Real Options and Strategic Interactions in Imperfectly Competitive Markets

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Abstract

As Smit and Trigeorgis (2004), Smit and Ankum (1993), and others have observed, an options-based approach to strategic investment needs to be considered from the perspective of competitive market structures. The models in this thesis show the synergies obtainable when real options ideas are embedded in industrial organisation-type frameworks. The results are insights that neither field, thus far, has captured independently.

Firstly, we present a real options model in an investment game of incomplete information in a duopoly where product market competition influences the value of the investment and entry times are endogenously determined. We show that type-asymmetry or the level of initial demand, independently, or together, as in extant models, are insufficient criteria upon which endogenous roles under uncertainty may be determined when firms have private information over their types. Rather, ex post market structures are determined by threshold functions whose images lie in the type-space of the firms. These results are discussed in detail along with numerical examples.

Secondly, whilst the advertising literature has particular focus on either the informative or persuasive effects of advertising efforts and views advertising investments as intertemporal expenditures, this thesis addresses the brand loyalty aspects of advertising in a new market which, asides from routine expenditure, requires a lumpy initial investment outlay in the development of a viable competitive advertising campaign. We view this as a real options investment with a differential game played at the advertising-efforts level.

Lastly, empirical analyses on the impact of celebrity endorsements have largely been inconclusive. In this thesis, we model investments in brand equity using celebrity endorsers with embedded options over the investment opportunity. The aim is to determine optimal strategies when the arrival of the investment itself follows a random process and when firms can update their beliefs about the risk profile of the celebrity.

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To my father, for giving me the greatest gift of all.

(Rest in peace Dad. 1943 – 2016)

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Chapter 1

Introduction

An investment is considered an act of incurring an immediate cost in the expectation of future reward. From this perspective, investment decisions are ubiquitous, constantly requiring the weighing up of expected rewards against costs, and economists have long been concerned with valuing them under various market conditions. The traditional NPV approach to valuing projects has been extensively utilised due to the ease with which it is can be applied to real assets. It involves discounting expected net cash flows at some predetermined discount rate reflecting the risks associated with the expected cash flows over time. The general rule being to investment if the NPV is positive, which suggests that the investment is worthless if its NPV is less than zero. For all of its benefits, there are a number of problems with using the NPV technique in valuing investments, one of which is its propensity to undervalue investments, and its inability to interpret and provide strategic considerations in the valuation of assets.

Now, nearly all investment decisions share three distinctive characteristics. To begin with, an investment may be partially or wholly irreversible: if we build a factory, we do not expect to recover our sunk costs should the production process fail. Operational flexibility, such as the ability to suspend production when prices fall, can determine the level of reversibility in our investment. Second, there is uncertainty over future rewards. In many industries, uncertainty can be found in the behaviour of rival firms, or the level of demand. Third, an investment can be postponed as we wait for better (but never perfect) information. If we were buying shares for example, we can invest now or we can wait until after the announcement on company performance. This information could change the time and even the amount we decide to invest. Traditional valuation methods, like the NPV often ignore or underplay the importance of the interaction of these three features in the valuation of an investment. But the real options approach is well suited to value investments under these conditions.

Real options analysis applies option valuation techniques to capital budgeting decisions. A real option itself, is the right but not the obligation to undertake some business decision; usually the option to make, abandon or expand a capital investment. Of particular importance to practitioners is the case for managerial flexibility in making costly investment decisions, which has played a key role in the revolution of the real options methods' application in practice. For example, the opportunity to invest in the expansion of a firm's factory, or alternatively to sell the factory, is a real option. It accounts for uncertainty by examining the trade-off at every point in time between the marginal benefit of waiting for a higher market price against the marginal cost of forgoing the revenues that could be gained from being active. Thus, real options is a dynamic tool for dealing with uncertainty, as it continually updates for changes in market conditions. This is in contrast to the more traditional technique of using the net present value (NPV) approach in valuing investments, which says that a firm should invest when the value of a unit of capital is at least as large as its purchase and installation costs. Uncertainty is accounted for by adjusting the discount rate or the cash flows. Unlike the real options approach, it is static and implicitly assumes that management will be "passive" with regards to their capital investment once committed. In the real options analysis, the ever-changing environment accounted for in the option itself requires management to be "active" and take decisions, such as when to expand or suspend production, where necessary.

In the case of uncertain demand, there is a possibility that firms cannot generate enough profit because the demand may fall, even if it is currently high. In the real options approach, management have the flexibility to wait until demand is high enough to compensate for that risk. Thus, in a monopoly setting, real options suggests delaying entry when compared with more traditional techniques of valuation.

The application of option concepts in valuing real assets have grown significantly in both theory and practice with its quantitative origins derived from the seminal work of Black and Scholes (1973) in pricing financial options. A host of other important contributions to the literature has created an entire toolbox of investment valuations using ideas from financial options applied to real assets. These include the binomial pricing approach, contingent claims analysis, valuation of compound options etc. (see for example the works of Cox et al. (1979), Majd and Pindyck (1987), and Geske (1979) amongst others).

In recent times, contributions to the real options literature have covered a number of different market conditions, for example, entry and exit strategies in an investment under uncertainty (see Dixit and Pindyck (1994)), strategic option value of waiting which extends the real options rule to examine preemption games (see Huisman and Kort (1999)). Additionally, Murto (2004) examines the optimal time to exit in a duopoly, while Goto et al. (2008) look at a non-pre-emptive duopoly where firms can freely suspend and restart production at a cost. Takashima et al. (2008) consider two cases of pre-emptive duopoly: one with and without the mothballing option (the ability to suspend production). Kijima and Shibata (2005) extends results in the duopoly case into an oligopoly market. It considers entry decisions with symmetric firms in a pre-emptive framework. It showed that in the oligopoly case, there are three types of equilibria, pre-emptive leader-follower equilibria, joint-investment equilibria, and their mixture. It found that the critical point is decreasing in the number of active firms in the market and always smaller than the optimal trigger point in the ordinary real options criterion, however, the critical point for pre-emption can never be smaller than the traditional NPV criterion. Bouis et al. (2009) extends the duopoly model of Dixit and Pindyck (1994) by allowing a number of symmetric firms ranging from three to n. Their findings reveal a new mechanism in the strategic real options literature which they called the *accordion effect*. Their work is perhaps the first to explore investment decisions in an oligopoly under the assumption of lumpy investment.

A unifying theme of these contributions to the literature on real options is that the market under consideration is characterised by perfection competition. The firms involved, either in the monopoly or duopoly case, are price-takers and product market competition does not play a part in the profit functions of the firms, and by extension, the value of the investment. A growing strand of the economics literature now focuses on an options approach to strategic investments from the perspective of competitive market structures. This requires embedding real options in strategic market games where expected payoffs are derived from the market structures and competitive reactions of firms, that is, either *à* la Cournot or Bertrand. This strand of the economics literature integrates real options and game-theoretic industrial organisation framework to develop competitive strategies for irreversible investments under uncertainty and operational flexibility. The work we present in this thesis sits within this strand of the economics literature. Specifically, we present models that capture some unique features that have yet to be studied in this context and by endowing firms with an option to delay making their investment decisions until a later time under uncertainty in these environments, we present investment strategies that are consistent with rational players in a game-theoretic setting.

1.1 Preliminary Details to the Models in the Thesis

When choosing to enter a market under the combined influence of irreversibility, demand uncertainty and, sometimes, incomplete information about rival firms, it is important to each firm involved to consider how much it is worth to them to hold an option giving them the right to delay until some more information about the uncertain element of market fundamentals is observed. For each firm, the large initial investment outlay required to enter the market as well as demand uncertainty provide incentives to delay until market conditions are favourable, but also carries with it, the risk of the firm becoming a second-mover in the market whilst losing out on profits in the periods it remained inactive. Preemption on the other hand, erodes the option value of waiting;

each firm, therefore faces a conflict between commitment and flexibility.

In this thesis, we first consider how incomplete information impacts equilibrium strategies in an investment game with embedded options and demand uncertainty. We specify what market structures may emerge in equilibrium contingent on the types of the firms, and what significance these have to the value of the investment.

Next, we examine a market environment where uncertainty over demand is not exogenously specified rather, the actions of the firms partially influences what level of demand may be faced in the future. We study this in an investment game of advertising campaigns, where both firms compete via prices and advertising efforts the differential game which follows an entry game in which an option has been embedded. This sort of market environment is encountered when product quality is a given, and firms practically compete via their brands.

Lastly, we study investments under uncertainty where the time of arrival of the investment follows a Poisson process, and asides from demand uncertainty, there exists uncertainty over the toxic risk of the investment. We allow the firms to have some preconceptions about the toxic potential of the investment, and then formulate investment strategies based on the various states the system may transition into, and the competitive strategies to adopt in equilibrium.

The work in this thesis provides extensions in part or in full to the following publications:

- Smit, T.H. and Trigeorgis, L. (2004). Flexibility and Commitment in Strategic Investment. In Real Options and Investment under Uncertainty. Classical Readings and Recent Contributions, pages 451–498, The MIT Press, Cambridge, Massachusetts, London, England. MIT Press.

- Hamilton, J. and Slutsky, S. (1990a). Endogenous Timing with Incomplete Information and with Observable Delay. Games and Economic Behavior, 39:282–291.

- Prasad, A. and Sethi, S.P. (2004). Competitive advertising under uncertainty: A stochastic differential game approach. Journal of Optimization Theory and Applications, 123(1):163–185.

- Doraszelski, U. and Markovich, S. (2007). Advertising dynamics and competitive advantage. The RAND Journal of Economics, 38(3):557–592.
- Huisman, K. J. and Kort, P. M. (2004). Strategic technology adoption taking into account future technological improvements: A real options approach. European Journal of Operational Research, 159(3):705–728.

1.2 Outline and Contributions of the Thesis

This prelude has introduced the central idea of this thesis and the strand of the economics literature within which it sits. What follows is an outline of the chapters in the thesis and a summary of the main contributions.

Chapter 2 provides a benchmark model which embeds real options in an investment game of incomplete information in a duopolistic market, where product market competition influences the state value of the investment, and entry times are endogenously determined. The model incorporates private information over types and unveils new features of strategic interactions in imperfectly competitive markets when firms are faced with the trade-off between commitment and flexibility under demand uncertainty. We illustrates in the model that type-asymmetry and/or initial demand level alone, as have been previously adopted in the literature, are insufficient criteria upon which endogenous roles under uncertainty may be determined when firms have private information over their types.

Rather, the ex post market structure is determined by threshold functions whose images lie in the type-space of the firms. These functions, therefore, specify, ex ante, the firms' optimal strategies, which may involve (anti)-coordination.

The model is extended to consider the plausible case where a firm is able to credibly "fool" its rival by masking its type. The threshold functions, and thus, ex post market structures obtained in equilibrium are found to be characteristically the same as with when types are truthfully revealed. Therefore, the competitive behaviour of firms remain the same whether or not there are industry regulations that make it illegal for firms to falsify, mask, or lie about their profits.

Chapter 3 constructs a strategic game with entry options. The entry game describes how ex ante symmetrically uninformed firms make investment decisions under uncertainty and with private information over the quality of each others' advertising campaigns (type). These types present both business-stealing and market expansion capabilities and strategically impacts the evolution of the firms' stocks of goodwill. The evolution of goodwill is not only influenced by advertising efforts (or expenditure), as has been severally studied in the literature, but more importantly, by the types of the firms which are derived through the firms' lumpy initial investment outlay (rather than arbitrarily specified). We obtain Markov Perfect Equilibria in the advertising efforts through an approximate dynamic programming procedure, and the sequential equilibria of the entry game over the support of the distribution of the firms' types. We demonstrate the importance of holding such an option to defer investment decisions until a later time when more information about one's rival may become available by solving for both one-sided and symmetric option games. Because a firm's type influences both its market expansion and business stealing capabilities, a stronger firm has a lower advertising intensity in the steady-state, however, there is no equilibrium in this game where the industry is occupied in the long run by symmetric firms as suggested by Doraszelski and Markovich (2007).

Chapter 4 presents a strategic interaction game of firms seeking to gain competitive advantage over one another by growing their brand equities. A firm may grow (deplete) its brand equity through successful (failed) investment in an innovation opportunity with uncertain outcomes. The innovation process is exogenous to the firms and requires a large initial investment outlay. There are two innovation opportunities and firms may adopt only one. Opportunity 1 may be exploited at some known time in the future. The arrival of opportunity 2 is, however, uncertain and follows a Poisson process. Time lag between these opportunities provide additional information — increasing the probability of a successful investment. Opportunity 2 is therefore preferred, but firms are restrained due to additional uncertainty regarding its arrival. Conventional knowledge, with price taking firms, suggest that strategic outcomes are entirely driven by the rate of arrival of opportunity 2.

What is new in this model is that, beyond the effects of the rate of arrival of opportunity 2, we explore how prior beliefs of the risk profile of the investments and posterior beliefs derived via Bayes' theorem helps with predicting market outcomes at any point in the future in an imperfectly competitive market. We find that the modification of firms' prior beliefs moderates the effect of the rate of arrival of the random investment opportunity, and plays a greater role in their investment strategies.

Chapter 5 provides a review of the overall contributions of this thesis and some ideas for future work.

Chapter 2

Endogenous Timing of Investments in a New Market under Uncertainty and Incomplete Information

2.1 Introduction

Firms are often called upon to make irreversible investment decisions in the face of uncertainty about future demand, and incomplete information over the competitiveness of the rivals with whom they might be competing for market share. There are benefits to delaying investment until a future period when the uncertain elements of market fundamentals become revealed. However, there are potential costs of waiting, both in terms of foregone market activity and in losing the opportunity to preempt ones rivals. There is thus, a trade-off between commitment and flexibility.

This trade-off is central to the derivation of optimal investment decisions in strategic investment problems under uncertainty. In a new market, for instance, there may be inherent uncertainty about the scale of future demand, cost functions of potential competitors, market price of commodities etc. An optimal investment strategy must, therefore, be based on a proper evaluation of the strategic value of flexibility against the benefits of early commitment. By committing to an irreversible investment at an early stage, a firm may obtain a first-mover advantage in the form of a lower production cost, earning monopoly rents, or emerging as the Stackelberg leader in the subsequent stage. These benefits may however, be eroded away if market conditions become unfavourable, as the firm cannot simply recover its initial investment outlay. As a result, a firm's ability to delay making such investment until a later time, when more information arrives, that fully or partially resolves some, or all, of the uncertain elements in the market, is immensely valuable.

In most industries, firms are often able to exercise this sort of flexibility when faced with investment opportunities under uncertainty. Which is why in practice, it is observed that firms do not invest in capital projects until price rises substantially above long-run average cost. This is in sharp contrast to the theoretical provision of the discounted cash flow (DCF) analysis or conventional net present value (NPV) approach to valuing investments. The DCF analysis specifies that an investment opportunity is viable whenever the discounted income flow is at least equal to the cost of investment (otherwise known as the *Marshallian trigger*). This trigger is, in general, less than what is observed in reality. The main shortcoming of this method is its inability to factor-in operating flexibility, i.e. the ability of management to make, revise or alter planned investment decisions as uncertainty gets resolved over time. It inherently assumes investment opportunities are "now-or-never" in nature, and hence, ignores the value of flexibility. To address this problem, an option-based valuation approach has been proposed as a tool capable of capturing managerial flexibility. It provides a dynamic decision making framework that affords firms the opportunity to delay investment decisions until such a time when more information becomes available that could influence both the timing and the level of investment.

The real options literature emphasises the value of this sort of flexibility and derives the optimal time to make an investment, when its value is determined, in part, by an exogenous stochastic variable e.g. the market price of a commodity. The literature presents various examples of flexible investment strategy in non-strategic (monopoly) and strategic (oligopoly) investments (e.g. Dixit and Pindyck (1994), Huisman and Kort (1999), Takashima et al. (2008), Masaaki and Takashi (2005)). The general idea is that the strategic option value of waiting is lower under preemption than in a monopoly. This is because preemption erodes the option value of waiting for more information. Firms in these models, and in many others in the real options literature, are assumed to be non-atomic, and have no real influence on the macro-structure of the market. The firms are assumed to be price takers competing in a perfectly competitive market, and intrinsic to these models is the assumption of the existence of a tradable asset whose price/value follows a known stochastic process e.g. a geometric Brownian motion (GBM), upon which the value of the option is then based. However, there exists capital investments in which the underlying asset cannot be dynamically spanned by a tradable asset or security, therefore, uncertainty in such investments cannot be modelled via a stochastic process such as the GBM, but rather, on the values of certain state variables that are not traded assets e.g. demand, costs, technology etc., which means that the value of the investment is partly determined by the outcomes of product market competition. As result, optimal investment strategies derived in the classic real options models do not naturally generalise to these industries.

The importance of considering product market competition in valuing investments of this nature is that most markets of interest are less than perfectly competitive, particularly if they require an initial large sunk cost, and payoff streams depend on post-investment market and information structures. For instance, a firm can consistently earn economic rents in such markets which are non-existent in a perfectly competitive market. The ability to sustain these economic rents over the life (finite or infinite) of a real asset plays a key role in determining the value of the asset. Whilst in a perfectly competitive market, the state value of an investment opportunity with an infinite life is determined by the maximum of the expected discounted cash inflows net initial investment outlay and the deferment value; in an imperfectly competitive market, the state value is determined by the outcome of the game which describes the market and information structure, i.e. (Bayesian) Nash-Cournot, Stackelberg, or Monopoly. Therefore, as observed, for example, by Smit and Ankum (1993) and Smit and Trigeorgis (2004), and others, an options-based approach to strategic investment needs to be considered from the perspective of competitive market structure.

The manner in which the burden of uncertainty has been introduced in the literature on investment decision models in imperfect markets warrants some consideration. Most authors introduce uncertainty in terms of insufficient strategic information, for example, as private information on cost functions or some other form of idiosyncratic shock that is peculiar to individual firms. The other common form is the generic uncertainty that affects all firms in an equal way e.g. the move of nature at the start of a game of imperfect information. Unfortunately, most of these models appear to indiscriminately introduce uncertainty in one of these forms, either in the bid to retain tractability, or just to focus on a specific problem. This presents a number of questions that beg for answers. For example, which form of uncertainty best describes reality? Which form most influences outcomes in the games, vis-*à*-vis first and second mover advantages? Or perhaps, is there an interactive effect that may be responsible for some of the counter-intuitive outcomes in existing models e.g. via signal distortions? Fortunately, the options valuation approach adopted in this model allows us to address these concerns in a manner that buttresses the impact of uncertainty (in either, or both forms) in investment games. Our aim is to model irreversible investment decisions by firms into imperfectly competitive markets where firms have incomplete information about their competitors' costs, and demand is uncertain.

2.2 Related Literature

Optimal investment strategies or role-choice in strategic investment programs under uncertainty in imperfectly competitive markets have received some following in the literature. Gal-Or (1987) demonstrates the role of strategic uncertainty in an exogenous leader-follower model with segmental private information about the level of demand.

The follower is able to accurately determine the leader's private information by inverting his output function. She shows that if the leader attempts to deviate from his equilibrium output (in order to "fool" the follower into presuming that market demand is low) and produces an output whose inverse image is outside the domain of definition of his signals (if this domain is bounded or has discontinuities), the follower may then believe he has more favourable information than the leader and therefore expand his output. This first-mover disadvantage under uncertainty is sustained even when the domain of definition of the leader's signals is unbounded and continuous. In effect, with partially correlated signals and moderate uncertainty, the leader supplying more, signals high demand, and then the follower supplies more as well. She also comments on the possibility of sustaining these first-mover disadvantages in an endogenous role choice model, but only gave specifications and did not pursue it further. Mailath (1993) presents a model that allows for endogenous sequencing in an asymmetric information game, where the more informed firm has the option to enter a market in one of two periods, but the less informed firm may only enter in the second period. The less informed firm is able to gain information about market profitability by observing the more informed firm's choice. The implications of signalling distortions in this game result in an equilibrium in which the more informed firm always enters in the first period, even when he could have earned a higher ex ante payoff by moving simultaneously with the uninformed firm. The focus is on the effect private information has on the choice of roles in an endogenous setting. However, having more information does not always confer leadership rights endogenously. The option available to each firm influences competitive strategies. Normann (2002), in fact, shows that when the less informed firm in Mailath (1993) has the opportunity to invest in period 1 as well, the Stackelberg equilibrium with the uninformed firm being the leader emerges as one of the equilibria surviving the $D1$ refinement.

It is curious that simultaneous-play outcome in the second period does not feature among the equilibria in these models. This may be due, in part, to the manner in which flexibility and uncertainty are modelled. Under generic uncertainty, with equal rights to enter the market at any one of two periods, Sadanand and Sadanand (1996) show that second period Cournot outcome persists in the set of equilibria, for all levels of risk in the distribution of demand.

It is pertinent to note that flexibility in these models carries no real option value, therefore, parametrisation of generic uncertainty and/or private information does not actually make it unprofitable to enter the market at any one of the entry periods, however large the level of uncertainty might be. This is not the case when initial investment outlays have to be sunk before production choices are made, because while profits may be earned (considering interior solutions alone) within the periods of output choices, the overall discounted stream of payoff less the investment outlay might not just be suboptimal, but result in an outright loss. Furthermore, investment decisions faced by firms in the business world very often require such lumpy investment outlays. Take, for instance, a pharmaceutical firm's decision to develop a new drug. The R&D phase of any drug discovery is, characteristically, capital intensive. The firm cannot simply recover sunk R&D costs in this endeavour, should it become unproductive. Or, in the event of a successful discovery, it remains uncertain if the drug will pass pre-clinical trials for approval or exactly how long it will take to get approved. Furthermore, other pharmaceuticals might be coming up with a similar drug. These, and other industry-specific forms of uncertainty bear upon investment opportunities in the real world. It often instructs decision-makers to exercise caution when making investment decisions under these circumstances (as the first-mover advantages and disadvantages are both very real).

Smit and Trigeorgis (2004)'s study quantifies the trade-off between commitment and flexibility in an investment game that incorporates real options in a strategic industrial organisation framework. By developing on Fudenberg and Tirole (1984), they show how demand uncertainty influences strategic interactions in environments where the investment is propriety or shared, competitor is tough or accommodating and whether the strategic variable is quantity or price. In the contrarian (quantity competition) case, the game proceeds in two stages. In the first stage, one of the firms has the opportunity to commit to a strategic capital investment that may give him a cost, or some other form of commitment, advantage over his competition in the second stage. The nature of this capital investment may make him a tough or accommodating incumbent in the second stage. The level of demand in the second stage is unknown, but follows a simple binomial process whose initial value is known in the first stage. In the second stage, both firms have the option to either invest in the first period or defer the decision to invest until the second period, and then decide to invest, or not, having observed the favourableness of the market condition. Equilibrium payoffs are earned in each of these periods and during the entire life of the investment. The value of the investment is derived from discounted cash flows less the initial investment outlay. They show how the level of demand in the first stage provides critical thresholds that determine the market structure in the second stage in the three cases where the strategic capital investment was, a) not made, b) shared, and c) proprietary.

The market environments in these cases can be thought of as being analogous to having, a) a less efficient pioneer firm, b) symmetric firms, and c) a more efficient pioneer firm, in a single-stage multi-period investment game. This analogy allows us to think of this model as one with endogenous sequencing, and see exactly what drives the choice of roles. The critical thresholds of demand in b involves a shift to the left of those in a, i.e. with equal standing in the market (as in b), the deferment threshold is lower for the pioneer firm than in a. Similarly, a higher demand level will, in case a, be required to offset the effect of the initial sunk cost and the pioneer firm's inefficiency, before entry may become profitable. Additionally, there exists a region of indeterminacy, where either firm may emerge as the leader or the follower. It is interesting, however, to note that in case c, for all levels of demand considered, this region collapses to a null set. Therefore, for all levels of observed demand, the pioneer, more efficient, firm never defers investment when the less efficient firm invests.

One of the main contributions of our model is to posit that cost asymmetry as depicted in the analogous framework above, under exogenous uncertainty, does not always preclude a more efficient firm from deferring when firms have private information about their cost function. More succinctly, cost asymmetry alone is not enough to determine endogenous roles under uncertainty. Competitive strategies in our model are driven by a pair of continuous functions of known market parameters (initial observed level of demand and the measure of uncertainty) whose co-domain is the set of types of the firms. The images of these functions determine critical values of types that specify the optimal strategy for each firm.

Interestingly, and contrary to the stipulations in Smit and Trigeorgis (2004) and Dewit and Leahy (2001) we find a non-degenerate region of types (even for some high levels of uncertainty), where an anti-coordination problem materialises. Cost asymmetry gives no leverage in this region, and it is never optimal to choose the same action. The optimal strategies are for either one of the firms to choose to move early while the other defers, and vice-versa. This will ordinarily be the case if marginal costs are not private information at the start of the game. Since each firm cannot observe its rival's cost, its ex ante scheme in this region will be in mixed strategies. By modelling private information into this analogous framework, we present a baseline model that allows us to establish how private information and exogenous uncertainty individually, and interdependently, influence the choice of strategies in investment games.

The rest of the chapter is structured as follows. Section 3 presents the model and assumptions, and section 4 describes equilibrium outputs, payoffs and value of the investment in each continuation game. Section 5 discusses the sequencing of actions based on the observed parameters of the model and section 6 contains extensions of the analysis that consider outcomes in a world where a firm is able to credibly lie about his type, and how the observed coordination problem might be addressed. Some concluding remarks follow. All derivations of equilibrium outputs, payoffs, expected values of the investment, and proofs are collected in Appendix 1.

2.3 The Model

The aim of this chapter is to model environments in which firms have limited information about the state of demand and the competitiveness of potential rivals in a market; firms' investment decisions, whilst irreversible once made, are not 'now or never'; and competition in the market upon entry is imperfect. To capture these key features we introduce a dynamic model in which demand evolves stochastically, firms have incomplete information about each other's variable costs, and in order to enter the market firms must undertake (large) sunk cost investment but have the freedom of choice over when to undertake this investment. As such, firms face two sources of uncertainty in the model when making their investment decision: uncertainty over the level of demand; and uncertainty about the competitiveness of the (potential) rival.

Suppose there are two risk-neutral firms $\mathcal{A}$ and $\mathcal{B}$ that are considering entering a market. The sunk cost expenditure required for each firm to enter the market is $\mathcal{K}$, which is the same for each firm and common knowledge. Conversely, there is incomplete information over variable costs: each firm has constant marginal cost drawn from a distribution F with support $[\underline{c}, \bar{c}]$, which is private information.

The investment game evolves over a sequence of periods $0, 1, 2, \ldots, n$, where period 0 is a pre-play period. Firms discount future payoffs at a rate of ρ. To capture the key feature of demand uncertainty but retain modelling simplicity, (inverse) demand is assumed to take the following structure, which is common knowledge. Inverse demand in period t is given by $P(Q_t, \Theta(t)) = \Theta(t) - Q_t$ where Q_t is the aggregate supply from participants in the market. In period 1 the level of demand is $\Theta(1) = \theta_1$ for sure, whilst in period 2 the level of demand evolves stochastically following a binomial distribution: $\Theta(2) = u\theta_1$ with probability p and $d\theta_1$ with probability $1 - p$, where $0 < d < 1 < u$ and $0 \leq p \leq 1$. From period 3 onwards demand has the same structure as period 2 demand, so once the uncertainty is resolved at the beginning of period 2 there is no further uncertainty about demand. At the end of each period the market clears, according to the total supply from market participants.

To maintain interest in the model, d will be parametrised to be such that, if a firm is making its decision to invest at a time when it knows demand is $d\theta_1$ then it would be unprofitable for any type of firm to enter, even if it did so as a monopolist. Conversely, if a firm is making a decision to invest when it knows demand is $u\theta_1$ this decision will not be inevitable and will depend on the firm's type and the level of information it has, which captures the feature that a firm may wish to delay entering the market to gain more information about the competitiveness of its rival because, upon doing so, it may optimally choose to remain out of the market even though demand conditions are favourable. This demand structure, while simple, aims to capture the key feature that firms may wish to delay investing until uncertainty about demand has been resolved, and more information about their rival's competitiveness has been gained.

The investment game described above incorporates a vital part of the structure of an extended game with observable delay as introduced by Hamilton and Slutsky (1990a)[1], in that there is a pre-play period 0, where the firms decide whether to sink the initial investment outlay $\mathcal{K}$ to enter the market in period 1 (I), and be subject to uncertain future demand; or whether to defer making this decision (D) until period 2 when demand uncertainty would have been resolved. Whilst the firms simultaneously and independently decides between I and D at this pre-play stage (period 0), their choices, once made, immediately become common knowledge. Additionally, when a firm enters the market it also has to decide on the level of output to produce given the period 0 choice of its rival. This is as far as the similarities between our model

[1]The extended game with observable delay is more suited to real-world cases where there is a lag between investment decisions and actual implementation. Hamilton and Slutsky (1990a) first propose this game as one of two extended games (the other being the extended game with action commitment) that endogenise the choice of roles in a duopoly with complete information. In the extended game with observable delay, firms simultaneous choose their adoption period in a pre-play stage (similar to period 0 in our model) and announces their choice before choosing an action. It is assumed that the firms are committed to whatever adoption period they choose in the pre-play stage. First period Cournot competition emerges when both firms have downward sloping reaction functions. Therefore, in a quantity competition, under further restrictions to payoff functions (as in Amir (1995)), the first mover advantage is eliminated.

and Hamilton and Slutsky (1990a) go. Indeed, our model departs from Hamilton and Slutsky (1990a)due to the real option endowed on each firm, which is not available to the firms in Hamilton and Slutsky (1990a); which is that: a firm who chooses, in period 0 to defers in period 1 has the opportunity to revisit this decision in period 2 when some (or all) of the uncertain elements of the market become revealed.

If both firms choose to invest in period 1, (I, I), then each firm is subject to both uncertainty over period 2 demand and their rival's cost. As such, in the first period firms compete in a game of Bayesian Cournot competition with demand level θ_1. At the end of this period the market clears and output and period 1 payoffs become common knowledge so each firm can deduce the others actual marginal cost. In period 2, therefore, the firms engage in a game of Cournot competition either with demand level $u\theta_1$ with probability p, or with demand level $d\theta_1$ with probability $1 - p$, which is the same from period 3 onwards.

If both firms choose to defer the investment decision until period 2, (D, D), then nothing happens in period 1, and the level of period 2's demand is realised as either $u\theta_1$ or $d\theta_1$ before firms decide whether or not to enter the market. If the level of demand is $d\theta_1$ then both firms choose not to invest in period 2 and in all subsequent periods. If the level of demand is $u\theta_1$, then both firms will be seeking to invest in the market after which they will engage in a bayesian Cournot game regarding outputs, but with updated beliefs about the support of their rival's marginal cost distribution since the (observed) act of delaying, in itself, reveals information about what the rival firm's marginal cost is unlikely to be. At the end of period 2 output choices and payoffs become common knowledge which reveals the rival's marginal cost, so from period 3 onwards firms engage in each period in a game of Cournot competition with demand level $u\theta_1$.

Consider now the case where one firm chooses to invest whilst the other defers its decision, i.e. (I, D) or (D, I). In either of these cases, the investing firm enjoys being a monopolist in period 1. Given its output and period 1 payoff becomes common knowledge at the end of period 1, the deferring firm becomes aware of its rival's type. After the realisation of period 2 demand, the deferring firm must now decide whether or not to invest. We assume that the firm who invested in period 1 has installed the required production capacity in period 1, so that it is positioned to take the role of a Stackelberg leader in period 2, whilst the the deferring firm, if it chooses to invest in period 2, has to first install its own capacity before producing outputs, therefore, it would assume the role of a Stackelberg follower. If the level of demand transpires to be $d\theta_1$ then it will not invest at this stage. If, on the other hand, it is $u\theta_1$ then it may consider investing. If it does so, it engages in a Stackelberg competition as the follower. Whilst the deferring firm learns the type of its rival from its period 1 activity, the early entrant does not have such accurate information over its rival's cost, but the act of delaying does reveal some information about its rival, so it should update its belief about the support of the deferring firm's marginal cost distribution. As such, the Stackelberg game is a game of asymmetric information in which the follower is perfectly informed. At the end of period 2, output and payoffs become common knowledge, both have installed capacity, and the incumbency advantage disappears, so in period 3, and all subsequent periods after that, the firms will either engage in Cournot competition if the firm that deferred its investment decision invested, otherwise, the early entrant maintains its position as the monopolist.

The benefits from investing early are that the firm receives profits from production in the first period and may, if its rival defers its investment decision and subsequently enters (given that demand rises), gain the advantage of being a Stackelberg leader in the second period. However, by doing so it exposes itself to losses should the level of demand fall.

Formally, we define the game as: $\mathbb{G} = (N, S, \pi)$, $N = \{\mathcal{A}, \mathcal{B}\}$ is the set of players. The inverse demand function at any period, $t = 1, 2, ...$, is given by $P(Q_t, \Theta(t)) = \Theta(t) - Q_t$, Q_t $(= Q_{t,\mathcal{A}} + Q_{t,\mathcal{B}})$ is the aggregate output, (and $Q_{t,\mathcal{A}}$ and $Q_{t,\mathcal{B}}$ are compact, convex intervals in $\mathbb{R}_+ \cup \{0\}$). The demand intercept, $\Theta(t)$, follows a simple binomial process with expected value $E\Theta(t)$, variance σ^2, and state space in $\mathbb{R}_+ \cup \{0\}$. The evolution of $\Theta(t)$ is similar to a Markov process[2] whose absorbing state is its value in period 2. Therefore, (see Figure 1) demand remains at its period-2 level for all subsequent periods after that.

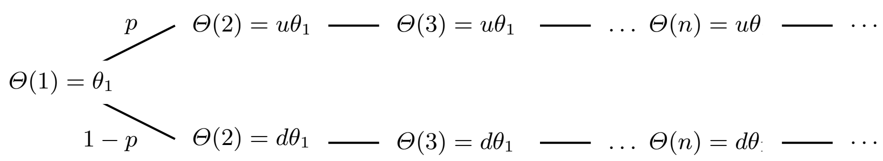


Figure 2.1: Binomial Process Depicting Stage 2 Evolution of Demand

Let $\mathcal{P} = \{I, D\}$, which is the set of available actions in period 0. I corresponds to sinking the investment cost $\mathcal{K}$ to enter the market in period 1. D corresponds to delaying the investment decision to period 2 at which point the uncertainty about demand will be resolved, in which case (due to the parameter restrictions we impose) the firm will invest to enter the market if and only if market demand is favourable ($\Theta(2) = u\theta_1$).

The set of strategies for player $i = \mathcal{A}, B$ is $S_i = \mathcal{P} \times \Xi_i$, where Ξ_i is the family of functions that map ζ_i into $Q_{t,i}$ for each period $t = 1, 2, \ldots$; and ζ_i is the set containing $\{(I, I), (I, D), (D, I) \times Q_{t,j}, (D, D)\}$. Define $s_i = (\sigma, \chi_{t,i}) \in S_i$, where $\sigma \in \mathcal{P}$ and $\chi_{t,i} \in \Xi_i$. Firm i's pure strategy in any period t involves the mapping $\chi_{t,i} : \zeta_i \to Q_{t,i}$, where $t = 1, 2, \ldots$. Having assumed that demand takes the same level as in period 2 from period 3 onwards, and noting that any relevant information will have been revealed by the beginning of period 3, it suffices to derive expressions of $\chi_{t,i}$ only for $t = 1, 2, 3$.

The appropriate solution concept for the extended game in this model, which ulti-

[2]In this setup, $Pr(\Theta(3) = \delta\theta_1 | \Theta(2) = \nu\theta_1, \Theta(1) = \theta_1) = Pr(\Theta(3) = \delta\theta_1 | \Theta(2) = \nu\theta_1)$, where δ and ν take values u or d and $0 < d < 1 < u$.

mately determines the value of the investment, is that of sequential equilibrium introduced by Kreps and Wilson (1982). The presence of non-singleton information sets in games of incomplete information of this kind, precludes subgame perfection, as there are no proper subgames. A sequential equilibrium requires sequential rationality in the strategies, and that the beliefs held by a firm at each information set it finds itself are consistent with the strategy that got it there. Sequential rationality in our game requires that, at each information set, the output choice of firm $\mathcal{A}$ is a best response to the output choice of firm $\mathcal{B}$, given firm $\mathcal{A}$'s belief about the support of the marginal cost distribution of the firm $\mathcal{B}$. Also, the belief held by firm $\mathcal{A}$ about the support of the marginal cost distribution of firm $\mathcal{B}$, at any information set consistent with the chosen strategy, must be derived by Bayes' rule. In contrast to the perfect Bayesian equilibrium concept, sequential equilibrium specifies how a firm should form beliefs when it reaches an out-of-equilibrium information set. Because firms announce their chosen entry times in period 0, out-of-equilibrium signals may be sent in terms of outputs as well as adoption periods. Therefore, to accurately obtain the expected value of the investment for period 0 choice, we require sequential rationality and consistency in the expected output choices of the firms in all periods. Thus, a strategy profile $(\lambda_1(s_{\mathcal{A}}), \lambda_2(s_{\mathcal{B}}))$, for this game, is a sequential equilibrium if for any $s_{\mathcal{A}}$ and $s_{\mathcal{B}}$, such that for all $(a, b) \in \mathcal{P}$, there exists a probability distribution $\tilde{F}$ over $c_{(\cdot)}$ such that (a, b) is chosen to maximize expected profits (and hence, the value of the investment) given $\chi_t^{\mathcal{A}}$ and $\chi_t^{\mathcal{B}}$; also, given the choice of adoption periods and given the firms' beliefs at each information set, $\chi_t^{\mathcal{A}}$ and $\chi_t^{\mathcal{B}}$ are chosen optimally; where the beliefs about the support of the distribution of each other's marginal cost held by the firms at each information set are obtained by Bayes' rule.

2.4 Continuation Game Analysis

At the beginning of period 0, firm $\mathcal{A}(\mathcal{B})$ faces only one source of uncertainty regarding its decision in period 1, i.e. the marginal cost of firm $\mathcal{B}(\mathcal{A})$, $c_{\mathcal{B}(\mathcal{A})}$. In period 2, however, (looking forward from period 0), each firm faces two sources of uncertainty, i.e., $\Theta(2)$ and its rival's marginal cost. The realisation of $\Theta(2)$ is common knowledge at the beginning of period 2, and marginal costs are revealed after period 1 or 2, *or not at all* (depending on period-0 choices and the realisation of $\Theta(2)$). The parameters u and d governing the evolution of $\Theta(2)$ are related to the variance by: $u = \exp(\sigma\sqrt{t})$ and $d = \exp(-\sigma\sqrt{t})$ (see Cox et al. (1979)). Payoffs in each period represent cash flows generated from output competition in the product market.

The investment decision of each firm, in period 0, depends on its calculation of the expected value of the investment opportunity at each entry period $t = 1, 2, ...$, given the possible actions of the rival firm and its expectation of $\Theta(t)$. This will be the entry choice that produces, in expectation, the highest value of the investment that exceeds the initial investment outlay by an amount equal to the value of keeping the investment option alive. The investment value, $\vartheta_{(\cdot)}^{(a,b)}$, for each possible outcome in period 0, is given by the sum of the expected profits in period 1 and the expected discounted cash flows of all future periods, minus the investment outlay, $\mathcal{K}$.

2.4.1 Simultaneous-move Equilibria

If both firms choose to sink $\mathcal{K}$ in period 1, they play a Bayesian-Cournot game in the early production period, and the basic Cournot in the second period and all other periods after that. Let $E_o(\cdot)$ denote the expected value of its argument given the information available in period 0 and let $q_{t,\mathcal{A}} \in Q_{t,\mathcal{A}} \subseteq \mathbb{R}_+ \cup \{0\}$ denote the output choice for firm $\mathcal{A}$ in period t. It follows that the firm may not find it profitable to produce outputs for all realizations of its marginal cost. As a matter of fact, we assume (as in Hurkens (2012)) that there exist some realizations of $c_\mathcal{A}$ for which the equilibrium output is 0. With this assumption, firm $\mathcal{A}$ produces $q^*_{1,\mathcal{A}} = \frac{1}{6}(2\theta_1 - \ddot{c} - 3c_\mathcal{A})$ in period 1 and earns $\pi^{*(I,I)}_{1,\mathcal{A}} = \frac{1}{36}(2\theta_1 - \ddot{c} - 3c_\mathcal{A})^2$, where $\ddot{c} = \int_{\underline{c}}^{\alpha} c \, \mathrm{d}F(c)$. This integral is taken over an updated support of the marginal cost distribution, i.e., if $\mathcal{A}$'s rival has chosen to invest early, $\mathcal{A}$ conjectures that its rival's marginal cost must be below some threshold α above which he will rather delay if $\mathcal{A}$ invests. Therefore, $\ddot{c}$ represents firm $\mathcal{A}$'s mean belief about his rival's marginal cost.

In period 2, having deduced its rival's (firm $\mathcal{B}$'s) marginal cost, firm $\mathcal{A}$ produces

$q^*_{2,\mathcal{A}} = \frac{1}{3}(\nu\theta_1 - 2c_\mathcal{A} + c_\mathcal{B})$ and earns $\pi^{*(I,I)}_{2,\mathcal{A}} = \frac{1}{9}(\nu\theta_1 - 2c_\mathcal{A} + c_\mathcal{B})^2$, where ν is either u or d, given the realization of $\Theta(2)$. These are *ex post* outputs and payoffs. *Ex ante*, the expected value of period-2 payoff is $E_o(\pi^{*(I,I)}_{2,\mathcal{A}}) = \frac{1}{9}(\sigma^2+\eta^2_{\ddot{c}}+(pu\theta_1+(1-p)d\theta_1+\ddot{c}-2c_\mathcal{A})^2)$ (see Appendix 1 for derivation). $\eta^2_{\ddot{c}}$ is the variance of the marginal cost derived from the updated support of its distribution. Period 3 and subsequent periods' payoffs follow accordingly. The ex ante expected value of the investment to this firm is therefore,

$$\vartheta^{(I,I)}_\mathcal{A} = \gamma_1 E_o(\pi^{*(I,I)}_{1,\mathcal{A}}) + \gamma_2 E_o(\pi^{*(I,I)}_{2,\mathcal{A}}) - \mathcal{K}, \tag{2.1}$$

where $\gamma_1 = 1/(1+\rho)$ and $\gamma_2 = \gamma_1/\rho$.

If both firms choose period 2, i.e. $(a,b) = (D,D)$, no production takes place in period 1. Output choices are made based upon the observed realisation of $\Theta(2)$, and only when $\Theta(2) = u\theta_1$. *Ex ante*, this happens with probability p, illustrating the fact that firms are not obligated to exercise their option to invest if they find it worthless. Marginal costs are still private information, but by choosing to defer, a firm, say $\mathcal{A}$, reveals information about its type. Its rival, firm $\mathcal{B}$, updates its own belief about the support of the distribution of firm $\mathcal{A}$'s marginal cost, i.e. $\mathcal{B}$ believes that $\mathcal{A}$'s true marginal cost must be greater than the lower bound of the prior support of the distribution. The updated lower bound corresponds to the value of $c_\mathcal{A}$, say β, below which $\mathcal{A}$ would never defer given that $\mathcal{B}$ defers in period 1. Bayesian updating, therefore, requires $\mathcal{B}$ to put probability zero on all types of $\mathcal{A}$ below β. We represent the updated mean belief of the firms' marginal costs by $\hat{c}$, which is equal to $\int_\beta^{\bar{c}} c\,\mathrm{d}F(c)$.

A Bayesian-Cournot game ensues in the second period, while the basic Cournot is played in subsequent periods. The ex ante expected value of the investment is given by,

$$\vartheta^{(D,D)}_\mathcal{A} = p\left(-\gamma_1\mathcal{K} + {\gamma_1}^2 E_o(\pi^{*(D,D)}_{2,\mathcal{A}}) + \gamma_1\gamma_2 E_o(\pi^{*(D,D)}_{3,\mathcal{A}})\right). \tag{2.2}$$

γ_1 and γ_2 are as previously specified. Period-2 payoff, $E_o(\pi^{*(D,D)}_{2,\mathcal{A}})$, is $((2u\theta_1 - \hat{c} - 3c_{\mathcal{A}})^2)/36$, and expected payoffs for each of the subsequent periods after is $((u\theta_1 + \hat{c} - 2c_{\mathcal{A}})^2)/9$.

2.4.2 Sequential-move Equilibria

Choosing (I, D), as with (D, D), also reveals information about the type of each firm. Suppose $\mathcal{A}$ chooses to enter early, and $\mathcal{B}$ defers, by choosing to defer, $\mathcal{B}$ sends a signal about the lower bound of the support of its marginal cost distribution, and $\mathcal{A}$ updates its belief about its rivals marginal cost accordingly. In the asymmetric information game played in period 2, $\mathcal{A}$'s marginal cost is revealed, but $\mathcal{A}$ still has incomplete information about its rival. However, $\mathcal{A}$ believes that $\mathcal{B}$'s true marginal cost must lie in the interval $[\alpha, \bar{c}] \subset [\underline{c}, \bar{c}]$, where α is the infimum of the the set of marginal costs for which $\mathcal{B}$ finds it unprofitable to invest early if $\mathcal{A}$ invests early.

Let $\breve{c}$ denote $\mathcal{A}$'s mean belief about the marginal cost of $\mathcal{B}$ based upon the updated support of the marginal cost. The first-mover's expected payoff stream is as follows: $E_o(\pi^{*M}_{1,\mathcal{A}}) = ((\theta_1 - c_{\mathcal{A}})^2)/4$, $E_o(\pi^{*(I,D)}_{2,\mathcal{A}}) = (p(u\theta_1 + \breve{c} - 2c_{\mathcal{A}})^2)/8 + ((1-p)(d\theta_1 - c_{\mathcal{A}})^2)/4$, and $E_o(\pi^{*(I,D)}_{3,\mathcal{A}}) = p((u\theta_1 + \breve{c} - 2c_{\mathcal{A}})^2)/9 + (1-p)(d\theta_1 - c_{\mathcal{A}})^2)/4$. The corresponding expected value for the first-mover is

$$\vartheta^{(I,D)}_{\mathcal{A}} = \left(\gamma_1 E_o(\pi^{*M}_{1,\mathcal{A}}) + \gamma_1^2 E_o(\pi^{*(I,D)}_{2,\mathcal{A}}) + 9\gamma_1\gamma_2 E_o(\pi^{*(I,D)}_{3,\mathcal{A}})\right) - \mathcal{K}. \tag{2.3}$$

Firm $\mathcal{B}$'s expected payoff stream if, and when, it enters is $E_o(\pi^{*(I,D)}_{2,\mathcal{B}}) = ((u\theta_1 - 2c_{\mathcal{B}} + 2\tilde{c} - \breve{c})^2)/16$ and $E_o(\pi^{*(I,D)}_{3,\mathcal{B}}) = ((u\theta_1 - 2c_{\mathcal{B}} + \tilde{c})^2)/9$, where $\tilde{c} = \int_{\underline{c}}^{\beta} c\,\mathrm{d}F(c)$ is $\mathcal{B}$'s mean belief of about $\mathcal{A}$'s marginal cost when he observes that $\mathcal{A}$ has chosen to invest early. The payoffs for all periods after period 3 are equivalent to that of period 3. Let the superscripts M indicates monopoly rent, then the follower's expected value for the investment is,

$$\vartheta^{(I,D)}_{\mathcal{B}} = p\left(-\gamma_1\mathcal{K} + {\gamma_1}^2 E_o(\pi^{*(I,D)}_{2,\mathcal{B}}) + \gamma_1\gamma_2 E_o(\pi^{*(I,D)}_{3,\mathcal{B}})\right). \tag{2.4}$$

When roles are reversed, these expected values are simply reversed as well.

It is not inconceivable that the sequential equilibria in this game may indeed be separating and perfectly revealing. In fact, with increasing regulatory requirements for adequate and timely reporting of financial activities, it might be difficult and/or illegal for a firm to misrepresent information about its costs and profits. The first part of our analyses assumes such environment. Therefore, in the sequential play outcome, for instance, the follower in period 2 observes first period (monopoly) payoff of the leader and can accurately infer his marginal cost (we have assumed the market clears after each period). Also, the follower's *ex ante* mean belief about the marginal cost of the leader uses the distribution's full support, hence $\tilde{c}$, in its best response function, and in the derivation of the value of the investment to firm $\mathcal{B}$ at period 0, as shown above. In effect, it does not matter what the first mover's exact cost is, in expectation, the follower's reaction is the same. We assume that, should demand rise in period 2, the market will be shared in a Stackelberg fashion, and also, that there exists a first-mover disadvantage should demand fall in period 2. In Section 2.6, we illustrate, as a possible extension to this model, how the possibility of misrepresenting one's type may change or influence outcomes in this game.

2.5 Endogenous Timing

In this section we analyse the timing decisions of firms in the light of the analysis undertaken above. We proceed as follows. Endogenous timing in the game is based on type, i.e. marginal cost. Therefore, the marginal cost draws of each firm at the beginning of the game determines what outcomes emerge endogenously. Table [2.1] represents the normal form of the extended game.

Table 2.1: Normal-form representation of the game

$\mathcal{A}/\mathcal{B}$	I	D
I	$\vartheta_{\mathcal{A}}^{(I,I)}, \vartheta_{\mathcal{B}}^{(I,I)}$	$\vartheta_{\mathcal{A}}^{(I,D)}, \vartheta_{\mathcal{B}}^{(I,D)}$
D	$\vartheta_{\mathcal{A}}^{(D,I)}, \vartheta_{\mathcal{B}}^{(D,I)}$	$\vartheta_{\mathcal{A}}^{(D,D)}, \vartheta_{\mathcal{B}}^{(D,D)}$

Lemma 1. *For moderate levels of uncertainty, u:*

(i) $\exists\, c_1 \in [\underline{c}, \bar{c}]$ *such that* $\vartheta^{(I,I)}_{i=\mathcal{A},\mathcal{B}} = \vartheta^{(D,I)}_{i=\mathcal{A},\mathcal{B}}$, *and*

(ii) $\exists\, c_2 \in [\underline{c}, \bar{c}]$ *such that* $\vartheta^{(I,D)}_{i=\mathcal{A},\mathcal{B}} = \vartheta^{(D,D)}_{i=\mathcal{A},\mathcal{B}}$

The variance of demand provides a measure of the level of uncertainty investors face. In binomial games of this kind, we are able to represent the variance in terms of the model parameter u, i.e. $\sigma^2 = (\ln(u))^2$. It therefore follows, that very high values of u indicates high levels of uncertainty, and so, high cost firms are much more wary of committing early. Very low values of u, on the other hand, diminishes the option value of waiting, in this case, some high cost firms might find it optimal to enter early. To determine optimal choices for values of u that fall within these extremes, we study the behaviour of the investment value functions.

Firstly, it is easy to see that $\vartheta^{(I,I)}$, $\vartheta^{(D,I)}$, $\vartheta^{(I,D)}$ and $\vartheta^{(D,D)}$ are strictly convex, monotone decreasing functions of c on the interval $\mathcal{I} := [\underline{c}, \bar{c}]$. This is because these value functions are monotone transformations of the individual equilibrium quantities derived within each period, which are themselves strictly decreasing and convex in the marginal cost. Secondly, they satisfy the following conditions:

a) $\vartheta^{(I,I)}(0) > \vartheta^{(D,I)}(0)$, $\vartheta^{(I,D)}(0) > \vartheta^{(D,D)}(0)$ and

b) $\left|\dfrac{\partial\vartheta^{(I,I)}}{\partial c}\right| > \left|\dfrac{\partial\vartheta^{(D,I)}}{\partial c}\right|$ and $\left|\dfrac{\partial\vartheta^{(I,D)}}{\partial c}\right| > \left|\dfrac{\partial\vartheta^{(D,D)}}{\partial c}\right|$ on $\mathcal{I}$.

Let $f_1 = |\partial\vartheta^{(I,I)}/\partial c| - |\partial\vartheta^{(D,I)}/\partial c|$ and $f_2 = |\partial\vartheta^{(I,D)}/\partial c| - |\partial\vartheta^{(D,D)}/\partial c|$. f_1 and f_2 are simple linear monotonic decreasing function of c, and are positive for all c in $\mathcal{I}$ for which $\vartheta^{(I,I)}$ and $\vartheta^{(I,D)}$ are non-negative (see Appendix 1 for details);

c) if ξ_0, ξ_1, ε_0 and ε_1 are respectively the "*zeros*" of $\vartheta^{(I,I)}$, $\vartheta^{(D,I)}$, $\vartheta^{(I,D)}$ and $\vartheta^{(D,D)}$ on $\mathcal{I}$, then $\xi_0 < \xi_1$, $\varepsilon_0 < \varepsilon_1$ and $\xi_0 < \varepsilon_0$ *a.s.*

The roots ξ_0, ξ_1, ε_0 and ε_1 of the value functions are themselves functions of the beliefs held by the firms during the course of the game, i.e. α and β (see Appendix 1 for details). Now, let $g_1(\alpha, \beta) = \xi_0 - \xi_1$ and $g_2(\alpha, \beta) = \varepsilon_0 - \varepsilon_1$; we show in Appendix 1 that g_1 and g_2 are negative everywhere on $\mathcal{I}$ for all values of α and β. Furthermore, $\xi_0 < \varepsilon_0$ on $\mathcal{I}$, and so, $\vartheta^{(I,I)}$ will always be less than $\vartheta^{(I,D)}$.

Now, since the functions $\vartheta^{(I,I)}$, $\vartheta^{(D,I)}$, $\vartheta^{(I,D)}$ and $\vartheta^{(D,D)}$ satisfy conditions (a) and (b), along with strict convexity and monotonicity, then, there must be two distinct points of intersection in $\mathcal{I}$ where $\vartheta^{(I,I)} = \vartheta^{(D,I)}$ and $\vartheta^{(I,D)} = \vartheta^{(D,D)}$ respectively.

Lemma 2. *The interval $[c_1, c_2]$ is non-degenerate.*

It is easily observed that if $\xi_0 < \varepsilon_0$ as in **L**emma 1, the points c_1 and c_2, corresponding to the intersections of the pair of value functions, do not coincide, moreover, $c_1 < c_2$.

Figure [2.2] illustrates the critical regions on the interval $\mathcal{I}$, within which one or more of the outcomes in period 0 is dominant. These outcomes are summarised in Table [2.2], and presented formally, in the propositions that follow:

Proposition 1. *(Simultaneous-play Equilibrium) If both firms independently draw marginal costs in the interval $[\underline{c}, c_1)$, there is a dominant strategy equilibrium involving both firms investing in period 1; also, if both firms independently draw marginal costs in the interval $(c_2, \bar{c}]$, there is a dominant strategy equilibrium in which both firms defer the decision to invest or not until period 2.*

Proposition 2. *(Sequential-play Equilibrium) If the firms' marginal cost draws lie in separate regions delineated by c_1 and c_2; there is a dominant strategy equilibrium in which the more efficient firm emerges as the endogenous first-mover.*

Proposition 3. *When either firm draws marginal cost in the interval (c_1, c_2), there is at least one cost realisation threshold in this interval for which there exists a unique mixed strategy Nash equilibrium such that below and above this cost realisation threshold, a firm would choose I or D.*

Proofs for Propositions [1] and [2] follow directly from **Lemma** 1 and **Lemma** 2, and from the the summary in Table [2.2]. Both Propositions describe equilibria in dominant strategies, such that a firm does not need to consider its rival's type or actions when formulating its own strategy. Indeed, it is clear that for all $c_i \in [\underline{c}, c_1)$, $I \in \mathcal{P}$ is a dominant strategy equilibrium since for either firm, $\vartheta_i^{(I,e)} > \vartheta_i^{(D,e)}$ for all $e \in \mathcal{P}$. Similarly, for all $c_i \in (c_2, \bar{c}]$, $D \in \mathcal{P}$ is a dominant strategy equilibrium since for either firm, $\vartheta_i^{(D,e)} > \vartheta_i^{(I,e)}$ for all $e \in \mathcal{P}$. The proof for Proposition [3] is in Appendix 1.

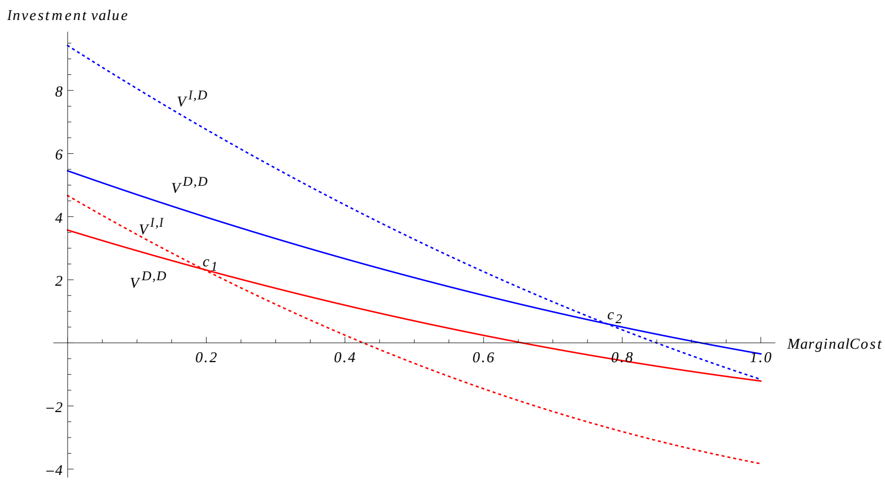


Figure 2.2: Investment Values for Period-0 Outcomes

$\mathcal{A}/\mathcal{B}$	$c_\mathcal{B} < c_1$	$c_1 < c_\mathcal{B} < c_2$	$c_\mathcal{B} > c_2$
$c_\mathcal{A} < c_1$	$\mathcal{I}, \mathcal{I}$	$\mathcal{I}, \mathcal{D}$	$\mathcal{I}, \mathcal{D}$
$c_1 < c_\mathcal{A} < c_2$	$\mathcal{D}, \mathcal{I}$	$\mathcal{I}, \mathcal{D}; \mathcal{D}, \mathcal{I}$	$\mathcal{I}, \mathcal{D}$
$c_\mathcal{A} > c_2$	$\mathcal{D}, \mathcal{I}$	$\mathcal{D}, \mathcal{I}$	$\mathcal{D}, \mathcal{D}$

Table 2.2: Type-based Equilibria

2.6 Discussion

As is evidenced from Table 1, the choice of roles in this game is governed by the firms' types through the critical thresholds c_1 and c_2. These thresholds are parametrised by the level of demand uncertainty and the beliefs of the firms. Quite unlike Smit and Trigeorgis (2004), investment timing is not solely determined by the level of demand,

and more so, a more efficient firm may not necessarily emerge endogenously as the first mover (as in case "c^*" of the analogous framework of Smit and Trigeorgis (2004), where the strategic investment is proprietary).

The belief-based equilibria we have derived in this chapter show that what outcomes emerge endogenously depends primarily on the side of the critical thresholds c_1 and c_2, the firms' marginal costs lie, the values of which are estimable at the start of the game. Simple cost(type) asymmetry is insufficient, therefore, to describe outcomes when there is private information about types under demand uncertainty. The import of private information in investment games of this nature is clearly non-trivia. For example, if both firms draw marginal costs in the intervals described in **Proposition** 1, then symmetric or not, the optimal outcome is simultaneous investment. On the other hand, no matter how close $c_{\mathcal{B}}$ might be to $c_{\mathcal{A}}$, if $c_{\mathcal{A}} < c_1 < c_{\mathcal{B}}$, then firm $\mathcal{B}$'s dominant strategy is to delay. A leader-follower equilibrium emerges endogenously only when their types are sufficiently asymmetric as in **Proposition** 2.

Proposition 3 shows an outcome that differ from that of Smit and Trigeorgis (2004) where a more efficient firm never chooses to defer if its rival invests. As we have shown, this interval is non-degenerate and does not collapse into a null set as their model specifies. Ex post, we may, therefore, find a more efficient firm emerging as the second-mover. The intuition behind this is that when a firm draws a type that falls in this interval, it realises that its dominant strategy is to defer if its rival invests, and to invest, if its rival defers. Unlike in the intervals $[\underline{c}, c_1)$ and $(c_2, \bar{c}]$ where a firm's dominant strategy is to invest and defer respectively, irrespective of what its rival chooses; in the interval (c_1, c_2), each firm's optimal strategy is conditional on its rival choosing the exact opposite. But firm in this interval has not knowledge of its rival's marginal cost, and therefore, must use a mixed strategy. We construct a possible solution to the (anti)-coordination problem in Appendix 1, which uses a threshold type in this interval to probabilistically specify optimal pure strategies below and above this threshold.

A firm's dominant strategies evolve across the type space, and the value of its option to defer investment increasing with the type it draws. For a firm with sufficiently large marginal cost, deferring commitment decision until the second period, at which time some, or all, of the uncertain elements of the game are resolved, becomes increasingly preferred. As the option value increases, the first-mover advantage decreases. Also, $\partial c_1/\partial u < 0$ and $\partial c_2/\partial u < 0$ for any given level of demand, meaning that the sub-spaces of the type-space where a first-mover advantage exists shrink with uncertainty for the a more efficient firm.

2.7 Concluding Remarks

This chapter presents a baseline investment game of incomplete information under uncertainty, where product market competition influences the state value of the investment, and timing is endogenously determined. We have shown that cost asymmetry is insufficient criterion upon which outcomes may be determined when there is private information over types (see Dewit and Leahy (2001), where cost asymmetry is used to determine market structure in an investment game with observable delay under demand uncertainty). Consequently, a first-mover advantage may not exist for sufficiently large draw of types, and when it does, it diminishes with type and with the level of uncertainty. The sequential equilibrium concept employed ensures that even when a firm defies the requirement of the game with observable delay (i.e. does not commit to its period 0 choice), the rival firm, finding itself on an off-equilibrium path is able to form beliefs consistent with how he may have arrived at this information set and update its belief about the defecting firm's type appropriately.

An important assumption that drives the results in this chapter is that the market clears in each period of production, and each period's payoffs are observed before the next period's output choices are made. However, an immediate extension to our model is to consider how the game might evolve if the first mover is able to credibly mask his type. In order to analyse this scenario, the firms may only be able to observe outputs and not payoffs in this particular world. This may be considered under two categories. One, is where the first mover can mask his type and the follower believes it. Two, is where the follower knows that the first mover may mask his type, and then modify his actions accordingly. Interestingly, the outcomes in both cases are similar. In the first case, we find that if the first mover shades his cost by the factor ϱ, where $0 < \varrho < 1$, then the benefit he derives from "fooling", as it were, his rival into considering him more efficient in the Cournot game in period 3 outweighs the temporary loss in revenue he would experience in period 1 and 2[3]. The net present value of this benefit is concave in the amount, ϱ, with which he shades his cost. In the second case, even though the potential follower realises that the first mover may be lying about his type, as a Stackelberg follower in period 2, his optimal action is to best-respond to the leader's output, whether it be a lie or not. Having deduced the cost relevant to the leaders output in the second period, the third period Cournot game proceeds accordingly at which point the true marginal cost of the first mover is then revealed. The Cournot competition in subsequent periods progress as with when both firms have been truthful all along.

Our conjecture is that, this additional incentive to move first (being able to benefit from lying) may not qualitatively alter the specifications of our game. The quantitative implication may be that it reduces the values of the critical thresholds that determine the outcomes of the game. We leave the determination of the value of the investment and the specifics of the ex post market structures that emerge endogenously when masking ones type is possible as the subject of future research.

[3]The scripts used to generate these results—and those for the simulations used to generate other numerical results in this thesis—were written in Mathematica *Version 9.0.1.0*, and available upon request.

Chapter 3

Competitive Strategies in an Investment Game of Advertising Campaigns

3.1 Introduction

Consumers encounter multiple products everyday, and often, these products have similar functions. Yet, they consistently find a way to choose one of several alternatives. Consumers do this for a number of reasons which, in many instances align with the intended outcomes of product advertisement. A consumer may choose a product because they have been reliably informed of its quality, or have an affinity for the brand it carries, or have previous experience using the product or do not care very much about exploring other alternatives. Firms, have historically, made attempts at cutting through this clutter of reasons by advertising their products, and they therefore, put a lot of value on their advertising strategies.

Advertising has informative properties which brings the awareness of a product to as many consumers as possible, and may result in an increase in industry demand. Indeed, there is sufficient empirical evidence in support of the positive relationship between advertising and increased industry demand; and there is also no shortage of theoretical analysis.

Asides from the market expansion property of advertising, there is also the persuasive effect which convinces a consumer that what they really want is a particular variety. This dual property of advertising have been shown to provide significant profit potential for firms through the modification of consumer perception about what they should be aware of and what they really want to (or should) pay for. Additionally, investments in advertising is similar to other kinds of capital investments in that it has carryover effects such that, today's advertisement continues to influence consumer choices several periods later. Nerlove and Arrow (1962) captures this idea by extending Dorfman and Steiner (1954)'s model to allow current advertising expenditure to affect future demand in their model through the concept of a stock of goodwill representing the effects of past and current advertising on demand. Goodwill evolves dynamically according to the firms' choices of advertising expenditures/efforts and, in reality, will involve an element of uncertainty. But just like most capital assets, by adopting a zero advertising policy, goodwill will decay over time, so firms advertise both to maintain and to increase their stocks of goodwill. It is no wonder, therefore, that firms invest vast sums of money in advertising on a regular basis. For example, whilst research and development is perhaps, the most competitive instrument in the Pharmaceutical industry, Pfizer's advertising expenditure has consistently tracked its research and development costs as shown in Figure 4.5, giving an indication of the importance of routine advertising to the company.

Over the last century, economists have employed empirical and theoretical methodologies to answer the question of how advertising works and to understand how consumers respond. A vast body of literature has been created focusing on either the informative (market expansion) or persuasive property of advertising, and in some cases, both (see Bagwell (2007) for a survey of the literature on the economics of advertising). The general observation is that the effects of advertising and implications for firms seem to vary depending on which view of the properties of advertising the researcher assumes, sometimes producing conflicting outcomes. Whilst a good portion of the literature on advertising have been devoted to the evolution of goodwill under

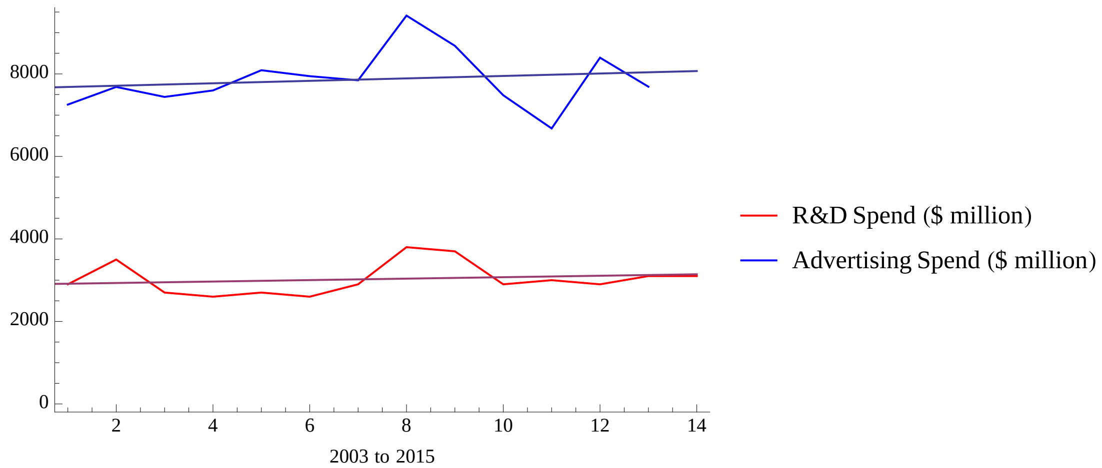


Figure 3.1: Pfizer's R&D and Advertising expenditure from 2003 to 2015.

different competitive market scenarios and specifying optimal advertising policies, little has yet been done to further align investment in advertising with other forms of capital assets in a manner that better conforms with more recent market reality. For instance, the analogue is incomplete if we consider the irreversible nature of investments in capital assets such as plants and R&D i.e. investments in advertising have hitherto been treated as being reversible because the literature refers mainly to the routine expenditure aspect of advertising to either gain or maintain goodwill; which is related to some level of advertising efforts, usually in the form κa_t, where $\kappa > 0$ measures the cost of advertising and a_t indicates number of television commercials, internet exposure, or newspaper pages etc.

In this chapter, we use a real options approach to extend the analogy between investments in advertising and in other capital assets, in a way that aligns better with more recent advertising ideology, and then we derive optimal advertising policies as well as investment strategies. Our consideration is that beyond the evolution of goodwill through depreciation and the effects of further advertising efforts, firms often have to incur a significant amount of investment outlay in developing the brand message around which their advertising campaign will be built. This usually involves a sunk cost, the size of which is not influenced by subsequent levels of advertising efforts (or expenditure). As a result, it is not enough to only consider the value of routine advertising expenditures but also the (irrecoverable) cost of developing advertising campaigns, which is separate from the cost of using (or reusing) one particular medium to spread the message e.g. television, newspaper etc. As the issue of the medium through which advertisements are made become relatively less vital, the case for the value of the message an advertising campaign carries become ever more important. And, perhaps, more than any other influencing factor is the advancement in technology and proliferation of social media platforms through which information can be sent millions of time over at no (real) cost to the owner of the message. Current understanding in practice, is that the evolution of advertising now involves the creation and use of subliminal messages to fuse a firm's brand into the narrative that appeals to consumers the most, in the words William Gelner, the chief creative officer of 180LA,

"The holy grail for advertising today is the same as it's always been: to rise above the fray of soulless sales pitches and become part of culture. Not just being recalled or remembered, but hitting a nerve and becoming both share-worthy and meaningful. The best brands get that. They aim higher".

Consider how John Lewis — a renowned retailing giant, who placed no adverts on television or Online until just eleven years ago — has captured the imagination of Christmas and consumers through its, now highly anticipated and successful, Christmas adverts since 2007. These adverts have become as much part of the festive season and British tradition as Christmas trees. In 2014, before spending nearly £6 million on television and newspaper spots, billboards etc., the retailer had spent £1 million over a period of six months to conceptualise, develop, and produce its intended message through the "Monty the Penguin" Christmas video, which drew about 22 million views on YouTube by the first week of January 2015. With social media playing a huge part in creating awareness and providing information, it has become more difficult to identify parallels between advertising efforts (in terms of routine expenditure e.g. television slots etc.) and the effectiveness of advertising. Indeed, the value or quality of the message of a campaign has become more important to the level of spread of the message, loyalty of existing consumers, and conversion with consumers of rival retailers.

It is worth noting, that by supposedly creating intense feeling of warmth and happiness through its emotional message, John Lewis was said to have made the nation "cry and buy," boosting its Christmas like-for-like sales by an average of 9.8% over the last seven years, and outperforming its highest performing rival by about 10% on average.

This sort of advertising campaign embodies informative, persuasive and, in fact, complementary[1] views of advertising, the value of which, as far as the author is aware, is yet to be collectively and explicitly studied in the economics literature relating to advertising or brand awareness. Therefore, in this chapter, we examine the value of investments in an advertisement campaign where the sentimental (or political, social engineering) potency of the campaign message signals brand superiority, instigates consumer loyalty, and creates new market frontiers for the firms. Firstly, we model investment in an advertising campaign as a real option where firms sink an initial investment outlay in order to participate (e.g. John Lewis' £1 million investment in their ad development). This investment is irreversible, there is uncertainty over expected returns (for instance, through direct sales volume), and the firms have an opportunity to delay in order to reconsider the decision of whether to create the ad campaign or delay. Secondly, we embed advertising effectiveness into the stocks of goodwill the firms enjoy through the potency of each firm's advertising campaign, such that the evolution of their individual stocks of goodwill reflect the market expansion and business stealing (persuasive) properties of advertising (as well as the complementary view).

The conceptual idea behind a firm's stock of goodwill is that it captures consumers' awareness of, and favourable disposition towards, a brand and its product. Friedman (1983) makes current state of goodwill the choice variable in his model so that a firm effectively asks, "given other firms' stocks of goodwill and my advertising effectiveness function, how much goodwill do I need to increase my profits," and therefore, how much advertising expenditure is required to achieve this goodwill level? In a linear-

[1]The complementary view of advertising relates to reinforcing the values consumers hold of a brand so that additional advertising consistent with these values reinforces consumers loyalty.

quadratic differential game set up, he studies the predatory and cooperative effects of advertising through the evolution of the stocks of goodwill of the firms. In his model, each firm's goodwill is an increasing concave function of its advertising expenditure at any time t, and the firms' stocks of goodwill signify the state of the game at any time. In general, advertising is predatory or cooperative depending on whether the advertising effectiveness function (a function of the vector of goodwill) is negative or positive. He observes that when the effectiveness function is positive, an increase in one firm's advertising expenditure, reflected in the size of its goodwill, increases the goodwill levels of all firms (cooperative effect), but when it is negative, an increase in one firm's goodwill decreases the goodwill levels of all other firms (predatory). Whilst these insights are interesting, his model ignores the dynamics through which advertising expenditure, or indeed efforts, organically create goodwill for a brand. A number of other extant models share Friedman (1983)'s view of cooperative advertising, which is sometimes referred to as informative advertising.

In the market environment of interest in this chapter, the effects of advertising expenditure is front-loaded into the creation of goodwill, so that we are able to study both informative and predatory effects simultaneously. Furthermore, as with most extant models in the literature, advertising expenditure in Friedman (1983)'s model relates only to periodical costs of advertising and ignores the cost of creation (or is not explicitly modelled). In a durable products' market scenario, Horsky and Mate (1988) use the probability with which advertising efforts increase a firm's market share to indicate how firms actually create potential demand. With a fixed market size, advertising can only be predatory in their model. We may think of the market share of each firm as a measure of its goodwill (though not explicitly described as such), which is assumed to grow stochastically according to a transition matrix.

The stochastic evolution of market share in their model is desirable in the market environment we examine here, but their model explicitly ignores market expansion possibilities which may significantly influence a firm's advertising expenditure, particularly when the firm lacks the capacity to steal business from its rival. Examining both market expansion possibilities as well as business stealing capabilities under uncertainty, Doraszelski and Markovich (2007)'s model looks at how advertising influences long term industry structure. They consider both goodwill (persuasive) and awareness advertising (informative) effect in separate models. Crucially, they consider conditions under which competitive advantages (i.e. asymmetries among firms) may be sustained in steady-state. Firms compete via prices and advertising efforts in both models, but goodwill levels of the firms evolve according to transition probabilities that depend on the cost and amount of advertising. The potential market is split into four disjoint sets: consumers who are randomly exposed to one or the other firm's advertising, consumers randomly exposed to both firms' advertising, and consumers who remain uninformed. Their approach provides useful insight into the population dynamics of the potential market and how advertising expenditures shuffles the size of each segment in every period. However, neither the awareness nor goodwill versions of their model addresses the situation where advertising may be simultaneously informative and persuasive. The construction of the system dynamics in these models effectively precludes such behaviour, and advertising effectiveness solely relies on the advertising efforts of the firm. Again, the quality of an ad campaign is not explicitly modelled or identified separately from the cost of transmission, wherewith they find that the industry may consist of symmetric firms in the steady-state if the market is large, firms were asymmetric from the get-go, and advertising is cheap. We contend in our model, by accounting for the value of the message in an ad campaign under a real options framework, that a symmetric outcome may not emerge even if advertising is cheap as long as the firms are asymmetric in the quality of their advertising campaigns and advertising is simultaneously informative and predatory.

An interesting feature of the market environment we examine is that the quality of advertising campaigns are rarely in dispute, just as it is not in contention who has given Britain the better Christmas advertising campaigns in the last decade — John Lewis. Consumers are able to figure out whose brand advertising connects more with their world view, feelings, or emotions. As a result the business-stealing effect of brand advertising plays-out somewhat differently than in extant models of advertising.

Whoever wins the brand affinity game through the quality of their advertising campaign has the dominant role in business-stealing efforts which does not solely depend on how many more times or through how many more mediums it advertises. The common idea with business-stealing effects in the literature is that either firm may be able to convert consumers loyal to rival brands through their own advertising depending on how effective it is, and it is often played out that the firm with the more effective advertising gets to convert more loyal consumers of its rival than its rival does of the firm's consumers. In brand loyalty advertising through the quality of the campaign, when a consumer comes into contact with the messages of both firms, since they are able to determine who appeals more to their sensibilities or emotions, the contest ends—the consumer switches. Therefore, the firm with the better ad campaign wins over as many consumers who engage in this comparison, which means it looses no consumers of its own to the rival with a weaker ad campaign.

Even more crucial in this environment is that business-stealing through ad campaign quality extends even to market expansion efforts. The informative or market expansion properties of advertising follow the notion of a public good in most previous studies, but looking through the prism of ad campaign quality, this is very much unlikely to be true, even with prima facie evidence. Take for example Dove's advertising campaigns, which have historically addressed political, social, mental, and psychological ideologies on beauty, and sometimes bordering on social engineering; because Dove's advertising is not specifically product-related rather about starting a conversation favourable to the brand, new consumers gravitate to the brand and therefore product, rather than simply being made aware of a product which may create demand for other generic products [2]. Therefore, in new market spaces, as the market grows, new consumers who engage with the firms' ad campaigns can also determine who has the better message and will gravitate to that firm, and the proportion of such consumers grow with the firm's advertising efforts. As a result, the firm with the weaker campaign therefore only grasps consumers who either have not encountered both campaigns, do not care, or do

[2] Of course, we do not dispute that some spillover effects may occur, we only contend here that the effects are considerably smaller in a market of pure brand advertising where product quality is either a given or a matter of degree rather than kind.

not want to engage in such comparisons. Which is why a lot of importance is placed on the development of the message a brand wishes to send out as is observed nowadays in practice.

To capture these characteristics, we allow firms make an irreversible investment in their brand by developing as compelling an ad campaign as possible, at the end of which a winner emerges, that is, the firm with the better, more mesmeric message. Consumers who engage both firms' ad campaigns are able to clearly discern whose is preferable. Consumers do not have switching costs in our model and so can freely choose any brand which captures their imaginations better irrespective of their previous affiliations. Naturally, this splits the population into four distinct sets similar to Doraszelski and Markovich (2007)'s, but the growth and switching dynamics in ours are different in the ways described below.

There are four segments of consumers in the market, the first two segments represent consumers who only receive either firm's message or perhaps do not have the will to engage in comparisons. There is also the segment of consumers who receive both messages and make a judgement concerning which of the campaigns appeals most to their sensibilities. Finally, there is the segment of consumers who do not receive either firm's message or are just not caught-up with the concept of brand loyalty. The size of the set of consumers who simply align with the campaign of one firm is determined jointly by the quality of the ad campaign and the advertising efforts of the firms. Therefore, the firm with the better quality will potentially reach a larger portion of these consumers and convert them to goodwill for its brand. This is how goodwill grows through the information property of advertising, which does not involve the firms' existing goodwill or market share, nor does it grow at a fixed or artificially constrained amount as in some studies. Consumers who receive both firms' campaigns and engage both of them will gravitate towards the firm with the better brand message form the segment of the population where a firm may steal potential business from its rival. That is, the firm with the better campaign draws these consumers over to its brand.

Similarly, the portion of existing consumers who engage with both brands' campaigns switch at no cost to them to the better brand and are lost to the firm with the weaker brand message. Lastly, the segment of consumers who receives neither message or do not engage either campaigns form the potential market in the next period.

Clearly, the firm with the better campaign can therefore steal current and potential business from its rival, and it is clear that the weaker brand will have to put a lot more effort into expanding the set of consumers who receives its own message in order to remain competitive. This characterisation of the market changes the evolution of goodwill markedly from those of extant models, but better captures the manner in which consumers interact with brands through their ad campaigns in markets where brand loyalty is the main competitive tool.

The construction of the system dynamics in our model reflects these features from which we derive each firm's optimal advertising policy and then the value of an investment in their campaigns. This framework allows us to specify equilibrium outcomes of the entry game two firms play with asymmetric types where their types dictate the evolution of the dynamical system within which they will compete via prices and advertising efforts. We are able to identify ex ante investment strategies under various threshold types of the firms and observe crucially, that while the firm with a higher type has a dominant strategy to invest early, the option value of a firm with a lower type increases the closer its type is to the threshold-type of a follower. Also, a firm's option to invest is not necessarily worthless if its type is below this threshold, indeed, there is a probability that this firm may emerge a monopolist if its rival, with a higher type, does not hold a similar option.

The rest of this chapter is structured as follows. Section 2 describes the model framework and the goodwill evolution for each firm based on their types and the strategy sets of the firms. In Section 3 we describe the numerical approach employed in solving the differential game that follows the entry game, and details of the construction of the basis function applied in the value function approximation. Section 4 describes the various continuation games and the steady-state outcomes in each case, as well as the main contributions of the model presented here. Section 5 contains our concluding remarks and possible extensions to the model.

3.2 The Model

This section describes the market environment we examine in this chapter. We consider a market with two, ex ante symmetric, risk neutral firms i and j. Both firms seek to introduce new products into the market and each consider (or is endowed with) an option to create as compelling an advertisement campaign as possible at some time t by sinking an initial investment outlay of $\mathcal{K}$ to get the word out about the new product. Our model targets product markets in which product quality is largely a question of degree rather than kind — such as Wernerfelt (1988) posits whilst drawing intuition to explain the existence of reputational economies of scope (albeit not explicitly modelled or further pursued). In these sorts of markets, product demand is built around a firm's reputation, or consumers' emotional affinity towards the firms' brands, which is induced by the potency of the messages their ad campaigns carry.

The potency of each firm's campaign message remains private information until both campaigns are launched. As a result, neither firm is aware of its rival's intended consumer conversation or, indeed, the level of innovation or degree of emotional response that the campaign is expected to generate. Whilst extant models in the literature have introduced advertising effectiveness somewhat arbitrarily[3], in this chapter, we model it as a product of the firms' investment in an advertisement campaign. Therefore, its influence goes beyond the impact on optimal advertising policies to determining the strategic value of an investment where the firms have an option not to engage the consumers through their ad campaigns or defer that choice until a later time when more information might become available.

Consumers' responses to the firms' products are heavily reliant on who wins the emotional game through the firms' investments in advertisement campaigns (here on out, where it creates no confusion, we will use "campaign" to mean "advertisement

[3]Often as an idiosyncratic multiple of advertising effort or cost without strategic investment implications.

campaign"). Neither firm is compelled to make this investment and either can wait until another time, say, $t+1$, before choosing to engage consumers with its own campaign message. Consumers' affinity for a firm is revealed in the stock of goodwill the firm possesses, so the relative potency of a campaign potentially increases the number of consumers who develop an affinity for the brand and thereby increasing the firm's stock of goodwill. The firms are strategic in their behaviour in the product market, competing via prices and advertising efforts, as a result, their stocks of goodwill evolve dynamically as well as stochastically through time.

Formally, suppose at some time $t=0$, each firm is endowed with an option to either enter a new product market at one of several investment windows starting at $t=1$ by creating an advertisement campaign, or defer until the next window. Upon entry, the firms compete strategically through prices and advertising efforts which, along with the potency of their campaigns, drive product market demand for goods. We denote the firms' stocks of goodwill at t as x_t^i and x_t^j; and potency of a firm's campaign is the measure of its rival's past and present stocks of goodwill it is able to convert into goodwill for its own brand through its campaign. Let us refer to this as the sentimental quality of the firms' advertising campaigns and denote them by real numbers β^i and β^j in the interval $[0,1]$. Of course, not all consumers may be willing to engage or compare both campaigns, but as many as are willing to do so find the better message incontrovertible. These consumers represent the portion of goodwill that the firm whose campaign has the better sentimental value converts into goodwill for itself irrespective of their previous affiliations. Therefore, $\beta^i \neq \beta^j$, and for the duration of the game, neither firm can make post-investment modifications to their messages.

Suppose after the launch of these campaigns, it is revealed that $\beta^i < \beta^j$, the numerical approximations of the stochastic differential equations governing the evolution of the firms' goodwills are given by

$$\mathsf{x}_{t+\Delta}^i = \mathsf{x}_t^i + \left(\lambda(\beta^i, u_t^i) - (\delta + \beta^j u^j)\mathsf{x}_t^i\right)\Delta + \sigma_{\mathsf{x}^i} W_t, \text{ and} \tag{3.1}$$

$$\mathsf{x}_{t+\Delta}^j = \mathsf{x}_t^j + \left(\lambda(\beta^j, u_t^j) - \delta\mathsf{x}_t^j + \beta^j u^j \mathsf{x}_t^i\right)\Delta + \sigma_{\mathsf{x}^j} W_t. \tag{3.2}$$

Together, the pair $(\mathsf{x}_t^i, \mathsf{x}_t^j) := \mathcal{X}_t$ represents the dynamical system that describes the state of the product market at any time t, and $u^{(\cdot)}$ represents the firm's advertising effort (or rate of advertising). The second term on the right hand side of each equation above denote the instantaneous drift functions, and $\sigma_{\mathsf{x}^{(\cdot)}}$ is the volatility function. In addition, $W_t = \sqrt{\Delta}\widetilde{Z}(t)$ is the Wiener process which is meant to explain uncertainty in the behaviour of consumers, where $\widetilde{Z}(t)$ are independent and identically distributed standard normal random variables.

Our specification of the evolution of goodwill in equations [3.1] and [3.2] has similar desirable properties as Vidale-Wolfe's classic concave response model in that goodwill is non-decreasing in own advertising efforts and non-increasing in rival's advertising efforts. However, where our specification differs is in the competitive nature of the potencies of the firms' campaign messages, and how they influence the informative and persuasive nature of advertising. The main feature separating our model from other goodwill (or marketshare) advertising models in the literature (for example, Prasad and Sethi (2004), Fershtman (1984), and Rao (1984)) where the predatory (or persuasive) effect of firms' advertising is depicted as mutual, is that the measure of brand loyalty, and therefore goodwill, that a firm is able to steal from its rival depends entirely on who wins the investment game of advertising campaigns. This is revealed in the relative values of β^i and β^j, such as in the equations above. Firm i cannot convince a consumer loyal to j to switch, because if at all the consumer comes in contact with i's campaign message, it is able to judge clearly that it is inferior to that of j, and so will choose to remain. However, any consumer loyal to i, who receives j's campaign message and chooses to engage both, will also figure out that j's message is superior and will switch at no cost. The central idea is that predatory effects, at least, in relation to brand advertising, is not a question of the relative competitive nature of advertising efforts or intensity of advertising effort, rather it is determined at the investment game level where the firms' $\beta^{(\cdot)}$'s are formulated, and subsequently revealed. It is not unreasonable that if a brand engages a consumer with a better message it does not have to transmit the message several times over again, indeed, an inferior message sent severally is unlikely to change the mind of a consumer who has come in contact with a superior message that connects better with their sensibilities or worldview.

In addition, the informative aspects of our specification also addresses an important feature of the market environment we model here, which is a firm's advertising effort may increase the size of the market as it reaches out to new consumers (awareness advertising), however, rather than the effect being a measure of their existing goodwill or marketshare, or indeed, artificially constrained, we allow firms create awareness of their brands in a way that is consistent with the potency of their messages and their advertising efforts. Specifically, these effects are revealed in the four segments of the market as follows: denote the entire potential market at time t, prior to investment, by $\mathcal{F}_t^{\varnothing}$. After the firms make their investments, say some time $t + \Delta$, $\mathcal{F}_t^{\varnothing}$ is split into four mutually disjoint sets: $\mathcal{F}_t^i$, $\mathcal{F}_t^j$, $\mathcal{F}_t^{i,j}$, and $\mathcal{F}_{t+\Delta}^{\varnothing}$. $\mathcal{F}_t^i$ is the portion of $\mathcal{F}_t^{\varnothing}$ who only receives i's message or, indeed, receives both, and perhaps does not care to engage in comparisons. This group represents the awareness effect of i's campaign which ultimately increases its goodwill at $t+\Delta$ (the same effect goes for $\mathcal{F}_t^j$). But $\mathcal{F}_t^{i,j}$ indicates the portion of the potential market at t who receives both firms' messages and engages in a comparison which, since $\beta^j > \beta^i$, j's brand will always win among this group. This portion is added on to j's goodwill at $t + \Delta$ along with those from $\mathcal{F}_t^j$. It is clear that the influence, whether on awareness or persuasion is impacted by the potencies of the firm's messages and not just the efforts they put into advertising, and this potency is a result of the investment that has gone into developing the message itself. Lastly, $\mathcal{F}_{t+\Delta}^{\varnothing}$ is the portion of untapped consumer potential, either these consumers are loyal to an outside brand or remain unaware of these two brands, or just do not care to engage in that market space. They form the potential market at $t+\Delta$ that both firms will target with their advertising and we note that for all t, there is a bijection between $\mathcal{F}_t^{\varnothing}$ and $\mathbb{N}$, therefore $\mathcal{F}_{t+\Delta}^{\varnothing}$ is countably infinite for all time steps, Δ, since $\mathcal{F}_{t+\Delta}^{\varnothing} \subset \mathcal{F}_t^{\varnothing}$.

Demand Function Characterisation

Given the evolution of goodwill described by equations [3.1] and [3.2], the firms now face the problem of choosing profit maximising prices in each period t. Let $c^{(\cdot)}$ and $p_t^{*(\cdot)}$

respectively represent the constant marginal cost of production and equilibrium price chosen by the firms at time t. Either firm, say i, will solve

$$\begin{aligned} \pi_t^i(\mathsf{x}_t^i, \mathsf{x}_t^j) &= (p_t^{*i} - c^i) g^i(\mathsf{x}_t^i, \mathbf{p}^*) \\ &= \max_{p_t^i} \left[(p_t^i - c^i) g^i(\mathsf{x}_t^i, p_t^i, p_t^j) \right], \end{aligned} \tag{3.3}$$

where π_t^i is firm i's profit from the realisation of x_t^i, and p_t^{*i} solves the maximisation problem in equation [3.3]; also, $\mathbf{p}^* = (p_t^{*i}, p_t^{*j})$ represents Nash equilibrium prices, and $g^i (= q^i(\mathsf{x}^i, p^i, p^j))$ is i's demand function. To facilitate computational tractability, it is preferable to use a relatively well-behaved demand function to specify g^i. We adopt one of the characterisations of a suitable demand function in Rao (1984), with desirable properties such as: demand has an inverse relationship with price, goodwill shifts demand function to the right, and firm's demand is a concave function of its own goodwill. Since we have restricted the number of firms in our model to two, Rao (1984)'s *Type 2* demand function:

$$g^i \triangleq \kappa^i(\mathsf{x}_t^i)[\tilde{a}^i - \tilde{b} p_t^i + \eta p_t^j], \tag{3.4}$$

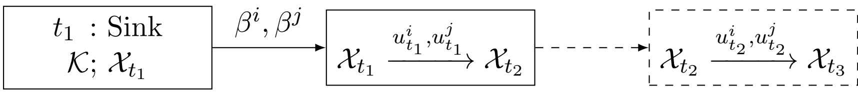
satisfies the requirements of our model, if $\eta > 0$, $\tilde{a}^i > c^i/2$, $(N-1)\eta < \sqrt{2}$ if N (number of firms) $= 2$ and $(N-1)\eta < 1$ if $N > 2$. $\kappa(\mathsf{x}_t)$ is how goodwill enters the firm's demand function and also how exogenous uncertainty, through W^t in the evolution of x, enters the demand faced by the firm in each period.

In the numerical evaluation of our model, we apply the relevant constraints for which the demand function set out above is well-behaved, that is, $(N-1)\eta < \sqrt{2}$. These restrictions guarantee that equilibrium outputs are decreasing functions of costs and ensures positive equilibrium prices. Furthermore, $\tilde{a}$ and the function $\kappa(\mathsf{x}_t)$ are sufficiently large so that the demand function is meaningful.

Timing and Competitive Strategies

Figure [3.2] is a simplified summary of the timeline of the investment game and shows how stocks of goodwill evolve. Starting at $t = 1$, the firms decide whether or not to

Figure 3.2: Timeline of the investment game



sink a sum $\mathcal{K}$ (a large initial investment outlay, which is the same for both firms, and is common knowledge at all times), in order to develop their ad campaigns and enter the new product market. Before making their investments and launching their campaigns, the potencies (or qualities) (i.e. the β's) of these campaigns remain private information. However, as soon as either firm launches its campaign (whether at $t = 1$ or $t = 2$), its β becomes revealed. The firms have initial stocks of goodwill x_1^i and x_1^j that are common knowledge at t_1. Let $\mathcal{A}_t$ represent the set of active firms in the market at time t. We use 'active' to denote a firm that has invested (or entered the market), therefore, $\mathcal{A}_0 = \varnothing$. The entry game bears a similarity with Hamilton and Slutsky (1990b)'s extended game with action commitment; that is, there is no pre-play stage where the firms may choose the timing of their investments, observe the results and then adhere to this timing in the second stage of the game. As with an action commitment type game, both firms choose simultaneously and without communication to either invest or defer at $t = 1$. In addition, a firm may only become active in the market at $t = 1$ by choosing to invest, launching its campaign, and producing outputs at $t = 1$. Where the entry game departs from Hamilton and Slutsky (1990b)'s extended game with action commitment, is that the deferring firm at $t = 1$ has to wait until the next period ($t = 2$) before it is able to revisit the decision to either invest or defer. This is how flexibility is introduced into the entry game. This flexibility along with the initial sunk cost ($\mathcal{K}$) and uncertainty over future payoffs through $\mathcal{X}_t$ constitute a real option.

Due to the importance of the potencies of the firms' campaign messages to the returns on their investments, and the fact that they remain private information until their campaigns are launched, either firm might, indeed, find it beneficial to defer its investment decision in order to gain some knowledge of its rival's type (potency). And depending on its rival's competitiveness (β), a firm may then decide whether or not to invest at the next investment window. However, as soon as a firm defers and learns of its rival's type, there no longer exists any benefit to further delaying the choice to invest – the firm either invests if it deems it profitable or quits, as β is the only remaining source of information asymmetry once one firm invests at $t = 1$. Hence, while the investment windows might be considered as infinite, it suffices to simply consider the investment windows at $t = 1$ and $t = 2$. As a result, the entry game ends at $t = 2$. We assume that a firm is able to produce outputs in the same period it enters the market (i.e. launches its campaign). Therefore, if both firms enter the market at $t = 1$, the entry game ends and both immediately begin to engage in a differential game over their choices of advertising efforts; however, if only one firm invests at $t = 1$ whilst the other defers, the entry game will end at $t = 2$ when the deferring firm would have either entered the market or quit. If this firm, again, defers (or quits) at $t = 2$, then the market will be defined as a monopoly; but if it chooses to invest at $t = 2$, the entry game ends, and a differential game begins in the same period.

Suppose $\mathcal{A}_2 = \{i, j\}$, then from $t = 2$ onwards the firms will compete via prices in each period while simultaneously engaging in a differential game over optimal advertising efforts, u^i and u^j. If $\mathcal{A}_2 = \{i\}$ or $\{j\}$, then, the active firm simply solves an optimal control problem in its advertising efforts. If $\mathcal{A}_1 = \emptyset$, then it means that both firms have chosen to defer their investment decisions until the next investment window $t = 2$. On the other hand, if $\mathcal{A}_1 = \{i\}$ or $\{j\}$, then the active firm's type is revealed to its rival who may then decide, based on this realisation whether to invest at $t = 2$ or quit the market altogether.

The strategy set of any firm, say j, at any investment window when $\mathcal{A} = \varnothing$, is the sequence: (Invest (I) $\times$ Optimal u^j $\times$ Equilibrium p^j, Defer (D)). If, however, j has not invested and $\mathcal{A} \neq \varnothing$, then its strategy set is the sequence: (Invest (I) $\times$ Optimal u^j $\times$ Equilibrium p^j, Quit (Q)).

3.3 Equilibrium Analysis

3.3.1 Approximate Dynamic Programming

Our model set up describes a dynamical system whose state ($\mathcal{X}_t = (\mathsf{x}_t^i, \mathsf{x}_t^j)$) is determined, at any given time, by a set of stochastic differential equations involving control variables, u^i and u^j, that probabilistically move the system from one state to another. A firm competing under these terms seeks an optimal path, or course of actions, that maximises the present value of its expected cashflows. While the outcome of the entry game determines the nature of the differential equations governing the evolution of the system, the possible outcomes of the differential game, in turn, instruct the firms' optimal choices regarding its option to invest. What we have, therefore, is an entry game whose outcome is influenced by a dynamic optimisation problem and vice-versa.

Approximate dynamic programming (ADP) provides an algorithmic framework that addresses stochastic optimisation problems of this sort described in our model, where the state space (goodwill) is too large to enumerate such that determining the probability of moving from one state to another is impossible to compute. Traditionally, for simpler problems, a dynamic programming approach only requires evaluating the value function of the system for every state in the state space, and then stepping backward in time using Bellman's equation. However, in the market environment we examine, the state variable $\mathcal{X} = (\mathsf{x}^i, \mathsf{x}^j)$, whilst having only two dimensions, each dimension may take any value in $\mathbb{R}^+$, therefore, there are up to $2^{\mathbb{R}^+}$ states to consider in each iteration. Similarly, the action space $\mathcal{U} = (u^i, w^j)$, whilst having only 2 dimensions, each dimension can also take values in $\mathbb{R}^+$, requiring the system to consider $2^{\mathbb{R}^+}$ possible actions. This amount of exploration in an optimisation problem quickly causes the problem to grow exponentially and computationally intractable using a standard dynamic programming approach[4]. ADP describes a number of numerical methods that may be used to derive *near* optimal solutions for complex sequential decision problems of this nature suffering from the curse of dimensionality.

[4]Bellman referred famously to this problem as the *Curse of Dimensionality*, see Bellman (1957).

Solution using ADP

Let J^j represents the present value of j's cashflows. J^j increases in the contributions obtained within each period based on j's choices of p^j and u^j, as well as the choices of i. These contributions are, in general, bounded below by 0. Since the main control variable in our model is advertising efforts, the goal is then to solve for the advertising policy that maximises J^j. We therefore have, for j, that $J^j = \max_{u^j} E\left(\sum_{t=1}^{\infty} \gamma \pi_t^j\right)$, subject to the evolution of x^j according to equations [3.1] and [2]; where π_t^j is time t's contribution which is also the profit earned at the time and γ denotes the discount factor. So if j ever becomes active in the market, it will have to choose, at any time t, u^j, and find the p that solves max $\left[(p^j - c)q^j(\mathcal{X}_t(u^i, u^j), p^i, p^j)\right]$ if i is also active, otherwise, it just solves $\max_p \left[(p - c)q(\mathcal{X}_t(u^j), p)\right]$.

To solve the optimisation problem required to evaluate J^j, the dynamic programming approach decomposes the whole sequence of decisions into just two components: the immediate decisions and its consequence, and a *value function* that encapsulates the consequences of all subsequent decisions, beginning with the position that results from the immediate decision. The Bellman equation below formulates this idea as a set of recursive equations linking these two components which needs to be solved at each time point to provide a set of policy or decisions to follow,

$$J_t^j(\mathcal{X}) = \max_{u_t^j} \Bigg(\underbrace{(\pi^j - \tau(u^j))}_{\text{immediate reward}} + \gamma \sum_{\mathsf{x}'} Pr\left(\mathsf{x}_{t+1} = \mathsf{x}' | \mathsf{x}_t^j, u_t^j, \mathsf{x}_t^i\right) \underbrace{J_{t+1}^j(\mathsf{x}')}_{\text{value function}} \Bigg). \tag{3.5}$$

If we suppose that j starts out with an optimal choice of u_t^j, it gets an 'immediate' reward represented by $(\pi_t^j - \tau(u_t^j))$ which is the first part of Equation [3.5] (recall that π is the profit function, while we use $\tau(u)$ to denote a (quadratic) cost function[5] that represents the cost of a u-amount of advertising (or advertising effort of u). Now, the initial choice of u_t^j will change the state of the system through the new value of the firms'

[5]We use a convex function to describe the cost of advertising efforts. Its use has some valid justifications which are well noted in the literature. One of the useful ones to our model framework is that the marginal benefit derived from an advertisement decreases as the number of messages increases. See, amongst other, Roberts and Samuelson (1988), Sorger (1989), & Espinosa and Mariel (2001).

goodwills, i.e. $\mathsf{x}_{t+1} = \mathsf{x}'$. The second part of the Bellman equation involves the present value of all future rewards assuming that j chooses the sequence $\{u_k^j\}_{t+1}^{\infty}$ optimally over the entire time horizon of the problem with the probability of each choice of the control variable moving the system from one state to another being $Pr(\mathsf{x}_{t+1}|\mathsf{x}_t, u_t)$. The sum of these two components provides the total present value of the investment. It is therefore important when considering an optimal choice of the initial value of u_t^j, that it not only maximises the initial reward at time t, but also steers the system along an optimal path in the evolution of the state variable. In essence, the optimality of the remaining control variables, $u_{t+1}, u_{t+2}, \ldots,$ is subsumed in the choice of u_t^j.

The optimal value of u^j depends on all the values of $J^j(\mathsf{x}')$ that appears, weighted by the appropriate probabilities, in the summation sign on the right of equation [3.5]. The procedure is often very easy to programme and compute with short time horizons or a finite state space. However, as the dimensions of the state space, control variable and/or time horizon increases, obtaining an optimal policy becomes challenging. The approximate dynamic programming approach offers a range of solution methods to tackle the computational challenges posed by high dimensional problems of this kind.

In consideration of the dimensions of the model parameters (i.e. unbounded state space and feasible set of the control variable) and, indeed, the absence of information regarding transition probabilities, the form of the standard Bellman equation that is better suited to the approximate dynamic programming method we implement in our model is the *expectation form*, that is

$$J_t^j(\mathcal{X}) = \max_{u_t^j} \left((\pi^j - \tau(u)) + \gamma E \left(J_{t+1}^j \left(\mathsf{x}_{t+1}^j(\mathsf{x}_t^j, u_t^j, W_{t+1}) \right) \right) \right). \tag{3.6}$$

Equilibrium in the differential game

It is implied in the build up to the model, and to the Bellman equation that solves the value function, that the firms' choices of advertising efforts at any time t depend only on "payoff-relevant" information. Specifically, if at some time $t = \tau$, the system is in state $\mathcal{X}_\tau = (\mathsf{x}_\tau^i, \mathsf{x}_\tau^j)$, a firm's choice of u_τ depends only on $\mathcal{X}_\tau$ through Equations [3.1] and [2], and its demand function. This is because, irrespective of how the system arrives at

$\mathcal{X}_\tau$, at some time τ, the only relevant information to the firm's payoff function is the current state at τ. This sort of solution approach is referred to as a Markov strategy.

Definition. *A Markov strategy for player i is a behavioural strategy, s^i, in which $s^i(\mathsf{x}_t^i, u_t^i) = s^i(\mathsf{x}_t^{*i}, u_t^i)$, if x_t^i and x_t^{*i} are equal and are final goodwill states of different histories.*

Without time dependences, the behavioural strategy s^i is a stationary Markov strategy, and in order to preclude non-plausible equilibria, we require each firm to seek a strategy profile consisting only of Markov strategies that is a Nash equilibrium regardless of the starting state (i.e. a Markov Perfect Equilibrium). This is analogous to a subgame-perfect equilibrium in an extended game. Therefore, seeking an advertising policy that maximises the value function in our model is precisely a question of looking for a Markov Perfect Equilibrium (MPE). We implement a ADP value iteration algorithm to numerically solve for the MPE that describes the firms' optimal advertising efforts through which we derive near-optimal values of their investment opportunities as prescribed by the solution to the initial entry game.

Basis Function and Value Function Approximation

For simple (or low dimensional) stochastic optimisation problems, equation [3.4] may be solved recursively through time in a closed and elegant way. Unfortunately, a vast majority of real world application problems do not fall into this category. Very often, we lack a formal model of the information process or transition function; and while, for some class of problems, the form of the objective function, J, may be known, this is not the case with the objective function that defines the firms' problem in this framework.

Using an approximate dynamic programme method to solve a problem requires an algorithmic strategy that steps *forward* through time, rather than backward in time (exactly computing the value function that is used to produce optimal decisions). When we step forward in time, we have not computed the value function, since the true form of J is not known, so we turn to an approximation. This involves replacing the value function J_t with a approximation of some form. In order to implement the process forward in time, we need to solve two problems: the first is that we need a way to randomly generate a sample of what might happen, and the second is that we need a way to make decisions (approximately).

We begin by identifying what the important features of the problem are, just as we would in a regression analysis. Suppose the system is in state $\mathcal{X}$, a feature is a function $\phi_f(\mathcal{X})$, (otherwise called a *basis function*) $f \in F$ that draws information from $\mathcal{X}$, where we may think of F as the set of features. In practice, creating features is an art form, depends largely on the specific problem, and will necessarily involve a trial-and-error process. After a satisfactory list of features are identified and their relationship specified, an approximation of the value function, say $\bar{J}$, may then be formulated. The approximation will look like

$$J \approx \bar{J}(\mathcal{X}|\theta) = \sum_{f \in F} \theta_f \phi_f(\mathcal{X}) = \phi(\mathcal{X})^T \theta, \tag{3.7}$$

where $\theta = (\theta_1, ..., \theta_I)^T$ is a vector of regression coefficients, and $\phi(\mathcal{X})$ is a column vector of the features. The idea is that we set up an algorithm that uses the value function approximation to make decisions whilst simultaneously updating the regression coefficients to improve the model's predictive power until no further reasonable improvements can be made. This is essentially a learning process in which the system uses simulated paths (or data) to figure out the best value function approximation for the problem.

Making use of $\bar{J}$, optimal advertising decisions are made as follows:

$$u_t = \text{argmax}_{u_t \in \mathcal{U}_t} \Big((\pi_t - \tau(u_t)) + \gamma E \left(\bar{J}_{t+1}(\mathsf{x}_{t+1}) \right) \Big), \tag{3.8}$$

where $\mathsf{x}_{t+1} = \hbar(\mathsf{x}_t, u_t, \omega_{t+1})$, $\mathcal{U}_t$ is the set of feasible actions at time t, and $\hbar(\cdot)$ is a transition function that takes the form of equation [3.1] or [3.2]. In each iteration, the algorithm finds the level of advertising effort that solves $\hat{v}^k = \max_{u \in \mathcal{U}^k} ((\pi_t - \tau(u))$ $+\gamma \sum_{f \in F} \theta_f^{k-1} \phi_f(\hbar(\mathsf{x}^k, u))$, and then recursively updates the value function by updating the coefficient vector θ.

ADP Algorithm: Recursive Least Squares Approach

Step 0. Initialization.

- Step 0a. Initialize $\bar{J}^0$.
- Step 0b. Initialize x^1.
- Step 0c. Set $k = 1$.
- Step 0d. Choose a sample path ω^k of the Wiener process, W.

Step 1. Solve

$$\hat{v}^k = \max_{u \in \mathcal{U}^k} \Big((\pi - \tau(u)) + \gamma \sum_{f \in F} \theta_f^{k-1} \phi_f(\hbar(\mathsf{x}^k, u)) \Big),$$

Let u^k be the value of u that solves the above equation.

Step 2. Recursively update the approximate value function to obtain θ^k.

- Step 2a. $\hat{\epsilon}^k = \bar{v}(\theta^{k-1}) - \hat{v}^k$
- Step 2b. $H^k = \frac{1}{\gamma^k} B^{k-1}$
- Step 2c. $\theta^k = \theta^{k-1} - H^k \phi_f^k \hat{\epsilon}^k$

Step 3. Compute $\mathsf{x}^k = \hbar(\mathsf{x}^k, u^k, W(\omega^k))$ to determine the next state, where $W(\omega^k)$ is a sample realisation of a possible transition.

Step 4. Increment k. If $k \leq K$, go to Step 1.

Step 5. Return the value function $\bar{J}_K$.

The algorithm uses the recursive least square approximation method to update the vector of coefficients θ by reducing the error represent by $\hat{\epsilon}$. B^{k-1} used in the algorithm is a $|F|$ by $|F|$ matrix, updated recursively using

$$B^k = B^k - \frac{1}{\gamma^k} \left(B^{k-1} \phi^k (\phi^n)^T B^{k-1} \right),$$

where $\gamma^k = 1 + (\phi^k)^T B^{k-1} \phi^k$ (see Powell (2007) for further and technical details on various ADP algorithms, prospects and limitations.)

We implement the above algorithm to find the MPE and the optimal coefficient vector that is used along with our chosen basis function to approximate the value function under different product market outcomes that may occur at the end of the entry game. Whilst the algorithm may suffer from convergence issues and, indeed, could be unstable, with careful choice of the basis function, and a specific form of the recursive least squares updating of the coefficient vector (as shown in the algorithm), we are able to improve the chances of convergence. Also, although it cannot be guaranteed that, in general, that the MPE is unique, our computations always led to the same value and policy function irrespective of the starting values.

Basis Function Construction

Constructing a basis function for a given problem is as much science as it is art. While some studies focus on constructing basis functions automatically, constructing a basis function using knowledge of some ramifications of the problem and experience of how the object being approximated behaves usually aids in deriving more accurate approximations specific to the problem at hand, much like intuiting the nature of the relationship between explanatory variables and the response variable through observable data in a regression analysis. We would not normally an investment value that increases infinitely, more so, goodwill does not grow infinitely either. What is observed often is goodwill having an initial surge through advertising, reaching some critical level where diminishing returns set in, and then settle to some value or — in our case, due to the stochastic element — within some interval over the long haul, unless other drastic measures are employed which could modify the dynamics of the firm's goodwill. But the analysis of such measures is outside the scope of our model.

With this consideration, we ran a number test simulations of possible trajectories with various functional forms that meet the criteria set out above. A choice formulation of the basis function derived from these, and which we then use in all the subsequent simulations that follow is shown in equation [3.7], that is, we assume that the value function, approximately, takes the form,

$$\bar{J}(\mathsf{x}|\theta) = \phi(\mathsf{x})^T\theta = (\ln(\mathsf{x}), \ln(\mathsf{x}^2), \ln(\mathsf{x}^3)) \cdot (\theta_1, \theta_2, \theta_3)^T. \tag{3.9}$$

3.4 Continuation Games Analyses

3.4.1 Monopoly

Suppose after the first investment window, $\mathcal{A}$ is a singleton. We interpret this to mean that one of the firms is active while the other is dormant – the active firm now acts as a monopolist. The monopolist maximises $(p-c)\kappa(\mathsf{x}^i)(a-bp)$ with respect to p. Let $\kappa(\mathsf{x}^i) = \kappa \cdot \mathsf{x}^i$, so, the firm's goodwill shifts its demand curve to the right. The parameters a and b are chosen carefully to reflect their equivalent values in the maximisation problem of the duopolist with differentiated products (more on this in the next section).

The monopolist's immediate reward is $\pi^M = \kappa\mathsf{x}^i(a-bc)^2/4b$, using the ADP algorithm we solve

$$\max_{u\in\mathcal{U}^k}\Big(\kappa\mathsf{x}^i(a-bc)^2/4b - \tau u^2 + \gamma\phi(\mathcal{X})^T\theta\Big); \tag{3.10}$$

subject to the evolution of i's goodwill. At each stage of the algorithm, we find the u^k that maximises equation [3.10]. The functional value such a u^k generates is then used in a least squares approach (details in the appendix) to update the values of the vector of parameters, θ, and subsequently update the firm's goodwill value to move the system into a new state.

At each iteration k, as long as the remaining[6] goodwill of j is positive, the dynamical system evolves according to

$$\Delta\mathsf{x}^i_t = (\lambda(\beta^i, u^i_t) - \delta\mathsf{x}^i_t + \beta^i u^i \mathsf{x}^j_t)\Delta + \sigma_{\mathsf{x}^i} W_t, \tag{3.11}$$

where $\lambda(\beta^i, u^i_t) = \alpha \cdot \beta^i \cdot u^i$. In this case, the active firm exploits all the remaining goodwill of firm j whose goodwill declines according to $\mathsf{x}^j_{t+1} = \mathsf{x}^j_t(1-(\beta^i u^i + \delta)\Delta)$.

[6]Since j is dormant, the evolution of its goodwill only depends on the actions of i and a constant rate of decline through the churn parameter δ.

When, however, j's goodwill value hits zero, the dynamical system of the active firm evolves according to

$$\Delta \mathsf{x}_t^i = (\lambda(\beta^i, u_t^i) - \delta \mathsf{x}_t^i)\Delta + \sigma_{\mathsf{x}^i} W_t \tag{3.12}$$

It is worth mentioning here that i's rival, j, is dormant at this time and cannot, therefore, take any strategic action (advertising) to improve its own goodwill. As a result, j's goodwill is decreasing in its churn parameter, δ, and in i advertising efforts, u^i.

After the algorithm's convergence criterion is met, the final value of the vector θ is used to calculate the monopolist's approximate value function as follows:

$$J^M = \left(\frac{\kappa \mathsf{x}_{(k=1)}(a - bc)^2}{4b} - \tau \cdot u^2_{(k=1)} \right. \\ \left. + \gamma \left(\theta_1^{(K)} \cdot \ln(\mathsf{x}_{(K)}) + \theta_2^{(K)} \cdot \ln(\mathsf{x}^2_{(K)}) + \theta_3^{(K)} \cdot \ln(\mathsf{x}^3_{(K)}) \right) \right). \tag{3.13}$$

Numerically derived advertising policy and value function

We recapitulate here that while the framework developed so far provides the benefit of introducing intricate details such as the potency of an ad campaign into the stochastic differential equation governing the evolution of a dynamical system, it precludes the usual analytical solution approach due to the high dimension of the variables involved. Instead, we rely on numerical methods to compute the MPE in each scenario that we consider in the following sections of this chapter. Starting with the single-active-firm case above, we solve equation [3.10] by simulating the processes described by equations [3.11] and [3.12], with [3.13] as an approximation of the value function, using the ADP algorithm described in the previous section.

Parameters used for simulation

$x_{k=1}$	30	γ	0.9091	α	5
Δ	1/12	σ	0.05	c	5
θ_1	5	δ	0.10	τ	10
θ_2	10	a	4	b	2
θ_3	8	κ	3	n	3000

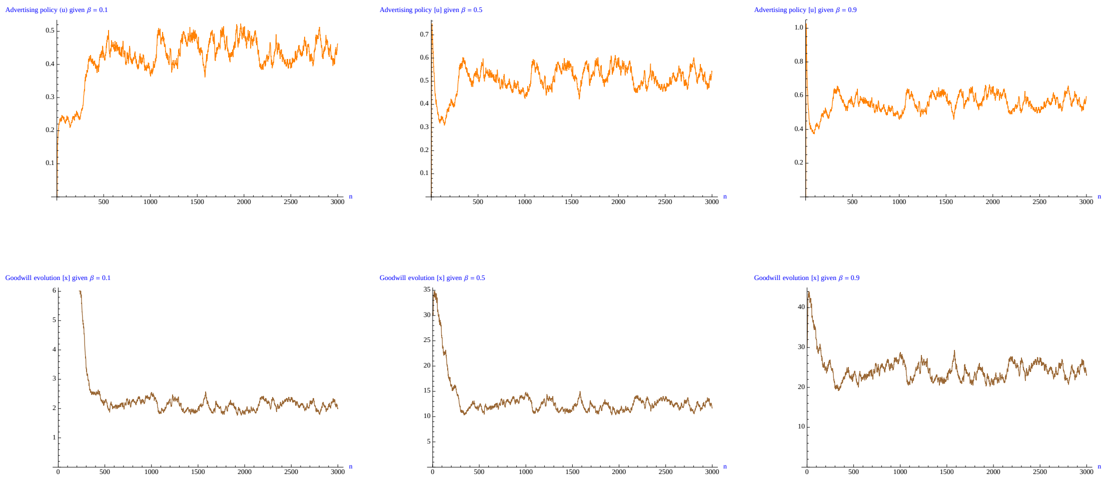


Figure 3.3: Steady-state advertising policy and goodwill

For consistency across the single firm and strategic scenarios of our model, the parameters a and b are derived from the competitive price-setting demand function in equation [3.4] in which $b = 1/\tilde{b}$ and $a = \tilde{a}(\tilde{b} + \eta)$. But $\eta = 0$ in the market for a single product, so if we let $\tilde{b}$ in equation [3.4] be 0.5 as in Rao (1984), then $b = 2$ and $a = \tilde{a} \cdot b = 4$ if we choose $\tilde{a} = 2$ (as we would when we run the simulations for the strategic case of our model). We have assumed a monthly time step in our simulation since it seems more practical that firms will assess their advertising budgets on a monthly basis rather than weekly or daily. Also, it is short enough a time for them to make adjustments and long enough to observe market reactions and review the benefits. The simulation is carried out over ($n =$)3000 iterations, and in each case, the learning parameters θ are updated to improve their predictive power and the accuracy of the approximate value function. Along with this, the optimal advertising policy and evolution of the stock of goodwill are derived. Figure 3.3 shows the steady-state advertising policy and goodwill for different values of the β. The steady-state reflects outcomes without the time dimension, and is used to determine the value of being in any given state.

The upper panels in Figure [3.3] show the monopolist's advertising policy for some levels of campaign potency i.e. $\beta = 0.1, 0.5$ and 0.9; while the lower panels show the evolution of the stock of goodwill. Since business stealing opportunities for the monopolist goes only as far as the value of initial goodwill the dormant firm has, its advertising efforts are predominantly informative. With low potency the monopolists seems to accelerate its advertising intensity at the outset, but then settles into a steady-state, whilst with a relatively high potency, it starts out dropping its intensity until the dormant firm's goodwill is dissipated and then is forced to raise its intensity enough to keep its goodwill from falling.

If we ignore, for a moment, the stochastic element of the dynamical system, it can be seen, through the amount of goodwill a unit of advertising intensity generates per period for the monopolist i.e. $\partial \Delta \mathsf{x}^i / \partial u^i = \beta^i(\alpha + \mathsf{x}^j)\Delta$, that the marginal contribution to the firm's goodwill from the informative part, $(\beta^i \Delta \alpha)$, is constant for any given level of β^i; but the marginal contribution to the firm's goodwill from the business-stealing part, $(\beta^i \Delta \mathsf{x}^j)$, is monotonically decreasing in the rival's goodwill level. Also, the dormant firm's goodwill is dissipated if $(\beta^i u^i + \delta)\Delta \to 1$[7], so that as $u^i \to (\Delta^{-1} - \delta)/(\beta^i)$ firm i can drastically reduce its rival's goodwill. It is clear that with a low β^i, i will have to ramp up its advertising intensity in order to contribute to its own goodwill and dissipate its rival's faster. But the intensity required to dissipate x^j is decreasing in the value of β^i, so we may not observe high intensity of advertising from a firm with a high ad campaign potency since the contribution of its informative aspects is much more significant than with a low β^i. However, as soon as the dormant firm's goodwill is dissipated, the marginal contribution of the firm's advertising to its goodwill, $(\beta^i \Delta \alpha)$, is constant in each period and only changes by the magnitude of β^i for different firms with varied levels of ad campaign potencies as is shown in the upper panels of Figure [3.3].

Furthermore, as the upper panels of Figure [3.3] show, as well as in Figure [3.4], the monopolist's stead-state advertising policies do not change nearly as much as its goodwill with higher levels of ad campaign potency. This is because, with a high value of β, advertising is still only informative, just more potent; but increased volume, with constant marginal cost does not translate into higher prices, in fact, each new consumer

[7]Note that $\mathsf{x}^j_{t+1} = \mathsf{x}^j_t(1 - (\beta^i u^i + \delta)\Delta) \to 0$, as $(\beta^i u^i + \delta)\Delta \to 1$.

brings their demand at the low price. This is consistent with the analytical outcomes of informative advertising where, with constant marginal cost, advertising will have no effect on the monopolist's price (see Bagwell (2007) on monopoly advertising). And as shown in Figure [3.4], β clearly plays a more important role in creating demand through goodwill with no additional running costs, unlike the $2\tau u_t$–expenditure that is required for any u_t–amount of advertising, a monopolist is not incentivised to increase its advertising intensity, which draws a penalty of τu^2 in the optimisation problem.

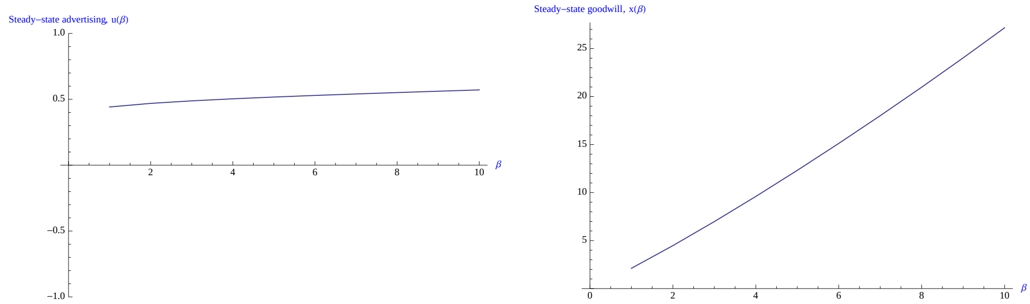


Figure 3.4: Steady-state u and x over the distributional support of β

In addition, Figure [3.5] shows that J^M (the present value of the investment as described in equation [3.13]) increases monotonically with β, therefore, there exists β^*, such that, for all $\beta \in (0, \beta^*] \subset (0, 1)$, we will have $V^M(\beta) = J^M(\beta) - \mathcal{K} \leq 0$ for some $\mathcal{K}$. Consequently, for such value of $\mathcal{K}$, a monopolist will not find it optimal to invest if its $\beta \in (0, \beta^*]$.

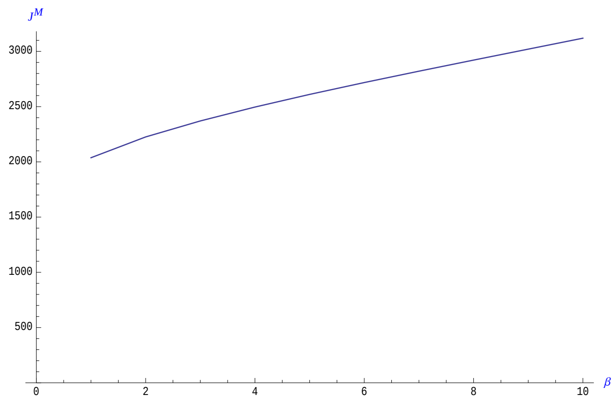


Figure 3.5: Approximate value of J^M over the distributional support of β

In the same vein, Table [3.1] shows the steady-state advertising, goodwill and present value function over the support of β.

Table 3.1: Approximate steady-state goodwill and advertising levels

	$\beta^i = 0.1$	$\beta^i = 0.2$	$\beta^i = 0.3$	$\beta^i = 0.4$	$\beta^i = 0.5$	$\beta^i = 0.6$	$\beta^i = 0.7$	$\beta^i = 0.8$	$\beta^i = 0.9$	$\beta^i = 1.0$
$\bar{u}^i$	0.44	0.47	0.49	0.50	0.52	0.53	0.54	0.55	0.56	0.57
$\bar{x}^i$	2.11	4.47	6.98	9.60	12.31	15.12	18.01	20.98	24.03	27.16
$J^M(\beta^i)$	2037.41	2225.64	2370.33	2496.12	2610.56	2717.85	2820.74	2921.12	3020.34	3119.4

3.4.2 Duopoly

The single-active-firm case is instructive, and provides a view towards how the dynamical system proposed in this chapter influences advertising efforts, which in turn, together with the campaign's potency, determine the value of the investment. However, the rival firm in the scenario above is dormant, so we now consider the case where both firms can be active in the market and what the industry structure will look like in equilibrium based on the relative potencies of the firms' advertising campaigns.

To fully analyse the continuation games in the extended form of the entry game, a number of competitive scenarios will be considered around the options available to the firms at each turn. We consider the following cases: a) both firms have no option to delay and, therefore, have to either sink the initial investment outlay or quit, implying that there is only one investment window; b) one of the firms holds an option to delay, while the other does not; and c) both firms hold an option to delay making their investment decisions until the next investment window.

The dynamical system does not change in any of these cases, however, the value of the investment will depend on the information available to either firm at the time of making its investment decision. As we will show, this will depend predominantly on the firm's type which is the relative potency of its campaign to its rival's. The value of the objective functional in each case informs the rewards obtainable when firms are locked together in the entry game from which we may determine equilibrium strategies based on the firms' types. With regards to the firms' types, this is an important form of operational efficiency (i.e. relatively more potent ad campaign), because it is not driven internally (e.g. as with lower marginal costs), rather externally through consumer feedback and needs to be exploited optimally through u_t for all time in order to derive the most benefit. Therefore, ex ante, the firms are incentivised to put their very best foot forward if at all they want to be competitive in the product market.

(a) Duopoly with no options to delay (simultaneous-play)

Suppose there are no options to defer the decision problem until a future time when the firms may have gathered additional information regarding the uncertain elements of the entry game, they each have to either enter the market if they find it optimal to do so, or quit. This is the classic "now-or-never" investment paradigm.

To begin with, we recall that at the time of choosing whether or not to invest, the potencies of the firms' campaigns are private information. Also, the ex post evolution of the goodwills that the firms will face (i.e. the system dynamics) depends on the relationship between their yet-to-be-revealed types. It has been assumed that the firms are ex ante symmetrically uninformed, so it suffices to solve the decision problem of one firm, say i, since j will be facing the exact same decision problem ex ante and form similar expectations about its rival.

At the start of the entry game, i cannot precisely predict how its stock of goodwill will evolve (either according to equation [3.1] or [3.2]), should it enter the market, because β^j is unknown. Its best guess therefore will involve forming expectations over the possible values of β^j.

To solve i's decision problem, we begin by stepping forward in time to the period where both firms have just become active in the market. After choosing price and advertising efforts optimally, the relationship between their types is revealed. Only two cases are feasible ex post, i.e. i realises that $\beta^i \gtrless \beta^j$. So let $\widetilde{J}^i$ denote i's ex post approximate value function if $\beta^i < \beta^j$, the evolution of the firms' respective goodwill follow equations [3.1] and [3.2]. On the other hand, should i observe that $\beta^i > \beta^j$, then the evolution of goodwill still follow equations [3.1] and [3.2], but with i assuming the role of j in the system dynamics. We denote i's approximate value function in this case with $\widehat{J}^i$.

In this section, we will demonstrate the importance of the amount of information available to both firms ex ante, and how, by stepping forward in time (but reasoning backwards), they can update this information and adjust their decision rules accordingly. For instance, suppose ex ante, that $\beta^i < \beta^j$, then i knows how its goodwill will evolve, but cannot tell by how much (or how fast) its own goodwill will be dissipated by j (i.e how close to 1 β^j is). The only obvious course of action for i, is to make an educated guess of what β^j could be. And since the distributional form of the β's is known ex ante (i.e. β^i, β^j are independent uniformly distribution random variables over the interval $(0,1)$), i's best guess of β^j is the expected value of β^j over the interval $(\beta^i, 1)$, which is $E(\beta^j) = (\beta^i + 1)/2$.

i is faced with a slightly different piece of information ex ante if it were to assume that $\beta^i > \beta^j$ ex post. This is because the evolution of the stocks of goodwill as in equations [3.1] and [3.2] now depends largely on β^i, so that i only needs to form expectation on the informative aspects of its rival's goodwill function i.e. $\lambda(\beta^j, u^j)$. And since $\beta^i > \beta^j$, i's best guess of this effect is the expectation of β^j over the interval $(0, \beta^i) = \beta^i/2$. Now, i can construct its ex ante objective functional under these two ex post scenarios as follows:

$$J_t^{D_i} = \max_{u_t^i} \left(\pi_t^{*D_i} - \tau \cdot (u_t^i)^2 + \gamma \left(Pr(\beta^j > \beta^i) E_t(\widetilde{J}_{t+1}^i) + Pr(\beta^j < \beta^i) E_t(\widehat{J}_{t+1}^i) \right) \right). \tag{3.14}$$

The time-stamps on elements of the objective functional in equation [3.14] indicate immediate and future rewards following from the expectation form of the objective functional in equation [3.6]. Also, π^{*D_i} is the profit a duopolist earns by engaging in a price-setting competition in a differentiated product market and is derived by solving

$$\begin{aligned} \max_{p^i} \pi^{D_i} &= \max_{p^j} \left((p^i - c^i)\kappa \cdot \mathsf{x}^i(\tilde{a} - \tilde{b}p^i + \eta p^j) \right), \\ \max_{p^j} \pi^{D_j} &= \max_{p^j} \left((p^j - c^j)\kappa \cdot \mathsf{x}^j(\tilde{a} - \tilde{b}p^j + \eta p^i) \right); \end{aligned} \tag{3.15}$$

simultaneously for Nash equilibrium prices p^{*i} and p^{*j}, substituting these back into the profit functions to derive π^{*D_i} and π^{*D_j}, where for i, $\pi^{*D_i} = (\kappa \mathsf{x}^i \tilde{b}(\tilde{a} + c(\eta - \tilde{b}))^2)/(\eta - 2\tilde{b})^2$. The payoff functions in equation [3.15] follow from the demand function specification in Rao (1984), where the value of η suggests whether the products are substitutes ($\eta > 0$) or are complements ($\eta < 0$). The demand function specified in the monopolist's case is easily inverted to generate the parameter values in the equation above if $-1 < \eta/b < 1$. In the same vein, since $b = 2$, then $\tilde{b} = 1/2$; we have assumed that the products are substitutes, therefore we set $\eta = 2/5$, which both satisfies the demand function conditions above and is efficient for the simulation of the paths for u^i and u^j in our ADP algorithm.

Information updating

Equation [3.14] is the present value of i's investment if both firms are active in the market. But it precludes the possibility of either one of the firms choosing not to make the investment (which in the 'no-options' case means the firm never gets to make the investment at all i.e quits the game altogether)[8]. In reality, given that the evolution of the firms' stocks of goodwill are influenced, indeed, determined by their types, should they simultaneously make the investment, their present value functions would also be influenced by their types. As a result, in the presence of an initial investment outlay, $\mathcal{K}$, the metric of concern to the firms, if they both decide to be active at $t = 1$ (where $\mathcal{K}$ is chosen to be less than $J^{D(\cdot)}$ for some types[9]), is the net present value of the investment, which we represent, for i, as $V^{D_i}(\beta; i) = J^{D_i} - \mathcal{K}$. Therefore, since the firms' (expected) net present values behave analogously to the present value (as will be shown below), for some value of $\mathcal{K}$, there exists a threshold-type $\tilde{\beta}$, such that $V^D(\tilde{\beta}) = 0$, so that for all $\beta^i < \tilde{\beta}$, $V^D(\beta^i) < 0$. It is therefore no longer the case that both firms will be active in the market irrespective of their types. The decision criteria will be: for all $\beta > \tilde{\beta}$, $V^D(\beta) > 0$, and the firm should invest, irrespective of its rival's type. However, if $\beta < \tilde{\beta}$, a firm should quit only if it believes its rival's type is above the threshold

[8]Recall that in this scenario, the firms either invests at $t = 1$ or not at all (i.e. the 'no-or-never' investment paradigm)

[9]This does not preclude the possibility of introducing an option even for the monopolist as we would later on in our analyses.

$\tilde{\beta}$. In essence, to be able to formulate an entry strategy in these circumstances, i has to consider that its rival will be solving a similar game, and while it has a dominant strategy to invest if $\beta^i > \tilde{\beta}$, the expected present value of its investment depends on j's type and, indeed, j's belief about i's type. The reasoning is that if $\beta^i > \tilde{\beta}$, we can be certain that the product market will be occupied by two firms if $\beta^j > \tilde{\beta}$, which means that i earns a present value of J^{D_i}, but if $\beta^j < \tilde{\beta}$, there is a chance that i may actually earn J^{M_i}, as its present value, since it is not in j's best interest to engage as a duopolist, so j should quit, but should only do so if it actually believes that $\beta^i > \tilde{\beta}$, which i cannot be certain of, and therefore should not rely on. In the spirit of Nash equilibrium, j's belief about i's strategy over the space of its type should be derived using Bayes' rule. Therefore, if $\beta^j < \tilde{\beta}$, then j should belief that i will invest with probability $Pr(\beta^i > \tilde{\beta})/[Pr(\beta^i > \tilde{\beta}) + Pr(\beta^i < \tilde{\beta})] = Pr(\beta^i > \tilde{\beta})$, since the history of the game before this time is a null set. Due to symmetry, i would hold an analogous belief about j and in this circumstance, therefore, i's expected net present value when $\beta^i > \tilde{\beta}$ is $Pr(\beta^j < \tilde{\beta})J^{M_i} + Pr(\beta^j > \tilde{\beta})J^{D_i} - \mathcal{K}(= \bar{V}_{\beta^i > \tilde{\beta}})$.

Simulation results for the duopoly

We solve for J^D using a modification of the algorithm presented earlier to account for strategic interactions where $\tilde{a} = a/b = 2$ and $\tilde{b} = 1/2$. For the sake of our analysis, suppose $\mathcal{K} = 650$, we therefore see from Table [3.2] and Figure [3.6] that there is indeed such a β for which $V^D(\beta) = J^D - \mathcal{K} = 0$, where $V^D(\beta)$ denote the expected net present value of the investment.

Table 3.2: Approximate steady-state goodwill, advertising levels and value function

	$\beta^i = 0.1$	$\beta^i = 0.2$	$\beta^i = 0.3$	$\beta^i = 0.4$	$\beta^i = 0.5$	$\beta^i = 0.6$	$\beta^i = 0.7$	$\beta^i = 0.8$	$\beta^i = 0.9$	$\beta^i = 1.0$
$\bar{u}^i$	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.21	0.22	0.22
$\bar{u}^j$	0.24	0.27	0.28	0.29	0.30	0.31	0.32	0.33	0.34	0.35
$\bar{x}^i$	6.34	7.72	9.22	10.83	12.55	14.37	16.30	18.34	20.48	22.73
$\bar{x}^j$	0.56	1.18	1.77	2.34	2.90	3.43	3.94	4.44	4.92	5.38
$J^D(\beta^i)$	150.19	222.28	294.80	369.03	445.40	524.06	605.24	689.18	776.21	866.69
$V^D(\beta^i)$	−499.81	−427.72	−355.20	−280.97	−204.61	−125.94	−44.76	39.18	126.21	216.69

We see from the right panel of Figure [3.6], that $\beta \approx 0.75$.

Figure [3.7] and [3.8] show the steady-state advertising policies and goodwill levels

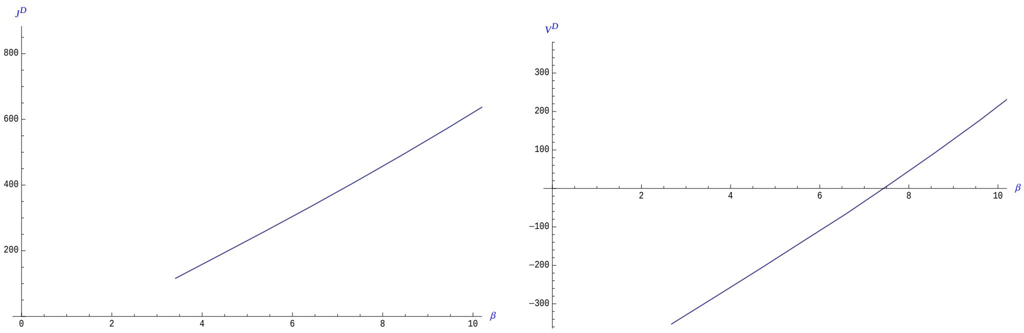


Figure 3.6: Present and net present value function for $\mathcal{K} = 650$

when $\beta^j < \beta^i = 0.2$ and $\beta^j < \beta^i = 0.8$. It is clear from the evolution diagrams that the firm with a less potent campaign (j) will need to ramp up its advertising intensity at the early stages of the game to offset the amount of goodwill it loses to i. So while j starts out with increasing its advertising intensity primarily to expand the market, i slows down its own intensity until it reaches a steady state. Since the differential in their campaign potencies does not change the system dynamics with respect to the nature of evolution, the effect of a much larger differential in their potencies is only reflected in how quickly j has to raise its advertising intensity to remain in the market, and in a higher level of goodwill in equilibrium, which in turn enters into the expected present value function of the investment, so that it is monotonically increasing in β.

Strategies when $\beta^i < \tilde{\beta}$ are even more ambiguous to prescribe than the counterfactual, because ex ante, i needs to think about all the possible ex post market outcomes given the relationship between β^i and β^j and the investment thresholds. The table below summarises the type-driven outcome of the entry game due to ex ante available information and information updating.

i/j	$\beta^j < \tilde{\beta}$	$\beta^j > \tilde{\beta}$
$\beta^i < \tilde{\beta}$	$\mathcal{I}, \mathcal{Q}; \mathcal{Q}, \mathcal{I}$	$\mathcal{Q}, \mathcal{I}$
$\beta^i > \tilde{\beta}$	$\mathcal{I}, \mathcal{Q}$	$\mathcal{I}, \mathcal{I}$

It does not follow immediately that i should simply quit if $\beta^i < \tilde{\beta}$, given that β^j could very well be below this threshold too. This is the problem presented in the table above. It is clear that when both firms' types are higher than the threshold, making the investment is the dominant strategy for either one of them. But if, for example, $\beta^i < \tilde{\beta}$,

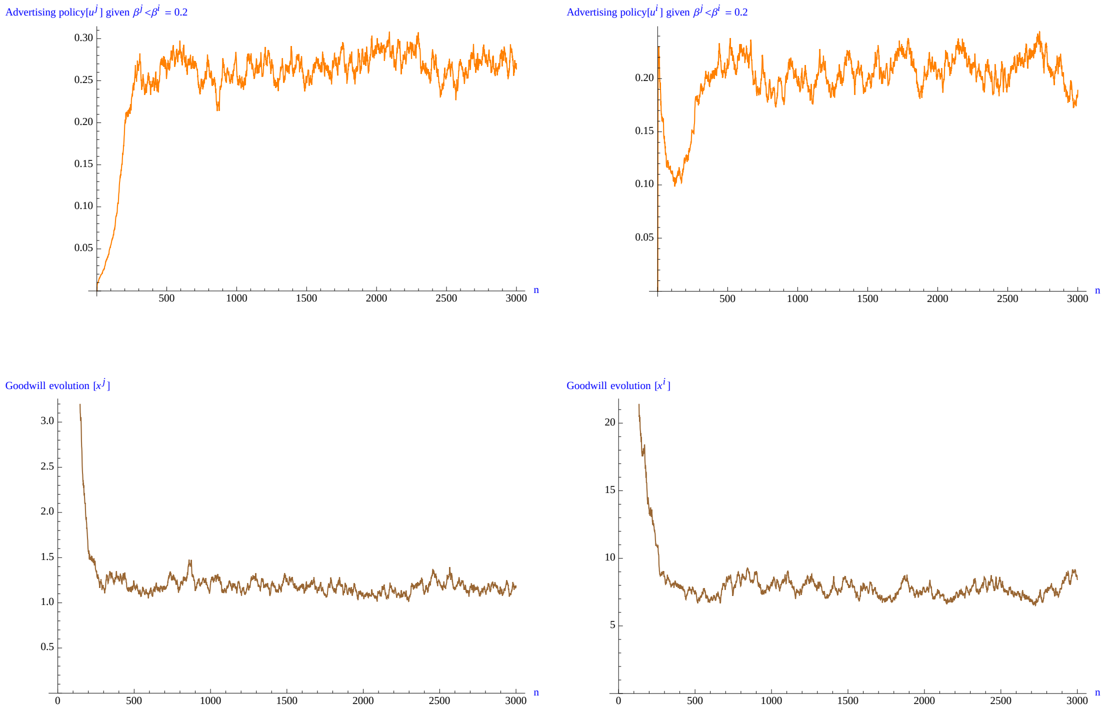


Figure 3.7: Steady-state advertising policy and goodwill with $\beta^j < \beta^i = 0.8$

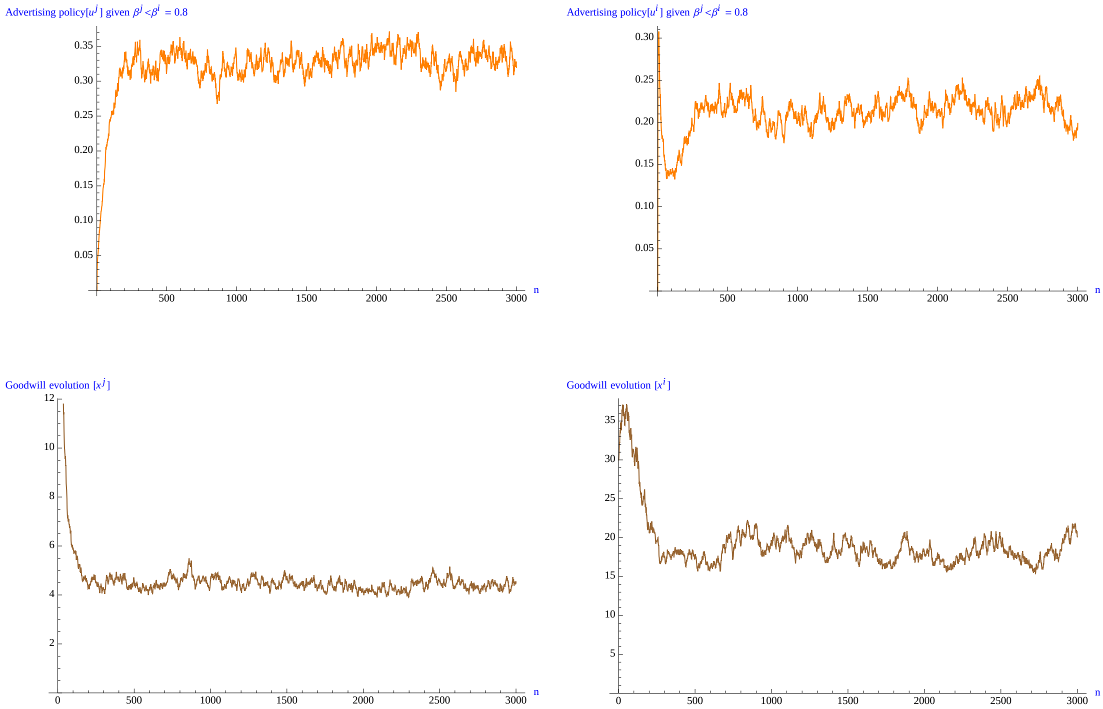


Figure 3.8: Steady-state advertising policy and goodwill with $\beta^j < \beta^i = 0.8$

i's optimal actions depend on which side of $\tilde{\beta}$ β^j lies. So if $\beta^j > \tilde{\beta}$, then i realises that j still has a dominant strategy to invest, and i's best choice is to quit. But if $\beta^i, \beta^j < \tilde{\beta}$, for some value of $\mathcal{K}$, our simulation results of the ex pose market shows that, it is not in either of their interests to choose the same action. The investment is worthless if they both invest since $V^{D_{i,j}} < 0$ when $\beta^i, \beta^j < \tilde{\beta}$. Also, if they both do not invest simultaneously, then at least one of them could have been profitable as a monopolist if he had made the investment, so the opportunity cost of quitting when your rival quits is the monopolist's net present value of the investment that is foregone. As a result, we have an *anti-coordination* problem where the only sensible course of action is for both firms to take opposite actions. If one invests, the other should quit, and vice-versa. This is, however, predicated on a presumed knowledge of which side of $\tilde{\beta}$ a firm's rival's type lies, an information that is not available ex ante, as it would still be private information. Ex ante therefore, there is no dominant strategy equilibrium in the entry game when either firm observes that its type falls below the threshold $\tilde{\beta}$; however, we can construct a strategy which depends on the realisation of a firm's type, such that the firm may adopt different pure strategies below and above a threshold type with the belief that its rival, in the same situation, would adopt a similar strategy and therefore create an equilibrium behaviour. This is a possible solution to the anti-coordination problem in this interval as illustrated below:

Proposition 4. *If β^i and β^j are less than $\tilde{\beta}$ the firms will want to anti-coordinate, but there is at least one type-threshold in this interval such that below and above this threshold, a firm would choose Q or I according to a unique mixed strategy Nash equilibrium.*

Proof: Suppose $\beta^i, \beta^j < \tilde{\beta}$, the normal form of the game that ensues is given by

i/j	$\mathcal{Q}$	$\mathcal{I}$
$\mathcal{Q}$	$0, 0$	$0, V^{M_j}$
$\mathcal{I}$	$V^{M_i}, 0$	V^{D_i}, V^{D_j}

Note that $\operatorname{sgn}(V^{D_i}) = -1$ since for either firm, entering a competitive market with campaign potency measured less than $\tilde{\beta}$ is not optimal given $\mathcal{K}$; however, $V^{M_i}, V^{M_j} > 0$

for all $\beta \in (0,1)$, there are therefore two pure Nash equilibria i.e. $(\mathcal{I}, \mathcal{Q})$ and $(\mathcal{Q}, \mathcal{I})$, so that it is in both firms' interests to anti-coordinate. Now suppose i thinks j, with a specific realisation of β^j in $[0, \tilde{\beta})$ would use a mixed strategy ξ and $(1-\xi)$ such that for a given $\beta \in [0, \tilde{\beta})$, j will play $\mathcal{Q}$ if $\beta^j < \beta < \tilde{\beta}$ and $\mathcal{I}$ if $\beta < \beta^j < \tilde{\beta}$. Then i will be indifferent between $\mathcal{I}$ and $\mathcal{Q}$ if its expected net present value is such that $\xi V^{M_i} + (1-\xi)V^{D_i} = 0$. So that if i's type is also below β then $\xi V^{M_i} + (1-\xi)V^{D_i} < 0$ and i will choose $\mathcal{Q}$, but if i's type is above β, then $\xi V^{M_i} + (1-\xi)V^{D_i} > 0$ and i will choose $\mathcal{I}$. Note that such a ξ would exist for a particular β, since if $\xi V^{M_i} + (1-\xi)V^{D_i} > 0$, then $\xi > -V^{D_i}/[V^{M_i} - V^{D_i}]$, so that $0 < \xi < 1$ since $\mathrm{sgn}\,(V^{D_i}) = -1$, $\mathrm{sgn}\,(V^{M_i} - V^{D_i}) = 1$, and $-V^{D_i} < V^{M_i}$. Therefore, for any particular β we may find a specific ξ which satisfies the inequalities above. □

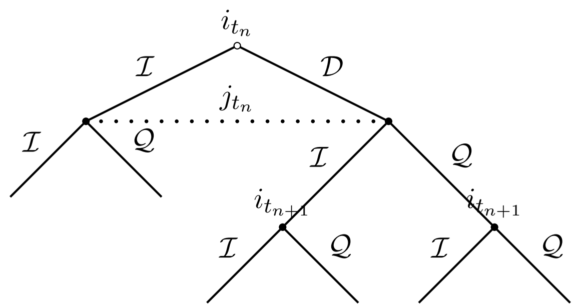
(b) Duopoly with a one-sided option

In practice, firms often have the opportunity to postpone their investment decisions for a number of reasons, one of which, in an uncertain competitive market, is the opportunity to gather better information about the uncertain elements of market dynamics before making any commitments. A firm with an option to delay is faced with the conflict between investing early and perhaps facing a firm with a higher type, or delaying and losing out on potential earnings. Holding such an option is clearly important and carries some value. Let us suppose, therefore, that one of the firms in our model has an option to defer its investment decision until a future investment window. In this section we explore the outcome of the entry game and the value of the investment opportunity to each firm.

Suppose at $t = t_n$, i holds an option to delay but j does not. i's choices are either to invest at t_n, or defer making that choice until the next investment window at t_{n+1}. j, on the other hand, is only faced with the choice to invest or quit at t_n. One of the benefits of holding an option in the entry game is that by delaying, the firm with the option has the opportunity to observe its rival's type after the first investment window by estimating the proportion of the goodwill it has lost to its rival. In this case, should the firm with the option consider that its type is lower, it might be better if it does not invest after all and not incur the initial sunk cost.

If $\mathcal{A}_{t_{n-1}} = \varnothing$, and i kills its option (i.e. i chooses to invest), the entry game reduces to the version studied in the last section (simultaneous-play), and the firms' competitive strategies and payoffs are as obtained previously. Accordingly, there exists a type threshold below which the investment is worthless. If i's type falls below this threshold, it would rather exercise its option to defer, if it does so, the value of the investment opportunity to i and j are derived by considering all the possible market structures that may emerge.

Suppose that i delays until t_{n+1} while j invests at t_n (since j has no option to defer), then j will earn monopoly rents in t_n, but its type becomes revealed to i, whereas, i's type remains private information . The extended-form of this entry game is as shown below.



Where $\mathcal{I} \to$ Invest, $\mathcal{D} \to$ Defer, and $\mathcal{Q} \to$ Quit.

In period t_{n+1}, i is faced with the choice to either invest or defer/quit, and will act as a second-mover if it chooses to enter the market since j has chosen earlier to invest rather than quit. We proceed by solving i's problem first.

Suppose j enters the market by investing at t_n but i defers until t_{n+1}. They will both earn duopoly rents in period t_{n+1} and forever. But by waiting until t_{n+1}, i loses any profits it could have earned in t_n. By delaying, however, i would have learned about j's type, i.e. β^j will no longer be private information, which then gives i the opportunity to decide whether or not to invest based on this new information in order to avoid costly entry against a more effective rival.

We consider i's choices at t_{n+1}: if j has invested at t_n, i will observe j's type, and can work out the profitability of engaging j in a duopoly competition and, therefore, the value of exercising its option to defer. The ex ante value to i of investing at t_{n+1}, given j is active, is obtained through the functional,

$$\Gamma^i = [Pr(\beta^j < \beta^i|\beta^j > \tilde{\beta}) + Pr(\beta^j < \beta^i|\beta^j < \tilde{\beta})]E_{t_n}\Big(\pi^{*D_i}_{t_{n+1}} - \tau\cdot(u^i_{t_{n+1}})^2 + \gamma\widetilde{J}^i_{t_{n+2}}\Big)$$
$$+ [Pr(\beta^j > \beta^i|\beta^j > \tilde{\beta}) + Pr(\beta^j > \beta^i|\beta^j < \tilde{\beta})]E_{t_n}\Big(\pi^{*D_i}_{t_{n+1}} - \tau\cdot(u^i_{t_{n+1}})^2 + \gamma\widehat{J}^i_{t_{n+2}}\Big), \tag{3.16}$$

where $\widetilde{J}^i$ and $\widehat{J}^i$ are as defined earlier, and as used in equation [3.14], but adjusted for the relevant time periods within this framework.

i's net present value of the investment at t_n, if it chooses to invest at t_{n+1} is given by $\gamma(\Gamma^i - \mathcal{K})$. At t_n, if j invests, it does so having incomplete information over i's type and will conjecture that, in expectation, for some values of $\mathcal{K}$, there exists a $\breve{\beta}^i$ such that if $\beta^i < \breve{\beta}^i$, then $\gamma(\Gamma^i(\beta^i) - \mathcal{K}) < 0$. From our simulation of the system dynamics for this scenario, using the parameters specified earlier, $\breve{\beta}^i \approx 0.38$ as shown in Table [3.3].

Table 3.3: Net present value at t_{n+1}

	$\beta^i = 0.1$	$\beta^i = 0.2$	$\beta^i = 0.3$	$\beta^i = 0.4$	$\beta^i = 0.5$	$\beta^i = 0.6$	$\beta^i = 0.7$	$\beta^i = 0.8$	$\beta^i = 0.9$	$\beta^i = 1.0$
Γ^i	566.18	606.59	631.87	650.85	666.29	679.49	691.14	701.69	711.42	720.55
$\gamma(\Gamma^i(\beta^i) - \mathcal{K})$	−76.20	−39.46	−16.48	0.77	14.81	26.81	37.40	46.99	55.84	64.14

j is aware that if i chooses to defer at t_n, it would only do so if $\beta^i < \tilde{\beta}$. But if i were to invest at t_{n+1} after j had invested at t_n, then $\beta^i > \breve{\beta}^i$, otherwise i simply quits. We note that should i observe that j quits at t_n, then irrespective of i's type, it invests at t_{n+1}. It follows that if $\beta^i < \breve{\beta}^i$, i defers at t_n, and should j invest at t_n (which it does if $\beta^j > \tilde{\beta}$), i quits and j remains a monopolist. Also, $\breve{\beta}^i < \tilde{\beta}$, so unlike the 'no options' case of the model considered earlier (where i uses a mixed strategy if $\beta^i < \tilde{\beta}$), if $\breve{\beta}^i < \beta^i < \tilde{\beta}$, then i has the option to defer and observe j's type before engaging t_{n+1}, but if $\beta^i < \breve{\beta}$, then i quits. The value of holding an option to delay will be the difference between the expected value from the mixed strategy equilibrium outcome in

Proposition 4 and equation [3.16] if β^i is such that $\breve{\beta}^i < \beta^i < \tilde{\beta}$.

For j, with no option to delay, the value of making the investment at t_n (if $\beta^j > \tilde{\beta}$) is

$$\Big[Pr(\beta^i > \tilde{\beta})J^{D_j} + Pr(\breve{\beta}^i < \beta^i < \tilde{\beta})(\pi^{M_j}_{t_n} - \tau \cdot (u^j_{t_n})^2 + \gamma \widetilde{J}_{n+1}) + Pr(\beta^i < \breve{\beta})J^{M_j}\Big] - \mathcal{K}. \tag{3.17}$$

In this case, i can decide at t_{n+1} whether or not to invest with full information about its rival's type. j is aware that i holds an option to delay, therefore, if $\beta^j < \tilde{\beta}$ and $\beta^i < \tilde{\beta}$, j could potentially be a monopolist in perpetuity if $\beta^i < \breve{\beta}$, but that could only happen if j had invested first at t_n, which it should if

$$\Big[Pr(\breve{\beta}^i < \beta^i < \tilde{\beta})(\pi^{M_j}_{t_n} + \gamma J^{D_j}_{t_{n+1}}) + Pr(\beta^i < \breve{\beta}^i)J^{M_j}_{t_n}\Big] - \mathcal{K} > 0; \tag{3.18}$$

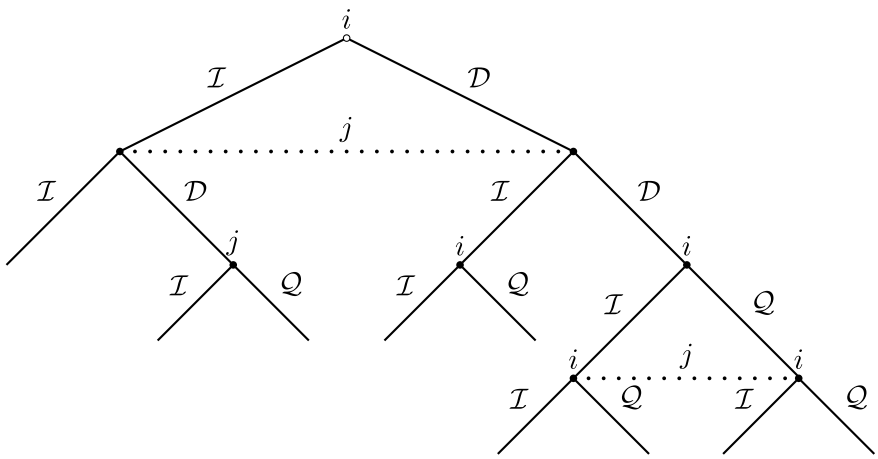
whenever $\beta^j, \beta^i < \tilde{\beta}$, otherwise, j quits.

(c) Duopoly with options

We now turn to the last scenario of the entry game with regards to the endowment of options. Here, we suppose both firms hold an option and may choose whether or not to exercise it at any of the investment windows. This case embodies all of the outcomes from our previous analyses and these outcomes are indicative of what to expect in the analogous decision branches which characterise the solution to this part of the game. To elaborate, suppose at t_{n-1}, $\mathcal{A}_{t_{n-1}} = \varnothing$, which means that at the end of the investment window in t_{n-1} there are no active firms in the market. Therefore, both firms will look to the next investment window to make their choices: either to kill their options or exercise them. We proceed by solving the problem of one firm, with contingent plans towards the re(actions) of the rival firm. If i looks to the next investment window at t_n to make its move, i can either invest or defer. if i chooses to invest, it does so with incomplete information over j's type and vice-versa. So if j also chooses to invest, this branch of the game is mathematically the same as when both firm have no option and i's objective functional is the equation [3.14]. And as we have shown previously, i will form expectation over the type for which j will choose to invest or not, and update its strategy accordingly, i.e. given its own type, i formulates its strategy by determining the type-threshold below which j defers (i.e. $\tilde{\beta}$).

Should j defer, however, the game then resembles the version studied earlier where one firm has an option and the other does not. In effect, by investing, i has killed its option whilst revealing its type to j, and the result is just as shown in the one-sided option duopoly game. On the other hand, if i chooses to defer at t_n, then it still has no information about its rival's type. However, if j sinks the initial investment outlay at t_n, while i defers, then j's type is revealed while i then decides at t_{n+1} whether or not to invest. This is just flipping the previous problem over. What solution holds for i in this case will also hold, in expectation, for j.

The last piece of detail to consider is if both firms defer at t_n, which simply implies that the entry game is reconsidered at t_{n+1}. However, we would point out that apart from the realisation that each firm's type must be below the threshold $\tilde{\beta}$, there are no further useful information to gather by deferring even further. It suffices, then to just assume that the game at t_{n+1} involves the choices: invest or quit; since if $\beta^i > \tilde{\beta}$, i would never have deferred in the first place. A solution in mixed strategies must then be found similar to the solution to the no options entry game solved earlier. The extended form of the game is shown in the figure below.



Where $\mathcal{I} \rightarrow$ Invest, $\mathcal{D} \rightarrow$ Defer, and $\mathcal{Q} \rightarrow$ Quit.

We have established that simultaneous investment at t_n is only profitable for any firm if its type is greater than $\tilde{\beta}$, and further, that with such type, a firm has a dominant strategy to invest irrespective of its rival's type. If, however, say $\beta^i < \tilde{\beta}$, firm i defers at t_n. i will then be able to observe β^j if j had invested at t_n, which it would if $\beta^j > \tilde{\beta}$. i's value of the investment under these conditions follow from equation [3.16] and Table [3.3]. The outcome of which is that i will invest at t_{n+1} if $\beta^i > \breve{\beta}^i$, otherwise, it quits.

Although j may have a dominant strategy to invest if $\beta^j > \tilde{\beta}$, j could very well observe a type below this threshold. Therefore, if $\breve{\beta}^j < \beta^j < \tilde{\beta}$, j's choices follow from that of i in the previous paragraph – j defers at t_n. We gather so far that it is in either of the firms' interest to defer at t_n and invest in the next window if its type lies in the interval $(\breve{\beta}, \tilde{\beta})$ given that it rival had invested at t_n.

In contrast to the one-sided option entry game, a firm with a type above $\breve{\beta}$ may not become a monopolist even if its rival does not invest at t_n, since in this case, the rival also holds an option to defer. A crucial point in the analysis is when both firms' types are below $\breve{\beta}$. Of course, ex ante, neither firm will invest early with the risk of facing a more competitive rival, and cannot decipher this until t_{n+1} when its rival may have invested. However, should they both differ, then there is no further strategic benefit to delaying anymore beyond t_{n+1} and the choices must then be to either invest at t_{n+1} or quit. Interestingly, the entry game at t_{n+1} is a replica of the entry game with neither firm holding an option to delay, except for a different timeline, and each realising that its rival's type, just as its own, lies in a subinterval $(0, \breve{\beta})$ of $(0, 1)$. A similar problem encountered in the 'no options' entry game will emerges i.e. the anti-coordination problem with a similar solution in mixed strategies adjusted for the new timeline.

3.5 Concluding Remarks

Business practices regarding advertising and brand equity development has evolved significantly from the days of concerns over the effects of advertising intensity. It is almost a rule of the game in the industry that more attention is paid to the quality of the messages companies' advertisement campaigns carry, and the sort of emotional, psychological or cultural response they expect these will generate. In fact, a high quality campaign promises lower penetration costs, as well as lower repeat advertising costs as shown in the steady-state advertising policies when β is either close to the lower or upper bound of $(0, 1)$. In particular, with a higher value of β the firm's only risk exposure relates to exogenous shocks through $\sigma_x W_t$ due to the magnitude of brand loyalty it has created and enjoys with its consumers at the expense of a weaker rival, who has to ramp up advertising intensity in order to stay active. Firms in these circumstances, therefore, have found it useful to invest heavily in the development of a compelling advertisement campaign that tells a narrative customers connect with on a subliminal level, sometimes associated with culture, thereby creating the opportunity to profitably charge premium prices.

This chapter departs from the existing literature on advertising by explicitly modelling the simultaneous effects of informative and persuasive advertising in the evolution of goodwill and the strategic implications of holding an option to invest. In effect, the outcome of the entry game impacts the dynamical system, which in turn determines the value of the initial investment made to enter the game through the revealed types of the firms.

The entry game outcomes depend entirely on the types of the firms. The importance of holding an option to defer, rather than a 'now-or-never' investment paradigm is pronounced in the one-sided option scenario studied, where a firm with an option may find it beneficial to invest later even with a lower type than its rival. It may very well be the case that this firm become a monopolist should it face a rival with a relatively low type who holds no such option. This situation does not arise when both have no options to defer or both equally hold an option to defer. We notice that a solution in mixed strategies is required to resolve the anti-coordination problem that arises at t_n and t_{n+1} in the respective investment windows in each of these scenarios.

We have analysed various market outcomes of an investment game of advertisement campaigns based on the relative potency of the campaign each firm brings into a new product market. One of our main observations is that whenever there is asymmetry in the types of the firms (which occurs with probability 1), there are no steady-state market outcomes with symmetric firms. The reason is that the steady-state size of the each firm is independent of is initial goodwill level, but rather on its type, which does not change through the course of the differential game. This contrasts some conventional positions in the literature, e.g. Doraszelski and Markovich (2007); but we note that in our model, the turning parameter influencing either informative or persuasive effects of advertising (β) is intrinsic to the firms rather than arbitrarily chosen or determined. Furthermore, there is non-zero probability of an outcome with a weaker firm emerging as a monopolist. Such a firm may, probabilistically, earn monopoly rents if it faces a rival holding an option to delay who also has low type. This is one of the outcomes from the anti-coordination problem the firms face in this scenario.

There are a couple of ways to extend the model framework presented here further. One of the interesting extensions would be to consider the case where firms can make improvements to their campaign messages based on market feedback. This will trigger incremental updates to their types. Another potential extension may be to consider how much firms will be willing to pay to obtain information about their rival's type, and how it influences the value of the option each firm holds. These are left for further research.

Chapter 4

Toxicity Risk in Brand Equity Investments through Celebrity Endorsements

4.1 Introduction

"Of course, there is ~~no~~ such [a] thing as bad publicity"

In what could only be described as a stance of defiance or a stroke of marketing genius, Adi Dassler got Jesse Owens, an African-American athlete competing in Hitler's 1936 "Aryan–race–superiority–propaganda" Olympic games, to wear his German-made track shoes. After hurling four gold medals and setting new world records, word quickly got out that the fastest man alive conquered the world in shoes made by an unknown man from a small town in Germany. In the year leading up to World War II, Adidas was already selling upwards of 200,000 pairs of shoes a year all around the world. Jesse Owens established the global presence of Adidas[1]. Such is the power of celebrity endorsements.

[1] In Barbara Smit's words (author of Drei Streifen gegen Puma (Three Stripes versus Puma)), "Owens's success cemented the good reputation of Dassler shoes among the world's most famous sportsmen. Letters from around the world landed on the brothers' desks, and the trainers of other national teams were all interested in their shoes."

The long tradition of enlisting celebrity endorsers to build brand equity has shown no signs of abating, and for good reasons: celebrities break through information clutter, create preferred product/brand narratives[2], initiate dialogues with intended audiences, and transfer fan base likability to a brand. Understanding how celebrity endorsements impact brand performance has been the pièce de résistance of a vast collection of theoretical/conceptual and empirical research papers in marketing, branding, and advertising[3]. Recent empirical evidence lends support to the direct impact the performances of professional athletes have on the stock returns (see, for example, Agrawal and Kamakura (1995) and Mathur et al. (1997)) and sales results of the brands they endorse (Elberse and Verleun (2012)). In the first attempt to analyse the impact of celebrity endorsement on sales Elberse and Verleun (2012)'s (and subsequently Chung et al. (2013)[4]) intervention model using present and historic data on athlete endorsement deals of publicly traded firms show jumps in sales and stock returns of endorsed products and their respective firms, for every major win of their athlete endorsers. The focus on sales outcomes (rather than just stock returns) is of particular interest, since sales figures more closely align with investment valuation. An important takeaway from their analysis is that endorsement related sales show considerable seasonality and variations, suggesting that marketers cannot really predict the nature of returns to expect from a celebrity endorsement. The overarching outcome being an easing-off in the rise of sales after each win and, over time, the relative stableness of stock returns after each spike.

Hitherto, the evidence on the true value of celebrity endorsements are at best ambiguous. Conventional wisdom is: for all its worth, celebrity endorsement is a nuanced and delicate venture. Nuanced, because it relies entirely on the perception of consumers which cannot be predicted, and tends to shift very quickly. It is delicate in that firms will typically be concerned with how a celebrity fits with their brands, ex-

[2]It is believed that Thierry Mugler's choice of Oscar Pistorius was partly to change attitudes toward disability and recast the image of what the perfect celebrity should look like.

[3]In Appendix 2 we present a summary of relevant conceptual frameworks from the literature.

[4]They show that celebrity endorsement creates the ability to charge price premiums e.g. Nike was able to profitably raise prices of its golf balls during Tiger Woods' endorsements while competitors had to cut prices.

isting consumer base, and target markets. They consider whether or not the business is mature enough to absorb any sudden negative shocks arising from an unfavourable perception of the celebrity-brand alliance; and amongst other pressing concerns, firms seeking celebrity endorsements cannot tell when the next best celebrity candidate will emerge and what level of returns to expect—as empirical evidence suggests. These considerations naturally pose a familiar positive economic question of investments under uncertainty, which is, "given the characteristic peculiarities of celebrity endorsements, *what is* the optimal investment strategy for firms seeking to build brand equity using celebrity endorsers?"

The aim of this chapter is to provide—as far as the author is aware—the first theoretical framework which utilises industrial organisation type ideas with embedded real options to examine market outcomes and optimal strategies of investments in brand equity through celebrity endorsements. Interestingly, several characteristics of investments in brand equity by enlisting celebrity endorsers bear very close similarities to those in models extensively analysed in industrial organisation and strategic real options literature, in particular, to the strand of the literature which studies investments and choice of roles in the adoption of new technologies/innovations[5].

As far as similarities go, for modelling purposes, the central ones between investments in new technologies and brand equity using celebrities are:

a) they are (partially) irreversible requiring a lumpy initial investment outlay. Just as firms cannot costlessly revert to a previous technology after adopting a new one, brands also cannot (fully) recover initial endorsement endowments whether or not the celebrity-brand relationship proves to be profitable. While sports endorsers tend to stagger initial endowments and often tie future rewards to performances, these investment outlays, once incurred, cannot be (fully or costlessly[6]) recovered;

[5] This concept originated from the IO literature going back to Reinganum (1981). The real options literature extended its application to accommodate market situations where there is value to delaying the decision to invest until a later time when more, however imperfect, information about the uncertain elements of the economic environment become available (as in Huisman and Kort (2004)).

[6] Brands have recourse to legally challenge the continuation of the terms of the investment where necessary, but this is also very expensive.

b) there is uncertainty over future profit flows arising from price fluctuations in perfectly competitive markets or strategic market outcomes in imperfectly competitive markets;

c) there is often an option to defer the investment decision until a later time when a better investment opportunity might become available (or some uncertainty may be resolved); and

d) in some market environments, there is an additional layer of uncertainty with respect to the arrival of the new opportunity, that is, at some given time, firms in the market cannot accurately predict when the new opportunity will emerge.

Therefore, in this chapter, we examine how firms should optimally formulate investment strategies when choosing to use celebrity endorsers under exogenous uncertainty over future profit flows as well as uncertainty over the arrival of future investment opportunities knowing they would be engaging in an imperfectly competitive product market post investment.

What sets our model apart from extant studies are that we allow the firms to incorporate their prior beliefs about the celebrities potentials, and by extension the value of the investment, into their investment decision mechanism; and we recognise the impact of toxicity in the form of certain unfavourable post investment market outcomes that may merge when either or both firms' investments turn toxic. Whilst the toxic risk of celebrity endorsers has been recognised and conceptualised in the branding (or marketing) literature, it has yet to be explicitly modelled into a economic decision framework of firms in a manner that may indeed restrain the firms' enthusiasm about the investment opportunity irrespective of their prior optimism.

4.1.1 Relationship with relevant literature

Consider two rival firms competing in a product market and facing a now-or-never-type opportunity to invest in their brand equities. Typically, post-investment benefits may be realised through direct cost savings (for example, a reduction in the cost of advertising), higher levels of product demand (if the investment makes the product more preferable), and often, the ability to profitably raise prices (where the investment leads to stronger product loyalty). The extent to which firms enjoy these benefits depends on the actions of their rivals which may result in lower (higher) profit levels, and therefore, lower (higher) investment value. So, even if a first mover advantage exists through early commitment, without pre-commitment arrangements, a firm cannot of its own accord be the first adopter; or indeed, be a follower if these had not been preset. This strategic uncertainty moderates whatever first mover advantage that exists. From Reinganum (1981), in this sort of environment, a firm may find that by delaying, it saves on the cost of making the investment if the new opportunity does not bring sufficient benefits. And even when the firms pre-commit to adoption times (as in an open-loop strategy), the possibility of sub-optimal investment returns following a lumpy investment outlay induces a 'diffused[7]' equilibrium.

Of course, Reinganum (1981)'s prediction of no simultaneous adoption only holds when firms can credibly threaten to stick with their pre-committed adoption times. This may be true of certain investment games (perhaps in the adoption of a new technology under strict regulatory requirements), but is clearly not general. A closed-loop equilibrium is preferred in some other kinds of investment games with perfect observability as in Fudenberg and Tirole (1985)'s. They look at an alternative, however, extreme case of preemption to Reinganum (1981)'s by modifying the information structure of the game so that a firm can observe its rival's decision and respond immediately. This dissipates any first mover advantages induced by the prospect of preemptive adoption. Notwithstanding, Reinganum (1981)'s diffusion equilibrium may be preserved if an early adopter may be able to earn monopoly rents for a longer span of time enough to counteract the initial higher cost of adoption, thereby incentivising early adoption leading, again, to a diffused equilibrium. With perfect observability, however, comes several Pareto comparable equilibria with respect to symmetric adoptions when a firm can respond immediately to an early adopter. Fudenberg and Tirole (1985) identify one such optimal joint adoption time that is sustainable in equilibrium where both firms adopt much later than all other equilibrium adoption times. The intuition being that,

[7]A diffused equilibrium is expected in a preemptive game where being a follower is more advantageous than early simultaneous investment.

if such equilibrium is sustainable, then it is in both firms' interest to delay adoption to a much later time[8], when the cost of investment is lower and thereby, earn more as duopolists. This restores a second mover advantage, which firms seeking to invest in their brands would welcome, particularly smaller brands in a highly concentrated market. In these sorts of markets, we would expect that smaller firms, who can ill-afford a non-beneficial investment (following a large initial outlay), would prefer to employ a feedback (close-loop) investment strategy, so that Fudenberg and Tirole (1985)'s joint adoption equilibrium becomes plausible. In effect, the firms may opt for a low risk–low return-type investments in their brand equities due to a second mover advantage which the potential learning opportunities that time presents either through better revelation of the toxic risk of a proposed celebrity endorser, or in fact, the firms' own ability to better manage such risks; but it still does not eliminate the risk as in the pure strategic competition case that Fudenberg and Tirole (1985) presents.

It is apparent that strategic uncertainty alone does not fully account for expected outcomes in investment games. For instance, high risk opportunities (potentially with high returns) as in celebrity endorsements, are often occasioned by more than the influence of strategic uncertainties. Celebrity endorsement deals may deliver abnormal returns or poor returns for a number of other reasons besides strategic competition in the product market. Indeed, factors exogenous to the firms play key roles in determining the profit potential of an investment opportunity. Therefore, besides strategic uncertainty over a firm's rival's actions, returns from an investment may involve exogenous uncertainties in the investment process itself and/or product market where the firms compete[9], which introduces noise into the predictive mechanism of models without exogenous uncertainty. On one hand, this sort of uncertainty moderates first mover advantages, but where firms perceive incremental benefits to a preemptive endorsement of a celebrity due to a learning process that resolves some utilisation uncertainties,

[8] They have assumed that the cost of investment decreases with time creating an incentive to delay.

[9] It could be that the enlisted celebrity endorser may lose traction with the fan base; the message to, and connection with, the target audience might be undesirable; and other shocks in the market could create unpredictable demand levels for branded products; or the brand-celebrity relation may not have been handled well enough to make it flourish.

adoption times are dispersed (as in Stenbacka and Tombak (1994))[10] whether or not the firms use pure or mixed strategies. However, a lower future adoption cost would create an incentive to follow (in a feedback strategy) and lessen dispersion in adoption times.

A practical shortfall of applying Stenbacka and Tombak (1994)'s model to a celebrity endorsement game is the presupposition that a firm has to commit and engage the celebrity first, that is, sink the initial investment outlay, before having the opportunity to improve its chances of a successful utilisation through experience. But it is not unexpected to find consumer preferences aligning within the same industry, in fact, we would anticipate that a firm who is yet to commit would be able to learn utilisation methods from its rival's own investment through observed market outcomes; or that a firm gain better understand of celebrity community enough to better understand their toxic inclinations. In our model, this sort of learning is realised in the firm's own prior regarding its chances at the second investment opportunity, which, as we will show in this chapter, is very valuable when the initial investment outlay is large ,and there is significant perception of toxicity at the first investment opportunity.

The innovation process in Hoppe (2000)'s assessment of adopting a new technology fits the pattern above. She assumes nature determines, ex ante, the probability of a good or bad investment opportunity which is common knowledge to the firms, in contrast to Stenbacka and Tombak (1994). The push and pull strain of first and second mover advantages on a firm's decision is now tied to the ex ante observed probability of a successful investment, where the follower is allowed to observe the true value of the investment after the first mover invests, which then informs the follower's decision whether or not to invest. The import being that a higher probability of a successful investment induces a preemptive strategy and rent equalisation (due to the desire to earn monopoly rent for as long as possible before the rival firm invests), but with a low probability of success, the opportunity to wait and learn becomes very valuable, turning a preemptive game into an attrition game – where each firm holds out on the investment for as long as it can. So under these circumstances, the opportunity to learn

[10] We make reference here to the common features of investments in technology and celebrity endorsement deals.

restores second mover advantage when the initial investment outlay is relatively large.

To a certain degree, these extant studies cast some light on the parallels between investments in brand equity through celebrity endorsement and the adoption of new innovation. But with celebrity (athlete) endorsers, for instance, there is the inevitability of performance decline, and with it, brand decay as the years go by. New and younger star athletes emerge and the marketing cycle starts all over again with brands seeking the most marketable of the lot. Brands, therefore, may either choose to renew an old contract or enter a new one with a younger, potentially more promising athlete[11]. This creates multiple investment opportunities, so that it would seem like a bad idea to invest at the very first opportunity if a little while later a much better opportunity becomes available. Whilst deciding whether to invest in a current opportunity, firms therefore would factor into their investment considerations the existence of a better opportunity in the near future and how that might alter the perceived value of the current opportunity. Where the time of the next investment opportunity is known, Huisman and Kort (2003) find conditions for the optimality of four investment strategies[12] identified in Grenadier and Weiss (1997). Here again, the benefit of learning bears some importance to a firm through lesser adoption cost in comparison to the first mover. What strategies are adopted depend on the time of arrival of the new innovation, which is understandable given the high direct and indirect costs of making either of the investments. Huisman and Kort (2004) extend, amongst others Hoppe (2000), Dixit and Pindyck (1994), and Huisman and Kort (2003), by incorporating

[11]One of the very curious features of investments in brand equity by enlisting celebrity athletes is that post active appearance, the brand an athlete has built, may sometimes, remain strong enough to eclipse those of rising active players. The most prominent case being Michael Jordan, who retired in 2003, but whose brand still raked in a phenomenal $2.6 billion in shoe sales in the US in 2014 and eight times the sale of the signatures shoes of the current NBA star, LeBron James. Jordan apparel and the international business have revenues over an additional $1 billion. The Jordan Brand commanded 58% market share of the $4.2 billion U.S. basketball shoe market in 2014, up from 54% in 2013. The Nike share jumps to 95.5% if you include Nike Basketball. The competition: Adidas (2.6% share), Under (1%) and Reebok (0.8%).

[12]The four strategies identified are: 1) *Compulsive strategy* – where a firm adopts every technology as it becomes available; 2) *Buy and Hold strategy* – the firm adopts the current technology and does nothing again; 3) *Leapfrog strategy* – the firm waits for a new technology to arrive and then adopts it when it does; and 4) *Laggard strategy* – the firm waits for a new technology to arrive and then adopts the one before it.

exogenous uncertainty in an investment game with two opportunities where the second is expected to be better than the first, however, the arrival of the second opportunity follows a Poisson process. The last assumption is important to the model we present in this chapter as we build upon Huisman and Kort (2004), whilst casting the game in an industrial organisation type framework – letting the firms engage one another post investment in an imperfectly competitive product market, which highlights how product market competition is influenced by the firm's investments in their brands. Investment timing predictions from Huisman and Kort (2004)'s model is predicated on the rate of arrival of the second investment opportunity. Which is, with a low arrival rate of the second opportunity (i.e. if it is expected to take longer to arrive) the prevailing strategy is to ignore it and both firms try to be the first to invest in the first opportunity. With a sufficiently high rate of arrival, the preemption game turns into an attrition game where both firms see who can hold out longer for the second opportunity, whilst foregoing profits that could have been earned by choosing to invest at the first opportunity. It is instructive that the choices here are real options so that the firms are not obligated to commit to any of the investment opportunities, which has more practical benefits over earlier now-or-never-type investment models in the literature.

This is, perhaps, as far at the parallels go with investments in new technologies and brand equity through celebrity endorsements. Their common features provide some insight into endogenous choice of roles when there is uncertainty concerning the arrival of future investment opportunity and the firms have an option to defer making their investment decisions until a later time. But celebrity endorsement deals are known to possess immense benefits as well as the potential to become toxic causing even greater damage to a firm's brand than could be explained by strategic or exogenous uncertainty. Random shocks such as celebrity scandals are characteristic of these sorts of investments, and depending on how toxic the scandal is, it is not uncommon that a firm may actually cease being going concern or that an entire product line shut down due to affiliation with the celebrity. So beyond uncertainty over future profit flows arising from price fluctuations and time of arrival of an investment opportunity (as in Huisman and

Kort (2004) and Chronopoulos and Siddiqui (2015)[13]), or strategic market outcomes in imperfectly competitive markets (as in Huisman and Kort (2003), Murto et al. (2004), Hoppe (2000), and Stenbacka and Tombak (1994)), investments in brand equity using celebrity endorsers can very quickly become toxic to both the brand and the endorser[14]. Therefore, whilst leveraging features from investments in technology adoption, in this chapter we introduce toxicity into the modelling framework, which creates additional product market structures that are absent in extant models. We analyse the probability of any of these market structures emerging by explicitly modelling future states of the product market as a stochastic process. The states the stochastic process may visit are dictated jointly by the outcomes of the firms' strategic interaction and, crucially, the presence of one or more (at most two) toxic investments. We then solve for the expected values of each investment opportunity over the state space of the stochastic process by deriving the transition probabilities from each state at each investment epoch.

The cost to a brand of being affiliated to a toxic celebrity is exacerbated in the present socio-economic atmosphere due to unrestrained global flow of information (thanks to a myriad of social media network platforms) and a growing–global–apprehension towards corporate behaviour and capitalist instincts. All of these have made already exposed celebrities overexposed and unceasingly more transparent, thereby, amplifying the potential minefield of scandals. Marketers have historical relied on a very simple model, which was, "High-profile celebrity (athlete or otherwise) + ubiquitous product plugging = huge profitability and brand equity," but throw in "toxicity risk," and the model collapses. Examples are abundant in Tiger Woods (and Nike), Lewis Armstrong (and Nike), Oscar Pistorious (and Mugler) etc. In today's markets, a firm ignores the potential toxicity of a celebrity endorser to its detriment – and major brands do not. While firms cannot predict if and when a celebrity endorser would get entangled in a toxic scandal, there are written and unwritten industry approaches to both prevent and manage brand toxicity due to celebrity affiliations. From mainstream data and other sort of data gathered through experience or other forms of interactions, firms would

[13]They study the relationship between price fluctuations and the optimal adoption of the strategies introduced in Grenadier and Weiss (1997).

[14]Our focus in this chapter is only on the impact of toxicity on the brand.

often have some sort of intuition (prior probability) of the toxicity risk a celebrity endorsement could pose. This prior probability is very important to the decisions these firms would make as it feeds into their expected returns and by extension the ex ante value of the investment opportunities. To our knowledge, accounting for toxicity in an investment game with Poisson arrivals, which we present in this chapter, has not occurred in the literature yet. Our approach is to allow a firm analyse the condition under which it would consider making an investment and then with its prior about the celebrity's viability we derive the posterior probability that a successful investment may be observed from this firm. With these posterior probabilities, we derive the transition probabilities from state to state with consideration for the actions of the rival firm and the potential outcomes of its choices.

Our results show that considering toxicity in an investment game with random investment arrival dramatically modifies what actions firms would consider optimal were toxicity absent as in extant studies. Hoppe (2000) did consider the probability that an investment is good or bad, those probabilities were, however static, and does not reflect how likelihoods modify prior beliefs in order to derive best possible ex ante predictions of a good or bad investment as described above. Additionally, we find that the random arrival of the second investment opportunity á la Huisman and Kort (2004) also moderates firms' actions in our model, but only in tandem with the firms own prior beliefs and the potential to learn from its rival's early investment (or simply through time). Crucially, we find that under a low (high) arrival rate of the second investment opportunity, we find an attrition (a preemption) type game turning quickly into a preemption (an attrition) game as the option value of waiting, revealed in the in the difference between prior beliefs for the first and second opportunities, increases. Notwithstanding, these outcomes may only be observed, whether with a low or high arrival rate, if the firms' initial priors are low. Should they be faced with a seemingly good prospect at the first go, whether or not the arrival rate of the second investment is low or high, we find that the firms have no incentive to delay and a preemption game prevails. Which suggests that with a relatively high prior regarding the first opportunity, a first mover advantage persists if firms do not perceive that there is substantive learning potential with waiting, whether or not a future opportunity arrives early.

The organisation of the chapter is as follows. The model is presented in Section 2. Section 3 describes the Markov jump process representing expected market structures post investment. In that section, we identify the strategies the firms may adopt and per period demand and payoff functions. Background to the Sequential equilibrium and transition matrices are presented in Section 4. In that section we also characterise the option value of waiting in the derivation of transition probabilities when a firm or its rival chooses to defer. Section 5 contains the analysis of the firms' net present valuation of the investments using posterior probabilities wherewith our main results are derived. Concluding and potential extensions to this model are contained in Section 6.

4.2 The Model

This section describes the economic environment that we model in this chapter. Consider two identical, risk neutral firms $\mathcal{A}$ and B. Both are value maximising firms in a market for branded product lines. Each has a fixed advertising budget to strike an endorsement deal in a bid to build its brand equity. An endorsement deal will come with a new branded product designed in conjunction with the celebrity endorser (preferably, an athlete[15]). With the branded product, a firm becomes active in a niche market space — the market for a unique celebrity branded product[16].

Investment opportunities arrive in our model in the same way they do in Huisman and Kort (2004)'s, but we take the view that, while a firm may be able to generate income from an investment opportunity forever, its budgetary planning horizon is usu-

[15] According to Elberse and Verleun (2012), "... athletes are excellent subjects to study for a number of reasons. First, sportspeople are among the most popular endorsers (meaning a relatively large sample of athlete endorsements can be assembled) and, unlike for many other kinds of celebrities, performance statistics for athletes are readily available and often fluctuate dramatically over relatively short periods of time (enabling a rich classification of the impact of endorsements and endorsers' achievements over the course of a partnership with a brand). Second, endorsements are a key source of income for sportspeople ... ."

[16] An example was Micheal Jordan's "Air Jordan" branded shoes, which he jointly designed with Nike following his endorsement in 1985.

ally finite. Which means in our model, we suppose the firms are aware that nature will provide an investment opportunity at some known time t_1 in the near future, and that nature may also provide another opportunity at a further time in the future which may or may not be within the firms' planning horizon. We use this construct to endogenise the firms' preconception about the rate of arrival of the second investment opportunity, making it a product of the firms' knowledge of historic cycle of investment opportunities in that industry, rather than it being arbitrary. This construct makes the assumption that each firm may only invest in one opportunity, within the planning horizon under consideration, come about organically rather than for analytical convenience as it appears to be in Huisman and Kort (2004).

The payoff potential and value of an investment in this environment is subject to the various sorts of uncertainties described earlier. Exogenous forces such as uncertainty in consumer demand or consumer willingness to pay for a product impacts expected profits, as well as strategic uncertainty arising from imperfect information at the start of the game. In the classic valuation literature, the standard investment principle is simple—calculate the net present value of an investment and see whether it is positive. The underlying assumption being that the present value of future streams of income is non-negative[17]. However, toxicity in investments using celebrity endorsers may mean a post-investment present value that is less than zero as a result of an exogenous random shock that consumers respond to. These shocks are created when celebrity endorsers become embroiled in activities or campaigns which consumers consider to be socially or culturally unacceptable. Thus, the celebrity quickly becomes toxic as any association with her is considered an endorsement of a poorly perceived behaviour. Any brand that is affiliated to such a toxic celebrity will not only lose its investment outlay, but may also see its per period profit fall below zero as consumers will perceive that any association with the celebrity will diminish their utility so much they risk negative utility should they choose to consume the firm's product. Therefore, they will actively

[17] A similar assumption and decision rule applies with the redefined net present value analysis (otherwise, real options analysis) where the opportunity cost of exercising the option to invest is included in the net present valuation (see Dixit and Pindyck (1994)).

avoid the product resulting in zero demand for the firm. A firm in this situation will quit the market altogether, and this changes the market structure over and above what strategic interaction alone may predict. In this regard, we will see both firms quit the market if they happen to have endorsed toxic celebrities, or if only one firm endorsed a toxic celebrity, the other firm becomes a monopolist. Celebrity toxicity risk, therefore, creates the potential that the market will transition from one state to another, at each investment epoch, depending on the investment choices of the firms as well as whether or not they remain active after making an investment.

The outcomes of the strategic variable in the game, i.e. product prices, also depend on the state of the market, which consist of demand level, number of firms in the market, and the willingness of consumers to pay for the product based on their relative perception of the value of the brand. Consumers are usually willing to pay a higher price for a product affiliated to a well regarded celebrity than others[18], therefore a successful non-toxic investment in a firm's brand equity could imply higher willingness to pay for the firm's product of that of its rival who has either not invested in its own brand equity or does but fails. At the time the firms consider whether or not to invest in their brands using celebrity endorsers, the additional value consumers may attribute to their brand following a successful non-toxic investment is unknown but may be modelled as a random variable with a known distribution.

The game commences when the firms become aware of the time nature presents the first investment opportunity and terminates at the end of their budgetary planning horizon. We model the arrival of the second investment opportunity within or outside this planning horizon as a Poisson process, which is a simple and commonly used stochastic process for modelling arrivals (departures) in (from) a system.

Formally, let $\mathit{\Gamma} = \{0, 1, \ldots, T\}$ be the set of time periods covering A and B's budgetary planning horizons. By $t_0 \in \mathit{\Gamma}$, we refer to any time period before nature presents the first investment opportunity at $t_1 \in (t_0, \ldots, T)$, while the second (and

[18] Empirical evidence in Chung et al. (2013) demonstrates consumers' willingness to pay a premium for Nike's golf balls during Tiger Woods' endorsement over those of other competitors – who had to cut prices.

last) investment opportunity may occur at some arbitrary time $t_2^* \in (t_1, T]$ or $[T, \infty)$[19]. At t_1 or t_2^* (in the event that the second investment opportunity arrives before the end of the game), the firms may each choose to either invest (I) or defer (D) making the investment decision. The firms discount future profits at rate $r(> 0)$ and whenever either firm decides to invest, it gives up a sunk cost, $\mathcal{K}_{t_{(\cdot)}}(> 0)$, where the subscript $t_{(\cdot)}$ indicates the time the sunk cost was expended. Meanwhile, these investments, once made, cannot be reversed whether or not they turn out to be successful. Now suppose at the beginning of the game nature reveals the probabilities that the first and second investment opportunities are non-toxic, say, p_1 and p_2 respectively. In addition, suppose we break up the interval $(t_1, T, \ldots, \infty)$ into disjoint intervals, each of length h, where h is small, we obtain the intervals $(t_1, t_1 + h), [t_1 + h, t_1 + 2h), [t_1 + 2h, t_1 + 3h), \ldots$. Suppose further that each interval corresponds to an independent Bernoulli trial, such that in each interval, independently of every other interval, an investment opportunity arrives with probability λh. This describes a Bernoulli process, $\{B(t) : t = t_1, t_1 + h, t_1 + 2h, t_1 + 3h, \ldots\}$, where $B(t)$ is the number of investment arrivals up to time t. $B(t)$ clearly corresponds to the notion of a process in which events occur randomly in time, with an intensity (or rate) that increases as λ increases – a conceptually simpler way to think about a Poisson process. Furthermore, this discrete-time version of the Poisson process more closely reflects how the investment opportunities of interest occur in practice. Therefore, given the definition of $B(t)$, the probability that an investment opportunity arrives in any given interval is λh, the probability that no investment arrives is $1 - \lambda h$, and the probability of two or more arrivals is 0, that is, $Pr(B(h) = 1) = \lambda h$, $Pr(B(h) = 0) = 1 - \lambda h$, and $Pr(B(h) \geq 2) = 0$. The precise Poisson process (N(t)) analogues to these probabilities are $Pr(N(h) = 1) \approx \lambda h + o(h)$, $Pr(N(h) = 0) \approx 1 - \lambda h + o(h)$, and $Pr(N(h) \geq 2) = o(h)$[20]. It is obvious that the number of arrivals before an arbitrary time $t \in (t_1, T]$, in the firms' planning horizon, is less than 1 if and only if the waiting time until the first arrival is greater than $T - t_1$. These waiting times are independent and identically distributed Exponential(λ)

[19] We use (T, ∞) only for analytical convenience, as new budgetary horizon could be defined beyond T, where an investment opportunity may be available.

[20] "$o(h)$" is Landau's $o(h)$ notation meaning any function of h that is of smaller order than h.

random variables, which for convenience and practicality, we replace with the analogous discrete-time Geometric random variables with parameter λh. The implication is that if we start observing the above Bernoulli process at some arbitrary time, not knowing how many arrivals have gone before or when the last arrival occurred, we still would know that the distribution of the time until the next arrival will be h times a Geometric random variable with parameter λh[21]. Under the Geometric distribution, therefore, the mean time until the first arrival in the interval $[t_1, T)$ is given by $1/\lambda$.

4.3 Strategies and state-to-state transitions

Associated with the firms' investment process starting at t_1 is an *embedded Markov (jump) chain* $(\mathcal{X}_n)_{n=0,1,2,3,...}$ whose state space represents market structures that may emerge post investment at either t_1 and/or t_2. Let $\mathcal{X}_0$ be the initial state at t_0, $\mathcal{X}_1$ the state entered on the first jump at t_1, and $\mathcal{X}_{2^*}$ the state entered on the second jump at t_2^*. The embedded chain will remain in its new state after t_1 at least for a short (random) while and transitions from state to state are governed by transition probabilities at t_1 and t_2^*, which are determined by the strategies of the firms and exogenous toxic shocks. Table 4.1 describes all the possible outcomes after each opportunity to invest, where $\mathcal{I}^1$ and $\mathcal{I}^2$ denote the first and second investments respectively; and I and D refer respectively to the "invest" and "defer" options available to the firms at each epoch. The dashes (–) in the table means that there longer is an opportunity to invest, which will be the case at $\mathcal{I}^2$ when both or either of the firms have invested at the first opportunity. The asterick ($*$) indicates outcomes when there is no arrival in the interval $(t_1, T]$, and finally, the superscript "$\varnothing$" represents a toxic investment.

From Table 4.1, we can now construct the investment strategies of the firms. Let $P_1 = (I_1, D_1)$ and $P_2 = (I_2, D_2)$, firm A's strategy is given by the cartesian product $P_1 \times P_2 = \{(I_1, I_2), (I_1, D_2), (D_1, I_2), (D_1, D_2)\}$. However, due to model assumptions, some of these actions are immaterial, so the set reduces to the following actual actions the firms may use to formulate their investment strategies:

[21]The memoryless property.

Table 4.1: Investment outcomes

	A	B	A	B	A	B	A	B	A	B	A	B
$\mathcal{I}^1$	I	I	I	D	I	D	I	D	I	$I^{\varnothing}$	$I^{\varnothing}$	$I^{\varnothing}$
$\mathcal{I}^2$	$-$	$-$	$-$	I	$-$	D	$-$	$*$	$-$	$-$	$-$	$-$
$\mathcal{I}^1$	I	D	I	D	$I^{\varnothing}$	D	D	D	D	D	D	D
$\mathcal{I}^2$	$-$	$I^{\varnothing}$	$-$	D	$-$	$*$	I	D	$I^{\varnothing}$	D	D	D
$\mathcal{I}^1$	$I^{\varnothing}$	D	D	D	D	D	D	D	D	D	I	I
$\mathcal{I}^2$	$-$	I	I	I	I	$I^{\varnothing}$	$I^{\varnothing}$	$I^{\varnothing}$	$*$	$*$	$*$	$*$

1. $(I_1, I_2) \equiv I_1 \rightarrow$ as a firm may only make one investment within the specified planning horizon $[t_0, T]$.

2. $(I_1, D_2) \equiv I_1 \rightarrow$ follows from (1) above: once a firm invests at the first opportunity, it has given up the option to make the choice whether or not to invest (t_1, T), where $t_1 > t_0$.

Following from the above, the strategy set of any firm is $\{I_1, (D_1, I_2), (D_1, D_2)\}$, where D_2 is equivalent to quitting the market altogether as there are no further investment opportunities after the last one arrives, should it arrive at all.

To complete the model description, we now characterise the state space of $\mathcal{X}$ and what combination of investment outcomes may instigate a transition from one state to another. We identify six states which $\mathcal{X}$ may enter and/or exit from, these are a) Monopoly *plus* ($M^\dagger$); b) Monopoly (M); c) Strong/Big vs. Weak/Small brand ($\dagger$); d) Strong vs. Strong ($\mathcal{S}^\dagger$); e) Status quo ($\mathcal{S}$); and e) Null market ($\mathcal{O}$). The investment process may therefore be fully described as an embedded Markov chain with finitely many jumps in $[t_1, T]$ and state space $\mathcal{E} := \{M^\dagger, M, \dagger, \mathcal{S}^\dagger, \mathcal{S}, \mathcal{O}\}$ where $M^\dagger, \mathcal{S}^\dagger$, and $\mathcal{O}$ are absorbing state. Formally, the embedded process is given by $\mathcal{X}_0 = \mathcal{X}(0)$, $\mathcal{X}_n = \mathcal{X}(\tau_n)$ if $\tau_n < \infty$ and $\mathcal{X}_n = \mathcal{O}$ if $\tau_n = \infty$ for all $n \in \mathbb{N}$, where $(\tau_n)_{n\in\mathbb{N}}$ denote the transition times and $\tau_n = \inf\{t > \tau_{n-1} : \mathcal{X}(t) \neq \mathcal{X}(\tau_{n-1})\}$, with $\tau_0 = 0$ and $\inf \emptyset = \infty$. The idea here, is that $\mathcal{X}$ stays in the same state in-between investment epochs, and the transition probability from state to state may be determined by the condition under which $\mathcal{X}$ would enter and/or exit any state at t_1 and/or t_2^*. These conditions are set out in the model statements below.

Model statement 1. *Assuming at* t_0*,* $\mathcal{X}_0$ *is in* $\mathcal{S}$*, it follows then, that at* t_1*, from the investment outcomes' table (Table* 4.1*), if the outcome is in the set:*

a. $\{(I, I^{\varnothing}), (I^{\varnothing}, I)\}$*,* $\mathcal{X}$ *enters* $M^{\dagger}$*;*

b. $\{(I^{\varnothing}, D), (D, I^{\varnothing})\}$*,* $\mathcal{X}$ *enters* M*;*

c. $\{(I, D), (D, I)\}$*,* $\mathcal{X}$ *enters "*$\dagger$*";*

d. (I, I)*,* $\mathcal{X}$ *enters* $\mathcal{S}^{\dagger}$*;*

e. $\{(D, D)\}$*,* $\mathcal{X}$ *enters* $\mathcal{S}$*; and*

f. $(I^{\varnothing}, I^{\varnothing})$*,* $\mathcal{X}$ *enters* $\mathcal{O}$*.*

Model statement 1 demonstrates how the transition of $\mathcal{X}$ is governed by the investment choices of the firms and the expected outcomes of these choices. If both firms choose to invest at t_1, the firm with the toxic investment becomes inactive in the market place for celebrity-endorsed branded products, while consumers will attribute higher value to the brand of the successful firm, resulting in a transition to $M^{\dagger}$ as in (a); if however, both of brands become toxic, then again, they both become inactive in the branded market place, and $\mathcal{X}$ enters $\mathcal{O}$ as in (f), where consumers attribute no value to their brands, more precisely, these generate negative brand equity — toxicity. On the other hand, if both firms get non-toxic celebrities, $\mathcal{X}$ transitions into $\mathcal{S}^{\dagger}$, and if both defer, $\mathcal{X}$ remains in $\mathcal{S}$, where it started from originally at t_0 (which is the case in (e) above. When the firms make non-identical choices, that is, one chooses to invest while the other defers, then should the investing firm become toxic, $\mathcal{X}$ enters M as in (b), but it enters $\dagger$, should the investing firm get a non-toxic celebrity ((c) above).

Model statement 2. *Until another opportunity arrives at* t_2^**,* $\mathcal{X}$ *will remain in its* t_1*-state as in* ***Model statement 1****, but if* $\mathcal{I}^2$ *arrives in* $(t_1, T]$*,* $\mathcal{X}$ *evolves as follows:*

a. If at t_1*,* $\mathcal{X}$ *entered* $M^{\dagger}$ *through* $\{(I, I^{\varnothing}), (I^{\varnothing}, I)\}$ *or entered* $\mathcal{S}^{\dagger}$ *through* (I, I)*, then it never leaves, as* $M^{\dagger}$ *and* $\mathcal{S}^{\dagger}$ *are absorbing states.*

b. *If* $\mathcal{X}$ *had entered* M *through* $\{(I^{\varnothing}, D), (D, I^{\varnothing})\}$ *at* t_1*, it switches to* $\mathcal{O}$ *at* t_2^* *through any outcome in the set* $\{(-, I^{\varnothing}), (I^{\varnothing}, -)\}$*, but remains in* M *through* $\{(-, D), (D, -)\}$*, but switches to* $M^{\dagger}$ $\{(-, I), (I, -)\}$*.*

c. *If* $\mathcal{X}$ *entered "*$\dagger$*" at* t_1*, it remains there if the outcome at* t_2^* *is in the set* $\{(D, *), (*, D)\}$*; it enters* $M^{\dagger}$ *if the outcome lie in* $\{(I^{\varnothing}, -), (-, I^{\varnothing})\}$*, and enters* $\mathcal{S}$ *with an outcome in the set* $\{(-, I), (I, -)\}$*.*

d. *If* $\mathcal{X}$ *had remained in* $\mathcal{S}$ *at* t_1 *(i.e. through* (D, D)*), then*

   i) *it enters* $M^{\dagger}$ *if the outcome at at* t_2^* *is* $\{(I, I^{\varnothing}), (I^{\varnothing}, I)\}$*;*

   ii) *it remains in* $\mathcal{S}$ *if the outcome at* t_2^* *is* (D, D)*;*

   iii) *it enters* $\mathcal{S}^{\dagger}$ *if the outcome at* t_2^* *is* (I, I)*;*

   iv) *it enters* M *if the outcome at* t_2^* *lie in* $\{(I^{\varnothing}, D), (D, I^{\varnothing})\}$*,*

   v) *it enters* $\dagger$ *if the outcome at* t_2^* *lie in* $\{(D, I), (I, D)\}$*, and*

   vi) *it enters* $\mathcal{O}$ *if the outcome at* t_2^* *is* $(I^{\varnothing}, I^{\varnothing})$*.*

e. *If* $\mathcal{X}$ *had entered* $\mathcal{O}$ *at* t_1*, it remains there permanently whether or not* $\mathcal{I}^2$ *arrives within the specified planning horizon, i.e.* $\mathcal{O}$ *is also an absorbing state.*

f. $\mathcal{S}^{\dagger}$ *is an absorbing state*

$\mathcal{X}$'s Markov property is evident from *Model Statement* 2: transitions at t_2 depend only on $\mathcal{X}$'s state at t_1; so there exists a transition probability function $f : \mathcal{E} \times \mathcal{E} \times T \to [0, 1]$, such that for the sequence of times $t_0 < t_1 < t_2^*$, $P(\mathcal{X}_{t_2^*} = j | \mathcal{X}_{t_1} = i_1, \mathcal{X}_{t_0} = i_0) = P(\mathcal{X}_{t_2^*} = j | \mathcal{X}_{t_1} = i_1) = f(\mathcal{X}_{t_1} = i_1, \mathcal{X}_{t_2^*} = j, t)$ for any j, i_1 and i_0 in $\mathcal{E}$. In other words, conditionally on the present state ($\mathcal{X}_{t_1} = i_1$), the future state ($\mathcal{X}_{t_2^*} = i_2$) does not depend on the past state ($\mathcal{X}_{t_0} = i_0$).

Transitions into states $M^{\dagger}$, and M share a common feature, which is that they are direct results of a toxic investment from either of the firms, and in the case of $\mathcal{O}$, from both. The difference then, only lie in the outcome of the strategic actions of the firms i.e. where the both invests in toxic celebrities, $\mathcal{X}$ enters $\mathcal{O}$; where a firm defers while its rival gets a toxic celebrity $\mathcal{X}$ enters M, but will enter $M^\dagger$ if that firm had invested too but got a non-toxic celebrity. The firms' knowledge about the interplay of the strategic aspects of the game and the effects of toxicity affords them the ex ante opportunity to analyse the game in full and identify optimal strategies as we show later in this chapter.

Together, *Model statements* 1 and 2 specify the collection of actions which trigger the transition of $\mathcal{X}$ within its state space. The probabilities that these actions are chosen are given by the one-step transition probabilities, $f(\mathcal{X}_{t_n}, \mathcal{X}_{t_{n+1}}, t) = f_{i,j}$, for all $n \geq 0$ and $\mathcal{X}_t \in \mathcal{E}$, as well as the elements of the transition matrix $\boldsymbol{F} = (f_{i,j})_{i,j\in\mathcal{E}}$. We determine the forms of these probabilities in Section 4.

4.3.1 Demand function and payoff structure

Consider the problem of a Manchester United fan who walks into a sports' apparel store to get a jersey. She is immediately faced with a number of alternative jerseys (ignoring those of specific players), for example, Chelsea FC's, FC Barcelona's etc. These alternatives represent her "choice set." They are mutually exclusive (choosing one jersey necessarily implies not choosing any of the other ones), exhaustive (in that she may only choose either a Manchester, Chelsea, Barcelona or none at all), and (the set of all club jerseys can be expected to be) finite. Subliminally, this shopper invokes a common decision mechanism to select an alternative from her choice set by evaluating each alternative through some sort of psychological stimuli or utility function (we abstract away from the specifics of how the shopper formulates this utility). This choice mechanism relies on the utility maximising behaviour of the shopper and demand models derived in this way are referred to as random utility models (RUMs). We take this perspective with the consumers' choice of either A's or B's product as follows.

Take any consumer in the product market facing a choice between the products of firms A and B, and would obtain a certain level of utility from each alternative. A random consumer η gets a utility value of $U_{\eta,a}$ if she chooses A and $U_{\eta,b}$ if B. Obviously, this consumer will choose the alternative that provides the greatest utility, so that η will choose A if and only if $U_{\eta,a} > U_{\eta,b}$.

While the consumer has full knowledge of her utility for each alternative, we may only observe some attributes of her utility, for example, her willingness to pay for a specific product. The heterogeneity of these attributes creates uncertainty, so that the observed attribute $J_{\eta,a} \neq U_{\eta,a}$ for A's product (the same holds for B), which comes from the fact that $U_{\eta,a} = J_{\eta,a} + \epsilon_{\eta,a}$, where $\epsilon_{\eta,a}$ is an Extreme Value Type I random variable that represents the error or the portion of the true utility that is unobserved/unknown. In our model, we consider the observable portion of a consumer's utility for a product as a measure of that brand's equity with the consumer, and interpret that as the maximum amount the consumer will be willing to part with to own a unit of the brand's product, more specifically, the net satisfaction the consumer derives from purchasing a unit of the product, i.e. $J_{(\cdot)} = u_{(\cdot)} - p_{(\cdot)}$, where u is the willingness of the consumer to pay, and p, the product price. This approach is well suited to economic problems involving endogeneity of prices in the presence of unobserved product characteristics and individual consumer preferences on which their utility depends, so that product market demand levels for each item in the market is derived as the aggregate outcome of consumer choices. Berry (1994) makes a very good case for some of the empirical advantages of using a discrete-choice modelling approach, similar to the RUM described above, to determine market outcomes in a differentiated oligopoly market.

Non-strategic demand and profit functions

Suppose there is only one potential brand available in the market place, say firm A's, and the choice facing a consumer is to purchase at most one unit of this brand's product or that of an outside product, O, for which the consumer has a utility of $\epsilon_{\eta,O}$. Then the probability that a randomly chosen consumer purchases from firm A is

$$\begin{aligned} Pr(A \text{ is chosen}) &= Pr(U_{\eta,a} > U_O) \\ &= Pr(J_{\eta,a} + \epsilon_{\eta,a} > \epsilon_{\eta,O}). \end{aligned} \tag{4.1}$$

Under the assumption that $\epsilon_{\eta,a}, \epsilon_{\eta,O}$ are independent and identically distributed Extreme Value Type I random variables, the probability in Equation 4.1 is precisely,

$$Pr(A \text{ is chosen}) = \frac{e^{J_{\eta,a}}}{1 + e^{J_{\eta,a}}} \triangleq S(p_a). \tag{4.2}$$

Firm A's profit maximisation problem is therefore

$$\max_{p_a>0} MS(p_a)p_a. \tag{4.3}$$

The per period equilibrium profit for firm A is given by

$$\pi_a^* = MS(p_a^*)p_a^* = W(e^{u_a-1})_{|M=1}, \tag{4.4}$$

where $W(\cdot)$ is a Lambert W function, $M(>0)$ is the measure of consumers in the market, and $S(\cdot)$ is the firm's market share. We assume, in the interest of parsimony, that marginal cost is zero and M is unity. See Appendix 2 for the derivations of the expressions in equations 4.2, 4.3, and 4.4.

Strategic demand and profit functions

Where a random consumer has to make a choice between A and B (and some outside good on offer), the market share of firm A is given by

$$S_a(p_a, p_b) = \frac{e^{J_{\eta,a}}}{1 + e^{J_{\eta,a}} + e^{J_{\eta,b}}}, \tag{4.5}$$

and per period profit of firm A in the Nash equilibrium of the product market is

$$\pi_a^* = MS_a(p_a^*, p_b^*)p_a^* = 1 + W\left(\frac{e^{u_a-1}}{1 + e^{u_b-p_b}}\right)_{|M=1}. \tag{4.6}$$

The demand and Nash equilibrium profit functions for firm B follow correspondingly from equations 4.5 and 4.6 (see Appendix 2 for further details).

4.4 Computing the equilibrium

We re-iterate, formally, that a non-toxic investment has the capacity to invoke a higher willingness to pay for the branded product: whereas the nominal willingness to pay, u_η, enters a random consumer's utility function as $U_\eta = u_\eta - p + \epsilon_\eta$; with regards to a non-toxic investment, we represent the consumer's willingness to pay as $\hat{u}_\eta \triangleq u_\eta + k$[22]. So suppose A's investment is non-toxic, the utility derived by an arbitrary consumer is therefore $U_a^s = u_a + k - p_a + \epsilon_a$, where $k \sim U(0, \kappa(u_\eta))$[23]. Firm A's strategic per period payoff, after this investment, will then be $\pi_a^* = S_a(k)(p_a^*(k), p_b^*)p_a^*(k)$, but the true functional form depends on whether firm B invests or defers, and should B choose to invest, whether or not its investment is toxic. The numerical value of π_a^*, on the other hand, further depends on whatever value k assumes within the support of its distribution. Which implies that firm A will consider investing if and only if its inference about the evolution of the system and the value of the counterfactual indicate that

$$V_{t,\mathcal{X}}(k|\text{B's choice}, \lambda)_{|\mathbf{I}} - \mathcal{K} > V_{t,\mathcal{X}}(\text{B's choice}, \lambda)_{|\mathbf{D}}; \tag{4.7}$$

where $V_{t,\mathcal{X}}(\cdot)$ is the firm's expected present value of whatever choice it makes at t, which depends also on the actions of the rival firm and the state of $\mathcal{X}$. Therefore V_t of a firm in a non-strategic environment will necessarily be different to that of a firm in a competitive market. A corresponding condition to the expression in [4.7] holds for B.

From a researcher's (or external observer's) point of view, the probability, therefore, that there is a successful investment, and that that investment came from firm A is

$$X_s^t = Pr\left(V_{t,\mathcal{X}}(k|\text{B's choice}, \lambda)_{|\mathbf{I}} - \mathcal{K} \geq V_{t,\mathcal{X}}(\text{B's choice}, \lambda)_{|\mathbf{D}}\right); \tag{4.8}$$

Equation 4.8 instructs the observer of the condition under which A would choose to invest, but it does not guarantee that a successful investment will be observed. We think of X_s^t, á la Bayes' theorem, as the likelihood of a successful investment by firm A (i.e. the available data which may modify initial opinion about the viability of the investment → prior belief). From Bayes' theorem, we know that the evolution of $\mathcal{X}$ will be specified, instead, by posterior probabilities which depend on the likelihood of success or failure (contingent on the expected strategic outcomes as in equation 4.8),

[22] Chung et al. (2013) show, through empirical analysis of Nike golf ball prices under Tiger Woods' endorsement, that a non-toxic celebrity endorsement can change the utility a consumer obtains from consuming a product.

[23] We assume here that the upper bound of k's distributional support is a function of a customer's willingness to pay.

and prior beliefs of pulling a toxic or non-toxic celebrity of the investment opportunity i.e. p_1 and $(1-p_1)$ respectively, which are determined by nature, and are common knowledge. We let p_1 denote the prior belief at t_1, and we use p_2 to denote the prior belief at t_2, while assuming that $0 < p_1 \leq p_2$[24]. The probability, therefore, that we see a successful investment from firm A is determined first by the likelihood that A chooses to invest, and that it succeeds at it. So let $X_a^t = Pr$(A invests and succeeds) $:= Pr$(success/invest), then

$$X_a^t = \frac{p_{1,2}X_s^t}{p_{1,2}X_s^t + (1-p_{1,2})X_f^t}, \tag{4.9}$$

where X_s^t is Pr(successful investment for A) $:= Pr$(invest/success), X_f^t is Pr(failed investment) $:= Pr$(invest/fail). In the same vein, let $Y_b^t := Pr$(B invests and succeeds) $= Pr$(success/invest), then

$$Y_b^t = \frac{p_{1,2}Y_s^t}{p_{1,2}Y_s^t + (1-p_{1,2})Y_f^t}, \tag{4.10}$$

where Y_s^t is Pr(successful investment for B) $:= Pr$(invest/success), Y_f^t is Pr(failed investment) $:= Pr$(invest/fail). $p_{1,2}$ is short-form for the priors at either t_1 or t_2.

These probabilities, and the mechanism through which they were derived, are critical to the evolution of $\mathcal{X}$, and by extension, the expected value of the investment that each firm may expect to earn subject to its actions and those of its rival.

As an illustration, consider a firm's expected present value as in equation [4.7], the mechanism involved in A's expected present value is as follows:

$$V_{t_1,\mathcal{X}}(k|\text{B's choice}, \lambda)_{|\mathbf{I}} = \left(\mathcal{G}\cdot Y_s^t + \mathcal{H}\cdot Y_f^t + \mathcal{L}(1-p_1Y_s^t-(1-p_1)Y_f^t)\right). \tag{4.11}$$

$\mathcal{G}$, $\mathcal{H}$, and $\mathcal{L}$ are the discounted expected payoffs under the various possible market scenarios that could emerge through the strategies of the firms. Their respective functional forms are $\delta_t(p_1\pi_{k_{a,b}}^{Duopoly} + (1-p_1)\mathit{0})$, $\delta_t(p_1\pi_{k_a}^{Monopoly} + (1-p_1)\mathit{0})$, and

[24]This condition introduces the concept of learning as is the case in Stenbacka and Tombak (1994).

$\delta_t(p_1\pi_{k_a}^{Duopoly}+(1-p_1)\mathit{0})$; where $k_{a,b}$ indicates the presence of the random variable k in A and B's consumers' utility functions following a successful investment in their brand equities, and $k_a(k_b)$ indicates that only firm $A(B)$'s consumers get the additional k in their utility functions. δ_t in the equations above, expressed as $(1-[\frac{1}{1+r}]^{t_2^*})/(1-(\frac{1}{1+r}))$ is the discount factor evaluated from t_1 through to the expected arrival time of the second investment opportunity — $t_2^*=1/\lambda$. We use δ to denote the discount factor evaluated from the expected arrival time of the second investment opportunity to infinity with the expression $1/(1-r)$, where r is the discount rate.

$\pi_{k_{a,b}}^{Duopoly}$ denotes firm A's per period expected payoff if it chooses to invest and its rival, B, also invests and succeeds. A earns this payoff with probability p_1, that is A succeeds, or earns $\mathit{0}$ with probability $(1-p_1)$ if it fails. Should B invest and fail, A earns $\pi_{k_a}^{Monopoly}$ with probability p_1 if it succeeds while it again earns $\mathit{0}$ with probability $(1-p_1)$ if it fails. Lastly, if B decides to defer, then A earns $\pi_{k_a}^{Duopoly}$ with probability p_1, but if it fails, it gets $\mathit{0}$.

Using the above illustration, and from equations [4.8] and [4.11], it follows that X_s^t can now be expressed as a non-linear function of Y_s^t and Y_f^t, and Y_s^t can also be expressed as a non-linear function of X_s^t and X_f^t. We then have two non-linear equations in four unknowns to be solved simultaneously. However, from equations [4.8], [4.9] and [4.10], the exhaustiveness of outcome possibilities and the distribution of k upon which they depend implies that $X_f^t=1-X_s^t$ and $Y_f^t=1-Y_s^t$, since

$$\begin{aligned}1-X_s^t &= Pr\left(V_{t,\mathcal{X}}(k|\text{B's choice},\lambda)_{|\mathbf{I}}-\mathcal{K}<V_{t,\mathcal{X}}(\text{B's choice})\right)\equiv X_f^t\\ &= Pr(\text{failed investment}):=Pr(\text{invest/fail}).\end{aligned} \tag{4.12}$$

This effectively means we can now express X_s^t as a non-linear function of just Y_s^t, and vice versa. We solve these equations numerically to derive the values of the unknowns.

The timeline of the game and possible evolution of $\mathcal{X}$ are shown in Figures [4.1] and [4.2] respectively.

Figure 4.1: Investment timeline

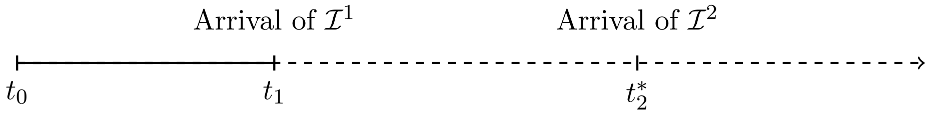
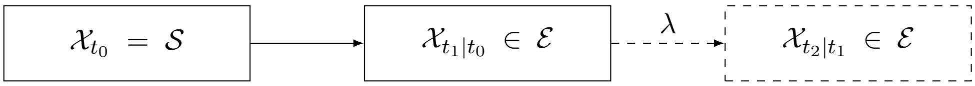


Figure 4.2: State-space evolution



4.4.1 Evolution of $\mathcal{X}$

The respective transition matrices at each of the investment epochs t_1 and t_2 are

$$\begin{array}{c} \\ \mathcal{S} \end{array}\begin{array}{c} \begin{array}{cccccc} M^\dagger & M & \dagger & \mathcal{S}^\dagger & \mathcal{S} & \mathcal{O} \end{array} \\ \begin{pmatrix} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 \end{pmatrix} \end{array} \quad \text{and} \quad \begin{array}{c|cccccc} & M^\dagger & M & \dagger & \mathcal{S}^\dagger & \mathcal{S} & \mathcal{O} \\ \hline M^\dagger & 1 & 0 & 0 & 0 & 0 & 0 \\ M & x_7 & x_8 & 0 & 0 & 0 & x_9 \\ \dagger & x_{10} & 0 & x_{11} & x_{12} & 0 & 0 \\ \mathcal{S}^\dagger & 0 & 0 & 0 & 1 & 0 & 0 \\ \mathcal{S} & x_{13} & x_{14} & x_{15} & x_{16} & x_{17} & x_{18} \\ \mathcal{O} & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

Figure [4.3] shows the transition dynamics of $\mathcal{X}$ at t_1 and t_2, which follow from the model statements in Section [3].

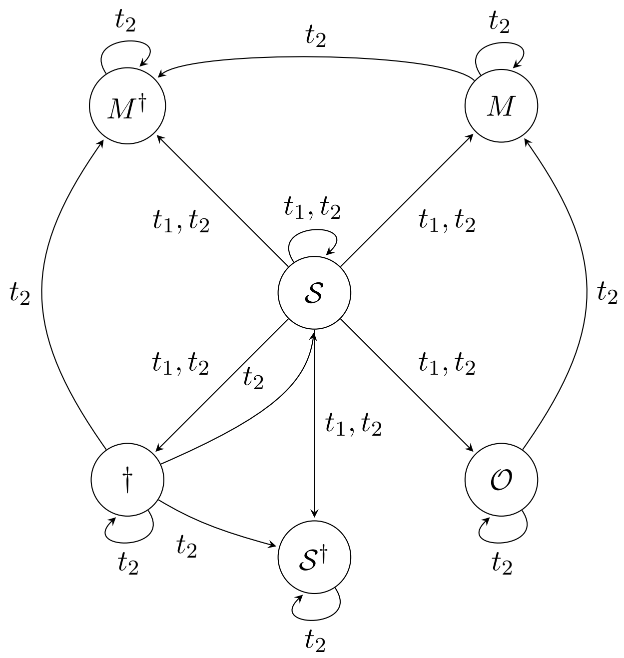
Transition probabilities at t_1

Claim 1. $x_1 = Pr(\mathcal{X}_{t_1} = M^\dagger | \mathcal{X}_{t_0} = \mathcal{S}) = X_a^{t_1}(1 - Y_b^{t_1}) + Y_b^{t_1}(1 - X_a^{t_1})$.

Let $\varkappa_t$ be the number of firms remaining in the system after any time t, so that $\varkappa_t = \{0, 1, 2\}$. It follows from *Model statement* 1 that $x_1 = Pr((I_1, I_1^{\varnothing}) \vee (I_1^{\varnothing}, I_1))$; which from equations [4.9] and [4.10] is $X_a^{t_1}(1 - Y_b^{t_1}) + Y_b^{t_1}(1 - X_a^{t_1})$.

Claim 2. $x_2 = Pr(\mathcal{X}_{t_1} = M | \mathcal{X}_{t_0} = \mathcal{S}) = (1 - X_a^{t_1}) \left(1 - p_1 \dfrac{Y_s^{t_1}}{Y_b^{t_1}}\right) + (1 - Y_b^{t_1}) \left(1 - p_1 \dfrac{X_s^{t_1}}{X_a^{t_1}}\right)$.

Figure 4.3: Transition rules



Under $\mathcal{X}$'s transition rules, transition to M from $\mathcal{S}$ at t_1 is achievable with $Pr((I_1^\varnothing, D_1) \vee (D_1, I_1^\varnothing))$. And from equation [4.9] (or equation [4.10] for B) the probability that A defers is $1-(p_1X_s^t+(1-p_1)X_f^t)$, which is equivalent to $1-p_1\dfrac{X_s^{t_1}}{X_a^{t_1}}$.

Claim 3. $x_3 = Pr(\mathcal{X}_{t_1} = \dagger | \mathcal{X}_{t_0} = \mathcal{S}) = X_a^{t_1}\left(1-p_1\dfrac{Y_s^{t_1}}{Y_b^{t_1}}\right) + Y_b^{t_1}\left(1-p_1\dfrac{X_s^{t_1}}{X_a^{t_1}}\right).$

Above transition is triggered with $Pr((I_1, D_1) \vee (D_1, I_1))$ — refer to *Model statement* [1]. The three claims following also derives easily from the model statement:

Claim 4. $x_4 = Pr(\mathcal{X}_{t_1} = \mathcal{S}^\dagger | \mathcal{X}_{t_0} = \mathcal{S}) = Pr(I_1, I_1) = X_a^{t_1}Y_b^{t_1}$*;*

Claim 5.
$$x_5 = Pr(\mathcal{X}_{t_1} = \mathcal{S} | \mathcal{X}_{t_0} = \mathcal{S}) = Pr(D_1, D_1) = \left(1-p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)\left(1-p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right)\text{; and}$$

Claim 6. $x_6 = Pr(\mathcal{X}_{t_1} = \mathcal{O} | \mathcal{X}_{t_0} = \mathcal{S}) = Pr((I_1^\varnothing, I_1^\varnothing)) = (1-X_a^{t_1})(1-Y_b^{t_1}).$

Transition probabilities at t_2

Claim 7. $x_7 = Pr(\mathcal{X}_{t_2} = M^\dagger | \mathcal{X}_{t_1} = M)$

$$= \lambda h \left(X_a^{t_2} \left(1 - Y_b^{t_1}\right) \underbrace{\left(1 - p_1 \frac{X_s^{t_1}}{X_a^{t_1}}\right)}_{A\ defers\ at\ t_1} + Y_b^{t_2} \left(1 - X_b^{t_1}\right) \underbrace{\left(1 - p_1 \frac{Y_s^{t_1}}{Y_b^{t_1}}\right)}_{B\ defers\ at\ t_1} \right)$$

For $\mathcal{X}$ to transition into $M^\dagger$ from M at t_2, it must be that one firm had kept its option alive at t_1 while its rival had invested in a toxic celebrity and has thus ceased to be active. If $\mathcal{I}^2$ arrives within any interval of length h before T, then it does so with probability, λh. Therefore, from *Model statements* [1] and [2], the following condition must be satisfied for the transition from M to $M^\dagger$ to occur: $\mathcal{I}^2$ arrives at some time $t_2 < T$, and $(I_1^{\varnothing}, D_1) \wedge (-, I_2) \vee (D_1, I_1^{\varnothing}) \wedge (I_2, -)$. Claim [7] formalises the probability that this condition is satisfied.

Claim 8. $x_8 = Pr(\mathcal{X}_{t_2} = M | \mathcal{X}_{t_1} = M)$

$$= (1 - Y_b^{t_1}) \left(1 - p_1 \frac{X_s^{t_1}}{X_a^{t_1}}\right) \left[\lambda h \left(1 - p_2 \frac{X_s^{t_2}}{X_a^{t_2}}\right) + (1 - \lambda h)\right]$$
$$+ (1 - X_a^{t_1}) \left(1 - p_1 \frac{Y_s^{t_1}}{Y_b^{t_1}}\right) \left[\lambda h \left(1 - p_2 \frac{Y_s^{t_2}}{Y_b^{t_2}}\right) + (1 - \lambda h)\right].$$

Claim 9. $x_9 = Pr(\mathcal{X}_{t_2} = \mathcal{O} | \mathcal{X}_{t_1} = M) = \lambda h \Big[(1 - Y_b^{t_1}) \left(1 - p_1 \frac{X_s^{t_1}}{X_a^{t_1}}\right) (1 - X_a^{t_2})$

$$+ (1 - X_a^{t_1}) \left(1 - p_1 \frac{Y_s^{t_1}}{Y_b^{t_1}}\right) (1 - Y_b^{t_2})\Big].$$

Claim 10. $x_{10} = Pr(\mathcal{X}_{t_2} = M^\dagger | \mathcal{X}_{t_1} = \dagger)$

$$= \lambda h \Big[X_a^{t_1} \left(1 - p_1 \frac{Y_s^{t_1}}{Y_b^{t_1}}\right) (1 - Y_b^{t_2}) + Y_b^{t_2} \left(1 - p_1 \frac{X_s^{t_1}}{X_a^{t_1}}\right) (1 - X_a^{t_2})$$
$$+ Y_b^{t_2} \left(1 - p_1 \frac{X_s^{t_1}}{X_a^{t_1}}\right) (1 - X_a^{t_2})\Big].$$

Claim 11. $x_{11} = Pr(\mathcal{X}_{t_2} = \dagger | \mathcal{X}_{t_1} = \dagger) = Y_b^{t_1} \left(1 - p_1 \frac{X_s^{t_1}}{X_a^{t_1}}\right)$

$$\times \Big[\lambda h \left(1 - p_2 \frac{X_s^{t_2}}{X_a^{t_2}}\right) + (1 - \lambda h)\Big].$$

Claim 12. $x_{12} = Pr(\mathcal{X}_{t_2} = \mathcal{S}^{\dagger}|\mathcal{X}_{t_1} = \dagger) = \lambda h\Big[Y_b^{t_1}\left(1 - p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)X_a^{t_2} + X_a^{t_1}\left(1 - p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right)Y_b^{t_2}\Big].$

Claim 13. $$x_{13} = Pr(\mathcal{X}_{t_2} = M^{\dagger}|\mathcal{X}_{t_1} = \mathcal{S}) = \left(1 - p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)\left(1 - p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right) \times\lambda h\Big[X_a^{t_2}(1 - Y_b^{t_2}) + Y_b^{t_2}(1 - X_a^{t_2})\Big].$$

Claim 14. $$x_{14} = Pr(\mathcal{X}_{t_2} = M|\mathcal{X}_{t_1} = \mathcal{S}) = \left(1 - p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)\left(1 - p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right) \times\lambda h\Big[\left(1 - p_2\frac{X_s^{t_2}}{X_a^{t_2}}\right)(1 - Y_b^{t_2}) + (1 - X_a^{t_2})\left(1 - p_2\frac{Y_s^{t_2}}{Y_b^{t_2}}\right)\Big].$$

Claim 15. $$x_{15} = Pr(\mathcal{X}_{t_2} = \dagger|\mathcal{X}_{t_1} = \mathcal{S}) = \left(1 - p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)\left(1 - p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right) \times\lambda h\Big[X_a^{t_2}\left(1 - p_2\frac{Y_s^{t_2}}{Y_b^{t_2}}\right) + Y_b^{t_2}\left(1 - p_2\frac{X_s^{t_2}}{X_a^{t_2}}\right)\Big].$$

Claim 16. $x_{16} = Pr(\mathcal{X}_{t_2} = \mathcal{S}^{\dagger}|\mathcal{X}_{t_1} = \mathcal{S}) = \lambda h \cdot X_a^{t_2}\left(1 - p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)Y_b^{t_2}\left(1 - p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right)$

Claim 17. $$x_{17} = Pr(\mathcal{X}_{t_2} = \mathcal{S}|\mathcal{X}_{t_1} = \mathcal{S}) = \left(1 - p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)\left(1 - p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right) \times\lambda h\Big[\left(1 - p_2\frac{X_s^{t_2}}{X_a^{t_2}}\right)\left(1 - p_2\frac{Y_s^{t_2}}{Y_b^{t_2}}\right)\Big].$$

Claim 18. $$x_{18} = Pr(\mathcal{X}_{t_2} = \mathcal{O}|\mathcal{X}_{t_1} = \mathcal{S}) = \left(1 - p_1\frac{X_s^{t_1}}{X_a^{t_1}}\right)\left(1 - p_1\frac{Y_s^{t_1}}{Y_b^{t_1}}\right) \times\lambda h(1 - X_a^{t_2})(1 - Y_b^{t_2}).$$

Sequential equilibrium

The foregoing illustrate how visitation to any state depends on the outcomes of the actions/choices the firms make at each investment epoch and the realisation of the stochastic element (k) indicating whether or not an investment is toxic. In our framework, the transition probabilities represent posterior probabilities that each of these states are visited and, collectively, they form the firms' belief system consistent with whatever choices are made at each investment epoch. This belief system is critical to the equilibrium strategies the firms are expected to adopt, and it presents an ex ante apprehension, that at any information set A encounters (e.g. $\mathcal{S}$ at t_1 and t_2), its strategy is a best response to B's strategies, given A's beliefs at that information set about B's probability measure over the set of actions available. These probabilities help a firm understand the history through which it may have arrived at a particular information set as well as beliefs about the future.

In the spirit of Nash equilibrium, our approach also require that the belief about the history that has occurred leading into an information set is derived using Bayes' rule and should be consistent with the firms' strategy profile. These conditions allows allow us to derive strategies that are consistent with the sequential equilibria of the game, which we specify in the following Section.

4.5 Numerical analysis

The model developed in this chapter provides us with the ability to fully characterise the optimal strategy in brand investment adoption times using celebrities in an imperfectly competitive product market. In this framework, it is not practical or, indeed, useful to make assumptions that place lower bounds on profit flows as in Huisman and Kort (2004) where per period profit follows a geometric Brownian motion. Profit flows in our model are jointly determined by competitive forces through Nash equilibrium prices and an exogenous uncertain element representing consumers' affinity for a brand through the celebrity it endorses (i.e. k, the random utility parameter which feeds into the firms' demand function). In contrast to the perfectly competitive market version that Huisman and Kort (2004) analyse, no realisation of k is available ex ante or at any time before the firms make their investment choices. In effect, firms have to make an investment before nature reveals the value of this uncertain element. However, the distribution of this parameter is common knowledge at the begin of the game.

Implicit in our model, and revealed in the transition states of $\mathcal{X}$, is the toxic risk investments in celebrities are susceptible to. With the infimum of per period profits being zero, a firm is never guaranteed a positive expected present value for its investment. As such, a firm may find that it makes no profits at all during the course of the game either through its own (in)actions or those of its rival's. For instance, where a firm invests in a toxic celebrity, it gets no boost in consumer willingness to pay for its product, ($k = 0$), and in addition, whatever utility value consumers had previously attributed to the firm's product will diminish in the face of toxicity, due to the firm's association with the celebrity. Therefore, depending on just how toxic the celebrity is, consumers may consider that consuming the firm's product is not only non-beneficially, but perhaps, socially costly, such that the products only provide negative utilities. Mathematically, this means, u (in Section 4.3.1) is less than 0, and therefore, per period profit for this firm (say A), in equilibrium given by $\pi_a^* = p_a^* S_a^*(p_a^*, p_b)$ is $W[(e^{u_a - 1})/(1 + e^{u_b - p_b})] = W[1/e^{\xi}(1 + e^{u_b - p_b})]$, where $e^{\xi} = e^{u-1}$ (where $u < 0$). Clearly, $1/e^{\xi}(1 + e^{u_b - p_b}) \to 0$ as $u \to -\infty$, so that $\pi^* = W(0) = 0$[25].

On the other hand, where a firm does not make an investment but its rival does, and succeeds, the measure of additional (consumer) willingness to pay for the rival firm's product due to its investment may be large enough to shrink the deferring firm's market share to zero. This is easily revealed in the deferring firm's expected per period profit as in $\pi = pS$, where $S = W[e^{u-1}/(1 + e^{u_b + k - p_b})]/(1 + W[e^{u-1}/(1 + e^{u_b + k - p_b})])$, and as $k \to \infty$ (in the extreme case), $S = W(0)/(1 + W(0)) = 0$, and therefore, $\pi = p \times 0 = 0$. As a result, the quantum of a firm's expected present value of making an investment (or seeking a celebrity's endorsement) could very well be zero[26]. In our model, the state space of $\mathcal{X}$ captures the toxic risk of investing in a celebrity, whilst the nature of the value of the investment over non-toxic states reflects the influence of uncertainty over k.

[25] Note that the Lambert $W(\cdot)$ function can be expressed as $W(x) = xe^{-W(x)}$, so that $W(0) = 0e^{-W(0)} = 0$ since $e^{-W(0)} = 1/e^{W(0)} \neq 0$.

[26] This is clearly a conservative outcome since we have assumed zero marginal costs. Should marginal costs be positive, then per period profits under these circumstances could be less than zero, and if this happens at anytime after t_2, then the expected present value of the investment will also be less than zero, so that $-\mathcal{K}$ will no longer be the lower bound of the expected net present value as is the case in the conservative version explored here. This simply escalates the toxicity risk of the investment/endorsement opportunities.

We have assumed in this model framework that the support of the distribution for k remains unchanged in all investment epochs, and rightly so, since irrespective of the potential of a celebrity athlete, for example, the toxicity risk does not improve. Should they get entangled in a scandal, they become just as toxic as other stars who have come before. Which essentially means that both investment opportunities carry similar expected ex post risk profiles with respect to k. However, experience or knowledge of the industry and/or celebrity endorser may improve the ex ante perception of success at different investment epochs. This experience/expert opinion or (imperfect) knowledge is captured by the prior probabilities of a successful investment–or that an investment does not turn out to be toxic–and they play a very key role in the transition of $\mathcal{X}$ and, by extension, the investment mechanism of the firms at each opportunity to invest. Our results are largely driven by the toxicity risk each firm faces with respect to its investment, and the fact that it also faces strategic uncertainty both of which are only revealed post investment (which are common features of competition in an imperfectly competitive market under exogenous uncertainty); except that, unlike outcomes in extant studies, toxicity risk moderates the firms' strategies, and therefore, the transition probabilities of $\mathcal{X}$ through which the values of the investments are analysed.

Our analysis begins with two symmetric firms — both starting out with the same consumer utility values and — with similar prior probabilities toward the toxicity risk each investment opportunity carries, and equal opportunity to either investment or defer at t_1. The transition matrices at t_1 and t_2 are derived from the firms' beliefs about the probability measure over each other's strategy space at t_1 and t_2. They further inform the firms' beliefs about the probability measure over the state space of $\mathcal{X}$, which represents the firms' best estimate of the chances that a particular market structure emerges ex post. These beliefs, together with expected payoffs from whatever market structure that ensues post t_1 and t_2 (or the end of the firms' investment window), are used to determine the true expected net present value of any strategy the firms may adopt at t_1, and subsequently the equilibrium of the game for any value of λ. We reiterate here that the set of actions that the firms may adopt is $\{I, D\}$ at either of the available investment opportunities, so we can represent the game using the normal form below:

		Firm B	
		I	D
Firm A	I	$V_a^{(I,I)}, V_b^{(I,I)}$	$V_a^{(I,D)}, V_b^{(I,D)}$
	D	$V_a^{(D,I)}, V_b^{(D,I)}$	$V_a^{(D,D)}, V_b^{(D,D)}$

Table 4.2: Normal form representation of the game at t_1

In deriving the functional form of the net present value functions, and subsequently, the solution of the game, we require that, given the state the firms find themselves before t_1, i.e. $\mathcal{S}$, and the initial decision they each take, the decision at t_2 is optimal with regards to the state resulting from their individual choices at t_1. The value functions above therefore constitute initial and contingency plans formulated at t_1 and supported by deriving ex post probabilities of potential t_1 and t_2 outcomes.

t_1 outcome is I, I

The transition matrix at t_1 contains posterior probabilities of the possible market structures that could emerge following the firms' choices, and with these, the firms may determine their expected per period payoffs, and the expected net present value of whatever combination of choices that lead to a particular market outcome. We have assumed that both firms are symmetric, therefore, it does not matter whose expected net present value we describe if both choose I at t_1; therefore, given that one firm chooses I, the other firm's expected net present value for choosing I at t_1, under the impression that it rival would only have chosen I if he thought it was optimal to do so, is

$$V^{(I,I)}(k) = \delta[(x_{10} =)Pr(\mathcal{X}_{t_1} = \mathcal{S}^{\dagger}|\mathcal{X}_{t_0} = \mathcal{S})]\pi^{i,i} - \mathcal{K}. \tag{4.13}$$

t_1 outcome is I, D

Suppose a firm's rival has chosen to defer at t_1, then its net present value for choosing to investment will be given by

$$\begin{aligned} V^{(I,D)}(k) = (\lambda h)^{\Delta}\big[&\delta_t Pr(\mathcal{X}_{t_1} = \dagger|\mathcal{X}_{t_0} = \mathcal{S})\pi^{i,d} - \mathcal{K} \\ &+ \delta_t \cdot \delta\left(Pr(\mathcal{X}_{t_2} = \mathcal{S}^{\dagger}|\mathcal{X}_{t_1} = \dagger)\pi^{i,i} + Pr(\mathcal{X}_{t_2} = \dagger|\mathcal{X}_{t_1} = \dagger)\pi^{i,d}\right)\big] \\ &+ (1-\lambda h)^{\Delta}\left(\delta \cdot Pr(\mathcal{X}_{t_1} = \dagger|\mathcal{X}_{t_0} = \mathcal{S})\pi^{i,d} - \mathcal{K}\right) \end{aligned} \tag{4.14}$$

Where $\Delta(= t_2^* - t_1)$ is the expected length of time of waiting until the second investment opportunity arrives, in the event that it does, which is also the length of time the per period profits earn beginning at t_1 would last until a new decisions are made at t_2^*.

t_1 outcome is D, I

Given that a firm's rival chooses I, the firm's expected net present value for choosing I at t_1 is

$$\begin{aligned} V^{(D,I)}(k) = (\lambda h)^{\Delta}\big[&\delta_t Pr(\mathcal{X}_{t_1} = \dagger|\mathcal{X}_{t_0} = \mathcal{S})\pi^{d,i} \\ &+ \delta_t\left(\delta\left(Pr(\mathcal{X}_{t_2} = \mathcal{S}^{\dagger}|\mathcal{X}_{t_1} = \dagger)\pi^{i,i} + Pr(\mathcal{X}_{t_2} = \dagger|\mathcal{X}_{t_1} = \dagger)\pi^{d,i}\right) - \mathcal{K}\right)\big] \\ &+ (1-\lambda h)^{\Delta}\left(\delta \cdot Pr(\mathcal{X}_{t_1} = \dagger|\mathcal{X}_{t_0} = \mathcal{S})\pi^{d,i} - \mathcal{K}\right) \end{aligned} \tag{4.15}$$

t_1 outcome is D, D

Should a firm choose to defer given that its rival also chooses to defer, then its expected net present value is given by

$$\begin{aligned} V^{(D,D)}(k) = (\lambda h)^{\Delta}\big[&\delta_t Pr(\mathcal{X}_{t_1} = \mathcal{S}|\mathcal{X}_{t_0} = \mathcal{S})\pi^{d,d} \\ &+ \delta_t\left(\delta\left(Pr(\mathcal{X}_{t_2} = \dagger|\mathcal{X}_{t_1} = \mathcal{S})\pi^{i,d} + Pr(\mathcal{X}_{t_2} = \mathcal{S}|\mathcal{X}_{t_1} = \mathcal{S})\pi^{d,d}\right) - \mathcal{K}\right)\big] \\ &+ (1-\lambda h)^{\Delta}\left(\delta \cdot Pr(\mathcal{X}_{t_1} = \mathcal{S}|\mathcal{X}_{t_0} = \mathcal{S})\pi^{d,d}\right) \end{aligned} \tag{4.16}$$

It is important to note that these expected net present value functions depend on k, the exogenous random parameter whose value is only revealed ex post. As a result, the firms' preferred strategy may be derived only in relation to an ex ante expected realisation of k, the best guess of which is the expected value of k over the support of its distribution i.e. $\kappa(u_\eta)/2$.

Parametrisation and equilibrium outcomes

We think of a period in this game as one year, i.e. we start observing the firm at some year 0 (t_0), and the first investment opportunity is presented to both firms a year after that i.e. t_1, which, in many sports, is the usual interval of time between investments and or endorsements of, for example, young athletes graduating from college. The prior probabilities of making a successful investment are important to our analysis and, therefore results, but not in the way Hoppe (2000) describes, where these prior probabilities are fixed, and firms have accurate knowledge of the value of the investment after observing a rival's investment, ultimately, increasing the second mover advantage, where the prior probability is very low. In our model, observing a rival's investment outcome, whilst beneficial in terms of learning best practices, does not confer perfect knowledge, only a higher expectation of improving a firm's chances of succeeding in future investments. It does not eliminate the toxicity risk the investment carries. In fact, how successful an investment turns out to be, depends on the value of the exogenous random variable k, which is only revealed after an investment is made, but each investment bears its own risks independent of the last, so that a future investment may have a different realisation of k from the former.

On the other hand, in Huisman and Kort (2004)'s model (which is more closely related to ours), it is implied that these prior probabilities are unity, also, the value of the stochastic component of their model never hits zero, so that the model outcomes are determined solely by the rate of arrival of the second investment opportunity. The arrival rate of the second investment opportunity also play a key role in what outcomes emerge from our model, but its impact is moderated by the prior probabilities, due the ability of the firms to analyse the posterior probabilities as we have done in this chapter.

For analytical convenience, we suppose $\kappa(u_\eta) = u_\eta$ is the upper bound of the support of the distribution of the exogenous random variable k. We solve numerically for the net present value functions as functions of k, from which we derive the equilibrium of the game, and present the main contribution of this chapter, which is summarised in the proposition below.

Proposition 5. *There are two critical regions of* λ*:*

(1) $\lambda \in [0, \lambda_1]$

- *if* p_1 *and* p_2 *are both sufficiently small (where* $p_1 \le p_2 < 1$*), the equilibrium is of the attrition type;*
- *if* p_1 *is sufficiently small and* $p_2 \to 1$*, the equilibrium is of the preemption type; and*
- *if* p_1 *is sufficiently large, for all values of* $p_2 \le 1$*, the equilibrium is of the preemption type.*

(2) $\lambda \in [\lambda_2, \infty]$*, where* $\lambda_1 < \lambda_2$*, then*

- *if* p_1 *and* p_2 *are both sufficiently small, the equilibrium is of the preemption type;*
- *if* p_1 *is sufficiently small and* $p_2 \to 1$*, the equilibrium is of the attrition type; and*
- *if* p_1 *is sufficiently large, for all values of* $p_2 \le 1$*, the equilibrium is of the preemption type.*

Proposition 5 presents some interesting implications of the prior beliefs held by firms at each investment epoch. For instance, when λ is low, then at t_1, both firms have very little expectations that a second investment opportunity will become available within their investment horizon, therefore, if the option value of waiting is relatively low (which is revealed in the value of p_2), the first panel of Figure [4.4] suggests that it is in both firms' best interest to defer at t_1[27]. Which first seems counterintuitive

[27] The horizontal axis of the figure shows the possible values of k, i.e. the exogenous random variable; and the vertical axis shows the range of values of the net present values of the investment.

since whichever firm invests at t_1 would potentially earn monopoly rents for much longer. However, the very low probability of making a success of the first investment opportunity overrides this benefit (since $V^{I,D} < 0$ for all but a small subset of $[0, \mathbf{K}]$). The second panel of Figure [4.4] demonstrates that as the value of p_2 increases (and close to perfection), the attrition game above turns into a preemption game. That is, the first mover advantage is restored as the option value of waiting increases. Again, this seems counterintuitive, but consider a firm with a low assessment of the investment opportunity at t_1 who thinks, should he succeed, he could potentially earn monopoly rents forever if the second investment opportunity never arrives (as it just might not with a very low λ). We find that there are values of k for which this firm would rather bite the bullet and commit with the hope that should its rival defer, it may never get a future shot at investing. Therefore, even when λ is low, there are sufficient values of k supporting $V^{I,D}$, such that the game could quickly turn from an attrition type to a preemption type based on the firm's prior beliefs about the toxicity of the investment opportunities.

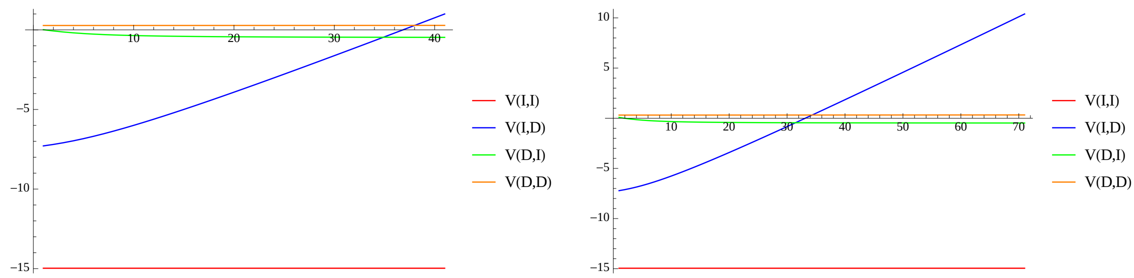


Figure 4.4: Solution to the investment game when $\lambda = 1/5$ and $p_1 = 1/3$

A slightly different scenario is observed when the prior probability of success is relatively large as in Figure [4.5]. With a very clear expectation to succeed at t_1 the firms' expectation of k has to be very low to make an early commitment seem unprofitable. In this case, the option value of waiting is relatively low since p_1 is already very high, then a firm would not expect to learn so much by delaying and waiting for a second opportunity which has a small chance of arriving. This again, leads to a preemption type game.

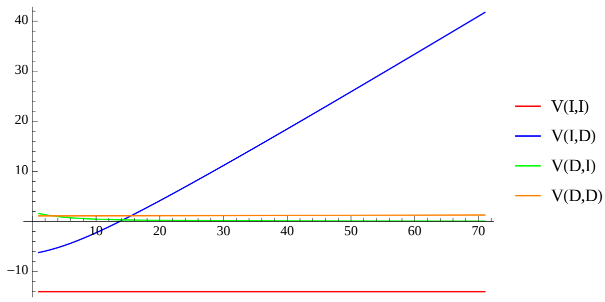


Figure 4.5: Solution to the investment game when $\lambda = 1/5$ and $p_1 = 2/3$

In the region where λ is relatively high, the outcomes are not too dissimilar. In fact, we find a preemption game turning rather quickly into an attrition game when the prior probability of success at t_2 is high relative to a low prior at t_1. Given that both firms reasonably expect the second opportunity to arrive soon, then with a higher p_2 it is more likely that the firms would want to wait to exploit it rather than prematurely hazard a more toxic investment at t_1. However, should the option value of waiting be also low, then a preemption equilibrium emerges. That is, with little to learn by waiting, a firm may consider it beneficial to sink its investment outlay early enough to earn as much monopoly rent as possible considering that, even in the event that the second opportunity does arrive early, its rival bears very similar risk at t_2 as he does at t_1. This is illustrated in the left and right panels of Figure [4.6], where the preferred outcome changes from $V^{I,D}$ to $V^{D,D}$ as $p_2 \to 1$.

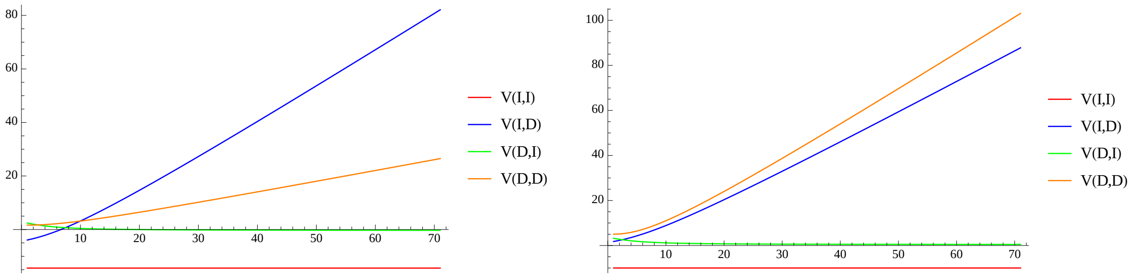


Figure 4.6: Solution to the investment game when $\lambda = 4/5$ and $p_1 = 1/3$

Finally, with λ being very high and p_1 also relatively high, there is very little to learn from waiting if the firms consider that they could make a good investment out of the first opportunity. This also leads to a preemption game irrespective of the value of

p_2 as shown in Figure [4.7].

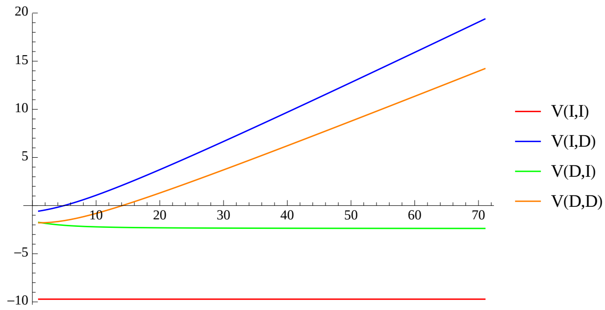


Figure 4.7: Solution to the investment game when $\lambda = 4/5$ and $p_1 = 2/3$

It is clear that in the preemption type games, there are two pure strategy Nash equilibria i.e. $\{(I, D); (D, I)\}$ and, therefore, another equilibrium in mixed strategies. The firms make decisions simultaneously and without communication at t_1, as a result, neither can guarantee being the first mover (leader) or, indeed, the follower. Furthermore, the strategy (I, I) is strictly dominated (as shown in Figures [4.4] to [4.7]), which results in an anti-coordination problem where it is in both firms' interests that they choose different actions. In a related case in Fudenberg and Tirole (1985) where $V^{I,D} > V^{I,I}$, the preemption type game results in a diffused equilibrium and no way could either of the firms make a mistake such that a simultaneous outcome emerges. This is partly due to the zero probability of a mistake which is expected naturally in continuous time; but also under the assumption that preemption gains are small within short periods in discrete time, in the mixed strategy equilibrium, the probability of a mistake converges to zero. However, the assumption of small gains within short periods does not quite hold in our model. In the time frame we have adopted, the gains of preemption can be significantly large depending on the value of k, and indeed λ, as illustrated in the Figures above, and in the same vein, the limit argument does not seem practical. Our conjecture here is that, $V^{D,I}$ involves an option at t_2, so, should the time arrive to make the investment, this firm is not obligated to follow through if it deems the investment to be unprofitable as a follower to a leader whose level of success depends on the realisation of k. Outside the risk of toxicity, we cannot guarantee that both firms will never invest simultaneously in these circumstances. One reason being the fact that a second investment opportunity might indeed not arrive, or arrive very late, so late that the deferring firm risks brand obsolescence due to the fickle nature of consumer loyalty.

Our main results demonstrate that λ is not, in itself, sufficient to prescribe equilibrium outcomes of investment opportunities where firms compete in imperfectly competitive product markets and the second opportunity's arrival follows a Poisson process. Our results push Huisman and Kort (2004)'s model out into an industrial organisation-type environment where the risks are higher due to toxicity, rather than the perfectly competitive product market where failure is not as debilitating as one would expect in practice when a celebrity endorsement goes awry.

4.6 Concluding remarks

We model the endogenous behaviour of firms facing a now-or-perhaps-later-type option to invest in their brand equities through celebrity endorsements. A firm may choose to investment in an opportunity currently available or wait until a future opportunity arrives at some random time following a Poisson process. The unique feature of our model is that such investments may not only turn out to be unprofitable, but in the extreme case, could be toxic, so that the firms or specific product lines become extinct—not just as a matter of preference, but necessity. This toxicity risk is factored into the firms' investment decisions through a stochastic process $\mathcal{X}_t$ that could visit certain states representing future market structures.

The framework we develop in this chapter allows firms to hold prior beliefs of their own regarding the potential of an investment opportunity. These probabilities are not static as in Hoppe (2000)'s model, rather firms are able to use the likelihood of an event occurring to modify their individual prior beliefs based on the states $\mathcal{X}_t$ is most likely to be in, given any combination of choices made at t_1 and/or t_2. With these modified (posterior) probabilities, the firms determine the net present value of whatever strategy is pursued at t_1. In the spirit of an action commitment type game, the firms need to invest first before becoming active and then realising whether or not an investment works, so we allow them to analyse the potential structures that could emerge given their rival's actions and a realisation of an exogenous stochastic variable representing the added value of an investment opportunity, which is reflected in the additional willingness of a consumer to pay for the firm's product (i.e. k). In effect, the firms use events' likelihoods to update their individual prior beliefs based on the states $\mathcal{X}_t$ could visit. With these, they determine the net present value of whatever strategy is implemented at t_1. Additionally, we are able to invoke sequential equilibrium through the determination of these posterior probabilities which simultaneously represent the firms' updated beliefs at any information set in which an optimal course of action is required consistent with the firms' beliefs of the probability measure over the available actions of its rival.

The main contribution of this chapter is the demonstration of the value of the firms' prior beliefs to the outcomes of the game. We find that being able to modify prior beliefs through the mechanism we specify in our model moderates the effect of the expectation of a future investment opportunity on the firms behaviour. Specifically, whether λ is high or low, we find parameter values that present the potential that an attrition game emerges in contrast to Huisman and Kort (2004), where the game changes from preemption to attrition depending on whether or not λ is low or high. While λ still plays an important role to the choice of actions in our model, we find that the value of the second mover advantage (which is the amount with which the prior at t_2 is higher than t_1's i.e. $\rho = p_2 - p_1$), plays an even greater role, as it quantifies the real value of waiting. Succinctly, when λ is high, and the firms reckon they could learn a lot from waiting—i.e. a relatively high value of ρ–then it is in their best interests to delay (if p_1 is relatively low), meaning there is a lot that they can learn between t_1 and t_2 about the potential risks of investing. However, when λ is low, their choices become more nuanced. The firms will weigh the risk of committing early against the benefit of being a monopolist forever as the second investment opportunity may never come even if the firms could learn so much more by waiting.

It is quite possible, and perhaps wise, that investors would be willing to invest in advance in any mechanism that enables them obtain useful information about their proposed endorsement in order to modify their priors and improve their investment decisions. We have not considered this scenario in our model, but it would be a useful extension to our work. This extension will likely involve another stage in the game (pre-play stage) requiring that the firms specify the amount of investments they are willing to make in order to buy/obtain additional, however imperfect, information about their proposed endorsers. This would possibly create asymmetry which might resolve the multiple equilibria and anti-coordination problem if the level of investment at the pre-play stage is common knowledge before the main game commences. Introducing asymmetry in general, presents interesting directions in which our model may be extended, for instance, firm with different values of initial brand equities at t_0, or more interestingly, firms with idiosyncratic priors which are private information, which in reality, would bring the model much closer to real world practice. However, the computational challenges of these sorts of extensions could be significant.

Lastly, it could be useful to consider multiple investment opportunities after t_1. But if the assumption that the firms may not invest more than once still holds, then, given that the investment opportunities after t_1 all follow a Poisson process, we do not envisage this extension, on its own, would present a significant improvement over the one we analyse here. Our conjecture is that since these other investment opportunities follows a Poisson process, we know that their distributional properties remain the same irrespective of how long the firms' have been waiting or, indeed, how many arrivals have been observed. Furthermore, toxicity risk implies that a toxic investment will lead the associated firm to quit the market altogether, so that it does not matter much whether the game continues a couple more periods with a monopolist serving the market.

Chapter 5

Conclusion

"What you don't know can ~~not~~ hurt you"

This thesis presents strategic real options models within an industrial organisation framework. The models extend our current knowledge regarding competitive market behaviours in the industrial organisation literature to market environments were investments are costly and irreversible, and returns are uncertain, but players have the right to delay until a later time to gather more information, and then revisit their decision problems — which are classic features of real option investments. However, where our models depart from other standard real options models is that profit potentials are dependent on the competitive market structures that emerge based on the optimal choices of the firms. On the other hand, sequential-play in these models are not parallels with the Stakelberg-type game, because there is the entry game i.e. whether to invest or defer; after which product market competition may commence *à* la Cournot or Bertrand.

By modelling real life investment problems, we show the importance of models of these type and why extant studies, have thus far, been unable to answer some of the key questions such investment opportunities raise. Crucially, the information a firm has or does not have is clearly a matter of serious importance in practice, and we have presented them as such in the models analysed in this thesis. Note that all the models presented in this thesis are theoretical models, not empirical models, but due to the complexity arising from the high dimensions of relevant variables, we resort to numerical methods in other to solve the games involved and derive the value of the investment. This is not a limitation in itself. Analytical solutions are often intuitive but also restrictive. The flexibility afforded through simulating the processes numerically means we do not have to artificially restrain relevant variables in the models to achieve analytical tractability whilst potentially limiting the scope of the model.

Chapter 2 of this thesis presents a general theory of investment games with incomplete information over types and uncertainty over demand in imperfectly competitive markets. Product market follows a Cournot-type game, but we solve for the sequential equilibrium of the entry/investment game (formulated as a game of observable delay) over the type-spaces of the firms. Crucially, we show that cost asymmetry does not explain all of the possible outcomes of the game as in extant models, rather a belief-based equilibrium is required to specify outcomes of entry game and optimal strategies of the firms. We demonstrate conditions under which a first-mover advantage exists in the entry game in relation to the value of the option held by each firm, and show that it is both firms' interest to play truthfully with respect to their types.

In Chapter 3, we extend the ideas developed in Chapter 2 to study product market competition characterised by a differential game. The firms now compete via two strategic variables, rather than one. But whilst the stochastic evolution of demand is exogenous to the firms in Chapter 2 (and in some extant models), outcomes from the differential game jointly influences market demand in the next period along with a stochastic element which represents aspects of the changing market not captured by the strategic interactions of the firms. However, to partake in the differential game, the firms first solve an entry game involving a lumpy initial investments in developing an advertising campaign through which they each try to win consumers' loyalties. Asides from its contribution to the real options, is presents important contributions to the literature on advertising and brand. We depart from conventional models by allowing adverting to simultaneously exhibit market expansion and persuasive effects but we endogenise the process through the quality of the campaign that each firm produces (types). Therefore, the dynamical system over which the differential game is played depends on the outcome of the entry game from which the firms' types are revealed.

The steady-state intensity of advertising from our simulations aligns with theoretical predictions, however, since the evolution of the dynamic system is driven not only by the intensity of advertising efforts but the relative values of the firms' types which are private information at the time of making their investment choices. This creates, within a subspace of the firm's type-space, the potential to anti-coordinate were a firm with a lower type may, with a positive but low probability end up a being a monopolist, if its rival hold an option to delay whereas itself does not.

Chapter 4 constructs a brand equity game through celebrity endorsements. Whilst celebrity endorsements have received a lot of attention in both marketing and economics community, little has been done theoretically to formulate a response to typical economic questions, such as, what is the optimal strategy when faced with investments of this nature. We adopt the economics framework regarding investment in new technology or innovation as far as its parallel goes with investments in celebrity endorsements. What uniquely separates these investments is their risk profile which, in the case of a celebrity, depends almost entirely on perception rather than any actual substance.

We extend the applications of Huisman and Kort (2004), who present a theoretically similar model framework, to show that uncertainty over the arrival of an innovation, or in our case, the next best/compatible celebrity endorser, is insufficient criterion upon which to base the investment behaviours of firms when they each hold some prior knowledge about the risk profile of the celebrity. What is important is that the firms are able to update this prior knowledge with likelihood information sourced in order to reassess their investment positions at a later time.

The overarching theme of this thesis is that by relaxing some assumptions around extant models in the real options literature and the industrial organisation literature, and creating a framework where there is synergy between these two areas, we are able to address investment problems in market environments where neither provides sufficient insight on its own.

Furthermore, the scope for further extensions of our models have been noted in each corresponding chapter where they are presented.

Chapter 6

Appendix

6.0.1 Appendix 1: Appendix to Chapter 2

Simultaneous-move Equilibria

If at the decision period (period-0), both firms choose to sink the investment outlay, $\mathcal{K}$, in period 1, i.e. $(a, b) = (I, I)$, then a Bayesian-Cournot game ensues in period 1 since marginal costs are still private information. Furthermore, as we have not made the common assumption that $q_t(c) > 0$ for all realisations of c, therefore, there exists some realisations of c for which the equilibrium output is 0 (see Hurkens (2012)). The firms, being ex ante symmetrically uninformed, and having independently drawn their marginal costs, will be maximising expected profits over an adjusted support of the marginal cost distribution as follows (we show this for firm $\mathcal{A}$):

$$\max_{q_{1,\mathcal{A}}(c_{\mathcal{A}})} \mathbb{E}(\pi_{1,\mathcal{A}}^{(I,I)}) = \max_{q_{1,\mathcal{A}}(c_{\mathcal{A}})} (q_{1,\mathcal{A}} (\theta_1 - q_{1,\mathcal{A}} - \mathbb{E}(q_{1,\mathcal{B}}(c_{\mathcal{B}})) - c_{\mathcal{A}})) \tag{A1}$$

For notational convenience, we will be using $q_{1,\mathcal{A}}$ for $q_{1,\mathcal{A}}(c_{\mathcal{A}})$ with the understanding that output decisions are functions of drawn marginal cost values. Firm $\mathcal{A}$'s expectation of $\mathcal{B}$'s output in (A1) is based on the belief that there is a marginal cost threshold value, α, above which $\mathcal{A}$, itself, will find it unprofitable to enter the market in period 1 (and hence produce zero output). As the firms are ex ante symmetric, firm $\mathcal{A}$ believes this to be true of its rival, firm $\mathcal{B}$. It therefore follows from (A1) that

$$\hat{q}_{1,\mathcal{A}} = \frac{1}{2}\left(\theta_1 - \mathbb{E}(q_{1,\mathcal{B}}(c_{\mathcal{B}})) - c_{\mathcal{A}}\right) \quad \text{and} \quad \hat{q}_{1,\mathcal{B}} = \frac{1}{2}\left(\theta_1 - \mathbb{E}(q_{1,\mathcal{A}}(c_{\mathcal{A}})) - c_{\mathcal{B}}\right) \tag{A2}$$

Ex ante, $\mathbb{E}(\hat{q}_{1,\mathcal{A}}) = \mathbb{E}(\hat{q}_{1,\mathcal{B}})$. Hence, $\mathbb{E}(\hat{q}_{1,\mathcal{B}}) = \frac{1}{2}\left(\theta_1 - \mathbb{E}(q_{1,\mathcal{A}}(c_{\mathcal{A}})) - \mathbb{E}(c_{\mathcal{B}})\right)$. Note that $\mathbb{E}(c_{\mathcal{B}}) = \int_{\underline{c}}^{\alpha} c_{\mathcal{B}} \mathrm{d}F(c_{\mathcal{B}}) := \ddot{c}$, where $\alpha \in (\underline{c}, \bar{c}]$. Expected outputs for independent draws of marginal costs in this interval is

$$\mathbb{E}(q_{1,\mathcal{B}}) = \tfrac{1}{2}\left(\theta_1 - \mathbb{E}(q_{1,\mathcal{A}}(c_{\mathcal{A}})) - \ddot{c}\right). \tag{A3}$$

Since expected outputs for both firms are taken over the same support, firm $\mathcal{B}$ should therefore expect that $\mathbb{E}(q_{1,\mathcal{B}}) = \mathbb{E}(q_{1,\mathcal{A}})$. Using this in (A3), we have

$$\mathbb{E}(q_{1,\mathcal{B}}) = \frac{\theta_1 - \ddot{c}}{3} = \mathbb{E}(q_{1,\mathcal{A}}) \tag{A4}$$

Substituting (A4) for the expectations in (A2), the equilibrium outputs for the firms are hence,

$$q^*_{1,\mathcal{A}} = \frac{1}{6}\left(2\theta_1 - \ddot{c} - 3c_{\mathcal{A}}\right) \qquad \text{and} \quad q^*_{1,\mathcal{B}} = \frac{1}{6}\left(2\theta_1 - \ddot{c} - 3c_{\mathcal{B}}\right) \tag{A5}$$

The corresponding expected equilibrium payoffs are,

$$\pi^{*(I,I)}_{1,\mathcal{A}} = \tfrac{1}{36}\left(2\theta_1 - \ddot{c} - 3c_{\mathcal{A}}\right)^2 \quad \text{and} \quad \pi^{*(I,I)}_{1,\mathcal{B}} = \tfrac{1}{36}\left(2\theta_1 - \ddot{c} - 3c_{\mathcal{B}}\right)^2. \tag{A6}$$

At the beginning of period 2, outputs and payoffs from period 1 would have been observed, therefore, the true marginal costs of each firm can be deduced. The demand level for period 2 is also observed at the start of the period. Having chosen (I, I) in period 0, the firms compete *a là* Cournot from period 2 onwards, having full information about the market parameters. Equilibrium outputs and payoffs in period 2 are

$$q^*_{2,\mathcal{A}} = \tfrac{1}{3}\left(\theta_2 - 2c_{\mathcal{A}} + c_{\mathcal{B}}\right) \qquad \text{and} \quad \pi^{*(I,I)}_{1,\mathcal{A}} = \tfrac{1}{9}\left(\theta_2 - 2c_{\mathcal{A}} + c_{\mathcal{B}}\right)^2. \tag{A7}$$

Note, however, that these are ex post outputs and payoffs, so θ_2 in (A7) is either $u\theta_1$ or $d\theta_1$ (because Θ follows a binomial process from period 2). In order to derive the expected net present value of the investment, we require ex ante expectations of the payoffs as follows

$$\begin{aligned}\mathbb{E}_o(\pi_{2,\mathcal{A}}^{*(I,I)}) &= \mathbb{E}_o\left[\frac{1}{9}\left(\Theta^{(2)} - 2c_{\mathcal{A}} + c_{\mathcal{B}}\right)^2\right] \\ &= \frac{1}{9}\left(\operatorname{Var}(\Theta^{(2)} + (c_{\mathcal{B}} - 2c_{\mathcal{A}})) + \left(\mathbb{E}_o\left(\Theta^{(2)} - 2c_{\mathcal{A}} + c_{\mathcal{B}}\right)\right)^2\right) \\ &= \frac{1}{9}\left[\sigma^2_{(\Theta^{(2)})} + \eta_{\ddot{c}} + (pu\theta_1 + (1-p)d\theta_1 + \ddot{c} - 2c_{\mathcal{A}})^2\right].\end{aligned} \tag{A8}$$

Where $\eta_{\ddot{c}}$ is the variance of the marginal cost distribution over the adjusted support. At period 0, the level of demand in period 2 is not known with certainty, and the marginal cost of the firm's rival is still private information, hence the expectations in (A8). Period 3, and subsequent periods' payoffs follow (A8), therefore, the expected net present value of the investment at the time of decision is

$$\begin{aligned}\vartheta_{\mathcal{A}}^{(I,I)} &= \frac{1}{1+\rho}\left(\frac{1}{36}\left(2\theta_1 - \ddot{c} - 3c_{\mathcal{A}}\right)^2\right) \\ &+ \frac{1}{9\rho(1+\rho)}\left(\sigma^2_{(\Theta^{(2)})} + \eta_{\ddot{c}} + (pu\theta_1 + (1-p)d\theta_1 + \ddot{c} - 2c_{\mathcal{A}})^2\right) - \mathcal{K}.\end{aligned} \tag{A9}$$

Putting $\gamma_1 = 1/(1+\rho)$ and $\gamma_2 = \gamma_1/\rho$ in (A9) gives the expression in (2.1).

When both firms keep the option to delay alive until period 2, (i.e. $(a, b) = (D, D)$), then, given that demand rises in period 2, (i.e. $\theta_2 = u\theta_1$ with probability p), they simultaneously enter the market. However, while marginal costs remain private information, each firm conjectures that the choice to delay implies that its rival's true marginal cost must lie in some interval $(\beta, \bar{c}]$, where β is the upper bound of the marginal cost distribution support below which first period entry is profitable. The firms, therefore, put zero probabilities on all types within this interval.

Although second period demand level has now been observed, a Bayesian-Cournot game is played again in period 2 since marginal costs are still private information. The basic Cournot game is then played in all other periods after 2. Ex ante expected output and payoff in period 2 are

$$q_{2,\mathcal{A}}^{*(D,D)} = \tfrac{1}{6}\left(2u\theta_1 - \hat{c} - 3c_{\mathcal{A}}\right) \qquad \text{and} \qquad \pi_{2,\mathcal{A}}^{*(D,D)} = \tfrac{1}{36}\left(2u\theta_1 - \hat{c} - 3c_{\mathcal{A}}\right)^2, \tag{A10}$$

where $\hat{c} = \int_{\beta}^{\bar{c}} c_{\mathcal{A}} \mathrm{d}F(c_{\mathcal{A}})$. Firm $\mathcal{B}$'s output and payoff are similarly determined.

Expected output and payoff in period 3 and all other periods after that are

$$q_{3,\mathcal{A}}^{*(D,D)} = \tfrac{1}{3}\left(u\theta_1 + \hat{c} - 2c_{\mathcal{A}}\right) \qquad \text{and} \qquad \pi_{3,\mathcal{A}}^{*(D,D)} = \tfrac{1}{9}\left(u\theta_1 + \hat{c} - 2c_{\mathcal{A}}\right)^2, \tag{A11}$$

and the value of the investment when the option is kept alive until period 2 is given by

$$\vartheta_{\mathcal{A}}^{(D,D)} = p(-k_1\mathcal{K} + \tfrac{\gamma_1^2}{36}\left(2u\theta_1 - \hat{c} - 3c_{\mathcal{A}}\right)^2 + \tfrac{\gamma_1\gamma_2}{9}\left(u\theta_1 + \hat{c} - 2c_{\mathcal{A}}\right)^2). \tag{A12}$$

γ_1 and γ_2 are as previously defined.

Sequential-move Equilibria

A number of scenarios play out when the firms choose to enter the market at different times. It suffices to consider the case for firm $\mathcal{A}$ entering early and firm $\mathcal{B}$ differing until period 2 before deciding to enter or not. After these choices are made and observed, firm $\mathcal{A}$ invests in period 1 and acts as a monopolist. His equilibrium output and payoff are

$$q_{1,\mathcal{A}}^{*(I,D)} = \tfrac{1}{2}\left(\theta_1 - c_{\mathcal{A}}\right) \qquad \text{and} \qquad \pi_{1,\mathcal{A}}^{*(I,D)} = \tfrac{1}{4}\left(\theta_1 - c_{\mathcal{A}}\right)^2. \tag{A13}$$

Firm $\mathcal{B}$ incurs no sunk cost and earns nothing in this period. Should demand rise in period 2, firm $\mathcal{B}$ exercises its right to enter the market and acts as a Stackelberg follower. Firm $\mathcal{A}$, having observed that $\mathcal{B}$ has chosen to defer, conjectures that $\mathcal{B}$'s marginal cost must lie in the interval $[\alpha, \bar{c}] \subset [\underline{c}, \bar{c}]$. $\mathcal{A}$ maximises its Stackelberg leader payoff given the expected reaction function of $\mathcal{B}$ as follows,

$$\begin{aligned}\max_{q_{2,\mathcal{A}}} \mathbb{E}(\pi_{2,\mathcal{A}}^{(I,D)}) &= \max_{q_{2,\mathcal{A}}} \{q_{2,\mathcal{A}} (u\theta_1 - q_{2,\mathcal{A}} - \mathbb{E}(\hat{q}_{2,\mathcal{B}}(c_{\mathcal{B}})) - c_{\mathcal{A}})\} \\ &= \max_{q_{2,\mathcal{A}}} \{q_{2,\mathcal{A}} \left(u\theta_1 - q_{2,\mathcal{A}} - \left[\frac{u\theta_1 - q_{2,\mathcal{A}} - \breve{c}}{2} \right] - c_{\mathcal{A}} \right) \}.\end{aligned} \tag{A14}$$

Where $\breve{c} = \int_{\alpha}^{\bar{c}} c_{\mathcal{B}} \mathrm{d}F(c_{\mathcal{B}})$. Solving (A14) and substituting into the follower's optimisation problem yield the expected equilibrium outputs:

$$q_{2,\mathcal{A}}^{*(I,D)} = \tfrac{1}{2} (u\theta_1 + \breve{c} - 2c_{\mathcal{A}}) \qquad \text{and} \quad q_{2,\mathcal{B}}^{*(I,D)} = \tfrac{1}{4} (u\theta_1 - 2c_{\mathcal{B}} + 2\tilde{c} - \breve{c}) \,. \tag{A15}$$

Their respective corresponding expected payoffs are

$$\pi_{2,\mathcal{A}}^{*(I,D)} = \tfrac{1}{8} (u\theta_1 + \breve{c} - 2c_{\mathcal{A}})^2 \qquad \text{and} \quad \pi_{2,\mathcal{B}}^{*(I,D)} = \tfrac{1}{16} (u\theta_1 - 2c_{\mathcal{B}} + 2\tilde{c} - \breve{c})^2. \tag{A16}$$

Marginal costs and demand level is now revealed and the basic Cournot game is played from period 3 onwards. The expected outputs and payoffs produced and earned respectively in each of these periods are:

$$q_{3,\mathcal{A}}^{*(I,D)} = \tfrac{1}{3} (u\theta_1 + c_{\mathcal{B}} - 2c_{\mathcal{A}}) \qquad \text{and} \quad q_{3,\mathcal{B}}^{*(I,D)} = \tfrac{1}{3} (u\theta_1 + c_{\mathcal{A}} - 2c_{\mathcal{B}}) \,, \tag{A17}$$

and

$$\pi_{3,\mathcal{A}}^{*(I,D)} = \tfrac{1}{9} (u\theta_1 + c_{\mathcal{B}} - 2c_{\mathcal{A}})^2 \qquad \text{and} \quad \pi_{3,\mathcal{B}}^{*(I,D)} = \tfrac{1}{9} (u\theta_1 + c_{\mathcal{B}} - 2c_{\mathcal{B}})^2. \tag{A18}$$

$\tilde{c}$ in the expressions above is $\int_{\underline{c}}^{\bar{c}} c_{\mathcal{A}} \mathrm{d}F(c_{\mathcal{A}})$ (i.e. the expected value of firm $\mathcal{A}$'s marginal cost over the full support of its distribution). After taking period-0 expectations of these payoff functions, the expected value of the investment for each firm is given by,

$$\vartheta_{\mathcal{A}}^{(I,D)} = \gamma_1 \frac{\theta_1 - c_{\mathcal{A}}}{4} + \gamma_1^2 \left(\frac{p(u\theta_1 + \check{c} - 2c_{\mathcal{A}})^2}{8} + \frac{(1-p)(d\theta_1 - c_{\mathcal{A}})^2}{4} \right) + 9\gamma_1\gamma_2 \left(\frac{p(\eta_{\check{c}} + (u\theta_1 + \check{c} - 2c_{\mathcal{A}})^2)}{9} + \frac{(1-p)(d\theta_1 - c_{\mathcal{A}})^2}{4} \right) - \mathcal{K} \tag{A19}$$

and

$$\vartheta_{\mathcal{B}}^{(I,D)} = p \left(-k_1\mathcal{K} + k_1^2 \frac{(4\eta_{\tilde{c}} + (u\theta_1 - 2c_{\mathcal{B}} + 2\tilde{c} - \check{c})^2)}{16} + k_1 k_2 \frac{(\eta_{\tilde{c}} + (u\theta_1 + \tilde{c} - 2c_{\mathcal{B}})^2)}{9} \right). \tag{A20}$$

Lemma 1

Given that the value functions themselves are monotone decreasing and convex over the support of the marginal cost distribution, it suffices, therefore, that along with conditions (a), (b) and (c), the claims in **Lemma** 1 hold. First, we show that f_1 and f_2 are positive everywhere on the support of the marginal cost distribution as follows.

(A9) and (A20) give the expected value of the investment when period 0 choices are respectively (I, I) and (D, I). Note that (A20) refers to the value of the investment to a firm who choses to defer, when its rival enters early. If $p = 1/2$ and the marginal cost expectations are evaluated over a continuous uniform distribution we obtain

$$\vartheta^{(I,I)} = \frac{\left(-\frac{\alpha}{2} - 3c + 2\theta_1\right)^2}{36(1+\rho)} + \frac{\frac{\alpha^2}{12} + \left(\frac{\alpha}{2} - 2c + \frac{u\theta_1}{2} + \frac{\theta}{2u}\right)^2 + \ln(u)^2}{9\rho(1+\rho)} - \mathcal{K} \tag{B1}$$

and

$$\vartheta^{(D,I)} = \frac{1}{2}\left(\frac{\left(\frac{1}{2}(-\alpha-1)+\beta-2c+u\theta_1\right)^2}{16(1+\rho)^2} + \frac{\left(\frac{\beta}{2}-2c+u\theta_1\right)^2}{9\rho(1+\rho)^2} - \frac{\mathcal{K}}{1+\rho}\right). \tag{B2}$$

Without loss of generality, let $[\underline{c}, \bar{c}] = [0, 1]$. This implies that $\alpha, \beta \in [0, 1]$.

$$\begin{aligned} f_1 &= \left|\frac{\partial\vartheta^{(I,I)}}{\partial c}\right| - \left|\frac{\partial\vartheta^{(D,I)}}{\partial c}\right| \\ &= \frac{1}{144u\rho(1+\rho)^2}\big(32\theta_1 + (32\theta_1 + u(90+53\alpha-18\beta+48\theta_1+14u\theta_1 \\ &\quad - 164c))\rho + 12u(\alpha+4\theta_1-6c)\rho^2 - 16u(2\alpha+\beta-4c)\big). \end{aligned} \tag{B3}$$

It is easy to see that f_1 is linear in c and differentiable on $[0, 1]$, Also, $f_1(0) > f_1(1) \geq 0$. Furthermore,

$$\frac{\partial f_1}{\partial c} = -\frac{(1+2\rho)(16+9\rho)}{36\rho(1+\rho)^2}. \tag{B4}$$

Notice that (B4) is independent of c. Also, $0 < \rho < 1$, therefore, given (B4), and the fact that $f_1(0) > f_1(1) \geq 0$, we see that f_1 is monotonic decreasing and non-negative for all c in $\mathcal{I}$, where $\vartheta^{(I,I)} \geq 0$.

Similarly, $f_2 = |\partial\vartheta^{(I,D)}/\partial c| - |\partial\vartheta^{(D,D)}/\partial c| > 0$ for all c in $\mathcal{I}$ where $\vartheta^{(I,D)} \geq 0$. f_2 is differentiable on $[0, 1]$ and $f_2(0) > f_2(1) \geq 0$. Moreover, it is linear and monotonic decreasing as shown in (B5).

$$\frac{\partial f_2}{\partial c} = -\frac{1+2\rho(2+\rho)}{4\rho(1+\rho)^2}. \tag{B5}$$

This verifies condition (b) of **L**emma 1.

For condition (c), we begin by noting that $g_1(\alpha, \beta) = (\vartheta^{(I,I)})^{-1}(0) - (\vartheta^{(D,I)})^{-1}(0)$ and $g_2(\alpha, \beta) = (\vartheta^{(I,D)})^{-1}(0) - (\vartheta^{(D,D)})^{-1}(0)$. Again, WLOG, we let $[\underline{c}, \bar{c}] = [0, 1]$.

Solving for the roots of $\vartheta^{(I,I)}$ and $\vartheta^{(D,I)}$, and taking the difference, we have

$$\begin{aligned}
g_1(\alpha,\beta) &= \frac{-3245 - 534\beta - 8\sqrt{15}\sqrt{49977 + 36\alpha - 86\alpha^2 - 1014\ln\left[\frac{3}{2}\right]^2}}{2028} \\
&\quad + \frac{525\alpha + 36\sqrt{10}\sqrt{3718 - (1+\alpha-\beta)^2}}{2028} \\
&\quad (\text{ set } \alpha = 1 \text{ and } \beta = 0 \text{ in the numerator of the above expression }) \\
&< -\frac{10156.58 + 7466.56}{2028} \\
&= -\frac{2690.02}{2028} \\
&< 0.
\end{aligned} \tag{B6}$$

Using $\alpha = 1$ and $\beta = 0$ in the numerator of the first line in (B6) minimises the absolute value of the negative part of it, whilst maximising the absolute value of positive part. This allows us to obtain the inequality in the line that followed. In the same vein, we have it that,

$$\begin{aligned}
g_2(\alpha,\beta) &= \frac{17989 + 445\alpha - \sqrt{55}\sqrt{2278765 - 267\alpha(34+9\alpha)}}{2968} \\
&\quad + \frac{-2147 - 83\beta + 2\sqrt{10}\sqrt{32933 - (-46+\beta)\beta}}{338} \\
&\quad (\text{ set } \alpha = 1 \text{ and } \beta = 0 \text{ in the numerators of the above expression }) \\
&< -\frac{11166.94}{2968} + \frac{18434}{2968} - \frac{2147}{338} + \frac{1147.75}{338} \\
&\approx 2.44 - 2.97 \\
&< 0.
\end{aligned} \tag{B7}$$

Again, putting $\alpha = 1$ and $\beta = 0$ minimises the absolute values of the negative terms of the first line of expression in (B7), whilst maximising the positive terms of it.

To conclude the verification of condition (c) of **L**emma 1, we let $h = \vartheta^{(I,I)})^{-1}(0) - \vartheta^{(I,D)})^{-1}(0)$, and show that $h < 0$ on $[0, 1]$.

We have

$$h = \frac{-2519591 - 5936\sqrt{15}\sqrt{49977 + 36\alpha - 86\alpha^2 - 1014\ln\left[\frac{3}{2}\right]^2}}{1504776} + \frac{143901\alpha + 507\sqrt{55}\sqrt{2278765 - 267\alpha(34 + 9\alpha)}}{1504776}. \tag{B8}$$

Recall that α represents the believe a firm holds about its rival's marginal cost, given his period-0 choice, and it lies in the interval $[0, 1]$. Moreover, h is quadratic but decreasing in α, therefore, it suffices to show that if $h(\alpha = 0) < 0$ and $h(\alpha = 1) < 0$, then h is negative everywhere on the interval $[0, 1]$. From (B8), $h(0) \approx -1974599.68$ and $h(1) \approx -1842439.30$.

Lastly, whenever θ takes on a value that produce a non-negative price, and α and β are chosen to minimise $\vartheta^{(I,I)}(0) - \vartheta^{(D,I)}(0)$ and $\vartheta^{(I,D)}(0) - \vartheta^{(D,D)}(0)$ as shown above, these expressions respectively yield $7482895/313632 + 100/99 \ln\left(\frac{3}{2}\right)^2$ and $5067665/17424$, which proves condition (a). □

Proposition 3

Propositions 1 and 2 establishes the existence of a dominant strategy equilibrium solution to the game when either firm draws marginal costs in $[\underline{c}, c_1)$ or $(c_2, \bar{c}]$. However, when a firm draws marginal cost in (c_1, c_2), its optimal action depends on the type of its adversary. For instance, if the firm knows that it to face an adversary whose marginal cost lies in $[\underline{c}$,$c_1)$ (for ease of notation, we refer to this interval as "Type–I"), who has a dominant strategy to play I, it is best to defer (i.e. play D); if however, it faces an adversary whose marginal cost lies in $(c_2, \bar{c}]$ ("Type–II"), who has a dominant strategy to play D, it is best invest (i.e. play I). But this firm is not endowed with the knowledge of its rival's type at the time it has to formulate its strategy – recall that firms have incomplete information over their marginal costs. This means that when a firm is formulating its strategy, it does so without knowing the type of its rival, and more so, cannot condition its action on the type of its rival. Furthermore, the firm realises that a rival with marginal cost in the interval (c_1, c_2), "Type–III", would be facing the same uncertainty – creating an anti-coordination problem.

It is clear from Figure [2.2], that a Type–III firm's action depends on the type of its rival, but while it does not know the type of its rival ex ante, the distribution function of the types is common knowledge, so that this firm knows that it faces

— a Type–I rival with probability $p_1 = \int_{\underline{c}}^{c_1} dF(c_{(\cdot)})$, who would be playing I;

— a Type–II rival with probability $p_2 = \int_{c_2}^{\bar{c}} dF(c_{(\cdot)})$, who would be playing D; and

— a Type–III rival with probability $1 - p_1 - p_2 = \int_{c_1}^{c_2} dF(c_{(\cdot)})$, who would be facing a similar decision problem.

Now, we have assumed that the firms are ex ante symmetric, therefore, we can hypothesise a strategy for one firm and check if, when the same strategy is also used by the other firm, it constitutes an equilibrium behaviour. Suppose therefore, that firm A is a Type–III firm and believes that any Type–III firm would use a mixed strategy that involves playing I with probability ϵ and D with probability $1 - \epsilon$. To be part of a mixed strategy Nash equilibrium, this needs to make any Type–III firm indifferent between playing I or D i.e. for any Type–III firm using this mixed strategy

$$\begin{aligned} p_1\vartheta^{I,I} + p_2\vartheta^{I,D} + (1 - p_1 - p_2)(\epsilon\vartheta^{I,I} + (1-\epsilon)\vartheta^{I,D}) \\ = p_1\vartheta^{D,I} + p_2\vartheta^{D,D} + (1 - p_1 - p_2)(\epsilon\vartheta^{D,I} + (1-\epsilon)\vartheta^{D,D}). \end{aligned}$$

Indeed, it may be possible to find an ϵ for which the above equality holds, but the same ϵ would not satisfy the above equation for all firms with types in the interval (c_1, c_2); as such, some of the firms would prefer I and some D, hence this mixed strategy Nash equilibrium breaks down.

One way to resolve the above problem is to specify that the strategy of a Type–III firm depends its cost realisation. For instance, suppose firm B realises it is a Type–III firm, and has a mean cost of $c_m = (c_1 + c_2)/2$, it will randomise over I and D according to a mixed strategy $(\epsilon, 1 - \epsilon)$, and play I if $c_1 < c_{\mathcal{B}} < c_m$, and D if $c_m < c_{\mathcal{B}} < c_2$.

Then for firm A who draws a cost in the interval (c_1, c_2), its payoff for playing I

will be,

$$R^1_{\mathcal{A}} = p_1\vartheta^{I,I}_{\mathcal{A}} + p_2\vartheta^{I,D}_{\mathcal{A}} + (1-p_1-p_2)\Big(\int_{c_1}^{c_m} \vartheta^{I,I}_{\mathcal{A}} dF(c_{\mathcal{B}}) + \int_{c_m}^{c_2} \vartheta^{I,D}_{\mathcal{A}} dF(c_{\mathcal{B}}) + f(c_m)[\epsilon\vartheta^{I,I}_{\mathcal{A}} + (1-\epsilon)\vartheta^{I,D}_{\mathcal{A}}]\Big); \tag{6.1}$$

and its payoff for playing D will be,

$$R^2_{\mathcal{A}} = p_1\vartheta^{D,I}_{\mathcal{A}} + p_2\vartheta^{D,D}_{\mathcal{A}} + (1-p_1-p_2)\Big(\int_{c_1}^{c_m} \vartheta^{D,I}_{\mathcal{A}} dF(c_{\mathcal{B}}) + \int_{c_m}^{c_2} \vartheta^{D,D}_{\mathcal{A}} dF(c_{\mathcal{B}}) + f(c_m)[\epsilon\vartheta^{D,I}_{\mathcal{A}} + (1-\epsilon)\vartheta^{D,D}_{\mathcal{A}}]\Big).$$

We now show that ϵ must be such that if $c_{\mathcal{A}} = c_m$ then $R^1_{\mathcal{A}} = R^2_{\mathcal{A}}$, which would mean that a firm whose marginal cost is the mean of c_1 and c_2, would be indifferent between I and D. Without loss of generalisation, we test this claim with an example, using model parameter values from Figure [2.2] i.e. $u = 1.3$, $c_1 = 0.78$, $c_2 = 0.2$, $\mathcal{K} = 5$, and $\theta = 2.5$. The figure below shows the values of $C_{\mathcal{A}}$ and $R^2_{\mathcal{A}}$ for all Type–III firms, i.e. firms with marginal costs in the interval $(0.2, 0.78)$.

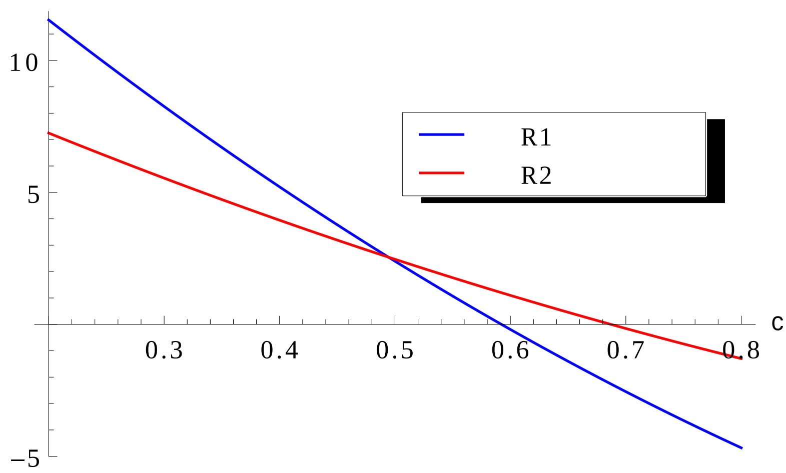


Figure 6: $c_m = 0.49$, $R^1 = R^2$, when $\epsilon = 0.85$

We see from Figure 6 above, that for firm A,

— if $c_1 < c_{\mathcal{A}} < c_m$, then $R^1 > R^2$, and

— if $c_m < c_{\mathcal{A}} < c_2$, then $R^2 > R^1$.

The above confirms that the threshold type strategy specified is consistent when adopted by either firm, as neither can do any better if either plays this strategy. Obviously, this only holds when $\epsilon = 0.85$, which means that a different value of ϵ will be required in order to specify a similar kind of strategy for other threshold types defined in (c_1, c_2).

6.0.2 Appendix 2: Appendix to Chapter 4

Consumer demand function

$$\begin{aligned}
(i) \quad P(A \text{ is chosen}) &= P(U_a > U_O) \\
&= P(J_a + \epsilon_a > \epsilon_O) \\
&= P(u_a - p_a + \epsilon_a > \epsilon_O) \\
&= \int_{-\infty}^{\infty} f(\epsilon_a) \left(\int_{-\infty}^{u_a - p_a + \epsilon_a} f(\epsilon_O) d\epsilon_O \right) d\epsilon_a \\
&= \int_{-\infty}^{\infty} f(\epsilon_a) e^{-e^{-(u_a - p_a + \epsilon_a)}} d\epsilon_a
\end{aligned} \tag{6.3}$$

Note that $F(\epsilon_a) = e^{-e^{-\epsilon_a}}$, and $f(\epsilon_a) = \dfrac{dF(\epsilon_a)}{d\epsilon_a} = e^{-(\epsilon_a)} e^{-e^{-\epsilon_a}}$. Using this in equation 6.3, we have

$$\begin{aligned}
P(A \text{ is chosen}) &= \int_{-\infty}^{\infty} e^{-(\epsilon_a)} e^{-e^{-\epsilon_a}} \cdot e^{-e^{-(u_a - p_a + \epsilon_a)}} d\epsilon_a \\
&= \frac{e^{u_a - p_a}}{e^0 + e^{u_a - p_a}} = \frac{e^{J_a}}{1 + e^{J_a}} \triangleq S(p_a) \\
&\quad (\text{Setting } J_a = u_a - p_a) .
\end{aligned} \tag{6.4}$$

(*ii*) *FOC* for the profit maximisation problem of a single firm in equation 4.3 is,

$$\begin{aligned}0 = \frac{d\pi}{dp_a} &= \frac{d}{dp_a}(p_a S)_{|M=1} \\ &= p_a \frac{dS}{dp} + S\frac{dp}{dp} \\ &= p_a \cdot \frac{d}{dp}\left[e^{J_a}(1+e^{J_a})^{-1}\right] + S \\ &= p_a \left[e^{2J_a}(1+e^{J_a})^{-2} - \frac{e^{J_a}}{1+e^{J_a}}\right] + S \\ &\quad \left(S = e^{J_a}(1+e^{J_a})^{-1}\right) \\ &= p_a\left[S(S-1)\right] + S\end{aligned} \tag{6.5}$$

From equation 6.5, $p_a^* = 1/(1-S)$. For analytical convenience in subsequent analysis, we introduce the Lambert W function in equation 4.4 as follows:

$$\begin{aligned}p^* &= \frac{1}{1-S} = 1 + e^J = 1 + e^{u-p^*} \\ &\quad \text{subtracting } u \text{ from both sides} \\ p^* - u &= e^{u-p^*} + 1 - u \\ u - 1 &= e^{u-p^*} - p^* + u \\ &\quad \text{taking exponentials of both sides} \\ e^{u-1} &= e^{[e^{u-p^*} - p^* + u]} \\ e^{u-1} &= e^{e^{u-p^*}} \cdot e^{u-p^*} \\ &\quad \text{let } e^{u-p^*} = W \text{ in the expression above} \\ We^W &= e^{u-1}(\doteq x)_{|W(x)e^{W}(x)=x}\end{aligned} \tag{6.6}$$

Therefore, $e^{u-p^*} = W(e^{u-1})$. Taking logarithms of both sides $\Rightarrow u - p^* = \ln W(e^{u-1})$; but $\ln W(\cdot) = \ln(\cdot) - W(\cdot)$, so that $p^* = 1 + W(e^{u-1})$.

Note that S is a function of p, and the equilibrium demand S^* is also a function of

p^* as in

$$
\begin{aligned}
S^* &= \frac{e^{u-p^*}}{1+e^{u-p^*}} \\
&= \frac{e^u}{e^{p^*}+e^u} \\
&\quad \big(\text{substituting } p^* \left(=1+W(e^{u-1})\right)\big) \\
&= \frac{e^u}{e\cdot e^{W(e^{u-1})}+e^u} \\
&= \frac{e^u}{e\cdot e^{-1}\frac{e^u}{W(e^{u-1})}+e^u} \\
&= \frac{1}{1+\frac{1}{W(e^{u-1})}} \\
&= \frac{W(e^{u-1})}{1+W(e^{u-1})}
\end{aligned} \tag{6.7}
$$

Finally, using equations 6.6 and $p^* = 1 + W(e^{u-1})$,

$$
\begin{aligned}
\pi^* &= p^* S(p^*) \\
&= \left[1+W(e^{u-1})\right]\frac{W(e^{u-1})}{1+W(e^{u-1})} \\
&= W(e^{u-1}).
\end{aligned} \tag{6.8}
$$

Deriving the expressions for S and π for each firm in the strategic case is similar to that of the non-strategic with an outside good. It follows accordingly, that

$$
\begin{aligned}
(iii) \quad P(A \text{ is chosen over } B) &= P(U_a > U_b) \\
&= P(J_a + \epsilon_a > J_b + \epsilon_b) \\
&\Big(\text{the algebra following the distribution of } \epsilon \text{ in} \\
&\text{equation (6.3) generalises to} > 2 \text{ alternatives} \\
&\text{such that } P(J_a + \epsilon_a > J_i + \epsilon_i, i = 2, \dots, n) = \\
&P\left[J_a + \epsilon_a > \max_{i=2,\dots,n}(J_i + \epsilon_i), a \neq i, \forall i\right] = \\
&\frac{e^{J_a}}{e^{J_a} + e^{J_2} + \cdots + e^{J_n}} = \frac{e^{J_a}}{e^{J_a} + \sum_{i=2}^{n} e^{J_a}}\Big) \\
&= \frac{e^{J_a}}{e^{J_a} + e^{J_b} + e^{J_O}} \\
&\Big(\text{Note that the consumer's utility for the outside} \\
&\text{good is given as } \epsilon_O \Rightarrow J_O = 0\Big) \\
&= \frac{e^{J_a}}{1 + e^{J_a} + e^{J_b}}
\end{aligned}
\tag{6.9}
$$

(iv) FOC for the profit maximisation problem in the strategic case is,

$$
\begin{aligned}
0 = \frac{\partial \pi}{\partial p_a} &= \frac{\partial \left(p_a S_a(p_a, p_b)\right)}{\partial p_a} \\
&= \frac{\partial}{\partial p_a}\left[p_a \frac{e^{J_a}}{1 + e^{J_a} + e^{J_b}}\right] \\
&= \frac{e^{u_a - p_a}}{1 + e^{u_a - p_a} + e^{u_b - p_b}}(1 - p_a) + \frac{e^{2(u_a - p_a)}}{(1 + e^{u_a - p_a} + e^{u_b - p_b})^2} p_a
\end{aligned}
\tag{6.10}
$$

The equilibrium price function is obtained from (6.10) as

$$\begin{aligned} p_a^* e^{u_a - p_a^*} &= e^{u_a - p_a^*}\left(1 + p_a^* \frac{e^{u_a - p_a^*}}{1 + e^{u_a - p_a^*} + e^{u_b - p_b}}\right) \\ &= \frac{1 + e^{J_a} + e^{J_b}}{1 + e^{J_b}} \\ &= 1 + \frac{e^{J_a}}{1 + e^{J_b}} \\ &= 1 + \frac{e^{u_a - p_a^*}}{\alpha};\ \text{Letting } \alpha = 1 + e^{J_b} \end{aligned} \tag{6.11}$$

Subtracting u from both sides in equation (6.11) above, we have

$$\begin{aligned} p^* - u_a &= 1 - u_a + \frac{e^{u_a - p_a^*}}{\alpha} \\ &\text{Taking exponentials from both sides:} \\ e^{u_a - 1} &= e^{\left(\frac{e^{u_a - p_a^*}}{\alpha} - p_a^* + u_a\right)} \\ &\text{Divide both sides by } \alpha: \\ \frac{e^{u_a - 1}}{\alpha} &= \left(\frac{e^{-p_a^* + u_a}}{\alpha}\right) \cdot e^{\left(\frac{e^{u_a - p_a^*}}{\alpha}\right)} \\ &\text{Let } \frac{e^{-p_a^* + u_a}}{\alpha} = W \text{ above}: \\ We^W &= \frac{e^{u_a - 1}}{\alpha}; \quad \text{which means } W = W\left(\frac{e^{u_a - 1}}{\alpha}\right) \\ &\text{Substituting for } W, \text{ and taking logarithms, we have} \\ \ln\left(\frac{e^{-p_a^* + u_a}}{\alpha}\right) &= \ln W\left(\frac{e^{u_a - 1}}{\alpha}\right) \\ p_a^* - u + \ln\alpha &= -\left(u - 1 - \ln\alpha - W\left(\frac{e^{u_a - 1}}{\alpha}\right)\right) \\ p_a^* &= 1 + W\left(\frac{e^{u_a - 1}}{\alpha}\right) \\ p_a^* &= 1 + W\left(\frac{e^{u_a - 1}}{1 + e^{u_b - p_b}}\right) \end{aligned} \tag{6.12}$$

In the same vein,

$$p_b^* = 1 + W\left(\frac{e^{u_b - 1}}{1 + e^{u_a - p_a}}\right) \tag{6.13}$$

For the equilibrium demand level in the strategic case,

$$\begin{aligned}
S_a^* &= \frac{e_a^J}{1+e^{J_a}+e^{J_b}}; \quad \text{again, letting } \alpha = 1+e^{J_b} \\
&= \frac{e^{u_a-p_a^*}}{\alpha+e^{u_a-p_a^*}} \\
&\quad \text{Substitution for } p_a^* \text{ from equation (6.13)} \\
&= \frac{e^u}{\alpha e^{W\left(\frac{e^{u-1}}{\alpha}\right)+1}+e^u} \\
&= \frac{1}{1+W\left(\frac{e^{u-1}}{\alpha}\right)^{-1}} \\
&= \frac{W\left(\frac{e^{u-1}}{1+e^{u_b-p_b}}\right)}{1+W\left(\frac{e^{u-1}}{1+e^{u_b-p_b}}\right)}
\end{aligned} \tag{6.14}$$

From equation (6.12) and (6.14), Nash equilibrium payoff for firm A is

$$\pi_a^* = p_a^* S_a^*(p_a^*, p_b) = W\left(\frac{e^{u-1}}{1+e^{u_b-p_b}}\right). \tag{6.15}$$

Conceptual frameworks on celebrity brand endorsements in the literature

Celebrity endorsements are delicate ventures. Whether or not they are intrinsically helpful investments for brands remain a long-standing debate among ad agencies. A faction of the debate puts the use of celebrities down to lethargic ad campaign strategies, while the other faction trumpets the many benefits associated with celebrity endorsements; each faction supporting its position with ample prima facie and, sometimes, empirical evidence. Investing in brand equity involves investing in the meanings, ideas, thoughts, concepts that are associated with a brand. Over the last couple of decades, celebrity endorsements have been one of the more common ways of achieving this. The advertising literature provides detailed analyses of the psychological construct of celebrity endorsement. Succinctly, consumers form memory nodes, respectively, of a celebrity and a brand with repeated advertising after the two are connected through the endorsement process. These nodes are then connected by an associative link that becomes stronger in the mind of the consumer with repeated advertised pairing of celebrity and brand. This consistent pairing of brand and celebrity potentially increases the likelihood that activating one stimulates the activation of the other.

The promotional influence of celebrity endorsements is hardly in dispute. Firms utilise celebrities in their advertisements with the hope that the celebrities' success will be transferred, through association, to their brands, and by extension, increase their companies' value (Nicolau and Santa-María (2013)). But critical to the endorsement process is "belongingness." It is much easier for consumers to create an associative link between two nodes (brand and celebrity) if both nodes share similar or related characteristics (McSweeney and Bierley (1984)). For instance, Kim et al. (1996) demonstrate the strengthening of consumer's belief about the speed of pizza service delivery when pizza service delivery is linked with race cars. Brands, therefore, tend to consider both the celebrity–brand fit, which improves the strength of association and the potential for forming an associative link in the first place amidst competing nodes, as well as the brand–celebrity–target audience fit (Till (1998)). These tests are of vital importance to marketers as the nature of returns/benefits the endorsement generates depend on the believability of the message the celebrity delivers on behalf of the brand (Langmeyer and Walker (1991)). An example is how Accenture, in 2003, exploited Tiger Woods' reputation of high performance and dependability to establish itself as a global management consulting and technology services company. These are characteristics customers expect from their consultants and advisors, and Accenture invested heavily in spreading that message. Accenture spent c. $50 million in advertising in the US alone in [Date], a staggering 83% of which featured Tiger Woods! It was the perfect fit until Tiger Woods' marital infidelity became a bigger headline. Congruity between brand and celebrity is widely investigated in the marketing literature (e.g. Till and Busler (1998), Langmeyer and Shank (1993), Kamins (1990), etc.), the general message being that match-up between brand and celebrity is crucial to returns on investments and, therefore, not to be taken lightly. Which, amongst other reasons, may explain why in practice, marketers spend considerable time in extensive consultations, working through their market segment match-up criteria before choosing a celebrity endorser.

Without a doubt, the benefits of using celebrities are apparent and their endorsements remain appealing to advertisers. However, these benefits do not come without significant risks, some of which could completely impair a brand if poorly managed. We recall that the value of a celebrity endorsement is derived through the associative link established between the brand and the celebrity. This link, however, permits all kinds of information – good and bad alike. More importantly, anecdotal evidence have shown how negative information about a celebrity endorser often has disproportionate influence on the brand they endorse. Survey analysis lends support to this observation revealing how bad news tend to gain more traction than good news of the same intensity with respect to consumer decision-making. Mizerski (1982) shows, using the attribution theory rationale, that the link between the attribution process and belief formation, at least, partially explains the prepotence of unfavourable information. The central idea being that consumers tend to attribute favourable information about an entity to causes other than the entity itself and therefore assigns a low probability to the believability of such information than they do an unfavourable information, which is perceived to have relatively fewer possible causes other than the entity in stark contract to favourable information. To this end, celebrity endorsers are generally considered high risk investments due to the increasing probability that they get caught up in some sort of scandal leading to bad publicity for both celebrity and the brands to which they are associated.

A fitting case in point is the relationship between Tiger Woods and Accenture. Accenture's "we know what it takes to be a Tiger" brand slogan since 2003 of Tiger Woods being the advertising face of Accenture quickly became toxic in the wake of his philandering. All Tiger-emblazoned posters and materials had to be taken down across the whole establishment in the bid to severe the associative link between the organisation and Tiger Woods due to the adverse effect the negative publicity would undoubtedly have on the organisation. As far as Accenture was concerned, Tiger Woods was damaged goods and "had to be taken into the woodshed" (in the words of Jon

Swallen – Senior Vice President for research for TNS Media Intelligence).

Taking all Accenture's Tiger Woods branded merchandise as complementary co-investments tied to the Tiger Woods brand, attribution theory suggests that that single negative information could completely wipe sales out. Knittel and Stango (2009)'s empirical analysis of market data in the early days of Tiger Woods' scandal shows significant decline in Tiger Woods' sponsors market value relative to organisations without his endorsement. Furthermore, products linked to the "Tiger brand" (i.e. from co-investments in merchandise) suffered substantial losses coming-off of the decline in the value of these assets and the brand's equity.

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