A Probabilistic Approach to Modelling Ultrasonic Shear Wave Propagation in Locally Anisotropic and Layered Heterogeneous Media

Alistair Stirling Ferguson

Faculty of Science

Department of Mathematics & Statistics

University of Strathclyde, Glasgow

June 8, 2022

Signed: Alistair Stirling Ferguson

Date: June 8, 2022

Acknowledgements

I am extremely grateful for the marvellous support given to me by Professor Tony Mulholland and Dr Katy Tant during my time at Strathclyde. The support given to me has went beyond my academic studies and I am very lucky to have had such a magnificent pair of supervisors.

I would like to thank Dr Mohammud Foondun for his help during the latter half of my studies. Not least, the office staff in the department of Mathematics and Statistics and the boys on level 10; Michael, Andrew and James.

Finally I would like to thank my parents Julie and Ian for their love and support during my studies.

Abstract

In this thesis the propagation of high-frequency elastic waves in a spatially heterogeneous, randomly layered material is reported upon. The material is locally anisotropic and a smooth function describes the spatial variation in the rotation of the associated slowness surface in the plane of wave propagation. The layer thicknesses and the rotation of their associated slowness curves follow a stochastic (Markovian) process. This situation is found in ultrasonic wave propagation in polycrystalline materials; for example, in the ultrasonic non-destructive testing of welds, additively manufactured metallic components and carbon fibre reinforced polymer (CFRP) composites. The model for wave propagation set out in later sections captures the attenuation and deformation of the input wave as it interacts with the internal material microstructure via multiple scattering. In early Chapters, a key parameter emerges (ν) which captures the degree of anisotropy in the medium and it is shown how this affects the transmitted and reflected energy. Using the differences in length scales between the ultrasound wavelength, the mean layer size, and the wave propagation distance, a small parameter is identified in the stochastic differential equations that emerge. Using these stochastic equations allows derivation of infinitesimal generators which encode information about the random processes in the wave propagation problem, which affords for studies into the probability density functions of the coherent wave via the use of Fokker-Planck equations. Later Chapters use diffusion approximations to study a broadband ultrasonic pulse via Ricatti equations; in particular a line source. The incoherent component of the wave is characterised via the autocorrelation of the reflection coefficient and an expression for the reflected intensity of the wave at different lateral observation points is derived.

Contents

Chapter 1

Introduction

1.0.1 Background and Motivation

Ultrasonic non-destructive evaluation (NDE) is an important technique for assessing the structural integrity of industrial infrastructure. It involves sending mechanical waves through the object of interest and analysing the resulting scattered field to determine if there exists any embedded defects [1], [2]. It is common practice when performing NDE on an unknown material to assume homogeneous material properties [3]. However, this is a physically unrealistic assumption in many materials of interest [4]. Wavelength size structures exist in many industrial materials which can lead to significant multiple scattering and the medium not being well characterised using homogenisation [3]. Modern engineering materials that are produced via additive manufacturing techniques [5], [6], [7] are becoming increasingly complex, and their structural integrity is essential for quality assurance. Waves propagating through such heterogeneous media experience scattering that attenuates the coherent input wave and transfers energy into small incoherent fluctuations [8]. The existence of such structures causes multiple scattering to occur between the coupling of the wave and the medium when they are commensurate in magnitude. A deterministic approach to studying horizontally polarised shear waves was reported in [9] where the symmetry axis of rotation was out of plane (where an austenitic steel weld was analysed) but with no consideration for random fluctuations in the wave amplitude produced by scattering from interactions with the internal material microstructure.

Recent studies using numerical simulations have outlined the effects of internal material microstructures on beam propagation [10], [11]. When the size of the internal grain microstructure is commensurate with the wavelength, amplitude fluctuations can be observed in the transmitted and repeated waveforms [12]. If these factors are ignored, deterministic models of wave propagation can give a poor representation of the transmitted and reflected waves and this in turn can negatively affect the ability to reliably detect and resolve flaws [13], [14]. Understanding such phenomena is key to developing NDE imaging methods based on the incoherent waves (coda waves) that result from ultrasonic wave propagation in such materials. Elastic wave propagation in anisotropic solids is also important in medical imaging [15], [16] and in migration problems in geophysics [17], [18], [19] where time reversal techniques are of interest.

Wave propagation in layered media is a special case in the sense that the coda wave that emerges is attenuated and has many fluctuations as it interacts with fine inhomogeneities in the media [20], [21]. This thesis will use the analytical framework for studying random wave propagation using stochastic differential equations, exploiting the separation of scales present in such problems [22], [23], [24], [25], [26], [27]. Using the plethora of analytical tools has led to the study of electromagnetic waves [28], [29] and elastic waves [30], [8], [9] in randomly layered media.

The motivation for studying elastic waves in heterogeneous media stems from [9], where a homogenised approach was used to study the structure of an austenitic steel weld. The model presented in [8] studies the problem of elastic shear wave propagation where the heterogeneity is modelled via a Markovian process. Since materials are naturally heterogeneous, it is more appropriate to study the physics of wave propagation is such media via a probabilistic approach. The application of this type of modelling is key to building tools that can predict the physics of wave propagation which can be helpful when trying to optimise the performance of imaging algorithms, many of which struggle with heterogeneous media. Coherent interferometry is one such imaging technique [31], [32], [33] that can cope with accurately detecting flaws in highly heterogeneous media via the use of statistical cross correlations.

1.0.2 Key Objectives

The goal in this thesis is to study ultrasonic wave propagation in materials where the separation of length scales in the problem (mean layer size, wavelength and wave propagation distance in the medium) produce a complex received wave which exhibits many fluctuations over a long time period, caused by its convoluted journey through the heterogeneous medium. The wave is so affected by its interactions with the medium that a homogenisation approach is inappropriate; that is, the received wave bears no resemblance to the input wave and so there does not exist an equivalent homogenised medium. This can occur when the propagation distance L is much larger than the wavelength (λ_3) of propagation which in turn is much larger than the layer sizes l ($L \gg \lambda_3 \gg l$) and the fluctuations in the material (the strength of the random fluctuations has amplitude σ) are large ($\sigma \sim 1$); the so called strongly heterogeneous regime [26]. It can also occur in the regime where $L_3 \gg \lambda_3 \sim l$ and $\sigma \ll 1$, which is the so called weakly heterogeneous regime.

The decay of a coherent wave has been studied in the literature for general acoustic and elastic systems. Articles [26], [25] consider acoustic systems in one dimension and more recently [8] considered a three dimensional elastic medium. In each case, stochastic differential equations (SDE's) are used to model the random fluctuations in the material properties [34]. The reason for employing random (stochastic) equations is that their solution captures complex physical phenomena which cannot be described using a deterministic (homogenised) model [35]. Each of these studies examined a wave travelling in a medium whose properties only varied in the direction of propagation, leading to a system of stochastic differential equations with a propagator matrix with certain symmetry properties. Later chapters in this thesis will set out problems using a similar approach, in order to study elastic shear wave propagation in a randomly layered, locally anisotropic medium with spatial fluctuations in the material microstructure.

1.0.3 Outline of Thesis

The aim of this thesis is to develop new models for ultrasonic wave propagation in heterogeneous, anisotropic elastic random media and study the affect that interactions between the wave and the media have on the reflected and transmitted wave. The thesis begins with a study of wave propagation in an austenitic steel weld, extending the homogenised model in [9]. The drawback of the homogenisation approach, is that there is no consideration for the random fluctuations in the material parameters which create the observed coda in the transmitted or received wave.

Chapter 2 presents a model for high frequency elastic shear waves in a layered material, in order to study the probability distribution of the transmitted energy. This class of material (which includes austenitic steel [36]) is of interest in the engineering aerospace industry where engineers would like to additively manufacture safety critical components (which creates heterogeneities) using this metal. In Chapter 3, a different class of material is studied, namely a CFRP (carbon fibre reinforced polymer). Due to the geometry of the problem, the form of the stress tensor changes and so the equations contain symmetries [37] which allows for a rich analytical study into the moments of the reflected and transmitted energy. A key parameter emerges which is identified as being related to the degree of anisotropy in the material. Chapter 4 expands on Chapter 3, considering the case where the source is a broadband pulse instead of a monochromatic elastic shear wave. The aim of this chapter is to study the autocorrelation function of the reflection coefficient; a key quantity which is important when studying the intensity of the reflected wave. Finally, Chapter 5 brings together elements of Chapters 2-4 to study a two dimensional problem where the broadband point source lies along the x_2 axis. This is of interest in the physical application of imaging with ultrasonic transducers, where a spatially distributed array of sensors [38], [39] is required to perform NDT imaging methods such as the Total Focusing Method [3], [40], [41], [42]. The inspection domain is formulated as a 2D grid, where the signals from each transmit and receive pair of ultrasonic transducers are summed and focused to build an internal image of the material. The work presented in this thesis is of interest to researchers in the NDT community as the anisotropic and heterogeneous [4], [43] nature of many safety critical materials (produced via modern additive manufacturing [5] methods or welding) hamper the detection and characterisation of flaws when using standard imaging approaches, based on homogenised material properties.

The original work in the thesis is stated below:

1. In Chapter 2 a model for the propagation of high-frequency monochromatic elastic waves in a spatially heterogeneous, randomly layered material is derived. The material is locally anisotropic and a smooth function describes the spatial variation in the rotation of the associated slowness surface in the plane of wave propagation. The layer thicknesses and the rotation of their associated slowness curves follow a stochastic (Markovian) process. Using the differences in length scales between the ultrasound wavelength, the mean layer size, and the wave propagation distance, a small parameter is identified in the stochastic differential equation that emerges. Its infinitesimal generator leads to a Fokker-Planck equation via limit theorems involving this small parameter. A weak form of the Fokker-Planck equation is derived and then solved via a finite element package. The numerical solution to the Fokker-Planck equation is used to compute statistical moments of the power transmission coefficient. Finally, a parametric study on the effect of the degree of anisotropy of the material on the transmitted energy is performed.

2. Chapter 3 considers the propagation (in the x_3 direction) of high frequency elastic waves in a layered material. Each layer is locally anisotropic and the layer thicknesses and slowness surface orientations follow a stochastic (Markovian) process. The wave displacement vector is in the out of plane direction (x_2) and the model focuses on the reflection and transmission of the wave at layer interfaces. The rotation of the slowness surface in each layer lies in the (x_1, x_2) plane and varies with the wave propagation direction (x_3) only. Expressions for the local and global coefficients for the reflected and transmitted wave amplitudes are derived and shown to satisfy energy conservation. The resulting stochastic differential equations lead to a self-adjoint infinitesimal generator and a Fokker-Planck equation emerges via limit theorems. Explicit expressions for

the moments of the probability distributions of the power transmission and reflection coefficients are then derived. The dependency of the mean and standard deviation of the power transmission coefficient on the depth of wave penetration, the localisation length, and the degree of anisotropy is then reported upon. This work is important in deepening the understanding of the ultrasonic non-destructive testing of carbon fibre reinforced polymer (CFRP) composites and polycrystalline materials.

3. Chapter 4 expands on the model presented in Chapter 3 by considering the source wave as a broadband pulse rather than a monochromatic plane wave. Ricatti equations are derived for the moments of the reflected signal of a high frequency, elastic shear wave propagating in a randomly layered elastic material; each layer is locally anisotropic and the layer thicknesses and associated slowness surface orientations are modelled by a stochastic Markovian process. A system of transport equations is formulated for the limit of the frequency autocorrelation function for the reflection coefficient. Analytical and numerical solutions for a specific case of the frequency autocorrelation function are produced via the use of a jump Markov process. The analytical results are compared with the probabilistic solution to the transport equations using a Monte Carlo technique.

4. Chapter 5 utilises the learnings from the models presented in earlier chapters to model the case where the input wave is a broadband pulse line source lying along the x_2 axis. The multifrequency problem is derived in terms of propagator matrices, each of which encode information about specific frequencies in the problem. The separation of scales used in this problem considers the strongly heterogeneous regime; that is where the strength of the random fluctuations between the wave and the medium is order one and the propagation distance is much larger than the wavelength, in order to produce significant scattering in the observed wave. The problem is formulated first in terms of stochastic matrix equations which are solved via a diffusion approximation theorem [26] to obtain limit expressions for the transmitted and reflected waves. The problem is then formulated in terms of a transport equation which, when solved, gives a solution for the frequency autocorrelation function of the reflection coefficient. Using the limit expressions derived earlier, a semi-analytical expression for the reflected intensity (of

the stress) is derived and plots show the effect of varying the lateral observation point (along the x_1 axis) of the reflected energy.

Chapter 2

Elastic Shear Wave Propagation in Locally Anisotropic Heterogeneous Media; Polycrystalline Rotations about the Out of Plane Axis

2.1 Nomenclature

Parameter			Equation
$\tilde{\alpha}$	Lumped non-dimensional parameter	[–]	(2.261)
$\tilde{\beta}$	Lumped non-dimensional parameter	[–]	(2.262)
$\tilde{\gamma}$	Lumped non-dimensional parameter	[–]	(2.263)
Γ_α	Lumped parameter	[–]	(2.236)
Γ_β	Lumped parameter	[–]	(2.240)
δ_1	Lumped parameter	[–]	(2.279)
δ_2	Lumped parameter	[–]	(2.280)
δ_4	Lumped parameter	[–]	(2.281)

Δ_1	Material expression	[–]	(2.275)
Δ_2	Material expression	[–]	(2.276)
Δ_3	Material expression	[–]	(2.277)
Δ_4	Material expression	[–]	(2.278)
ε	Small dimensionless parameter	[–]	(2.248)
ζ	Lumped parameter	[L^{-1}]	(2.48)
η	Material expression	[$M^{-1}LT$]	(2.37)
$\hat{\eta}$	Material expression	[$M^{-1}LT$]	(2.38)
θ	Rotation angle of material slowness surface	[–]	(2.220)
$\bar{\theta}$	Typical slowness angle	[–]	(2.220)
κ_1	Wavenumber in x_1	[L^{-1}]	(2.25)
Λ	Eigenvalues of matrix $\mathbf{M}$	[L^{-1}]	(2.41)
λ_3	Wavelength in x_3 direction	[L]	(2.283)
μ	Stress tensor component in half-space	[$ML^{-1}T^{-2}$]	(2.58)
ν	Ratio of wave numbers (Anisotropy parameter)	[–]	(2.259)
ξ	Velocity in x_3 direction	[LT^{-1}]	(2.13)
$\check{\xi}$	Velocity in x_3 direction (frequency domain)	[L]	(2.17)
$\hat{\xi}$	Velocity in x_3 direction (frequency wavenumber domain)	[L^2]	(2.25)
ϖ	Dimensionless parameter	[–]	(2.248)
ρ	Constant material density	[ML^{-3}]	(2.1)
$\boldsymbol{\sigma}_{1,2,3}$	Pauli spin matrices	[–]	(2.294)
σ	Random process amplitude	[–]	(2.220)
σ_{66}	Amplitude term	[–]	(2.226)

σ_{64}	Amplitude term	[–]	(2.232)
$\hat{\sigma}_{64}$	Amplitude term	[–]	(2.236)
σ_{44}	Amplitude term	[–]	(2.229)
$\hat{\sigma}_{44}$	Amplitude term	[–]	(2.240)
σ_{ψ}	Amplitude term	[–]	(2.244)
ς	Lumped parameter	[L^{-1}]	(2.34)
$\hat{\varsigma}$	Lumped parameter	[L^{-1}]	(2.35)
τ	Power transmission coefficient	[–]	(2.400)
τ_{ij}	Material stress tensor	[$ML^{-1}T^{-2}$]	(2.2)
$\check{\tau}$	Stress (frequency domain)	[$ML^{-1}T^{-1}$]	(2.20)
$\hat{\tau}$	Stress (frequency wavenumber domain)	[MT^{-1}]	(2.31)
Υ	Symmetric correlation integral	[–]	(2.318)
$\Upsilon^{(AS)}$	Anti-symmetric correlation integral	[–]	(2.319)
ϕ	Lumped phase parameter	[–]	(2.266)
$\chi^{\varepsilon}_{1,2,3,4}$	Propagator matrix functions	[–]	(2.289)
ψ	Lumped parameter	[$ML^{-3}T^{-1}$]	(2.32)
$\hat{\psi}$	Lumped parameter	[$ML^{-3}T^{-1}$]	(2.33)
$\bar{\hat{\psi}}$	Lumped parameter	[$ML^{-3}T^{-1}$]	(2.242)
ω	Angular frequency	[T^{-1}]	(2.17)
a	Propagator function	[–]	(2.347)
$\boldsymbol{a}$	PDE coefficient matrix	[–]	(2.406)
A_1	Propagator matrix evolution equation coefficient	[–]	(2.322)
A_2	Propagator matrix evolution equation coefficient	[–]	(2.323)
A_3	Propagator matrix evolution equation coefficient	[–]	(2.324)

A_4	Propagator matrix evolution equation coefficient	[–]	(2.325)
A_5	Propagator matrix evolution equation coefficient	[–]	(2.326)
$\hat{a}$	Forward wave mode in frequency domain	[–]	(2.271)
$\mathbf{a}^\star$	PDE coefficient matrix	[–]	(2.429)
$\mathbf{b}^\star$	PDE coefficient matrix	[–]	(2.429)
b	Propagator function	[–]	(2.347)
$\hat{b}$	Backward wave mode in frequency domain	[–]	(2.272)
$\boldsymbol{b}$	Drift vector	[–]	(2.405)
$\mathbf{C}$	Correlation integral matrix	[–]	(2.315)
$\hat{c}_{44}$	Lumped parameter	[$\mathrm{M^{-1}LT}$]	(2.238)
$\bar{\hat{c}}_{44}$	Lumped parameter	[$\mathrm{M^{-1}LT}$]	(2.239)
$\hat{c}_{64}$	Lumped parameter	[$\mathrm{L^{-1}}$]	(2.234)
$\bar{\hat{c}}_{64}$	Lumped parameter	[$\mathrm{L^{-1}}$]	(2.235)
c_3	Mean wave velocity in x_3 direction	[$\mathrm{LT^{-1}}$]	(2.246)
c_{44}	Stress tensor component (random angle)	[$\mathrm{ML^{-1}T^{-2}}$]	(2.222)
$\bar{c}_{44}$	Stress tensor component	[$\mathrm{ML^{-1}T^{-2}}$]	(2.228)
$\tilde{c}_{44}$	Non-dimensional Stress tensor component	[–]	(2.253)
c_{64}	Stress tensor component (random angle)	[$\mathrm{ML^{-1}T^{-2}}$]	(2.223)
$\bar{c}_{64}$	Stress tensor component	[$\mathrm{ML^{-1}T^{-2}}$]	(2.231)
$\tilde{c}_{64}$	Non-dimensional Stress tensor component	[–]	(2.253)
c_{66}	Stress tensor component (random angle)	[$\mathrm{ML^{-1}T^{-2}}$]	(2.221)
$\bar{c}_{66}$	Stress tensor component	[$\mathrm{ML^{-1}T^{-2}}$]	(2.225)
$\tilde{c}_{66}$	Non-dimensional Stress tensor component	[–]	(2.256)
$c_{1,2}$	Arbitrary constants	[–]	(2.153)

c_{ijkl}	Stress tensor	$[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$	(2.6)
D_2^s	Cosine lumped parameter	[–]	(2.380)
D_2^c	Sine lumped parameter	[–]	(2.381)
$\mathbf{D}$	Diagonalised factor of matrix $\mathbf{M}$	$[\mathrm{M}^{-1}\mathrm{L}^{-1}]$	(2.43)
e_{kl}	Symmetric strain tensor	[–]	(2.3)
$\hat{f}$	Incident wave amplitude	[–]	(2.67)
F	Stress tensor component	$[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$	(2.221)
$\mathcal{G}$	Substitution parameter	[–]	(2.408)
g	Propagator function	[–]	(2.347)
$\mathbf{H}$	Coupling matrix in wave amplitude evolution equation	$[\mathrm{L}^{-1}]$	(2.128)
$\mathbf{H}^{\varepsilon}$	Random matrix	[–]	(2.287)
h	Propagator function	[–]	(2.347)
j	Propagator function	[–]	(2.347)
$\hat{\mathbf{J}}_0$	Lumped parameter	[–]	(2.83)
$\hat{\mathbf{J}}_1$	Lumped parameter	[–]	(2.94)
k	Propagator function	[–]	(2.347)
$\hat{\mathbf{K}}$	Linking matrix	[–]	(2.101)
$\mathcal{L}_{\mathcal{G},p}$	Infinitesimal generator	[–]	(2.411)
L_3	Typical propagation distance in x_3	[L]	(2.246)
l	Mean layer size	[L]	(2.247)
$\mathbf{M}$	Stress-strain coupling matrix	[]	(2.40)
M	Coupling matrix	[]	(2.265)
m	Fluctuations in the crystal orientation	[–]	(2.220)
N	Stress tensor component	$[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$	(2.221)

p	Propagator function	[–]	(2.347)
$\mathbf{P}$	Propagator matrix	[–]	(2.156)
$\mathbf{P}^{\varepsilon}$	Propagator matrix	[–]	(2.289)
P	Probability density function	[–]	(2.428)
q	Propagator function	[–]	(2.347)
Q	Lumped parameter	[–]	(2.430)
$\mathbf{Q}$	Factor of matrix $\mathbf{M}$ (first row)	[L^{-1}]	(2.42)
$\mathbf{Q}$	Factor of matrix $\mathbf{M}$ (second row)	[$\mathrm{ML}^{-3}\mathrm{T}^{-1}$]	(2.42)
$r_0^{\pm}$	Lumped parameter	[–]	(2.84)
$r_1^{\pm}$	Lumped parameter	[–]	(2.95)
R_0	Local reflection coefficient	[–]	(2.86)
R_1	Local reflection coefficient	[–]	(2.96)
$\hat{R}$	Local reflection coefficient	[–]	(2.193)
$\hat{\mathcal{R}}$	Local reflection coefficient	[–]	(2.103)
T_0	Local transmission coefficient	[–]	(2.87)
T_1	Local transmission coefficient	[–]	(2.97)
$\hat{T}$	Local transmission coefficient	[–]	(2.201)
$\hat{\mathcal{T}}$	Local transmission coefficient	[–]	(2.104)
t	Time	[T]	(2.1)
u	Wave displacement vector	[L]	(2.7)
v	Test function	[–]	(2.437)

2.2 Introduction

This Chapter focuses on the propagation of ultrasound waves in elastic media which has a heterogeneous microstructure (that is, the material properties vary on a length scale commensurate with the wavelength) and this microstructure varies in a random manner from one sample to another. This property is present in a plethora of engineering materials, and in particular welds and additively manufactured metals [9], [44]. It is common practise when performing NDE (non-destructive evaluation) on an unknown material to assume homogeneous material properties. However, this is a physically unrealistic assumption in many materials of interest [45], [46], [47]. Wavelength-size grain-like structures exist in many industry relevant materials and waves propagating through such heterogeneous media experience scattering that converts the coherent input wave into small incoherent fluctuations which distort and attenuate the wave [8], [48], [49] which exits the material. There is interest within the non-destructive testing community to better understand ultrasonic wave propagation through random media. There are many examples of components in which a heterogeneous microstructure exists and interacts with the propagating wave [4], [50], [51] to such an extent that the received wave has a significant incoherent component and bears little resemblance to the input waveform. This means that the medium cannot be characterised using homogenisation. The received wave will vary from one sample/weld-site to the next and so it makes sense to describe the wave properties as a distribution and to use a probabilistic framework to model this phenomenon. Only via computationally prohibitive Monte Carlo [52], [53], [54] simulations can deterministic models [55], [55] of wave propagation provide a similar characterisation of wave propagation in such materials.

The form of the elastic tensor used here stems from [9] where the authors investigated the propagation of elastic shear waves in a class of anisotropic material motivated by non-destructive testing and geophysics problems. The deterministic model outlined in [9] describes elastic waves propagating in heterogeneous locally anisotropic media. This Chapter adopts this formulation of the constitutive law for a specific material (austenitic steel) and the associated rotation of the slowness surface θ about the x_2 axis, which is

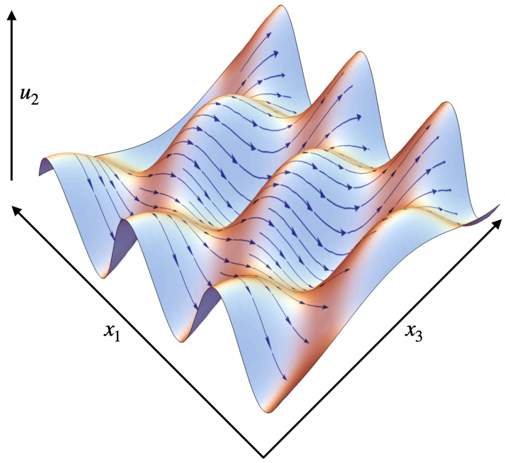

Figure 2.1: Elastic shear wavefront depiction moving in the (x_1, x_3) plane with displacement vector $\mathbf{u} = (0, u_2(x_1, x_3), 0)^T$. The arrows depict the orientation of the underlying crystalline material.

linked to the directionality of the cooling process [36]. As the metal cools in the weld, the crystalline structures elongate and align with the direction of the thermal gradients (see Figure 2.2). This crystal structure inside the material is characterised by the grain angle $\theta(x_3)$. In this Chapter however, a stochastic representation of the slowness surface is presented in the form of additive noise to capture the random fluctuations in the crystalline alignment in the material caused by cooling.

Propagation of monochromatic shear waves in a randomly stratified transversely isotropic elastic media was studied in [8], where the wave vibrates in the plane of propagation and the wave amplitude is only dependent on x_3. The rotation of the slowness surface was however around the x_3 axis; this class of material is also considered in [56]. In addition, the displacement vector $\mathbf{u} = (u_1(x_3), u_2(x_3), 0)$ vibrated the particles in both the x_1 and x_2 directions but the amplitude did not depend on the lateral direction x_1. The associated slowness surface then led to a set of wave-mode equations which had a conjugate symmetry which led to a simplified set of governing equations.

This Chapter considers the case where the slowness surface rotates about the x_2 axis with a fixed (random) rotation $\theta(x_3)$ in each layer of the material. In addition, the amplitude of the shear wave is now allowed to vary in the lateral direction x_2. This change leads to a richer and more complex model as illustrated by the intractable Fokker-Planck equation which emerges.

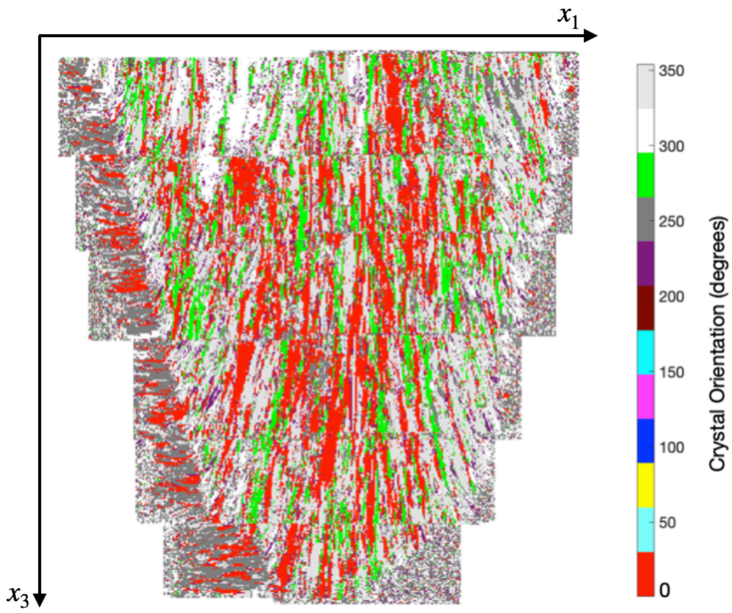

Figure 2.2: Fanned grain structure of a heterogeneous weld structure. This image shows a simplified representation of the material properties as measured destructively using electron backscatter diffraction (EBSD) measurements, which shows clusters of grains of wavelength size. The colours correspond to different crystal orientations (θ) inside the weld. Image generated from data obtained in [57] and [58].

As the metal cools in the weld, the crystalline structures elongate and align with the direction of the thermal gradients (see Figure 2.2). This crystal structure inside the material is characterised by the grain angle $\theta(x_3)$.

Section 2.3 introduces the governing elastic wave equation, the wave-mode equations and the physical geometry of the problem. Sections 2.4 - 2.7 study the problem with no stochastic fluctuations in the material parameters; the simplified problems in these sections will allow for the formulation of key pieces of mathematical machinery needed later on, such as the propagator formulation of the wave-mode problem. Section 2.8 introduces a stochastic (Markovian) process to model the variations in the local orientation of the slowness surface. In the weakly heterogeneous scaling regime a small parameter emerges in the governing random wave-mode evolution equations. A diffusion approximation is then used to derive stochastic differential equations in order to study the probability distribution of the transmitted energy. A Fokker-Planck equation is derived for the probability density function for the transmitted energy which then is solved numerically via finite elements [59]. Section 2.9 contains a discussion of the power transmission coefficient and a parametric study on an austenitic steel weld.

2.3 Governing Elastic Wave Equations

The elastodynamic equation of wave motion is [9]

$$\rho\, u_{j,tt} = \tau_{jk,k}, \tag{2.1}$$

where ρ is the density of the medium, u is the displacement, and τ is the stress. The general stress-strain law is of the form

$$\tau_{ij} = c_{ijkl}(\theta(x_3))e_{kl}, \tag{2.2}$$

with

$$e_{kl} = \frac{1}{2}\,(u_{k,l} + u_{l,k}), \tag{2.3}$$

where c_{ijkl} is the stiffness tensor, and e_{kl} is the direct strain. Define

$$c_{pqrs} = a_{ip}\; a_{jq}\; a_{kr}\; a_{ls}\; \bar{\bar{c}}_{ijkl}, \tag{2.4}$$

where a_{ij} describes a rotation θ about the x_2 axis given by

$$a_{ij} = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}, \tag{2.5}$$

where θ is dependent on x_3 and the locally transversely isotropic matrix $\bar{\bar{c}}_{ijkl}$ is of the form

$$\bar{\bar{c}}_{ijkl} = \bar{\bar{c}}_{mn} = \begin{bmatrix} \bar{\bar{c}}_{11} & \bar{\bar{c}}_{12} & \bar{\bar{c}}_{13} & 0 & 0 & 0 \\ \bar{\bar{c}}_{12} & \bar{\bar{c}}_{11} & \bar{\bar{c}}_{13} & 0 & 0 & 0 \\ \bar{\bar{c}}_{13} & \bar{\bar{c}}_{13} & \bar{\bar{c}}_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\bar{c}}_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{\bar{c}}_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{\bar{c}}_{66} \end{bmatrix}. \tag{2.6}$$

Austenite welds are a commonly used material in the engineering world, exhibiting material properties that align with equation (2.6). Considering a horizontally polarised shear wave, u_j can be expressed as

$$u_j = (0, u_2(x_1, x_3), 0). \tag{2.7}$$

The rotation matrix (equation 2.5) links with the stress tensor (equation (2.6)) by picking out the non-zero stress components when the choice is made for the type of wave polarisation, which comes from equation (2.7). Therefore, equations (2.2) and (2.3) give

$$\tau_{21} = c_{66}(x_3)u_{2,1} + c_{46}(x_3)u_{2,3}, \tag{2.8}$$

$$\tau_{23} = c_{46}(x_3)u_{2,1} + c_{44}(x_3)u_{2,3}, \tag{2.9}$$

where (noting that $c_{64}(x_3) = c_{46}(x_3)$)

$$c_{66}(x_3) = \cos^2\theta(x_3)\bar{\bar{c}}_{66} + \sin^2\theta(x_3)\bar{\bar{c}}_{44} \tag{2.10}$$

$$c_{64}(x_3) = -\cos\theta(x_3)\sin\theta(x_3)\bar{\bar{c}}_{66} + \sin\theta(x_3)\cos\theta(x_3)\bar{\bar{c}}_{44} \tag{2.11}$$

$$c_{44}(x_3) = \sin^2\theta(x_3)\bar{\bar{c}}_{66} + \cos^2\theta(x_3)\bar{\bar{c}}_{44}. \tag{2.12}$$

Defining the velocity

$$\xi = u_{2,t}, \tag{2.13}$$

then equation (2.1) can be written

$$\rho \, \xi_{,t} = \tau_{21,1} + \tau_{23,3}, \tag{2.14}$$

and differentiating (2.8) and (2.9) with respect to t gives

$$\tau_{21,t} = c_{66} \, \xi_{,1} + c_{46} \, \xi_{,3}, \tag{2.15}$$

$$\tau_{23,t} = c_{46} \, \xi_{,1} + c_{44} \, \xi_{,3}. \tag{2.16}$$

Note that since $\theta(x_3)$, then $c_{ij}(x_3)$, $\xi(t, x_1, x_3)$, $\tau_{21}(t, x_1, x_3)$, and $\tau_{23}(t, x_1, x_3)$ become spatially dependent. Now define the Fourier transforms with respect to time with positive sign convention as

$$\check{\xi}(\omega, \underline{x}) = \int \xi_2(t, x_1, x_3) \, e^{i\omega t} \, dt, \tag{2.17}$$

$$\check{\tau}_{21}(\omega, \underline{x}) = \int \tau_{21}(t, x_1, x_3) \, e^{i\omega t} \, dt, \tag{2.18}$$

$$\check{\tau}_{23}(\omega, \underline{x}) = \int \tau_{23}(t, x_1, x_3) \, e^{i\omega t} \, dt. \tag{2.19}$$

Then equations (2.14), (2.15) and (2.16) become

$$-\rho i\omega\check{\xi} = \check{\tau}_{21,1} + \check{\tau}_{23,3}, \tag{2.20}$$

$$-i\omega\check{\tau}_{21} = c_{66}\check{\xi}_{,1} + c_{46}\check{\xi}_{,3}, \tag{2.21}$$

$$-i\omega\check{\tau}_{23} = c_{44}\check{\xi}_{,3} + c_{46}\check{\xi}_{,1}. \tag{2.22}$$

Defining the spatial Fourier transforms with respect to x_1 gives

$$\hat{\tau}_{21}(\omega, \kappa_1, x_3) = \int \check{\tau}_{21}(\omega, x_1, x_3) e^{i\kappa_1 x_1} dx_1, \tag{2.23}$$

$$\hat{\tau}_{23}(\omega, \kappa_1, x_3) = \int \check{\tau}_{23}(\omega, x_1, x_3) e^{i\kappa_1 x_1} dx_1, \tag{2.24}$$

$$\hat{\xi}(\omega, \kappa_1, x_3) = \int \check{\xi}(\omega, x_1, x_3) e^{i\kappa_1 x_1} dx_1. \tag{2.25}$$

Applying these to equations (2.20), (2.21), (2.22) gives

$$-\rho i\omega\hat{\xi}(\omega,\kappa_1,x_3) = -i\kappa_1\hat{\tau}_{21}(\omega,\kappa_1,x_3) + \hat{\tau}_{23,3}(\omega,\kappa_1,x_3), \tag{2.26}$$

$$-i\omega\hat{\tau}_{21}(\omega,\kappa_1,x_3) = -i\kappa_1 c_{66}(x_3)\hat{\xi}(\omega,\kappa_1,x_3) + c_{64}(x_3)\hat{\xi}_{,3}(\omega,\kappa_1,x_3), \tag{2.27}$$

$$-i\omega\hat{\tau}_{23}(\omega,\kappa_1,x_3) = c_{44}(x_3)\hat{\xi}_{,3}(\omega,\kappa_1,x_3) - i\kappa_1 c_{64}(x_3)\hat{\xi}(\omega,\kappa_1,x_3). \tag{2.28}$$

From equations (2.27) and (2.28)

$$\hat{\xi}_{,3} = \frac{-i\omega\hat{\tau}_{21}}{c_{64}} + \frac{i\kappa_1 c_{66}}{c_{64}}\hat{\xi} = \frac{-i\omega}{c_{44}}\hat{\tau}_{23} + \frac{i\kappa_1 c_{64}}{c_{44}}\hat{\xi}, \tag{2.29}$$

hence

$$\hat{\tau}_{21}(\omega,\kappa_1,x_3) = \frac{c_{64}(x_3)}{c_{44}(x_3)}\hat{\tau}_{23}(\omega,\kappa_1,x_3) - \left(\frac{\kappa_1 c_{64}(x_3)^2}{\omega c_{44}(x_3)} - \frac{\kappa_1 c_{66}(x_3)}{\omega}\right)\hat{\xi}(\omega,\kappa_1,x_3). \tag{2.30}$$

$$\hat{\tau}_{23,3}(\omega,\kappa_1,x_3) = \psi(x_3)\hat{\xi}(\omega,\kappa_1,x_3) + \varsigma(x_3)\hat{\tau}_{23}(\omega,\kappa_1,x_3), \tag{2.31}$$

where

$$\psi(x_3) = -i\omega\left(\rho - \frac{\kappa_1^2 c_{66}(x_3)}{\omega^2} + \frac{\kappa_1^2 c_{64}(x_3)^2}{\omega^2 c_{44}(x_3)}\right) = -i\omega\hat{\psi}(x_3), \tag{2.32}$$

$$\hat{\psi}(x_3) = \rho - \frac{\kappa_1^2 c_{66}(x_3)}{\omega^2} + \frac{\kappa_1^2 c_{64}(x_3)^2}{\omega^2 c_{44}(x_3)}, \tag{2.33}$$

and

$$\varsigma(x_3) = i\frac{\kappa_1 c_{64}(x_3)}{c_{44}(x_3)} = i\hat{\varsigma}(x_3), \tag{2.34}$$

$$\hat{\varsigma}(x_3) = \frac{\kappa_1 c_{64}(x_3)}{c_{44}(x_3)}. \tag{2.35}$$

Then, from equation (2.28)

$$\begin{aligned}\hat{\xi}_{,3}(\omega,\kappa_1,x_3) &= \varsigma(x_3)\hat{\xi}(\omega,\kappa_1,x_3) - \frac{i\omega}{c_{44}(x_3)}\hat{\tau}_{23}(\omega,\kappa_1,x_3)\\ &= \varsigma(x_3)\hat{\xi}(\omega,\kappa_1,x_3) - \eta(x_3)\hat{\tau}_{23}(\omega,\kappa_1,x_3),\end{aligned} \tag{2.36}$$

where

$$\eta(x_3) = i\frac{\omega}{c_{44}(x_3)} = i\hat{\eta}(x_3), \tag{2.37}$$

$$\hat{\eta}(x_3) = \frac{\omega}{c_{44}(x_3)}. \tag{2.38}$$

Putting equations (2.31) and (2.36) into matrix form gives

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix} = \begin{bmatrix}\varsigma(x_3) & -\eta(x_3)\\ \psi(x_3) & \varsigma(x_3)\end{bmatrix}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}, \tag{2.39}$$

where the subscript on $\hat{\tau}_{23}$ has been dropped to simplify the notation. Let

$$\mathbf{M} = \begin{bmatrix}\varsigma(x_3) & -\eta(x_3)\\ \psi(x_3) & \varsigma(x_3)\end{bmatrix}, \tag{2.40}$$

whose eigenvalues (using equations (2.32), (2.34), (2.35), (2.37), (2.38) and (2.41)) are given by

$$\begin{aligned}\Lambda^{\pm}(x_3) &= \varsigma(x_3) \pm i\sqrt{\psi(x_3)\eta(x_3)},\\ &= i\left(\frac{c_{64}\kappa_1}{c_{44}} \pm \omega\sqrt{\frac{\hat{\psi}}{c_{44}}}\right)\\ &= i(\hat{\varsigma} \pm \zeta)\\ &= i\lambda^{\pm},\end{aligned} \tag{2.41}$$

with corresponding eigenvectors $\left[\, i\sqrt{\psi(x_3)\eta(x_3)}\;\; \psi(x_3)\,\right]^T$, $\left[\, -i\sqrt{\psi(x_3)\eta(x_3)}\;\; \psi(x_3)\,\right]^T$. Letting

$$\mathbf{Q}(x_3) = \begin{bmatrix} i\sqrt{\psi(x_3)\eta(x_3)} & -i\sqrt{\psi(x_3)\eta(x_3)}\\ \psi(x_3) & \psi(x_3)\end{bmatrix}, \tag{2.42}$$

$$\mathbf{D}(x_3) = \begin{bmatrix}\Lambda^{+}(x_3) & 0\\ 0 & \Lambda^{-}(x_3)\end{bmatrix}, \tag{2.43}$$

then $\mathbf{M} = \mathbf{Q}(x_3)\mathbf{D}(x_3)\mathbf{Q}^{-1}(x_3)$. Therefore equation (2.39) can be written as

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix} = \mathbf{Q}(x_3)\mathbf{D}(x_3)\mathbf{Q}^{-1}(x_3)\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}. \tag{2.44}$$

It will prove helpful in defining an ansatz for this system of equations to first consider the problem where the material is homogeneous. The consideration of this simplified problem will prove instructive and shape the treatment of the heterogeneous case in later sections. Temporarily assuming $\mathbf{Q}$ is independent of x_3 (so the material is homogeneous), then

$$\frac{\partial}{\partial x_3}\left(\mathbf{Q}^{-1}\begin{bmatrix}\hat{\xi}\\ \hat{\tau}\end{bmatrix}\right) = \mathbf{D}\left(\mathbf{Q}^{-1}\begin{bmatrix}\hat{\xi}\\ \hat{\tau}\end{bmatrix}\right), \tag{2.45}$$

where

$$\mathbf{Q}^{-1} = \frac{1}{2i\sqrt{\eta}\psi^{\frac{3}{2}}}\begin{bmatrix}\psi & i\sqrt{\psi\eta}\\ -\psi & i\sqrt{\psi\eta}\end{bmatrix}, \tag{2.46}$$

which can be written as

$$\mathbf{Q}^{-1} = \frac{1}{2i\zeta\psi}\begin{bmatrix}\psi & i\zeta\\ -\psi & i\zeta\end{bmatrix}, \tag{2.47}$$

where

$$\zeta = \sqrt{\psi\eta}. \tag{2.48}$$

By choosing new variables

$$\begin{bmatrix}\hat{b}(\omega,\kappa_1,x_3)\\ \hat{a}(\omega,\kappa_1,x_3)\end{bmatrix} = \left(\zeta^{1/2}\psi^{1/2}\right)\mathbf{Q}^{-1}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}, \tag{2.49}$$

and rearranging gives

$$\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix} = \begin{bmatrix}i\zeta^{1/2}\psi^{-1/2} & -i\zeta^{1/2}\psi^{-1/2}\\ \psi^{1/2}\zeta^{-1/2} & \psi^{1/2}\zeta^{-1/2}\end{bmatrix}\begin{bmatrix}\hat{b}(\omega,\kappa_1,x_3)\\ \hat{a}(\omega,\kappa_1,x_3)\end{bmatrix}, \tag{2.50}$$

which in equation form gives

$$\hat{\xi}(\omega, \kappa_1, x_3) = i\zeta^{1/2}\psi^{-1/2}\hat{b}(\omega, \kappa_1, x_3) - i\zeta^{1/2}\psi^{-1/2}\hat{a}(\omega, \kappa_1, x_3), \tag{2.51}$$

$$\hat{\tau}(\omega, \kappa_1, x_3) = \psi^{1/2}\zeta^{-1/2}\hat{b}(\omega, \kappa_1, x_3) + \psi^{1/2}\zeta^{-1/2}\hat{a}(\omega, \kappa_1, x_3). \tag{2.52}$$

Notice that ζ and ψ both have degree of $\pm 1/2$; this will be helpful later when balancing the equations to satisfy energy conditions at each interface. Writing equation (2.45) in decoupled form

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{b}(\omega, \kappa_1, x_3) \\ \hat{a}(\omega, \kappa_1, x_3)\end{bmatrix} = \mathbf{D}\begin{bmatrix}\hat{b}(\omega, \kappa_1, x_3) \\ \hat{a}(\omega, \kappa_1, x_3)\end{bmatrix}, \tag{2.53}$$

shows that it has solution

$$\hat{a}(\omega, \kappa_1, x_3) = \hat{a}(\omega, \kappa_1)e^{\Lambda^- x_3}, \tag{2.54}$$

$$\hat{b}(\omega, \kappa_1, x_3) = \hat{b}(\omega, \kappa_1)e^{\Lambda^+ x_3}. \tag{2.55}$$

Equations (2.51) and (2.52) can then be rewritten (using equation (**??**))

$$\hat{\xi}(\omega, \kappa_1, x_3) = i\zeta^{1/2}\psi^{-1/2}\hat{b}(\omega, \kappa_1)e^{i\lambda^+ x_3} - i\zeta^{1/2}\psi^{-1/2}\hat{a}(\omega, \kappa_1)e^{i\lambda^- x_3}, \tag{2.56}$$

$$\hat{\tau}(\omega, \kappa_1, x_3) = \psi^{1/2}\zeta^{-1/2}\hat{b}(\omega, \kappa_1)e^{i\lambda^+ x_3} + \psi^{1/2}\zeta^{-1/2}\hat{a}(\omega, \kappa_1)e^{i\lambda^- x_3}. \tag{2.57}$$

2.4 Single Interface between two Homogeneous and Isotropic Half-Spaces

Consider two homogeneous, locally isotropic half-spaces separated by an interface at $x_3 = 0$ and

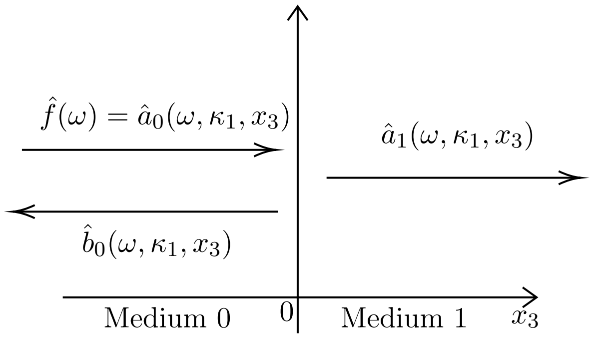

Figure 2.3: Single interface at $x_3 = 0$ between two homogeneous, isotropic half-spaces.

assume the density is constant in each half-space, with material properties

$$c_{66} = \begin{cases} \mu_0 & \text{if } x_3 < 0, \\ \mu_1 & \text{if } x_3 > 0, \end{cases} \tag{2.58}$$

$$c_{64} = \begin{cases} 0 & \text{if } x_3 < 0, \\ 0 & \text{if } x_3 > 0, \end{cases}$$

$$c_{44} = \begin{cases} \mu_0 & \text{if } x_3 < 0, \\ \mu_1 & \text{if } x_3 > 0, \end{cases}$$

$$\psi = \begin{cases} \psi_0 = -i\omega\hat{\psi}_0 = -i\omega\left(\rho_0 - \dfrac{\kappa_1^2\mu_0}{\omega^2}\right) & \text{if } x_3 < 0, \\ \psi_1 = -i\omega\hat{\psi}_1 = -i\omega\left(\rho_1 - \dfrac{\kappa_1^2\mu_1}{\omega^2}\right) & \text{if } x_3 > 0, \end{cases}$$

$$\zeta = \begin{cases} \zeta_0 = \sqrt{\mu_0\psi_0} & \text{if } x_3 < 0, \\ \zeta_1 = \sqrt{\mu_1\psi_1} & \text{if } x_3 > 0. \end{cases}$$

The interface at $x_3 = 0$ has continuity conditions for the modes given by

$$\hat{\xi}_0(x_3 = 0) = \hat{\xi}_1(x_3 = 0), \tag{2.59}$$

$$\hat{\tau}_0(x_3 = 0) = \hat{\tau}_1(x_3 = 0). \tag{2.60}$$

Using equations (2.56) and (2.57), it follows that

$$\hat{a}_1(x_3 = 0) = r^+\hat{a}_0(x_3 = 0) + r^-\hat{b}_0(x_3 = 0), \tag{2.61}$$

$$\hat{b}_1(x_3 = 0) = r^- \hat{a}_0(x_3 = 0) + r^+ \hat{b}_0(x_3 = 0), \tag{2.62}$$

or in matrix form

$$\begin{bmatrix} \hat{a}_1(x_3 = 0) \\ \hat{b}_1(x_3 = 0) \end{bmatrix} = \mathbf{J}(\omega) \begin{bmatrix} \hat{a}_0(x_3 = 0) \\ \hat{b}_0(x_3 = 0) \end{bmatrix}, \tag{2.63}$$

where

$$\mathbf{J}(\omega) = \begin{bmatrix} r^+(\omega) & r^-(\omega) \\ r^-(\omega) & r^+(\omega) \end{bmatrix}. \tag{2.64}$$

and

$$r^{\pm}(\omega) = \frac{1}{2}\left(\sqrt{\frac{\psi_0\zeta_1}{\psi_1\zeta_0}} \pm \sqrt{\frac{\psi_1\zeta_0}{\psi_0\zeta_1}}\right) = \frac{\psi_0\zeta_1 \pm \psi_1\zeta_0}{2\sqrt{\psi_0\psi_1\zeta_0\zeta_1}}. \tag{2.65}$$

Note that

$$\det\{\mathbf{J}(\omega)\} = (r^+)^2 - (r^-)^2 = 1. \tag{2.66}$$

The matrix $\mathbf{J}(\omega)$ can be thought of as an interface propagator, since it propagates the modes from the left half-space to the right half-space. Taking into account boundary conditions at the interface, the system can be written as

$$\begin{bmatrix} \hat{a}_1(x_3 = 0) \\ 0 \end{bmatrix} = \mathbf{J} \begin{bmatrix} \hat{f}(\omega) \\ \hat{b}_0(x_3 = 0) \end{bmatrix}, \tag{2.67}$$

where $\hat{f}(\omega)$ is the transmitted wave travelling to the right in the left half-space. Solving this system gives

$$\hat{b}_0(x_3 = 0) = \mathcal{R}\hat{f}(\omega), \tag{2.68}$$

$$\hat{a}_1(x_3 = 0) = \mathcal{T}\hat{f}(\omega), \tag{2.69}$$

where $\mathcal{R}$ and $\mathcal{T}$ are the reflection and transmission coefficients of the interface, given by

$$\mathcal{R} = -\frac{r^-}{r^+} = \frac{\psi_1\zeta_0 - \psi_0\zeta_1}{\psi_1\zeta_0 + \psi_0\zeta_1}, \tag{2.70}$$

$$\mathcal{T} = \frac{1}{r^+} = \frac{2\sqrt{\psi_0\psi_1\zeta_0\zeta_1}}{\psi_0\zeta_1 + \psi_1\zeta_0}. \tag{2.71}$$

These coefficients satisfy the energy conservation relation

$$\mathcal{R}^2 + \mathcal{T}^2 = 1, \tag{2.72}$$

which states that the sum of the energies of transmitted and reflected waves equals the energy of the incoming wave; the energy put into the system equals the energy travelling out of the system.

2.5 A Transversely Isotropic, Homogeneous Layer Between Two Homogeneous and Isotropic Half-Spaces

Consider the case of a transversely isotropic homogeneous layer with a single slowness surface rotation θ, and thickness L, embedded between two homogeneous and isotropic half-spaces. Assume that in each region the density of the medium is constant. For brevity the κ_1 dependency is omitted from the modes.

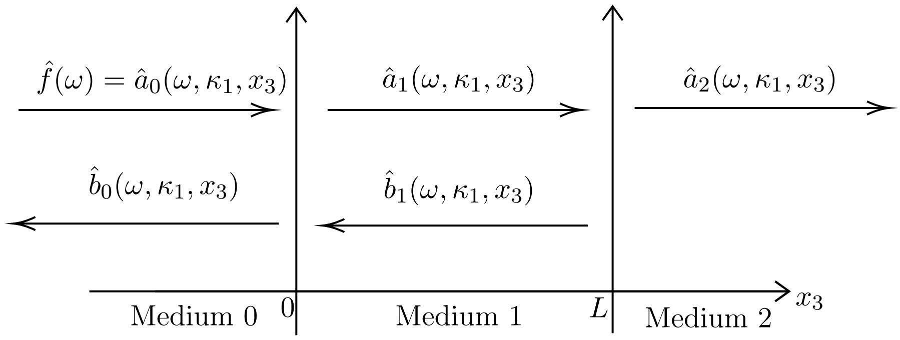

Figure 2.4: Single layer of a locally anisotropic but homogeneous material, sandwiched between two homogeneous and isotropic half-spaces.

The material properties are (where μ_n is a constant)

$$c_{66} = \begin{cases} \mu_0 & \text{if } x_3 < 0, \\ c_{66} & \text{if } x_3 \in [0, L], \\ \mu_2 & \text{if } x_3 > L, \end{cases} \tag{2.73}$$

$$c_{64} = \begin{cases} 0 & \text{if } x_3 \in (-\infty, 0) \cup (L, \infty), \\ c_{64} & \text{if } x_3 \in [0, L], \end{cases}$$

$$c_{44} = \begin{cases} \mu_0 & \text{if } x_3 < 0, \\ c_{44} & \text{if } x_3 \in [0, L], \\ \mu_2 & \text{if } x_3 > L, \end{cases}$$

$$\psi = \begin{cases} \psi_0 = -i\omega\hat{\psi}_0 = -i\omega\left(\rho_0 - \dfrac{\kappa_1^2\mu_0}{\omega^2}\right) \text{ if } x_3 < 0, \\ \psi_1 = -i\omega\hat{\psi}_1 = -i\omega\left(\rho_1 - \dfrac{\kappa_1^2 c_{66}}{\omega^2} + \dfrac{\kappa_1^2 c_{64}^2}{\omega^2 c_{44}}\right) \text{ if } x_3 \in [0, L], \\ \psi_2 = -i\omega\hat{\psi}_2 = -i\omega\left(\rho_2 - \dfrac{\kappa_1^2\mu_2}{\omega^2}\right) \text{ if } x_3 > L, \end{cases}$$

$$\zeta = \begin{cases} \zeta_0 = \sqrt{\mu_0 \psi_0} & \text{if } x_3 < 0, \\ \zeta_1 = \sqrt{\eta \psi_1} & \text{if } x_3 \in [0, L], \\ \zeta_2 = \sqrt{\mu_2 \psi_2} & \text{if } x_3 > 0. \end{cases}$$

As before, conditions of continuity of velocity and stress are imposed at the interface $x_3 = 0$

$$\hat{\xi}_0 = \hat{\xi}_1, \tag{2.74}$$

$$\hat{\tau}_0 = \hat{\tau}_1, \tag{2.75}$$

and at the interface $x_3 = L$

$$\hat{\xi}_1 = \hat{\xi}_2, \tag{2.76}$$

$$\hat{\tau}_1 = \hat{\tau}_2, \tag{2.77}$$

where the subscript indicates the solution in region $j = 0, 1, 2$. Equations (2.56) and (2.74) give

$$i\zeta_0^{1/2}\psi_0^{-1/2}\hat{b}_0 - i\zeta_0^{1/2}\psi_0^{-1/2}\hat{a}_0 = i\zeta_1^{1/2}\psi_1^{-1/2}\hat{b}_1 - i\zeta_1^{1/2}\psi_1^{-1/2}\hat{a}_1, \tag{2.78}$$

and similarly, equations (2.57) and (2.75) give

$$\psi_0^{1/2}\zeta_0^{-1/2}\hat{b}_0 + \psi_0^{1/2}\zeta_0^{-1/2}\hat{a}_0 = \psi_1^{1/2}\zeta_1^{-1/2}\hat{b}_1 + \psi_1^{1/2}\zeta_1^{-1/2}\hat{a}_1. \tag{2.79}$$

Solving these simultaneous equations gives

$$\begin{aligned} \hat{b}_1(\omega) = {} & \frac{1}{2}\left(\sqrt{\frac{\psi_0\zeta_1}{\psi_1\zeta_0}} - \sqrt{\frac{\psi_1\zeta_0}{\psi_0\zeta_1}}\right)\hat{a}_0(\omega) \\ & + \frac{1}{2}\left(\sqrt{\frac{\psi_0\zeta_1}{\psi_1\zeta_0}} + \sqrt{\frac{\psi_1\zeta_0}{\psi_0\zeta_1}}\right)\hat{b}_0(\omega), \end{aligned} \tag{2.80}$$

$$\hat{a}_1(\omega) = \frac{1}{2}\left(\sqrt{\frac{\psi_0\zeta_1}{\psi_1\zeta_0}} + \sqrt{\frac{\psi_1\zeta_0}{\psi_0\zeta_1}}\right)\hat{a}_0(\omega) + \frac{1}{2}\left(\sqrt{\frac{\psi_0\zeta_1}{\psi_1\zeta_0}} - \sqrt{\frac{\psi_1\zeta_0}{\psi_0\zeta_1}}\right)\hat{b}_0(\omega). \tag{2.81}$$

These conditions at the first interface can be written

$$\begin{bmatrix}\hat{a}_1(\omega)\\ \hat{b}_1(\omega)\end{bmatrix} = \hat{\mathbf{J}}_0(\omega)\begin{bmatrix}\hat{a}_0(\omega)\\ \hat{b}_0(\omega)\end{bmatrix}, \tag{2.82}$$

where

$$\hat{\mathbf{J}}_0(\omega) = \begin{bmatrix} r_0^+(\omega) & r_0^-(\omega)\\ r_0^-(\omega) & r_0^+(\omega)\end{bmatrix}, \tag{2.83}$$

and

$$r_0^{\pm}(\omega) = \frac{1}{2}\left(\sqrt{\frac{\psi_0\zeta_1}{\psi_1\zeta_0}} \pm \sqrt{\frac{\psi_1\zeta_0}{\psi_0\zeta_1}}\right). \tag{2.84}$$

As before

$$\det\left\{\hat{\mathbf{J}}_0(\omega)\right\} = (r_0^+(\omega))^2 - (r_0^-(\omega))^2 = 1. \tag{2.85}$$

Similarly to equations (2.70) and (2.71), the local reflection and transmission coefficients at $x_3 = 0$ can be defined by

$$R_0 = -\frac{r_0^-}{r_0^+} = \frac{\psi_1\zeta_0 - \psi_0\zeta_1}{\psi_0\zeta_1 + \psi_1\zeta_0}, \tag{2.86}$$

$$T_0 = \frac{1}{r_0^+} = \frac{2\sqrt{\psi_0\zeta_0\psi_1\zeta_1}}{\psi_0\zeta_1 + \psi_1\zeta_0}. \tag{2.87}$$

Using equation (2.85) it is clear that

$$R_0^2 + T_0^2 = \frac{(r_0^-)^2 + 1}{(r_0^+)^2} = 1. \tag{2.88}$$

Repeating the above at the $x_3 = L$ interface in equations (2.56) and (2.76) gives

$$
\begin{aligned}
& i\zeta_1^{1/2}\psi_1^{-1/2}\hat{b}_1(\omega)e^{i\lambda_1^+ L} - i\zeta_1^{1/2}\psi_1^{-1/2}\hat{a}_1(\omega)e^{i\lambda_1^- L} \\
&= i\zeta_1^{1/2}\psi_2^{-1/2}\hat{b}_2(\omega)e^{i\lambda_2^+ L} - i\zeta_2^{1/2}\psi_2^{-1/2}\hat{a}_2(\omega)e^{i\lambda_2^- L}.
\end{aligned}
\tag{2.89}
$$

Using equations (2.57) and (2.77) gives

$$
\begin{aligned}
& \psi_1^{1/2}\zeta_1^{-1/2}\hat{b}_1(\omega)e^{i\lambda_1^+ L} + \psi_1^{1/2}\zeta_1^{-1/2}\hat{a}_1(\omega)e^{i\lambda_1^- L} \\
&= \psi_2\zeta_2^{-1/2}\hat{b}_2(\omega)e^{i\lambda_2^+ L} + \psi_2^{1/2}\zeta^{-1/2}\hat{a}_2(\omega)e^{i\lambda_2^- L}.
\end{aligned}
\tag{2.90}
$$

Solving these gives

$$
\begin{aligned}
\hat{b}_2 = (\omega) = & \frac{1}{2}\left(\sqrt{\frac{\psi_1\zeta_2}{\psi_2\zeta_1}} - \sqrt{\frac{\psi_2\zeta_1}{\psi_1\zeta_2}}\right)\hat{a}_1(\omega)e^{i\lambda_1^- L} \\
& + \frac{1}{2}\left(\sqrt{\frac{\psi_1\zeta_2}{\psi_2\zeta_1}} + \sqrt{\frac{\psi_2\zeta_1}{\psi_1\zeta_2}}\right)\hat{b}_1(\omega)e^{i\lambda_1^+ L},
\end{aligned}
\tag{2.91}
$$

$$
\begin{aligned}
\hat{a}_2 = (\omega) = & \frac{1}{2}\left(\sqrt{\frac{\psi_1\zeta_2}{\psi_2\zeta_1}} + \sqrt{\frac{\psi_2\zeta_1}{\psi_1\zeta_2}}\right)\hat{a}_1(\omega)e^{i\lambda_1^- L} \\
& + \frac{1}{2}\left(\sqrt{\frac{\psi_1\zeta_2}{\psi_2\zeta_1}} - \sqrt{\frac{\psi_2\zeta_1}{\psi_1\zeta_2}}\right)\hat{b}_1(\omega)e^{i\lambda_1^+ L}.
\end{aligned}
\tag{2.92}
$$

Hence the modes at the second interface in the layer can be written

$$
\begin{bmatrix} \hat{a}_2(\omega) \\ \hat{b}_2(\omega) \end{bmatrix} = \hat{\mathbf{J}}_1(\omega) \begin{bmatrix} \hat{a}_1(\omega) \\ \hat{b}_1(\omega) \end{bmatrix},
\tag{2.93}
$$

where

$$
\hat{\mathbf{J}}_1(\omega) = \begin{bmatrix} r_1^+(\omega)e^{i\lambda_1^- L} & r_1^-(\omega)e^{i\lambda_1^+ L} \\ r_1^-(\omega)e^{i\lambda_1^- L} & r_1^+(\omega)e^{i\lambda_1^+ L} \end{bmatrix},
\tag{2.94}
$$

and similar to equation (2.84)

$$r_1^{\pm}(\omega) = \frac{1}{2}\left(\sqrt{\frac{\psi_1\zeta_2}{\psi_2\zeta_1}} \pm \sqrt{\frac{\psi_2\zeta_1}{\psi_1\zeta_2}}\right). \tag{2.95}$$

The local transmission and reflection coefficients can be defined for the second interface at $x_3 = L$ via

$$R_1 = -\frac{r_1^-}{r_1^+} = \frac{\psi_2\zeta_1 - \psi_1\zeta_2}{\psi_1\zeta_2 + \psi_2\zeta_1}, \tag{2.96}$$

$$T_1 = \frac{1}{r^+} = \frac{2\sqrt{\psi_1\psi_2\zeta_1\zeta_2}}{\psi_1\zeta_2 + \psi_2\zeta_1}. \tag{2.97}$$

where

$$(r_1^+(\omega))^2 - (r_1^-(\omega))^2 = 1, \tag{2.98}$$

with the conservation of energy property

$$R_1^2 + T_1^2 = \frac{(r_1^-)^2 + 1}{(r_1^+)^2} = 1. \tag{2.99}$$

Using equations (2.82) and (2.93) the equations for the modes at each interface can be combined to provide a global propagator matrix for the layer

$$\begin{bmatrix} \hat{a}_2(\omega) \\ \hat{b}_2(\omega) \end{bmatrix} = \hat{\mathbf{K}}(\omega) \begin{bmatrix} \hat{a}_0(\omega) \\ \hat{b}_0(\omega) \end{bmatrix}, \tag{2.100}$$

where

$$
\begin{aligned}
\hat{\mathbf{K}}(\omega) &= \hat{\mathbf{J}}_1(\omega)\hat{\mathbf{J}}_0(\omega) \\
&= \begin{bmatrix} r_1^+(\omega)e^{i\lambda_1^- L} & r_1^-(\omega)e^{i\lambda_1^+ L} \\ r_1^-(\omega)e^{i\lambda_1^- L} & r_1^+(\omega)e^{i\lambda_1^+ L} \end{bmatrix} \begin{bmatrix} r_0^+(\omega) & r_0^-(\omega) \\ r_0^-(\omega) & r_0^+(\omega) \end{bmatrix} \\
&= \begin{bmatrix} r_0^+(\omega)r_1^+(\omega)e^{i\lambda_1^- L} & r_0^-(\omega)r_1^+(\omega)e^{i\lambda_1^- L} \\ \quad + r_0^-(\omega)r_1^-(\omega)e^{i\lambda_1^+ L} & \quad + r_0^+(\omega)r_1^-(\omega)e^{i\lambda_1^+ L} \\ r_0^+(\omega)r_1^-(\omega)e^{i\lambda_1^- L} & r_0^-(\omega)r_1^-(\omega)e^{i\lambda_1^- L} \\ \quad + r_0^-(\omega)r_1^+(\omega)e^{i\lambda_1^+ L} & \quad + r_0^+(\omega)r_1^+(\omega)e^{i\lambda_1^+ L} \end{bmatrix} \qquad (2.101) \\
&= \begin{bmatrix} \hat{U}(\omega) & \hat{G}(\omega) \\ \hat{V}(\omega) & \hat{H}(\omega) \end{bmatrix}.
\end{aligned}
$$

By applying the boundary conditions $\hat{a}_0(\omega) = \hat{f}(\omega)$ and $\hat{b}_2(\omega) = 0$, then equation (2.100) can be written as

$$
\begin{bmatrix} \hat{a}_2(\omega) \\ 0 \end{bmatrix} = \begin{bmatrix} \hat{U}(\omega) & \hat{G}(\omega) \\ \hat{V}(\omega) & \hat{H}(\omega) \end{bmatrix} \begin{bmatrix} \hat{f}(\omega) \\ \hat{b}_0(\omega) \end{bmatrix}. \qquad (2.102)
$$

This gives

$$
\hat{b}_0(\omega) = -\frac{\hat{V}}{\hat{H}}\hat{f}(\omega) = \hat{\mathcal{R}}(\omega)\hat{f}(\omega), \qquad (2.103)
$$

where $\hat{\mathcal{R}}$ is the global reflection coefficient and

$$
\hat{a}_2(\omega) = \hat{U}\hat{f}(\omega) + \hat{G}\hat{b}_0 = \frac{\hat{U}\hat{H} - \hat{G}\hat{V}}{\hat{H}} = \hat{\mathcal{T}}(\omega)\hat{f}(\omega), \qquad (2.104)
$$

where $\hat{\mathcal{T}}$ is the global transmission coefficient. Energy conservation can be verified by calculating

$$
\begin{aligned}
|\hat{\mathcal{R}}|^2 + |\hat{\mathcal{T}}|^2 &= \frac{1 + |\hat{V}|^2}{|\hat{H}|^2} \qquad (2.105) \\
&= \frac{1 + (r_0^+ r_1^- e^{i\lambda_1^- L} + r_0^- r_1^+ e^{i\lambda_1^+ L})(r_0^+ r_1^- e^{-i\lambda_1^- L} + r_0^- r_1^+ e^{-i\lambda_1^+ L})}{(r_0^- r_1^- e^{i\lambda_1^- L} + r_0^+ r_1^+ e^{i\lambda_1^+ L})(r_0^- r_1^- e^{-i\lambda_1^- L} + r_0^+ r_1^+ e^{-i\lambda_1^+ L})} \\
&= \frac{1 + (r_0^+)^2(r_1^-)^2 + (r_0^-)^2(r_1^+)^2 + (r_0^- r_0^+)(r_1^- r_1^+)e^{2i\zeta_1 L} + (r_0^- r_0^+)(r_1^- r_1^+)e^{-2i\zeta_1 L}}{(r_0^-)^2(r_1^-)^2 + (r_0^+)^2(r_1^+)^2 + (r_0^- r_0^+)(r_1^- r_1^+)e^{2i\zeta_1 L} + (r_0^- r_0^+)(r_1^- r_1^+)e^{-2i\zeta_1 L}},
\end{aligned}
$$

The following identity can be shown to hold (via algebraic checking in Mathematica)

$$1 + (r_0^+)^2(r_1^-)^2 + (r_0^-)^2(r_1^+)^2 = (r_0^-)^2(r_1^-)^2 + (r_0^+)^2(r_1^+)^2, \tag{2.106}$$

and so the conservation of energy

$$|\hat{\mathcal{R}}|^2 + |\hat{\mathcal{T}}|^2 = 1, \tag{2.107}$$

is preserved.

2.6 Wave Propagation in a Heterogeneous Layer

This Section will consider a heterogeneous layer whose material parameters vary continuously in space. Using an ansatz guided by observations in Section 2.3 for a homogeneous medium. For the generated right-moving modes $\hat{a}(\omega, x_3)$ and left-moving modes $\hat{b}(\omega, x_3)$, it is assumed that the wave mode functions now have spatial dependence [60]. Inspired by equations (2.56) and (2.57), consider the following ansatz for the generated right-moving modes $\hat{a}(\omega, \kappa_1, x_3)$ and left-moving modes $\hat{b}(\omega, \kappa_1, x_3)$

$$\begin{aligned} \hat{\xi}(\omega, \kappa_1, x_3) = {} & i\sqrt{\bar{\zeta}/\bar{\psi}}\, \hat{b}(\omega, \kappa_1, x_3) e^{i\bar{\lambda}^+ x_3} \\ & - i\sqrt{\bar{\zeta}/\bar{\psi}}\, \hat{a}(\omega, \kappa_1, x_3) e^{i\bar{\lambda}^- x_3}, \end{aligned} \tag{2.108}$$

$$\begin{aligned} \hat{\tau}(\omega, \kappa_1, x_3) = {} & \sqrt{\bar{\psi}/\bar{\zeta}}\, \hat{b}(\omega, \kappa_1, x_3) e^{i\bar{\lambda}^+ x_3} \\ & + \sqrt{\bar{\psi}/\bar{\zeta}}\, \hat{a}(\omega, \kappa_1, x_3) e^{i\bar{\lambda}^- x_3}. \end{aligned} \tag{2.109}$$

Note that their prefactors are independent of x_3 (quantities which are not dependent on x_3 are denoted by a bar). Combining equations (2.108) and (2.109) gives

$$\hat{a}(\omega, \kappa_1, x_3) = \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\psi}}\, \hat{\tau}(\omega, \kappa_1, x_3) + i\sqrt{\bar{\psi}/\bar{\zeta}}\, \hat{\xi}(\omega, \kappa_1, x_3)\right) e^{-i\bar{\lambda}^- x_3}, \tag{2.110}$$

and

$$\hat{b}(\omega,\kappa_1,x_3) = \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\psi}}\,\hat{\tau}(\omega,\kappa_1,x_3) - i\sqrt{\bar{\psi}/\bar{\zeta}}\,\hat{\xi}(\omega,\kappa_1,x_3)\right)e^{-i\bar{\lambda}^{+}x_3}. \tag{2.111}$$

Recall the governing equations from the linear system given by equation (2.39)

$$\frac{\partial\hat{\xi}}{\partial x_3} = \varsigma(x_3)\hat{\xi} - \eta(x_3)\hat{\tau}, \tag{2.112}$$

$$\frac{\partial\hat{\tau}}{\partial x_3} = \psi(x_3)\hat{\xi} + \varsigma(x_3)\hat{\tau}. \tag{2.113}$$

Taking the spatial derivative of (2.110) gives

$$\frac{\partial\hat{a}}{\partial x_3} = \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\psi}}\,\frac{\partial\hat{\tau}}{\partial x_3} - i\bar{\lambda}^{-}\sqrt{\bar{\zeta}/\bar{\psi}}\,\hat{\tau} + i\sqrt{\bar{\psi}/\bar{\zeta}}\,\frac{\partial\hat{\xi}}{\partial x_3} + \bar{\lambda}^{-}\sqrt{\bar{\psi}/\bar{\zeta}}\,\hat{\xi}\right)e^{-i\bar{\lambda}^{-}x_3}. \tag{2.114}$$

Inserting (2.112) and (2.113) into (2.114) gives

$$\frac{\partial\hat{a}(\omega,\kappa_1,x_3)}{\partial x_3} = \left(\Gamma_1(x_3)\hat{\xi}(\omega,\kappa_1,x_3) + \Gamma_2(x_3)\hat{\tau}(\omega,\kappa_1,x_3)\right)e^{-i\bar{\lambda}^{-}x_3}, \tag{2.115}$$

where

$$\Gamma_1(x_3) = \frac{1}{2}\left(\psi(x_3)\sqrt{\bar{\zeta}/\bar{\psi}} + i\varsigma(x_3)\sqrt{\bar{\psi}/\bar{\zeta}} + \bar{\lambda}^{-}\sqrt{\bar{\psi}/\bar{\zeta}}\right), \tag{2.116}$$

$$\Gamma_2(x_3) = \frac{1}{2}\left(\varsigma(x_3)\sqrt{\bar{\zeta}/\bar{\psi}} - i\eta(x_3)\sqrt{\bar{\psi}/\bar{\zeta}} - i\bar{\lambda}^{-}\sqrt{\bar{\zeta}/\bar{\psi}}\right). \tag{2.117}$$

Inserting equations (2.108) and (2.109) into equation (2.115) gives

$$\frac{\partial\hat{a}(\omega,\kappa_1,x_3)}{\partial x_3} = \hat{a}(\omega,\kappa_1,x_3)\Delta_1(x_3) + \hat{b}(\omega,\kappa_1,x_3)\Delta_2(x_3), \tag{2.118}$$

where

$$\Delta_1(x_3) = \sqrt{\bar{\psi}/\bar{\zeta}}\,\Gamma_2(x_3) - i\sqrt{\bar{\zeta}/\bar{\psi}}\,\Gamma_1(x_3), \tag{2.119}$$

$$\Delta_2(x_3) = \left(\sqrt{\bar{\psi}/\bar{\zeta}}\,\Gamma_2(x_3) + i\sqrt{\bar{\zeta}/\bar{\psi}}\,\Gamma_1(x_3)\right)e^{2i\bar{\zeta}x_3}. \tag{2.120}$$

Repeating the same calculation for the backward mode (by differentiating equation (2.111)) gives

$$\frac{\partial \hat{b}(\omega,\kappa_1,x_3)}{\partial x_3} = \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\psi}}\,\frac{\partial \hat{\tau}}{\partial x_3} - i\bar{\lambda}^{+}\sqrt{\bar{\zeta}/\bar{\psi}}\,\hat{\tau} - i\sqrt{\bar{\psi}/\bar{\zeta}}\,\frac{\partial \hat{\xi}}{\partial x_3} - \bar{\lambda}^{+}\sqrt{\bar{\psi}/\bar{\zeta}}\,\hat{\xi}\right)e^{-i\bar{\lambda}^{+}x_3}. \tag{2.121}$$

Inserting equations (2.112) and (2.113) gives

$$\frac{\partial \hat{b}(\omega,\kappa_1,x_3)}{\partial x_3} = \Big(\Gamma_3(x_3)\hat{\xi}(\omega,\kappa_1,x_3) + \Gamma_4(x_3)\hat{\tau}(\omega,\kappa_1,x_3)\Big)e^{-i\bar{\lambda}^{+}x_3}, \tag{2.122}$$

where

$$\Gamma_3(x_3) = \frac{1}{2}\left(\psi(x_3)\sqrt{\bar{\zeta}/\bar{\psi}} - i\varsigma(x_3)\sqrt{\bar{\psi}/\bar{\zeta}} - \bar{\lambda}^{+}\sqrt{\bar{\psi}/\bar{\zeta}}\right), \tag{2.123}$$

$$\Gamma_4(x_3) = \frac{1}{2}\left(\varsigma(x_3)\sqrt{\bar{\zeta}/\bar{\psi}} + i\eta(x_3)\sqrt{\bar{\psi}/\bar{\zeta}} - i\bar{\lambda}^{+}\sqrt{\bar{\zeta}/\bar{\psi}}\right). \tag{2.124}$$

Substituting in equations (2.108) and (2.109) into equation (2.122) then gives

$$\frac{\partial \hat{b}(\omega,\kappa_1,x_3)}{\partial x_3} = \hat{a}(\omega,\kappa_1,x_3)\Delta_3(x_3) + \hat{b}(\omega,\kappa_1,x_3)\Delta_4(x_3), \tag{2.125}$$

where

$$\Delta_3(x_3) = \left(\Gamma_4(x_3)\sqrt{\bar{\psi}/\bar{\zeta}} - i\Gamma_3(x_3)\sqrt{\bar{\zeta}/\bar{\psi}}\right)e^{-2i\bar{\zeta}x_3}, \tag{2.126}$$

$$\Delta_4(x_3) = \left(\Gamma_4(x_3)\sqrt{\bar{\psi}/\bar{\zeta}} + i\Gamma_3(x_3)\sqrt{\bar{\zeta}/\bar{\psi}}\right). \tag{2.127}$$

The complex mode amplitudes $\hat{a}(\omega,\kappa_1,x_3)$ and $\hat{b}(\omega,\kappa_1,x_3)$ can be combined to show that they satisfy the linear system

$$\begin{aligned}\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{a}(\omega,x_3)\\ \hat{b}(\omega,x_3)\end{bmatrix} &= \begin{bmatrix}\Delta_1(x_3) & \Delta_2(x_3)\\ \Delta_3(x_3) & \Delta_4(x_3)\end{bmatrix}\begin{bmatrix}\hat{a}(\omega,x_3)\\ \hat{b}(\omega,x_3)\end{bmatrix}\\ &= \mathbf{H}\begin{bmatrix}\hat{a}(\omega,x_3)\\ \hat{b}(\omega,x_3)\end{bmatrix}.\end{aligned} \tag{2.128}$$

From equations (2.32), (2.34), (2.37) and (2.41) then

$$\Delta_1(x_3) = \frac{i}{2}\Bigg(2(\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}} + \bar{\zeta}) - \left(\frac{\bar{\zeta}}{\bar{\hat{\psi}}}\right)\hat{\psi}(x_3) - \frac{\omega\bar{\hat{\psi}}}{\bar{\zeta}}\hat{\eta}(x_3)\Bigg), \tag{2.129}$$

$$\Delta_2(x_3) = \frac{ie^{2i\bar{\zeta}x_3}}{2}\Bigg(\left(\frac{\bar{\zeta}}{\bar{\hat{\psi}}}\right)\hat{\psi}(x_3) - \left(\frac{\omega\bar{\hat{\psi}}}{\bar{\zeta}}\right)\hat{\eta}(x_3)\Bigg), \tag{2.130}$$

$$\Delta_3(x_3) = \frac{ie^{-2i\bar{\zeta}x_3}}{2}\Bigg(\left(\frac{\omega\bar{\hat{\psi}}}{\bar{\zeta}}\right)\hat{\eta}(x_3) - \left(\frac{\bar{\zeta}}{\bar{\hat{\psi}}}\right)\hat{\psi}(x_3)\Bigg), \tag{2.131}$$

and

$$\Delta_4(x_3) = \frac{i}{2}\Bigg(2(\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}} - \bar{\zeta}) + \left(\frac{\bar{\zeta}}{\bar{\hat{\psi}}}\right)\hat{\psi}(x_3) + \left(\frac{\omega\bar{\hat{\psi}}}{\bar{\zeta}}\right)\hat{\eta}(x_3)\Bigg). \tag{2.132}$$

Note here useful relations about the expressions in the linear system (2.128) that will be helpful in later sections: the complex conjugate of Δ_2 is related to Δ_3 via

$$\overline{\Delta}_2(x_3) = \Delta_3(x_3), \tag{2.133}$$

where the bar here denotes the complex conjugate. Also

$$\begin{aligned}
\mathrm{Tr}(\mathbf{H}_\omega) &= \Delta_1(x_3) + \Delta_4(x_3) \qquad (2.134)\\
&= \frac{i}{2}\Bigg(2(\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}} + \bar{\zeta}) - \left(\frac{\bar{\zeta}}{\bar{\hat{\psi}}}\right)\hat{\psi}(x_3) - \frac{\omega\bar{\hat{\psi}}}{\bar{\zeta}}\hat{\eta}(x_3)\Bigg)\\
&+ \frac{i}{2}\Bigg(2(\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}} - \bar{\zeta}) + \left(\frac{\bar{\zeta}}{\bar{\hat{\psi}}}\right)\hat{\psi}(x_3) + \left(\frac{\omega\bar{\hat{\psi}}}{\bar{\zeta}}\right)\hat{\eta}(x_3)\Bigg)\\
&= 2i(\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}}). \qquad (2.135)
\end{aligned}$$

Using equation (2.128) then, since $\overline{\Delta}_1 = -\Delta_1$ from equation (2.129) and $\overline{\Delta}_4 = -\Delta_4$ from equation (2.132), then (where the bar denotes the complex conjugate)

$$\frac{\partial|\hat{a}|^2}{\partial x_3} = \hat{a}\frac{\partial\bar{\hat{a}}}{\partial x_3} + \bar{\hat{a}}\frac{\partial\hat{a}}{\partial x_3} = \overline{\Delta}_2\,\hat{a}\,\bar{\hat{b}} + \Delta_2\,\bar{\hat{a}}\,\hat{b}, \tag{2.136}$$

$$\frac{\partial|\hat{b}|^2}{\partial x_3} = \hat{b}\frac{\partial\bar{\hat{b}}}{\partial x_3} + \bar{\hat{b}}\frac{d\hat{b}}{\partial x_3} = \overline{\Delta}_3\bar{\hat{a}}\,\hat{b} + \Delta_3\,\hat{a}\,\bar{\hat{b}}. \tag{2.137}$$

Hence, from equations (2.133), (2.136) and (2.137) the complex mode amplitudes have the energy conservation property [8]

$$\frac{\partial}{\partial x_3}\left(|\hat{a}(\omega, x_3)|^2 - |\hat{b}(\omega, x_3)^2|\right) = 0. \tag{2.138}$$

The matching conditions at the interface $x_3 = 0$ and $x_3 = L$ are given by equations (2.74) to (2.77). In a similar manner to the derivation of equations (2.82) and (2.93) the modes in each layer are coupled via

$$\begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix} = \hat{\mathbf{J}}_0(\omega) \begin{bmatrix} \hat{a}_0(\omega, 0) \\ \hat{b}_0(\omega, 0) \end{bmatrix}, \tag{2.139}$$

$$\begin{bmatrix} \hat{a}_1(\omega, L) \\ \hat{b}_1(\omega, L) \end{bmatrix} = \hat{\mathbf{J}}_1(\omega) \begin{bmatrix} \hat{a}(\omega, L) \\ \hat{b}(\omega, L) \end{bmatrix}, \tag{2.140}$$

where

$$\hat{\mathbf{J}}_0(\omega) = \begin{bmatrix} r_0^+(\omega) & r_0^-(\omega) \\ r_0^-(\omega) & r_0^+(\omega) \end{bmatrix}, \tag{2.141}$$

$$\hat{\mathbf{J}}_1(\omega) = \begin{bmatrix} r_1^+(\omega)e^{i\bar{\lambda}^- L} & r_1^-(\omega)e^{i\bar{\lambda}^+ L} \\ r_1^-(\omega)e^{i\bar{\lambda}^- L} & r_1^+(\omega)e^{i\bar{\lambda}^+ L} \end{bmatrix}, \tag{2.142}$$

and

$$r_0^{\pm}(\omega) = \frac{1}{2}\left(\sqrt{\frac{\psi_0\bar{\zeta}}{\bar{\psi}\zeta_0}} \pm \sqrt{\frac{\bar{\psi}\zeta_0}{\psi_0\bar{\zeta}}}\right), \tag{2.143}$$

$$r_1^{\pm}(\omega) = \frac{1}{2}\left(\sqrt{\frac{\bar{\psi}\zeta_1}{\psi_1\bar{\zeta}}} \pm \sqrt{\frac{\psi_1\bar{\zeta}}{\bar{\psi}\zeta_1}}\right). \tag{2.144}$$

Note that $\zeta = \sqrt{\psi\eta} = \sqrt{(i\omega/c_{44})(-i\omega\hat{\psi})} = \sqrt{\omega^2\hat{\psi}/c_{44}}$ is real and positive (provided that $\hat{\psi} = \rho\frac{\kappa_1^2 c_{66}^2}{\omega^2} + \frac{\kappa_1^2 c_{64}^2}{\omega^2 c_{44}} > 0$) and $\bar{\psi}/\psi_1 = \hat{\bar{\psi}}/\hat{\psi}_1$ is real and positive, hence $r_{0,1}^{\pm}(\omega)$ and $\hat{\mathbf{J}}_{0,1}$ are real. In the half-space $x_3 < 0$, $\hat{a}_0(\omega, 0) = \hat{f}(\omega)$ and in the half-space $x_3 > L$,

$\hat{b}_1(\omega, L) = 0$. This gives

$$\begin{bmatrix} \hat{a}(\omega,0) \\ \hat{b}(\omega,0) \end{bmatrix} = \hat{\mathbf{J}}_{\mathbf{0}}(\omega) \begin{bmatrix} \hat{f}(\omega) \\ \hat{b}_0(\omega,0) \end{bmatrix}, \quad \begin{bmatrix} \hat{a}_1(\omega,L) \\ 0 \end{bmatrix} = \hat{\mathbf{J}}_{\mathbf{1}}(\omega) \begin{bmatrix} \hat{a}(\omega,L) \\ \hat{b}(\omega,L) \end{bmatrix}. \tag{2.145}$$

By eliminating $\hat{b}_0(\omega, 0)$ and $\hat{a}_1(\omega, L)$, the boundary conditions for the forward and backward modes in equation (2.128), since $(r_0^+)^2 - (r_0^-)^2 = 1$ are

$$r_0^+(\omega)\hat{a}(\omega,0) - r_0^-(\omega)\hat{b}(\omega,0) = \hat{f}(\omega), \tag{2.146}$$

$$r_1^-(\omega)e^{i\bar{\lambda}^- L}\hat{a}(\omega,L) + r_1^+(\omega)e^{i\bar{\lambda}^+ L}\hat{b}(\omega,L) = 0. \tag{2.147}$$

Equations (2.146) and (2.147) can be rewritten as

$$\hat{a}(\omega,0) + R_0\hat{b}(\omega,0) = T_0\hat{f}(\omega), \tag{2.148}$$

$$R_1 e^{i\bar{\lambda}^- L}\hat{a}(\omega,L) - e^{i\bar{\lambda}^+ L}\hat{b}(\omega,L) = 0, \tag{2.149}$$

where the reflection and transmission coefficients at the interfaces $x_3 = 0$ and $x_3 = L$ are defined as

$$R_j = -\frac{r_j^{(-)}}{r_j^{(+)}}, \quad T_j = \frac{1}{r_j^{(+)}}, \quad j = 0, 1, \tag{2.150}$$

where R_j, T_j are real. A similar calculation to that used to derive equation (2.99) shows at each interface energy is conserved

$$R_j^2 + T_j^2 = 1, \quad j = 0, 1. \tag{2.151}$$

2.6.1 Deriving The Propagator Matrix

Suppose that

$$\begin{bmatrix} \chi_1(\omega,x_3) \\ \chi_3(\omega,x_3) \end{bmatrix} \text{ and } \begin{bmatrix} \chi_2(\omega,x_3) \\ \chi_4(\omega,x_3) \end{bmatrix}, \tag{2.152}$$

form two linearly independent solutions of equation (2.128), then the general solution with arbitrary constants c_1 and c_2 is given by

$$\begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = c_1 \begin{bmatrix} \chi_1(\omega, x_3) \\ \chi_3(\omega, x_3) \end{bmatrix} + c_2 \begin{bmatrix} \chi_2(\omega, x_3) \\ \chi_4(\omega, x_3) \end{bmatrix} = \begin{bmatrix} \chi_1(\omega, x_3) & \chi_2(\omega, x_3) \\ \chi_3(\omega, x_3) & \chi_4(\omega, x_3) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{2.153}$$

Hence, at $x_3 = 0$

$$\begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix} = \begin{bmatrix} \chi_1(\omega, 0) & \chi_2(\omega, 0) \\ \chi_3(\omega, 0) & \chi_4(\omega, 0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \tag{2.154}$$

Equation (2.153) can be written as

$$\begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = \begin{bmatrix} \chi_1(\omega, x_3) & \chi_2(\omega, x_3) \\ \chi_3(\omega, x_3) & \chi_4(\omega, x_3) \end{bmatrix} \begin{bmatrix} \chi_1(\omega, 0) & \chi_2(\omega, 0) \\ \chi_3(\omega, 0) & \chi_4(\omega, 0) \end{bmatrix}^{-1} \begin{bmatrix} \chi_1(\omega, 0) & \chi_2(\omega, 0) \\ \chi_3(\omega, 0) & \chi_4(\omega, 0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \tag{2.155}$$

and by defining the propagator matrix

$$\mathbf{P}_\omega(\omega, x_3) = \begin{bmatrix} \chi_1(\omega, x_3) & \chi_2(\omega, x_3) \\ \chi_3(\omega, x_3) & \chi_4(\omega, x_3) \end{bmatrix} \begin{bmatrix} \chi_1(\omega, 0) & \chi_2(\omega, 0) \\ \chi_3(\omega, 0) & \chi_4(\omega, 0) \end{bmatrix}^{-1}, \tag{2.156}$$

equation (2.155) can be rewritten as

$$\begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = \mathbf{P}(\omega, x_3) \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}. \tag{2.157}$$

Differentiating equation (2.157) with respect to x_3 gives

$$\frac{\partial}{\partial x_3} \begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = \frac{\partial \mathbf{P}(\omega, x_3)}{\partial x_3} \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}. \tag{2.158}$$

Using equation (2.128), then

$$\mathbf{H} \begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = \frac{\partial \mathbf{P}(\omega, x_3)}{\partial x_3} \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}. \tag{2.159}$$

and from equation (2.157) it follows that

$$\mathbf{HP}\begin{bmatrix}\hat{a}(\omega,0)\\ \hat{b}(\omega,0)\end{bmatrix} = \frac{\partial \mathbf{P}(\omega,x_3)}{\partial x_3}\begin{bmatrix}\hat{a}(\omega,0)\\ \hat{b}(\omega,0)\end{bmatrix}, \tag{2.160}$$

and so

$$\frac{\partial \mathbf{P}(\omega,x_3)}{\partial x_3} = \mathbf{HP}(\omega,x_3). \tag{2.161}$$

From equation (2.157) it can be deduced that $\mathbf{P}(\omega,0)=\mathbf{I}$, and from equation (2.156)

$$\mathbf{P}_\omega(\omega,x_3) = \begin{bmatrix}\chi_1(\omega,x_3) & \chi_2(\omega,x_3)\\ \chi_3(\omega,x_3) & \chi_4(\omega,x_3)\end{bmatrix}. \tag{2.162}$$

Such propagator matrices in the literature [26] have a symmetry with elements that are complex conjugate pairs. From equations (2.128) and (2.152) then

$$\frac{d\chi_1}{dx_3} = \Delta_1\chi_1 + \Delta_2\chi_3, \tag{2.163}$$
$$\frac{d\chi_3}{dx_3} = \Delta_3\chi_1 + \Delta_4\chi_3. \tag{2.164}$$

Suppose that the complex conjugates $(\overline{\chi}_3,\overline{\chi}_1)^T$ also satisfy equation (2.128), then

$$\frac{d\bar{\chi}_1}{dx_3} = \Delta_3\bar{\chi}_3 + \Delta_4\bar{\chi}_1, \tag{2.165}$$
$$\frac{d\bar{\chi}_3}{dx_3} = \Delta_1\bar{\chi}_3 + \Delta_2\bar{\chi}_1. \tag{2.166}$$

From equations (2.163) then, (2.165) holds if and only if

$$\overline{\Delta}_1\bar{\chi}_1 + \overline{\Delta}_2\bar{\chi}_3 = \Delta_3\bar{\chi}_3 + \Delta_4\bar{\chi}_1, \tag{2.167}$$

$$\overline{\Delta}_3\bar{\chi}_1 + \overline{\Delta}_4\bar{\chi}_3 = \Delta_1\bar{\chi}_3 + \Delta_2\bar{\chi}_1, \tag{2.168}$$

that is

$$\overline{\Delta}_2(x_3) = \Delta_3(x_3), \tag{2.169}$$

$$\overline{\Delta}_1(x_3) = \Delta_4(x_3). \tag{2.170}$$

The first condition holds from equation (2.133) but the latter condition is violated and so the symmetry cannot be exploited in the model equations.

2.6.2 Analytical Solution For the Propagator Equation

Equation (2.161) can be solved by recasting it in the form of a linear first-order differential system with variable coefficients

$$\begin{aligned}
\frac{\partial \chi_1(x_3)}{\partial x_3} &= \Delta_1(x_3)\chi_1(x_3) + \Delta_2(x_3)\chi_3(x_3), \\
\frac{\partial \chi_2(x_3)}{\partial x_3} &= \Delta_1(x_3)\chi_2(x_3) + \Delta_2(x_3)\chi_4(x_3), \\
\frac{\partial \chi_3(x_3)}{\partial x_3} &= \Delta_3(x_3)\chi_1(x_3) + \Delta_4(x_3)\chi_3(x_3), \\
\frac{\partial \chi_4(x_3)}{\partial x_3} &= \Delta_3(x_3)\chi_2(x_3) + \Delta_4(x_3)\chi_4(x_3).
\end{aligned} \tag{2.171}$$

This can be rewritten in matrix form as

$$\boldsymbol{\chi}'(x_3) = \mathbf{A}(x_3)\,\boldsymbol{\chi}(x_3), \quad \boldsymbol{\chi} = (\chi_1, \chi_2, \chi_3, \chi_4)^\top \tag{2.172}$$

where

$$\mathbf{A} = \begin{pmatrix} \Delta_1 & 0 & \Delta_2 & 0 \\ 0 & \Delta_1 & 0 & \Delta_2 \\ \Delta_3 & 0 & \Delta_4 & 0 \\ 0 & \Delta_3 & 0 & \Delta_4 \end{pmatrix}. \tag{2.173}$$

The solutions are given by

$$\boldsymbol{\chi}(x_3) = e^{\mathbf{B}(x_3)}\,\boldsymbol{\chi}_0, \quad \boldsymbol{\chi}_0 = (1, 0, 0, 1)^\top, \tag{2.174}$$

where the initial condition $\mathbf{P}(\omega, 0) = \mathbf{I}$ was used, $e^{\mathbf{B}(x_3)}$ denotes the matrix exponential of $\mathbf{B}(x_3) = \int_0^{x_3} \mathbf{A}(\varphi)\, \mathrm{d}\varphi$, and the matrix exponential is defined as

$$e^{\mathbf{B}(x_3)} = \sum_{k=0}^{\infty} \frac{(\mathbf{B}(x_3))^k}{k!}. \tag{2.175}$$

Writing

$$\mathbf{B} = \begin{pmatrix} d_1 & 0 & d_2 & 0 \\ 0 & d_1 & 0 & d_2 \\ d_3 & 0 & d_4 & 0 \\ 0 & d_3 & 0 & d_4 \end{pmatrix}, \qquad d_n(x_3) = \int_0^{x_3} \Delta_n(\varphi)\, \mathrm{d}\varphi\,, \tag{2.176}$$

the solutions χ_n are then found to be

$$\boldsymbol{\chi}(x_3) = \frac{e^{(d_1+d_4)/2}}{2\sqrt{\mathcal{D}}} \begin{pmatrix} (d_1 - d_4 + \sqrt{\mathcal{D}})e^{\sqrt{\mathcal{D}}/2} - (d_1 - d_4 - \sqrt{\mathcal{D}})e^{-\sqrt{\mathcal{D}}/2} \\ 2\,(e^{\sqrt{\mathcal{D}}/2} - e^{-\sqrt{\mathcal{D}}/2})\, d_2 \\ 2\,(e^{\sqrt{\mathcal{D}}/2} - e^{-\sqrt{\mathcal{D}}/2})\, d_3 \\ (d_1 - d_4 + \sqrt{\mathcal{D}})e^{-\sqrt{\mathcal{D}}/2} - (d_1 - d_4 - \sqrt{\mathcal{D}})e^{\sqrt{\mathcal{D}}/2} \end{pmatrix}, \tag{2.177}$$

where $\mathcal{D} = (d_1 - d_4)^2 + 4d_2d_3$. By inspection it can be seen that there is a lack of complex conjugate symmetry in the solutions. Furthermore, this solution to the propagator equation relates the reflection/transmission coefficients at different points in the random media, which can be seen in equation (2.399). The parametric dependencies contained within $\mathcal{D}$ reflect the material the wave is propagating through.

2.6.3 Solving the Propagator Equation Using Jacobi's Formula

Taking the determinant of the propagator matrix and equations (2.135), (2.161) and applying Jacobi's formula [61] gives

$$\frac{d(\det\{\mathbf{P}_\omega\})}{dx_3} = \mathrm{Tr}(\mathbf{H}_\omega)\det\{\mathbf{P}_\omega\} = 2i(\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}}) \times \det\{\mathbf{P}_\omega\}, \tag{2.178}$$

That is

$$\begin{aligned}\frac{d\ln(\det\{\mathbf{P}_\omega\})}{dx_3} &= 2i(\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}}) \\ &= 2iQ(x_3), \end{aligned} \tag{2.179}$$

where $Q(x_3)$ has been temporarily set to equal $\hat{\varsigma}(x_3) - \bar{\hat{\varsigma}}$. Note that from equations (2.135) and (2.161) $\bar{\hat{\varsigma}}$ is independent of x_3. Integrating between 0 and x_3 gives

$$\frac{\det\{\mathbf{P}_\omega(x_3)\}}{\det\{\mathbf{P}_\omega(0)\}} = \exp\left(2i\int_0^{x_3} Q(s)\,ds\right). \tag{2.180}$$

The initial condition $\mathbf{P}_\omega(x_3 = 0) = \mathbf{I}$ gives

$$\det\{\mathbf{P}_\omega(x_3)\} = \exp\left(2i\int_0^{x_3} Q(s)\,ds\right), \tag{2.181}$$

as the solution of equation (2.178). Note that $\det\{\mathbf{P}\}$ is not a constant and depends on x_3.

2.6.4 Wave Modes From the Propagator

From equation (2.157)

$$\begin{bmatrix}\hat{a}(\omega, L)\\ \hat{b}(\omega, L)\end{bmatrix} = \mathbf{P}_\omega(0, L)\begin{bmatrix}\hat{a}(\omega, 0)\\ \hat{b}(\omega, 0)\end{bmatrix}. \tag{2.182}$$

Using equation (2.162) the wave modes are

$$\hat{a}(\omega, L) = \chi_1\hat{a}(\omega, 0) + \chi_2\hat{b}(\omega, 0), \tag{2.183}$$

$$\hat{b}(\omega, L) = \chi_3\hat{a}(\omega, 0) + \chi_4\hat{b}(\omega, 0), \tag{2.184}$$

which can be rearranged to give

$$\hat{a}(\omega, 0) = \frac{\hat{b}(\omega, L)\chi_2 - \hat{a}(\omega, L)\chi_4}{\chi_2\chi_3 - \chi_1\chi_4}, \tag{2.185}$$

$$\hat{b}(\omega, 0) = \frac{\hat{a}(\omega, L)\chi_3 - \hat{b}(\omega, L)\chi_1}{\chi_2\chi_3 - \chi_1\chi_4}. \tag{2.186}$$

The boundary condition given by equation (2.149) can be written as

$$\hat{a}(\omega, L) = \frac{e^{i\bar{\lambda}^+ L}\hat{b}(\omega, L)}{R_1 e^{i\bar{\lambda}^- L}} = \frac{e^{2i\bar{\zeta}L}\hat{b}(\omega, L)}{R_1}, \tag{2.187}$$

Inserting equations (2.185) and (2.186) into the other boundary equation (2.148) gives

$$\frac{\hat{a}(\omega, L)(R_0\chi_3 - \chi_4) + \hat{b}(\omega, L)(\chi_2 - R_0\chi_1)}{\chi_2\chi_3 - \chi_1\chi_4} = T_0\hat{f}(\omega). \tag{2.188}$$

Inputting equation (2.187) into equation (2.188) and rearranging gives

$$\hat{a}(\omega, L) = \frac{e^{2i\bar{\zeta}L}(\chi_2\chi_3 - \chi_1\chi_4)T_0\hat{f}(\omega)}{e^{2i\bar{\zeta}L}(\chi_3 R_0 - \chi_4) + R_1(\chi_2 - \chi_1 R_0)}, \tag{2.189}$$

$$\hat{b}(\omega, L) = \frac{(\chi_2\chi_3 - \chi_1\chi_4)R_1 T_0\hat{f}(\omega)}{e^{2i\bar{\zeta}L}(\chi_3 R_0 - \chi_4) + R_1(\chi_2 - \chi_1 R_0)}. \tag{2.190}$$

Substituting equations (2.189) and (2.190) into equations (2.185) and (2.186) gives

$$\hat{a}(\omega, 0) = \frac{(e^{2i\bar{\zeta}L}\chi_4 - \chi_2 R_1)T_0\hat{f}(\omega)}{e^{2i\bar{\zeta}L}(\chi_4 - \chi_3 R_0) + R_1(\chi_1 R_0 - \chi_2)}, \tag{2.191}$$

$$\hat{b}(\omega, 0) = \frac{(e^{2i\bar{\zeta}L}\chi_3 - \chi_1 R_1)T_0\hat{f}(\omega)}{e^{2i\bar{\zeta}L}(\chi_3 R_0 - \chi_4) + R_1(\chi_2 - \chi_1 R_0)}. \tag{2.192}$$

This representation provides analytic expressions for the wave modes at the interfaces between the heterogeneous layer and the two half-spaces.

2.7 The Riccati Equation for Local Reflection Coefficient

Consider the local reflection coefficient of a layer of heterogeneous material occupying $0 \le x_3 \le L$, with a wave incident from the homogeneous half-space $x_3 < 0$, given by

$$\hat{R}(\omega, x_3) = \frac{\hat{b}(\omega, x_3)}{\hat{a}(\omega, x_3)}. \tag{2.193}$$

The functions $\hat{a}(\omega, x_3)$ and $\hat{b}(\omega, x_3)$ are the right and left-going wave modes respectively. The boundary condition given by equation (2.149) at $x_3 = L$ can then be written

$$\hat{R}(\omega, L) = R_1 e^{-2i\bar{\zeta}L}. \tag{2.194}$$

By differentiating equation (2.193) with respect to x_3 and using equation (2.128), it can be seen that the local reflection coefficient satisfies a Ricatti equation [62] for $x_3 \in [0, L]$

$$\begin{aligned}\frac{d\hat{R}}{dx_3} &= \frac{\hat{a}(\omega, x_3)(\partial(\hat{b}(\omega, x_3))/\partial x_3) - \hat{b}(\omega, x_3)(\partial(\hat{a}(\omega, x_3))/\partial x_3)}{\hat{a}^2(\omega, x_3)} \\ &= \hat{R}(\Delta_4(x_3) - \Delta_1(x_3)) - \hat{R}^2\Delta_2(x_3) + \Delta_3(x_3).\end{aligned} \tag{2.195}$$

From the boundary condition given by equation (2.148) at $x_3 = 0$, and equation (2.193), then

$$\hat{b}(\omega, 0) = \frac{T_0\hat{R}(\omega, 0)}{1 + R_0\hat{R}(\omega, 0)}\,\hat{f}(\omega), \tag{2.196}$$

and from equation (2.103)

$$\hat{b}_0(\omega, 0) = \hat{\mathcal{R}}(\omega)\hat{f}(\omega). \tag{2.197}$$

Now using the matrix equation (2.139) gives

$$\hat{\mathcal{R}}(\omega) = \frac{T_0\hat{b}(\omega, 0)}{\hat{f}(\omega)} + R_0, \tag{2.198}$$

which can be rewritten using equation (2.196) to obtain

$$\hat{\mathcal{R}}(\omega) = \frac{\hat{R}(\omega, 0)(T_0^2 + R_0^2) + R_0}{1 + R_0\hat{R}(\omega, 0)}. \tag{2.199}$$

Finally, since $T_0^2 + R_0^2 = 1$, then

$$\hat{\mathcal{R}}(\omega) = \frac{\hat{R}(\omega, 0) + R_0}{1 + R_0\hat{R}(\omega, 0)}, \tag{2.200}$$

which is the global reflection coefficient of the heterogeneous slab where R_0 is the local reflection coefficient at $x_3 = 0$. The Ricatti equation (2.195) is a nonlinear terminal value problem for the local reflection coefficient and presents an alternative method to obtain the global reflection and transmission coefficients of the layer, replacing the linear two-point boundary value problem given by equations (2.128), (2.148) and (2.149). This is a backward Ricatti equation which must be solved in reverse from $x_3 = L$ to $x_3 = 0$, starting from the terminal condition given by equation (2.194), and this presents a problem later in the stochastic case. One way to circumvent this (as outlined in [26]) is to consider reflection from waves incident from a homogeneous half-space on the right of the slab. The local reflection coefficient will therefore satisfy a forward Ricatti equation with a terminal condition at $x_3 = -L$.

The local reflection coefficient can be found by solving equation (2.195) subject to the boundary condition (2.194). A nonlinear terminal value problem for the local transmission coefficient can also be derived to give an analytical expression for the transmitted wave. Define the local transmission coefficient for the random medium as

$$\hat{T}(\omega, x_3) = \frac{T_1 \hat{a}(\omega, L)}{\hat{a}(\omega, x_3)}, \quad 0 \leq x_3 \leq L. \tag{2.201}$$

Differentiating with respect to x_3 and using equation (2.128) gives

$$\begin{aligned} \frac{d\hat{T}}{dx_3} &= \frac{-T_1 \hat{a}(\omega, L)}{\hat{a}^2(\omega, x_3)} \frac{\partial}{\partial x_3}\Big(\hat{a}(\omega, x_3)\Big) \\ &= -\hat{T}(\omega, x_3)\Big(\Delta_1(x_3) + \Delta_2(x_3)\hat{R}(\omega, x_3)\Big), \end{aligned} \tag{2.202}$$

which forms a Ricatti equation in the interval $x_3 \in [0, L]$ for the local transmission coefficient, with terminal condition at $x_3 = L$

$$\hat{T}(\omega, L) = T_1. \tag{2.203}$$

The boundary condition given by equation (2.148) at the interface $x_3 = 0$ and equation (2.201) gives

$$\frac{T_1 \hat{a}(\omega, L)}{\hat{T}(\omega, 0)} + R_0 \hat{b}(\omega, 0) = T_0 \hat{f}(\omega). \tag{2.204}$$

Combining the local reflection coefficient given by equation (2.193) and the local transmission coefficient given by equation (2.201) at $x_3 = 0$ gives

$$\hat{R}(\omega, 0) = \frac{\hat{b}(\omega, 0)}{\hat{a}(\omega, 0)} = \frac{\hat{b}(\omega, 0)\hat{T}(\omega, 0)}{T_1 \hat{a}(\omega, L)}. \tag{2.205}$$

Inserting (2.205) in equation (2.204) gives

$$\frac{T_1 \hat{a}(\omega, L)}{\hat{T}(\omega, 0)} + \frac{R_0 T_1 \hat{R}(\omega, 0)\hat{a}(\omega, L)}{\hat{T}(\omega, 0)} = T_0 \hat{f}(\omega). \tag{2.206}$$

An expression for the forward mode at $x_3 = L$ in the medium is then given by

$$\hat{a}(\omega, L) = \frac{T_0 \hat{T}(\omega, 0)}{T_1 + R_0 T_1 \hat{R}(\omega, 0)} \hat{f}(\omega). \tag{2.207}$$

Using matrix equation (2.145) and rearranging for the transmitted wave gives

$$\hat{a}_1(\omega, L) = \left(\frac{(r_1^+)^2 - (r_1^-)^2}{r_1^+} \right) e^{-\bar{\lambda} L} \hat{a}(\omega, L) = T_1 e^{-i\bar{\lambda} L} \hat{a}(\omega, L). \tag{2.208}$$

The transmitted wave is therefore $\hat{a}_1(\omega, L) = \hat{\mathcal{T}}(\omega)\hat{f}(\omega)$, which gives

$$\hat{\mathcal{T}}(\omega) = \frac{T_0 \hat{T}(\omega, 0)}{1 + R_0 \hat{R}(\omega, 0)} e^{-i\bar{\lambda} L}, \tag{2.209}$$

where $\hat{\mathcal{T}}$ is the global transmission coefficient for the slab.

2.7.1 Energy Conservation

An expression for the energy in the layer in terms of the local reflection and transmission coefficients is given by $|\hat{R}(\omega, x_3)|^2 + |\hat{T}(\omega, x_3)|^2$. Taking the derivative with respect to x_3 gives

$$\frac{d}{dx_3}\left(|\hat{R}(\omega, x_3)|^2 + |\hat{T}(\omega, x_3)|^2\right) = \frac{d\hat{R}}{dx_3}\overline{\hat{R}} + \frac{d\overline{\hat{R}}}{dx_3}\hat{R} + \frac{d\hat{T}}{dx_3}\overline{\hat{T}} + \frac{d\overline{\hat{T}}}{dx_3}\hat{T}. \tag{2.210}$$

For simplicity, define the energy by $Q(x_3)$. From equations (2.195) and (2.202), its derivative can be rewritten as

$$
\begin{aligned}
\frac{d}{dx_3}(Q(x_3)) &= \hat{R}\overline{\hat{R}}(\Delta_4 - \Delta_1) - \hat{R}^2\overline{\hat{R}}\Delta_2 + \overline{\hat{R}}\Delta_3 + \hat{R}\overline{\hat{R}}(\overline{\Delta}_4 - \overline{\Delta}_1) - \overline{\hat{R}}^2\hat{R}\overline{\Delta}_2 + \hat{R}\overline{\Delta}_3 \\
&\quad - \hat{T}\overline{\hat{T}}(\Delta_1 + \Delta_2\hat{R}) - \hat{T}\overline{\hat{T}}(\overline{\Delta}_1 + \overline{\Delta}_2\overline{\hat{R}}), \\
&= |\hat{R}|^2(\Delta_4 - \Delta_1) - \hat{R}|\hat{R}|^2\Delta_2 + \overline{\hat{R}}\Delta_3 + |\hat{R}|^2(\overline{\Delta}_4 - \overline{\Delta}_1) - \overline{\hat{R}}|\hat{R}|^2\overline{\Delta}_2 + \hat{R}\overline{\Delta}_3 \\
&\quad - |\hat{T}|^2(\Delta_1 + \Delta_2\hat{R}) - |\hat{T}|^2(\overline{\Delta}_1 + \overline{\Delta}_2\overline{\hat{R}}), \\
&= |\hat{R}|^2(\Delta_4 + \overline{\Delta}_4 - (\Delta_1 + \overline{\Delta_1})) - |\hat{R}|^2(\hat{R}\Delta_2 + \overline{\hat{R}}\overline{\Delta}_2) + \overline{\hat{R}}\Delta_3 + \hat{R}\overline{\Delta}_3 \\
&\quad - |\hat{T}|^2((\Delta_1 + \overline{\Delta}_1) + \Delta_2\hat{R} + \overline{\Delta}_2\overline{\hat{R}}).
\end{aligned} \tag{2.211}
$$

Recall from equations (2.129) and (2.132) that $\Delta_1 = -\overline{\Delta}_1$ and $\Delta_4 = -\overline{\Delta}_4$, so

$$
\frac{dQ(x_3)}{dx_3} = \hat{R}\left(\overline{\Delta}_3 - \Delta_2\left(|\hat{R}|^2 + |\hat{T}|^2\right)\right) + \overline{\hat{R}}\left(\Delta_3 - \overline{\Delta}_2\left(|\hat{R}|^2 + |\hat{T}|^2\right)\right). \tag{2.212}
$$

From equation (2.133), $\overline{\Delta}_2 = \Delta_3$ so

$$
\begin{aligned}
\frac{dQ(x_3)}{dx_3} &= \hat{R}\overline{\Delta}_3\left(1 - \left(|\hat{R}|^2 + |\hat{T}|^2\right)\right) + \overline{\hat{R}}\Delta_3\left(1 - \left(|\hat{R}|^2 + |\hat{T}|^2\right)\right) \\
&= \left(1 - Q(x_3)\right)\left(\hat{R}\overline{\Delta}_3 + \overline{\hat{R}}\Delta_3\right).
\end{aligned} \tag{2.213}
$$

Defining

$$
F(x_3) = -\left(\hat{R}\overline{\Delta}_3 + \overline{\hat{R}}\Delta_3\right), \tag{2.214}
$$

then

$$
\frac{dQ(x_3)}{dx_3} = \left(Q(x_3) - 1\right)F(x_3), \tag{2.215}
$$

and hence

$$
Q(x_3) = 1 + A\exp\left\{\int^{x_3} F(s)\, ds\right\}. \tag{2.216}
$$

From equations (2.194) and (2.203)

$$
Q(L) = |\hat{R}(\omega, L)|^2 + |\hat{T}(\omega, L)|^2 = |R_1 e^{-2i\bar{\zeta}L}|^2 + |T_1|^2 = 1, \tag{2.217}
$$

by equation (2.151). Hence $A = 0$ and so

$$Q(x_3) = |\hat{R}(\omega, x_3)|^2 + |\hat{T}(\omega, x_3)|^2 = 1, \quad 0 \leq x_3 \leq L. \tag{2.218}$$

The global energy can be expressed using equations (2.200) and (2.209) as

$$\begin{aligned}
|\hat{\mathcal{R}}(\omega)|^2 + |\hat{\mathcal{T}}(\omega)|^2 &= \left| \frac{R_0 + \hat{R}(\omega,0)}{1 + R_0\hat{R}(\omega,0)} \right|^2 + \left| \frac{T_0\hat{T}(\omega,0)}{1 + R_0\hat{R}(\omega,0)} e^{-i\bar{\lambda}L} \right|^2 \\
&= \left(\frac{R_0 + \hat{R}(\omega,0)}{1 + R_0\hat{R}(\omega,0)} \right) \left(\frac{R_0 + \overline{\hat{R}}(\omega,0)}{1 + R_0\overline{\hat{R}}(\omega,0)} \right) \\
&+ \left(\frac{T_0\hat{T}(\omega,0)}{1 + R_0\hat{R}(\omega,0)} \right) \left(\frac{T_0\overline{\hat{T}}(\omega,0)}{1 + R_0\overline{\hat{R}}(\omega,0)} \right) \\
&= \frac{R_0^2 + R_0(\hat{R}(\omega,0) + \overline{\hat{R}}(\omega,0)) + |\hat{R}(\omega,0)|^2 + T_0^2|\hat{T}(\omega,0)|^2}{(1 + R_0\hat{R}(\omega,0))(1 + R_0\overline{\hat{R}}(\omega,0))} \\
&= \frac{R_0^2 + R_0(\hat{R}(\omega,0) + \overline{\hat{R}}(\omega,0)) + |\hat{R}(\omega,0)|^2 + (1 - R_0^2)(1 - |\hat{R}(\omega,0)|^2)}{(1 + R_0\hat{R}(\omega,0))(1 + R_0\overline{\hat{R}}(\omega,0))},
\end{aligned}$$

since $R_0, T_0 \in \mathbb{R}$. Hence

$$\begin{aligned}
|\hat{\mathcal{R}}(\omega)|^2 + |\hat{\mathcal{T}}(\omega)|^2 &= \frac{1 + R_0\hat{R}(\omega,0) + R_0\overline{\hat{R}}(\omega,0) + (R_0\hat{R}(\omega,0))(R_0\overline{\hat{R}}(\omega,0))}{(1 + R_0\hat{R}(\omega,0))(1 + R_0\overline{\hat{R}}(\omega,0))} \\
&= \frac{(1 + R_0\hat{R}(\omega,0))(1 + R_0\overline{\hat{R}}(\omega,0))}{(1 + R_0\hat{R}(\omega,0))(1 + R_0\overline{\hat{R}}(\omega,0))} \\
&= 1.
\end{aligned} \tag{2.219}$$

So the energy is conserved and this also implies that the complex-valued global reflection and transmission coefficients are uniformly bounded in absolute value by one.

2.8 Randomly Layered Anisotropic Medium

Assume that the local orientation of the slowness surface $\theta(x_3)$ varies randomly over the interval $x_3 \in [0, L]$ according to the additive noise formulation

$$\theta(x_3) = \bar{\theta} + \sigma m(x_3/l), \quad x_3 \in [0, L], \tag{2.220}$$

where $\bar{\theta} \sim 1$ is the mean angle and $m(x_3/l)$ is a stationary stochastic process (an ergodic Markov process on a compact state space) with mean zero. The random process $m(x_3/l)$ takes values in R^1, on an interval which is closed and bounded. From equations (2.4) and (2.5) the stress tensor components can be rewritten

$$c_{66} = \sin^2(\bar{\theta} + \sigma m(x_3/l))F + \cos^2(\bar{\theta} + \sigma m(x_3/l))N, \tag{2.221}$$
$$c_{44} = \sin^2(\bar{\theta} + \sigma m(x_3/l))N + \cos^2(\bar{\theta} + \sigma m(x_3/l))F, \tag{2.222}$$

and

$$c_{64} = \sin(\bar{\theta} + \sigma m(x_3/l))\cos(\bar{\theta} + \sigma m(x_3/l))(F - N), \tag{2.223}$$

where $N = \bar{\bar{c}}_{66}$ and $F = \bar{\bar{c}}_{44}$ when $\bar{\theta} = 0$. Taking a Taylor series in σ gives

$$c_{66} = \bar{c}_{66}(1 + \sigma_{66} m(x_3/l)) + \mathcal{O}(\sigma^2), \tag{2.224}$$

where

$$\bar{c}_{66} = N\cos^2\bar{\theta} + F\sin^2\bar{\theta}, \tag{2.225}$$
$$\sigma_{66} = \frac{\sigma \sin 2\bar{\theta}(F - N)}{\bar{c}_{66}}, \tag{2.226}$$

and

$$c_{44} = \bar{c}_{44}(1 + \sigma_{44} m(x_3/l)) + \mathcal{O}(\sigma^2), \tag{2.227}$$

where

$$\bar{c}_{44} = N \sin^2 \bar{\theta} + F \cos^2 \bar{\theta}, \tag{2.228}$$

$$\sigma_{44} = \frac{\sigma \sin 2\bar{\theta}(N - F)}{\bar{c}_{44}}, \tag{2.229}$$

and

$$c_{64} = \bar{c}_{64}(1 + \sigma_{64} m(x_3/l)) + \mathcal{O}(\sigma^2), \tag{2.230}$$

where

$$\bar{c}_{64} = \frac{(F - N)\sin 2\bar{\theta}}{2}, \tag{2.231}$$

$$\sigma_{64} = 2\sigma \cot 2\bar{\theta}. \tag{2.232}$$

Equation (2.227) (to order σ) gives

$$\frac{1}{c_{44}} = \frac{1 - \sigma_{44} m(x_3/l)}{\bar{c}_{44}} + \mathcal{O}(\sigma^2). \tag{2.233}$$

Equation (2.34) becomes

$$\begin{aligned} \varsigma &= \frac{i\kappa_1 \bar{c}_{64}}{\bar{c}_{44}}\Big(1 + \sigma_{64} m(x_3/l)\Big)\Big(1 - \sigma_{44} m(x_3/l)\Big) \\ &= \frac{i\kappa_1 \bar{c}_{64}}{\bar{c}_{44}}\Big(1 + (\sigma_{64} - \sigma_{44}) m(x_3/l)\Big) + \mathcal{O}(\sigma^2) \\ &= \bar{\hat{c}}_{64}(1 + \hat{\sigma}_{64} m(x_3/l)). \end{aligned} \tag{2.234}$$

where

$$\bar{\hat{c}}_{64} = \frac{i\kappa_1 \bar{c}_{64}}{\bar{c}_{44}}, \tag{2.235}$$

$$\hat{\sigma}_{64} = \sigma_{64} - \sigma_{44} = \sigma\left(2\cot 2\bar{\theta} + \frac{\sin 2\bar{\theta}(F - N)}{\bar{c}_{44}}\right) = \sigma \Gamma_\alpha, \tag{2.236}$$

and

$$\Gamma_\alpha = 2\cot 2\bar{\theta} + \frac{\sin 2\bar{\theta}(F-N)}{\bar{c}_{44}}. \tag{2.237}$$

Equation (2.37) can now be written (to order σ) as

$$\eta = \bar{\hat{c}}_{44}(1+\hat{\sigma}_{44}m(x_3/l)), \tag{2.238}$$

where

$$\bar{\hat{c}}_{44} = \frac{i\omega}{\bar{c}_{44}}, \tag{2.239}$$

and

$$\hat{\sigma}_{44} = -\sigma_{44} = \sigma\Gamma_\beta. \tag{2.240}$$

Equation (2.32) can now be rewritten (to order σ) using equation (2.230)

$$\begin{aligned}
\psi(\omega,\kappa_1) &= -i\omega\left(\rho - \frac{\kappa_1^2 c_{66}}{\omega^2} + \frac{\kappa_1^2 c_{64}^2}{\omega^2 c_{44}}\right) \\
&= -i\omega\left(\rho + \frac{\kappa_1^2}{\omega^2}\left(\frac{c_{64}^2}{c_{44}} - c_{66}\right)\right) \\
&= -i\omega\left(\rho + \frac{\kappa_1^2}{\omega^2}\left(\frac{\bar{c}_{64}^2}{\bar{c}_{44}}(1+\sigma_{64}m)^2(1-\sigma_{44}m) - \bar{c}_{66}(1+\sigma_{66}m)\right)\right) + \mathcal{O}(\sigma^2) \\
&= -i\omega\left(\rho + \frac{\kappa_1^2}{\omega^2}\left(\frac{\bar{c}_{64}^2}{\bar{c}_{44}}(1+(2\sigma_{64}-\sigma_{44})m) - \bar{c}_{66}(1+\sigma_{66}m)\right)\right) \\
&= \bar{\hat{\psi}}(1+\sigma_\psi m),
\end{aligned} \tag{2.241}$$

where

$$\bar{\hat{\psi}}(\omega,\kappa_1) = \frac{-i(\bar{c}_{64}^2 - \bar{c}_{66}\bar{c}_{44})\kappa_1^2 - i\rho\omega^2\bar{c}_{44}}{\bar{c}_{44}\omega}, \tag{2.242}$$

$$\sigma_\psi(\omega,\kappa_1) = \frac{\kappa_1^2(\bar{c}_{64}^2(2\sigma_{64}-\sigma_{44}) - \bar{c}_{44}\bar{c}_{66}\sigma_{66})}{(\bar{c}_{64}^2 - \bar{c}_{44}\bar{c}_{66})\kappa_1^2 + \rho\omega^2\bar{c}_{44}}. \tag{2.243}$$

Using equations (2.226), (2.229) and (2.232)

$$\sigma_\psi = \sigma \Gamma_\gamma, \tag{2.244}$$

where

$$\Gamma_\gamma = \frac{\kappa_1^2(\bar{c}_{64}^2(4\cot 2\bar{\theta} + \sin 2\bar{\theta}(F-N)/\bar{c}_{44}) - \bar{c}_{44}\sin 2\bar{\theta}(F-N))}{(\bar{c}_{64}^2 - \bar{c}_{44}\bar{c}_{66})\kappa_1^2 + \rho\omega^2\bar{c}_{44}}. \tag{2.245}$$

2.8.1 Weakly Heterogeneous Scaling Regime

Denote the wavelength by λ_3, the wave propagation distance by L_3 and the mean layer size in the media by l. In what follows $L_3 \gg \lambda_3 \sim l$ and the amplitude of the fluctuations in the spatially varying material properties $0 < \sigma \ll 1$; this is the so called weakly heterogeneous regime [26]. This setting causes fluctuations to build up behind the transmitted wave, producing an incoherent coda wave containing the bulk of the wave energy. Now non-dimensionalise the governing equations via the transformations

$$\tilde{x}_3 = \frac{x_3}{L_3}, \quad \tilde{\omega} = \frac{L_3\,\omega}{c_3}, \quad \text{and} \quad \tilde{\kappa}_1 = \kappa_1 L_3, \tag{2.246}$$

where c_3 is the mean shear wave speed in the x_3 direction. $\tilde{x}_3$ can be interpreted as a ratio of distances in the propagation direction, $\tilde{\omega}$ as a ratio of the propagation distance to the typical wavelength in the propagation direction and $\tilde{\kappa}_1$ as a ratio of propagation distance per wavelength in the x_1 direction. Define two dimensionless parameters ε and ϖ to capture the length scale differences via

$$0 < \epsilon \ll 1, \quad \frac{L_3}{l} = \frac{1}{\varepsilon^2}, \quad \tilde{\omega} = \frac{\omega L_3}{c_3} = \frac{\varpi}{\varepsilon}. \tag{2.247}$$

The ratio ϖ/ε is the propagation distance measured in units of wavelength. These relations can be combined to write

$$\varepsilon = \sqrt{\frac{l}{L_3}}, \quad \text{and} \quad \varpi = \frac{\omega}{c_3}\sqrt{lL_3}. \tag{2.248}$$

The non-dimensional velocity and stress fields take the form

$$\tilde{\xi}(\tilde{\omega}, \tilde{\kappa_1}, \tilde{x_3}) = \frac{1}{c_3}\hat{\xi}\left(\frac{c_3\tilde{\omega}}{L_3}, \frac{\omega_3\tilde{\kappa}_1}{c_3}, L_3\tilde{x}_3\right), \quad \tilde{\tau}(\tilde{\omega}, \tilde{\kappa_1}, \tilde{x_3}) = \frac{1}{\rho c_3^2}\hat{\tau}\left(\frac{c_3\tilde{\omega}}{L_3}, \frac{\omega_3\tilde{\kappa}_1}{c_3}, L_3\tilde{x}_3\right). \tag{2.249}$$

The non-dimensional stress and velocity equations are then (from equations (2.31), (2.36), (2.238) and (2.241))

$$\tilde{\xi}_{,3} = L_3\bar{\hat{c}}_{64}\left(1 + \hat{\sigma}_{64} m\left(\frac{\tilde{x}_3}{\varepsilon^2}\right)\right)\tilde{\xi} - \rho c_3 L_3\bar{\hat{c}}_{44}\left(1 + \hat{\sigma}_{44} m\left(\frac{\tilde{x}_3}{\varepsilon^2}\right)\right)\tilde{\tau}, \tag{2.250}$$

$$\tilde{\tau}_{,3} = \frac{L_3}{\rho c_3}\bar{\hat{\psi}}\left(1 + \hat{\sigma}_{\psi} m\left(\frac{\tilde{x}_3}{\varepsilon^2}\right)\right)\tilde{\xi} + L_3\bar{\hat{c}}_{64}\left(1 + \hat{\sigma}_{64} m\left(\frac{\tilde{x}_3}{\varepsilon^2}\right)\right)\tilde{\tau}. \tag{2.251}$$

These equations have three non-dimensional lumped parameters. From equation (2.235)

$$L_3\bar{\hat{c}}_{64} = i\frac{\kappa_1 L_3 \bar{c}_{64}}{\bar{c}_{44}} = i\tilde{\kappa}_1\frac{\tilde{c}_{64}}{\tilde{c}_{44}}, \tag{2.252}$$

where

$$\tilde{c}_{64} = \frac{\bar{c}_{64}}{\rho c_3^2}, \quad \text{and} \quad \tilde{c}_{44} = \frac{\bar{c}_{44}}{\rho c_3^2}. \tag{2.253}$$

Also, from equations (2.239) and (2.246)

$$\rho c_3 L_3\bar{\hat{c}}_{44} = \rho c_3 L_3\frac{i\omega}{\bar{c}_{44}} = i\frac{\rho c_3^2}{\bar{c}_{44}}\frac{\omega L_3}{c_3} = \frac{i\tilde{\omega}}{\tilde{c}_{44}}. \tag{2.254}$$

Finally, from equation (2.242)

$$\frac{L_3}{\rho c_3}\bar{\hat{\psi}} = -i\tilde{\omega}\left(1 + \frac{\tilde{\kappa}_1^2}{\tilde{\omega}^2}\left(\frac{\tilde{c}_{64}^2}{\tilde{c}_{44}} - \tilde{c}_{66}\right)\right), \tag{2.255}$$

where

$$\tilde{c}_{66} = \frac{\bar{c}_{66}}{\rho c_3^2}. \tag{2.256}$$

Equations (2.250) and (2.251) then become

$$\tilde{\xi}_{,3} = i\tilde{\kappa}_1 \frac{\tilde{c}_{64}}{\tilde{c}_{44}} \left(1 + \hat{\sigma}_{64} m\left(\frac{\tilde{x}_3}{\varepsilon^2}\right)\right)\tilde{\xi} + i\tilde{\omega}\left(\frac{-1}{\tilde{c}_{44}}\right)\left(1 + \hat{\sigma}_{44} m\left(\frac{\tilde{x}_3}{\varepsilon^2}\right)\right)\tilde{\tau}, \tag{2.257}$$

$$\tilde{\tau}_{,3} = -i\tilde{\omega}\left(1 + \frac{\tilde{\kappa}_1^2}{\tilde{\omega}^2}\left(\frac{\tilde{c}_{64}^2}{\tilde{c}_{44}} - \tilde{c}_{66}\right)\right)\left(1 + \hat{\sigma}_{\psi} m\left(\frac{L_3}{l}\tilde{x}_3\right)\right)\tilde{\xi} + i\tilde{\kappa}_1 \frac{\tilde{c}_{64}}{\tilde{c}_{44}}\left(1 + \hat{\sigma}_{64} m\left(\frac{\tilde{x}_3}{\varepsilon^2}\right)\right)\tilde{\tau}, \tag{2.258}$$

where $\tilde{c}_{44}, \tilde{c}_{64}, \tilde{c}_{66} \sim 1$. The prefactor in equation (2.258) is

$$\frac{\tilde{\kappa}_1}{\tilde{\omega}} = \frac{\kappa_1 L_3}{(L_3\omega/c_3)} = \frac{\kappa_1}{\kappa_3} = \nu, \text{ say} \tag{2.259}$$

the ratio of wave numbers in the (x_1, x_3) directions. For a monochromatic wave the ratio of wave-numbers is equal to the ratio of slowness (the inverse of the phase velocity) values which in turn (for constant density materials) is the degree of anisotropy of the medium as governed by the stiffness tensor (equation (2.6)). As the crystal orientation θ changes, the phase velocities in the x_1 and x_3 direction change and hence, the wave-numbers in these directions change commensurately. In Figure 2.5 a plot of the ratio of the wave-numbers (ν) versus the crystal orientation is shown. Hence, ν is related to the degree of anisotropy of the material in the (x_1, x_3) plane and hence to the degree of asphericity of the associated slowness surface.

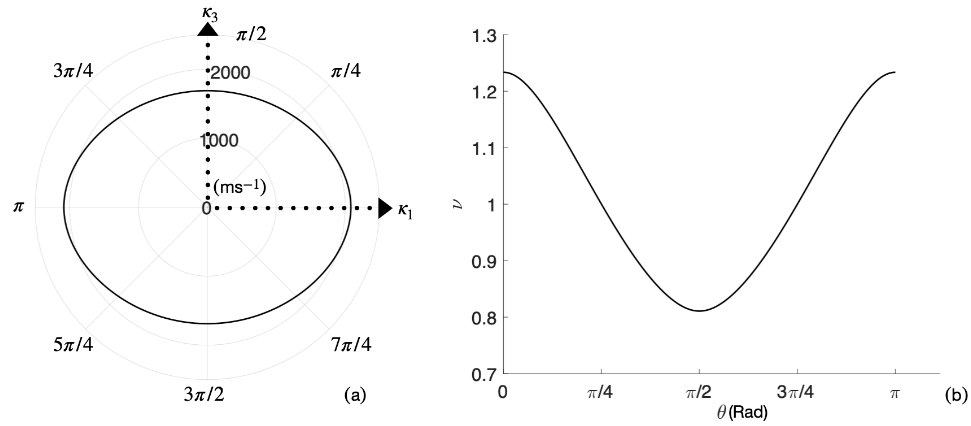

Figure 2.5: A wavenumber surface plot for austenitic steel showing its aspherical nature due to the anisotropic stiffness tensor. As the surface rotates through θ radians, the ratio of the wavenumbers κ_1 and κ_3 vary (b). Plot of ν (equation (2.259)) versus the rotation of the stiffness tensor θ using values from Figure 2.5 (b).

Using equations (2.257), (2.258), (2.247) and (2.259) gives

$$\frac{\partial}{\partial \tilde{x}_3}\begin{bmatrix}\tilde{\xi}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\\ \tilde{\tau}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\end{bmatrix} = i\frac{\varpi}{\varepsilon}\begin{bmatrix}\tilde{\alpha}(1+\sigma\Gamma_\alpha m(\tilde{x}_3/\varepsilon^2)) & \tilde{\beta}(1+\sigma\Gamma_\beta m(\tilde{x}_3/\varepsilon^2))\\ \tilde{\gamma}(1+\sigma\Gamma_\gamma m(\tilde{x}_3/\varepsilon^2)) & \tilde{\alpha}(1+\sigma\Gamma_\alpha m(\tilde{x}_3/\varepsilon^2))\end{bmatrix}\begin{bmatrix}\tilde{\xi}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\\ \tilde{\tau}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\end{bmatrix}, \tag{2.260}$$

where the new non-dimensional variables are defined as

$$\tilde{\alpha} = \nu\frac{\tilde{c}_{64}}{\tilde{c}_{44}}, \tag{2.261}$$

$$\tilde{\beta} = -\frac{1}{\tilde{c}_{44}}, \tag{2.262}$$

$$\tilde{\gamma} = -\left(1+\nu^2\left(\frac{\tilde{c}_{64}^2}{\tilde{c}_{44}} - \tilde{c}_{66}\right)\right), \tag{2.263}$$

where equation (2.245) now becomes

$$\Gamma_\gamma = \frac{\nu^2(\sin 2\bar{\theta}(F-N)(\bar{c}_{44}^2 - \bar{c}_{64}^2) - 4\bar{c}_{44}\bar{c}_{64}^2\cot 2\bar{\theta})}{\bar{c}_{44}(\nu^2(\bar{c}_{66}\bar{c}_{44} - \bar{c}_{64}^2) - \bar{c}_{44}\rho c_3^2)}, \tag{2.264}$$

and where $\tilde{\alpha}$, $\tilde{\beta}$ and $\tilde{\gamma}$ are $\mathcal{O}(1)$. From now on the tildes are dropped and temporarily setting $m(x_3/\varepsilon^2) = 0$ in equation (2.260) gives

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{\xi}(\nu,x_3)\\ \hat{\tau}(\nu,x_3)\end{bmatrix} = i\frac{\varpi}{\varepsilon}\begin{bmatrix}\alpha & \beta\\ \gamma & \alpha\end{bmatrix}\begin{bmatrix}\hat{\xi}(\nu,x_3)\\ \hat{\tau}(\nu,x_3)\end{bmatrix} = \mathbf{M}\begin{bmatrix}\hat{\xi}(\nu,x_3)\\ \hat{\tau}(\nu,x_3)\end{bmatrix}. \tag{2.265}$$

The eigenvalues of $\mathbf{M}$ are $\Lambda^{\pm} = i\omega/\varepsilon(\alpha \pm \sqrt{\gamma\beta})$ and the eigenvectors are $[\sqrt{\beta\gamma},\, \gamma]^T$ and $[-\sqrt{\beta\gamma},\, \gamma]^T$. Let

$$\phi = \sqrt{\beta\gamma} = \sqrt{\frac{1+\nu^2\left(\frac{\tilde{c}_{64}^2}{\tilde{c}_{44}} - \tilde{c}_{66}\right)}{\tilde{c}_{44}}}, \tag{2.266}$$

Now consider the following ansatz for the generated right-moving modes $\hat{a}^{\varepsilon}(\nu,x_3)$ and left-moving modes $\hat{b}^{\varepsilon}(\nu,x_3)$

$$\begin{bmatrix}\hat{\xi}(\nu,x_3)\\ \hat{\tau}(\nu,x_3)\end{bmatrix} = \begin{bmatrix}\sqrt{\zeta/\gamma} & -\sqrt{\zeta/\gamma}\\ \sqrt{\gamma/\zeta} & \sqrt{\gamma/\zeta}\end{bmatrix}\begin{bmatrix}\hat{b}^{\varepsilon}(\nu,x_3)\\ \hat{a}^{\varepsilon}(\nu,x_3)\end{bmatrix}, \tag{2.267}$$

where

$$\zeta = \sqrt{\alpha\gamma}, \tag{2.268}$$

and

$$\hat{a}^{\varepsilon}(\nu,x_3) = \hat{a}^{\varepsilon}(\nu,x_3)e^{\Lambda^{(-)}x_3}, \tag{2.269}$$

$$\hat{b}^{\varepsilon}(\nu,x_3) = \hat{b}^{\varepsilon}(\nu,x_3)e^{\Lambda^{(+)}x_3}. \tag{2.270}$$

This produces a new system whose coefficients are zero centered random variables

$$\hat{a}^{\varepsilon}(\nu,x_3) = \frac{1}{2}\left(\sqrt{\zeta/\gamma}\,\hat{\tau}(\nu,x_3) - \sqrt{\gamma/\zeta}\,\hat{\xi}(\nu,x_3)\right)e^{-\Lambda^{(-)}x_3}, \tag{2.271}$$

$$\hat{b}^{\varepsilon}(\nu,x_3) = \frac{1}{2}\left(\sqrt{\zeta/\gamma}\,\hat{\tau}(\nu,x_3) + \sqrt{\gamma/\zeta}\,\hat{\xi}(\nu,x_3)\right)e^{-\Lambda^{(+)}x_3}. \tag{2.272}$$

Taking partial derivatives in x_3 gives

$$\frac{\partial\hat{a}^{\varepsilon}}{\partial x_3}(\nu,x_3) = \frac{i\varpi}{2\varepsilon}\left(\Delta_1(\nu,x_3/\varepsilon^2)\hat{a}^{\varepsilon}(\nu,x_3) + \Delta_2(\nu,x_3/\varepsilon^2)\hat{b}^{\varepsilon}(\nu,x_3)\right), \tag{2.273}$$

and

$$\frac{\partial \hat{b}^\varepsilon}{\partial x_3}(\nu, x_3) = \frac{i\varpi}{2\varepsilon}\bigg(\Delta_3(\nu, x_3/\varepsilon^2)\hat{a}^\varepsilon(\nu, x_3) + \Delta_4(\nu, x_3/\varepsilon^2)\hat{b}^\varepsilon(\nu, x_3)\bigg), \tag{2.274}$$

where

$$\Delta_1(\nu, x_3/\varepsilon^2) = \sigma m(x_3/\varepsilon^2)\delta_1(\nu), \tag{2.275}$$

$$\Delta_2(\nu, x_3/\varepsilon^2) = \sigma m(x_3/\varepsilon^2)\delta_2(\nu)e^{\frac{2i\varpi}{\varepsilon}\phi x_3}, \tag{2.276}$$

$$\Delta_3(\nu, x_3/\varepsilon^2) = -\sigma m(x_3/\varepsilon^2)\delta_2(\nu)e^{-\frac{2i\varpi}{\varepsilon}\phi x_3}, \tag{2.277}$$

$$\Delta_4(\nu, x_3/\varepsilon^2) = \sigma m(x_3/\varepsilon^2)\delta_4(\nu), \tag{2.278}$$

and

$$\delta_1(\nu) = 2\alpha\Gamma_\alpha - \phi\big(\Gamma_\beta + \Gamma_\gamma\big), \tag{2.279}$$

$$\delta_2(\nu) = \phi\big(\Gamma_\gamma - \Gamma_\beta\big), \tag{2.280}$$

$$\delta_4(\nu) = 2\alpha\Gamma_\alpha + \phi\big(\Gamma_\gamma + \Gamma_\beta\big). \tag{2.281}$$

Note here that α, β and γ are given by equations (2.261), (2.262) and (2.263) and so are independent of x_3. The wave-mode amplitude evolution equations are therefore

$$\frac{d}{dx_3}\begin{bmatrix}\hat{a}^\varepsilon(\nu, x_3)\\ \hat{b}^\varepsilon(\nu, x_3)\end{bmatrix} = i\frac{\varpi\sigma}{2\varepsilon}m\bigg(\frac{x_3}{\varepsilon^2}\bigg)\begin{bmatrix}\delta_1(\nu) & \delta_2(\nu)e^{i\frac{2\varpi}{\varepsilon}\phi x_3}\\ -\delta_2(\nu)e^{-i\frac{2\varpi}{\varepsilon}\phi x_3} & \delta_4(\nu)\end{bmatrix}\begin{bmatrix}\hat{a}^\varepsilon(\nu, x_3)\\ \hat{b}^\varepsilon(\nu, x_3)\end{bmatrix}. \tag{2.282}$$

Here $L_3 \gg \lambda_3$ and so from equations (2.246) and (2.247)

$$1 \ll \frac{L_3}{\lambda_3} = \frac{\omega L_3}{2\pi c_3} = \frac{\tilde{\omega}}{2\pi} = \frac{\varpi}{2\pi\varepsilon}. \tag{2.283}$$

Here $\lambda_3 \sim l$ and so from equation (2.248)

$$1 \sim \frac{l}{\lambda_3} = \frac{\omega l}{2\pi c_3} = \frac{\omega L_3}{2\pi c_3}\frac{l}{L_3} = \frac{\varpi}{2\pi\varepsilon}\varepsilon^2. \tag{2.284}$$

Hence, equation (2.284) implies (with $0 < \varepsilon \ll 1$) that

$$\varpi \sim \frac{1}{\varepsilon}, \tag{2.285}$$

and so equation (2.283) holds. Since $0 < \sigma \ll 1$ then set $\sigma = \varepsilon$. This gives the evolution equation (2.282) in the form

$$\frac{d}{dx_3}\begin{bmatrix}\hat{a}^\varepsilon(\nu, x_3)\\ \hat{b}^\varepsilon(\nu, x_3)\end{bmatrix} = \frac{1}{\varepsilon}\mathbf{H}^\varepsilon\left(\nu, \frac{x_3}{\varepsilon^2}, m\left(\frac{x_3}{\varepsilon^2}\right)\right)\begin{bmatrix}\hat{a}^\varepsilon(\nu, x_3)\\ \hat{b}^\varepsilon(\nu, x_3)\end{bmatrix}, \tag{2.286}$$

where

$$\mathbf{H}^\varepsilon\left(\nu, \frac{x_3}{\varepsilon^2}, m\left(\frac{x_3}{\varepsilon^2}\right)\right) = \frac{i}{2}m(x_3/\varepsilon^2)\begin{bmatrix}\delta_1(\nu) & \delta_2(\nu)e^{i\frac{2\phi x_3}{\varepsilon^2}}\\ -\delta_2(\nu)e^{-i\frac{2\phi x_3}{\varepsilon^2}} & \delta_4(\nu)\end{bmatrix}. \tag{2.287}$$

The random fluctuations are assumed to have the form $m(x_3) = g(Y(x_3))$, where Y is a homogeneous in x_3 Markov process with values in a compact space [26], and that this process is strongly ergodic and satisfies the Fredholm alternative [8], and the real bounded function g satisfies the centering condition $\mathbb{E}[g(Y(0))] = 0$. Equation (2.286) can be recast into an initial value problem using

$$\begin{bmatrix}\hat{a}^\varepsilon(\nu, x_3)\\ \hat{b}^\varepsilon(\nu, x_3)\end{bmatrix} = \mathbf{P}^\varepsilon(\nu, x_3)\begin{bmatrix}\hat{a}^\varepsilon(\nu, 0)\\ \hat{b}^\varepsilon(\nu, 0)\end{bmatrix}, \tag{2.288}$$

where the propagator matrix

$$\mathbf{P}^\varepsilon(\nu, x_3) = \begin{bmatrix}\chi_1^\varepsilon(\nu, x_3) & \chi_2^\varepsilon(\nu, x_3)\\ \chi_3^\varepsilon(\nu, x_3) & \chi_4^\varepsilon(\nu, x_3)\end{bmatrix}, \tag{2.289}$$

is formed from eigensolutions of equation (2.286), and $\mathbf{P}^\varepsilon(x_3 = 0) = \mathbf{I}$. It follows that

$$\frac{\partial \mathbf{P}^\varepsilon}{\partial x_3}(\nu, x_3) = \mathbf{H}^\varepsilon\left(\nu, \frac{x_3}{\varepsilon^2}, m\left(\frac{x_3}{\varepsilon^2}\right)\right)\mathbf{P}^\varepsilon(\nu, x_3). \tag{2.290}$$

2.8.2 Diffusion Approximation Theorem

Expanding the right hand side of equation (2.286) gives

$$
\begin{aligned}
\frac{d}{dx_3}\begin{bmatrix}\hat{a}^\varepsilon(\nu,x_3)\\ \hat{b}^\varepsilon(\nu,x_3)\end{bmatrix} &= \frac{i}{\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\begin{bmatrix}\alpha\Gamma_\alpha & 0\\ 0 & \alpha\Gamma_\alpha\end{bmatrix}\begin{bmatrix}\hat{a}^\varepsilon(\nu,x_3)\\ \hat{b}^\varepsilon(\nu,x_3)\end{bmatrix}\\
&- \frac{i}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\phi(\Gamma_\beta+\Gamma_\gamma)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\hat{a}^\varepsilon(\nu,x_3)\\ \hat{b}^\varepsilon(\nu,x_3)\end{bmatrix}\\
&+ \frac{1}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\phi(\Gamma_\beta-\Gamma_\gamma)\cos\left(\frac{2\phi x_3}{\varepsilon^2}\right)\begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}\begin{bmatrix}\hat{a}^\varepsilon(\nu,x_3)\\ \hat{b}^\varepsilon(\nu,x_3)\end{bmatrix}\\
&+ \frac{1}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\phi(\Gamma_\beta-\Gamma_\gamma)\sin\left(\frac{2\phi x_3}{\varepsilon^2}\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\hat{a}^\varepsilon(\nu,x_3)\\ \hat{b}^\varepsilon(\nu,x_3)\end{bmatrix}.
\end{aligned} \tag{2.291}
$$

This can be written in terms of the propagator matrix (2.289), so the random matrix ordinary differential equation is in the form

$$
\frac{d}{dx_3}\mathbf{P}^\varepsilon(\nu,x_3) = \frac{1}{\varepsilon}\mathbf{F}\left(\mathbf{P}^\varepsilon(\nu,x_3), m\left(\frac{x_3}{\varepsilon^2}\right), \frac{x_3}{\varepsilon^2}\right), \tag{2.292}
$$

where

$$
\begin{aligned}
\frac{1}{\varepsilon}\mathbf{F}\left(\mathbf{P}^\varepsilon(\nu,x_3), m\left(\frac{x_3}{\varepsilon^2}\right), \frac{x_3}{\varepsilon^2}\right) &= \frac{i}{\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\alpha\Gamma_\alpha\mathbf{I}\,\mathbf{P}^\varepsilon(\nu,x_3)\\
&- \frac{i}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\phi(\Gamma_\beta+\Gamma_\gamma)\,\boldsymbol{\sigma}_3\mathbf{P}^\varepsilon(\nu,x_3)\\
&+ \frac{1}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\phi(\Gamma_\beta-\Gamma_\gamma)\cos\left(\frac{2\phi x_3}{\varepsilon^2}\right)\boldsymbol{\sigma}_2\mathbf{P}^\varepsilon(\nu,x_3)\\
&+ \frac{1}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\phi(\Gamma_\beta-\Gamma_\gamma)\sin\left(\frac{2\phi x_3}{\varepsilon^2}\right)\boldsymbol{\sigma}_1\mathbf{P}^\varepsilon(\nu,x_3),
\end{aligned} \tag{2.293}
$$

and $\boldsymbol{\sigma}_1, \boldsymbol{\sigma}_2, \boldsymbol{\sigma}_3$ are the Pauli spin matrices

$$
\boldsymbol{\sigma}_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}. \tag{2.294}
$$

The matrix field $\mathbf{F}$ can be written as

$$\mathbf{F} = \sum_{p=1}^{4} g_p(m,\tau)\,\mathbf{h}_p\mathbf{P}^{\varepsilon}, \tag{2.295}$$

where $\tau = x_3/\varepsilon^2$ and

$$\mathbf{h} = \begin{bmatrix} i\alpha\Gamma_\alpha\mathbf{I} \\ -\frac{i\phi(\Gamma_\beta+\Gamma_\gamma)}{2}\boldsymbol{\sigma}_3 \\ \frac{\phi(\Gamma_\beta-\Gamma_\gamma)}{2}\boldsymbol{\sigma}_2 \\ \frac{\phi(\Gamma_\beta-\Gamma_\gamma)}{2}\boldsymbol{\sigma}_1 \end{bmatrix}, \quad \mathbf{g}(m,\tau) = \begin{bmatrix} m \\ m \\ m\cos(2\phi\tau) \\ m\sin(2\phi\tau) \end{bmatrix}. \tag{2.296}$$

From equations (2.259), (2.262), (2.263) and (2.266) ϕ does not depend on frequency, and is a function of $\rho, c_3, \bar{\theta}, N$ and F. Equation (2.293) is independent of frequency with the scaling requirement that $\omega \sim \varepsilon^{-2}$ in the weakly heterogeneous regime, due to the separation of scales. The correlation matrix $\mathbf{C} = (C_{pq})_{p,q=1,2,3,4}$ is computed using the covariance of the random process m. The correlation integrals can be assembled in a matrix, together with symmetric (S) and anti-symmetric (AS) elements. This can be computed using the covariance of the random process m. So

$$C_{pq} = 2\lim_{Z_0\to\infty}\frac{1}{Z_0}\int_0^{Z_0}\int_0^{\infty}\mathbb{E}\Big[g^{(p)}(m(0),\tau)g^{(q)}(m(x_3),\tau+x_3)\Big]dx_3d\tau \quad p,q = 1,\ldots,4, \tag{2.297}$$

where Z_0 is the phase. Starting with the diagonal elements it follows that

$$\begin{aligned} C_{11} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]dx_3dx, \\ &= 2\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]dx_3 = \Upsilon(0), \end{aligned} \tag{2.298}$$

$$\begin{aligned} C_{22} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]dx_3dx, \\ &= 2\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]dx_3 = \Upsilon(0), \end{aligned} \tag{2.299}$$

$$C_{33} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)\cos(x)m(x_3)\cos(x+2\phi x_3)\Big]dx_3dx,$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(x)\cos(x+2\phi x_3)dx_3dx \\
&= \frac{1}{\pi}\int_0^{2\pi}\cos^2(x)dx\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(2\phi x_3)dx_3 \\
&- \frac{1}{\pi}\int_0^{2\pi}\cos(x)\sin(x)dx\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(2\phi x_3)dx_3 \\
&= \int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(2\phi x_3)dx_3 = \frac{1}{2}\Upsilon(\phi),
\end{aligned} \tag{2.300}
$$

and

$$
\begin{aligned}
C_{44} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(x)\sin(x+2\phi x_3)dx_3dx \\
&= \frac{1}{\pi}\int_0^{2\pi}\sin^2(x)dx\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(2\phi x_3)dx_3 \\
&- \frac{1}{\pi}\int_0^{2\pi}\cos(x)\sin(x)dx\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(2\phi x_3)dx_3 \\
&= \int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(2\phi x_3)dx_3 = \frac{1}{2}\Upsilon(\phi).
\end{aligned} \tag{2.301}
$$

The off-diagonal elements of the first row are given by

$$
C_{12} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]dx_3dx = \Upsilon(0), \tag{2.302}
$$

$$
\begin{aligned}
C_{13} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(x+2\phi x_3)dx_3dx \\
&= \frac{1}{\pi}\int_0^{2\pi}\cos(x)dx\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(2\phi x_3)dx_3 \\
&- \frac{1}{\pi}\int_0^{2\pi}\sin(x)dx\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(2\phi x_3)dx_3 \\
&= 0,
\end{aligned} \tag{2.303}
$$

and

$$
C_{14} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(x+2\phi x_3)dx_3dx \tag{2.304}
$$

$$
= 0. \tag{2.305}
$$

similarly, the second row can be completed to give

$$C_{21} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] dx_3 dx = \Upsilon(0), \tag{2.306}$$

$$C_{23} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \cos(x + 2\phi x_3) dx_3 dx = 0, \tag{2.307}$$

and

$$C_{24} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \sin(x + 2\phi x_3) dx_3 dx = 0. \tag{2.308}$$

The elements of the third row are then

$$C_{31} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \cos(x) dx_3 dx = 0, \tag{2.309}$$

$$C_{32} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \cos(x) dx_3 dx = 0, \tag{2.310}$$

$$\begin{aligned} C_{34} &= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \cos(x) \sin(x + 2\phi x_3) dx_3 dx \\ &= \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \sin(2\phi x_3) dx_3 = \frac{1}{2}\Upsilon^{(AS)}(\phi), \end{aligned} \tag{2.311}$$

and those of the fourth row are

$$C_{41} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \sin(x) dx_3 dx = 0, \tag{2.312}$$

$$C_{42} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \sin(x) dx_3 dx = 0, \tag{2.313}$$

and

$$\begin{aligned} C_{43} &= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \sin(x) \cos(x + 2\phi x_3) dx_3 dx \\ &= -\int_0^{\infty} \mathbb{E}\Big[m(0)m(x_3)\Big] \sin(2\phi x_3) dx_3 = -\frac{1}{2}\Upsilon^{(AS)}(\phi). \end{aligned} \tag{2.314}$$

$$\mathbf{C} = \begin{bmatrix} \Upsilon(0) & \Upsilon(0) & 0 & 0 \\ \Upsilon(0) & \Upsilon(0) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\Upsilon(\phi) & \frac{1}{2}\Upsilon^{(AS)}(\phi) \\ 0 & 0 & -\frac{1}{2}\Upsilon^{(AS)}(\phi) & \frac{1}{2}\Upsilon(\phi) \end{bmatrix}, \tag{2.315}$$

$$\mathbf{C}^{(S)} = \begin{bmatrix} \Upsilon(0) & 0 & 0 & 0 \\ 0 & \Upsilon(0) & 0 & 0 \\ 0 & 0 & \frac{1}{2}\Upsilon(\phi) & 0 \\ 0 & 0 & 0 & \frac{1}{2}\Upsilon(\phi) \end{bmatrix}, \tag{2.316}$$

and

$$\mathbf{C}^{(AS)} = \begin{bmatrix} 0 & \Upsilon(0) & 0 & 0 \\ \Upsilon(0) & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}\Upsilon^{(AS)}(\phi) \\ 0 & 0 & -\frac{1}{2}\Upsilon^{(AS)}(\phi) & 0 \end{bmatrix}, \tag{2.317}$$

where the correlation integrals are defined as

$$\Upsilon(\phi) = 2\int_0^\infty \mathbb{E}\Big[m(0)m(x_3)\Big]\cos(2\phi x_3)dx_3, \tag{2.318}$$

and

$$\Upsilon^{(AS)}(\phi) = 2\int_0^\infty \mathbb{E}\Big[m(0)m(x_3)\Big]\sin(2\phi x_3)dx_3. \tag{2.319}$$

The quantity $\Upsilon(\phi)$ is a non-negative real number, as it proportional to the power spectral density of the stationary random process m. Details of how to compute the correlation integrals numerically for a given material are in [63]. Now the diffusion approximation theorem ([26], page 161) can be used to show that $\mathbf{P}^\varepsilon(\nu, x_3)$ converges in distribution to $\mathbf{P}(\nu, x_3)$ where $\mathbf{P}(\nu, x_3)$ is the solution of the Stratonovich stochastic differential equation

$$d\mathbf{P}(\nu, x_3) = i\sqrt{\Upsilon(0)}\Big(\alpha\Gamma_\alpha\mathbf{P}(\nu, x_3)\circ dW_1(x_3) - \frac{\phi}{2}(\Gamma_\beta + \Gamma_\gamma)\boldsymbol{\sigma}_3\mathbf{P}(\nu, x_3)\circ dW_2(x_3)\Big)$$

$$
+ \frac{\phi}{2\sqrt{2}}(\Gamma_\beta - \Gamma_\gamma)\sqrt{\Upsilon(\phi)}\Big(\boldsymbol{\sigma}_2\mathbf{P}(\nu, x_3) \circ dW_3(x_3) + \boldsymbol{\sigma}_1\mathbf{P}(\nu, x_3) \circ dW_4(x_3)\Big)
$$

$$
+ \frac{1}{2}\Big(\Upsilon(0)\alpha\Gamma_\alpha\phi(\Gamma_\beta + \Gamma_\gamma) + i\frac{\Upsilon^{AS}(\phi)}{4}\phi^2(\Gamma_\beta - \Gamma_\gamma)^2\Big)\boldsymbol{\sigma}_3\mathbf{P}(\nu, x_3)dx_3. \quad (2.320)
$$

Using equations (2.289) and (2.294) this can be written

$$
d\begin{bmatrix}\chi_1(\nu, x_3) & \chi_2(\nu, x_3) \\ \chi_3(\nu, x_3) & \chi_4(\nu, x_3)\end{bmatrix} = iA_1\begin{bmatrix}\chi_1(\nu, x_3) & \chi_2(\nu, x_3) \\ \chi_3(\nu, x_3) & \chi_4(\nu, x_3)\end{bmatrix} \circ dW_1(x_3)
$$

$$
+ iA_2\begin{bmatrix}\chi_1(\nu, x_3) & \chi_2(\nu, x_3) \\ -\chi_3(\nu, x_3) & -\chi_4(\nu, x_3)\end{bmatrix} \circ dW_2(x_3)
$$

$$
+ iA_3\begin{bmatrix}-\chi_3(\nu, x_3) & -\chi_4(\nu, x_3) \\ \chi_1(\nu, x_3) & \chi_2(\nu, x_3)\end{bmatrix} \circ dW_3(x_3)
$$

$$
+ A_3\begin{bmatrix}\chi_3(\nu, x_3) & \chi_4(\nu, x_3) \\ \chi_1(\nu, x_3) & \chi_2(\nu, x_3)\end{bmatrix} \circ dW_4(x_3)
$$

$$
+ \big(A_4 + iA_5\big)\begin{bmatrix}\chi_1(\nu, x_3) & \chi_2(\nu, x_3) \\ -\chi_3(\nu, x_3) & -\chi_4(\nu, x_3)\end{bmatrix} dx_3, \quad (2.321)
$$

where

$$
A_1 = \alpha\Gamma_\alpha\sqrt{\Upsilon(0)}, \quad (2.322)
$$

$$
A_2 = -\frac{\phi(\Gamma_\beta + \Gamma_\gamma)}{2}\sqrt{\Upsilon(0)}, \quad (2.323)
$$

$$
A_3 = \frac{\phi}{2\sqrt{2}}(\Gamma_\beta - \Gamma_\gamma)\sqrt{\Upsilon(\phi)}, \quad (2.324)
$$

$$
A_4 = \frac{1}{2}\Big(\alpha\Gamma_\alpha\phi(\Gamma_\beta + \Gamma_\gamma)\Upsilon(0)\Big), \quad (2.325)
$$

and

$$
A_5 = \frac{1}{8}\Big(\phi^2(\Gamma_\beta - \Gamma_\gamma)^2\Upsilon^{AS}(\phi)\Big). \quad (2.326)
$$

Expanding equation (2.321) gives four coupled stochastic differential equations in Stratonovich form

$$
\begin{aligned}
d\chi_1 &= iA_1\chi_1 \circ dW_1 + iA_2\chi_1 \circ dW_2 - iA_3\chi_3 \circ dW_3 + A_3\chi_3 \circ dW_4 \\
&\quad + (A_4 + iA_5)\chi_1\, dx_3, && (2.327)\\
d\chi_2 &= iA_1\chi_2 \circ dW_1 + iA_2\chi_2 \circ dW_2 - iA_3\chi_4 \circ dW_3 + A_3\chi_4 \circ dW_4 \\
&\quad + (A_4 + iA_5)\chi_2\, dx_3, && (2.328)\\
d\chi_3 &= iA_1\chi_3 \circ dW_1 - iA_2\chi_3 \circ dW_2 + iA_3\chi_1 \circ dW_3 + A_3\chi_1 \circ dW_4 \\
&\quad - (A_4 + iA_5)\chi_3\, dx_3, && (2.329)\\
d\chi_4 &= iA_1\chi_4 \circ dW_1 - iA_2\chi_4 \circ dW_2 + iA_3\chi_2 \circ dW_3 + A_3\chi_2 \circ dW_4 \\
&\quad - (A_4 + iA_5)\chi_4\, dx_3. && (2.330)
\end{aligned}
$$

Introduce the polar coordinate parameterisation for the elements of the propagator matrix via

$$\chi_1(x_3) = a(x_3)e^{ib(x_3)}, \tag{2.331}$$

$$\chi_2(x_3) = g(x_3)e^{ih(x_3)}, \tag{2.332}$$

$$\chi_3(x_3) = j(x_3)e^{ik(x_3)}, \tag{2.333}$$

$$\chi_4(x_3) = p(x_3)e^{iq(x_3)}. \tag{2.334}$$

In the Stratonovich framework the standard chain rule applies, and so

$$d\chi_1 = \frac{\partial \chi_1}{\partial a}da + \frac{\partial \chi_1}{\partial b}db = (da + iadb)e^{ib}, \tag{2.335}$$

$$d\chi_2 = \frac{\partial \chi_2}{\partial g}dg + \frac{\partial \chi_2}{\partial h}dh = (dg + igdh)e^{ih}, \tag{2.336}$$

$$d\chi_3 = \frac{\partial \chi_3}{\partial j}dj + \frac{\partial \chi_3}{\partial k}dk = (dj + ijdk)e^{ik}, \tag{2.337}$$

$$d\chi_4 = \frac{\partial \chi_4}{\partial p}dp + \frac{\partial \chi_4}{\partial q}dq = (dp + ipdq)e^{iq}. \tag{2.338}$$

Using (2.327) and (2.335) gives the two equations

$$da = A_3 j \sin(k-b) \circ dW_3 + A_3 j \cos(k-b) \circ dW_4 + A_4 a dx_3, \tag{2.339}$$

$$db = A_1 \circ dW_1 + A_2 \circ dW_2 - A_3 \frac{j}{a} \cos(k-b) \circ dW_3 + A_3 \frac{j}{a} \sin(k-b) \circ dW_4 + A_5 dx_3. \tag{2.340}$$

Similarly, using equations (2.328) and (2.336) gives

$$dg = A_3 p \sin(q-h) \circ dW_3 + A_3 p \cos(q-h) \circ dW_4 + A_4 g dx_3, \tag{2.341}$$

$$dh = A_1 \circ dW_1 + A_2 \circ dW_2 - A_3 \frac{p}{g} \cos(q-h) \circ dW_3 + A_3 \frac{p}{g} \sin(q-h) \circ dW_4 + A_5 dx_3. \tag{2.342}$$

Nextly, using equations (2.329) and (2.337) gives

$$dj = -A_3 a \sin(b-k) \circ dW_3 + A_3 a \cos(b-k) \circ dW_4 - A_4 j dx_3, \tag{2.343}$$

$$dk = A_1 \circ dW_1 - A_2 \circ dW_2 + A_3 \frac{a}{j} \cos(b-k) \circ dW_3 + A_3 \frac{a}{j} \sin(b-k) \circ dW_4 - A_5 dx_3. \tag{2.344}$$

Finally, using equations (2.330) and (2.338) gives

$$dp = -A_3 g \sin(h-q) \circ dW_3 + A_3 g \cos(h-q) \circ dW_4 - A_4 p dx_3, \tag{2.345}$$

$$dq = A_1 \circ dW_1 - A_2 \circ dW_2 + A_3 \frac{g}{p} \cos(h-q) \circ dW_3 + A_3 \frac{g}{p} \sin(h-q) \circ dW_4 - A_5 dx_3. \tag{2.346}$$

The equations for the radial and phase parts of the propagator functions can be written in matrix form as

$$d\begin{bmatrix} a \\ b \\ g \\ h \\ j \\ k \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_3 j \sin(k-b) & A_3 j \cos(k-b) \\ A_1 & A_2 & -A_3(j/a)\cos(k-b) & A_3(j/a)\sin(k-b) \\ 0 & 0 & A_3 p \sin(q-h) & A_3 p \cos(q-h) \\ A_1 & A_2 & -A_3(p/g)\cos(q-h) & A_3(p/g)\sin(q-h) \\ 0 & 0 & -A_3 a \sin(b-k) & A_3 a \cos(b-k) \\ A_1 & -A_2 & A_3(a/j)\cos(b-k) & A_3(a/j)\sin(b-k) \\ 0 & 0 & -A_3 g \sin(h-q) & A_3 g \cos(h-q) \\ A_1 & -A_2 & A_3(g/p)\cos(h-q) & A_3(g/p)\sin(h-q) \end{bmatrix} \circ d\begin{bmatrix} xW_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} A_4 a \\ A_5 \\ A_4 g \\ A_5 \\ -A_4 j \\ -A_5 \\ -A_4 p \\ -A_5 \end{bmatrix} dx_3. \tag{2.347}$$

Using Jacobi's formula and equation (2.290) gives

$$\frac{d\det\{\mathbf{P}^\varepsilon\}}{dx_3} = \mathrm{Tr}(\mathbf{H}^\varepsilon)\det\{\mathbf{P}^\varepsilon\}, \tag{2.348}$$

where equations (2.279), (2.281) and (2.287) give $\mathrm{Tr}(\mathbf{H}^\varepsilon) = 2i\alpha\Gamma_\alpha m(x_3/\varepsilon^2)$. Solving this differential equation yields

$$\det\{\mathbf{P}^\varepsilon\} = \exp\left(2i\alpha\Gamma_\alpha \int_0^{\frac{x_3}{\varepsilon^2}} m(s)ds\right), \tag{2.349}$$

and hence

$$|\det\{\mathbf{P}^\varepsilon(x_3)\}| = 1, \tag{2.350}$$

and so equation (2.289) implies

$$|\chi_1\chi_4 - \chi_2\chi_3| = 1. \tag{2.351}$$

This conservation of energy relationship can be written as

$$\cos(b+q-h-k) = \frac{a^2p^2 + g^2j^2 - 1}{2apgj} = D_1(a,g,j,p), \tag{2.352}$$

and so

$$q - h = \cos^{-1}(D_1) + (k - b), \tag{2.353}$$

and so

$$\cos(q - h) = D_1 \cos(k - b) - \sqrt{1 - D_1^2}\sin(k - b), \tag{2.354}$$
$$\sin(h - q) = \sqrt{1 - D_1^2}\cos(k - b) + D_1 \sin(k - b). \tag{2.355}$$

Equation (2.347) contains the Stratonovich subsystem

$$d\begin{bmatrix} g \\ h \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_3 p \sin(q-h) & A_3 p \cos(q-h) \\ A_1 & A_2 & -A_3 \frac{p}{g}\cos(q-h) & A_3 \frac{p}{g}\sin(q-h) \\ 0 & 0 & A_3 g \sin(q-h) & A_3 g \cos(q-h) \\ A_1 & -A_2 & A_3 \frac{g}{p}\cos(q-h) & -A_3 \frac{g}{p}\sin(q-h) \end{bmatrix} \circ d \begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} A_4 g \\ A_5 \\ -A_4 p \\ -A_5 \end{bmatrix} dx_3, \tag{2.356}$$

Using Jacobi's formula and equation (2.293) (following the same approach as in equations (2.157) - (2.161))

$$\frac{d\det\{\mathbf{P}^\varepsilon\}}{dx_3} = \mathrm{Tr}(\mathbf{H}^\varepsilon)\det\{\mathbf{P}^\varepsilon\}. \tag{2.357}$$

Equations (2.279), (2.281) and (2.287) then give $\mathrm{Tr}(\mathbf{H}^\varepsilon) = 2i\alpha\Gamma_\alpha m(x_3/\varepsilon^2)$. Solving this separable differential equation yields the solution for the determinant of the propagator matrix

$$\begin{aligned} \det\{\mathbf{P}^\varepsilon\} &= \det\{\mathbf{P}^\varepsilon(\omega, x_3 = 0)\}\exp\left(2i\alpha\Gamma_\alpha \int_0^{\frac{x_3}{\varepsilon^2}} m(s)ds\right) \\ &= \exp\left(2i\alpha\Gamma_\alpha \int_0^{\frac{x_3}{\varepsilon^2}} m(s)ds\right). \end{aligned} \tag{2.358}$$

Hence

$$|\det\{\mathbf{P}^\varepsilon(\omega, x_3)\}| = 1, \tag{2.359}$$

and equation (2.289) implies

$$|\chi_1\chi_4 - \chi_2\chi_3| = 1. \tag{2.360}$$

From equations (2.331) - (2.334), the conservation of energy relationship can be written as

$$\begin{aligned}\left|ae^{ib}pe^{iq} - ge^{ih}je^{ik}\right| &= |ap\cos(b+q) - gj\cos(h+k) + i(ap\sin(b+q) - gj\sin(h+k))| \\ &= \Big(ap\cos(b+q) - gj\cos(h+k)\Big)^2 \\ &+ \Big(ap\sin(b+q) - gj\sin(h+k)\Big)^2 = 1,\end{aligned} \tag{2.361}$$

which implies that

$$a^2p^2 + g^2j^2 - 2apgj\cos(b+q-h-k) = 1, \tag{2.362}$$

which can be rewritten as

$$\cos(b+q-h-k) = \frac{a^2p^2 + g^2j^2 - 1}{2apgj} = D_1(a,g,j,p). \tag{2.363}$$

Equation (2.363) also implies that

$$q - h = \cos^{-1}(D_1) + (k-b). \tag{2.364}$$

So from equation (2.364)

$$\cos(q-h) = D_1\cos(k-b) - \sqrt{1-D_1^2}\sin(k-b), \tag{2.365}$$

$$\sin(h-q) = \sqrt{1-D_1^2}\cos(k-b) + D_1\sin(k-b). \tag{2.366}$$

Furthermore, from equation (2.363)

$$q = \cos^{-1}(D_1) + k - b + h = D_2(a,b,g,h,j,k,p). \tag{2.367}$$

Applying the chain rule to equation (2.367) then gives

$$\begin{aligned} dq &= \frac{\partial D_2}{\partial a}da + \frac{\partial D_2}{\partial b}db + \frac{\partial D_2}{\partial g}dg + \frac{\partial D_2}{\partial h}dh + \frac{\partial D_2}{\partial j}dj + \frac{\partial D_2}{\partial k}dk + \frac{\partial D_2}{\partial p}dp \\ &= -\frac{1}{\sqrt{1-D_1^2}}\left(\frac{\partial D_2}{\partial a}da + \frac{\partial D_2}{\partial g}dg + \frac{\partial D_2}{\partial j}dj + \frac{\partial D_2}{\partial p}dp\right) - db + dh + dk. \end{aligned} \tag{2.368}$$

To obtain this result, equation (2.368) was substituted into the left hand side of equation (2.346), using also equations (2.363), (2.365) and (2.366). One can show through the help of Mathematica that equation (2.346) holds and is consistent with the energy conservation relation (2.361), allowing the system of eight coupled SDE's to be reduced to seven. Note that this new system of seven SDE's decouples first into multiple coupled systems; $d(a, b, j, k)$, $d(g, h, p)$ and then $d(g, p)$.

2.8.3 Analysing the Diffusion Markov Process

Equations (2.341), (2.342), (2.345) and (2.346) give the Stratonovich system

$$d\begin{bmatrix} g \\ h \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_3 p \sin(q-h) & A_3 p \cos(q-h) \\ A_1 & A_2 & -A_3 \frac{p}{g}\cos(q-h) & A_3 \frac{p}{g}\sin(q-h) \\ 0 & 0 & A_3 g \sin(q-h) & A_3 g \cos(q-h) \\ A_1 & -A_2 & A_3 \frac{g}{p}\cos(q-h) & -A_3 \frac{g}{p}\sin(q-h) \end{bmatrix} \circ d\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} + \begin{bmatrix} A_4 g \\ A_5 \\ -A_4 p \\ -A_5 \end{bmatrix} dx_3. \tag{2.369}$$

To convert system (2.369) into Itô form, first rewrite equation (2.369) using

$$\boldsymbol{\sigma} = \begin{bmatrix} 0 & 0 & A_3 p \sin(q-h) & A_3 p \cos(q-h) \\ A_1 & A_2 & -A_3 \frac{p}{g}\cos(q-h) & A_3 \frac{p}{g}\sin(q-h) \\ 0 & 0 & A_3 g \sin(q-h) & A_3 g \cos(q-h) \\ A_1 & -A_2 & A_3 \frac{g}{p}\cos(q-h) & -A_3 \frac{g}{p}\sin(q-h) \end{bmatrix}, \tag{2.370}$$

$$\boldsymbol{x} = \begin{bmatrix} g \\ h \\ p \\ q \end{bmatrix}, \quad d\boldsymbol{W} = d\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} A_4 g \\ A_5 \\ -A_4 p \\ -A_5 \end{bmatrix}, \tag{2.371}$$

to obtain

$$d\boldsymbol{x}_i = \boldsymbol{\sigma}_{ip} \circ d\boldsymbol{W}_p + \boldsymbol{b}_i dx_3, \quad i = 1, 2, 3, 4. \tag{2.372}$$

In Itô form (see Section 6.7.2 [26])

$$d\boldsymbol{x}_i = \boldsymbol{\sigma}_{ip} d\boldsymbol{W}_p + \underbrace{\frac{1}{2}\sum_{j=1}^{4} \boldsymbol{\sigma}_{jp}\boldsymbol{\sigma}_{ip,j} dx_3}_{\text{Modified Drift}} + \boldsymbol{b}_i dx_3. \tag{2.373}$$

Calculating the modified drift (from equation (2.373)) for $i = 1, 2, 3, 4$ gives

$$\begin{aligned}
\frac{1}{2}\sum_{j=1}^{4} \boldsymbol{\sigma}_{jp}\boldsymbol{\sigma}_{1p,j} dx_3 &= \frac{1}{2}(\sigma_{11}\sigma_{11,1} + \sigma_{21}\sigma_{11,2} + \sigma_{31}\sigma_{11,3} + \sigma_{41}\sigma_{11,4} \\
&+ \sigma_{12}\sigma_{12,1} + \sigma_{22}\sigma_{12,2} + \sigma_{32}\sigma_{12,3} + \sigma_{42}\sigma_{12,4} \\
&+ \sigma_{13}\sigma_{13,1} + \sigma_{23}\sigma_{13,2} + \sigma_{33}\sigma_{13,3} + \sigma_{43}\sigma_{13,4} \\
&+ \sigma_{14}\sigma_{14,1} + \sigma_{24}\sigma_{14,2} + \sigma_{34}\sigma_{14,3} + \sigma_{44}\sigma_{14,4}) dx_3 \\
&= \frac{A_3^2}{2}\left(\frac{p^2}{g} + 2g\right) dx_3, \qquad (2.374) \\
\frac{1}{2}\sum_{j=1}^{4} \boldsymbol{\sigma}_{jp}\boldsymbol{\sigma}_{2p,j} dx_3 &= \frac{1}{2}(\sigma_{11}\sigma_{21,1} + \sigma_{21}\sigma_{21,2} + \sigma_{31}\sigma_{21,3} + \sigma_{41}\sigma_{21,4} \\
&+ \sigma_{12}\sigma_{22,1} + \sigma_{22}\sigma_{22,2} + \sigma_{32}\sigma_{22,3} + \sigma_{42}\sigma_{22,4} \\
&+ \sigma_{13}\sigma_{23,1} + \sigma_{23}\sigma_{23,2} + \sigma_{33}\sigma_{23,3} + \sigma_{43}\sigma_{23,4} \\
&+ \sigma_{14}\sigma_{24,1} + \sigma_{24}\sigma_{24,2} + \sigma_{34}\sigma_{24,3} + \sigma_{44}\sigma_{24,4}) \\
&= 0, \qquad (2.375) \\
\frac{1}{2}\sum_{j=1}^{4} \boldsymbol{\sigma}_{jp}\boldsymbol{\sigma}_{3p,j} dx_3 &= \frac{1}{2}(\sigma_{11}\sigma_{31,1} + \sigma_{21}\sigma_{31,2} + \sigma_{31}\sigma_{31,3} + \sigma_{41}\sigma_{31,4} \\
&+ \sigma_{12}\sigma_{32,1} + \sigma_{22}\sigma_{32,2} + \sigma_{32}\sigma_{32,3} + \sigma_{42}\sigma_{32,4} \\
&+ \sigma_{13}\sigma_{33,1} + \sigma_{23}\sigma_{33,2} + \sigma_{33}\sigma_{33,3} + \sigma_{43}\sigma_{33,4} \\
&+ \sigma_{14}\sigma_{34,1} + \sigma_{24}\sigma_{34,2} + \sigma_{34}\sigma_{34,3} + \sigma_{44}\sigma_{34,4}) \\
&= \frac{A_3^2}{2}\left(2p + \frac{g^2}{p}\right), \qquad (2.376)
\end{aligned}$$

and

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{4}\boldsymbol{\sigma}_{jp}\boldsymbol{\sigma}_{4p,j}dx_3 = \frac{1}{2}(&\sigma_{11}\sigma_{41,1} + \sigma_{21}\sigma_{41,2} + \sigma_{31}\sigma_{41,3} + \sigma_{41}\sigma_{41,4} \\
&+ \sigma_{12}\sigma_{42,1} + \sigma_{22}\sigma_{42,2} + \sigma_{32}\sigma_{42,3} + \sigma_{42}\sigma_{42,4} \\
&+ \sigma_{13}\sigma_{43,1} + \sigma_{23}\sigma_{43,2} + \sigma_{33}\sigma_{43,3} + \sigma_{43}\sigma_{43,4} \\
&+ \sigma_{14}\sigma_{44,1} + \sigma_{24}\sigma_{44,2} + \sigma_{34}\sigma_{44,3} + \sigma_{44}\sigma_{44,4}) \\
&= 0. \qquad (2.377)
\end{aligned}
$$

Equation (2.369) in Itô form reads

$$
\begin{aligned}
d\begin{bmatrix} g \\ h \\ p \\ q \end{bmatrix} &= \begin{bmatrix} 0 & 0 & A_3 p\sin(q-h) & A_3 p\cos(q-h) \\ A_1 & A_2 & -A_3\frac{p}{g}\cos(q-h) & A_3\frac{p}{g}\sin(q-h) \\ 0 & 0 & A_3 g\sin(q-h) & A_3 g\cos(q-h) \\ A_1 & -A_2 & A_3\frac{g}{p}\cos(q-h) & -A_3\frac{g}{p}\sin(q-h) \end{bmatrix} d\begin{bmatrix} W_1 \\ W_2 \\ W_3 \\ W_4 \end{bmatrix} \\
&+ \begin{bmatrix} A_4 g + \frac{A_3^2}{2}\left(\frac{p^2}{g} + 2g\right) \\ A_5 \\ -A_4 p + \frac{A_3^2}{2}\left(2p + \frac{g^2}{p}\right) \\ -A_5 \end{bmatrix} dx_3. \qquad (2.378)
\end{aligned}
$$

Introduce the orthogonal Itô transform to separate this system

$$
\begin{bmatrix} W_3^\star \\ W_4^\star \end{bmatrix} = \int_0^{x_3} \begin{bmatrix} D_2^s & D_2^c \\ -D_2^c & D_2^s \end{bmatrix} d\begin{bmatrix} W_3 \\ W_4 \end{bmatrix}, \qquad (2.379)
$$

where, from equations (2.365) and (2.366)

$$
D_2^s = -\sqrt{1-D_1^2}\cos(k-b) - D_1\sin(k-b) = \sin(q-h), \qquad (2.380)
$$

and

$$
D_2^c = D_1\cos(k-b) - \sqrt{1-D_1^2}\sin(k-b) = \cos(q-h). \qquad (2.381)
$$

Note that

$$(D_2^s)^2 + (D_2^c)^2 = 1, \quad \Big[D_2^s, D_2^c\Big] \cdot \Big[-D_2^c, D_2^s\Big] = 0, \tag{2.382}$$

which ensures the transform is orthogonal. Furthermore, it can be shown that the transform is justified by showing that the correlation is zero, and the variance of each component is zero [34]. Start by defining the stochastic integral

$$\int_a^b X(z)dW = \sum_i X(z_i)\Big(W(z_{i+1}) - W(z_i)\Big) = \sum_i X(z_i)\Delta W_i, \tag{2.383}$$

where

$$\Delta W_i = W(z_{i+1}) - W(z_i), \tag{2.384}$$

and W is a Wiener process. By definition W_i has independent increments

$$\mathbb{E}[\Delta W_i \Delta W_j] = \delta_{ij}, \tag{2.385}$$

and is normally distributed

$$W_i - W_j \sim \mathcal{N}(0, z_i - z_j), \quad z_j \leq z_i, \tag{2.386}$$

so that

$$\mathbb{E}[(W_i - W_j)^2] = z_i - z_j, \tag{2.387}$$

and

$$\mathbb{E}[(\Delta W_i)^2] = z_{i+1} - z_i. \tag{2.388}$$

Now consider the stochastic integral

$$
\begin{aligned}
\mathbb{E}\Big[\Big(\int_a^b X(z)dW\Big)\Big(\int_a^b Y(z)dW\Big)\Big] &= \mathbb{E}\Big[\Big(\sum_i X(z_i)\Delta W_i\Big)\Big(\sum_j Y(z_j)\Delta W_j\Big)\Big] \\
&= \mathbb{E}\Big[X(z_1)\Delta W_1\Big(Y(z_1)\Delta W_1 + Y(z_2)\Delta W_2 + ...\Big) \\
&+ X(z_2)\Delta W_2\Big(...\Big) + ...\Big] \\
&= \mathbb{E}\Big[\sum_i X(z_i)Y(z_i)(\Delta W_i)^2\Big] \quad \text{(using equation (2.385))} \\
&= \sum_i \mathbb{E}\Big[X(z_i)Y(z_i)(\Delta W_i)^2\Big] \\
&= \sum_i \mathbb{E}\Big[X(z_i)Y(z_i)\Big](z_{i+1} - z_i) \quad \text{(using equation (2.388))} \\
&= \mathbb{E}\Big[\sum_i X(z_i)Y(z_i)(z_{i+1} - z_i)\Big] \\
&= \mathbb{E}\Big[\int_a^b X(z)Y(z)dz\Big]. \qquad (2.389)
\end{aligned}
$$

For two independent Wiener processes [34]

$$\mathbb{E}[W_1 W_2] = 0, \qquad (2.390)$$

which implies

$$\mathbb{E}\Big[\Big(\int_a^b X(z)dW_1\Big)\Big(\int_a^b Y(z)dW_2\Big)\Big] = 0. \qquad (2.391)$$

Hence, it can be shown that the transform (2.379) satisfies

$$
\begin{aligned}
\mathbb{E}[W_3^\star W_4^\star] &= \mathbb{E}\Big[\Big(\int_0^{x_3} D_2^s dW_3 + D_2^c dW_4\Big)\Big(\int_0^{x_3} D_2^s dW_4 - D_2^c dW_3\Big)\Big] \\
&= \mathbb{E}\Big[\int_0^{x_3} D_2^s dW_3 \int_0^{x_3} D_2^c dW_4 - \int_0^{x_3} D_2^c dW_4 \int_0^{x_3} D_2^c dW_3 \\
&+ \int_0^{x_3} D_2^c dW_4 \int_0^{x_3} D_2^s dW_4 - \int_0^{x_3} D_2^c dW_3 \int_0^{x_3} D_2^s dW_3\Big] \\
&= \mathbb{E}\Big[\int_0^{x_3} D_2^c dW_4 \int_0^{x_3} D_2^s dW_4 - \int_0^{x_3} D_2^c dW_3 \int_0^{x_3} D_2^s dW_3\Big] \quad \text{(using equation (2.391))}
\end{aligned}
$$

$$= \mathbb{E}\left[\int_0^z D_2^c D_2^s dz - \int_0^z D_2^c D_2^s dz\right] \quad \text{(using equation (2.389))}$$

$$= 0. \tag{2.392}$$

This transform can be written

$$dW_3^\star = \sin{(q-h)} dW_3 + \cos{(q-h)} dW_4, \tag{2.393}$$

$$dW_4^\star = -\cos{(q-h)} dW_3 + \sin{(q-h)} dW_4, \tag{2.394}$$

so the system of Itô SDE's (2.378) can be rewritten as

$$d\begin{bmatrix} g \\ h \\ p \\ q \end{bmatrix} = \begin{bmatrix} 0 & 0 & A_3 p & 0 \\ A_1 & A_2 & 0 & A_3 \frac{p}{g} \\ 0 & 0 & A_3 g & 0 \\ A_1 & -A_2 & 0 & -A_3 \frac{g}{p} \end{bmatrix} d \begin{bmatrix} W_1 \\ W_2 \\ W_3^\star \\ W_4^\star \end{bmatrix} + \begin{bmatrix} A_4 g + \frac{A_3^2}{2}\left(\frac{p^2}{g} + 2g\right) \\ A_5 \\ -A_4 p + \frac{A_3^2}{2}\left(2p + \frac{g^2}{p}\right) \\ -A_5 \end{bmatrix} dx_3. \tag{2.395}$$

2.8.4 Moments of the Power Transmission Coefficient

Consider an incident unit pulse impinging a slab of layered random media from the left at $x_3 = 0$ (the domain of the layered media is $x_3 \in [0, L]$) with a radiation condition in the homogeneous half-space $x_3 \in (0, \infty)$. Then the mode amplitudes governed by equations (2.271) and (2.272) satisfy the boundary conditions

$$\hat{a}^\varepsilon(0) = 1, \quad \hat{b}^\varepsilon(L) = 0. \tag{2.396}$$

The linear system (2.288) then gives

$$\begin{bmatrix} \hat{a}^\varepsilon(L) \\ 0 \end{bmatrix} = \begin{bmatrix} \chi_1^\varepsilon(L) & \chi_2^\varepsilon(L) \\ \chi_3^\varepsilon(L) & \chi_4^\varepsilon(L) \end{bmatrix} \begin{bmatrix} 1 \\ \hat{b}^\varepsilon(0) \end{bmatrix}, \tag{2.397}$$

and the reflection and transmission coefficients can then be written as

$$R_\omega^\varepsilon(L) = \hat{b}^\varepsilon(\omega, 0), \quad \text{and} \quad T_\omega^\varepsilon(L) = \hat{a}^\varepsilon(L), \tag{2.398}$$

and so

$$R_\omega^\varepsilon(L) = -\frac{\chi_3^\varepsilon(L)}{\chi_4^\varepsilon(L)}, \quad T_\omega^\varepsilon(L) = \frac{\chi_1^\varepsilon(L)\chi_4^\varepsilon(L) - \chi_2^\varepsilon(L)\chi_3^\varepsilon(L)}{\chi_4^\varepsilon(L)}. \tag{2.399}$$

The transmission coefficient $T_\omega^\varepsilon(L)$ converges in distribution (as $\varepsilon \to 0$) to $T_\omega(L)$ and the power transmission coefficient ($\tau_\omega(L)$ say) is given by

$$\tau_\omega(L) = |T_\omega(L)|^2 = |\chi_4(L)|^{-2} = \frac{1}{p^2}, \tag{2.400}$$

where the conservation of energy relation (2.360) has been used. Note that the initial condition of the power transmission coefficient

$$\tau_\omega(L = 0) = 1, \tag{2.401}$$

follows from that fact that $\mathbf{P}(x_3 = 0) = \mathbf{I}$. The moments of the power transmission coefficient can then be calculated via

$$\mathbb{E}[(\tau_\omega(L))^n] = \int_1^{p_\infty} \int_0^{g_\infty} \frac{P(L, g, p)}{p^{2n}} \, dg \, dp, \tag{2.402}$$

where $P(L, g, p)$ is the probability density function associated with p at $x_3 = L$. From equation (2.395) it is clear that (dg, dp) and (dh, dq) decouple into a two independent subsystems. Studying the statistics of the power transmission coefficient requires solving the (dg, dp) system

$$d\begin{bmatrix} g \\ p \end{bmatrix} = \begin{bmatrix} A_3 p & 0 \\ A_3 g & 0 \end{bmatrix} d\begin{bmatrix} W_3^\star \\ W_4^\star \end{bmatrix} + \begin{bmatrix} A_4 g + \frac{A_3^2}{2}\left(\frac{p^2}{g} + 2g\right) \\ -A_4 p + \frac{A_3^2}{2}\left(2p + \frac{g^2}{p}\right) \end{bmatrix} dx_3. \tag{2.403}$$

The infinitesimal generator of the Markov process (g, p) is calculated via

$$\mathcal{L}_{g,p} = \frac{1}{2}\sum_{i,j=1}^{2} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{2} \boldsymbol{b}_i \frac{\partial}{\partial x_i}, \quad \boldsymbol{a}_{ij} = \boldsymbol{\sigma}_{ik}\boldsymbol{\sigma}_{kj}. \tag{2.404}$$

By defining

$$\boldsymbol{\sigma} = \begin{bmatrix} A_3 p & 0 \\ A_3 g & 0 \end{bmatrix}, \quad \boldsymbol{b} = \begin{bmatrix} A_4 g + \frac{A_3^2}{2}\left(\frac{p^2}{g} + 2g\right) \\ -A_4 p + \frac{A_3^2}{2}\left(2p + \frac{g^2}{p}\right) \end{bmatrix}, \tag{2.405}$$

it follows that

$$\boldsymbol{a} = \begin{bmatrix} A_3^2 p^2 & A_3^2 pg \\ A_3^2 pg & A_3^2 g^2 \end{bmatrix}. \tag{2.406}$$

Thus, the infinitesimal generator $\mathcal{L}_{g,p}$ can be written

$$\begin{aligned} \mathcal{L}_{g,p} &= \frac{1}{2}\left(a_{11}\frac{\partial^2}{\partial g^2} + 2a_{12}\frac{\partial^2}{\partial g \partial p} + a_{22}\frac{\partial^2}{\partial p^2}\right) + b_1\frac{\partial}{\partial g} + b_2\frac{\partial}{\partial p} \\ &= \frac{1}{2}\left(A_3^2 p^2\frac{\partial^2}{\partial g^2} + 2A_3^2 pg\frac{\partial^2}{\partial g \partial p} + A_3^2 g^2\frac{\partial^2}{\partial p^2}\right) + \left(A_4 g + \frac{A_3^2}{2}\left(\frac{p^2}{g} + 2g\right)\right)\frac{\partial}{\partial g} \\ &+ \left(\frac{A_3^2}{2}\left(2p + \frac{g^2}{p}\right) - A_4 p\right)\frac{\partial}{\partial p}. \end{aligned} \tag{2.407}$$

Now introduce the substitution

$$\mathcal{G} = g^2, \tag{2.408}$$

which gives

$$\frac{\partial}{\partial g} = \frac{\partial \mathcal{G}}{\partial g}\frac{\partial}{\partial \mathcal{G}} = 2g\frac{\partial}{\partial \mathcal{G}} = 2\mathcal{G}^{1/2}\frac{\partial}{\partial \mathcal{G}}, \tag{2.409}$$

$$\frac{\partial^2}{\partial g^2} = \frac{\partial}{\partial g}\left(2g\frac{\partial}{\partial \mathcal{G}}\right) = 2\frac{\partial \mathcal{G}}{\partial g}\frac{\partial}{\partial \mathcal{G}}\left(\mathcal{G}^{1/2}\frac{\partial}{\partial \mathcal{G}}\right) = 2\frac{\partial}{\partial \mathcal{G}} + 4\mathcal{G}\frac{\partial^2}{\partial \mathcal{G}^2}. \tag{2.410}$$

Inserting equations (2.409) and (2.410) into the generator (2.407) gives

$$\begin{aligned} \mathcal{L}_{\mathcal{G},p} = 2A_3^2 p^2 \mathcal{G}\frac{\partial^2}{\partial \mathcal{G}^2} + \left(2A_3^2 p^2 + \mathcal{G}(2A_4 + 2A_3^2)\right)\frac{\partial}{\partial \mathcal{G}} + 2A_3^2 p\mathcal{G}\frac{\partial^2}{\partial \mathcal{G}\partial p} \\ + \frac{A_3^2}{2}\mathcal{G}\frac{\partial^2}{\partial p^2} + \left(\frac{A_3^2}{2}\left(2p + \frac{\mathcal{G}}{p}\right) - A_4 p\right)\frac{\partial}{\partial p}. \end{aligned} \tag{2.411}$$

To compute the adjoint of equation (2.411), consider each term in turn. The first term is

$$\mathcal{L}_1 = 2A_3^2 p^2 \mathcal{G} \frac{\partial^2}{\partial \mathcal{G}^2}. \tag{2.412}$$

In the case where $\psi(\mathcal{G}, p)$ and $\phi(\mathcal{G}, p)$ are two compactly supported test functions

$$\begin{aligned} 2A_3^2 \int_0^{\mathcal{G}_\infty} \int_1^{p_\infty} \psi p^2 \mathcal{G} \frac{\partial^2 \phi}{\partial \mathcal{G}^2}\, dp d\mathcal{G} &= -2A_3^2 \int_0^{\mathcal{G}_\infty} \int_1^{p_\infty} \left(\psi p^2 + \frac{\partial \psi}{\partial \mathcal{G}} p^2 \mathcal{G} \right) \frac{\partial \phi}{\partial \mathcal{G}}\, dp d\mathcal{G} \\ &= 2A_3^2 \int_0^{\mathcal{G}_\infty} \int_1^{p_\infty} \left(\frac{\partial \psi}{\partial \mathcal{G}} p^2 + \frac{\partial^2 \psi}{\partial \mathcal{G}^2} p^2 \mathcal{G} + \frac{\partial \psi}{\partial \mathcal{G}} p^2 \right) \phi\, dp d\mathcal{G}, \end{aligned} \tag{2.413}$$

provided that

$$\left[\frac{\partial \phi}{\partial \mathcal{G}} \psi \mathcal{G} \right]_{\mathcal{G}=0}^{\mathcal{G}=\mathcal{G}_\infty} = 0,$$

and

$$\left[\phi \left(\psi + \frac{\partial \psi}{\partial \mathcal{G}} \mathcal{G} \right) \right]_{\mathcal{G}=0}^{\mathcal{G}=\mathcal{G}_\infty} = 0.$$

That is, $\phi(\mathcal{G} = 0) = \phi(\mathcal{G} = \mathcal{G}_\infty) = 0$, and $\partial\psi/\partial\mathcal{G}(\mathcal{G} = 0) = \partial\psi/\partial\mathcal{G}(\mathcal{G} = \mathcal{G}_\infty) = 0$, hence

$$\mathcal{L}_1^\star = 2A_3^2 \left(p^2 \mathcal{G} \frac{\partial^2}{\partial \mathcal{G}^2} + 2p^2 \frac{\partial}{\partial \mathcal{G}} \right). \tag{2.414}$$

The second term in equation (2.411) is

$$\mathcal{L}_2 = \left(2A_3^2 p^2 + \mathcal{G}(2A_4 + 2A_3^2) \right) \frac{\partial}{\partial \mathcal{G}}. \tag{2.415}$$

Computing the inner product gives

$$\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\psi\Big(2A_3^2p^2+\mathcal{G}(2A_4+2A_3^2)\Big)\phi_{\mathcal{G}}\,dpd\mathcal{G}=-\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\Big((2A_4+2A_3^2)\psi + (2A_3^2p^2+\mathcal{G}(2A_4+2A_3^2))\psi_{\mathcal{G}}\Big)\phi\,dpd\mathcal{G}, \tag{2.416}$$

and so

$$\mathcal{L}_2^{\star}=-\Big(2A_3^2p^2+\mathcal{G}(2A_4+2A_3^2)\Big)\frac{\partial}{\partial\mathcal{G}}+(2A_4+2A_3^2)I_d. \tag{2.417}$$

Next, in equation (2.411)

$$\mathcal{L}_3=2A_3^2p\mathcal{G}\frac{\partial^2}{\partial\mathcal{G}\partial p}, \tag{2.418}$$

and so

$$\begin{aligned}2A_3^2\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\psi p\mathcal{G}\phi_{\mathcal{G}p}\,dpd\mathcal{G}&=-2A_3^2\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\Big(\psi p+\psi_{\mathcal{G}}p\mathcal{G}\Big)\phi_p\,dpd\mathcal{G}\\&=2A_3^2\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\Big(\psi+\psi_p p+\psi_{\mathcal{G}p}p\mathcal{G}+\psi_{\mathcal{G}}\mathcal{G}\Big)\phi\,dpd\mathcal{G},\end{aligned} \tag{2.419}$$

provided that $\phi(p=1)=\phi(p=p_\infty)=0$, $\psi(p=1)=\psi(p=p_\infty)=$ and $\partial\psi/\partial\mathcal{G}(p=1)=\partial\psi/\partial\mathcal{G}(p=p_\infty)=0$. Hence

$$\mathcal{L}_3^{\star}=2A_3^2\Big(p\frac{\partial}{\partial p}+p\mathcal{G}\frac{\partial^2}{\partial\mathcal{G}\partial p}+\mathcal{G}\frac{\partial}{\partial\mathcal{G}}+I_d\Big). \tag{2.420}$$

Considering the fourth term in equation (2.411)

$$\mathcal{L}_4=\frac{A_3^2}{2}\mathcal{G}\frac{\partial^2}{\partial p^2}, \tag{2.421}$$

so

$$\frac{A_3^2}{2}\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\mathcal{G}\psi\phi_{pp}\,dpd\mathcal{G} = -\frac{A_3^2}{2}\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\mathcal{G}\psi_p\phi_p\,dpd\mathcal{G}$$
$$= \frac{A_3^2}{2}\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\mathcal{G}\psi_{pp}\phi\,dpd\mathcal{G}, \tag{2.422}$$

provided that $\partial\phi/\partial p(p=1) = \partial\phi/\partial p(p=p_\infty) = 0$ and $\partial\psi/\partial p(p=1) = \partial\psi/\partial p(p=p_\infty) = 0$. Hence

$$\mathcal{L}_4^\star = \frac{A_3^2}{2}\mathcal{G}\frac{\partial^2}{\partial p^2}. \tag{2.423}$$

Next in equation (2.411)

$$\mathcal{L}_5 = \left(\frac{A_3^2}{2}\left(2p+\frac{\mathcal{G}}{p}\right) - A_4 p\right)\frac{\partial}{\partial p}, \tag{2.424}$$

so

$$\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\left(\frac{A_3^2}{2}\left(2p+\frac{\mathcal{G}}{p}\right) - A_4 p\right)\psi\phi_p\,dpd\mathcal{G}$$
$$= -\int_0^{\mathcal{G}_\infty}\int_1^{p_\infty}\left(\left(\frac{A_3^2}{2}\left(2-\frac{\mathcal{G}}{p^2}\right) - A_4\right)\psi + \left(\frac{A_3^2}{2}\left(2p+\frac{\mathcal{G}}{p}\right) - A_4 p\right)\psi_p\right)\phi\,dpd\mathcal{G}, \tag{2.425}$$

hence

$$\mathcal{L}_5^\star = -\left(\frac{A_3^2}{2}\left(2p+\frac{\mathcal{G}}{p}\right) - A_4 p\right)\frac{\partial}{\partial p} - \left(\frac{A_3^2}{2}\left(2-\frac{\mathcal{G}}{p^2}\right) - A_4\right)I_d. \tag{2.426}$$

Using equations (2.412) - (2.426) the adjoint operator of (2.411) can be written as

$$\mathcal{L}_{\mathcal{G},p}^\star = 2A_3^2p^2\mathcal{G}\frac{\partial^2}{\partial\mathcal{G}^2} + \left(2A_3^2p^2 - 2\mathcal{G}A_4\right)\frac{\partial}{\partial\mathcal{G}} + \left(2A_3^2p - \frac{A_3^2}{2}\left(2p+\frac{\mathcal{G}}{p}\right) + A_4 p\right)\frac{\partial}{\partial p}$$
$$+ 2A_3^2\mathcal{G}p\frac{\partial^2}{\partial\mathcal{G}\partial p} + \frac{A_3^2}{2}\mathcal{G}\frac{\partial^2}{\partial p^2} + \left(\frac{A_3^2\mathcal{G}}{2p^2} + 3A_3^2 + 3A_4\right)I_d. \tag{2.427}$$

The Fokker-Planck equation for the pair of processes ($\mathcal{G}$, p) is then

$$\frac{\partial P}{\partial L}(L, \mathcal{G}, p) = \mathcal{L}^{\star}_{\mathcal{G},p} P(L, \mathcal{G}, p), \quad P(L=0, \mathcal{G}, p) = \delta(\mathcal{G})\delta(p-1). \tag{2.428}$$

By defining the following

$$\mathbf{a}^{\star} = \begin{bmatrix} 2A_3^2 p^2 \mathcal{G} & A_3^2 \mathcal{G} p \\ A_3^2 \mathcal{G} p & \frac{A_3^2}{2}\mathcal{G} \end{bmatrix}, \quad \mathbf{b}^{\star} = \begin{bmatrix} 2A_3^2 p^2 - 2\mathcal{G} A_4 \\ 2A_3^2 p - A_3^2(2p + \mathcal{G}/p)/2 + A_4 p \end{bmatrix}, \quad \nabla = \begin{bmatrix} \partial/\partial \mathcal{G} \\ \partial/\partial p \end{bmatrix}, \tag{2.429}$$

and

$$Q = \frac{A_3^2 \mathcal{G}}{2p^2} + 3A_3^2 + 3A_4, \tag{2.430}$$

the Fokker-Planck equation (2.428) can be rewritten as

$$\frac{\partial P}{\partial L}(L, \mathcal{G}, p) = (\mathbf{a}^{\star}\nabla + \mathbf{b}^{\star}) \cdot \nabla P(L, \mathcal{G}, p) + QP(L, \mathcal{G}, p). \tag{2.431}$$

Equation (2.431) can be rewritten by using the identity

$$(\mathbf{a}^{\star}\nabla + \mathbf{b}^{\star}) \cdot \nabla = \nabla \cdot (\mathbf{a}^{\star}\nabla) + \mathbf{c}^{\star} \cdot \nabla, \tag{2.432}$$

where

$$\mathbf{c}^{\star} = \mathbf{b}^{\star} - \nabla^T \mathbf{a} = \begin{bmatrix} -A_3^2 \mathcal{G} - 2\mathcal{G} A_4 \\ A_3^2 p - A_3^2(2p + \mathcal{G}/p)/2 + A_4 p \end{bmatrix}. \tag{2.433}$$

The Fokker-Planck equation may then be written as

$$\frac{\partial P}{\partial L}(L, \mathcal{G}, p) = \nabla \cdot \mathbf{a}^{\star} \nabla P(L, \mathcal{G}, p) + \mathbf{c}^{\star} \cdot \nabla P(L, \mathcal{G}, p) + QP(L, \mathcal{G}, p), \tag{2.434}$$

with Dirac delta initial condition

$$P(L=0, \mathcal{G}, p) = \delta(\mathcal{G})\delta(p-1). \tag{2.435}$$

2.8.5 Weak Form of Fokker Planck Equation

The Fokker-Planck equation (2.434) is expressed in its weak form to obtain a numerical solution for the probability density function $P(L, \mathcal{G}, p)$. This numerical solution is obtained here using the FeniCS software package [59] in Python. With Neumann boundary conditions and test function $v(\mathbf{x})$ ($\mathbf{x} = (\mathcal{G}, p)$, $\Omega = [0,1] \times [1, \infty]$). First, the length derivative is approximated via

$$\frac{P^{n+1} - P^n}{\Delta L} \approx (\nabla \cdot \mathbf{a}^{\star} \nabla + \mathbf{c}^{\star} \cdot \nabla + Q) P^{n+1}. \tag{2.436}$$

By multiplying by a test function $v(\mathbf{x})$ and integrating over the domain Ω

$$\begin{aligned} &\int_{\Omega} \Big(v(\mathbf{x}) P^{n+1} - \Delta L \Big(v(\mathbf{x}) \nabla \cdot \mathbf{a}^{\star} \nabla P^{n+1} + v(\mathbf{x}) \mathbf{c}^{\star} \cdot \nabla P^{n+1} + v(\mathbf{x}) Q P^{n+1} \Big) \Big) d\mathbf{x} \\ &= \int_{\Omega} v(\mathbf{x}) P^n d\mathbf{x}. \end{aligned} \tag{2.437}$$

To formulate the weak form of the problem, integrate by parts the second term to obtain

$$\int_{\Omega} v(\mathbf{x}) \nabla \cdot \mathbf{a}^{\star} \nabla P d\mathbf{x} = \int_{\partial\Omega} v(\mathbf{x}) \mathbf{n} \cdot \mathbf{a}^{\star} \nabla P ds - \int_{\Omega} \nabla v(\mathbf{x}) \cdot \mathbf{a}^{\star} \nabla P d\mathbf{x}. \tag{2.438}$$

hence, the weak form of the problem is

$$a(P, v) = L(v), \tag{2.439}$$

where

$$a(P, v) = \int_{\Omega} \Big(v(\mathbf{x}) P + \Delta L \, S(P) \Big) d\mathbf{x}, \tag{2.440}$$

$$L(v) = \int_{\Omega} v(\mathbf{x}) P \, d\mathbf{x}, \tag{2.441}$$

with

$$S(P) = \nabla v(\mathbf{x}) \cdot \mathbf{a}^{\star} \nabla P - v(\mathbf{x}) \mathbf{c}^{\star} \cdot \nabla P - v(\mathbf{x}) Q P. \tag{2.442}$$

This weak form of the problem is now solved directly using FeniCS [59] for discretised length steps (ΔL). A rectangular mesh is used for the domain Ω. A convergence study for optimal mesh discretisation suggested a coarse mesh in the $\mathcal{G}$ direction and a very fine mesh in the p direction.

2.9 Results

The properties of austenite (see Table 2.1) were used to obtain the diffusion coefficients given by equations (2.322) to (2.326) which appear in the Fokker-Planck equation (2.439). The numerical solution of the the Fokker-Planck equation (2.434) provides the probability density function $P(L, \mathcal{G}, p)$ which is then used to compute the statistical moments of the power transmission coefficients in equation (2.402). A frequency ($\tilde{\omega} \sim \varepsilon^{-2}$) and a mean wave speed of $c_3 = 4500\,\mathrm{ms}^{-1}$ were used together with the stiffness tensor constants in Table 2.1. In Figure 2.6 the mean power transmission coefficient versus the

Elastic Material Constants					
	$\bar{\bar{c}}_{11}$	$\bar{\bar{c}}_{33}$	$\bar{\bar{c}}_{44}$	$\bar{\bar{c}}_{66}$	$\bar{\bar{c}}_{13}$
Austenite	217.1 GPa	263.2 GPa	82.4 GPa	128.4 GPa	144.4 GPa

Table 2.1: Table of material constants for Austenite [36] with density $\rho = 8100\,\mathrm{kgm}^{-3}$.

depth into the random medium L is plotted. As the degree of anisotropy ν is increased, an increased decay in the amplitude of the coherent wave is observed. Figure 2.6 also shows the mean and variance in the power transmission coefficient.

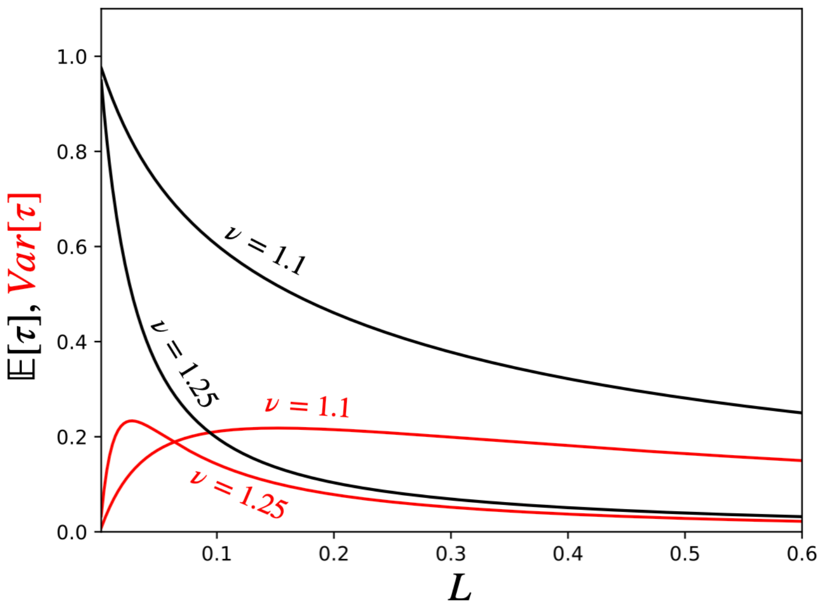

Figure 2.6: Plots of the mean transmission coefficient (see equation (2.402)) (black) with the associated variance (red) for the degree of anisotropy $\nu = (1.1, 1.25)$ versus L. The non-dimensionalised penetration depth L. The material parameters are given in Table 2.1. Eventually this will asymptote to zero, and at that stage the process has self averaged. This means that very thick materials will have a narrow probability density function and so the mean can be used to characterise the material. Therefore homogenisation could be applied in such cases, however, in intermediate material thicknesses, uncertainty quantification is needed.

For an infinitesimally thin material (characterised by $L = 0$) the energy of the wave is fully transmitted with no uncertainty. As the thickness increases, then the mean transmission coefficient decreases. This is more marked for materials with a higher degree of anisotropy. At the same time the uncertainty increases and peaks at a depth of material that varies with the degree of anisotropy. It can be seen that this high uncertainty persists for a large range of lengths L for $\nu = 1.1$.

2.9.1 Concluding Remarks

A probabilistic model of a monochromatic horizontally polarised shear wave propagating in a randomly layered heterogeneous medium constructed of locally anisotropic layers has been constructed and studied. The spatial scaling regime is such that the internal microstructure of the medium interacts with the probing wave to produce an incoherent coda wave. The orientation of the anisotropic material varies randomly from layer to layer according to a Markov process. Using elastodynamic equations, expressions for the forward and backward wave modes, which describe the reflected and transmitted energy for the input wave were derived. Via a series of transformations, a system of stochastic differential equations was then derived for a propagator formulation of this wave-mode problem. Utilising a limit theorem from stochastic analysis a linear partial differential equation (Fokker-Planck equation [64]) for the probability density function associated with the transmitted power was derived which was solved via a finite element package in Python [59]. Its numerical solution enabled an investigation into the effect that the material parameters have on the decay of energy in the coherent part of the transmitted probing wave.

In particular the statistics of the transmitted energy through a class of austenitic steel welds was reported upon. Varying the degree of anisotropy parameter ν had a significant impact on the attenuation of the coherent energy. By capturing the randomness present in materials such as austenite welds, this model could be used by experimental scientists in the NDT community. This model could be extended for a broadband pulse to estimate optimal frequency ranges for a probing ultrasonic wave in order to image (with good resolution) to a certain depth in a given random media. The model presented in this Chapter could also be used in finite element simulations of ultrasonic wave propagation, to generate attenuation factors [65] for elastic wave propagation in such layered materials without the need for explicitly including the layer geometry in the simulation; calculations for the correlation integrals can be obtained from experimental images [63].

The next chapter studies horizontally polarised elastic shear waves in a randomly layered media, but with the key difference that the form of the stress tensor will change. This change creates a symmetry in the governing equations that allow for a special parameterisation, which allows for more analytical headway in terms of computing the moments of the reflected and transmitted energy, without the need for the finite element method.

Chapter 3

Elastic Shear Wave Propagation in Locally Anisotropic Heterogeneous Media; Polycrystalline Rotations about the Lateral Direction

3.1 Nomenclature

Parameter			Equation
α	Lumped parameter	$[\mathrm{M}^{-1}\mathrm{LT}]$	(3.31)
$\bar{\alpha}$	Lumped parameter	$[\mathrm{M}^{-1}\mathrm{LT}]$	(3.122)
β	Lumped parameter	$[\mathrm{ML}^{-3}\mathrm{T}^{-1}]$	(3.32)
$\bar{\beta}$	Lumped parameter	$[\mathrm{ML}^{-3}\mathrm{T}^{-1}]$	(3.126)
Γ_α	Lumped parameter	[–]	(3.124)
Γ_β	Lumped parameter	[–]	(3.128)
γ_1	Material expression	$[\mathrm{M}^{\frac{1}{2}}\mathrm{L}^{-2}\mathrm{T}^{-\frac{1}{2}}]$	(3.54)
γ_2	Material expression	$[\mathrm{M}^{-\frac{1}{2}}\mathrm{L}^{-2}\mathrm{T}^{\frac{1}{2}}]$	(3.55)

γ_ν	Propagator parameterisation function	[–]	(3.214)
Δ_1	Material expression	$[L^{-1}]$	(3.57)
Δ_2	Material expression	$[L^{-1}]$	(3.58)
Δ_3	Material expression	$[L^{-1}]$	(3.61)
Δ_4	Material expression	$[L^{-1}]$	(3.62)
δ_1	Lumped parameter	[–]	(3.167)
δ_2	Lumped parameter	[–]	(3.168)
ε	Small dimensionless parameter	[–]	(3.132)
ζ	Lumped parameter	$[L^{-1}]$	(3.38)
η	Auxiliary (power transmission coefficient) process	[–]	(3.277)
θ	Rotation angle of material slowness surface	[–]	(3.2)
ϑ	Lumped parameter	[–]	(3.119)
κ_1	Wavenumber in x_1	$[L^{-1}]$	(3.24)
Λ	Eigenvalues of matrix $\mathbf{M}$	$[M^{-1}L^{-1}]$	(3.34)
λ_3	Wavelength in x_3 direction	[L]	(3.161)
μ	Legendre function of the first kind parameter	[–]	(3.283)
ν	Ratio of wave numbers	[–]	(3.142)
ξ	Velocity in x_3 direction	$[LT^{-1}]$	(3.16)
ϖ	Dimensionless parameter	[–]	(3.132)
ρ	Constant material density	$[ML^{-3}]$	(3.1)
ϱ	Lumped parameter	$[ML^{-1}T^{-2}]$	(3.114)
$\boldsymbol{\sigma}_{1,2,3}$	Pauli spin matrices	[–]	(3.174)
$\boldsymbol{\sigma}$	Stochastic differential equation matrix coefficient	[–]	(3.224)
σ	Random process amplitude	[–]	(3.107)

τ_{jk}	Material stress tensor	$[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$	(3.4)
τ	Power transmission coefficient	[–]	(3.250)
$\tilde{\Upsilon}^{AS}$	Non-dimensional anti-symmetric correlation integral	[–]	(3.194)
$\tilde{\Upsilon}$	Non-dimensional symmetric correlation integral	[–]	(3.193)
φ	Lumped parameter	[–]	(3.116)
ϕ_ν	Propagator parametrisation function	[–]	(3.214)
ϕ	Lumped phase parameter	[–]	(3.169)
$\chi_{1,2,3,4}$	Propagator matrix functions	$[\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-\frac{3}{2}}]$	(3.86)
χ	Propagator matrix function	[–]	(3.214)
$\varkappa$	Propagator matrix function	[–]	(3.215)
ψ_ν	Propagator parameterisation function	[–]	(3.215)
ω	Angular frequency	$[\mathrm{T}^{-1}]$	(3.20)
A	Independent stress tensor component	$[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$	(3.3)
A_p	Poisson process intensity	[–]	(3.213)
A_1	Propagator matrix evolution equation coefficients	[–]	(3.205)
A_2	Propagator matrix evolution equation coefficients	[–]	(3.206)
A_3	Propagator matrix evolution equation coefficients	[–]	(3.207)
$\hat{a}$	Forward wave mode in frequency domain	$[\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-\frac{3}{2}}]$	(3.41)
$\hat{b}$	Backward wave mode in frequency domain	$[\mathrm{M}^{\frac{1}{2}}\mathrm{T}^{-\frac{3}{2}}]$	(3.41)
$\boldsymbol{b}$	Drift vector	[–]	(3.226)
$\mathbf{C}$	Correlation integral matrix	[–]	(3.195)
C	Stress tensor component C_{33} (when the symmetry axis points in the x_1 direction)	$[\mathrm{ML}^{-1}\mathrm{T}^{-2}]$	(3.3)
c_3	Mean wave velocity in x_3 direction	$[\mathrm{LT}^{-1}]$	(3.130)

c_{44}	Stress tensor component (Voigt notation)	$[\mathrm{ML^{-1}T^{-2}}]$	(3.11)
$\tilde{c}_{44}$	Non-dimensional stress tensor component	[–]	(3.137)
c_{66}	Stress tensor component (Voigt notation)	$[\mathrm{ML^{-1}T^{-2}}]$	(3.10)
$\tilde{c}_{66}$	Non-dimensional stress tensor component	[–]	(3.139)
$c_{1,2}$	Arbitrary constants	[–]	(3.65)
c_{ijkl}	Stress tensor	$[\mathrm{ML^{-1}T^{-2}}]$	(3.3)
$\mathbf{D}$	Diagonalised factor of matrix $\mathbf{M}$	$[\mathrm{M^{-1}L^{-1}}]$	(3.37)
d_1	Lumped parameter	[–]	(3.145)
d_2	Lumped parameter	[–]	(3.146)
e_{kl}	Symmetric strain tensor	[–]	(3.4)
F	Stress tensor component (C_{13} when the symmetry axis points in the x_1 direction)	$[\mathrm{ML^{-1}T^{-2}}]$	(3.3)
$g^{(i)}$	Coefficients of propagator matrix equation	[–]	(3.176)-(3.178)
$\mathbf{H}$	Coupling matrix in wave amplitude evolution equation	$[\mathrm{L^{-1}}]$	(3.63)
$\mathbf{H}^{\varepsilon}$	Random matrix	[–]	(3.166)
$\mathbf{h}_i$	Matrix coefficients in propagator matrix expansion .	[–]	(3.176)-(3.178)
$J^{(n)(\mu)}$	Legendre function weighted moments	[–]	(3.291)
$K^{(n)}(\mu)$	Product of Legendre function parameter (and moment indices)	[–]	(3.313)
$\mathcal{L}$	Infinitesimal generator	[–]	(3.237)
$J^{(n)}(\mu)$	Legendre function weighted moments	[–]	(3.291)
L_3	Typical propagation distance in x_3	[L]	(3.130)
L_{loc}	Localization length	[L]	(3.266)
l	Mean layer size	[L]	(3.131)
$\mathbf{M}$	Stress-strain coupling matrix	$[\mathrm{M^{-1}LT}]$	(3.33)

m	Fluctuations in the crystal orientation	[–]	(3.107)
N	Stress tensor component (when the symmetry axis points in the x_1 direction)	[ML^{-1}T^{-2}]	(3.3)
$\mathbf{P}$	Propagator matrix	[–]	(3.67)
$\mathbf{P}$	Propagator matrix	[–]	(3.171)
$P_{-\frac{1}{2}+i\mu}$	Legendre function of the first kind	[–]	(3.284)
p	Probability density function of the power transmission coefficient	[–]	(3.289)
$\mathbf{Q}$	Factor of matrix $\mathbf{M}$ (first row)	[M^{-1}L^{-1}]	(3.36)
$\mathbf{Q}$	Factor of matrix $\mathbf{M}$ (second row)	[M^{-1}L^{-3}T^{-1}]	(3.36)
ϱ	Lumped parameter	[ML^{-1}T^{-2}]	(3.114)
R	Power reflection coefficient	[–]	(3.315)
R_ν	Reflection coefficient	[–]	(3.245)
R	Number of rays	[–]	(3.211)
$r^{\pm}_{0,1}$	Boundary interface linking expressions	[–]	(3.90)
S	Stress tensor component (C_{44} when the symmetry axis points in the x_1 direction)	[ML^{-1}T^{-2}]	(3.3)
$\boldsymbol{s}$	Symmetry axis vector	[–]	(3.2)
T_ν	Transmission coefficient	[–]	(3.245)
t	Time	[T]	(3.1)
$\mathbf{u}$	Three dimensional wave displacement vector	[L]	(3.6)
$\mathbf{V}$	Eigenvector (first entry) of matrix $\mathbf{M}$	[M^{-1}L^{-1}]	(3.35)
$\mathbf{V}$	Eigenvector (second entry) of matrix $\mathbf{M}$	[M^{-1}L^{-3}T^{-1}]	(3.35)
v	Transformed power reflection coefficient	[–]	(3.321)
$W_n(x_3)$	Independent Brownian motion	[–]	(3.203)

$Z(x_3)$	Transformed power transmission coefficient	[−]	(3.255)

3.2 Introduction

This Chapter aims to study length scale regimes where the received wave is complex, exhibiting many fluctuations over a long time period caused by its convoluted journey through the heterogeneous medium. In this case the wave is so affected by its interactions with the medium that a homogenisation approach is inappropriate. This can occur when the propagation distance L is much larger than the wavelength (λ_3) of propagation which in turn is much larger than the layer sizes l ($L \gg \lambda_3 \gg l$) and the fluctuations in the material are large ($\sigma \sim 1$); the so called strongly heterogeneous regime [26]. It can also occur in the regime where $L_3 \gg \lambda_3 \sim l$ and $\sigma \ll 1$, which is the so called *weakly* heterogeneous regime and it is this latter case that will be examined in this Chapter.

The decay of a coherent wave has been studied in the literature for general acoustic and elastic systems. Articles [26], [25] and [66] consider acoustic systems in one dimension and more recently [8] considered a three dimensional elastic medium. In each case, stochastic differential equations (SDE's) are used to model the random fluctuations in the material properties [34]. Each of these studies examined a wave travelling in a medium whose properties only varied in the direction of propagation, leading to a system of stochastic differential equations with a propagator matrix with certain symmetry properties. This Chapter uses a similar approach to consider a shear wave propagating in an elastic medium with random fluctuations in the material microstructure. The effect that the localisation length and the degree of anisotropy of the host material on the attenuation of the transmitted wave, is studied.

3.3 Governing Equations

This Chapter focuses on studying waves in layered polycrystalline media [67] composed of a single anisotropic material. The media is partitioned into a tessellation of grains and the orientation of the material varies from one grain to the next. Since the material is anisotropic, this variation in orientation affects the speed at which the incident wave travels through each grain and hence the wave experiences a spatially heterogeneous medium. This local variation in wave speed can be described using a slowness curve which can be derived analytically from the Christoffel equation [68], [69]. The governing elastic wave equation can be written

$$\rho \frac{\partial^2 u_i}{\partial t^2} = \sum_{k=1}^{3} \frac{\partial \tau_{jk}}{\partial x_k}, \quad i = 1, 2, 3, \tag{3.1}$$

where the displacement vector is denoted $\boldsymbol{u} = (u_1(t, \boldsymbol{x}), u_2(t, \boldsymbol{x}), u_3(t, \boldsymbol{x}))$ and ρ is the density of the material (assumed to be constant) and τ_{jk} is the material stress tensor. The elastic tensor for a transversely anisotropic medium has five independent stress tensor components namely C_{11}, C_{33}, C_{13}, C_{66}, C_{44} with $\boldsymbol{s} = (s_1, s_2, s_3)$ as the symmetry axis vector defined by

$$\boldsymbol{s} = \begin{bmatrix} \cos\theta(x_3) \\ \sin\theta(x_3) \\ 0 \end{bmatrix}, \tag{3.2}$$

then, in contrast to Chapter 2, the stress tensor can be written [36] as

$$\begin{aligned} c_{ijkl} =& (A - 2N)\delta_{ij}\delta_{kl} + N\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \\ &+ (F - A + 2N)\left(\delta_{ij}s_k s_l + \delta_{kl}s_i s_j\right) \\ &+ (S - N)\left(\delta_{ik}s_j s_l + \delta_{il}s_j s_k + \delta_{jk}s_i s_l + \delta_{jl}s_i s_k\right) \\ &+ (A + C - 2F - 4S)s_i s_j s_k s_l, \end{aligned} \tag{3.3}$$

where $A = C_{33}$, $C = C_{11}$, $F = C_{13}$, $N = C_{44}$, $S = C_{66}$ when $\theta = 0$; when the stiffness matrix [70] symmetry axis points in the x_1 direction.

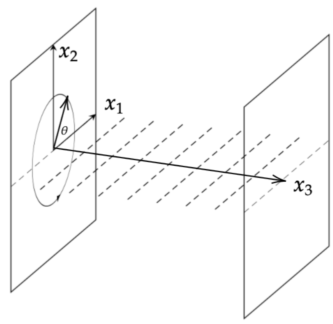

Figure 3.1: For each value of x_3 the material is anisotropic in the (x_1, x_2) plane. The degree of anisotropy is dictated by the material's slowness surface and $\theta(x_3)$ describes its rotation in the (x_1, x_2) plane. The direction of wave propagation is fixed and lies in the (x_1, x_3) plane. In each layer (shown by the dashed lines) the material properties are constant.

This form assumes that the anisotropy is spatially varying only in the x_3 direction as in [26], so that $\boldsymbol{s}$ lies in the (x_1, x_2) plane, making an angle $\theta(x_3)$ with the x_1 axis as shown in Figure 1. The elastic tensor relates the symmetric strain and stress tensors via Hooke's law

$$\tau_{ij} = \sum_{k,l=1}^{3} c_{ijkl} e_{kl}, \tag{3.4}$$

where the symmetric strain tensor is given by

$$e_{kl} = \frac{1}{2} \left(\frac{\partial u_k}{\partial x_l} + \frac{\partial u_l}{\partial x_k} \right). \tag{3.5}$$

3.3.1 Wave Parameterisation

Consider a horizontally polarised shear wave [9] with displacement vector

$$u_j = (0, u_2(x_1, x_3), 0), \tag{3.6}$$

whereby the medium vibrates in a direction perpendicular to the (x_1, x_3) plane. The wave direction is then in the (x_1, x_3) plane. Using this parameterisation in equation (3.5) it can be observed that the only non-vanishing strains are e_{12}, e_{21}, e_{23} and e_{32}.

The symmetric stress tensor can then be written as

$$\tau_{ij} = \frac{1}{2}\left(c_{ij12}\frac{\partial u_2}{\partial x_1} + c_{ij21}\frac{\partial u_2}{\partial x_1} + c_{ij23}\frac{\partial u_2}{\partial x_3} + c_{ij32}\frac{\partial u_2}{\partial x_3}\right). \tag{3.7}$$

From equation (3.7), it is clear by inspection that the only non-vanishing stresses are

$$\tau_{21} = \tau_{12} = \frac{1}{2}\left(c_{1212}\frac{\partial u_2}{\partial x_1} + c_{1221}\frac{\partial u_2}{\partial x_1} + c_{1223}\frac{\partial u_2}{\partial x_3} + c_{1232}\frac{\partial u_2}{\partial x_3}\right), \tag{3.8}$$

$$\tau_{32} = \frac{1}{2}\left(c_{3212}\frac{\partial u_2}{\partial x_1} + c_{3221}\frac{\partial u_2}{\partial x_1} + c_{3223}\frac{\partial u_2}{\partial x_3} + c_{3232}\frac{\partial u_2}{\partial x_3}\right), \tag{3.9}$$

and from equation (3.3)

$$c_{2121} = c_{1212} = c_{66}(x_3) = S + (A + C - 2F - 4S)\cos^2(\theta(x_3))\sin^2(\theta(x_3)), \tag{3.10}$$

$$c_{3223} = c_{3232} = c_{44}(x_3) = N + (S - N)\sin^2(\theta(x_3)), \tag{3.11}$$

$$c_{1223} = c_{1232} = c_{46} = c_{64} = 0. \tag{3.12}$$

The stresses in equations (3.8) and (3.9) simplify to (dropping the explicit x_3 dependency for now)

$$\tau_{21} = c_{66}\frac{\partial u_2}{\partial x_1}, \tag{3.13}$$

$$\tau_{32} = c_{44}\frac{\partial u_2}{\partial x_3}, \tag{3.14}$$

and so the elastic wave equation can be rewritten as

$$\begin{aligned}\rho\frac{\partial^2 u}{\partial t^2} &= \frac{\partial \tau_{12}}{\partial x_1} + \frac{\partial \tau_{32}}{\partial x_3} \\ &= \frac{\partial}{\partial x_1}\left(c_{66}\frac{\partial u_2}{\partial x_1}\right) + \frac{\partial}{\partial x_3}\left(c_{44}\frac{\partial u_2}{\partial x_3}\right).\end{aligned} \tag{3.15}$$

The velocity in the x_2 direction is defined as

$$\xi = \frac{\partial u_2}{\partial t} = u_{2,t}, \tag{3.16}$$

and so the elastic wave equation (3.15) can then be written as

$$\rho\xi_{,t} = \tau_{21,1} + \tau_{32,3}. \tag{3.17}$$

Finally, taking partial derivatives with respect to time in equations (3.13) and (3.14) gives

$$\tau_{21,t} = c_{66}\xi_{,1}, \tag{3.18}$$

$$\tau_{32,t} = c_{44}\xi_{,3}. \tag{3.19}$$

3.3.2 Frequency Wavenumber Domain

Now take Fourier transforms in time and space (x_1 direction with respect to a wavenumber κ_1) of the governing stress and velocity equations which aids the analysis; in this frequency-wavenumber domain the stress and velocity are denoted by $\hat{\tau}$ and $\hat{\xi}$ respectively. Define the temporal Fourier transform of a function $f(t, x_1, x_3)$ by

$$\check{f}(\omega, x_1, x_3) = \int f(t, x_1, x_3)e^{i\omega t}dt, \tag{3.20}$$

where ω is the angular frequency. Applying this transform to equations (3.17), (3.18) and (3.19) gives

$$-\rho i\omega\check{\xi} = \check{\tau}_{21,1} + \check{\tau}_{32,3}, \tag{3.21}$$

$$-i\omega\check{\tau}_{21} = c_{66}\check{\xi}_{,1}, \tag{3.22}$$

$$-i\omega\check{\tau}_{32} = c_{44}\check{\xi}_{,3}. \tag{3.23}$$

Define the spatial Fourier transform with respect to a wavenumber κ_1 in the x_1 direction by

$$\hat{f}(\omega, \kappa_1, x_3) = \int \check{f}(\omega, x_1, x_3)e^{i\kappa_1 x_1}dx_1. \tag{3.24}$$

Applying this transform to equations (3.21), (3.22) and (3.23) then gives

$$-\rho i\omega\hat{\xi}(\omega,\kappa_1,x_3) = -i\kappa_1\hat{\tau}_{21}(\omega,\kappa_1,x_3) + \hat{\tau}_{32,3}(\omega,\kappa_1,x_3), \tag{3.25}$$

$$-i\omega\hat{\tau}_{21}(\omega,\kappa_1,x_3) = -i\kappa_1 c_{66}\hat{\xi}(\omega,\kappa_1,x_3), \tag{3.26}$$

$$-i\omega\hat{\tau}_{32}(\omega,\kappa_1,x_3) = c_{44}\hat{\xi}_{,3}(\omega,\kappa_1,x_3). \tag{3.27}$$

Inserting equation (3.26) into equation (3.25) gives

$$\hat{\tau}_{32,3} = i\left(\frac{\kappa_1^2 c_{66} - \rho\omega^2}{\omega}\right)\hat{\xi}, \tag{3.28}$$

and from equation (3.27)

$$\hat{\xi}_{,3} = -\frac{i\omega}{c_{44}}\hat{\tau}_{32}. \tag{3.29}$$

Equations (3.28) and (3.29) can be rewritten in matrix form (note that the subscripts are dropped for notational convenience) to obtain a system of velocity and stress evolution equations which read

$$\begin{aligned}\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix} &= \begin{bmatrix}0 & -i\omega/c_{44}\\ i(\kappa_1^2 c_{66}-\rho\omega^2)/\omega & 0\end{bmatrix}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}\\ &= \begin{bmatrix}0 & -i\alpha\\ -i\beta & 0\end{bmatrix}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix},\end{aligned} \tag{3.30}$$

where

$$\alpha = \frac{\omega}{c_{44}}, \tag{3.31}$$

and

$$\beta = \frac{\rho\omega^2 - \kappa_1^2 c_{66}}{\omega}, \tag{3.32}$$

where it is assumed that $\beta > 0$. Now let

$$\mathbf{M} = \begin{bmatrix} 0 & -i\alpha \\ -i\beta & 0 \end{bmatrix}, \tag{3.33}$$

which has eigenvalues

$$\Lambda^{\pm} = \pm i\sqrt{\beta\alpha}, \tag{3.34}$$

and the associated eigenvectors

$$\mathbf{V}^{\pm} = [\mp\sqrt{\beta\alpha},\, \beta]^T. \tag{3.35}$$

Letting

$$\mathbf{Q} = \begin{bmatrix} -\sqrt{\alpha\beta} & \sqrt{\alpha\beta} \\ \beta & \beta \end{bmatrix} = \begin{bmatrix} -\zeta & \zeta \\ \beta & \beta \end{bmatrix}, \tag{3.36}$$

and

$$\mathbf{D} = \begin{bmatrix} \Lambda^{+} & 0 \\ 0 & \Lambda^{-} \end{bmatrix} = \begin{bmatrix} i\zeta & 0 \\ 0 & -i\zeta \end{bmatrix}, \tag{3.37}$$

where

$$\zeta = \sqrt{\alpha\beta}, \tag{3.38}$$

it follows that $\mathbf{M} = \mathbf{QDQ}^{-1}$, and therefore equation (3.30) can be written as

$$\frac{\partial}{\partial x_3} \begin{bmatrix} \hat{\xi} \\ \hat{\tau} \end{bmatrix} = \mathbf{QDQ}^{-1} \begin{bmatrix} \hat{\xi} \\ \hat{\tau} \end{bmatrix}. \tag{3.39}$$

To tackle this system of equations, it is instructive to first consider a series of problems where the materials involved are homogeneous so ζ and β do not depend on x_3. To signify this, they will be denoted by $\bar{\zeta}$ and $\bar{\beta}$. Examination of this simplified problem will then shape the treatment of the heterogeneous case. Temporarily assuming $\mathbf{Q}$ is independent of x_3 (so that the material is homogeneous), equation (3.39) can be written as

$$\frac{\partial}{\partial x_3}\left(\bar{\mathbf{Q}}^{-1}\begin{bmatrix}\hat{\xi}\\ \hat{\tau}\end{bmatrix}\right) = \bar{\mathbf{D}}\left(\bar{\mathbf{Q}}^{-1}\begin{bmatrix}\hat{\xi}\\ \hat{\tau}\end{bmatrix}\right). \tag{3.40}$$

Now introduce new variables $\hat{a}$ and $\hat{b}$ which are right and left-going wave modes respectively, via the transformation

$$\begin{bmatrix}\hat{a}(\omega,\kappa_1,x_3)\\ \hat{b}(\omega,\kappa_1,x_3)\end{bmatrix} = \left(\bar{\zeta}^{1/2}\bar{\beta}^{1/2}\right)\bar{\mathbf{Q}}^{-1}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}. \tag{3.41}$$

Equation (3.40) can then be written as

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{a}(\omega,\kappa_1,x_3)\\ \hat{b}(\omega,\kappa_1,x_3)\end{bmatrix} = \begin{bmatrix}i\bar{\zeta} & 0\\ 0 & -i\bar{\zeta}\end{bmatrix}\begin{bmatrix}\hat{a}(\omega,\kappa_1,x_3)\\ \hat{b}(\omega,\kappa_1,x_3)\end{bmatrix}. \tag{3.42}$$

which can be solved to give

$$\hat{a}(\omega,\kappa_1,x_3) = \hat{a}(\omega,\kappa_1)e^{i\bar{\zeta}x_3}, \tag{3.43}$$

$$\hat{b}(\omega,\kappa_1,x_3) = \hat{b}(\omega,\kappa_1)e^{-i\bar{\zeta}x_3}. \tag{3.44}$$

By rearranging equation (3.41)

$$\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix} = \begin{bmatrix}-\sqrt{\bar{\zeta}/\bar{\beta}} & \sqrt{\bar{\zeta}/\bar{\beta}}\\ \sqrt{\bar{\beta}/\bar{\zeta}} & \sqrt{\bar{\beta}/\bar{\zeta}}\end{bmatrix}\begin{bmatrix}\hat{a}(\omega,\kappa_1,x_3)\\ \hat{b}(\omega,\kappa_1,x_3)\end{bmatrix}, \tag{3.45}$$

The stress and velocity equations can then be written as

$$\hat{\xi}(\omega,\kappa_1,x_3) = \sqrt{\bar{\zeta}/\bar{\beta}}\left(\hat{b}(\omega,\kappa_1)e^{-i\bar{\zeta}x_3} - \hat{a}(\omega,\kappa_1)e^{i\bar{\zeta}x_3}\right), \tag{3.46}$$

and

$$\hat{\tau}(\omega, \kappa_1, x_3) = \sqrt{\bar{\beta}/\bar{\zeta}}\Big(\hat{b}(\omega, \kappa_1)e^{-i\bar{\zeta}x_3} + \hat{a}(\omega, \kappa_1)e^{i\bar{\zeta}x_3}\Big). \tag{3.47}$$

3.4 Wave Propagation in a Heterogeneous Layer

This Section examines a heterogeneous layer, using an ansatz guided by observations in Section 3.3 for a homogeneous medium. For the generated right-moving modes $\hat{a}(\omega, x_3)$ and left-moving modes $\hat{b}(\omega, x_3)$, it is assumed that the wave mode functions now have spatial dependence [60]

$$\hat{\xi}(\omega, \kappa_1, x_3) = \sqrt{\bar{\zeta}/\bar{\beta}}\left(\hat{b}(\omega, \kappa_1, x_3)e^{-i\bar{\zeta}x_3} - \hat{a}(\omega, \kappa_1, x_3)e^{i\bar{\zeta}x_3}\right), \tag{3.48}$$

$$\hat{\tau}(\omega, \kappa_1, x_3) = \sqrt{\bar{\beta}/\bar{\zeta}}\left(\hat{b}(\omega, \kappa_1, x_3)e^{-i\bar{\zeta}x_3} + \hat{a}(\omega, \kappa_1, x_3)e^{i\bar{\zeta}x_3}\right). \tag{3.49}$$

Combining these two equations gives

$$\hat{a}(\omega, \kappa_1, x_3) = \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\rho}}\,\hat{\tau}(\omega, \kappa_1, x_3) - \sqrt{\bar{\rho}/\bar{\zeta}}\,\hat{\xi}(\omega, \kappa_1, x_3)\right)e^{-i\bar{\zeta}x_3}, \tag{3.50}$$

and

$$\hat{b}(\omega, \kappa_1, x_3) = \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\rho}}\,\hat{\tau}(\omega, \kappa_1, x_3) + \sqrt{\bar{\rho}/\bar{\zeta}}\,\hat{\xi}(\omega, \kappa_1, x_3)\right)e^{i\bar{\zeta}x_3}. \tag{3.51}$$

Taking the spatial derivative of equation (3.50) gives

$$\frac{\partial \hat{a}}{\partial x_3} = \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\beta}}\left(\frac{\partial \hat{\tau}}{\partial x_3} - i\bar{\zeta}\hat{\tau}\right) - \sqrt{\bar{\beta}/\bar{\zeta}}\left(\frac{\partial \hat{\xi}}{\partial x_3} - i\bar{\zeta}\hat{\xi}\right)\right)e^{-i\bar{\zeta}x_3}. \tag{3.52}$$

From the spatially dependent linear system given in equation (3.30) this can be rewritten as

$$\frac{\partial \hat{a}}{\partial x_3}(\omega, \kappa_1, x_3) = \frac{ie^{-i\bar{\zeta}x_3}}{2}\Big(\gamma_1(x_3)\hat{\xi}(\omega, \kappa_1, x_3) + \gamma_2(x_3)\hat{\tau}(\omega, \kappa_1, x_3)\Big), \tag{3.53}$$

where

$$\gamma_1(x_3) = \beta\sqrt{\bar{\zeta}/\bar{\beta}} + \bar{\zeta}\sqrt{\bar{\beta}/\bar{\zeta}}\,, \tag{3.54}$$

$$\gamma_2(x_3) = \alpha\sqrt{\bar{\beta}/\bar{\zeta}} - \bar{\zeta}\sqrt{\bar{\zeta}/\bar{\beta}}\,. \tag{3.55}$$

Inserting equations (3.48) and (3.49) into equation (3.53) gives

$$\frac{\partial \hat{a}}{\partial x_3}(\omega, \kappa_1, x_3) = \hat{a}(\omega, \kappa_1, x_3)\Delta_1(x_3) + \hat{b}(\omega, \kappa_1, x_3)\Delta_2(x_3), \tag{3.56}$$

where

$$\Delta_1(x_3) = \frac{i}{2}\left(\gamma_2\sqrt{\bar{\beta}/\bar{\zeta}} - \gamma_1\sqrt{\bar{\zeta}/\bar{\beta}}\right) = \frac{i}{2}\left(\alpha\frac{\bar{\beta}}{\bar{\zeta}} - \beta\frac{\bar{\zeta}}{\bar{\beta}} - 2\bar{\zeta}\right), \tag{3.57}$$

$$\Delta_2(x_3) = \frac{i}{2}\left(\gamma_2\sqrt{\bar{\beta}/\bar{\zeta}} + \gamma_1\sqrt{\bar{\zeta}/\bar{\beta}}\right)e^{-2i\bar{\zeta}x_3} = \frac{i}{2}\left(\alpha\frac{\bar{\beta}}{\bar{\zeta}} + \beta\frac{\bar{\zeta}}{\bar{\beta}}\right)e^{-2i\bar{\zeta}x_3}. \tag{3.58}$$

Repeating the same calculation for the left-going mode (by differentiating equation (3.51)) gives

$$\frac{\partial \hat{b}(\omega, \kappa_1, x_3)}{\partial x_3} = \frac{ie^{i\bar{\zeta}x_3}}{2}\bigg(\gamma_1(x_3)\hat{\xi}(\omega, \kappa_1, x_3) - \gamma_2(x_3)\hat{\tau}(\omega, \kappa_1, x_3)\bigg). \tag{3.59}$$

Using equations (3.48) and (3.49), equation (3.59) can be rewritten as

$$\frac{\partial \hat{b}(\omega, \kappa_1, x_3)}{\partial x_3} = \hat{a}(\omega, \kappa_1, x_3)\Delta_3(x_3) + \hat{b}(\omega, \kappa_1, x_3)\Delta_4(x_3), \tag{3.60}$$

where

$$\Delta_3(x_3) = \frac{i}{2}\left(-\gamma_2\sqrt{\bar{\beta}/\bar{\zeta}} - \gamma_1\sqrt{\bar{\zeta}/\bar{\beta}}\right)e^{2i\bar{\zeta}x_3}, \tag{3.61}$$

$$\Delta_4(x_3) = \frac{i}{2}\left(-\sqrt{\bar{\beta}/\bar{\zeta}}\,\gamma_2 + \sqrt{\bar{\zeta}/\bar{\beta}}\,\gamma_1\right). \tag{3.62}$$

Equations (3.56) and (3.60) can be combined to show that the complex mode amplitudes $\hat{a}(\omega, \kappa_1, x_3)$ and $\hat{b}(\omega, \kappa_1, x_3)$ satisfy the linear system

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{a}(\omega, x_3)\\ \hat{b}(\omega, x_3)\end{bmatrix} = \begin{bmatrix}\Delta_1(x_3) & \Delta_2(x_3)\\ \Delta_3(x_3) & \Delta_4(x_3)\end{bmatrix}\begin{bmatrix}\hat{a}(\omega, x_3)\\ \hat{b}(\omega, x_3)\end{bmatrix}$$
$$= \mathbf{H}\begin{bmatrix}\hat{a}(\omega, x_3)\\ \hat{b}(\omega, x_3)\end{bmatrix}, \tag{3.63}$$

where the wavenumber dependence κ_1 is dropped for brevity. Observe from equations (3.57), (3.58), (3.61) and (3.62) that $\Delta_1 = \overline{\Delta_4}$, $\Delta_2 = \overline{\Delta_3}$ (where the bar here denotes the complex conjugate) and hence $\mathrm{Tr}(\mathbf{H}) = 0$.

3.4.1 Propagator Matrix Formulation

Suppose that

$$\begin{bmatrix}\chi_1(\omega, x_3)\\ \chi_3(\omega, x_3)\end{bmatrix} \text{ and } \begin{bmatrix}\chi_2(\omega, x_3)\\ \chi_4(\omega, x_3)\end{bmatrix}, \tag{3.64}$$

form two linearly independent solutions of equation (3.63), then the general solution with arbitrary constants c_1 and c_2 is

$$\begin{bmatrix}\hat{a}(\omega, x_3)\\ \hat{b}(\omega, x_3)\end{bmatrix} = c_1\begin{bmatrix}\chi_1(\omega, x_3)\\ \chi_3(\omega, x_3)\end{bmatrix} + c_2\begin{bmatrix}\chi_2(\omega, x_3)\\ \chi_4(\omega, x_3)\end{bmatrix} = \begin{bmatrix}\chi_1(\omega, x_3) & \chi_2(\omega, x_3)\\ \chi_3(\omega, x_3) & \chi_4(\omega, x_3)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}, \tag{3.65}$$

which can be written as

$$\begin{bmatrix}\hat{a}(\omega, x_3)\\ \hat{b}(\omega, x_3)\end{bmatrix} = \begin{bmatrix}\chi_1(\omega, x_3) & \chi_2(\omega, x_3)\\ \chi_3(\omega, x_3) & \chi_4(\omega, x_3)\end{bmatrix}\begin{bmatrix}\chi_1(\omega, 0) & \chi_2(\omega, 0)\\ \chi_3(\omega, 0) & \chi_4(\omega, 0)\end{bmatrix}^{-1}\begin{bmatrix}\chi_1(\omega, 0) & \chi_2(\omega, 0)\\ \chi_3(\omega, 0) & \chi_4(\omega, 0)\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}. \tag{3.66}$$

Defining the propagator matrix

$$\mathbf{P}(\omega, x_3) = \begin{bmatrix}\chi_1(\omega, x_3) & \chi_2(\omega, x_3)\\ \chi_3(\omega, x_3) & \chi_4(\omega, x_3)\end{bmatrix}\begin{bmatrix}\chi_1(\omega, 0) & \chi_2(\omega, 0)\\ \chi_3(\omega, 0) & \chi_4(\omega, 0)\end{bmatrix}^{-1}, \tag{3.67}$$

equation (3.66) can be rewritten as

$$\begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = \mathbf{P}(\omega, x_3) \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}. \tag{3.68}$$

Now differentiating equation (3.68) with respect to x_3 gives

$$\frac{\partial}{\partial x_3} \begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = \frac{\partial \mathbf{P}(\omega, x_3)}{\partial x_3} \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}, \tag{3.69}$$

and so, from equation (3.63)

$$\mathbf{H} \begin{bmatrix} \hat{a}(\omega, x_3) \\ \hat{b}(\omega, x_3) \end{bmatrix} = \frac{\partial \mathbf{P}(\omega, x_3)}{\partial x_3} \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}. \tag{3.70}$$

Substituting in equation (3.68), this can be rewritten

$$\mathbf{HP} \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix} = \frac{\partial \mathbf{P}(\omega, x_3)}{\partial x_3} \begin{bmatrix} \hat{a}(\omega, 0) \\ \hat{b}(\omega, 0) \end{bmatrix}, \tag{3.71}$$

to finally give

$$\frac{\partial \mathbf{P}(\omega, x_3)}{\partial x_3} = \mathbf{HP}(\omega, x_3). \tag{3.72}$$

From equation (3.68) it can also be deduced that $\mathbf{P}(\omega, 0) = \mathbf{I}$, and so using equation (3.67) it follows that

$$\mathbf{P}(\omega, x_3) = \begin{bmatrix} \chi_1(\omega, x_3) & \chi_2(\omega, x_3) \\ \chi_3(\omega, x_3) & \chi_4(\omega, x_3) \end{bmatrix}. \tag{3.73}$$

The propagator matrix in equation (3.72) mathematically describes the evolution of energy (wave modes) through the medium embedded between two homogeneous half spaces. This formulation into a boundary value problem allows us to relate the coefficients which describe the reflection and transmission of energy. Taking the determinant of the propagator matrix given by equation (3.72) and applying Jacobi's formula [61] gives

$$\frac{d(\det\{\mathbf{P}\})}{dx_3} = \mathrm{Tr}(\mathbf{H})\det\{\mathbf{P}\} = 0, \tag{3.74}$$

since $\mathrm{Tr}(\mathbf{H}) = 0$ and so $\det\{\mathbf{P}(\omega, x_3)\}$ is independent of x_3. Given that $\mathbf{P}(\omega, 0) = \mathbf{I}$ then $\det\{\mathbf{P}(\omega, 0)\} = 1$, and hence

$$\det\{\mathbf{P}(\omega, x_3)\} = 1. \tag{3.75}$$

From equations (3.63) and (3.64) then

$$\frac{d\chi_1}{dx_3} = \Delta_1\chi_1 + \Delta_2\chi_3, \tag{3.76}$$

and

$$\frac{d\chi_3}{dx_3} = \Delta_3\chi_1 + \Delta_4\chi_3. \tag{3.77}$$

Suppose that $(\overline{\chi}_3, \overline{\chi}_1)^T$ also satisfies equation (3.63) then it follows that

$$\frac{d\bar{\chi}_1}{dx_3} = \Delta_3\bar{\chi}_3 + \Delta_4\bar{\chi}_1, \tag{3.78}$$

and

$$\frac{d\bar{\chi}_3}{dx_3} = \Delta_1\bar{\chi}_3 + \Delta_2\bar{\chi}_1. \tag{3.79}$$

From equations (3.76) and (3.78) then

$$\begin{aligned} \overline{\Delta}_1\bar{\chi}_1 + \overline{\Delta}_2\bar{\chi}_3 &= \Delta_3\bar{\chi}_3 + \Delta_4\bar{\chi}_1, \\ \overline{\Delta}_3\bar{\chi}_1 + \overline{\Delta}_4\bar{\chi}_3 &= \Delta_1\bar{\chi}_3 + \Delta_2\bar{\chi}_1, \end{aligned} \tag{3.80}$$

that is

$$\overline{\Delta}_2(x_3) = \Delta_3(x_3), \tag{3.81}$$

$$\overline{\Delta}_1(x_3) = \Delta_4(x_3). \tag{3.82}$$

$\det\{\mathbf{P}(\omega, x_3)\} = \mathbf{I}$ implies that the initial condition for the first eigensolution is $\begin{bmatrix}1, 0\end{bmatrix}^T$ and $\begin{bmatrix}0, 1\end{bmatrix}^T$ for the second eigensolution of equation (3.63). If $\begin{bmatrix}\chi_1, \chi_3\end{bmatrix}^T$ is the first eigensolution of (3.63) then

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\chi_1, \chi_3\end{bmatrix}^T = \begin{bmatrix}\Delta_1 & \Delta_2 \\ \overline{\Delta}_2 & \overline{\Delta}_1\end{bmatrix}\begin{bmatrix}\chi_1, \chi_3\end{bmatrix}^T, \tag{3.83}$$

with $\begin{bmatrix}\chi_1(x_3 = 0), \chi_3(x_3 = 0)\end{bmatrix}^T = \begin{bmatrix}1, 0\end{bmatrix}^T$. Taking conjugates

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\overline{\chi}_1, \overline{\chi}_3\end{bmatrix}^T = \begin{bmatrix}\overline{\Delta}_1 & \overline{\Delta}_2 \\ \Delta_2 & \Delta_1\end{bmatrix}\begin{bmatrix}\overline{\chi}_1, \overline{\chi}_3\end{bmatrix}^T, \tag{3.84}$$

implies that

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\overline{\chi}_3, \overline{\chi}_1\end{bmatrix}^T = \begin{bmatrix}\Delta_1 & \Delta_2 \\ \overline{\Delta}_2 & \overline{\Delta}_1\end{bmatrix}\begin{bmatrix}\overline{\chi}_3, \overline{\chi}_1\end{bmatrix}^T = \mathbf{H}\begin{bmatrix}\overline{\chi}_3, \overline{\chi}_1\end{bmatrix}^T, \tag{3.85}$$

so $\begin{bmatrix}\bar{\chi}_3, \bar{\chi}_1\end{bmatrix}^T$ is the second eigensolution and it satisfies the initial condition $\begin{bmatrix}0, 1\end{bmatrix}^T$ so $\chi_3(0) = 0$, $\chi_1(0) = 1$. Both of these conditions hold and so the propagator matrix has conjugate symmetries. To simplify the notation, set $\chi_1 = \chi$ and $\chi_3 = \varkappa$ and hence

$$\mathbf{P}(\omega, x_3) = \begin{bmatrix}\chi(\omega, x_3) & \overline{\varkappa}(\omega, x_3) \\ \varkappa(\omega, x_3) & \overline{\chi}(\omega, x_3)\end{bmatrix}, \tag{3.86}$$

where the conservation of energy relation $|\chi|^2 - |\varkappa|^2 = 1$ holds.

3.5 Deriving Reflection and Transmission Coefficients

Now take the governing equations and consider the problem where a layer of heterogeneous material of length L is embedded between two homogeneous half spaces. Reflection and transmission coefficients are defined using the governing equations, which quantify the energy of the wave as it passes through the material. First start with the governing equations for velocity $\hat{\xi}$ and stress $\hat{\tau}$ in the frequency domain given by equations (3.48)-(3.49) and apply conditions of continuity of velocity and stress with the left homogeneous half space and heterogeneous medium interface. Quantities in the left homogeneous half space are denoted with a 'zero' subscript and quantities inside the heterogeneous medium by a 'one' subscript, hence

$$\hat{\xi}_0(\omega, \kappa_1, x_3)\Big|_{x_3=0} = \hat{\xi}_1(\omega, \kappa_1, x_3)\Big|_{x_3=0}, \tag{3.87}$$

$$\hat{\tau}_0(\omega, \kappa_1, x_3)\Big|_{x_3=0} = \hat{\tau}_1(\omega, \kappa_1, x_3)\Big|_{x_3=0}. \tag{3.88}$$

Using equation (3.45) to write these two equations in terms of $\hat{a}$ and $\hat{b}$ gives

$$\begin{bmatrix} \hat{a}_1(\omega, \kappa_1, x_3 = 0) \\ \hat{b}_1(\omega, \kappa_1, x_3 = 0) \end{bmatrix} = \begin{bmatrix} r_0^+ & r_0^- \\ r_0^- & r_0^+ \end{bmatrix} \begin{bmatrix} \hat{a}_0(\omega, \kappa_1, x_3 = 0) \\ \hat{b}_0(\omega, \kappa_1, x_3 = 0) \end{bmatrix}, \tag{3.89}$$

where

$$r_0^{\pm} = \frac{1}{2}\left(\sqrt{\frac{\bar{\beta}_0\bar{\zeta}_1}{\bar{\zeta}_0\bar{\beta}_1}} \pm \sqrt{\frac{\bar{\zeta}_0\bar{\beta}_1}{\bar{\beta}_0\bar{\zeta}_1}}\right). \tag{3.90}$$

Similarly at the $x_3 = L$ interface, where the heterogeneous medium meets the right half space

$$\hat{\xi}_1(\omega, \kappa_1, x_3)\Big|_{x_3=L} = \hat{\xi}_2(\omega, \kappa_1, x_3)\Big|_{x_3=L}, \tag{3.91}$$

and

$$\hat{\tau}_1(\omega, \kappa_1, x_3)\Big|_{x_3=L} = \hat{\tau}_2(\omega, \kappa_1, x_3)\Big|_{x_3=L}. \tag{3.92}$$

Again equation (3.45) is used to link the modes across the right interface

$$\begin{bmatrix} \hat{a}_2(\omega, \kappa_1, x_3 = L) \\ \hat{b}_2(\omega, \kappa_1, x_3 = L) \end{bmatrix} = \begin{bmatrix} r_1^+ & r_1^- e^{-i(\bar{\zeta}_1 + \bar{\zeta}_2)L} \\ r_1^- e^{i(\bar{\zeta}_1 + \bar{\zeta}_2)L} & r_1^+ \end{bmatrix} \begin{bmatrix} \hat{a}_1(\omega, \kappa_1, x_3 = L) \\ \hat{b}_1(\omega, \kappa_1, x_3 = L) \end{bmatrix}, \tag{3.93}$$

where

$$r_1^{\pm} = \frac{1}{2}\left(\sqrt{\frac{\bar{\beta}_1\bar{\zeta}_2}{\bar{\zeta}_1\bar{\beta}_2}} \pm \sqrt{\frac{\bar{\zeta}_1\bar{\beta}_2}{\bar{\beta}_1\bar{\zeta}_2}}\right). \tag{3.94}$$

Note that is it assumed that the ζ terms are real and positive and hence $r_{0,1}^{\pm}(\omega)$ are real. In the half space $x_3 < 0$, the right-moving input wave is defined as $\hat{a}_0(\omega, 0) = \hat{f}(\omega)$ and in the half space $x_3 > L$, the left travelling wave mode does not exist, hence $\hat{b}_2(\omega, L) = 0$. Hence, from equation (3.89)

$$r_0^+\hat{a}_1 = (r_0^+)^2\hat{f} + r_0^- r_0^+\hat{b}_0, \tag{3.95}$$

$$r_0^-\hat{b}_1 = (r_0^-)^2\hat{f} + r_0^- r_0^+\hat{b}_0, \tag{3.96}$$

which implies that

$$r_0^+\hat{a}_1 - r_0^-\hat{b}_1 = \left((r_0^+)^2 - (r_0^-)^2\right)\hat{f} = \hat{f}, \tag{3.97}$$

since $(r_0^+)^2 - (r_0^-)^2 = 1$. Hence

$$\hat{a}_1 - \frac{r_0^-}{r_0^+}\hat{b}_1 = \frac{1}{r_0^+}\hat{f}. \tag{3.98}$$

From Figure 3.2, identify $T_0 = 1/r_0^+$ as a transmission coefficient from region 0 to region

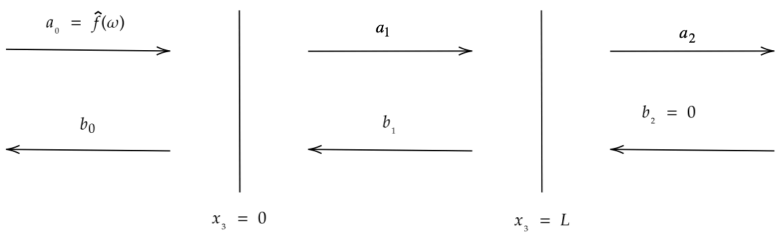

Figure 3.2: Schematic showing the directions of the partial waves $\hat{a}_i$ and $\hat{b}_i$ in the left hand half space ($i = 0$), the single layer ($i = 1$), and the right hand half space ($i = 2$).

1, and $R_0 = -r_0^-/r_0^+$ as a reflection coefficient at the same interface. Equation (3.98) then becomes $\hat{a}_1 + R_0\hat{b}_1 = T_0\hat{f}$. The local transmission and reflection coefficients at an interface describe locally, the fraction of energy propagating through the heterogeneous layer interface. The medium is not dissipative and so the energy in the system must be conserved. This corresponds to the sum of reflection and transmission coefficients equalling one. These definitions can be extended to the other interface to give

$$T_j = \frac{1}{r_j^+}, \quad R_j = -\frac{r_j^-}{r_j^+}, \quad j = 0, 1. \tag{3.99}$$

Noting that $(r_j^+)^2 - (r_j^-)^2 = 1, \quad j = 0, 1$, the conservation of energy relation reads

$$R_j^2 + T_j^2 = 1, \quad j = 0, 1, \tag{3.100}$$

holds. By eliminating $\hat{b}_0(\omega, 0)$ in equation (3.89) the boundary conditions for the forward and backward modes in equation (3.63) are

$$\hat{a}_1(\omega, 0) + R_0 \hat{b}_1(\omega, 0) = T_0 \hat{f}(\omega), \tag{3.101}$$

$$R_1 \hat{a}_1(\omega, L) e^{i(\bar{\zeta}_1 + \bar{\zeta}_2)L} - \hat{b}_1(\omega, L) = 0. \tag{3.102}$$

3.5.1 Reflection and Transmission Equations

Consider the local reflection and transmission coefficients for a layer of heterogeneous material occupying $0 \leq x_3 \leq L$, with a wave incident from a homogeneous half space $x_3 < 0$, defined by

$$\hat{R}(\omega, x_3) = \frac{\hat{b}(\omega, x_3)}{\hat{a}(\omega, x_3)}, \quad \text{and} \quad \hat{T}(\omega, x_3) = \frac{T_1 \hat{a}(\omega, L)}{\hat{a}(\omega, x_3)}. \tag{3.103}$$

The functions $\hat{a}(\omega, x_3)$ and $\hat{b}(\omega, x_3)$ are the wave modes given by equations (3.50) and (3.51), which propagate inside the heterogeneous medium. The boundary condition given by equationsw (3.102) and (3.103) at $x_3 = L$ can then be written

$$\hat{R}(\omega, L) = R_1 e^{i(\bar{\zeta}_1 + \bar{\zeta}_2)L}, \quad \hat{T}(\omega, L) = T_1. \tag{3.104}$$

By differentiating equation (3.103) with respect to x_3 and using equation (3.63), observe that the local reflection coefficient satisfies the Ricatti equation for $x_3 \in [0, L]$

$$\begin{aligned} \frac{d\hat{R}}{dx_3} &= \frac{\hat{a}(\omega, x_3)\partial_{x_3}\hat{b}(\omega, x_3) - \hat{b}(\omega, x_3)\partial_{x_3}\hat{a}(\omega, x_3)}{\hat{a}^2(\omega, x_3)} \\ &= \hat{R}(\Delta_4(x_3) - \Delta_1(x_3)) - \hat{R}^2\Delta_2(x_3) + \Delta_3(x_3). \end{aligned} \quad (3.105)$$

Similarly

$$\begin{aligned} \frac{d\hat{T}}{dx_3} &= \frac{-T_1\hat{a}(\omega, L)}{\hat{a}^2(\omega, x_3)}\partial_{x_3}(\hat{a}(\omega, x_3)) \\ &= -\hat{T}(\omega, x_3)\Big(\Delta_1(x_3) + \Delta_2(x_3)\hat{R}(\omega, x_3)\Big). \end{aligned} \quad (3.106)$$

3.6 Randomly Layered Anisotropic Medium

The stochastic model employed to describe the heterogeneities gives rise to fluctuations which build up behind the wave, producing a distorted coda wave exiting the material. As the angle θ is varied in the (x_1, x_2) plane, the stress tensor components of the material

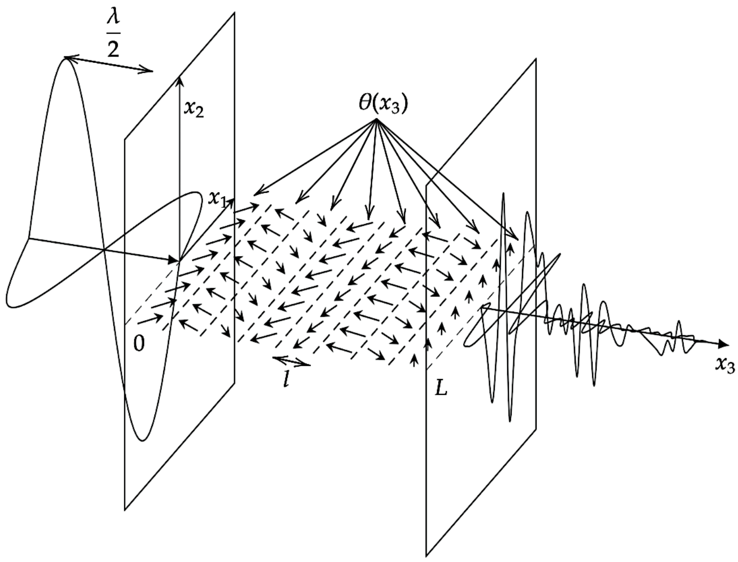

Figure 3.3: Elastic shear wave with wavelength λ incident on a randomly layered material with layer size l and slab length L. The small arrows indicate the local fibre orientation $\theta(x_3)$ (the symmetry axis vector and hence θ, lie in the (x_1, x_2) plane, as shown in Figure 3.1) of the anisotropic material in each layer. Upon exiting the material the transmitted wave has very little energy in the coherent wave and has a long coda wave.

are given by equations (3.10) and (3.11). Assume that the angle $\theta(x_3)$ varies randomly over the interval $x_3 \in [0, L]$ according to

$$\theta(x_3) = \bar{\theta} + \sigma m(x_3/l), \quad x_3 \in [0, L]. \tag{3.107}$$

where $\bar{\theta} \sim 1$ is the mean angle, $m(x_3/l)$ is a stationary stochastic process (an ergodic Markov process on a compact state space) with mean zero, l is a typical layer size inside the material and σ is a dimensionless (no physical dimension) small parameter ($0 < \sigma \ll 1$) which controls the strength of the random process $m(x_3/l)$. Recall the symmetry vector (3.2) lies in the vertical "in-layer" (x_1, x_2) plane. The stress tensor components can be rewritten

$$c_{66} = S + (A + C - 2F - 4S)\cos^2\left(\bar{\theta} + \sigma m(x_3/l)\right)\sin^2\left(\bar{\theta} + \sigma m(x_3/l)\right), \tag{3.108}$$

$$c_{44} = N + (S - N)\sin^2\left(\bar{\theta} + \sigma m(x_3/l)\right). \tag{3.109}$$

To assist in the analysis that follows, linearise this dependency of the material properties with respect to the random process $m(x_3/l)$. As such the range of validity of the analysis is restricted to the regime where equations (3.108) and (3.109) are approximately linear with respect to $m(x_3/l)$; given the dependency on θ in equations (3.108) and (3.109) this will be around $\bar{\theta} = \pi/8$. Taking a Taylor series in σ gives

$$\sin^2\left(\bar{\theta} + \sigma m(x_3/l)\right) = \sin^2\bar{\theta} + 2\sigma m(x_3/l)\cos\bar{\theta}\sin\bar{\theta} + \mathcal{O}(\sigma^2), \tag{3.110}$$

$$\cos^2\left(\bar{\theta} + \sigma m(x_3/l)\right) = \cos^2\bar{\theta} - 2\sigma m(x_3/l)\cos\bar{\theta}\sin\bar{\theta} + \mathcal{O}(\sigma^2), \tag{3.111}$$

which gives

$$\sin^2\left(\bar{\theta} + \sigma m(x_3/l)\right)\cos^2\left(\bar{\theta} + \sigma m(x_3/l)\right) = \cos^2\bar{\theta}\sin^2\bar{\theta} + \sigma m(x_3/l)\cos 2\bar{\theta}\sin 2\bar{\theta} + \mathcal{O}(\sigma^2). \tag{3.112}$$

Substituting equation (3.112) into equation (3.108) then gives

$$\begin{aligned} c_{66} &= (S + \varrho \cos^2 \bar{\theta} \sin^2 \bar{\theta})\left(1 + \sigma \frac{\varrho \sin(2\bar{\theta}) \cos(2\bar{\theta})}{(S + \varrho \cos^2 \bar{\theta} \sin^2 \bar{\theta})} m(x_3/l)\right) + \mathcal{O}(\sigma^2) \\ &= \bar{c}_{66}\left(1 + \varphi m(x_3/l)\right), \end{aligned} \tag{3.113}$$

where

$$\varrho = A + C - 2F - 4S, \tag{3.114}$$

$$\bar{c}_{66} = S + \varrho \cos^2 \bar{\theta} \sin^2 \bar{\theta}, \tag{3.115}$$

$$\varphi = \sigma \frac{\varrho \sin(2\bar{\theta}) \cos(2\bar{\theta})}{\bar{c}_{66}}. \tag{3.116}$$

Similarly, substituting (3.110) into (3.109) gives

$$\begin{aligned} c_{44} &= \left(N + (S - N) \sin^2 \bar{\theta}\right)\left(1 + \sigma \frac{(S - N) \sin 2\bar{\theta}}{\left(N + (S - N) \sin^2 \bar{\theta}\right)} m(x_3/l)\right) + \mathcal{O}(\sigma^2) \\ &= \bar{c}_{44}\left(1 + \vartheta m(x_3/l)\right), \end{aligned} \tag{3.117}$$

where

$$\bar{c}_{44} = N + (S - N) \sin^2 \bar{\theta}, \tag{3.118}$$

$$\vartheta = \sigma \frac{(S - N) \sin 2\bar{\theta}}{\bar{c}_{44}}. \tag{3.119}$$

Since $\bar{\theta} \sim 1$ and $0 < \sigma \ll 1$ then $0 < |\varphi|, |\vartheta| \ll 1$. Since it will appear in the following analysis, then to order σ

$$\frac{1}{c_{44}} = \frac{1 - \vartheta m(x_3/l)}{\bar{c}_{44}} + \mathcal{O}(\sigma^2). \tag{3.120}$$

Equation (3.31) is now

$$\alpha = \frac{\omega}{\bar{c}_{44}}\left(1 - \vartheta m(x_3/l)\right) + \mathcal{O}(\sigma^2)$$

$$= \bar{\alpha}(1 - \vartheta m(x_3/l)), \tag{3.121}$$

where

$$\bar{\alpha} = \frac{\omega}{\bar{c}_{44}}, \tag{3.122}$$

$$\vartheta = -\sigma\Gamma_\alpha, \tag{3.123}$$

$$\Gamma_\alpha = \frac{(N-S)\sin 2\bar{\theta}}{\bar{c}_{44}}. \tag{3.124}$$

Equation (3.32) can then be written

$$\begin{aligned}
\beta &= \frac{\kappa_1^2 \bar{c}_{66}(1 + \varphi m(x_3/l)) - \rho\omega^2}{\omega} + \mathcal{O}(\sigma^2) \\
&= \frac{(\kappa_1^2 \bar{c}_{66} - \rho\omega^2)}{\omega}\left(1 + \frac{\varphi\kappa_1^2\bar{c}_{66}}{(\kappa_1^2\bar{c}_{66} - \rho\omega^2)} m(x_3/l)\right) \\
&= \bar{\beta}(1 + \varsigma m(x_3/l)),
\end{aligned} \tag{3.125}$$

where

$$\bar{\beta} = \frac{\kappa_1^2\bar{c}_{66} - \rho\omega^2}{\omega}, \tag{3.126}$$

$$\varsigma = \frac{\bar{c}_{66}\varphi\kappa_1^2}{(\kappa_1^2\bar{c}_{66} - \rho\omega^2)} = \sigma\Gamma_\beta, \tag{3.127}$$

$$\Gamma_\beta = \frac{\kappa_1^2 \varrho \sin(2\bar{\theta})\cos(2\bar{\theta})}{\left(\kappa_1^2\bar{c}_{66} - \rho\omega^2\right)}. \tag{3.128}$$

Inserting (3.121) and (3.125) into the stress-velocity evolution, given by equation (3.30) can be written

$$\frac{\partial}{\partial x_3}\begin{bmatrix} \hat{\xi}(\omega, \kappa_1, x_3) \\ \hat{\tau}(\omega, \kappa_1, x_3) \end{bmatrix} = \begin{bmatrix} 0 & -i\bar{\alpha}(1 - \vartheta m(x_3/l)) \\ i\bar{\beta}(1 + \varsigma m(x_3/l)) & 0 \end{bmatrix} \begin{bmatrix} \hat{\xi}(\omega, \kappa_1, x_3) \\ \hat{\tau}(\omega, \kappa_1, x_3) \end{bmatrix}. \tag{3.129}$$

3.6.1 The Dimensionless Elastic Wave Equations

To put the system of governing equations (3.129) in dimensionless form, choose the dimensionless variables

$$\tilde{x}_3 = \frac{x_3}{L_3}, \quad \tilde{\omega} = \frac{L_3\,\omega}{c_3}, \quad \text{and} \quad \tilde{\kappa}_1 = \kappa_1 L_3, \tag{3.130}$$

where L_3 is a typical propagation distance in x_3 and c_3 is the mean wave speed in the x_3 direction. One can interpret $\tilde{x}_3$ as a ratio of distances in the propagation direction, $\tilde{\omega}$ as a ratio of the propagation distance to the typical wavelength in the propagation direction and $\tilde{\kappa}_1$ as a ratio of propagation distance per wavelength in the x_1 direction. Two further dimensionless parameters ε and ϖ are defined in order to capture the length scale differences in the problem, via

$$\varepsilon \ll 1, \quad \frac{L_3}{l} = \frac{1}{\varepsilon^2}, \quad \text{and} \quad \varpi = \varepsilon\tilde{\omega}, \tag{3.131}$$

where ε $(0 < \varepsilon \ll 1)$ and ϖ depend on the regime (strongly or weakly heterogeneous) being investigated. These relations can be combined to give

$$\varepsilon = \sqrt{\frac{l}{L_3}}, \quad \text{and} \quad \varpi = \frac{\omega}{c_3}\sqrt{lL_3}. \tag{3.132}$$

The non-dimensional velocity and stress fields take the form

$$\tilde{\xi}(\tilde{\omega}, \tilde{\kappa_1}, \tilde{x_3}) = \frac{1}{c_3}\hat{\xi}\left(\frac{c_3\tilde{\omega}}{L_3}, \kappa_1 L_3, L_3\tilde{x_3}\right), \quad \tilde{\tau}(\tilde{\omega}, \tilde{\kappa_1}, \tilde{x_3}) = \frac{1}{\rho c_3^2}\hat{\tau}\left(\frac{c_3\tilde{\omega}}{L_3}, \kappa_1 L_3, L_3\tilde{x}_3\right), \tag{3.133}$$

where ρ is the (assumed to be constant in space) density of the material. From equation (3.129) the non-dimensional stress and velocity equations read

$$\frac{1}{L_3}\frac{\partial}{\partial\tilde{x}_3}\begin{bmatrix} c_3\tilde{\xi}(\tilde{\omega}, \tilde{\kappa}_1, \tilde{x}_3) \\ \rho c_3^2\tilde{\tau}(\tilde{\omega}, \tilde{\kappa}_1, \tilde{x}_3)\end{bmatrix} = \begin{bmatrix} 0 & -i\bar{\alpha}(1 - \vartheta m(\tilde{x}_3/\varepsilon^2)) \\ i\bar{\beta}(1 + \varsigma m(\tilde{x}_3/\varepsilon^2)) & 0 \end{bmatrix}\begin{bmatrix} c_3\tilde{\xi}(\tilde{\omega}, \tilde{\kappa}_1, \tilde{x}_3) \\ \rho c_3^2\tilde{\tau}(\tilde{\omega}, \tilde{\kappa}_1, \tilde{x}_3)\end{bmatrix}, \tag{3.134}$$

which simplifies to

$$\frac{\partial}{\partial \tilde{x}_3}\begin{bmatrix}\tilde{\xi}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\\ \tilde{\tau}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\end{bmatrix} = \begin{bmatrix} 0 & -i(L_3\rho c_3)\bar{\alpha}(1-\vartheta m(\tilde{x}_3/\varepsilon^2)) \\ i(L_3/(\rho c_3))\bar{\beta}(1+\varsigma m(\tilde{x}_3/\varepsilon^2)) & 0\end{bmatrix}\begin{bmatrix}\tilde{\xi}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\\ \tilde{\tau}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\end{bmatrix}. \tag{3.135}$$

The amplitude terms in equation (3.135) can be put into non-dimensional form via

$$L_3\rho c_3\bar{\alpha} = L_3\rho c_3\left(\frac{\omega}{\bar{c}_{44}}\right) = \left(\frac{\omega L_3}{c_3}\right)\left(\frac{\rho c_3^2}{\bar{c}_{44}}\right) = \frac{\tilde{\omega}}{\tilde{c}_{44}}, \tag{3.136}$$

where

$$\tilde{c}_{44} = \frac{\bar{c}_{44}}{\rho c_3^2}. \tag{3.137}$$

Then

$$\begin{aligned}\frac{L_3}{\rho c_3}\bar{\beta} &= \left(\frac{L_3}{\rho c_3}\right)\left(\frac{\kappa_1^2\bar{c}_{66}-\rho\omega^2}{\omega}\right)\\
&= \left(\frac{L_3}{\rho\omega c_3}\right)\left(\frac{\tilde{\kappa}_1^2\rho c_3^2\tilde{c}_{66}}{L_3^2}-\rho\omega^2\right)\\
&= \left(\frac{L_3\omega}{c_3}\right)\left(\frac{1}{\rho\omega^2}\right)\left(\frac{\rho c_3^2\tilde{\kappa}_1^2\tilde{c}_{66}}{L_3^2}-\frac{\rho c_3^2\tilde{\omega}^2}{L_3^2}\right)\\
&= \tilde{\omega}\left(\frac{1}{\rho\omega^2}\right)\left(\frac{\rho c_3^2}{L_3^2}\right)\left(\tilde{\kappa}_1^2\tilde{c}_{66}-\tilde{\omega}^2\right)\\
&= \tilde{\omega}\left(\frac{c_3}{\omega L_3}\right)^2\left(\tilde{\kappa}_1^2\tilde{c}_{66}-\tilde{\omega}^2\right)\\
&= \frac{1}{\tilde{\omega}}\left(\tilde{\kappa}_1^2\tilde{c}_{66}-\tilde{\omega}^2\right)\\
&= \tilde{\omega}\left(\frac{\tilde{\kappa}_1^2}{\tilde{\omega}^2}\tilde{c}_{66}-1\right),\end{aligned} \tag{3.138}$$

where

$$\tilde{c}_{66} = \frac{\bar{c}_{66}}{\rho c_3^2}, \tag{3.139}$$

and ρ and c_3 are chosen to ensure that $\tilde{c}_{44}, \tilde{c}_{66} \sim 1$. The dimensionless stress velocity evolution equation (3.135) is then

$$\frac{\partial}{\partial \tilde{x}_3}\begin{bmatrix}\tilde{\xi}\\ \tilde{\tau}\end{bmatrix} = \begin{bmatrix}0 & -i(\tilde{\omega}/\tilde{c}_{44})(1-\vartheta m(\tilde{x}_3/\varepsilon^2))\\ i\tilde{\omega}((\tilde{\kappa}_1^2/\tilde{\omega}^2)\tilde{c}_{66}-1)(1+\varsigma m(\tilde{x}_3/\varepsilon^2)) & 0\end{bmatrix}\begin{bmatrix}\tilde{\xi}\\ \tilde{\tau}\end{bmatrix}. \tag{3.140}$$

3.6.2 Normalising Equation Amplitudes

The prefactors in the linear system (3.140) are related via ν

$$\tilde{\kappa}_1 = \nu\tilde{\omega}, \tag{3.141}$$

where

$$\nu = \frac{\tilde{\kappa}_1}{\tilde{\omega}} = \frac{\kappa_1 L_3}{(L_3\omega/c_3)} = \frac{\kappa_1}{\kappa_3}, \tag{3.142}$$

is the ratio of wave numbers in the (x_1, x_3) directions. For a monochromatic wave the ratio of wave-numbers is equal to the ratio of slowness (the inverse of the phase velocity) values which in turn (for constant density materials) is the degree of anisotropy of the medium as governed by the stiffness tensor (see equation (3.3)). As the crystal orientation θ changes, the phase velocities in the x_1 and x_3 direction change and hence, the wave-numbers in these directions change commensurately - see Figure 2.5 and the discussion in Chapter 2.8.1. Hence equation (3.140) is

$$\frac{\partial}{\partial \tilde{x}_3}\begin{bmatrix}\tilde{\xi}\\ \tilde{\tau}\end{bmatrix} = \begin{bmatrix}0 & -i(\tilde{\omega}/\tilde{c}_{44})(1-\vartheta m(\tilde{x}_3/\varepsilon^2))\\ i\tilde{\omega}(\nu^2\tilde{c}_{66}-1)(1+\varsigma m(\tilde{x}_3/\varepsilon^2)) & 0\end{bmatrix}\begin{bmatrix}\tilde{\xi}\\ \tilde{\tau}\end{bmatrix}, \tag{3.143}$$

which can be written compactly as

$$\frac{\partial}{\partial \tilde{x}_3}\begin{bmatrix}\tilde{\xi}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\\ \tilde{\tau}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\end{bmatrix} = i\frac{\varpi}{\varepsilon}\begin{bmatrix}0 & d_1(1+\sigma\Gamma_\alpha m(\tilde{x}_3/\varepsilon^2))\\ d_2(1+\sigma\Gamma_\beta m(\tilde{x}_3/\varepsilon^2)) & 0\end{bmatrix}\begin{bmatrix}\tilde{\xi}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\\ \tilde{\tau}(\tilde{\omega},\tilde{\kappa}_1,\tilde{x}_3)\end{bmatrix} \tag{3.144}$$

where

$$d_1 = -\frac{1}{\tilde{c}_{44}}, \tag{3.145}$$

$$d_2 = \nu^2 \tilde{c}_{66} - 1, \tag{3.146}$$

and equations (3.128) and (3.142) give

$$\Gamma_\beta = \frac{\nu^2 \varrho \sin 2\bar{\theta} \cos 2\bar{\theta}}{\nu^2 \bar{c}_{66} - \rho c_3^2}, \tag{3.147}$$

where d_1, d_2 are $\mathcal{O}(1)$. Now consider the deterministic case, where $m = 0$ to remove the spatial variation in the material properties. Then from equation (3.144)

$$\frac{\partial^2 \hat{\xi}}{\partial x_3^2} = \frac{\partial}{\partial x_3}\left(\frac{i\varpi}{\varepsilon} d_1 \hat{\tau}\right) = -\frac{\varpi^2}{\varepsilon^2} d_1 d_2 \hat{\xi}, \tag{3.148}$$

so (with $d_1 d_2 > 0$)

$$\hat{\xi} = \hat{a}^\varepsilon e^{i\varpi/\varepsilon\sqrt{d_1 d_2} x_3} + \hat{b}^\varepsilon e^{-i\varpi/\varepsilon\sqrt{d_1 d_2} x_3}, \tag{3.149}$$

where $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$ are constants. Hence

$$\frac{\partial \hat{\tau}}{\partial x_3} = \frac{i\varpi}{\varepsilon} d_2 \hat{\xi} = \frac{i\varpi d_2}{\varepsilon}\left(\hat{a}^\varepsilon e^{i\varpi/\varepsilon\sqrt{d_1 d_2} x_3} + \hat{b}^\varepsilon e^{-i\varpi/\varepsilon\sqrt{d_1 d_2} x_3}\right), \tag{3.150}$$

and so

$$\hat{\tau} = \sqrt{d_2/d_1}\left(\hat{a}^\varepsilon e^{i\varpi/\varepsilon\sqrt{d_1 d_2} x_3} - \hat{b}^\varepsilon e^{-i\varpi/\varepsilon\sqrt{d_1 d_2} x_3}\right). \tag{3.151}$$

This suggests the following ansatz for the solution to equation (3.144) where $\hat{a}^\varepsilon$ and $\hat{b}^\varepsilon$ now depend on x_3

$$\hat{\xi} = \hat{a}^\varepsilon(x_3) e^{i\varpi/\varepsilon\sqrt{d_1 d_2} x_3} + \hat{b}^\varepsilon(x_3) e^{-i\varpi/\varepsilon\sqrt{d_1 d_2} x_3}, \tag{3.152}$$

and

$$\hat{\tau} = \sqrt{d_2/d_1}\left(\hat{a}^\varepsilon(x_3)e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} - \hat{b}^\varepsilon(x_3)e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}\right). \tag{3.153}$$

Now substitute equations (3.152) and (3.153) into equation (3.144) to give

$$\begin{aligned}
\frac{\partial\hat{\xi}}{\partial x_3} &= \frac{\partial\hat{a}^\varepsilon}{\partial x_3}e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} + i\frac{\varpi}{\varepsilon}\sqrt{d_1d_2}\hat{a}^\varepsilon e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} + \frac{\partial\hat{b}^\varepsilon}{\partial x_3}e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3} \\
&\quad - \frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\hat{b}^\varepsilon e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3} \\
&= \frac{i\varpi}{\varepsilon}d_1\left(1+\sigma\Gamma_\alpha m\right)\left(\sqrt{d_2/d_1}\left(\hat{a}^\varepsilon e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} - \hat{b}^\varepsilon e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}\right)\right).
\end{aligned} \tag{3.154}$$

That is

$$\begin{aligned}
\frac{\partial\hat{a}^\varepsilon}{\partial x_3}e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} + \frac{\partial\hat{b}^\varepsilon}{\partial x_3}e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3} &= \hat{a}^\varepsilon\left(-\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2} + \frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}(1+\sigma\Gamma_\alpha m)\right)e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} \\
&\quad + \hat{b}^\varepsilon\left(\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2} - \frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}(1+\sigma\Gamma_\alpha m)\right)e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3} \\
&= \hat{a}^\varepsilon\left(\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\sigma\Gamma_\alpha m\right)e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} \\
&\quad - \hat{b}^\varepsilon\left(\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\sigma\Gamma_\alpha m\right)e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}.
\end{aligned} \tag{3.155}$$

Similarly

$$\begin{aligned}
\frac{\partial\hat{\tau}}{\partial x_3} &= \sqrt{d_2/d_1}\left(\frac{\partial\hat{a}^\varepsilon}{\partial x_3}e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} + \frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\hat{a}^\varepsilon e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} - \frac{\partial\hat{b}^\varepsilon}{\partial x_3}e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}\right. \\
&\quad \left. + \frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\hat{b}^\varepsilon e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}\right) \\
&= \frac{i\varpi}{\varepsilon}d_2\left(1+\sigma\Gamma_\beta m\right)\left(\hat{a}^\varepsilon e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} + \hat{b}^\varepsilon e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}\right).
\end{aligned} \tag{3.156}$$

That is

$$\begin{aligned}
\frac{\partial\hat{a}^\varepsilon}{\partial x_3}e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} - \frac{\partial\hat{b}^\varepsilon}{\partial x_3}e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3} &= \hat{a}^\varepsilon\left(-\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2} + \frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}(1+\sigma\Gamma_\beta m)\right)e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3} \\
&\quad + \hat{b}^\varepsilon\left(\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}(1+\sigma\Gamma_\beta m) - \frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\right)e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}
\end{aligned}$$

$$= \hat{a}^\varepsilon\left(\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\sigma\Gamma_\beta m\right)e^{i\varpi/\varepsilon\sqrt{d_1d_2}x_3}$$

$$+\,\hat{b}^\varepsilon\left(\frac{i\varpi}{\varepsilon}\sqrt{d_1d_2}\sigma\Gamma_\beta m\right)e^{-i\varpi/\varepsilon\sqrt{d_1d_2}x_3}. \qquad (3.157)$$

Adding (3.155) and (3.157) gives

$$\frac{\partial\hat{a}^\varepsilon}{\partial x_3} = \left(i\frac{\varpi}{2\varepsilon}\sqrt{d_1d_2}\sigma(\Gamma_\alpha+\Gamma_\beta)m\right)\hat{a}^\varepsilon + \left(i\frac{\varpi}{2\varepsilon}\sqrt{d_1d_2}\sigma(\Gamma_\beta-\Gamma_\alpha)m\right)e^{-2i\varpi/\varepsilon\sqrt{d_1d_2}x_3}\hat{b}^\varepsilon. \qquad (3.158)$$

Subtracting gives

$$\frac{\partial\hat{b}^\varepsilon}{\partial x_3} = \left(i\frac{\varpi}{2\varepsilon}\sqrt{d_1d_2}\sigma(\Gamma_\alpha-\Gamma_\beta)m\right)\hat{a}^\varepsilon e^{2i\varpi/\varepsilon\sqrt{d_1d_2}x_3} + \left(-\,i\frac{\varpi}{2\varepsilon}\sqrt{d_1d_2}\sigma(\Gamma_\beta+\Gamma_\alpha)m\right)\hat{b}^\varepsilon. \qquad (3.159)$$

This gives rise to the linear system

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{a}^\varepsilon\\ \hat{b}^\varepsilon\end{bmatrix} = \frac{i\varpi\sigma}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\begin{bmatrix}\sqrt{d_1d_2}(\Gamma_\alpha+\Gamma_\beta) & \sqrt{d_1d_2}(\Gamma_\beta-\Gamma_\alpha)e^{-2i\frac{\varpi}{\varepsilon}\sqrt{d_1d_2}x_3}\\ \sqrt{d_1d_2}(\Gamma_\alpha-\Gamma_\beta)e^{2i\frac{\varpi}{\varepsilon}\sqrt{d_1d_2}x_3} & -\sqrt{d_1d_2}(\Gamma_\alpha+\Gamma_\beta)\end{bmatrix}\begin{bmatrix}\hat{a}^\varepsilon\\ \hat{b}^\varepsilon\end{bmatrix}. \qquad (3.160)$$

Given the system of scaled dimensionless wave-mode evolution equations above, it is appropriate now to proceed with solving this system via a diffusion approximation. Scaling arguments will be imposed which will produce a problem with multiple length scales. This provides an intuitive description of the wave-medium dynamics. This formulation will prove useful in describing the statistics of the flow of energy through the random slab embedded between two homogeneous half spaces.

3.7 Weakly Heterogeneous Regime

The weakly heterogeneous regime is a high frequency regime (wavelength λ_3 is small relative to the slab length L_3, but commensurate with the microscopic layer length l) where the amplitude of the fluctuations in the medium properties is weak. The strength of the random fluctuations in equation (3.107) is controlled by σ where $0 < \sigma \ll 1$. The propagation distance is relatively large so that significant multiple scattering builds up. This regime corresponds to $L_3 \gg \lambda_3 \sim l$, $0 < \sigma \ll 1$. From equation (3.131)

$$1 \ll \frac{L_3}{\lambda_3} = \frac{\omega L_3}{2\pi c_3} = \frac{\tilde{\omega}}{2\pi} = \frac{\varpi}{2\pi\varepsilon}, \tag{3.161}$$

and from equation (3.132)

$$\frac{l}{\lambda_3} = \frac{\omega l}{2\pi c_3} = \frac{\omega L_3}{2\pi c_3}\frac{l}{L_3} = \frac{\varpi}{2\pi\varepsilon}\varepsilon^2 = \frac{\varpi\varepsilon}{2\pi} \sim 1. \tag{3.162}$$

Equation (3.162) implies (with $0 < \varepsilon \ll 1$) that

$$\varpi \sim \frac{1}{\varepsilon}, \tag{3.163}$$

and so (3.161) holds. Since $0 < \sigma \ll 1$, then take $\sigma = \varepsilon$. Recall from equation (3.141) that the random and dimensionless linear system has been normalised, allowing application of a diffusion approximation theorem. In the weakly heterogeneous regime, equations (3.130), (3.131), (3.141) and (3.163) imply that

$$\tilde{\omega} \sim \frac{1}{\varepsilon^2}. \tag{3.164}$$

These parameter choices ensure that the material fluctuation amplitude σ, is small, the layer size l is much smaller than the propagation distance L_3, the system is in a high frequency regime $L_3 \gg \lambda_3$, and the propagation distance is large enough so the wave experiences significant scattering $L_3 \sim L_{loc}$. This choice of parameter scaling transforms the evolution equation (3.160) to read

$$\frac{d}{dx_3}\begin{bmatrix}\hat{a}^\varepsilon \\ \hat{b}^\varepsilon\end{bmatrix} = \frac{1}{\varepsilon}\mathbf{H}^\varepsilon\left(\frac{x_3}{\varepsilon^2}, m\left(\frac{x_3}{\varepsilon^2}\right)\right)\begin{bmatrix}\hat{a}^\varepsilon \\ \hat{b}^\varepsilon\end{bmatrix}, \tag{3.165}$$

where

$$\mathbf{H}^{\varepsilon}\left(\frac{x_3}{\varepsilon^2}, m\left(\frac{x_3}{\varepsilon^2}\right)\right) = \frac{im(x_3/\varepsilon^2)}{2}\begin{bmatrix} \delta_1 & -\delta_2 e^{-i\frac{\phi x_3}{\varepsilon^2}} \\ \delta_2 e^{i\frac{\phi x_3}{\varepsilon^2}} & -\delta_1 \end{bmatrix}, \tag{3.166}$$

$$\delta_1 = \sqrt{d_1 d_2}(\Gamma_\alpha + \Gamma_\beta), \tag{3.167}$$

$$\delta_2 = \sqrt{d_1 d_2}(\Gamma_\alpha - \Gamma_\beta), \tag{3.168}$$

$$\phi = 2\sqrt{d_1 d_2}. \tag{3.169}$$

Assumptions that allow the use of limit theorems in stochastic calculus require that the random fluctuations have the form $m(x_3) = g(Y(x_3))$, where Y is a homogeneous in x_3 Markov process with values in a compact space [26]. Also assume that this process is strongly ergodic and satisfies the Fredholm alternative [8], and the real bounded function g satisfies the centering condition $\mathbb{E}[g(Y(0))] = 0$. Equation (3.165) can be recast into an initial value problem associated with a propagator equation. That is

$$\begin{bmatrix} \hat{a}^{\varepsilon}(\nu, x_3) \\ \hat{b}^{\varepsilon}(\nu, x_3) \end{bmatrix} = \mathbf{P}^{\varepsilon}(\nu, x_3)\begin{bmatrix} \hat{a}^{\varepsilon}(\nu, 0) \\ \hat{b}^{\varepsilon}(\nu, 0) \end{bmatrix}, \tag{3.170}$$

where the propagator matrix

$$\mathbf{P}^{\varepsilon}(\nu, x_3) = \begin{bmatrix} \chi^{\varepsilon}(\nu, x_3) & \overline{\varkappa^{\varepsilon}}(\nu, x_3) \\ \varkappa^{\varepsilon}(\nu, x_3) & \overline{\chi^{\varepsilon}}(\nu, x_3) \end{bmatrix}, \tag{3.171}$$

is formed from eigensolutions of equation (3.165), where $\mathbf{P}^{\varepsilon}(\omega, x_3 = 0) = \mathbf{I}$, and there is the conserved quantity $|\chi^{\varepsilon}|^2 - |\varkappa^{\varepsilon}|^2 = 1$.

3.7.1 Diffusion Approximation Theorem

To proceed, now apply the diffusion-approximation theorem in its linear form [26] to obtain the asymptotic distribution of the propagator matrix $\mathbf{P}^{\varepsilon}$. Equation (3.170) is written using the propagator formulation (used in equation (3.72)) to obtain the random matrix equation

$$\frac{d\mathbf{P}^{\varepsilon}}{dx_3}(\nu, x_3) = \frac{1}{\varepsilon}\mathbf{H}^{\varepsilon}\left(\frac{x_3}{\varepsilon^2}, m\left(\frac{x_3}{\varepsilon^2}\right)\right)\mathbf{P}^{\varepsilon}(\nu, x_3). \tag{3.172}$$

Splitting this into its real and imaginary parts gives

$$\begin{aligned}\frac{d\mathbf{P}^\varepsilon}{dx_3}(\nu,x_3) &= \frac{i}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\delta_1\,\boldsymbol{\sigma}_3\mathbf{P}^\varepsilon(\nu,x_3)\\ &\quad-\frac{1}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\delta_2\sin\left(\frac{\phi x_3}{\varepsilon^2}\right)\boldsymbol{\sigma}_1\mathbf{P}^\varepsilon(\nu,x_3)\\ &\quad+\frac{1}{2\varepsilon}m\left(\frac{x_3}{\varepsilon^2}\right)\delta_2\cos\left(\frac{\phi x_3}{\varepsilon^2}\right)\boldsymbol{\sigma}_2\mathbf{P}^\varepsilon(\nu,x_3),\end{aligned} \tag{3.173}$$

where $\boldsymbol{\sigma}_1,\boldsymbol{\sigma}_2,\boldsymbol{\sigma}_3$ are the Pauli spin matrices

$$\boldsymbol{\sigma}_1=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad \boldsymbol{\sigma}_2=\begin{bmatrix}0&-i\\i&0\end{bmatrix},\quad \boldsymbol{\sigma}_3=\begin{bmatrix}1&0\\0&-1\end{bmatrix}. \tag{3.174}$$

The right hand side of equation (3.173) can be written as

$$\frac{1}{\varepsilon}\sum_{p=1}^{3}g^{(p)}(m(\tau),\tau)\,\mathbf{h}_p\mathbf{P},\quad \tau=\frac{x_3}{\varepsilon^2}, \tag{3.175}$$

where

$$\mathbf{h}_1=i\frac{\delta_1}{2}\boldsymbol{\sigma}_3,\qquad g^{(1)}(m,\tau)=m \tag{3.176}$$

$$\mathbf{h}_2=-\frac{\delta_2}{2}\boldsymbol{\sigma}_1,\qquad g^{(2)}(m,\tau)=m\sin(\phi\tau) \tag{3.177}$$

$$\mathbf{h}_3=\frac{\delta_2}{2}\boldsymbol{\sigma}_2,\qquad g^{(3)}(m,\tau)=m\cos(\phi\tau). \tag{3.178}$$

From equations (3.145), (3.146) and (3.169) ϕ does not depend on frequency, and is a function of $\rho, c_3, \bar{\theta}$, ν and the five independent stresses in equation (3.3). Equation (3.173) is independent of frequency with the scaling requirement that $\omega\sim\varepsilon^{-2}$. The diffusion approximation Theorem [26] (Theorem 6.4) requires the correlation integral matrix $\mathbf{C}=(C_{pq})_{p,q=1,2,3}$. This can be computed using the covariance of the random process m, hence

$$C_{pq}=2\lim_{Z_0\to\infty}\frac{1}{Z_0}\int_0^{Z_0}\int_0^{\infty}\mathbb{E}\Big[g^{(p)}(m(0),\tau)g^{(q)}(m(x_3),\tau+x_3)\Big]dx_3d\tau \tag{3.179}$$

$$= \frac{\phi}{\pi} \int_0^{\frac{2\pi}{\phi}} \int_0^{\infty} \mathbb{E}\Big[g^{(p)}(m(0), \tau) g^{(q)}(m(x_3), \tau + x_3) \Big] dx_3 d\tau \tag{3.180}$$

$$= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[g^{(p)}(m(0), y) g^{(q)}(m(x_3), y + x_3) \Big] dx_3 dy. \tag{3.181}$$

Proceed by calculating each of the nine entries of $\mathbf{C}$. Start with the diagonal elements

$$\begin{aligned} C_{11} &= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] dx_3 dy, \\ &= 2 \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] dx_3 = \Upsilon(0), \end{aligned} \tag{3.182}$$

and

$$C_{22} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0) \cos(y) m(x_3) \cos(y + \phi x_3) \Big] dx_3 dy, \tag{3.183}$$

$$\begin{aligned} &= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \cos(y) \cos(y + \phi x_3) dx_3 dy \\ &= \frac{1}{\pi} \int_0^{2\pi} \cos^2(y) dy \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \cos(\phi x_3) dx_3 \\ &- \frac{1}{\pi} \int_0^{2\pi} \cos(y) \sin(y) dy \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \sin(\phi x_3) dx_3 \\ &= \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \cos(\phi x_3) dx_3 = \frac{1}{2} \Upsilon(\phi), \end{aligned} \tag{3.184}$$

and

$$\begin{aligned} C_{33} &= \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \sin(y) \sin(y + \phi x_3) dx_3 dy \\ &= \frac{1}{\pi} \int_0^{2\pi} \sin^2(y) dy \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \cos(\phi x_3) dx_3 \\ &- \frac{1}{\pi} \int_0^{2\pi} \cos(y) \sin(y) dy \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \sin(\phi x_3) dx_3 \\ &= \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \cos(\phi x_3) dx_3 = \frac{1}{2} \Upsilon(\phi). \end{aligned} \tag{3.185}$$

Now the off-diagonal elements are

$$C_{12} = C_{13} = \frac{1}{\pi} \int_0^{2\pi} \int_0^{\infty} \mathbb{E}\Big[m(0) m(x_3) \Big] \cos(y + \phi x_3) dx_3 dy \tag{3.186}$$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_0^{2\pi}\cos(y)dy\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(\phi x_3)dx_3 \\
&- \frac{1}{\pi}\int_0^{2\pi}\sin(y)dy\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(\phi x_3)dx_3 \\
&= 0. \qquad (3.187)
\end{aligned}
$$

Now the second row can be completed using the same approach to give

$$
C_{21} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(y+\phi x_3)dx_3dy = 0, \qquad (3.188)
$$

$$
\begin{aligned}
C_{23} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(y)\cos(y+\phi x_3)dx_3dy \\
&= -\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(\phi x_3)dx_3 = -\frac{1}{2}\Upsilon^{(AS)}(\phi). \qquad (3.189)
\end{aligned}
$$

Now the third row

$$
C_{31} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(y)dx_3dy = 0, \qquad (3.190)
$$

$$
\begin{aligned}
C_{32} &= \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(y)\sin(y+\phi x_3)dx_3dy \\
&= \int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(\phi x_3)dx_3 = \frac{1}{2}\Upsilon^{(AS)}(\phi). \qquad (3.191)
\end{aligned}
$$

The correlation matrix $\mathbf{C}$ can be written

$$
\mathbf{C} = \begin{bmatrix} \Upsilon(0) & 0 & 0 \\ 0 & \frac{1}{2}\Upsilon(\phi) & -\frac{1}{2}\Upsilon^{AS}(\phi) \\ 0 & \frac{1}{2}\Upsilon^{AS}(\phi) & \frac{1}{2}\Upsilon(\phi) \end{bmatrix}, \qquad (3.192)
$$

where

$$
\Upsilon(\phi) = 2\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\cos(\phi x_3)\,dx_3, \qquad (3.193)
$$

$$
\Upsilon^{AS}(\phi) = 2\int_0^{\infty}\mathbb{E}\Big[m(0)m(x_3)\Big]\sin(\phi x_3)\,dx_3. \qquad (3.194)
$$

The quantity $\Upsilon(\phi)$ is a non-negative real number, and is proportional to the power spectral density of the stationary random process m (the Fourier cosine transform of the autocorrelation function at frequency zero, and is consequently zero from Section 6.3.6 in [26]). The symmetric (S) and anti-symmetric (AS) elements can be assembled in separate matrices as

$$\mathbf{C}^S = \begin{bmatrix} \Upsilon(0) & 0 & 0 \\ 0 & \frac{1}{2}\Upsilon(\phi) & 0 \\ 0 & 0 & \frac{1}{2}\Upsilon(\phi) \end{bmatrix}, \quad \mathbf{C}^{AS} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{2}\Upsilon^{AS}(\phi) \\ 0 & \frac{1}{2}\Upsilon^{AS}(\phi) & 0 \end{bmatrix}. \tag{3.195}$$

Now the diffusion approximation Theorem 6.4 in [26] can be used to show that $\mathbf{P}^\varepsilon(\nu, x_3)$ converges in distribution to $\mathbf{P}(\nu, x_3)$, that is

$$\begin{bmatrix} \chi^\varepsilon(\nu, x_3) & \overline{\varkappa^\varepsilon}(\nu, x_3) \\ \varkappa^\varepsilon(\nu, x_3) & \overline{\chi^\varepsilon}(\nu, x_3) \end{bmatrix} \xrightarrow{\varepsilon\to 0} \begin{bmatrix} \chi(\nu, x_3) & \overline{\varkappa}(\nu, x_3) \\ \varkappa(\nu, x_3) & \overline{\chi}(\nu, x_3) \end{bmatrix}, \tag{3.196}$$

is the solution of the Stratonovich stochastic differential equation

$$d\mathbf{P}(\nu, x_3) = \sum_{l=1}^{3} \tilde{\mathbf{h}}_l \mathbf{P}(\nu, x_3) \circ dW_l(x_3) + \frac{1}{2}\sum_{p,q=1}^{3} \mathbf{C}_{pq}^{(AS)} \mathbf{h}_q \mathbf{h}_p \mathbf{P}(\nu, x_3) dx_3, \tag{3.197}$$

where ($\circ$) denotes the Stratonovich integral. Note here that

$$\tilde{\mathbf{h}}_l = \sum_{p=1}^{3} \tilde{\sigma}_{lp} \mathbf{h}_p, \quad \tilde{\sigma}_{lp} = (\mathbf{C}_{lp}^S)^{1/2}, \tag{3.198}$$

and $W_l(x_3)$ are independent standard Brownian motions. Hence from (3.195)

$$\tilde{\mathbf{h}}_1 = \tilde{\sigma}_{11}\mathbf{h}_1 = i\frac{\delta_1\sqrt{\Upsilon(0)}}{2}\boldsymbol{\sigma}_3, \tag{3.199}$$

$$\tilde{\mathbf{h}}_2 = \tilde{\sigma}_{22}\mathbf{h}_2 = -\frac{\delta_2\sqrt{\Upsilon(\phi)}}{2\sqrt{2}}\boldsymbol{\sigma}_1, \tag{3.200}$$

$$\tilde{\mathbf{h}}_3 = \tilde{\sigma}_{33}\mathbf{h}_3 = \frac{\delta_2\sqrt{\Upsilon(\phi)}}{2\sqrt{2}}\boldsymbol{\sigma}_2. \tag{3.201}$$

The drift term can be calculated to be

$$\frac{1}{2}\sum_{p,q=1}^{3}\mathbf{C}_{pq}^{(AS)}\mathbf{h}_q\mathbf{h}_p\mathbf{P}dx_3 = \frac{1}{2}\Big(C_{23}^{AS}\mathbf{h}_3\mathbf{h}_2 + C_{32}^{AS}\mathbf{h}_2\mathbf{h}_3\Big)\mathbf{P}dx_3$$
$$= -i\frac{\delta_2^2\Upsilon^{AS}(\phi)}{8}\boldsymbol{\sigma}_3\mathbf{P}dx_3. \tag{3.202}$$

From equations (3.197) to (3.202)

$$\begin{aligned} d\mathbf{P}(\nu,x_3) = {} & i\frac{\delta_1\sqrt{\Upsilon(0)}}{2}\boldsymbol{\sigma}_3\mathbf{P}(\nu,x_3)\circ dW_1(x_3) \\ & - \frac{\delta_2\sqrt{\Upsilon(\phi)}}{2\sqrt{2}}\boldsymbol{\sigma}_1\mathbf{P}(\nu,x_3)\circ dW_2(x_3) \\ & + \frac{\delta_2\sqrt{\Upsilon(\phi)}}{2\sqrt{2}}\boldsymbol{\sigma}_2\mathbf{P}(\nu,x_3)\circ dW_3(x_3) \\ & - i\frac{\delta_2^2\Upsilon^{AS}(\phi)}{8}\boldsymbol{\sigma}_3\mathbf{P}(\nu,x_3)dx_3 \end{aligned} \tag{3.203}$$

Equation (3.203) can be expanded in full to give

$$\begin{aligned} d\begin{bmatrix}\chi(\nu,x_3) & \overline{\varkappa}(\nu,x_3)\\ \varkappa(\nu,x_3) & \overline{\chi}(\nu,x_3)\end{bmatrix} = {} & iA_1\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\chi(\nu,x_3) & \overline{\varkappa}(\nu,x_3)\\ \varkappa(\nu,x_3) & \overline{\chi}(\nu,x_3)\end{bmatrix}\circ dW_1(x_3) \\ & - A_2\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\chi(\nu,x_3) & \overline{\varkappa}(\nu,x_3)\\ \varkappa(\nu,x_3) & \overline{\chi}(\nu,x_3)\end{bmatrix}\circ dW_2(x_3) \\ & + iA_2\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\begin{bmatrix}\chi(\nu,x_3) & \overline{\varkappa}(\nu,x_3)\\ \varkappa(\nu,x_3) & \overline{\chi}(\nu,x_3)\end{bmatrix}\circ dW_3(x_3) \\ & - iA_3\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}\chi(\nu,x_3) & \overline{\varkappa}(\nu,x_3)\\ \varkappa(\nu,x_3) & \overline{\chi}(\nu,x_3)\end{bmatrix}dx_3, \end{aligned} \tag{3.204}$$

where

$$A_1 = \frac{\delta_1\sqrt{\Upsilon(0)}}{2}, \tag{3.205}$$
$$A_2 = \frac{\delta_2\sqrt{\Upsilon(\phi)}}{2\sqrt{2}}, \tag{3.206}$$
$$A_3 = \frac{\delta_2^2\Upsilon^{AS}(\phi)}{8}. \tag{3.207}$$

From the system (3.204) the two propagator SDEs read

$$d\chi = iA_1\chi \circ dW_1 - A_2\varkappa \circ (dW_2 + idW_3) - iA_3\chi dx_3, \tag{3.208}$$

$$d\varkappa = -iA_1\varkappa \circ dW_1 + A_2\chi \circ (idW_3 - dW_2) + iA_3\varkappa dx_3, \tag{3.209}$$

with initial conditions $\chi(\nu, x_3 = 0) = 1$ and $\varkappa(\nu, x_3 = 0) = 0$, together with the conservation of energy relation

$$|\chi|^2 - |\varkappa|^2 = 1. \tag{3.210}$$

Note that this is automatically satisfied by equations (3.208)-(3.209).

3.7.2 Numerical Simulation of Correlation Integrals

Now, a material is simulated in order to show how to calculate the correlation integrals (3.193) and (3.194). The symmetric correlation integral $\Upsilon(\phi)$ may be written in discretised form as

$$\Upsilon_N(\phi) = 2\sum_{i=1}^{N}\left(\sum_{r=1}^{R}\sum_{j=1}^{K}\frac{(m_1^r)^j(m_i^r)^j}{RK}\right)\cos(\phi x_3^i)\Delta x_3, \tag{3.211}$$

where $(m_i^r)^j$ is the orientation of the crystalline material in realisation j, along ray r and at arc length position $(x_3)^i$. These orientations are drawn from a uniform distribution over $[-1, 1]$. From equations (3.145), (3.146) and (3.169), since $\tilde{c}_{44}, \tilde{c}_{66} \approx 1$, then $\phi \approx 2$. The mean layer length $l = \lambda_3$ is chosen accordingly to fall into the weakly heterogeneous regime $L_3 \gg \lambda_3 \sim l$. Assume that the layer sizes

$$(X_1, X_2 - X_1, ..., X_n - X_{n-1}, ...) \tag{3.212}$$

generate a sequence of independent random variables that follow an exponential distribution with intensity parameter l

$$\mathbb{P}[X_n - X_{n-1} \leq x_3] = 1 - e^{-lx_3}, \tag{3.213}$$

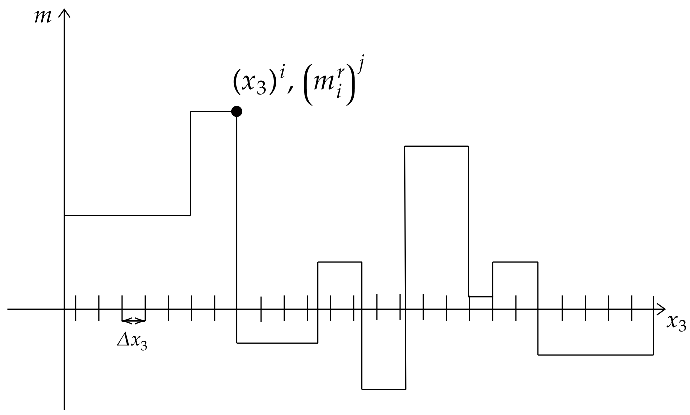

Figure 3.4: Visual representation of a single ray realisation traversing a layered material. Each random variable $(m_i^r)^j$ represents the crystal orientation in the material at the point $(x_3)^i$. Δx_3 is the discretisation step size used to evaluate the integral (3.211), i represents the step number along the discretised ray, and the length of the ray is $L_r = N\Delta x_3$.

This ensures that the layer lengths are constructed from an exponential distribution and therefore form a Markov process [26]. In this scaled non-dimensional framework, the propagation distance is denoted by $\tilde{L}$ and $l = 1$. Typical values of the correlation integrals for varying values of number of rays R and number of realisations K are shown in Table (3.1). This concludes that the integrals are of order one. Note that $\Upsilon(\phi)$ is a non-negative real number (which is expected) since it is proportional to the power spectral density of the random process m [34].

Numerical Correlation Integral Values			
$(R,\, K)$	$\Upsilon(\phi)$	$\Upsilon^{AS}(\phi)$	$\Upsilon(0)$
$(100, 10)$	2.05	1.44	1.60
$(200, 10)$	1.53	0.95	2.31
$(400, 10)$	1.05	1.08	2.45
$(800, 10)$	1.06	0.90	2.45

Table 3.1: Numerical estimates for the values of the correlation integrals in equation (3.195). Material values from Table 3.2 for Graphite Epoxy were used. The discretised integrals are calculated for different combinations of R and K.

3.7.3 Parameterising Solution On a Hyperbola

Now a substitution is used to obtain a reduced, closed system of SDEs. First parameterise the entries of the propagator matrix via

$$\chi(\nu, x_3) = \cosh\left(\frac{\gamma_\nu(x_3)}{2}\right) e^{i\phi_\nu(x_3)}, \tag{3.214}$$

$$\varkappa(\nu, x_3) = \sinh\left(\frac{\gamma_\nu(x_3)}{2}\right) e^{i(\phi_\nu(x_3)+\psi_\nu(x_3))}, \tag{3.215}$$

where $\gamma_\nu(x_3) \in [0, \infty)$, $\phi_\nu(x_3)$, $\psi_\nu(x_3) \in \mathbb{R}$, and $\gamma_\nu(x_3 = 0) = \phi_\nu(x_3 = 0) = \psi_\nu(x_3 = 0) = 0$. Using the standard chain rule for the derivatives in the Stratonovich regime gives

$$d\chi = \frac{\partial \chi}{\partial \gamma_\nu} d\gamma_\nu + \frac{\partial \chi}{\partial \phi_\nu} d\phi_\nu, \tag{3.216}$$

$$d\varkappa = \frac{\partial \varkappa}{\partial \gamma_\nu} d\gamma_\nu + \frac{\partial \varkappa}{\partial \phi_\nu} d\phi_\nu + \frac{\partial \varkappa}{\partial \psi_\nu} d\psi_\nu. \tag{3.217}$$

Substituting equations (3.214) to (3.217) in equations (3.208) and (3.209) admits the parameterised form

$$d\gamma_\nu = -2A_2\bigg(\cos(\psi_\nu) \circ dW_2 - \sin(\psi_\nu) \circ dW_3 \bigg), \tag{3.218}$$

$$d\psi_\nu = -2A_1 \circ dW_1 + \frac{2A_2}{\tanh(\gamma_\nu)}\bigg(\cos(\psi_\nu) \circ dW_3 + \sin(\psi_\nu) \circ dW_2 \bigg) + 2A_3\, dx_3, \tag{3.219}$$

and

$$d\phi_\nu = A_1 \circ dW_1 - A_2 \tanh(\gamma_\nu/2)\bigg(\sin(\psi_\nu) \circ dW_2 + \cos(\psi_\nu) \circ dW_3 \bigg) - A_3\, dx_3, \tag{3.220}$$

which can be written in matrix form as

$$\begin{bmatrix} d\phi_\nu \\ d\psi_\nu \\ d\gamma_\nu \end{bmatrix} = \begin{bmatrix} A_1 & -A_2 \tanh(\gamma_\nu/2)\sin(\psi_\nu) & -A_2 \tanh(\gamma_\nu/2)\cos(\psi_\nu) \\ -2A_1 & 2A_2 \sin(\psi_\nu)/\tanh(\gamma_\nu) & 2A_2 \cos(\psi_\nu)/\tanh(\gamma_\nu) \\ 0 & -2A_2 \cos(\psi_\nu) & 2A_2 \sin(\psi_\nu) \end{bmatrix} \circ \begin{bmatrix} dW_1 \\ dW_2 \\ dW_3 \end{bmatrix} + \begin{bmatrix} -A_3 \\ 2A_3 \\ 0 \end{bmatrix} dx_3, \tag{3.221}$$

which can be rewritten as

$$d\boldsymbol{x} = \boldsymbol{\sigma} \circ \boldsymbol{W} + \boldsymbol{b} dx_3, \tag{3.222}$$

where

$$\boldsymbol{x} = \begin{bmatrix} d\phi_\nu, d\psi_\nu, d\gamma_\nu \end{bmatrix}^T, \tag{3.223}$$

$$\boldsymbol{\sigma} = \begin{bmatrix} A_1 & -A_2 \tanh(\gamma_\nu/2)\sin(\psi_\nu) & -A_2 \tanh(\gamma_\nu/2)\cos(\psi_\nu) \\ -2A_1 & 2A_2 \sin(\psi_\nu)/\tanh(\gamma_\nu) & 2A_2 \cos(\psi_\nu)/\tanh(\gamma_\nu) \\ 0 & -2A_2 \cos(\psi_\nu) & 2A_2 \sin(\psi_\nu) \end{bmatrix}, \tag{3.224}$$

$$\boldsymbol{W} = \begin{bmatrix} W_1,\ W_2,\ W_3 \end{bmatrix}^T \tag{3.225}$$

and

$$\boldsymbol{b} = \begin{bmatrix} -A_3, 2A_3, 0 \end{bmatrix}^T. \tag{3.226}$$

Equation (3.222) can be converted to Itô form by calculating the modified drift [34] via

$$\sigma_{ip} \circ dW_p = \sigma_{ip} dW_p + \frac{1}{2} \sum_{j=1}^{3} \sigma_{jp} \sigma_{ip,j}\, dx_3, \quad p = 1, 2, 3. \tag{3.227}$$

Note that the only non-vanishing modified drift term is present in the γ_ν process, namely

$$\frac{1}{2}\sum_{j=1}^{3}\sigma_{jp}\sigma_{3p,j}\,dx_3 = \frac{2A_2^2}{\tanh(\gamma_\nu)}dx_3. \tag{3.228}$$

Equation (3.222) can be rewritten in Itô form as

$$d\gamma_\nu = -2A_2\Bigg(\cos(\psi_\nu)dW_2 - \sin(\psi_\nu)dW_3\Bigg) + \frac{2A_2^2}{\tanh(\gamma_\nu)}dx_3, \tag{3.229}$$

$$d\psi_\nu = \frac{2A_2}{\tanh(\gamma_\nu)}\Bigg(\cos(\psi_\nu)dW_3 + \sin(\psi_\nu)dW_2\Bigg) - 2A_1dW_1 + 2A_3\,dx_3, \tag{3.230}$$

and

$$d\phi_\nu = -A_2\tanh(\gamma_\nu/2)\Bigg(\sin(\psi_\nu)dW_2 + \cos(\psi_\nu)dW_3\Bigg) + A_1dW_1 - A_3\,dx_3. \tag{3.231}$$

This system can be reduced further by introducing two (auxiliary) random processes via

$$\begin{bmatrix} W_4 \\ W_5 \end{bmatrix} = \int_0^{x_3} \begin{bmatrix} \cos(\psi_\nu) & \sin(\psi_\nu) \\ \sin(\psi_\nu) & -\cos(\psi_\nu) \end{bmatrix} d\begin{bmatrix} W_3 \\ W_2 \end{bmatrix}, \tag{3.232}$$

so that the system (3.229) to (3.231) can be transformed to give

$$d\gamma_\nu = 2A_2\,dW_5 + \frac{2A_2^2}{\tanh(\gamma_\nu)}dx_3, \tag{3.233}$$

$$d\psi_\nu = -2A_1dW_1 + \frac{2A_2}{\tanh(\gamma_\nu)}\,dW_4 + 2A_3\,dx_3, \tag{3.234}$$

and

$$d\phi_\nu = A_1dW_1 - A_2\tanh(\gamma_\nu/2)\,dW_4 - A_3\,dx_3. \tag{3.235}$$

In matrix form (3.233) to (3.235) is given by

$$d\begin{bmatrix}\phi_\nu\\ \psi_\nu\\ \gamma_\nu\end{bmatrix} = \begin{bmatrix}A_1 & -A_2\tanh(\gamma_\nu/2) & 0\\ -2A_1 & 2A_2/\tanh(\gamma_\nu) & 0\\ 0 & 0 & 2A_2\end{bmatrix}\begin{bmatrix}dW_1\\ dW_4\\ dW_5\end{bmatrix} + \begin{bmatrix}-A_3\\ 2A_3\\ 2A_2^2/\tanh(\gamma_\nu)\end{bmatrix} dx_3. \quad (3.236)$$

Now that the matrix equations are in Itô form, the infinitesimal generator [26] of the Markov process $(\gamma_\nu,\ \phi_\nu,\ \psi_\nu)$ can be calculated as

$$\mathcal{L} = \frac{1}{2}\sum_{i,\,j=1}^{3} a_{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i\frac{\partial}{\partial x_i}, \qquad \sum_{k=1}^{3} a_{ij} = \sigma_{ik}\sigma_{jk}. \quad (3.237)$$

The only non-zero entries in the a_{ij} functions are

$$a_{11} = A_1^2 + A_2^2\tanh^2(\gamma_\nu/2), \quad (3.238)$$

$$a_{12} = a_{21} = -2A_1^2 - \frac{2A_2^2\tanh(\gamma_\nu/2)}{\tanh(\gamma_\nu)}, \quad (3.239)$$

$$a_{22} = 4A_1^2 + \frac{4A_2^2}{\tanh^2(\gamma_\nu)}, \quad (3.240)$$

$$a_{33} = 4A_2^2, \quad (3.241)$$

and so the generator can be written as

$$\begin{aligned}\mathcal{L}_{\gamma_\nu,\phi_\nu,\psi_\nu} &= 2A_2^2\left[\frac{\partial^2}{\partial\gamma_\nu^2} + \frac{1}{\tanh(\gamma_\nu)}\frac{\partial}{\partial\gamma_\nu}\right]\\ &+ A_2^2\left[\frac{1}{2}\tanh^2(\gamma_\nu/2)\frac{\partial^2}{\partial\phi_\nu^2} - 2\frac{\tanh(\gamma_\nu/2)}{\tanh(\gamma_\nu)}\frac{\partial^2}{\partial\phi_\nu\partial\psi_\nu} + \frac{2}{\tanh^2(\gamma_\nu)}\frac{\partial^2}{\partial\psi_\nu^2}\right]\\ &+ A_1^2\left[\frac{1}{2}\frac{\partial^2}{\partial\phi_\nu^2} - 2\frac{\partial^2}{\partial\phi_\nu\partial\psi_\nu} + 2\frac{\partial^2}{\partial\psi_\nu^2}\right]\\ &+ A_3\left[2\frac{\partial}{\partial\psi_\nu} - \frac{\partial}{\partial\phi_\nu}\right]. \end{aligned} \quad (3.242)$$

The radial process γ_ν (so focusing on the amplitude in transformations (3.214) and (3.215)) has infinitesimal generator

$$\mathcal{L}_{\gamma_\nu} = 2A_2^2\left(\frac{\partial^2}{\partial\gamma_\nu^2} + \frac{1}{\tanh(\gamma_\nu)}\frac{\partial}{\partial\gamma_\nu}\right) = \frac{\delta_2^2\Upsilon(\phi)}{4}\left(\frac{\partial^2}{\partial\gamma_\nu^2} + \frac{1}{\tanh(\gamma_\nu)}\frac{\partial}{\partial\gamma_\nu}\right). \quad (3.243)$$

3.7.4 Unit Monochromatic Wave Impinging From Left half space

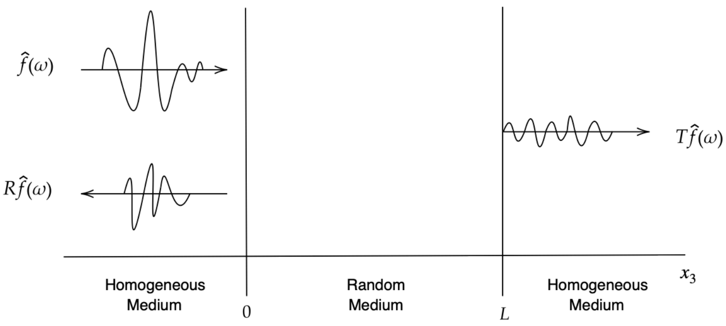

Figure 3.5: Incident wave $\hat{f}(\omega)$ incoming from the left half space at $x_3 = 0$. R and T are the reflection and transmission coefficients respectively. These coefficients determine the amount of energy being transmitted and reflected at the boundaries of the slab occupying $x_3 \in [0, L]$.

Assume now that the random medium is embedded between two homogeneous half spaces with a unit wave impinging at $x_3 = 0$ (see Figure 3.5). The boundary conditions for equation (3.170) are prescribed by

$$\hat{a}(\nu, x_3 = 0) = 1, \quad \hat{b}(\nu, x_3 = L) = 0, \tag{3.244}$$

which translates into a unit wave travelling from the left and no left-going wave from the right at $x_3 = L$. The reflection and transmission coefficients are given by

$$R_\nu(\nu, L) = \hat{b}(\nu, 0), \quad T_\nu = \hat{a}(\nu, L). \tag{3.245}$$

Using equation (3.170) with (3.245), gives the system

$$\begin{bmatrix} \hat{a}(\nu, L) \\ \hat{b}(\nu, L) \end{bmatrix} = \begin{bmatrix} T_\nu(L) \\ 0 \end{bmatrix} = \mathbf{P}(\nu, L) \begin{bmatrix} 1 \\ R_\nu(L) \end{bmatrix}. \tag{3.246}$$

That is

$$T_\nu(L) = \chi + \overline{\varkappa}(L) R_\nu(L), \tag{3.247}$$

$$0 = \varkappa(L) + \overline{\chi}(L) R_\nu(L), \tag{3.248}$$

and so

$$R_\nu(\nu, L) = -\frac{\varkappa(\nu, L)}{\overline{\chi}(\nu, L)}, \quad \text{and} \quad T_\nu(\nu, L) = \frac{1}{\overline{\chi}(\nu, L)}. \tag{3.249}$$

Now define the power transmission coefficient and use equations (3.214) and (3.249) to get

$$\tau(L) = |T_\nu(\nu, L)|^2 = \cosh^{-2}\left(\frac{\gamma_\nu(L)}{2}\right), \tag{3.250}$$

to describe the amplitude of the energy transmitted through the medium. Integrating equation (3.235) gives

$$\int_0^L d\phi_\nu(x_3) = A_1 \int_0^L dW_1(x_3) - A_2 \int_0^L \tanh\left(\gamma_\nu(x_3)/2\right) dW_4(x_3) - A_3 \int_0^L dx_3, \tag{3.251}$$

with $\phi_\nu(0) = 0$, $W_1(0) = 0$ and so

$$\phi_\nu(L) = A_1 W_1(L) - A_2 \int_0^L \tanh\left(\gamma_\nu(x_3)/2\right) dW_4(x_3) - A_3 L, \tag{3.252}$$

so that

$$\exp\{i\phi_\nu(L)\} = \exp\left\{ iA_1 W_1(L) - iA_2 \int_0^L \tanh\left(\gamma_\nu(x_3)/2\right) dW_4(x_3) - iA_3 L \right\}. \tag{3.253}$$

Since $d(\cosh(\gamma_\nu/2)^{-1}) = -1/2(\tanh(\gamma_\nu/2)/\cosh(\gamma_\nu/2)) \circ d\gamma_\nu$, then equation (3.233) gives

$$d(\cosh(\gamma_\nu/2)^{-1}) = -\frac{1}{2}\left(\frac{\tanh(\gamma_\nu/2)}{\cosh(\gamma_\nu/2)}\right) \circ \left(2A_2 dW_5 + \frac{2A_2^2}{\tanh(\gamma_\nu)} dx_3\right). \tag{3.254}$$

Multiplying both sides by $\cosh(\gamma_\nu/2)$, introducing $Z(x_3)$ for brevity and integrating with respect to x_3 (from 0 to L) gives

$$Z(L) = \ln\left(\cosh(\gamma_\nu(L)/2)^{-1}\right) = -A_2 \int_0^L \tanh\left(\gamma_\nu(x_3)/2\right) \circ dW_5(x_3)$$

$$- A_2^2 \int_0^L \frac{\tanh(\gamma_\nu(x_3)/2)}{\tanh(\gamma_\nu(x_3))} dx_3. \tag{3.255}$$

Since

$$\tanh^2(\gamma_\nu/2) = 1 - \cosh^{-2}(\gamma_\nu/2) = 1 - \tau = 1 - e^{2Z(x)}, \tag{3.256}$$

then

$$Z(L) = -A_2 \int_0^L \left(1 - e^{2Z(x_3)}\right)^{\frac{1}{2}} \circ dW_5(x_3) - \frac{A_2^2}{2} \int_0^L \left(2 - e^{2Z(x_3)}\right) dx_3, \tag{3.257}$$

via the identity

$$\frac{\tanh(\gamma_\nu/2)}{\tanh(\gamma_\nu)} = \frac{1}{2}(1 + \tanh(\gamma_\nu/2)^2) = \frac{1}{2}(2 - \tau) = \frac{1}{2}(2 - e^{2Z(x_3)}). \tag{3.258}$$

Now convert equation (3.257) into Itô form via

$$a = -A_2(1 - e^{2Z(x_3)})^{\frac{1}{2}}, \quad \frac{\partial a}{\partial Z} = A_2(1 - e^{2Z(x_3)})^{-\frac{1}{2}} e^{2Z(x_3)}, \tag{3.259}$$

which gives the correction term

$$\frac{a}{2} \frac{\partial a}{\partial Z} = -\frac{A_2^2}{2} e^{2Z(x_3)}, \tag{3.260}$$

which is added to equation (3.257) to obtain

$$Z(L) = -A_2 \int_0^L (1 - e^{2Z(x_3)})^{\frac{1}{2}} dW_5(x_3) - A_2^2 L. \tag{3.261}$$

Taking exponentials of both sides gives

$$\frac{1}{\cosh(\gamma_\nu(L)/2)} = \exp\left\{-A_2 \int_0^L \tanh(\gamma_\nu(x_3)/2) dW_5(x_3) - A_2^2 L\right\}. \tag{3.262}$$

3.7.5 Localisation Length Definition

Since $\lim_{x\to\infty}(\tanh(x)) = 1$, the power transmission coefficient in equation (3.250) tends asymptotically as $L \to \infty$ to

$$\tau(L) \sim \exp\left\{-2A_2 dW_5(L) - 2A_2^2 L\right\}. \tag{3.263}$$

$$\tau(L) \sim \exp\left\{-2\tilde{A}_2^2 \tilde{L}\right\} \sim \exp\left\{-\frac{\tilde{L}}{\tilde{L}_{loc}}\right\}, \tag{3.264}$$

where the non-dimensionalised localisation length is

$$\tilde{L}_{loc} = \frac{1}{2\tilde{A}_2^2 L_3} = \frac{L_{loc}}{L_3}. \tag{3.265}$$

$\tilde{L}_{loc}$ is a non-dimensional parameter which controls the rate of the decay of energy through the random slab. From equation (3.206) the non-dimensionalised localisation length is written as

$$\tilde{L}_{loc}(\nu, c_{ijkl}, \rho, \theta(x_3)) = \frac{L_{loc}}{L_3} = \frac{1}{2\tilde{A}_2^2 L_3} = \frac{4}{\delta_2^2 \tilde{\Upsilon}(\phi)} = \frac{4}{\delta_2^2(\nu, c_{ijkl}, \rho)\tilde{\Upsilon}(\nu, c_{ijkl}, \rho, \theta(x_3))}, \tag{3.266}$$

where $\tilde{\Upsilon}(\phi) = \Upsilon(\phi)/L_3$ is the non-dimensional correlation integral. This non-dimensional parameter $\tilde{L}_{loc}$ controls the rate of the decay of energy through the random slab.

3.7.6 Martingale Representation of Power Transmission Coefficient

Using equations (3.253) and (3.262), the transmission coefficient (3.249) can be written as

$$\begin{aligned} T_\nu(\nu, L) &= \frac{e^{i\phi_\nu}}{\cosh(\gamma_\nu/2)} \\ &= \exp\left\{iA_1 W_1(L) - A_2^2 L - iA_3 L - A_2 \int_0^L \tanh(\gamma_\nu(x_3)/2)[dW_5(x_3) + idW_4(x_3)]\right\} \\ &= \exp\left\{iA_1 W_1(L) - A_2^2 L - iA_3 L\right\} \end{aligned}$$

$$\times \exp\Big\{-A_2 \int_0^L \tanh{(\gamma_\nu(x_3)/2)}[dW_5(x_3) + idW_4(x_3)]\Big\}$$
$$= T^M(\nu, L)M(\nu, L), \tag{3.267}$$

where

$$T^M(\nu, L) = \exp\big\{iA_1W_1(L) - A_2^2L - iA_3L\big\}, \tag{3.268}$$
$$M(\nu, L) = \exp\Big\{-A_2 \int_0^L \tanh{(\gamma_\nu(x_3)/2)}[dW_5(x_3) + idW_4(x_3)]\Big\}. \tag{3.269}$$

Since γ_ν is deterministic (with $\gamma_\nu \in [0, \infty)$) then

$$\int_0^L \tanh{(\gamma_\nu(x_3))}dW_4 \sim N\Big(0, \int_0^L \tanh^2 \gamma_\nu(x_3)/2dx_3\Big), \tag{3.270}$$
$$\int_0^L \tanh{(\gamma_\nu(x_3))}dW_5 \sim N\Big(0, \int_0^L \tanh^2 \gamma_\nu(x_3)/2dx_3\Big). \tag{3.271}$$

Since these two integrals are independent real valued random numbers with common variance, $M(\nu, L)$ has mean

$$\mathbb{E}[M(\nu, L)] = 1. \tag{3.272}$$

3.7.7 Fokker-Planck Equation for the Power Transmission Coefficient

From equation (3.250)

$$\frac{\partial\tau}{\partial\gamma_\nu} = -\cosh{(\gamma_\nu/2)}^{-3}\sinh{(\gamma_\nu/2)} = -\cosh{(\gamma_\nu/2)}^{-2}\tanh{(\gamma_\nu/2)} = -\tau\tanh{(\gamma_\nu/2)}$$
$$= -\tau\sqrt{1-\tau}. \tag{3.273}$$

Hence, using equation (3.258)

$$\frac{1}{\tanh{(\gamma_\nu)}}\frac{\partial}{\partial\gamma_\nu} = \frac{1}{\tanh{(\gamma_\nu)}}\frac{\partial\tau}{\partial\gamma_\nu}\frac{\partial}{\partial\tau}$$
$$= -\frac{\tanh{(\gamma_\nu/2)}}{\tanh{(\gamma_\nu)}}\tau\frac{\partial}{\partial\tau}$$
$$= -\frac{1}{2}(2-\tau)\tau\frac{\partial}{\partial\tau}. \tag{3.274}$$

Then from equation (3.273)

$$\begin{aligned}
\frac{\partial^2}{\partial\gamma_\nu^2} &= \frac{\partial}{\partial\gamma_\nu}\bigg(-(1-\tau)^{\frac{1}{2}}\tau\frac{\partial}{\partial\tau}\bigg) \\
&= \frac{\partial\tau}{\partial\gamma_\nu}\frac{\partial}{\partial\tau}\bigg(-(1-\tau)^{\frac{1}{2}}\tau\frac{\partial}{\partial\tau}\bigg) \\
&= -\tau(1-\tau)^{\frac{1}{2}}\bigg(\frac{1}{2}(1-\tau) - \frac{1}{2}\tau\frac{\partial}{\partial\tau} - (1-\tau)^{\frac{1}{2}}\frac{\partial}{\partial\tau} - (1-\tau)^{\frac{1}{2}}\tau\frac{\partial^2}{\partial\tau^2}\bigg) \\
&= \tau^2(1-\tau)\frac{\partial^2}{\partial\tau^2} + (\tau - \frac{3}{2}\tau^2)\frac{\partial}{\partial\tau} \\
&= -\frac{\tau^2}{2}\frac{\partial}{\partial\tau} + \tau(1-\tau)\frac{\partial}{\partial\tau} + (1-\tau)\tau^2\frac{\partial^2}{\partial\tau^2}.
\end{aligned} \tag{3.275}$$

The infinitesimal generator, as given by equation (3.243), of the power transmission coefficient, can now be written (dropping tildes)

$$\mathcal{L}_\tau = \frac{1}{L_{loc}}\bigg(\tau^2(1-\tau)\frac{\partial^2}{\partial\tau^2} - \tau^2\frac{\partial}{\partial\tau}\bigg). \tag{3.276}$$

Now introduce the transformation

$$\eta = \frac{2-\tau}{\tau}, \tag{3.277}$$

which takes values in $[1,\infty)$. The infinitesimal generator is then given by

$$\mathcal{L}_\eta = \frac{1}{L_{loc}}\frac{\partial}{\partial\eta}\bigg((\eta^2-1)\frac{\partial}{\partial\eta}\bigg). \tag{3.278}$$

A calculation via inner products can show that the operator $\mathcal{L}_\eta$ is self adjoint. Taking two test functions, say $\phi(\eta)$ and $\psi(v)$, the adjoint $\mathcal{L}_\eta^\star$ is defined by

$$\int \psi\mathcal{L}_\eta\phi\, d\eta = \int \phi\mathcal{L}_\eta^\star\psi\, dv. \tag{3.279}$$

Noting that

$$\mathcal{L}_\eta = \frac{1}{L_{loc}}\bigg(\frac{\partial}{\partial\eta}\bigg((\eta^2-1)\frac{\partial}{\partial\eta}\bigg)\bigg) = \frac{1}{L_{loc}}\bigg((\eta^2-1)\frac{\partial^2}{\partial\eta^2} + 2\eta\frac{\partial}{\partial\eta}\bigg), \tag{3.280}$$

so

$$
\begin{aligned}
\frac{1}{L_{loc}}\int\left(\psi(\eta^2-1)\phi_{\eta\eta}+2\eta\psi\phi_\eta\right)d\eta &= \frac{1}{L_{loc}}\int\Bigg(\psi_{\eta\eta}(\eta^2-1)+2\eta\psi_\eta+2\eta\psi_\eta+2\psi \\
&\quad -2\psi-2\eta\psi_\eta\Bigg)\phi\, d\eta \\
&= \frac{1}{L_{loc}}\int\Big((\eta^2-1)\psi_{\eta\eta}+2\eta\psi_\eta\Big)\phi\, d\eta \\
&= \frac{1}{L_{loc}}\int\phi\mathcal{L}_\eta^\star\psi\, d\eta \\
&= \frac{1}{L_{loc}}\int\left(\phi(\eta^2-1)\psi_{\eta\eta}+2\eta\phi\psi_\eta\right)d\eta, \qquad (3.281)
\end{aligned}
$$

which shows that $\mathcal{L}_\eta = \mathcal{L}_\eta^\star$ is self-adjoint. Hence, the Fokker-Planck equation for the probability density function of $\eta(x_3 = L)$ is

$$\frac{\partial p}{\partial L}(L,\eta) = \frac{1}{L_{loc}}\frac{\partial}{\partial\eta}\left(\eta^2-1\right)\frac{\partial p}{\partial\eta}(L,\eta), \quad \eta > 1, \tag{3.282}$$

with initial condition $p(L = 0, \eta) = \delta(\eta - 1)$, where δ denotes the Dirac delta function. The Legendre function of the first kind $P_{-\frac{1}{2}+i\mu}(\eta)$, $\eta > 1$, $\mu > 0$, satisfies the Legendre differential equation [71]

$$\frac{d}{d\eta}(\eta^2-1)\frac{d}{d\eta}P_{-\frac{1}{2}+i\mu}(\eta) = -\left(\mu^2+\frac{1}{4}\right)P_{-\frac{1}{2}+i\mu}(\eta), \tag{3.283}$$

where

$$P_{-\frac{1}{2}+i\mu}(\eta) = \frac{\sqrt{2}}{\pi}\cosh(\pi\mu)\int_0^\infty \frac{\cos(\mu x)}{\sqrt{\cosh(x)+\eta}}dx. \tag{3.284}$$

and initial condition $P_{-\frac{1}{2}+i\mu}(\eta) = 1$. Equation (3.282) can be solved analytically by the use of the Mehler-Fock transform [72] given by [26]

$$\check{f}(\mu) = \int_1^\infty f(\eta)P_{-\frac{1}{2}+i\mu}(\eta)\, d\eta, \tag{3.285}$$

with inverse transform

$$f(\eta) = \int_0^\infty \check{f}(\mu)\mu \tanh(\mu\pi) P_{-\frac{1}{2}+i\mu}(\eta)\, d\mu. \tag{3.286}$$

Applying the Mehler-Fock transform to equation (3.282) gives

$$\frac{\partial \check{p}}{\partial L}(L,\mu) = \frac{1}{L_{loc}} \int_1^\infty \frac{\partial p}{\partial L}(L,\eta) P_{-\frac{1}{2}+i\mu}(\eta)\, d\eta,$$

and integrating by parts

$$\begin{aligned}
&= \frac{1}{L_{loc}}\left(\left[(\eta^2-1)\frac{\partial p}{\partial \eta}(L,\eta) P_{-\frac{1}{2}+i\mu}(\eta) \right]_1^\infty - \int_1^\infty \frac{\partial P}{\partial \eta}(L,\eta)(\eta^2-1)\frac{\partial P_{-\frac{1}{2}+i\mu}}{\partial \eta}(\eta)\, d\eta \right) \\
&= \frac{1}{L_{loc}}\left(- \left[p(L,\eta)(\eta^2-1)\frac{\partial P_{-\frac{1}{2}+i\mu}}{\partial \eta}(\eta) \right]_1^\infty \right. \\
&\left. + \int_1^\infty p(L,\eta)\frac{\partial}{\partial \eta}\left((\eta^2-1)\frac{\partial P_{-\frac{1}{2}+i\mu}}{\partial \eta}(\eta) \right) d\eta \right) \\
&= \frac{1}{L_{loc}} \int_1^\infty p(L,\eta)\frac{\partial}{\partial \eta}\left[(\eta^2-1)\frac{\partial P_{-\frac{1}{2}+i\mu}}{\partial \eta}(\eta) \right] d\eta \\
&= -\frac{1}{L_{loc}}\left(\mu^2 + \frac{1}{4} \right) \int_1^\infty p(L,\eta) P_{-\frac{1}{2}+i\mu}(\eta)\, d\eta \\
&= -\frac{1}{L_{loc}}\left(\mu^2 + \frac{1}{4} \right) \check{p}(L,\mu),
\end{aligned} \tag{3.287}$$

with initial condition $\check{p}(L=0,\mu) = 1$. This PDE can be seen to have, by inspection, a solution

$$\check{p}(L,\mu) = \exp\left\{ -\left(\mu^2 + \frac{1}{4} \right) \frac{L}{L_{loc}} \right\}. \tag{3.288}$$

Now, apply the inverse Mehler-Fock transform (3.286) to obtain the probability density function (PDF) of the process $\eta(L)$ given by

$$p(L,\eta) = \int_0^\infty \mu \tanh(\mu\pi) P_{-\frac{1}{2}+i\mu}(\eta) \exp\left\{ -\left(\mu^2 + \frac{1}{4} \right) \frac{L}{L_{loc}} \right\} d\mu. \tag{3.289}$$

Equipped with the probability density function for the power transmission coefficient, the moments can be computed as follows

$$\begin{aligned}\mathbb{E}[\tau^n(L)] &= \int_1^\infty \tau^n(L)p(L,\eta)\,d\eta \\ &= \int_1^\infty \left(\frac{2}{1+\eta}\right)^n p(L,\eta)\,d\eta \\ &= \int_1^\infty \left(\frac{2}{1+\eta}\right)^n \\ &\times \int_0^\infty \mu\tanh{(\mu\pi)}P_{-\frac{1}{2}+i\mu}(\eta)\exp\left\{-\left(\mu^2+\frac{1}{4}\right)\frac{L}{L_{loc}}\right\}d\mu d\eta. \end{aligned} \tag{3.290}$$

By defining the family of functions

$$J^{(n)}(\mu) = \int_1^\infty \left(\frac{2}{1+\eta}\right)^n P_{-\frac{1}{2}+i\mu}(\eta)\,d\eta, \quad n\in\mathbb{N}, \tag{3.291}$$

the moments can be written as

$$\mathbb{E}[\tau^n(L)] = \int_0^\infty \mu\tanh{(\mu\pi)}J^{(n)}(\mu)\exp\left\{-\left(\mu^2+\frac{1}{4}\right)\frac{L}{L_{loc}}\right\}d\mu. \tag{3.292}$$

For $n=1$, using equation (3.284) gives

$$\begin{aligned}J^{(1)}(\mu) &= 2\int_1^\infty \frac{P_{-\frac{1}{2}+i\mu}(\eta)}{1+\eta}\,d\eta \\ &= \frac{2\sqrt{2}}{\pi}\cosh{(\pi\mu)}\int_0^\infty \cos{(\mu x)}\left(\int_1^\infty \frac{1}{(\eta+1)\sqrt{\cosh{(x)}+\eta}}\,d\eta\right)dx. \end{aligned}\tag{3.293}$$

Now evaluate

$$\int_1^\infty \frac{1}{(\eta+1)\sqrt{\cosh{(x)}+\eta}}\,d\eta, \tag{3.294}$$

and let

$$u = \frac{\sqrt{\cosh x+\eta}}{\sqrt{\cosh x-1}}, \quad \eta+1 = (\cosh x-1)(u^2-1), \tag{3.295}$$

so that

$$\frac{\partial u}{\partial \eta} = \frac{1}{2}\frac{(\cosh x + \eta)^{-\frac{1}{2}}}{(\cosh x - 1)^{\frac{1}{2}}}, \quad d\eta = 2\sqrt{\cosh x - 1}\sqrt{\cosh x + \eta}\, du. \tag{3.296}$$

Hence

$$\begin{aligned}\int_1^\infty \frac{d\eta}{(1+\eta)\sqrt{\cosh x + \eta}} &= \int_{\frac{\sqrt{\cosh x+1}}{\sqrt{\cosh x-1}}}^\infty \frac{2\sqrt{\cosh x - 1}}{(\cosh x - 1)(u^2 - 1)}\, du \\ &= \frac{2}{\sqrt{\cosh x - 1}}\left[\ln\sqrt{\frac{u-1}{u+1}}\right]_{\frac{\sqrt{\cosh x+1}}{\sqrt{\cosh x-1}}}^\infty .\end{aligned} \tag{3.297}$$

Now

$$\sqrt{\cosh(x) - 1} = \sqrt{\frac{e^x + e^{-x} - 2}{2}} = \sqrt{\frac{(e^{x/2} - e^{-x/2})^2}{2}} = \sqrt{2}\sinh(x/2), \tag{3.298}$$

and

$$\begin{aligned}\ln\left|\sqrt{\frac{\frac{\sqrt{\cosh x+1}}{\sqrt{\cosh x-1}} - 1}{\frac{\sqrt{\cosh x+1}}{\sqrt{\cosh x-1}} + 1}}\right| &= \frac{1}{2}\ln\left|\frac{\sinh x - \cosh x + 1}{\sinh x + \cosh x - 1}\right| \\ &= \frac{1}{2}\ln\left|\frac{-2e^{-x} + 2}{2e^x - 2}\right| \\ &= \frac{1}{2}\ln\left|\frac{e^{-x}(e^x - 1)}{e^x - 1}\right| = -x/2.\end{aligned} \tag{3.299}$$

Hence

$$\int_1^\infty \frac{d\eta}{(1+\eta)\sqrt{\cosh x + \eta}} = \frac{x}{\sqrt{2}\sinh(x/2)}. \tag{3.300}$$

$$J^{(1)}(\mu) = \frac{2}{\pi}\cosh(\pi\mu)\int_0^\infty \frac{x\cos(\mu x)}{\sinh(x/2)}\, dx = \frac{\cosh(\pi\mu)}{\pi}\int_{-\infty}^\infty \frac{x\cos(\mu x)}{\sinh(x/2)}\, dx. \tag{3.301}$$

Evaluating the last integral above gives

$$\int_{-\infty}^\infty \frac{x\cos(\mu x)}{\sinh(x/2)}\, dx = \frac{\partial}{\partial \mu}\int_{-\infty}^\infty \frac{\sin(\mu x)}{\sinh(x/2)}\, dx = \frac{\partial}{\partial \mu} Im \int_{-\infty}^\infty \frac{e^{i\mu x}}{\sinh(x/2)}\, dx. \tag{3.302}$$

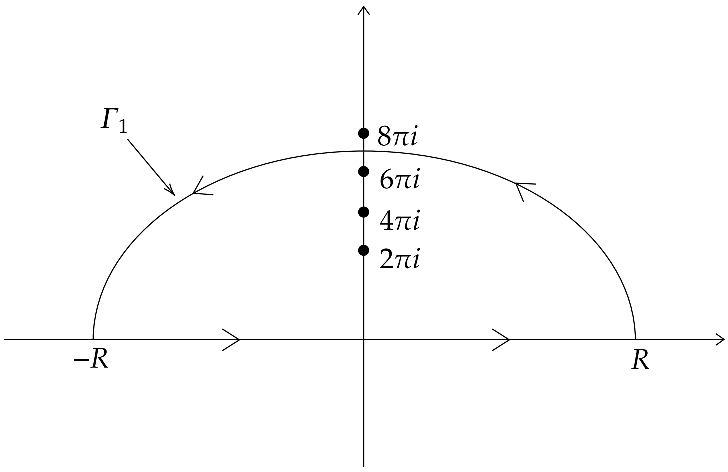

Figure 3.6: Contour region used to integrate $J^n(\mu)$ in the complex plane.

Using Cauchy's residue theorem [73] gives

$$\oint \frac{e^{i\mu x}}{\sinh(x/2)}\,dx = 2\pi i \sum_{n=1}^{\infty} Res\left(\frac{e^{i\mu x_n}}{\sinh\left(\frac{x_n}{2}\right)}\right), \quad x_n = 2\pi i n, \tag{3.303}$$

where the contour is shown in Figure 3.6. Note that this integral can be split into two parts, using this contour, as

$$\lim_{R\to\infty}\oint = \lim_{R\to\infty}\int_{\Gamma_1} + \int_{-R}^{R}. \tag{3.304}$$

Note however that

$$\lim_{R\to\infty}\int_{\Gamma_1} \frac{e^{i\mu x}}{\sinh(x/2)}\,dx = \lim_{R\to\infty}\int_{\Gamma_1} \frac{e^{i\mu Re^{it}}}{\sinh(Re^{it}/2)} iRe^{it}\,dt,$$

where $x = Re^{it}, \quad dx = iRe^{it}dt$. Hence

$$\begin{aligned}\lim_{R\to\infty}\int_{\Gamma_1} \frac{e^{i\mu x}}{\sinh(x/2)}\,dx &= \lim_{R\to\infty}\int_{\Gamma_1} \frac{R(ie^{it}e^{i\mu R\cos t})}{\underbrace{e^{R(\mu\sin t + \cos(t/2))}}_{\to 0\, as\, R\to\infty}(e^{iR\sin(t/2)} - e^{-R\cos(t/2)((iR/2)\cos t + i\sin t)})}\,dt \\ &= 0, \end{aligned} \tag{3.305}$$

hence, from equations (3.303) and (3.305)

$$\lim_{R\to\infty}\int_{-R}^{R}\frac{e^{i\mu x}}{\sinh{(x/2)}}dx = 2\pi i\sum_{i=1}^{\infty}\mathrm{Res}\left(\frac{e^{i\mu x_n}}{\sinh{(x_n/2)}}\right), \quad x_n = 2\pi i n. \tag{3.306}$$

By multiplying the right-hand side by $(x-x_n)$ and using L'Hôpital's rule [74] to evaluate the limit as $x\to x_n$ gives

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{e^{i\mu x}}{\sinh{(x/2)}}\,dx &= 2\pi i\sum_{n=1}^{\infty}\frac{2e^{-2\pi n\mu}}{\cosh{(\pi n i)}}\\ &= 4\pi i\sum_{n=1}^{\infty}(-1)^n e^{-2\pi n\mu}\\ &= -4\pi i(e^{2\pi\mu}+1)^{-1}.\end{aligned} \tag{3.307}$$

So, from the integral (3.302)

$$\frac{\partial}{\partial\mu}\int_{-\infty}^{\infty}\frac{\sin{(\mu x)}}{\sinh{(x/2)}}dx = \frac{\partial}{\partial\mu}\mathrm{Im}\left(\frac{-4\pi i}{e^{2\pi\mu}+1}\right) = 8\pi^2(e^{\pi\mu}+e^{-\pi\mu})^{-2} = 2\pi^2\,\mathrm{sech}^2\,(\pi\mu). \tag{3.308}$$

From equations (3.301) and (3.302) then

$$J^{(1)}(\mu) = \frac{2\pi}{\cosh{(\pi\mu)}}. \tag{3.309}$$

Now consider $J^{(n)}(\mu)$ for $n>2$, so it holds that (via integration by parts and equation (3.283)) from equation (3.291) that

$$\begin{aligned}J^{(n)}(\mu) &= -\left(\mu^2+\frac{1}{4}\right)^{-1}\int_1^{\infty}\left(\frac{2}{1+\eta}\right)^n\frac{d}{d\eta}\left[(\eta^2-1)\frac{dP_{-\frac{1}{2}+i\mu}}{d\eta}(\eta)\right]d\eta\\ &= -\frac{1}{\mu^2+\frac{1}{4}}\left(\left[\frac{2^n}{(1+\eta)^n}(\eta^2-1)\frac{\partial P_{-\frac{1}{2}+i\mu}}{\partial\eta}(\eta)\right]_1^{\infty}+\int_1^{\infty}\frac{2^n n(\eta^2-1)}{(1+\eta)^{n+1}}\frac{\partial P_{-\frac{1}{2}+i\mu}}{\partial\eta}(\eta)\,d\eta\right)\\ &= -\frac{1}{\mu^2+\frac{1}{4}}\left(\int_1^{\infty}\frac{n2^n(\eta-1)}{(1+\eta)^n}\frac{\partial P_{-\frac{1}{2}+i\mu}}{\partial\eta}(\eta)\,d\eta\right)\\ &= -\frac{1}{\mu^2+\frac{1}{4}}(\left[\frac{n2^n(\eta-1)}{(1+\eta^n)}P_{-\frac{1}{2}+i\mu}(\eta)\right]_1^{\infty}\end{aligned}$$

$$
\begin{aligned}
&- \int_1^\infty n2^n \left(\frac{1}{(1+\eta)^n} - \frac{n(\eta-1)}{(1+\eta)^{n+1}} \right) P_{-\frac{1}{2}+i\mu}(\eta)\, d\eta \Bigg) \\
&= \frac{1}{\mu^2+\frac{1}{4}} \int_1^\infty \left(\frac{n2^n}{(1+\eta)^n} - \frac{n^2 2^n}{(1+\eta)^n} + \frac{n^2 2^n}{(1+\eta)^n} - \frac{n^2 2^n(\eta-1)}{(1+\eta)^{n+1}} \right) P_{-\frac{1}{2}+i\mu}(\eta\, d\eta) \\
&= \frac{1}{\mu^2+\frac{1}{4}} \int_1^\infty \left(\frac{2^n n(1-n)}{(1+\eta)^n} + \frac{n^2 2^n + n^2 2^n \eta - n^2 2^n \eta + n^2 2^n}{(1+\eta)^{n+1}} \right) P_{-\frac{1}{2}+i\mu}(\eta)\, d\eta \\
&= \frac{1}{\mu^2+\frac{1}{4}} \int_1^\infty \left(\frac{2^n n(1-n)}{(1+\eta)^n} + \frac{2^{n+1} n^2}{(1+\eta)^{n+1}} \right) P_{-\frac{1}{2}+i\mu}(\eta)\, d\eta \\
&= \frac{n(1-n)}{\mu^2+\frac{1}{4}} \int_1^\infty \frac{2^n P_{-\frac{1}{2}+i\mu}(\eta)}{(1+\eta)^n} d\eta + \frac{n^2}{\mu^2+\frac{1}{4}} \int_1^\infty \frac{2^{n+1}}{(1+\eta)^{n+1}} P_{-\frac{1}{2}+i\mu}\, d\eta \\
&= \frac{n(1-n)}{\mu^2+\frac{1}{4}} J^n(\mu) + \frac{n^2}{\mu^2+\frac{1}{4}} J^{(n+1)}(\mu). \qquad (3.310)
\end{aligned}
$$

Rearranging equation (3.310) gives the recurrence relation

$$
J^{(n+1)}(\mu) = \frac{J^{(n)}(\mu)}{n^2} \left(\mu^2 + (n^2 - 1/2)^2 \right). \qquad (3.311)
$$

It is straight forward to see then that

$$
\begin{aligned}
J^{(n)}(\mu) &= \frac{2\pi}{\cosh(\pi\mu)} \prod_{j=1}^{n-1} \frac{1}{j^2} \left(\mu^2 + (j - 1/2)^2 \right) \\
&= \frac{2\pi}{\cosh(\pi\mu)} K^{(n)}(\mu), \qquad (3.312)
\end{aligned}
$$

where

$$
K^{(n)}(\mu) = \prod_{j=1}^{n-1} \frac{1}{j^2} \left(\mu^2 + \left(j - \frac{1}{2} \right)^2 \right), \quad K^{(1)}(\mu) = 1. \qquad (3.313)
$$

Inserting equation (3.312) into equation (3.292) gives the integral solution for the moments of the power transmission coefficient as

$$
\mathbb{E}[\tau^n(L)] = e^{-\frac{L}{4L_{loc(\nu)}}} \int_0^\infty \frac{2\pi\mu \sinh(\mu\pi)}{\cosh^2(\mu\pi)} e^{-\frac{\mu^2 L}{L_{loc(\nu)}}} K^{(n)}(\mu) d\mu. \qquad (3.314)
$$

3.7.8 Moments of the Power Reflection Coefficient

A similar calculation can be performed for the reflection coefficient (as defined in equation (3.249)). Using equations (3.214) and (3.215) then

$$R_\nu(\nu, L) = -\tanh(\gamma_\nu(L))/2)e^{i(\psi_\nu(L)+2\phi_\nu(L))}, \tag{3.315}$$

and the reflection power transmission coefficient, or reflected energy, is defined as

$$|R_\nu(\nu, L)|^2 = \tanh^2(\gamma_\nu(L)/2) = R, \tag{3.316}$$

say. Using the chain rule and equation (3.256) then

$$\begin{aligned} \frac{\partial}{\partial \gamma_\nu} = \frac{\partial R}{\partial \gamma_\nu}\frac{\partial}{\partial R} &= \tanh(\gamma_\nu/2)\operatorname{sech}^2(\gamma_\nu/2)\frac{\partial}{\partial R} \\ &= \sqrt{R}(1-R)\frac{\partial}{\partial R}. \end{aligned} \tag{3.317}$$

Furthermore, using equation (3.258)

$$\begin{aligned} \frac{1}{\tanh(\gamma_\nu)}\frac{\partial}{\partial \gamma_\nu} &= \frac{\tanh(\gamma_\nu/2)}{\tanh(\gamma_\nu)}\operatorname{sech}^2(\gamma_\nu/2)\frac{\partial}{\partial R} \\ &= \frac{1}{2}(1+\tanh^2(\gamma_\nu/2))\operatorname{sech}^2(\gamma_\nu/2)\frac{\partial}{\partial R} \\ &= \frac{1}{2}(1-R^2)\frac{\partial}{\partial R}. \end{aligned} \tag{3.318}$$

From equation (3.317)

$$\begin{aligned} \frac{\partial^2}{\partial \gamma_\nu^2} &= \frac{\partial}{\partial \gamma_\nu}\left(\sqrt{R}(1-R)\frac{\partial}{\partial R}\right) \\ &= \frac{\partial R}{\partial \gamma_\nu}\frac{\partial}{\partial R}\left(\sqrt{R}(1-R)\frac{\partial}{\partial R}\right) \\ &= \sqrt{R}(1-R)\left(\frac{1-3R}{2\sqrt{R}}\frac{\partial}{\partial R} + \sqrt{R}(1-R)\frac{\partial^2}{\partial R^2}\right) \\ &= \frac{1}{2}(1-R)(1-3R)\frac{\partial}{\partial R} + R(1-R)^2\frac{\partial^2}{\partial R^2}. \end{aligned} \tag{3.319}$$

Inserting equations (3.318) and (3.319) into equation (3.243), the infinitesimal generator for the power reflection coefficient R reads

$$\mathcal{L}_R = \frac{1}{L_{loc}}\left[R(1-R)^2\frac{\partial^2}{\partial R^2} + (1-R)^2\frac{\partial}{\partial R}\right]. \tag{3.320}$$

To simplify the analysis, introduce the transformation

$$v = \frac{1+R}{1-R}, \quad R = \frac{v-1}{v+1}. \tag{3.321}$$

Since $R(L=0)=0$ and takes values in $[0,1]$, v takes values in $[1,\infty)$. Using the chain rule

$$\frac{\partial}{\partial R} = \frac{\partial v}{\partial R}\frac{\partial}{\partial v} = \frac{2}{(1-R)^2}\frac{\partial}{\partial v} = \frac{(v+1)^2}{2}\frac{\partial}{\partial v}. \tag{3.322}$$

Again, the chain rule gives

$$\begin{aligned}\frac{\partial^2}{\partial R^2} &= \frac{\partial}{\partial R}\left(\frac{(v+1)^2}{2}\frac{\partial}{\partial v}\right)\\ &= \frac{\partial v}{\partial R}\frac{\partial}{\partial v}\left(\frac{(v+1)^2}{2}\frac{\partial}{\partial v}\right)\\ &= \frac{(v+1)^3}{2}\frac{\partial}{\partial v} + \frac{(v+1)^4}{4}\frac{\partial^2}{\partial v^2}.\end{aligned} \tag{3.323}$$

Furthermore

$$R(1-R)^2\frac{\partial^2}{\partial R^2} = 2(v-1)\frac{\partial}{\partial v} + (v^2-1)\frac{\partial^2}{\partial v^2}, \tag{3.324}$$

and

$$(1-R)^2\frac{\partial}{\partial R} = 2\frac{\partial}{\partial v}. \tag{3.325}$$

The infinitesimal generator is them from equation (3.320)

$$\mathcal{L}_v = \frac{1}{L_{loc}}\frac{\partial}{\partial v}\left[(v^2-1)\frac{\partial}{\partial v}\right], \tag{3.326}$$

which is self-adjoint by the same calculation in equation (3.281). The Fokker-Planck equation for the probability density function of the power reflection coefficient $p(L, v)$ then reads

$$\frac{\partial p}{\partial L}(L, v) = \frac{1}{L_{loc}} \frac{\partial}{\partial v}\left[(v^2 - 1)\frac{\partial p}{\partial v}(L, v)\right], \quad v > 1,$$

with initial condition $p(L = 0, v) = \delta(v - 1)$, where δ denotes the Dirac delta function. Note that this is the same form of Fokker-Planck equation as seen in equation (3.282) in the analysis of the power transmission coefficient. Noting the solution for the probability density function for the power transmission coefficient in equation (3.289), he solution for the probability density function of the power reflection coefficient can be written as

$$p(L, v) = \int_0^\infty \mu \tanh(\mu\pi) P_{-\frac{1}{2}+i\mu}(v) \exp\left\{-\left(\mu^2 + \frac{1}{4}\right)\frac{L}{L_{loc}}\right\} d\mu. \tag{3.327}$$

The moments of the power reflection coefficient are thus given by

$$\begin{aligned}\mathbb{E}[R^n(L)] &= \int_0^1 R^n(L) p(L, R)\, dR \\ &= \int_1^\infty \left(\frac{v-1}{v+1}\right)^n p(L, v)\, dv.\end{aligned} \tag{3.328}$$

Starting with the first moment ($n = 1$) gives

$$\begin{aligned}\mathbb{E}[R(L)] &= \int_1^\infty \left(\frac{v-1}{v+1}\right) p(L, v)\, dv \\ &= \int_1^\infty \left(1 - \frac{2}{v+1}\right) p(L, v)\, dv \\ &= \int_1^\infty p(L, v)\, dv - \int_1^\infty \left(\frac{2}{1+v}\right) p(L, v)\, dv \\ &= 1 - \int_1^\infty \left(\frac{2}{1+v}\right) p(L, v)\, dv.\end{aligned} \tag{3.329}$$

The second integral term in equation (3.329) is precisely the integral in equation (3.290) (with $n = 1$) which was solved in the transmission case. Thus equation (3.314) gives

$$\mathbb{E}[R(L)] = 1 - e^{-\frac{L}{4L_{loc}}} \int_0^\infty \frac{\mu \sinh(\mu\pi)}{\cosh^2(\mu\pi)} e^{-\frac{\mu^2 L}{L_{loc}}}\, d\mu,$$

$$= 1 - \mathbb{E}[\tau(L)], \tag{3.330}$$

where $K^{(n)}(\mu)$ is defined by equation (3.313), and $K^{(n=1)}(\mu) = 1$. Note that

$$\mathbb{E}[\tau(L)] + \mathbb{E}[R(L)] = 1, \tag{3.331}$$

and so energy is conserved. Now moving to the second moment ($n = 2$)

$$\begin{aligned}
\mathbb{E}[R^2(L)] &= \int_1^\infty \left(\frac{v-1}{v+1}\right)^2 p(L,v)\, dv \\
&= \int_1^\infty \left(1 - \frac{2}{v+1}\right)^2 p(L,v)\, dv \\
&= \int_1^\infty p(L,v)\, dv + \int_1^\infty \left(\frac{2}{1+v}\right)^2 p(L,v),\, dv - 2\int_1^\infty \left(\frac{2}{1+v}\right) p(L,v)\, dv \\
&= 1 + \mathbb{E}[\tau^2(L)] - 2\mathbb{E}[\tau(L)].
\end{aligned} \tag{3.332}$$

Note that standard deviation of the reflected energy

$$\begin{aligned}
\mathrm{Std}[R(L)] &= \sqrt{\mathbb{E}[R^2(L)] - \mathbb{E}[R(L)]^2} \\
&= \sqrt{1 + \mathbb{E}[\tau^2(L)] - 2\mathbb{E}[\tau(L)] - (1 - \mathbb{E}[\tau(L)])} \\
&= \sqrt{\mathbb{E}[\tau^2(L)] - \mathbb{E}[\tau(L)]^2} = Std[\tau(L)],
\end{aligned} \tag{3.333}$$

is equal to the standard deviation of the transmitted energy.

3.8 Results: Material Study

Now the mean power transmission coefficient given by equation (3.314) is analysed as a function of key parameters, including the propagation distance through the material $\tilde{L}$ and ν (the ratio between the wave numbers in the x_1 and x_3 directions) which is the degree of anisotropy parameter. The directionality of the fibres in the random medium is controlled by the angle $\theta(x_3)$ in equation (3.107) which varies spatially. The wavelength of the monochromatic shear wave is chosen to be commensurate with the typical layer width (see Figure 3.3). Carbon fibre reinforced polymers (CFRP's) are a common class of materials in the engineering world and have material properties that align with the model; specifically the heterogeneous and locally anisotropic nature of layered CFRP's. The stiffness tensor components for a CFRP were given by IHI. For this study, a frequency $\tilde{\omega} \sim \varepsilon^{-2}$ and a mean wave speed of $c_3 = 2000\,\mathrm{ms}^{-1}$ were used together with the elastic constants in Table 3.2.

Elastic Material Constants (in GPa) ($\mathrm{ML}^{-1}\mathrm{T}^{-2} \times 10^9$)					
	C_{11}	C_{33}	C_{44}	C_{66}	C_{13}
CFRP	146.53	12.25	2.55	4.00	6.67

Table 3.2: Table of material constants for a CFRP obtained from IHI with geometry [70] and density $\rho = 1500\mathrm{kgm}^{-3}$.

The decrease in $\tilde{L}_{loc}$ (see equation (3.266)) as the degree of anisotropy parameter ν increases is shown in Figure 3.7.

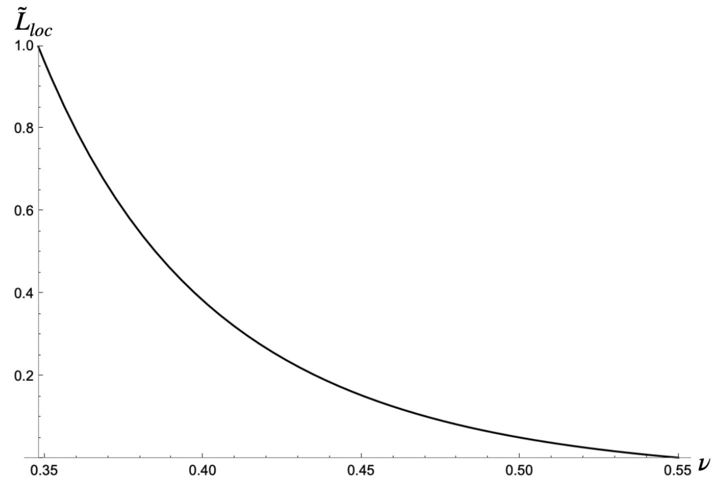

Figure 3.7: Non-dimensional localisation length $\tilde{L}_{loc}$ (given by equation (3.266)) as a function of the material anisotropy parameter ν (equation (3.142)). The material properties are given in Table 3.2.

This change in the localisation length ($\tilde{L}_{loc}$) affects the decay rate of the mean power transmission coefficient $\mathbb{E}[\tau(\tilde{L}, \nu)]$ (equation (3.314)) as shown in Figure 3.8. An increase in the ratio of wave numbers (degree of anisotropy) causes greater energy decay with the depth of wave penetration into the material $\tilde{L}$.

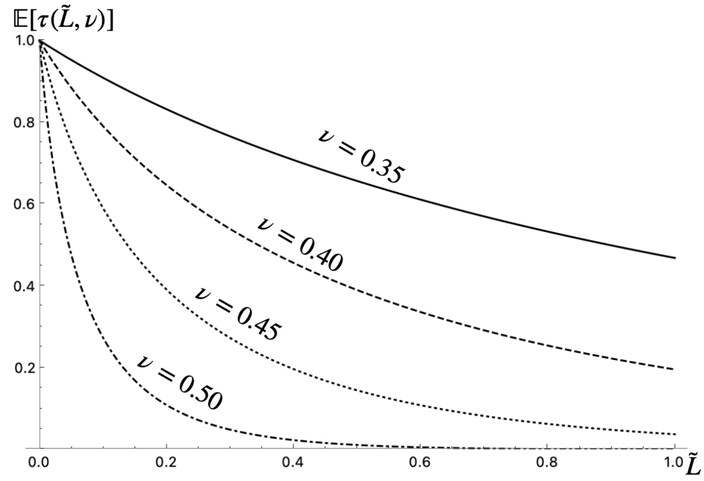

Figure 3.8: Plots of the mean of the power transmission coefficient (see equation (3.314)) as the non-dimensionalised penetration depth $\tilde{L}$ varies for different values of the material anisotropy parameter ν (equation (3.142)). The material properties are given in Table 3.2.

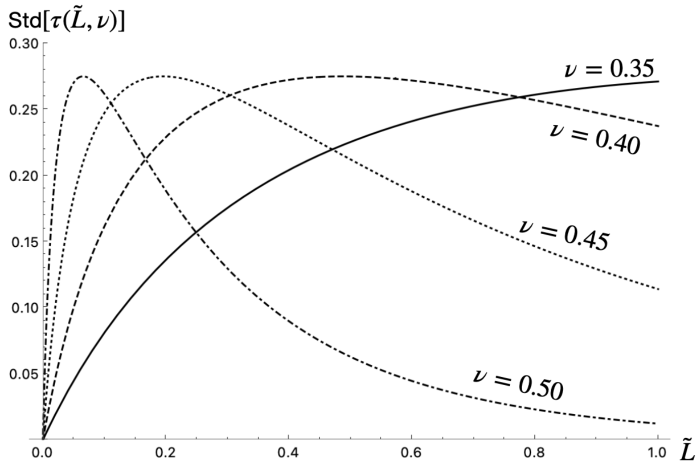

Figure 3.9: Plots of the standard deviation of the power transmission coefficient (equation 3.314)) as the non-dimensionalised wave propagation depth $\tilde{L}$ varies for different values of material anisotropy parameter ν (equation (3.142)). The material properties are given in Table 3.2.

The effect of changing the anisotropy parameter (ν) is also seen on the standard deviation of the power transmission coefficient. Figure 3.9 shows the sharpening (variance tending towards zero) of the peak of the standard deviation as ν increases and the self-averaging of the wave for large propagation distance $\tilde{L}$. Hence the materials with a more marked anisotropy parameter (greater deviation from the isotropic case where $\nu = 1$) require a larger propagation distance before any self-averaging occurs.

Conclusion

A model of a monochromatic horizontally polarised shear wave propagating through a layered random and heterogeneous medium, composed of a polycrystalline material where the orientation varied from one layer to the next, has been constructed. A suitable scaling regime was studied whereby the microstructure produced a transmitted wave which had an attenuated, coherent wave front followed by an incoherent coda wave. The random fluctuations present within the anisotropic material were modelled via a local rotation of the corresponding slowness surface as a function of the wave propagation direction x_3. A system of stochastic differential equations were derived which were subsequently solved in order to access the statistical properties of the energy transmitted and reflected through this layered random medium.

A horizontally polarised shear wave parameterisation $\mathbf{u} = (0, u_2(x_1, x_3), 0)$ was chosen and the transmitted energy (attenuation) of the amplitude of the coherent wave then depended on the level of anisotropy present in the layered material. This variation in the wave attenuation on the anisotropy of the material has relevance to the use of ultrasonic arrays for medical imaging and nondestructive testing applications. An expression for the localisation length was also derived, which depends on correlation integrals which can be numerically evaluated for specific material microstructures. For example, a 20% change in ν could lead to a threefold reduction in the amplitude of the transmitted coherent wave. The work presented here could be expanded to model a pulse, which could be used by experimentalists to decide on an appropriate frequency of ultrasonic wave to emit for a given material microstructure and required depth of wave penetration.

There is a delicate balance between parameters in order to achieve the correct separation of scales. It has been shown that the decay rate of energy in the medium $\tilde{L}_{loc}$ depends on the ratio of wave numbers ν in the (x_1, x_3) plane. This is important when dealing with array applications in NDT since ν can change, affecting how the energy is transmitted in the medium. The analytical expressions predict the energy decay rate and how this is affected by ν.

Chapter 4 takes the model presented in this Chapter, but considers the source as a broadband pulse. This allows study of the autocorrelation function of the reflection coefficient which is a key parameter in studying imaging schemes such as the Total Focusing Method [3], [43]. The forthcoming Chapters will consider the incoherent part of the wave, whereas previous Chapters have considered the coherent wave.

Chapter 4

Multi-Frequency Elastic Wave Propagation in a Randomly Layered Media

4.1 Nomenclature

Parameter			Equation
α	Lumped parameter	$[\mathrm{M^{-1}LT^2}]$	(4.18)
$\bar{\alpha}$	Lumped parameter	$[\mathrm{M^{-1}LT^2}]$	(4.35)
β	Lumped parameter	$[\mathrm{ML^{-3}}]$	(4.19)
$\bar{\beta}$	Lumped parameter	$[\mathrm{ML^{-3}}]$	(4.32)
$\gamma^{(1)}$	Lumped parameter	$[\mathrm{L^{-1}T^{-1}}]$	(4.80)
$\gamma^{(2)}$	Lumped parameter	$[\mathrm{L^{-1}T^{-1}}]$	(4.81)
$\gamma^{(3)}$	Lumped parameter	$[\mathrm{L^{-1}T^{-1}}]$	(4.82)
Γ_α	Lumped parameter	[–]	(4.36)
Γ_β	Lumped parameter	[–]	(4.33)
δ	Dirac delta function	[–]	(4.3)
δ_1	Lumped slowness parameter	$[\mathrm{TL^{-1}}]$	(4.39)
δ_2	Lumped slowness parameter	$[\mathrm{TL^{-1}}]$	(4.40)
ε	Small dimensionless parameter	[–]	(4.9)
θ	Rotation angle of material slowness surface	[–]	(4.24)
$\bar{\theta}$	Mean slowness angle	[–]	(4.23)
κ_1	Wavenumber in x_1	$[\mathrm{L^{-1}}]$	(4.10)
λ_3	Wavelength in x_3 direction	[L]	
ξ	Velocity in x_3 direction	$[\mathrm{LT^{-1}}]$	(4.6)
$\hat{\xi}$	Velocity (frequency wave-number domain) in x_3 direction	$[\mathrm{L^2}]$	(4.13)
ρ	Constant material density	$[\mathrm{ML^{-3}}]$	(4.1)
ϱ	Lumped stress parameter	$[\mathrm{ML^{-1}T^{-2}}]$	(4.30)

$\tilde{\boldsymbol{\sigma}}$	Symmetric correlation integral matrix	[$L^{1/2}$]	(4.87)
σ	Random process amplitude	[–]	(4.23)
τ_{jk}	Material stress	[$ML^{-1}T^{-2}$]	(4.1)
$\tau_{(21,23)}$	Material stresses	[$ML^{-1}T^{-2}$]	(4.6)
$\hat{\tau}_{(21,23)}$	Material (frequency wave-number domain) stresses	[MT^{-1}]	(4.13)
τ	Power transmission coefficient	[–]	(4.47)
$\tilde{\tau}$	Non-dimensional time	[–]	(4.107)
$\Upsilon(\omega)$	Symmetric correlation integral	[L]	(4.45)
$\Upsilon^{(AS)}(\omega)$	Anti-symmetric correlation integral	[L]	(4.46)
φ	Lumped parameter	[–]	(4.29)
ϕ	Slowness parameter	[TL^{-1}]	(4.22)
$\chi^{\varepsilon}_{(1,2)}$	Complex propagator function	[–]	(4.42)
ω	Angular frequency	[T^{-1}]	(4.9)
$\hat{a}$	Frequency wave-number wave-mode	[$M^{1/2}T^{-1/2}L$]	(4.38)
A	Material stress	[$ML^{-1}T^{-2}$]	(4.3)
A^{ε}	Time domain wave-mode	[$MT^{-3/2}L$]	(4.56)
A_1	Lumped parameter	[$L^{-1/2}$]	(4.44)
A_2	Lumped parameter	[$L^{-1/2}$]	(4.44)
A_3	Lumped parameter	[L^{-1}]	(4.44)
$\hat{b}$	Frequency wave-number wave-mode	[$M^{1/2}T^{-1/2}L$]	(4.38)
c_{ijkl}	Material stress tensor	[$ML^{-1}T^{-2}$]	(4.3)
$c_{44}(x_3)$	Material stress	[$ML^{-1}T^{-2}$]	(4.24)
$\bar{c}_{44}$	Material stress	[$ML^{-1}T^{-2}$]	(4.25)
$c_{66}(x_3)$	Material stress	[$ML^{-1}T^{-2}$]	(4.27)

$\bar{c}_{66}$	Material stress	$[\mathrm{ML^{-1}T^{-2}}]$	(4.28)
$\mathbf{C}$	Correlation integral matrix	[L]	(4.86)
e_{kl}	Symmetric strain tensor	[–]	(4.4)
$\hat{f}(\omega)$	Source term	$[\mathrm{M^{1/2}T^{-1/2}L}]$	(4.51)
F	Material stress	$[\mathrm{ML^{-1}T^{-2}}]$	(4.3)
$F(V_{p,q})$	Transport function	$[\mathrm{L^{-1}T^{-1}}]$	(4.84)
$g^{(1)}$	Random function	[–]	(4.76)
$g^{(2)}$	Random function	[–]	(4.77)
$g^{(3)}$	Random function	[–]	(4.78)
$g^{(3)}$	Random function	[–]	(4.78)
G	Transport function	$[\mathrm{L^{-1}T^{-1}}]$	(4.79)
$\mathbf{H}^{\varepsilon}$	Random matrix	$[\mathrm{L^{-1}}]$	(4.38)
$\mathbf{I}$	Identity matrix	[–]	(4.42)
$I(t)$	Intensity	$[\mathrm{M^2T^{-3}L^2}]$	(4.58)
L	Material length	[L]	(4.52)
L_{loc}	Localisation length	[L]	(4.48)
$\mathcal{L}_{\mathcal{W}}$	Differential operator	[–]	(4.100)
m	Markov process	[–]	(4.23)
M	Martingale	$[\mathrm{T^{-1}}]$	(4.117)
N	Material stress	$[\mathrm{ML^{-1}T^{-2}}]$	(4.3)
N_{x_3}	Jump Markov process	[–]	(4.113)
$\mathbf{P}^{\varepsilon}$	Propagator matrix	[–]	(4.41)
R^{ε}	Reflection coefficient	[–]	(4.54)
S	Material stress	$[\mathrm{ML^{-1}T^{-2}}]$	(4.3)

$\boldsymbol{s}$	Rotation vector	[–]	(4.2)
t	Time	[–]	(4.1)
T^{ε}	Transmission coefficient	[–]	(4.55)
$\mathcal{T}$	Time variable	[T]	(4.113)
u	Displacement vector	[L]	(4.5)
U	Solution of transport equation	[T^{-1}]	(4.116)
$U^{\varepsilon}_{p,q}$	Product of reflection coefficients	[–]	(4.66)
$V^{\varepsilon}_{p,q}$	Fourier transform	[T^{-1}]	(4.70)
W	Brownian motion	[L$^{1/2}$]	(4.44)
$\mathcal{W}_p$	Transport equation solution	[T^{-1}]	(4.98)
$\widetilde{\mathcal{W}}^{\infty}_p$	Non-dimensional limiting transport equation solution	[–]	(4.107)
$x_{1,2,3}$	Coordinate system	[L]	(4.1)
$\tilde{x}_3$	Non-dimensional x_3	[–]	(4.107)
X	SDE solution	[T^{-1}]	(4.89)

4.2 Introduction

Modern engineering materials that are produced via additive manufacturing techniques [5] are becoming increasingly complex and their structural integrity is critical in ensuring they can be used in situ. Non-destructive evaluation (NDE) techniques are important for evaluating the health and safety of such components. One NDE method which will be studied in this Chapter, uses high frequency ultrasonic waves to probe the material; analysis of the coda wave which emerges allows characterisation of the material microstructure and identification and location of any flaws or defects. Current imaging algorithms assume that the material of interest is isotropic and homogeneous [3], with no consideration of inhomogeneities and anisotropies in the material. The existence of inhomogeneities can cause multiple scattering to occur particularly when the length scales of the material inhomogeneities are commensurate with the wavelength of the incoming wave.

This Chapter studies the regime where the propagation distance is much larger than the wavelength of the incoming pulse ($L_3 \gg \lambda_3$), the mean layer size is commensurate with the wavelength ($\lambda_3 \sim l$) and the change in elasticity from one region to another (strength of random fluctuations) is small ($0 < \sigma \ll 1$); this regime is known as the weakly heterogeneous regime [26]. This regime produces a received wave that is attenuated and distorted as a result of its interaction with the internal material microstructure. The work in this Chapter builds on probabilistic models for elastic wave propagation inside randomly layered media for a monochromatic wave [26], [56], [8] to study a broadband, high frequency ultrasonic pulse via analytical and computational [52] means. By studying a set of transport equations, it is possible to derive solutions for the frequency autocorrelation function for the reflection coefficient.

Section 4.3 introduces the governing elastodynamic equation for a broadband, high frequency ultrasonic pulse. Section 4.3.2 introduces the stochastic model to describe the spatial changes in the slowness surface of the material. Section 4.3.3 introduces the governing evolution equations for the stress and velocity, and derives a set of matrix stochastic differential equations. Section 4.4 derives a Ricatti equation for the reflection coefficient which then leads to a system of transport equations. An analytical solution to the transport equations for the frequency autocorrelation function is presented in Section 4.4.1 and then a numerical solution is presented in Section 4.4.2. A discussion of the main results of the Chapter (including validation between analytical and Monte Carlo solutions for the frequency autocorrelation function) is presented in Section 4.5.

4.3 Multi-Frequency Formulation for Pulse

4.3.1 Scaling Governing Equations

The governing elastodynamic equation can be written

$$\rho u_{i,tt} = \tau_{jk,k}, \tag{4.1}$$

where the displacement vector u_i in the i^{th} direction is $u(t, x_1, x_3)$, ρ is the density of the material (assumed to be constant) and τ_{jk} is the material stress tensor. The elastic tensor for a transversely anisotropic medium has five independent stress tensor components (in Voigt notation) namely C_{11}, C_{33}, C_{13}, C_{66}, C_{44} with $\boldsymbol{s} = (s_1, s_2, s_3)$ as the symmetry axis vector defined by

$$\boldsymbol{s} = (\cos\theta(x_3),\, \sin\theta(x_3),\, 0)^{\intercal}, \tag{4.2}$$

where $\theta(x_3)$ is the angle of anisotropy which is related to the slowness surface of the material which varies in x_3 (from layer to layer). The stress tensor can then be written [36] as

$$\begin{aligned} c_{ijkl} =& (A - 2N)\delta_{ij}\delta_{kl} + N\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \\ &+ (F - A + 2N)\left(\delta_{ij}s_k s_l + \delta_{kl}s_i s_j\right) \\ &+ (S - N)\left(\delta_{ik}s_j s_l + \delta_{il}s_j s_k + \delta_{jk}s_i s_l + \delta_{jl}s_i s_k\right) \\ &+ (A + C - 2F - 4S)s_i s_j s_k s_l, \end{aligned} \tag{4.3}$$

where $A = C_{33}$, $C = C_{11}$, $F = C_{13}$, $N = C_{44}$, $S = C_{66}$ when $\theta = 0$. The elastic tensor relates the symmetric strain and stress tensors via Hooke's law, where the symmetric strain tensor is given by

$$e_{kl} = (u_{k,l} + u_{l,k})/2. \tag{4.4}$$

Focusing on shear wave propagation in the (x_1, x_3) plane, the wave parameterisation

$$u_j = (0, u_2(x_1, x_3), 0), \tag{4.5}$$

produces the three governing elastic wave equations

$$\rho\xi_{,t}(t, x_1, x_3) = \tau_{21,1}(t, x_1, x_3) + \tau_{32,3}(t, x_1, x_3), \tag{4.6}$$

$$\tau_{21,t}(t, x_1, x_3) = c_{66}\xi_{,1}(t, x_1, x_3), \tag{4.7}$$

$$\tau_{32,t}(t, x_1, x_3) = c_{44}\xi_{,3}(t, x_1, x_3), \tag{4.8}$$

where the velocity $\xi(t, x_1, x_3) = u_{2,t}(t, x_1, x_3)$. Now assume that the coupling between the random internal microstructure and probing wave is weak; the so called weakly heterogeneous regime [75]. The wavelength (λ_3) in the x_3 direction of the probing wave is assumed to be commensurate with the layer size l and much smaller than the propagation distance L. To characterise this regime, a small parameter ε is introduced, where the width of the input pulse is order ε^2 [26]. Using a scaled temporal (with frequency ω) and spatial (with wavenumber κ_1) Fourier transform $(0 < \varepsilon \ll 1)$

$$\breve{g}(\omega, x_1, x_3) = \frac{1}{\varepsilon^2}\int g(t, x_1, x_3)e^{\frac{i\omega t}{\varepsilon^2}}dt, \tag{4.9}$$

$$\hat{g}(\omega, \kappa_1, x_3) = \frac{1}{\varepsilon^2}\int \breve{g}(\omega, x_1, x_3)e^{\frac{i\kappa_1 x_1}{\varepsilon^2}}dx_1, \tag{4.10}$$

and their inverse transforms

$$g(t, x_1, x_3) = \frac{1}{2\pi}\int \breve{g}(\omega, x_1, x_3)e^{-\frac{i\omega t}{\varepsilon^2}}d\omega, \tag{4.11}$$

$$\breve{g}(\omega, x_1, x_3) = \frac{1}{2\pi}\int \hat{g}(\omega, \kappa_1, x_3)e^{-\frac{i\kappa_1 x_1}{\varepsilon^2}}d\kappa_1, \tag{4.12}$$

equations (4.6), (4.7) and (4.8) read

$$
\begin{aligned}
-\frac{\rho i\omega}{\varepsilon^2}\hat{\xi}(\omega,\kappa_1,x_3) &= -\frac{i\kappa_1}{\varepsilon^2}\hat{\tau}_{21}(\omega,\kappa_1,x_3)+\hat{\tau}_{32,3}(\omega,\kappa_1,x_3), && (4.13)\\
-\frac{i\omega}{\varepsilon^2}\hat{\tau}_{21}(\omega,\kappa_1,x_3) &= -\frac{i\kappa_1}{\varepsilon^2}c_{66}(x_3)\hat{\xi}(\omega,\kappa_1,x_3), && (4.14)\\
-\frac{i\omega}{\varepsilon^2}\hat{\tau}_{32}(\omega,\kappa_1,x_3) &= c_{44}(x_3)\hat{\xi}_{,3}(\omega,\kappa_1,x_3). && (4.15)
\end{aligned}
$$

Inserting equation (4.14) into equation (4.13) gives

$$
\hat{\tau}_{32,3}(\omega,\kappa_1,x_3) = \frac{i\omega}{\varepsilon^2}\left(\frac{\kappa_1^2 c_{66}(x_3)-\rho\omega^2}{\omega^2}\right)\hat{\xi}(\omega,\kappa_1,x_3). \tag{4.16}
$$

Equations (4.15) and (4.16) give the system

$$
\begin{aligned}
\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix} &= \frac{i\omega}{\varepsilon^2}\begin{bmatrix}0 & -1/c_{44}(x_3)\\ (\kappa_1^2 c_{66}(x_3)-\rho\omega^2)/\omega^2 & 0\end{bmatrix}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}\\
&= \frac{i\omega}{\varepsilon^2}\begin{bmatrix}0 & -\alpha(x_3)\\ -\beta(\omega,\kappa_1,\rho,x_3) & 0\end{bmatrix}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}, \qquad (4.17)
\end{aligned}
$$

where

$$
\alpha(x_3) = \frac{1}{c_{44}(x_3)}, \tag{4.18}
$$

and

$$
\beta(\omega,\kappa_1,\rho,x_3) = \frac{\rho\omega^2-\kappa_1^2 c_{66}}{\omega^2}, \tag{4.19}
$$

where it is assumed that $\beta > 0$. The stress and velocity equations can be written in terms of the wave modes ($\hat{a}$,$\hat{b}$)

$$
\hat{\xi}(\omega,\kappa_1,x_3) = \sqrt{\phi/\bar{\beta}}\left(\hat{b}(\omega,\kappa_1)e^{-\frac{i\omega\phi x_3}{\varepsilon^2}} - \hat{a}(\omega,\kappa_1)e^{\frac{i\omega\phi x_3}{\varepsilon^2}}\right), \tag{4.20}
$$

and

$$\hat{\tau}(\omega,\kappa_1,x_3)=\sqrt{\bar{\beta}/\phi}\left(\hat{b}(\omega,\kappa_1)e^{-\frac{i\omega\phi x_3}{\varepsilon^2}}+\hat{a}(\omega,\kappa_1)e^{\frac{i\omega\phi x_3}{\varepsilon^2}}\right), \tag{4.21}$$

where

$$\phi=\sqrt{\overline{\alpha\beta}}, \tag{4.22}$$

and the bar denotes a quantity which is independent of x_3. The input wave is at $x_3=0$ and travels to the left. The mode $\hat{a}$ travels in the direction of increasing x_3 and $\hat{b}$ travels in the direction of decreasing x_3.

4.3.2 Randomly Layered Material

Consider an elastic shear wave with wavelength λ incident on a randomly layered material with layer size l and slab length L. The slab is contained in $x_3 \in [-L,0]$ sandwiched between two half-spaces (which are isotropic, homogeneous and have a shear velocity equal to mean shear velocity in the random slab) where $x_3 \in (-\infty,-L)$ and $x_3 \in (0,\infty)$. The exiting transmitted wave has little energy in the coherent wave and has a long coda wave. Assume that the angle $\theta(x_3)$ in equation (4.3) (which relates to the material slowness surface) varies randomly according to

$$\theta(x_3)=\bar{\theta}+\sigma m(x_3/l), \quad x_3\in[-L,0], \tag{4.23}$$

where $\bar{\theta}\sim 1$ is the mean angle, $m(x_3/l)$ is a stationary stochastic process (an ergodic Markov process on a compact state space) with mean zero, l is a typical layer size inside the material and σ is a dimensionless and small parameter ($0<\sigma\ll 1$) which controls the strength of the random process $m(x_3/l)$. Taking series expansions in σ gives

$$c_{44}=\bar{c}_{44}(1+\vartheta m(x_3/l))+\mathcal{O}(\sigma^2), \tag{4.24}$$

where

$$\bar{c}_{44} = N + (S - N)\sin^2\bar{\theta}, \tag{4.25}$$

$$\vartheta = \sigma\frac{(S-N)\sin(2\bar{\theta})}{\bar{c}_{44}}. \tag{4.26}$$

Also

$$c_{66} = \bar{c}_{66}(1 + \varphi m(x_3/l)) + \mathcal{O}(\sigma^2), \tag{4.27}$$

where

$$\bar{c}_{66} = S + \varrho\cos^2\bar{\theta}\sin^2\bar{\theta}, \tag{4.28}$$

$$\varphi = \sigma\frac{\varrho\sin(2\bar{\theta})\cos(2\bar{\theta})}{\bar{c}_{66}}, \tag{4.29}$$

$$\varrho = A + C - 2F - 4S, \tag{4.30}$$

noting that $\bar{c}_{44}$ is an effective material stiffness and A, C, F, N and S are the stiffness tensor constants defined in equation (4.3). From equation (4.19)

$$\beta = \bar{\beta}(1 + \sigma\Gamma_\beta m(x_3/l)), \tag{4.31}$$

where

$$\bar{\beta}(\omega, \kappa_1, \rho, \bar{c}_{66}) = \frac{\rho\omega^2 - \kappa_1^2\bar{c}_{66}}{\omega^2}, \tag{4.32}$$

$$\Gamma_\beta(\omega, \kappa_1, \bar{\theta}, \varrho) = \frac{\kappa_1^2\varrho\sin\left(2\bar{\theta}\right)\cos(2\bar{\theta})}{\kappa_1^2\bar{c}_{66} - \rho\omega^2}. \tag{4.33}$$

From equation (4.18)

$$\alpha = \bar{\alpha}(1 + \sigma\Gamma_\alpha m(x_3/l)), \tag{4.34}$$

where

$$\bar{\alpha}(\bar{c}_{44}) = \frac{1}{\bar{c}_{44}}, \tag{4.35}$$

$$\Gamma_\alpha(\bar{c}_{44}, \bar{\theta}) = \frac{(N-S)\sin(2\bar{\theta})}{\bar{c}_{44}}. \tag{4.36}$$

System (4.17) can then be written as

$$\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix} = \frac{i\omega}{\varepsilon^2}\begin{bmatrix}0 & -\bar{\alpha}(1+\sigma\Gamma_\alpha m(x_3/l))\\ -\bar{\beta}(1+\sigma\Gamma_\beta m(x_3/l)) & 0\end{bmatrix}\begin{bmatrix}\hat{\xi}(\omega,\kappa_1,x_3)\\ \hat{\tau}(\omega,\kappa_1,x_3)\end{bmatrix}. \tag{4.37}$$

4.3.3 Weakly Heterogenous Regime

In the weakly heterogeneous regime, equation (4.37) can be written in terms of the forward ($\hat{a}^\varepsilon(\omega,\kappa_1,x_3)$) and backward ($\hat{b}^\varepsilon(\omega,\kappa_1,x_3)$) (equations (4.20) and (4.21)) wave-modes (which now depend on ω, κ_1 and x_3) as

$$\begin{aligned}\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{a}^\varepsilon(\omega,\kappa_1,x_3)\\ \hat{b}^\varepsilon(\omega,\kappa_1,x_3)\end{bmatrix} &= \frac{i\omega}{2\varepsilon}m(x_3/\varepsilon^2)\begin{bmatrix}\delta_1 & -\delta_2 e^{-2i\omega\phi x_3/\varepsilon^2}\\ \delta_2 e^{2i\omega\phi x_3/\varepsilon^2} & -\delta_1\end{bmatrix}\begin{bmatrix}\hat{a}^\varepsilon(\omega,\kappa_1,x_3)\\ \hat{b}^\varepsilon(\omega,\kappa_1,x_3)\end{bmatrix}\\ &= \frac{1}{\varepsilon}\mathbf{H}^\varepsilon(x_3/\varepsilon^2, m(x_3/\varepsilon^2))\begin{bmatrix}\hat{a}^\varepsilon(\omega,\kappa_1,x_3)\\ \hat{b}^\varepsilon(\omega,\kappa_1,x_3)\end{bmatrix},\end{aligned} \tag{4.38}$$

where

$$\delta_1 = \phi(\Gamma_\alpha + \Gamma_\beta), \tag{4.39}$$

$$\delta_2 = \phi(\Gamma_\alpha - \Gamma_\beta). \tag{4.40}$$

Equation (4.38) can be recast into an initial value problem [56], [60] via a propagator equation

$$\begin{bmatrix}\hat{a}^\varepsilon(\omega,\kappa_1,x_3)\\ \hat{b}^\varepsilon(\omega,\kappa_1,x_3)\end{bmatrix} = \mathbf{P}^\varepsilon(\omega,\kappa_1,x_3)\begin{bmatrix}\hat{a}^\varepsilon(\omega,\kappa_1,-L)\\ \hat{b}^\varepsilon(\omega,\kappa_1,-L)\end{bmatrix}, \tag{4.41}$$

where the propagator matrix is (which propagates the wave from left to right in the domain $x_3 \in [-L, 0]$) defined by

$$\mathbf{P}^\varepsilon(\omega, \kappa_1, x_3) = \begin{bmatrix} \chi_1^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_2^\varepsilon(\omega, \kappa_1, x_3)} \\ \chi_2^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_1^\varepsilon(\omega, \kappa_1, x_3)} \end{bmatrix}, \quad \mathbf{P}^\varepsilon(\omega, \kappa_1, x_3 = -L) = \mathbf{I}. \tag{4.42}$$

However, in Figure 4.1 the initial wave is coming from the right at $x_3 = 0$ and travels right to left. In this initial value problem, the initial condition is at $x_3 = -L$ even though the input wave is at $x_3 = 0$. The problem is setup in this way in order to derive a Ricatti equation for the reflection coefficient. Furthermore, it is possible to write

$$\frac{d\mathbf{P}^\varepsilon}{dx_3}(\omega, \kappa_1, x_3) = \frac{1}{\varepsilon}\mathbf{H}^\varepsilon\left(\frac{x_3}{\varepsilon^2}, m\left(\frac{x_3}{\varepsilon^2}\right)\right)\mathbf{P}^\varepsilon(\omega, \kappa_1, x_3), \quad \mathbf{P}^\varepsilon(\omega, \kappa_1, x_3 = -L) = \mathbf{I}, \tag{4.43}$$

to which a diffusion approximation theorem [26] is applied (in the limit $\varepsilon \to 0$) to give the matrix stochastic differential equations

$$\begin{aligned} d\begin{bmatrix} \chi_1^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_2^\varepsilon(\omega, \kappa_1, x_3)} \\ \chi_2^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_1^\varepsilon(\omega, \kappa_1, x_3)} \end{bmatrix} &= iA_1 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \chi_1^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_2^\varepsilon(\omega, \kappa_1, x_3)} \\ \chi_2^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_1^\varepsilon(\omega, \kappa_1, x_3)} \end{bmatrix} \circ dW_1(x_3) \\ &- A_2 \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \chi_1^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_2^\varepsilon(\omega, \kappa_1, x_3)} \\ \chi_2^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_1^\varepsilon(\omega, \kappa_1, x_3)} \end{bmatrix} \circ dW_2(x_3) \\ &+ iA_2 \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \chi_1^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_2^\varepsilon(\omega, \kappa_1, x_3)} \\ \chi_2^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_1^\varepsilon(\omega, \kappa_1, x_3)} \end{bmatrix} \circ dW_3(x_3) \\ &- iA_3 \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \chi_1^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_2^\varepsilon(\omega, \kappa_1, x_3)} \\ \chi_2^\varepsilon(\omega, \kappa_1, x_3) & \overline{\chi_1^\varepsilon(\omega, \kappa_1, x_3)} \end{bmatrix} dx_3, \end{aligned} \tag{4.44}$$

where $A_1 = \omega\delta_1\sqrt{\Upsilon(0)}/2$, $A_2 = \omega\delta_2\sqrt{\Upsilon(\omega)}/(2\sqrt{2})$ and $A_3 = \omega^2\delta_2^2\Upsilon^{(AS)}(\omega)/8$ and the correlation integrals are defined as

$$\Upsilon(\omega) = 2\int_0^\infty \mathbb{E}\Big[m(0)m(x_3)\Big]\cos(2\omega\phi x_3)\,dx_3, \tag{4.45}$$

$$\Upsilon^{(AS)}(\omega) = 2\int_0^\infty \mathbb{E}\Big[m(0)m(x_3)\Big]\sin(2\omega\phi x_3)\,dx_3. \tag{4.46}$$

It can be shown that the power transmission coefficient can be written as [56]

$$\tau(L) \sim \exp\{-2A_2^2 L\} \sim \exp\left\{-\frac{L}{L_{loc}}\right\}, \tag{4.47}$$

where the localisation length is defined as

$$L_{loc}(\omega) = \frac{1}{2A_2^2} = \frac{4}{\omega^2 \delta_2^2 \Upsilon(\omega)} = \frac{4}{\omega^2 \phi^2 (\Gamma_\alpha - \Gamma_\beta)^2 \Upsilon(\omega)}. \tag{4.48}$$

4.4 Ricatti Formulation

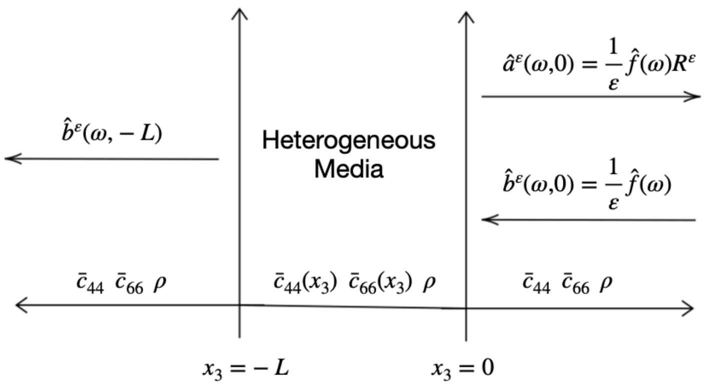

Figure 4.1: Boundary conditions for a multi-frequency wave (broadband pulse) impinging on the random layered, heterogeneous medium from the right at $x_3 = 0$. The random medium $x_3 \in [-L, 0]$ is sandwiched between two isotropic and homogeneous half spaces $x_3 \in (-\infty, -L) \cup (0, \infty)$.

In the weakly heterogeneous regime, the input pulse $f(t)$ has a support which [26] is of order ε^2. Its amplitude is scaled appropriately in order to ensure the input pulse has energy of order one. It is given by

$$\frac{1}{\varepsilon} f\left(\frac{t}{\varepsilon^2}\right), \tag{4.49}$$

so that

$$\int_{-\infty}^{\infty} \left(\frac{1}{\varepsilon} f\left(\frac{t}{\varepsilon^2}\right)\right)^2 dt = \int_{-\infty}^{\infty} f^2(p)\, dp, \tag{4.50}$$

which is finite. The boundary conditions for the left-going wave entering the slab at $x_3 = 0$ is then (see Figure 4.1)

$$\hat{b}^\varepsilon(\omega, \kappa_1, 0) = \frac{1}{\varepsilon}\hat{f}(\omega), \quad \hat{a}^\varepsilon(\omega, \kappa_1, -L) = 0. \tag{4.51}$$

The transmitted and reflected fields can be written as

$$\hat{a}^\varepsilon(\omega, \kappa_1, 0) = \frac{1}{\varepsilon}\hat{f}(\omega)\, R^\varepsilon(\omega, \kappa_1, 0), \quad \hat{b}^\varepsilon(\omega, \kappa_1, -L) = \frac{1}{\varepsilon}\hat{f}(\omega)\, T^\varepsilon(\omega, \kappa_1, -L). \tag{4.52}$$

Inserting these boundary conditions into equation (4.41) suggests the following relationship for the reflection $R^\varepsilon(\omega, \kappa_1, x_3)$ and transmission $T^\varepsilon(\omega, \kappa_1, x_3)$ coefficients at a general x_3 position

$$\begin{bmatrix} R^\varepsilon(\omega, \kappa_1, x_3) \\ 1 \end{bmatrix} = \mathbf{P}^\varepsilon(\omega, \kappa_1, x_3) \begin{bmatrix} 0 \\ T^\varepsilon(\omega, \kappa_1, x_3) \end{bmatrix}. \tag{4.53}$$

The transmission and reflection coefficients can then be written in terms of the propagator matrix (4.42) elements as

$$R^\varepsilon(\omega, \kappa_1, x_3) = \frac{\overline{\chi_2^\varepsilon(\omega, \kappa_1, x_3)}}{\chi_1^\varepsilon(\omega, \kappa_1, x_3)}, \tag{4.54}$$

$$T^\varepsilon(\omega, \kappa_1, x_3) = \frac{1}{\chi_1^\varepsilon(\omega, \kappa_1, x_3)}. \tag{4.55}$$

The reflected field in the time domain (in the right-hand half-space) is defined via the inverse Fourier transform

$$A^\varepsilon(t, 0) = \hat{a}^\varepsilon\left(\frac{t}{\varepsilon^2}, 0\right) = \frac{1}{2\pi}\int \hat{a}^\varepsilon(\omega, 0) e^{-i\omega t/\varepsilon^2}\, d\omega = \frac{1}{2\pi\varepsilon}\int \hat{f}(\omega) R^\varepsilon(\omega, 0) e^{-i\omega t/\varepsilon^2}\, d\omega. \tag{4.56}$$

The intensity of the reflected wave is related to its frequency domain autocorrelation. To calculate this, the reflected field intensity is determined at two close frequencies

ω^+, ω^- via

$$A^\varepsilon(t,0)^2 = A^\varepsilon(t,0)\overline{A^\varepsilon(t,0)}$$
$$= \frac{1}{4\pi^2\varepsilon^2}\left(\int R^\varepsilon_{\omega^+}\hat{f}(\omega^+)e^{-i\omega^+ t/\varepsilon^2}\,d\omega^+\right)\left(\int \overline{R^\varepsilon_{\omega^-}}\hat{f}(\omega^-)e^{i\omega^- t/\varepsilon^2}\,d\omega^-\right). \quad (4.57)$$

Hence, the mean intensity of the reflected wave is

$$I(t) = \mathbb{E}[A^\varepsilon(t,0)^2] = \frac{1}{4\pi\varepsilon^2}\int\int \mathbb{E}[R^\varepsilon_{\omega^+}\overline{R^\varepsilon_{\omega^-}}]\hat{f}(\omega^+)\overline{\hat{f}(\omega^-)}e^{i(\omega^- - \omega^+)t/\varepsilon^2}\,d\omega^+ d\omega^-. \quad (4.58)$$

Here it is natural to introduce a change of variables to remove the fast phase $(\omega^- - \omega^+)t/\varepsilon^2$ via

$$\omega^+ = \omega + \frac{\varepsilon^2 h}{2}, \quad \omega^- = \omega - \frac{\varepsilon^2 h}{2}, \quad (4.59)$$

which gives

$$I(t) = \mathbb{E}[A^\varepsilon(t,0)^2] = \frac{1}{4\pi^2}\int\int \mathbb{E}[R^\varepsilon_{\omega^+}\overline{R^\varepsilon_{\omega^-}}]\hat{f}(\omega+\varepsilon^2 h/2)\overline{\hat{f}(\omega-\varepsilon^2 h/2)}e^{-iht}\,d\omega dh$$
$$\approx \frac{1}{4\pi^2}\int\int \mathbb{E}[R^\varepsilon_{\omega^+}\overline{R^\varepsilon_{\omega^-}}]\left|\hat{f}(\omega)\right|^2 e^{-iht}\,d\omega dh, \quad (4.60)$$

where $\hat{f}(\omega+\varepsilon^2 h/2) \approx \hat{f}(\omega-\varepsilon^2 h/2)$. Taking derivatives of equations (4.54) and (4.55) gives two Ricatti equations for the transmission and reflection coefficients

$$\frac{dR^\varepsilon}{dx_3} = \overline{\frac{1}{\chi_1^\varepsilon}\frac{d\chi_2^\varepsilon}{dx_3} - \frac{\chi_2^\varepsilon}{(\chi_1^\varepsilon)^2}\frac{d\chi_1^\varepsilon}{dx_3}}, \quad (4.61)$$
$$\frac{dT^\varepsilon}{dx_3} = -\overline{\frac{1}{(\chi_1^\varepsilon)^2}\frac{d\chi_1^\varepsilon}{dx_3}}, \quad (4.62)$$

with initial conditions (recall equation (4.42)) $R^\varepsilon(\omega,-L) = 0$, $T^\varepsilon(\omega,-L) = 1$. Now, equations (4.42) and (4.43) give

$$\frac{d\chi_1^\varepsilon}{dx_3} = \frac{i\omega}{2\varepsilon}m(x_3/\varepsilon^2)\left(\delta_1\chi_1^\varepsilon - \delta_2 e^{-\frac{2i\omega\phi x_3}{\varepsilon^2}}\chi_2^\varepsilon\right), \quad (4.63)$$
$$\frac{d\chi_2^\varepsilon}{dx_3} = \frac{i\omega}{2\varepsilon}m(x_3/\varepsilon^2)\left(\delta_2 e^{\frac{2i\omega\phi x_3}{\varepsilon^2}}\chi_1^\varepsilon - \delta_1\chi_2^\varepsilon\right). \quad (4.64)$$

Inserting these in equation (4.61) gives

$$\frac{dR^\varepsilon}{dx_3} = \frac{i\omega}{2\varepsilon} m(x_3/\varepsilon^2)\Big(- \delta_2 e^{\frac{-2i\omega\phi x_3}{\varepsilon^2}} + 2R^\varepsilon \delta_1 - (R^\varepsilon)^2 \delta_2 e^{\frac{2i\omega\phi x_3}{\varepsilon^2}}\Big). \tag{4.65}$$

As the above Ricatti equation is nonlinear, it is necessary to consider the complete family of moments in order to get a closed system of equations to allow calculation of the mean and variance of the reflection coefficient intensity. For p, $q \in \mathbb{Z}^+$ let

$$U^\varepsilon_{p,q}(\omega, h, x_3) = (R^\varepsilon_{\omega^+})^p (\overline{R^\varepsilon_{\omega^-}})^q. \tag{4.66}$$

Taking derivatives

$$\frac{\partial U^\varepsilon_{p,q}}{\partial x_3} = pU^\varepsilon_{p-1,q} \frac{\partial R^\varepsilon_{\omega^+}}{\partial x_3} + qU^\varepsilon_{p,q-1} \frac{\partial \overline{R^\varepsilon_{\omega^-}}}{\partial x_3}. \tag{4.67}$$

Inserting equation (4.65) into (denoting $\delta^\pm = \delta(\omega \pm \varepsilon^2 h/2)$ and $\phi^\pm = \phi(\omega \pm \varepsilon^2 h/2)$ from equations (4.39) to (4.22)) equation (4.67) gives

$$\begin{aligned} \frac{\partial U^\varepsilon_{p,q}}{\partial x_3} &= pU^\varepsilon_{p-1,q} \frac{i\omega^+ m(x_3/\varepsilon^2)}{2\varepsilon} \Big[- \delta_2^+ e^{\frac{-2i\omega^+\phi^+ x_3}{\varepsilon^2}} + 2R^\varepsilon_{\omega^+}\delta_1^+ - (R^\varepsilon_{\omega^+})^2 \delta_2^+ e^{\frac{2i\omega^+\phi^+ x_3}{\varepsilon^2}} \Big] \\ &\quad + qU^\varepsilon_{p,q-1} \frac{i\omega^- m(x_3/\varepsilon^2)}{2\varepsilon} \Big[\delta_2^- e^{\frac{2i\omega^-\phi^- x_3}{\varepsilon^2}} - 2\overline{R^\varepsilon_{\omega_-}}\delta_1^- + (\overline{R^\varepsilon_{\omega^-}})^2 \delta_2^- e^{\frac{-2i\omega^-\phi^- x_3}{\varepsilon^2}} \Big]. \end{aligned} \tag{4.68}$$

Since $\omega^+ \approx \omega^-$, then

$$\begin{aligned} \frac{\partial U^\varepsilon_{p,q}}{\partial x_3} &= \frac{i\omega m(x_3/\varepsilon^2)}{\varepsilon} \delta_1 (p-q) U^\varepsilon_{p,q} \\ &\quad + \frac{i\omega\delta_2 m(x_3/\varepsilon^2)}{2\varepsilon} e^{2i\phi\omega x_3/\varepsilon^2} \Big(qU^\varepsilon_{p,q-1} e^{-i\phi h x_3} - pU^\varepsilon_{p+1,q} e^{i\phi h x_3} \Big) \\ &\quad + \frac{i\omega\delta_2 m(x_3/\varepsilon^2)}{2\varepsilon} e^{-2i\phi\omega x_3/\varepsilon^2} \Big(qU^\varepsilon_{p,q+1} e^{i\phi h x_3} - pU^\varepsilon_{p-1,q} e^{-i\phi h x_3} \Big), \end{aligned} \tag{4.69}$$

with $U^\varepsilon_{p,q}(\omega, h, x_3 = -L) = \mathbf{1}_0(p)\mathbf{1}_0(q)$. Now let

$$V^\varepsilon_{p,q}(\omega, \tau, x_3) = \frac{1}{2\pi} \int e^{-ih(\tau - \phi(p+q)x_3)} U^\varepsilon_{p,q}(\omega, h, x_3) dh$$

$$= \frac{1}{2\pi} \int e^{-ih\tau} e^{ih\phi(p+q)x_3} U^{\varepsilon}_{p,q}(\omega, h, x_3) dh, \tag{4.70}$$

where τ is time. Taking a derivative of equation (4.70) in x_3 gives

$$\begin{aligned}
\frac{\partial V^{\varepsilon}_{p,q}}{\partial x_3} &= \frac{1}{2\pi} \int e^{-ih\tau} \left[ih\phi(p+q) e^{ih\phi(p+q)x_3} U^{\varepsilon}_{p,q} + e^{ih\phi(p+q)x_3} \frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3} \right] dh \\
&= ih\phi(p+q) \left(\frac{1}{2\pi} \int e^{-ih\tau} e^{ih\phi(p+q)x_3} U^{\varepsilon}_{p,q}\, dh \right) + \frac{1}{2\pi} \int e^{-ih\tau} e^{ih\phi(p+q)x_3} \frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3}\, dh \\
&= ih\phi(p+q) V^{\varepsilon}_{p,q} + \frac{1}{2\pi} \int e^{-ih\tau} e^{ih\phi(p+q)x_3} \frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3}\, dh \\
&= -(p+q)\phi \frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} + \frac{1}{2\pi} \int e^{-ih\tau} e^{ih\phi(p+q)x_3} \frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3}\, dh. \qquad (4.71)
\end{aligned}$$

Since

$$V^{\varepsilon}_{p,q} = -\frac{1}{ih} \frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau}, \tag{4.72}$$

then

$$\frac{1}{2\pi} \int e^{-ih\tau} e^{ih\phi(p+q)x_3} \frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3}\, dh = \phi(p+q) \frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} + \frac{\partial V^{\varepsilon}_{p,q}}{\partial x_3}. \tag{4.73}$$

Applying the transform (4.70) to equation (4.69) gives

$$\begin{aligned}
\frac{\partial V^{\varepsilon}_{p,q}}{\partial x_3} &= -\phi(p+q) \frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} + \frac{i\omega\delta_1 m(x_3/\varepsilon^2)}{\varepsilon} (p-q) V^{\varepsilon}_{p,q} \\
&\quad + \frac{i\omega\delta_2 m(x_3/\varepsilon^2)}{2\varepsilon} e^{2i\phi\omega x_3/\varepsilon^2} \left(q V^{\varepsilon}_{p,q-1} - p V^{\varepsilon}_{p+1,q} \right) \\
&\quad + \frac{i\omega\delta_2 m(x_3/\varepsilon^2)}{2\varepsilon} e^{-2i\phi\omega x_3/\varepsilon^2} \left(q V^{\varepsilon}_{p,q+1} - p V^{\varepsilon}_{p-1,q} \right), \qquad (4.74)
\end{aligned}$$

with initial condition $V^{\varepsilon}_{p,q}(\omega, \tau, x_3 = -L) = \delta(\tau) \mathbf{1}_0(p) \mathbf{1}_0(q)$. Expanding equation (4.74) gives (noting how the transform removes the phase terms)

$$\begin{aligned}
\frac{\partial V^{\varepsilon}_{p,q}}{\partial x_3} &= -\phi(p+q) \frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} + \frac{i\omega\delta_1 m(x_3/\varepsilon^2)}{\varepsilon} (p-q) V^{\varepsilon}_{p,q} \\
&\quad + \frac{\omega\delta_2 m(x_3/\varepsilon^2)}{2\varepsilon} \sin\left(\frac{2\phi\omega x_3}{\varepsilon^2} \right) \left(p V^{\varepsilon}_{p+1,q} - q V^{\varepsilon}_{p,q-1} - p V^{\varepsilon}_{p-1,q} + q V^{\varepsilon}_{p,q+1} \right)
\end{aligned}$$

$$+\frac{i\omega\delta_2 m(x_3/\varepsilon^2)}{2\varepsilon}\cos\left(\frac{2\phi\omega x_3}{\varepsilon^2}\right)\left(qV^\varepsilon_{p,q-1} - pV^\varepsilon_{p+1,q} + qV^\varepsilon_{p,q+1} - pV^\varepsilon_{p-1,q}\right). \quad (4.75)$$

Define

$$g^{(1)}(m(x_3/\varepsilon^2), x_3/\varepsilon^2) = m(x_3/\varepsilon^2), \quad (4.76)$$

$$g^{(2)}(m(x_3/\varepsilon^2), x_3/\varepsilon^2) = m(x_3/\varepsilon^2)\sin\left(\frac{2\phi\omega x_3}{\varepsilon^2}\right), \quad (4.77)$$

$$g^{(3)}(m(x_3/\varepsilon^2), x_3/\varepsilon^2) = m(x_3/\varepsilon^2)\cos\left(\frac{2\phi\omega x_3}{\varepsilon^2}\right), \quad (4.78)$$

$$G(V^\varepsilon_{p,q}) = -\phi(p+q)\frac{\partial V^\varepsilon_{p,q}}{\partial \tau}, \quad (4.79)$$

and

$$\gamma^{(1)}(V^\varepsilon_{p,q}) = i\omega\delta_1(p-q)V^\varepsilon_{p,q}, \quad (4.80)$$

$$\gamma^{(2)}(V^\varepsilon_{p,q}) = -\frac{\omega\delta_2}{2}\left(qV^\varepsilon_{p,q-1} - pV^\varepsilon_{p+1,q} - qV^\varepsilon_{p,q+1} + pV^\varepsilon_{p-1,q}\right), \quad (4.81)$$

$$\gamma^{(3)}(V^\varepsilon_{p,q}) = \frac{i\omega\delta_2}{2}\left(qV^\varepsilon_{p,q-1} - pV^\varepsilon_{p+1,q} + qV^\varepsilon_{p,q+1} - pV^\varepsilon_{p-1,q}\right). \quad (4.82)$$

Equation (4.75) can then be written in the form

$$\frac{dV^\varepsilon_{p,q}}{dx_3} = \frac{1}{\varepsilon}F\left(V^\varepsilon_{p,q}(x_3), m(x_3/\varepsilon^2), x_3/\varepsilon^2\right) + G\big(V^\varepsilon_{p,q}(x_3)\big), \quad (4.83)$$

where

$$F(V_{p,q}, m(x_3/\varepsilon^2), x_3/\varepsilon^2) = \sum_{j=1}^{3}\gamma^{(j)}(V_{p,q})g^{(j)}(m(x_3/\varepsilon^2), x_3/\varepsilon^2). \quad (4.84)$$

The correlation integral matrix $\mathbf{C} = (C_{ij})_{i,j=1,2,3}$ is computed using the covariance of the random process m via

$$C_{ij} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\infty}\mathbb{E}\Big[g^{(i)}(m(0), y)g^{(j)}(m(x_3), y+x_3)\Big]dx_3 dy, \quad (4.85)$$

that is

$$\mathbf{C} = \begin{bmatrix} \Upsilon(0) & 0 & 0 \\ 0 & \frac{1}{2}\Upsilon(\omega) & -\frac{1}{2}\Upsilon^{(AS)}(\omega) \\ 0 & \frac{1}{2}\Upsilon^{(AS)}(\omega) & \frac{1}{2}\Upsilon(\omega) \end{bmatrix}, \tag{4.86}$$

where the correlation integrals are given by equations (4.45) and (4.46). The symmetric part of the correlation matrix is $\mathbf{C}^{(S)} = \frac{1}{2}(\mathbf{C} + \mathbf{C}^T) = \tilde{\boldsymbol{\sigma}}^2$, say where

$$\tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \sqrt{\Upsilon(0)} & 0 & 0 \\ 0 & \sqrt{\Upsilon(\omega)/2} & 0 \\ 0 & 0 & \sqrt{\Upsilon(\omega)/2} \end{bmatrix}, \tag{4.87}$$

and define

$$\boldsymbol{\sigma}_l(V_{p,q}) = \tilde{\boldsymbol{\sigma}}_{ll}\gamma^{(l)}(V_{p,q}). \tag{4.88}$$

Now a limit theorem [26] (p140, p157) is applied (as $\varepsilon \to 0$) to equation (4.83) to give the stochastic differential equation

$$dX_i = \sum_{l=1}^{3} \boldsymbol{\sigma}_l(X_i)\, dW_{(l-1)}(x_3) + \left(\frac{1}{2} \sum_{k,m=1}^{3} \sum_{j=1}^{5} \mathbf{C}_{km}\gamma^{(k)}(X_j) \frac{\partial \gamma^{(m)}(X_i)}{\partial X_j} + G(X_i) \right) dx_3, \tag{4.89}$$

where $X = \big(V_{p,q},\, V_{p,q-1},\, V_{p+1,q},\, V_{p,q+1},\, V_{p-1,q}\big)$. The focus is now on the SDE for $V_{p,q}$ and so $i = 1$. Expanding the second expression on the right hand side term by term then for $k = m = 1$ gives

$$\begin{aligned} \frac{1}{2}\sum_{j=1}^{5} \Upsilon(0)\gamma^{(1)}(X_j)\frac{\partial \gamma^{(1)}(V_{p,q})}{\partial X_j} &= \frac{\Upsilon(0)}{2}\gamma^{(1)}(X_1) i\omega\delta_1(p-q) \\ &= \frac{\Upsilon(0)}{2}\Big(i\omega\delta_1(p-q)\Big)^2 V_{p,q} \\ &= -\frac{\Upsilon(0)\omega^2\delta_1^2(p-q)^2}{2} V_{p,q}. \end{aligned} \tag{4.90}$$

For $k = m = 2$

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{5}\frac{\Upsilon(\omega)}{2}\gamma^{(2)}(X_j)\frac{\partial\gamma^{(2)}(V_{p,q})}{\partial X_j} &= \frac{\Upsilon(\omega)}{4}\bigg(\gamma^{(2)}(X_2)\bigg(-q\frac{\omega\delta_2}{2}\bigg)+\gamma^{(2)}(X_3)\bigg(p\frac{\omega\delta_2}{2}\bigg)\\
&+\gamma^{(2)}(X_4)\bigg(q\frac{\omega\delta_2}{2}\bigg)+\gamma^{(2)}(X_5)\bigg(-p\frac{\omega\delta_2}{2}\bigg)\bigg)\\
&=\frac{\Upsilon(\omega)\omega^2\delta_2^2}{16}\bigg(q(q-1)V_{p,q-2}-qpV_{p+1,q-1}-q(q-1)V_{p,q}\\
&+pqV_{p-1,q-1}-pqV_{p+1,q-1}+p(p+1)V_{p+2,q}+pqV_{p+1,q+1}\\
&-p(p+1)V_{p,q}-q(q+1)V_{p,q}+pqV_{p+1,q+1}+q(q+1)V_{p,q+2}\\
&-pqV_{p-1,q+1}+pqV_{p-1,q-1}-p(p-1)V_{p,q}-pqV_{p-1,q+1}\\
&+p(p-1)V_{p-2,q}\bigg), \qquad (4.91)
\end{aligned}
$$

$k = m = 3$

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{5}\frac{\Upsilon(\omega)}{2}\gamma^{(3)}(X_j)\frac{\partial\gamma^{(3)}(V_{p,q})}{\partial X_j} &= \frac{\Upsilon(\omega)}{4}\bigg(\gamma^{(3)}(X_2)\bigg(\frac{i\omega\delta_2 q}{2}\bigg)+\gamma^{(3)}(X_3)\bigg(\frac{-i\omega\delta_2 p}{2}\bigg)\\
&+\gamma^{(3)}(X_4)\bigg(\frac{i\omega\delta_2 q}{2}\bigg)+\gamma^{(3)}(X_5)\bigg(\frac{-i\omega\delta_2 p}{2}\bigg)\bigg)\\
&=\frac{\Upsilon(\omega)\omega^2\delta_2^2}{16}\bigg(-q(q-1)V_{p,q-2}+qpV_{p+1,q-1}-q(q-1)V_{p,q}\\
&+qpV_{p-1,q-1}+pqV_{p+1,q-1}-p(p+1)V_{p+2,q}+pqV_{p+1,q+1}\\
&-p(p+1)V_{p,q}-q(q+1)V_{p,q}+qpV_{p+1,q+1}-q(q+1)V_{p,q+2}\\
&+pqV_{p-1,q+1}+pqV_{p-1,q-1}-p(p-1)V_{p,q}+pqV_{p-1,q+1}\\
&-p(p-1)V_{p-2,q}\bigg). \qquad (4.92)
\end{aligned}
$$

For $k = 3$, $m = 2$

$$
\begin{aligned}
\frac{1}{2}\sum_{j=1}^{5}\frac{\Upsilon^{(AS)}(\omega)}{2}\gamma^{(3)}(X_j)\frac{\partial\gamma^{(2)}(V_{p,q})}{\partial X_j} &= \frac{\Upsilon^{(AS)}(\omega)}{4}\bigg(\gamma^{(3)}(X_2)\bigg(\frac{-\omega\delta_2 q}{2}\bigg)+\gamma^{(3)}(X_3)\bigg(\frac{\omega\delta_2 p}{2}\bigg)\\
&+\gamma^{(3)}(X_4)\bigg(\frac{\omega\delta_2 q}{2}\bigg)+\gamma^{(3)}(X_5)\bigg(\frac{-\omega\delta_2 p}{2}\bigg)\bigg)\\
&=\frac{i\Upsilon^{(AS)}(\omega)\delta_2^2\omega^2}{16}\bigg(-q(q-1)V_{p,q-2}+qpV_{p+1,q-1}
\end{aligned}
$$

$$- q(q-1)V_{p,q} + qpV_{p-1,q-1}$$

$$+ pqV_{p+1,q-1} - p(p+1)V_{p+2,q} + pqV_{p+1,q+1}$$

$$- p(p+1)V_{p,q} + q(q+1)V_{p,q} - qpV_{p+1,q+1}$$

$$+ q(q+1)V_{p,q+2} - qpV_{p-1,q+1} - pqV_{p-1,q-1}$$

$$+ p(p-1)V_{p,q} - pqV_{p-1,q+1} + p(p-1)V_{p-2,q}\Big). \tag{4.93}$$

For $k = 2$, $m = 3$

$$
\begin{aligned}
-\frac{1}{2}\sum_{j=1}^{5}\frac{\Upsilon^{(AS)}(\omega)}{2}\gamma^{(2)}(X_j)\frac{\partial\gamma^{(3)}(V_{p,q})}{\partial X_j} &= -\frac{\Upsilon^{(AS)}(\omega)}{4}\Bigg(\gamma^{(2)}(X_2)\left(\frac{i\omega\delta_2 q}{2}\right) \\
&+ \gamma^{(2)}(X_3)\left(\frac{-i\omega\delta_2 p}{2}\right) \\
&+ \gamma^{(2)}(X_4)\left(\frac{i\omega\delta_2 q}{2}\right) + \gamma^{(2)}(X_5)\left(\frac{-i\omega\delta_2 p}{2}\right)\Bigg) \\
&= \frac{i\Upsilon^{(AS)}(\omega)\delta_2^2\omega^2}{16}\Big(q(q-1)V_{p,q-2} - qpV_{p+1,q-1} \\
&- q(q-1)V_{p,q} + qpV_{p-1,q-1} \\
&- pqV_{p+1,q-1} + p(p+1)V_{p+2,q} + pqV_{p+1,q+1} \\
&- p(p+1)V_{p,q} + q(q+1)V_{p,q} - qpV_{p+1,q+1} \\
&- q(q+1)V_{p,q+2} + qpV_{p-1,q+1} - pqV_{p-1,q-1} \\
&+ p(p-1)V_{p,q} + pqV_{p-1,q+1} - p(p-1)V_{p-2,q}\Big),
\end{aligned} \tag{4.94}
$$

so

$$
\begin{aligned}
&\frac{1}{2}\sum_{k,m=1}^{3}\sum_{j=1}^{5}\mathbf{C}_{km}\gamma^{(k)}(X_j)\frac{\partial\gamma^{(m)}(V_{p,q})}{\partial X_j} \\
&= \frac{-\Upsilon(0)\omega^2\delta_1^2(p-q)^2}{2}V_{p,q} + \frac{\Upsilon(\omega)\omega^2\delta_2^2}{4}\Big(qpV_{p-1,q-1} + qpV_{p+1,q+1} - p^2V_{p,q} - q^2V_{p,q}\Big) \\
&+ \frac{i\Upsilon^{(AS)}(\omega)\omega^2\delta_2^2}{4}\Big(qV_{p,q} - pV_{p,q}\Big)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\Upsilon(\omega)\omega^2\delta_2^2}{4} qp \Big(V_{p+1,q+1} + V_{p-1,q-1} \Big) \\
&+ \frac{\omega^2}{4} V_{p,q} \Big(-2\Upsilon(0)\delta_1^2(p-q)^2 - (p^2+q^2)\Upsilon(\omega)\delta_2^2 + i\Upsilon^{(AS)}(\omega)\delta_2^2(q-p) \Big) \\
&= \frac{\Upsilon(\omega)\omega^2\delta_2^2}{4} qp \Big(V_{p+1,q+1} + V_{p-1,q-1} - 2V_{q,p} \Big) \\
&+ \frac{\omega^2}{4} V_{p,q} \Big(-2\Upsilon(0)\delta_1^2(p-q)^2 - (p-q)^2\Upsilon(\omega)\delta_2^2 + i\Upsilon^{(AS)}(\omega)\delta_2^2(q-p) \Big). \qquad (4.95)
\end{aligned}
$$

Hence, equation (4.89) reads

$$
\begin{aligned}
dV_{p,q} &= -\phi(p+q)\frac{\partial V_{p,q}}{\partial \tau}\, dx_3 + \sqrt{\Upsilon(0)} i\omega\delta_1(p-q)V_{p,q}\, dW_0(x_3) \\
&- \frac{\sqrt{\Upsilon(\omega)}\omega\delta_2}{2\sqrt{2}} \Big(qV_{p,q-1} - pV_{p+1,q} - qV_{p,q+1} + pV_{p-1,q} \Big)\, dW_1(x_3) \\
&+ \frac{i\sqrt{\Upsilon(\omega)}\omega\delta_2}{2\sqrt{2}} \Big(qV_{p,q-1} - pV_{p+1,q} + qV_{p,q+1} pV_{p-1,q} \Big)\, dW_2(x_3) \\
&+ \frac{\Upsilon(\omega)\omega^2\delta_2^2}{4} qp \Big(V_{p+1,q+1} + V_{p-1,q-1} - 2V_{p,q} \Big)\, dx_3 \\
&+ \frac{\omega^2}{4} V_{p,q} \Big(-2\Upsilon(0)\delta_1^2(p-q)^2 - (p-q)^2\Upsilon(\omega)\delta_2^2 + i\Upsilon^{(AS)}(\omega)\delta_2^2(q-p) \Big)\, dx_3.
\end{aligned} \qquad (4.96)
$$

Taking expectations of the SDE (4.96) gives

$$
\begin{aligned}
\frac{\partial \mathbb{E}[V_{p,q}]}{\partial x_3} &= -\phi(p+q)\frac{\partial \mathbb{E}[V_{p,q}]}{\partial \tau} + \frac{\Upsilon(\omega)\omega^2\delta_2^2}{4} qp \Big(\mathbb{E}[V_{p+1,q+1}] + \mathbb{E}[V_{p-1,q-1}] - 2\mathbb{E}[V_{p,q}] \Big) \\
&+ \frac{\omega^2}{4}\mathbb{E}[V_{p,q}] \Big(-2\Upsilon(0)\delta_1^2(p-q)^2 - (p-q)^2\Upsilon(\omega)\delta_2^2 + i\Upsilon^{(AS)}(\omega)\delta_2^2(q-p) \Big).
\end{aligned} \qquad (4.97)
$$

To study the moments of the autocorrelation function $V_{p,q}$, consider the diagonal elements ($p = q$) and let

$$
\mathcal{W}_p(\omega,\tau,x_3) = \lim_{\varepsilon\to 0} \mathbb{E}\Big[V^{\varepsilon}_{p,p}(\omega,\tau,x_3) \Big]. \qquad (4.98)
$$

This satisfies the transport equation

$$\frac{\partial \mathcal{W}_p}{\partial x_3} + 2\phi p \frac{\partial \mathcal{W}_p}{\partial \tau} = (\mathcal{L}_{\mathcal{W}} \mathcal{W})_p, \quad x_3 \geq -L, \quad p \in \mathbb{N}, \quad \tau \in \mathbb{R}, \tag{4.99}$$

where, using equation (4.48)

$$(\mathcal{L}_{\mathcal{W}} \mathcal{W})_p = \frac{2p^2}{L_{loc}} \left(\frac{1}{2} \mathcal{W}_{p+1} + \frac{1}{2} \mathcal{W}_{p-1} - \mathcal{W}_p \right), \tag{4.100}$$

with initial condition

$$\mathcal{W}_p(\omega, \tau, x_3 = -L) = \delta(\tau) \mathbf{1}_0(p). \tag{4.101}$$

The mean intensity given by equation (4.60) has an expectation in terms of the reflection coefficient at $x_3 = 0$ which is linked to the moments via equation (4.66) via

$$\mathbb{E}\left[R^{\varepsilon}_{\omega^+}(\omega, 0) \overline{R^{\varepsilon}_{\omega^-}(\omega, 0)} \right] = \mathbb{E}[U^{\varepsilon}_{11}(\omega, \tau, x_3 = 0)]. \tag{4.102}$$

From equation (4.70) (at $x_3 = 0$) the Fourier transform from h to τ is given by

$$V^{\varepsilon}_{11}(\omega, \tau, 0) = \frac{1}{2\pi} \int e^{-ih\tau} U^{\varepsilon}_{11}(\omega, h, 0)\, dh, \tag{4.103}$$

and the inverse Fourier transform from τ to h by

$$U^{\varepsilon}_{11}(\omega, h, 0) = \frac{1}{2\pi} \int e^{ih\tau} V^{\varepsilon}_{11}(\omega, \tau, 0)\, d\tau. \tag{4.104}$$

Hence, from equation (4.98)

$$\mathbb{E}\left[R^{\varepsilon}_{\omega^+}(\omega, 0) \overline{R^{\varepsilon}_{\omega^-}(\omega, 0)} \right] \overset{\varepsilon \to 0}{=} \int \mathcal{W}_1(\omega, \tau, 0) e^{ih\tau}\, d\tau. \tag{4.105}$$

4.4.1 Analytical Transport Equation Solution for $L \to \infty$

In the case for an infinite propagation distance ($L \to \infty$) the derivative $\partial \mathcal{W}_p / \partial x_3 = 0$ (since the slab occupies the entire half space $\mathcal{W}_p$ is insensitive to x_3 [26]) so from equation

(4.99)

$$\frac{\partial \mathcal{W}_p^{\infty}}{\partial \tau} = \frac{p}{2\phi L_{loc}(\omega)}\Big(\mathcal{W}_{p+1}^{\infty} + \mathcal{W}_{p-1}^{\infty} - 2\mathcal{W}_p^{\infty}\Big). \tag{4.106}$$

Let the non-dimensional terms be defined as

$$\tilde{x}_3 = \frac{x_3 + L}{L_{loc}(\omega)}, \quad \tilde{\tau} = \frac{\tau}{\phi L_{loc}(\omega)}, \quad \mathcal{W}_p^{\infty} = \frac{\widetilde{\mathcal{W}}_p^{\infty}(\tilde{\tau}, \tilde{x}_3)}{\phi L_{loc}(\omega)}. \tag{4.107}$$

The scaled, dimensionless transport equations (4.106) can then be written as

$$\frac{\partial \widetilde{\mathcal{W}}_p^{\infty}}{\partial \tilde{\tau}} = \frac{p}{2}\Big(\widetilde{\mathcal{W}}_{p+1}^{\infty} + \widetilde{\mathcal{W}}_{p-1}^{\infty} - 2\widetilde{\mathcal{W}}_{p+1}^{\infty}\Big), \tag{4.108}$$

with solution

$$\widetilde{\mathcal{W}}_p^{\infty} = \frac{2p\tilde{\tau}^{(p-1)}}{(2+\tilde{\tau})^{(p+1)}}\mathbf{1}_{[0,\infty)}(\tilde{\tau}). \tag{4.109}$$

Hence, for $p = 1$ (as required in equation (4.105))

$$\lim_{L\to\infty} \mathcal{W}_1(\omega, \tau, x_3) = \frac{2\phi L_{loc}(\omega)}{(2\phi L_{loc}(\omega) + \tau)^2}\mathbf{1}_{[0,\infty)}(\tau). \tag{4.110}$$

4.4.2 Probabilistic Framework for Transport Equations

Equation (4.99) is a forward Kolmogorov equation and in order to use the solution theory [26] (p113), this equation needs to be transformed to a backward Kolmogorov equation. The adjoint of the operator $(2\phi p\, \partial/\partial\tau)$ is $(-2\phi p\, \partial/\partial\tau)$ so the backward Kolmogorov equation is

$$\frac{\partial \mathcal{W}_p}{\partial x_3} - 2\phi p\frac{\partial \mathcal{W}_p}{\partial \tau} = \big(\mathcal{L}_{\mathcal{W}}\mathcal{W}\big)_p, \tag{4.111}$$

subject to the terminal condition $\mathcal{W}_p(x_3 = -L) = \delta(\tau)\mathbf{1}_0(p)$. The left hand side of equation (4.99) can be written as a total derivative via

$$\frac{d\mathcal{W}_p}{dx_3}(x_3, \tau(x_3)) = \frac{\partial \mathcal{W}_p}{\partial x_3} + \frac{\partial \tau}{\partial x_3}\frac{\partial \mathcal{W}_p}{\partial \tau}, \tag{4.112}$$

and so $\partial\tau/\partial x_3 = -2\phi p$. Define the continuous time jump Markov process [75] as N_{x_3}, equipped with a state space which is the integers $\mathbb{N}$ paired with the infinitesimal generator $\mathcal{L}_{\mathcal{W}}$ from equation (4.100). Define the second Markov process τ_{x_3}, then via (4.112)

$$\frac{\partial \mathcal{T}_{x_3}}{\partial x_3} = -2\phi p N_{x_3}, \tag{4.113}$$

where the terminal condition is now $\mathcal{T}_{x_3}(x_3 = -L) = \tau$. This implies that

$$\begin{aligned}\int_{-L}^{x_3} d\mathcal{T}_{x_3} &= -2\phi p \int_{-L}^{x_3} N_{x_3'}\, dx_3' \\ \Big[\mathcal{T}_{x_3}\Big]_{-L}^{x_3} &= -2\phi p \int_{-L}^{x_3} N_{x_3'}\, dx_3',\end{aligned} \tag{4.114}$$

and so

$$\mathcal{T}_{x_3} = \tau - 2\phi p \int_{-L}^{x_3} N_{x_3'}\, dx_3'. \tag{4.115}$$

Equation (4.99) can be written $\partial\mathcal{W}_p/\partial x_3 + \mathcal{L}(\mathcal{W}_p) = 0$ where $\mathcal{L} = -2\phi p \partial/\partial\tau - (\mathcal{L}_{\mathcal{W}}\mathcal{W})_p$ subject to the terminal condition $\mathcal{W}_p(x_3 = 0) = U(p,\tau) = \mathbf{1}_0(p)\delta(\tau)$ where the Markov process $Y(x_3) = [N_{x_3}, \mathcal{T}_{x_3}] = [p,\tau]$ at $x_3 = -L$. Equation (4.99) then has solution [26] (eqn. 6.11)

$$\begin{aligned}\mathcal{W}_p(\tau, x_3) &= \mathbb{E}\Big[U(N_{x_3}, \mathcal{T}_{x_3})\Big| N_{x_3}(x_3 = -L) = p,\, \mathcal{T}_{x_3}(x_3 = -L) = \tau\Big] \\ &= \mathbb{E}\Big[U\Big(N_{x_3}, \tau - 2\phi p \int_{-L}^{x_3} N_{x_3'} dx_3{}'\Big)\Big| N_{x_3}(x_3 = -L) = p\Big].\end{aligned} \tag{4.116}$$

This solution can be justified by using the martingale representation ([26] Appendix 6.9)

$$\begin{aligned}M(x_3) = {}& \mathcal{W}_p(x_3, N_{x_3}(x_3), \mathcal{T}_{x_3}(x_3)) - \mathcal{W}_p(-L, N_{x_3}(-L), \mathcal{T}_{x_3}(-L)) \\ & - \int_{-L}^{x_3} \left(\frac{\partial}{\partial x_3'} + \mathcal{L}\right) \mathcal{W}_p(s, N_{x_3'}(s), \mathcal{T}_{x_3'}(s))\, ds.\end{aligned} \tag{4.117}$$

Then

$$M(x_3) = \mathcal{W}_p(x_3, N_{x_3}(x_3), \mathcal{T}_{x_3}(x_3)) - \mathcal{W}_p(-L, N_{x_3}(-L), \mathcal{T}_{x_3}(-L)), \qquad (4.118)$$

as the integral is zero. So the martingale at $x_3 = 0$ is

$$M(0) = \mathcal{W}_p(0, N_{x_3}(0), \mathcal{T}_{x_3}(0)) - \mathcal{W}_p(-L, N_{x_3}(-L), \mathcal{T}_{x_3}(-L)). \qquad (4.119)$$

The martingale property

$$\mathbb{E}[M(x_3 + h)|Y(x_3'),\, x_3' \le x_3] = M(x_3), \quad h > 0, \qquad (4.120)$$

implies that, by taking expectations, gives

$$\begin{aligned}
&\mathbb{E}[\mathcal{W}_p(0, N_{x_3}(0), \mathcal{T}_{x_3}(0))|N_{x_3}(x_3'), \mathcal{T}_{x_3}(x_3'), x_3' \le 0] - \mathcal{W}_p(-L, N_{x_3}(-L), \mathcal{T}_{x_3}(-L)) \\
&= \mathcal{W}_p(0, N_{x_3}(0), \mathcal{T}_{x_3}(0)) - \mathcal{W}_p(-L, N_{x_3}(-L), \mathcal{T}_{x_3}(-L)), \qquad (4.121)
\end{aligned}$$

so

$$\mathcal{W}_p(0, N_{x_3}(0), \mathcal{T}_{x_3}(0)) = \mathbb{E}[u(N_{x_3}(0), \mathcal{T}_{x_3}(0))|N_{x_3}(x_3'), \mathcal{T}_{x_3}(x_3'), x_3' \le 0]. \qquad (4.122)$$

Using the terminal condition $\mathcal{W}_p(0, N_{x_3}(0), \mathcal{T}_{x_3}(0)) = U(N_{x_3}(0), \mathcal{T}_{x_3}(0))$ admits the solution

$$\mathcal{W}_p(0, N_{x_3}(0), \mathcal{T}_{x_3}(0)) = \mathbb{E}[U(N_{x_3}, \mathcal{T}_{x_3})|N_{x_3}(-L) = p,\, \mathcal{T}_{x_3}(-L) = \tau], \qquad (4.123)$$

as shown in equation (4.116). Using the terminal condition $U(p, \tau) = \mathbf{1}_0(p)\delta(\tau)$ and integrating equation (4.116) with respect to time τ at $x_3 = 0$, gives

$$\begin{aligned}
\int_{\tau_0}^{\tau_1} \mathcal{W}_p(\omega, \tau, 0)\, d\tau &= \int_{\tau_0}^{\tau_1} \mathbb{E}\left[U\left(N_{x_3}(0), \tau - 2\phi p \int_{-L}^{0} N_{x_3'}\, dx_3'\right)\middle| N_{x_3}(-L) = p\right] d\tau \\
&= \int_{\tau_0}^{\tau_1} \mathbb{E}\left[\mathbf{1}_0(p)\delta\left(\tau - 2\phi p \int_{-L}^{0} N_{x_3'}\, dx_3'\right)\middle| N_{x_3}(-L) = p\right] d\tau
\end{aligned}$$

$$= \mathbb{P}\bigg(N_{x_3}(0) = 0,\, 2\phi p \int_{-L}^{0} N_{x_3'}\, dx_3' \in [\tau_0, \tau_1] \bigg| N_{x_3}(-L) = p \bigg), \tag{4.124}$$

where the expectation of the indicator function has been exchanged for a probability. Equation (4.124) can be used to calculate values for $\mathcal{W}_p(\omega, \tau, 0)$ using a Monte Carlo technique. Using the Gillespie algorithm [52] at state p a random clock with exponential distribution and intensity parameter $2p^2/L_{loc}$ starts running. When the clock strikes, the process jumps to state $p+1$ or $p-1$ with probability 1/2. The process counts $p = 0$ as an absorbing state (the process stops), so $N_0 = 0$. To compute this, a large number of jump processes N_{x_3} (trajectories) are generated (realisations) which are used to compute the integral

$$2\phi p \int_{-L}^{0} N_{x_3'}\, dx_3',$$

which has dimensions of time. This integral computes the lifetime of each jump process before reaching the absorbing state. To compute a probability, the data is binned into corresponding $d\tau$ intervals. To compute the function $\mathcal{W}_p(\omega, \tau, 0)$ the values for the probability are divided by the $d\tau$ discretisation interval. The limiting solution ($L \to \infty$) provided by equation (4.110) is shown as the red line which is in good agreement with the Monte Carlo solution (carried out using Python). Also note that the Monte Carlo simulation is in agreement with this analytical result at $\tau = 0$. As the frequency increases the localisation length equation (4.48) decreases so the solution for $\mathcal{W}_1(\omega, \tau = 0, 0)$ changes with the localisation length, and can be seen algebraically from equation (4.110) which is in agreement with the Monte Carlo simulation. The drop off in $\mathcal{W}_1$ as τ increases is expected as it is related to the recieved energy at the surface from the incoherent wave and, can be seen from the form of the solution in equation (4.124). Figure (4.3) shows the second moment $\mathcal{W}_2$. Notice how for small values of τ there is a peak in the signal, emphasising that a homogenisation approach here is not valid and would not capture the physics of the multiple scattering interactions between wave and medium. It is only for long times that the process starts to self average. The

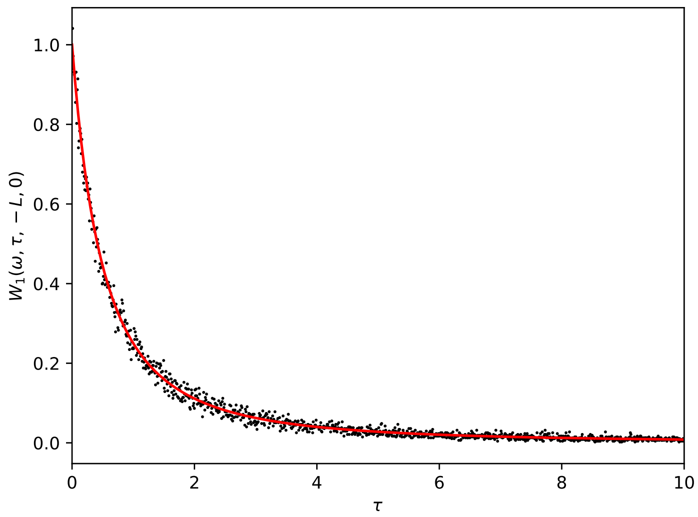

Figure 4.2: Plot of the frequency domain autocorrelation function of the reflection coefficient (which is proportional to its mean intensity) $\mathcal{W}_1(\omega, \tau, -L, 0)$ (the Monte Carlo simulation is depicted by the black dots, where each dot represents a realisation the of the solution $\mathcal{W}_1(\omega, \tau, -L, 0)$) and the limit case $\mathcal{W}_1^{\infty}(\omega, \tau, 0)$ (red line) from equation (4.110) versus time τ where $\phi = 1$ and $L_{loc} = 0.5$.

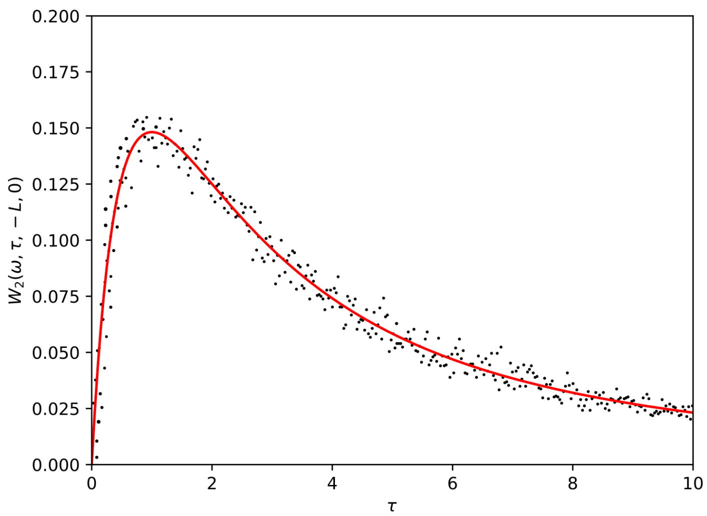

Figure 4.3: Plot of the frequency domain autocorrelation function (second moment) of the reflection coefficient $\mathcal{W}_2(\omega, \tau, -L, 0)$ (the Monte Carlo simulation is depicted by the black dots, where each dot represents a realisation the of the solution $\mathcal{W}_2(\omega, \tau, -L, 0)$) and the limit case $\mathcal{W}_2^{\infty}(\omega, \tau, 0)$ (red line) from equation (4.110) versus time τ where $\phi = 1$ and $L_{loc} = 0.5$.

mean intensity of the reflected signal is then given by equations (4.60) and (4.105) to be

$$\begin{aligned} I(t) &= \frac{1}{4\pi^2} \int\int\int \mathcal{W}_1(\omega, \tau, 0) \left| \hat{f}(\omega) \right|^2 e^{ih(\tau - t)}\, d\omega dh\, d\tau \\ &= \frac{1}{4\pi^2} \int\int \mathcal{W}_1(\omega, \tau, 0) 2\pi\delta(\tau - t) \left| \hat{f}(\omega) \right|^2 d\omega\, d\tau \\ &= \frac{1}{2\pi} \int \mathcal{W}_1(\omega, t, 0) \left| \hat{f}(\omega) \right|^2 d\omega. \end{aligned} \tag{4.125}$$

4.5 Discussion

This Chapter presented a probabilistic description of a broadband ultrasound pulse (multi-frequency) that has been reflected from a randomly layered and locally anisotropic, solid material. Through the use of a diffusion approximation theorem, the moments of the frequency autocorrelation function of the reflection coefficient were studied. The reflected wave is key to understanding the scattered incoherent part of the elastic wave. Formulation of the problem began in Chapter 3, where the SDE system for the propagator equations presented a route to deriving a Ricatti equation for the reflection coefficient. The analysis then considered the frequency autocorrelation function of two nearby frequencies which produced a tractable system of SDE's via the use of a diffusion approximation theorem. The diagonal elements of the evolution equation for the autocorrelation function were then studied analytically. The solution to this equation satisfied a transport equation, which was framed in a probabilistic setting and was solved using a continuous time jump Markov process. Furthermore, the case for an infinitely thick component ($L \to \infty$) of medium was studied, and analytical solutions for the autocorrelation function were presented. Comparison was made between this analytical solution and the numerical simulation of the transport equations, via a Monte Carlo technique. The numerical and analytical solution were in good agreement with one another, which is shown in Figure 4.2. Figure 4.3 shows the second moment $\mathcal{W}_2(\omega, \tau, -L, 0)$. The variance rises to a peak for small time values and it then takes time to gradually asymptote down to a low level. This is further evidence that a homogenisation approach is not appropriate for randomly layered media, as the variance in the signal due to interactions between the wave and the medium are significant.

The analysis presented here is relevant to NDT engineers who wish to study the performance characteristics of imaging algorithms when using ultrasonic waves in randomly layered elastic media, such as welds or additively manufactured metals. This work could also be used in finite element codes where one wishes to estimate attenuation factors in randomly layered media without the computational expense of explicitly including the physics of the multiple scattering interactions between the layered geometry with the input wave in the simulation.

The next Chapter follows on from the analysis here, studying a broadband pulse via an extended line source along the x_2 axis. The reader will become aware of the importance of studying the frequency autocorrelation function (and the transport equations) in regard to obtaining analytical expressions for the reflected mean intensity in the coming Chapter.

Chapter 5

Moments of the Reflected Intensity of a Line Source

5.1 Nomenclature

Parameter			Equation
α	Lumped parameter	$[\mathrm{M}^{-1}\mathrm{LT}^{2}]$	(5.30)
$\bar{\alpha}$	Lumped parameter	$[\mathrm{M}^{-1}\mathrm{LT}^{2}]$	(5.35)
β	Lumped parameter	$[\mathrm{ML}^{-3}]$	(5.31)
$\bar{\beta}$	Lumped parameter	$[\mathrm{ML}^{-3}]$	(5.36)
γ	Correlation integral	[L]	(5.99)
γ_1	Lumped parameter	$[\mathrm{M}^{1/2}\mathrm{T}^{1/2}\mathrm{L}^{-2}]$	(5.41)
γ_2	Lumped parameter	$[\mathrm{M}^{-1/2}\mathrm{T}^{3/2}]$	(5.42)
Γ_1	Lumped parameter	$[\mathrm{TL}^{-1}]$	(5.71)
Γ_2	Lumped parameter	$[\mathrm{TL}^{-1}]$	(5.73)
Δ_1	Lumped parameter	$[\mathrm{TL}^{-1}]$	(5.44)
Δ_2	Lumped parameter	$[\mathrm{TL}^{-1}]$	(5.45)
δ	Dirac delta function	[–]	(5.3)
ε	Small dimensionless parameter	[–]	(5.16)
ζ	Slowness function	$[\mathrm{TL}^{-1}]$	(5.32)
θ	Rotation angle of material slowness surface	[–]	(5.2)
θ_j	Propagator function	[–]	(5.106)
$\bar{\theta}$	Mean slowness angle	[–]	(5.11)
κ_1	Slowness vector component in x_1 direction	$[\mathrm{TL}^{-1}]$	(5.16)
ξ	Velocity in x_3 direction	$[\mathrm{LT}^{-1}]$	(5.8)
$\hat{\xi}^{\varepsilon}$	Velocity (frequency slowness domain) in x_3 direction	$[\mathrm{L}^{2}]$	(5.16)
ρ	Constant material density	$[\mathrm{ML}^{-3}]$	(5.1)
σ	Matrix	[]	(5.134)

τ	Stress in x_3 direction	$[ML^{-1}T^{-2}]$	(5.6)
$\hat{\tau}$	Stress (frequency slowness domain) in x_3 direction	$[MT^{-1}]$	(5.17)
τ	Power transmission coefficient	[–]	(5.157)
$\tilde{\tau}$	Non-dimensional time	[–]	(5.249)
ϕ_j	Propagator function	[–]	(5.106)
$\chi^{\varepsilon}_{(1,2)}$	Complex propagator functions	[–]	(5.57)
ψ_j	Propagator function	[–]	(5.106)
ω	Angular frequency	$[T^{-1}]$	(5.16)
$\hat{a}^{\varepsilon}$	Frequency slowness wave-mode	$[M^{1/2}T^{-1/2}L]$	(5.38)
$\hat{b}^{\varepsilon}$	Frequency slowness wave-mode	$[M^{1/2}LT^{-1/2}]$	(5.39)
A	Non-dimensional paramater	[–]	(5.223)
c_{ijkl}	Stress tensor	$[ML^{-1}T^{-2}]$	(5.3)
$C_{p,q}$	Correlation integral	[]	(5.96)
c_{66}	Stress term	$[ML^{-1}T^{-2}]$	(5.9)
c_{44}	Stress term	$[ML^{-1}T^{-2}]$	(5.10)
$\bar{c}_{66}$	Stress term	$[ML^{-1}T^{-2}]$	(5.11)
$\bar{c}_{44}$	Stress term	$[ML^{-1}T^{-2}]$	(5.12)
e_{kl}	Strain tensor	[–]	(5.4)
$\mathbf{F}^{\varepsilon}$	Forcing function	[]	(5.15)
$\hat{f}(\omega)$	Input wave	$[MT^{-1}]$	(5.16)
$\mathbf{G}$	Propagator function	[–]	(5.77)
$\mathbf{g}_i^{(n)}$	Vector	[]	(5.79)
$\mathbf{h}_n$	Vector	[]	(5.79)
$\mathbf{g}_i^{(n)}$	Vector	[]	(5.79)

$\mathbf{H}^{\varepsilon}$	Random matrix	[]	(5.56)
$\mathcal{K}$	Slowness value	[TL^{-1}]	(5.245)
L_{loc}	Localisation length	[L]	(5.158)
$\mathcal{L}_{\theta_j}$	Radial infinitesimal generator	[–]	(5.159)
$\mathbf{P}^{\varepsilon}$	Propagator matrix	[–]	(5.57)
$\mathbf{P}^{\varepsilon}_{M}$	Block propagator matrix	[–]	(5.80)
R^{ε}	Reflection coefficient	[–]	(5.175)
$\boldsymbol{s}$	Symmetry axis vector	[–]	(5.2)
T^{ε}	Transmission coefficient	[–]	(5.174)
t	Time	[T]	(5.1)
$U^{\varepsilon}_{p,q}$	Product of reflection coefficients	[–]	(5.182)
u_j	Displacement vector	[L]	(5.5)
$\mathbb{V}_{p,q}$	Limiting solution function	[]	(5.214)
$V^{\varepsilon}_{p,q}$	Scaled Fourier transform	[]	(5.190)
$\bar{v}$	Mean shear velocity	[LT^{-1}]	(5.225)
$\mathcal{W}_p$	Transport equation solution	[T^{-1}]	(5.220)
$\widetilde{\mathcal{W}}^{\infty}_{p}$	Non-dimensional transport equation solution	[–]	(5.249)
$\tilde{x}_3$	Non-dimensional space	[–]	(5.249)
λ	Slowness	[TL^{-1}]	(5.181)
$\boldsymbol{\Omega}_M$	Frequency matrix	[]	(5.95)
χ	Length	[L]	(5.190)

5.2 Introduction

This Chapter studies the regime where the propagation distance L is much larger than the wavelength λ_3 of propagation which in turn is much larger than the layer sizes l ($L \gg \lambda_3 \gg l$) and the fluctuations in the material properties are order one. This regime is introduced in Section 5.3. Using the same approach as in Chapter 4 via a set of transport equations, the focus is to study a broadband, high frequency ultrasonic point source and derive an expression for the reflected intensity as a function of the lateral direction x_1.

Section 5.3 introduces the governing equations in the strongly heterogeneous regime for a broadband pulse. Section 5.4 derives expressions for the broadband frequency pulse and then derives an expression for the localisation length - a key parameter in characterising the attenuation of the wave. Sections 5.5 and 5.6 derive transport equations which are then used to obtain an expression for the reflected intensity of the wave as x_1 varies; plots are then presented showing how changing x_1 affects the reflected intensity. An overview of the results are then presented in Section 5.7.

5.3 Governing Elastic Wave Equations in the Strongly Heterogeneous Regime

The governing elastodynamic equation can be written

$$\rho u_{i,tt} = \tau_{jk,k} + \mathbf{F}, \tag{5.1}$$

where the displacement vector u_i in the i^{th} direction is $u(t, x_1, x_3) = \mathbf{u}(t, \mathbf{x})$, the density of the material is ρ (assumed to be constant), τ_{jk} is the material stress tensor and $\mathbf{F}$ is the forcing function. The elastic tensor for a transversely anisotropic medium has five independent stress tensor components (in Voigt notation) namely C_{11}, C_{33}, C_{13}, C_{66}, C_{44} with $\boldsymbol{s} = (s_1, s_2, s_3)$ as the symmetry axis vector defined by

$$\boldsymbol{s} = (\cos\theta(x_3),\, \sin\theta(x_3),\, 0)^{\intercal}, \tag{5.2}$$

where $\theta(x_3)$ is the angle of anisotropy which is related to the slowness surface of the material which varies in x_3 (from layer to layer). The stress tensor can then be written [36] as

$$\begin{aligned} c_{ijkl} =& (A-2N)\delta_{ij}\delta_{kl} + N\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \\ &+ (F-A+2N)\left(\delta_{ij}s_k s_l + \delta_{kl}s_i s_j\right) \\ &+ (S-N)\left(\delta_{ik}s_j s_l + \delta_{il}s_j s_k + \delta_{jk}s_i s_l + \delta_{jl}s_i s_k\right) \\ &+ (A+C-2F-4S)s_i s_j s_k s_l, \end{aligned} \tag{5.3}$$

where $A = C_{33}$, $C = C_{11}$, $F = C_{13}$, $N = C_{44}$, $S = C_{66}$ when $\theta = 0$. The elastic tensor relates the symmetric strain and stress tensors via Hooke's law, where the symmetric strain tensor is given by

$$e_{kl} = (u_{k,l} + u_{l,k})/2. \tag{5.4}$$

Focusing on shear wave propagation in the (x_1, x_3) plane, the wave parameterisation

$$u_j = (0, u_2(x_1, x_3), 0), \tag{5.5}$$

gives the set of governing equations (with $\xi = u_{2,t}$)

$$\tau_{21,t} = c_{66}\xi_{,1}, \tag{5.6}$$

$$\tau_{32,t} = c_{44}\xi_{,3}, \tag{5.7}$$

and

$$\rho\xi_{,t} = \tau_{21,1} + \tau_{32,3} + F_2. \tag{5.8}$$

Given the wave displacement parameterisation (5.5), the only non-zero stresses are

$$c_{2121} = c_{1212} = c_{66}(x_3) = S + (A+C-2F-4S)\cos^2(\theta(x_3))\sin^2(\theta(x_3)), \tag{5.9}$$

$$c_{3223} = c_{3232} = c_{44}(x_3) = N + (S - N)\sin^2(\theta(x_3)). \tag{5.10}$$

Now consider the length scale regime where the received wave exhibits many fluctuations over a long time period caused by its convoluted journey through the heterogeneous medium. In this case the wave is so affected by its interactions with the medium that a homogenisation approach is inappropriate. This can occur when the propagation distance L is much larger than the wavelength (λ_3) of propagation which in turn is much larger than the layer sizes l ($L \gg \lambda_3 \gg l$) and the fluctuations in the material (the percentage change in mechanical impedance from one layer to the next is significant) are order one; the so called strongly heterogeneous regime [26]. In this regime a large propagation distance is required to build up significant multiple scattering. From equations (5.9) and (5.10), define the mean stress tensor components

$$\bar{c}_{66} = S + (A + C - 2F - 4S)\cos^2(\bar{\theta})\sin^2(\bar{\theta}), \tag{5.11}$$

$$\bar{c}_{44} = N + (S - N)\sin^2(\bar{\theta}), \tag{5.12}$$

where $\bar{\theta} \sim 1$. Equation (5.7) suggests that the random variation in the c_{44} elastic tensor component is expressed in terms of its reciprocal via

$$\frac{1}{c_{44}(x_3)} = \begin{cases} \frac{1}{\bar{c}_{44}}(1 + m(x_3/\varepsilon^2)) & \text{if } x_3 \in [-L, 0], \\ \frac{1}{\bar{c}_{44}} & \text{if } x_3 \in (-\infty, -L) \cup (0, \infty). \end{cases} \tag{5.13}$$

and from equation (5.6)

$$c_{66}(x_3) = \begin{cases} \bar{c}_{66}(1 + m(x_3/\varepsilon^2)) & \text{if } x_3 \in [-L, 0], \\ \bar{c}_{66} & \text{if } x_3 \in (-\infty, -L) \cup (0, \infty), \end{cases} \tag{5.14}$$

where a matched medium sandwiches the random medium - that is, the random medium $x_3 \in [-L, 0]$ is sandwiched between two half-spaces (see Figure 5.1) with mean elastic properties $\bar{c}_{44}$, $\bar{c}_{66}$ and constant density ρ. The Markov process is denoted by m (the same form as in Section 2.8.1). The source term is written

$$\mathbf{F}^\varepsilon = \left(0, \varepsilon^q f\left(\frac{t}{\varepsilon}\right)\delta\left(\frac{x_1}{\varepsilon}\right)\delta(x_3), 0\right), \tag{5.15}$$

which is a line source along the x_2 axis. Applying ε-scaled Fourier transforms of the governing equations (5.7) and (5.8) via

$$\hat{\xi}^\varepsilon(\omega, \kappa_1, x_3) = \int_{x_1}\int_t e^{i\frac{\omega}{\varepsilon}(t-\kappa_1 x_1)}\xi^\varepsilon(t, x_1, x_3)\, dt dx_1, \tag{5.16}$$

$$\hat{\tau}_{21}^\varepsilon(\omega, \kappa_1, x_3) = \int_{x_1}\int_t e^{i\frac{\omega}{\varepsilon}(t-\kappa_1 x_1)}\tau_{21}^\varepsilon(t, x_1, x_3)\, dt dx_1, \tag{5.17}$$

$$\hat{\tau}_{32}^\varepsilon(\omega, \kappa_1, x_3) = \int_{x_1}\int_t e^{i\frac{\omega}{\varepsilon}(t-\kappa_1 x_1)}\tau_{32}^\varepsilon(t, x_1, x_3)\, dt dx_1, \tag{5.18}$$

with inverse transforms

$$\xi^\varepsilon(t, x_1, x_3) = \frac{1}{(2\pi\varepsilon)^2}\int_{\kappa_1}\int_\omega e^{-i\frac{\omega}{\varepsilon}(t-\kappa_1 x_1)}\hat{\xi}^\varepsilon(\omega, \kappa_1, x_3)\omega\, d\omega d\kappa_1, \tag{5.19}$$

$$\tau_{21}^\varepsilon(t, x_1, x_3) = \frac{1}{(2\pi\varepsilon)^2}\int_{\kappa_1}\int_\omega e^{-i\frac{\omega}{\varepsilon}(t-\kappa_1 x_1)}\hat{\tau}_{21}^\varepsilon(\omega, \kappa_1, x_3)\omega\, d\omega d\kappa_1, \tag{5.20}$$

$$\tau_{32}^\varepsilon(t, x_1, x_3) = \frac{1}{(2\pi\varepsilon)^2}\int_{\kappa_1}\int_\omega e^{-i\frac{\omega}{\varepsilon}(t-\kappa_1 x_1)}\hat{\tau}_{32}^\varepsilon(\omega, \kappa_1, x_3)\omega\, d\omega d\kappa_1. \tag{5.21}$$

The governing equations become

$$-\frac{i\omega}{\varepsilon}\hat{\tau}_{21} = \frac{i\omega}{\varepsilon}\kappa_1 c_{66}\hat{\xi}^\varepsilon, \tag{5.22}$$

$$-\frac{i\omega}{\varepsilon}\hat{\tau}_{32}^\varepsilon = c_{44}\frac{\partial\hat{\xi}^\varepsilon}{\partial x_3}, \tag{5.23}$$

$$-\frac{i\omega\rho}{\varepsilon}\hat{\xi}^\varepsilon = \frac{i\omega}{\varepsilon}\kappa_1\hat{\tau}_{21}^\varepsilon + \frac{\partial\hat{\tau}_{32}^\varepsilon}{\partial x_3} + \varepsilon\hat{F}(\omega)\delta(x_3), \tag{5.24}$$

where the unscaled Fourier transform is denoted by

$$\hat{f}(\omega) = \int e^{i\omega u} f(u) du. \tag{5.25}$$

Eliminating $\hat{\tau}_{21}^{\varepsilon}$ from equations (5.22) - (5.24) gives the reduced system

$$\frac{\partial \hat{\xi}^{\varepsilon}}{\partial x_3} + \frac{i\omega}{\varepsilon c_{44}} \hat{\tau}_{32}^{\varepsilon} = 0, \tag{5.26}$$

$$\frac{\partial \hat{\tau}_{32}^{\varepsilon}}{\partial x_3} + \frac{i\omega}{\varepsilon} \left(\rho - \kappa_1^2 c_{66} \right) \hat{\xi}^{\varepsilon} = -\varepsilon^{q+2} \hat{f}(\omega) \delta(x_3). \tag{5.27}$$

The continuity of stress and velocity at $x_3 = 0$ produce the jumps defined by

$$[\hat{\tau}_{32}^{\varepsilon}]_0 \coloneqq \hat{\tau}_{32}^{\varepsilon}(\omega, \kappa_1, 0^+) - \hat{\tau}_{32}^{\varepsilon}(\omega, \kappa_1, 0^-) = -\varepsilon \hat{f}(\omega), \tag{5.28}$$

$$[\hat{\xi}^{\varepsilon}]_0 \coloneqq \hat{\xi}^{\varepsilon}(\omega, \kappa_1, 0^+) - \hat{\xi}^{\varepsilon}(\omega, \kappa_1, 0^-) = 0. \tag{5.29}$$

To simplify the algebra

$$\alpha(x_3) = \frac{1}{c_{44}(x_3)}, \tag{5.30}$$

$$\beta(x_3, \kappa_1) = \rho - \kappa_1^2 c_{66}(x_3), \tag{5.31}$$

$$\zeta(x_3, \kappa_1) = \sqrt{\beta(x_3)\alpha(x_3)}, \tag{5.32}$$

are defined. Let

$$\hat{\xi}^{\varepsilon}(\omega, \kappa_1, x_3) = \sqrt{\bar{\zeta}/\bar{\beta}} \left(\hat{b}^{\varepsilon}(\omega, \kappa_1, x_3) e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}} - \hat{a}^{\varepsilon}(\omega, \kappa_1, x_3) e^{\frac{i\omega\bar{\zeta}x_3}{\varepsilon}} \right), \tag{5.33}$$

$$\hat{\tau}_{32}^{\varepsilon}(\omega, \kappa_1, x_3) = \sqrt{\bar{\beta}/\bar{\zeta}} \left(\hat{b}^{\varepsilon}(\omega, \kappa_1, x_3) e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}} + \hat{a}^{\varepsilon}(\omega, \kappa_1, x_3) e^{\frac{i\omega\bar{\zeta}x_3}{\varepsilon}} \right), \tag{5.34}$$

where

$$\bar{\alpha} = \frac{1}{\bar{c}_{44}}, \tag{5.35}$$

$$\bar{\beta} = \rho - \kappa_1^2 \bar{c}_{66}, \tag{5.36}$$

$$\bar{\zeta} = \sqrt{\bar{\beta}\bar{\alpha}}, \tag{5.37}$$

are the effective material properties, with $\bar{c}_{44}$ and $\bar{c}_{66}$ defined in equations (5.12), (5.11) respectively. These expressions will be used in Section 5.6 to derive an expression for the temporal/spatial distribution of the stress at the surface due to the reflected wave.

The right and left-going waves are then

$$\hat{a}^\varepsilon(\omega,\kappa_1,x_3) = \frac{1}{2}\bigg(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}_{32}^\varepsilon - \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon\bigg)e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}}, \tag{5.38}$$

$$\hat{b}^\varepsilon(\omega,\kappa_1,x_3) = \frac{1}{2}\bigg(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}_{32}^\varepsilon + \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon\bigg)e^{\frac{i\omega\bar{\zeta}x_3}{\varepsilon}}. \tag{5.39}$$

Taking derivatives in x_3 and using equations (5.26) and (5.27) gives

$$\begin{aligned}\frac{\partial\hat{a}^\varepsilon}{\partial x_3} &= \frac{1}{2}\bigg(\sqrt{\bar{\zeta}/\beta}\frac{\partial\hat{\tau}_{32}^\varepsilon}{\partial x_3} - \sqrt{\beta/\bar{\zeta}}\frac{\partial\hat{\xi}^\varepsilon}{\partial x_3}\bigg)e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}} - \frac{i\omega\bar{\zeta}}{2\varepsilon}\bigg(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}_{32}^\varepsilon - \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon\bigg)e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}}\\
&= \frac{1}{2}\bigg(-\sqrt{\bar{\zeta}/\bar{\beta}}\frac{i\omega\beta}{\varepsilon}\hat{\xi}^\varepsilon + \sqrt{\bar{\beta}/\bar{\zeta}}\frac{i\omega\alpha}{\varepsilon}\hat{\tau}_{32}^\varepsilon\bigg)e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}} - \frac{i\omega\bar{\zeta}}{2\varepsilon}\bigg(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}_{32}^\varepsilon - \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon\bigg)e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}}\\
&= \frac{i\omega}{2\varepsilon}\bigg(\bigg(-\sqrt{\bar{\zeta}/\bar{\beta}}\beta + \bar{\zeta}\sqrt{\bar{\beta}/\bar{\zeta}}\bigg)\hat{\xi}^\varepsilon + \bigg(\alpha\sqrt{\bar{\beta}/\bar{\zeta}} - \bar{\zeta}\sqrt{\bar{\zeta}/\bar{\beta}}\bigg)\hat{\tau}_{32}^\varepsilon\bigg)e^{-\frac{i\omega\bar{\zeta}x_3}{\varepsilon}}\\
&= \frac{i\omega}{2\varepsilon}\bigg(\gamma_1(x_3)\hat{\xi}^\varepsilon(\omega,\kappa_1,x_3) + \gamma_2(x_3)\hat{\tau}_{32}^\varepsilon(\omega,\kappa_1,x_3)\bigg),\end{aligned} \tag{5.40}$$

where

$$\gamma_1(x_3) = -\sqrt{\bar{\zeta}/\bar{\beta}}\beta + \bar{\zeta}\sqrt{\bar{\beta}/\bar{\zeta}}, \tag{5.41}$$

$$\gamma_2(x_3) = \alpha\sqrt{\bar{\beta}/\bar{\zeta}} - \bar{\zeta}\sqrt{\bar{\zeta}/\bar{\beta}}. \tag{5.42}$$

Using equations (5.33) and (5.34)

$$\begin{aligned}\frac{\partial\hat{a}^\varepsilon}{\partial x_3} &= \frac{i\omega}{2\varepsilon}\bigg(\gamma_1(x_3)\sqrt{\bar{\zeta}/\bar{\beta}}\Big(\hat{b}^\varepsilon e^{-\frac{i\omega\zeta x_3}{\varepsilon}} - \hat{a}^\varepsilon e^{\frac{i\omega\zeta x_3}{\varepsilon}}\Big)\\
&\quad + \gamma_2(x_3)\sqrt{\bar{\beta}/\bar{\zeta}}\Big(\hat{b}^\varepsilon e^{-\frac{i\omega\zeta x_3}{\varepsilon}} + \hat{a}^\varepsilon e^{\frac{i\omega\zeta x_3}{\varepsilon}}\Big)\bigg)e^{-\frac{i\omega\zeta x_3}{\varepsilon}}\\
&= \frac{i\omega}{2\varepsilon}\bigg(\bigg(-\gamma_1\sqrt{\bar{\zeta}/\bar{\beta}} + \gamma_2\sqrt{\bar{\beta}/\bar{\zeta}}\bigg)\hat{a}^\varepsilon + \bigg(\gamma_1\sqrt{\bar{\zeta}/\bar{\beta}} + \gamma_2\sqrt{\bar{\beta}/\bar{\zeta}}\bigg)e^{-\frac{2i\omega\bar{\zeta}x_3}{\varepsilon}}\hat{b}^\varepsilon\bigg),\end{aligned} \tag{5.43}$$

where

$$-\gamma_1\sqrt{\bar{\zeta}/\bar{\beta}} + \gamma_2\sqrt{\bar{\beta}/\bar{\zeta}} = (\bar{\zeta}/\bar{\beta})\beta - \bar{\zeta} + \alpha(\bar{\beta}/\bar{\zeta}) - \bar{\zeta} = \beta\sqrt{\bar{\alpha}/\bar{\beta}} + \alpha\sqrt{\bar{\beta}/\bar{\alpha}} - 2\sqrt{\bar{\alpha}\bar{\beta}} = \Delta_1(x_3), \tag{5.44}$$

and

$$\Big(\gamma_1\sqrt{\bar\zeta/\bar\beta}+\gamma_2\sqrt{\bar\beta/\bar\zeta}\Big)e^{-\frac{2i\omega\bar\zeta x_3}{\varepsilon}} = \Big(-\beta(\bar\zeta/\bar\beta)+\bar\zeta+\alpha(\bar\beta/\bar\zeta)-\bar\zeta\Big)e^{-\frac{2i\omega\bar\zeta x_3}{\varepsilon}}$$
$$= \Big(-\beta\sqrt{\bar\alpha/\bar\beta}+\alpha\sqrt{\bar\beta/\bar\alpha}\Big)e^{-\frac{2i\omega\sqrt{\bar\alpha\bar\beta}}{\varepsilon}}$$
$$= \Delta_2(x_3)e^{-\frac{2i\omega\bar\zeta}{\varepsilon}}. \tag{5.45}$$

Then equation (5.43) can be written

$$\frac{\partial \hat a^\varepsilon}{\partial x_3} = \frac{i\omega}{2\varepsilon}\Big(\Delta_1(x_3)\hat a^\varepsilon + \Delta_2(x_3)e^{-\frac{2i\omega\bar\zeta x_3}{\varepsilon}}\hat b^\varepsilon\Big). \tag{5.46}$$

Similarly, the backward mode evolution equation is obtained from equation (5.39) and equations (5.26), (5.27), (5.33), (5.34)

$$\begin{aligned}
\frac{\partial \hat b^\varepsilon}{\partial x_3} &= \frac{1}{2}\Big(\sqrt{\bar\zeta/\bar\beta}\frac{\partial \hat\tau_{32}^\varepsilon}{\partial x_3} + \sqrt{\bar\beta/\bar\zeta}\frac{\partial \hat\xi^\varepsilon}{\partial x_3}\Big)e^{\frac{i\omega\bar\zeta x_3}{\varepsilon}} + \frac{i\omega\bar\zeta}{2\varepsilon}\Big(\sqrt{\bar\zeta/\bar\beta}\hat\tau_{32}^\varepsilon + \sqrt{\bar\beta/\bar\zeta}\hat\xi^\varepsilon\Big)e^{\frac{i\omega\bar\zeta x_3}{\varepsilon}} \\
&= \frac{1}{2}\Big(-\sqrt{\bar\zeta/\bar\beta}\frac{i\omega\beta}{\varepsilon}\hat\xi^\varepsilon - \sqrt{\bar\beta/\bar\zeta}\frac{i\omega\alpha}{\varepsilon}\hat\tau_{32}^\varepsilon + \frac{i\omega\bar\zeta}{\varepsilon}\Big(\sqrt{\bar\zeta/\bar\beta}\hat\tau_{32}^\varepsilon + \sqrt{\bar\beta/\bar\zeta}\hat\xi^\varepsilon\Big)e^{\frac{i\omega\bar\zeta x_3}{\varepsilon}}\Big) \\
&= \frac{i\omega}{2\varepsilon}\Big(\Big(\bar\zeta\sqrt{\bar\beta/\bar\zeta} - \beta\sqrt{\bar\zeta/\bar\beta}\Big)\hat\xi^\varepsilon + \Big(\bar\zeta\sqrt{\bar\zeta/\bar\beta} - \alpha\sqrt{\bar\beta/\bar\zeta}\Big)\hat\tau_{32}^\varepsilon\Big)e^{\frac{i\omega\bar\zeta x_3}{\varepsilon}} \\
&= \frac{i\omega}{2\varepsilon}\Big(\gamma_1(x_3)\hat\xi^\varepsilon - \gamma_2(x_3)\hat\tau_{32}^\varepsilon\Big)e^{\frac{i\omega\bar\zeta x_3}{\varepsilon}} \\
&= \frac{i\omega}{2\varepsilon}\Big(\gamma_1\sqrt{\bar\zeta/\bar\beta}\Big(\hat b^\varepsilon e^{-\frac{i\omega\bar\zeta x_3}{\varepsilon}} - \hat a^\varepsilon e^{\frac{i\omega\bar\zeta x_3}{\varepsilon}}\Big) - \gamma_2\sqrt{\bar\beta/\bar\zeta}\Big(\hat b^\varepsilon e^{-\frac{i\omega\bar\zeta x_3}{\varepsilon}} + \hat a^\varepsilon e^{\frac{i\omega\bar\zeta}{\varepsilon}}\Big)\Big)e^{\frac{i\omega\bar\zeta x_3}{\varepsilon}} \\
&= \frac{i\omega}{2\varepsilon}\Big(\Big(-\gamma_1\sqrt{\bar\zeta/\bar\beta} - \gamma_2\sqrt{\bar\beta/\bar\zeta}\Big)\hat a^\varepsilon e^{\frac{2i\omega\bar\zeta x_3}{\varepsilon}} + \Big(\gamma_1\sqrt{\bar\zeta/\bar\beta} - \gamma_2\sqrt{\bar\beta/\bar\zeta}\Big)\hat b^\varepsilon\Big) \\
&= \frac{i\omega}{2\varepsilon}\Big(-\Delta_2(x_3)\hat a^\varepsilon e^{\frac{2i\omega\bar\zeta x_3}{\varepsilon}} - \Delta_1(x_3)\hat b^\varepsilon\Big).
\end{aligned} \tag{5.47}$$

The continuity of stress and velocity at $x_3 = 0$ produce the jumps defined by

$$[\hat\tau_{32}^\varepsilon]_0 \coloneqq \hat\tau_{32}^\varepsilon(\omega,\kappa_1,0^+) - \hat\tau_{32}^\varepsilon(\omega,\kappa_1,0^-) = -\varepsilon\hat f(\omega), \tag{5.48}$$

$$[\hat\xi^\varepsilon]_0 \coloneqq \hat\xi^\varepsilon(\omega,\kappa_1,0^+) - \hat\xi^\varepsilon(\omega,\kappa_1,0^-) = 0. \tag{5.49}$$

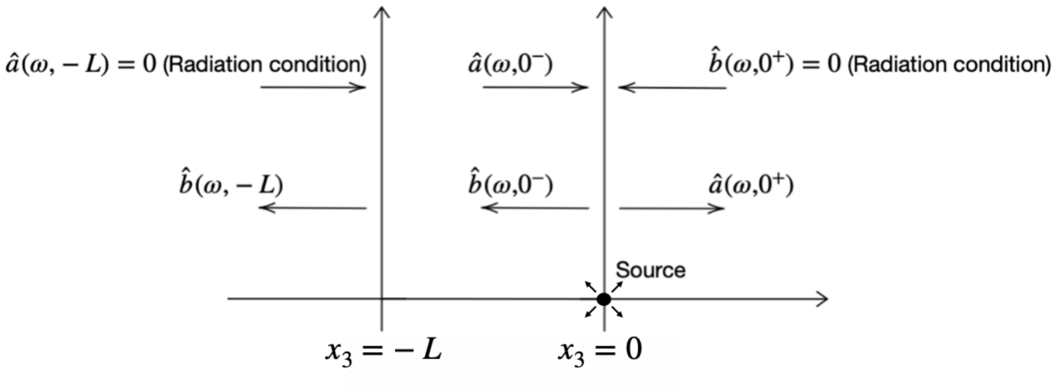

Figure 5.1: Boundary conditions for the $\hat{a}^\varepsilon$, $\hat{b}^\varepsilon$ wave modes in the random medium $x_3 \in [-L, 0]$, where the source is $\mathbf{F}$ at $x_3 = x_1 = 0$.

From equation (5.38) the jump at $x_3 = 0$ in the right-going mode is

$$\begin{aligned}
[\hat{a}^\varepsilon]_{x_3=0} &= \hat{a}^\varepsilon(\omega, \kappa_1, 0^+) - \hat{a}^\varepsilon(\omega, \kappa_1, 0^-) \\
&= \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}^\varepsilon_{32}(0^+) - \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon(0^+)\right) - \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}^\varepsilon_{32}(0^-) - \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon(0^-)\right) \\
&= \frac{1}{2}\sqrt{\bar{\zeta}/\bar{\beta}}[\tau^\varepsilon_{32}]_{x_3=0} - \frac{1}{2}\sqrt{\bar{\beta}/\bar{\zeta}}[\hat{\xi}^\varepsilon]_{x_3=0}. \qquad (5.50)
\end{aligned}$$

Using equations (5.48) and (5.49) then

$$[\hat{a}^\varepsilon]_{x_3=0} = -\frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}. \qquad (5.51)$$

Similarly, from equation (5.39), the jump in the left-going mode is

$$\begin{aligned}
[\hat{b}^\varepsilon]_{x_3=0} = \hat{b}^\varepsilon(0^+) - \hat{b}^\varepsilon(0^-) &= \frac{1}{2}\left(\sqrt{\bar{\zeta}/\bar{\beta}}[\hat{\tau}^\varepsilon_{32}]_0 + \sqrt{\bar{\beta}/\bar{\zeta}}[\hat{\xi}^\varepsilon]_0\right) \qquad (5.52) \\
&= -\frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}. \qquad (5.53)
\end{aligned}$$

Hence, by imposing a radiation condition of $\hat{b}^\varepsilon(\omega, 0^+) = 0$ in the right-hand half-space (see Figure 5.1)

$$\hat{b}^\varepsilon(0^-) = \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}. \qquad (5.54)$$

Similarly, a radiation conition for the left-going waves in the left-hand half-space is

$$\hat{a}^{\varepsilon}(-L) = 0. \tag{5.55}$$

From equations (5.46) and (5.47) one may write the wave-mode evolution system as

$$\begin{aligned}\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{a}^{\varepsilon}(\omega,\kappa_1,x_3)\\ \hat{b}^{\varepsilon}(\omega,\kappa_1,x_3)\end{bmatrix} &= \frac{i\omega}{2\varepsilon}\begin{bmatrix}\Delta_1 & \Delta_2 e^{-2i\omega\bar{\zeta}x_3/\varepsilon}\\ -\Delta_2 e^{2i\omega\bar{\zeta}x_3/\varepsilon} & -\Delta_1\end{bmatrix}\begin{bmatrix}\hat{a}^{\varepsilon}(\omega,\kappa_1,x_3)\\ \hat{b}^{\varepsilon}(\omega,\kappa_1,x_3)\end{bmatrix}\\ &= \mathbf{H}^{\varepsilon}(\omega,\kappa_1,x_3)\begin{bmatrix}\hat{a}^{\varepsilon}(\omega,\kappa_1,x_3)\\ \hat{b}^{\varepsilon}(\omega,\kappa_1,x_3)\end{bmatrix},\end{aligned} \tag{5.56}$$

where $\mathrm{Tr}[\mathbf{H}^{\varepsilon}] = 0$. The propagator matrix takes the form

$$\mathbf{P}^{\varepsilon}(\omega,\kappa_1,x_3) = \begin{bmatrix}\chi_1^{\varepsilon}(\omega,\kappa_1,x_3) & \overline{\chi_2^{\varepsilon}(\omega,\kappa_1,x_3)}\\ \chi_2^{\varepsilon}(\omega,\kappa_1,x_3) & \overline{\chi_1^{\varepsilon}(\omega,\kappa_1,x_3)}\end{bmatrix}, \quad \mathbf{P}^{\varepsilon}(x_3 = -L) = \mathbf{I}, \quad |\chi_1^{\varepsilon}| - |\chi_2^{\varepsilon}| = 1, \tag{5.57}$$

and

$$\begin{bmatrix}\hat{a}^{\varepsilon}(\omega,\kappa_1,x_3)\\ \hat{b}^{\varepsilon}(\omega,\kappa_1,x_3)\end{bmatrix} = \mathbf{P}^{\varepsilon}(\omega,\kappa_1,x_3)\begin{bmatrix}\hat{a}^{\varepsilon}(\omega,\kappa_1,-L)\\ \hat{b}^{\varepsilon}(\omega,\kappa_1,-L)\end{bmatrix}. \tag{5.58}$$

Note that the propagator matrix takes the solution at $x_3 = -L$ to some $x_3 \geq -L$. This is the same as Section 4.3.3 and, as the source is not at $x_3 = -L$ but rather $x_3 = 0$, it models wave motion moving towards the source (see discussion in Section 2.7). In the next section this framework will be used to study the multi-frequency version to facilitate the study of the autocorrelation of the reflection coefficients that are in the subsequent sections. From equations (5.54) and (5.55) then

$$\begin{bmatrix}\hat{a}^{\varepsilon}(0^-)\\ \frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}\end{bmatrix} = \begin{bmatrix}\chi_1^{\varepsilon}(0) & \overline{\chi_2^{\varepsilon}(0)}\\ \chi_2^{\varepsilon}(0) & \overline{\chi_1^{\varepsilon}(0)}\end{bmatrix}\begin{bmatrix}0\\ \hat{b}^{\varepsilon}(-L)\end{bmatrix}. \tag{5.59}$$

In Section 5.5 this will be used to establish forms for the reflection and transmission coefficients. Hence

$$\begin{aligned}\hat{b}^{\varepsilon}(\omega,\kappa_1,-L) &= \frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2\overline{\chi_1^{\varepsilon}(0)}} \\ &= \frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}T^{\varepsilon}_{\omega,\kappa_1}(\omega,\kappa_1,x_3), \end{aligned} \tag{5.60}$$

where the transmission coefficient is defined as

$$T^{\varepsilon}_{\omega,\kappa_1}(\omega,\kappa_1,x_3) = \frac{1}{\overline{\chi_1^{\varepsilon}(\omega,\kappa_1,x_3)}}, \tag{5.61}$$

and so equations (5.57) to (5.60) imply the mode energy conservation relation

$$\begin{aligned}\left|\hat{a}^{\varepsilon}(0^-)\right|^2 + \left|\hat{b}^{\varepsilon}(-L)\right|^2 &= \hat{a}^{\varepsilon}(0^-)\overline{\hat{a}^{\varepsilon}(0^-)} + \hat{b}^{\varepsilon}(-L)\overline{\hat{b}^{\varepsilon}(-L)} \\ &= \left(\frac{\overline{\chi_2^{\varepsilon}(0)}\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2\overline{\chi_1^{\varepsilon}(0)}}\right)\left(\frac{\chi_2^{\varepsilon}(0)\varepsilon\overline{\hat{f}(\omega)}\sqrt{\bar{\zeta}/\bar{\beta}}}{2\chi_1^{\varepsilon}(0)}\right) \\ &+ \left(\frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2\overline{\chi_1^{\varepsilon}(0)}}\right)\left(\frac{\varepsilon\overline{\hat{f}(\omega)}\sqrt{\bar{\zeta}/\bar{\beta}}}{2\chi_1^{\varepsilon}(0)}\right) \\ &= \left(\frac{\overline{\chi_2^{\varepsilon}(0)}\chi_2^{\varepsilon}(0)}{\overline{\chi_1^{\varepsilon}(0)}\chi_1^{\varepsilon}(0)} + \frac{1}{\overline{\chi_1^{\varepsilon}(0)}\chi_1^{\varepsilon}(0)}\right)\left|\frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}\right|^2 \\ &= \left(\frac{|\chi_2^{\varepsilon}(0)|^2+1}{|\chi_1^{\varepsilon}(0)|^2}\right)\left|\frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}\right|^2. \end{aligned} \tag{5.62}$$

Using equation (5.57) implies that

$$\left|\hat{a}^{\varepsilon}(0^-)\right|^2 + \left|\hat{b}^{\varepsilon}(-L)\right|^2 = \left|\frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}\right|^2. \tag{5.63}$$

From equations (5.33), (5.55) and (5.60) then the transmitted velocity at $x_3 = -L$ is

$$\hat{\xi}^{\varepsilon}(\omega,\kappa_1,-L) = \frac{1}{2}\frac{\bar{\zeta}(\kappa_1)}{\bar{\beta}(\kappa_1)}\varepsilon\hat{f}(\omega)e^{-i\omega\bar{\zeta}(\kappa_1)L/\varepsilon}T^{\varepsilon}_{\omega,\kappa_1}(-L,0), \tag{5.64}$$

and from equation (5.34) the transmitted stress is

$$\hat{\tau}_{32}^{\varepsilon}(\omega, \kappa_1, -L) = \frac{1}{2}\varepsilon \hat{f}(\omega) e^{-i\omega\bar{\zeta}(\kappa_1)L/\varepsilon}\, T_{\omega,\kappa_1}^{\varepsilon}(-L, 0). \tag{5.65}$$

Now taking inverse Fourier transforms in time and space using

$$\xi^{\varepsilon}(t, x_1, x_3) = \frac{1}{(2\pi\varepsilon)^2} \int\int e^{-i\omega(t-\kappa_1 x_1)/\varepsilon}\, \hat{\xi}^{\varepsilon}(\omega, \kappa_1, x_3)\omega \, d\omega d\kappa_1, \tag{5.66}$$

with $t = t_0 + \varepsilon s$, being a time window centered on t_0 (t_0 is the mean arrival time given by $t_0 = L_3/\bar{v}$ where $\bar{v}$ is the mean shear velocity given by $\bar{v} = \sqrt{\bar{c}_{44}/\rho}$) and on the scale ε of the source pulse, with s being the time variable, then

$$\begin{aligned}
\xi^{\varepsilon}(t_0 + \varepsilon s, x_1, -L) = \frac{1}{(2\pi\varepsilon)^2} \int_{\omega}\int_{\kappa_1} & e^{-i\omega(t_0+\varepsilon s-\kappa_1 x_1-\bar{\zeta}(\kappa_1)L)/\varepsilon}\, T_{\omega,\kappa_1}^{\varepsilon}(-L, 0) \\
& \times \left(\frac{1}{2}\frac{\bar{\zeta}(\kappa_1)}{\bar{\beta}(\kappa_1)}\varepsilon \hat{f}(\omega)\right)\omega d\omega d\kappa_1 \\
& = \int_{\omega}\int_{\kappa_1} \Psi(\omega, \kappa_1, s) d\omega d\kappa_1,
\end{aligned} \tag{5.67}$$

noting that there is no x_2 dependency and hence, only a double integral appears. Also, the prefactor is $1/(2\pi\varepsilon)^2$ and the integrand has a factor of ω.

5.4 A Multifrequency Formulation for a Broadband Pulse Source

This Section focuses on studying the multi-frequency stochastic equations governing wave propagation through the random medium. Since the source will be a broadband pulse, it is necessary to study a discrete set of wavelength and slowness components, where each component has an associated propagator equation. This formulation is key to studying the frequency autocorrelation function for the reflected energy. Equations (5.11), (5.14), (5.31) and (5.36) give

$$\beta(x_3, \kappa_1) = \rho - \kappa_1^2 c_{66}(x_3)$$

$$
\begin{aligned}
&= \rho - \kappa_1^2 \bar{c}_{66}(1 + m(x_3/\varepsilon^2)) \\
&= \rho - \kappa_1^2 \bar{c}_{66} - \kappa_1^2 \bar{c}_{66} m(x_3/\varepsilon^2) \\
&= \bar{\beta} - \kappa_1^2 \bar{c}_{66} m(x_3/\varepsilon^2). \qquad (5.68)
\end{aligned}
$$

Equations (5.12), (5.13), (5.30) and (5.35) give

$$
\begin{aligned}
\alpha(x_3) &= \frac{1}{\bar{c}_{44}}(1 + m(x_3/\varepsilon^2)) \\
&= \bar{\alpha}(1 + m(x_3/\varepsilon^2)). \qquad (5.69)
\end{aligned}
$$

Equations (5.35), (5.36) and (5.44) give

$$
\begin{aligned}
\Delta_1(x_3, \kappa_1) &= \sqrt{\bar{\alpha}/\bar{\beta}}\left(\bar{\beta} - \kappa_1^2 \bar{c}_{66} m(x_3/\varepsilon^2)\right) + \sqrt{\bar{\beta}/\bar{\alpha}}\bar{\alpha}(1 + m(x_3/\varepsilon^2)) - 2\sqrt{\bar{\alpha}\bar{\beta}} \\
&= \sqrt{\bar{\alpha}/\bar{\beta}}\left(\bar{\beta} - \kappa_1^2 \bar{c}_{66}\right) m(x_3/\varepsilon^2) \\
&= \sqrt{\bar{\alpha}/\bar{\beta}}(\rho - 2\kappa_1^2 \bar{c}_{66}) m(x_3/\varepsilon^2) \\
&= \Gamma_1(\kappa_1) m(x_3/\varepsilon^2), \qquad (5.70)
\end{aligned}
$$

where

$$
\Gamma_1(\kappa_1) = \sqrt{\bar{\alpha}/\bar{\beta}}(\rho - 2\kappa_1^2 \bar{c}_{66}), \qquad (5.71)
$$

which has dimensions of slowness. Equations (5.35), (5.36), (5.45), (5.68) and (5.69) give

$$
\begin{aligned}
\Delta_2(x_3) &= -\sqrt{\bar{\alpha}/\bar{\beta}}(\bar{\beta} - \kappa_1^2 \bar{c}_{66} m(x_3/\varepsilon^2)) + \sqrt{\bar{\beta}/\bar{\alpha}}\bar{\alpha}(1 + m(x_3/\varepsilon^2)) \\
&= -\sqrt{\bar{\alpha}\bar{\beta}} + \sqrt{\bar{\alpha}/\bar{\beta}}\kappa_1^2 \bar{c}_{66} m(x_3/\varepsilon^2) + \sqrt{\bar{\alpha}\bar{\beta}} + \sqrt{\bar{\alpha}\bar{\beta}} m(x_3/\varepsilon^2) \\
&= \sqrt{\bar{\alpha}/\bar{\beta}}(\kappa_1^2 \bar{c}_{66} + \bar{\beta}) m(x_3/\varepsilon^2) \\
&= \Gamma_2(\kappa_1) m(x_3/\varepsilon^2), \qquad (5.72)
\end{aligned}
$$

where

$$\Gamma_2(\kappa_1) = \rho\sqrt{\bar{\alpha}/\bar{\beta}}. \tag{5.73}$$

In this multi-frequency problem, for each discrete frequency ω_j where $j = 1, \ldots, m$ there is a corresponding set of discrete slowness values $\kappa_1^{j,l}$ where $l = 1, \ldots, n$. To simplify the notation, introduce the set of indices $i = (j, l)$ where $j = 1, \ldots, m$ and $l = 1, \ldots, n = [(1,1), (2,1), \ldots, (m,1), (1,2), (2,2), \ldots, (1,n), (2,n), \ldots, (m,n)]$ where $|i| = mn$. Equation (5.56) can be written for each $(\omega_i,\, \kappa_1^i)$ frequency-slowness pair as $i = 1, \ldots$ as

$$\begin{aligned}
\frac{\partial}{\partial x_3}\begin{bmatrix}\hat{a}^\varepsilon(x_3,\kappa_1^i,\omega_i)\\ \hat{b}^\varepsilon(x_3,\kappa_1^i,\omega_i)\end{bmatrix} &= \frac{i\omega_i}{2\varepsilon} m(x_3/\varepsilon^2)\begin{bmatrix}\Gamma_1(\kappa_1^i) & \Gamma_2(\kappa_1^i)e^{-2i\omega_i\bar{\zeta}(\kappa_1^i)x_3/\varepsilon}\\ -\Gamma_2(\kappa_1^i)e^{2i\omega_i\bar{\zeta}(\kappa_1^i)x_3/\varepsilon} & -\Gamma_1(\kappa_1^i)\end{bmatrix}\\
&\quad\times\begin{bmatrix}\hat{a}^\varepsilon(x_3,\kappa_1^i,\omega_i)\\ \hat{b}^\varepsilon(x_3,\kappa_1^i,\omega_i)\end{bmatrix}\\
&= \frac{1}{\varepsilon}\mathbf{H}^\varepsilon_{(\omega_i,\kappa_1^i)}\left(\frac{x_3}{\varepsilon}, m\left(\frac{x_3}{\varepsilon^2}\right)\right)\begin{bmatrix}\hat{a}^\varepsilon(x_3,\kappa_1^i,\omega_i)\\ \hat{b}^\varepsilon(x_3,\kappa_1^i,\omega_i)\end{bmatrix},
\end{aligned} \tag{5.74}$$

where

$$\mathbf{H}^\varepsilon_{(\omega_i,\kappa_1^i)}\left(\frac{x_3}{\varepsilon}, m\left(\frac{x_3}{\varepsilon^2}\right)\right) = \begin{bmatrix}\Gamma_1(\kappa_1^i) & \Gamma_2(\kappa_1^i)e^{-2i\omega_i\bar{\zeta}(\kappa_1^i)x_3/\varepsilon}\\ -\Gamma_2(\kappa_1^i)e^{2i\omega_i\bar{\zeta}(\kappa_1^i)x_3/\varepsilon} & -\Gamma_1(\kappa_1^i)\end{bmatrix}. \tag{5.75}$$

The propagator matrix $\mathbf{P}(\omega^i, \kappa_1^i, x_3)$ in equation (5.57) satisfies equation (5.74), hence

$$\begin{aligned}
\frac{\partial}{\partial x_3}\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3) &= \frac{1}{\varepsilon}\mathbf{H}^\varepsilon_{\omega^i,\kappa_1^i}\left(\frac{x_3}{\varepsilon}, m\left(\frac{x_3}{\varepsilon^2}\right)\right)\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3)\\
&= \frac{i\omega^i}{2\varepsilon}m(x_3/\varepsilon^2)\Gamma_1(\kappa_1^i)\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3)\\
&\quad+\frac{\omega^i}{2\varepsilon}m(x_3/\varepsilon^2)\Gamma_2(\kappa_1^i)\sin\left(\frac{2\omega^i\bar{\zeta}(\kappa_1^i)x_3}{\varepsilon}\right)\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3)
\end{aligned}$$

$$
-\frac{i\omega^i}{2\varepsilon} m(x_3/\varepsilon^2)\Gamma_2(\kappa_1^i)\cos\left(\frac{2\omega^i\bar{\zeta}(\kappa_1^i)x_3}{\varepsilon}\right)\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3)
$$

$$
=\frac{1}{\varepsilon}\mathbf{G}\bigg(\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3), m(x_3/\varepsilon^2), x_3/\varepsilon\bigg), \tag{5.76}
$$

where $\tau = x_3/\varepsilon$, so

$$
\mathbf{G}\bigg(\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3), m(x_3/\varepsilon^2), x_3/\varepsilon\bigg) = \frac{\omega^i}{2}\sum_{p=0}^{2} g_i^{(p)}\bigg(m(x_3/\varepsilon^2), x_3/\varepsilon\bigg)\mathbf{h}_p\mathbf{P}^\varepsilon(\omega^i,\kappa_1^i,x_3), \tag{5.77}
$$

where $\boldsymbol{\sigma}_{0,1,2}$ are the Pauli spin matrices

$$
\boldsymbol{\sigma}_1 = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{bmatrix}0 & -i\\ i & 0\end{bmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}, \tag{5.78}
$$

where

$$
\mathbf{h}_n = \begin{bmatrix} i\boldsymbol{\sigma}_3 \\ \boldsymbol{\sigma}_1 \\ -\boldsymbol{\sigma}_2 \end{bmatrix} \quad \mathbf{g}_i^{(n)}(m,\tau) = \begin{bmatrix} m\Gamma_1(\kappa_1^i) \\ m\Gamma_2(\kappa_1^i)\sin\left(2\omega^i\bar{\zeta}(\kappa_1^i)\tau\right) \\ m\Gamma_2(\kappa_1^i)\cos\left(2\omega^i\bar{\zeta}(\kappa_1^i)\tau\right) \end{bmatrix}, \tag{5.79}
$$

where $j = 1,\ldots,m_l$, $l = 1,\ldots,n$ and $M = \sum_{l=1}^{n} m_n$. To deal with the multi-frequency problem in full, a $2M \times 2M$ multi-frequency, multi-slowness propagator matrix is introduced via

$$
\mathbf{P}_M^\varepsilon(x_3) = \begin{bmatrix} \mathbf{P}^\varepsilon(\omega_1,\kappa_1^1,x_3) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \ldots & \mathbf{0} & \mathbf{P}^\varepsilon(\omega_M,\kappa_1^M,x_3) \end{bmatrix}
$$

$$
= \begin{bmatrix} \mathbf{P}_1^\varepsilon & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{P}_M^\varepsilon \end{bmatrix}. \tag{5.80}
$$

Then system (5.76) can be written as

$$
\frac{\partial}{\partial x_3}\mathbf{P}_M^\varepsilon(x_3) = \begin{bmatrix} \mathbf{U} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{V} \end{bmatrix}, \tag{5.81}
$$

where

$$
\begin{aligned}
\mathbf{U} = {} & m\Gamma_1(\kappa_1^1)\omega_1\mathbf{h}_0\mathbf{P}_1^\varepsilon + m\Gamma_2(\kappa_1^1)\sin\bigl(2\omega_1\bar{\zeta}x_3/\varepsilon\bigr)\omega_1\mathbf{h}_1\mathbf{P}_1^\varepsilon \\
& + m\Gamma_2(\kappa_1^1)\cos\bigl(2\omega_1\bar{\zeta}x_3/\varepsilon\bigr)\omega_1\mathbf{h}_2\mathbf{P}_1^\varepsilon,
\end{aligned} \tag{5.82}
$$

and

$$
\begin{aligned}
\mathbf{V} = {} & m\Gamma_1(\kappa_1^M)\omega_M\mathbf{h}_0\mathbf{P}_1^\varepsilon + m\Gamma_2(\kappa_1^M)\sin\bigl(2\omega_M\bar{\zeta}x_3/\varepsilon\bigr)\omega_M\mathbf{h}_1\mathbf{P}_M^\varepsilon \\
& + m\Gamma_2(\kappa_1^1)\cos\bigl(2\omega_M\bar{\zeta}x_3/\varepsilon\bigr)\omega_M\mathbf{h}_2\mathbf{P}_M^\varepsilon.
\end{aligned} \tag{5.83}
$$

That is

$$
\begin{aligned}
\frac{\partial}{\partial x_3}\mathbf{P}_M^\varepsilon(x_3) = {} & \frac{1}{2\varepsilon}\begin{bmatrix} m\Gamma_1(\kappa_1^1)\omega_1\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{h}_0 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix}\mathbf{P}_M^\varepsilon(x_3) \\
& + \frac{1}{2\varepsilon}\begin{bmatrix} m\Gamma_2(\kappa_1^1)\sin\bigl(\frac{2\omega_1\bar{\zeta}x_3}{\varepsilon}\bigr)\omega_1\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{h}_1 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix}\mathbf{P}_M^\varepsilon(x_3)
\end{aligned}
$$

$$
+\frac{1}{2\varepsilon}\begin{bmatrix} m\Gamma_2(\kappa_1^1)\cos\big(\frac{2\omega_1\bar{\zeta}x_3}{\varepsilon}\big)\omega_1\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix}\begin{bmatrix} \mathbf{h}_2 & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix}\mathbf{P}_M^{\varepsilon}(x_3)
$$

$$
+\dots
$$

$$
+\frac{1}{2\varepsilon}\begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & m\Gamma_1(\kappa_1^1)\omega_1\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{h}_0 \end{bmatrix}\mathbf{P}_M^{\varepsilon}(x_3)
$$

$$
+\frac{1}{2\varepsilon}\begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & m\Gamma_2(\kappa_1^M)\sin\big(\frac{2\omega_M\bar{\zeta}x_3}{\varepsilon}\big)\omega_M\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{h}_1 \end{bmatrix}\mathbf{P}_M^{\varepsilon}(x_3)
$$

$$
+\frac{1}{2\varepsilon}\begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & m\Gamma_2(\kappa_1^M)\cos\big(\frac{2\omega_M\bar{\zeta}x_3}{\varepsilon}\big)\omega_M\mathbf{I} \end{bmatrix}\begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{h}_2 \end{bmatrix}\mathbf{P}_M^{\varepsilon}(x_3), \tag{5.84}
$$

which can be written succinctly as

$$
\mathbf{A} = \operatorname{diag}(\mathbf{U}_1, \dots, \mathbf{U}_M)\,, \tag{5.85}
$$

where

$$
\begin{aligned} \mathbf{U}_j = {} & m\Gamma_1(\kappa_1^j)\omega_j\mathbf{h}_0\mathbf{P}_j^{\varepsilon} \\ & + m\Gamma_2(\kappa_1^j)\sin\big(2\omega_j\bar{\zeta}x_3/\varepsilon\big)\omega_j\mathbf{h}_1\mathbf{P}_j^{\varepsilon} \\ & + m\Gamma_2(\kappa_1^j)\cos\big(2\omega_j\bar{\zeta}x_3/\varepsilon\big)\omega_j\mathbf{h}_2\mathbf{P}_j^{\varepsilon}, \end{aligned} \tag{5.86}
$$

for $j = 1, \dots, M$. That is

$$\frac{\partial}{\partial x_3}\mathbf{P}^{\varepsilon}_M(x_3) = \frac{1}{2\varepsilon}\begin{bmatrix} \omega_1\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ \mathbf{0} & \dots & \dots & \mathbf{0} \end{bmatrix}\left(g^{(0)}\mathbf{h}^{(0)} + g^{(1)}\mathbf{h}^{(1)} + g^{(2)}\mathbf{h}^{(2)}\right)\mathbf{P}^{\varepsilon}_M(x_3)$$

$$+ \dots \tag{5.87}$$

$$+ \frac{1}{2\varepsilon}\begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \omega_M\mathbf{I} \end{bmatrix}\left(g^{(3M)}\mathbf{h}^{(3M)} + g^{(3M+1)}\mathbf{h}^{(3M+1)} + g^{(3M+2)}\mathbf{h}^{(3M+2)}\right),$$

where

$$\mathbf{h}^{(3j)} = \begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{h}_0 & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{cases} i & \text{if } m = n = 2j+1, \\ -i & \text{if } m = n = 2j+2, \\ 0 & \text{otherwise,} \end{cases} \tag{5.88}$$

$$\mathbf{h}^{(3j+1)} = \begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{h}_1 & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{cases} 1 & \text{if } m = 2j+1,\ n = 2j+2, \\ 1 & \text{if } m = 2j+2,\ n = 2j+1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.89}$$

$$\mathbf{h}^{(3j+2)} = \begin{bmatrix} \mathbf{0} & \dots & \dots & \mathbf{0} \\ \vdots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \mathbf{h}_2 & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} \end{bmatrix} = \begin{cases} i & \text{if } m = 2j+1,\ n = 2j+2, \\ -i & \text{if } m = 2j+2,\ n = 2j+1, \\ 0 & \text{otherwise,} \end{cases} \tag{5.90}$$

and so

$$g^{(3j)}(m,\tau) = m\Gamma_1(\kappa_1^j), \tag{5.91}$$

$$g^{(3j+1)}(m,\tau) = m\Gamma_2(\kappa_1^j)\sin\left(2\omega_j\bar{\zeta}(\kappa_1^j)\tau\right), \tag{5.92}$$

$$g^{(3j+2)}(m,\tau) = m\Gamma_2(\kappa_1^j)\cos(2\omega_j\bar{\zeta}(\kappa_1^j)\tau), \tag{5.93}$$

where $m, n = 1, \dots, 2M$. Hence

$$\frac{\partial}{\partial x_3}\mathbf{P}_M^{\varepsilon}(x_3) = \frac{1}{2\varepsilon}\sum_{p=0}^{3M+2}\mathbf{\Omega}_M g^{(p)}(m,\tau)\mathbf{h}^{(p)}\mathbf{P}_M^{\varepsilon}(x_3), \tag{5.94}$$

where

$$\mathbf{\Omega}_M = \begin{bmatrix} \omega_1\mathbf{I} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \omega_2\mathbf{I} & \ddots & \vdots \\ \vdots & \ddots & \ddots & \mathbf{0} \\ \mathbf{0} & \dots & \mathbf{0} & \omega_M\mathbf{I} \end{bmatrix}. \tag{5.95}$$

The correlation matrix $\mathbf{C} = (C_{p,q})_{p,q=0,\dots,3M+2}$ is (from [26] pp. 157)

$$C_{pq} = 2\lim_{Z_0\to 0}\frac{1}{Z_0}\int_0^{Z_0}\int_0^{\infty}\mathbf{E}\Big[g^{(p)}(m(0),\tau)g^{(q)}(m(x_3),\tau)\Big]dx_3 d\tau, \tag{5.96}$$

which is diagonal given the orthogonality of $\cos(2\omega_j\bar{\zeta}(\kappa_1^j)\tau)$ and $\sin(2\omega_j\bar{\zeta}(\kappa_1^j)\tau)$ and the periodicity of Z_0. Now calculate the elements of the correlation matrix

$$\begin{aligned} C_{00} &= 2\lim_{Z_0\to\infty}\frac{1}{Z_0}\int_0^{\infty}\mathbf{E}\Big[m(0)\Gamma_1(\kappa_1^0)m(x_3)\Gamma_1(\kappa_1^0)\Big]dx_3 d\tau \\ &= 2\Gamma_1^2(\kappa_1^0)\int_0^{\infty}\mathbf{E}[m(0)m(x_3)]dx_3 \\ &= \Gamma_1^2(\kappa_1^0)\gamma, \end{aligned} \tag{5.97}$$

$$\begin{aligned} C_{01} &= 2\lim_{Z_0\to\infty}\frac{1}{Z_0}\int_0^{\infty}\mathbf{E}\Big[m(0)\Gamma_1(\kappa_1^0)m(x_3)\Gamma_2(\kappa_1^1)\sin(2\omega_1\bar{\zeta}\tau)\Big]dx_3 d\tau, \\ &= 0, \end{aligned} \tag{5.98}$$

since $Z_0 = \pi/(\omega_0\bar{\zeta})$, and $\int_0^{Z_0}\sin(2\pi\tau/Z_0)d\tau = 0$. Next

$$C_{11} = 2\lim_{Z_0\to\infty}\frac{1}{Z_0}\int_0^{\infty}\mathbf{E}\Big[m(0)\Gamma_2(\kappa_1^1)\sin(2\omega_1\bar{\zeta}\tau)m(x_3)\Gamma_2(\kappa_1^1)\sin(2\omega_1\bar{\zeta}\tau)\Big]dx_3 d\tau,$$

since $\int_0^{Z_0} \sin^2(2\pi\tau/Z_0)d\tau = Z_0/2$, then

$$C_{11} = \Gamma_2^2(\kappa_1^1)\int_0^\infty \mathbb{E}[m(0)m(x_3)]dx_3 = \frac{\Gamma_2^2(\kappa_1^1)\gamma}{2},$$

where

$$\gamma = 2\int_0^\infty \mathbb{E}[m(0)m(x_3)]dx_3. \tag{5.99}$$

In a similar way $C_{11} = C_{22}$, and in fact

$$\gamma_p = \Gamma_2^2(\kappa_1^p)\int_0^\infty \mathbb{E}[m(0)m(x_3)]dx_3 = \frac{\Gamma_2^2(\kappa_1^p)\gamma}{2}.$$

Hence

$$C_{pq} = \frac{\gamma_p \delta_{pq}}{2}. \tag{5.100}$$

From [26] (since $\mathbf{C}^{AS} = \mathbf{0}$) $\mathbf{P}_M^\varepsilon$ converges in distribution (as $\varepsilon \to 0$) to

$$d\mathbf{P}_M^\varepsilon(x_3) = \sum_{p=0}^{3M+2} \sqrt{C_{pp}}\boldsymbol{\Omega}_M \mathbf{h}^{(p)}\mathbf{P}_M \circ dW_p(x_3), \tag{5.101}$$

so

$$\begin{aligned} d\mathbf{P}_M(x_3) = {} & \frac{\sqrt{\gamma}}{2}\sum_{j=0}^{M}\Gamma_1\boldsymbol{\Omega}_M\mathbf{h}^{(3j)}\mathbf{P}_M(x_3)\circ dW_j(x_3) + \frac{\Gamma_2\sqrt{\gamma}}{2\sqrt{2}}\sum_{j=0}^{M}\boldsymbol{\Omega}_M\mathbf{h}^{(3j+1)}\mathbf{P}_M(x_3)\circ d\overline{W}_j(x_3) \\ & + \frac{\Gamma_2\sqrt{\gamma}}{2\sqrt{2}}\sum_{j=0}^{M}\boldsymbol{\Omega}_M\mathbf{h}^{(3j+2)}\mathbf{P}_M(x_3)\circ d\widetilde{W}(x_3), \end{aligned} \tag{5.102}$$

where W_j, $\overline{W}_j$ and $d\widetilde{W}_j$ with $(j = 0, \ldots, M)$ are $3M+1$ independent standard Brownian motions. The block structure of $\mathbf{P}_M(x_3)$ from (5.80) shows that for each (ω^j, κ_1^j) pair the matrix $\mathbf{P}(\omega^j, \kappa_1^j, x_3)$ satisfies the stochastic differential equation

$$d\mathbf{P}(\omega^j, \kappa_1^j, x_3) = \frac{\sqrt{\gamma}}{2}\Gamma_1(\kappa_1^j)\omega_j\mathbf{h}_0\mathbf{P}(\omega^j, \kappa_1^j, x_3)\circ dW_j(x_3)$$

$$
\begin{aligned}
&+ \frac{\sqrt{\gamma}}{2\sqrt{2}} \Gamma_2(\kappa_1^j)\omega_j \mathbf{h}_1 \mathbf{P}(\omega^j, \kappa_1^j, x_3) \circ d\overline{W}_j(x_3) \\
&+ \frac{\sqrt{\gamma}}{2\sqrt{2}} \Gamma_2(\kappa_1^j)\omega_j \mathbf{h}_2 \mathbf{P}(\omega^j, \kappa_1^j, x_3) \circ d\widetilde{W}_j(x_3), \qquad (5.103)
\end{aligned}
$$

where

$$
\mathbf{P}(\omega, \kappa_1^j, x_3) = \begin{bmatrix} \chi_1(\omega, \kappa_1^j, x_3) & \overline{\chi_2(\omega, \kappa_1^j, x_3)} \\ \chi_2(\omega, \kappa_1^j, x_3) & \overline{\chi_1(\omega, \kappa_1^j, x_3)} \end{bmatrix}, \quad |\chi_1| - |\chi_2| = 1. \qquad (5.104)
$$

Now parameterise the equations via

$$
\chi_{1j} = \chi_1(\omega_j, \kappa_1^j, x_3) = \cosh\left(\theta(\omega_j, \kappa_1^j, x_3)/2\right) e^{i\phi(\omega_j, \kappa_1^j, x_3)}, \qquad (5.105)
$$

$$
\chi_{2j} = \chi_2(\omega_j, \kappa_1^j, x_3) = \sinh\left(\theta(\omega_j, \kappa_1^j, x_3)/2\right) e^{i(\psi(\omega_j, \kappa_1^j, x_3) + \phi(\omega_j, \kappa_1^j, x_3))}. \qquad (5.106)
$$

Applying the chain rule gives two equations

$$
d\chi_1 = \frac{\partial \chi_1}{\partial \theta_j} d\theta_j + \frac{\partial \chi_1}{\partial \phi_j} d\phi_j, \qquad (5.107)
$$

$$
d\chi_2 = \frac{\partial \chi_2}{\partial \theta_j} d\theta_j + \frac{\partial \chi_2}{\partial \phi_j} d\phi_j + \frac{\partial \chi_2}{\partial \psi_j} d\psi_j. \qquad (5.108)
$$

That is

$$
d\chi_1 = \left(\frac{\sinh(\theta_j/2)}{2} d\theta_j + i \cosh(\theta_j/2) d\phi_j \right) e^{i\phi_j}, \qquad (5.109)
$$

$$
d\chi_2 = \left(\frac{\cosh(\theta_j/2)}{2} + i \sinh(\theta_j/2) \big(d\phi_j + d\psi_j \big) \right) e^{i(\psi_j + \phi_j)}. \qquad (5.110)
$$

From equations (5.78), (5.103) and (5.104) the stochastic differential equations for the propagator matrix in Stratonovich form can be written as

$$
\begin{aligned}
d \begin{bmatrix} \chi_1 & \overline{\chi_2} \\ \chi_2 & \overline{\chi_1} \end{bmatrix} &= i \frac{\sqrt{\gamma}\, \Gamma_1(\kappa_1^j)\omega_j}{2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} \chi_1 & \overline{\chi_2} \\ \chi_2 & \overline{\chi_1} \end{bmatrix} \circ dW_j(x_3) \\
&+ \frac{\sqrt{\gamma}\, \Gamma_2(\kappa_1^j)\omega_j}{2\sqrt{2}} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \chi_1 & \overline{\chi_2} \\ \chi_2 & \overline{\chi_1} \end{bmatrix} \circ d\overline{W}_j(x_3)
\end{aligned}
$$

$$- i\frac{\sqrt{\gamma}\,\Gamma_2(\kappa_1^j)\omega_j}{2\sqrt{2}} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \chi_1 & \overline{\chi_2} \\ \chi_2 & \overline{\chi_1} \end{bmatrix} \circ d\widetilde{W}_j(x_3). \tag{5.111}$$

To simplify the algebra, introduce the constants

$$A_{1,j} = A_1 = \frac{\sqrt{\gamma}\,\Gamma_1(\kappa_1^j)\omega_j}{2}, \tag{5.112}$$

$$A_{2,j} = A_2 = \frac{\sqrt{\gamma}\,\Gamma_2(\kappa_1^j)\omega_j}{2\sqrt{2}}, \tag{5.113}$$

and simplify equation (5.111) to give

$$\begin{aligned} d\begin{bmatrix} \chi_1 & \overline{\chi_2} \\ \chi_2 & \overline{\chi_1} \end{bmatrix} &= iA_1 \begin{bmatrix} \chi_1 & \overline{\chi_2} \\ -\chi_2 & \overline{\chi_1} \end{bmatrix} \circ dW_j(x_3) \\ &+ A_2 \begin{bmatrix} \chi_2 & \overline{\chi_1} \\ \chi_1 & \overline{\chi_2} \end{bmatrix} \circ d\overline{W}_j(x_3) \\ &- iA_2 \begin{bmatrix} -\chi_2 & -\overline{\chi_1} \\ \chi_1 & \overline{\chi_2} \end{bmatrix} \circ d\widetilde{W}_j(x_3). \end{aligned} \tag{5.114}$$

Using equation (5.114) gives two coupled Stratonovich equations

$$d\chi_1 = iA_1\chi_1 \circ dW_j(x_3) + A_2\chi_2 \circ (d\overline{W}_j(x_3) + id\widetilde{W}_j(x_3)), \tag{5.115}$$

$$d\chi_2 = -iA_1\chi_2 \circ dW_j(x_3) + A_2\chi_1 \circ (d\overline{W}_j(x_3) - id\widetilde{W}_j(x_3)). \tag{5.116}$$

Using equations (5.105), (5.106), (5.109) and (5.115) gives

$$\begin{aligned} &\sinh(\theta_j/2)e^{i\phi_j}/2d\theta_j + i\cosh(\theta_j/2)e^{i\phi_j}d\phi_j \\ &= iA_1\cosh(\theta_j/2)e^{i\phi_j} \circ dW_j(x_3) + A_2\sinh(\theta_j/2)e^{i(\phi_j+\psi_j)} \circ (d\overline{W}(x_3) + id\widetilde{W}(x_3)). \end{aligned} \tag{5.117}$$

Dividing by $\sinh(\theta_j/2)e^{i\phi_j}/2$ yields the equation

$$d\theta_j + 2i\coth(\theta_j/2)d\phi_j$$

$$
\begin{aligned}
&= 2iA_1 \coth(\theta_j/2) \circ dW_j(x_3) + 2A_2 e^{i\psi_j} \circ (d\overline{W}_j(x_3) + id\widetilde{W}_j(x_3)) \\
&= 2iA_1 \coth(\theta_j/2) \circ dW_j(x_3) + 2A_2(\cos(\psi_j) \circ d\overline{W}_j(x_3) - \sin(\psi_j) \circ d\widetilde{W}_j(x_3)) \\
&+ 2iA_2(\sin(\psi_j) \circ d\overline{W}(x_3) + \cos(\psi_j) \circ d\widetilde{W}_j(x_3)), \qquad (5.118)
\end{aligned}
$$

and separating real and imaginary parts gives the equations

$$
d\theta_j = 2A_2(\cos(\psi_j) \circ d\overline{W}_j(x_3) - \sin(\psi_j) \circ d\widetilde{W}_j(x_3)), \qquad (5.119)
$$

$$
d\phi_j = A_1 \circ dW_j(x_3) + A_2(\sin(\psi_j) \circ d\overline{W}(x_3) + \cos(\psi_j) \circ d\widetilde{W}_j(x_3)) \tanh(\theta_j/2). \qquad (5.120)
$$

Next, applying the chain rule gives

$$
\begin{aligned}
d\chi_2 &= \frac{\partial \chi_2}{\partial \theta_j} d\theta_j + \frac{\partial \chi_2}{\partial \psi_j} d\psi_j + \frac{\partial \chi_2}{\partial \phi_j} d\phi_j \\
&= \left(\frac{1}{2} \cosh(\theta_j/2) d\theta_j + i \sinh(\theta_j/2) d\psi_j + i \sinh(\theta_j/2) d\phi_j \right) e^{i(\psi_j + \phi_j)}, \qquad (5.121)
\end{aligned}
$$

so

$$
\begin{aligned}
&d\theta_j + 2i \tanh(\theta_j/2)(d\psi_j + d\phi_j) \\
&= \left(-iA_1 \sinh(\theta_j/2) e^{i(\psi_j + \phi_j)} \circ dW_j(x_3) + A_2 \cosh(\theta_j/2) e^{i\phi_j} \circ \left(d\overline{W}_j(x_3) - id\widetilde{W}_j(x_3)\right)\right) \\
&\times \frac{2e^{-i(\psi_j + \phi_j)}}{\cosh(\theta_j/2)}, \qquad (5.122)
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&d\theta_j + 2i \tanh(\theta_j/2)(d\psi_j + d\phi_j) \\
&= -2iA_1 \tanh(\theta_j/2) \circ dW_j(x_3) + 2A_2 e^{-i\psi_j} \circ (d\overline{W}_j(x_3) - id\widetilde{W}_j(x_3)) \\
&= -2iA_1 \tanh(\theta_j/2) \circ dW_j(x_3) + 2A_2(\cos(\psi_j) - i \sin(\psi_j)) \circ (d\overline{W}_j(x_3) - id\widetilde{W}_j(x_3)) \\
&= -2iA_1 \tanh(\theta_j/2) \circ dW_j(x_3) + 2A_2\Big(\cos(\psi_j) \circ d\overline{W}_j(x_3) - \sin(\psi_j) \circ d\widetilde{W}_j(x_3) \\
&\qquad - i\big(\sin(\psi_j) \circ d\overline{W}_j(x_3) + \cos(\psi_j) \circ d\widetilde{W}_j(x_3)\big)\Big), \qquad (5.123)
\end{aligned}
$$

hence

$$
\begin{aligned}
&d\psi_j + d\phi_j \\
&= -A_2\Big(\sin(\psi_j) \circ d\overline{W}_j(x_3) + \cos(\psi_j) \circ d\widetilde{W}_j(x_3)\Big) \coth(\theta_j/2) - A_1 \circ dW_j(x_3).
\end{aligned} \tag{5.124}
$$

Now, subtracting equation (5.120) and using the identity $2/\tanh(\theta_j/2) = \coth(\theta_j/2) + \tanh(\theta_j/2)$ gives the equation

$$
d\psi_j = -2A_1 \circ dW_j(x_3) - \frac{2A_2}{\tanh\theta_j}\Big(\sin(\psi_j) \circ d\overline{W}_j + \cos(\theta_j/2) \circ d\widetilde{W}_j(x_3)\Big). \tag{5.125}
$$

Combining equations (5.119), (5.120) and (5.125) into matrix format gives

$$
d\begin{bmatrix} \phi_j \\ \psi_j \\ \theta_j \end{bmatrix} = \begin{bmatrix} A_1 & A_2 \sin(\psi_j)\tanh(\theta_j/2) & A_2\cos(\psi_j)\tanh(\theta_j/2) \\ -2A_1 & 2A_2\sin(\psi_j)/\tanh\theta_j & -2A_2\cos(\psi_j)/\tanh(\theta_j) \\ 0 & 2A_2\cos(\psi_j) & -2A_2\sin(\psi_j) \end{bmatrix} \circ d\begin{bmatrix} W_j(x_3) \\ \overline{W}_j(x_3) \\ \widetilde{W}_j(x_3) \end{bmatrix}. \tag{5.126}
$$

Now compute the modified drift to transform the equation from Stratonovich into Itô form via

$$
d_i = \frac{1}{2}\sum_{j=1}^{3} \sigma_{jp}\sigma_{ip,j}dx_3, \quad i = 1, 2, 3, \tag{5.127}
$$

which gives

$$
d_1 = 0 \tag{5.128}
$$

$$
d_2 = 0, \tag{5.129}
$$

$$
d_3 = \frac{2A_2^2}{\tanh(\theta_j)}. \tag{5.130}
$$

Introducing the auxiliary processes

$$
dW_j^\star(x_3) = \sin(\psi_j)d\overline{W}_j(x_3) + \cos(\psi_j)d\widetilde{W}_j(x_3), \tag{5.131}
$$

$$d\widetilde{W}^{\star}(x_3) = -\cos(\psi_j)d\overline{W}_j(x_3) + \sin(\psi_j)d\widetilde{W}_j(x_3), \tag{5.132}$$

system (5.126) in Itô form can be written as

$$d\begin{bmatrix}\phi_j\\ \psi_j\\ \theta_j\end{bmatrix} = \begin{bmatrix}A_1 & 0 & A_2\tanh(\theta_j/2)\\ -2A_1 & 0 & -2A_2/\tanh(\theta_j)\\ 0 & -2A_2 & 0\end{bmatrix} d\begin{bmatrix}W_j(x_3)\\ \widetilde{W}_j^{\star}(x_3)\\ W_j^{\star}(x_3)\end{bmatrix} + \begin{bmatrix}0\\ 0\\ 2A_2^2/\tanh(\theta_j)\end{bmatrix} dx_3. \tag{5.133}$$

Now use system (5.133) to compute the infinitesimal generator for the processes $(\phi_j, \psi_j, \theta_j)$, where

$$\boldsymbol{\sigma} = \begin{bmatrix}A_1 & 0 & A_2\tanh(\theta_j/2)\\ -2A_1 & 0 & -2A_2/\tanh(\theta_j)\\ 0 & -2A_2 & 0\end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix}0\\ 0\\ 2A_2^2/\tanh(\theta_j)\end{bmatrix}, \quad \boldsymbol{x} = \begin{bmatrix}\phi_j\\ \psi_j\\ \theta_j\end{bmatrix}, \tag{5.134}$$

via

$$\mathcal{L}_{(\phi_j,\psi_j,\theta_j)} = \frac{1}{2}\sum_{i,j=1}^{3} a_{ij}\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{i=1}^{3} b_i \frac{\partial}{\partial x_i}, \quad a_{ij} = \sigma ik\sigma_{jk}, \tag{5.135}$$

where

$$\boldsymbol{a} = \begin{bmatrix}A_1^2 + A_2^2\tanh(\theta_j/2) & -2A_1^2 - 2A_2^2\frac{\tanh(\theta_j/2)}{\tanh(\theta_j)} & 0\\ -2A_1^2 - 2A_2^2\frac{\tanh(\theta_j/2)}{\tanh(\theta_j)} & 4A_1^2 + \frac{4A_2^2}{\tanh(\theta_j)^2} & 0\\ 0 & 0 & 4A_2^2\end{bmatrix}. \tag{5.136}$$

The infinitesimal generator can then be written as

$$\mathcal{L}_{(\phi_j,\psi_j,\theta_j)} = \frac{1}{2}\left(a_{11}\frac{\partial^2}{\partial\phi_j^2} + 2a_{12}\frac{\partial^2}{\partial\phi_j\psi_j} + a_{22}\frac{\partial^2}{\partial\psi_j^2} + a_{33}\frac{\partial^2}{\partial\theta_j^2}\right) + b_3\frac{\partial}{\partial\theta_j}. \tag{5.137}$$

To study the transmission and reflection coefficients, the infinitesimal generator for θ is

$$\begin{aligned}\mathcal{L}_{\theta_j} &= \frac{a_{33}}{2}\frac{\partial^2}{\partial\theta_j^2} + b_3\frac{\partial}{\partial\theta_j} \\ &= 2A_2^2\left(\frac{\partial^2}{\partial\theta_j^2} + \frac{1}{\tanh(\theta_j)}\frac{\partial}{\partial\theta_j}\right) \\ &= \frac{\gamma\Gamma_2^2(\kappa_1^j)\omega_j^2}{4}\left(\frac{\partial^2}{\partial\theta_j^2} + \frac{1}{\tanh(\theta_j)}\frac{\partial}{\partial\theta_j}\right),\end{aligned} \tag{5.138}$$

which could then be used to give a Fokker-Planck equation to study the probability distribution of the process θ_j.

5.4.1 Deriving the Localisation Length

Now use equation (5.120) and integrate in x_3 along the length of the random medium L, to obtain

$$\int_{-L}^{0} d\phi_j = A_1\int_{-L}^{0} dW_j + A_2\int_{-L}^{0} \tanh(\theta_j/2)dW_j^\star, \tag{5.139}$$

so

$$\phi_j(0) = A_1 W_j(0) + A_2\int_{-L}^{0} \tanh(\theta_j/2)dW_j^\star. \tag{5.140}$$

Since

$$\frac{d}{dx}\left(\frac{1}{\cosh(x)}\right) = -\frac{\tanh(x)}{\cosh(x)}, \tag{5.141}$$

then

$$\begin{aligned}d\left(\frac{1}{\cosh(\theta_j/2)}\right) &= -\frac{\tanh(\theta_j/2)}{2\cosh(\theta_j/2)}\circ d\theta_j, \\ &= -\frac{\tanh(\theta_j/2)}{2\cosh(\theta_j/2)}\circ\left(-2A_2 d\widetilde{W}_j(x_3) + \frac{2A_2^2}{\tanh(\theta_j)}dx_3\right).\end{aligned} \tag{5.142}$$

Equation (5.142) is in the form

$$dy = y dx, \tag{5.143}$$

where

$$y = \cosh^{-1}(\theta_j/2), \quad x = -\frac{1}{2}\tanh(\theta_j/2)\bigg(- 2A_2 d\widetilde{W}_j(x_3) + \frac{2A_2^2}{\tanh(\theta_j)}dx_3\bigg), \tag{5.144}$$

and so

$$y = y_{-L}e^x, \quad \theta_j(x_3 = -L) = 0, \quad y_{-L} = \cosh(\theta_j(-L)/2) = 1, \tag{5.145}$$

so

$$\frac{1}{\cosh(\theta_j(0)/2)} = \exp\bigg\{A_2\int_{-L}^{0}\tanh(\theta_j/2)\circ d\widetilde{W}_j^\star - A_2^2\int_{-L}^{0}\frac{\tanh(\theta_j/2)}{\tanh(\theta_j)}dx_3\bigg\}. \tag{5.146}$$

Defining

$$\begin{aligned} Z(0) = \ln(\cosh(\theta_j(0)/2)^{-1}) &= A_2\int_{-L}^{0}\tanh(\theta_j/2)\circ d\widetilde{W}_j^\star(x_3) - A_2^2\int_{-L}^{0}\frac{\tanh(\theta_j/2)}{\tanh(\theta_j)}dx_3 \\ &= A_2\int_{-L}^{0}(1-e^{2Z})^{\frac{1}{2}}\circ d\widetilde{W}_j^\star(x_3) - \frac{A_2^2}{2}\int_{-L}^{0}(2-e^{2Z})dx_3, \end{aligned} \tag{5.147}$$

convert this equation into Itô form via

$$a = A_2(1-e^{2Z})^{\frac{1}{2}}, \quad \frac{\partial a}{\partial Z} = -A_2(1-e^{2Z})^{-\frac{1}{2}}e^{2Z}, \tag{5.148}$$

where the drift term is

$$\frac{a}{2}\frac{\partial a}{\partial Z} = -\frac{A_2^2 e^{2Z}}{2}. \tag{5.149}$$

Now writing

$$Z(0) = A_2 \int_{-L}^{0} (1 - e^{2Z})^{\frac{1}{2}} \circ d\widetilde{W}_j^\star(x_3) - A_2^2 \int_{-L}^{0} dx_3 + \frac{A_2^2}{2} \int_{-L}^{0} e^{2Z} dx_3 - \frac{A_2^2}{2} \int_{-L}^{0} e^{2Z} dx_3$$
$$= A_2 \int_{-L}^{0} (1 - e^{2Z})\frac{1}{2} \circ d\widetilde{W}_j^\star(x_3) + A_2^2 L, \tag{5.150}$$

implies that

$$\frac{1}{\cosh(\theta_j(0)/2)} = \exp\Big\{ A_2 \int_{-L}^{0} \tanh(\theta_j/2) \circ d\widetilde{W}_j(x_3) - A_2^2 L \Big\}. \tag{5.151}$$

The transmission coefficient is written (from equations (5.107) and (5.120)) as

$$T_j(-L,0) = \frac{1}{\overline{\chi_{1j}(-L,0)}} \tag{5.152}$$
$$= \frac{e^{i\phi_j}}{\cosh(\theta_j/2)}$$
$$= \exp\Big\{ iA_1 W_j(0) + iA_2 \int_{-L}^{0} \tanh(\theta_j/2) dW_j^\star(x_3) \Big\} \exp\Big\{ A_2 \int_{-L}^{0} \tanh(\theta_j/2) d\widetilde{W}_j^\star - A_2^2 L \Big\}$$
$$= \exp\{ iA_1 W_j(0) - A_2^2 L \} \exp\Big\{ A_2 \int_{-L}^{0} \tanh(\theta_j/2) \Big(d\widetilde{W}_j^\star + i dW_j^\star \Big) \Big\}$$
$$= \widetilde{T}_{(\omega_j,\kappa_1^j)}(-L,0) M_{(\omega_j,\kappa_1^j)}(-L,0), \tag{5.153}$$

where

$$\widetilde{T}_{(\omega_j,\kappa_1^j)}(-L,0) = \exp\{ iA_1 W_j(0) - A_2^2 L \}, \tag{5.154}$$
$$M_{(\omega_j,\kappa_1^j)}(-L,0) = \exp\Big\{ A_2 \int_{-L}^{0} \tanh(\theta_j/2) \Big(d\widetilde{W}_j^\star + i dW_j^\star \Big) \Big\}. \tag{5.155}$$

Since

$$\frac{1}{\cosh(\theta_j(-L)/2)} = \exp\Big\{ A_2 \int_{-L}^{0} \tanh(\theta_j/2) d\widetilde{W}_j^\star - A_2^2 L \Big\}, \tag{5.156}$$

the power transmission coefficient is

$$\tau(0) \sim \frac{1}{\cosh(\theta_j(0)/2)^2}$$

$$= \exp\big\{-2A_2^2 L\big\}, \tag{5.157}$$

and since $\tau \sim \exp\Big\{-L/L_{loc}(\omega_j, \kappa_1^j)\Big\}$, then

$$L_{loc}(\omega_j, \kappa_1^j) = \frac{1}{2A_2^2}, \tag{5.158}$$

(which characterises the decay of energy in the coherent wave) so the infinitesimal generator (equation (5.138)) can be written as

$$\mathcal{L}_{\theta_j} = \frac{1}{L_{loc}(\omega_j, \kappa_1^j)} \left(\frac{\partial^2}{\partial \theta_j^2} + \frac{1}{\tanh(\theta_j)} \frac{\partial}{\partial \theta_j} \right). \tag{5.159}$$

The tools required to describe the full multi-frequency problem have now been derived. The multi-frequency propagator formulation will allow for a study into the statistics of the reflected wave in the next section.

5.5 Statistics of the Reflected Wave

From equation (5.15) the forcing function is written

$$\begin{aligned} \mathbf{F}^\varepsilon &= \left(0, \varepsilon^q f\left(\frac{t}{\varepsilon}\right) \delta\left(\frac{x_1}{\varepsilon}\right) \delta(x_3), 0\right) \\ &= \left(0, \varepsilon^{q+1} f\left(\frac{t}{\varepsilon}\right) \delta(x_1)\delta(x_3), 0\right). \end{aligned} \tag{5.160}$$

Setting $q = -1$ means that the forcing function $\mathbf{F}^\varepsilon$ (5.160) reads

$$\begin{aligned} \mathbf{F}^\varepsilon &= \left(0, f\left(\frac{t}{\varepsilon}\right) \delta(x_1)\delta(x_3), 0\right), \\ &= \Big(0, f(u)\delta(x_1)\delta(x_3), 0\Big), \end{aligned} \tag{5.161}$$

where $u = t/\varepsilon$, so

$$\int_{-\infty}^{\infty} \varepsilon^2 f^2(u)\, du < \infty. \tag{5.162}$$

From equation (5.51) the wave mode boundary conditions are

$$[\hat{a}^\varepsilon(\omega,\kappa_1,x_3)]_{x_3=0} = \hat{a}^\varepsilon(\omega,\kappa_1,0^+) - \hat{a}^\varepsilon(\omega,\kappa_1,0^-), \tag{5.163}$$

which are related via equations (5.33) and (5.34)

$$\hat{\xi}^\varepsilon(\omega,\kappa_1,x_3) = \sqrt{\bar{\zeta}/\bar{\beta}}\Big(\hat{b}^\varepsilon(\omega,\kappa_1,x_3)e^{-i\omega\bar{\zeta}x_3/\varepsilon} - \hat{a}^\varepsilon(\omega,\kappa_1,x_3)e^{i\omega\bar{\zeta}x_3/\varepsilon}\Big), \tag{5.164}$$

$$\hat{\tau}^\varepsilon_{32}(\omega,\kappa_1,x_3) = \sqrt{\bar{\beta}/\bar{\zeta}}\Big(\hat{b}^\varepsilon(\omega,\kappa_1,x_3)e^{-i\omega\bar{\zeta}x_3/\varepsilon} + \hat{a}^\varepsilon(\omega,\kappa_1,x_3)e^{i\omega\bar{\zeta}x_3/\varepsilon}\Big). \tag{5.165}$$

The jump at $x_3 = 0$ can be written as

$$[\hat{a}^\varepsilon(\omega,\kappa_1,x_3)]_{x_3=0} = \frac{-\varepsilon\hat{f}(\omega)}{2}\sqrt{\bar{\zeta}/\bar{\beta}}, \tag{5.166}$$

which can be used to obtain the wave-mode equations

$$\hat{a}^\varepsilon(\omega,\kappa_1,x_3) = \frac{1}{2}\Big(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}^\varepsilon_{32}(\omega,\kappa_1,x_3) - \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon(\omega,\kappa_1,x_3)\Big)e^{-i\omega\bar{\zeta}x_3/\varepsilon}, \tag{5.167}$$

$$\hat{b}^\varepsilon(\omega,\kappa_1,x_3) = \frac{1}{2}\Big(\sqrt{\bar{\zeta}/\bar{\beta}}\hat{\tau}^\varepsilon_{32}(\omega,\kappa_1,x_3) + \sqrt{\bar{\beta}/\bar{\zeta}}\hat{\xi}^\varepsilon(\omega,\kappa_1,x_3)\Big)e^{i\omega\bar{\zeta}x_3/\varepsilon}, \tag{5.168}$$

Similarly, the boundary condition for the backward modes gives

$$[\hat{b}^\varepsilon(\omega,\kappa_1,x_3)]_{x_3=0} = -\frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}. \tag{5.169}$$

From the boundary conditions (see Figure 5.1)

$$\hat{b}^\varepsilon(\omega,\kappa_1,0^-) = \frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}. \tag{5.170}$$

From the boundary value problem (5.59) together with the boundary conditions in Figure 5.1, the wave-mode boundary conditions can be written as a system of equations

$$\begin{bmatrix}\hat{a}^\varepsilon(x_3=0^-)\\ \hat{b}^\varepsilon(x_3=0^-)\end{bmatrix} = \begin{bmatrix}\hat{a}^\varepsilon(0^-)\\ \frac{\varepsilon\hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}\end{bmatrix} = \begin{bmatrix}\chi_1^\varepsilon(0) & \overline{\chi_2^\varepsilon(0)}\\ \chi_2^\varepsilon(0) & \overline{\chi_1^\varepsilon(0)}\end{bmatrix}\begin{bmatrix}0\\ \hat{b}^\varepsilon(-L)\end{bmatrix}. \tag{5.171}$$

Solving the system gives

$$
\begin{aligned}
\hat{b}^{\varepsilon}(-L) &= \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2\overline{\chi_1^{\varepsilon}(0)}} \\
&= \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2} T_{\omega_j,\kappa_1^j}(-L,0),
\end{aligned}
\tag{5.172}
$$

and

$$
\begin{aligned}
\hat{a}^{\varepsilon}(0^-) &= \overline{\chi_2^{\varepsilon}(0)}\hat{b}^{\varepsilon}(-L) \\
&= \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2} \frac{\overline{\chi_2^{\varepsilon}(0)}}{\overline{\chi_1^{\varepsilon}(0)}} \\
&= \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2} R^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0),
\end{aligned}
\tag{5.173}
$$

where the transmission and reflection coefficients are defined as

$$
T^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0) = \frac{1}{\overline{\chi_1^{\varepsilon}(0)}}, \tag{5.174}
$$

$$
R^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0) = \frac{\overline{\chi_2^{\varepsilon}(0)}}{\overline{\chi_1^{\varepsilon}(0)}}. \tag{5.175}
$$

Note that

$$
\begin{aligned}
\hat{a}^{\varepsilon}(0^+) &= \hat{a}^{\varepsilon}(0^-) - \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2} \\
&= \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}} R^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0)}{2} - \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2} \\
&= \frac{\varepsilon \hat{f}(\omega)\sqrt{\bar{\zeta}/\bar{\beta}}}{2}\Big(R^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0) - 1\Big).
\end{aligned}
\tag{5.176}
$$

At $x_3 = -L$, $\mathbf{P}^{\varepsilon}_{(\omega,\kappa_1)} = \mathbf{I}$, so the initial condition for $R^{\varepsilon}_{(\omega,\kappa_1,-L)} = 0$. Using equations (5.56) and (5.58) gives

$$
\frac{\partial}{\partial x_3}\begin{bmatrix} \chi_1^{\varepsilon}(\omega,\kappa_1,x_3) & \overline{\chi_2^{\varepsilon}(\omega,\kappa_1,x_3)} \\ \chi_2^{\varepsilon}(\omega,\kappa_1,x_3) & \overline{\chi_1^{\varepsilon}(\omega,\kappa_1,x_3)} \end{bmatrix}
$$

$$= \frac{i\omega}{2\varepsilon} \begin{bmatrix} \Delta_1 & \Delta_2 e^{-2i\omega\bar{\zeta}x_3/\varepsilon} \\ -\Delta_2 e^{2i\omega\bar{\zeta}x_3/\varepsilon} & -\Delta_1 \end{bmatrix} \begin{bmatrix} \chi_1^\varepsilon(\omega,\kappa_1,x_3) & \overline{\chi_2^\varepsilon(\omega,\kappa_1,x_3)} \\ \chi_2^\varepsilon(\omega,\kappa_1,x_3) & \overline{\chi_1^\varepsilon(\omega,\kappa_1,x_3)} \end{bmatrix}$$

$$= \mathbf{H}^\varepsilon_{(\kappa_1,\omega)} \mathbf{P}^\varepsilon_{(\omega,\kappa_1)}, \tag{5.177}$$

where $\mathrm{Tr}\Big[\mathbf{H}^\varepsilon_{(\kappa_1,\omega)}\Big] = 0$. The conjugate pair evolution equations then read

$$\frac{\partial \overline{\chi_1^\varepsilon}}{\partial x_3} = \frac{i\omega}{2\varepsilon}\Big(- \overline{\chi_2^\varepsilon}\Delta_2 e^{2i\omega\bar{\zeta}x_3\varepsilon} - \overline{\chi_1^\varepsilon}\Delta_1\Big), \tag{5.178}$$

$$\frac{\partial \overline{\chi_2^\varepsilon}}{\partial x_3} = \frac{i\omega}{2\varepsilon}\Big(\overline{\chi_2^\varepsilon}\Delta_1 + \overline{\chi_1^\varepsilon}\Delta_2 e^{-2i\omega\bar{\zeta}x_3/\varepsilon}\Big). \tag{5.179}$$

The derivative of the reflection coefficient (5.175) is

$$\frac{dR^\varepsilon_{(\omega,\kappa_1)}}{dx_3} = \overline{\frac{1}{\chi_1^\varepsilon}\frac{d\chi_2^\varepsilon}{dx_3} - \frac{\chi_2^\varepsilon}{(\chi_1^\varepsilon)^2}\frac{d\chi_1^\varepsilon}{dx_3}}. \tag{5.180}$$

Now, using equations (5.178) and (5.179) one can obtain a Ricatti equation which is written as

$$\frac{dR^\varepsilon_{(\omega,\kappa_1)}}{dx_3} = \frac{i\omega}{2\varepsilon}\Big(2\Delta_1 R^\varepsilon_{(\omega,\kappa_1)} + (R^\varepsilon_{(\omega,\kappa_1)})^2 \Delta_2 e^{2i\omega\bar{\zeta}x_3/\varepsilon} + \Delta_2 e^{-2i\omega\bar{\zeta}x_3/\varepsilon}\Big). \tag{5.181}$$

The product of reflection coefficients (which is of interest in order to study the autocorrelation function of the reflection coefficient) are defined as

$$U^\varepsilon_{p,q}(\omega,\kappa_1,h,\lambda,x_3) = \Big(R^\varepsilon_{(\omega+\varepsilon h/2,\kappa_1+\varepsilon\lambda/2)}\Big)^p \Big(\overline{R^\varepsilon_{(\omega-\varepsilon h/2,\kappa_1-\varepsilon\lambda/2)}}\Big)^q, \quad p,q \in \mathbb{N}. \tag{5.182}$$

In order to study the frequency dependent autocorrelation function for the reflection coefficient, consider two close frequencies (and slownesses) $\omega \pm \varepsilon h/2$, $\kappa_1 \pm \varepsilon\lambda/2$. From equations (5.35), (5.36) and (5.37)

$$\bar{\zeta}(\kappa_1) = \sqrt{\overline{\alpha\beta}} = \sqrt{\frac{\rho - \kappa_1^2 \bar{c}_{66}}{\bar{c}_{44}}},$$

so

$$\bar{\zeta}(\kappa_1 \pm \varepsilon\lambda/2) = \sqrt{\frac{\rho - (\kappa_1 \pm \varepsilon\lambda/2)^2 \bar{c}_{66}}{\bar{c}_{44}}}. \tag{5.183}$$

Taking an expansion in small ε gives

$$\begin{aligned}\bar{\zeta}(\kappa_1 \pm \varepsilon\lambda/2) &= \sqrt{\frac{\rho - \kappa_1^2 \bar{c}_{66}}{\bar{c}_{44}}} \mp \varepsilon\left(\frac{\bar{c}_{66}\kappa_1\lambda}{2\bar{c}_{44}\sqrt{\frac{\rho-\kappa_1^2\bar{c}_{66}}{\bar{c}_{44}}}}\right) + \mathcal{O}(\varepsilon^2)\\ &= \bar{\zeta}(\kappa_1) \mp \varepsilon\left(\frac{\bar{c}_{66}\kappa_1\lambda}{2\bar{c}_{44}\bar{\zeta}(\kappa_1)}\right) + \mathcal{O}(\varepsilon^2).\end{aligned} \tag{5.184}$$

This means that the exponential arguments in equation (5.181) become

$$\begin{aligned}\frac{2i(\omega + \varepsilon h/2)\bar{\zeta}(\kappa_1 + \varepsilon\lambda/2)x_3}{\varepsilon} &= \frac{2ix_3(\omega + \varepsilon h/2)}{\varepsilon}\left(\bar{\zeta}(\kappa_1) - \frac{\varepsilon\bar{c}_{66}\kappa_1 h}{2\bar{c}_{44}\bar{\zeta}(\kappa_1)}\right) + \mathcal{O}(\varepsilon^2)\\ &= \frac{2i\omega\bar{\zeta}(\kappa_1)x_3}{\varepsilon} + 2i\omega x_3\left(\frac{-\varepsilon h\bar{c}_{66}\kappa_1\lambda}{4\omega\bar{c}_{44}\bar{\zeta}(\kappa_1)} + \frac{h\bar{\zeta}(\kappa_1)}{2\omega} - \frac{\bar{c}_{66}\kappa_1\lambda}{2\bar{c}_{44}\bar{\zeta}(\kappa_1)}\right) + \mathcal{O}(\varepsilon^2)\\ &= \frac{2i\omega\bar{\zeta}(\kappa_1)x_3}{\varepsilon} + i\omega x_3\left(\frac{h\bar{\zeta}(\kappa_1)}{\omega} - \frac{\bar{c}_{66}\kappa_1\lambda}{\bar{c}_{44}\bar{\zeta}(\kappa_1)}\right) + \mathcal{O}(\varepsilon),\end{aligned} \tag{5.185}$$

and in a similar fashion for the opposite sign

$$\frac{2i(\omega - \varepsilon h/2)\bar{\zeta}(\kappa_1 - \varepsilon\lambda/2)x_3}{\varepsilon} = \frac{2i\omega\bar{\zeta}(\kappa_1)x_3}{\varepsilon} + i\omega x_3\left(\frac{\bar{c}_{66}\kappa_1\lambda}{\bar{c}_{44}\bar{\zeta}(\kappa_1)} - \frac{h\bar{\zeta}(\kappa_1)}{\omega}\right) + \mathcal{O}(\varepsilon). \tag{5.186}$$

Taking a derivative in x_3 (where $\omega^{(\pm)} = \omega \pm \varepsilon h/2$, $\kappa_1^{(\pm)} = \kappa_1 \pm \varepsilon\lambda/2$) gives

$$\begin{aligned}\frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3} &= \frac{i\omega^{(+)}pU^{\varepsilon}_{p-1,q}}{2\varepsilon}\left(2\Delta_1^{(+)}R^{\varepsilon}_{\omega^{(+)},\kappa_1^{(+)}} + (R^{\varepsilon}_{\omega^{(+)},\kappa_1^{(+)}})^2\Delta_2^{(+)}e^{2i\omega^{(+)}\bar{\zeta}^{(+)}x_3/\varepsilon} + \Delta_2^{(+)}e^{-2i\omega^{(+)}\bar{\zeta}^{(+)}x_3/\varepsilon}\right)\\ &\quad - \frac{i\omega^{(-)}pU^{\varepsilon}_{p,q-1}}{2\varepsilon}\left(2\Delta_1^{(-)}\overline{R^{\varepsilon}_{\omega^{(-)},\kappa_1^{(-)}}} + (\overline{R^{\varepsilon}_{\omega^{(-)},\kappa_1^{(-)}}})^2\Delta_2^{(-)}e^{-2i\omega^{(-)}\bar{\zeta}^{(-)}x_3/\varepsilon} + \Delta_2^{(-)}e^{+2i\omega^{(-)}\bar{\zeta}^{(-)}x_3/\varepsilon}\right)\\ &= \frac{i\omega^{(+)}pU^{\varepsilon}_{p-1,q}}{2\varepsilon}\Bigg(2\Delta_1^{(+)}R^{\varepsilon}_{\omega^{(+)},\kappa_1^{(+)}} + (R^{\varepsilon}_{\omega^{(+)},\kappa_1^{(+)}})^2\Delta_2^{(+)}e^{2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon}e^{ih\bar{\zeta}(\kappa_1)x_3 - i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}\\ &\quad + \Delta_2^{(+)}e^{-2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon}e^{-ih\bar{\zeta}(\kappa_1)x_3 + i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}\Bigg)\end{aligned}$$

$$- \frac{i\omega^{(-)} p U^\varepsilon_{p,q-1}}{2\varepsilon} \Bigg(2\Delta_1^{(-)} R^\varepsilon_{\omega^{(-)},\kappa_1^{(-)}} + (R^\varepsilon_{\omega^{(-)},\kappa_1^{(-)}})^2 \Delta_2^{(-)} e^{-2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} e^{ih\bar{\zeta}(\kappa_1)x_3 - i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}$$
$$+ \Delta_2^{(-)} e^{2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} e^{-ih\bar{\zeta}(\kappa_1)x_3 + i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))} \Bigg), \tag{5.187}$$

which can be simplified to

$$\frac{\partial U^\varepsilon_{p,q}}{\partial x_3} = \frac{i\omega p U^\varepsilon_{p-1,q}}{2\varepsilon} \Bigg(2\Gamma_1 m(x_3/\varepsilon^2) U^\varepsilon_{1,0} + U^\varepsilon_{2,0}\Gamma_2 m(x_3/\varepsilon^2) e^{2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} e^{ih\bar{\zeta}(\kappa_1)x_3 - i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}$$
$$+ \Gamma_2 m(x_3/\varepsilon^2) e^{-2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} e^{-ih\bar{\zeta}(\kappa_1)x_3 + i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))} \Bigg)$$
$$- \frac{i\omega q U^\varepsilon_{p,q-1}}{2\varepsilon} \Bigg(2\Gamma_1 m(x_3/\varepsilon^2) U^\varepsilon_{0,1} + U^\varepsilon_{0,2}\Gamma_2 m(x_3/\varepsilon^2) e^{-2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} e^{ih\bar{\zeta}(\kappa_1)x_3 - i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}$$
$$+ \Gamma_2 m(x_3/\varepsilon^2) e^{2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} e^{-ih\bar{\zeta}(\kappa_1)x_3 + i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))} \Bigg), \tag{5.188}$$

which simplifies to

$$\frac{\partial U^\varepsilon_{p,q}}{\partial x_3} = \frac{i\omega\Gamma_1(\kappa_1) m(x_3/\varepsilon^2)}{\varepsilon} \big(pU^\varepsilon_{p,q} - qU^\varepsilon_{p,q} \big)$$
$$+ \frac{i\omega\Gamma_2(\kappa_1) m(x_3/\varepsilon^2)}{2\varepsilon} e^{2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} \Big(pU^\varepsilon_{p+1,q} e^{ih\bar{\zeta}(\kappa_1)x_3 - i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}$$
$$- qU^\varepsilon_{p,q-1} e^{-ih\bar{\zeta}(\kappa_1)x_3 + i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))} \Big)$$
$$+ \frac{i\omega\Gamma_2(\kappa_1) m(x_3/\varepsilon^2)}{2\varepsilon} e^{-2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} \Big(pU^\varepsilon_{p-1,q} e^{-ih\bar{\zeta}x_3 + i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}$$
$$- qU^\varepsilon_{p,q+1} e^{ih\bar{\zeta}(\kappa_1)x_3 - i\omega\bar{c}_{66}\kappa_1\lambda x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1))} \Big), \tag{5.189}$$

with initial condition $U^\varepsilon_{p,q}(\omega, \kappa_1, h, \lambda, x_3, x_3 = -L) = \mathbf{1}_0(p)\mathbf{1}_0(q)$. Now introduce the Fourier transform (A is non-dimensional and will be defined later in equation (5.223))

$$V^\varepsilon_{p,q} = \frac{A\omega}{4\pi^2} \int\int e^{-ih(A\tau - (p+q)x_3\bar{\zeta}(\kappa_1))} e^{i\lambda\omega(\chi - (p+q)\bar{c}_{66}\kappa_1 x_3)/(\bar{c}_{44}\bar{\zeta}(\kappa_1))} U^\varepsilon_{p,q}\, dh d\lambda. \tag{5.190}$$

Taking derivatives

$$\frac{\partial V^\varepsilon_{p,q}}{\partial x_3} = \frac{A\omega}{4\pi^2} \int\int e^{-ih(A\tau - (p+q)\bar{\zeta}(\kappa_1)x_3)} e^{i\omega\lambda(\chi - (p+q)x_3\bar{c}_{66}\kappa_1/(\bar{c}_{44}\bar{\zeta}(\kappa_1))} \Big(ih(p+q)\bar{\zeta}(\kappa_1) U^\varepsilon_{p,q}$$

$$
\begin{aligned}
&- \frac{i\lambda\omega(p+q)\bar{c}_{66}\kappa_1}{\bar{c}_{44}\bar{\zeta}(\kappa_1)} U^{\varepsilon}_{p,q} + \frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3}\bigg) \, dh d\lambda \\
&= \left(ih(p+q)\bar{\zeta}(\kappa_1) - \frac{i\lambda\omega(p+q)\bar{c}_{66}\kappa_1}{\bar{c}_{44}\bar{\zeta}(\kappa_1)} \right) V^{\varepsilon}_{p,q} \\
&+ \frac{A\omega}{4\pi^2} \int \int e^{-ih(A\tau - (p+q)\bar{\zeta}(\kappa_1)x_3)} e^{i\lambda\omega(\chi - (p+q)\bar{c}_{66}\kappa_1 x_3/(\bar{c}_{44}\bar{\zeta}(\kappa_1)))} \frac{\partial U^{\varepsilon}_{p,q}}{\partial x_3} \, dh d\lambda,
\end{aligned}
\tag{5.191}
$$

and

$$
\frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} = -ihAV^{\varepsilon}_{p,q}, \tag{5.192}
$$

$$
\frac{\partial V^{\varepsilon}_{p,q}}{\partial \chi} = i\lambda\omega V^{\varepsilon}_{p,q}. \tag{5.193}
$$

Applying the Fourier transform (5.190) to equation (5.189) gives

$$
\begin{aligned}
\frac{\partial V^{\varepsilon}_{p,q}}{\partial x_3} &= -\frac{(p+q)\bar{\zeta}(\kappa_1)}{A} \frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} - \frac{(p+q)\bar{c}_{66}\kappa_1}{\bar{c}_{44}\bar{\zeta}(\kappa_1)} \frac{\partial V^{\varepsilon}_{p,q}}{\partial \chi} \\
&+ \frac{i\omega\Gamma_1(\kappa_1)m(x_3/\varepsilon^2)}{\varepsilon}(p-q)V^{\varepsilon}_{p,q} \\
&+ \frac{i\omega\Gamma_2(\kappa_1)m(x_3/\varepsilon^2)}{2\varepsilon} e^{2i\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon} \left(pV^{\varepsilon}_{p+1,q} - qV^{\varepsilon}_{p,q-1} \right) \\
&+ \frac{i\omega\Gamma_2(\kappa_1)m(x_3/\varepsilon^2)}{2\varepsilon} e^{-2i\omega\bar{\zeta}(\kappa_1)/2\varepsilon} \left(pV^{\varepsilon}_{p-1,q} - qV^{\varepsilon}_{p,q+1} \right),
\end{aligned}
\tag{5.194}
$$

and expanding the complex exponentials gives

$$
\begin{aligned}
\frac{\partial V^{\varepsilon}_{p,q}}{\partial x_3} &= -\frac{(p+q)\bar{\zeta}(\kappa_1)}{A} \frac{\partial V^{\varepsilon}_{p,q}}{\partial \tau} - \frac{(p+q)\bar{c}_{66}\kappa_1}{\bar{c}_{44}\bar{\zeta}(\kappa_1)} \frac{\partial V^{\varepsilon}_{p,q}}{\partial \chi} \\
&+ \frac{i\omega\Gamma_1(\kappa_1)m(x_3/\varepsilon^2)}{\varepsilon}(p-q)V^{\varepsilon}_{p,q} \\
&- \frac{i\omega\Gamma_2(\kappa_1)m(x_3/\varepsilon^2)}{2\varepsilon} \cos\left(\frac{2\omega\bar{\zeta}(\kappa_1)x_3}{\varepsilon} \right) \left(qV^{\varepsilon}_{p,q-1} + qV^{\varepsilon}_{p,q+1} - pV^{\varepsilon}_{p+1,q} - pV^{\varepsilon}_{p-1,q} \right) \\
&+ \frac{\omega\Gamma_2(\kappa_1)m(x_3/\varepsilon^2)}{2\varepsilon} \sin\left(\frac{2\omega\bar{\zeta}(\kappa_1)x_3}{\varepsilon} \right) \left(qV^{\varepsilon}_{p,q-1} - pV^{\varepsilon}_{p+1,q} - qV^{\varepsilon}_{p,q+1} + pV^{\varepsilon}_{p-1,q} \right).
\end{aligned}
\tag{5.195}
$$

Equation (5.195) can be written as

$$\frac{d\mathbf{P}^\varepsilon}{dx_3} = \frac{1}{\varepsilon}F\big(\mathbf{P}^\varepsilon(x_3), m(x_3)\big) + G\big(\mathbf{P}^\varepsilon(x_3), m(x_3)\big), \tag{5.196}$$

where

$$G(x^\varepsilon(x_3), Y^\varepsilon(x_3)) = -\frac{(p+q)\bar{\zeta}(\kappa_1)}{A}\frac{\partial V^\varepsilon_{p,q}}{\partial \tau} - \frac{(p+q)\bar{c}_{66}\kappa_1}{\bar{c}_{44}\bar{\zeta}(\kappa_1)}\frac{\partial V^\varepsilon_{p,q}}{\partial \chi}, \tag{5.197}$$

Now define

$$g^{(1)} = m(x_3/\varepsilon^2), \tag{5.198}$$

$$g^{(2)} = m(x_3/\varepsilon^2)\sin\big(2\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon\big), \tag{5.199}$$

$$g^{(3)} = m(x_3/\varepsilon^2)\cos\big(\omega\bar{\zeta}(\kappa_1)x_3/\varepsilon\big), \tag{5.200}$$

$$F^{(1)}_{p,q}(X) = i\omega\Gamma_1(\kappa_1)(p-q)V^\varepsilon_{p,q}, \tag{5.201}$$

$$F^{(2)}_{p,q}(X) = \frac{\omega\Gamma_2(\kappa_1)}{2}\big(qV^\varepsilon_{p,q-1} - pV^\varepsilon_{p+1,q} - qV^\varepsilon_{p,q+1} + pV^\varepsilon_{p-1,q}\big), \tag{5.202}$$

$$F^{(3)}_{p,q}(X) = -\frac{i\omega\Gamma_2(\kappa_1)}{2}\big(qV^\varepsilon_{p,q-1} + qV^\varepsilon_{p,q+1} - pV^\varepsilon_{p+1,q} - pV^\varepsilon_{p-1,q}\big), \tag{5.203}$$

where

$$F(x,y,\mathcal{Q}) = \sum_{p=1}^{n=3} F^{(p)}(x)g^{(p)}(y,\mathcal{Q}), \quad x = V_{p,q},\ y = m(x_3/\varepsilon^2),\ \mathcal{Q} = x_3/\varepsilon^2. \tag{5.204}$$

The correlation matrix is

$$\mathbf{C} = \begin{bmatrix} \gamma & 0 & 0 \\ 0 & \frac{\gamma}{2} & 0 \\ 0 & 0 & \frac{\gamma}{2} \end{bmatrix}, \quad \tilde{\boldsymbol{\sigma}} = \begin{bmatrix} \sqrt{\gamma} & 0 & 0 \\ 0 & \sqrt{\frac{\gamma}{2}} & 0 \\ 0 & 0 & \sqrt{\frac{\gamma}{2}} \end{bmatrix}, \tag{5.205}$$

and

$$\sigma_{il}(x) = \sum_{p=1}^{n=3} \tilde{\boldsymbol{\sigma}}_{lp} F^{(p)}_i(x). \tag{5.206}$$

This gives the SDE

$$dV_i(x_3) = \sum_{l=1}^{n=3} \sigma_{il}(x)\, dW_l(x_3) + \left(\frac{1}{2} \sum_{k,m=1}^{n=3} \sum_{j=1}^{d} C_{pq} F_j^{(p)}(x) \frac{\partial F_i^{(m)}(x)}{\partial x_j} + G_i(x) \right) dx_3. \tag{5.207}$$

The only parts that contribute are $k = m = 1$, $k = m = 2$, $k = m = 3$ since the anti-symmetric parts of the correlation matrix (5.205) are zero. For $k = m = 1$

(5.208)

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^{d} \gamma F_j^{(1)} \frac{\partial F_{p,q}^{(1)}}{\partial x_j} &= \frac{\gamma}{2} F_{p,q}^{(1)} \frac{\partial}{\partial V_{p,q}} \Big(i\omega \Gamma_1(\kappa_1)(p-q) V_{p,q} \Big) \\
&= \frac{\gamma}{2} \Big(i\omega \Gamma_1(\kappa_1)(p-q) V_{p,q} \Big) \Big(i\omega \Gamma_1(\kappa_1)(p-q) \Big) \\
&= -\frac{\gamma \omega^2 \Gamma_1^2(\kappa_1)(p-q)^2}{2} V_{p,q},
\end{aligned} \tag{5.209}$$

and for $k = m = 2$

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^{d} \frac{\gamma}{2} F_j^{(2)} \frac{\partial F_{p,q}^{(2)}}{\partial x_j} = \frac{\gamma \omega^2 \Gamma_2^2(\kappa_1)}{16} \quad & \Big(q(q-1)V_{p,q-2} - qpV_{p+1,q-1} - q(q-1)V_{p,q} + qpV_{p-1,q-1} \\
& - pqV_{p+1,q-1} + p(p+1)V_{p+2,q} + pqV_{p+1,q+1} - p(p+1)V_{p,q} \\
& - q(q+1)V_{p,q} + qpV_{p+1,q+1} + q(q+1)V_{p,q+2} - qpV_{p-1,q+1} \\
& + pqV_{p-1,q-1} - p(p-1)V_{p,q} - pqV_{p-1,q+1} + p(p-1)V_{p-2,q} \Big).
\end{aligned} \tag{5.210}$$

For $k = m = 3$

$$\begin{aligned}
\frac{1}{2} \sum_{j=1}^{d} \frac{\gamma}{2} F_j^{(3)} \frac{\partial F_{p,q}^{(3)}}{\partial x_j} = \frac{\gamma \omega^2 \Gamma_2^2(\kappa_1)}{16} \quad & \Big(- q(q-1)V_{p,q-2} - q(q-1)V_{p,q} + qpV_{p+1,q-1} + qpV_{p-1,q-1} \\
& - q(q+1)V_{p,q} - q(q+1)V_{p,q+2} + qpV_{p+1,q+1} + qpV_{p-1,q+1} \\
& + pqV_{p+1,q-1} + pqV_{p+1,q+1} - p(p+1)V_{p+2,q} - p(p+1)V_{p,q}
\end{aligned}$$

$$+ pqV_{p-1,q-1} + pqV_{p-1,q+1} - p(p-1)V_{p,q} - p(p-1)V_{p-2,q}\Big). \tag{5.211}$$

Adding equations (5.211), (5.210) and (5.211) gives

$$\frac{1}{2}\sum_{k,m=1}^{n=3}\sum_{j=1}^{d} C_{pq}F_j^{(p)}(x)\frac{\partial F_i^{(m)}(x)}{\partial x_j}dx_3$$
$$= \frac{\gamma\omega^2\Gamma_2^2(\kappa_1)}{4}\Big(pq\big(V_{p-1,q-1}+V_{p+1,q+1}\big) - V_{p,q}(p^2+q^2) - 2V_{p,q}(p-q)^2\Big)dx_3,$$

and using the identity $-p^2 - q^2 = -(p-q)^2 - 2pq$, gives

$$= \frac{\gamma\omega^2\Gamma_2^2(\kappa_1)}{4}\Big(pq\big(V_{p-1,q-1}+V_{p+1,q+1}-2V_{p,q}\big) - 3V_{p,q}(p-q)^2\Big)dx_3. \tag{5.212}$$

The resulting SDE in Stratonovich form is then

$$\begin{aligned}
dV_{p,q} = &-(p+q)\bar{\zeta}(\kappa_1^j)\frac{\partial V_{p,q}}{\partial \tau}dx_3 - \frac{(p+q)\bar{c}_{66}\kappa_1^j}{\bar{c}_{44}\bar{\zeta}(\kappa_1^j)}\frac{\partial V_{p,q}}{\partial \chi}dx_3 \\
&+ i\omega_j\Gamma_1(\kappa_1^j)\sqrt{\gamma}(p-q)V_{p,q}\,dW_0(x_3) \\
&+ i\frac{\omega_j\Gamma_2(\kappa_1^j)\sqrt{\gamma}}{2\sqrt{2}}\Big(qV_{p,q-1} + qV_{p,q+1} - pV_{p+1,q} - pV_{p-1,q}\Big)d\widetilde{W}_j(x_3) \\
&+ \frac{\omega_j\Gamma_2(\kappa_1^j)\sqrt{\gamma}}{2\sqrt{2}}\Big(qV_{p,q-1} - pV_{p+1,q} - qV_{p,q+1} + pV_{p-1,q}\Big)d\overline{W_j}(x_3) \\
&+ \frac{\omega_j\Gamma_2(\kappa_1^j)\sqrt{\gamma}}{4}\Big(pq\big(V_{p-1,q-1}+V_{p+1,q+1}-2V_{p,q}\big) - 3V_{p,q}(p-q)^2\Big)dx_3.
\end{aligned} \tag{5.213}$$

Now set

$$\mathbb{E}\big[V_{p,q}^{\varepsilon}(\omega,\tau,\chi,\kappa_1,x_3)\big] \xrightarrow{\varepsilon\to 0} \mathbb{V}_{p,q}(\omega,\tau,\chi,\kappa_1,x_3), \tag{5.214}$$

to obtain

$$\begin{aligned}
\frac{\partial \mathbb{V}_{p,q}}{\partial x_3} = &-\frac{(p+q)\bar{\zeta}(\kappa_1)}{A}\frac{\partial \mathbb{V}_{p,q}}{\partial \tau} - \frac{(p+q)\bar{c}_{66}\kappa_1^j}{\bar{c}_{44}\bar{\zeta}(\kappa_1^j)}\frac{\partial \mathbb{V}_{p,q}}{\partial \chi} \\
&+ \frac{1}{L_{loc}(\omega_j,\kappa_1^j)}\Big(pq\big(\mathbb{V}_{p-1,q-1}+\mathbb{V}_{p+1,q+1}-2\mathbb{V}_{p,q}\big) - 3\mathbb{V}_{p,q}(p-q)^2\Big),
\end{aligned} \tag{5.215}$$

with initial condition $\mathbb{V}_{p,q}(\omega,\tau,\chi,\kappa_1,x_3=-L)=\mathbf{1}_0(p)\mathbf{1}_0(q)\delta(\tau)\delta(\chi)$. For simplicity, now focus on the symmetric problem where $p=q$, which satisfies the transport equation

$$\frac{\partial \mathbb{V}_{p,p}}{\partial x_3}=-\frac{2p\bar{\zeta}(\kappa_1)}{A}\frac{\partial \mathbb{V}_{p,p}}{\partial \tau}-\frac{2p\bar{c}_{66}\kappa_1^j}{\bar{c}_{44}\bar{\zeta}(\kappa_1^j)}\frac{\partial \mathbb{V}_{p,p}}{\partial \chi}$$
$$+\frac{p^2}{L_{loc}(\omega_j,\kappa_1^j)}\Big(\mathbb{V}_{p-1,p-1}+\mathbb{V}_{p+1,q+1}-2\mathbb{V}_{p,p}\Big), \tag{5.216}$$

with initial condition $\mathbb{V}_{p,p}(\omega,\tau,\chi,\kappa_1,x_3=-L)=\mathbf{1}_0(p)\delta(\tau)\delta(\chi)$. The localisation length (equation (5.159)) is

$$L_{loc}(\omega_j,\kappa_1^j)=\frac{4}{\gamma\Gamma_2^2(\kappa_1^j)\omega_j^2}. \tag{5.217}$$

The probabilistic solution [26] to the transport equations is (see Chapter 4.4.2)

$$\mathbb{V}_p(\omega_j,\kappa_1^j,\tau,\chi,-L,x_3)=\mathbb{E}\Bigg[\mathbf{1}_0(N_{x_3})\delta\Big(\tau-\frac{2\bar{\zeta}(\kappa_1^j)}{A}\int_{-L}^{x_3}N_{x_3'}dx_3'\Big)$$
$$\times\,\delta\Big(\chi-\frac{2\bar{c}_{66}\kappa_1^j}{\bar{c}_{44}\bar{\zeta}(\kappa_1^j)}\int_{-L}^{x_3}N_{x_3'}dx_3'\Big)\Bigg|N_{-L}=p\Bigg], \tag{5.218}$$

with initial condition $\mathbb{V}_p(\omega_j,\kappa_1^j,\tau,\chi,-L)=\mathbf{1}_0(p)\delta(\tau)\delta(\chi)$. The solution can be written as

$$\mathbb{V}_p(\omega_j,\kappa_1^j,\tau,\chi,-L,x_3)=\mathcal{W}_p(\omega_j,\kappa_1^j,\tau,-L,x_3)\delta\Big(\chi-\frac{2\bar{c}_{66}\kappa_1^j}{\bar{c}_{44}\bar{\zeta}(\kappa_1^j)}\Big), \tag{5.219}$$

where

$$\mathcal{W}_p(\omega_j,\kappa_1^j,\tau,-L,x_3)=\mathbb{E}\Bigg[\mathbf{1}_0(N_{x_3})\delta\Big(\tau-\frac{2\bar{\zeta}(\kappa_1^j)}{A}\int_{-L}^{x_3}N_{x_3'}dx_3'\Big)\Bigg|N_{-L}=p\Bigg]. \tag{5.220}$$

The inverse transform of equation (5.190) gives

$$U_{p,q}^{\varepsilon}(\omega,\kappa_1,h,\lambda,-L,x_3)=\int\int e^{ih(A\tau-(p+q)x_3\bar{\zeta}(\kappa_1))}e^{-i\lambda\omega(\chi-(p+q)\bar{c}_{66}\kappa_1x_3)/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}$$
$$\times\,V_{p,q}^{\varepsilon}(\omega,\kappa_1,\tau,\chi,-L,x_3)\,d\tau d\chi. \tag{5.221}$$

Taking the mean and applying the limit $\varepsilon \to 0$ gives

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[U^{\varepsilon}_{p,q}(\omega,\kappa_1,h,\lambda,-L,x_3)\Big] = \int\int e^{ih(A\tau-(p+q)x_3\bar{\zeta}(\kappa_1))} e^{-i\lambda\omega(\chi-(p+q)\bar{c}_{66}\kappa_1 x_3)/(\bar{c}_{44}\bar{\zeta}(\kappa_1))}$$
$$\times \mathcal{W}_p(\omega,\kappa_1,\tau,-L,x_3)\delta\bigg(\chi - \frac{2\bar{c}_{66}\kappa_1}{\bar{c}_{44}\bar{\zeta}(\kappa_1)}\bigg)\, d\tau d\chi. \tag{5.222}$$

Setting

$$A = \bar{\zeta}(\kappa_1)^2\bar{v}^2, \tag{5.223}$$

where, from equation (5.37)

$$\bar{\zeta}(\kappa_1) = \sqrt{\alpha\beta} = \sqrt{\frac{\rho}{\bar{c}_{44}}\bigg(1-\frac{\kappa_1^2\bar{c}_{66}}{\rho}\bigg)}$$
$$= \frac{1}{\bar{v}}\sqrt{\bigg(1-\frac{\kappa_1^2\bar{c}_{66}}{\rho}\bigg)}, \tag{5.224}$$

where

$$\bar{v} = \sqrt{\frac{\bar{c}_{44}}{\rho}}, \tag{5.225}$$

the Dirac delta functions in the solution (5.218) to equation (5.216), are only activated when

$$\frac{1}{\bar{\zeta}(\kappa_1)} = \frac{\tau\bar{v}^2}{2}, \tag{5.226}$$

so in equation (5.222), the Dirac delta function is only activated when

$$\chi = \frac{\bar{c}_{66}\kappa_1\tau\bar{v}^2}{\bar{c}_{44}}, \tag{5.227}$$

so equation (5.222) can be simplified to read

$$\lim_{\varepsilon\to 0} \mathbb{E}\Big[U^{\varepsilon}_{p,q}(\omega,\kappa_1,h,\lambda,-L,x_3)\Big] = \int \exp\big\{ih(A\tau-(p+q)x_3\bar{\zeta}(\kappa_1))\big\}$$

$$\times \exp\left\{-i\lambda\omega\left(\frac{\bar{c}_{66}\kappa_1\tau\bar{v}^2}{\bar{c}_{44}} - (p+q)\frac{\bar{c}_{66}\kappa_1 x_3}{\bar{c}_{44}\bar{\zeta}(\kappa_1)}\right)\right\}$$
$$\times \mathcal{W}_p(\omega,\kappa_1,\tau,-L,x_3)\, d\tau.$$

Hence

$$\lim_{\varepsilon\to 0}\mathbb{E}\left[U^{\varepsilon}_{p,q}(\omega,\kappa_1,h,\lambda,-L,x_3)\right] = \int \exp\left\{i\tau\left(h\bar{\zeta}(\kappa_1)^2\bar{v}^2 - \frac{\lambda\omega\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}\right)\right\}\mathcal{W}_p(\omega,\kappa_1,\tau,-L,x_3)\, d\tau$$
$$\times \exp\left\{2ipx_3\left(\frac{\lambda\omega\bar{c}_{66}\kappa_1}{\bar{c}_{44}\bar{\zeta}(\kappa_1)} - h\bar{\zeta}(\kappa_1)\right)\right\}, \tag{5.228}$$

in the case when $p = q$. The next section will focus on the case where $p = q = 1$. The solution in equation (5.228) will be used in conjunction with the expression for the stress (equation (5.34)) to derive an expression for the reflected intensity as a function of the lateral observation point x_1.

5.6 Mean Reflected Stress Intensity from a Line Source

From equation (5.34) the stress is written as

$$\hat{\tau}^{\varepsilon}(\omega,\kappa_1,0^+) = \sqrt{\bar{\beta}/\bar{\zeta}}\left(\hat{b}^{\varepsilon}(\omega,\kappa_1,0^+) + \hat{a}^{\varepsilon}(\omega,\kappa_1,0^+)\right)$$
$$= \sqrt{\bar{\beta}/\bar{\zeta}}\,\hat{a}^{\varepsilon}(\omega,\kappa_1,0^+) = \frac{\varepsilon\hat{f}(\omega)\left(R^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0) - 1\right)}{2}, \tag{5.229}$$

using the boundary condition $\hat{b}^{\varepsilon}(\omega,0^+) = 0$ (see Figure 5.1). Considering the direct emission from the point source (see [26]), now define the point of reception from the reflection intensity as $\mathbf{x} = (x_1,0,0)$ and use the Fourier transform (5.18) to express the stress in the time domain

$$\tau^{\varepsilon}(t,x_1,0^+) = \frac{1}{(2\pi\varepsilon)^2}\int\int e^{-\frac{i\omega(t-\kappa_1 x_1)}{\varepsilon}}\left(\frac{\varepsilon R^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0)\hat{f}(\omega)}{2}\right)\omega\, d\omega d\kappa_1,$$
$$= \frac{1}{8\pi^2\varepsilon}\int\int e^{-\frac{i\omega(t-\kappa_1 x_1)}{\varepsilon}} R^{\varepsilon}_{\omega_j,\kappa_1^j}(-L,0)\hat{f}(\omega)\omega\, d\omega d\kappa_1. \tag{5.230}$$

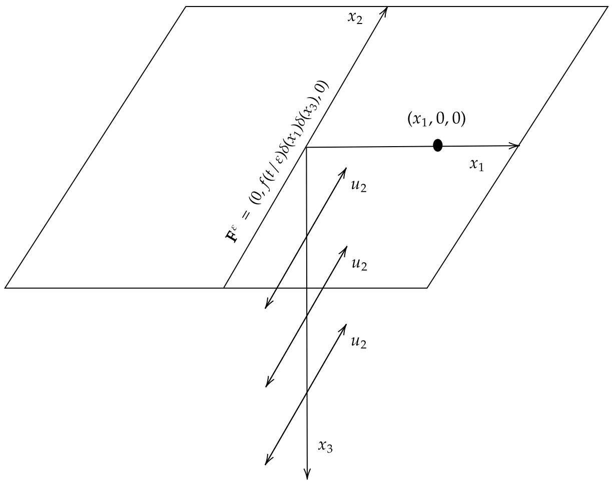

Figure 5.2: Three-dimensional schematic of the wave propagation problem where a line source is on the surface of the medium (at $x_3 = 0$) with a reception point $(x_1, 0, 0)$ where the reflected intensity of the stress field is measured.

The mean reflected intensity $\mathbb{E}[\tau^\varepsilon(t, x_1, x_3)^2]$ of the stress at the surface $x_3 = 0$ is of interest from a non-destructive testing scenario where engineers record the reflected power in a sensor array to enable them to image and characterise flaws in safety critical materials [76]. Consider the intensity at two close frequencies ω, ω' and slownesses κ_1, κ_1'. This gives an integral representation for the mean intensity which reads

$$\mathbb{E}[\tau^\varepsilon(t, x_1, 0)^2] = \frac{1}{64\pi^4\varepsilon^2} \int\int\int\int e^{-\frac{it(\omega-\omega')}{\varepsilon}} e^{\frac{ix_1(\omega\kappa_1 - \omega'\kappa_1')}{\varepsilon}} \times \mathbb{E}\Big(R^\varepsilon_{\omega_j,\kappa_1^j}(-L,0)\overline{R^\varepsilon_{\omega_j',(\kappa_1^j)'}(-L,0)}\Big)\hat{f}(\omega)\overline{\hat{f}(\omega)}\omega\omega'\, d\omega d\kappa_1 d\omega' d\kappa_1'. \tag{5.231}$$

Studying the two close slownesses and frequencies suggests the change of variables

$$\omega' = \omega - \varepsilon h, \tag{5.232}$$

$$\kappa_1' = \kappa_1 - \varepsilon\lambda, \tag{5.233}$$

where

$$d\omega' d\kappa_1' = \varepsilon^2 dh d\lambda, \quad \omega\omega' = \omega^2 + \mathcal{O}(\varepsilon^2). \tag{5.234}$$

The exponential arguments become

$$-\frac{it}{\varepsilon}\big(\omega - (\omega - \varepsilon h)\big) = -ith, \tag{5.235}$$

and

$$\begin{aligned}\frac{ix_1}{\varepsilon}\big(\omega\kappa_1 - (\omega - \varepsilon h)(\kappa_1 - \varepsilon\lambda)\big) &= ix_1(\overbrace{-\varepsilon h\lambda}^{\to 0} + h\kappa_1 + \omega\lambda) \\ &= ix_1(h\kappa_1 + \omega\lambda).\end{aligned} \tag{5.236}$$

This reduces equation (5.231) to

$$\begin{aligned}\mathbb{E}[\tau^\varepsilon(t, x_1, 0)^2] = \frac{1}{64\pi^4} \int & \exp\{-iht + ix_1(h\kappa_1 + \omega\lambda)\} \\ & \times \mathbb{E}\Big[R^\varepsilon_{\omega,\kappa_1} \overline{R^\varepsilon_{(\omega-\varepsilon h),(\kappa_1-\varepsilon\lambda)}}\Big] \Big|\hat{f}(\omega)\Big|^2 \omega^2 \, d\omega d\kappa_1 dh d\lambda.\end{aligned} \tag{5.237}$$

Equation (5.228) with $x_3 = 0$ gives

$$\begin{aligned}&\lim_{\varepsilon\to 0} \mathbb{E}\Big[U^\varepsilon_{1,1}(\omega, \kappa_1, h, \lambda, -L, x_3)\Big] \\ &= \int \exp\Big\{ i\tau\Big(h\bar{\zeta}(\kappa_1)^2\bar{v}^2 - \frac{\lambda\omega\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}\Big)\Big\} \mathcal{W}_1(\omega, \kappa_1, \tau, -L, x_3)\, d\tau.\end{aligned} \tag{5.238}$$

Taking the limit of equation (5.237) as $\varepsilon \to 0$, and inserting equation (5.238) at $x_3 = 0$ gives

$$\begin{aligned}\lim_{\varepsilon\to 0} \mathbb{E}[\tau^\varepsilon(t, x_1, 0)^2] = \frac{1}{64\pi^4} \int & \exp\big\{ih\big(x_1\kappa_1 - t + \tau\bar{\zeta}(\kappa_1)^2\bar{v}^2\big)\big\} \exp\Big\{i\lambda\Big(x_1\omega - \frac{\tau\omega\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}\Big)\Big\} \\ & \times \mathcal{W}_1(\omega, \kappa_1, \tau, -L, x_3 = 0)\Big|\hat{f}(\omega)\Big|^2 \omega^2 \, d\omega d\kappa_1 dh d\lambda d\tau.\end{aligned} \tag{5.239}$$

The Dirac delta function can be written in integral form [77] as

$$\delta(x-a) = \frac{1}{2\pi}\int e^{i(x-a)}dt, \quad a > 0, \tag{5.240}$$

so

$$\exp\Big\{ih\big(x_1\kappa_1 - t + \tau\bar{\zeta}(\kappa_1)^2\bar{v}^2\big)\Big\}\exp\left\{i\lambda\left(x_1\omega - \frac{\tau\omega\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}\right)\right\} \tag{5.241}$$

$$= 4\pi^2\delta\left(x_1\kappa_1 - t + \tau\bar{\zeta}(\kappa_1)^2\bar{v}^2\right)\delta\left(x_1\omega - \frac{\tau\omega\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}\right), \tag{5.242}$$

which means equation (5.239) can be written as

$$\begin{aligned}\lim_{\varepsilon\to 0}\mathbb{E}[\tau^{\varepsilon}(t,x_1,0)^2] = \frac{1}{16\pi^2}\int \delta\left(x_1\kappa_1 - t + \tau\bar{\zeta}(\kappa_1)^2\bar{v}^2\right)\delta\left(x_1\omega - \frac{\tau\omega\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}\right)\\ \times \mathcal{W}_1(\omega,\kappa_1,\tau,-L,x_3=0)\left|\hat{f}(\omega)\right|^2\omega^2\,d\omega d\kappa_1 d\tau.\end{aligned} \tag{5.243}$$

The first delta function gives

$$\begin{aligned}\delta\left(x_1\omega - \frac{\tau\omega\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}\right) &= \delta\left(\frac{\tau\omega\bar{c}_{66}\bar{v}^2}{\bar{c}_{44}}\left(\kappa_1 - \frac{\bar{c}_{44}x_1}{\bar{v}^2\tau\bar{c}_{66}}\right)\right)\\ &= \frac{\bar{c}_{44}}{\bar{v}^2\tau\omega\bar{c}_{66}}\delta\left(\kappa_1 - \frac{\bar{c}_{44}x_1}{\bar{v}^2\tau\bar{c}_{66}}\right),\end{aligned} \tag{5.244}$$

which is centered around the values

$$\kappa_1 = \frac{\bar{c}_{44}x_1}{\bar{v}^2\tau\bar{c}_{66}} := \mathcal{K}, \quad x_1 = \frac{\tau\bar{c}_{66}\kappa_1\bar{v}^2}{\bar{c}_{44}}. \tag{5.245}$$

The second delta can be written (using equations (5.224) and (5.225))

$$\begin{aligned}\delta\big(x_1\kappa_1 - t + \tau\bar{\zeta}(\kappa_1)^2\bar{v}^2\big) &= \delta\left(t - \tau\left(1 - \frac{\kappa_1^2\bar{c}_{66}}{\rho} - \frac{\tau\bar{c}_{66}\kappa_1^2\bar{c}_{44}}{\bar{c}_{44}\rho}\right)\right)\\ &= \delta(t-\tau),\end{aligned} \tag{5.246}$$

so $t = \tau$. This simplification of the delta functions reduces equation (5.243) to read

$$\lim_{\varepsilon\to 0}\mathbb{E}[\tau^{\varepsilon}(t,x_1,0^+)^2] = \frac{\bar{c}_{44}}{16\pi^2\bar{v}^2t\bar{c}_{66}}\int \mathcal{W}_1\Big(\omega,\mathcal{K},t,-L,x_3=0\Big)\times\left|\hat{f}(\omega)\right|^2\omega\,d\omega. \tag{5.247}$$

5.6.1 Solution For $L \to \infty$

From equation (5.216) (with $p = q$) that

$$\frac{\partial \mathcal{W}_p}{\partial x_3} + \frac{2p}{\bar{v}^2\bar{\zeta}(\kappa_1)}\frac{\partial \mathcal{W}_p}{\partial \tau} = \frac{p^2}{L_{loc}(\omega,\kappa_1)}\Big(\mathcal{W}_{p-1} + \mathcal{W}_{p+1} - 2\mathcal{W}_p\Big),$$

with initial condition $\mathcal{W}_p(\omega, \kappa_1, x_3 = -L) = \mathbf{1}(p)\delta(\tau)$. For $L \to \infty$ the spatial derivative is set to zero (see Section 4.4.1) and the equation becomes

$$\frac{\partial \mathcal{W}_p^\infty}{\partial \tau} = \frac{\bar{v}^2\bar{\zeta}(\kappa_1)p}{2L_{loc}(\omega,\kappa_1)}\Big(\mathcal{W}_{p-1}^\infty + \mathcal{W}_{p+1}^\infty - 2\mathcal{W}_p^\infty\Big). \tag{5.248}$$

Now introduce the non-dimensional parameters

$$\tilde{x}_3 = \frac{x_3 + L}{L_{loc}(\omega,\kappa_1)}, \quad \tilde{\tau} = \frac{\tau\bar{\zeta}(\kappa_1)\bar{v}^2}{L_{loc}(\omega,\kappa_1)}, \quad \widetilde{\mathcal{W}_p^\infty} = \frac{L_{loc}(\omega,\kappa_1)}{\bar{\zeta}(\kappa_1)\bar{v}^2}\mathcal{W}_p^\infty. \tag{5.249}$$

Similar to the treatment of equation (4.106) then

$$\mathcal{W}_p^\infty(\omega, t, \kappa_1, x_3 = 0) = \frac{\bar{\zeta}(\kappa_1)\bar{v}^2}{L_{loc}(\omega,\kappa_1)}P_p^\infty\left(\frac{t\bar{\zeta}(\kappa_1)\bar{v}^2}{L_{loc}(\omega,\kappa_1)}\right), \tag{5.250}$$

where

$$P_p^\infty(\tilde{\tau}) = \frac{2p\tilde{\tau}^{(p-1)}}{(2+\tilde{\tau})^{(p+1)}}. \tag{5.251}$$

Assuming that $\bar{c}_{44} \approx \bar{c}_{66}$ then, using equations (5.224), (5.225) and (5.245)

$$\begin{aligned}\bar{\zeta}(\mathcal{K}) &= \frac{1}{\bar{v}}\sqrt{1 - \frac{\bar{c}_{66}}{\rho}\left(\frac{\bar{c}_{44}x_1}{\bar{c}_{66}\bar{v}^2 t}\right)^2} \\ &\approx \frac{1}{\bar{v}}\sqrt{1 - \frac{\rho}{\bar{c}_{44}}\frac{x_1^2}{t^2}} \\ &= \frac{1}{\bar{v}}\sqrt{\frac{\rho}{\bar{c}_{44}}\left(\frac{\bar{c}_{44}}{\rho} - \frac{x_1^2}{t^2}\right)} \\ &= \frac{1}{\bar{v}^2 t}\sqrt{\bar{v}^2t^2 - x_1^2},\end{aligned} \tag{5.252}$$

so the limiting solution ($L \to \infty$) can be written as

$$\begin{aligned}
\mathcal{W}_1^{\infty}(\omega, t, \mathcal{K}, 0) &= \frac{\bar{\zeta}(\mathcal{K})\bar{v}^2}{L_{loc}(\omega, \mathcal{K})} P_1^{\infty}\left(\frac{t\bar{\zeta}(\mathcal{K})\bar{v}^2}{L_{loc}(\omega, \mathcal{K})}\right) \\
&= \frac{\bar{\zeta}(\mathcal{K})\bar{v}^2}{L_{loc}(\omega, \mathcal{K})} \frac{2}{\left(2 + \frac{t\bar{\zeta}(\mathcal{K})\bar{v}^2}{L_{loc}(\omega, \mathcal{K})}\right)^2} \\
&= \frac{2\bar{v}^2\sqrt{\bar{v}^2t^2 - x_1^2}}{\bar{v}^2 t L_{loc}(\omega, \mathcal{K})} \frac{1}{\left(2 + \frac{\bar{v}^2 t\sqrt{\bar{v}^2t^2 - x_1^2}}{\bar{v}^2 t L_{loc}(\omega, \mathcal{K})}\right)^2} \\
&= \frac{2L_{loc}(\omega, \mathcal{K})\sqrt{\bar{v}^2t^2 - x_1^2}}{t\big(2L_{loc}(\omega, \mathcal{K}) + \sqrt{\bar{v}^2t^2 - x_1^2}\big)^2}.
\end{aligned} \tag{5.253}$$

From equation (5.217) the localisation length is defined as

$$L_{loc}(\omega, \mathcal{K}) = \frac{4}{\gamma\Gamma_2^2(\mathcal{K})\omega^2}, \tag{5.254}$$

where equation (5.73) gives $\Gamma_2(\mathcal{K}) = \rho\sqrt{\bar{\alpha}/\bar{\beta}}$, and from equations (5.35) and (5.36) $\bar{\alpha} = 1/\bar{c}_{44}$ and $\bar{\beta} = \rho - \kappa_1^2\bar{c}_{44}$. Assuming that $\bar{c}_{44} \approx \bar{c}_{66}$ then

$$\begin{aligned}
\Gamma_2(\kappa_1) &\approx \frac{\rho}{\sqrt{\bar{c}_{44}}\big(\rho - \kappa_1^2\bar{c}_{44}\big)} \\
&= \sqrt{\frac{\rho}{\bar{c}_{44}}} \frac{\sqrt{\rho}}{\sqrt{\rho - \kappa_1^2\bar{c}_{44}}} \\
&= \frac{\sqrt{\rho}}{\bar{v}\big(\rho - \kappa_1^2\bar{c}_{44}\big)},
\end{aligned} \tag{5.255}$$

so

$$\begin{aligned}
\Gamma_2(\mathcal{K}) &= \frac{1}{\bar{v}}\sqrt{\frac{\rho}{\rho - \frac{\bar{c}_{44}^3 x_1^2}{\bar{c}_{66}^2 t^2\bar{v}^2}}} \\
&\approx \frac{1}{\bar{v}}\sqrt{\frac{\rho}{\rho - \frac{\bar{c}_{44} x_1^2}{t^2\bar{v}^2}}} \\
&= \frac{1}{\bar{v}}\sqrt{\frac{t^2\bar{v}^2}{t^2\bar{v}^2 - x_1^2}},
\end{aligned} \tag{5.256}$$

hence, using equations (5.254) and (5.256), the localisation length can be written

$$L_{loc}(\omega, \mathcal{K}) = \frac{4(t^2\bar{v}^2 - x_1^2)}{\gamma t^2 \omega^2}. \tag{5.257}$$

The mean reflected intensity can then be written using equations (5.247), (5.252), (5.253) and (5.257) as

$$\lim_{\varepsilon \to 0} \mathbb{E}[\tau^\varepsilon(t, x_1, 0)^2] = \frac{\gamma(\bar{v}^2 t^2 - x_1^2)^{3/2}}{2\pi^2 \bar{v}^2} \int \frac{\omega^3 \left|\hat{f}(\omega)\right|^2}{\left[8(\bar{v}^2 t^2 - x_1^2) + \gamma\omega^2 t^2 \sqrt{\bar{v}^2 t^2 - x_1^2}\right]^2} d\omega, \tag{5.258}$$

and with the non-dimensional scalings

$$\tilde{x}_1 = \frac{x_1}{L_3}, \quad \tilde{\omega} = \frac{\omega}{\omega_0}, \quad \tilde{t} = \frac{t}{t_0}, \quad \tilde{v} = \frac{\bar{v}}{v_0}, \quad \hat{f}(\tilde{\omega}) = \frac{\hat{f}(\omega)}{\rho v_0 L_3^2}, \quad \tilde{\tau}^\varepsilon = \frac{\tau^\varepsilon}{\rho\omega_0^2 L_3^2}, \tag{5.259}$$

the non-dimensionalised intensity reads

$$\lim_{\varepsilon \to 0} \mathbb{E}[(\tilde{\tau}^\varepsilon(\tilde{t}, \tilde{x}_1, 0))^2] = \frac{\tilde{\gamma}(\tilde{v}^2 \tilde{t}^2 - \tilde{x}_1^2)^{3/2}}{2\pi^2 \tilde{v}^2} \int \frac{\tilde{\omega}^3 \left|\hat{f}(\tilde{\omega})\right|^2}{\left(8(\tilde{v}^2 \tilde{t}^2 - \tilde{x}_1^2) + \tilde{\gamma}\tilde{\omega}^2 t^2 \sqrt{\tilde{v}^2 \tilde{t}^2 - \tilde{x}_1^2}\right)^2} d\tilde{\omega}. \tag{5.260}$$

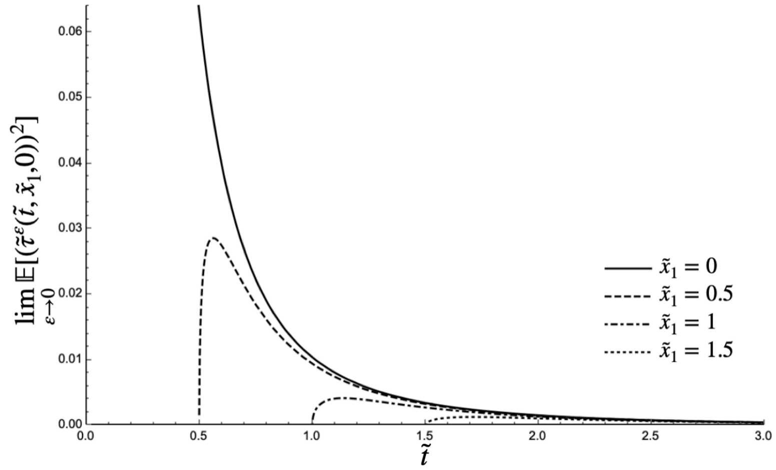

Figure 5.3: The non-dimensional mean intensity $\lim_{\varepsilon\to 0} \mathbb{E}[(\tilde{\tau}^{\varepsilon}(\tilde{t},\tilde{x}_1,0))^2]$ (equation (5.260)) as a function of time $\tilde{t}$ for different values of observation point $\tilde{x}_1$. The mean wave speed is $\tilde{v} = 1$, correlation integral $\tilde{\gamma} = 10$ and the input wave is $\hat{f}(\tilde{\omega}) = \tilde{\omega}^2 e^{-\tilde{\omega}^2}$.

Figure 5.3 shows how the received wave obeys the mean velocity $\bar{v}$ along the lateral x_1 direction. This Figure could be used to produce coverage maps via the TFM algorithm [3] or analytical assessment of TFM uncertainty quantification. Given the observation point x_1, the only waves which will contribute to the intensity in equation (5.258) will have a slowness $\mathcal{K} = (\bar{c}_{44}x_1)/(\bar{v}^2\tau\bar{c}_{66})$. This expression is valid (see Chapter 4.4.2) when

$$t \leq \frac{2L}{\bar{v}^2\bar{\zeta}(\mathcal{K})}, \tag{5.261}$$

which can be simplified to condition

$$\bar{v}t\sqrt{1 - \frac{x_1^2\rho}{t^2\bar{c}_{66}}} \leq 2L, \tag{5.262}$$

if $\bar{c}_{44} \approx \bar{c}_{66}$ then the condition becomes

$$0 \leq \sqrt{\bar{v}^2t^2 - x_1^2} \leq 2L, \tag{5.263}$$

which emphasises the importance of the incoherent part of the wave in the autocorrelation function as the wave has finite mean velocity $\bar{v}$ over time t.

5.7 Concluding Remarks

A model of a horizontally polarised elastic shear wave propagating in a randomly layered material has been constructed. The orientation of the internal material microstructure varies from one layer to the next, modelled by a Markov process. The problem was formulated in the strongly heterogeneous regime via a local rotation of the material's slowness surface as a function of the wave propagation direction x_3.

In Section 5.3 a diffusion approximation was used to derive the solution to the wave propagator equation (as $\varepsilon \to 0$) to obtain the moments of the transmission/reflection coefficients. In Section 5.4.1 an expression for the localisation length of the random medium was derived; the localisation length characterises the energy decay of the coherent wave due to interactions/multiple scattering inside the medium. In Section 5.5 a Ricatti equation for the reflection coefficient was derived, allowing a study into the frequency autocorrelation function for the reflected wave. This formulation of the problem presented a set of transport equations, capturing the moments of the product of reflection coefficients (see equation (5.182)).

A semi-analytical solution was obtained for the moments of the autocorrelation function in terms of a jump Markov process. In Section 5.6 the intensity of the reflected wave was studied. Using the solution to the transport equations in Section 5.5, an expression was obtained for the reflected intensity of the wave at an observation point x_1 on the surface ($x_3 = 0$) of the material. Considering the limiting case ($L \to \infty$) of the problem, an analytical solution was derived. Plots of the reflected intensity for varying values of x_1 are shown in Figure 5.3. The limit expression for the reflected intensity derived here could be used to create coverage maps, utilising the Total Focusing Method.

Chapter 6

Conclusions

The focus of this thesis has been to study elastic wave propagation in layered random media. The main motivation to study such problems in this area of mathematics stems from the field of non-destructive testing, which has became ever more important in recent years as the cost of in-situ testing has decreased and the complexity of safety critical engineering materials has increased. The detailed microstructure present in modern engineering materials create fine heterogeneities which will interact with a probing mechanical wave. Since the exact variation in material geometry is not known for a specific material sample, these variations are replaced with a random process whose statistical properties align with that of the medium. The received wave will vary from one sample to the next and so it makes sense to describe the wave properties as a distribution and to use a probabilistic framework for modelling purposes. Only via computationally prohibitive Monte Carlo simulations can deterministic models of wave propagation provide a similar characterisation of wave propagation in such materials. The governing stochastic differential equations capture the multiple scattering effects caused by coupling between the wave and the medium, enabling an analytical approach to be undertaken.

6.0.1 Results

Monochromatic elastic shear wave propagation in an austenitic steel weld was examined in Chapter 2. A weak form of the Fokker-Planck equation was derived and subsequently solved via a finite element package. The numerical solution to the Fokker-Planck equation was used to compute statistical moments of the power transmission coefficient. This led to a parametric study on the effect of the degree of anisotropy (of an austenitic steel weld sample) on the transmitted energy. Chapter 3 again considered monochromatic shear wave propagation, but in a different class of material. Using a different form of stress tensor allowed for an extensive study into the moments of transmission and reflection coefficients without the need for a finite element approach to solve the probabilistic (Fokker-Planck) equation. A key parameter which emerged in the governing equations related the degree of anisotropy in the medium to the decay in the energy of the coherent wave. It was shown that a change in this wave anisotropy parameter (ν) affected the standard deviation of the power transmission coefficient; an important quantity when investigating NDT array imaging applications.

Chapter 4 extended the analysis in Chapter 3 by modelling a source with multiple frequencies; a broadband pulse. This allowed a study of the moments of the autocorrelation function of the reflection coefficient. The main result was the agreement between the analytical and Monte-Carlo solution of the final transport equation for the moments of the frequency autocorrelation function. This model also provides estimates for attenuation factors in randomly layered media without the computational cost of including such factors into a numerical method. The work presented in Chapter 5 studies multi-frequency horizontally polarised elastic shear waves in a randomly layered medium. This model extended the work presented in previous chapters to account for a line source. The equations which followed showed how the material slowness surface affected the travel time through a two-dimensional medium. Using limit theorems to solve the resulting stochastic differential equations, gave a solution for the intensity of the reflected wave at different points in space along the surface of the medium. This solution could be used to calculate coverage maps of a material by summing travel times at different observation points.

6.0.2 Future Work

The main aim is to take the model in Chapter 5 and produce coverage maps using the derived expressions for the reflected energy. This is the first step in studying performance characteristics of the Total Focusing Method [3]. The goal would then be to perform an experimental study on a relevant material (such as austenitic steel [44]) to compare the model derived in this thesis with experimental results. It would be interesting to use the model to perform flaw detection in a highly heterogeneous randomly layered medium. Furthermore, it would be interesting to use the models presented here to study time reversal [78], [79], [80] with the aim of applying it to detection of flaws in layered, solid media [81], [82], [51].

Bibliography

[1] B. Raj, T. Jayakumar, and B. P. C. Rao. “Non-destructive testing and evaluation for structural integrity”. In: *Integrity Of Engineering Components* Vol. 20 (1995), pp. 5–38.

[2] K. M. M. Tant, A. J. Mulholland, and A. Gachagan. “A model-based approach to crack sizing with ultrasonic arrays”. In: *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control* Vol. 62 (2015), pp. 915–926.

[3] B. W. Drinkwater and P. D Wilcox. “Ultrasonic arrays for non-destructive evaluation: A review”. In: *NDT & E International, Issue 7.* Vol. 39 (2006), pp. 525–541.

[4] C. Li, D. Pain, P. D. Wilcox, and B. W. Drinkwater. “Imaging composite material using ultrasonic arrays”. In: *NDT & E International* Vol. 53 (2013), pp. 8–17.

[5] L. Yang, K. Hsu, B. Baughman, D. Godfrey, and F. Medina. *Additive Manufacturing of Metals: The Technology, Materials, Design and Production.* Springer, (2017).

[6] Y.Kok, X. Tan, P. Wang, M. L.S. Nai, N. Hiang. Loh, E. Liu, and S. B. Tor. “Anisotropy and heterogeneity of microstructure and mechanical properties in metal additive manufacturing: A critical review”. In: *Materials & Design* Vol. 139 (2018), pp. 565–586.

[7] V. A. Popovich, E. Borisov, A. A. Popovich, V. Sufiiarov, D. V. Masaylo, and L. Alzina. “Functionally graded Inconel 718 processed by additive manufacturing: Crystallographic texture, anisotropy of microstructure and mechanical properties”. In: *Materials & Design* Vol. 114 (2017), pp. 441–449.

[8] J. Garnier and K. Sølna. "Apparent Attenuation of Shear Waves Propagating through a Randomly Anisotropic Medium". In: *Stochastics and Dynamics, No. 4.* Vol. 16 (2015).

[9] I. D. Abrahams and G. R. Wickham. "The Propagation of Elastic Waves in a Certain Class of Inhomogeneous Anisotropic Materials. I. The Refraction of a Horizontally Polarized Shear Wave Source". In: *London, The Royal Society Publishing, Issue 1898.* Vol. 14 (2006).

[10] G. Ghoshal and J. A. Turner. "Numerical model of longitudinal wave scattering in polycrystals". In: *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control* Vol. 56 (2009).

[11] A. V. Pamel, G. Sha, S. I. Rokhlin, and M. J. S. Lowe. "Finite-element modelling of elastic wave propagation and scattering within heterogeneous media". In: *Proc. R. Soc. A.* Vol. 473 (2017).

[12] R. B. Thompson, F. J. Margetan, P. Haldipur, L. Yu, A. Li, P. Panetta, and H.Wasan. "Scattering of elastic waves in simple and complex polycrystals". In: *Wave Motion, Issue 5.* Vol. 45 (2008), pp. 655, 674.

[13] K. M. M. Tant, E. Galetti, A. J. Mulholland, A. Curtis, and A. Gachagan. "Effective grain orientation mapping of complex and locally anisotropic media for improved imaging in ultrasonic non-destructive testing". In: *Inverse Problems in Science and Engineering* Vol. 28 (2020), pp. 1694–1718.

[14] Z. Fan, A. Mark, M. J. S. Lowe, and P. J. Withers. "Nonintrusive estimation of anisotropic stiffness maps of heterogeneous steel welds for the improvement of ultrasonic array inspection". In: *IEEE Transactions on Ultrasonics, Ferroelectrics, and Frequency Control* Vol. 62 (2015), pp. 1530–1543.

[15] J. W. Hunt, M. Arditi, and F. S. Foster. "Ultrasound Transducers for Pulse-Echo Medical Imaging". In: *IEEE Transactions on Biomedical Engineering, No. 8.* (1983), pp. 453–481.

[16] P. N. T. Wells and H. Liang. “Medical ultrasound: imaging of soft tissue strain and elasticity”. In: *Journal of The Royal Society Interface* Vol. 8 (2011), pp. 1521–1549.

[17] M. V. de Hoop and A. T. de Hoop. “Elastic wave up/down decomposition in inhomogeneous and anisotropic media: an operator approach and its approximations”. In: *Wave Motion, Issue 1.* Vol. 20 (1994), pp. 57–82.

[18] Y. Ricard, S. Durand, J. P. Montagner, and F. Chambat. “Is there seismic attenuation in the mantle”. In: *Earth and Planetary Science Letters* Vol. 388 (2014), pp. 257–264.

[19] Z. Feng, B. Guo, and G. T. Schuster. “Multiparameter deblurring filter and its application to elastic migration and inversion”. In: *Geophysics* (2018).

[20] R. F. O’Doherty and N. A. Anstey. “Reflections on Amplitudes”. In: *Geophysical Prospecting* Vol. 19 (1971), pp. 430–458.

[21] J. W. Hosken, S. K. Mayo, and A. T. Walden. “The influence of reflection coefficient statistics on the seismic method: scattering attenuation and transmission wavelets”. In: *Proceedings of the Royal Society of London. Series A: Mathematical and Physical Sciences* Vol. 439 (1992), pp. 1–23.

[22] M. Asch, W. Kohler, G. Papanicolaou, M. Postel, and B. S. White. “Frequency Content of Randomly Scattered Signals”. In: *SIAM Rev.* Vol. 33 (1991), pp. 519–625.

[23] L. Ryzhik, G. Papanicolaou, and J. B. Keller. “Transport equations for elastic and other waves in random media”. In: *Wave Motion* Vol. 24 (1996), pp. 327–370.

[24] W. Kohler and G. Papanicolaou. “Power statistics for wave propagation in one-dimension and comparison with radiative transport theory. II”. In: *Journal of Mathematical Physics* Vol. 15 (1973), pp. 2186–2197.

[25] G. Papanicolaou. “Wave Propagation in a One-Dimensional Random Medium”. In: *SIAM Journal on Applied Mathematics, No. 1.* Vol. 21 (1971).

[26] J. P. Fouque, J. Garnier, G. Papanicolaou, and K. Sølna. *Wave Propagation and Time Reversal in Randomly Layered Media.* New York, Springer, (2007).

[27] R. Burridge, G. Papanicolaou, and B. S. White. "Statistics for pulse reflection from a randomly layered medium". In: *SIAM Journal on Applied Mathematics* Vol. 47 (1987), pp. 146–168.

[28] J. Garnier and K. Sølna. "Paraxial Coupling of Electromagnetic Waves in Random Media". In: *Multiscale Model. Simul.* Vol. 7 (2009), pp. 1928–1955.

[29] L. Borcea and J. Garnier. "Polarization effects for electromagnetic wave propagation in random media". In: *arXiv: Analysis of PDEs* (2015).

[30] S. Chevrot. "Finite-frequency vertical tomography: A new method for high-resolution imaging of upper mantle anistropy". In: *Geophysical Journal International, Issue 2.* (2006), pp. 641–657.

[31] L. Borcea, G. C Papanicolaou, and C. Tsogka. "Interferometric array imaging in clutter". In: *Inverse Problems* Vol. 21 (2005), pp. 1419–1460.

[32] L. Borcea, G. Papanicolaou, and C. Tsogka. "Adaptive Time-Frequency Detection and Filtering for Imaging in Heavy Clutter". In: *SIAM J. Imaging Sci.* Vol. 4 (2011), pp. 827–849.

[33] L. Borcea, J. Garnier, G. Papanicolaou, and C. Tsogka. "Coherent interferometric imaging, time gating, and beamforming". In: *Inverse Problems* Vol. 27 (2011), p. 065008.

[34] R. F. Bass. *Stochastic Processes.* Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, (2011).

[35] J. B. Keller. "Stochastic equations and wave propagation in random media". In: (1964).

[36] M. Spies. "Elastic waves in homogeneous and layered transversely isotropic media: Plane waves and Gaussian wave packets. A general approach." In: *The Journal of the Acoustical Society of America, Issue 1748.* Vol. 95 (1994).

[37] G. Papanicolaou and S. Weinryb. "A functional limit theorem for waves reflected by a random medium". In: *Applied Mathematics and Optimization* Vol. 30 (1994), pp. 307–334.

[38] S. W. Smith, H. G. Pavy, and O. T. von Ramm. "High-speed ultrasound volumetric imaging system. I. Transducer design and beam steering". In: *IEEE Transactions on Ultrasonics, Ferroelectrics and Frequency Control* Vol. 38 (1991), pp. 100–108.

[39] L. Zhao, E. Laredo, O. Ryan, A. Yazdkhasti, H. Kim, R. Ganye, T. K. Horiuchi, and M. Yu. "Ultrasound beam steering with flattened acoustic metamaterial Luneburg lens". In: *arXiv: Applied Physics* (2019).

[40] C. Holmes, B. W. Drinkwater, and P. D. Wilcox. "Post-processing of the full matrix of ultrasonic transmit-receive array data for non-destructive evaluation". In: *NDT & E International* Vol. 38 (2005), pp. 701–711.

[41] J. O. Davies, F. Simonetti, M. J. S. Lowe, and P. Cawley. "Review of Synthetically Focused Guided Wave Imaging Techniques With Application to Defect Sizing". In: *Quantitative Nondestructive Evaluation* Vol. 820 (2006), pp. 142–149.

[42] J. Lambert, A. Pedron, G. Gens, F. Bimbard, L. Lacassagne, and E. Iakovleva. "Performance evaluation of total focusing method on GPP and GPU". In: *Proceedings of the 2012 Conference on Design and Architectures for Signal and Image Processing* (2012), pp. 1–8.

[43] J. Zhang, B. W. Drinkwater, P. D. Wilcox, and A. J. Hunter. "Defect detection using ultrasonic arrays: The multi-mode total focusing method". In: *NDT & E International* Vol. 43 (2010), pp. 123–133.

[44] J. A. Brooks and A. Thompson. "Microstructural development and solidification cracking susceptibility of austenitic stainless steel welds". In: *International Materials Reviews* Vol. 36 (1991), pp. 16–44.

[45] K. M. M. Tant, E. Galetti, A. J. Mulholland, A. Curtis, and A. Gachagan. "A transdimensional Bayesian approach to ultrasonic travel-time tomography for non-destructive testing". In: *Inverse Problems Vol 34.* (2018).

[46] Y. Zhang, M. Brady, and S. M. Smith. "Segmentation of brain MR images through a hidden Markov random field model and the expectation-maximization algorithm". In: *IEEE Transactions on Medical Imaging* Vol. 20 (2001), pp. 45–57.

[47] J. J. Lewandowski and M. Seifi. "Metal Additive Manufacturing: A Review of Mechanical Properties". In: *Annual Review of Materials Research* Vol. 46 (2016), pp. 151–186.

[48] J. Garnier and K. Sølna. "On effective attenuation in multiscale composite media". In: *Waves in Random and Complex Media* Vol. 25 (2015), pp. 482–505.

[49] J. Garnier and G. Papanicolaou. "A review of wave propagation in random media". In: *Cambridge University Press* (2016).

[50] L. Borcea, F. González del Cueto, G. Papanicolaou, and C. Tsogka. "Filtering Random Layering Effects in Imaging". In: *Multiscale Model. Simul.* Vol. 8 (2010), pp. 751–781.

[51] J. P. Fouque and O. V. Poliannikov. "Time reversal detection in one-dimensional random media". In: *Inverse Problems* Vol. 22 (2006), pp. 903–922.

[52] D. T. Gillespie. "Exact stochastic simulation of coupled chemical reactions". In: *The Journal of Physical Chemistry* Vol. 81.25 (1977), pp. 2340–2361.

[53] S. P. Meyn and R. L. Tweedie. "Markov Chains and Stochastic Stability". In: *Communications and Control Engineering Series*. 1993.

[54] D. J. Higham. "An Algorithmic Introduction to Numerical Simulation of Stochastic Differential Equations". In: *SIAM Rev.* Vol. 43 (2001), pp. 525–546.

[55] W. J. Parnell and I. D. Abrahams. "Dynamic homogenization in periodic fibre reinforced media. Quasi-static limit for SH waves." In: *Wave Motion* Vol. 43 (2006), pp. 474–498.

[56] A. Ferguson, A. J. Mulholland, K. M. M. Tant, and M. Foondun. "Ultrasonic Wave Propagation in Randomly Layered Heterogeneous Media (Under Review)". In: *Wave Motion* (2022).

[57] G. Harvey, A. Tweedie, C. Carpentier, and P. Reynolds. "Finite Element Analysis of Ultrasonic Phased Array Inspections on Anisotropic Welds". In: *AIP Conf Proc, San Diego, California* Vol. 1335 (2010), pp. 827–834.

[58] C. Carpentier C. Nageswaran and Y. Y. Tse. "Microstructural quantification, modelling and array ultrasonics to improve the inspection of austenitic welds". In: *Insight-Non-Destructive Testing and Condition Monitoring, No. 12* Vol. 51 (2009), pp. 660–666.

[59] M. S. Alnæs, J. Blechta, J. Hake, A. Johansson, B. Kehlet, A. Logg, C. Richardson, J. Ring, M. E. Rognes, and G. N. Wells. "The FEniCS Project Version 1.5". In: *Archive of Numerical Software* Vol. 3.100 (2015).

[60] J. Garnier and K. Sølna. "Parabolic and White-Noise Approximations For Elastic Waves In Random Media". In: *Wave Motion, No. 4.* Vol. 46 (2009), pp. 237–254.

[61] J. Magnus and H. Neudecker. *Matrix Differential Calculus with Applications in Statistics and Econometrics.* New Jersey, Wiley, (2019).

[62] S. Bittanti, A. J. Laub, and J. C. Willems. *The Riccati Equation.* Springer Berlin Heidelberg, (1991).

[63] A. Ferguson, A. J. Mulholland, and K. M. M. Tant. "Modelling of Ultrasonic Waves in Layered Elastic Heterogeneous Materials". In: *2021 IEEE International Ultrasonics Symposium (IUS).* 2021, pp. 1–4.

[64] J. N. Elgin. "The Fokker-Planck Equation: Methods of Solution and Applications". In: *Journal of Modern Optics* Vol. 31 (1984), pp. 1206–1207.

[65] M. Brajanovski, B. Gurevich, and M. Schoenberg. "A model for P-wave attenuation and dispersion in a porous medium permeated by aligned fractures". In: *Geophysical Journal International* Vol. 163 (2005), pp. 372–384.

[66] G. Papanicolaou and S. Weinryb. "A Functional Limit Theorem for Waves Reflected by a Random Medium". In: *Applied Mathematics and Optimization* Vol. 30 (1994), pp. 307–334.

[67] D. P. G. Russo and J. H. Harris. “Wave propagation in anisotropic thin-film optical waveguides”. In: *Journal of the Optical Society of America* Vol. 63 (1973), pp. 138–145.

[68] J. D. Achenbach. *Wave Propagation In Elastic Solids.* North Holland, (1973).

[69] I. D. Tsvankin. “Anisotropic parameters and P-wave velocity for orthorhombic media”. In: *Geophysics* Vol. 62 (1997), pp. 1292–1309.

[70] V. Munoz, M. Perrin, M. Pastor, H. Welemane, A. Cantarel, and M. Karama. “Determination of the elastic properties in CFRP composites: comparison of different approaches based on tensile tests and ultrasonic characterization”. In: *Advances in Aircraft and Spacecraft Science, No. 3* Vol. 2 (2015), pp. 249–260.

[71] D. Zwillinger. *Handbook of Differential Equations, 3rd ed.* Boston, MA: Academic Press, (1997).

[72] V. A. Ditkin and A. P. Prudnikow. “Integral Transforms and Operational Calculus”. In: *Pergamon Press, New York* (1965).

[73] D. S. Mitrinovic and J. D. Kevckic. “The Cauchy Method of Residues: Theory and Applications”. In: 1984.

[74] H. Anton, I. C. Bivens, and S. Davis. “Single and Multivariable Calculus: Early Transcendentals”. In: *Whitman Mathematics* (2002).

[75] Berger M.A. *Markov Jump Processes. In: An Introduction to Probability and Stochastic Processes.* Springer, New York, NY, (1993).

[76] Z. Zhou, W. Li, Y. Li, and S. Li. “On ultrasonic array methods for additive-manufactured titanium alloy.” In: *International Symposium on Structural Health Monitoring and Nondestructive Testing* (2018).

[77] D. W. Lozier. “NIST Digital Library of Mathematical Functions”. In: *Annals of Mathematics and Artificial Intelligence* Vol. 38 (2004), pp. 105–119.

[78] L. Borcea, G. Papanicolaou, C. Tsogka, and J. G. Berryman. “Imaging and time reversal in random media”. In: *Inverse Problems* Vol. 18 (2002), pp. 1247–1279.

[79] G. Papanicolaou. “Imaging and time reversal in random media”. In: *Journal of the Acoustical Society of America* Vol. 110 (2001), pp. 2633–2633.

[80] G. Papanicolaou, L. Ryzhik, and K. Sølna. “The Parabolic Wave Approximation And Time Reversal”. In: *Matematica Contemporanea* (2002).

[81] N. O’Donoughue and J. M. F. Moura. “Gaussian Target Detection in Multipath Clutter With Single-Antenna Time Reversal”. In: *IEEE Transactions on Signal Processing* Vol. 61 (2013), pp. 3733–3744.

[82] J. P. Fouque, J. Garnier, A. Nachbin, and K. Sølna. “Imaging of a Dissipative Layer in a Random Medium Using a Time Reversal Method”. In: *Proceedings MC2QMC* (2004).