idea of

---

3. Aristotle, Physics, 233a.

4. Aristotle, Physics, 233a. The same argument occurs in Leibniz: "Ne craignez point, monsieur, la tortue que les Pyrrhoniens faisaient aller aussi vite qu'Achille. Un éspace divisible sans fin se passe dans un temps aussi divisible sans fin."

5. R.M. Blake, "The Paradox of Temporal Process" in Journal of Philosophy, 23 (1926), pp. 645-654.

completing the summation of this series:

> It will be said, there are an infinite number of these fractional parts, and for them all to be added each to each and one after the other would obviously take forever. Not so, I reply. This process of successive synthesis would, indeed, take forever to complete itself, if we were to assume that the addition of each fractional increment to the preceding must take not less than a certain definite minimum of duration. But it must be pointed out that such an assumption would be wholly arbitrary, and by no means necessarily demanded by the nature of temporal process as such. [6]

Time, or "temporal process" is going to be divided because he refuses to accept that there is a certain definite minimum of duration, and his debt to Aristotle becomes more obvious a few lines later, where he makes his strategy more precise: "... if the addition of each increment (of the series) is itself, at each successive stage, a process of less and less duration and if this lessening proceed without limit, then the addition of all the increment each to each shows no sign whatever of taking forever."[7] It will, he says, take a definitely limited duration. So, as in Aristotle, time is to be the subject of infinite divisibility as is the space which the protagonists traverse. If the first "step" takes 1/2 second, the next a 1/4 second, and so on, "the enumeration in order of each of the members of the infinite series will all be accomplished in precisely one second,"[8] and, even if at any stage of the process, so long as the process is going on and is not finished, "if the first half of the total process

---

6. ibid. p.649.

7. ibid. p.650.

8. ibid. p.651.

takes one half minute, for example, the next one quarter minute, etc., the entire process will ... quite plainly take just one minute."[9]

This same device, Blake insists, will overcome Zeno's point that motion cannot even get started. There is no first increment in the series ... 1/8, 1/4, 1/2, but if we say that Achilles runs the first 1/16 of the race in 1/16 of the available time, or, more generally, that Achilles runs 1/x of the total distance in 1/x of the total time, then the problem is completely solved.

If we still entertain doubts about this device which Blake is using (and I certainly do) and ask how a process that has no last increment can come to an end, Blake replies that "It will get completed when all the increments have been added each to each."[10] If we are still dubious and ask when this process of adding increments will come to an end, Blake breezily assures us that it will end "When all the increments have been added each to each; and in any concrete case, if I know the rate at which the process is proceeding, I can specify just when this will be."[11]

Ostensibly, this last paragraph heralds disaster for Blake. If a process has always another increment, then the increments can never be totally added together; the successive synthesis of

---

9. ibid., p. 652.

10. ibid., p. 652.

11. ibid., p. 653.

parts will never get the series completed, and the answer to the question of when the process will be completed is simply never. However, this variable of dividing time is very disturbing because I can see precisely what Blake is up to: no matter how tiny an increment is added, the time allotted to that addition is a perfect match to the size of that increment.[12] The fact that there is no discernible last member does not mean that the series lasts forever for that is to be guilty of a confusion "between infinitude of division and that of extent."[13] This is precisely Aristotle's criticism of Zeno although the argument is now clearer.

The clearest exposition of this idea is contained in the article "Achilles and the Tortoise" by Wilmot V. Metcalf.[14] He begins by pointing out that, to a mathematician, a lot of

---

12. See R.B. Winn, "On Zeno's Paradox of Motion" in Journal of Philosophy, 29, (1932), p.401, where this point is made very forcefully indeed:

> ... the problem involves two decreasing magnitudes: that of distance and that of time. And because they are being continually divided by one and the same number, the process is necessarily infinite. As long as we pay attention to the changing distance alone, no solution can be found. But as soon as we come to understand that each decrease of distance is paralleled by a corresponding decrease of time ... it becomes evident that we are dealing here with a constant, created by division of a spatial magnitude of a temporal magnitude.

13. R. Taylor, "Mr. Black on Temporal Paradoxes" in Analysis, 12, (1951-52), p.43.

14. Wilmot V. Metcalf, "Achilles and the Tortoise" in Mind, 51, (1942).

discussions on Zeno's paradox of Achilles and the tortoise are irrelevant. The reason for this, he maintains, is that the "paradox" is not a paradox at all but a fallacy in mathematical reasoning. In order to enlighten us we are asked to discard Achilles and the tortoise: they are "irrelevant complications."[15] We substitute for them two material points A and T[16] which move in the same direction on the same straight line. A travels at one mile per hour while T travels at 1/2 mile per hour. T gets a start of one mile. According to Metcalf the paradox will now read: "While A is covering the one mile, the original distance between T and A, T moves 1/2 mile further along and the distance between them is 1/2 mile; while A covers this 1/2 mile, T moves 1/4 mile and the distance between them is 1/4 mile, etc. We thus obtain an infinite series, 1, 1/2, 1/4, 1/8, ... Since no term of the series can ever become zero, no matter how long the process continues, the distance between A and T can never become zero, i.e. A can never overtake T."[17] I, for one, fail to see much more light being cast on the paradox by the substitution of capital

---

15. ibid., p.89.

16. This is strange: I have absolutely no idea why Metcalf should want to do this unless, like the machine builders of the previous chapter, he feels that humanity tends to cloud the issue.

17. Wilmot V. Metcalf, ibid., p.89.

letters for the names of the protagonists. It gives Metcalf's work an entirely spurious air of mathematical soundness, but more of this later.

Metcalf now attempts to restate the paradox, bringing in the notion of time. "During the hour required for A to move the one mile, T moves 1/2 mile further on. Therefore at the end of the first hour the distance between them is 1/2 mile. During the next half hour which A requires to move this 1/2 mile, T moves 1/4 mile, and therefore at the end of 1½ hours the distance between them is 1/4 mile. During the 1/4 hour required for A to ... We thus have an infinite series of space intervals, the terms of which express the different distances between A and T at different times."[18]

The mathematical conclusion from all this is not that A will never catch T but that "A cannot cover the sum of the series of space intervals separating the two in less time than the sum of the series of time intervals required for A to cover the corresponding space intervals."[19] Although this seems to be saying the trivial "He will take as long as he takes", it is simply a recouching of Aristotle's solution where two interrelated

18. ibid., p.89.

19. ibid., p.90.

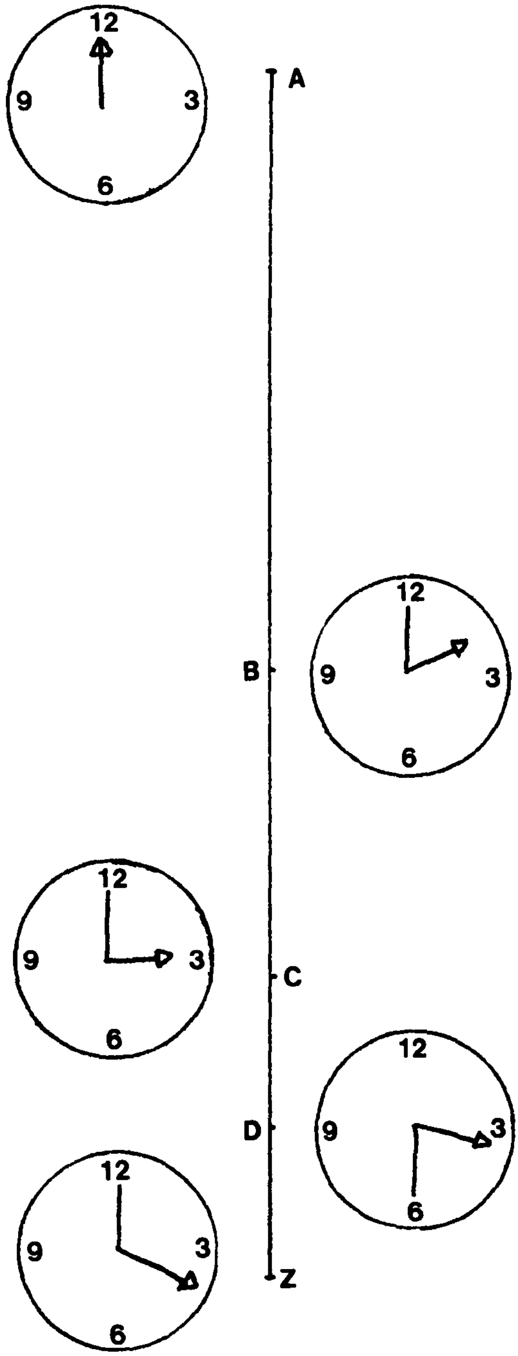
infinite series, one of space intervals and the other of time intervals, intervals, each of which converges to a limit. The problem arises because "Zeno states his puzzle shrewdly to draw attention away from the time series."[20] It is interesting that the total penury of this strategy has never been noted from the time of Aristotle, and that there has been absolutely no evolution within it throughout the centuries.[21]

Let me now try to show how this strategy is useless. If we subtract the main common points from the above discussions we derive the following situation:

---

20. ibid., p.90.

21. Although it could probably be argued that Aristotle set the argument out so well that development would have been impossible.



To run from A to B in this example takes two hours; from B to C takes one hour, etc. Ryle maintained[22] that we always know where Z is because it is AB + BC + 2CD from A, or, every interval already traversed plus the interval last traversed, from A. The same thing can be seen in operation here with regard to time. The finishing time is two hours (from A to B) + one hour (from B to C) + 2 (1/2 hour) (two times C to D), or, four hours. This seems reasonable enough but it is nevertheless all very questionable.

Firstly, the general assumption that half the distance will be covered in half the allotted time is an immense question-beggar,[23] and cannot be casually accepted in the way Aristotle and all the others have done. The point at issue is that Achilles will not catch the tortoise and that the Runner will fail to get to the finishing tape. It seems strange strategy to base

---

22. G. Ryle, see Chapter Four.

23. There is also an assumption that Achilles and the Runner move at constant speeds. If Achilles runs in such a way that the first half mile is run in four minutes, the next quarter mile in three minutes, the next eighth mile in two minutes, etc. Bostock (Chapter Four) has assured us that the run will take for ever. However, let us not be concerned with trivial arguments such as this. It must be mentioned though that there seems to me to be yet another assumption about the behaviour of the tortoise throughout this race. He, naturally, must also run at a constant speed in order to get some sort of correlation between space traversed and time taken. Let Achilles run at <u>three</u> miles per hour and the tortoise, with a one mile start, run at <u>one</u> mile per hour. The bland assumption that Achilles will cover half the required distance in half the time is really too glib by far in this example. It is an example which Zeno could have chosen. The paradox <u>seems</u> simpler than it is because Achilles moves quicker than the tortoise by a factor of 2.

all one's calculations on the assumption that Achilles will catch the tortoise and that the Runner will get to the finishing tape. It only makes sense to speak of taking "half the allotted time" if, in fact, one has already assumed what the total time will be! Naturally, if we assume that the course will be completed in a certain set time it is relatively simple to show the arithmetic of the process, but these writers then go from this to the illegitimate device of using the arithmetic of the process (derived from the assumption that the course will be completed) to show that the course will be completed. There is a profound circularity in this argument.

Secondly, this refutation fails to do justice to the subtleties of Zeno's argument, confirming that Achilles will take an infinite amount of time to catch the tortoise and that the Runner will take an infinite amount of time to cross the Stadium to the finishing tape. Zeno could speak thus: "This notion of dividing time into bits is very interesting but have you considered what follows from it? How many bits will you have to divide this temporal continuum into? Clearly an infinite number. Now, let us consider this infinite number of sub-divisions more closely. Do they have duration or not? If they have no duration, then Achilles will catch the tortoise instantly the starting-gun is fired, and the Runner will flash instantly from one side of the Stadium to the other. Is this what you want to say? Given your dependence on the testimony of the senses this would seem unlikely. You must therefore agree that these sub-divisions or bits of time have duration. But this means that the total period of time consists of an infinite number of sub-divisions, all of which occupy time. Therefore, the total temporal interval must be infinite in duration. As well as this, the traversal or any sub-interval of the spatial interval will consume a "bit" of time: as there are an infinite number of spatial sub-intervals there must also be an infinite time taken for the total traversal."[24]

Thirdly, I wish to consider the time series 1 hour + 1/2 hour + 1/4 hour + ...

In the previous chapter it was clearly shown that the limit to an operation does not form part of the series leading to that limit. We saw that the series 1 + 1/2 + 1/4 ... is endless, and to say that the limit of this series is 2 is to utter a flat contradiction. The notion of a limit was shown to be a useful mathematical _tool_, a device which nevertheless cannot conceivably be shown to give truth: one can add members of the series together until the cows come home, yet one will never attain the limit. This is not because we do not live long enough or fail to speak or run fast enough, but because it is logically impossible. Now, what Aristotle is presenting to us is _another_ geometric series,

---

24. This is to use the Epicheirema on time which is constructed in Appendix One. Aristotle maintained that Zeno had Epicheiremata on time which were lost, so that I do not think that anything illegitimate is being done on my part.

with the difference that this series is in time rather than in space. Therefore, the arguments which were destructive to a series in space must be equally palpable against a series in time! This is because both series have the same logical structure.

According to Aristotle, Metcalf, et al., we have the following temporal sequence: half the time plus quarter the time plus eighth the time, etc. For ease of illustration let me put this into diagrammatic form as:

$$1/2t + 1/4t + 1/8t + \ldots$$

where (t) stands for time. How can this attain its limit (t) if, as has already been shown,

$$1/2s + 1/4s + 1/8s + \ldots$$

where (s) stands for space, cannot attain its limit (s)?

What this solution has done is to saddle us with _two_ geometric progressions _neither_ of which can attain their limit. To marry them together seems to me to compound the problem rather than simplify it. "... recourse to an infinite series of time intervals has the effect of duplicating the difficulty associated with the infinite sequence of distances."[25] This is really disastrous for the Aristotelian solution, and it is based entirely on the fact that "Time intervals of size diminishing according to

---

25. A. Ushenko, "Zeno's Paradoxes" in _Mind_, (55), 1946, p.157.

this pattern (i.e. geometrically) do not add up to their limit 1 (unit of time)."[26, 27]

These arguments have been based entirely on the premise that Zeno overlooked the fact that the time of the run is infinitely divisible. Otherwise he would have realized that a finite time, similarly divided as a dichotomic progression, would yield a sequence of temporal subintervals each giving the runner just the right amount of time to complete each interval of the run. But, it could conceivably be the case that Zeno had Epicheiremata on

---

26. J.M. Martin and C.B. Hinton, "Achilles and the Tortoise", in Analysis, 14, (1954), p.65.

27. To show these difficulties more clearly, let us abandon motion as a temporary irrelevance. By so doing we can consider in isolation the infinite series of time-intervals and see that it is amenable, by the argument of the Stadium Paradox, to the same sorts of difficulty which we found in trying to traverse space, e.g. "A second cannot elapse without the antecedent completion of the first half of the second: but this first half second requires the completion of the first quarter second, etc." or "A time interval cannot elapse before the completion of half that time interval: the completion of the remainder cannot elapse before ..." Hence Whitehead's remark that the real problem is time and that motion is an irrelevancy.

Time and its infinite divisibility, so that it is highly unlikely that Zeno would have committed the very elementary error of which he is accused: "Since Zeno could not have ignored the elementary fact that speed is the time-rate of the change of distance, he must have believed that the time would decrease as unendingly as would the distance."[28] Vlastos, is in fact, saying that Zeno was aware of what Aristotle's argument would be, but realized that it was a futile piece of reasoning. What must also be borne in mind is that for Parmenides and Zeno, Time cannot pass (it gives existence to that which is-not), so that to use the infinite divisibility of Time as a deus ex machina is to argue illegitimately.

I wish now to consider the course taken by David S. Schwayder[29] where he tries to show that an error has been made in our consideration of the problem of Achilles and the tortoise. He is convinced that the presence of the infinite which is bringing such difficulties is entirely illegitimate. His startling suggestion is that "direct intercourse with the infinite is not required to get clear about Achilles and the tortoise."[30] He maintains that the infinite comes only into our representation of the race, our manner of describing the race, and so he proposes to describe the

---

28. G. Vlastos, "Zeno's Race Course" in Journal of the History of Philosophy, (4), 1966.

29. David S. Schwayder, "Achilles Unbound" in Journal of Philosophy, 52, (1955), pp.449-459.

30. ibid., p.453.

race in language which has no place open for the notion of the infinite.

He begins by saying that his will be a clear and direct answer to Zeno and is an "obvious" solution. How trippingly we enter the lion's den!

"Assuming that Achilles will catch the tortoise, we know that he must do it within some definite amount of time."[31] This is an incredibly bizarre beginning: we have to assume the very point at issue, but let us press on. His next point is that if we are given sufficient information about the initial distance between Achilles and the tortoise and the respective velocities of each, then we can "easily calculate" when the tortoise will be caught.[32] We are to assume that Achilles runs at m feet per second, the tortoise runs at n feet per second, and that the tortoise has a start of a feet. Achilles, it seems will catch the tortoise in

31. ibid., p. 453, my underlining.

32. I am not sure if Schwayder is saying anything new here. It is not the case that Zeno also gives us respective velocities and the initial start which the tortoise is to get?

"precisely" a/m-n seconds.[33]

According to Schwayder, what Zeno has shown is that for any time before Achilles has passed the tortoise, we can specify later times at which Achilles will still not have caught the tortoise. Given any point where Achilles is short of his goal. Zeno shows us how to choose the next point so as to leave Achilles still short of his goal.[34] This, however, is just to say that if Achilles has not caught the tortoise he has still some way to go. This is more tautology, although well disguised as pseudo-logical reasoning.

What precisely has Schwayder shown so far? Only that the time taken by Achilles to catch the tortoise, if he is able to catch the tortoise, can be calculated. Note that his whole proof rests on the assumption that Achilles does catch the tortoise.

---

33. This is all very plausible, but what would Schwayder make of this example? Achilles runs at ten feet per second, the tortoise runs at one foot per second, and the tortoise has three hundred feet of a start. Achilles will catch the tortoise, according to the formula, at 300/9 seconds, or, 33.333333 ... seconds after the start of the race. But what precisely is this time after which Achilles will catch the tortoise? To answer this we have to invoke the theory of limits, saying that 33.333333 ... seconds has a limit, and that this limit is the contact time of Achilles and the tortoise. But this device was dismissed in the previous chapter!

34. David S. Schwayder, ibid., p.453.

Schwayder now goes on to clarify his strategy by saying[35] that we are safe in betting Achilles, at any stage before he has caught the tortoise, that he will not catch the tortoise in any time less than (a/m-n) seconds after the start of the race. We could make "howsoever many similar bets we wished, so long as Achilles had not reached his more sluggardly opponent."[36] Zeno has confused us because he contrives to lengthen each of the successive temporal intervals which he plots for Achilles; he "slows down the passage of time" so as to make each interval ($t_{n+1} - t_n$) just as long as the previous one. I can see what he is getting at, but, just to mesmerise us further, we are told that "Zeno-time" resembles in reverse the "absolute time scale" which (for good measure) is based on the exponential law of radio-active decay.[37] Comment on this is surely superfluous. Aristotle has pointed out that we must reduce the time intervals to correspond with the shortening spatial intervals traversed. This, cloaked in an inky scientific blackness, is precisely the point which Schwayder is making: "The paradox requires us to use Zeno-time (constant time intervals) when we do not do so; so there is no problem."[38] Unfortunately there is a problem: Schwayder is saying the same thing as Aristotle, therefore the criticisms

---

35. ibid., p.454.

36. ibid., p.454.

37. ibid., p.454.

38. ibid., p.454.

applied to Aristotle must also apply in equal measure to Schwayder.

Schwayder proceeds, and here he breaks new ground. He has accepted (p.451) that Zeno has shown that we may think of any finite bounded interval as composed of an infinite set of sub-intervals, and he has also said (pp.453-455) that we can specify any number of points before which Achilles would not have caught the tortoise and we can also specify "as many such points as we wish."[39] He then goes on to assert that however many we specify, "it will still be only a finite number of points."[40]

Schwayder's plan should be transparently clear; it is a good plan. If there are a finite number of points between A and Z, and if each step covers a finite number of points, then we can use simple arithmetic to solve the paradox. However, we must ask ourselves this question: has he shown that there is a finite number of points within any spatial interval? His proof seems to be that the number of points is finite because we cannot specify each of an infinite number of points. This, however, is far from showing that there are a finite number of points in any spatial interval, because it is a statement about us rather than about space. Besides, Schwayder has already concluded (p.453) that Zeno shows us how to choose the next point so as to leave Achilles still short of his goal always. How then can Schwayder also conclude

---

39. ibid., p.455.

40. ibid., p.455.

that here are a finite number of points in a line? These are contraries.[41]

Thinking that he has shown the Achilles paradox to be a finite problem, Schwayder goes on to ask how infinity ever came into the reckoning at all. He says that the mathematical representation of the race which is implied by Zeno's statement of the paradox does not give a miniature model of that situation. By this he means that the sequence (1/2, 1/4, 1/8, ...) is not somehow "a condensation of a process into numerals or figures: that the line of figures, is, as it were, a replica reproduction of the race. What we have is a mathematical representation of the race."[42] These numbers are part of the logical machinery with which we describe what would have happened in the race but do not somehow add up to starts, steps, and stops. Infinity comes in as a characteristic of the numbering system and not as a characteristic of what we describe.

This suggestion is very good and will be covered more fully in a later chapter, but a little preliminary work on Schwayder's idea will not go amiss. Is it the case that we need use numbers ("the logical counters we use to describe the process") at all in describing the paradox? We need only use numbers if we make certain Schwayderian assumptions which we then use as a means of

---

41. There could, after all, be no points in a line; this will be discussed more fully at a later stage.

42. David S. Schwayder, ibid., p.456.

breaking down the paradox. Let me clarify what I mean. Schwayder has assumed speeds expressed as number symbols, distances expressed as number symbols, and the logic of algebra and arithmetic to attempt to undermine the paradox. That is to say, he is using the numbering system to overturn the paradox while simultaneously maintaining that the paradox arises through using the numbering system! Something is wrong with this approach.

The Achilles paradox need not even mention numbers at all: couched at its most simple it can be stated as follows: "For Achilles to catch the tortoise he must pass through some point already passed by the tortoise." Where are the numbers?

In conclusion, Schwayder has failed to show that we do not need to use the notion of infinity in solving the Achilles paradox. Infinity is there in the sense of a growing or potential infinite, based on the infinite divisibility of space. To try to overcome this problem by utilizing Aristotle's notion of dividing time to match the divisions in space has been shown to be ineffective.

We have examined the notions of dividing space and dividing time: let us now turn to an examination of what the protagonists in the paradoxes are doing as the runs proceed, i.e. we now turn to the dynamic aspects of the operations.

Chapter Six

The Dilemma of Dynamics

Scientists have had disappointingly little to say on Zeno's paradoxes, which is strange as the paradoxes seem to me to be a subject on which physics would have a lot to offer. One should never generalize from personal experience, but most scientists whom I have sought out on the subject have been rather dismissive, inclining to the geometric progressions as a solution. A philosopher who has written extensively on Zeno's paradoxes is Adolf Grünbaum and his most interesting and approachable solution to Zeno is contained in his "Modern Science and Zeno's Paradoxes of Motion".[1] This work, which is based on an examination of dynamics, has been described as "the finest work ever done on Zeno's Paradoxes"[2] and, in it, Grünbaum intends to show that certain kinds of super-task, such as those performed by the infinity machines of Black and Thomson, are logically possible.

The fundamental problem, which these machines have been thought to demonstrate quite conclusively, is that a process involving an infinite sequence of "acts" or operations cannot be completed. In order to show how an infinite sequence of tasks, a supertask, can be done, Grünbaum proposes to reopen the discussion on the infinity machines considered earlier. It will be remembered that Black "invented" various machines which were able to pass a marble from position A to position B an infinite

---

1. Grünbaum, Adolf, "Modern Science and Zeno's Paradoxes of Motion" in The Philosophy of Time, ed, Gale, Richard M. (Doubleday Anchor, 1967).

2. Gale, Richard M., ibid., p.394.

number of times in two minutes. Thomson had a lamp which could be switched on and off an infinite number of times in a finite time interval. As if this were not enough, Grünbaum intends to consider yet another two magnificent constructions: firstly, a machine which will print the entire decimal representation of π such that the first digit is printed in the first half minute, the second in the next quarter minute, the third in the next eighth minute, etc. Secondly, a machine which will recite the entire series of fractions between 1 and 0.

Grünbaum's first step is to consider the motion of Achilles, which he calls the Legato Motion, and the activity of this wonderful machinery, which he calls the Staccato Motion. If he can conflate these by showing that Achilles and the infinity machines are doing exactly the same sort of thing, and if he can show how the infinity machines can perform their supertasks, then he will have shown how Achilles can catch the Tortoise and how the Runner can traverse the Stadium.

The Legato Motion is the traversal of a Z-sequence[3] in unit time by a runner who runs continuously at uniform unit velocity. The runner will therefore traverse smoothly the first Z-interval in half a unit of time, the second Z-interval in quarter of a unit of time, etc. The infinity machines of Black and Thomson, on the other hand, use Staccato Motion, in that they operate discontinuously. The machine takes quarter of a unit of time to

---

3. A Z-sequence is a sequence of the form 1/2, 1/4, 1/8, etc., each interval of which is called a Z-interval.

traverse the first Z-interval of length ½ and thereupon rests for another quarter unit of time; it then takes 1/8 unit of time to traverse the second Z-interval of length ¼ and rests for an 1/8 unit of time, etc. By substituting another runner for the infinity machine which is running discontinuously, Grünbaum hopes to show that these motions can be conflated. Imagine that both the Legato runner and the Staccato runner depart jointly and run parallel courses. The Staccato runner traverses each Z-interval in half the time required by his Legato colleague but waits for him to catch up before setting off over the next Z-interval. They "arrive jointly at their final destination after a finite time",[4] the reason for this being that, while considering the running within the Z-sequence, the Staccato runner's velocity is twice that of his Legato competitor, his overall average velocity is equal to that of his fellow. Because the motions have the same average velocity they can be conflated.

It seems to me that Grünbaum's is an extremely doubtful position on two counts. Firstly, it begs the very point at issue which is that they do not arrive at their final destination. Secondly, a little thought will show that they do not arrive (if they can arrive at all) jointly. At every junction in the Z-sequence the Staccato runner arrives first and waits for his colleague to catch up. It follows then that they cannot arrive jointly anywhere, although they can jointly be somewhere, e.g. at a junction in the Z-sequence. The average velocity of the

---

4. Grünbaum, loc. cit., p.463.

Staccato runner is surely calculated between when he sets off from the start and when he arrives at the finish (assuming, just for now, that he can arrive at the finish). To have the same average velocity as the Legato runner the calculation must be from when the Staccato runner sets off at the start until when the Legato runner arrives at the finish. Therefore, they do not have the same average velocity, and if it is this which is to be used as the basis for showing that the Infinity Machines and Achilles are doing the same thing, then Grünbaum has failed before his argument gets to its main thrust. The main point of his argument, however, is of such importance and interest that I propose to let this objection pass.

It is critical to notice that the Staccato runner cannot do anything while waiting for the Legato runner to catch up. To clarify this point, let us imagine that the Staccato runner has read Ryle's "Dilemmas" and, in order to show where he had a rest, decides to plant a flag each time he stops. The erection of a flag at each of the junctions in the Z-sequence (let us call them Z-stops) will clearly require him to move his own limbs and translate the flagpole each time through some positive distance, however small that distance is. One does not have to be Zeno to see that, at the end of the Z-run, the Staccato runner will have effected a spatially infinite total displacement of his own limbs and of the flagpoles, i.e. if he raises the flag by two feet in order to generate the momentum necessary to jab it into the ground at each Z-stop, the total distance covered by his arms in raising and lowering the flagpoles will be two feet multiplied by the number of times he did it. This is $\aleph_0$ times, giving an infinite total distance which is impossible. As well as this, the successive vertical velocities required by his limbs would increase boundlessly with time as he has less and less time in which to raise and lower the flagpoles to generate the necessary momentum for penetration. As he plants the last flag his limbs will be moving infinitely fast, which is impossible. In order to accomplish this feat he would have to expand an infinite amount of energy, which is impossible. This, then, is the dynamic problem attached to Zeno's paradoxes and it is this which Grünbaum proposes to confront.

The first machine which he considers is the π-machine: this machine prints the first number of the decimal expansion of π in the first half minute, the second digit in the next quarter minute, etc., and, in order to overcome the problems discovered in the case of the flag planting Staccato runner, Grünbaum makes the following stipulations:

1. The heights from which the press descends to the paper to print the digits must not be equal. If they were this machine would be nothing but a mechanical flag-planter with all its attendant problems. The heights must form a geometrically decreasing series converging to zero. The height of the press above the paper must be half the height of the press which it follows. In this way the machine traverses ever smaller distances in the same proportion as the ever smaller time intervals and as the number expands.

2. Frege has said (Ch.4, note 35) that an operation like this will require infinite amounts of paper and ink. Grünbaum states that the width of the successive numerals printed must converge to zero. Were they to remain that same size, then the horizontal line on which $\aleph_0$ digits were printed would have to be infinitely long. The numerals, then, must also form a geometrically decreasing series converging to zero.

If these stipulations are satisfied, then Grünbaum maintains that the decimal expansion of π will be printed in a finite time interval on a strip of paper of finite length:

> Under the fundamental restruction of my first proviso regarding the heights of descent, the π-printing machine no more requires an infinite time than do either the Legato or the Staccato runner. And, given my second requirement concerning the widths of the successive digits, the spatial array of the digits no more requires an infinite space than the unending progression of Z-intervals which collectively fit into the space of a finite unit interval ... [5]

This is a very interesting notion, but before considering it in more detail, I wish to examine his treatment of Thomson's lamp and the marble-transferring machines of Black. With regard to Thomson's lamp, it will be recollected that the lamp possessed the kind of on-off button which is activated by jabbing. If the lamp is off, a jab on the button will light it; if the lamp is on, a jab on the button will extinguish it. Thomson suggested that the problem after an infinite number of jabbings was that we could not know whether the lamp was on or off. Grünbaum points out that this is not the real problem: if each successive jab at the

---

5. Grünbaum, loc. cit., p.471.

switching button involved the same spatial displacement, then we would be like the flag-planter in that we would have jabbed over an infinite distance with limbs which eventually moved at infinite speed (whatever that is, (but certainly faster than the speed of light discussed in the introduction)). There therefore has to be the following stipulations:

1. "the successive spatial displacements involved in the consecutive jabs of the switching button must have lengths forming a suitably decreasing sequence which converges to zero."[6]

The argument here is clearly the same as with the π-printing machine and it should be obvious how Grünbaum is going to tackle the Marble-transferring infinity machines of Max Black. The stipulation is:

1. The distances through which the marbles have to be transferred cannot remain constant. Were they to remain constant the arms of the machine would swing through an infinite distance and move infinitely fast. "... The distance through which the successive marble transfers would have to be effected would decrease in proportion to the available successive times and thus would suitably converge to zero ..."[7]

Let me now try to relate these to the Stadium and Achilles and the Tortoise. We can say (in the style of Grünbaum) that what Achilles and Runner do is exactly the same as these machines do if we obey Grünbaum's stipulations. The machines have less

---

6. ibid., p.479.

7. ibid., p.479.

and less to do in each time- and space-interval of the supertask (otherwise they would be doing that which is dynamically impossible) and this is precisely the same for Achilles and the Runner! That the time-intervals match the space-intervals is neither here nor there: what _is_ significant is that Achilles has less to do in each spatial and temporal interval than he had to do in the preceding spatial and temporal interval so that he does _not_ have to expend an infinite amount of energy. He does not take the same number of steps in each spatial interval and, as the intervals become less than one's step's worth, his legs are swinging less and less within each interval. This does not mean that his steps are getting smaller but that smaller segments of each step are being isolated within each spatial interval. He does half his _work_ in the first space- and time-interval, quarter his _work_ in the next space- and time-interval, etc., so that he can be perfectly human and still accomplish his task: he simply has to expend a _finite_ amount of energy and not indulge in anything which is dynamically unsound.

This refutation is very tempting but it can be shown that it is fundamentally flawed. If we examine each of his stipulations we can see that they are all of a one: let us isolate the aspect of the task which is causing the dynamic problem and perform an operation on it such that it no longer gives cause for concern. The operation he chooses is in each case the same. In the examples of the π-machine, the Thomson lamp and the marble-transferring machine we see that he invokes the following progressions:

1. (h + 1/2h + 1/4h + etc.) where h is the height of the press.

2. (p + 1/2p + 1/4p + etc.) where p is the size of the print.

3. (j + 1/2j + 1/4j + etc.) where j is the length of the jab on the lamp.

4. (l + 1/2l + 1/4l + etc.) where l is the length of the swing of the marble-transferring arm.

and, arguing from these to Zeno:

5. (w + 1/2w + 1/4w + etc.) where w is the work done by Achilles in each spatial and temporal interval.

These can all be reduced to one idea: that the energy expended in each interval decreases in a geometric progression to zero.

It can be deduced from Chapter Five that Grünbaum's attempts to bypass the dynamical difficulties of the paradoxes have led him to commit what is, in principle, the same error as Aristotle. If the progression (1/2, 1/4, 1/8, etc.) will not attain its limit zero, then it does not really matter what symbol we attach to the numbers: the progression will still never reach zero. We saw earlier that if neither (1/2d + 1/4d + 1/8d + etc.), where d equals distance covered, nor (1/2t + 1/4t + 1/8t + etc.), where t equals time taken, reached their limit, then there was absolutely no reason to suppose that putting them into clinker would enable the limit to be reached. The substitution of the different letter "w" (for work done) for "d" or "t" and setting it in clinker with whichever of "d" or "t" is left, gets us nowhere. Grünbaum has used the very device which is causing the problem (that a geometric progression does not attain its limit) to try to overcome the problem. He therefore unwittingly begs the question and this part of his article must be seen as mistaken, perhaps not on dynamical grounds but certainly on logical grounds.

Grünbaum, however, goes on to consider the Peano Machine, which is a mechanical device capable of reciting the sequence of fractions 1/2, 1/3, 1/4, ... It departs from the number 1 which it expresses as 1/1 (one over one) and within one minute arrives at the number 0. Let us consider the progression of points 1/n (n = 1,2,3, ...) within this unit interval, a progression which contains 1 (1/1) but no 0. The machine works as follows: for every one of these points 1/n (n = 1,2,3, ...), when reaching that point it begins to recite the number 1/n, and completes the recitation of 1/n by the time it arrives at the next point in the progression. Thus for every fraction 1/n, the device takes (1/n - 1/n+1) of a minute to be recited, e.g. 1/2 takes (1/2 - 1/3) minutes to be recited, 1/3 takes (1/3 - 1/4) minutes to be recited, etc.

If we allow the use of English names for the fractions to be recited (e.g. one over sixty four or one over eight thousand and four hundred and twenty two), then the names which increase in syllable content as the denominators increase, would increase boundlessly. Human physiology being what it is, this boundless increase ensures that we cannot articulate the members of the series of fractions (1/2, 1/3, 1/4 ...) in such a way that the enumeration of each member takes a proportion of the duration of the previous member. We would have to move our lips infinitely fast, and clearly the same would hold true of the Peano Machine.

To overcome this difficulty, Grünbaum therefore stipulates that:

1. Non-English names have to be used, names which decrease in syllable content in proportion to the size of the number and converge to zero, so that "the successive distances traversed by the mechanical lips as they perform their recitation would decrease in proportion to the available time."[8]

There are two obvious objections to the first part of this stipulation. Firstly, if the numbers are to decrease in syllable size on the way to zero, what size is the first fraction to be? Clearly the fraction is 1/2, but how many syllables is it to be constructed from? That cannot be answered, so that this notion is a definite non-starter. Secondly, if there is no fraction for 0 (1/?), then how can it be attained? We are in exactly the same position as the Magic Leprechaun which popped into non-existence if it attained the limit of its journey. Grünbaum, then, is asking the impossible.

Let us turn to the actual method of producing the noise which is spoken by the mechanical lips. The voice is activated by a vibrating membrane and "each of the distinct sound names or noises requires at least one vibration of the voice membrane. But the time available for the utterance of these successive noises converges to zero. Hence the <u>frequency</u> of the noises and also of the membrane must increase indefinitely."[9] In other words, each

---

8. Grünbaum, loc. cit., p.475

9. ibid., p.475.

number will be sounded higher in pitch than the number before. Let us point out right away that this "increase indefinitely" is a physical impossibility. There is a speed beyond which nothing can go: it is called the speed of light and acts as a barrier in nature to the increasing speed of the voice membrane. Grünbaum clearly realizes this because he brings in the notion of amplitude. As the frequency increases, the amplitudes have to "decrease in such a way that the total energy expended is finite."[10] It would be interesting to see an oscillograph's recording of the sine wave he has in mind for the last fraction! It would have infinite frequency with zero amplitude. What it would sound like is unimaginable. Grünbaum does not know what he is talking about because frequency is dependent on amplitude for its very legitimacy.

The notion of pitch is, however, quite exciting. Let us abandon all this talk of names which have less syllables than the name before as just so much rubbish: what is important is his idea of pitch. Let us imagine that the machine emits a humming sound. This sound forms a true musical note which is transformed by rising in pitch to correspond to the progression of fractions. The Peano Machine will therefore count the progression by sounding an upward glissando. We must be careful about this: if each fraction sounds a tone higher than the fraction before, we are, very shortly, going to be far beyond the hearing of any human, animal or machine. I am not sure if the expression "infinitely

---

10. ibid., p.475.

high in pitch" actually means anything but it seems to me to be rather redundant to set up the Peano Machine and not be able to know if it is sounding or not. We need a note which is to be the last note sounded <u>and</u> which is to be audible. What we could say is this: let the first note sound tenor C and the last note be soprano C two octaves higher. How are we to divide the two octaves? Clearly if we use semitones, quarter tones, or microtones we will soon run out of intervals to which we can allot a fraction. We must do the following: let the tuning be equally tempered and the first pitch sound tenor C, the second fraction sound one octave higher, i.e. middle C, the third fraction sound one half octave higher, i.e. F sharp, etc. We thus get the following pitches:

These pitch intervals, however, will reduce to zero in exactly the same way as the energy intervals of the other machines. Unfortunately, then, soprano C cannot be attained. Although this is the only way in which using musical pitch can be made plausible, it means that it falls for precisely the same reasons as the other arguments.

This attempt by Grünbaum to overcome the dynamic difficulties inherent in the paradoxes seems to me to be the most cogent discussion so far, but, although it has the ring of truth, the logic of the earlier arguments means that the method suggested by Grünbaum must be rejected.

However, Grünbaum has yet another suggestion to make: motion, force and energy have caused all the difficulties associated with the paradoxes. If we could explain change without reference to force and motion then a solution may just present itself. Accordingly, I propose to examine just what is meant by "infinity" and, from this, to proceed to a discussion of Relativistic Kinematics.

Chapter Seven

Infinity and Kinematics

"God made the integers; all else is the work of man."
(Kronecker)

The paradoxes can be seen as a manifestation of the confusions over precisely what is meant by the word "infinite", and it has been said that once we are "clear about what different things are intended by "infinite" the problem will resolve itself."[1]

It is my contention that the basic confusion running through Zeno's description of the Stadium and Achilles and the Tortoise is a confusion between the necessary and sufficient conditions for defining continuity. What Zeno does is to take a necessary condition as a sufficient condition. This necessary condition is the infinite divisibility of a continuum into rational fractions. The infinite divisibility which Zeno prescribes is the infinity of a series the elements of which can be counted one after the other but which has no end. The elements are the successive positions of the Runner in the Stadium or the successive relative positions of Achilles and the Tortoise, and the argument assumes that these positions must be passed through one after the other in the order of positions that can be counted. If we grant Zeno this way of putting his arguments then his conclusions are inescapable because we allow him to divide space and time into an infinite number of rational fractions, and this division has no last term.

But, this condition of defining continuity, that it consists of rational fractions having no last term, is not sufficient, because Zeno's analysis of the conditions is inadequate: the infinity involved in an adequate analysis of continuity is of a different type from the type which he assumes. There are <u>different</u>

---

1. D. Schwayder, loc. cit. p.449.

types of powers of infinity and it is unfortunate that most subsequent commentators have followed Aristotle's acceptance of Zeno's analysis. We have seen that Aristotle noted two kinds of infinity, infinity with respect to divisibility and with respect to extremities. Zeno may well have been confused between these but Aristotle's solution is of no help in solving the paradoxes for each of these kinds of infinity is of the same ordinal type and has the same transfinite cardinal number. Both of Aristotle's kinds of infinity are made up of elements that can be counted one after the other without end. Every discussion encountered in the last few chapters has clearly been carried on in terms of Zeno's analysis and Aristotelian theory, and, because of this, has been doomed to failure.

The reason why I can reject Zeno's analysis of continuity is that the infinite divisibility of a continuum into rational fractions is not sufficient to define continuity: it "does not define a continuum and will not yield one."[2] However, we have said that Zeno's analysis gives a <u>necessary</u> condition for the continuum: if it can be shown that this condition is not, in fact, necessary, then Zeno (and the continuum) is in deep trouble. This is attempted by Fred I. Dretske who intends to count to infinity,[3] (equivalent to counting the rational fractions and, thereby, violating Zeno's condition.)

---

2. H.N. Lee, "Are Zeno's Paradoxes based on a Mistake?", in <u>Mind</u>, 74, (1965), p.564.

3. Fred I. Dretske, "Counting to Infinity", in <u>Analysis</u>, 25, (1964-65), pp. 99-101.

The opening to this article is tremendously dramatic: George announces that he is going to count all the natural numbers. When informed that there is an infinitude of natural numbers, George replies that he will therefore be counting to infinity. Can he do it? Dretske thinks that he can if certain conditions are allowed. "George possesses certain physical limitations which prevent him from counting to infinity; there are no other factors, either logical or conceptual, which disallow it."[4] In other words, the "everyday" definition of an infinite class as one whose members cannot be counted in any finite time, however long, is to be challenged.

For this claim to make any sense, Dretske must somehow assume that there is an actual infinite number, and this becomes more apparent when we pass to his proof which is given in three steps:

(a) If George never stops counting, he will count to infinity.

(b) If George stops counting, then some purely contingent circumstance (such as death) interfered.

(c) If no purely contingent circumstances interfere, then George will count to infinity (and Achilles will catch the tortoise, etc.).

I propose to examine these three stages rather closely.

(a) is interesting because, despite the self-contradictory nature of the statement, it has a proof. The proof, however, is based on an assumption of which either Dretske is unaware or does not acknowledge. If George counts the numbers at the rate of one per

---

4. ibid., p.99.

second then, for any finite number N, George will count it N seconds after he started counting. Hence, for all finite numbers N, George will count N after the elapse of N seconds. Since there are an infinite number of numbers "we can say that George will count to infinity in the sense that he will count each and every one of the finite numbers."[5]

He will reach the hundredth number after the elapse of one hundred seconds, he will reach the millionth number after the elapse of one million seconds, he will reach infinity after the elapse of an infinite number of seconds. This is to say that George will count to infinity in an infinite time. But what does George do when he gets there, when he reaches infinity? Could he, for example, quaff a beer? No, because to get to infinity Dretske has told us that "George <u>never</u> stops counting."[6] If he never stops counting then he never gets there: to deny that is to be foolish.

(b) also has a proof in which is to be found the following: If George stops counting (perhaps his voice gets tired), then "there must be some finite numbers which he did not count."[7] This statement can easily be missed because it is contained in diversions about the contingency of death (and lots of other equally amazing claims). The red herring laid out by Dretske for us is to argue

---

5. ibid., p.99.

6. ibid., p.99, my underlining.

7. ibid., p.99.

about mortality being a necessary condition of humanity, but this is peripheral. The most important of what he says is the statement which I have quoted, that if I stop on the grand count to infinity, I will have some numbers left to count. Dretske is, in fact, saying that if I count to 1,000,000 and then have a well-deserved rest, I will have (infinity-1,000,000) numbers still to count. It implies that if George (purely contingently) dies after counting for one thousand years, we should all hang our heads, saying, "Poor old George, he almost made it: only (infinity - 1,000 years) to go." This is strange: is it not more likely that we would say "Poor old George, no nearer than when he started"? On the count to infinity, if he stops, there will still be an infinitude of numbers to count. (This, in fact, is the basis for my contention that Dretske thinks that there is a transfinite number).

(c) In discussing this part of his argument, Dretske says that "It is true that at any stage of his task George will not (yet) have counted some numbers."[8] He then goes on to say: "... that fact (that he still has numbers to say) is not relevant to whether he will count to infinity; it only shows that he never will have counted to infinity."[9] In order to justify this, we are told that this is the form of statement where "George will do X" does not imply that at some later time it can be said "George did X". How

---

8. ibid., p.100.

9. ibid., p.100.

could it, for George never stops counting! Given that George never stops counting, and that it can never be said that "George finally made it to infinity", one wonders what the point is of saying that "George will count to infinity"? The statement seems to me to be completely redundant. At best, the statement should be transformed into "George hopes to count to infinity." (Presumably accompanied by much head-shaking and tapping of the temples.)

There is something going on in the word "counting" which is rather strange. I'm not all that sure myself about this but "counting" seems to contain an ambiguity. We can distinguish between "task" words and "achievement" words. For example, "I am going home" would be construed by me as a "task", whereas "I arrive home" would be an "achievement". Once an achievement word has been uttered, it cannot be negated without contradiction. Let us imagine that these two propositions are uttered at some time (X):

(a) I am going home.

(b) I arrive home.

At some time later (Y) the following propositions could be uttered:

(a) I did not go home.

(b) I did not arrive home.

Now, it is my contention that someone could utter "I am going home" at time (X) <u>and</u> still be able to say "I did not go home" at time (Y). This is because "going home" is a task. But, someone who utters "I arrive home" at time (X) and then "I did not arrive home" at time (Y) will have uttered a contradiction. He will have said

"I did and did not arrive home." This is because "arriving home" is an achievement.[10]

Let us now return to "counting": is "Counting to infinity" a task or an achievement? It is hard to tell precisely. "I counted to infinity" is even more unclear: it could mean that I started counting off the integers, i.e. a task. I could, therefore, by the logic of my previous argument, say that I did not count to infinity, i.e. "I counted to infinity (task) but did not count (all the way) to infinity (task)." I could not say "I counted to infinity (achievement) but did not count to infinity (achievement)". That would be self-contradictory. I could also not say "I counted to infinity (achievement) but did not count to infinity (task)". How else could I count to infinity? But I could say "I counted to infinity (task) but did not count to infinity (achievement)".

Dretske has said that we can say "George is counting to infinity". This must always be a task. However, we can never say that "George counted to infinity." It is unclear whether he means a task or an achievement.

---

10. Consider one more example: Someone, in the act of seducing the girl next door utters (hopefully to himself) "I am seducing the girl next door." Her mother arrives, activity ceases, and he has to say later that he did not seduce the girl next door. No contradiction has been uttered. However, assume his ambitions to be less grand. In the act of <u>kissing</u> the girl next door, he thinks to himself "Gosh! I'm kissing the girl next door." Even if her mother arrives he cannot later say that he did not kiss the girl next door. The seduction is a task: the kiss is an achievement.

Dretske assumes that transfinite numbers, if they exist at all, belong to the arithmetical series: but how many philosophers would concede that infinity can be reached through the ordinal sequence originating in the first integer (or any integer, come to that)? Unless transfinite numbers belong to the arithmetical series and unless, consequently, they can be reached through the ordinal sequence, the finite and the infinite remain disparate, and the operations created through the study of finite numbers are not applicable to transfinite numbers. That said, how does it affect Zeno?

In the Stadium and Achilles Paradoxes, the analysis of motion involving space and time is made in terms of a rational series only; as we saw, infinite divisibility into rational fractions (1/2, 1/4, 1/8, ...). But these rational fractions make up a denumerable dense series, and there are gaps within such a series which cannot be filled by other rational fractions, however small. Even if there is an infinitude of these rational fractions there will still be gaps. Zeno's definition of continuity is inadequate because it leaves out the irrationals. In a continuous series, irrational numbers have ordinal positions that do not coincide with any possible rational fraction. The gaps in a denumerable dense series are inescapable because the elements of such a series are discrete, but the elements of a continuous series are not discrete. Zeno was correct in saying that no motion could be made up from a sum of discrete and denumerable positions, but this is not how motion is made up. If we wish to analyse the motion of the Runner in the Stadium or Achilles' racing the tortoise, the model required is the theory of the linear continuum developed by Cantor.

The class consisting of all natural numbers, 1,2,3, ... and the class consisting of the square of all natural numbers, 1,4,9,16,.. are examples of infinite classes. They satisfy the following loose but intuitive definition of the sort which Zeno uses: "An infinite class is one whose members cannot be counted in any finite period of time, however long." But, it may occur to us that there are more members of the first class than there are members of the second. The reasoning for this is obvious: it is true that all members of the second class are members of the first, while there are members of the first class which are not members of the second, namely 2,3,5,6,7, ... Could it not be said then that even though both classes have an infinite number of members, we feel that the infinitude of members in the first class is greater than the infinitude of members in the second class? Galileo[11] came to the conclusion that all we can say about these classes is that they are both infinite. The relations "equal to", "greater than" and "less than" could not be applied to infinite classes and Galileo dismissed the problem as a paradox. Georg Cantor concluded that this paradox could only be resolved by attributing to infinite sets some specific property not possessed by finite sets, and the paradox itself, that the subset (square numbers) has as many terms as the set (natural numbers) provided such a property. Thus Cantor _defined_ an infinite set as one whose elements could be

---

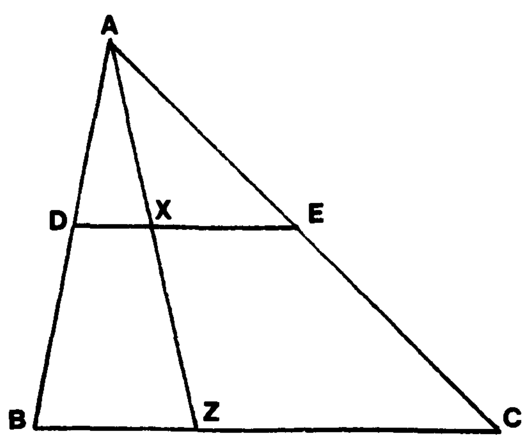
11. Galileo, _Dialogues concerning Two New Sciences_, (New York, 1914), pp. 31-33.

placed in a one-to-one correspondence with the elements of one of its own subsets. The simplest and most fundamental of all infinite classes seems to be the class consisting of all natural numbers. This class can be donated by A_1, its transfinite number. Consequently, we denote by A_1 the number of any class whose members can be put into a one-to-one correspondence with the members of this particular class, the series of natural numbers. We can set up a relationship between numbers in the following way:

1	2	3	4	5	6	....	n
1	4	9	16	25	36	....	n^2
2	4	6	8	10	12	....	2n
1	3	5	7	9	11	....	2n - 1

The class of all even numbers has the same transfinite number A_1 as the class of all odd numbers and the class of all square numbers, and in each of these cases there is the same transfinite number which we gave to the class of all natural numbers! We can now recouch our previous loose definition of "infinite" and define an infinite class as "one which can be put into a one-to-one correspondence with part of itself."

If we apply this idea to geometry we can get very startling results as well as seeing the application of Cantor's work to Zeno's paradoxes.



Consider the triangle ABC. By observation and construction it can be seen that DE is shorter than BC. How many more points are there on BC than there are on DE? AXZ is a straight line from A intersecting DE at X and BC at Z with the position of Z being determined by the position of X (because AZ is a straight line). Thus X and Z can be paired, i.e. a one-to-one correspondence can be set up between them. Imagine now the line AZ to swing so that X and Z move along their respective lines. This shows that every point on DE can be uniquely paired with its corresponding point on BC, and vice versa. Therefore there are as many points on DE as there are on BC! This, as has been seen, can only occur if there are an infinite number of points in each set; (here the sets are considered to be BC (the major set), and DE (the subset)).[12] Therefore DE and BC contain an infinite number of points. The number of points in each line segment is the same and, by extension, the number of points in any line segment, irrespective of its length, can always be placed in a one-to-one correspondence with the points of any other line segment. Therefore, there must be an infinite number of points in any line segment. Indeed, "if we wish to carry the argument still further we can show that there

---

12. Any worries which are entertained about the way in which DE can be called a subset of BC, can be overcome. Drop parallel lines from D and E to the line BC. The interval which they isolate on BC must be the same size as DE. The geometry of reflections does the rest.

are as many points in a line one inch long as there is in all of three-dimensional space."[13]

It can be deduced from this that a linear continuum is a dense series; between every two elements there is another element. As this interpolation of elements has no defined stopping place, the number of elements in a dense series is infinite and no element has an immediate predecessor or successor. This according to Russell[14] is as far as we have to go to see how Zeno is wrong. But Russell, too, is wrong, because the denseness (or compactness) of a series does not define continuity. The series of rational numbers is a dense series, yet every element is discrete. Within a continuous series, however, the elements are not discrete. The difference between a dense series and a continuous series is its denumerability. The elements of a dense series are denumerable, that is, they can be put into a one-to-one correspondence with the series of positive integers. But the elements of a continuous series are not denumerable, that is, they cannot be put into a one-to-one correspondence with the positive integers.[15]

---

13. E.P. Northrop, Riddles in Mathematics, (London, 1959), p.153: the proof of this is given by Cantor in Journal für Mathematik, vol. 84, (1878), pp. 242-258.

14. Russell, Our Knowledge of the External World, (London, 1914,) p.132.

15. E.V. Huntington, The Continuum and Other Types of Serial Order, (New York, 1955), section 58.

From this we can deduce that the elements of a dense series and the elements of a continuous series are both infinite in number, but that the types of infinity are different. The elements of a dense series are discrete and the set of elements is denumerable, while the elements of a continuous series are not discrete and the set of elements is not denumerable. The cardinal number of the dense series is the first transfinite number which Cantor called aleph-sub-zero. This is the cardinal number of Zeno's series. The cardinal number of a continuous series is the <u>second</u> transfinite and is aleph-sub-one.[16]

In terms of this analysis, the paradoxes of Achilles and the Tortoise and the Stadium stem from the failure to discriminate between a denumerable infinity composed of discrete elements and a non-denumerable infinity composed of continuous elements. The first transfinite cardinal number is not distinguished from the next transfinite cardinal number. Zeno has used the wrong model of analysis for his argument. He has presented us with a pseudo-continuum. A pseudo-continuum has spaces and this, of course, leads him to the paradoxes of the Flying Arrow and the Moving Rows.

Before considering these two paradoxes let us see just what has been achieved by this Cantorean Analysis. The first two paradoxes of space are refuted: any spatial interval is finite even though it can be seen as an infinite number of points.

---

16. or 2 to the aleph-sub-zero power: these may be the same number: see H.N. Lee, loc. cit.

Similarly, any paradox of time is refuted because any temporal interval is finite even though it can be seen as an infinite number of instants. This also means that the Runner in the Stadium or Achilles chasing the Tortoise, even though they must pass over an infinite number of points, need not necessarily have to pass over an infinite distance. The intuitive feeling that an infinite number of points in a line must indicate a line of infinite length is now redundant: there is nothing inherently wrong in the notion of traversing an infinite number of points.

However, the Achilles and Stadium Paradoxes have to do with motion as much as space and time; can motion be discussed in terms of our Cantorean analysis of the spatio-temporal continuum? Yes, this is done through Relativistic Kinematics.

An attempt to use Relativistic Kinematics against Zeno occurs in two articles by Andrew Ushenko.[17, 18] He suggests that a horizontal line representing Space be drawn with a vertical line representing Time drawn to cut it. A path drawn parallel to the Time-axis represents a body at rest: a path drawn parallel to the Space-axis represents a body changing its location instantaneously. These paths are to be called world-lines.

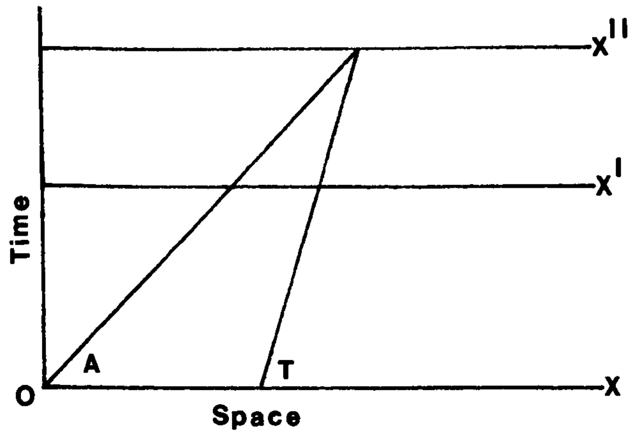
The Space-axis OX represents the track at the moment the race starts: this is because the track is extended in space at the

---

17. Ushenko, A. "The Final Solution of the Paradox of the Race" in Journal of Philosophy, (29), 1932, pp. 241-242.

18. Ushenko, A. "Zeno's Paradoxes" in Mind, (55), 1946.

moment the race begins. The Time-axis is divided into hours. Ushenko's diagram appears below:



The X-axis represents the track at noon (when the race begins), the different line X^{I} represents the track at time one hour later, and the different line X^{II} represents the track two hours later when the two protagonists are supposed to meet. Any time we choose to consider within these two hours will necessitate the drawing of a horizontal X-line. There will be an X-line corresponding to each of the infinitude of instants within the two hour-interval. Ushenko says that "the language of space-time does not allow us to speak of the <u>same</u> track, i.e. the same line, at two different moments."[19] It follows from this that, after the first hour, "the event A is replaced by the event A^{I}."[20] The path of A, then, has been transformed into a <u>sequence</u> of events. In the Relativistic Kinematics of world-lines, there is no

---

19. ibid., p.160

20. ibid., p.160

transition from one event to another, "the events are simply there,"[21] and therefore there is no agent to perform a transition. This is clearly based on Cantorean analysis of the continuum where no two points are in propinquity. A transition from one event to another could only occur where points were neighbours. This notion has been discarded because it provides an inadequate analysis of the continuum. If we are to use real numbers to describe the continuum, then we have to abandon the notion of transition. Indeed, this carries off with it the notion of "agent": Achilles does not have to travel over an infinite series of distances along the track, the X-axis, because he does not travel at all. The dynamic aspect of Nature is completely ignored: Ushenko argues[22] that the world-lines of Achilles and the Tortoise (or the Runner and the Tape) are routes of events and that the geometry of the world-lines of Achilles and the Tortoise (or the Runner and the Tape) are such that they intersect.[23]

Let us investigate more closely the notion that objects are not transformed, by motion, growth, etc. but that they consist, rather, in a sequence of events. In this way we will discover (from another direction, as it were) whether physical events have the ordinal structure of the continuum.

In Relativity theory it is events rather than objects which are Nature's basic entities. Relativity theory also employs

---

21. ibid., p.160.
22. Ushenko, A., "The Final Solution of the Paradox of the Race" in Journal of Philosophy, (20), 1946.
23. Although the drawing of world-lines hardly overcomes the logical difficulties besetting Ushenko's assumption.

Cantor's analysis to say that the motion of a body in space-time is a continuum of events, i.e. a linear physical movement, such as that of the Runner, consists of a non-discrete and non-denumerable set of events, each of which occurs and none of which "moves."[24] And, like the points and instants of space and time, "The events of which it (Relativity theory) makes assertions cannot be sensed."[25]

In order to amplify this point it should be noted that in everyday thought, we consider that in "getting" from one event to another we "move" consecutively through the intervening, and discernibly distinct, events. Relativistic Kinematics rejects this, however. In the "experiential" context, when asking how a certain event "came about" or "became", we expect as an answer

---

24. This is not to identify the occurrence of the event with the path of the movement (as Bergson tries to do): movement is a set of events ordered by the relation "later than" and not simply by the relation "to the right of." That is to say, in a Relativistic continuum of events, time relations are as significant as space relations. More importantly, it should be noted that we do not use our consciousness for the determining of a temporal order among physical events. If we determine temporal order by reference to a mind (as Whitehead wants to do), i.e. by "past, present, or future", we could not say that these events constitute a linear Cantorean continuum. "Past, present and future" are dependent on one's position in space at observation. This means that two separate events are topologically simultaneous if they are not causally connected, i.e. they have no timelike separation as this depends on the observer. Timelike separation can occur only if the two events are causally connected, as this is totally independent of observation. (although see next page).

25. Grünbaum, A. "Relativity and the Atomicity of Becoming" in The Review of Metaphysics, (1950), p.163.

an enumeration of consecutive intervening actual sensed events. We have seen that this is the foundation of Zeno's paradoxes, but Relativistic Kinematics holds that:

1. there are no consecutive events, because of the infinitude of points in the linear continuum and infinitude of instants in the temporal continuum,
2. these events (the way things become) cannot be sensed.

These two assertions completely cut away Zeno's ground. If he were to ask how an object gets from one event to another,[26] the reply would be simple: Zeno would still be labouring under the expectation that the answer to all questions relating to sensible change will be enumerative. Change, however, is not like that: it is simply a dense set of non-denumerable point-events which are not arbitrary, but which are connected by causality.[27] Clearly, if the notion of "agent" is, as we saw earlier, now redundant, this must be a very special sort of causality. Unfortunately, given that the events cannot be sensed, its precise nature must remain enigmatic.

---

26. Given the concept of the world-line, this would seem to be a silly question: objects can neither move nor persist because they are collections of point-events. There can be no temporal nor spatial motion. Grünbaum, in fact, could be saying almost the same as Zeno with the Flying Arrow; events occur at instants. The world-lines are just "there" and motion is an illusion.

27. Grünbaum, A., ibid., p.174.

We can deduce the following, however: given that the events postulated by Relativistic Kinematics form a dense Cantorean set with transfinite cardinal number, aleph-sub-one, then we can say that they are without duration, and also that a continuum of them does have positive duration. No next instant is required for "becoming" and processes of finite duration can be resolved into non-extensive events.[28] If these events take place, then change results: the events are simply there, but they do not advance into time for there would then be more than one event: an event does not move: "... physical events as such (do not) come into being at all ... they merely occur tenselessly in a network of relations of time-like separation ..."[29]

---

28. Although see P.W. Bridgeman, "Some Implications of Recent Points of View in Physics", in Revue Internationale de Philosophie, Vol III, No.10, (1949): "... if I literally thought of a line as consisting of an assemblage of points ... paradox would present itself."

29. Grünbaum, A., "Modern Science and Zeno's Paradoxes of Motion" in Philosophy of Time, ed. R.M. Gale, (New York, 1967) p.442: I feel, however, that as the gaps in explanation in terms of efficient causality are filled in, the idea that all causal links are reduceable to that of strict necessity grows, which effectively means that all chains of causality shrivel up.

Achilles and the Tortoise (or the Runner) are to be seen as sets of causally connected events forming a dense series in which there is always an infinity of events between any two which I wish (conceptually) to isolate. Individual events, which make up the protagonists, do not move, but are ordered non-consecutively (because there are always events between) by "later than" (based on causal connection). The paradoxes of motion fail to get off the ground because it is held that physical movement is a sum of immobilities! As well as this, the denumerability of point-events, which may be what Zeno has in mind in his description, has been seen to be inadequate as an analysis of what happens in movement.

Well, can Zeno respond to the above arguments and attempt to defend his position? He could start by asserting that the notion underlying everything which Ushenko and Grünbaum have said is that motion is constructed out of point-events, (and that any spatial interval is constructed out of points and that any temporal interval is constructed out of instants). This would be a very poor assertion to make. He could go on to say that "mathematical space and time ... have this property of compactness, though whether actual space and time have it is a further question, dependent upon empirical evidence, and probably incapable of being answered with certainty."[30] Indeed, Cantor seems to be on Zeno's side when he says that "in order to achieve a more satisfactory description of nature, the ultimate or genuinely simple elements of matter must

---

30. Russell, B., Our Knowledge of the External World, (London, 1914), p.132.

be postulated as actually infinite in number and ... spatially they must be regarded as entirely without extension."[31]

In other words, the best way to go about either the philosophical or the mathematical task of dealing with a continuum is to interpret elements, parts or events as the result of analysis, not to interpret a continuum as a synthesis of separately given elements, parts or events. Clearly, if we think of a point in space or time as a division, i.e. giving us a "to the right of" and "to the left of" and a "before" and "after", the whole notion of construction is superfluous.[32] If motion were constructed from point-events then we could never perceive the movement. The reason for this is simple: if the point-event does not occupy time could light illuminate what the point-event contained? No![33] Therefore, rather than claim that the motion we perceive consists of an aggregation of darkness (or something equally strange), we hold that points, instants, and point-events are devices placed by us on the world in order to help explain it.

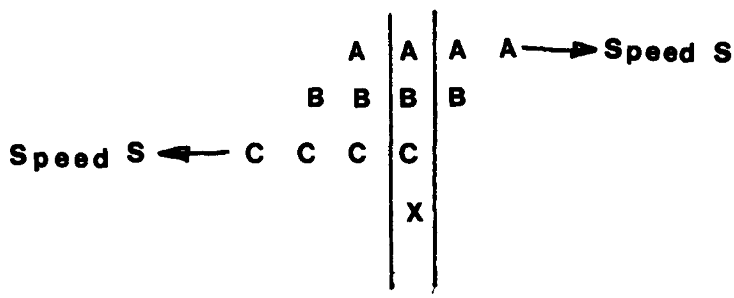
---

31. Cantor, G., Gesammelte Abhandlungen, ed. E. Zermelo, (Berlin, 1932), p.275.

32. An analysis of "the point" occurs several pages hence.

33. This emerged during discussion with Dr. Robin Preston of the Department of Natural Philosophy in the University of Strathclyde, although Grünbaum ("Modern Science and Zeno's Paradoxes of Motion" in Philosophy of Time, (New York, 1967), ed. R. Gale) maintains (p.430) that "... the interval of physical space ... is conceived as literally a linear mathematical continuum of points." The reason for the absence of light is that, if no movement can take place, then there can be no light wave.

Could Zeno use the Paradox of the Moving Rows as an argument against Kinematics? Let us try it and see what happens. Zeno speaks: "I wish to conceptually isolate one of these point-events which you speak of. Clearly, from your logic, change is impossible within this one point-event. Change is impossible anyway, so perhaps it is better to say that there can be no "later than" <u>within</u> this point-event, for otherwise it could be further decomposed into two or more point-events. Let there be three rows of objects, As, Bs, and Cs. The Bs are stationary, while the As move to the right and the Cs move to the left, both at speed S.[34] Let us isolate column X, the instant in time in which an observation is made of a point in space, i.e. a point-event.



Relative to the Bs, the As are moving at speed S, while relative to the As, the Cs are moving at speed 2S. Therefore, for

---

34. According to Relativistic Kinematics, of course, they are not moving at all, but that is not important: it is enough for the purpose of this argument that they <u>seem</u> to an observer to be moving.

everything that occurs in column X (or is observed) between the As and Bs, twice as much occurs (or is observed) between the As and Cs. But column X represents a point-event, i.e. an absolute minimum in which change is impossible! But if column X is so fine in width that no movement is recorded between the As and Bs, it must record movement between the As and Cs. If column X is so fine that no movement is recorded between the As and Cs, it must record some movement between the As and Bs! Therefore (says Zeno), I have deduced that:

1. a point-event is not a minimum, or
2. the more quickly moving object has more point-events on its world-line than the stationary or more slowly moving object.

From 1 it follows that if point-events compose the continuum and they are not minima, then motion is impossible, as the Paradox of the Flying Arrow demonstrates. From 2 it follows that a positive finite number of point-events can be predicated of any world-line. It then follows that the Runner cannot reach his destination because the set of point-events is no longer continuous and non-denumerable, but dense, denumerable and consecutive. This, you may recollect, is what I have maintained all along."

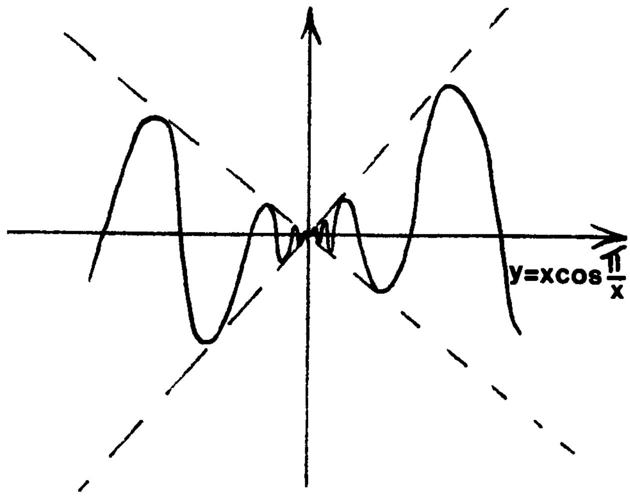
Were Zeno to argue thus he would be mistaken: he would be guilty of confusing change with rates of change. In the point-event change has vanished but the rates of change have not changed. Given the definition of infinity with which we are now operating then it does not matter what relative speeds are; the point-event is indivisible because it occurs when the limits of any interval of space and time is reduced so that the interval gets shorter and shorter. The limits of the interval never become neighbours (because there are an infinity of points in any spatial interval and an infinity of instants in any temporal interval) but they coincide. This coincidence is the point-event.[35] As well as this, Zeno's talk of more point-events one one world-line than another is wrong because for any point on Achilles' world-line I can set it in a one-to-one correspondence with a point in the Tortoise's world-line. This is because of our more adequate definition of infinity. There is the same number of points on every world-line, i.e. all infinite number . The point-event is derived from the Cantorean definition of infinity so that if Zeno is proposing to argue his case, he must direct himself to Cantor and not to the point-event. Can he do so?

Cantor's case, although it is paradoxical, seems to have all possible sources of attack covered. Zeno is prohibited from using arithmetic against it. However, he can use an extremely

---

35. See Aristotle, Physics, VI, iii, 234a20-25, and Russell, Our Knowledge of the External World, pp. 134, 136 and 174, for arguments supporting the above contention.

interesting strategy against Cantor. Consider, for example, some of the peculiar attributes of this graph:



The interval (0,1) on the X-axis is cut by the curve an infinite number of times; this it does by definition because no-one could actually draw this curve. In the same way, the interval (-1,0) on the X-axis is cut by the curve an infinite number of times. Now it is manifestly implausible to suggest that the curve cuts the X-axis interval (0,1) the same number of times as it cuts the X-axis interval (-1,1), although Cantor claims that the number of times will be the same. His reason is that the interval (0,1) is a subinterval of the interval (-1,1) and, given his definition of infinity, he is correct. But, his definition of infinity must be incorrect: I do not have to engage in counting the infinite number of times that the curve cuts the X-axis to see that he is wrong. (Although I could count one of the extra times it cuts within the interval (-1,0). All I have to do is look. I can see that it cuts the X-axis more times within the interval (-1,1) than it does within the interval (0,1). (I will make more of this in my final chapter.)

Secondly, if we examine in even the most cursory manner the Achilles and Stadium paradoxes, we will find that Zeno is not simply saying that there are an infinite number of points in a line, but that each of these points must be traversed in sequence and in succession.[36] Russell's attempt to cope with this is paltry: "... it is not essential to the existence of a collection, or even to knowledge and reasoning concerning it, that we should be able to pass its terms in review one by one."[37] This is utterly inadequate because it does not meet the point of the difficulty: Achilles' course of running is not a matter of knowledge or reasoning, nor is it a matter of the existence of an infinite collection of distances. It is absolutely essential to the *traversing* of a collection that its terms are passed (reviewed) one by one. However, if the continuum *is* a dense set of points, then there is never (by definition) a *next* point because always there is an infinitude of points between any two under consideration. Where, then, could the protagonists run to? To the next point? No, for this is *ex hypothesi*, impossible. If Cantor is correct, then Zeno is correct.

However, let us assume that Cantor *is* correct. Zeno has therefore abandoned the Paradoxes of Plurality and accepted that an infinite number of points can, without absurdity, be contained within a finite interval. Zeno can *still* ask, "What has this to do with the Achilles and Stadium Paradoxes? In fact, because we

---

36. Which would be equivalent to actually drawing the curve $y=x \cos\frac{\pi}{x}$

37. B. Russell, *Our Knowledge of the External World*, (London,1922), p.187.

can now state conclusively that the racecourse is continuous, it is therefore infinitely divisible. It is even more definite now that Achilles will not catch the Tortoise and that the Runner will not get to the end of the Stadium. Because there are no indivisible minima, the distance between the Runner and the Tape must always decrease as I describe. The Runner still has to run halfway to the tape, halfway from the halfway point to the tape, and so on. By accepting that my Paradoxes of Plurality have been refuted, I have confirmed that the Paradoxes of Motion must be accepted."

Chapter Eight

A Brief Encounter with Relativity

"That whereas we are sore let and hindered in running the race that is set before us, thy bountiful grace and mercy may speedily deliver us."

(Prayer Book, 1662)

Consider the Achilles paradox, but with the addition of one extra hypothesis. Instead of observing the race from a static frame of reference, I wish to observe it from a moving frame of reference, one which is moving in a very particular sort of way. Achilles and the tortoise are ready to run as usual, but, somehow, the tortoise has gained enormous strength. It is so strong, in fact, that I have attached a howdah to its shell and propose to climb aboard to observe the race. The race begins and answers Zeno's description of how the protagonists behave. Note however, that, relative to me (the observer), the tortoise is stationary. As far as I am concerned there is only one moving participant and that is Achilles. Irrespective of how the speeds of the runners change with reference to a stationary object, and irrespective of whether the tortoise stops for a breather, walks backwards, or sprints relative to a stationary observer, Achilles is the only protagonist who moves relative to me. We can no longer say that Achilles has to run to the starting-point of the tortoise because, relative to me, the tortoise has no starting-point!

Therefore, we can completely side-step Zeno's hypnotic description of the race: because the observer also moves in such a way that he is stationary relative to the tortoise, Achilles no longer has to chase hopelessly from starting-point to starting-point, getting ever closer to, but never reaching, his destination.

Does this now mean that Relativity has shown us that Achilles will catch the tortoise? "Not on your life," Zeno would reply. "I admit that you seem to have done something to my description of the race, rendering it redundant, but notice this: Achilles still has to get halfway to the relatively stationary tortoise, and then halfway from _that_ point to the relatively stationary tortoise, and so on. Relativity may overcome my Achilles paradox, but it will prove harmless against my Stadium paradox." And he would be correct.

The logic of this entire argument seems to be faultless, but I must confess to more than a vestige of unease about it. We know that the Paradox of the Stadium and the Achilles Paradox have to do with infinite series. However, they have to do with different kinds of infinite series, series with a difference which is quite radical. Bennett[1] analyses various kinds of infinite series, and the two which are germane to this issue are these:

(a) an infinite series with two termini such as

0, 1/2, 3/4, 7/8, ... 1,

and

(b) an infinite series with only one terminus, such as

0, 1, 2, 3, 4, ...

An examination of the Stadium Paradox will make it obvious that it is an example of (a): there is the Runner's starting-point and there is the finishing tape which he must strive to reach. On the other hand, the Achilles Paradox is a sophisticated example of (b): there are two "united" starting-points, but, unless one begs the question, no finishing point.

If it is the case that the Achilles Paradox can be reduced to the Stadium Paradox, there is a parallel implication that an infinite series with only one terminus can be reduced to an infinite

---

1. D. Bennett, _Kant's Dialectic_ (Cambridge, 1981), Chs. 7 and 8.

series <u>with two termini</u>! Here we have a Cerberus: we cut off one head only to have another, even more problematical, grow in its place.

This is a terrible quandary and we cannot leave things as they are. The Relativity argument is, I think, impeccable, so let us enquire again into infinite series (b). In a sense, Achilles and the Tortoise <u>do</u> have finishing points: Achilles will be finished when he catches the tortoise and the tortoise will be finished when it is caught by Achilles. However, the problem is that this will occur only if Achilles catches the tortoise. How can we construct an appropriate kind of series without begging the very point at issue? What kind of status does this second terminus have? Clearly it is not the same sort of status as the end terminus in series (a).

Let me invent the notion of a quasi-limit, a limit which may or may not exist; even if it does exist we may not <u>know</u> that it exists. We could transform (b) into (c) 0, 1, 2, 3, 4, ... E (where E is the next prime number above 1,000,000).

If there is a prime number above 1,000,000 then that is where the series stops, and so for Achilles and the Tortoise. If there is no prime number above 1,000,000 then the series keeps going, and so for Achilles and the Tortoise.

Therefore the Achilles Paradox can be shown to be like the Stadium Paradox in having two termini, even if the end terminus is not quite like the end terminus of the Stadium.

An interesting comment has been made by a colleague on the Relativity strategy. To side-step the paradox is not to eliminate it. The paradox remains, waiting to be refuted. All that has been achieved is a failure to square up to it. I take this point as valid, but propose now to examine whether the device of altering the description of the race will prove effective in overturning the paradoxes.

Part Three

Periphrastic Refutations

It can be held that the paradoxes of Zeno tell us nothing about the world: the world really _is_ as we experience it and is a _fait accompli_. The interest of the paradoxes lies in the way language seems to confuse various aspects of that world. "These (philosophical problems) are ... not empirical problems; they are solved, rather by looking into the workings of our language, and that in such a way as to make us recognise those workings ... The problems are solved, not by giving new information, but by arranging what we have always known. Philosophy is a battle against the bewitchment of our intelligence by means of language."[1] Zeno's paradoxes seem to be an excellent example of bewitchment through language, and to get to grips with them it would seem that one should almost disregard the world and attend to language. To this end, several philosophers have therefore sought to overcome the paradoxes by examining closely the words and grammar in which they are couched, and, by redefining and restructuring the problem areas, render the paradoxes impotent.

Given that I have failed to detect any grand strategy in this approach (as emerged in the Geometric Progression Refutation), it is best to think of this approach as a series of minor skirmishes against Zeno.

---

1. Wittgenstein, _Philosophical Investigations_, (Oxford, 1972),109.

Chapter Nine

Circumlocuting the Problem

"Language originated before philosophy and that's what is the matter with philosophy."

(Lichtenberg)

One of the most hypnotic aspects of the Paradoxes of Motion is that they are couched in "ordinary" language. This is not to say that they describe an ordinary phenomenon (motion) in an ordinary sort of way; but a layman, ignorant of necessary propositions, the logic of commands, modus tollens, and the rest of the paraphernalia with which philosophy confronts Zeno, will still be able to see precisely what he is driving at. The basic strategy of this chapter is to move away from "ordinary" language and to transform those propositions which lead to paradox in such a way that the paradox is circumvented. Were this to be successful it would indeed be a triumph: however, it will be seen that the price to pay would often be a mystifying philosophical darkness where once Zeno stood in daylight.

An interesting and subtle attempt at refutation of the paradoxes by trying to circumlocute Zeno's structures is to be found in a very interesting, if confusing, article by Hinton and Martin.[1] In this article are advanced various refutations which purport to overturn both the Achilles and Stadium paradoxes. Of these, the most original and significant occur in the early part of their paper.

They begin by couching the Stadium and Achilles paradoxes in their own words:

---

1. J.M. Hinton and C.B. Martin, "Achilles and the Tortoise", in Analysis, 14, (1953-54), pp. 56-68.

A: An object O cannot reach a point S, at a given distance from its point of departure, until it has reached a point R, at ½ that distance. But it cannot reach R until it has reached Q, at ¼ the distance, and so on, ad infinitum. Therefore it can never reach its destination. (Variant: "... it can never start.")

and

B: Suppose A moves at 10 times the speed of T, and T has a handicap of a given distance. Then, by the time A has covered that distance, T has covered a further 1/10 of it; and by the time A has covered that 1/10, T has covered a further 1/100th, and so on ad infinitum. Therefore A can never overtake T."[2]

Although it may seem pedantic, it is of great importance to note that their version of the Stadium paradox is incomplete. Their version of the Stadium is but one of two alternative formulations of the paradox. The "Therefore" which they employ seems to be less entailed by their couching of the paradox than its variant (which is Zeno's entailment.) The version that more strictly leads to the conclusion that the destination can never be reached rests on the premise that half the remainder remains always to be covered. This means that, straight off, we can conjecture that if Hinton and Martin come up with a refutation, it is either a refutation of only the first version of the paradox, or it is a refutation based on a confusion between the two versions of the paradox. This confusion reflects these authors' increasingly buccaneering approach to the paradoxes.

Hinton and Martin propose to solve these paradoxes by using two methods, the first of which is to treat the accounts of the two paradoxes as a series of necessary propositions from which Zeno's conclusions do not follow.

---

2. ibid., p.57.

In both A and B the premise-sentence is said to consist of an infinite series of sentences of the form "X cannot reach S until it has reached R, X cannot reach R until it has reached Q, and so on." Hinton and Martin decide that this is but an elliptical way of saying "It's logically impossible that X pass through R to S but does not reach R before S" or, more pertinently to their objective, "The conjunction of (X passes through R to S) and (X doesn't reach R before S) is self-contradictory."[3] The same applies to "(X passes through Q to R) and (X doesn't reach Q before R)", and so on. The writers' conclusion is that it does not follow from these necessary propositions that it is logically impossible for motion to occur:

"But these necessary truths don't entail the conclusion; the strict-implication sentences don't even look as though they express propositions which entail the conclusion."[4]

"However long we go on, we shall just be pointing out the self-contradictoriness of a series of conjuctions which nobody wants to assert."[5]

Observe closely how far the ground has been shifted from Zeno's premise: we begin with Hinton and Martin's version of Zeno's premise. They then deduce how they think their version of the premise should be rewritten. This new version of their

3. ibid., p.59

4. ibid., p.59

5. ibid., p.59 (my underlining).

premise is then *transformed* into the series of necessary propositions which are open to assault. To assist our getting to grips with this abstraction, let us chart the evolution of Zeno's premise:

(1) "X cannot reach S until it has reached R. X cannot reach R until it has reached Q, etc."

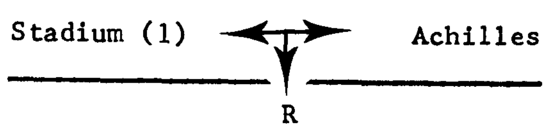
(2) "It's logically impossible that X pass through R to S but does not reach R before S. It's logically impossible that X pass through Q to R but does not reach Q before R, etc."

(3) "The conjunction of (X passes through R to S) and (X doesn't reach R before S) is self-contradictory. The conjunction of (X passes through Q to R) and (X doesn't reach Q before R) is self-contradictory, etc."

An initial response to this is to point out that this is not how the Achilles paradox works: this is only a series of transpositions of the *first* version of the Stadium. The Achilles paradox should read:

(1a) "X cannot reach S until it has reach R. X cannot reach S until it has passed R and reached T."

That is, in the Achilles paradox we are dealing with points *to the right of R*, while in the first version of the Stadium paradox we are dealing with points *to the left of R*:



The chapter on Relativity showed that the Achilles can be transformed into the second version of the Stadium. Therefore, what Hinton and Martin have to say does not apply to it either.

However, now that we know where their argument is actually directed, we can consider whether their alternative version of Stadium (1) is acceptable.

We can note immediately that there seems to be a gross artificiality in what Hinton and Martin are doing. They could, perhaps, respond by saying that there is also a gross artificiality in what Zeno is doing. I would not accept this and my grounds for rejection are these: if we take an everyday sort of request, such as for railway information, can we meaningfully reduce statement of the sort Zeno makes into necessary propositions? Suppose that Zeno is asked how to get from Glasgow to Cardiff by train. His reply is this: "You cannot get to Cardiff without passing through Hereford. You cannot get to Hereford without passing through Birmingham. That's how you get to Cardiff."

This is equivalent to Zeno's premise (1).[6] I could, perhaps, and with great unwillingness, accept that I'm saying Hinton and Martin's premise (2):
"It's logically impossible that you go through Hereford to Cardiff without reaching Hereford before you reach Cardiff; it's logically impossible that you pass through Birmingham to Hereford without reaching Birmingham before you reach Hereford."

---

6. A multiplication of the "passing through" will generate the paradox.

But anyone receiving this mystery as a reply to the question "How do I get to Cardiff from Glasgow?" would surely be rather nonplussed. But Hinton and Martin do not consider even _this_ abomination to be sufficient: I should actually say something like their premise (3):

"Ah, you want to know how to get from Glasgow to Cardiff by train? Well, if I say (You pass through Hereford to get to Cardiff) and (You won't get to Hereford before you get to Cardiff), I'll have said something self-contradictory. And if I say (You pass through Birmingham to get to Hereford) and (You won't get to Birmingham before you get to Hereford), I'll have said something else self-contradictory."

This is a recipe for a punch on the nose, especially if uttered in Glasgow Central Station. Hinton and Martin are correct in stating that this is a conjunction "which nobody wants to assert." It is too much work, it is basically incomprehensible, and, if correct, does great damage to some of our ideas of what is going on in description. Zeno is giving descriptions of actions: if these can be inflated as Hinton and Martin want to do, then every set of instructions or every description is really a set of necessary propositions in disguise. This is unacceptable: if true, then no-one would want to describe _anything_.

_If_ we want to recouch "X cannot reach S until it has reached R," what is wrong with "R must be reached before S can be reached."? If I continue by saying "And Q must be reached before R can be reached, and so on", there clearly _is_ an implication of sorts that I will not get started.

To return to the authors' first version of the Stadium: were this to be couched in the manner of the second version of the paradox which shows that a destination cannot be attained, then there would be a very strong implication, possibly even a strict entailment, that the destination could not be reached. To explain: "X cannot reach Z until it has reached A. Having reached A, X cannot reach Z until it has reached B. Having reached B, it cannot reach Z until ...", is a constant reinforcement of the notion that Z is unattainable.

The second of their two methods of refutation is also based on a rather idiocyncratic interpretation: "Instead of interpreting "O can't reach S until it has reached R" and its fellows, as strict-implication sentences, we might interpret them as commands."[7] The colourful way in which this is couched contains the following: (We are moving a chess or ludo piece to a goal), "we ... are just about to place it on S, the finishing line, when we are told: "No! You can't, may not, must not put it on point S until you have first put it on point R. We shrink back ..."[8]

If, for whatever reason, we decided to couch the paradox as a series of "Don't" commands, how do we overcome the paradox? Simple: "... nothing obliges the object to obey them."[9] This is astonishing, and they further compound this by saying, "He cannot both obey these commands and win the race; but nothing authorizes

7. ibid., p.59

8. ibid., pp. 59-60.

9. ibid., p.60.

the commands."[10] And why does nothing authorize the commands? Because "... the infinite series of strict implications provides no grounds for inferring that the object does or must obey the commands."[11]

It is a peculiar form of civil disobedience which forms the basis for this strategy, so let us ask if it is the case that nothing obliges the object to obey these commands, that we say to Zeno, "I refuse to accept that you have any authority over me." Clearly, there is some kind of connection between an action being an obedient action and a command being a command. If I own a totally unruly type of dog which does whatever it wants whenever the notion is upon it, biting holes in the upholstery, urinating on the curtains, terrorising my neighbour's cat, and so on, we say that that animal is out of control: the concept of "command" in relation to that dog is redundant. This gives rise to the interesting speculation that, if there is a command which no-one obeys, in what way is it a command? One can argue from the polar concept of total disobedience into concluding that there can be no concept of disobedience nor obedience. What, then, would a command be?

Wittgenstein[12] makes a similar but rather more subtle point about commands. His notion is that there are analogies between propositions and commands. Analogous to the truth value of

---

10. ibid., p.61
11. ibid., p.60. Where did this conflation spring from?
12. G.E. Moore, "Wittgenstein's Lectures in 1930-33", in Mind, (62), 1953, pp. 12-13.

propositions is the legitimacy of commands. (It is clearly the case that in certain social situations and societal institutions such as the classroom, the Armed Forces, industry, and so on, the concept of "commanding" is perfectly legitimate.) Analogous to the process of verification of a proposition is obedience to the command, i.e. obedience is the verification of the legitimacy of the command. Equally important, although not mentioned by Wittgenstein, is that the punishment attached to disobedience is also a verification of the legitimacy of the command. His example of a man playing the piano from a score is very subtle. The player is "guided" by the symbols on the paper which act as a command, and he justifies his behaviour by reference to the score. If he plays correctly "there is a similarity between what he does on the piano and the score ..."[13] Clearly, one cannot defy the commands of the musical score, because, if one is to play the game of music, one must accept that there is a tacit rule: obey the score. In the same way in other social situations it is surely not the case that nothing obliges the object to obey commands.

Zeno has clearly run rings around Hinton and Martin. This talk of "commands" is the final clutch at straws. Having got to the notion of commands (by a curious process which I cannot quite fathom), which they admit (p.60), if obeyed, will lead to Zeno's conclusion, their suggestion that we simply defy Zeno by refusing to obey him is barren: one is reminded of Bostock[14] and his

---

13. ibid., pp. 12-13.

14. See Chapter Four.

injunction to start running and keep running until the destination is reached. The only difference is that now we run with our hands over our ears so that we do not hear Zeno's interminable "Don't!"

As a way of understanding what is wrong with Zeno's argument this is worthless: as a prescription for getting from place to place it does have some worth: it tells never to listen to Zeno.

Let us turn now from necessary propositions and commands to the more interesting strategy of "guidance", hinted at by Wittgenstein. R.M. Gale, in a very interesting paper[15], maintains that there is an action type formulation of the Stadium in which there would be a conceptual absurdity in an agent running the unit distance by going through the described infinite set of sub-runs.[16] His argument opens with a description of what is meant by "action". An action, he says, is something done intentionally by an agent, and more especially he proposes to consider actions whose agent can supply us with the recipe which guided the performance. Zeno has given us the recipe to guide the actions in the Stadium: "Before you run the whole distance you must run half the distance, and before that, half of that, etc." It is true that if we follow this recipe we will not get to the end of the racecourse. The

---

15. Richard M. Gale, "Has the Present any Duration?". Paper read at 69th Annual Meeting of the Western Division of the American Philosophical Association, reprinted in Nous, 5, 1971, pp 39 47.

16. ibid., p.45.

recipe is absurd, says Gale, and the absurdity follows from the concept of an "action-guiding recipe." An action-guiding recipe is "a finite-step recipe which will

(i) specify some initial action;

(ii) indicate what subsequent actions have to be performed and in what discrete order, or supply a rule for the performance of these actions; and

(iii) specify the traits possessed by the final action in this sequence."[17]

Given this concept of an action-guiding recipe, it becomes conceptually absurd for a person to run through a unit distance by following Zeno's recipe that he must begin by running half the distance, half that distance, etc.

Why is it absurd? Because it fails on step (i) of Gale's scheme in that no initial action is specified. This means that we cannot get to (ii), the recursive rule, or, (iii), a specification of the final action. This is a very good argument and I cannot find anything wrong with it when it is applied to the first version of Zeno's Stadium. "We are going to have a run, but we are going to run in such a way that we cannot start to run" certainly has an air of absurdity about it, if not, in fact, the air of self-contradiction. I entirely agree with Gale that this is described in a conceptually absurd way, but does his argument have any application wider than to that specific version of the Stadium?

Consider the alternative version where one is instructed to run half the distance, and then half the remaining distance, and

---

17. ibid., p.45.

so on. This is impervious to Gale's attack because Zeno could couch this paradox in a way that satisfies the criteria of his action-guiding recipe:

(i) Blink[18] at the instant you start running towards the finishing tape. (This is the specification of the initial action).

(ii) Blink whenever you are halfway between the finishing post and the point where you last blinked. (This is the recursive rule.)

(iii) Stop after you breast the tape. (This specifies the trait possessed by the final action - breasting the tape.)

We can, as before, recouch this as "We are going to have a run, but we are going to run in such a way that we won't stop running." There is no absurdity in this. Gale claims[19] that the runner would not know when to cease his endeavours in this version of the paradox because the recipe would not specify the traits of the final action. Well, Zeno has just told him when to cease his efforts: when he breasts the tape. Gale seems to be saying that there is a last step if I run across some finite interval, yet,

---

18. The "blink" serves to indicate where one is on the racetrack: any other way would be equally acceptable. There is no mystery attached to the "blink".

19. ibid., p.46, footnote.

by Zeno's description, it cannot be described. Therefore Zeno's description must be absurd. Zeno's response, as we have seen, can be that there _is_ a description of the last step, to wit, "breasting the tape", but it is up to the runner to do this while obeying the recursive rule. Gale wants the description to be absurd _analytically_, but Zeno, by describing the last step, can avoid this criticism. Gale has correctly dismissed the first version of the Stadium paradox, but we are still left with the second. We are now aware that there is an extremely close relation between the second version of the Stadium and the Achilles paradox: does it, too, remain inviolate? Yes, because it can also be described in such a way as to satisfy the scheme of the action-guiding recipe:

(i) Both Achilles and the tortoise blink at the instant they start to run.

(ii) Achilles must blink where the tortoise last blinked and the tortoise must blink whenever Achilles blinked.

(iii) Stop everything when Achilles catches the tortoise.

Gale's analysis will damage only one of the paradoxes, although that achievement cannot be praised too highly.

Chapter Ten

"Infinity" and "Motion"

"A bait is used to catch fish. When you have got the fish, you can forget about the bait. Words are used to express meaning: when you understand the meaning you can forget about the words."

(Chuang-Tse)

The paradoxes can be seen as a manifestation of confusions over the notion of "the infinite", and it is thought that once we are "clear about what different things are intended by "infinite" the problem will resolve itself."[1] It would be impossible, and very probably rewardless to attempt to catalogue every nuance that the word "infinity" possesses. The most pertinent notions, however, have been grouped by TeHennepe[2], who claims that "the only genuine solution is a linguistic one based on an analysis of the term "infinite"."[3]

This sounds more impressive than it actually turns out to be. It "is paradoxical to describe a finite time or distance as an infinite series of diminishing magnitudes", and the solution he proposes to offer is "simply to recognize that it is paradoxical (because contradictory) to describe a finite magnitude as an infinite series of diminishing magnitudes, and then to refuse such a description."[4]

---

1. D. Schwayder, "Achilles Unbound" in Journal of Philosophy, 52, (1955), p.449.
2. E. TeHennepe, "Language Reform and Philosophical Imperialism: Another Round with Zeno" in Analysis, 23, (1962-63 and Supp.), pp. 43-49.
3. ibid., p.43.
4. ibid., p.44.

TeHennepe's approach has the authority of Aristotle behind it. Aristotle, who distinguished between an actual infinite and a potential infinite, contended that the infinite exists only _in potentia_. He declares (Physics VIII, 8) that it is inadmissable to describe continuous motion as a series of moves or as covering a series of part distances. The mid-points in any motion so described are only potential[5]: To describe them as actual is to imply that each mid-point is the end of a first part and the beginning of a second part, and so on. This leads to Zeno's paradox so that he, like TeHennepe, refuses to describe a continuous motion as a series of moves like the terms of an infinite series: "In this case, therefore, where the motion of a thing is continuous, it is impossible to use this form of expression." Aristotle goes on to say

> ... to the question whether it is possible to pass through an infinite number of units either of time or of distance we must reply that in a sense it is and in a sense it is not. If the units are actual, it is not possible; if they are potential, it is possible. For in the course of a continuous motion the traveller has traversed an infinite number of units in an accidental sense but not in an unqualified sense: for though it is an accidental characteristic of distance to be an infinite number of half distances, this is not its real and essential character.[6]

There is an interesting mixture of ideas contained in this extract. Aristotle accepts that Zeno is correct if the points are

---

5. This notion is further clarified by Bergson: see Chapter 12.

6. Aristotle, _Physics_, VIII, 263b3-9.

actual[7] and he wishes to substitute "potential" as a means of overcoming the paradoxes. However, it seems to me that there are problems in this notion of a potentially infinite number of points in an interval.

If the distance AZ consists of (can be reduced to, can be divided into) a series of points (or dense set of point-events), are these points actual or potential? Grünbaum claims actual while Aristotle claims potential, which goes to show how difficult the whole thing is. What Zeno would wish to know is this: what happens if an object occupies a point? Is it an actual or a potential point? I'm not sure what it would mean to say "I am standing here on a potential point" or "I am potentially standing here" (if these are the same thing). Zeno would want to say that an object can only occupy an actual point. From this he can point out that the Runner occupies the points of the line through traversing the interval (not necessarily by stopping at any

---

7. Here is a trifle to consider: Let the runner run across the interval AZ. According to Aristotle it can be done because, as the motion is continuous, the Runner will not stop at mid-point B and then restart his run. But what if the Runner blinks at B and again when he has crossed half of the remaining distance, etc. Will he complete his run? No! This gives rise to the interesting idea that the attainment of the destination is dependent on whatever else the Runner is doing, or not doing, en route.

particular point, but simply by moving through the points). From this it seems to follow that, in the process of the run, the Runner actualizes each point through which he passes. If this is so, then Aristotle must acknowledge that the run remains uncompleted.

There is another notion going on in the Aristotle extract, a notion which becomes clearer when we consider Kant, whose "Critique of Pure Reason" is also cited as authority by TeHennepe. Kant says:

> We are not, however, entitled to say of a whole which is divisible to infinity, that it is made up of infinitely many parts. For although all parts are contained in the intuition of the whole, the whole division is not so contained, but consists only in the continuous decomposition, that is, in the regress itself, whereby the series first becomes actual. Since this regress is infinite, all the members or parts at which it arrives are contained in the given whole, viewed as an aggregate. But the whole series of the division is not so contained, for it is a successive infinite and never whole, and cannot, therefore, exhibit an infinite multiplicity, or any combination of an infinite multiplicity in a whole. [8]

A lot is going on here, the main thrust of which is that we are not entitled to say that a whole which is divisible to infinity is therefore made up of an infinite number of parts. TeHennepe calls this statement of Kant "terse and incisive", and what he

---

8. Kant, Critique of Pure Reason, (London, 1929), p.459: See also pp. 391-392, "Since ... the unconditioned is necessarily contained in the absolute totality of the regressive synthesis of the manifold of appearance ... reason here adopts the method of starting from the idea of totality ... (the regressive synthesis) is without limits or beginning, i.e. is infinite, and is given in its entirety."

(Kant) is saying is this: if 1 + 1/2 + 1/4 + ... does not equal 2, then 2 cannot be described as 1 + 1/2 + 1/4 ... For TeHennepe, however, this makes us "clear about the contradiction involved in describing a finite magnitude as an infinite series of diminishing magnitudes."[9]

But, this is absolutely Zeno's point! How could Aristotle and TeHennepe miss it? Zeno is not describing a finite magnitude: he is saying that 2 cannot be described as 1 + 1/2 + 1/4 + ..., and that this is the reason why the Runner never gets to his destination and Achilles does not catch the tortoise. It seems inapt to do as TeHennepe suggests: "... to recognise that it is paradoxical (because contradictory) to describe a finite magnitude as an infinite series of diminishing magnitudes, and then to refuse such a description." (Remember that for Zeno there are no finite magnitudes, only zero or infinite magnitudes). That is not what Zeno is doing: he is not describing a finite magnitude, but describing a race in such a way that it cannot equal the finite magnitude. We know where the finishing tape is!

There are no grounds for rejecting this on the basis of contradiction: Zeno is not describing a finite quantity in such a way that it is not the same finite quantity, e.g. "The course is one hundred yards long. There are three feet to the yard, therefore the course is two hundred and forty seven feet long." That could be rejected on the grounds of contradiction.

---

9. TeHennepe, loc. cit., p.46.

Another approach is to investigate what is going on in the concept of "motion", and, by redefining it or clarifying it, attempt to skirt Zeno's problems. Ushenko[10] gives a very elegant and terse exposition of his ideas on "motion": "... in reality the motion of a body is simply the fact that at different moments it happens to be at different places ... they (Achilles and the tortoise) meet when they both happen to be at the same place at the same time."

What concerns me in this extract is the "happen to be": it is not the case that Achilles just "happens to be" next to the tortoise; he has to run in order to reach the tortoise. In this exposition, the actual moving is thrown out of the window. This may be no bad thing because it is the actual moving which leads to all the problems, but it certainly smacks of unfair play. The

---

10. Ushenko, "The Final Solution of Zeno's Paradox of the Race", in Journal of Philosophy, (20), 1932, p.241. The same point is made by Shamsi, Towards a Definitive Solution of Zeno's Paradoxes, (Karachi, 1973): he says that "motion" is for an object O to be in position A at time X and to be in position B at time Y, where $A \neq B$ and $X \neq Y$. From these ideas we can give reasonable definitions of extension and duration: if an object O is at position A at time X and at position B at time X, then it must have extension enough to cover both A and B: if an object O is at position A at time X and at position A at time Y, then it must have duration enough to endure from X until Y.

underlying idea in Ushenko's notion shows a connection to the metaphysical writers soon to be considered: "motion" is considered to be an intellectual construction placed by us on the world in order to account for changes of position. Underlying this idea is a close relation to the Coherence Theory of Truth: objects (we think) do not flash in and out of existence, so that we posit motion to account for their being at different places at different times. Bearing in mind my comment about unfair play, let us apply this device to the paradoxes. At time X, the Runner is at the starting-line, while at time Y, he is at the finishing-line. Objects in the world do not suddenly disappear only to reappear some distance away a little time later, therefore the Runner must have moved.

Although this is an interesting strategy, is it either tenable or acceptable? Certainly I think that it seems reasonably watertight (unless one wants to discuss ultimate electronic particles which do seem to flash in and out of existence). However, note that it is essentially static in character; nothing actually happens. We are presented with a state of affairs at time X and another state of affairs at time Y, and I use the word "state" properly. Between these states of affairs is the dynamic process by which one gets from A to B within the time X and Y. This is utterly disregarded and so it cannot be accepted as applicable to Zeno's paradoxes. The strategy this device employs is a question-beggar in that it assumes that motion is taking place. I can still ask Zeno's question of how the Runner manages to perform his stupendous task. Neither Ushenko nor Shamsi will provide an answer.

A second approach based on redefining "motion" can be found in the extremely interesting, if rather dark, article by Jenny Teichman entitled "Incompatible Predicates."[11] In this article she is arguing against Berkeley and, en passant, the problems about perception raised by Russell in the first few pages of his "Problems of Philosophy". She is speaking of statis predicates and what follows if objects always remained the same colour, irrespective of light conditions, remained the same shape, irrespective of our perspective changing as we move around them, and always remained the same size, irrespective of whether we approach them or recede from them. She comes to this conclusion: "... (if predicates were static) Motion would be Unreal, the world would be static."[12] She seems to be saying[13] that motion is objects "growing" in stature as we approach, or, rather, our approaching towards objects is their growing in stature, and our

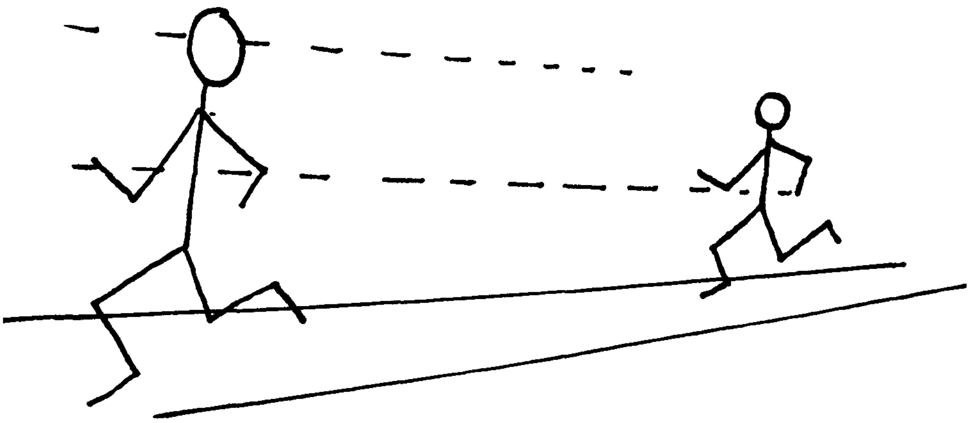
---

11. J. Teichmann, "Incompatible Predicates" in Analysis, 26, (1965-66), pp.57-58: My colleagues feel that this argument should not be included as it is too off the point. Teichmann does not mention Zeno and her phenomenological approach seems to be of little relevance. However, it is possible that her strategy could (will) be used to attach Zeno, and I feel, therefore, that its inclusion (and scotching) are important.

12. ibid., p.58.

13. Although it must be stressed in fairness to a splendid article that this is not Mrs. Teichmann's point in writing.

withdrawing from objects is their diminishing in stature. To apply this to the Stadium paradox; the runner sees a faint line in the distance; it is the finishing tape. It gradually assumes a greater clarity, becoming more solid in colour and clearer in definitions; it seems to stretch to either side, becoming longer and it also becomes thicker; the posts from which the tape is stretched grow in height and girth. Therefore the runner is moving towards the finishing line. (Or else the posts and tape are moving towards the runner). For example, an observer suitably placed beside a racetrack will see this:



If the observer knows that he is stationary, then he can deduce that a runner is approaching him.

This conflation of "growing in stature" and "moving" is very problematic because of Relativity. Consider, for example, the Achilles paradox: imagine that we are the swift Achilles. The tortoise is a tiny object in the distance and begins to grow. We (Achilles) are therefore approaching the tortoise. At some stage the tortoise will stop growing and start to diminish in size, and here the problems begin. Is the tortoise diminishing in size because I have passed him by, or has the tortoise put on a sudden spurt and is now racing ahead? There is no way in which this can be answered. That is, I still don't know if I have caught the tortoise. If one tries to do so by considering points of reference outside the actual protagonists of the race, these points of reference, given a little ingenuity, can be rendered impotent. Besides, Zeno can ask, "I'm interested in the tortoise or the tape growing in size. Let us assume it starts of the dimension X. How can it get to dimension Y without first getting halfway from dimension X to dimension Y, and so on? That is, the motion of the Runner is dependent, in this analysis of Teichmann, on the growing of the tape. But that growing is equally open to my paradox. Teichmann cannot therefore be accepted."

Any attempt to eradicate the moving from "motion" must be construed as unfair play, while any attempt to redefine "motion" which keeps in the actual moving must fall because of the paradox. Here is a dilemma best left. It is time to turn to metaphysical refutations of Zeno's paradoxes.

Part Four

Metaphysical Refutations

Metaphysics has become increasingly démodé with the passage of this century. There is, for me, a fascination in the edifices which have been constructed by metaphysical thinkers such as Bergson and Whitehead, because there is a supreme optimism in their work, an optimism that important deductions can be made about Reality, the universe as a whole, rather than the fragmented bits and pieces of other disciplines.

It may be felt that the first chapter in this section is not quite metaphysical. This, of course, depends on what the reader means by "metaphysics". I think that it does show a metaphysical approach because, although there is no grand Weltanschaung, there is, nevertheless, an extremely interesting and distinctive subtlety in the arguments employed. In the chapters on Bergson, James and Whitehead we do find an attempt to set forth, on the basis of reasoning alone, a final and ultimate account of the real nature of things. These metaphysical theories are characterised by the following peculiar attribute: facts cannot be cited in their support nor can they be used against them. This attribute, together with the splendid rhetoric in which these theories are invariably couched, means that philosophical criticism is rather frustrated. Warnock[1] speaks of these theories as being like onions: we strip away the metaphorical dress and find that we are left with a chimera. He may well be right.

---

1. G.J. Warnock, English Philosophy Since 1900, (London, 1969,)p.5.

Chapter Eleven

THE POINT

"Details are always vulgar"

(Wilde)

Much controversy has been generated over the vexed questions of what sort of "points" Zeno had in mind in sentences such as "The Runner must first run to the halfway point," or "Achilles must first run to the point from which the tortoise started in his race." Given that Zeno is attempting to defend the integrity of the continuous sphere of his master Parmenides, and that he is therefore trying to demonstrate the non-existence of "the point", it may seem, at first, a rather bizarre strategy to try to base a refutation on a subtle distinction between <u>kinds</u> of points. This notwithstanding, at least three very interesting attempts use the notion of the point as the weak area in the paradoxes of Achilles and the Runner, and, by close analysis of the notion of "point", try to show that Zeno can be overturned.

The first paper which I propose to consider and which utilises this mode of attack is by J.O. Wisdom.[1]

Wisdom opens the main section of his paper by stating, as I hope to have shown, that an infinite convergent geometric series cannot for logical reasons describe a distance in a physical race. Zeno's description of the race between Achilles and the Tortoise is plausible in that it seems reasonable to ask Achilles to carry out the infinite number of tasks stated in the following premise: "As Achilles tries to catch the tortoise he must first run to the tortoise's starting-point, by which time the tortoise has moved

---

1. J.O. Wisdom "Achilles on a Physical Racecourse" in <u>Analysis</u>, 12, (1951-52), pp. 67-72.

to a new point: Achilles must run to this ... etc." Wisdom then proceeds to investigate the source of this plausibility. The thesis which he proposes to put forward is based on a distinction between physical points and mathematical points, and that a mathematical description of a physical distance is an incorrect description.

> A physical point, unlike a mathematical point, has some size, though this may be as small as we please. But however small a physical point, since it has some size greater than zero, an infinity of them cannot be packed into a finite distance. In particular an infinity of physical points cannot be packed to correspond to the mathematical points described by an infinite geometric series. Hence an infinite geometric series is inapplicable to a physical distance, i.e. a physical race cannot be described by repeated bisection, or Zeno's premise (about Achilles' progress) is false. [2]

This is an attractive method of approach, but several questions immediately arise. Firstly, if I can make the physical points as small as I please, why can't I make them infinitely small? Clearly I cannot physically make them infinitely small: if I press my pencil on to paper I will produce a point, but pencils, needles, hypodermics, etc., are too gross to make an infinitely small point. Is that what he means? I am unsure but presumably I could, in mind at least, make a point as small as I please. If I could do this, i.e. make (in some way or other) infinitely small points, then could I pack an infinity of them into a finite distance? Zeno will say no. No matter how small, even infinitely small, they will still have some dimension.

---

2. ibid., p.72: my own refutation is similar to this although I steer clear of the complexities raised by the notion of "point".

Therefore an infinitude of infinitesimals cannot be packed into a finite interval. Perhaps when the physical point reaches a certain degree of smallness it becomes a mathematical point? Wisdom says, in fact, that when physical points are reduced to zero they are no longer physical but mathematical, i.e. they seem to correspond to the Euclidean definition of a mathematical point. He therefore seems to reject the notion of the infinitely small, and as this seems to be a good strategy, I will not cavil at his move. Wisdom is therefore asserting that there are two sorts of points: physical and mathematical. No matter how much I reduce the dimensions of a point, as long as there is some dimension, it is physical: as soon as I speak of a point having no dimension, then I am speaking about a mathematical point. More significantly, this thesis of Wisdom's is not only compatible with what Zeno is saying, but is actually the same as Zeno said! Wisdom has distinguished between "physical points" and "mathematical points". The former, he repeatedly stresses, always have some size: "physical points are ... unlike mathematical points that have no size."[3] From this statement we can deduce that physical points, no matter how small, are always small bodies and "so long as they are physical and thus greater than zero, (they) cannot be packed into a finite distance."[4] This is being put forward as an insight into the problem, but, not only did Zeno understand this, it forms the basis for the first argument against plurality!

---

3. ibid., p.70.

4. ibid., p.70.

Zeno says "points must be indivisible and therefore without magnitude ...... any point will have some magnitude ..."[5] What Zeno failed to understand, and what Wisdom clearly indicates, is that "points with magnitude" and "points without magnitude" are not the same sort of points. There is an important distinction between them: mathematical points will be without magnitude whereas physical points will have magnitude. When it is seen, as Wisdom sees, that Zeno is confused, the paradoxes of plurality fall.

Wisdom then distinguishes between "physical distance" and "mathematical distance". I find what he says here unclear but I think that his deduction is that "mathematical distance" is a representation of "physical distance".

"The premise 'Achilles' distance is 1 + ½ + ¼ + ... etc.' contains no contradiction if 'Achilles' distance' is short for 'Achilles' mathematical distance.' But the premise really is 'Achilles' physical distance is 1 + ½ + ¼ + ... etc."[6] He is therefore saying that there are two kinds of distance, real physical distance which can be traversed, and mathematical distance, which, because it is numbers representing the world, can have nothing move in it, and that Zeno got the two mixed up. Wisdom says (p.70) that there is no contradiction if Zeno is speaking of Achilles' mathematical distance, but also that Achilles does not

---

5. See chapter, Zeno the Eleatic.

6. J.O. Wisdom, ibid., p.70: (this is actually another representation of the Stadium paradox.)

physically cross this kind of distance since he "cannot run on a racecourse consisting of mathematical points."[7]

Let us ask a bit more about these mathematical points. Wisdom has said that they are distinguished from physical points in that they have no size; they are physical points "reduced to zero." If we now bring in his notion of a distance then something very peculiar happens. If we have a finite distance composed of points (which have size) and steadily[8] reduce the size of these points (adding on more to keep the distance of the line the same), then at the instant before the points become zero the line is a physical line, but when the points become zero, the "physical line" somehow becomes transmogrified into a "mathematical line." This is very difficult.

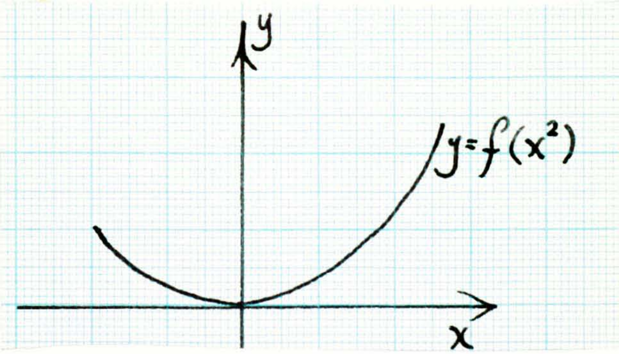
What is the precise ontological status of these mathematical points and lines? It will be agreed, I hope, that the physical representation of a physical object, and the physical object itself, have the same ontological status. An example of this would be the blueprint drawings of a tanker and the tanker itself. However, a mathematical object and its physical representation must be

---

7. ibid., p. 70: does this carry with it the queer imputation that a mathematical distance is made up from mathematical points?

8. Not a geometric reduction which would cause another paradox.

ontologically disparate. Consider, in geometry the difference between the physical representation



and the Platonic curve which both the algebraic equation and the graph try to represent. The objects of Platonic curves are certain spatial relations which simply _are_, whatever we may do, and do not come into existence in virtue of our equations or constructions.

If Wisdom means that the zero point is really a symbol, number, or whatever, _representing_ a physical point, it does not exist physically in space. Clearly then, a physical racecourse could not be constructed out of mathematical points. But, if mathematical points are symbols, then I fail to see how a mathematical distance, also a symbol, could be _constructed_ out of mathematical points, as Wisdom seems to imply.

The distinction which Wisdom is making between kinds of points is useful, but I cannot see how it happens that, as a point reduces to zero, it suddenly becomes a symbol. I suspect that I am confused about the precise relationship between a point and the symbol for that point, a distance and the symbol for that distance. However, as Zeno can avoid Wisdom's thesis quite easily, I will leave my confusion aside.

Wisdom has maintained throughout that the refutation is based on the fact that "physical distance cannot be split up into an infinity of parts"[9] because physical points always have some size. He says (p.70) that "even if we make the points extremely small, this cannot be done." I have shown a sense in which this is true - our senses and our instruments are too bulky and indiscriminating to split anything up this way. There is, however, no logical or intellectual difficulty in doing so. Wisdom may say that by using logic and intellect I have slipped into the notion of mathematical points. Taylor's[10] riposte to this would be that Wisdom, if he is cogently to maintain this, must also conclude that between 0 and 1 there are only a finite number of atoms to be packed. Surely no-one contends that it is not the case that there are infinitely many fractions contained in this interval?

The notion, never mentioned in his paper, but which I think is underlying it, is that we cannot make a mathematical point, nor can we run in the way Zeno requires (over very small distances) because we are too bulky. A situation must arise where we are required to cover distances so small that we are unsure whether we are in the realm of physical points or mathematical points. These points are imperceptible.

Wisdom has made another attempt, based on this notion of the point as imperceptible, to solve the Achilles paradox.[11] There

---

9. ibid., p.69

10. Richard Taylor, "Mr. Wisdom on Temporal Paradoxes", <u>Analysis</u>, 13 (1952-53), p.17.

11. J.O. Wisdom, "Why Achilles does not fail to catch the Tortoise" in <u>Mind</u>, (1941), pp. 58-73.

is no evidence in the paper to confirm this, but at the back of Wisdom's mind may lie the following passage from Aristotle: "things come into being from things that exist and are present, but owing to their minute size are imperceptible to us."[12]

The paper is very difficult, but Wisdom begins to make his main attack on page 63: "... in assigning the position of a body we often regard the body, even when it is of considerable size, as being at a point: When we say that London and Liverpool are 200 miles apart, we talk as if a line 200 miles long joined two points, London and Liverpool." We use 200 miles as a practical notion but we tend unwittingly to think of this as an absolute notion, that London is 200 miles from Liverpool. A length, Wisdom says, is "in practice ..... measured to within some degree of accuracy,"[13] and he gives as examples measuring Mount Everest to the nearest foot, measuring the distance to the Sun to the nearest million miles, measuring the fit of a piston rod to the nearest thousandth of an inch. "In no case does the notion of an absolute length play a practical part."[14]

We may object by saying that even though we cannot measure more accurately than instruments allow, distances and intervals still have absolute length. Not so, says Wisdom; to assign a length is to assign a number, and a number that has a finite number of integers and that belongs to a certain scale of measurement. If the scale "does not give a number more completely

---

12. Aristotle, Physics, 187a36.
13. J.O. Wisdom, loc. cit., p.64.
14. ibid., p.64.

than to a certain degree of accuracy, an absolutely complete number cannot be assigned to the distance, so that the interval has no absolute length."[15] To ask what a distance is is to ask what is its length and that can only be given within certain limits of error. The absolute "is a psychological conception different from a very good approximation and having no counterpart in reality."[16]

Before we ask where this is leading us, let us ask if we can accept what Wisdom says. He seems to be saying that we cannot know precisely the distance from the Earth to the Sun, so that to ask what the distance is is tacitly to accept whatever our instruments say it is, even if we know that they are always in error. From this he seems to go on to say that the very notion of absolute distance is somehow illegitimate, something only occurring in thought and not in the real world.[17] But there is an absolute distance from the Earth to the Sun: quite simply, it is the distance from the Earth to the Sun. This may seem to be rather simple-minded, but I think that Wisdom is confusing the map with the territory. Irrespective of what our instruments have

---

15. ibid., p.64.

16. ibid., p.66.

17. Presumably by this way of thinking, the old chronicler got it wrong when he related, "That there might be no Abuse in Measures, he ordained a Measure made by the length of his own Arm, which is called a Yard." Whatever Wisdom says, Henry I's arm, irrespective of its length, was a yard.

to say about distance, this will not affect the real distance in the real world. In fact, it seems to me that it is when we move from the real world into the world of mathematics that problems arise, (although more of this in my last chapters).

What has this to do with Zeno? It is a devilishly cunning device.

Wisdom has tried to cast doubt on the notion of absolute length. Length, he maintains, only makes sense with reference to the means used to measure. If, then, we can measure length only to one thousandth of a metre, there must come a point when, according to our instruments, Achilles has reached the tortoise: measuring to one thousandth of a metre, our instruments do not reveal a gap between Achilles and the tortoise by which it leads. And, an interval too small to measure or observe is "without meaning."[18]

In Wisdom's method we have to realize that our perceptions of space, time and motion have lower or minimal threshholds. Since, therefore, we cannot become aware of Achilles approaching the tortoise according to Zeno's rules, it is impossible to obey the rules and they are therefore nonsensical or operationally insignificant.

---

18. J.O. Wisdom, ibid., p.72. See also Vlastos, "Zeno's Race Course" in Journal of the History of Philosophy, (4), 1966, p.107: "Achilles is in a position to make the difference between him and the tortoise less than any assignable quantity, however small - a perfectly good way of overtaking him."

If, his argument goes, we cannot perceive so exactly as would be necessary actually to see Achilles overtake the tortoise, then it is immediately apparent that Achilles will pass the tortoise during one of the minimal perceptual intervals. This I take to mean that we do not see Achilles pass the tortoise because he is doing something we cannot be aware of. He seems to pass the tortoise when we're not looking (or can't see). Maybe we could look at the race through binoculars or a microscope so that our perceptual threshholds were enhanced in such a way that we were able to see things we couldn't see before. The problem then becomes not one of perception but one of technology. Wisdom has covered this possibility, by saying that "no matter how much we improve our instruments, the improvement could not keep pace with the diminution of the term $a/2^n$," (the distance between Achilles and the Tortoise).[19]

However, even despite his confusions between map and territory, I think that Wisdom is totally on the wrong track. The problem is not simply a problem of perception, it is also a problem of conception. By this I mean that, after any one of Zeno's steps, it would always be logically possible that our perceptions would be sufficiently acute to discern the movement in the next step described by Zeno. In this way we would have the same problem once again, but couched in different terms.

Besides, despite what the instruments say or fail to say, that does not mean that Achilles has caught the tortoise. Wisdom

19. ibid., p.66.

maintains that if the instruments fail to record a spatial interval between Achilles and the tortoise, then Achilles has caught the tortoise. This is because the distance between them is operationally insignificant.

To show how an error has been made, I propose to introduce a necessary condition for Achilles' truly catching the tortoise: imagine that the tortoise has an explosive device attached to its shell. Achilles wears a magnetic belt and, only when he reaches the tortoise, the magnetism of the belt sets off the explosive. The tortoise will blow up. Clearly, even if Wisdom's instruments fail to record that Achilles has not caught the tortoise, we know that it will fail to explode, because Achilles has <u>not</u> caught the tortoise. Also, if the argument presumed in Wisdom's refutation of Zeno were to be correct, it fails to be clear how it could ever be meaningful to speak of Achilles passing the tortoise, since we can never observe the exact moment and point of passing, (it being below the threshhold of our perception).

There is a hint in this paper that we cannot obey Zeno's rules governing the race, the reason being that we are too gross and undiscriminating to take heed of the imperceptibles. This is amplified by L.E. Thomas in his paper "Achilles and the Tortoise."[20]

Thomas states that a solution to Zeno's paradoxes will not be found by attacking the problem as though it were one of explaining how a body may move from one point to another in an

---

20. L.E. Thomas "Achilles and the Tortoise" in <u>Analysis</u>, 12 (1951-52)

infinite series in a finite time.[21] The paradoxes arise, he maintains, from the apparent incompatibility of two pieces of knowledge: (a) Achilles does overtake the Tortoise, and (b) space and time are infinitely divisible. This infinite divisibility of the line on which they are running shows that there are an infinite number of points between Achilles' starting point and the starting point of the tortoise so that not only does Achilles never overtake the tortoise, Achilles does not even approach the tortoise.[22] This seems to me to be unobjectionable[23] because this is precisely the point that Zeno is trying to make; not simply that a runner can never catch another, for that does nothing to preserve the sphere of Parmenides, but that runners can never move at all.

In the first move to overturn Zeno, Thomas asks us to imagine a "completely timeless but spatial Achilles all set at point X to start the race. Then such an Achilles does not wholly exist at point X."[24] What seems to be going on here is this: Achilles is a man and cannot be accomodated at a point, therefore to imagine him at a point is to imagine a cross-section of him. Zeno's arguments, therefore, cannot apply in the real world.

---

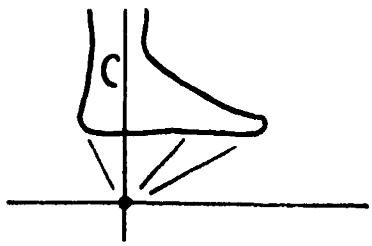
21. ibid., p.92.

22. ibid., p.92.

23. This is the device I use to counter the geometric series solution.

24. ibid., p.93.

This is an interesting suggestion, but I really wish that Thomas had not brought in the notion of timelessness at point X. He will rectify this anon but it clutters up the proceedings more than somewhat, because it is difficult to know just what sort of process we should initiate to suppose something existing in space yet not in time.[25] It is even more difficult to know what I should have in my mind, when I have completed the imagining. This, however, is irrelevant to his main point which is clearly worthy of consideration: it is that if Zeno claims that Achilles is at point X, then the situation as Zeno imagines it is:

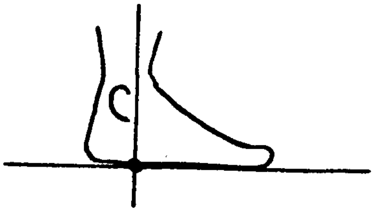


Thomas points out, quite correctly, that the whole of Achilles cannot be accomodated at a point: at a point there would only be

---

25. The same sort of problem as confronted the critics of the sphere of Parmenides. Walker in Kant, (London, 1978), pp. 34-41, tries to make sense of experience without time. His argument is complex and proceeds almost entirely by analogy. The main point on which it founders is its requirement that an inhabitant of an entirely static world (because change implies time) should be able to make deductions about that part of the world which cannot be seen. But (a) how could we know it was there, and (b) does not deduction imply time?

a cross-section of him. Thomas sees Zeno's description of Achilles as:

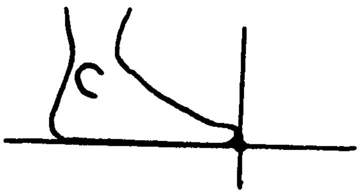


Thomas' version would imply that if Achilles were to bend over from the waist then his upper torso would disappear. This is clearly damaging to Zeno. If Thomas' version of Zeno's position is accurate then the only parts of Achilles that would be more or less permanently on view during the race would be his pelvic region. Thomas deduces that Achilles never exists totally at one spatial point but at a place which includes infinitely many spatial points.

What could Zeno say to this charge that it is impossible for Achilles to occupy a single point?

He would say something like the following:

"Your representation of my position looks to be correct but is actually wrong. I know that you can't get the whole of Achilles on to a single point, but what I really mean is this:



When I say that Achilles is at point X I do not mean that the whole of Achilles is at point X, only that the foremost part of him (in my diagram his great toe) abuts on to point X. In this way, Mr. Thomas, your theory that Achilles is too gross to fit at a point, is rendered redundant. He simply has to move the foremost part of his feet less and less.

Your criticism also strikes me as extremely questionable, because if, as you say, Achilles' feet occupy an infinite number of points, then rather strange things follow. If these points have no magnitude, and Achilles must have feet the same size as the number of points they occupy, then his feet must occupy no space at all. That is, he has no feet! If, on the other hand, these points do have magnitude, and there are an infinite number of them under his feet, then he must have infinitely large feet."[26]

Thomas' suggestion will not work.

He now allows time back in the form of "instant t". He tries to draw an analogy between space and time by saying that the temporal Achilles can no more exist at an instant than the spatial Achilles can exist at a point. This seems a very strange analogy to make: does Thomas wish to say that at an instant only a temporal cross-section of Achilles exists? Thomas bases his idea on the following statement which, he maintains, also generates the paradox: we are guilty of supposing "that any entity that exists for any

---

26. It is the first argument against plurality which is being used against Thomas.

length of time is essentially complete at any given moment in that period."[27]

This is a doubtful point to make. If he means "complete in all parts" by "essentially complete", then it seems to me to be axiomatic that I exist with all my parts at any one moment, just as much as at any two moments, or just as much as over a period of days or weeks. If Thomas were to be correct, then it would follow that every living creature would be essentially incomplete because it still has some of its life to live. I do not particularly like the notion of being essentially complete only after I die.

To say that a cross-section in space of Achilles is not the same as Achilles seems perfectly acceptable. (We have been able to circumvent this criticism, however). The temporal aspects of the fallacy, as Thomas sees them, are very peculiar. He has failed to show how "Achilles for duration one second" is qualitatively different from "Achilles for duration two seconds." In what way would one be more complete than the other? We can also imagine, I suppose, objects that exist for only an instant. Many elementary particles, some of which have the relatively <u>long</u> life of a million millionth of a second, are the sorts of thing which I have in mind. Are we to say that these objects are incomplete because they last only for one instant? But that is all they last for!

Thomas has become entangled, failing utterly to show that we must look at Achilles' action in an organic way through an analysis

---

27. ibid., p.94.

of points and instants in order to overturn the paradox.

So far I have shown that those who chose this approach have made errors, but I have still to show how any approach based on analysis of the point will fail. This can be done quite simply by recouching the Achilles and the Stadium paradoxes in such a way that there is no mention of points, mathematical, physical or any other kind.

"If an object is going to move from A to B it must first pass through part of that distance. Before it succeeds in doing that it must pass through part of that interval. We can say this for ever, without saying anything which is obviously untrue."

To attempt to deny this would be even more bizarre, for a denial would imply that an object could get from A to B without first going part of the way. But if this could be done, then vast distances could be traversed in a twinkling because the intervening distance need not be traversed. There is no mention in this of points, or whether they have dimension or not. There is only one kind of distance involved; what Wisdom would term "physical distance", and only one thing is said of it, that to have crossed it, some smaller part of it must first have been crossed. This, far from being contradictory, is an obvious truth.

This, then, is a road leading nowhere.

There appear to be ten conceivable alternatives as to the composition of magnitudes. They might be composed of:

1) a finite number of finite parts

2) a finite number of indefinite parts

3) a finite number of infinitesimal parts

4) an indefinite number of finite parts

5) an indefinite number of indefinite parts

6) an indefinite number of infinitesimal parts

7) an infinite number of finite parts

8) an infinite number of indefinite parts

9) an infinite number of infinitesimal parts

10) no parts at all

The only options which are at all tenable are (1), (9) and (10). (1) fails because minima are indeterminate and given any determinate magnitude, a smaller one can be supposed. (9) is extremely contentious given what modern mathematicians have had to say on infinite, and will be examined in Chapter Fifteen. (10) is probably the most hopeful avenue for our investigation to now take. It defeats Zeno's main presupposition by denying that a length is composed of all possible smaller lengths. "To say that a foot is composed of, or contains, or can be divided into, twelve inches is a handy but inaccurate mode of speech. It is more correct to say that the length of a foot is to the length of an inch as 12 is to 1."[28]

Chihara claims that although "We do say such things as "Achilles is at the halfway point" ... to say this is not to claim that there are two things halfway to the finish, namely

---

28. A.D. Ritchie, "Notes" in Mind, (1941), p.311.

Achilles and a point. One is claiming only that Achilles is only halfway to the finish."[29]

This idea leads us now to a discussion of more metaphysical writers, Bergson, James and their notions of Space and Time.

---

29. C.S. Chihara, "On the Possibility of Completing an Infinite Process" in Philosophical Review, 74, (1965), p.84.

Chapter Twelve

Bergson and the Mental Camera

"You should live in the world so that it may hang about you like a loose garment."

(Lord Halifax)

Russell has said that, on the Cantorean view of space and time, motion is continuous. This is because of the fact that between any two positions occupied by a moving body at any two instants "there are an infinite number of positions still nearer together" which means that the moving body "never jumps from one position to another, but always passes by gradual transition through an infinite number of intermediaries."[1] We may "find it hard to avoid supposing that, when the arrow is in flight, there is a next position occupied at the next moment ... in fact there is no next position and no next moment and once this is imaginatively realized, the difficulty is seen to disappear."[2]

Bergson[3] points out that time and motion are not really continuous at all on Russell's view, since their elements remain distinct and external to one another. The moving arrow still is at each of the positions along the course it traverses; the entire course is exhausted by the positions at which the arrow is in turn, and the entire time of the flight is exhausted by the moments at which it is at each of the positions along the path of the flight; the arrow does not move insofar as it is at a position; yet to move does take time; hence, since there is no time at or during which the

---

1. B. Russell, Our Knowledge of the External World. (London, 1922) p.142.

2. ibid., pp. 179-180; this is a typical Russellian device: if we fail to agree with him, it is because our imagination is not up to scratch.

3. H. Bergson, An Introduction to Metaphysics, (New York, 1912), pp. 48-49.

arrow is not at some position, there is no time at or during which it moves; hence the arrow does not move at all. Bergson, therefore, claims that Zeno's conclusion is the same whether the positions along the arrow's course and the moments of its flight are finite or infinite in number.

Bergson, however, is unruffled by this conclusion because he feels that metaphysical argument, rather than logico-mathematical argument, can refute Zeno's paradoxes. What is it he has to offer? He considered Zeno to have sown the seeds of the paradoxes by failing to make the intellectual distinction between motion and the space traversed by moving:

> Once more it is conceptual thinking which is responsible for the confusion between movement, the symbol of duration, and the spatial distance traversed; and it is enough, in order to refute Zeno's arguments, to clear up this confusion by distinguishing between indivisible or divisible movement characterised solely by a series of actions which are themselves indivisible, and homogeneous and amorphous space which is indefinitely divisible ... now, in this continuity (of movement) thought isolates states clearly distinguished one from another." (my translation and underlining.)[4]

This idea of the intellect being the cause of the paradoxicality of motion is scattered throughout Bergson's writings and those of his followers:

---

4. Unsigned review of Bergson's philosophy in Revue de Philosophie, II, 6, (1902), pp. 828-832.

> To Zeno's argument which is purely intellectual, his adversary will respond by using movement itself; he starts walking ... He would have said: "I walk. I do not make up movement as you make up the path it describes. I take steps and each one is indivisible; after two or three I will have crossed the gap. Action is always a bond which contains the indivisible. You break up the result but not the movement itself. More so, give up letting understanding rule your action, or you will run the risk of never doing anything." (my translation)[5]

Bergson has it that "... the difficulties and contradictions which have arisen around the question of motion collapse when one thinks of motion as a simple thing (i.e. without parts)."[6] The intellect "solidifies" our perceptions of change, but "... this discontinuity is an illusion, an illusion analogous to that which would occur if a self-conscious cinematograph could take a series of snapshots of a movement and which, on looking at the series of motionless photographs, would come to the conclusion that the moving scene is but the total of immobilities." (my translation)[7]

The intellect, then, crystallises motion rather like a camera taking photographs, isolating states clearly distinguished one from another in that which is indivisible. Bergson explains how this cinematographic capacity of thought arises. Let us consider the activity "Becoming". "Becoming" is an activity which is extremely varied: there can be qualitative movements such as in the subtly

---

5. J. Grivet, "La theorie de la personne d'après Henri Bergson." in Les Etudes, 1911, (pp. 449-485).

6. Bergson, "A propos de l'evolution de l'intelligence géometrique in Revue de Metaphysique et de Morale, XVI, 1, (1908).

7. Speech in 1911 at University of Oxford, reprinted in Mélanges as "Perception of Change".

changing colours of a sunset (Oh look, it's become quite golden!), there can be evolutionary movements as in the growing up of a child (Gosh, you have become a big boy), and there can be extensive movements such as in the action of walking. But, "... these three kinds of movement themselves qualitative, evolutionary, extensive - differ profoundly."[8] However, our intelligence, language and perception abstracts from these very different "Becomings" the idea of "Becoming in general", a notion which is always the same and always unconscious.[9] What is it we do with this notion of "Becoming in general"? To it we join in each particular case one image that represents a state and which serves to distinguish the different becomings. Bergson has it that we substitute for specific change a composition of "a specified and definite state with change general and undefined."[10] What he seems to be saying is something like this: in considering a sunset, or a child growing up, or movement across some spatial interval, I have in my mind an abstract notion of "Becoming in general" to which I ally (in my mind's eye) a static image of sundown, a little boy, or whatever. There is the abstract notion plus the mental photograph.

The parallel with the cinematograph manifests itself. This instrument is seen as being exactly like the human intellect in that it consists of a series of snapshots. If we simply looked

---

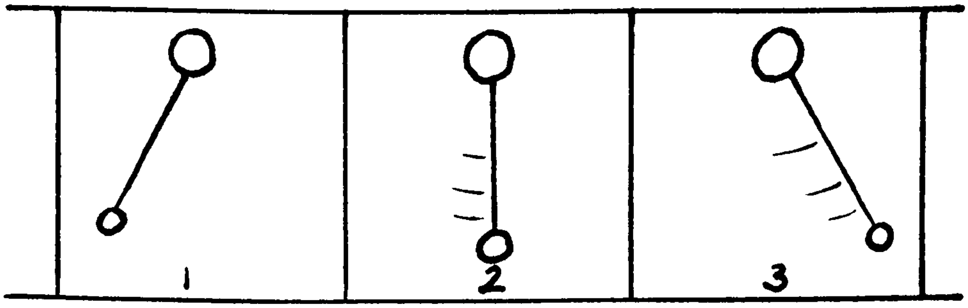
8. Bergson, Creative Evolution. (London, 1954), p.321.

9. ibid., p.321.

10. ibid., p.321. (my underlining).

at the photographs or frames we could never get animation: "with immobility set beside immobility, even endlessly, we could never make movement."[11] In our knowledge, he maintains, we do the same sort of thing. The case against Zeno is beginning to take shape, for, "We take snapshots, as it were, of the passing reality ... the mechanism of our ordinary knowledge is of a cinematographical kind."[12]

What happens is that I take a series of views of the continuity of a particular becoming which I connect together by "Becoming in general". However, this "Becoming in general" symbolizes a certain transition of which I have taken some snapshots: "... of the transition itself it teaches me nothing."[13] The application to Zeno is now clear:



Imagine this to be a filmstrip of a pendulum swinging from position one to position three. Bergson claims that, because of the way in which we think, we cannot reconstruct the movement

11. ibid., p.322: this is possibly directed against Russell and Cantor.

12. ibid., pp. 322-323.

13. ibid., p.324.

between the frames. To try to understand what occurs between frames one and two I may interpose an extra frame (use a faster camera), but "I may begin as often as I will, I may set views alongside of view for ever. I shall obtain nothing else. The application of the cinematographical method therefore leads to a perpetual recommencement during which the mind is never able to satisfy itself."[14.]

The intellect can never be able to reconstruct movements, and herein lies the clue to the refutation of the paradoxes. William James[15] makes the same point: "Perception changes pulsewise, but the pulses continue each other and melt their bounds. In conceptual translation, however, a continuum can only stand for elements with other elements between them ad infinitum, all separately conceived; and such an infinite series can never by exhausted by successive addition." If we accept that the intellect is never able to reconstruct movement, what are we to do? The answer is profound: "... install yourself within change and you will grasp at once both change itself and the successive states in which it might at any instant be immobilized."[16]

Bergson writes of installing oneself in change, but how is this to be done? Clearly it cannot be through a process of intellection because we have been told that the mechanism of our

---

14. ibid., pp. 323-324.
15. W. James, Some Problems of Philosophy, (New York, 1919)pp.87-88.
16. Bergson, op. cit., pp. 324-325.

ordinary knowledge is of a cinematographical kind. It must be in some non-rational way; it is an instinctive kind of knowledge which Bergson has in mind and which he calls intuition: "... instinct is akin to that power of direct insight we call intuition. It is this power which in our view philosophy must make use of to seize again the simplicity of reality that is in a manner distorted in the intelligent view of things."[17] In intuition, then, lies the possibility of a refutation of intellectual puzzles such as the paradoxes of Zeno: "Your acquaintance with reality grows literally with buds or drops of perception. Intellectually and on reflection you can divide these into components, but as immediately given, they come totally or not at all."[18]

If we can break away from the habits of thinking and perceiving that have become natural to us and so get back to direct immediate perception of change and mobility, we will perceive that all change and all movement is absolutely indivisible.[19]

This, then, is intuition, the immediate grasp or feeling that precedes discursive thought; it is the medium through which we can discover what reality is.

---

17. H. Wildon Carr, Henri Bergson: The Philosophy of Change, (London, 1912), p.45.

18. W. James, op. cit., p.155.

19. D. Ballsillie, Professor Bergson's Philosophy. (London, 1912), p.156.

The foregoing has been very complex: let me therefore summarise it briefly before going on to consider if Zeno could respond.

If we rely purely upon our intellectual faculty we can never grasp what reality, especially motion, is. Reality has a ceaseless flow, yet its perpetual change must ever evade our intellectual categories. To know reality we must have recourse to intuition, the means "whereby we plunge into the stream or flux of Becoming and apprehend it from inside."[20] Change and time as represented by abstractions, according to the intellectual method, consist of stages in relations of succession, but the fact does not happen by stages and is not held together by these relations. Chihara[21] captures Bergson's point when he says that "To complete a journey, one must simply _perform_ a task which can be _analyzed ad infinitum_."

Bergson assumes that the ultimate features of reality are those which are directly revealed to us in our pre-analytic experiences. We must let reality directly reveal itself, for any imposition of concepts or symbols upon a revealing experience will distort the nature of reality. All the means by which our intellect operates, i.e. language and concepts, "viciously distort the true nature of the given, or reality."[22]

---

20. E.W.F. Tolmin, _Great Philosophers of the West_, (1959), p.266.

21. C.S. Chihara, loc. cit., p.86.

22. R.M. Gale, op. cit., p.391.

It will probably have been noticed that almost every quotation so far from Bergson and his followers has been in the form of a flat assertion. There is simply no argument, something which makes the task of criticism rather difficult. In an extremely virulent attack on Bergson, Santayana has stated that he is persuasive without argument, managing to envelope everything in a penumbra of emotional suggestion: "In essence, it (Bergson's achievement) is myth or fable; but in the texture and degree of its fabulousness it differs notably from the performance of previous metaphysicians."[23] Russell states that "... a large part of Bergson's philosophy, probably the part to which most of its popularity is due, does not depend on argument, and cannot be upset by argument. His imaginative picture of the world, regarded as a poetic effort, is in the main not capable of either proof or disproof."[24] It seems clear that, however much we strip Bergson's ideas of their brilliant metaphorical dress and try to divorce them from his literary style, Zeno will be left with a problem. We must not rest content with insults but we cannot argue against Bergson's grounds for his conclusions in an attempt to cast doubt on those conclusions. This is because Bergson does not offer any grounds. However, with regard to one of the paradoxes, The Flying Arrow, we can see Bergson's remarkable method in operation.

---

23. G. Santayana, Winds of Doctrine, (London, 1940), p.58.

24. B. Russell, History of Western Philosophy, p.764.

He seems to be denying the premise which was isolated in Chapter Four as being the generator of the paradoxes of motion: "On a journey from A to Z, we must first go from A to B, the mid-point of AZ, and thence to C, the mid-point of BZ, and so on." To deny this seems strange, but Bergson has said that

> ... the truth is that the arrow thrown at point A falls at point B, this movement from A to B is absolutely indivisible. Certainly, mathematicians have the greatest interest in considering the question another way, and in supposing that movement is divisible. But, in reality, it is a simple and indivisible act ... in reality, these positions (the points of the flight) do not exist ... Truly, movement is an indivisible thing. (my translation)[25]

Movement of the arrow is not to be seen as a series of static states: this is caused by the cinematographical attributes of the intellect which analyzes change in terms of successive states of a single object which differ qualitatively or quantitatively. This concept of change has been dubbed the "at-at" theory since it analyzes the movement of a thing in terms of its being <u>at</u> a different place <u>at</u> one time than it is <u>at</u> <u>at</u> a later time. Bergson is trying to replace this with a "from-to" theory. Relative knowledge of a movement involves a noting of difference at different times: intuitive knowledge of the same movement "involves putting ourselves inside one of the objects and experiencing its movements internally"[26] in the same way as we

---

25. Bergson, <u>Mélanges</u>, p.1221.

26. Richard M. Gale, <u>Philosophy of Time</u>, p.390. This is, in fact, to draw an extremely interesting distinction between <u>movement</u> and <u>action</u>. The notion of "action" antedate the notion of "movement" in that "action" is the result of a process of intellection on movement, considering ancillary concepts such as "intention" and "consequence".

experience the movement of our limbs. Bergson wants us to intuit the "from-to" aspect of movement, how Achilles gets from one place to another.

Bergson maintains, then, that the arrow paradox falsely reduces the flight of the arrow to a sequence of static positions along the path of its flight. He takes the entire flight to be a single and "absolutely indivisible" event: the arrow is never at a given place at a given time, for then Zeno's paradox would occur. This, perhaps, assumes a lot, because arrows cannot tell us what their movement feels like to them. It could be that, for the arrow, the movement feels disjointed, shuddering and awkward. However, this line of speculation is crude and futile: I propose to abandon it because what Bergson says contains something very interesting. What is the word "at" actually doing in Zeno's account of the Flying Arrow?

"At" has a connotation of being "stationary": one thinks of expressions such as "at rest", "at prayer", and so on. This conflation of being "at" and being "stationary" is, I think, not the most subtle of linguistic snakes in the grass, but is quite strong in the common unanalytical mind. It is, however, far from obvious that there is a necessary connection between A's being at X and A's being stationary. Consider, for example, the following conversation:

M: "Where were you for your summer holiday?"

N: "I was at the Olympic Games."

M: "Oh, any part in particular?"

N: "Yes, I was at the rowing regatta."

M: "Did you have a good view?"

N: "Yes, I was at the end of the course."

M: "That's interesting. Where were you?"

N: "Well, I was actually at the finishing line."

Clearly, we feel that M is giving N less and less room for manoeuvre, but it is only at N's last utterance that the notion of being stationary arises. But if we think about it a bit more closely, it is we who are doing the work and not the word "at": why should N's being at the finishing line imply that he is stationary? As we continue focussing more and more precisely there is a feeling almost of being "boxed in", but this "boxing in" could be totally illusory. If we are giving a series of spatial "at"-type descriptions, none of which imply stationariness, what, then, is the magical occurrence which finally renders N stationary? It seems to me that Zeno is a bit confused here and that Bergson has spotted the confusion.

Zeno may accuse me of being unfair because I have given only half of his argument. He has more to say about the "at-ness" of the arrow, for he is not only focussing the place; he is also focussing the time. Consider this conversation which has to do with temporal focus:

M: "Do you remember the 1967 football season?"

N: "Sure I do. That's the year Glasgow Celtic won the European Cup."

M: "What a game. That was a day to remember."

N: "Remember the first half? That's when Jimmy Johnstone made his famous run down the wing, cutting to the centre of the penalty area past three Inter Milan defenders."

M: "What a shot! A terrific goal, and right on the half hour!"

N: "But I didn't think it would go in. Just before it crossed the line that big Italian full-back leaped across ..."

Again, the feeling of being "boxed in" starts to become manifest. Personally, I do not feel it as strongly as in the spatial example, but this may be because we tend to pay more regard to space rather than to time in our intercourse with the world. However, it cannot be denied that, when the focussing of space and time is taken in tandem, what Zeno says becomes very plausible. When we get to the point in our descriptions, the logical extreme, where the space and time under consideration are so small as to be indivisible, Zeno will say that the possibility of motion must be denied: stationariness must result from this focus.[27]

---

27. How would a judge react to this?

Advocate: "What were you doing at the precise time at which the crime was committed?"

Witness: "By the Paradox of the Flying Arrow, at the precise time at which the crime was committed I suppose that I could not have been doing anything!"

However, this plausibility does not imply that we have to accept the Paradox of the Flying Arrow. The paradox assumes its force when we consider that the focussings have reached indivisible minima in space and time. Let us now turn our attention to this. Bergson has said that the arrow in its flight cannot be said to be at a particular point, ("they do not exist.") We find this out if we "install ourselves within change". For those who, like me, experience some difficulty in following these instructions, and who, unlike me, remain unconvinced that the word "at" needs a bit more analysis than Zeno gives it, let us see if we can give logical grounds for showing that the arrow is never at a point.

The weapon which I propose to use in order to give firm ground to Bergson's assertion is a modification of Zeno's Paradox of the Moving Rows: I term this the Paradox of the Two Flying Arrows.

Two archers, A and B, are going to fire arrows at a target some distance away. Archer A has a new shiny carbon-fibre bow while archer B has a bow made from a tatty old branch of yew. A's arrows fly at twice the speed of those fired by B. We can show, by Zeno's analysis, that B's arrow is at a particular point at a particular moment throughout its entire flight and that these points and moments are minima: by his reasoning, then, B's arrow is stationary. But what is this sudden movement? A has released his arrow.

Let us consider B's arrow at a particular minimum moment. His arrow was said to be at a particular minimum point. But A's arrow will pass through this particular "minimum" point in half the time allotted to B. The minimum moment is not a minimum after all. Let us now consider A's arrow at a particular minimum point. His arrow is said by Zeno to be in a particular minimum moment of its flight. But B's arrow will, in this same "minimum" moment, pass through only half the spatial interval of A's arrow. The minimum point is not a minimum after all.

The argument can go on _ad infinitum_: we, however, need not, for we can see that the arrow cannot be at rest after all, in that there are no minima in which it can be entrapped. We have come a long way from Bergson, but we have shown that we cannot hold, without inconsistency, both the Paradox of the Flying Arrow and the Paradox of the Moving Rows. At least one must be rejected: I reject the Paradox of the Flying Arrow because it is based on an inadequate analysis of the word "at", and because when a "from - to" analysis of motion is substituted for an "at - at" analysis, the paradox dissolves.

This has created another clearing in Zeno's jungle, but will this new, and highly successful, analysis prove effective against the Stadium Paradox?

In our couching of the Stadium Paradox there is more than a hint of the mental camera, the "at - at" analysis: we say that, in the run from A to Z, the runner will, _at_ some time, be _at_ the halfway point, B, then, later, _at_ C, the threequarter point, and so on. It must be noted, however, that the Stadium Paradox does not set out to show that the Runner is always static: its point is that he always _moves_. This implies, _a priori_, that a "from - to" analysis of the progress of the Runner may well help make

Zeno's point for him.

The object of "from - to" analysis is to show the indivisibility of motion: "... the truth is that ... this movement from A to B is absolutely indivisible", and the Stadium Paradox certainly has an implication that the runner is "at" various points of the course "at" various times. In support of this we could say that there is a natural division inherent in the movement of the runner, the steps he takes. But Bergson, who cannot allow this, (it cuts away his objections to the Paradox of the Flying Arrow), will say this:[28]

> ... all movement is articulated inwardly. It is either an indivisible bound (which may occupy, nevertheless, a very long duration) or a series of indivisible bounds.

This underlined disjunct makes it clear that Bergson considers that a movement can contain a sequence of events, yet that each of these events or elements is itself indivisible. In the case of Achilles' movement each of his steps would be an indivisible bound.

There are at least two responses to this. Here is Santayana's:[29]

> ... He (Bergson) says that ... to solve the riddle about Achilles and the Tortoise we need no mathematics of the infinitesimal, but only to ask Achilles how he accomplishes the feat. Achilles will reply that in so many strides he would do it; and we may be surprised to learn that these strides are indivisible, so that, apparently, Achilles could not have stumbled in the middle of one, and taken only half of it.

---

28. Bergson, Creative Evolution, pp. 327-328 (my underlining).

29. Santayana, Winds of Doctrine, p.79.

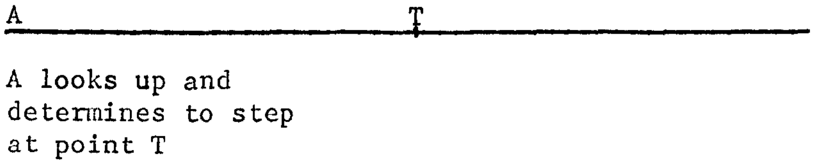
Bergson could point out that the mental camera is in action here and that Santayana has committed the cinematographical error. Santayana has "gone back", as it were, to make an intellectual analysis of what could occur. However, Santayana has not made a stupid quibble: he is removing the notion of intention from action and is therefore able to describe the stumble as another sort of action. But this is going to be more trouble than it is worth: firstly, Bergson will merely say that this stumbling is itself an indivisible action, and, secondly, Zeno will say that he is not interested in actions at all. Zeno is interested in movement, and to bring in "action" with its attendant clutter of motive, consequence, agent, and so on is totally off the point. The paradoxes, after all, are in support of Parmenides' deduction that motion is impossible.

A second response which could be made is more rewarding. Suppose Achilles,[30] sure of victory, makes a little bet with himself. "I will run in such a way that my foot always lands on the place where the tortoise was when I last looked up, which was the place where the tortoise was when I looked up the time before." The diagram will help to explain this:

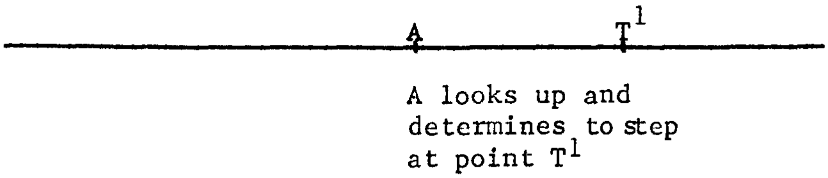
---

30. I revert to Achilles and the tortoise only to obtain more verisimilitude; the example can easily be altered to illustrate the Runner in the Stadium.

(a)



(b)



Each of his steps will be "absolutely indivisible" according to Bergson, but there is no possible way in which Achilles can catch the tortoise. The notion of the "absolute indivisibility" of motion, of the elements in an action or of actions in a sequence is *no guarantee that the tortoise will be overtaken*.

We have therefore, through Bergson, shown how the Paradox of the Flying Arrow can be refuted. We are still left with the Moving Rows and with the Stadium. It seems that some method, other than the "from-to" analysis, must be used.

Chapter Thirteen

James and Whitehead

"If in your thought you must measure time
into seasons, let each season encircle
all the other seasons, and let today
embrace the past with remembrance and
the future with longing."

(Kahlil Gibran)

Both James and Whitehead are dissatisfied with solutions such as those examined in Part II of this study, the way familiar to them through the work of Russell. They are both dissatisfied for the same sort of reason and their ways of overcoming their dissatisfaction are essentially the same. Over the last several chapters we have moved far from Zeno and into areas of language and psychology. The scene must be re-set and I propose to do this by examining the problem as seen by James and Whitehead.

James distinguishes between two types of infinite, which he calls the "standing" infinite and the "growing" infinite. To this first class belong infinities which are already completed, such as the time which has already passed up to the present day. He feels that the difficulties of the "standing" infinite, the kind isolated by Zeno in the Paradoxes of Space, are resolved by modern mathematics (presumably Cantorean analysis). However, the kind of infinite analysed by Zeno in the Paradoxes of Motion, the "growing" infinite, is still problematical:

> Zeno's ...... dialectic holds good wherever, before an end can be reached, a succession of terms, endless by definition, must needs have been successively counted out. This is the case with every process of change, however small; with every event which we conceive as unrolling itself continuously. What is continuous must be divisible ad infinitum; and from division to division here you can not proceed by addition .... and touch a farther limit. You can indeed define what the limit ought to be, but you can not reach it by this process. That Achilles should occupy in succession 'all' the points in a single continuous inch of space, is as inadmissable a conception as that he should count the series of whole numbers, 1, 2, 3, 4, etc., to infinity and reach an end.[1]

---

1. James, Some Problems of Philosophy, pp. 170-171.

and later,

"..... whoso actually traverses a continuum, can do so by no process continuous in the mathematical sense. Be it short or long, each point must be occupied in due order of succession; and if the points are necessarily infinite, then their end cannot be reached, for the "remainder" in this kind of process, is just what one cannot "neglect". ... its limit, if "successive synthesis" were the only way of reaching it, could simply not be reached."[2]

Clearly, this is the fundament of Zeno's paradoxes: no temporal process whatever can ever by completed, for every process of this sort consists of successive stages which must occur in order. The earlier half of the process must occur before the later half, the earlier half of the later half before the later half of the later half, etc. As Zeno points out, no matter how far the process may go forward, there will always be, at every stage, a remainder which is yet to occur. The process will never be completed. In order to be completed, the final segment of any process must occur; and this is impossible because no process contains any final segment. However, if we look at Zeno's paradoxes of space we can see that the first half of every segment is itself a segment which must be completed before the whole segment can be completed. The first half of the half segment will also be a segment, etc. "Hence there is no segment, no part of which occurs until after the completion of every other segment."[3]

---

2. ibid., pp. 182-184.
3. R.M. Blake, "The Paradox of Temporal Process", Journal of Philosophy. (23), 1926, p.647.

James' version of the paradox is about how any process can ever be completed. This we saw earlier is but one version of Zeno's paradox of the Runner although it is a correct interpretation of the Achilles paradox. Whitehead concerns himself with the other interpretation of the Runner paradox: how a process can ever get started. His writing is characteristically obscure: "..... what becomes has duration, but no duration can become until a former duration (part of the former) has antecedently come into being ...... The same argument also applies to this smaller duration, and so on. Also the infinite regress of these durations converges to nothing - and even on the Aristotelian view there is no first moment."[4]

His difficulty is therefore the opposite of that of James and may be more clearly recouched as follows. No process through time can ever begin to occur, because every such process consists of successive stages which must occur in order. The earlier half of the process must occur before the later half. This is an obvious truth but we need not stop there; the earlier half of the earlier must occur before its later half, and so on. No matter how near any stage of the process may be to the beginning, it cannot be the beginning, because no stage of the process can occur until another process (part of itself) has already been completed; whatever stage of a process is later than another is no genuine beginning of that process. Consequently, the process can never make a beginning. In order to begin, a process' initial

---

4. A.N. Whitehead, Science and the Modern World, p.180.

segment must occur; this is impossible for no process possesses an initial segment. The initial segment of a process would be that segment, no part of which occurs after the completion of any other part. However, the later half of every segment is itself a segment which cannot occur until after the completion of the earlier half. From this it is easily deduced that there is no segment, no part of which occurs after the completion of any other segment.

So far, James and Whitehead have given an excellent account of Zeno's paradoxes, but what are they going to do in the face of his dialectic? Both adopt the same kind of remarkable solution.

According to James, we will avoid Zeno's difficulties if we abandon the notion of temporal processes as continuous. We must, says James, "treat real processes of change no longer as being continuous, but as taking place by finite, not infinitesimal, steps, like the successive drops by which a cask of water is filled, when whole drops fall into it at once or nothing."[5] This metaphor is not all that clear, and Whitehead is equally vague, although the solution is essentially the same: "Temporalisation is not another continuous process. It is an atomic succession."[6] "Time is sheer succession of epochal durations. The epochal duration is not realized *via* its *successive* divisible parts, but is given *with* its parts."[7] I am unhappy about the obscurity of this last

---

5. James, *Some Problems of Philosophy*, p.172
6. Whitehead, *Science and the Modern World*, p.179.
7. ibid., pp. 177-178.

sentence and will consider it later.

One assumes, almost tacitly, that these finite steps occur in a serial order. This assumption is, I think, wrong, but let us make this assumption and investigate whether this notion of finite steps of time will help to overcome Zeno.

James speaks of "finite steps" while Whitehead speaks of "epochal durations". Do these steps or durations contain parts and, if they do, are these parts mutually simultaneous or mutually successive? Both James and Whitehead seem to agree that these durations contain parts. James likens the "finite steps" to "whole drops" of water; they are "drops, buds, steps, or whatever we please to term them, of change, coming wholly when they do come or coming not at all."[8] Notice that, despite whatever we choose to call them, they are "wholes" which clearly has the implication that they have parts. He says also that they take place by "finite, not infinitesimal steps". We must allow these steps to have parts, because, if these durations have no parts, we are led straight back to Parmenides. Let us grant then that, for James, the steps in a temporal process have parts. Whitehead is more clear: "...... time is the succession of elements in themselves divisible".[9] "Time is sheer succession of epochal durations ..... the divisibility and extensiveness if within the given duration."[10]

---

8. James, op.cit., p.185.

9. Whitehead, op. cit., p.179.

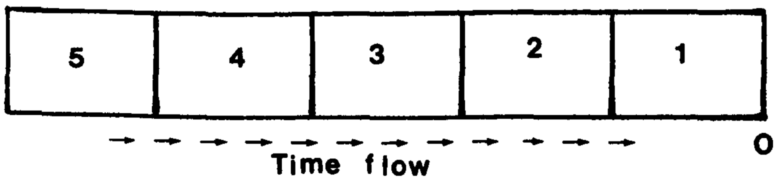
10. ibid., p.177.

For Whitehead and James, then, the "finite steps" or "epochal durations" contain parts. Problems start to arise, however, when we turn to the second question of whether these parts are simultaneous or successive. Each "finite step" or "epochal duration" is a part of the temporal process and one would suppose that there must be earlier and later segments. James, in fact, speaks of drops or steps or buds "of change" and a finite step of change would seem to mean a change of finite durations, involving successive processes within the step. The only alternative is that these steps are infinitesimal and this he has categorically denied. Whitehead's "epochal durations" are durations "involving a definite lapse of time, and not merely an instantaneous moment."[11] But consider this: if the parts of a finite step are to be taken as occurring in serial order, then there is nothing new in this view at all: it is, in fact, a confirmation of Zeno's view, for his paradox will spring out again within each finite step. To complete a finite step of change one would have to complete the earlier half of the finite step of change before the later half of the finite step of change. Now, depending on how we word the argument from here (completing the earlier half of the earlier half, etc., or completing the earlier half of the later half, etc.) we arrive back precisely at our starting point. We have made absolutely no progress at all towards a solution, and must alter our initial assumption that the parts occur in serial order.

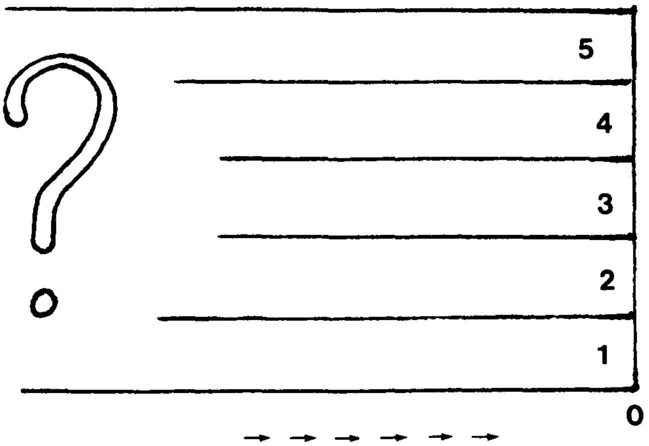
---

11. ibid., p.179.

If we turn to the more obscure parts noted in the quotations from James and Whitehead we see that they therefore have in mind the notion that the parts of a finite step are not successive, but simultaneous. James ways that the finite steps come "wholly when they do come" or do not come at all: his metaphor says that they are like "whole drops" of water which fall into the cask"at once". Whitehead says "the epochal duration is not realized <u>via</u> its <u>successive</u> divisible parts, but is given <u>with</u> its parts." This seems to mean that the parts of a duration all occur simultaneously. This is very strange, and despite Bergson's warnings about trying to draw pictures of time, I feel that I must try to make sense of this notion by encapsulating it in a diagram. Should the diagram be worthless, no matter, for problems loom which are not caused by any picture. The natural reaction to statements such as "Temporal process consists of finite steps" is to visualize something like this:



The observer cannot experience step 5 of the process until he has experienced all of the preceding steps. This serial order, leads directly to Zeno's problem. James and Whitehead, on the other hand, seem to want us to visualize something like the following:



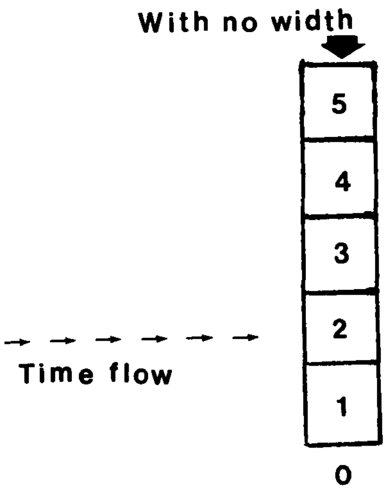
The parts come "all at once" or "not at all". The problem which possibly could arise is apparent in my diagram. What is going on to the left of the numbers? Should there by anything there at all?

This conception of simultaneity, however, is held to solve the paradox of temporal processes. The difficulty about getting started was that "Every part of time involves some small part of it, and so on." This series, it was seen, regresses back ultimately to nothing, "since the initial moment is without duration and merely marks the relation of contiguity to an earlier time."[12] Now, however, in the notion of "epochal duration" we have a genuine first step in the process. It is a step of finite duration and not a mere durationless boundary. At the same time, it is not a step in which the paradox can break out because this step is not realized via its successive parts. All the parts come at once.

---

12. ibid., p.179.

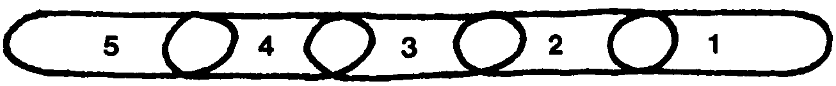
It is as if my diagram must be resolved into:



This seems then to be a solution, but how can a whole, every part of which is simultaneous with every other part, be called a "duration" or be said to involve a "definite lapse of time"? How would time lapse? Time, according to Whitehead involves parts occurring in succession. If each of these parts is again also a time, it too must involve parts occurring in succession. But this is precisely what Whitehead and James deny. They ask us to conceive of a step of finite duration in which nothing is earlier and nothing is later. I find this rather difficult.

If the parts consist of simultaneities, and these parts all occur at once, we are led to the difficulties associated with Bergson: how can change occur? To say that the parts consist of simultaneities and they occur in succession is nonsense. If the parts consist of successions and they occur in succession, Zeno's paradoxes arise. To say that the parts consist of successions and the parts occur simultaneously is strange, but is something like this that they mean.

Starting again from the premise that conceptual thinking gives a positively false picture of the flux, James is impressed by the consideration that in the time-flow each moment must be absolutely next to the moments before and after; this utter nextness is caused by a kind of "penetration". No part there is not really next to its neighbours; which means that there is literally nothing in between; which means again that no part goes exactly so far and no farther that no part absolutely excludes another, but that they compenetrate and are cohensive ..."[13] "Perception changes pulsewise, but the pulses continue each other and melt their bounds."[14] This again, seems to assume the serial order which has caused all the problems, an assumption which we now think should be abandoned.



Every present moment is therefore a fusion of past and future, and, if past and future are composed of moments similar to present moments, then all moments of time appear to be completely fused or telescoped together.

Does this notion of "compenetration" help? It means that there are no barriers between moments which have to be crossed, and so no possibility of Zeno using the paradox of the Arrow or

---

13. James, A Pluralistic Universe, p.271.

14. James, Some Problems of Philosophy, p.87.

the Moving Rows to show that motion is impossible. It also means that there can be neither a first nor a last moment, for "no part ... is not really next to its neighbours." This seems to imply that each moment must have a neighbour on either side, a neighbour with which it "compenetrates". If this is a correct deduction, then it follows that time is infinite. This is because there can be no first moment (it would lack a previous neighbour) nor a last moment (it would lack a subsequent neighbour). This may or may not be a problem; I do not think, however, that it is of any consequence to Zeno.

Does "compenetration", which overcomes the problem of barriers between moments, exclude the possibility of divisibility? It is the notion of divisibility into segments which generates the paradoxes, yet I fail to see how the notion that any moments "melts" its bounds with another moment guarantees its indivisibility, and hence its immunity from Zeno's attack. A possible means of escaping this is to posit a more complex degree of "compenetration" such as is shown below.

Here, irrespective of what happens to any one moment, e.g. the diagonally striped moment, examination of the diagram will show that there is still a compenetration which allows a move from moment to moment skirting the diagonally striped moment (which, for the hypothesis, is undergoing a "Zeno" attack). However, it is not a "compenetration" with its neighbour: James has said categorically that "No part there is not really next to its neighbours." There is no mention of the "next-neighbour-but-one."

I am not altogether sure that rendering the notion of "compenetration" even more complex than James wants it to be will guarantee the indivisibility we look for, and I think we shall abandon the notion of serial order as being more trouble than it is worth.

Let us turn instead to Whitehead who is altogether deeper and more fruitful than my interpretation would seem to suggest.

In common and everyday usage there is hardly any definiteness as to what may be regarded as a single state or time which is referred to by the word "now". No matter how small the moment is that we choose, there is always the possibility of change or process within it. This leads directly to Zeno's problem for we cannot get from one part of the process to another without first having covered all the infinitely many other parts in between the states in question. However, if we are able to reduce our moments of time to be so small as to admit of no change or process within them, then, again, we are left with Zeno's problem: how can events and processes be made up of elements which have no duration, or, how is it possible to pass through an infinity of intermediate states?

Whitehead[15] jettisons these antimonies by rejecting the notion that things are made up from parts, (atoms, molecules, or whatever),

---

15. How powerful such a seemingly simple conceptual shift can be!

and that there ultimate "bits" from which the universe and things in it are constructed. It is this "building up" which generates the paradoxes and Whitehead proposes to offer something far more radical about what a "part" is.

I intend to begin by investigating his concept of temporal parts, moments in time. Normally the mathematical presentation of time is of a continuum which can be divided without limit. If we imagine this time-line, then we can also imagine the "points" of such a line as instantaneous moments in the line. Whitehead asks if there is anything in nature to correspond to such an instantaneous moment and his answer is "No". There are ways of deriving the concept of a dimensionless point in space or an unenduring moment in time but these are not derivations from nature: this was Zeno's mistake.

The world, according to Whitehead, is a fabric of interconnected and interdependent events: larger events have smaller events as constituents.[16] The least unit of the real he calls an "actual occasion" which is, says Whitehead, "the limiting type of an event with only one member."[17] Since they are the least units of reality, actual occasions do not materialize gradually; they come into existence all at once or not at all. He quotes James to explain his view: "Either your experience is of no content, of no change, or it is a perceptible amount of content or change. Your acquaintance with reality grows literally by buds or drops

---

16. This cuts across my Sam Snead example.

17. Whitehead, Process and Reality, p.113.

of perception. Intellectually and on reflection you can divide these into components, but as immediately given, they come totally or not at all."[18]

As a result we can say that units of temporality come in minimal but extended chunks. Extended because, were there no extension, there would be no change and therefore no temporal unit. The abstract "time" which we drive from temporality is indefinitely divisible: the temporality from which it is derived is not.

However, this must inevitably lead to Zeno's paradoxes: as soon as the notion of "chunks" of time is considered, the problem of how we get from one to the other is raised.

But we can reinterpret "larger events have smaller events as constituents". Whitehead does not mean in this that "Larger events are built up from smaller events", but "From the notion of larger events we can get the notion of smaller events, eventually arriving at the "moment"."

The "moment" seems to be a sort of limit of natural relations and gives "sets of physical properties ..... (which) should not be regarded as in nature, that is, they should not be regarded as features of any actual event."[19] The moment provides us with an "abstractive set", and nature cannot be built up out of abstractions. We get the notion of abstractive elements by

---

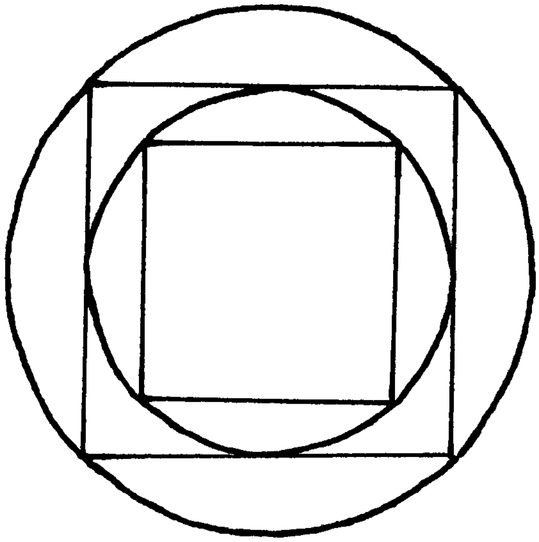
18. ibid., p.81.

19. Nathaniel Lawrence, Whitehead's Philosophical Development, (Berkley, 1956), p.81.

considering the relation between abstractive sets, especially the relation of "covering":

> I define covering as follows: an abstractive set p covers an abstractive set q when every member of p contains as its parts some members of q. It is evident that if any event e contains as a part any member of the set q, then owing to the transitive property of extension every succeeding member of the small end of q is part of e. In such a case I will say that the abstractive set q "inheres in" the event e. Thus when an abstractive set p covers an abstractive set q, the abstractive set q inheres in every member of p.[20]

This is very dark, but Lawrence uses the following diagram as an instance of "covering" sets. "The circles represent one set, the squares represent another. They converge to a common centre."[21]



"Covering" is also synonymous with "enclosing":

> ..... consider a set of enclosure objects which is such that (1) of any two of its members one encloses the other, and (2) there is no member which is enclosed by all the others, and (3) there is no enclosure - objects, not a member of the set which is enclosed by every member of

---

20. Whitehead, The Concept of Nature, (Cambridge, 1923), p.83.

21. Lawrence, op. cit., pp. 82-83.

> the set. As we pass along the series from larger to smaller members, evidently we progress towards an ideal simplicity to any degree of approximation to which we like to proceed ..... the series is a route of approximation.[22]

This seems to me to lead to a possible problem which I will consider later.

"Thus an abstractive element is the group of routes of approximation to a definite intrinsic character of ideal simplicity to be found as a limit among natural facts."[23] A moment is therefore "an abstraction of all nature at an instant."[24] To try to clarify this, one further quotation from Whitehead can be cited: "A moment is a limit to which we approach as we confine attention to durations of minimum extension. Natural relations among the ingredients of a duration gain in complexity as we consider durations of increasing temporal extension. Accordingly there is an approach to ideal simplicity as we approach an ideal diminution of extension."[25] The main property of a moment, its instantaneousness, is "a complex logical concept of a procedure in thought by which constructed logical entities are produced for the sake of the simple expression in thought of the properties of nature."[26] The temporal continuum, then, if I

---

22. Whitehead, The Aims of Education, (London, 1929), p.210.
23. Whitehead, The Concept of Nature, p.84.
24. Whitehead, Principles of Human Knowledge, p.112. There is a distinct hint of question-begging in this quotation.
25. Whitehead, The Concept of Nature, p.57.
26. ibid., pp. 56-57.

have penetrated the mists correctly, is not made up of moments at all. We derive, the notion of "temporal part" from the notion of "temporal whole."

What Whitehead has to say on the notion of "point of space" will shed some illumination on the above. His work on this problem springs from the question "How can a point be defined in terms of lines?" This clearly illustrates that the motivation has once more been turned on its head. The notion of lines being constructed from points is what leads to Zeno's paradoxes, but, as Whitehead thinks that linear reals are closer than points to the ultimate physical existents,[27] lines are ontologically prior to points.

Whitehead, as we saw, pictures nature as an interrelated structure of events with each event possessing volume and duration and having to each other the part and whole relation of extension.[28]

The notion which Whitehead uses to enable us to arrive at instants applies also to ideal points, convergence with diminution of extent. Starting with some large enough event, e.g. the coronation of a monarch, we can analyse it into a convergent series of successively smaller events "rather like the children's toy consisting of boxes each fitting the other."[29] The difference is that, unlike the toy, there is in the case of events no terminal

---

27. Wolfe Mays, <u>Whitehead's Philosophy of Science and Metaphysics</u> p.42.

28. ibid., p.53.

29. ibid., p.53.

event, only a fictional limit.[30] In my coronation example, the successively smaller events could possibly read something like this:

1. Grand panoply of coronation.
2. Spectacle of military personnel in coronation.
3. Spectacle of mounted soldiers in coronation.
4. Behaviour of mounted soldiers' band in coronation.
5. Behaviour of percussion instrumentalists in mounted soldiers' band.
6. Behaviour of Glockenspiel player in band.
7. A Glockenspiel.
8. etc.

As this abstractive set converges, there is a progressive diminution in the extent, both spatial and temporal, of the events considered, so that we finally arrive at the ideal of an event, so restricted as to be without extension in space and time. This ideal restricted event is an "event-particle" or a "point-flash".

Whitehead on Euclid is particularly revealing on this: "Euclid has expressed for all time the general ideal of a point, as being without parts and without magnitude. It is this character of being an absolute minimum which we want to get at and to express in terms of the extrinsic characters of the abstractive sets which make up a point. Furthermore, points which are thus arrived at represent the ideal of events without

---

30. Whitehead, The Concept of Nature, p.79.

any extension, though there are in fact no such entities as these ideal events."[31] Euclid defines planes, surfaces and volumes directly or indirectly in terms of points: Whitehead begins with experienced events and ultimately defines points in terms of events.

For Whitehead, the "point-flash" or "event-particle" is a synonym for "point". This is so because "each event-particle is as much an instant of time as it is a point of space. I have called it an instantaneous point-flash."[32] This "point-flash" is an ideal event and we "must not think of the world as ultimately built up of event-particles."[33] "... the point-object in time and the point-object in space, and the double point-object both in time and space, must be conceived as intellectual constructions, although they are in space and time. The fundamental fact is the sense-object, extended both in time and space, with the fundamental relation of whole-to-part to other such objects, <u>and subject to the law of convergence to simplicity as we proceed in thought through a series of successively contained parts</u>."[34] A close connection between space and time is here readily discerned.

The points of instantaneous space, the name he gives to the space discovered in experience,[35] are event-particles and these

---

31. ibid., pp. 85-86.

32. ibid., p.173.

33. ibid., p.172.

34. Whitehead, <u>Aims of Education</u>, p.205. (my underlining)

35. Whitehead, <u>The Concept of Nature</u>, p.177 "..... the space which we see as we look about is instantaneous space."

event particles are ideal limits to events. However, he also says that these event-particles are abstractions, but abstractions that are truly in nature.[36]

Before we attempt to unravel the extreme difficulties uncovered in this examination of James and Whitehead, let us recapitulate the problems which have arisen. We have been told that our acquaintance with reality "grows by buds of perception" and yet "arrives simultaneously or not at all"; we have been told that points and instants are intellectual constructions and yet are truly in space and time; we have been told that we arrive at "part" from the notion of "whole" with no indication of how we identify something as being a "whole", if perceptions arrive simultaneously, then what is supposed to happen after their arrival? Are they to be replaced by a new set of perceptions, and, if so, how do we cross the spatio-temporal interval from one set of perception to the next one without invoking Grünbaum's point-events discussed earlier?

No-one could be blamed for abandoning Whitehead in the face of these difficulties, but, despite the logical contradictions which appear to permeate his writings, an unravelling of these problems will produce an incredibly rich, powerful and convincing argument.

Whitehead has spoken of "focusing": "There is the set with its members growing indefinitely smaller and smaller as we proceed in thought towards the smaller end of the series; but there is no

---

36. ibid., p.173.

absolute minimum of any sort which is finally reached."[37] We will use this as our springboard from which we make the leap into his apparent inconsistencies. The problem which arises here is "How do we get to the big end? Where do we get the original big set from?

There is no problem in getting to the big end; to say that what I perceive is part of something greater, which, in turn, is part of something greater still, is an intellectual problem and of no concern whatsoever to Whitehead. Granted, it is what Bergson appears to be saying, and, were Whitehead to be saying the same thing, he would be sadly wrong. Individuation is a fact of experience. To understand more fully Whitehead's argument let us ask what is going on when I open my eyes to gaze upon something. I see, initially, a panorama, visual field, perspectival field, or what Whitehead calls "instantaneous space". There is no intellectual struggle involved here in deciding where I start "collecting" the information in my visual field; the field is presented in the act of intercourse between me and the world. And, if it does not come at once, then it does not come at all. If it does not come at all, then there is something wrong with my perceptual apparatus. Note that I am not building up the panorama by looking at bits and pieces and then engaging in some act of visual or intellectual synthesis. The visual field at the big end is given all at once or not at all.

---

37. ibid., p.172.

How then can we account for our acquaintance with reality growing by buds of perception? There are two things which Whitehead could have in mind, one more unlikely than the other. To deal with the more unlikely one first: I can make the visual field grow by moving my eyeballs, turning my head, my shoulders, and so on. There is a degree of intellectual construction going on here in that part of my original visual field (say the rightmost part) is replaced by a new leftmost part as I swivel my eyes in that direction. By "holding on" to the rightmost part in my memory, I can make it, in a sense, propinquitous with the new visual field. Hence, I suppose, I could say that my perception has "grown".

However, the notion of "focusing" again provides a more adequate explanation of the difficulties stemming from our acquaintance with reality growing by buds of perception. I am presented with my visual field (as we saw, simultaneously or not at all) and can now begin to operate on it. I can make "cuts", as it were, in my visual field. For example, I can isolate red things, abandon angular things, and so on. Presumably, on a more sophisticated conceptual level, I can go straight into isolating cats, plates, and so on. This process, although labouring to write of, is extremely quick.

If I did not know what a cat was but had some definition, e.g. an object with four legs, small, covered in fur which makes mewing noises and tends to change shape with extreme rapidity, I would eventually focus on the cat. Initially, I would perhaps focus on the dining table, the occasional table, the baby's toy cat and the real cat, because they all had four legs. I would then abandon the dining table as being large, the occasional table as not having any fur, the toy cat as not being able to change shape rapidly, and am therefore left with my small cat which tends to rush around at high speed, mews at passing tomcats, and so on.

I have started with the visual field of instantaneous space, yet my perception has grown. The buds of perception are not in a series, one after the other, but a focus.[38]

Zeno has asked how it is possible to pass from one momentary state to another, if one must first pass through an infinity of intervening states. This is clearly based on an assumption of space and time forming some sort of serial order. Whitehead has shown that this can be rejected by interpreting moments and parts in terms of his "abstractive sets". He has perceived infinite classes of events fitting into each other in the manner of Chinese boxes which do not converge to any minimal event (the point is a "fiction") although their properties converge to a limiting definiteness.

These parts are in nature, (but we have to look for, or "focus" on them, but nature is not constructed from these parts). We construct the parts by operating on our perception of instantaneous space.

With this we have arrived at a position where we can tackle the paradoxes.

---

38. Consider: "Here! If you look more carefully you'll see that the cat has a little white patch on its timmy! We look, and, "Bless me, so it has!"

Part Five

Conclusions

Chapter Fourteen

Zeno's Achievement

"There are thousands who can see that a statement is nonsense and yet are quite unable to disprove it formally."

(Lichtenberg)

Philosophy is one of the human intellect's most destructive weapons because there is very little which can survive sustained critical philosophical analysis. Philosophy would indeed be a petulant and sterile discipline were its only object to be a destroyer. To have any worth, philosophy must also be willing to construct. I have shown how many refutations of Zeno have proved unsatisfactory (at least, to me) and it would be churlish and not a little cowardly if I failed to hazard a refutation of my own. I know, of course, that the philosopher puts up ideas only to have them tilted at, but that should be no deterrent. If one's ideas cannot be knocked down then a triumph has been achieved. Modesty forbids me to expect that these conclusions will be a triumph, but I hope that what I have to offer in refutation will not collapse upon the slightest scrutiny.

Refutations in the traditional style where, by a terse series of supposedly cutting deductions, Zeno is reduced to tatters, seem to me to be utterly pointless. It indicates a degree of presumptive showiness which is altogether unwarranted against an opponent as magnificent as Zeno, and also runs the risk of merely indicating something which we all knew anyway: that Achilles will catch his tortoise and that the Runner can run home for his supper. I grant that a proof of these items of everyday knowledge may provide reassurance to a certain sort of philosophical mind, but, unless there is an <u>explanation</u> of how Zeno has worked his wizardry, a refutation of this kind is rather barren.

Our initial conclusion would seem to be that we are stuck with the Stadium Paradox and the Paradox of the Moving Rows every other paradox having been refuted. This conclusion, however, would be inappropriate because there is a sense in which Zeno could be said not to have believed in his deductions: surely an Eleatic Monist is not asserting that, once monism is adopted, the acolyte no longer encounters the world of everyday life. There is no evidence to suggest that Zeno climbed a tree and forgot about the world: he still had to preserve himself against careering chariots, thieves and robbers, and deliver lectures to lots of people at the Grand Panathanea. How then are we to react to Zeno's deduction that "Motion and Plurality are impossible?" Grünbaum[1] suggests that this is not Zeno's aim at all: the paradoxes "were designed to show that the science of geometry is beset by a paradox and that any attempt to provide a mathematical description of motion becomes ensnared in contradictions." This helps to explain the seemingly aberrant behaviour of Parmenides and Zeno, because a "translation" of "Motion and Plurality are impossible" should therefore be rendered "Any attempt to explain Motion and Plurality will result in paradoxes." But, what sort of explanation is it which Zeno has in mind? Clearly there seems to be little which is problematic in saying "I was at the front entrance of the zoo at midday: I was at the back entrance of the zoo at 2.00 p.m. I must therefore have walked through or round

---

1. A. Grünbaum, "Modern Science and Refutations of the Paradoxes of Zeno", in Salmon, Zeno's Paradoxes, (New York, 1970),p.164.

the zoo." Explanations of this sort are not what Zeno has in mind. We have seen, however, that when we try to explain this explanation in his terms, something very strange happens. His conclusion is that any adequate explanation of the behaviour of objects of ordinary experience, obtained by a conjunction of mathematics and common sense, must always result in contradictions.

This is revealed in his strategy. Let us call the position he defends (Sx), i.e. there is but a single x. The position he attacks is the pluralist hypothesis that things are Many. From it he will try to deduce that there is at least one x which is qualified in the same respect by opposite predicates; in this case that Reality is continuous and is composed of atomic points and instants. The pluralist hypothesis therefore will contain a contradiction and must be abandoned.

Those who assert a plurality are led through specific attacks on the conjunction of mathematics, i.e. the assigning of numbers as ranking co-ordinates derived from the mathematical continuum, and common sense, i.e. expressing our intuitions of space and time in a natural language as spatial and temporal orderings of cages in a zoo, runners in a race, etc. What Zeno has shown is that when mathematical co-ordinates are assigned to the spatial and temporal orderings of individual physical objects, then contradictions and infinite regresses must result. Therefore, the paradoxes have revealed an inherent inconsistency which results if mathematics is employed as an explanatory framework for common sense experience.

In order to see fully the bifurcation which Zeno has achieved, we must ask what precisely is meant by the "common sense way" of describing and explaining the world and relationships between things and the "mathematical way". "Common Sense" is the expression employed when we describe and explain in a natural language our everyday experience[2] of phenomena such as shopping in the supermarket, flying in an aeroplane, etc. Mathematical descriptions and explanations are not like this at all. For example, if I take one button and put it together with two buttons I get three buttons. One reason for this is that I have learned that the number one plus the number two equals the number three. But, the numbers one, two and three, together with the notions of adding and equalling, are not properties of the buttons and numbers are not flat round things which can have thread pulled through them, nor can you fit a number into a buttonhole. So we can see that some properties of numbers pertain to objects while others do not: and it is this characteristic of numbers that causes the dilemmas highlighted by Zeno.

I have stated that the problem has arisen because not all mathematical characteristics apply to objects, this must be clarified. Arithmetical counting and geometrical measuring as

---

2. I thereby exclude peculiarities such as ghosts, bogles, whigmaleeries, and so on, which some people claim to have experienced.

mathematical procedures are applicable to objects, because we can add things together or measure their magnitude. However, some mathematical properties clearly do not pertain to physical objects, e.g. the geometrical point and transfinite cardinality.

We count three buttons by adding one button plus one button plus one button equals three buttons, but we cannot count one button plus one orange plus one parrot as adding up to the same thing unless we disregard the particular differences between them and focus only on the numbers assigned them. When numbers, as ranking or ordering co-ordinates, are assigned to objects, we imply that the mathematical properties or relations among the co-ordinates reflect and describe those properties found among the objects which we mean to describe and explain.

This brings a very curious feature to light, and it is here that the relevance of Zeno manifests itself. When we assign numbers to a single physical object, e.g. "one button", the physical objects are said each to be a single thing, and each number that can be assigned to a physical object is itself a single number ("one"). Both the physical object and the number can be called a "one". But, when we say that something is "one ", e.g. that there is only one Saturn which is one of the planets in the Solar System, if we do not stipulate that we mean "is one by perception" (the only one with rings) or "is one by counting", a certain amount of confusion is bound to arise.[3] The same

---

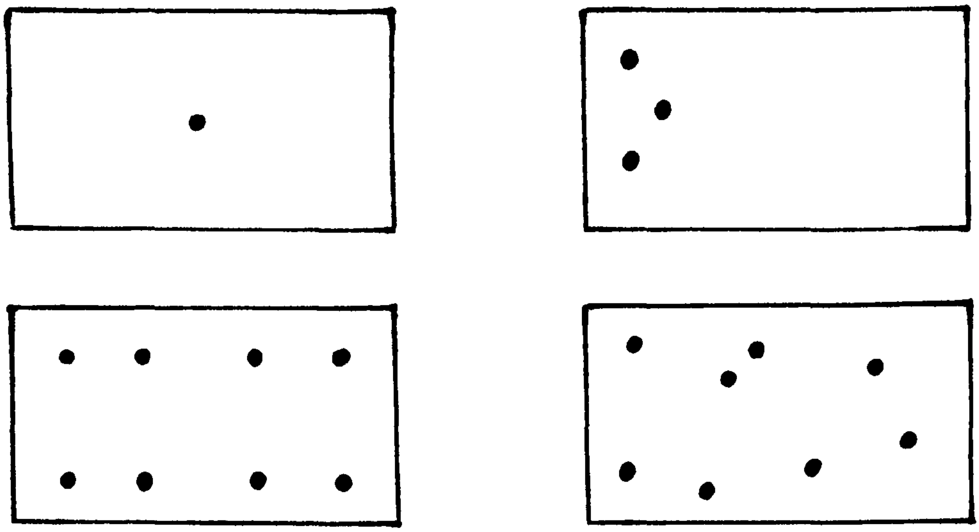
3. This was found in Murphy, Zeno's Paradoxes: Tasks, Super-tasks and Undecidability, M.A. Thesis, 1975, California State University at Long Beach.

confusion arises with regard to "many".

Philosophically, there is a difference between saying that something is (a) "one (many) by counting", and (b) "one (many) by perception but what this difference is is not always easy to say. Suppose while standing in a football stadium I ask my companion for the number of spectators he thought were present. His reply might be of the ordinary kind: "There are many (plenty of, lots of) people." If I then ask how he knows this, he would probably say: "Well, I looked." If I then ask him how many referees there were, he would correctly respond: "One," and if I ask him how he knows, he would say "I looked."

Usually, when we use the terms "one" or "many" as answers to the question "How many?", we assume tacitly that the answer is derived from some mathematical counting procedure, but common sense tells us that, in order to know whether something is "one" (the number of moons circling Earth) or "many" (the number of leaves on the tree in my garden), we need only look and not count at all! If I protested to my football companion by saying that his answers presupposed that there was some definite number which could replace them and hence to presuppose that his "There are many people" has been reached by some (unconscious) counting procedure, he would respond, "Hold on a minute: I know when I'm counting and I know when I'm looking. When I look at the crowd and see many people, I'm not counting; I'm looking." Look at the following diagrams to discover how many dots each box contains.

Is the answer found by looking or counting?



The difficulty which Zeno highlights with "one" and "many" is that when we say something is "one (many) by nature", our context is common sense (and derived from looking) and not mathematics. This is because mathematics does not study the nature of things, only number. Simply stated, unless the context in which the words "one" and "many" are used is made clear, a confusion about whether it is their nature or their number which we are talking about will arise. Though the specification of context as mathematical or common sense can clear up the difficulties surrounding the use of the terms "one" and "many", it does not give an answer as to whether we arrived at either term by a mathematical procedure or by looking. A solution to this problem I leave to someone else.

I have suggested that Zeno's conclusion is that the conjunction of mathematics and common sense must always result in contradictions and have given an example of the problems with "one" and "many". Benacerraf[4], in his discussion of Thomson's lamp agrees that "we have what appears to be a conceptual mismatch. Sequences of tasks do not exhibit the characteristics of sequences that lead themselves to proofs of infinity." This mismatch seems to me to result from the introduction of the mathematical notion of the limit of an infinite sequence as if it were interchangeable with the common sense notion of "completion." What Zeno has shown is that (a) the mathematical limit and its attendant infinite sequence, and (b) the common sense notion of "completion", are not freely interchangeable with one another.

Let us examine the mathematical notion of the limit of an infinite sequence to see why this should be. At one time it was thought that the principle, "What is true up to the limit is true at the limit" was correct. But, as we have seen, there need not be a continuity between the properties of an infinite sequence and its limit. Consider, for example, the infinite sequence of

---

4. Benacerraf, loc. cit. See also Black, "Achilles and the Tortoise", loc. cit., p.101: "We create the illusion of the infinite tasks by the kind of mathematics that we use to describe space, time and motion."

regular polygons and its limit, the circle. Each of the polygons has two identifying properties, straight lines and angles, which the limit shape, the circle, lacks. Thus, one of the identifying characteristics of the mathematical notion of the limit of an infinite sequence is that some of the properties of that infinite sequence <u>may vanish</u> at its limit.

But, if we consider the common sense notion of "completion", we can safely say that what is true of each component as regards its identifying properties, will be true upon completion. That is, none of the identifying properties of the objects acted upon, unless transformed,[5] will disappear upon completion. For example, if I am writing a letter, on the completion of my task, none of the identifying properties of the letter, (words, colours, salutations, etc.), will have vanished. If they had disappeared, my task would be <u>incomplete</u>. Hence, there must be a continuity of properties supposed by the common sense notion of "completion".

---

5. For example, in baking a cake, sugar, eggs and flour are transformed. This transformation is different from the polygons being "transformed" into the circle. I could examine a cake and deduce that it was transformed from eggs, sugar, etc. Could I examine a circle and deduce that it was transformed from a sequence of polygons?

Therefore, the two different senses of completion, mathematical and common sense, cannot be freely substituted, one for the other, as if they were identical.

As well as this, there is yet <u>another</u> problem highlighted by Zeno which accounts for our conceptual mismatch: as well as the problem of presupposing a continuity of properties at the limit of an infinite sequence of tasks, we can develop a <u>further</u> comparison between the mathematical limit of an infinite series and the common sense notion of "completion".

Suppose (a) that I was commanded to draw a line from 0 to 1 by a single stroke thus:

This is a finished task.

Now suppose (b) that I was commanded to draw a line from 0 to the halfway point, from this point to the threequarter point, etc., thus:

This is an unfinished sequence of finished tasks.

With the first command (a), there is only a single stroke performed, whereas the command (b), there is an infinite number of strokes performed. How do I know that this is the case? I can see it! Confusion arises in supposing that to complete (a) is to complete (b), and in supposing that to apply the mathematical limit to (b) yields (a). The fallacy is due to the dual usage of the expression "a line is drawn from point A to point B". Let us call this expression X. In describing (a) and each individual stage of (b), the expression X means:

X = a line whose end points are (A,B),

(i.e. a stroke terminating at A and B).

When we say that a stroke from 0 to 1 is also a stroke from 0 to ½, ½ to ¾, etc., we have changed the meaning of the expression X, for stroke 0 to 1, i.e. command (a) does not terminate at ½, ¾, ... Therefore a stroke drawn from 0 to 1 cannot be said to consist of the strokes from 0 to ½, ½ to ¾, etc., because what constitutes a stage of operation (b), the termination of a stroke at various points, is precisely what is lacking in the first operation (a). Now, if operation (b) is a description of a super-task, and operation (a) is a description of a normal task, then (a) and (b) are not identical, in the sense that (a) is completable while (b) is not.[6]

Therefore it has been established that there are at least two important differences which have emerged. The first difference was that the properties of an infinite sequence are not always preserved at its limit, which the common sense notion of "completion" presupposes. The second was that a description of an infinite sequence of tasks shows at least one subset of members (T), whereas

6. The notion of "identity" which I use is this: two sets are identical if and only if all the members of one set are members of the other. But (b) has a subset of members (T), (terminating strokes) which (a) lacks. Thus (a) and (b) are not identical nor does one imply the other.

a common sense description of the notion of "completion" lacks (T). Hence it is the failure to recognize the non-identity between the two senses of completion that help to support the conceptual mis-match exposed by Zeno. We can therefore say that the mathematical "explanation" of how the Runner crosses the Stadium is inappropriate.

A further failure of mathematics when it is applied to common sense has been exposed by Thomson in his attempts to refute the paradoxes. It will be recollected that he had a lamp whose on-off switch took the values +1 when the lamp was on and -1 when it was off. We can apply this feature to Bostock's Bouncing Ball to make the failure even clearer. Let us suppose that the ball is travelling either up or down: we can represent the up-down feature by the sequence (+1 -1 + 1 - 1 ...). At the time of completion of the ball's bouncing, when it is supposed to be no longer moving, is it up or down, i.e. +1 or -1? Common sense would tell us that it was down, i.e. -1, but the mathematics of infinite series tells us that the falling ball always returns in the opposite direction. How then can the ball stop moving? If there is a lack of continuity between an infinite spatial or temporal decreasing sequence and its limit, then it is undecidable which value obtains when the ball is no longer in motion.[7] But this is just the sort of information needed to give a common sense description of the ball's performance, an account of the completability of the infinite number of bouncings the ball must

---

7. Thomson suggested the value (½), reminiscent of the Grand Old Duke of York.

perform. If mathematics cannot decide whether the ball is up or down at the completion point, then there is something lacking in mathematics. In fact, mathematics could not then decide whether the ball does complete an infinite number of bouncings.

What Zeno's Paradoxes have shown is therefore not that Motion and Plurality are impossible: what they have shown is that a mathematical analysis of common sense experiences will be fatally flawed in that it leads to paradox. The language of common sense cannot be translated into the language of mathematics.

This, however, is not all that emerges. The Calculational refutation which we have encountered illustrates a motive which has pervaded (or even defined) the whole of modern philosophy: a desire for definiteness in our conceptions and our beliefs. Russell's defence of mathematical analysis of physical continua, against older "intuitionist" conceptions, is an avowed instance of that motive. This, however, results in a genuine question being obscured by a simplistic facade. Zeno's question has to do with the legitimacy of enshrining plurality and motion as fundamental categories of thought. Calculational refutations of his paradoxes commit ignoratio elenchi: they are addressed to the very different question of whether we can devise a formalism for handling these notions in an operationally determinate way, once these categories are admitted as intelligible. What calculational approaches do is challenge the traditional metaphysical distinction between categorial-level and operational-level concepts. They assume, for example, that the meaningfulness of the point-aggregate concept of extension is directly attested by the existence of a physically applicable formalism built around that concept, but this claim depends on denying that there is an independently meaningful concept of extension against which the adequacy of this formal one can be measured. This, as we saw, is precisely what Russell, Ushenko and Grünbaum undertook to prove in their efforts to give the calculational solution a philosophical grounding. They did this by making explicit and consciously defending the Pluralist presuppositions without which there could be no theoretical justification for taking their formalism as self-sufficiently and exclusively meaningful.

This creates problems for the interpretation of our ordinary experience of concrete phenomena, because what is underpinning their argument is the thesis that we cannot claim to have an idea of any complex whole, unless it is the literal product of an actual synthesizing of antecedently given separate parts. (These parts must be available beforehand and therefore exhaustively discoverable by subsequent analysis.) We have to abandon the notion that we come to have ideas of complexes given by wholesale assimilation in casual non-synthetic ways. This reduction entails quite directly that, whether we are consciously aware of it or not, the idea of a whole is the idea of a very specific set of parts, the assemblage of which literally gives us the notion of the whole, insofar as we have a determinate contentful idea at all. However, our study of Zeno has shown that words such as "one", "many", "several", etc. are used through wholesale assimilation, (I had a couple of gins in a pub in which there were many people and one barmaid.)

As well as this Grünbaum misses the point which raised the issue in the first place. His kind of refutation is an essentially quantitative resolution to a difficulty which is essentially qualitative. It concerns a generic contrast rather than a matter of degree of accomplishment in the task of analysis or synthesis.

That generic contrast is between two very different sorts of internal structure which may characterize wholes: connectedness versus juxtaposition.[8] Procedures which are designed to reveal the juxtapositional type of structure in which all relations are external to the constituents are logically excluded from giving us the integral type that is normally presupposed in those notions of concrete complexes on which analysis is done. It should also be noted that this pattern of explication, as well as Zeno's monism, leaves the phenomenal world in the status of utterly inexplicable appearance.

Philosophers are extremely loath to say what Philosophy is, but few, I imagine, would cavil at the suggestion that Philosophy is concerned with a search for certainty. This may be rather a rash generalization about Philosophy but it certainly seems to have been the inspiration of many philosophers.

---

8. These are the very structures which emerged in the discussion of James and Whitehead where we saw the problems which emerge if we consider event-particles, etc., to occur in serial order. Whitehead's notion that parts arrive later (in a sense) than wholes and that they compenetrate is extremely damaging to the juxtapositional type of structure advocated by Grünbaum.

The paradigm held up to Philosophy has often been mathematics, and it seems to me that this lies behind the work of philosophers such as Russell, Ushenko and Grünbaum. If propositions about the world can be reduced to mathematical or quasi-mathematical propositions, then the certainty to which we aspire can be attained. Zeno, however, has shown that this certainty is not so easily to be gained, because mathematics is inadequate to explain processes in the world.

The paradoxes, then, are based on certain mathematical procedures being illegitimately allied to common sense experiences, and draw their seductiveness from our mistaken belief that these procedures have real world counterparts.

Chapter Fifteen

A Mathematical Response

"To see a World in a Grain of Sand,
And a Heaven in a Wild Flower,
Hold Infinity in the palm of your hand,
And Eternity in an hour."

(Blake)

It is clear to me that Zeno has made his point: standard mathematics is unable to cope with the logic of his arguments. This enquiry has shown how the Cantor-Russell analysis of the continuum removes the Paradoxes of Plurality,[1] while elementary Relativity Theory can reduce the Achilles Paradox to the second version of the Stadium Paradox.[2] It has also shown how both the first version of the Stadium Paradox[3] and the Paradox of the Flying Arrow[4] can be rejected. However, this enquiry, as yet, has failed to show how either the second version of the Stadium Paradox or the Paradox of the Moving Rows can be refuted using standard mathematics. Are we to give Zeno the laurel wreath of victory and then retire with our confidence in the majestic edifice of mathematics shattered? This would, indeed, be an unmitigated disaster, but if we are led to this precipice then I suppose we must be prepared to jump.

Mathematics, however, can respond, even though in a highly controversial manner. Vlastos[5] gives a hint charged with tremendous interest. Each sub-interval of the athlete's run from A to Z, i.e. AB, BC, CD, etc., he calls a Z-run. By making these Z-runs,

---

1. See Chapter Seven.
2. See Chapter Eight.
3. See Chapter Nine.
4. See Chapter Twelve.
5. G. Vlastos, "Zeno's Race Course", in Journal of the History of Philosophy, (4), 1966.

the athlete can get so close to Z that "no arithmetical sense could be given to the statement that he has not reached Z."[6] His reason for this statement is that "If no quantity could express the difference, δ , between the sum of nZ-intervals and AZ (for δ could be made smaller than any ϵ that might be chosen), what mathematical sense could there be in saying that there is a difference?"[7]

What Vlastos is hinting at is the infinitesimal, a highly controversial little thing whose controversy arises from the paradoxicality of its definition: an infinitesimal is a number that is infinitely small yet greater than zero, i.e. it is not zero, but is, nevertheless, smaller than any fraction. Archimedes asserted that there is no such thing as the infinitesimal because even a small non-zero number (no matter how small) will, if it is added to itself a sufficient number of times, become a large number. This property, that the number grows when added to itself, is called the Archimedean property of real numbers. An infinitesimal, if it existed, would be a non-Archimedean number: it would be a number greater than zero but which would nevertheless remain less than 1 no matter how many times it was finitely added to itself. This seems so counter to the common sense that Archimedes formally denied[8] the existence of the infinitesimal.

---

6. ibid., p. 104.
7. ibid., p. 104.
8. See P.J. Davis and R. Hersch, The Mathematical Experience, (Penguin, 1981), p.239, for an account of this and of other mathematicians such as Nicholas of Cusa who used infinitesimals in their calculations.

Let us return to the series which has generated the major part of this thesis:

$$1/2, 1/4, 1/8, 1/16, \ldots$$

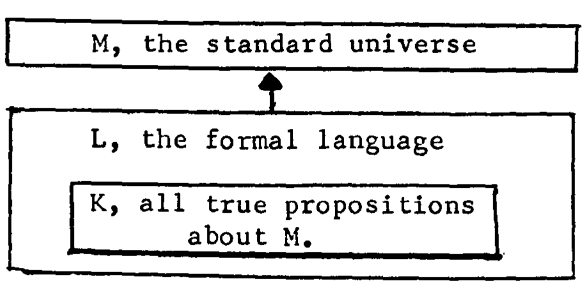
We can couch this as "there will always be a fraction smaller than the last fraction so far in the series". This contains the notion that, no matter how far we go along the series, there is still further to go. In the common sense, we would ascribe an endlessness to a series such as the above: indeed, we have seen that it is a queer sort of endlessness in that, no matter how far we go along the series, we are no closer to the end of the series than when we started.

However, it has been shown in Chapter Seven that the defining characteristic of an infinite series or class (in this case synonymous) is <u>not</u> that it is impossible to count its members successively in a finite time: the defining characteristic is that it is a class (or series) which can be put into a one-to-one correspondence with part of itself. This carries the notion of the infinite series existing, in some sense, <u>as a completed set</u>. In other words, it seems that by using the new definition of "infinity" analysed in Chapter Seven, we can (somehow) transform "there will always be a fraction smaller than the last fraction so far in the series" into "there is a fraction smaller than any other fraction."

There seems to be something funny going on here, because I am sure that most people still feel a lot of conceptual sympathy for the "endlessness" kind of infinity, but, if there is a way in which the notion of infinitesimals can be rendered intellectually coherent (by this is meant that any intellectual revulsion that we may feel in the presence of the concept of the infinitesimal is outweighed by the intellectual acceptance brought about by the logical rigour of the proof of the infinitesimal's existence), we clearly have a new and extremely fascinating sort of weapon with which to confront Zeno, and it is just such a rendering which was achieved by the logician Abraham Robinson in a process which he called "Non-standard Analysis".

In understanding Robinson, the first thing to get clear is his distinction between a standard universe and a non-standard universe. This distinction is used to make more precise just what kind of "existence" is implied by the Cantorean definition of "infinity". It also enables one en passant to hang on to the "endlessness" thesis, insofar as what is completed in one kind of universe (the non-standard) is seen to be endless in another kind of universe (the standard).

His investigation begins with the finite real numbers and the rest of the calculus (as known to standard mathematicians), to be called "the standard universe", designated as M. The formal language which we use to speak about M is to be designated L. Any sentence in L is a proposition about M and takes a truth value, i.e. propositions about M are either true or false. The set of all true sentences in L is called K, and we say that M is a model for K.



What this means, if we put it more simply, is that M is a mathematical structure such that every sentence in K (when referring to M) is true. However, this is by definition: we cannot effectively know K (because if we did know all of K, then we would be like an omniscient God in knowing the answer to every possible question in analysis, even those which have still to be formulated). The main thrust of Robinson's argument is that, which M is the "standard" model for K, there are M* which are still "models" for K:

> ... there are objects in M* and relations between objects in M* such that if the symbols in L are reinterpreted to apply to these pseudo objects and pseudorelations in the appropriate way, then every sentence in K is still true, although with a different meaning. [9]

Let us now consider these universes, M and M*, with M the standard universe for mathematics and the calculus; what can we discover about M*, the non-standard universe?

Robinson took the "completeness" theorem of Gödel:

> A set of sentence is logically consistent if and only if the sentences have a model, that is, if and only if there is a "universe" in which they are all true.

To this he allied the "compactness" theorem proved by Anatoli Malcev and Leon Henkin:

> Of a collection of sentences in L, if in the standard universe every finite subset of the collection is true, then there exists a non-standard universe where the entire collection is true at once.

By combination we can deduce that:

---

9. ibid., p.248.

> If every finite subset of a collection of sentences of L is true in the standard universe, then every finite subset is logically consistent. Therefore the entire collection of sentences is logically consistent (because any deduction can make use of only a finite number of premises.) Therefore there is a non-standard universe where the entire collection is simultaneously true.

Let us examine where this is taking us with regard to the series of fractions. Gödel's "completeness" theorem states that the series of sentences

(a) "C is less than 1/2."

(b) "C is less than 1/4."

etc.

is consistent only if there is a "universe" where they are <u>all</u> true. The "compactness" theorem states that if every subset of the form "C is less than 1/x" is true, there is a "universe" (the non-standard) where they are all simultaneously true.

Consider now the following series, relating it to the completeness-compactness combination:

(a) "C is a number bigger than zero but less than 1/2"

(b) "C is a number bigger than zero but less than 1/3"

(c) "C is a number bigger than zero but less than 1/4"

etc.

This is an infinite collection of sentences each of which can be written in formal language L. With reference to the standard universe R of real numbers, every finite subset is <u>true</u>. This is because if we have <u>finitely</u> many sentences of the form "C is bigger than zero but less than 1/n", then <u>one</u> of the sentences (the last) will contain only the number 1/n. For example, consider this series:

(a) "1/5 is bigger than zero but less than 1/2"

(b) "1/5 is bigger than zero but less than 1/3"

(c) "1/5 is bigger than zero but less than 1/4"

(d) "1/5 is bigger than zero and equal to 1/5"

However, if we consider the entire infinite set of these sentences, it is false with reference to the standard real numbers. This is because no matter how small a positive real number C we choose, 1/n will be smaller than C if n is big enough, i.e. there is no last fraction. (This was Vlastos' point.)

However, from the "compactness" theorem we can see that there is a non-standard universe containing pseudoreal numbers R*, including a positive pseudoreal number C smaller than any number of the form 1/n. This is because, in this non-standard world, the entire collection of sentences of the form "C is bigger than zero but less than 1/n" is true simultaneously. That is to say, C is an infinitesimal, bigger than zero, but smaller than every 1/n!

The Archimedean property of R (a real number), that the number grows when added to itself, can be expressed by using an infinite set of sentences of L as follows:

$$C > 1$$

$$C+C > 1$$

$$C+C+C > 1$$

etc.

That is, no matter how small C is, we get to a point where an addition of enough C's will be greater than 1, i.e. for each positive element C of R, all but a finite number of these sentences will be true. There will be a finite number of false propositions of the form "C+ ... > 1", and an infinite number of true propositions of the form "C+ ... > 1". From this it follows that the completeness theorem cannot apply, (the entire collection cannot be true simultaneously).

This is not true of the infinitesimals in R*. In the non-Archimedean world of the infinitesimal, all of these sentences of the form "C+ ... > 1" are false, i.e. no finite sum of C's can exceed 1, no matter how many terms we take. This fact, that the Archimedean property is true in the standard world but false in the non-standard world, "proves that the property (being greater than zero but less than any fraction) cannot be expressed by a sentence of L."[10] This is because infinitely many sentences would be involved!

Remember that our objective is not to prove the existence of the infinitesimal: what we are trying to do is legitimize the notion of the infinitesimal so that it can be meaningfully introduced into discourse. Has this been achieved? I think that it has, because, although forbidden through Archimedean mathematics and intuition by Zeno, we have here a new kind of universe of discourse which seems to be logically tight. From the "completeness" theorem of Gödel and the "compactness" theorem of Malcev and Henkin, there is no conclusion about infinitesimals

---

10. ibid., P.250.

to be drawn except those which Robinson has shown. His non-standard analysis makes the infinitesimal method precise for the first time, transferring it from the realm of instinct and intuition to logic and intellectual clarity. If we now apply the concept of the infinitesimal to the unrefuted paradoxes we are amazed to see how quickly they collapse.

The Paradox of the Moving Rows: the main object of this paradox is to show that what seems to be the smallest part of space or time is not the smallest part of space or time, because it can be divided by two: twice as much happens in one row relative to a stationary row as happens in a third row relative to the stationary row. However, we can now say that the smallest point in space or time is the infinitesimal (which we can prove by the completeness-compactness combination) and that division by two is to perform quite illegitimately, an Archimedean operation on a non-standard number. As Archimedean operations do not transform non-standard numbers the Paradox of the Moving Rows collapses.

The Paradox of the Stadium (second version): let the termini of the racetrack be 0 and 1. The points of the racetrack, by the completeness-compactness combination, can be reduced to non-Archimedean infinitesimals. The finite addition of these infinitesimals, which means that we cannot pass (in the standard universe) from 0 to 1, is an Archimedean operation and therefore quite illegitimate. Yet this is the very thing which Zeno wants us to accept. The intervals of time required by the athlete can, by the completeness-compactness combination, be reduced to non-Archimedean infinitesimals. (A propos Chapter Six, "The Dilemma of Dynamics", the movements of the athlete, swinging arms, etc., can, pari passu, also be reduced to infinitesimals.) This gives a one-to-one correspondence in R* between the infinitesimals of the racetrack and the infinitesimals of the time taken, (and the infinitesimals of the swinging arms, etc.) The traversing of the interval 0 to 1 can therefore be thought of as the addition of an infinite number of infinitesimals, and the time taken as the addition of an infinite number of infinitesimals. Any real number R will not be altered by the addition of an infinitesimal from the universe of R*: similarly, a finite addition of infinitesimals in R* will not lead us to a real number in R. However, the non-standard world of pseudoreal non-Archimedean numbers shows how it is logically coherent for an infinite number of infinitesimals to be added each to each in R* and thus to take us from 0 to 1 in R! The second version of the Stadium paradox has been refuted.

We can, however, apply similar reasoning to the rest of the paradoxes, i.e. non-standard analysis is capable of overturning all of the pertinent paradoxes.[11]

---

11. The Paradox of Space and the Paradox of the Millet Seed are naturally excluded in that they are not instances of the "growing" infinite.

The Paradoxes of Plurality fall because non-standard analysis shows clearly how an infinite number of infinitesimals, when added together, do not give an infinite total. The total arrived at when adding together an infinite number of infinitesimals is reassuringly finite. The Paradox of Achilles and the Tortoise falls because we cannot say that the tortoise is ahead if the interval between the protagonists, by the completeness-compactness combination, is infinitesimal. This is because the addition of an infinitesimal from R* to a number in R will not transform that number. The first version of the Stadium Paradox falls because the first movement is infinitesimal, taking us to a position greater than zero. However, to divide it by 2 is to attempt to transform a non-Archimedean number by Archimedean means. The Paradox of the Flying Arrow falls when non-standard analysis shows us that the place where the arrow is in any moment of its flight is in R*. To argue from this to its being stationary in R is quite wrong.

We have arrived, but what is the price which we have had to pay for our journey? These two antinomies, (the Paradox of the Moving Rows and the second version of the Stadium Paradox), have caused nothing less than "a repudiation of our conceptual heritage."[12] The price has been an insight into how standard mathematics breaks down when confronted with motion; for the common unversed man, a trivial aspect of our everyday existence.

---

12. W.V. Quine, op. cit., p.11.

As well as this, the price contains a suspension of intuition, as logic has remorselessly driven us into a strange and remote mathematical world, a world of numbers which are smaller than any other number, yet the only world which has been found capable of confounding Zeno. Perhaps this is Zeno's greatest heritage: that he leads us to places where we do not want to go and makes us look at our cultural heritage, not with amazement at our achievements, but with an appropriate modesty incurred because this "silly problem"[13] has caused such utter confusion since its creation.

13. See Introduction, note one.

Appendix One

A Reconstruction of Zeno's Paradoxes of Time

"Time is the only true purgatory"

(Samuel Butler)

We do not know if Zeno constructed any Paradoxes concerning Time: none has survived. This is very unfortunate because his view of time is extremely significant. His view, however, can be deduced from the style of the Paradoxes of Plurality and Motion, taken with the deductions arrived at by Parmenides in his poem.

This, then, is how I think Zeno almost certainly ran the lost Paradoxes of Time.

(a) If a period of time is composed of instants, how many instants are there in the period? There must be a finite number because there are just as many instants as there are, and neither more nor less than this. There must also be an infinite number of instants, because each instant will have an earlier and later part, each having duration. The earlier part and the later part will each be instants. The earlier part of each instant will have an earlier part and a later part, each having duration and each also being instants, etc. There are therefore an infinite number and a finite number of instants in any period of time.

(b) If a period of time is composed of instants, do these instants have duration or not? If they have no duration, then any instant added to any period of time will not extend it, for any duration cannot gain in duration by that which has no duration. But if the period of time is composed of instants none of which has duration, then there can be no such thing as a period of time.

If these instants have duration, then each instant can be subdivided into parts, each also having duration, and if you take any such part, the same argument applies. In fact, the same argument applies always, so that there is never a subdivision so small that it cannot be further divided. Thus we get an infinite number of parts each having duration. Therefore, a period of time must endure always.

Zeno's conclusion is that time is not composed of particles but is a plenum. He can show, by the first version of the Stadium Paradox, that time cannot pass: we are therefore in the "eternal now" as posited by Parmenides.

Bibliography

ADKINS, C.J. Equilibrium Thermodynamics, (London, 1969).

ARISTOTLE. De Anima

ARISTOTLE. De Caelo

ARISTOTLE. Metaphysics

ARISTOTLE. Nichomachean Ethics

ARISTOTLE. Physics

BALSILLIE, D. Professor Bergson's Philosophy, (London, 1912).

BENACERRAF, P. "Tasks, Supertasks and the Modern Eleatics" in Journal of Philosophy, (59), 1962.

BERGSON, H. Creative Evolution, (London, 1954).

BERGSON, H. An Introduction to Metaphysics, (New York, 1912).

BERGSON, H. "A propos de l'evolution de l'intelligence géometrique" in Revue de Metaphysique et de Morale, XVI, 1, (1908).

BERGSON, H. Mélanges, (Paris, 1972).

"Bergson", article in Encyclopedia of Philosophy, (New York, 1972).

"Bergson", a review in Revue de Philosophie, II, 6, (1902).

BLACK, Max. "Achilles and the Tortoise" in Analysis, (11), 1951.

BLAKE, R.M. "The Paradox of Temporal Process" in Journal of Philosophy, (23), 1926.

BOOTH, N.B. "Were Zeno's arguments a reply to attacks upon Parmenides?" in Phronesis, 2, (1957).

BOOTH, N.B. "Were Zeno's arguments directed against the Pythagoreans?" in Phronesis, 2, (1957).

BOSTOCK, D. "Aristotle, Zeno and the Potential Infinite" in Proceedings of the Aristotelian Society, (1972-73).

BREHIER, E. The Hellenic Age, Vol. 1., (Chicago, 1965).

BRIDGEMAN, P.W. "Some Implications of Recent Points of View in Physics" in Revue Internationale de Philosophie, Vol. III, No. 10, (1949).

BURNET, J. Greek Philosophy: Thales to Plato, (London, 1962).

CANTOR, G. "Contributions to the Founding of the Theory of Transfinite Numbers in MIDONOCK, H. The Treasury of Mathematics, Vol. 2, (London, 1965).

CANTOR, G. Gesammelte Abhandlungen ed. E. Zermelo, (Berlin, 1932).

CHAPPELL, V.C. "Time and Zeno's Arrow" in Journal of Philosophy, (59), 1962.

CHATTIMATTAM, J.B. Conciousness and Reality, (London, 1971).

CHIHARA, C.S. "On the possibility of completing an infinite process in Philosophical Review, (74), 1965.

COPLESTON, F. A History of Philosophy, Vol. 1., (New York, 1962).

CORNFORD, F. Before and after Socrates, (Cambridge, 1968).

CORNFORD, F.M. Plato and Parmenides, (London, 1958).

COURANT, R. and ROBBINS, H. What is Mathematics? (New York, 1969).

COUTURAT. De l'Infini Mathematique, (Paris, 1896).

DAVIE, G.E. The Democratic Intellect

DE LONG, H. A Profile of Mathematical Logic, (Massachusetts, 1970).

DESCARTES, R. The Philosophical Works of Descartes, Vol.1., trans. Haldane and Ross, (Cambridge, 1970).

DRETSKE, F.I. "Counting to Infinity" in Analysis, 25, (1964-65).

EHRENBERG, V. From Solon to Socrates, (London, 1968).

FREEMAN, K. Ancilla to the Pre-Socratic Philosophers, (Oxford, 1948)

FREEMAN, K. The Pre-Socratic Philosophers, (Oxford, 1946).

GALE, R.M. "Has the Present any Duration?" in Nous, (5), 1971.

GALE, R.M. (ed). The Philosophy of Time, (New York, 1967).

GALLILEO, G. Dialogues concerning Two New Sciences, (New York, 1914)

GRIVET, J. "La Théorie de la Personne d'après Henri Bergson" in Les Etudes, (1911).

GRÜNBAUM, A. "Messrs. Black and Taylor on Temporal Paradoxes" in Analysis (13), 1952.

GRÜNBAUM, A. "Modern Science and Refutation of the Paradoxes of Zeno" in Zeno's Paradoxes, ed. W.C. Salmon.

GRÜNBAUM, A. "Modern Science and Zeno's Paradoxes of Motion" in Zeno's Paradoxes, ed. W.C. Salmon.

GRUNBAUM, A. "Modern Science and Zeno's Paradoxes of Motion" in The Philosophy of Time, ed. Richard M. Gale. (This is a fuller treatment than the article of the same name in W.C. Salmon.)

GRÜNBAUM, A. "Relativity and the Atomicity of Becoming" in Review of Metaphysics, (4), 1950.

GRÜNBAUM, A. "Zeno's Metrical Paradox of Extension" in Zeno's Paradoxes, ed. W.C. Salmon.

GUTHRIE, G.K. History of Greek Philosophy, Vol.2., (Cambridge 1979)

"Heraclitus", article in Encyclopedia Brittanica, (1964).

HINCKLEY, D.B. Zeno of Elea and Plato's "Parmenides" Ph.D. Thesis, The University of Texas at Austin (1977)

HINTON, J.M. and MARTIN, C.B. "Achilles and the Tortoise" in Analysis, (14), 1954.

HUNTINGTON, E.V. The Continuum and Other Types of Serial Order (New York, 1955).

JAMES, W. Some Problems of Philosophy, (New York, 1919).

JEFFREYS, H. "Does a contradiction entail every proposition?" in Mind, 1942.

JONES, P.C. "Achilles and the Tortoise" in MIND, (55), 1946.

KANT, I. Critique of Pure Reason, (London, 1970).

KING, H.R. "Aristotle and the Paradoxes of Zeno" in Journal of Philosophy, (46), 1949.

KIRK, G.S. and RAVEN, J.E. The Pre Socratic Philosophers, (Cambridge, 1962).

KORNER, S. The Philosophy of Mathematics, (London, 1960).

LAWRENCE, N. Alfred North Whitehead, (New York, 1974).

LAWRENCE, N. Whitehead's Philosophical Development, (Berkley, 1956)

LECLERC, I. (ed) The Relevance of Whitehead, (London, 1961).

LEE, H.D.P. Zeno of Elea, (Amsterdam, 1967).

LEE, H.N. "Are Zeno's Paradoxes based on a Mistake?" in Mind, 74, (1965).

LOVEJOY, A.O. "The Problem of Time in Recent French Philosophy" in Philosophical Review, (21), 1912.

McCOLLOUGH, J. Zeno, Extension, and the Problem of Method. Ph.D. Thesis, EMORY UNIVERSITY, (1970).

McKELLAR STEWART, J. A Critical Exposition of Bergson's Philosophy (London, 1911).

MAYS, W. Whitehead's Philosophy of Science and Metaphysics, (The Hague, 1977).

METCALF, W.V. "Achilles and the Tortoise" in Mind, 51, (1942).

MURPHY, L.J. Zeno's Paradoxes: Tasks, Super-tasks and Undecidability, M.A. Thesis, Californian State University, Long Beach (1975).

NORTHROP, E.P. Riddles in Mathematics, (London, 1959). Publ: The English Universities Press Ltd., (London, 1959).

OWEN, G.E.L. "Zeno and the Mathematicians" in Proceedings of the Aristotelian Society, (58), 1957-1958.

PEIERLS, R.E. "Relativity" in Problems of Space and Time, ed. J.C.C. Smart, (New York, 1964).

PHILLIPS, E.D. "Parmenides on Thought and Being" in Philosophical Review, (64), 1955.

PIPPARD, A.B. The Elements of Classical Thermodynamics, (Cambridge, 1966).

PLATO. Alcibiades

PLATO. Cratylus

PLATO. Parmenides

PLATO. Phaedrus

PLUTARCH. The Rise and Fall of Athens, (London, 1969).

QUINE, W.V. The Ways of Paradox, (New York, 1966).

REICHMANN, W.J. The Spell of Mathematics, (London, 1967).

RITCHIE, A.D. "Notes" in MIND, (1941).

ROGERS, E.M. Physics for the Inquiring Mind, (New Jersey, 1960).

ROSS, Sir D. Aristotle, (London, 1964).

RUSSELL, B. History of Western Philosophy, (London, 1961).

RUSSELL, B. "The Limits of Empiricism" in Proceedings of the Aristotelian Society, (36), 1935-36.

RUSSELL, B. Mysticism and Logic, (New York, 1957).

RUSSELL, B. Our Knowledge of the External World, (London, 1914).

RUSSELL, B. Wisdom of the West, (London, 1970).

RYLE, G. Dilemmas, (Cambridge, 1954).

SALMON, W.C. (ed) Zeno's Paradoxes, (New York, 1970).

SANTAYANA, G. Winds of Doctrine, (London, 1940).

SCHWAYDER, D. "Achilles Unbound" in Journal of Philosophy, (52), (1955).

SHAMSI, F.A. Towards a Definitive Solution of Zeno's Paradoxes (Karachi, 1973).

SHIMONY, A. "Resolution of the Paradox" in Zeno's Paradoxes, ed. W.C. Salmon.

SMART, J.J.C. (ed) Problems of Space and Time, (New York, 1964).

SMITH, D.E. History of Mathematics, Vol.II, (New York, 1958).

SMITH, H.B. "Mr. Blake and the Paradox of Zeno" in Journal of Philosophy, (34), 1923.

SNEAD, S. The Education of a Golfer, (New York, 1964).

STACE, W. A Critical History of Greek Philosophy, (London, 1920).

STEPHEN, K. The Misuse of Mind (publ: Kegan, Paul, Trench Trubner and Co. Ltd., 1922).

TAYLOR, R. "Mr. Black on Temporal Paradoxes" in Analysis, (12), 1951-52.

TAYLOR, R. "Mr. Wisdom on Temporal Paradoxes" in Analysis, (13), 1952-53.

TE HENNEPE, E. "Language Reform and Philosophical Imperdialism: another round with Zeno" in Analysis, (23), 1963.

TEICHMANN, J. "Incompatible Predicates" in Analysis, (26), (1965-66).

THOMAS, L.E. "Achilles and the Tortoise" in Analysis, (12), (1951-52).

THOMSON, J.F. "Tasks and Supertasks" in Analysis, (15), 1954.

TOLSTOY, L. War and Peace, Vol. 3., (Heron Books, n.d.).

TOMLIN, E.W.F. Great Philosophers of the West, (Arrow Books, 1959).

USHENKO, A. "The Final Solution of the Paradox of the Race" in Journal of Philosophy, (29),(1932).

USHENKO, A. "Zeno's Paradoxes" in MIND, (55), 1946.

VLASTOS, G. "Zeno's Race Course" in Journal of the History of Philosophy, (4), 1966.

WARNOCK, G.J. English Philosophy since 1900, (London, 1969).

WATLING, J. "The Sum of an Infinite Series" in Analysis, (13), 1952.

WHITEHEAD, A.N. The Aims of Education and other essays, (London, 1929).

WHITEHEAD, A.N. The Concept of Nature, (Cambridge, 1920).

WHITEHEAD, A.N. Process and Reality, (New York, 1929).

WHITEHEAD, A.N. Science and the Modern World, (London, 1975).

WILDON CARR, H. Henri Bergson: The Philosophy of Change, (London, 1912).

WINN, R.B. "On Zeno's Paradox of Motion" in Journal of Philosophy, (29), 1932.

WINN, R.B. "Reflections on Infinity" in Journal of Philosophy, (29), 1932.

WISDOM, J.O. "Achilles on a Physical Racecourse" in Analysis (12), 1951-52.

WISDOM, J.O. "Why Achilles does not fail to catch the Tortoise" in MIND, (50), 1941.

WITTGENSTEIN, L. Philosophical Investigations, (Oxford, 1972).

WITTGENSTEIN, L. Philosophical Remarks, (Oxford, 1975).

WITTGENSTEIN, L. Tractatus Logico-Philosophicus, (London, 1961.)

"Zeno", article in Encyclopedia Brittanica, (1964).

"Zeno of Elea", article in Encyclopedia of Philosophy, (New York, 1972).

"Zénon d'Elée", article in Dictionaire de Science Philosophique, (Paris, 1852).