University of Strathclyde
Department of Mathematics and Statistics

Mathematical Analysis of Discrete Coagulation-Fragmentation Equations

by
Ann Louise Smith

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Abstract

The work contained in this thesis is a mathematical analysis of discrete coagulation-fragmentation equations. Initially semigroup techniques are used to investigate the existence and uniqueness of a global, non-negative strong solution to a mass-conserving model. Minimal restrictions are imposed on the fragmentation rate but it is required that the coagulation rate is uniformly bounded. It is also shown how a technique described by Ziff and McGrady in [58] leads to an explicit solution for a particular pure fragmentation equation. The semigroup techniques are then applied to a pure fragmentation model with discrete mass loss to derive similar existence and uniqueness results. A multi-component model with reformation terms is then investigated. Finally, the coagulation-fragmentation equation with an added time-dependent source term is analysed. Existence and uniqueness of a strong solution can also be proved in this case provided the source term satisfies certain conditions.

Contents

Chapter 1

Introduction and Literature Review

Suppose that we have a system of particles in which each individual particle can be characterised by a discrete quantity such as mass, size or age. (We shall call this quantity size for the moment.) The evolution with time of the distribution of each size class in such a system can be described by a population balance equation. This type of equation models processes such as coagulation and fragmentation in which the numbers in each size class either increase or decrease. In [17, Chapter 13] a population balance equation (PBE) is used by Belleni-Morante to describe a population of bacteria in a culture. The author uses a method first employed by Kato in [32] to determine when a solution to the PBE exists. Kato's method has since been extended by Voigt [55], Arlotti [6] and Banasiak [9] to give a general perturbation result, often referred to as the Kato-Voigt perturbation theorem. We shall use these more recent perturbation results to establish the well-posedness of a number of coagulation-fragmentation equations.

1.1 The Coagulation-Fragmentation Equation

In this thesis we shall consider a system of clusters of particles which are initially described by their mass. We shall later move on to the situation where a cluster can be characterised by two variables, namely mass and diameter. We shall describe different models in which we allow a mixture of processes to occur. On the one hand clusters can break up or fragment. On the other hand, two clusters will also be allowed to join together or coagulate. We shall also introduce the processes of surface recession and reformation.

We begin by describing the general discrete coagulation-fragmentation equation. Suppose a cluster of particles is characterised by its mass. We consider each individual particle to have mass one and we call this a *monomer*. A cluster of n monomers, or n-mer, will thus have mass n. Let $u_n(t)$, $n = 1, 2, \ldots$ denote the number concentration of clusters having mass n. The discrete coagulation-fragmentation equation is given by

$$\begin{aligned} \frac{du_n(t)}{dt} &= -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \\ &\quad + \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j} u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j} u_n(t) u_j(t) \qquad (1.1) \end{aligned}$$

where, for all n and j, a_n, $b_{n,j}$ and $k_{n,j}$ are non-negative constants. Note that the last summation is to be defined as zero when $n = 1$. Here a_n is the fragmentation rate of an n-mer, $b_{n,j}$ is a distribution which gives the average number of n-mers produced after the break up of a j-mer where $j > n$ and $k_{n,j}$ is the coagulation rate of an n-mer with a j-mer.

We shall now give an interpretation of the terms in (1.1). The first term on the right-hand side of (1.1) gives the loss of n-mers due to them fragmenting. The second term gives the gain in n-mers due to larger clusters fragmenting. The third term on the right-hand side of (1.1) represents the gain in n-mers due to two smaller clusters coalescing. Note that the factor of $\frac{1}{2}$ ensures that we do not double count the process of an $(n-j)$-mer coalescing with a j-mer and the same j-mer coalescing with the same $(n-j)$-mer. The last term gives the loss of n-mers due to them coalescing with other clusters. If we set $a_n = 0$ for all n we have the pure coagulation equation which was first looked at by Smoluchowski in [53] and similarly if we set $k_{n,j} = 0$ for all n, j we have the pure fragmentation equation.

Note that the system described above allows for multiple fragmentation to occur, i.e. a fragmenting cluster can break into two or more pieces. In some of the previous studies of the coagulation-fragmentation equation, which will be discussed below, only binary fragmentation has been considered. Naturally we require our system to be mass-conserving (or density-conserving) unless we have a built-in mass loss mechanism. This leads to the assumptions

$$\sum_{n=1}^{j-1} nb_{n,j} = j, \quad j = 2, 3, \dots, \text{ and } a_1 = 0. \tag{1.2}$$

The total mass of the system is given by

$$M(t) = \sum_{n=1}^{\infty} nu_n(t), \quad t \geq 0 \tag{1.3}$$

and for mass to be conserved we require that

$$\sum_{n=1}^{\infty} nu_n(t) = \sum_{n=1}^{\infty} nu_n(0), \quad t > 0. \tag{1.4}$$

As will be discussed later, we shall rewrite (1.1) as an Abstract Cauchy Problem (ACP) in an appropriate Banach space. The norm on the Banach space is chosen to correspond to the summation used to calculate $M(t)$.

1.2 Previous Work: The Discrete Model

We shall now describe some of the work that has been carried out previously. Many of the early engineering papers investigate exact solutions to the discrete coagulation-fragmentation equation without any consideration of the criteria required for existence or uniqueness of solutions. In [52], Simha looks at the pure fragmentation equation for polymers. He considers monomers to be joined in a chain and N_j represents the number of molecules consisting of j monomers having $j - 1$ links. He also places an upper bound n on the size of molecules so that the system of equations is finite-dimensional. The author makes use of the rate constant $k_i^{(j)}$ to characterise breakage. Simha considers three different situations: random breakage, surface recession and a fragmentation rate which depends on the number of links in the chain. In the case of random breakage, $k_i^{(j)} = k$ for all

i, j and the fragmentation equation becomes

$$\frac{dN_j}{dt} = 2k \sum_{i=j+1}^{n} N_i - k(j-1)N_j. \tag{1.5}$$

Note that if we set $k = 1$ above then we have the same equation as is discussed by Ziff and McGrady in [58]. We shall give details of how a general solution to (1.5) can be obtained in Chapter 3. In [52], a solution is given for homogeneous initial conditions. In the case of surface recession the equation becomes

$$\frac{dN_j}{dt} = 2k^{(j+1)}N_{j+1} - 2k^{(j)}N_j, \ \ n \geq j \geq 2. \tag{1.6}$$

We shall discuss the case where surface recession terms are combined with fragmentation terms with built-in mass loss in Chapter 4.

In [7] the fragmentation equation for polymers is again considered, this time by Bak and Bak. The cases of bonds breaking with equal probability and of molecules only splitting in the middle are considered. The generating function

$$G(x,t) = \sum_{n=1}^{\infty} c_n(t)x^n \tag{1.7}$$

is used to solve the rate equations, where $c_n(t)$ denotes the concentration of molecules with molecular weight nM, M constant. The uniqueness of density-conserving mild solutions to the discrete coagulation-fragmentation equation is considered in [8] by Ball and Carr. The authors work with a finite-dimensional version of (1.1) with binary fragmentation and prove that this system has a unique non-negative, density-conserving mild solution. Ball and Carr then go on to show, by taking limits of solutions of the truncated system, that there exists a mild solution to the infinite-dimensional coagulation-fragmentation equation for a coagulation rate $k_{n,j}$ (in our notation) satisfying

$$k_{n,j} \leq K(n+j), \ \ \forall n, j, \ \ K \text{ constant},$$

and this solution is density-conserving. They then go on to describe conditions under which all solutions conserve density and show, by imposing growth conditions on the coagulation and fragmentation rates, that there exists a unique mild solution to the infinite-dimensional coagulation-fragmentation equation. Finally they prove that if the coagulation rate is uniformly bounded, i.e. $k_{n,j} \leq K$ for all n, j, then there exists a unique mild, density-conserving solution. In [40], the existence and uniqueness of mild, density-conserving solutions for the discrete coagulation-fragmentation equation with multiple fragmentation is investigated by Laurençot in a similar manner to [8]. Note that we shall prove a similar result in Chapter 3 for strong solutions to the full coagulation-fragmentation equation with multiple fragmentation using semigroup theory. Our theory does not rely on taking limits of truncated systems and establishes the existence and uniqueness of a *strong* solution.

The coagulation-fragmentation equation with collisional breakage is studied in [41] by Laurençot and Wrzosek. Collisional breakage refers to the fragmentation of a cluster after collision with another cluster. The collisional breakage rate is proportional to the number densities of the size classes of the two colliding clusters and allows for mass transfer between two colliding clusters. The probability of colliding clusters with mass i and mass j merging into one cluster is introduced and the coagulation-fragmentation involves a collision rate, $a_{i,j}$, for clusters. The existence of mild solutions for general collision rates is shown by taking limits of a truncated system. These solutions were not shown to be density-conserving but it can be proved that density is non-increasing. The authors then show the existence of density conserving solutions when the collision rate satisfies

$$a_{i,j} \leq A(i+j), \quad \forall i, j, \ A \text{ constant.}$$

The large-time behaviour of the system is then studied for certain cases.

In [21] and [23] Carr and da Costa investigate the asymptotic behaviour of solutions to the discrete coagulation-fragmentation with binary fragmentation with certain assumptions made on the fragmentation rate. Again, solutions are taken to be limits of mild solutions of the truncated coagulation-fragmentation equation. The authors then study equilibrium solutions, i.e. time-independent solutions, for these conditions. In [26] da Costa also studies the equilibrium solutions of a finite-dimensional version of Smoluchowski's pure coagulation equation.

1.3 Previous Work: The Continuous Model

The coagulation-fragmentation equation can also be written in a continuous form. We shall now discuss some of the previous studies of the continuous equation. Semigroup theory has not been used much to analyse the discrete coagulation-fragmentation equation but it has been applied to the continuous model. As with the discrete case, many of the early engineering papers focused on finding exact solutions for particular rate coefficients without worrying about existence and uniqueness results. In [28], the multiple fragmentation equation with an added continuous mass loss term is studied by Edwards, Cai and Han. Exact and asymptotic solutions are found for certain rate coefficients. Note that this paper also describes a discrete fragmentation equation with discrete mass loss, namely,

$$\frac{du_n(t)}{dt} = -nu_n(t) + 2\sum_{j=n+1}^{\infty} u_j(t) \tag{1.8}$$

in our notation. We shall deduce the form of the unique strong solution to (1.8) in Chapter 4 and show why the exact solution given in [28] is not a strong solution.

Aizenman and Bak [1] study the existence and uniqueness of solutions to the full coagulation-fragmentation equation for rate coefficients of a specific type using semigroup techniques and contraction mapping arguments. The authors also show uniform convergence to equilibrium solutions for a general class of initial data.

In [46], McLaughlin, Lamb and McBride consider a truncated multiple fragmentation equation and show via semigroup theory the existence and uniqueness of a strong mass-conserving solution for a locally bounded fragmentation rate, i.e. $a(x) \leq C_n$ for all $x \in (0, n],\ n > 0$ where the sequence $\{C_n\}$ may be unbounded. They then take limits of the truncated solutions and show that there is a unique strong, mass-conserving solution to the infinite-dimensional system. The results in [46] are then extended in [47] to show existence and uniqueness of a solution to the full coagulation-fragmentation equation, with a uniformly bounded coagulation rate, using a contraction mapping argument. McLaughlin et al. then go on [48] to study a non-autonomous multiple fragmentation equation, i.e. the fragmentation rate depends on time. The method of proving the existence of a unique strong solution for the truncated system and then taking limits is again used. This time, solutions are given in the form of evolution families.

In more recent work, the Kato-Voigt perturbation theorem has been used to prove the existence of unique strong solutions to the coagulation-fragmentation equation. The application of this theorem removes the need to look at the truncated system first. In [38], Lamb shows that the solution given by the application of the Kato-Voigt perturbation theorem is indeed the same as the solution found in the truncation/limit approach. In [12], Banasiak and Lamb use semigroup techniques to show existence of a strong solution to the pure fragmentation equation with continuous and discrete mass loss. The results in [12] are then extended in [13] to show the global existence and uniqueness of strong solutions to the full coagulation-fragmentation equation with continuous and discrete mass loss. Banasiak and Lamb also use semigroup theory in [15] to investigate existence and uniqueness results for the continuous coagulation-fragmentation model in a more abstract setting. The same authors also analyse coagulation, fragmentation and growth processes in a size-structured population in [14].

1.4 Previous Work: Numerical Techniques

It should be noted that some numerical techniques have been developed for solving the coagulation-fragmentation equation. Although we shall not pursue this, we shall briefly describe some of the literature. The non-linearity of the coagulation-fragmentation equation makes it particularly difficult to write down exact solutions. Over the years, numerical techniques for solving the coagulation-fragmentation equation have been used to combat this problem and give an idea as to how the system behaves under certain conditions. The main obstacle when solving the coagulation-fragmentation equation numerically is verifying that the numerical technique is providing you with the correct solution. Such verifications can be carried out by comparing numerical solutions to experimental data or comparing them to the exact solutions known for particular cases. In [20], a method of characteristics is developed to solve the partial differential equation satisfied by the moment generating function

$$\phi(x,t) = \sum_{k=1}^{\infty} x^k N_k(t), \quad |x| \leq 1$$

where N_k denotes the number density of clusters of k particles. The authors consider a product coagulation rate, i.e. $k_{n,j} = Knj$, K constant, and a constant fragmentation rate. Kumar and Ramkrishna developed the Fixed Pivot Technique for solving population balance models using discretisation in [37]. The Cell Average Technique, which is shown to be more accurate than the Fixed Pivot Technique for solving the pure coagulation equation, is described in [36]. Kostoglou extends the Cell Average Technique in [33] to create an even more accurate method for solving the coagulation equation numerically. Kumar et al. then develop the Cell Average Technique to solve multi-dimensional aggregation equations in [35]. These methods split the cluster size range into a finite number of intervals and assume that the clusters in each interval are of the same size. These methods are known as internally consistent since the balance equations for each interval are constructed in such a way as to ensure mass conservation (or any other integral property of the system) is maintained.

1.5 The Shattering and Gelation Phenomena

In some particular cases of the pure coagulation and pure fragmentation equations, the system appears to lose mass even though no mass-loss has been built into the model. These phenomena are known as gelation in the coagulation case and shattering in the fragmentation case. Gelation can occur when clusters coagulate at a 'fast' rate to form an infinitely large cluster in finite time and it appears that mass has been lost from the system. Shattering occurs essentially from the opposite effect to gelation, i.e. clusters fragment into infinitely small dust particles which are not detected. Note that setting $a_1 = 0$ in the discrete equation will eliminate shattering. Investigations into the conditions under which these phenomena occur have been carried out in [20], [22], [26], [30], [42], [45], [54] and [59].

In [22], Carr and da Costa obtain a condition for instantaneous gelation in the discrete coagulation equation, namely, if the coagulation rate satisfies

$$n^\alpha + j^\alpha \le k_{n,j} \le (nj)^\beta \quad \text{for } \beta > \alpha > 1$$

(in our notation) then any non-zero size distribution has zero (instantaneous) gelation time. In [20], Brunelle, Owens and van Roessel consider the discrete coagulation-fragmentation equation with a constant binary fragmentation rate, b, and the bilinear coagulation rate

$$k_{n,j} = (\alpha + \beta n)(\alpha + \beta j) \quad \text{for } \alpha, \beta \in \mathbb{R}.$$

Characteristic equations are used to help calculate the gelation time and post-gelation mass when $\alpha = 0$, $\beta > 0$ and $b = 0$ and when $\alpha = 0$, $\beta > 0$ and $b > 0$. In [26] the existence of an infinite family of gelling solutions of a truncated version of the Smoluchowski coagulation equation is proved for a general coagulation rate by da Costa. In [54], van Dongen proves that, for a homogeneous coagulation rate $k(i,j)$ satisfying, for $j >> i$,

$$k(ai, aj) = a^{\lambda} k(i,j) = a^{\lambda} k(j,i),$$
$$k(i,j) \sim i^{\mu} j^{\nu}, \quad j \to \infty, i \text{ fixed}, \ \lambda = \mu + \nu,$$

instantaneous gelation occurs if and only if $\nu > 1$.

The shattering effect in the continuous fragmentation equation is considered by McGrady and Ziff in [45]. The fragmentation rate and distribution are given by

$$a(y) = y^{\beta+1}$$

and

$$b(x|y) = f(y) x^{\nu}$$

respectively, where

$$f(y) = \frac{\nu + 2}{y^{\nu+1}}, \quad \nu > -2$$

is a mass conservation condition. It is deduced that, for $\beta < -1$ and for all admissible values of ν, the mass of the system is time-dependent. This reflects a loss of mass due to the production of infinitely small particles.

In [30], Ernst and Pagonabarraga compare the shattering effect for collisional and linear continuous fragmentation models for the fragmentation rate

$$a(x) = x^{\alpha}.$$

It is shown that in the collisional breakage model, shattering always occurs at a finite time, $t \neq 0$. In the linear fragmentation model, shattering occurs instantaneously, i.e. at $t = 0$, for $\alpha < 0$. Banasiak applies semigroup theory to ascertain conditions under which shattering occurs in the continuous pure fragmentation model in [10].

We will not concern ourselves with the phenomena described above as we shall be looking for solutions which exhibit the physical properties built into the model. For example, if mass is to be conserved, we are interested in mass-conserving solutions. If our system has built-in mass loss, then we require that we can calculate the mass lost exactly.

1.6 Plan of Action

The main focus of this thesis is to investigate the existence of unique strong, non-negative, conservative solutions to various forms of the coagulation-fragmentation equation. We shall make use of the theory of semigroups of operators in a similar fashion to [12], [13], [14] and [15], to show under which conditions these solutions exist. In Chapter 2 we begin by introducing some preliminary results from semigroup theory and we outline the procedure we shall follow to show the existence of unique strong solutions. In Chapter 3 we analyse the coagulation-fragmentation equation with mass-conservation. Initially we look at the linear fragmentation terms before the semilinear coagulation terms are added in. We shall prove that for general a_n and $b_{n,j}$ (satisfying the mass-conservation condition) and a uniformly bounded coagulation rate, i.e. $k_{n,j} \leq k$ for all n, j, where k is a constant, that there exists a unique globally defined, non-negative, mass-conserving strong solution to the corresponding ACP. We also use a technique described by Ziff and McGrady in [58] to determine the exact solution for the pure fragmentation equation with $a_n = n - 1$ and $b_{n,j} = \frac{2}{j-1}$. In Chapter 4 the pure fragmentation equation with discrete mass loss in investigated. Again, we can show that for general a_n and $b_{n,j}$ there exists a unique strong solution. Moreover, we can calculate the expected mass loss. We also look at the case where there is an added surface recession process. In Chapter 5 we look at a multi-component model with reformation terms introduced by Wattis in [56]. We make some modifications to the model and for each of the new models we can prove the existence of unique strong solutions for a general fragmentation rate and a uniformly bounded coagulation rate. Finally, in Chapter 6 we apply the theory found in [49] to show that a strong solution to the coagulation-fragmentation with a time-dependent source term exists provided that the source term is continuously differentiable with respect to time.

Chapter 2

Preliminary Results and Definitions

We shall begin by introducing the basic theory that will be applied throughout this thesis. As mentioned in the previous chapter, the main aim is to use the theory of semigroups of operators to show that there exist unique, non-negative solutions to a variety of coagulation-fragmentation equations. A C_0-semigroup (strongly continuous semigroup) is defined as follows.

Definition 2.1. *A C_0-semigroup of bounded linear operators on a Banach space X is a family*

$$\{T(t)\}_{t\geq 0} \subseteq B(X)$$

such that

(i) $T(0) = I$, the identity operator on X

(ii) $T(s)T(t) = T(s+t)$ for all $s, t \geq 0$

(iii) for each fixed $f \in X$, $T(t)f \to f$ as $t \to 0^+$.

We can find a growth bound for the norms of the operators $\{T(t)\}_{t\geq 0}$.

Theorem 2.2. *Let X be a complex Banach space and let $\{T(t)\} \subseteq B(X)$ be a C_0-semigroup. Then there exist constants $\omega \in \mathbb{R}$ and $M \geq 1$ such that*

$$\|T(t)\| \leq Me^{\omega t} \qquad (t \geq 0). \tag{2.1}$$

We write $C_0(M,\omega)$ to denote the class of semigroups satisfying (2.1).

Proof: See [29, Proposition I.5.5].

□

It is now possible to prove the following.

Lemma 2.3. *Let $\{T(t)\} \subseteq B(X)$ be a C_0-semigroup of class $C_0(M,\omega)$ and, for $t \geq 0$, define $S(t) \in B(X)$ by*

$$S(t) = e^{-\omega t} T(t). \tag{2.2}$$

Then the family $\{S(t)\}_{t\geq 0}$ is a C_0-semigroup of class $C_0(M,0)$.

Proof: See [43, pp. 42-43].

□

Definition 2.4. *Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup of bounded linear operators on a Banach space X. The family $\{T(t)\}_{t\geq 0}$ is called*

(i) a C_0-semigroup of ***isometries*** *if $\|T(t)f\| = \|f\|$ for all $t \geq 0$, $f \in X$*

(ii) a C_0-semigroup of ***contractions*** *if $\|T(t)\| \leq 1$ for all $t \geq 0$.*

The infinitesimal generator of a C_0-semigroup is defined by

Definition 2.5. *Let $\{T(t)\}_{t\geq 0} \subseteq B(X)$ be a C_0-semigroup of bounded linear operators on a Banach space X and for each $t > 0$, let $Q_t \in B(X)$ be defined by*

$$Q_t = \frac{T(t) - I}{t}. \tag{2.3}$$

The ***infinitesimal generator*** *of $\{T(t)\}$ is the operator $Q : X \supseteq D(Q) \to X$ defined by*

$$D(Q) = \{\, f \in X : Q_t f \text{ tends to a limit (in } X\text{) as } t \to 0_+\}$$

$$Qf = \lim_{t\to 0_+} Q_t f. \tag{2.4}$$

We shall need the following definition of the resolvent operator.

Definition 2.6. *Let X be a complex Banach space and let $A : X \supseteq D(A) \to X$ be a linear operator.*

(i) The ***resolvent set****, $\rho(A)$, of A is the set of complex numbers*

$$\rho(A) = \left\{\lambda \in \mathbb{C} : (\lambda I - A)^{-1} \in B(X)\right\}. \tag{2.5}$$

(ii) The ***spectrum****, $\sigma(A)$, of A is the complement of $\rho(A)$,*

$$\sigma(A) = \mathbb{C} - \rho(A). \tag{2.6}$$

(iii) For $\lambda \in \rho(A)$, we write

$$R(\lambda, A) \equiv (\lambda I - A)^{-1} \in B(X). \tag{2.7}$$

We call $R(\lambda, A)$ the ***resolvent operator*** *of A (at λ).*

The following theorem, developed by Hille and Yosida, gives us the conditions under which Q is the infinitesimal generator of a C_0-semigroup of contractions.

Theorem 2.7 (The Hille-Yosida Theorem)**.** *Let X be a complex Banach space and let $Q : X \supseteq D(Q) \to X$. Then Q is the infinitesimal generator of a C_0-semigroup of contractions on X if and only if*

(i) Q is a closed linear operator and $D(Q)$ is a dense linear subspace of X

and

(ii) $\rho(Q)$ contains $\{\lambda \in \mathbb{R} : \lambda > 0\}$ and $\|R(\lambda, Q)\| \leq \frac{1}{\lambda}$ $\forall \lambda > 0$

or

(ii)' $\rho(Q)$ contains $\{\lambda \in \mathbb{C} : Re\ \lambda > 0\}$ and $\|R(\lambda, Q)\| \leq \frac{1}{Re\ \lambda}$ for $\lambda \in \mathbb{C}$ with $Re\ \lambda > 0$.

Proof: See [43, p. 65].

□

The Hille-Yosida theorem was then extended to give conditions under which Q is the infinitesimal generator of a general C_0-semigroup.

Theorem 2.8 (The Hille-Yosida-Phillips-Miyadera-Feller Theorem)**.** *Let X be a complex Banach space and let $Q : X \supseteq D(Q) \to X$ be a linear operator. Then Q is the infinitesimal generator of a C_0-semigroup of class $C_0(M, \omega)$ on X, $M \geq 1, \omega \in \mathbb{R}$, if and only if*

(i) Q is a closed linear operator and $D(Q)$ is a dense linear subspace of X

and

(ii) $\rho(Q)$ contains all real numbers λ such that $\lambda > \omega$ and

$$\|R(\lambda, Q)^n\| \leq \frac{M}{(\lambda - \omega)^n} \qquad \lambda > \omega, \quad n = 1, 2, \ldots \tag{2.8}$$

or

(ii)' $\rho(Q)$ contains all complex numbers λ with $Re\lambda > \omega$ and

$$\|R(\lambda, Q)^n\| \leq \frac{M}{(Re\lambda - \omega)^n} \qquad Re\ \lambda > \omega, \quad n = 1, 2, \ldots. \tag{2.9}$$

Proof: See [43, pp. 69-70].

□

Throughout the thesis we shall be studying abstract Cauchy problems (ACPs) which often take the form

$$\begin{aligned} \frac{du}{dt} &= Au(t), \quad t > 0, \\ u(0) &= u_0 \in X, \end{aligned} \tag{2.10}$$

where $A : X \supseteq D(A) \to X$ is a linear operator and X is a complex Banach space. Let us define a solution to the ACP (2.10) by the following

Definition 2.9. *Let X be a Banach space, $A : X \supseteq D(A) \to X$ be a linear operator and let $u_0 \in X$. A function $u : [0, \infty) \to X$ is a solution of the ACP (2.10) if*

(i) u is continuous on $[0, \infty)$

(ii) u is continuously differentiable on $(0, \infty)$

(iii) $u(t) \in D(A)$ *for* $t > 0$

(iv) the equations in (2.10) are satisfied.

It is important to show that our ACPs are well-posed, i.e. a solution exists, this solution is unique and it depends continuously on the initial data. The following results give conditions under which such solutions exist.

Theorem 2.10. *In the ACP (2.10) let* A *be the infinitesimal generator of a* C_0*-semigroup* $\{T(t)\}_{t\geq 0} \subseteq B(X)$ *and let* $u_0 \in D(A)$*. Then the problem has a unique solution* u *(in the sense of Definition 2.9) given by*

$$u(t) = T(t)u_0, \qquad t \geq 0. \tag{2.11}$$

Proof : See [43, p. 111].

□

Theorem 2.11. *Let* X *be a Banach space,* A *be a linear operator and let* $\{g_n\}_{n=1}^{\infty}$ *be a sequence in* $D(A)$ *converging to zero with respect to the norm on* X *. Let* u_n *be the unique solution (in the sense of Definition 2.9) of the ACP*

$$\begin{aligned} \frac{du}{dt} &= Au, \qquad t > 0, \\ u(0) &= g_n. \end{aligned} \tag{2.12}$$

Then $\{u_n\}_{n=1}^{\infty}$ *converges uniformly to zero on any interval of the form* $0 \leq t \leq t_0$ *with* $t_0 > 0$*.*

Proof : See [43, p. 114]

□

A non-homogeneous form for the ACP (2.10) would be

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + f(t), \qquad t > 0, \\ u(0) &= u_0 \end{aligned} \tag{2.13}$$

where $f : [0, \infty) \to X$ is a given vector-valued function.

A solution $u : [0, \infty) \to X$ of the ACP (2.13) is as in *Definition 2.9* (modified appropriately). The form for the solution is given in the following theorem.

Theorem 2.12. *Let X be a Banach space and let $A : X \supseteq D(A) \to X$ be the infinitesimal generator of a C_0-semigroup $\{T(t)\}_{t\geq 0} \subseteq B(X)$. If the function $f : [0, \infty) \to X$ is continuously differentiable and $u_0 \in D(A)$ then the ACP (2.13) has a unique solution given by*

$$u(t) = T(t)u_0 + \int_0^t T(t-s)f(s)ds \qquad t \geq 0. \tag{2.14}$$

Proof : See [43, pp. 116-117].

□

We shall now give some results which are true when working in a Banach space of the type $Y = L^1(\Omega, \mu)$ where (Ω, μ) is a measure space. Particular classes of C_0-semigroups can be defined as follows.

Definition 2.13. *(See [11, p.159]) Let $\{T(t)\}_{t\geq 0}$ be a C_0-semigroup on $Y = L^1(\Omega, \mu)$ with the usual norm, where (Ω, μ) is a measure space.*

*(i) $\{T(t)\}_{t\geq 0}$ is a **substochastic** semigroup if, for each $t \geq 0$, $T(t) \geq 0$ (i.e. $T(t)f \in Y^+$ for all $f \in Y^+$) and $\|T(t)\| \leq 1$.*

*(ii) $\{T(t)\}_{t\geq 0}$ is a **stochastic** semigroup if, in addition, $\|T(t)f\| = \|f\|$ for all $t \geq 0$ and $f \in Y^+$*

where Y^+ is defined by

$$Y^+ := \{f \in Y : f \geq 0\},$$

and similarly for other sets.

We shall also require the following result for multiplication semigroups.

Lemma 2.14 (Multiplication Semigroup)**.** *Let $q : \Omega \to \mathbb{C}$ be a measurable function such that $-Re\, q$ is (essentially) bounded above and define*

$$T_Q(t)f := e^{-qt}f, \;\; f \in Y, t \geq 0,$$

where $Y = L^1(\Omega, \mu)$. Then $\{T_Q(t)\}_{t\geq 0}$ is a C_0-semigroup on Y and has infinitesimal generator, Q, given by the multiplication operator

$$Qf := -qf, \;\; D(Q) := \{f \in Y : qf \in Y\}.$$

Proof: See [29, p.65].

□

Example 2.15. *Suppose we define an operator A on Y by*

$$Af := -af, \quad D(A) := \{f \in Y : af \in Y\},$$

where $a : \Omega \to \mathbb{R}^+$. Then A is now the infinitesimal generator of the substochastic semigroup $\{T_A(t)\}_{t\geq 0}$ on Y given by

$$T_A(t)f = e^{-at}f, \quad f \in Y, t \geq 0.$$

Later on we shall apply this result for the specific case of sequences $\{a_n\}$ and a weighted l^1 space.

We shall use the following decomposition for real-valued functions throughout the thesis.

Notation 2.16. *Let f be a real-valued function in $Y = L^1(\Omega, \mu)$. Then we may express this as*

$$f = f^+ - f^- \tag{2.15}$$

where $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$.

Let $u : [0, \infty) \to Y$. Throughout this thesis we shall be studying the existence and uniqueness of solutions to various forms of the Abstract Cauchy Problem

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + Bu(t) + Ku(t) + N(t) \\ \lim_{t\to 0^+} u(t) &= f \end{aligned} \tag{2.16}$$

where A is the generator of a substochastic semigroup on some space Y, B is a linear perturbation of A, K is a nonlinear operator on Y and $N(t)$ is a continuously differentiable function of t in Y. In each of the different variations of the ACP (2.16) that we study we follow the same basic procedure to deduce that there exists a unique strong solution to our ACP. This routine is outlined below.

Firstly we shall define what is meant by a mild and a strong solution of an

ACP of the form

$$\begin{aligned} \frac{du(t)}{dt} &= Gu(t) + F(t, u(t)) \\ \lim_{t \to 0^+} u(t) &= f \end{aligned} \tag{2.17}$$

where G is the generator of a substochastic semigroup $\{T_G(t)\}_{t\geq 0}$ on Y and $F : [0, \infty) \times Y \to Y$.

Definition 2.17. *A mild solution of (2.17) is a continuous solution* $u : [0, \infty) \to Y$ *of*

$$u(t) = T_G(t)f + \int_0^t T_G(t-s)F(s, u(s))ds. \tag{2.18}$$

Definition 2.18. *A function* u *is a strong (strict) solution of (2.17) on* $[0, \infty)$ *if* u *is continuous on* $[0, \infty)$*, continuously differentiable on* $(0, \infty)$*,* $u(t) \in D(G)$ *for* $t \in [0, \infty)$ *and (2.17) is satisfied on* $[0, \infty)$*.*

In order to establish whether a unique strong solution of any version of (2.16) exists we begin by studying the linear part, i.e.

$$\frac{du(t)}{dt} = Au(t) + Bu(t), \quad t > 0, \tag{2.19}$$

$$\lim_{t \to 0^+} u(t) = f \in D(A + B). \tag{2.20}$$

The following theorem is used throughout the thesis to show that there exists a smallest extension of $A + B$ which generates a substochastic semigroup.

Theorem 2.19 (Kato-Voigt Perturbation Theorem)**.** *Let* $(A, D(A)), (B, D(B))$ *be two linear operators in the Banach space* Y *such that*

(i) $(A, D(A))$ *generates a substochastic semigroup* $\{T_A(t)\}_{t\geq 0}$ *on* Y*,*

(ii) $D(B) \supseteq D(A)$ *and* $Bf \geq 0$ *for all* $f \in [D(B)]^+$*,*

(iii) for all $f \in [D(A)]^+$, $\int_\Omega (Af + Bf)d\mu =: -c(f) \leq 0.$ (2.21)

Then there exists a smallest extension G *of* $A+B$ *which generates a substochastic semigroup* $\{T_G(t)\}_{t\geq 0}$ *on* Y*.*

Proof: See [11, Corollary 5.17].

□

Note 2.20. *The substochastic semigroup $\{T_G(t)\}$ is the smallest substochastic semigroup generated by an extension of $(A+B)$ in terms of an order relation on X. If G' is another extension of $(A+B)$ then*

$$T_{G'}(t)f - T_G(t)f \geq 0 \text{ for all } f \in D(G)^+.$$

We shall make use of the following theory found in [11, Section 6.3] to prove that, in certain circumstances, $G = \overline{A+B}$, the closure of the operator $(A+B, D(A))$. Firstly we shall need some preliminary results involving extensions of the operators in the model. As above, let $Y = L^1(\Omega, \mu)$ where (Ω, μ) is a measure space.

Definition 2.21. *The sets of functions E and E_f are defined by*

$$E := L_0(\Omega, d\mu) = \text{the set of } \mu\text{-measurable functions defined on } \Omega \text{ and taking values in } \bar{\mathbb{R}}, \text{ the set of extended real numbers;} \tag{2.22}$$

$$E_f := \text{the subspace of } E \text{ consisting of functions that are finite almost everywhere.} \tag{2.23}$$

It follows that $Y \subset E_f \subset E$.

Let $\mathbb{A}$, $\mathbb{B}$ and $\mathbb{L}$ denote extensions of A, B and $R(1, A)$ respectively. We require these extensions to have domains and ranges in E_f. We also need $\mathbb{B}$ and $\mathbb{L}$ to be positive operators on their domains. These extensions are obtained in the following way.

Definition 2.22. *Let $F \subset E$ be defined by*

$$F := \{f \in E : \text{ for any non-negative and non-decreasing sequence } (f_n) \text{ satisfying } \sup_n f_n = |f| \text{ we have } \sup_n R(1, A)f_n \in Y\}. \tag{2.24}$$

We now make the extra assumptions that $f \in D(B) \Leftrightarrow f^+, f^- \in D(B)$ and, for any two non-decreasing sequences $(f_n), (g_n) \in D(B)^+$

$$\sup_n f_n = \sup_n g_n \implies \sup_n Bf_n = \sup_n Bg_n.$$

These additional assumptions are satisfied in the cases where we apply these extension techniques.

Definition 2.23. *Let $H \subset Y$ be defined by*

$$H := \{f \in Y : \text{ if } (f_n) \text{ is any non-negative, non-decreasing sequence of elements of } D(B) \text{ such that } \sup_n f_n = |f| \text{ then } \sup_n Bf_n < \infty \text{ a.e.}\}. \quad (2.25)$$

It can be shown that $D(A) \subseteq H \subseteq Y \subseteq F \subseteq E$. We can now state the following properties ([11, Lemma 6.17]):

(i) $f \in F^+$ and $0 \le g \le f \Rightarrow g \in F^+$,

(ii) $F \subset E_f$,

(iii) $f \in F^+$ and $f = \sup f_n = \sup g_n$, where $(f_n), (g_n) \subset Y^+$ are both non-decreasing $\Rightarrow \sup R(1,A)f_n = \sup R(1,A)g_n$.

We shall now define appropriate extended mappings.

Definition 2.24. *Let $\mathbb{B}$, $\mathbb{L}$ be defined by*

$$\mathbb{B} : D(\mathbb{B})^+ \to E_f^+; \;\; D(\mathbb{B}) = H;$$

$$\mathbb{B}f := \sup_n Bf_n, \quad f \in D(\mathbb{B})^+,$$

$$\mathbb{L} : F^+ \to Y^+;$$

$$\mathbb{L}f := \sup_n R(1,A)f_n, \quad f \in F^+,$$

where $0 \le f_n \le f_{n+1}$ $\forall n$ and $\sup_n f_n = f$. Note that $\mathbb{B}$ and $\mathbb{L}$ are well-defined due to the assumption made on B and part (iii) above.

As explained in [11, Theorem 2.64], these mappings can be extended to positive linear operators on all of $D(\mathbb{B})$ and F respectively. This follows from the fact that if $Q : X^+ \to Y^+$ is an additive operator, i.e. $Q(f+g) = Qf + Qg$ for all $f, g \in X^+$, then Q extends uniquely to a positive linear operator from $X \to Y$ by

$$Qf = Q(f^+ - f^-) := Qf^+ - Qf^-,$$

where f^+ and f^- are the positive and negative parts of $f \in X$ respectively as in (2.15).

It can be shown that $\mathbb{L}$ is invertible ([11, Lemma 6.18]) and this leads to the following definition of the extended operator $\mathbb{A}$.

Definition 2.25. *The operator $\mathbb{A}$ is defined by $\mathbb{A} : D(\mathbb{A}) \to F$;*

$$\mathbb{A} := f - \mathbb{L}^{-1} f, \quad D(\mathbb{A}) = \mathbb{L}F \subset Y.$$

From [11, pp. 171-172] we have

(i) $\mathbb{A}f = Af \;\; \forall f \in D(A)$,

(ii) $\mathbb{A}f \in Y \Leftrightarrow f \in D(A)$,

(iii) $\mathbb{B}f = Bf \;\; \forall f \in D(B)$,

(iv) $\mathbb{L}f = R(1; A)f \;\; \forall f \in Y$.

The following result can now be proved.

Theorem 2.26. *If for any $g \in F^+$ such that $-g + \mathbb{B}\mathbb{L}g \in Y$ and $c(\mathbb{L}g)$, with c as in (2.21), exists such that*

$$\int_\Omega \mathbb{L}g d\mu + \int_\Omega (-g + \mathbb{B}\mathbb{L}g) d\mu \geq -c(\mathbb{L}g) \tag{2.26}$$

then $G = \overline{A + B}$.

Proof. [11, p. 176].

□

After analysing the linear part involving the operators A and B we shall then move on to the nonlinear part $F(t, u(t))$, which is dealt with using theory found in [49]. The main theorems and definition we shall require are given below. Note that we are now working in a general Banach space X and not just in an L^1 space of the form Y. We shall firstly define a local Lipschitz condition on $F(t, u(t))$.

Definition 2.27. *[49, p.185] We say that $F(t, u)$ is locally Lipschitz continuous in u, uniformly in t on bounded intervals if, for every $t' \geq 0$ and constant $r \geq 0$, there is a constant $L(r, t')$ such that*

$$\|F(t, u) - F(t, v)\| \leq L(r, t')\|u - v\| \tag{2.27}$$

holds for all $u, v \in X$ with $\|u\| \leq r$, $\|v\| \leq r$ and $t \in [0, t']$.

The following theorem then gives conditions for a mild solution to the full ACP (2.17).

Theorem 2.28. *[49, Theorem 1.4, p.185.] Let $F : [0, \infty) \times X \to X$ be continuous in t for $t \geq 0$ and locally Lipschitz continuous in u, uniformly in t on bounded intervals. If G is the infinitesimal generator of a C_0-semigroup $\{T_G(t)\}_{t \geq 0}$ on X then, for every $f \in X$, there is a $t_{max} \leq \infty$ such that the initial value problem (2.17) has a unique mild solution u on $[0, t_{max})$. Moreover, if $t_{max} < \infty$ then*

$$\lim_{t \to t_{max}} \|u(t)\| = \infty.$$

Proof: [49, p.186].

□

We shall state the following which gives a sufficient condition for the mild solution to (2.17) to be a strong solution.

Theorem 2.29. *[49, Theorem 1.5, p.187] Let G be the infinitesimal generator of a C_0-semigroup $\{T_G(t)\}$ on X. If $F : [0, \infty) \times X \to X$ is continuously differentiable from $[0, \infty) \times X$ into X then the mild solution of (2.17) with $f \in D(G)$ is a strong solution.*

Proof: [49, pp.187-188].

□

We shall make use of the following definition of the total derivative of a function.

Definition 2.30. *[5, p.346] A function* $g : [0, \infty) \times X \to X$ *is said to be differentiable at* $(t_0, \phi_0) \in [0, \infty) \times X$ *if there exists a linear operator* $S_{(t_0,\phi_0)} : [0, \infty) \times X \to X$ *such that*

$$g(t_0 + t, \phi_0 + \phi) = g(t_0, \phi_0) + S_{(t_0,\phi_0)}(t, \phi) + \|(t, \phi)\|_{[0,\infty)\times X} E_{(t_0,\phi_0)}(t, \phi),$$

where the error term $E_{(t_0,\phi_0)}(t, \phi) \to 0$ *in* X *as* $(t, \phi) \to (0, 0)$ *in* $[0, \infty) \times X$. *The operator* $S_{(t_0,\phi_0)}$ *is usually denoted by* $g'(t_0, \phi_0)$. *Note that*

$$\|(t, \phi)\|_{[0,\infty)\times X} = |t| + \|\phi\|_X.$$

In Chapters 3 - 5 we consider the case when $N(t) \equiv 0$ for all $t \geq 0$, i.e. we only need to deal with the operator K. We shall need the following specific form of definition 2.30 in order to deduce some results for the reduced semilinear ACP.

Definition 2.31 (Fréchet Differentiable)**.** *Let* X *be a Banach space and let* $K : D(K) \subseteq X \to X$. *Suppose that* $D_0(K)$ *is an open subset of* $D(K)$ *and consider* $c, c + \delta \in D_0(K)$ *for all sufficiently small* δ. *If a linear operator* $K_c \in B(X)$ *exists such that*

$$K(c + \delta) = K(c) + K_c\delta + R(c, \delta), \tag{2.28}$$

where the remainder R *satisfies*

$$\lim_{\|\delta\|\to 0} \left\{ \frac{\|R(c, \delta)\|}{\|\delta\|} \right\} = 0, \tag{2.29}$$

then we say that K *is Fréchet differentiable at* $c \in D_0(K)$ *and* K_c *is the Fréchet derivative of* K *at* c. *If the operator* K *is Fréchet differentiable at any* $c \in D_0(K)$ *then* K *is said to be Fréchet differentiable on* $D_0(K)$.

We shall use the following two theorems to deduce the existence of a strong solution to the ACP (2.16) with $N(t) \equiv 0$.

Theorem 2.32. *[19, Theorem 3.30] Let* X *be a Banach space and assume that*

(i) G *is the infinitesimal generator of a* C_0*-semigroup on* X,

(ii) $K : D(K) \to X$ *satisfies the local Lipschitz condition*

$$\|K(c) - K(d)\| \leq C\|c - d\| \tag{2.30}$$

for all $c, d \in \overline{B}(f, r) \subseteq D(K)$*, where* C *and* r *are positive constants (*r *suitably small),*

(iii) K *is Fréchet differentiable at any* $c \in B(f, r)$ *and the Fréchet derivative* K_c *is such that* $\|K_c d\| \leq C_1\|d\|$ *for all* $c \in B(f, r)$ *and* $d \in X$*, where* C_1 *is a positive constant,*

(iv) the Fréchet derivative is continuous with respect to $c \in B(f, r)$*, i.e.*

$$\|K_{c_1}d - K_{c_2}d\| \to 0 \text{ as } \|c_1 - c_2\| \to 0, \ \ c_1, c_2 \in B(f, r) \tag{2.31}$$

for any given $d \in X$*,*

(v) $f \in D(G)$*.*

Under these assumptions, the continuous solution on $[0, t_1]$ *of (2.18) belongs to* $D(G)$ *for all* $t \in [0, t_1]$ *and is the strong solution of the semilinear ACP (2.16) with* $N(t) \equiv 0$*.*

Proof: This is described in [19, pp. 132-134].

□

Note 2.33. *This is just a special case of Theorem 2.29.*

The following inequality is used in Chapter 3.

Lemma 2.34 (Gronwall's Inequality)**.** *Let* ϕ_0 *be a non-negative constant and* $h(t)$ *be a continuous non-negative function defined over* $[t_1, t_2]$*. Then any continuous non-negative function* $\phi = \phi(t)$ *that satisfies the inequality*

$$\phi(t) \leq \phi_0 + \int_{t_1}^{t} h(s)\phi(s)ds, \quad t \in [t_1, t_2],$$

is such that

$$0 \leq \phi(t) \leq \psi(t) \ \ \forall t \in [t_1, t_2]$$

where $\psi(t)$ is the unique continuous solution of the equation

$$\psi(t) = \phi_0 \exp\left(\int_{t_1}^{t} h(s)ds\right).$$

Proof: See [17, Lemma 3.2].

□

Another result which we shall use to justify taking limits inside summations is the following.

Theorem 2.35 (Dominated Convergence Theorem)**.** *Let (Ω, μ) be a measure space and let $\{f_n\}$ be a sequence of integrable functions which converges almost everywhere to a real-valued measurable function f. If there exists an integrable function g such that $|f_n| \le g$ for all n, then f is integrable and*

$$\int_\Omega f d\mu = \lim_{n\to\infty} \int_\Omega f_n d\mu. \tag{2.32}$$

Proof: See [16, pp. 44-45].

□

Throughout the thesis we shall make use of the following theorem to justify interchanging the order of summations.

Theorem 2.36 (Fubini's Theorem)**.** *[51, p. 140] Let (Ω_1, M_1, μ_1) and (Ω_2, M_2, μ_2) be σ-finite measure spaces and let f be a $(M_1 \times M_2)$-measurable function on $\Omega_1 \times \Omega_2$ and*

- *for each $x \in \Omega_1$ we define $f_x(y) = f(x, y)$ on Ω_2,*
- *for each $y \in \Omega_2$ we define $f^y(x) = f(x, y)$ on Ω_1,*

for a function f on $\Omega_1 \times \Omega_2$.

(i) If $0 \le f \le \infty$ and if

$$\phi(x) = \int_{\Omega_2} f_x d\mu_2, \quad \psi(y) = \int_{\Omega_1} f^y d\mu_1 \quad (x \in \Omega_1,\ y \in \Omega_2)$$

then ϕ is M_1-measurable and ψ is M_2-measurable, and

$$\int_{\Omega_1} \phi d\mu_1 = \int_{\Omega_1\times\Omega_2} f d(\mu_1 \times \mu_2) = \int_{\Omega_2} \psi d\mu_2. \tag{2.33}$$

Note that the first and last integrals in (2.33) can also be written as

$$\int_{\Omega_1} d\mu_1(x) \int_{\Omega_2} f(x,y) d\mu_2(y) = \int_{\Omega_2} d\mu_2(y) \int_{\Omega_1} f(x,y) d\mu_1(x). \tag{2.34}$$

(ii) If f is $(M_1 \times M_2)$-measurable and if

$$\int_{\Omega_1} d\mu_1(x) \int_{\Omega_2} |f(x,y)| d\mu_2(y) < \infty \tag{2.35}$$

then the iterated integrals (2.34) are finite and equal.

In Chapter 5 we will require to change the order of four summations/integrals and so we will need an extended version of Fubini's Theorem. The following shows how we can extend Theorem 2.36 to a product of three measures. We wish to prove

Theorem 2.37. *Let (Ω_1, M_1, μ_1), (Ω_2, M_2, μ_2) and (Ω_3, M_3, μ_3) be σ-finite measure spaces and let f be a $M_1 \times M_2 \times M_3$-measurable function on $\Omega_1 \times \Omega_2 \times \Omega_3$. Then, if $0 \le f \le \infty$, we have that*

$$\begin{aligned}
&\int_{\Omega_1\times\Omega_2\times\Omega_3} f(x,y,z)(d\mu_1 \times d\mu_2 \times d\mu_3) \\
= &\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3} f(x,y,z) d\mu_3 d\mu_2 d\mu_1 = \int_{\Omega_3}\int_{\Omega_1}\int_{\Omega_2} f(x,y,z) d\mu_2 d\mu_1 d\mu_3 \\
= &\int_{\Omega_2}\int_{\Omega_3}\int_{\Omega_1} f(x,y,z) d\mu_1 d\mu_3 d\mu_2 = \int_{\Omega_1}\int_{\Omega_3}\int_{\Omega_2} f(x,y,z) d\mu_2 d\mu_3 d\mu_1 \\
= &\int_{\Omega_3}\int_{\Omega_2}\int_{\Omega_1} f(x,y,z) d\mu_1 d\mu_2 d\mu_3 = \int_{\Omega_2}\int_{\Omega_1}\int_{\Omega_3} f(x,y,z) d\mu_3 d\mu_1 d\mu_2.
\end{aligned}$$

Also, we can prove a result analogous to (2.35) for three measures.

Proof: Let $\mathcal{X} = \Omega_1 \times \Omega_2$, $\mathcal{Y} = \Omega_1 \times \Omega_3$ and $\mathcal{Z} = \Omega_2 \times \Omega_3$. We have that $(\mathcal{X}, \mathcal{M}_\mathcal{X}, \nu_\mathcal{X})$, $(\mathcal{Y}, \mathcal{M}_\mathcal{Y}, \nu_\mathcal{Y})$ and $(\mathcal{Z}, \mathcal{M}_\mathcal{Z}, \nu_\mathcal{Z})$ are σ-finite measure spaces with $\mathcal{M}_\mathcal{X} = M_1 \times M_2$, $\mathcal{M}_\mathcal{Y} = M_1 \times M_3$, $\mathcal{M}_\mathcal{Z} = M_2 \times M_3$ and $\nu_\mathcal{X} = \mu_1 \times \mu_2$, $\nu_\mathcal{Y} = \mu_1 \times \mu_3$, $\nu_\mathcal{Z} = \mu_2 \times \mu_3$. If $0 \le f \le \infty$, let

$$\phi_1(\underline{x}) = \int_{\Omega_3} f_{\underline{x}} d\mu_3, \qquad \underline{x} \in \mathcal{X},$$

$$\phi_2(\underline{y}) = \int_{\Omega_2} f_{\underline{y}} d\mu_2, \qquad \underline{y} \in \mathcal{Y},$$

$$\phi_3(\underline{z}) = \int_{\Omega_1} f_{\underline{z}} d\mu_1, \qquad \underline{z} \in \mathcal{Z},$$

where

- for each $\underline{x} = (x, y) \in \mathcal{X}$ we define $f_{\underline{x}}$ on Ω_3 by $f_{\underline{x}}(z) = f(x, y, z)$
- for each $\underline{y} = (x, z) \in \mathcal{Y}$ we define $f_{\underline{y}}$ on Ω_2 by $f_{\underline{y}}(y) = f(x, y, z)$
- for each $\underline{z} = (y, z) \in \mathcal{Z}$ we define $f_{\underline{z}}$ on Ω_1 by $f_{\underline{z}}(x) = f(x, y, z)$.

Then ϕ_1 is $\mathcal{M}_{\mathcal{X}}$-measurable, ϕ_2 is $\mathcal{M}_{\mathcal{Y}}$-measurable and ϕ_3 is $\mathcal{M}_{\mathcal{Z}}$-measurable and we can apply Theorem 2.36 to deduce that

$$\int_{\mathcal{X}} \phi_1 d\nu_{\mathcal{X}} = \int_{\mathcal{Y}} \phi_2 d\nu_{\mathcal{Y}} = \int_{\mathcal{Z}} \phi_3 d\nu_{\mathcal{Z}} = \int_{\Omega_1\times\Omega_2\times\Omega_3} f(x, y, z) d(\mu_1 \times \mu_2 \times \mu_3), \quad (2.36)$$

where the first three integrals in (2.36) can be written as

$$\begin{aligned} & \int_{\mathcal{X}} d\nu_{\mathcal{X}}(x, y) \int_{\Omega_3} f(x, y, z) d\mu_3(z) \\ = & \int_{\mathcal{Y}} d\nu_{\mathcal{Y}}(x, z) \int_{\Omega_2} f(x, y, z) d\mu_2(y) \\ = & \int_{\mathcal{Z}} d\nu_{\mathcal{Z}}(y, z) \int_{\Omega_1} f(x, y, z) d\mu_1(x). \end{aligned} \quad (2.37)$$

We also have that if f is $M_1 \times M_2 \times M_3$-measurable and if

$$\int_{\mathcal{X}} d\nu_{\mathcal{X}}(x, y) \int_{\Omega_3} |f(x, y, z)| d\mu_3(z) < \infty$$

then the three iterated integrals in (2.37) are finite and equal. To get the required result (2.36) for three measure spaces we need to apply Theorem 2.36 a second time. First we shall now look at the integral

$$\int_{\mathcal{X}} \phi_1(\underline{x}) d\nu_{\mathcal{X}} = \int_{\Omega_1 \times \Omega_2} \phi_1(x, y) d(\mu_1 \times \mu_2)$$

where

$$\phi_1(x, y) = \int_{\Omega_3} f_{(x,y)} d\mu_3 \geq 0$$

since μ_3 is a positive measure. We can write

$$\psi_1(x) = \int_{\Omega_2} (\phi_1)_x d\mu_2, \qquad x \in X$$

$$\psi_2(y) = \int_{\Omega_1} (\phi_1)_y d\mu_1, \qquad y \in Y$$

where

- for each $x \in \Omega_1$ we define $(\phi_1)_x$ on Ω_2 by $(\phi_1)_x(y) = \phi_1(x, y)$,
- for each $y \in \Omega_2$ we define $(\phi_1)_y$ on Ω_1 by $(\phi_1)_y(x) = \phi_1(x, y)$.

Then ψ_1 is M_1-measurable, ψ_2 is M_2-measurable and

$$\int_{\Omega_1} \psi_1 d\mu_1 = \int_{\Omega_1 \times \Omega_2} \phi_1 d(\mu_1 \times \mu_2) = \int_{\Omega_2} \psi_2 d\mu_2 \tag{2.38}$$

by Theorem 2.36. Again the first and last integrals in (2.38) can be written as the iterated integrals

$$\int_{\Omega_1} d\mu_1(x) \int_{\Omega_2} \phi_1(x, y) d\mu_2(y) = \int_{\Omega_2} d\mu_2(y) \int_{\Omega_1} \phi(x, y) d\mu_1(x). \tag{2.39}$$

We can also apply the final result in Fubini's Theorem to say that if $\phi(x, y)$ is $(M_1 \times M_2)$-measurable and if

$$\int_{\Omega_1} d\mu_1(x) \int_{\Omega_2} |\phi(x, y)| d\mu_2(y) < \infty$$

then the iterated integrals in (2.39) are finite and equal.

Similar results can be obtained for

$$\int_{\mathcal{Y}} \phi_2(\underline{y}) d\nu_{\mathcal{Y}} = \int_{\Omega_1 \times \Omega_3} \phi_2(x, z) d(\mu_1 \times \mu_3)$$

and

$$\int_{\mathcal{Z}} \phi_3(\underline{z}) d\nu_{\mathcal{Z}} = \int_{\Omega_2 \times \Omega_3} \phi_3(y, z) d(\mu_2 \times \mu_3).$$

We can now put all of our results together to get

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2 \times \Omega_3} f(x, y, z) d(\mu_1 \times \mu_2 \times \mu_3) \\ = & \int_{\Omega_1 \times \Omega_2} \int_{\Omega_3} f(x, y, z) d\mu_3(z) d(\mu_1 \times \mu_2)(x, y) \\ = & \int_{\Omega_2 \times \Omega_1} \int_{\Omega_3} f(x, y, z) d\mu_3(z) d(\mu_2 \times \mu_1)(y, x) \\ = & \int_{\Omega_1} \int_{\Omega_2} \int_{\Omega_3} f(x, y, z) d\mu_3(z) d\mu_2(y) d\mu_1(x) \\ = & \int_{\Omega_2} \int_{\Omega_1} \int_{\Omega_3} f(x, y, z) d\mu_3(z) d\mu_1(x) d\mu_2(y), \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2 \times \Omega_3} f(x, y, z) d(\mu_1 \times \mu_2 \times \mu_3) \\ = & \int_{\Omega_1 \times \Omega_3} \int_{\Omega_2} f(x, y, z) d\mu_2(y) d(\mu_1 \times \mu_3)(x, z) \\ = & \int_{\Omega_3 \times \Omega_1} \int_{\Omega_2} f(x, y, z) d\mu_2(y) d(\mu_3 \times \mu_1)(z, x) \\ = & \int_{\Omega_1} \int_{\Omega_3} \int_{\Omega_2} f(x, y, z) d\mu_2(y) d\mu_3(z) d\mu_1(x) \\ = & \int_{\Omega_3} \int_{\Omega_1} \int_{\Omega_2} f(x, y, z) d\mu_2(y) d\mu_1(x) d\mu_3(z) \end{aligned}$$

and

$$\begin{aligned} & \int_{\Omega_1 \times \Omega_2 \times \Omega_3} f(x, y, z) d(\mu_1 \times \mu_2 \times \mu_3) \\ = & \int_{\Omega_2 \times \Omega_3} \int_{\Omega_1} f(x, y, z) d\mu_1(x) d(\mu_2 \times \mu_3)(y, z) \end{aligned}$$

$$
\begin{aligned}
&= \int_{\Omega_3 \times \Omega_2} \int_{\Omega_1} f(x,y,z) d\mu_1(x) d(\mu_3 \times \mu_2)(z,y) \\
&= \int_{\Omega_2} \int_{\Omega_3} \int_{\Omega_1} f(x,y,z) d\mu_1(x) d\mu_3(z) d\mu_2(y) \\
&= \int_{\Omega_3} \int_{\Omega_2} \int_{\Omega_1} f(x,y,z) d\mu_1(x) d\mu_2(y) d\mu_3(z).
\end{aligned}
$$

We can also say that if f is $(M_1 \times M_2 \times M_3)$-measurable and if one of the iterated integrals is absolutely convergent, then all of the above iterated integrals are finite and equal. □

It is possible to use techniques similar to those above to extend Fubini's theorem to four measure spaces. This would result in 4! iterated integrals, all of which are equal under appropriate conditions. We shall require the version for four spaces for some of our calculations in Chapter 5.

Chapter 3

The Discrete Coagulation-Fragmentation Equation with Mass Conservation

We shall begin by investigating the discrete coagulation-fragmentation (C-F) equation with mass conservation. The basic form of the C-F equation that we are concerned with is

$$\begin{aligned} \frac{d}{dt}u_n(t) &= -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \\ &\quad + \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j} u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j} u_n(t) u_j(t)\,, \qquad (3.1) \\ u_n(0) &= f_n\,, \qquad (n = 1, 2, 3, ...)\,, \qquad (3.2) \end{aligned}$$

where $u_n(t)$ is the number concentration of n-mers at time $t \geq 0$. Physical conditions lead to the following assumptions:

(A1) $a_1 = 0$

(A2) $b_{n,j} = 0$ for $j \leq n$.

We shall also assume that $k_{n,j} = k_{j,n}$. To ensure that we have mass conservation we also assume that

(A3) $\sum_{n=1}^{j-1} n b_{n,j} = j\,, \quad (j = 2, 3, \ldots)$.

Mass is clearly conserved since by a formal argument we have

$$\frac{d}{dt}M(t) = \frac{d}{dt}\left(\sum_{n=1}^{\infty} nu_n(t)\right) = \sum_{n=1}^{\infty} n\frac{d}{dt}u_n(t), \tag{3.3}$$

where $M(t) = \sum_{n=1}^{\infty} nu_n(t)$. Since

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\left(-a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t)\right) \\
= &-\sum_{n=1}^{\infty} na_n u_n(t) + \sum_{j=2}^{\infty} a_j u_j(t) \sum_{n=1}^{j-1} nb_{n,j} \\
= &-\sum_{n=1}^{\infty} na_n u_n(t) + \sum_{j=2}^{\infty} ja_j u_j(t) \qquad \text{by (A1)} \\
= &\ 0, \qquad (3.4)
\end{aligned}$$

and

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\left(\frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j} u_{n-j}(t)u_j(t) - \sum_{j=1}^{\infty} k_{n,j} u_n(t)u_j(t)\right) \\
= &\ \frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty} nk_{n-j,j} u_{n-j}(t)u_j(t) - \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nk_{n,j} u_n(t)u_j(t) \\
= &\ \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}(l+j)k_{l,j} u_l(t)u_j(t) - \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nk_{n,j} u_n(t)u_j(t) \\
= &\ \frac{1}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty} lk_{l,j} u_l(t)u_j(t) + \frac{1}{2}\sum_{l=1}^{\infty}\sum_{j=1}^{\infty} jk_{j,l} u_j(t)u_l(t) \\
&\quad -\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nk_{n,j} u_n(t)u_j(t) \\
= &\ 0, \qquad (3.5)
\end{aligned}$$

it follows that $\frac{dM(t)}{dt} = 0$, i.e. $M(t) = M(0) = M$ for all $t \geq 0$.

Note that most of the following results have been published in [44]. We shall present an extended version of [44] in this chapter. We shall begin by analysing the pure fragmentation equation (i.e. $k_{n,j} = 0$ for all $n, j = 1, 2, \ldots$) for a general sequence $\{a_n\}_{n=1}^{\infty}$ of non-negative numbers. We shall then move on to the specific cases when $\{a_n\}_{n=1}^{\infty}$ is bounded and $\{a_n\}_{n=1}^{\infty}$ is monotonic increasing. We choose to look at the monotonic increasing case since it is feasible to assume that larger clusters of particles will break up at a greater rate than smaller clusters.

3.1 Fragmentation: The General Case

When coagulation is removed, the pure fragmentation equation takes the following form:

$$\frac{d}{dt}u_n(t) = -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \qquad (n = 1, 2, 3, ...). \tag{3.6}$$

Since we are looking for a mass-conserving solution there is a natural Banach space X in which to study the abstract Cauchy problem (ACP) corresponding to (3.6).

Definition 3.1. *Let X be the space of all real infinite sequences $\{f_n\}_{n=1}^{\infty}$ such that*

$$\|f\| = \sum_{n=1}^{\infty} n|f_n| < \infty\,. \tag{3.7}$$

The expression (3.7) defines a norm with respect to which X becomes a Banach space. Note that X is a weighted l^1 space. Also, X is a particular case of $Y = L^1(\Omega, \mu)$ used in Chapter 2 where $\Omega = \mathbb{N}$ is the set of positive integers and, for any subset $\mathbb{M}$ of $\mathbb{N}$,

$$\mu(\mathbb{M}) = \begin{cases} \sum_{m\in\mathbb{M}} m & \text{if } \mathbb{M} \text{ is finite} \\ \infty & \text{if } \mathbb{M} \text{ is infinite.} \end{cases} \tag{3.8}$$

We shall make use of the following operators defined in X.

Definition 3.2. *Define operators A and B in X by*

$$[Af]_n = -a_n\, f_n; \;\; D(A) = \left\{ f \in X : \sum_{n=1}^{\infty} n a_n |f_n| < \infty \right\}, \tag{3.9}$$

$$[Bf]_n = \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j; \quad D(B) = \left\{ f \in X : \sum_{n=1}^{\infty} n \Big| \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \Big| < \infty \right\}. \quad (3.10)$$

Lemma 3.3. *As sets $D(A) \subseteq D(B)$.*

Proof: Let $f \in D(A)$. Then

$$\begin{aligned} \|Bf\| &\leq \sum_{n=1}^{\infty} n \sum_{j=n+1}^{\infty} a_j \, b_{n,j} \, |f_j| = \sum_{j=2}^{\infty} a_j \, |f_j| \left(\sum_{n=1}^{j-1} n \, b_{n,j} \right) \\ &= \sum_{j=1}^{\infty} j \, a_j \, |f_j| = \|Af\| < \infty \quad \text{(by (A1) and (A3))}, \end{aligned}$$

so that $f \in D(B)$ and $D(A) \subseteq D(B)$. □

Notation 3.4. *We shall denote the set $D(A)$ simply by D.*

When the pure fragmentation equation (3.6) is converted into an ACP we obtain:

Problem 3.5. *(ACP for (3.6).) Find a function $u : [0, \infty) \to X$ such that*

$$\frac{d}{dt} u(t) = Au(t) + Bu(t) \quad (t > 0) \quad (3.11)$$

$$\lim_{t \to 0+} u(t) = f \in D\,. \quad (3.12)$$

Later we shall show that a unique strict solution to (3.11) and (3.12) will always exist when $\{a_n\}$ is either bounded or monotonic increasing. However, for more general sequences $\{a_n\}$, we have to modify Problem 3.5 slightly by replacing the operator sum $A + B$ that appears on the right-hand side of (3.11) by an appropriate extension G (namely, the closure of $(A + B, D)$). In this case the corresponding ACP has a unique strict solution for any $f \in D(G)$.

Note that we also want a solution to be non-negative and mass-conserving. Thus we require

$$f \in D^+ \Rightarrow u(t) \in D^+ \quad \text{for all } t > 0\,, \quad (3.13)$$

$$\|u(t)\| = \|f\| \quad \text{for all } t > 0\,. \quad (3.14)$$

One approach to dealing with the case of a general sequence $\{a_n\}$ is to consider the operator C in X defined by

$$\begin{aligned} [Cf]_n &= [Af]_n + [Bf]_n\,; \\ D(C) &= \left\{ f \in X : \sum_{n=1}^{\infty} n \Big| [Cf]_n \Big| < \infty \right\}. \end{aligned} \tag{3.15}$$

We can think of $D(C)$ as the maximal domain of existence of the right-hand side of (3.11) while D is the minimal domain (in which Af and Bf separately belong to X). If we replace (3.11) by

$$\frac{d}{dt}u(t) = Cu(t) \tag{3.16}$$

we obtain an ACP involving C. By a method similar to [17, Chapter 13], it can be proved that a restriction of C exists that generates a strongly continuous semigroup on X. Consequently, *this* ACP has a unique, strict solution which can also be shown to be non-negative and mass-conserving for suitably restricted initial data f. However, this solution does not necessarily satisfy (3.13). The method in [17] uses results related to the Kato-Voigt perturbation theorem but we choose to use this theorem directly.

Theorem 3.6. *Let X, A, B be as in Definitions 3.1 and 3.2. Then there exists a smallest extension G of $A + B$ which generates a substochastic semigroup $\{T_G(t)\}_{t\geq 0}$ on X.*

Proof: As stated earlier, X is the space $L^1(\mathbb{N}, \mu)$ with $\Omega = \mathbb{N}$ and μ given by (3.8). Consequently, we can apply the Kato-Voigt theorem (Theorem 2.19) to the ACP (3.11) and (3.12). We must therefore check each of the conditions in Theorem 2.19.

(i) From Lemma 2.14 it is clear that A generates the substochastic semigroup $\{T_A(t)\}_{t\geq 0}$ on X, where

$$[T_A(t)f]_n = e^{-a_n t} f_n \qquad (n = 1, 2, ...). \tag{3.17}$$

(ii) Certainly $D(B) \supseteq D(A)$ by Lemma 3.3. Also it is immediate that $Bf \geq 0$ for all $f \in D(B)^+$.

(iii) For $f \in D(A)^+$,

$$\int_\Omega (Af + Bf)d\mu = \sum_{n=1}^{\infty} n \left(-a_n f_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j \right) = 0 =: -c(f)$$

by the calculations in the proof of Lemma 3.3.

The result therefore follows from Theorem 2.19. □

We shall now obtain a precise characterisation of G namely, $G = \overline{A + B}$. In order to do this we need to show that we can apply the results involving extended operators described in Chapter 2. In what follows we shall use the notation of Definitions 2.21 - 2.25. As mentioned previously, the appropriate space to work in is $X = L^1(\mathbb{N}, \mu)$. If we let l denote the space of all sequences, then $l = E_f \subset E$. (The inclusion is strict because E can contain sequences with an arbitrary number of infinite entries, whereas a sequence in E_f must contain no infinite entries, since the only set with measure 0 is the empty set.) Also

$$F = \{f \in l : \left\{ \frac{f_n}{1 + a_n} \right\}_{n=1}^{\infty} \in X\}$$

and therefore

$$f \in F^+ \iff \sum_{n=1}^{\infty} \frac{n f_n}{1 + a_n} < \infty, \;\; f_n \geq 0.$$

Further we have

$$[\mathbb{L}f]_n = \frac{f_n}{1 + a_n}, \quad f \in F;$$
$$[\mathbb{A}f]_n = f_n - (1 + a_n)f_n = -a_n f_n,$$
$$D(\mathbb{A}) = \mathbb{L}F = \{f \in X : f = \left\{ \frac{g_n}{1 + a_n} \right\}_{n=1}^{\infty}, \, g \in F\};$$
$$[\mathbb{B}f]_n = \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j,$$
$$D(\mathbb{B}) = H = \Big\{f \in X : \text{for any non-negative, non-decreasing sequence } \{f^n\} \text{ in } D(B) \text{ such that } \sup_n f^n = |f|, \text{ i.e. } \sup_n f^n_m = |f_m| \; \forall m, \text{ we have } \sup_n Bf^n < \infty, \text{ i.e. } \sup_n Bf^n_m < \infty \; \forall m \Big\},$$

Theorem 3.7. *In the context of Theorem 3.6, $G = \overline{A+B}$, the closure of the operator $(A+B, D)$.*

Proof: We shall use Theorem 2.26 with $c(f) := 0$. We must verify that, for any $g \in F^+$ such that $-g + \mathbb{B}\mathbb{L}g \in X$, we have

$$\sum_{n=1}^{\infty} n(\mathbb{L}g)_n + \sum_{n=1}^{\infty} n\left(-g_n + (\mathbb{B}\mathbb{L}g)_n\right) \geq 0. \tag{3.18}$$

Let

$$f_n = (\mathbb{L}g)_n = (1+a_n)^{-1} g_n, \;\; n = 1, 2, \ldots,$$

so that $f \in X^+$. Since

$$(\mathbb{L}g)_n - g_n = (1+a_n)^{-1} g_n - g_n = -a_n f_n, \tag{3.19}$$

equation (3.18) holds if, for any $f \in X^+$ such that $\mathbb{A}f + \mathbb{B}f \in X$, we have

$$\sum_{n=1}^{\infty} n\left(-a_n f_n + (\mathbb{B}f)_n\right) \geq 0. \tag{3.20}$$

Now

$$\begin{aligned} &\sum_{n=1}^{\infty} n\left(-a_n f_n + (\mathbb{B}f)_n\right) \\ = &\lim_{N\to\infty} \sum_{n=1}^{N} n \left(-a_n f_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j\right) \end{aligned} \tag{3.21}$$

and we know that, for any finite $N \in \mathbb{N}$,

$$\sum_{n=1}^{N} n a_n f_n < \infty \text{ and } \sum_{n=1}^{N} n(\mathbb{B}f)_n < \infty,$$

the latter following from the fact that $\mathbb{B}f \in l^+$. Therefore (3.21) can be written as

$$\lim_{N\to\infty} \left(-\sum_{n=1}^{N} n a_n f_n + \sum_{n=1}^{N} \sum_{j=n+1}^{\infty} n a_j b_{n,j} f_j\right). \tag{3.22}$$

Now

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{j=n+1}^{\infty} n a_j b_{n,j} f_j \\
= &\sum_{j=2}^{N}\left(\sum_{n=1}^{j-1} n b_{n,j}\right) a_j f_j + \sum_{j=N+1}^{\infty}\sum_{n=1}^{N} n a_j b_{n,j} f_j \\
= &\sum_{j=2}^{N} j a_j f_j + S_N
\end{aligned}$$

where

$$S_N = \sum_{j=N+1}^{\infty}\sum_{n=1}^{N} n a_j b_{n,j} f_j \geq 0 \quad \forall N.$$

Thus from (3.22) we have

$$\begin{aligned}
&\lim_{N\to\infty}\left(-\sum_{n=1}^{N} n a_n f_n + \sum_{n=1}^{N}\sum_{j=n+1}^{\infty} n a_j b_{n,j} f_j\right) \\
= &\lim_{N\to\infty}\left(-\sum_{n=1}^{N} n a_n f_n + \sum_{n=1}^{N} n a_n f_n + S_N\right) \\
= &\lim_{N\to\infty} S_N \geq 0.
\end{aligned}$$

Hence we have satisfied the conditions of Theorem 2.26 and we can conclude that $G = \overline{A+B}$. □

From [11, p. 159], if $G = \overline{A+B}$, then for $u(t) \in D(G)$ there exists a sequence $(u^n(t))_{n\in\mathbb{N}}$ of elements of $D(A)$ such that $u^n(t) \to u(t)$ and $(A+B)u^n(t) \to Gu(t)$ as $n \to \infty$ in X. This leads to

$$\int_\Omega Gu(t)d\mu = \lim_{n\to\infty}\int_\Omega (A+B)u^n(t)d\mu = 0$$

If $f \in D(G)^+$, then $u(t) = T_G(t)f \in D(G)^+$ for any $t \geq 0$ and we have

$$\frac{d}{dt}\|u(t)\| = \int_\Omega \frac{du(t)}{dt} d\mu = \int_\Omega Gu(t)d\mu = 0$$

i.e. in our space $X = L^1(\mathbb{N}, \mu)$ we have

$$\frac{d}{dt}\|u(t)\| = \sum_{n=1}^{\infty} n \left[\frac{du(t)}{dt}\right]_n = \sum_{n=1}^{\infty} n[Gu(t)]_n = 0.$$

It follows immediately from Theorem 3.7, in conjunction with the above discussion, that the ACP

$$\frac{d}{dt}u(t) = Gu(t) \quad (t > 0) \tag{3.23}$$

$$\lim_{t\to 0+} u(t) = f\,, \tag{3.24}$$

has a unique, strict, non-negative and mass-conserving solution $u : [0, \infty) \to D(G)^+$ for each $f \in D(G)^+$ and hence for each $f \in D(A)^+$. This solution is given by $u(t) = T_G(t)f$, with $G = \overline{A + B}$. In the next section we shall show that, under certain additional constraints on the sequence $\{a_n\}$, the set $D(A)^+$ is invariant under $T_G(t)$.

3.2 Fragmentation: Particular Cases

Let X, A, B be as in Definitions 3.1 and 3.2 and let $G = \overline{A + B}$, as in Theorem 3.7. Since $D \equiv D(A) \subseteq D(G)$ we know that

$$f \in D^+ \Rightarrow u(t) \in D(G)^+ \text{ for all } t > 0.$$

However we are not able to deduce in general that (3.13) holds. We shall now look at two special cases where progress can be made.

Case 1: $\{a_n\}$ bounded.

Suppose that $|a_n| \le M$ (M a positive constant) for all $n \ge 1$. Then

$$\|Af\| = \sum_{n=1}^{\infty} n\, a_n\, |f_n| \le M \sum_{n=1}^{\infty} n|f_n| = M\|f\| < \infty. \tag{3.25}$$

In this case $D(A) = X$ and A is a bounded linear operator on X. Also, by the calculations in the proof of Lemma 3.3, we obtain

$$\|Bf\| \leq \sum_{n=1}^{\infty} n \sum_{j=n+1}^{\infty} a_j b_{n,j} |f_j| = \|Af\| < \infty \tag{3.26}$$

for any $f \in X$. Hence B is also a bounded linear operator on X (with $D(B) = D(A) = X$).

It follows that $G = A + B$ and $\{T_G(t)\}_{t\geq 0} = \{e^{t(A+B)}\}_{t\geq 0}$ is a uniformly continuous (and hence strongly continuous) semigroup on X with $D(G) = X$. Theorems 3.6 and 3.7, together with the comment thereafter, guarantee that $\{T_G(t)\}_{t\geq 0}$ is stochastic. Since $u(t) = T_G(t)f$ is the unique solution of Problem 3.5 under the given conditions, we deduce that (3.13) holds.

We have therefore proved

Theorem 3.8. *In the case when $\{a_n\}$ is bounded, Problem 3.5 has a unique strict, non-negative, mass-conserving solution for each $f \in X$, given by*

$$u(t) = e^{t(A+B)} f \qquad (t \geq 0).$$

Case 2: $\{a_n\}$ monotonic increasing.

We shall adapt an argument used by Banasiak in [10, Example 6.4] for a continuous fragmentation equation, with kernels

$$a(x) = x^{\alpha},\ \alpha > 0 \text{ and } b(x,y) = (\nu+2)x^{\nu}/y^{\nu+1},\ -2 < \nu \leq 0,$$

to prove the following result.

Theorem 3.9. *Let X, A, B and G be as in Theorem 3.6 and, in addition to assumptions (A1)-(A3), let $\{a_n\}$ be a monotonic increasing sequence. Then D $(= D(A))$ is invariant under the semigroup $\{T_G(t)\}_{t\geq 0}$.*

Proof: We proceed in a number of steps.

Step 1 Let D be equipped with the graph norm

$$\begin{aligned} \|f\|_A &= \|f\| + \|Af\| \qquad (f \in D) \\ &= \sum_{n=1}^{\infty} (n + na_n)|f_n|\,, \end{aligned}$$

and define operators A_D and B_D on D by

$$(A_D f)_n = -a_n\, f_n\ ;\ D(A_D) = \{f \in D : A_D f \in D\}\,, \tag{3.27}$$

$$(B_D f)_n = \sum_{j=n+1}^{\infty} a_j\, b_{n,j}\, f_j\ ;\ D(B_D) = \{f \in D : B_D f \in D\}\,. \tag{3.28}$$

Thus A_D (respectively B_D) is the part of A (respectively B) in D.

Then A_D generates a substochastic semigroup $\{T(t)\}_{t\geq 0}$ on the Banach space $(D, \|\cdot\|_A)$ with

$$[T(t)f]_n = e^{-a_n t} f_n \quad \text{for } f \in D, t \geq 0. \tag{3.29}$$

Step 2 $D(A_D) \subseteq D(B_D)$ and $f \in [D(A_D)]^+ \Rightarrow B_D f \in [D(A_D)]^+$.

For $f \in D(A_D)$, by calculations similar to those in the proof of Lemma 3.3,

$$\begin{aligned} \|B_D f\|_A &= \|B_D f\| + \|A(B_D f)\| \\ &\leq \sum_{n=1}^{\infty} n \left(\sum_{j=n+1}^{\infty} a_j b_{n,j} |f_j| \right) + \sum_{n=1}^{\infty} n a_n \sum_{j=n+1}^{\infty} a_j b_{n,j} |f_j| \\ &\leq \sum_{j=2}^{\infty} a_j |f_j| \left(\sum_{n=1}^{j-1} n b_{n,j} \right) + \sum_{j=2}^{\infty} a_j^2 |f_j| \left(\sum_{n=1}^{j-1} n b_{n,j} \right) \\ &\qquad \text{(since} \{a_n\}_{n=1}^{\infty} \text{ is monotonic increasing)} \\ &= \sum_{j=2}^{\infty} a_j |f_j| j + \sum_{j=2}^{\infty} a_j^2 |f_j| j \\ &= \| A_D f \| + \| A(A_D f) \| = \| A_D f \|_A < \infty\,. \end{aligned}$$

Hence $D(A_D) \subseteq D(B_D)$. Positivity of B_D is immediate from its definition.

Step 3 Verify that the appropriate version of (2.21) is satisfied.

For $f \in [D(A_D)]^+$, the left-hand side of (2.21) becomes

$$\begin{aligned}
& \sum_{n=1}^{\infty}(n+na_n)[-a_nf_n + \sum_{j=n+1}^{\infty} a_jb_{n,j}f_j] \\
= & -\sum_{n=1}^{\infty}(n+na_n)a_nf_n + \sum_{n=1}^{\infty}(n+na_n)\sum_{j=n+1}^{\infty} a_jb_{n,j}f_j \\
= & -\sum_{n=1}^{\infty}(n+na_n)a_nf_n + \sum_{j=2}^{\infty} a_jf_j\sum_{n=1}^{j-1}(n+na_n)b_{n,j} \\
\leq & -\sum_{n=1}^{\infty}(n+na_n)a_nf_n + \sum_{j=2}^{\infty}(j+ja_j)a_jf_j = 0
\end{aligned}$$

since $a_1 = 0$ and, for $n = 1, ..., j-1$ (where $j \geq 2$)

$$\sum_{n=1}^{j-1} nb_{n,j} + \sum_{n=1}^{j-1} na_nb_{n,j} \leq (1+a_j)\sum_{n=1}^{j-1} nb_{n,j} = j(1+a_j)$$

by (A3) and monotonicity.

Step 4 Apply Theorem 2.19.

Steps 1-3 show that the hypotheses of Theorem 2.19 are satisfied. Hence there exists an extension G_D, say, of $(A_D+B_D, D(A_D))$ which generates a substochastic semigroup $\{T_{G_D}(t)\}_{t\geq 0}$ on $(D, \|\cdot\|_A)$.

When $f \in X$ has bounded support (so that $f_n = 0$ for all sufficiently large n), $f \in D(A_D)$ and $T_{G_D}(t)f = T_G(t)f$ for all $t \geq 0$, where $\{T_G(t)\}_{t\geq 0}$ is the stochastic semigroup on X generated by $G = \overline{A+B}$. By continuity and density,

$$T_{G_D}(t)f = T_G(t)f \text{ for all } f \in (D, \|\cdot\|_A) \text{ and } t \geq 0\,.$$

Thus $T_{G_D}(t)$ is the restriction of $T_G(t)$ to $(D, \|\cdot\|_A)$ and hence D^+ is invariant under the semigroup $\{T_G(t)\}_{t\geq 0}$. □

Remark 3.10. *Since*

$$\|f\| \leq \|f\| + \|Af\| = \|f\|_A \quad (f \in D)$$

$(D, \|\cdot\|_A)$ *is continuously imbedded in* $(X, \|\cdot\|)$. *Hence* G_D *is the part of* G *in* $(D, \|\cdot\|_A)$ *so that*

$$\begin{aligned} G_D f &= Gf;\ D(G_D) = \{f \in D(G) \cap D : Gf \in D\} \\ i.e.\quad G_D f &= (A+B)f;\ D(G_D) = \{f \in D : (A+B)f \in D\}\,. \end{aligned} \tag{3.30}$$

Note this means that when $\{a_n\}_{n=1}^{\infty}$ is a monotonic increasing sequence we have that the solution to $\dot{u} = Gu$ is also the solution to $\dot{u} = (A+B)u$ for $f \in D(A)$. Hence we are actually solving the original problem.

Note 3.11. *In Section 3.4 we shall analyse in detail a particular form of monotonic increasing fragmentation rate, namely,*

$$a_n = n^{\alpha} - 1, \quad \alpha > 0.$$

3.3 Uniformly Bounded Coagulation

We now study the full coagulation-fragmentation equation (3.1). By analogy with Problem 3.5, the corresponding (nonlinear) ACP takes the form

$$\frac{d}{dt}u(t) = Au(t) + Bu(t) + Ku(t) \tag{3.31}$$

$$\lim_{t \to 0+} u(t) = f \in D \tag{3.32}$$

where the coagulation operator K is given by

$$[Kf]_n = \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j} f_{n-j} f_j - \sum_{j=1}^{\infty} k_{n,j} f_n f_j\,, \quad f \in X\,. \tag{3.33}$$

As mentioned previously we shall assume that the coagulation kernel is symmetric. In addition to this we also assume $k_{n,j}$ is uniformly bounded, i.e. $k_{n,j} = k_{j,n}$ and there exists a constant k such that

(A4) $\quad k_{n,j} \le k \quad$ for all $n, j = 1, 2, \ldots$.

Definition 3.12. *Given the coagulation operator* K *in (3.33), define* $\tilde{K}$ *on* $X \times X$

by

$$(\tilde{K}[c,d])_n = \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}c_{n-j}d_j - \sum_{j=1}^{\infty} k_{n,j}c_n d_j \tag{3.34}$$

where $c, d \in X$.

Theorem 3.13. *Under Assumption (A4),* $\tilde{K}$ *defines a bilinear, continuous form mapping* $X \times X$ *into* X *and* $\|\tilde{K}[c,d]\| \leq 2k\|c\|\|d\|$.

Proof: It is convenient to write

$$(\tilde{K}[c,d])_n = (\tilde{K}_1[c,d])_n - (\tilde{K}_2[c,d])_n$$

where

$$(\tilde{K}_1[c,d])_n = \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}c_{n-j}d_j\,, \quad (\tilde{K}_2[c,d])_n = \sum_{j=1}^{\infty} k_{n,j}c_n d_j\,. \tag{3.35}$$

We first prove that $\tilde{K}$ maps $X \times X$ into X. Indeed

$$\begin{aligned} \|\tilde{K}_1[c,d]\| &= \sum_{n=1}^{\infty} n|(\tilde{K}_1[c,d])_n| \\ &\leq \frac{k}{2}\sum_{n=1}^{\infty} n \sum_{j=1}^{n-1} |c_{n-j}|\;|d_j| \;\text{ by (A4)} \\ &= \frac{k}{2}\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty} n|c_{n-j}|\;|d_j| \end{aligned}$$

$$\begin{aligned} &= \frac{k}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}(\,l|c_l|\,|d_j| + j|c_l|\,|d_j|\,) \\ &\leq \frac{k}{2}\|c\|\sum_{j=1}^{\infty} j|d_j| + \frac{k}{2}\|d\|\sum_{l=1}^{\infty} l|c_l| \\ &= k\|c\|\,\|d\| < \infty\,. \end{aligned} \tag{3.36}$$

Similarly,

$$\|\tilde{K}_2[c,d]\| \leq k\|c\|\,\|d\|. \tag{3.37}$$

From (3.36) and (3.37) it follows that

$$\|\tilde{K}[c,d]\| \leq 2k\|c\|\,\|d\|. \tag{3.38}$$

This shows that $\tilde{K}[c,d] \in X$ and that $\tilde{K}[\cdot,\cdot]$ is bounded, and hence continuous, in each argument separately. It is a routine matter to show that $\tilde{K}$ is bilinear. □

We now use Theorem 3.13 to derive properties of K given by (3.33).

Theorem 3.14. *Under Assumption (A4),*

(i) $K : X \to X$,

(ii) K *is locally Lipschitz on* X,

(iii) K *is Fréchet differentiable on* X.

Proof: The proof follows similar lines to that given in [39, Section 3] for a continuous coagulation-fragmentation equation.

(i) By (3.38),

$$\|Kc\| = \|\tilde{K}[c,c]\| \leq 2k\|c\|^2 < \infty \quad \forall c \in X\,. \tag{3.39}$$

(ii) For $c, d \in X$,

$$\begin{aligned} \|Kc - Kd\| &= \|\tilde{K}[c-d,c] + \tilde{K}[d,c-d]\| \ \text{(by bilinearity)} \\ &\leq 2k\|c-d\|\,\|c\| + 2k\|d\|\,\|c-d\| \ \text{(by (3.38))} \\ &= 2k\|c-d\|(\|c\| + \|d\|)\,. \end{aligned} \tag{3.40}$$

If we fix $f \in X$, then $\forall c, d \in \bar{B}(f,r) := \{g \in X : \|g - f\| \leq r\}$,

$$\|Kc - Kd\| \leq C(f,r)\|c - d\|$$

where $C(f,r) = 4k(\|f\| + r)$.

(iii) Let $c, \delta \in X$. Then

$$K[c+\delta] = Kc + \tilde{K}[c,\delta] + \tilde{K}[\delta,c] + K\delta\,.$$

For fixed $c, \tilde{K}[c,\cdot] + \tilde{K}[\cdot, c]$ is in $B(X)$ by (3.38) and

$$\|\tilde{K}[c,\delta] + \tilde{K}[\delta, c]\| \leq \|\tilde{K}[c,\delta]\| + \|\tilde{K}[\delta, c]\| \leq 4k\|c\|\,\|\delta\| \ \forall \delta \in X.$$

Also, for $\delta \neq 0$,

$$\frac{\|K\delta\|}{\|\delta\|} \leq 2k\|\delta\| \to 0 \text{ as } \|\delta\| \to 0\,.$$

Therefore K is Fréchet differentiable at each $c \in X$ and the Fréchet derivative $K_{[c]}$ at c is given by

$$K_{[c]}d = \tilde{K}[c,d] + \tilde{K}[d,c] \quad \forall d \in X. \tag{3.41}$$

□

Theorem 3.15. *The Fréchet derivative $K_{[c]}$ is continuous with respect to c.*

Proof: For $c_1, c_2 \in X$ we have

$$\begin{aligned} \|K_{[c_1]}d - K_{[c_2]}d\| &\leq \|\tilde{K}[c_1 - c_2, d]\| + \|\tilde{K}[d, c_1 - c_2]\| \\ &\leq 4k\|c_1 - c_2\|\,\|d\| \ \text{ (by (3.38))} \\ &\to 0 \text{ as } \|c_1 - c_2\| \to 0. \end{aligned}$$

□

Having established the behaviour of the coagulation operator K subject to Assumption (A4), we now combine this with the results obtained earlier on the fragmentation equation.

In the general case when the sequence $\{a_n\}$ is constrained only by Assumption (A1), we know that $G = \overline{A+B}$ generates a positive semigroup of isometries denoted by $\{T_G(t)\}_{t\geq 0}$. In view of Theorems 3.14 and 3.15, we may deduce from Theorem 2.32 that, for any $f \in D(G)$, the semilinear ACP

$$\frac{d}{dt}u(t) = Gu(t) + Ku(t) \tag{3.42}$$

$$\lim_{t\to 0+} u(t) = f \in D(G) \tag{3.43}$$

has a uniquely defined strongly differentiable solution

$$u : [0, t_0) \to B(f, r) := \{g \in X : \|g - f\| < r\}$$

for suitably small positive constants t_0 and r. Our aim is to show that this solution exists globally in time. First we establish that $u(t) \in D(G)^+$ whenever $f \in D(G)^+$. To do this, we apply an elegant argument used, in a different context, by Belleni-Morante in [18, Chapter 8]. This hinges on recognising that the solution u of (3.42) and (3.43) is also the unique strongly differentiable solution of

$$\frac{d}{dt}u(t) = \Big(G[u(t)] - \alpha u(t)\Big) + \Big(\alpha u(t) + K[u(t)]\Big)$$

for any $\alpha \in \mathbb{R}$. Hence u is the unique solution of the integral equation

$$u(t) = e^{-\alpha t}T_G(t)f + \int_0^t e^{-\alpha(t-s)}T_G(t-s)K^\alpha[u(s)]ds\,, \quad 0 \le t \le t_0,$$

where $K^\alpha := K + \alpha I$.

Lemma 3.16. *Let $f \in D(G)^+$ and let $\alpha \ge k(\|f\| + r)$. Then $K^\alpha c \in X^+$ for all $c \in B(f, r)^+$.*

Proof: We have

$$(K^\alpha c)_n = \alpha c_n + (\tilde{K}_1[c, c])_n - (\tilde{K}_2[c, c])_n = \alpha c_n + (K_1 c)_n - (K_2 c)_n.$$

Now

$$(K_1 c)_n = \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j}c_{n-j}c_j \ge 0 \quad \forall c \in X^+\,.$$

Also, if $c \in B(f, r)^+$ then

$$(K_2 c)_n = \sum_{j=1}^{\infty} k_{n,j}c_n c_j \le k\|c\|c_n \le k(\|f\| + r)c_n.$$

Hence

$$\alpha c_n - (K_2 c)_n \ge \alpha c_n - k(\|f\| + r)c_n \ge 0 \quad \text{if} \quad \alpha \ge k(\|f\| + r)$$

and therefore $K^\alpha c \in X^+$ for all $c \in B(f,r)^+$ if $\alpha \geq k(\|f\| + r)$.

$\square$

Theorem 3.17. *Let $f \in D(G)^+$ and let $u : [0, t_0) \to B(f,r)$ be the unique strict solution of (3.42) and (3.43). Then there exists $t_1 \in [0, t_0)$ such that $u(t) \in X^+$ for all $t \in [0, t_1)$.*

Proof: Let $Y = C([0,t_1], X)$ with norm $\|v\|_Y := \max\{\|v(t)\| : 0 \leq t \leq t_1\}$, let

$$\Sigma := \{v \in Y : v(t) \in \bar{B}(f, r_1)^+ \ \forall t \in [0, t_1]\}\,, \qquad \text{where } 0 < r_1 < r\,,$$

and define an operator Q on Σ by

$$(Qv)(t) := e^{-\alpha t} T_G(t) f + \int_0^t e^{-\alpha(t-s)} T_G(t-s) K^\alpha[v(s)] ds\,, \quad 0 \leq t \leq t_1\,,$$

for $\alpha \geq k(\|f\| + r)$. Note that t_1 is chosen later.

Firstly we shall show that $Q(\Sigma) \subset Y$. Let $v \in \Sigma$ and $t, t_2 \in [0, t_1]$. Then

$$\begin{aligned} \|(Qv)(t) - (Qv)(t_2)\| \;\leq\; & \|e^{-\alpha t} T(t) f - e^{-\alpha t_2} T(t_2) f\| \\ & + \Big\| \int_0^t e^{-\alpha(t-s)} T(t-s) K_\alpha[v(s)] ds \\ & - \int_0^{t_2} e^{-\alpha(t_2-s)} T(t_2 - s) K_\alpha[v(s)] ds \Big\|. \end{aligned}$$

The first term on the right-hand side gives

$$\begin{aligned} & \|e^{-\alpha t} T(t) f - e^{-\alpha t_2} T(t_2) f\| \\ = \; & \|e^{-\alpha t} T(t) f - e^{-\alpha t_2} T(t) f + e^{-\alpha t_2} T(t) f - e^{-\alpha t_2} T(t_2) f\| \\ \leq \; & \|e^{-\alpha t} T(t) f - e^{-\alpha t_2} T(t) f\| + \|e^{-\alpha t_2} T(t) f - e^{-\alpha t_2} T(t_2) f\| \\ \leq \; & |e^{-\alpha t} - e^{-\alpha t_2}| \|T(t) f\| + e^{-\alpha t_2} \|T(t) f - T(t_2) f\| \\ \leq \; & |e^{-\alpha t} - e^{-\alpha t_2}| \|f\| + e^{-\alpha t_2} \|T(t) f - T(t_2) f\| \\ & \qquad (\text{ since } \{T(t)\}_{t \geq 0} \text{ is a substochastic semigroup}) \\ \to \; & 0 \ \text{ as } t \to t_2. \end{aligned}$$

The second term on the right-hand side gives

$$
\begin{aligned}
& \| \int_0^t e^{-\alpha(t-s)} T(t-s) K_\alpha[v(s)] ds - \int_0^{t_2} e^{-\alpha(t_2-s)} T(t_2-s) K_\alpha[v(s)] ds \| \\
\leq \; & \| \int_0^t e^{-\alpha(t-s)} T(t-s) K_\alpha[v(s)] ds - \int_0^t e^{-\alpha(t_2-s)} T(t-s) K_\alpha[v(s)] ds \| \\
& + \| \int_0^t e^{-\alpha(t_2-s)} T(t-s) K_\alpha[v(s)] ds - \int_0^{t_2} e^{-\alpha(t_2-s)} T(t_2-s) K_\alpha[v(s)] ds \| \\
= \; & \| \int_0^t \left(e^{-\alpha(t-s)} - e^{-\alpha(t_2-s)} \right) T(t-s) K_\alpha[v(s)] ds \| \\
& + \| e^{-\alpha t_2} \int_{t_2}^t e^{\alpha s} \left(T(t-s)[v(s)] - T(t_2-s)[v(s)] \right) ds \| \\
\to \; & 0 \text{ as } t \to t_2.
\end{aligned}
$$

Altogether we have

$$
\|(Qv)(t) - (Qv)(t_0)\| \to 0 \text{ as } t \to t_2.
$$

Hence $Q(\Sigma) \subset Y$, as required.

It is also clear that $(Qv)(t) \in X^+ \; \forall t \in [0, t_1]$ since $T_G(t) : X^+ \to X^+$ and $K^\alpha[v(s)] \in X^+$ for all $v(s) \in \bar{B}(f, r_1)^+$. Now let $v, w \in \Sigma$. Then

$$
\begin{aligned}
& \|(Qv)(t) - (Qw)(t)\| \\
\leq \; & \int_0^t e^{-\alpha(t-s)} \|T_G(t-s)\| \, \|K^\alpha[v(s)] - K^\alpha[w(s)]\| ds \\
\leq \; & \int_0^t e^{-\alpha(t-s)} \|K^\alpha[v(s)] - K^\alpha[w(s)]\| ds \\
= \; & \int_0^t e^{-\alpha(t-s)} \|\alpha(v(s) - w(s)) + K[v(s)] - K[w(s)]\| ds \\
\leq \; & \int_0^t e^{-\alpha(t-s)} \Big\{ \alpha \|v(s) - w(s)\| + 2k(\|v(s)\| + \|w(s)\|) \|v(s) - w(s)\| \Big\} ds \\
& \text{(by (3.40))} \\
\leq \; & [\alpha + C(f, r)] \int_0^t e^{-\alpha(t-s)} \|v(s) - w(s)\| ds \, ,
\end{aligned}
$$

where $C(f,r) = 4k(\|f\| + r)$, as before. So

$$\|Qv - Qw\|_Y \le [\alpha + C(f,r)]\, \|v - w\|_Y \, t_1 \, .$$

Also

$$\begin{aligned} \|(Qv)(t) - f\,\| &\le \|\, e^{-\alpha t} T_G(t) f - f\| + \int_0^t e^{-\alpha(t-s)} \|T_G(t-s) K^\alpha [v(s)]\| ds \\ &\le \|e^{-\alpha t} T_G(t) f - f\| + \int_0^t e^{-\alpha(t-s)} \|K^\alpha [v(s)]\| ds \, , \end{aligned}$$

since $\{T_G(t)\}_{t\ge 0}$ is a substochastic semigroup. Now

$$\begin{aligned} \|K^\alpha[v(s)]\| &= \|K^\alpha[v(s)] - K^\alpha f + K^\alpha f\| \\ &\le \|K^\alpha[v(s)] - K^\alpha f\| + \|K^\alpha f\| \\ &\le [\alpha + C(f,r)]\|v(s) - f\| + \alpha\|f\| + \|Kf\| \;\text{ by (3.40)} \\ &\le [\alpha + C(f,r)] r_1 + \alpha\|f\| + \|Kf\| \, . \end{aligned}$$

Therefore

$$\|(Qv)(t) - f\| \le \|e^{-\alpha t} T_G(t) f - f\| + \Big\{[\alpha + C(f,r)] r_1 + \alpha\|f\| + \|Kf\|\Big\} t_1 \, .$$

Let

$$q(t_1) := \frac{1}{r_1} \max_{0\le t\le t_1} \Big\{ \|\, e^{-\alpha t} T_G(t) f - f\| \Big\} + \frac{1}{r_1}\Big\{[\alpha + C(f,r)] r_1 + \alpha\|f\| + \|Kf\|\Big\} t_1 \, .$$

Then, for all $v, w \in \Sigma$,

$$\|(Qv)(t) - f\| \le r_1 q(t_1) \text{ and } \|Qv - Qw\|_Y \le q(t_1)\|v - w\|_Y .$$

Now $q(t_1) \to 0+$ as $t_1 \to 0+$. Hence we can choose t_1 so that $0 < q(t_1) < 1$ in which case we have $Q(\Sigma) \subset \Sigma$ and Q is a contraction. Hence there exists a unique solution $u \in \Sigma$ of $u = Qu$ so that the integral equation has a unique solution $u \in C([0, t_1], X^+)$.

□

Theorem 3.18. *Let the maximal interval of existence of the strict solution of*

(3.42) and (3.43) be $[0, \hat{T})$. *Then* $u(t) \in X^+$ *for all* $t \in [0, \hat{T})$ *whenever* $f \in D(G)^+$.

Proof: Let $T_0 \in (0, \hat{T})$ be arbitrarily fixed and define

$$\tau_{\max} := \sup\{0 < \tau < T_0 \ : \ u(t) \in X^+ \ \forall t \in [0, \tau]\} .$$

Suppose $\tau_{\max} < T_0$ and consider the semilinear problem

$$\frac{d}{dt} v(t) = Gv(t) + Kv(t) \, , \ t > 0 \, ; \quad v(0) = u(\tau_{\max}) \, .$$

By continuity, $v(0) = u(\tau_{\max}) \in X^+$, and so, arguing as before, we deduce that there exists $t_0 \in (0, T_0 - \tau_{\max})$ such that a unique (mild) non-negative solution of this problem exists on $[0, t_0]$. Since $v(t) = u(t + \tau_{\max})$, this means that $u(t) \in X^+$ for all $t \in [0, \tau_{\max} + t_0]$, which is a contradiction. Hence $u(t) \in X^+$ for all $t \in [0, \hat{T})$ whenever $f \in X^+$.

□

Theorem 3.19. *The solution to the ACP (3.42) and (3.43) exists globally in time.*

Proof: From Theorem 2.28, all we need to show is that the unique, local solution u does not blow up in finite time. Since u is strongly differentiable and non-negative, we have

$$\frac{d}{dt}\|u(t)\| = \frac{d}{dt}\sum_{n=1}^{\infty} n u_n(t) = \sum_{n=1}^{\infty} n[Gu(t)]_n + \sum_{n=1}^{\infty} n[Ku(t)]_n \, .$$

By the discussion on pages 39-40 and by the same calculations as in (3.5) we have that the derivative is equal to zero, and therefore

$$\|u(t)\| = \|u(0)\| = \|f\|, \ \ \forall t \in [0, \hat{T}).$$

Hence the local solution cannot blow up in finite time so we have global existence of a strict non-negative solution to our ACP (3.42) and (3.43).

□

As was the case with the fragmentation equation for general $\{a_n\}$, it should be noted that $f \in D^+$ does not guarantee that the solution $u(t)$ of (3.42) and (3.43) is in D^+ for all $t > 0$. In the following section we discuss a specific class of fragmentation kernels for which it *is* possible to prove that $u(t)$ remains in D^+ for all t.

3.4 Power Law Fragmentation

We now consider the specific case of (3.31) and (3.32) when the fragmentation rate a_n is defined by the power law

$$a_n = n^\alpha - 1\,, \qquad \alpha \in \mathbb{R}\,.$$

Note that when $\alpha = 0$, $a_n = 0$ for all n and we are dealing with pure coagulation. Also, when $\alpha < 0$ the sequence $\{a_n\}$ is bounded and so we have the existence and uniqueness of a non-negative strict solution to (3.31) and (3.32) for any $f \in D = X$. Consequently, we shall restrict our attention to the case when $\alpha > 0$.

If, initially, we consider only fragmentation, then since $\{a_n\}$ is a monotonic increasing sequence, $D(A)$ is invariant under the semigroup $\{T_G(t)\}_{t\geq 0}$, where $G = \overline{(A + B, D)}$ by Theorem 3.9. Moreover, the graph norm $\|\cdot\|_A$ now takes the form

$$\|f\|_A = \sum_{n=1}^{\infty} \{n + n(n^\alpha - 1)\}\,|f_n| = \sum_{n=1}^{\infty} n^{\alpha+1}|f_n|\,. \tag{3.44}$$

It is therefore convenient to introduce the following spaces.

Definition 3.20. *For $\alpha > 0$, define the space $(X_\alpha, \|\cdot\|_\alpha)$ by*

$$X_\alpha \;:=\; \left\{ f = \{f_n\}_{n=1}^{\infty} \;:\; \sum_{n=1}^{\infty} n^\alpha |f_n| < \infty \right\} \tag{3.45}$$

$$\|f\|_\alpha \;:=\; \sum_{n=1}^{\infty} n^\alpha |f_n|\,,\; f \in X_\alpha\,.$$

(Note that our original space X corresponds to $\alpha = 1$.)

Lemma 3.21. *For each $\alpha > 0$, $X_{\alpha+1}$ is compactly embedded in X_α. We shall denote this by $X_{\alpha+1} \hookrightarrow\hookrightarrow X_\alpha$.*

Proof: Define $J : X_{\alpha+1} \to X_\alpha$ by $Jf = f$, $f \in X_{\alpha+1}$. It is clear that the embedding J is a bounded, linear mapping from $X_{\alpha+1}$ into X_α since an easy calculation shows that $\|Jf\|_\alpha \le \|f\|_{\alpha+1}$ for all $f \in X_{\alpha+1}$. To prove that J is compact, we introduce operators $\{J_r\}_{r=1}^{\infty}$ defined on $X_{\alpha+1}$ by

$$(J_r f)_n = \begin{cases} f_n & n = 1, ..., r \\ 0 & n \ge r+1. \end{cases}$$

Then, for $f \in X_{\alpha+1}$,

$$\|J_r f\|_\alpha = \sum_{n=1}^{r} n^\alpha |f_n| \le \sum_{n=1}^{\infty} n^{\alpha+1}|f_n| = \|f\|_{\alpha+1}$$

so that $J_r \in B(X_{\alpha+1}, X_\alpha)$ for each $r = 1, 2, ...$ Moreover, each J_r is a finite rank operator with $R(J_r) =$ span $\{e_1, ..., e_r\}$ (where e_n is the standard canonical basis vector with zero entries in all positions except for the n^{th} which is 1) and hence compact. Finally, for $f \in X_{\alpha+1}$ with $f \neq 0$,

$$\begin{aligned} \|Jf - J_r f\|_\alpha &= \sum_{n=r+1}^{\infty} n^\alpha |f_n| = \sum_{n=r+1}^{\infty} \frac{1}{n} n^{\alpha+1}|f_n| \\ &\le \frac{1}{r+1} \sum_{n=r+1}^{\infty} n^{\alpha+1}|f_n| \le \frac{1}{r+1}\|f\|_{\alpha+1}\,. \end{aligned}$$

Hence $\|J - J_r\| \le \frac{1}{r+1} \to 0$ as $r \to \infty$, where $\|\cdot\|$ denotes the norm in $B(X_{\alpha+1}, X_\alpha)$. It is a standard theorem ([34, Theorem 8.1-5]) that the norm limit of a sequence of compact operators is a compact operator. Thus J is compact.

□

From the derivation of (3.44), we see that $(D, \|\cdot\|_A) = (X_{\alpha+1}, \|\cdot\|_{\alpha+1})$ for the case when

$$(Af)_n = -(n^\alpha - 1) f_n \quad (\alpha > 0)\,. \tag{3.46}$$

Corollary 3.22. *For the operator A in (3.46) and $G = \overline{(A+B, D)}$, $X_{\alpha+1}$ is invariant under $\{T_G(t)\}_{t\ge 0}$.*

Proof: This follows from Theorem 3.9 and the preceding remarks.

□

Note also that the proof of Theorem 3.9 and equation (3.30) show that, for

$\{a_n\}_{n=1}^{\infty} = \{n^\alpha - 1\}_{n=1}^{\infty}$ and $t \geq 0$, the restrictions $T_{G_{\alpha+1}}(t)$ of $T_G(t)$ to $X_{\alpha+1}$ form a substochastic semigroup on $X_{\alpha+1}$ with generator $G_{\alpha+1}$, where

$$G_{\alpha+1} f := (A+B)f;\ \ D(G_{\alpha+1}) := \{f \in X_{\alpha+1} : (A+B)f \in X_{\alpha+1}\}\,. \tag{3.47}$$

Our aim now is to obtain a similar result for the semilinear ACP (3.42) and (3.43), that is, to show that if f is a suitably restricted element in $D = X_{\alpha+1}$, then $u(t) \in X_{\alpha+1}$ for all $t > 0$, where u is the uniquely defined strict solution of (3.42) and (3.43). This will then imply that (3.31) and (3.32) has a unique strict solution. To accomplish this, we shall study the ACP

$$\frac{d}{dt}v(t) = G_{\alpha+1}v(t) + Kv(t)\,, \quad t > 0\,; \quad \lim_{t\to 0+} v(t) = f, \tag{3.48}$$

in the space $X_{\alpha+1}$ where $G_{\alpha+1}$ is given by (3.47) and $f \in D(G_{\alpha+1})$.

First we note that, for any $\beta > 0$,

$$\begin{aligned}
\sum_{n=1}^{\infty} n^\beta (Kf)_n &= \frac{1}{2}\sum_{n=1}^{\infty} n^\beta \sum_{j=1}^{n-1} k_{n-j,j} f_{n-j} f_j - \sum_{n=1}^{\infty} n^\beta \sum_{j=1}^{\infty} k_{n,j} f_n f_j \\
&= \frac{1}{2}\sum_{j=1}^{\infty} \sum_{n=j+1}^{\infty} n^\beta k_{n-j,j} f_{n-j} f_j - \sum_{n=1}^{\infty} n^\beta f_n \sum_{j=1}^{\infty} k_{n,j} f_j \\
&= \frac{1}{2}\sum_{j=1}^{\infty} \sum_{l=1}^{\infty} (l+j)^\beta k_{l,j} f_l f_j - \sum_{n=1}^{\infty} n^\beta f_n \sum_{j=1}^{\infty} k_{n,j} f_j \\
&= \frac{1}{2}\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} (n+j)^\beta k_{n,j} f_n f_j - \frac{1}{2}\sum_{n=1}^{\infty} n^\beta f_n \sum_{j=1}^{\infty} k_{n,j} f_j \\
&\qquad - \frac{1}{2}\sum_{j=1}^{\infty} j^\beta f_j \sum_{n=1}^{\infty} k_{j,n} f_n \\
&= \frac{1}{2}\sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \Big\{(n+j)^\beta - n^\beta - j^\beta\Big\} k_{n,j} f_n f_j \\
&\qquad\qquad \text{(since } k_{n,j} \text{ is symmetric).}
\end{aligned}$$

We shall also make use of the inequality

$$(n+j)^\beta \leq 2^\beta (n^\beta + j^\beta),\ \ \beta > 0 \text{ and } \ n, j = 1, 2, \ldots\,. \tag{3.49}$$

Inequality (3.49) follows easily from the fact that

$$n \geq j \implies (n+j)^\beta \leq 2^\beta n^\beta < 2^\beta(n^\beta + j^\beta)$$

and similarly for $j \geq n$. This then leads to

$$\sum_{n=1}^{\infty} n^\beta (Kf)_n \leq \frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(2^\beta - 1)(n^\beta + j^\beta)k_{n,j}f_n f_j$$

and, in particular,

$$\sum_{n=1}^{\infty} n^{\alpha+1}(Kf)_n \leq \frac{1}{2}\sum_{j=1}^{\infty}\sum_{n=1}^{\infty}(2^{\alpha+1} - 1)(n^{\alpha+1} + j^{\alpha+1})k_{n,j}f_n f_j\,. \tag{3.50}$$

We now consider properties of the restriction $K_{\alpha+1}$ of K to the Banach space $X_{\alpha+1} \hookrightarrow\hookrightarrow X$. Corresponding bilinear forms $\tilde{K}_{\alpha+1}, \tilde{K}^{(1)}_{\alpha+1}$ and $\tilde{K}^{(2)}_{\alpha+1}$ are defined by analogy with (3.35). In particular

$$\left(\tilde{K}_{\alpha+1}[c,d]\right)_n \;=\; \left(\tilde{K}^{(1)}_{\alpha+1}[c,d]\right)_n \;-\; \left(\tilde{K}^{(2)}_{\alpha+1}[c,d]\right)_n.$$

Lemma 3.23. *Let $C_\alpha = 2^{\alpha+1}k$ and $D_\alpha = C_\alpha + k$ where k is the constant in (A4). Then $\forall c, d \in X_{\alpha+1}$,*

$$\begin{aligned}
\|\tilde{K}^{(1)}_{\alpha+1}[c,d]\|_{\alpha+1} &\leq C_\alpha \|c\|_{\alpha+1}\|d\|_{\alpha+1}, & (3.51)\\
\|\tilde{K}^{(2)}_{\alpha+1}[c,d]\|_{\alpha+1} &\leq k \|c\|_{\alpha+1}\|d\|_{\alpha+1}, & (3.52)\\
\|\tilde{K}_{\alpha+1}[c,d]\|_{\alpha+1} &\leq D_\alpha \|c\|_{\alpha+1}\|d\|_{\alpha+1}. & (3.53)
\end{aligned}$$

Proof: For (3.51), we have

$$\begin{aligned}
\|\tilde{K}^{(1)}_{\alpha+1}[c,d]\|_{\alpha+1} &\leq \frac{k}{2}\sum_{n=1}^{\infty} n^{\alpha+1}\sum_{j=1}^{n-1}|c_{n-j}||d_j| \\
&= \frac{k}{2}\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty} n^{\alpha+1}|c_{n-j}||d_j| \\
&= \frac{k}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}(l+j)^{\alpha+1}|c_l||d_j|
\end{aligned}$$

$$
\begin{aligned}
&\leq \; 2^\alpha k \sum_{j=1}^{\infty} \sum_{l=1}^{\infty} [l^{\alpha+1}|c_l||d_j| + j^{\alpha+1}|c_l||d_j|] \quad \text{by (3.49)} \\
&= \; 2^\alpha k \left[\sum_{j=1}^{\infty} |d_j| \|c\|_{\alpha+1} + \sum_{l=1}^{\infty} |c_l| \|d\|_{\alpha+1} \right] \\
&\leq \; C_\alpha \|c\|_{\alpha+1} \|d\|_{\alpha+1}.
\end{aligned}
$$

The proof of inequality (3.52) is similar and (3.53) then follows immediately. □

Theorem 3.24. *Let $K_{\alpha+1}$ denote the restriction of K to $X_{\alpha+1}$. Then*

(i) $K_{\alpha+1} : X_{\alpha+1} \to X_{\alpha+1}$*,*

(ii) $K_{\alpha+1}$ *is locally Lipschitz on* $X_{\alpha+1}$*,*

(iii) $K_{\alpha+1}$ *is Fréchet differentiable on* $X_{\alpha+1}$ *.*

Proof. The proof is very similar to that given in Theorem 3.14 for the case $\alpha = 0$.

(i) It follows directly from (3.53) that, for $c \in X_{\alpha+1}$,

$$\|K_{\alpha+1} c\|_{\alpha+1} \leq D_\alpha \|c\|_{\alpha+1}^2 .$$

(ii) On using the bilinearity of $\tilde{K}_{\alpha+1}$ together with (3.53), we obtain

$$\|K_{\alpha+1} c - K_{\alpha+1} d\|_{\alpha+1} \leq D_\alpha \|c - d\|_{\alpha+1} (\|c\|_{\alpha+1} + \|d\|_{\alpha+1})$$

for all $c, d \in X_{\alpha+1}$. Consequently, if we fix $f \in X_{\alpha+1}$ then

$$\|K_{\alpha+1} c - K_{\alpha+1} d\|_{\alpha+1} \leq C(f, r) \|c - d\|_{\alpha+1} \; \forall c, d \in \bar{B}(f, r)$$

where

$$\bar{B}(f, r) = \{g \in X_{\alpha+1} : \|g - f\|_{\alpha+1} \leq r\}$$

and

$$C(f, r) = 2 D_\alpha (r + \|f\|_{\alpha+1}) .$$

(iii) Arguing as in the proof of Theorem 3.14(iii), we find that the Fréchet derivative of $K_{\alpha+1}$ at any given $c \in X_{\alpha+1}$ is given by

$$(K_{\alpha+1})_{[c]}d = \tilde{K}_{\alpha+1}[c,d] + \tilde{K}_{\alpha+1}[d,c] \qquad \forall d \in X_{\alpha+1}\,.$$

Note also that $(K_{\alpha+1})_{[c]}$ is continuous with respect to c.

□

As before, we can now use Theorem 2.32 to deduce that the ACP (3.48) has a uniquely defined strict solution

$$v : [0, t_0) \to B(v_0, r) := \{g \in X_{\alpha+1} : \|g - v_0\|_{\alpha+1} < r\}$$

for suitably small t_0 and r. Moreover, analogues of Lemma 3.16 and Theorems 3.17 and 3.18, with $X, K, \|f\|$ replaced by $X_{\alpha+1}, K_{\alpha+1}$ and $\|f\|_{\alpha+1}$ can be proved by identical arguments. Consequently, if $[0, \hat{T})$ denotes the maximal interval of existence of the strict solution v of (3.48), then $v(t) \in X^+_{\alpha+1}$ for $t \in [0, \hat{T})$ whenever $f \in X^+_{\alpha+1}$.

Since $X_{\alpha+1}$ is compactly imbedded in $X = X_1$, and $G_{\alpha+1}$ and $K_{\alpha+1}$ are restrictions of G and K respectively, it follows that v also satisfies the integral equation

$$v(t) = T_G(t)f + \int_0^t T_G(t-s)K[v(s)]ds \text{ in } X\,.$$

Also, if f is in $D(G_{\alpha+1})^+$ then f is also in $D(A)^+ \subseteq D(G)^+$ and so we deduce that v agrees on $[0, \hat{T})$ with the unique (globally defined) strict solution u of the ACP (3.42) and (3.43) posed in the space X. Hence, if $f \in D(G_{\alpha+1})^+$ then $u(t) \in D(A)^+$ for all $t \in [0, \hat{T})$. It remains to show that $\hat{T} = \infty$.

Theorem 3.25. *The solution v of (3.48) is global in time.*

Proof: For each $n = 1, 2, 3, \ldots$, we have

$$\begin{aligned} v_n(t) &= f_n + \int_0^t \left[\Big(G_{\alpha+1}[v(s)]\Big)_n + \Big(K_{\alpha+1}[v(s)]\Big)_n \right] ds \\ &= f_n + \int_0^t \left[\Big(A[v(s)]\Big)_n + \Big(B[v(s)]\Big)_n + \Big(K[v(s)]\Big)_n \right] ds\,, \end{aligned}$$

since, on $X_{\alpha+1}$, $G_{\alpha+1} = A + B$ and $K_{\alpha+1} = K$. Consequently,

$$\begin{aligned}
\|v(t)\|_{\alpha+1} &\leq \|f\|_{\alpha+1} \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha+1} \int_0^t \left\{ \sum_{j=n+1}^{\infty} b_{n,j}(j^\alpha - 1)v_j(s) - (n^\alpha - 1)v_n(s) \right\} ds \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha+1} \int_0^t \Big(K[v(s)] \Big)_n ds \\
&\leq \|f\|_{\alpha+1} \\
&\quad + \int_0^t \left\{ \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n^{\alpha+1} b_{n,j}(j^\alpha - 1)v_j(s) \right. \\
&\qquad\qquad \left. - \sum_{n=1}^{\infty} n^{\alpha+1}(n^\alpha - 1)v_n(s) \right\} ds \\
&\quad + \int_0^t \left(\frac{k}{2} \sum_{j=1}^{\infty} \sum_{n=1}^{\infty} \Big(2^{\alpha+1} - 1\Big) \Big(n^{\alpha+1} + j^{\alpha+1}\Big) v_n(s) v_j(s) \right) ds
\end{aligned}$$

by (3.50). If we take the second term on the right-hand side and change the order of summation we obtain

$$\begin{aligned}
& \int_0^t \left\{ \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n^{\alpha+1} b_{n,j}(j^\alpha - 1)v_j(s) - \sum_{n=1}^{\infty} n^{\alpha+1}(n^\alpha - 1)v_n(s) \right\} ds \\
= & \int_0^t \left\{ \sum_{j=1}^{\infty} \sum_{n=1}^{j-1} n^{\alpha+1} b_{n,j}(j^\alpha - 1)v_j(s) - \sum_{n=1}^{\infty} n^{\alpha+1}(n^\alpha - 1)v_n(s) \right\} ds \\
\leq & \int_0^t \left\{ \sum_{j=1}^{\infty} j^{\alpha+1}(j^\alpha - 1)v_j(s) - \sum_{n=1}^{\infty} n^{\alpha+1}(n^\alpha - 1)v_n(s) ds \right\} = 0 \,,
\end{aligned}$$

where we have used (A3). So overall we get

$$\begin{aligned}
\|v(t)\|_{\alpha+1} \leq \|f\|_{\alpha+1} + \int_0^t \Bigg\{ \frac{k}{2}(2^{\alpha+1} - 1)\Big(& \sum_{j=1}^{\infty} \|v(s)\|_{\alpha+1} v_j(s) \\
& + \sum_{n=1}^{\infty} \|v(s)\|_{\alpha+1} v_n(s) \Big) ds
\end{aligned}$$

$$
\begin{aligned}
&= \|f\|_{\alpha+1} + \int_0^t k(2^{\alpha+1}-1)\sum_{n=1}^{\infty} v_n(s)\|v(s)\|_{\alpha+1} ds \\
&\leq \|f\|_{\alpha+1} + \int_0^t k(2^{\alpha+1}-1)\|v(s)\|_{\alpha+1}\sum_{n=1}^{\infty} n v_n(s) ds \\
&= \|f\|_{\alpha+1} + \int_0^t k(2^{\alpha+1}-1)\|v(s)\|_{\alpha+1}\|f\| ds \text{ (by mass conservation)}
\end{aligned}
$$

$$
\begin{aligned}
\|v(t)\|_{\alpha+1} &\leq \|f\|_{\alpha+1} \exp\Big(\int_0^t k(2^{\alpha+1}-1)\|f\| ds\Big) \ \forall t \in [0, \hat{T}) \\
&\qquad \text{(by Gronwall's inequality, see Lemma 2.34)} \\
&= \|f\|_{\alpha+1} \exp\Big(k(2^{\alpha+1}-1)\|f\|t\Big) < \infty \text{ for all } t < \infty\,.
\end{aligned}
$$

Hence $\|v(t)\|_{\alpha+1}$ cannot blow up in finite time and therefore v is defined globally in time.

□

As an immediate consequence, we deduce that, when $a_n = n^\alpha - 1$, the solution $u(t)$ of (3.42) and (3.43) remains in $D(A)^+ = X^+_{\alpha+1}$ for all $t > 0$ whenever $f \in D(G_{\alpha+1})^+$.

3.5 Closed Form Solutions

We shall now show that the explicit solution to a specific version of the discrete pure fragmentation problem given by Ziff and McGrady in their paper [58] is the same as the unique strict solution that has been shown to exist by our semigroup analysis in Section 3.1. The specific equation we consider is obtained from (3.6) by setting $a_n = n - 1$ and $b_{n,j} = \frac{2}{j-1}$. This leads to

$$
\frac{du_n(t)}{dt} = -(n-1)u_n(t) + 2\sum_{j=n+1}^{\infty} u_j(t). \tag{3.54}
$$

Note that this represents a binary fragmentation process since

$$
\sum_{n=1}^{j-1} b_{n,j} = 2 \quad \text{for } j \geq 2.
$$

In [58], Ziff and McGrady describe a technique for obtaining a solution to (3.54). We shall provide full details here of how their solution was obtained. The first step is to consider the case of monodisperse initial conditions i.e.

$$u_n(0) = \delta_{nk} = \begin{cases} 1, & n = k \\ 0, & n \neq k, \end{cases} \tag{3.55}$$

where $k \in \mathbb{N}$ is fixed. Note that $u_n(t) = 0$ for $n > k$ since our initial condition means we are starting with only clusters of size k and, as fragmentation is the only process that occurs, larger particles will never be created. So $u_k(t)$ satisfies the initial-value problem

$$\frac{du_k(t)}{dt} = -(k-1)u_k(t), \quad u_k(0) = 1,$$

from which it follows immediately that

$$u_k(t) = e^{-(k-1)t}. \tag{3.56}$$

Similarly u_{k-1} satisfies

$$\frac{du_{k-1}(t)}{dt} = -(k-2)u_{k-1}(t) + 2u_k(t), \quad u_{k-1}(0) = 0, \tag{3.57}$$

i.e.

$$\frac{du_{k-1}(t)}{dt} + (k-2)u_{k-1}(t) = 2e^{-(k-1)t}, \quad u_{k-1}(0) = 0,$$

and therefore

$$u_{k-1}(t) = -2e^{-(k-1)t} + 2e^{-(k-2)t}. \tag{3.58}$$

Continuing with this procedure, we obtain

$$\frac{du_{k-2}(t)}{dt} = -(k-3)u_{k-2}(t) + 2u_{k-1}(t) + 2u_k(t), \quad u_{k-2}(0) = 0 \tag{3.59}$$

which leads to

$$u_{k-2}(t) = e^{-(k-1)t} - 4e^{-(k-2)t} + 3e^{-(k-3)t} \tag{3.60}$$

and, similarly

$$u_{k-3}(t) = 2e^{-(k-2)t} - 6e^{-(k-3)t} + 4e^{-(k-4)t}. \tag{3.61}$$

On obtaining corresponding expressions for $u_{k-4}(t)$, $u_{k-5}(t)$, a noticeable pattern emerges which indicates that

$$u_n(t) = \begin{cases} e^{-(k-1)t}, & n = k \\ (1-n+k)e^{-(n-1)t} + 2(n-k)e^{-nt} + (k-n-1)e^{-(n+1)t}, & n < k. \end{cases} \tag{3.62}$$

We shall now go on to prove by induction that this form for the solution is correct. We have shown that the form for the cases $n = k, k-1, k-2, k-3$ in (3.62) is correct. Now assume this form is true for $n = k, k-1, k-2, \ldots, k-i$, where $i \leq k-2$. For $n = k-i-1$ we have

$$\frac{du_{k-i-1}(t)}{dt} = -(k-i-2)u_{k-i-1}(t) + 2\sum_{j=k-i}^{n} u_j(t), \quad u_{k-i-1}(0) = 0. \tag{3.63}$$

By assumption, the right-hand side can be written as

$$\begin{aligned} &-(k-i-2)u_{k-i-1}(t) + 2\left((1+i)e^{-(k-i-1)t} - 2ie^{-(k-i)t}\right. \\ &\quad +(i-1)e^{-(k-i+1)t} + ie^{-(k-i)t} - 2(i-1)e^{-(k-i+1)t} \\ &\qquad \left. +(i-2)e^{-(k-i+2)t} + \ldots + 2e^{-(k-2)t} - 2e^{-(k-1)t} + e^{-(k-1)t}\right). \end{aligned}$$

This is a telescoping sum which reduces to

$$-(k-i-2)u_{k-i-1}(t) + 2(1+i)e^{-(k-i-1)t} - 2ie^{-(k-i)t}.$$

Consequently, u_{k-i-1} satisfies

$$\begin{aligned} \frac{du_{k-i-1}(t)}{dt} &= -(k-i-2)u_{k-i-1}(t) + 2(1+i)e^{-(k-i-1)t} - 2ie^{-(k-i)t}, \\ u_{k-i-1}(0) &= 0, \end{aligned}$$

and on solving this first order linear ODE, we obtain

$$u_{k-i-1}(t) = -2(1+i)e^{-(k-i-1)t} + ie^{-(k-i)t} + (2+i)e^{-(k-i-2)t}, \tag{3.64}$$

which agrees with (3.62) when n is replaced by $k-i-1$. Hence by induction the formula holds for all $n = 0, 1, 2, \ldots, k$.

From the linearity of the problem, a likely candidate for the corresponding solution in the case of general initial conditions $u_n(0) = f_n$, $n = 1, 2, 3, \ldots$, can then be obtained by superposition as is described by Ziff and McGrady in [58]. To create a particular solution for general initial conditions we take a linear combination of the above solutions for the monodisperse initial conditions. Hence

$$\begin{aligned} u_n(t) &= f_n e^{-(n-1)t} + \sum_{k=n+1}^{\infty} f_k \{ e^{-(n-1)t} - e^{-(n+1)t} \\ &\quad + (k-n)[e^{-(n-1)t} - 2e^{-nt} + e^{-(n+1)t}] \} \\ &= f_n e^{-(n-1)t} + \sum_{k=n+1}^{\infty} f_k e^{-(n-1)t} \{ 1 - e^{-2t} + (k-n)(1-e^{-t})^2 \}. \end{aligned} \tag{3.65}$$

Our general theory establishes that the ACP associated with this choice of a_n, $b_{n,j}$ has a strongly differentiable solution $u(t) = S_G(t)f$ with $f \in D(G)$ where $\{S_G(t)\}_{t\geq 0}$ is the stochastic semigroup generated by $G = \overline{A+B}$. Since $\{a_n\}_{n=1}^{\infty}$ is a monotonic increasing sequence we have further that $S_G(t)f \in D(A)$ $\forall t \geq 0$ whenever $f \in D(A)$. We shall now show that $S_G(t)f$ is given explicitly by the formula (3.65) found by Ziff and McGrady.

Firstly, we shall show that (3.65) is indeed a solution of (3.54). We note that on rearranging (3.65) we have

$$\begin{aligned} u_n(t) &= f_n e^{-(n-1)t} + e^{-(n-1)t} \left(1 - e^{-2t} - n(1-e^{-t})^2 \right) \sum_{k=n+1}^{\infty} f_k \\ &\quad + e^{-(n-1)t} (1-e^{-t})^2 \sum_{k=n+1}^{\infty} k f_k. \end{aligned} \tag{3.66}$$

Since $f \in X$, each of the infinite series appearing in (3.66) is absolutely convergent. We obtain

$$
\begin{aligned}
u_n'(t) &= -(n-1)f_n e^{-(n-1)t} \\
&\quad +[-(n-1)e^{-(n-1)t}]\left(1-e^{-2t}-n(1-e^{-t})^2\right)\sum_{k=n+1}^{\infty} f_k \\
&\quad +e^{-(n-1)t}\left(2e^{-2t}-2n(1-e^{-t})e^{-t}\right)\sum_{k=n+1}^{\infty} f_k \\
&\quad -(n-1)e^{-(n-1)t}(1-e^{-t})^2\sum_{k=n+1}^{\infty} kf_k \\
&\quad +2e^{-(n-1)t}(1-e^{-t})e^{-t}\sum_{k=n+1}^{\infty} kf_k \\
&= -(n-1)f_n e^{-(n-1)t} \\
&\quad +\left[-(n-1)e^{-(n-1)t}+(n-1)e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} f_k \\
&\quad +n(n-1)\left(e^{-(n-1)t}-2e^{-nt}+e^{-(n+1)t}\right)\sum_{k=n+1}^{\infty} f_k \\
&\quad +\left[2e^{-(n+1)t}-2n(e^{-nt}-e^{-(n+1)t})\right]\sum_{k=n+1}^{\infty} f_k \\
&\quad -(n-1)\left(e^{-(n-1)t}-2e^{-nt}+e^{-(n+1)t}\right)\sum_{k=n+1}^{\infty} kf_k \\
&\quad +\left(2e^{-nt}-2e^{-(n+1)t}\right)\sum_{k=n+1}^{\infty} kf_k \\
&= -(n-1)f_n e^{-(n-1)t} \\
&\quad +\left[(n-1)^2e^{-(n-1)t}-2n^2e^{-nt}+(n+1)^2e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} f_k \\
&\quad +\left[-(n-1)e^{-(n-1)t}+2ne^{-nt}-(n+1)e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} kf_k.
\end{aligned}
\tag{3.67}
$$

Now we shall substitute (3.65) into the right-hand side of (3.54). We have

$$-(n-1)f_n e^{-(n-1)t} \tag{3.68}$$

$$-(n-1)\sum_{k=n+1}^{\infty} e^{-(n-1)t}\left\{1-e^{-2t}+(k-n)(1-e^{-t})^2\right\} f_k \tag{3.69}$$

$$+2\sum_{j=n+1}^{\infty}\left[f_j e^{-(j-1)t}\right.$$

$$\left.+\sum_{k=j+1}^{\infty} e^{-(j-1)t}\left\{1-e^{-2t}+(k-j)(1-e^{-t})^2\right\} f_k\right]. \tag{3.70}$$

We can rearrange the expression in (3.69) to get

$$\begin{aligned} & -(n-1)\left[e^{-(n-1)t}-e^{-(n+1)t}-n(e^{-(n-1)t}-2e^{-nt}+e^{-(n+1)t})\right]\sum_{k=n+1}^{\infty} f_k \\ & -(n-1)\left[e^{-(n-1)t}-2e^{-nt}+e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} kf_k \\ = & \left[(n-1)^2 e^{-(n-1)t}-2n(n-1)e^{-nt}+(n^2-1)e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} f_k \\ & +\left[-(n-1)e^{-(n-1)t}+2(n-1)e^{-nt}-(n-1)e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} kf_k. \end{aligned} \tag{3.71}$$

Interchanging the order of summation in (3.70) produces

$$\begin{aligned} & 2\sum_{j=n+1}^{\infty} e^{-(j-1)t} f_j \\ & +2(1-e^{-2t})\sum_{k=n+2}^{\infty}\sum_{j=n+1}^{k-1} e^{-(j-1)t} f_k \\ & +2(1-e^{-t})^2\sum_{k=n+1}^{\infty}\sum_{j=n+1}^{k} e^{-(j-1)t}(k-j) f_k. \end{aligned}$$

Using the substitution $r=(k-j)$ we get

$$
\begin{aligned}
& (1-e^{-2t})\sum_{k=n+2}^{\infty} e^{-(k-1)t} f_k \sum_{r=1}^{k-n-1} e^{rt} \\
& +(1-e^{-t})^2 \sum_{k=n+1}^{\infty} e^{-(k-1)t} f_k \sum_{r=0}^{k-n-1} r e^{rt} \\
= \;& (1-e^{-2t})\sum_{k=n+2}^{\infty} e^{-(k-1)t} f_k \left(\frac{e^t(1-e^{(k-n-1)t})}{1-e^t}\right) \\
& +(1-e^{-t})^2 \sum_{k=n+1}^{\infty} e^{-(k-1)t} f_k \frac{d}{dt}\left(\frac{1-e^{(k-n)t}}{1-e^t}\right) \\
= \;& \left(\frac{e^t(1-e^{-2t})}{1-e^t}\right)\sum_{k=n+2}^{\infty} f_k \left(e^{-(k-1)t}-e^{-nt}\right) \\
& +\left(\frac{(1-e^{-t})^2}{(1-e^t)^2}\right)\sum_{k=n+1}^{\infty} f_k e^{-(k-1)t}\left(-(k-n)e^{(k-n)t}\right. \\
& \qquad\qquad \left.+(k-n-1)e^{(k-n+1)t}+e^t\right) \\
= \;& -(1+e^{-t})\left(\sum_{k=n+2}^{\infty} e^{-(k-1)t} f_k - e^{-nt}\sum_{k=n+2}^{\infty} f_k\right) \\
& +e^{-2t}\sum_{k=n+1}^{\infty} f_k \left[(k-n)(e^t-1)e^{-(n-1)t} - e^{-(n-2)t} + e^{-(k-2)t}\right] \\
= \;& -(1+e^{-t})\left(\sum_{k=n+2}^{\infty} e^{-(k-1)t} f_k - e^{-nt}\sum_{k=n+2}^{\infty} f_k\right) \\
& +e^{-(n+1)t}(e^t-1)\sum_{k=n+1}^{\infty}(k-n)f_k - e^{-nt}\sum_{k=n+1}^{\infty} f_k \\
& +e^{-t}\sum_{k=n+1}^{\infty} e^{-(k-1)t} f_k \\
= \;& -(1+e^{-t})\left(\sum_{k=n+1}^{\infty} e^{-(k-1)t} f_k - e^{-nt}\sum_{k=n+1}^{\infty} f_k - e^{-nt}f_{n+1} + e^{-nt}f_{n+1}\right) \\
& +e^{-(n+1)t}(e^t-1)\sum_{k=n+1}^{\infty}(k-n)f_k - e^{-nt}\sum_{k=n+1}^{\infty} f_k \\
& +e^{-t}\sum_{k=n+1}^{\infty} e^{-(k-1)t} f_k.
\end{aligned}
$$

After further manipulation we see that (3.70) simplifies to

$$
\begin{aligned}
& 2\left((n+1)e^{-(n+1)t} - ne^{-nt}\right)\sum_{k=n+1}^{\infty} f_k \\
& +2\left(e^{-nt} - e^{-(n+1)t}\right)\sum_{k=n+1}^{\infty} kf_k. \qquad (3.72)
\end{aligned}
$$

Combining (3.68), (3.71) and (3.72) we get

$$
\begin{aligned}
& -(n-1)e^{-(n-1)t}f_n \\
& + \left[(n-1)^2 e^{-(n-1)t} - 2n(n-1)e^{-nt} + (n^2-1)e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} f_k \\
& +2\left((n+1)e^{-(n+1)t} - ne^{-nt}\right)\sum_{k=n+1}^{\infty} f_k \\
& + \left[-(n-1)e^{-(n-1)t} + 2(n-1)e^{-nt} - (n-1)e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} kf_k \\
& +2\left(e^{-nt} - e^{-(n+1)t}\right)\sum_{k=n+1}^{\infty} kf_k \\
= \; & -(n-1)e^{-(n-1)t}f_n \\
& + \left[(n-1)^2 e^{-(n-1)t} - 2n^2 e^{-nt} + (n+1)^2 e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} f_k \\
& + \left[-(n-1)e^{-(n-1)t} + 2ne^{-nt} - (n+1)e^{-(n+1)t}\right]\sum_{k=n+1}^{\infty} kf_k \qquad (3.73)
\end{aligned}
$$

which agrees with (3.67).

We shall use formula (3.65) to define a family of operators $\{S(t)\}_{t\geq 0}$ on X by

$$
\begin{aligned}
[S(t)f]_n \; = \; & e^{-(n-1)t}f_n \\
& + e^{-(n-1)t}\sum_{j=n+1}^{\infty}[1 - e^{-2t} + (1 - 2e^{-t} + e^{-2t})(j-n)]f_j
\end{aligned}
$$

$$
\begin{aligned}
&= e^{-(n-1)t} f_n \\
&\quad + e^{-(n-1)t} \sum_{j=n+1}^{\infty} [2(1-e^{-t}) + (1-e^{-t})^2 (j-n-1)] f_j \\
&\qquad\qquad f \in X, t \geq 0.
\end{aligned} \tag{3.74}
$$

The next step is to show that $S(t) \in B(X)$ for all $t \geq 0$ and that $S(t)f$ is strongly continuous with respect to t on $[0, \infty)$ for each $f \in X$. It is easy to see that when $t = 0$, $[S(0)f]_n = f_n$, $n = 1, 2, \ldots$, and so $S(0) = I$, the identity operator on X. For $t > 0$ and $f \in X$,

$$
\begin{aligned}
\|S(t)f\| &= \sum_{n=1}^{\infty} n \Big| (S(t)f)_n \Big| \\
&\leq \sum_{n=1}^{\infty} n e^{-(n-1)t} |f_n| + 2(1-e^{-t}) \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n e^{-(n-1)t} |f_j| \\
&\qquad + (1-e^{-t})^2 \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n e^{-(n-1)t} (j-n-1) |f_j|.
\end{aligned} \tag{3.75}
$$

We shall now look at each of the three terms on the right-hand side of (3.75). The first term gives us

$$
\sum_{n=1}^{\infty} n e^{-(n-1)t} |f_n| \leq \sum_{n=1}^{\infty} n |f_n| = \|f\|
$$

since $e^{-(n-1)t} \leq 1$ for $n = 1, 2, \ldots$. The second term gives us

$$
\begin{aligned}
2(1-e^{-t}) \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n e^{-(n-1)t} |f_j| &\leq 2(1-e^{-t}) e^{t} \sum_{n=1}^{\infty} n e^{-nt} \sum_{j=1}^{\infty} j |f_j| \\
&= \frac{2}{(1-e^{-t})} \|f\|
\end{aligned}
$$

since

$$
\sum_{n=1}^{\infty} n e^{-nt} = -\frac{d}{dt} \left(\sum_{n=0}^{\infty} e^{-nt} \right) = -\frac{d}{dt} \left(\frac{1}{1-e^{-t}} \right) = \frac{e^{-t}}{(1-e^{-t})^2}. \tag{3.76}
$$

The third term on the right-hand side of (3.75) gives us

$$\begin{aligned}
& (1-e^{-t})^2 \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n e^{-(n-1)t} (j-n-1)|f_j| \\
= \;& e^t (1-e^{-t})^2 \sum_{n=1}^{\infty} n e^{-nt} \sum_{j=n+1}^{\infty} (j-n-1)|f_j| \\
= \;& e^t (1-e^{-t})^2 \sum_{n=1}^{\infty} n e^{-nt} \sum_{j=0}^{\infty} j |f_{j+n+1}| \\
\leq \;& e^t (1-e^{-t})^2 \sum_{n=1}^{\infty} n e^{-nt} \sum_{j=0}^{\infty} (j+n+1) |f_{j+n+1}| \\
\leq \;& \|f\| \quad \text{by (3.76).}
\end{aligned}$$

Hence $S(t) \in B(X)$ $\forall t \geq 0$, with

$$\|S(t)f\| \leq \left(2 + \frac{2}{1-e^{-t}}\right) \|f\|. \tag{3.77}$$

If we take $f \in X^+$ then we can obtain a sharper result than inequality (3.77). In this case we have that

$$\begin{aligned}
\|S(t)f\| = \;& e^t \left(\sum_{n=1}^{\infty} n e^{-nt} f_n + 2(1-e^{-t}) \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} n e^{-nt} f_j \right. \\
& \left. + (1-e^{-t})^2 \sum_{n=1}^{\infty} \sum_{j=n+1}^{\infty} e^{-nt} n (j-n-1) f_j \right) \\
= \;& e^t \left(\sum_{n=1}^{\infty} n e^{-nt} f_n + 2(1-e^{-t}) \sum_{j=2}^{\infty} \left(\sum_{n=1}^{j-1} n e^{-nt} \right) f_j \right. \\
& \left. + (1-e^{-t})^2 \sum_{j=2}^{\infty} \sum_{n=1}^{j-1} (j n e^{-nt} - n^2 e^{-nt} - n e^{-nt}) f_j \right) \\
= \;& e^t \left(\sum_{n=1}^{\infty} n e^{-nt} f_n - \sum_{j=2}^{\infty} 2(1-e^{-t}) \frac{d}{dt} \left(\frac{1-e^{-jt}}{1-e^{-t}} \right) f_j \right. \\
& \left. - (1-e^{-t})^2 \sum_{j=2}^{\infty} j \frac{d}{dt} \left(\frac{1-e^{-jt}}{1-e^{-t}} \right) f_j \right.
\end{aligned}$$

$$-(1-e^{-t})^2 \sum_{j=2}^{\infty} \frac{d^2}{dt^2}\left(\frac{1-e^{-jt}}{1-e^{-t}}\right) f_j$$

$$+(1-e^{-t})^2 \sum_{j=2}^{\infty} \frac{d}{dt}\left(\frac{1-e^{-jt}}{1-e^{-t}}\right) f_j\Bigg) \tag{3.78}$$

since

$$\sum_{n=1}^{j-1} ne^{-nt} = -\frac{d}{dt}\left(\sum_{n=0}^{j-1}(e^{-t})^n\right) = -\frac{d}{dt}\left(\frac{1-e^{-jt}}{1-e^{-t}}\right)$$

and

$$\sum_{n=1}^{j-1} n^2 e^{-nt} = \frac{d^2}{dt^2}\left(\frac{1-e^{-jt}}{1-e^{-t}}\right).$$

Expanding these terms and cancelling we get

$$\begin{aligned}
& e^t\left(\sum_{k=1}^{\infty} ke^{-kt} f_k + e^{-t}\sum_{k=2}^{\infty} kf_k - \sum_{k=2}^{\infty} ke^{-kt} f_k\right) \\
= \;& e^t\left(e^{-t} f_1 + e^{-t}\sum_{k=2}^{\infty} kf_k\right) \\
= \;& \sum_{k=1}^{\infty} kf_k = \|f\|.
\end{aligned} \tag{3.79}$$

Thus we have shown that $\|S(t)f\| = \|f\|$ for all $f \in X^+$. It is also clear that $S(t)f \in X^+ \; \forall f \in X^+$. Using the decomposition (2.15) we can now deduce that

$$\begin{aligned}
\|S(t)f\| &= \sum_{n=1}^{\infty} n\big|[S(t)f]_n\big| = \sum_{n=1}^{\infty} n\big|S(t)(f^+)_n - S(t)(f^-)_n\big| \\
&= \sum_{n=1}^{\infty} nS(t)(f^+)_n + \sum_{n=1}^{\infty} nS(t)(f^-)_n = \|f^+\| + \|f^-\| \\
&= \|f\|
\end{aligned} \tag{3.80}$$

so $\|S(t)f\| = \|f\| \; \forall f \in X$. Finally we need to show that $S(t)$ is strongly continuous on $[0,\infty)$, i.e.

$$\|S(t)f - S(s)f\| \to 0 \text{ as } t \to s. \tag{3.81}$$

Let

$$v_n(t) = e^{-(n-1)t} f_n \tag{3.82}$$

$$w_n(t) = 2g(t)e^{-(n-1)t} \sum_{j=n+1}^{\infty} f_j \tag{3.83}$$

$$z_n(t) = (g(t))^2 e^{-(n-1)t} \sum_{j=n+1}^{\infty} (j-n-1) f_j \tag{3.84}$$

where $g(t) = 1 - e^{-t}$.

We shall first establish continuity on $(0, \infty)$. Let $s > 0$ be fixed and consider (3.82). Then

$$\begin{aligned}
\|v(t) - v(s)\| &\leq \sum_{n=1}^{\infty} n|e^{-(n-1)t} - e^{-(n-1)s}||f_n| \\
&\leq \|f\| \sum_{n=1}^{\infty} |e^{-(n-1)t} - e^{-(n-1)s}| \\
&\leq \|f\| \left(\sum_{n=1}^{\infty} e^{-(n-1)t} + \sum_{n=1}^{\infty} e^{-(n-1)s} \right) \\
&\leq 2\|f\| \sum_{n=1}^{\infty} e^{-(n-1)\epsilon} \quad \text{for } s, t \in [\epsilon, T].
\end{aligned}$$

It can be shown that this summation is convergent by the ratio test and thus we can now apply the dominated convergence theorem (see Theorem 2.35) to justify taking the limit as $t \to s$ inside the summation. It follows that $v(t) \to v(s)$ in X as $t \to s$.

Next we shall look at (3.83) where

$$\begin{aligned}
\|w(t) - w(s)\| &\leq 2 \sum_{n=1}^{\infty} n|g(t)e^{-(n-1)t} - g(s)e^{-(n-1)s}| \sum_{j=n+1}^{\infty} |f_j| \\
&\leq 2\|f\| \sum_{n=1}^{\infty} n|g(t)e^{-(n-1)t} - g(s)e^{-(n-1)s}| \\
&\leq 4\|f\| \sum_{n=1}^{\infty} ne^{-(n-1)\epsilon} \quad \text{for } s, t \in [\epsilon, T],
\end{aligned}$$

since $|g(t)| \leq 1$ and $|g(s)| \leq 1$. The ratio test can again be used to show that the summation is convergent and we can apply the dominated convergence theoreom to deduce that $w(t) \to w(s)$ as $t \to s$.

Similarly for (3.84) we have

$$\begin{aligned}
\|z(t) - z(s)\| &\leq \sum_{n=1}^{\infty} n|(g(t))^2 e^{-(n-1)t} - (g(s))^2 e^{-(n-1)s}| \sum_{j=n+1}^{\infty} (j - n - 1)|f_j| \\
&\leq \sum_{n=1}^{\infty} n|(g(t))^2 e^{-(n-1)t} - (g(s))^2 e^{-(n-1)s}| \sum_{j=1}^{\infty} j|f_j| \\
&\leq 2\|f\| \sum_{n=1}^{\infty} n e^{-(n-1)\epsilon} \quad \text{for } s, t \in [\epsilon, T],
\end{aligned}$$

since $|g(t)| \leq 1$ and $|g(s)| \leq 1$. Thus $z(t) \to z(s)$ as $t \to s$ upon using Theorem 2.35. Putting these results together we have that $S(t)f \to S(s)f$ as $t \to s$ for $s > 0$.

To establish one-sided continuity at 0, we consider

$$\begin{aligned}
\|S(t)f - f\| &\leq \sum_{n=1}^{\infty} n|e^{-(n-1)t} - 1||f_n| + 2\sum_{n=1}^{\infty} n(g(t)e^{-(n-1)t} - g(0)) \sum_{j=n+1}^{\infty} |f_j| \\
&\quad + \sum_{n=1}^{\infty} n((g(t))^2 e^{-(n-1)t} - (g(0))^2) \sum_{j=n+1}^{\infty} (j - n - 1)|f_j| \\
&= \sum_{n=1}^{\infty} n|e^{-(n-1)t} - 1||f_n| + 2\sum_{n=1}^{\infty} n(1 - e^{-t})e^{-(n-1)t} \sum_{j=n+1}^{\infty} |f_j| \\
&\quad + \sum_{n=1}^{\infty} n(1 - e^{-t})^2 e^{-(n-1)t} \sum_{j=n+1}^{\infty} (j - n - 1)|f_j|. \qquad (3.85)
\end{aligned}$$

Changing the order of summation gives us

$$
\begin{aligned}
& \sum_{n=1}^{\infty} n|e^{-(n-1)t}-1||f_n| + 2\sum_{j=2}^{\infty}\sum_{n=1}^{j-1} n(1-e^{-t})e^{-(n-1)t}|f_j| \\
& \quad + \sum_{j=2}^{\infty}\sum_{n=1}^{j-1} n(1-e^{-t})^2 e^{-(n-1)t}(j-n-1)|f_j| \\
\leq\; & \sum_{n=1}^{\infty} n(1-e^{-(n-1)t})|f_n| + 2(1-e^{-t})\sum_{j=2}^{\infty} j|f_j| \\
& \quad +(1-e^{-t})^2\sum_{j=2}^{\infty} j^2|f_j|
\end{aligned}
$$

since $e^{-(n-1)t} \leq 1$ for all $t \geq 0$ and $n < j$. Hence we have that $\|S(t)f - f\| \to 0$ as $t \to 0^+$ for each $f \in D(A)$. Since $D(A)$ is dense in X, it follows that $S(t)f \to f$ in X as $t \to 0^+$ for each $f \in X$.

We have now shown that $u(t) = S(t)f$ satisfies the system of ODEs (3.54) pointwise for each $f \in X$. Moreover $S(t)f \in X^+$ whenever $f \in X^+$, $\|S(t)f\| = \|f\|$ for all $t \geq 0$, showing that mass is conserved, and the family of operators $\{S(t)\}_{t\geq 0}$ is strongly continuous in t on $[0, \infty)$. The last step is to show that $S(t) = S_G(t)$ on X. We shall make use of the family of operators $\{P_N\}_{N=1}^{\infty}$ where for $f \in X$, $P_N f = \{f_1, f_2, \ldots, f_N, 0, \ldots\}$. Clearly $P_N \in B(X)$ for each N, with $\|P_N\| \leq 1$ and $P_N f \to f$ in X as $N \to \infty$.
Moreover, a strong solution of the ACP

$$\frac{du^N(t)}{dt} = Gu^N(t),\ t > 0,\ \lim_{t\to 0^+} u^N(t) = P_N f$$

is given by

$$u^N(t) = S(t)P_N f.$$

By uniqueness of solutions,

$$S_G(t)P_N f = S(t)P_N f \tag{3.86}$$

and, letting $N \to \infty$, we obtain the required result

$$S_G(t)f = S(t)f, \quad \forall f \in X. \tag{3.87}$$

Thus we conclude that an explicit formula for the stochastic semigroup associated with the system of ODEs (3.54) is given by the solution obtained by Ziff and McGrady in [58].

Chapter 4

The Discrete Fragmentation Equation with Mass Loss

We shall now consider the case where mass can be lost during a fragmentation event. In our model we have chains of monomers, e.g. n-mers. During a fragmentation event a monomer can be annihilated either at the surface or in the interior of these n-mers, which results in the loss of mass.

4.1 The General Mass Loss Case with Bond Annihilation

The fragmentation equation (3.6) still models the mass loss case but now a_1 need not be 0 so we discard (A1). We will still need assumption (A2) but this time we have mass loss so we require the condition

(A3)* $\sum_{n=1}^{j-1} nb_{n,j} = (1-\lambda_j)j$ for $j \geq 2$

where $0 \leq \lambda_j \leq 1$ is the discrete mass loss fraction. With no coagulation occurring, the calculation which produced (3.4) now leads to

$$\dot{M}(t) = -a_1 u_1(t) - \sum_{j=2}^{\infty} j\lambda_j a_j u_j(t). \tag{4.1}$$

This collapses to $\dot{M} = 0$ when $a_1 = 0$ and $\lambda_j = 0$ for $j \geq 2$.

Again, the natural Banach space X in which to study the problem is the weighted l^1 space defined in Definition 3.1. In analogy with Problem 3.5 we wish to solve

Problem 4.1. *Find a function $u : [0,\infty) \to X$ such that*

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + Bu(t) \qquad\qquad t > 0 \\ \lim_{t\to 0+} u(t) &= f \in D(A), \end{aligned} \tag{4.2}$$

where A and B are defined as in Definition 3.2.

We have already shown that $D(A) \subseteq D(B)$ in Lemma 3.3 for the case when $\lambda_j = 0$. Similarly, for $f \in D(A)$ and $\lambda_j \neq 0$ we have

$$\begin{aligned} \|Bf\| &\le \sum_{n=1}^{\infty} n \sum_{j=n+1}^{\infty} a_j\, b_{n,j}\, |f_j| = \sum_{j=2}^{\infty} a_j\, |f_j| \left(\sum_{n=1}^{j-1} n\, b_{n,j} \right) \\ &= \sum_{j=1}^{\infty} (1-\lambda_j) j\, a_j\, |f_j| \le \sum_{j=1}^{\infty} j a_j |f_j| = \|Af\| < \infty, \end{aligned} \tag{4.3}$$

i.e. $D(A) \subseteq D(B)$. Again, we require our solution to be non-negative so we need $f \in D(A)^+ \implies u(t) \in D(A)^+$.

First we shall look at the case of a general sequence $\{a_n\}$. We can prove the following for our operators A and B

Theorem 4.2. *Let X, A and B be as in (3.1), (3.9) and (3.10). Then there exists a smallest extension G of $A+B$ which generates a substochastic semigroup $\{T_G(t)\}_{t\ge 0}$ on X.*

Proof: This is more or less the same as the proof for Theorem 3.6 apart from part (iii) where now we have

$$\int_{\Omega} (Af + Bf)d\mu = -a_1 f_1 - \sum_{j=2}^{\infty} j\lambda_j a_j f_j =: -c(f) \le 0 \tag{4.4}$$

for $f \in D(A)^+$. □

Theorem 4.3. *In the context of Theorem 4.2, $G = \overline{A+B}$, the closure of the operator $(A+B, D(A))$.*

Proof: Along similar lines to the proof of Theorem 3.7, we wish to apply Theorem 2.26 but this time with

$$c(f) := a_1 f_1 + \sum_{j=2}^{\infty} j\lambda_j a_j f_j.$$

Let all spaces and operators be defined as in the proof of Theorem 3.7. We wish to show that for any $g \in F^+$ such that $-g + \mathbb{B}\mathbb{L}g \in X$ we have

$$\sum_{n=1}^{\infty} n(\mathbb{L}g)_n + \sum_{n=1}^{\infty} n\left(-g_n + (\mathbb{B}\mathbb{L}g)_n\right) \geq -a_1 f_1 - \sum_{j=2}^{\infty} j\lambda_j a_j f_j. \tag{4.5}$$

We follow the same procedure as in the proof of Theorem 3.7 to reach the point

$$\begin{aligned} &\sum_{n=1}^{\infty} n\left(-a_n f_n + (\mathbb{B}f)_n\right) \\ &= \lim_{N\to\infty} \left(-\sum_{n=1}^{N} n a_n f_n + \sum_{n=1}^{N} \sum_{j=n+1}^{\infty} n a_j b_{n,j} f_j \right), \end{aligned} \tag{4.6}$$

i.e. (3.21) and (3.22). We can now write

$$\begin{aligned} &\sum_{n=1}^{N} \sum_{j=n+1}^{\infty} n a_j b_{n,j} f_j \\ &= \sum_{j=2}^{N} a_j f_j \sum_{n=1}^{j-1} n b_{n,j} + \sum_{j=N+1}^{\infty} \sum_{n=1}^{N} n a_j b_{n,j} f_j \\ &= \sum_{j=2}^{N} j(1-\lambda_j) a_j f_j + S_N \\ &= \sum_{j=1}^{N} j a_j f_j - a_1 f_1 - \sum_{j=2}^{N} j\lambda_j a_j f_j + S_N \end{aligned}$$

where S_N is the same as in the proof of Theorem 3.7. Substituting into (4.6) we end up with

$$\lim_{N\to\infty} \left(-a_1 f_1 - \sum_{j=2}^{N} j\lambda_j a_j f_j + S_N \right). \tag{4.7}$$

As shown in the proof of Theorem 3.7, $\lim_{N\to\infty} S_N$ is non-negative. Putting everything together we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} n(\mathbb{L}g)_n + \sum_{n=1}^{\infty} n\left(-g_n + (\mathbb{B}\mathbb{L}g)_n\right) \\
= \quad & -a_1 f_1 - \sum_{j=2}^{\infty} j\lambda_j a_j f_j + \lim_{N\to\infty} S_N \\
\geq \quad & -a_1 f_1 - \sum_{j=2}^{\infty} j\lambda_j a_j f_j
\end{aligned}$$

and thus we have shown that $G = \overline{A+B}$. □

Along similar lines to the comments after Theorem 3.7, we can deduce that the ACP

$$\frac{d}{dt}u(t) = Gu(t) \quad (t > 0) \tag{4.8}$$

$$\lim_{t\to 0+} u(t) = f\,, \tag{4.9}$$

has a unique strict, non-negative solution $u : [0,\infty) \to D(G)^+$ for each $f \in D(G)^+$ and hence for each $f \in D(A)^+$. This solution is given by $u(t) = T_G(t)f$, with $G = \overline{A+B}$.

From the discussion on pages 39-40 we can show that the C_0-semigroup $\{T_G(t)\}_{t\geq 0}$ is substochastic and the solution is not mass-conserving (unless $\lambda_j = 0$ $\forall j$ and $a_1 = 0$) since

$$\begin{aligned}
\frac{d}{dt}\|u(t)\| &= \sum_{k=1}^{\infty} k\dot{u}_k(t) \ \ (\text{since } u(t) \in X^+) \\
&= -\sum_{k=1}^{\infty} ka_k u_k(t) + \sum_{k=1}^{\infty} k \sum_{j=k+1}^{\infty} a_j b_{k,j} u_j(t) \\
&= -\sum_{k=1}^{\infty} ka_k u_k(t) + \sum_{j=2}^{\infty} a_j u_j(t) \underbrace{\sum_{k=1}^{j-1} kb_{k,j}}_{=(1-\lambda_j)j}
\end{aligned}$$

$$
\begin{aligned}
&= -\sum_{k=1}^{\infty} ka_k u_k(t) + \sum_{j=2}^{\infty}(1-\lambda_j) ja_j u_j(t) \\
&= -a_1 u_1(t) - \sum_{j=2}^{\infty} \lambda_j ja_j u_j(t) \\
&\leq 0. \qquad (4.10)
\end{aligned}
$$

Note that the calculations leading to (4.10) show rigorously that the mass loss is exactly that predicted by the formal calculation (4.1).

4.2 Particular Cases

Now we shall look at how the additional mass loss condition affects the results for the particular cases discussed in Section 3.2.

Case 1: $\{a_n\}$ bounded.

First, consider the case where the sequence $\{a_n\}_{n=1}^{\infty}$ is bounded i.e. $a_n \leq M$ for $n = 1, 2, 3, \ldots$. We have already shown in (3.25) that $\forall f \in D(A) = X$,

$$\|Af\| \leq M\|f\| < \infty.$$

Also for $f \in X$ we have

$$
\begin{aligned}
\|Bf\| &\leq \sum_{j=1}^{\infty} ja_j|f_j| \\
&= \|Af\| \leq M\|f\| < \infty,
\end{aligned}
$$

by similar calculations to those in (3.26). Hence we have $D(B) = D(A) = X$. By arguments similar to those preceding Theorem 3.8 we can prove the following.

Theorem 4.4. *In the case when $\{a_n\}$ is bounded, Problem 4.1 has a unique strict, non-negative solution for each $f \in X$, given by*

$$u(t) = e^{t(A+B)}f \qquad (t \geq 0).$$

Case 2: $\{a_n\}$ monotonic increasing.

Now we shall look again at the case when $\{a_n\}$ is a monotonic increasing sequence, but not necessarily bounded.

Theorem 4.5. *Let X, A, B and G be as in Theorem 4.2 and, in addition to assumptions (A2) and (A4), let $\{a_n\}$ be a monotonic increasing sequence. Then $D(A)$ is invariant under the semigroup $\{T_G(t)\}_{t\geq 0}$.*

Proof: The details are very similar to those in the proof of Theorem 3.9 and thus we shall only highlight the minor differences.

Step 1 This is exactly as in Theorem 3.9.

Step 2 For $f \in D(A_D)$

$$\begin{aligned} \|B_D f\|_A &\leq \sum_{j=2}^{\infty}(1-\lambda_j)ja_j|f_j| + \sum_{j=2}^{\infty}(1-\lambda_j)ja_j^2|f_j| \\ &\leq \sum_{j=2}^{\infty} ja_j|f_j| + \sum_{j=2}^{\infty} ja_j^2|f_j| \\ &\leq \|A_D\| + \|A(A_D f)\| \\ &= \|A_D f\|_A < \infty. \end{aligned}$$

Step 3 For $f \in D(A_D)^+$

$$\begin{aligned} &\sum_{n=1}^{\infty}(n+na_n)[-a_n f_n + \sum_{j=n+1}^{\infty} a_j b_{n,j} f_j] \\ \leq\ & -\sum_{n=1}^{\infty}(n+na_n)a_n f_n + \sum_{j=2}^{\infty}(1-\lambda_j)j(1+a_j)a_j f_j \\ =\ & -(1+a_1)a_1 f_1 - \sum_{j=2}^{\infty}\lambda_j j(1+a_j)a_j f_j \\ \leq\ & 0. \end{aligned}$$

Step 4 This is the same as in Theorem 3.9. Note that the discussion following Step 4 in Theorem 3.9 also applies here. □

Note 4.6. *We can now add in coagulation terms as we did in Section 3.3. The* *analysis of these terms will be identical to that shown previously so we have omitted the details.*

4.3 The Specific Case of Cai, Edwards and Han

In [28] Cai, Edwards and Han look at the specific case when $a_n = n$ and $b_{n,j} = \frac{2}{j}$ so that we have

$$\frac{du_n(t)}{dt} = -nu_n(t) + 2\sum_{j=n+1}^{\infty} u_j(t). \tag{4.11}$$

Note that in the specific case studied in Chapter 3 for the mass conservation model we looked at $a_n = n-1$ and $b_{n,j} = \frac{2}{j-1}$. If instead we substitute the value $b_{n,j} = \frac{2}{j}$ into (A3)* we see that

$$\begin{aligned}\sum_{n=1}^{j-1} nb_{n,j} &= \sum_{n=1}^{j-1} \frac{2n}{j} = \frac{2}{j}\sum_{n=0}^{j-1} n \\ &= \frac{2}{j}\frac{(j-1)j}{2} = j-1. \end{aligned} \tag{4.12}$$

We know that $(1-\lambda_j)j = (j-1)$ and thus $\lambda_j = \frac{1}{j}$. Substituting $a_n = n$ and $b_{n,j} = \frac{2}{j}$ into (4.1) we see that the mass loss in this case is

$$\begin{aligned}\dot{M}(t) &= -a_1u_1(t) - \sum_{j=2}^{\infty} \lambda_j j a_j u_j(t) = -u_1(t) - \sum_{j=2}^{\infty} ju_j(t) \\ &= \sum_{j=1}^{\infty} ju_j(t) = -M(t). \end{aligned} \tag{4.13}$$

If we now solve the ODE (4.13) we get

$$M(t) = M(0)e^{-t}, \quad t \geq 0. \tag{4.14}$$

Notice that we can write

$$\begin{aligned}\frac{du_n(t)}{dt} &= -nu_n(t) + 2\sum_{j=n+1}^{\infty} u_j(t) \\ &= -(n-1)u_n(t) - u_n(t) + 2\sum_{j=n+1}^{\infty} u_j(t) \end{aligned} \tag{4.15}$$

so that

$$\frac{du_n(t)}{dt} + u_n(t) = -(n-1)u_n(t) + 2\sum_{j=n+1}^{\infty} u_j(t)$$

i.e.

$$\frac{d}{dt}(e^t u_n(t)) = -(n-1)e^t u_n(t) + 2\sum_{j=n+1}^{\infty} e^t u_j(t).$$

If we substitute $v_n(t) = e^t u_n(t)$ we get

$$\frac{dv_n(t)}{dt} = -(n-1)v_n(t) + 2\sum_{j=n+1}^{\infty} v_j(t) \tag{4.16}$$

which is in fact the same form as equation (3.54) which was discussed in Section 3.5. Since the solution v_n to (4.16) is given by (3.65), it follows that u_n is given by

$$u_n(t) = u_n(0)e^{-nt} + \sum_{j=n+1}^{\infty} u_j(0)e^{-nt}\{1 - e^{-2t} + (j-n)(1-e^{-t})^2\}. \tag{4.17}$$

The ACP associated with (4.15) is now

$$\begin{aligned}\frac{du(t)}{dt} &= (A+B)u(t) - u(t) = (A+B-I)u(t) \\ \lim_{t\to 0^+} u(t) &= f \end{aligned} \tag{4.18}$$

where I is the identity operator on X and

$$[Af]_n := -(n-1)f_n \tag{4.19}$$

and

$$[Bf]_n := 2 \sum_{j=n+1}^{\infty} f_j \tag{4.20}$$

are defined on appropriate domains. From the work carried out in Section 3.5 we know that $G = \overline{(A+B, D(A))}$ is the generator of a stochastic semigroup $\{T_G(t)\}_{t\geq 0}$ on X. Also, from the general theory in Section 4.1 we know that $G_1 = \overline{(A+B-I, D(A))}$ is also the generator of a substochastic semigroup $\{T_{G_1}(t)\}_{t\geq 0}$ on X. We wish to show that

$$T_{G_1}(t) = e^{-t}T_G(t) \quad \forall t \geq 0. \tag{4.21}$$

Notice that

$$\|T_{G_1}(t)f\| = \|e^{-t}T_G(t)f\| = e^{-t}\|T_G(t)f\| = e^{-t}\|f\| \text{ for } f \in D(G) \tag{4.22}$$

which agrees with (4.14). To show that (4.21) holds we need to show that $\{T_{G_1}(t)\}$ and $\{e^{-t}T_G(t)\}$ have the same infinitesimal generator. In order to do this we need to prove the following

Lemma 4.7. *Let A and B be as in (4.19) and (4.20). Then*

$$\overline{A+B-I} = \overline{A+B} - I.$$

Proof: Let $f \in D(\overline{A+B}) = D(G)$. Then, there exists a sequence $\{f^n\} \subset D(A)$ such that

$$f^n \to f \ \text{ in } X$$

and

$$(A+B)f^n \to g = \overline{(A+B)}f \ \text{ in } X.$$

It follows that

$$(A+B-I)f^n \to \overline{(A+B)}f - f = g - f \ \text{ in } X$$

and hence $f \in D(\overline{A+B-I})$, with

$$\overline{(A+B-I)}f = \overline{(A+B)}f - f = (\overline{A+B} - I)f.$$

Therefore

$$\overline{(A+B-I)}f = (\overline{A+B}-I)f \quad \forall f \in D(\overline{A+B}).$$

Now let $f \in D(\overline{A+B-I})$. Then, there exists a sequence $\{f^n\} \subset D(A)$ such that

$$f^n \to f \quad \text{in } X$$

and

$$(A+B-I)f^n \to g = \overline{(A+B-I)}f \quad \text{in } X.$$

It follows that

$$(A+B)f^n = (A+B-I)f^n + f^n \to \overline{(A+B-I)}f + f = g + f \quad \text{in } X$$

and hence $f \in D(\overline{A+B})$ with

$$\overline{(A+B)}f = \overline{(A+B-I)}f + f = g + f.$$

Therefore

$$(\overline{A+B}-I)f = \overline{(A+B-I)}f \quad \forall f \in D(\overline{A+B-I}).$$

We have thus shown that $\overline{A+B}-I = \overline{A+B-I}$. □

Note 4.8. *For the remainder of this section we shall define the operators A, B and G to be those associated with the ACP resulting from the ODE (4.11). Therefore*

$$[Af]_n = -nf_n, \quad D(A) = \{f \in X : \sum_{n=1}^{\infty} n^2|f_n| < \infty\} \tag{4.23}$$

and

$$[Bf]_n = 2\sum_{j=n+1}^{\infty} f_j, \quad D(B) = \{f \in X : \sum_{n=1}^{\infty} n\Big|\sum_{j=n+1}^{\infty} f_j\Big| < \infty\} \tag{4.24}$$

and $G = \overline{A+B}$ with A and B defined as in (4.23) and (4.24).

Our rigorous analysis shows that the only strong solution of the ACP

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + Bu(t), \quad t > 0 \\ \lim_{t\to 0^+} u(t) &= 0 \end{aligned} \tag{4.25}$$

is $u(t) \equiv 0$. However, in [28] Cai et. al introduce the alternative solution

$$v_k(t) = (1 - e^{-t})^2 e^{-kt}, \quad k = 1, 2, \ldots. \tag{4.26}$$

Clearly $v(0) = 0$. Also

$$\begin{aligned} \sum_{k=1}^{\infty} k^2 |v_k(t)| &= \sum_{k=1}^{\infty} k^2 |(1 - e^{-t})^2 e^{-kt}| \\ &= \sum_{k=1}^{\infty} k^2 (1 - e^{-t})^2 e^{-kt} \\ &= (1 - e^{-t})^2 \sum_{k=1}^{\infty} k^2 e^{-kt} \\ &= (1 - e^{-t})^2 \frac{d^2}{dt^2} \left(\sum_{k=0}^{\infty} e^{-kt} \right) \\ &= (1 - e^{-t})^2 \frac{d^2}{dt^2} \left(\frac{1}{1 - e^{-t}} \right) \\ &= e^{-t} + \frac{2e^{-2t}}{1 - e^{-t}} < \infty \qquad \forall t > 0, \end{aligned}$$

showing that $v(t) \in D(A)$ for all $t > 0$. Note that, for each $k \in \mathbb{N}$ and $t > 0$,

$$\begin{aligned} [Av(t) + Bv(t)]_k &= -kv_k(t) + 2 \sum_{j=k+1}^{\infty} v_j(t) \\ &= -ke^{-kt}(1 - e^{-t})^2 + 2 \sum_{j=k+1}^{\infty} e^{-jt}(1 - e^{-t})^2 \\ &= -ke^{-kt}(1 - e^{-t})^2 + 2(1 - e^{-t})^2 \frac{e^{-(k+1)t}}{1 - e^{-t}} \\ &= -ke^{-kt}(1 - e^{-t})^2 + 2e^{-(k+1)t}(1 - e^{-t}) \\ &= \dot{v}_k(t) \end{aligned} \tag{4.27}$$

so that (4.11) is satisfied pointwise. Furthermore, for $t > 0$,

$$\begin{aligned}
& \lim_{h\to 0}\sum_{k=1}^{\infty}\Big|\frac{v_k(t+h)-v_k(t)}{h} - [Av(t)]_k - [Bv(t)]_k\Big| \\
= & \lim_{h\to 0}\sum_{k=1}^{\infty}\Big|\dot{v}_k(\theta_k(h)) - \dot{v}_k(t)\Big|, \; t \le \theta_k(h) \le t+h \\
= & \; 0
\end{aligned} \tag{4.28}$$

using arguments similar to those in Chapter 3. However, we can show that

$$\begin{aligned}
\|v_k(t)\| &= \sum_{k=1}^{\infty} k(1-e^{-t})^2 e^{-kt} \\
&= (1-e^{-t})^2 \sum_{k=1}^{\infty} ke^{-kt} \\
&= (-\frac{d}{dt}\sum_{k=0}^{\infty} e^{-kt})(1-e^{-t})^2 \\
&= -\frac{d}{dt}(\frac{1}{1-e^{-t}})(1-e^{-t})^2 \\
&= e^{-t} \\
&\not\to 0 \;\; \text{as} \;\; t \to 0^+,
\end{aligned}$$

i.e. (4.26) is not strongly continuous at $t = 0$ in X and hence it is not a strong solution of (4.25).

4.4 The General Mass Loss Case with Surface Recession

We shall now go on to look at the general discrete fragmentation equation with mass loss as above but with added surface recession terms. We shall now consider the case where not only is mass lost during fragmentation but clusters also experience recession of particles at their surface. This type of equation could be used to model processes such as evaporation. We shall investigate the case where monomers are lost at the surface of clusters at a specific rate. The equation we wish to analyse is

$$\frac{du_n(t)}{dt} = -a_n u_n(t) + \sum_{j=n+2}^{\infty} a_j b_{n,j} u_j(t) + c_{n+1} u_{n+1}(t) - c_n u_n(t) \tag{4.29}$$

where $u_n(t)$, a_n and $b_{n,j}$ have their usual meanings and c_n is non-negative and represents the surface recession rate of an n-mer. Note that the summation now starts at $j = n+2$ instead of $j = n+1$ as in the previous case. This is due to the fact that if a cluster of mass $n+1$ was to break up to produce a cluster of mass n and a monomer then this would be equivalent to a surface recession event.

We can rewrite (4.29) as

$$\begin{aligned} \frac{du_n(t)}{dt} &= -(a_n + c_n)u_n(t) + \sum_{j=n+1}^{\infty} \beta_{n,j} u_j(t) \\ &= -\alpha_n u_n(t) + \sum_{j=n+1}^{\infty} \beta_{n,j} u_j(t). \end{aligned} \tag{4.30}$$

where

$$\alpha_n = a_n + c_n$$

and

$$\beta_{n,j} = \begin{cases} c_{n+1}, & j = n+1, \\ a_j b_{n,j}, & j = n+2, n+3, \ldots. \end{cases}$$

We shall apply the mass loss condition (A3)* here but, as a consequence of the above discussion, the summation will run from $n = 1$ to $j-2$. Note that equation (4.30) is in the form of our usual discrete fragmentation equation but with a_n replaced by α_n and $a_j b_{n,j}$ replaced by $\beta_{n,j}$. In analogy to Problems 3.5 and 4.1 we wish to solve

Problem 4.9. *Find a function $u : [0, \infty) \to X$, where X is as in Definition 3.1, such that*

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + Bu(t) \qquad t > 0 \\ \lim_{t \to 0+} u(t) &= f \in D(A), \end{aligned} \tag{4.31}$$

where

$$[Af]_k = -\alpha_k f_k; \quad D(A) = \left\{ f \in X : \sum_{k=1}^{\infty} k\alpha_k |f_k| < \infty \right\} \tag{4.32}$$

and

$$[Bf]_k = \sum_{j=k+1}^{\infty} \beta_{n,j} f_j; \quad D(B) = \left\{ f \in X : \sum_{k=1}^{\infty} k \left| \sum_{j=k+1}^{\infty} \beta_{n,j} f_j \right| < \infty \right\}. \tag{4.33}$$

Similarly to Lemma 3.3 we can prove the following

Lemma 4.10. *$D(B) \supseteq D(A)$.*

Proof: Let $f \in D(A)$. Then

$$\begin{aligned}
\|Bf\| &\leq \sum_{n=1}^{\infty} n \sum_{j=n+1}^{\infty} \beta_j |f_j| \\
&= \sum_{n=1}^{\infty} n \sum_{j=n+2}^{\infty} a_j b_{n,j} |f_j| + \sum_{n=1}^{\infty} n c_{n+1} |f_{n+1}| \\
&\leq \sum_{j=3}^{\infty} \sum_{n=1}^{j-2} n a_j b_{n,j} |f_j| + \sum_{n=1}^{\infty} n c_n |f_n| \\
&= \sum_{j=3}^{\infty} a_j |f_j| (1 - \lambda_j) j + \sum_{n=1}^{\infty} n c_n |f_n| \\
&\leq \sum_{j=1}^{\infty} j a_j |f_j| + \sum_{j=1}^{\infty} j c_j |f_j| \\
&= \sum_{j=1}^{\infty} j (a_j + c_j) |f_j| \\
&= \|Af\|.
\end{aligned}$$

Hence we have $D(B) \supseteq D(A)$.

□

We are now in a position to prove

Theorem 4.11. *Let X, A and B be as in (3.1), (4.32) and (4.33). Then there exists a smallest extension G of $A+B$ which generates a substochastic semigroup*

$\{T_G(t)\}_{t\geq 0}$ *on* X.

Proof: Again we wish to apply Theorem 2.19 and the details are very similar to those in the proof of Theorem 3.6 so we will only highlight the differences here.

(i) The semigroup $\{T_A(t)\}_{t\geq 0}$ generated by A is now given by

$$[T_A(t)f]_n = e^{-\alpha_n t} f_n \qquad (n = 1, 2, ...). \tag{4.34}$$

(ii) We proved that $D(B) \supseteq D(A)$ in Lemma 4.10. It is also clear that $Bf \geq 0$ for all $f \in [D(B)]^+$.

(iii) For $f \in D(A)^+$ we have

$$\begin{aligned}
&\int_\Omega (Af + Bf)d\mu \\
= \;& \sum_{n=1}^{\infty} n \left(-\alpha_n u_n(t) + \sum_{j=n+1}^{\infty} \beta_{n,j} u_j(t) \right) \\
= \;& -\sum_{n=1}^{\infty} n\alpha_n f_n + \underbrace{\sum_{n=1}^{\infty} n \sum_{j=n+2}^{\infty} a_j b_{n,j} f_j}_{=\sum_{j=3}^{\infty}\sum_{n=1}^{j-2} n a_j b_{n,j} f_j} + \sum_{n=1}^{\infty} n c_{n+1} f_{n+1} \\
= \;& -\sum_{n=1}^{\infty} n a_n f_n - \sum_{n=1}^{\infty} n c_n f_n + \sum_{j=3}^{\infty} (1-\lambda_j) j a_j f_j + \sum_{n=1}^{\infty} n c_{n+1} f_{n+1} \\
= \;& -a_1 f_1 - 2a_2 f_2 - \sum_{n=3}^{\infty} n\lambda_n a_n f_n + \sum_{n=1}^{\infty} n\left(c_{n+1} f_{n+1} - c_n f_n\right) =: -c(f) \\
\leq \;& 0
\end{aligned}$$

since

$$\begin{aligned}
&\sum_{n=1}^{\infty} n\left(c_{n+1} f_{n+1} - c_n f_n\right) \\
\leq \;& \sum_{n=1}^{\infty} \left((n+1)c_{n+1} f_{n+1} - n c_n f_n\right) \\
= \;& -c_1 f_1 \\
\leq \;& 0.
\end{aligned}$$

$\square$

Theorem 4.12. *In the context of Theorem 4.11,* $G = \overline{A+B}$*, the closure of the operator* $(A+B, D)$*.*

Proof: This is very similar to the proofs of Theorems 3.7 and 4.3. We wish to apply Theorem 2.26 but this time with

$$c(f) := a_1 f_1 + 2a_2 f_2 + \sum_{n=3}^{\infty} n\lambda_n a_n f_n - \sum_{n=1}^{\infty} n\left(c_{n+1} f_{n+1} - c_n f_n\right).$$

Let all spaces and operators be defined as in the proof of Theorem 3.7 but with a_n replaced by α_n and $a_j b_{n,j}$ replaced by $\beta_{n,j}$. We wish to show that for any $g \in F^+$ such that $-g + \mathbb{B}\mathbb{L}g \in X$ we have

$$\begin{aligned} & \sum_{n=1}^{\infty} n(\mathbb{L}g)_n + \sum_{n=1}^{\infty} n\left(-g_n + (\mathbb{B}\mathbb{L}g)_n\right) \\ \geq \; & -a_1 f_1 - 2a_2 f_2 - \sum_{n=3}^{\infty} n\lambda_n a_n f_n + \sum_{n=1}^{\infty} n\left(c_{n+1} f_{n+1} - c_n f_n\right). \end{aligned} \tag{4.35}$$

We follow the same procedure found in the proof of Theorems 3.7 and 4.3 to reach the point

$$\begin{aligned} & \sum_{n=1}^{\infty} n\left(-\alpha_n f_n + (\mathbb{B}f)_n\right) \\ = \; & \lim_{N\to\infty} \left(-\sum_{n=1}^{N} n\alpha_n f_n + \sum_{n=1}^{N} \sum_{j=n+1}^{\infty} n\beta_{n,j} f_j \right). \end{aligned} \tag{4.36}$$

We can now write

$$\begin{aligned} & \sum_{n=1}^{N} \sum_{j=n+1}^{\infty} n\beta_{n,j} f_j \\ = \; & \sum_{n=1}^{N} \sum_{j=n+1}^{N} n\beta_{n,j} f_j + \sum_{n=1}^{N} \sum_{j=N+1}^{\infty} n\beta_{n,j} f_j \\ = \; & \sum_{n=1}^{N} n c_{n+1} f_{n+1} + \sum_{n=1}^{N} \sum_{j=n+2}^{N} n a_j b_{n,j} f_j + S_N \end{aligned}$$

$$
\begin{aligned}
&= \sum_{n=1}^{N} nc_{n+1}f_{n+1} + \sum_{j=3}^{N} a_j f_j \sum_{n=1}^{j-2} nb_{n,j} + S_N \\
&= \sum_{n=1}^{N} nc_{n+1}f_{n+1} + \sum_{n=3}^{N} (1-\lambda_n) na_n f_n + S_N
\end{aligned}
$$

where $S_N = \sum_{n=1}^{N} \sum_{j=N+1}^{\infty} n\beta_{n,j} f_j \geq 0 \ \forall N$. Substituting into (4.36) we have

$$
\begin{aligned}
& \lim_{N\to\infty} \left(-\sum_{n=1}^{N} n\alpha_n f_n + \sum_{n=1}^{N} nc_{n+1}f_{n+1} + \sum_{n=3}^{N} (1-\lambda_n) na_n f_n + S_N \right) \\
&= \lim_{N\to\infty} \left(-a_1 f_1 - 2a_2 f_2 - \sum_{n=3}^{N} n\lambda_n a_n f_n \right. \\
&\qquad \left. + \sum_{n=1}^{N} n \left(c_{n+1}f_{n+1} - c_n f_n \right) + S_N \right) \qquad (4.37)
\end{aligned}
$$

by the calculations in part (*iii*) of the proof of Theorem 4.11. Note that since S_N is again non-negative, $\lim_{N\to\infty} S_N$ is also non-negative. Putting everything together we have that

$$
\begin{aligned}
& \sum_{n=1}^{\infty} n(\mathbb{L}g)_n + \sum_{n=1}^{\infty} n \left(-g_n + (\mathbb{BL}g)_n \right) \\
&= -a_1 f_1 - 2a_2 f_2 - \sum_{n=3}^{\infty} n\lambda_n a_n f_n + \sum_{n=1}^{\infty} n \left(c_{n+1}f_{n+1} - c_n f_n \right) + \lim_{N\to\infty} S_N \\
&\geq -a_1 f_1 - 2a_2 f_2 - \sum_{n=3}^{\infty} n\lambda_n a_n f_n + \sum_{n=1}^{\infty} n \left(c_{n+1}f_{n+1} - c_n f_n \right)
\end{aligned}
$$

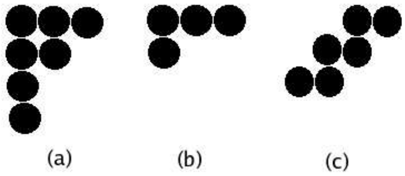
as required. □

We have shown that there exists a unique strong solution to the pure fragmentation equation with mass-loss through bond annihilation and surface recession for a general fragmentation rate. We can also add in our usual coagulation terms and, under assumption (A4), can show that there exists a unique strong, non-negative solution to the full C-F equation.

Chapter 5

A Multi-Component Coagulation-Fragmentation Equation with Reformation Terms

In the previous chapters we have considered a system of particles with evolving size and we have obtained existence and uniqueness results for strong solutions. Now we shall investigate a system in which not only the cluster size can change but also the shape profile, namely the diameter of a cluster of particles, can evolve. We shall analyse the model described by Wattis in [56] to obtain existence and uniqueness results. This model builds in coagulation and reformation events which we shall later modify to include also fragmentation and a greater range of reformation events. In the model discussed in [56], Wattis considers a system where the particles have been scaled so that a monomer of mass one has diameter one. The clusters are grouped by their mass and diameter and so we represent the number concentration of clusters of mass n and diameter j by $u_{n,j}$. In one version of the model in [56] a cluster with mass n is considered to have diameter j where $1 \leq j \leq n$. In our investigations we shall consider particles to join together in a lattice formation. We shall take the diameter of a particular formation to be the maximal extension in a direction of an axis. For example,



in diagram (a) the diameter would be 4, in diagram (b) the diameter would be 3 and in diagram (c) the diameter would be 4.

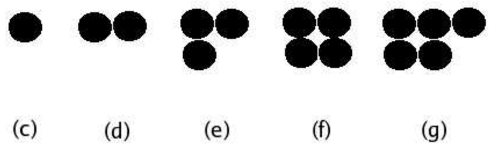
In this situation the minimum physically possible diameter is always going to be

$$d_{min2D}(n) = \lceil \sqrt{n} \rceil$$

where $\lceil x \rceil$ represents the smallest integer $\geq x$. This is the case if we are in a 2D system. If we consider 3D, the minimum diameter will be given by

$$d_{min3D}(n) = \lceil \sqrt[3]{n} \rceil.$$

The following diagrams illustrate why this is true for the 2D case:



(c) $n = 1$ so $d_{min2D}(1) = \lceil \sqrt{1} \rceil = 1$. (d) $n = 2$ so $d_{min2D}(2) = \lceil \sqrt{2} \rceil = 2$.
(e) $n = 3$ so $d_{min2D}(3) = \lceil \sqrt{3} \rceil = 2$. (f) $n = 4$ so $d_{min2D}(4) = \lceil \sqrt{4} \rceil = 2$.
(g) $n = 5$ so $d_{min2D}(5) = \lceil \sqrt{5} \rceil = 3$.

It is a case of building squares as the mass increases since a square is the most compact configuration that particles in a lattice can take. Similar considerations apply in 3D but we are working with cubes instead of squares.

In [56] mass and diameter are considered to be additive in a coagulation event, i.e.

$$u_{n,j} + u_{r,s} \rightarrow u_{n+r,j+s}. \tag{5.1}$$

Let $d_{min}(n)$ represent the minimum possible cluster in either 2D or 3D. Then, the general model is given by

$$\begin{aligned}
\frac{du_{n,n}}{dt} &= F_{n,n} - L_{n,n} - \gamma_{n,n}u_{n,n}, \\
\frac{du_{n,j}}{dt} &= F_{n,j} - L_{n,j} + \gamma_{n,j+1}u_{n,j+1} - \gamma_{n,j}u_{n,j}, \quad d_{min}(n)+1 \leq j < n \\
\frac{du_{n,j}}{dt} &= F_{n,j} - L_{n,j} + \gamma_{n,j+1}u_{n,j+1}, \quad j = d_{min}(n) < n
\end{aligned} \tag{5.2}$$

where $n = 1, 2, \ldots,$ and

$$F_{n,j} = \frac{1}{2}\sum_{r=1}^{n-1}\sum_{s=1}^{j-1} k_{n-r,r,j-s,s}u_{n-r,j-s}u_{r,s} \tag{5.3}$$

is the gain in clusters of mass n, diameter j due to the coagulation of clusters $u_{r,s}$ and $u_{n-r,j-s}$. Note that $F_{n,j} = 0$ for $n = 1$ or $j = 1$ since there cannot be a gain in clusters of mass one or diameter one due to a coagulation event. Similarly

$$L_{n,j} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s}u_{n,j}u_{r,s} \tag{5.4}$$

is the loss of clusters of mass n, diameter j due to the coagulation of clusters $u_{n,j}$ and $u_{r,s}$. The coefficient $k_{r,n,s,j}$ represents the rate of coagulation of clusters $u_{r,s}$ and $u_{n,j}$. The reformation coefficient $\gamma_{n,j}$ gives the rate at which a cluster $u_{n,j}$ will reshape to create a cluster with smaller diameter, $u_{n,j-1}$. Clearly, since the maximum diameter of a cluster of mass n is n, we have $u_{n,j} = 0$ for $j > n$.

5.1 Wattis' Model

We shall begin by looking at the case that is studied in [56]. Here the coagulation kernel $k_{r,n,s,j}$ is considered to be constant, k, say. Also the reformation coefficient $\gamma_{n,j}$ is independent of n and is given by $\gamma(j-1)$ with γ constant. We shall first consider the case $d_{min}(n) = 1$, as in [56]. If we put these assumptions into model

(5.2) we get

$$\begin{aligned}\frac{du_{n,j}(t)}{dt} &= \frac{1}{2}\sum_{r=1}^{n-1}\sum_{s=1}^{j-1} ku_{r,s}(t)u_{n-r,j-s}(t) - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} ku_{r,s}(t)u_{n,j}(t) \\ &\quad +\gamma j u_{n,j+1}(t) - \gamma(j-1)u_{n,j}(t) \end{aligned} \tag{5.5}$$

for $j \leq n$. Note that if $n = j > 1$ the third term will disappear, since $u_{n,n+1} = 0$, which coincides with the first equation in (5.2). Observe that if $n = j = 1$ then (5.5) will reduce to $\dot{u}_{1,1} = -L_{1,1}$ which still agrees with the first equation in (5.2). If $j = d_{min}(n) = 1$ the fourth term disappears, thus matching the third equation in (5.2) (notice the first term also disappears since $F_{n,1} = 0$). For $2 \leq j \leq n$ the equation (5.5) is the same as the second equation in (5.2). Therefore (5.5) covers all cases in (5.2) so that we study (5.5) instead.

The total mass and total diameter of the system are given respectively by

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nu_{n,j}(t) = \sum_{n=1}^{\infty}\sum_{j=1}^{n} nu_{n,j}(t), \quad \text{since } u_{n,j}(t) = 0 \text{ for } j > n, \tag{5.6}$$

and

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} ju_{n,j}(t) = \sum_{n=1}^{\infty}\sum_{j=1}^{n} ju_{n,j}(t). \tag{5.7}$$

In the case where no mass loss occurs, we require

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nu_{n,j}(t) = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} nu_{n,j}(0), \quad \forall t > 0. \tag{5.8}$$

As both total mass and diameter should be finite quantities, the natural space to work in is the Banach space $Y = L^1(\Omega, \mu)$ given by the following

Definition 5.1. *Let Y be the Banach space of all real doubly-infinite sequences $\{f_{n,j}\}_{n,j=1}^{\infty}$, with norm*

$$\|f\| = \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n|f_{n,j}| < \infty. \tag{5.9}$$

Here $\Omega = \mathbb{N} \times \mathbb{N}$ *and* $\mu = \mu_1 \times \mu_2$ *where*

$$\mu_1(M_1) = \begin{cases} \displaystyle\sum_{n \in M_1} n & \text{if } M_1 \subseteq \mathbb{N} \text{ is finite} \\ \infty & \text{if } M_1 \text{ is infinite} \end{cases}$$

and

$$\mu_2(M_2) = \begin{cases} \displaystyle\sum_{j \in M_2} 1 & \text{if } M_2 \subseteq \mathbb{N} \text{ is finite} \\ \infty & \text{if } M_2 \text{ is infinite} \end{cases}$$

so that

$$\mu(M) = \begin{cases} \displaystyle\sum_{(n,j) \in M} n & \text{if } M \subseteq \mathbb{N} \times \mathbb{N} \text{ is finite} \\ \infty & \text{if } M \text{ is infinite.} \end{cases} \tag{5.10}$$

Since we wish to impose the condition $u_{n,j} = 0$ for $j > n$ we shall work in the subspace X of Y defined by

$$X := \{f \in Y : f_{n,j} = 0 \text{ for } j > n\}, \tag{5.11}$$

i.e. the space of doubly-infinite lower triangular matrices. It is possible to prove the following

Lemma 5.2. *The space X given by (5.11) is a closed subspace of Y and hence is also a Banach space.*

Proof: Let $\{f^k\}$ be a sequence in X such that for each k, $f^k_{n,j} = 0$ when $j > n$. Suppose that

$$f^k \to f \text{ in } Y.$$

Then,

$$\|f^k - f\| = \sum_{n=1}^{\infty} n \sum_{j=1}^{\infty} |f^k_{n,j} - f_{n,j}| \to 0 \text{ as } k \to \infty$$

which implies that

$$|f^k_{n,j} - f_{n,j}| \to 0 \ \ \forall n, j.$$

Hence we must have

$$|0 - f_{n,j}| = 0 \ \ \forall j > n,$$

i.e. $f_{n,j} = 0$ for $j > n$ and so $f \in X$. The result now follows.

□

Note that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{n} n|f_{n,j}| \geq \sum_{n=1}^{\infty}\sum_{j=1}^{n} j|f_{n,j}| \text{ since } j \leq n. \tag{5.12}$$

At first we shall look at the reformation part of (5.5) given by

$$\frac{du_{n,j}(t)}{dt} = -\gamma(j-1)u_{n,j}(t) + \gamma j u_{n,j+1}(t). \tag{5.13}$$

In analogy with Problems 3.5, 4.1 and 4.9 we can transform (5.13) into the following ACP:

Problem 5.3. *Find a function $u : [0,\infty) \to X$, where X is as in (5.11), such that*

$$\begin{aligned} \frac{du(t)}{dt} &= Au(t) + Bu(t) \qquad\qquad t > 0 \\ \lim_{t\to 0+} u(t) &= f \in D(A), \end{aligned} \tag{5.14}$$

where the operators A and B are defined by

$$[Af]_{n,j} = -\gamma(j-1)f_{n,j}, \;\; [Bf]_{n,j} = \gamma j f_{n,j+1}, \tag{5.15}$$

with

$$D(A) = \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} n(j-1)|f_{n,j}| < \infty\}, \tag{5.16}$$

$$D(B) = \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} nj|f_{n,j+1}| < \infty\}. \tag{5.17}$$

It is clear that A maps $D(A)$ into X and likewise B maps $D(B)$ into X. Notice that by a simple change of variable in the domain given by (5.16) we have

$$D(A) = \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n-1} nj|f_{n,j+1}| < \infty\}.$$

We can also write

$$D(B) = \{f \in X : \sum_{n=1}^{\infty} \sum_{j=1}^{n-1} nj|f_{n,j+1}| < \infty\} \tag{5.18}$$

since $f_{n,n+1} = 0$ for $f \in X$. It follows that $D(A) = D(B)$.

We are now in a position to prove

Theorem 5.4. *Let X, A and B be as in (5.11), (5.15), (5.16) and (5.17). Then there exists a smallest extension G of $A + B$ which generates a substochastic semigroup $\{T_G(t)\}_{t\geq 0}$ on X.*

Proof: Again we wish to apply Theorem 2.19.

(i) A is the generator of the substochastic semigroup $\{T_A(t)\}_{t\geq 0}$ on X given by

$$[T_A(t)f]_{n,j} = e^{-\gamma(j-1)t} f_{n,j} \qquad \forall n, j. \tag{5.19}$$

(ii) We have shown that $D(B) = D(A)$. It is also clear that

$$[Bf]_{n,j} = \gamma j f_{n,j+1} \geq 0 \quad \forall f \in D(B)^+.$$

(iii) As discussed previously, X is a closed subspace of $Y = L^1(\mathbb{N} \times \mathbb{N}, \mu)$ with μ defined as in (5.10). For $f \in D(A)^+$ we have

$$\begin{aligned} \int_\Omega (Af + Bf) d\mu &= \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} n \left[-\gamma(j-1) f_{n,j} + \gamma j f_{n,j+1}\right] \\ &= -\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} n\gamma(j-1) f_{n,j} + \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} n\gamma j f_{n,j+1} \\ &= -\gamma \sum_{n=1}^{\infty} \sum_{j=1}^{\infty} n(j-1) f_{n,j} + \gamma \sum_{n=1}^{\infty} \sum_{l=1}^{\infty} n(l-1) f_{n,l} \\ &= 0 =: -c(f). \end{aligned} \tag{5.20}$$

Thus all of the conditions of Theorem 2.19 are satisfied. □

Along similar lines to Theorem 3.7 we wish to apply Theorem 2.26 with $c(f) := 0$ to show that $G = \overline{A + B}$. As mentioned previously, the appropriate space to now work in is X defined by (5.11). If we let l denote the space of all double sequences (0 when $j > n$), then $l = E_f \subset E$. (This is a strict inclusion because E can contain sequences with an arbitrary number of infinite entries, whereas a sequence in E_f must contain no infinite entries, since the only set with measure 0 is the empty set.) Also

$$F = \{f \in l : \left(\frac{f}{1+\gamma(j-1)}\right)_{n,j} \in X\}$$

and

$$\begin{aligned}
&[\mathbb{L}f]_{n,j} = \frac{f_{n,j}}{1+\gamma(j-1)}, \quad f \in F;\\
&[\mathbb{A}f]_{n,j} = f_{n,j} - (1+\gamma(j-1))f_{n,j} = -\gamma(j-1)f_{n,j},\\
&D(\mathbb{A}) = \mathbb{L}F = \{f \in X : f_{n,j} = \frac{g_{n,j}}{1+\gamma(j-1)},\ g \in F\};\\
&[\mathbb{B}f]_{n,j} = \gamma j f_{n,j+1},\\
&D(\mathbb{B}) = H = \{f \in X : \text{for any non-negative, non-decreasing sequence } \{f^n\}\\
&\qquad\qquad \text{in } D(B) \text{ such that } \sup_n f^n = |f|, \text{ i.e. } \sup_n f^n_{m,j} = |f_{m,j}|\ \forall m, j,\\
&\qquad\qquad \left.\text{we have } \sup_n Bf^n < \infty, \text{ i.e. } \sup_n Bf^n_{m,j} < \infty\ \forall m, j\right\}.
\end{aligned}$$

We can now prove the following.

Theorem 5.5. *In the context of Theorem 5.4, $G = \overline{A+B}$, the closure of the operator $(A+B, D(A))$.*

Proof: We must verify that for any $g \in F^+$ such that $-g + \mathbb{B}\mathbb{L}g \in X$ we have

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n(\mathbb{L}g)_{n,j} + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n\left(-g_{n,j} + (\mathbb{B}\mathbb{L}g)_{n,j}\right) \geq 0. \qquad (5.21)$$

Let

$$f_{n,j} = (\mathbb{L}g)_{n,j} = (1+\gamma(j-1))^{-1}g_{n,j}$$

so that $f \in X^+$. Then equation (5.21) holds if, for any $f \in X^+$ such that

$\mathbb{A}f + \mathbb{B}f \in X$, we have

$$\sum_{n=1}^{\infty} \sum_{j=1}^{\infty} n \left(-\gamma(j-1) f_{n,j} + (\mathbb{B}f)_{n,j}\right) \geq 0 \tag{5.22}$$

since

$$\begin{aligned}
(\mathbb{L}g)_{n,j} - g_{n,j} &= (1+\gamma(j-1))^{-1} g_{n,j} - g_{n,j} \\
&= -\frac{\gamma(j-1)}{1+\gamma(j-1)} g_{n,j} \\
&= -\gamma(j-1) f_{n,j}.
\end{aligned}$$

Now (5.22) can be written as

$$\begin{aligned}
&\lim_{N\to\infty} \sum_{n=1}^{N} \sum_{j=1}^{n} n \left(-\gamma(j-1) f_{n,j} + \gamma j f_{n,j+1}\right) \\
&= \lim_{N\to\infty} \left(-\sum_{n=1}^{N} \sum_{j=1}^{n} n\gamma(j-1) f_{n,j} + \sum_{n=1}^{N} \sum_{j=1}^{n} n\gamma j f_{n,j+1} \right).
\end{aligned} \tag{5.23}$$

Also,

$$\sum_{n=1}^{N} \sum_{j=1}^{n} n\gamma j f_{n,j+1} = \sum_{n=1}^{N} \sum_{l=1}^{n} n\gamma(l-1) f_{n,l}$$

on using the change of variable $j+1 \to l$. Thus from (5.23) we have

$$\lim_{N\to\infty} \left(-\sum_{n=1}^{N} \sum_{j=1}^{n} n\gamma(j-1) f_{n,j} + \sum_{n=1}^{N} \sum_{j=1}^{n} n\gamma(j-1) f_{n,j} \right) = 0. \tag{5.24}$$

Hence we have satisfied the conditions of Theorem 2.26 and we can conclude that $G = \overline{A+B}$.

□

Note 5.6. *Following Theorem 5.5 and the discussion on pages 39-40 we can say that the semigroup $\{T_G(t)\}_{t\geq 0}$ generated by G is stochastic since $c(f) := 0$.*

We can now deduce that the ACP

$$\frac{d}{dt}u(t) = Gu(t) \quad (t > 0) \tag{5.25}$$
$$\lim_{t\to 0+} u(t) = f\,, \tag{5.26}$$

has a unique strict, non-negative and mass-conserving solution $u : [0, \infty) \to D(G)^+$ for each $f \in D(G)^+$ and hence for each $f \in D(A)^+$. This solution is given by $u(t) = T_G(t)f$, with $G = \overline{A + B}$.

Note 5.7. *Since the coagulation terms will remain the same for each variation of the model described in this section we shall analyse them at the end to avoid repetition.*

5.2 The Modified Model

We shall now modify the above model so that we have a general reformation rate $\gamma_{n,j}$. We shall still consider the case where $d_{min}(n) = 1$ for all n and $u_{n,j} = 0$ if $j > n$. For the model to make physical sense we will need

$$\gamma_{n,1} = 0 \quad \text{for} \quad n = 1, 2, \dots\ . \tag{5.27}$$

We shall begin again by considering the reformation terms only

$$\frac{du_{n,j}(t)}{dt} = -\gamma_{n,j}u_{n,j}(t) + \gamma_{n,j+1}u_{n,j+1}(t). \tag{5.28}$$

We are now looking to solve

Problem 5.8. *Find a function $u : [0, \infty) \to X$ such that*

$$\frac{d}{dt}u(t) = Au(t) + Bu(t) \quad (t > 0) \tag{5.29}$$
$$\lim_{t\to 0+} u(t) = f \in D(A)\,,$$

where

$$[Af]_{n,j} = -\gamma_{n,j}f_{n,j}, \;\; [Bf]_{n,j} = \gamma_{n,j+1}f_{n,j+1}, \tag{5.30}$$

with

$$\begin{aligned} D(A) &= \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} n\gamma_{n,j}|f_{n,j}| < \infty\}, \\ D(B) &= \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} n\gamma_{n,j+1}|f_{n,j+1}| < \infty\}. \end{aligned} \tag{5.31}$$

We can again prove that $D(A) = D(B)$ by a simple change of variable.

We now wish to prove

Theorem 5.9. *Let X, A and B be as in (5.11), (5.30) and (5.31). Then there exists a smallest extension G of $A+B$ which generates a substochastic semigroup $\{T_G(t)\}_{t\geq 0}$ on X.*

Proof: Again we need to check that the conditions of Theorem 2.19 are satisfied. Since the details are very similar to those in the proof of Theorem 3.6 we shall only highlight the differences here. The substochastic semigroup $\{T_A(t)\}_{t\geq 0}$ on X generated by A is given by

$$[T_A(t)f]_{n,j} = e^{-\gamma_{n,j}t} f_{n,j} \qquad \forall n, j. \tag{5.32}$$

We know that $D(B) = D(A)$ and it is clear that $[Bf]_{n,j} \geq 0 \ \ \forall f \in D(B)^+$. Also for $f \in D(A)^+$

$$\begin{aligned} \int_\Omega (Af + Bf)d\mu &= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n[-\gamma_{n,j}f_{n,j} + \gamma_{n,j+1}f_{n,j+1}] \\ &= -\sum_{n=1}^{\infty}\sum_{j=1}^{n} n\gamma_{n,j}f_{n,j} + \sum_{n=1}^{\infty}\sum_{l=1}^{n} n\gamma_{n,l}f_{n,l} \\ &= 0 =: -c(f) \end{aligned}$$

by (5.27) and since $f_{n,n+1} = 0$.

□

We can now go on to prove the following

Theorem 5.10. *In the context of Theorem 5.9, $G = \overline{A+B}$, the closure of the operator $(A + B, D(A))$.*

Proof: This is essentially the same as the proof of Theorem 5.5 with $\gamma(j-1)f_{n,j}$ replaced with $\gamma_{n,j}f_{n,j}$ and $\gamma j f_{n,j+1}$ replaced with $\gamma_{n,j+1}f_{n,j+1}$ and so the details are omitted. $\square$

We now make a further extension to the model by examining the case where any cluster with diameter greater than j can reform to produce a cluster with diameter j. For this we will need to replace the third term, $\gamma_{n,j+1}u_{n,j+1}$, in (5.28) by

$$\sum_{s=j+1}^{n} \gamma_{n,s}p_{n,j,s}u_{n,s} \tag{5.33}$$

where $\gamma_{n,s}$ is the reformation rate of a cluster with mass n, diameter s and $p_{n,j,s}$ is the probability that a cluster with mass n, diameter s will reform to produce a cluster with mass n, diameter j. Note that again we have $u_{n,j} = 0$ for $j > n$ and $d_{min}(n) = 1$. Naturally, since $p_{n,j,s}$ is a probability, we require

$$\sum_{j=1}^{s-1} p_{n,j,s} = 1, \quad 0 \leq p_{n,j,s} \leq 1. \tag{5.34}$$

Note that $p_{n,j,j} = 0$ since we assume that a reformation event always changes the diameter of a cluster. Also, for physical reasons we require $p_{n,j,s} = 0$ if $j > n$ or $s > n$. For each $n = 1, 2, \ldots$ and $j \leq n$ the evolution of clusters of mass n and diameter j due to reformation is then described by

$$\frac{du_{n,j}(t)}{dt} = \sum_{s=j+1}^{\infty} \gamma_{n,s}p_{n,j,s}u_{n,s}(t) - \sum_{l=1}^{j-1} \gamma_{n,j}p_{n,l,j}u_{n,j}(t). \tag{5.35}$$

This time we wish to solve

Problem 5.11. *Find a function $u : [0, \infty) \to X$ such that*

$$\begin{aligned} &\frac{d}{dt}u(t) = Au(t) + Bu(t) \quad (t > 0) \\ &\lim_{t \to 0+} u(t) = f \in D(A) \end{aligned} \tag{5.36}$$

where

$$[Af]_{n,j} = -\sum_{l=1}^{j-1} \gamma_{n,j} p_{n,l,j} f_{n,j} = -\gamma_{n,j} f_{n,j}, \quad [Bf]_{n,j} = \sum_{s=j+1}^{\infty} \gamma_{n,s} p_{n,j,s} f_{n,s}, \tag{5.37}$$

with

$$\begin{aligned} D(A) &= \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} n\gamma_{n,j}|f_{n,j}| < \infty\}; \\ D(B) &= \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} n\Big|\sum_{s=j+1}^{\infty} \gamma_{n,s} p_{n,j,s} f_{n,s}\Big| < \infty\}. \end{aligned} \tag{5.38}$$

We follow the same procedure as before and hence we start by proving the following

Lemma 5.12. *As sets $D(A) \subseteq D(B)$.*

Proof: For $f \in D(A)$

$$\begin{aligned} \|Bf\| &\le \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n \sum_{s=j+1}^{\infty} \gamma_{n,s} p_{n,j,s} |f_{n,s}| \\ &= \sum_{n=1}^{\infty}\sum_{s=2}^{\infty}\sum_{j=1}^{s-1} n\gamma_{n,s} p_{n,j,s} |f_{n,s}| \\ &= \sum_{n=1}^{\infty}\sum_{s=2}^{\infty} n\gamma_{n,s}|f_{n,s}| \quad \text{on using (5.34)} \\ &= \|Af\| \quad \text{on using (5.27)} \\ &< \infty. \end{aligned}$$

Hence $D(A) \subseteq D(B)$.

□

We are now in a position to prove

Theorem 5.13. *Let X, A and B be as in (5.11), (5.37) and (5.38). Then there exists a smallest extension G of $A+B$ which generates a substochastic semigroup $\{T_G(t)\}_{t\ge 0}$ on X.*

Proof: Again the details are similar to those in the proof of Theorem 5.9 and so we shall only point out the differences here.

(i) This is exactly the same.

(ii) $D(B) \supseteq D(A)$ was shown in Lemma 5.12 and it is clear that $[Bf]_{n,j} \geq 0 \quad \forall f \in D(B)^+$.

(iii) From the calculations in the proof of Lemma 5.12 we have $\int_\Omega (Af + Bf) d\mu = 0$.

□

Next we shall prove

Theorem 5.14. *In the context of Theorem 5.13, $G = \overline{A+B}$, the closure of the operator $(A + B, D(A))$.*

Proof: Again the details are essentially the same as those given in the proof of Theorem 5.5 with $\gamma(j-1)f_{n,j}$ replaced with $\gamma_{n,j}f_{n,j}$ and $\gamma j f_{n,j+1}$ replaced by $\sum_{s=j+1}^{\infty} \gamma_{n,s}p_{n,j,s}f_{n,s}$. We follow the same procedure to reach the point

$$\begin{aligned}
&\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n\left(-\gamma_{n,j}f_{n,j} + (\mathbb{B}f)_{n,j}\right) \\
&= \lim_{N\to\infty}\sum_{n=1}^{N}\sum_{j=1}^{n} n\left(-\gamma_{n,j}f_{n,j} + \sum_{s=j+1}^{\infty}\gamma_{n,s}p_{n,j,s}f_{n,s}\right). \qquad (5.39)
\end{aligned}$$

We know that

$$\begin{aligned}
&\sum_{n=1}^{N}\sum_{j=1}^{n} n \sum_{s=j+1}^{\infty}\gamma_{n,s}p_{n,j,s}f_{n,s} \\
&= \sum_{n=1}^{N}\sum_{s=2}^{\infty}\sum_{j=1}^{s-1} n\gamma_{n,s}p_{n,j,s}f_{n,s} \\
&= \sum_{n=1}^{N}\sum_{s=1}^{n} n\gamma_{n,s}f_{n,s}
\end{aligned}$$

on using (5.34) and (5.27) and since $f_{n,s} = 0$ for $s > n$. Thus we can deduce that the limit in (5.39) is 0 and hence we have proved the required result. □

As a final extension, suppose we look at the model (5.28) with general reformation rate and added fragmentation terms. For $n = 1, 2, \ldots,$ and $j \leq n$ the evolution of clusters of mass n and diameter j due to reformation and fragmentation is described by

$$\begin{aligned}
\frac{du_{n,j}(t)}{dt} &= -a_{n,j}u_{n,j}(t) + \sum_{r=n+1}^{\infty}\sum_{s=j}^{\infty} a_{r,s}b_{n,j,r,s}u_{r,s}(t) \\
&\quad +\gamma_{n,j+1}u_{n,j+1}(t) - \gamma_{n,j}u_{n,j}(t) \\
&= -(a_{n,j} + \gamma_{n,j})u_{n,j}(t) + \sum_{r=n+1}^{\infty}\sum_{s=j}^{\infty} a_{r,s}b_{n,j,r,s}u_{r,s}(t) + \gamma_{n,j+1}u_{n,j+1}(t) \\
&= -\alpha_{n,j}u_{n,j}(t) + \sum_{r=n}^{\infty}\sum_{s=j}^{\infty} \beta_{r,s}u_{r,s}(t) \qquad (5.40)
\end{aligned}$$

where

$$\alpha_{n,j} = a_{n,j} + \gamma_{n,j} \qquad (5.41)$$

$$\beta_{r,s} = \begin{cases} \gamma_{n,j+1} \text{ when } r = n, s = j+1 \\ 0 \text{ when } r = n, s = j, j+2, j+3 \ldots \\ a_{r,s}b_{n,j,r,s} \text{ otherwise.} \end{cases} \qquad (5.42)$$

Here $a_{n,j}$ is the general fragmentation rate of a cluster with mass n, diameter j and $b_{n,j,r,s}$ is the number of clusters of mass n, diameter j produced due to the break-up of a cluster of mass r, diameter s. Again $\gamma_{n,j}$ is the general reformation rate of a cluster with mass n and diameter j. Notice that equation (5.40) is very similar to equation (4.30) which represented a system of fragmenting clusters incorporating mass loss and surface recession, except this time we have a multicomponent equation. For physical reasons we require

$$a_{1,1} = 0, \qquad (5.43)$$

$$b_{n,j,r,s} = 0 \quad \text{for} \quad j > n \qquad (5.44)$$

and

$$\sum_{n=1}^{r-1}\sum_{j=1}^{s} nb_{n,j,r,s} = r. \qquad (5.45)$$

The latter relates to conservation of mass in the system.

The problem we wish to solve this time is

Problem 5.15. *Find a function* $u : [0, \infty) \to X$ *such that*

$$\begin{aligned} &\frac{d}{dt}u(t) = Au(t) + Bu(t) \quad (t > 0) \\ &\lim_{t\to 0+} u(t) = f \in D(A)\,. \end{aligned} \tag{5.46}$$

where

$$[Af]_{n,j} = -\alpha_{n,j} f_{n,j}, \quad [Bf]_{n,j} = \sum_{r=n}^{\infty}\sum_{s=j}^{r} \beta_{r,s} f_{r,s}, \tag{5.47}$$

with

$$\begin{aligned} D(A) &= \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} n\alpha_{n,j}|f_{n,j}| < \infty\}, \\ D(B) &= \{f \in X : \sum_{n=1}^{\infty}\sum_{j=1}^{n} \Big|\sum_{r=n}^{\infty}\sum_{s=j}^{\infty} n\beta_{r,s} f_{r,s}\Big| < \infty\}. \end{aligned} \tag{5.48}$$

Note that in all previous cases it has been clear that A maps $D(A)$ into X and B maps $D(B)$ into X. Now we are working with more complicated operators and so we shall prove

Lemma 5.16. *Let* X *be as in (5.11) and* $(A, D(A))$ *and* $(B, D(B))$ *be defined as in (5.47) and (5.48). Then* A *maps* $D(A)$ *into* X *and* B *maps* $D(B)$ *into* X.

Proof: It is clear that A maps $D(A)$ into X since, for $f \in D(A)$,

$$[Af]_{n,j} = -\alpha_{n,j} f_{n,j} = -a_{n,j} f_{n,j} + \gamma_{n,j} f_{n,j} = 0 \text{ for } n < j.$$

For $f \in D(B)$

$$\begin{aligned} [Bf]_{n,j} &= \sum_{r=n}^{\infty}\sum_{s=j}^{\infty} \beta_{r,s} f_{r,s} \\ &= \underbrace{\gamma_{n,j+1} f_{n,j+1}}_{=0 \text{ for } n<j} + \sum_{r=n+1}^{\infty}\sum_{s=j}^{\infty} a_{r,s} \underbrace{b_{n,j,r,s}}_{=0 \text{ for } n<j} f_{r,s} \\ &= 0 \quad \text{for } n < j. \end{aligned}$$

Thus B maps $D(B)$ into X. □

Following the same strategy as before we can now prove

Lemma 5.17. *As sets $D(B) \supseteq D(A)$.*

Proof: For $f \in D(A)$

$$\begin{aligned}
\|Bf\| &\leq \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{r=n}^{\infty}\sum_{s=j}^{\infty} n\beta_{r,s}|f_{r,s}| \\
&= \sum_{n=1}^{\infty}\sum_{j=1}^{\infty}\sum_{r=n+1}^{\infty}\sum_{s=j}^{\infty} na_{r,s}b_{n,j,r,s}|f_{r,s}| + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n\gamma_{n,j+1}|f_{n,j+1}| \\
&= \sum_{r=2}^{\infty}\sum_{s=1}^{\infty}\sum_{n=1}^{r-1}\sum_{j=1}^{s} na_{r,s}b_{n,j,r,s}|f_{r,s}| + \sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n\gamma_{n,j+1}|f_{n,j+1}| \\
&= \sum_{r=2}^{\infty}\sum_{s=1}^{\infty} ra_{r,s}|f_{r,s}| + \sum_{n=1}^{\infty}\sum_{l=2}^{\infty} n\gamma_{n,l}|f_{n,l}| \\
&= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} ra_{r,s}|f_{r,s}| + \sum_{n=1}^{\infty}\sum_{l=1}^{\infty} n\gamma_{n,l}|f_{n,l}| \\
&= \|Af\|
\end{aligned}$$

on using (5.27), (5.43) and (5.45). □

We are now in a position to prove

Theorem 5.18. *Let X, A and B be as in (5.11), (5.47) and (5.48). Then there exists a smallest extension G of $A+B$ which generates a substochastic semigroup $\{T_G(t)\}_{t\geq 0}$ on X.*

Proof: Again this is very similar to previous cases and so only brief details are given. We wish to show that our operators satisfy the conditions of Theorem 2.19

(i) The semigroup generated by A on X is

$$[T_A(t)f]_{n,j} = e^{-\alpha_{n,j}t} f_{n,j} \qquad \forall j, n. \tag{5.49}$$

(ii) $D(B) \supseteq D(A)$ was shown in Lemma 5.17 and it is clear that $[Bf]_{n,j} \geq 0 \;\; \forall f \in D(B)^+$.

(iii) For $f \in D(A)^+$, $\int_\Omega (Af + Bf)d\mu = 0$ from the calculations in the proof of Lemma 5.17.

Thus all of the conditions of Theorem 2.19 are satisfied. □

We can finally prove

Theorem 5.19. *In the context of Theorem 5.18, $G = \overline{A + B}$, the closure of the operator $(A + B, D(A))$.*

Proof: The details are essentially the same as those found in all previous cases and thus have been omitted.

□

Note 5.20. *For each of the above cases we have shown that the corresponding ACP of the form*

$$\begin{aligned} \frac{du(t)}{dt} &= Gu(t), \ \ t > 0, \\ \lim_{t \to 0^+} u(t) &= f \end{aligned}$$

has a unique strict, non-negative and mass-conserving solution $u : [0, \infty) \to D(G)^+$ for each $f \in D(G)^+$ and hence for each $f \in D(A)^+$. This solution is given by $u(t) = T_G(t)f$, with $G = \overline{A + B}$.

As mentioned previously the coagulation terms that will be added to each of the above cases will be the same and so we only need to analyse them once. We shall adopt the same technique used in Section 3.3 for the regular one-component coagulation terms to analyse the multicomponent coagulation terms. The non-linear ACP which we wish to solve for each A and B above takes the form

$$\frac{d}{dt}u(t) = Au(t) + Bu(t) + Ku(t) \tag{5.50}$$

$$\lim_{t \to 0+} u(t) = f \in D(A) \tag{5.51}$$

where the coagulation operator K is given by

$$[Kf]_{n,j} = \frac{1}{2}\sum_{r=1}^{n-1}\sum_{s=1}^{j-1} k_{n-r,r,j-s,s} f_{n-r,j-s} f_{r,s} - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s} f_{n,j} f_{r,s}. \tag{5.52}$$

Note that we can show that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n[Kf]_{n,j} = 0$$

by calculations similar to those leading to (3.5). We will place similar restrictions on $k_{n,r,j,s}$ as we did in Section 3.3, namely $k_{n,r,j,s} = k_{r,n,s,j}$ and there exists a constant k such that

(A4)* $\quad k_{n,r,j,s} \leq k \quad$ for all $n, r = 1, 2, \ldots$ and $j \leq n, s \leq r$.

Definition 5.21. *We define $\tilde{K}$ on $X \times X$ by*

$$\left(\tilde{K}[f,g]\right)_{n,j} = \frac{1}{2}\sum_{r=1}^{n-1}\sum_{s=1}^{j-1} k_{n-r,r,j-s,s} f_{n-r,j-s} g_{r,s} - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s} f_{n,j} g_{r,s} \tag{5.53}$$

with $f, g \in X$.

In analogy with Theorem 3.13 we can prove

Theorem 5.22. *Under Assumption (A4)*, $\tilde{K}$ defined in (5.53) is a bilinear, continuous form mapping $X \times X$ into X.*

Proof: Again, it is convenient to write

$$\left(\tilde{K}[f,g]\right)_{n,j} = \left(\tilde{K}_1[f,g]\right)_{n,j} - \left(\tilde{K}_2[f,g]\right)_{n,j}$$

where

$$\begin{aligned}\left(\tilde{K}_1[f,g]\right)_{n,j} &= \frac{1}{2}\sum_{r=1}^{n-1}\sum_{s=1}^{j-1} k_{n-r,r,j-s,s} f_{n-r,j-s} g_{r,s},\\ \left(\tilde{K}_2[f,g]\right)_{n,j} &= \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s} f_{n,j} g_{r,s}.\end{aligned}$$

Note that $(\tilde{K}_1[f,g])_{n,j} = 0$ if either $n = 1$ or $j = 1$.

We can show that $\left(\tilde{K}_1[f,g]\right)_{n,j} = 0$ for $j > n$, $f, g \in X$. Indeed if $s > r$ then $g_{r,s} = 0$ while if $s \leq r$ then $j - s > n - r$ so that $f_{n-r,j-s} = 0$. Thus all terms in the sum defining $\left(\tilde{K}_1[f,g]\right)_{n,j}$ are zero when $j > n$. Clearly $\left(\tilde{K}_2[f,g]\right)_{n,j} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s} f_{n,j} g_{r,s} = 0$ for $j > n$ since $f_{n,j} = 0$ for $j > n$.

Also, for $f, g \in X$ we have

$$
\begin{aligned}
\|\tilde{K}_1[f,g]\| &= \sum_{n=2}^{\infty}\sum_{j=2}^{\infty} n\left|\left(\tilde{K}_1[f,g]\right)_{n,j}\right| \\
&\le \frac{k}{2}\sum_{n=2}^{\infty}\sum_{j=2}^{\infty}\sum_{r=1}^{n-1}\sum_{s=1}^{j-1} n|f_{r,s}||g_{n-r,j-s}| \\
&= \frac{k}{2}\sum_{n=2}^{\infty}\sum_{r=1}^{n-1}\sum_{j=2}^{\infty}\sum_{s=1}^{j-1} n|f_{r,s}||g_{n-r,j-s}| \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{n=r+1}^{\infty}\sum_{s=1}^{\infty}\sum_{j=s+1}^{\infty} n|f_{r,s}||g_{n-r,j-s}| \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{l=1}^{\infty}\sum_{s=1}^{\infty}\sum_{m=1}^{\infty}(l+r)|f_{r,s}||g_{l,m}| \\
&\qquad \text{(on putting } l = n-r,\ \ m = j-s) \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{l=1}^{\infty}\sum_{s=1}^{\infty}\sum_{m=1}^{\infty}\left(l|f_{r,s}||g_{l,m}| + r|f_{r,s}||g_{l,m}|\right) \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}|f_{r,s}|\|g\| + \frac{k}{2}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty}|g_{l,m}|\|f\| \\
&\le k\|f\|\|g\|. \qquad (5.54)
\end{aligned}
$$

Similarly we have

$$
\|\tilde{K}_2[f,g]\| \le k\|f\|\|g\|.
$$

So we have $K : X \times X \to X$. Also, we have that $\tilde{K}[\cdot,\cdot]$ is bounded, and hence continuous, in each argument separately. It is easily checked through calculations similar to those found in the proof of Theorem 3.13 that $\tilde{K}$ is bilinear. □

Notice that the bounds on $\tilde{K}_1$ and $\tilde{K}_2$ found in the proof above are identical to those found in Section 3.3. This means that all that the results following Theorem 3.13 can be reproduced identically here and thus we omit the details. We can now state the following

Theorem 5.23. *There exists a global, strong, non-negative, mass-conserving solution to the ACP*

$$\frac{d}{dt}u(t) = Gu(t) + Ku(t) \tag{5.55}$$

$$\lim_{t\to 0+} u(t) = f \in D(G) \tag{5.56}$$

where G is any of the generators from this chapter.

Proof: This follows from the results analogous to those in Section 3.3. □

5.3 Restrictions

We now consider the case discussed in the introduction to this chapter where in 2D

$$d_{min2D}(n) = \lceil \sqrt{n} \rceil$$

and in 3D

$$d_{min3D}(n) = \lceil \sqrt[3]{n}. \rceil$$

As before, let $d_{min}(n)$ represent the minimum possible diameter of a cluster in either 2D or 3D. For all the models described in Sections 5.1 and 5.2 (and the original Wattis model) to make physical sense we shall require the further restriction that

$$u_{n,j} = 0 \quad \text{if } j < d_{min}(n).$$

Also, we must be careful when considering the reformation coefficients. We have to make sure that our coefficients do not allow for the formation of an 'illegally-sized' particle. For example, in the original Wattis model where we have

$$\gamma_{n,j} = \gamma(j-1) \text{ with } \gamma \text{ a constant}$$

we need to specify that

$$\gamma_{n,j} = \begin{cases} \gamma(j-1) & \text{if } d_{min}(n) \le j-1 \le n, \\ 0 & \text{if } j-1 < d_{min}(n). \end{cases}$$

In our first modified model with general $\gamma_{n,j}$ we would need to have

$$\gamma_{n,j} = 0 \quad \text{if } j - 1 < d_{min}(n).$$

In the second modified case it is not so simple. Here a particle of size n with any diameter s larger than j can reform to produce a particle of size n, diameter j. The added restriction here would be

$$p_{n,j,s} = 0 \;\; \text{if } j < d_{min}(n) \text{ or } s < d_{min}(n).$$

Adding in these extra restrictions will not affect the above existence/uniqueness results.

5.4 Non-Additive Diameter

Now we shall consider the case where the resulting diameter after a coagulation event is not necessarily additive, but can in fact take on a range of values. Consequently, after the coagulation of a cluster with mass n and diameter j with a cluster with mass r and diameter s the resulting diameter l, say, could take any value in the range $min\{j, s\} \leq l \leq j + s$. Note that we would still have the restrictions $j \leq n$ and $s \leq r$.

Previously the coagulation part of the evolution equation took the form

$$\frac{du_{n,j}(t)}{dt} = \frac{1}{2}\sum_{r=1}^{n-1}\sum_{s=1}^{j-1} k_{n-r,r,j-s,s}u_{n-r,j-s}(t)u_{r,s}(t) - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s}u_{n,j}(t)u_{r,s}(t). \tag{5.57}$$

If we are now assuming that the diameter is no longer additive the first term in (5.57) will change. We will now have a gain in clusters $u_{n,j}$ due to the coagulation of clusters $u_{r,l}$ and $u_{n-r,m}$ where $l < r$ and $m < n - r$. Consider the probability function, $0 \leq p'_{n,r,j,l,m} \leq 1$, which gives the probability that a cluster of diameter j will be produced by the coagulation of a cluster with mass n and diameter l with a cluster with mass r and diameter m. Note that $p'_{n,r,j,l,m} = 0$ if $j < min\{l, m\}$

or $j > l + m$. Also

$$\sum_{j=1}^{\infty} p'_{n,r,j,l,m} = 1 \quad \text{for each } n, r, l, m \tag{5.58}$$

and $p'_{n,r,j,l,m} = p'_{r,n,j,m,l}$ for all j. The term

$$\frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} k_{n-r,r,l,m} p'_{n-r,r,j,l,m} u_{n-r,l} u_{r,m} \tag{5.59}$$

will replace the first term in (5.57). Note that the second term will not change since the loss of clusters $u_{n,j}$ due to the coagulation of a cluster $u_{n,j}$ with another cluster will always result in a cluster with mass greater than n since mass is additive. The new evolution equation for coagulation is thus

$$\begin{aligned} \frac{du_{n,j}(t)}{dt} &= \frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} k_{n-r,r,l,m} p'_{n-r,r,j,l,m} u_{n-r,l}(t) u_{r,m}(t) \\ &\quad - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s} u_{n,j}(t) u_{r,s}(t), \end{aligned} \tag{5.60}$$

where we still have $d_{min}(n) = 1$ for all n. Again, we can show that

$$\sum_{n=1}^{\infty}\sum_{j=1}^{\infty} n[Kf]_{n,j} = 0.$$

We will analyse these coagulation terms using the same strategy as in the previous section. Again, we shall assume that $k_{n,r,j,s}$ is symmetric in the sense that $k_{n,r,j,s} = k_{r,n,s,j}$ and is uniformly bounded by a constant k.

Let

$$\begin{aligned} [Kf]_{n,j} &= \frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} k_{n-r,r,l,m} p'_{n-r,r,j,l,m} f_{n-r,l} f_{r,m} \\ &\quad - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s} f_{n,j} f_{r,s} \end{aligned} \tag{5.61}$$

and define $\tilde{K}$ on $X \times X$ by

$$\left(\tilde{K}[f,g]\right)_{n,j} = \frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} k_{n-r,r,l,m}p'_{n-r,r,j,l,m}f_{n-r,l}g_{r,m} - \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s}f_{n,j}g_{r,s} \tag{5.62}$$

with $f, g \in X$. Similarly to Definition 5.21 we have

Definition 5.24. *We define $\tilde{K}$ on $X \times X$ by*

$$\left(\tilde{K}[f,g]\right)_{n,j} = \left(\tilde{K}_1[f,g]\right)_{n,j} - \left(\tilde{K}_2[f,g]\right)_{n,j} \tag{5.63}$$

with

$$\left(\tilde{K}_1[f,g]\right)_{n,j} = \frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} k_{n-r,r,l,m}p'_{n-r,r,j,l,m}f_{n-r,l}g_{r,m} \tag{5.64}$$

and

$$\left(\tilde{K}_2[f,g]\right)_{n,j} = \sum_{r=1}^{\infty}\sum_{s=1}^{\infty} k_{n,r,j,s}f_{n,j}g_{r,s}. \tag{5.65}$$

Note that $\left(\tilde{K}_1[f,g]\right)_{n,j} = 0$ for $n = 1$. We can now prove

Theorem 5.25. *Under Assumption (A4)***, $\tilde{K}$ defined in (5.63) is a bilinear, bicontinuous form mapping $X \times X$ into X.*

Proof: We can show that $\left(\tilde{K}_1[f,g]\right)_{n,j} = 0$ for $j > n$. We have $min\{l,m\} \leq j \leq l+m$ so that $n < j \leq l+m$ which implies that $n-l < m$. If we have $m > r$ then $g_{r,m} = 0$. On the other hand if $r \geq m$ then $n-l < r$ i.e. $n-r < l$, thus $f_{n-r,l} = 0$. It is easily shown that $\left(\tilde{K}_2[f,g]\right)_{n,j} = 0$ for $j > n$ since $f_{n,j} = 0$ for $j > n$.

For $f, g \in X$ we have

$$\begin{aligned}
\|\tilde{K}_1[f,g]\| &= \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} n|(\tilde{K}_1[f,g])_{n,j}| \\
&= \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} n\left|\frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} k_{n-r,r,l,m}p'_{n-r,r,j,l,m}f_{n-r,l}g_{r,m}\right| \\
&\leq \frac{k}{2}\sum_{n=2}^{\infty}\sum_{j=1}^{\infty}\sum_{r=1}^{n-1}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} np'_{n-r,r,j,l,m}|f_{n-r,l}||g_{r,m}|
\end{aligned}$$

$$
\begin{aligned}
&= \frac{k}{2}\sum_{n=2}^{\infty}\sum_{r=1}^{n-1}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} np'_{n-r,r,j,l,m}|f_{n-r,l}||g_{r,m}| \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{n=r+1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} np'_{n-r,r,j,l,m}|f_{n-r,l}||g_{r,m}| \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} (s+r)p'_{s,r,j,l,m}|f_{s,l}||g_{r,m}| \\
&\qquad\qquad \text{(on putting } s = n - r\text{)} \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} \left(sp'_{s,r,j,l,m}|f_{s,l}||g_{r,m}| + rp'_{s,r,j,l,m}|f_{s,l}||g_{r,m}|\right) \\
&= \frac{k}{2}\sum_{r=1}^{\infty}\sum_{s=1}^{\infty}\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} (s|f_{s,l}||g_{r,m}| + r|f_{s,l}||g_{r,m}|) \text{ by (5.58)} \\
&= \frac{k}{2}\left(\sum_{s=1}^{\infty}\sum_{l=1}^{\infty}|f_{s,l}|\|g\| + \sum_{r=1}^{\infty}\sum_{m=1}^{\infty}\|f\||g_{r,m}|\right) \\
&\leq k\|f\|\|g\|.
\end{aligned}
$$

Similarly we have

$$\|\tilde{K}_2[f,g]\| \leq k\|f\|\|g\|.$$

This shows that $\tilde{K} : X \times X \to X$ and that $\tilde{K}[\cdot,\cdot]$ is bounded, and hence continuous, in each argument separately. It is easily shown through calculations similar to those found in the proof of Theorem 3.13 that $\tilde{K}$ is bilinear. □

Notice that the bounds on $\tilde{K}_1$ and $\tilde{K}_2$ found in the proof above are again identical to those found in Section 3.3. This leads to the following result.

Theorem 5.26. *There exists a global, strong, non-negative, mass-conserving solution to the ACP*

$$\frac{d}{dt}u(t) = Gu(t) + Ku(t) \tag{5.66}$$
$$\lim_{t\to 0+} u(t) = f \in D(G) \tag{5.67}$$

where G is any of the generators from this chapter.

We now consider the minimum diameter $d_{min}(n)$ of a cluster with mass n

and diameter j to be given by either $d_{min2D}(n)$ or $d_{min3D}(n)$, depending on how many dimensions we are working in, so that $u_{n,j} = 0$ for $1 < j < d_{min}(n)$. Thus we would be working in the Banach space X as in (5.11) but with the added restriction $f_{n,j} = 0$ for $1 < j < d_{min}(n)$. We need to check that the above calculations still follow through. Our evolution equation would now be

$$\begin{aligned}\frac{du_{n,j}(t)}{dt} &= \frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=d_{min}(n-r)}^{\infty}\sum_{m=d_{min}(r)}^{\infty} kp'_{n-r,r,j,l,m}u_{n-r,l}(t)u_{r,m}(t) \\ &\quad -\sum_{r=1}^{\infty}\sum_{s=d_{min}(r)}^{\infty} ku_{n,j}(t)u_{r,s}(t). \end{aligned} \tag{5.68}$$

This time

$$\left(\tilde{K}_1[f,g]\right)_{n,j} = \frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=d_{min}(n-r)}^{\infty}\sum_{m=d_{min}(r)}^{\infty} kp'_{n-r,r,j,l,m}f_{n-r,l}g_{r,m} \tag{5.69}$$

and

$$\left(\tilde{K}_2[f,g]\right)_{n,j} = \sum_{r=1}^{\infty}\sum_{s=d_{min}(r)}^{\infty} kf_{n,j}g_{s,r}. \tag{5.70}$$

For $f, g \in X$ we have

$$\begin{aligned}\|\tilde{K}_1[f,g]\| &= \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} n|(\tilde{K}_1[f,g])_{n,j}| \\ &= \sum_{n=2}^{\infty}\sum_{j=1}^{\infty} n|\frac{1}{2}\sum_{r=1}^{n-1}\sum_{l=d_{min}(r)}^{\infty}\sum_{m=d_{min}(n-r)}^{\infty} kp'_{n-r,r,j,l,m}f_{r,l}g_{n-r,m}| \\ &\leq \frac{k}{2}\sum_{n=2}^{\infty}\sum_{j=1}^{\infty}\sum_{r=1}^{n-1}\sum_{l=d_{min}(r)}^{\infty}\sum_{m=d_{min}(n-r)}^{\infty} p'_{n-r,r,j,l,m}|f_{r,l}||g_{n-r,m}| \\ &\leq k\|f\|\|g\| \end{aligned}$$

since

$$\sum_{l=1}^{\infty}\sum_{m=1}^{\infty} f_{n-r,l}g_{r,m} = \sum_{l=d_{min}(n-r)}^{\infty}\sum_{m=d_{min}(r)}^{\infty} f_{n-r,l}g_{r,m}.$$

Similarly

$$\|\tilde{K}_2[f,g]\| \leq k\|f\|\|g\|.$$

Thus we have the same bounds on our $\tilde{K}_1$ and $\tilde{K}_2$ as we have throughout this chapter and thus our Theorem 5.26 still holds.

5.5 Discrete Mass and Continuous Diameter

In the previous sections we have looked at a multi-component C-F model in which a cluster of particles is identified by both its mass and diameter, each of which is assumed to take only positive integer values. Now we examine the case where we have a cluster described by a discrete mass variable but a continuous diameter variable. We let $u_n(y)$ represent the concentration of clusters with mass $n = 1, 2, \ldots$ and diameter $y \in \mathbb{R}^+$. We shall consider the situation where the mass and diameter of a monomer have been scaled so that a particle of mass one has diameter one. Thus the largest diameter we can have in a cluster of n particles will be when the particles are joined in a single straight line and thus $u_n(y) = 0$ for $y > n$. We shall consider the coagulation equation for discrete mass and continuous diameter, which is given by

$$\begin{aligned} \frac{\partial u_n(y,t)}{\partial t} &= \frac{1}{2}\sum_{j=1}^{n-1}\int_{z=0}^{y} k_{n-j,j}(y-z,z)u_{n-j}(y-z,t)u_j(z,t)dz \\ &\quad -\sum_{j=1}^{\infty}\int_{z=0}^{\infty} k_{n,j}(y,z)u_n(y,t)u_j(z,t)dz \end{aligned} \tag{5.71}$$

where $k_{n,j}(y,z)$ is the coagulation rate for a cluster with mass n and diameter $y \le n$ with a cluster with mass j and diameter $z \le j$. Note that the first term is zero when $n = 1$ due to the empty sum and we have that

$$k_{n,j}(y,z) = k_{j,n}(z,y) \le k \quad \text{for all } n, j = 1, 2, \ldots \text{ and } y \le n, z \le r.$$

The total mass and total diameter of the system are given respectively by

$$\sum_{n=1}^{\infty}\int_{0}^{\infty} nu_n(y)dy = \sum_{n=1}^{\infty}\int_{0}^{n} nu_n(y)dy \tag{5.72}$$

and

$$\sum_{n=1}^{\infty}\int_{0}^{\infty} yu_n(y)dy = \sum_{n=1}^{\infty}\int_{0}^{n} yu_n(y)dy. \tag{5.73}$$

The natural space to work in is the Banach space X of all infinite sequences of real functions $\{f_n(y)\}_{n=1}^{\infty},\ y \in \mathbb{R}^+$ such that $f_n(y) = 0$ for $y > n$ with norm

$$\|f\| = \sum_{n=1}^{\infty}\int_{0}^{\infty} n|f_n(y)|dy = \sum_{n=1}^{\infty}\int_{0}^{n} n|f_n(y)|dy < \infty.$$

If we wish to prove a theorem analogous to Theorem 5.22, then we are required to show, for example, that

$$\|\frac{1}{2}\sum_{j=1}^{n-1}\int_{z=0}^{y} k_{n-j,j}(y-z,z)f_{n-j}(y-z)f_j(z)dz\| < \infty$$

or

$$\|\sum_{j=1}^{\infty}\int_{z=0}^{\infty} k_{n,j}(y,z)f_n(y)f_j(z)dz\| < \infty.$$

In order to prove these results we will need to be able to interchange the order of the summation and the integral. For example

$$\begin{aligned}
&\left\|\frac{1}{2}\sum_{j=1}^{n-1}\int_{z=0}^{y} k_{n-j,j}(y-z,z)f_{n-j}(y-z)f_j(z)dz\right\| \\
\leq\ & \frac{1}{2}\sum_{n=1}^{\infty}\int_{y=0}^{n}\sum_{j=1}^{n-1}\int_{z=0}^{y} kn|f_{n-j}(y-z)||f_j(z)|dzdy \\
=\ & \frac{k}{2}\sum_{n=1}^{\infty}\sum_{j=1}^{n-1}\int_{y=0}^{n}\int_{z=0}^{y} n|f_{n-j}(y-z)||f_j(z)|dzdy \\
=\ & \frac{k}{2}\sum_{j=1}^{\infty}\sum_{n=j+1}^{\infty}\int_{z=0}^{n}\int_{y=z}^{n} n|f_{n-j}(y-z)||f_j(z)|dydz \\
=\ & \frac{k}{2}\sum_{j=1}^{\infty}\sum_{l=1}^{\infty}\int_{z=0}^{n}\int_{x=0}^{n-z} (j+l)|f_j(z)||f_l(x)|dxdz \\
& \qquad\qquad \text{on putting } l = n-j \text{ and } x = y-z
\end{aligned}$$

$$\begin{aligned}
&\le \frac{k}{2}\sum_{j=1}^{\infty}\int_{z=0}^{n}\sum_{l=1}^{\infty}\int_{x=0}^{n}(j+l)|f_j(z)||f_l(x)|dxdz \\
&= \frac{k}{2}\sum_{l=1}^{\infty}\int_{x=0}^{n}\|f\||f_l(x)|dx + \frac{k}{2}\sum_{j=1}^{\infty}\int_{z=0}^{n}\|f\||f_j(z)|dz \\
&\le k\|f\|^2 < \infty \quad \text{since } f \in X. \qquad (5.74)
\end{aligned}$$

As mentioned in Chapter 2, we must apply Fubini's Theorem extended to a product of four measures to enable us to carry out this change in order of summation/integration . Let $\Omega_1 = \Omega_3 = \mathbb{N}$, $\Omega_2 = \Omega_4 = \mathbb{R}^+$, let $\mu_2 = \mu_4$ be the usual Lebesgue measure, let μ_3 be the usual counting measure on $\mathbb{N}$ and define μ_1 on $\mathbb{N}$ by

$$\mu_1(M) = \sum_{m\in M} m, \quad M \subset \mathbb{N}.$$

Then we can write

$$\frac{k}{2}\sum_{n=1}^{\infty}\int_{y=0}^{n}\sum_{j=1}^{n-1}\int_{z=0}^{y} n|f_{n-j}(y-z)||f_j(z)|dzdy$$

as

$$\frac{k}{2}\int_{\Omega_1}\int_{\Omega_2}\int_{\Omega_3}\int_{\Omega_4}\mathcal{X}_F(y)\mathcal{X}_G(j)\mathcal{X}_H(z)|f_{n-j}(y-z)||f_j(z)|d\mu_4(z)d\mu_3(j)d\mu_2(y)d\mu_1(n)$$

where $\mathcal{X}_F$, $\mathcal{X}_G$ and $\mathcal{X}_H$ are the characteristic functions of the sets $F = [0, n]$, $G = \{1, 2, 3, \ldots, n-1\}$ and $H = [0, y]$ respectively.

We can now apply the four-space version of Fubini's Theorem. By a simple change of variable we can write $f_{n-j}(y-z)f_j(z)$ as $g_{n,j}(y, z) = f_j(z)f_n(y)$ and this is $(\Omega_1 \times \Omega_2 \times \Omega_3 \times \Omega_4)$-measurable. Also, since one of the iterated integrals, namely,

$$\frac{k}{2}\sum_{j=1}^{\infty}\int_{z=0}^{\infty}\sum_{l=1}^{\infty}\int_{x=0}^{\infty}(j+l)|f_j(z)||f_l(x)|dxdz$$

can be shown to be finite then all of the iterated integrals are finite and equal. Hence, we can justify changing the order of summations/integrals and carry out all of the required analysis to show that the operator K defined by

$$
\begin{aligned}
[Kf]_n(y) &= \frac{1}{2}\sum_{j=1}^{n-1}\int_{z=0}^{y} k_{n-j,j}(y-z,z)u_{n-j}(y-z,t)u_j(z,t)dz \\
&\quad -\sum_{j=1}^{\infty}\int_{z=0}^{\infty} k_{n,j}(y,z)u_n(y,t)u_j(z,t)dz \qquad (5.75)
\end{aligned}
$$

satisfies the conditions of Theorem 2.32. It is possible to show upon pairing these coagulation terms with any of the reformation terms described previously in this chapter, but within the discrete/continuous setting, that there exists a unique non-negative solution to the appropriate ACP.

We have shown that there exists a unique strong, non-negative solution to various versions of the discrete multi-component coagulation equation with reformation terms. We also have a strong solution when the equation is extended to include fragmentation terms. We were then able to verify that if we wished to have diameter as a continuous variable and mass as a discrete variable, we would still be able to carry out our usual semigroup analysis to deduce the existence of a strong solution. In the final chapter we shall return to our regular one-component C-F equation. We shall investigate what restrictions are required on a source term $N(t)$ in order for there to be a unique strong solution to the full equation.

Chapter 6

The Coagulation-Fragmentation Equation with a Time-Dependent Source Term

Thin films are increasingly being used in physical applications such as optical coatings and semiconductor devices. There have been many studies into the way in which thin films grow. Submonolayer growth is an important stage in the development of thin films and is studied in the form of coagulation and fragmentation equations with monomer input in, for example, [2], [3], [4], [31] and [50]. In these studies there is a 'capture zone' around particle clusters and correlations between size of clusters and the corresponding capture zones are taken into account when looking at the long-term behaviour of cluster-size distributions. An investigation into similarity solutions for the Becker-Döring system with a time-dependent input of monomers of power-like type is carried out in [24] and [57]. In [25] the Becker-Döring system is again studied but this time with a constant input of monomers. The long-term behaviour of the Smoluchowski equations is investigated in [27] where the concentration of monomers is to be kept at a constant level by an input source.

In our regular one-component C-F equation we can consider what would happen if we added in a source term (dependent on time) which introduced new particle clusters. If we wished to include a source for each cluster size our C-F equation would now look like

$$
\begin{aligned}
\frac{du_n(t)}{dt} &= -a_n u_n(t) + \sum_{j=n+1}^{\infty} a_j b_{n,j} u_j(t) \\
&\quad + \frac{1}{2}\sum_{j=1}^{n-1} k_{n-j,j} u_{n-j}(t) u_j(t) - \sum_{j=1}^{\infty} k_{n,j} u_n(t) u_j(t) \\
&\quad + [N(t)]_n \,, \qquad (6.1) \\
u_n(0) &= f_n \,, \qquad (n = 1, 2, 3, ...)
\end{aligned}
$$

where $u_n(t)$, a_n, $b_{n,j}$ and $k_{n,j}$ have their usual meanings and $[N(t)]_n$ is non-negative for each n and represents the concentration of n-mers added to the system from some source. We shall assume that our system without the source term is mass-conserving and we shall again impose conditions $(A1) - (A4)$. Note that if we only had a source of monomers we would have

$$
[N(t)]_n = \begin{cases} g(t) & \text{for } n = 1 \\ 0 & \text{otherwise.} \end{cases}
$$

If we transform (6.1) into an ACP in the usual way we have

$$
\begin{aligned}
\frac{du(t)}{dt} &= Au(t) + Bu(t) + Ku(t) + N(t), \qquad (6.2) \\
u(0) &= f
\end{aligned}
$$

where A, B and K are as in Definition 3.2 and (3.33) defined in the Banach space X of Definition 3.1. We also require that $N(t) \in X \ \forall t \geq 0$.

From our analysis in Chapter 3 of the ACP

$$
\begin{aligned}
\frac{du(t)}{dt} &= Au(t) + Bu(t), \qquad (6.3) \\
u(0) &= f
\end{aligned}
$$

we know that $G = \overline{A + B}$ is the generator of a C_0-semigroup, $\{T_G(t)\}_{t \geq 0}$, on X. So now we can look at the ACP

$$\begin{aligned} \frac{du(t)}{dt} &= Gu(t) + Ku(t) + N(t), \qquad (6.4) \\ u(0) &= f. \end{aligned}$$

For convenience we shall rewrite this as

$$\begin{aligned} \frac{du(t)}{dt} &= Gu(t) + F(t, u(t)) \qquad (6.5) \\ u(0) &= f \end{aligned}$$

where $F(t, u(t)) = Ku(t) + N(t)$.

We shall now apply the theory described in Chapter 2 to determine when a solution of (6.5) is a strong solution.

Theorem 6.1. *Let $N : [0, \infty) \to X$ be continuous and non-negative. Then the ACP (6.5) has a unique mild solution $u : [0, t_0) \to X$ for every $f \in X$.*

Proof: We show that the function $F : [0, \infty) \times X \to X$ defined by

$$F(t, u) = Ku + N(t), \qquad (6.6)$$

satisfies the conditions of Theorem 2.28. For $u \in X$ and $t_1, t_2 \geq 0$

$$\begin{aligned} \|F(t_1, u) - F(t_2, u)\| &= \|Ku + N(t_1) - Ku - N(t_2)\| \\ &= \|N(t_1) - N(t_2)\| \\ &\to 0 \text{ as } t_1 \to t_2 \end{aligned}$$

since N is continuous on $[0, \infty)$.

Also, from the proof of Theorem 3.14 we have, for $u, v \in X$,

$$\begin{aligned} \|F(t, u) - F(t, v)\| &= \|Ku + N(t) - Kv - N(t)\| \\ &= \|Ku - Kv\| \\ &\leq 2k\|u - v\|(\|u\| + \|v\|). \end{aligned}$$

Hence,

$$\|F(t,u) - F(t,v)\| \le L(r,t')\|u - v\|, \ \ \forall u, v \in \bar{B}(f,r),$$

where

$$L(r,t') = C(f,r) = 4k(\|f\| + r)$$

as in Chapter 3. The result now follows from Theorem 2.28. □

The next aim is to establish a similar result for a strong solution. This will require imposing a further restriction on N to ensure that the conditions of Theorem 2.29 are satisfied, i.e. that $F(t,u(t))$ is continuously differentiable from $[0,\infty) \times X$ into X. We can now go on to prove the following

Theorem 6.2. *Let $N : [0,\infty) \to X$ be continuously differentiable and non-negative. Then the ACP (6.5) has a unique strong solution $u : [0,t_0) \to B(f,r)$ for each $f \in D(G)$.*

Proof: The assumption that N is differentiable allows us to write

$$\begin{aligned} F(t_0 + t, \phi_0 + \phi) &= K(\phi_0 + \phi) + N(t_0 + t) \\ &= K\phi_0 + \tilde{K}[\phi_0, \phi] + \tilde{K}[\phi, \phi_0] + K\phi \\ &\quad + \underbrace{N(t_0) + N'(t_0)t + tE_{t_0}(t)}_{\text{from [5, p. 346]}} \end{aligned} \tag{6.7}$$

$$\begin{aligned} &= K\phi_0 + \underbrace{K_{\phi_0}(\phi)}_{\text{Fréchet Derivative}} + K\phi \\ &\quad + N(t_0) + N'(t_0)t + tE_{t_0}(t) \\ &= K\phi_0 + N(t_0) + K_{\phi_0}(\phi) + N'(t_0)t + K\phi + tE_{t_0}(t) \\ &= F(t_0, \phi_0) + S_{(t_0,\phi_0)}(t,\phi) + K\phi + tE_{t_0}(t), \end{aligned}$$

where $\tilde{K}$ is as in the proof of Theorem 3.13 and $S_{(t_0,\phi_0)}(t,\phi) = K_{\phi_0}(\phi) + N'(t_0)t$. It is easily checked that $S_{(t_0,\phi_0)}(t,\phi)$ is bilinear in t and ϕ. From its definition $N'(t_0)t$ is a linear function of t. We also have that $K_{\phi_0}(\phi)$ is linear in ϕ by Definition 2.31.

We now need to check the conditions on the error term. From Definition 2.30 we have that

$$\|(t,\phi)\|_{[0,\infty)\times X} E_{(t_0,\phi_0)}(t,\phi) = K\phi + tE(t_0)(t).$$

Thus

$$E_{(t_0,\phi_0)}(t,\phi) = \frac{K\phi + tE(t_0)(t)}{\|(t,\phi)\|} = \frac{K\phi + J_{t_0}(t)}{\|(t,\phi)\|}.$$

where $J_{t_0}(t) = tE(t_0)(t)$. From its definition $J_{t_0}(t) = o(|t|)$ i.e.

$$\frac{\|J_{t_0}(t)\|}{|t|} \to 0 \quad \text{as} \quad |t| \to 0.$$

Hence, given $\frac{\epsilon}{2} > 0$ there exists $\delta > 0$ such that

$$\|J_{t_0}(t)\| < \frac{\epsilon}{2}|t| \quad \text{for } 0 < |t| < \delta.$$

If $\|(t,\phi)\| < \delta$, then $|t| < \delta$ and $\|\phi\| < \delta$ and

$$\begin{aligned} \|K\phi + J_{t_0}(t)\| &\leq \|K\phi\| + \|J_{t_0}(t)\| \\ &\leq 2k\|\phi\|^2 + \frac{\epsilon}{2}|t| \\ &\leq (2k\|\phi\| + \frac{\epsilon}{2})(\|\phi\| + |t|) \end{aligned}$$

by (3.39). Therefore, for $(t,\phi) \neq (0,0)$,

$$\frac{\|K\phi + J_{t_0}(t)\|}{|t| + \|\phi\|} \leq 2k\|\phi\| + \frac{\epsilon}{2} \leq 2k\delta + \frac{\epsilon}{2}.$$

If we choose $\delta < \frac{\epsilon}{4k}$, then

$$\frac{\|K\phi + J_{t_0}(t)\|}{|t| + \|\phi\|} < \epsilon,$$

i.e.

$$\frac{\|K\phi + J_{t_0}(t)\|}{|t| + \|\phi\|} \to 0 \quad \text{as } (t,\phi) \to (0,0)$$

as required.

Since we now know the total derivative $S_{(t_0,\phi_0)}$ of $F(t_0,\phi_0)$ we can check whether this is continuous. If we take $t_0, t_1 \geq 0$ and $\phi_0, \phi_1 \in X$ then

$$\begin{aligned}
& \|S_{(t_0,\phi_0)}(t,\phi) - S_{(t_1,\phi_1)}(t,\phi)\|_{[0,\infty)\times X} \\
\leq\ & \|(N'(t_0) - N'(t_1))t\|_X + \|K_{\phi_0}(\phi) - K_{\phi_1}(\phi)\|_X \\
\leq\ & |t|\|N'(t_0) - N'(t_1)\|_X + \|K_{\phi_0}(\phi) - K_{\phi_1}(\phi)\|_X \\
\to\ & 0 \text{ as } (t_1, \phi_1) \to (t_0, \phi_0) \qquad (6.8)
\end{aligned}$$

since N' is continuous on $[0, \infty)$ and the Fréchet derivative K_ψ is continuous with respect to $\psi \in X$. □

Theorem 6.3. *The unique strong solution $u : [0, t_0) \to B(f, r)$ of ACP (6.5) is non-negative for all $f \in D(G)^+$.*

Proof: The arguments are similar to those found in the proof of Theorem 3.17. □

Finally, to show that this solution exists globally in time we need to show that it does not blow up in finite time. Since we know that the unique solution is strong we have that

$$\begin{aligned}
\frac{d}{dt}\|u(t)\| &= \underbrace{\sum_{n=1}^{\infty} n[Gu(t)]_n + \sum_{n=1}^{\infty} n[Ku(t)]_n}_{=0} + \sum_{n=1}^{\infty} n[N(t)]_n \\
&= \|N(t)\|, \qquad t \geq 0, \qquad (6.9)
\end{aligned}$$

i.e.

$$\begin{aligned}
\|u(t)\| &= \int_0^t \|N(s)\|\, ds \\
&\leq M_t \int_0^t 1\, ds \\
&= M_t t, \qquad (6.10)
\end{aligned}$$

since $\|N(s)\| \leq M_T\ \forall s \in [0, T]$ for some $T \geq 0$ by continuity where M_t, M_T are constants. Thus we do not have finite-time blow-up and our unique strong, non-negative solution exists globally in time.

Note 6.4. *In the specific case of monomer input discussed at the beginning of* *this chapter we have that*

$$N(t) = (g(t), 0, 0, \ldots).$$

It can be shown that if $N(t)$ has a finite number of non-zero components and each of these components is continuously differentiable, then $N(t)$ is continuously differentiable. Thus, in order for the solution to the corresponding solution to be strong we require that $g(t)$ is continuously differentiable.

We have shown that our regular mass-conserving C-F equation with assumptions (A1) - (A4) and an added time-dependent source-term has a unique, non-negative solution provided that N is continuously differentiable. A possible further extension would be to consider a multi-component system, similar to that in Chapter 5, with an added source-term. It is easy to see that we would require the same condition of continuous differentiability for the source-term in order for a unique strong solution to exist. We would also expect similar results if we allowed mass to be lost during fragmentation events.

Chapter 7

Conclusion

In this thesis, semigroup-based techniques for showing existence and uniqueness of strong solutions to the continuous coagulation-fragmentation equation have been extended to the discrete coagulation-fragmentation equation. In Chapters 3-5 the Kato-Voigt theorem has been used along with perturbation results to show that unique, non-negative strong solutions exist for various versions of the equation. In Chapter 3 the system was mass-conserving, in Chapter 4 the system had discrete mass loss built into it and in Chapter 5 we considered a multi-component version of the equation. In Chapter 6 conditions under which a unique strong solution to the coagulation-fragmentation equation with an added time-dependent source term exists have been established. In all cases we imposed minimal restrictions on the fragmentation rate but we required the coagulation rate to be uniformly bounded. The work in Chapters 3 and 4 extends existing results for the continuous model to the discrete model. Explicit solutions to particular cases of the pure fragmentation equation were also investigated. Although we have made some progress in applying semigroup techniques to a variety of coagulation-fragmentation models, there are still possibilities to extend our work futher. In particular, to the best of our knowledge, the semigroup approach has not previously been applied to a continuous version of the multi-component model. Another possible extension, as mentioned previously, would be to add a source term to the multi-component model.

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