On The Möbius Function And Topology Of The Permutation Poset

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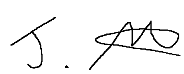
This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

Part II of this thesis has been published here [Smi14a]. Parts III and IV have been submitted to journals and preprints are available at [Smi14b] and [Smi15], respectively. All three papers are a result of the author's original work and are the sole work of the author.

Signed: 

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Abstract

A *permutation* is an ordering of the letters $1, \ldots, n$. A permutation σ *occurs as a pattern* in a permutation π if there is a subsequence of π whose letters appear in the same relative order of size as the letters of σ, such a subsequence is called an *occurrence*. The set of all permutations, ordered by pattern containment, is a poset. In this thesis we study the behaviour of the Möbius function and topology of the permutation poset.

The first major result in this thesis is on the Möbius function of intervals $[1, \pi]$, such that $\pi = \pi_1 \pi_2 \ldots \pi_n$ has exactly one *descent*, where a descent occurs at position i if $\pi_i > \pi_{i+1}$. We show that the Möbius function of these intervals can be computed as a function of the positions and number of *adjacencies*, where an adjacency is a pair of letters in consecutive positions with consecutive increasing values.

We then alter the definition of adjacencies to be a maximal sequence of letters in consecutive positions with consecutive increasing values. An occurrence is *normal* if it includes all letters except (possibly) the first one of each of all the adjacencies. We show that the absolute value of the Möbius function of an interval $[\sigma, \pi]$ of permutations with a fixed number of descents equals the number of normal occurrences of σ in π. Furthermore, we show that these intervals are shellable, which implies many useful topological properties.

Finally, we allow adjacencies to be increasing or decreasing and apply the same definition of normal occurrence. We present a result that shows the Möbius function of any interval of permutations equals the number of normal occurrences plus an extra term. Furthermore, we conjecture that this extra term vanishes for a significant proportion of intervals.

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Part I

Introduction

Chapter 1

Motivation and Overview

1.1 Motivation

The main focus of this thesis is the study of permutation patterns. A *permutation* π is an ordering of the letters $1, \ldots, n$ and σ *occurs as a pattern* in π if there is a subsequence of π whose letters appear in the same relative order of size as the letters of σ. For example, 132 occurs as a pattern in 25143 as the subsequence 254. If a pattern σ does not occur in a permutation π we say that π *avoids* σ. Furthermore, we define the *permutation poset* consisting of all permutations with the partial order $\sigma \leq \pi$ if σ occurs as a pattern in π.

Permutation patterns have been studied implicitly for a long time. For example, in 1768 Leonhard Euler introduced the *Eulerian numbers* $A_{n,k}$ which count the number of length n permutations with k *descents*, where a descent occurs any time a letter is smaller in value than the letter directly preceding it. A descent can be viewed as an occurrence of the pattern 21 appearing in consecutive letters. The modern study of permutation patterns can be traced to Knuth's 1968 work [Knu68], where it was shown that permutations that avoid 231 can be sorted using a stack. This work was expanded on throughout the 1970s and 1980s in papers such as [Knu70], [Rog78], [Rot81] and [SS85]. An excellent review of the field's current state can be found in [Kit11]. And, many open problems, some of which are answered in this thesis, can be found in [Ste13].

The *Möbius function* was first introduced by August Ferdinand Möbius in 1832, in the paper [Möb32], as a function on an integer n giving a value 0 if n has a repeated prime factor and $(-1)^k$ if n is a product of k distinct primes. In 1964, in his seminal paper [Rot64], Gian-Carlo Rota introduced the systematic study of the Möbius function for arbitrary posets. Note that the original definition of the Möbius function of the integer n is equivalent to the Möbius function of the poset of divisors of n. The Möbius function of a poset turns out to have many interesting properties and applications and has been extensively studied since, see [Sta12, Section 3].

One interesting consequence of Rota's paper [Rot64] is the development of poset topology. That is, the study of the topology of simplicial complexes constructed from posets. Poset topology has links to a variety of mathematical fields such as commutative algebra, group theory, representation theory, combinatorics and topology; for an excellent overview of the field see [Wac07].

Due to the wealth of knowledge associated with poset topology and the Möbius function of posets, two questions that are often asked for any given poset are 'What is its topology?' and 'How does its Möbius function behave?'. In this thesis we give some answers to these questions for the permutation poset.

1.2 Outline and List of Results

This thesis is in four parts. The first part is an introduction to the relevant areas and the latter three parts are three independent, but related, papers. Part I is split into the following chapters:

In Chapter 2 we introduce the definitions required to study the permutation poset and define the Möbius function. Moreover, we present some previous results on the Möbius function of intervals of permutations. We also give an overview of the results on the Möbius function presented in this thesis.

In Chapter 3 we introduce order complexes, which are the simplicial complexes we use to study the topology of posets. We then give a brief overview of the topology necessary to study these complexes. Finally, we give a summary of the results presented in this thesis on the topology, and in particular on the Möbius function, of the permutation poset.

Part II contains the published paper "On the Möbius Function of Permutations with One Descent", see [Smi14a]. In this paper we present a formula for the Möbius function of intervals of permutations of the form $[1, \pi]$, such that π has exactly one descent. We prove this formula using an inductive argument on the recursive formula for the Möbius function.

Part III contains the paper "Intervals of Permutations with a Fixed Number of Descents are Shellable", which is to appear in *Discrete Math-*

ematics, see [Smi14b]. In this paper we present a formula for intervals of permutations with a fixed number of descents and show that these intervals are *shellable*, a property with strong topological implications. We prove this by presenting an order isomorphism to posets of words with subword order, which are shown to be shellable, and for which an efficient formula of the Möbius function is presented, in [Bjö90]. This result is used to present a simpler proof of the result in Part II.

Part IV contains the paper "A Formula for the Möbius Function of the Permutation Poset Based on a Topological Decomposition", which has been submitted for publication, see [Smi15]. This paper introduces a formula for the Möbius function of all intervals of the permutation poset. To prove this formula we use topological methods to construct the interval (σ, π) from simpler posets for which the Möbius function can be computed easily.

We now give an overview of the main results in Parts II - IV:

Part II

- If a permutation π contains three consecutive letters with values $x(x+1)(x+2)$, for any x, then $\mu(1, \pi) = 0$ (Lemma 32).
- Complete classification of the permutations that occur as patterns in a permutation with exactly one descent and no adjacencies (Lemma 34).
- Complete classification of $\mu(1, \pi)$ when π has exactly one descent (Theorem 36).
- The Möbius function $\mu(1, \pi)$ is unbounded on the permutation poset (Corollary 46).

Part III

- There is an order preserving bijection between intervals of permutations with a fixed number of descents and certain intervals of words with subword order (Theorem 56).
- Intervals of permutations with a fixed number of descents are shellable (Corollary 57).
- The Möbius function of an interval of permutations with a fixed number of descents equals the number of normal embeddings, with sign determined by the rank of the interval (Proposition 61).
- If a permutation π has exactly one descent, then $\mu(1, \pi) = -\mu(21, \pi)$ (Lemma 64).
- Complete classification of $\mu(1, \pi)$ when π has exactly one descent, with an alternate proof to the one presented in Part II (Theorem 65).
- A formula for the Möbius function of intervals $[\sigma, \pi]$, such that σ and π both have exactly one descent and π has no adjacencies (Proposition 67). This is a proof of Conjecture 47 in Part II.
- If π is a permutation with exactly one descent, then the order complex $\Delta(1, \pi)$ is homotopy equivalent to a suspension of $\Delta(21, \pi)$ (Theorem 69).
- We conjecture that the intervals $[1, \pi]$ are shellable if π has exactly one descent and avoids 456123 and 356124 (Conjecture 71).

- If π has exactly one descent and avoids 456123 and 356124, then the interval $[1, \pi]$ has no disconnected subintervals of rank 3 or more (Proposition 75).

Part IV

- A two term formula for the Möbius function of all intervals of permutations (Theorem 94).
- A result linking the Möbius function of two posets connected by a poset fibration satisfying a certain condition (Proposition 97).
- The expected number of letters in the tails of the adjacencies of a permutation tends to 2 as the length of the permutation tends to infinity (Lemma 99).
- Consider an interval $[\sigma, \pi]$. If for all $\lambda \in [\sigma, \pi)$ there is a letter of π that is not in any occurrence of λ in π, then the Möbius function of $[\sigma, \pi]$ equals the number of normal embeddings, with sign determined by the rank of the interval (Proposition 102).

Chapter 2

The Permutation Poset and its Möbius Function

Permutations have been extensively studied for a long time. This is largely due to their versatility in encoding data from a wide range of origins. While there exists a wealth of knowledge associated with permutations there are still many open questions to be answered.

Permutations can be written in many forms. In this thesis we consider permutations using one-line notation. As such, we formally define a permutation as follows:

Definition 1. *A* permutation π *on the set* $\{\alpha_1, \ldots, \alpha_n\}$ *is a sequence of distinct letters* $\pi_1 \ldots \pi_n$ *such that* $\pi_i \in \{\alpha_1, \ldots, \alpha_n\}$*. We say* π *has* length n *which we denote* $|\pi|$*.*

Example 2. *The sequence* 54316728 *is a permutation on the set* $\{1, \ldots, 8\}$ *of length* 8*.*

Remark 3. *The definition of length given here is not to be confused with the definition of length used in the study of permutations in relation to Coxeter groups.*

In this thesis we largely focus on patterns in permutations. As such, we are interested in the relative differences between the letters of a permutation. Therefore, given a permutation on some set other than $\{1, \ldots, n\}$, we reduce said permutation to its smallest form whilst maintaining the pattern of the permutation, in the following way:

Definition 4. *Let π be a permutation on the set $\{j_1, \ldots, j_n\}$. Define the* reduced form *of π, denoted $red(\pi)$, as the sequence obtained by replacing the i-th smallest letter in π with i, for all $1 \leq i \leq n$.*

Example 5. *The reduced form of $\pi = 25473$ is $red(\pi) = 14352$.*

Throughout this thesis we consider a permutation to be in its reduced form unless otherwise stated. As such, we consider two permutations with the same reduced form to be equivalent.

There are numerous ways of considering patterns in permutations, see [Kit11] for a good overview. We focus on the classical permutation patterns which are defined as follows:

Definition 6. *Consider two permutations σ and $\pi = \pi_1 \ldots \pi_n$. We say there is an* occurrence *of σ in π if there is a subsequence $\pi_{j_1} \ldots \pi_{j_m}$ of π such that the reduced form of the subsequence equals σ, that is, $red(\pi_{j_1} \ldots \pi_{j_m}) = \sigma$.*

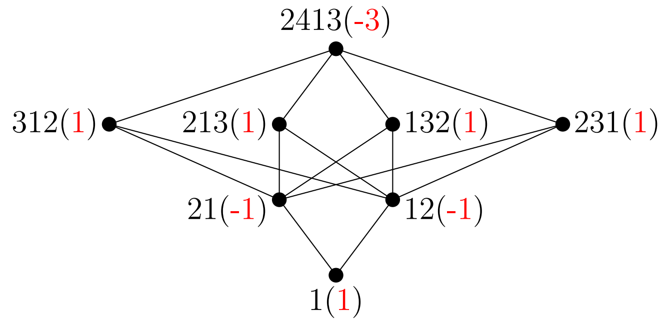


Figure 2.1: The interval [1,2413] with the Möbius function $\mu(1, \pi)$ in red.

We can consider occurrence as a binary relation between permutations. It is quite straightforward to show that this relation is reflexive, antisymmetric and transitive. As such, we can construct a poset of permutations in the following way:

Definition 7. *The* permutation poset*, denoted $\mathcal{P}$, consists of all permutations with the order relation $\sigma \leq \pi$ if there is an occurrence of σ in π. Define an* interval of permutations *$[\sigma, \pi]$ as the induced subposet $\{\lambda \in \mathcal{P} \,|\, \sigma \leq \lambda \leq \pi\} \subset \mathcal{P}$ and the* interior *of $[\sigma, \pi]$ as $(\sigma, \pi) := \{\lambda \in \mathcal{P} \,|\, \sigma < \lambda < \pi\} \subset \mathcal{P}$.*

The permutation poset has a very complicated structure and has been the focus of much study in recent years. Many questions remain open in relation to the permutation poset, such as the behaviour of the *Möbius function* on $\mathcal{P}$, which is defined as follows:

Definition 8. *Given an interval $[\sigma, \pi]$ the* Möbius function *is defined recursively as $\mu(\sigma, \sigma) = 1$ and for all $\lambda \in (\sigma, \pi]$:*

$$\mu(\sigma, \lambda) = -\sum_{\sigma \leq \tau < \lambda} \mu(\sigma, \tau). \tag{2.1}$$

See Figure 2 for an example of an interval of permutations and its Möbius function.

2.1 Previous Results on the Möbius Function of the Permutation Poset

In the past ten years there has been some progress towards understanding the Möbius function of the permutation poset. Efficient formulas have been found for the Möbius function of certain classes of permutations. However, the proportion of intervals for which we have such a formula tends to zero as the *rank* of the interval increases, where the rank of an interval $I = [\sigma, \pi]$ is defined $\mathrm{rk}(I) = |\pi| - |\sigma|$.

One result that we can deduce straight away follows from the trivial bijections, which are defined thus:

Definition 9. *Consider a permutation $\pi = \pi_1 \ldots \pi_n$. The following bijections are the three* trivial bijections*:*

- *The* reverse *of π, denoted $r(\pi)$, is $\pi_n \pi_{n-1} \ldots \pi_1$.*
- *The* complement *of π, denoted $c(\pi)$, is $\pi'_1 \ldots \pi'_n$, where $\pi'_i = n+1-\pi_i$.*
- *The* inverse *of π, denoted $i(\pi)$, is obtained by setting the π_j-th letter of $i(\pi)$ as j.*

Example 10. *Consider the permutation $\pi = 51243$. Applying the trivial bijections to π gives $r(\pi) = 34215$, $c(\pi) = 15423$ and $i(\pi) = 23541$.*

We can also compose trivial bijections, for example $r(c(\pi)) = 32451$ *and* $c(i(\pi)) = 43125$.

It is straightforward to show that the trivial bijections are order isomorphisms, this implies the following result on the Möbius function.

Lemma 11. *Consider two permutations* σ *and* π*. Let* f *be any trivial bijection or composition of trivial bijections, then:*

$$\mu(\sigma, \pi) = \mu(f(\sigma), f(\pi)).$$

In order to simplify the problem of finding an efficient formula for the Möbius function of intervals of $\mathcal{P}$ we can split permutations into smaller parts and consider each part separately. There are many different ways that we can split permutations, one that has given numerous interesting results is splitting a permutation using direct sums.

Definition 12. *The* direct sum $\sigma \oplus \pi$ *of two permutations* σ *and* π *is the concatenation of* σ *with* π'*, where* π' *is obtained by increasing each letter of* π *by* $|\sigma|$*. Similarly, the* skew sum $\sigma \ominus \pi$ *is the concatenation of* σ' *and* π*, where* σ' *is obtained by increasing each letter of* σ *by* $|\pi|$*.*

Example 13. *Consider the permutations* $\sigma = 2413$ *and* $\pi = 12435$*, then* $\sigma \oplus \pi = 241356879$ *and* $\sigma \ominus \pi = 796812435$*.*

Using direct sums, and skew sums, we can split permutations into smaller parts. Moreover, we can consider special permutations which can be written as the direct sum of certain classes of permutations.

Definition 14. *A permutation is* decomposable *if it can be written as the direct sum* $\pi_1 \oplus \cdots \oplus \pi_t$*, for some* $t > 1$*. A* layered *permutation is the direct sum of decreasing permutations. A permutation is* separable *if it can be written using only direct sums, skew sums, parentheses and the singleton permutation* 1*.*

Example 15. *The permutation* 213645 *is decomposable because it can be written* $213 \oplus 312$*, and the permutation* 2413 *is indecomposable. The permutation* 1326547 *is layered as it can be written* $1 \oplus 21 \oplus 321 \oplus 1$*. The permutation* 34125 *is separable because it can be written* $((1 \oplus 1) \ominus (1 \oplus 1)) \oplus 1$*.*

A decomposable permutation can be written uniquely in its finest form $\pi_1 \oplus \cdots \oplus \pi_t$ such that each π_i, which we call a *component* of π, is indecomposable.

The first major result on the Möbius function of $\mathcal{P}$ was presented in [SV06] where a formula for *layered* permutations was given. This formula is based on counting *normal embeddings*. Many of the formulas for the Möbius function of $\mathcal{P}$, and of other posets, use this idea of normal embeddings to give an efficient formula for the Möbius function. However, although they all follow a similar theme, the definition of normal varies in each case. We give our own definition of normal in Section 2.2, but first we define embeddings, which are in one-to-one correspondence with occurrences:

Definition 16. *Consider permutations* $\sigma \leq \pi$*. An* embedding η *of* σ *in* π *is a sequence of the same length as* π *such that the nonzero positions in* η

are the positions of an occurrence of σ in π and the removal of all the zeroes leaves an occurrence of σ.

Example 17. *For $\sigma = 132$ and $\pi = 2314765$ the sequence 0300065 is an embedding of σ in π.*

Remark 18. *In Part II the notation of embeddings is slightly altered such that removal of all the zeroes leaves the reduced form of σ. As we consider two permutations with the same reduced form to be equivalent we can use either definition.*

The next major results on this topic appeared in [BJJS11], where formulas for decomposable and separable permutations are given. The formula for separable permutations is also based on normal embeddings. A simplification of the formula for decomposable permutations is given in [MS15]. Both of the formulas for decomposable permutations are recursive and bottom out in indecomposable permutations.

2.2 Overview of Results in this Thesis on the Möbius Function of the Permutation Poset

Many of the results we present in Parts II - IV are focused on providing an efficient formula for computing the Möbius function of intervals of permutations.

As seen in Section 2.1, most of the results on the Möbius function of intervals of $\mathcal{P}$ are for intervals of decomposable permutations. A formula for the Möbius function of intervals of indecomposable permutations has proven more difficult to develop. In Part II we present the first result that gives an efficient formula for the Möbius function of intervals of a class of permutations that contains a significant proportion of indecomposable permutations. This is the class of permutations with at most one *descent*, where a descent occurs at position i in $\pi = \pi_1 \dots \pi_n$ if $\pi_i > \pi_{i+1}$. In Part II we give a full classification of the Möbius function of intervals $[1, \pi]$ when π has exactly one descent. This provides the first proof that the function $\mu(1, \pi)$ is unbounded on the permutation poset. The proof of this classification is an inductive proof on the length of π. Furthermore, we present a pair of conjectures on the Möbius function of $[\sigma, \pi]$, where either σ or π has no adjacencies and both have exactly one descent. We prove one of these conjectures in Part III, while the other one remains open.

In Part III we prove a more general result on intervals of permutations with a fixed number of descents. The proof of this result is based on an order isomorphism from intervals of permutations to intervals of words with subword order, for which an efficient formula for the Möbius function is presented in [Bjö90]. The formula for words with subword order that is used in Part III is based on counting the number of normal embeddings. We adapt this definition of normal embeddings for permutations using increasing adjacencies, which are defined as follows:

Definition 19. *An* adjacency *in a permutation is a maximal sequence, of length $\ell \geq 1$, of increasing or decreasing consecutively valued letters in* *consecutive positions. The* tail *of an adjacency of length at least* 2 *is all but the first letter of the adjacency. An adjacency of length* 1 *does not have a tail.*

Example 20. *The permutation* $\pi = 2314765$ *has adjacencies* 23*,* 1*,* 4 *and* 765 *and the tails are* 3 *and* 65*.*

In Part III we say an embedding η of σ in π is normal if the positions of all the letters in all the tails of the increasing adjacencies in π are nonzero in η. The result from [Bjö90] implies the Möbius function of an interval of permutations $[\sigma, \pi]$ equals the number of normal embeddings with sign depending only on the rank of the interval. We apply this result to give a much shorter proof of the main theorem, and to prove one of the conjectures, of Part II.

Note that any increasing adjacency in π becomes a decreasing adjacency in the reverse permutation $r(\pi)$. By Lemma 11 we know that the reverse operation preserves the value of the Möbius function, which appears to be intrinsically linked to the notion of normal embeddings. In Part IV we extend the definition of normal to consider both increasing and decreasing adjacencies.

Definition 21. *An embedding* η *of* σ *in* π *is* normal *if the letters in the positions of all the tails of the adjacencies in* π *are nonzero in* η*. We denote the number of normal embeddings of* σ *in* π *as* $NE(\sigma, \pi)$*.*

Example 22. *For* $\sigma = 132$ *and* $\pi = 2314765$ *the sequence* 0300065 *is the only normal embedding of* σ *in* π*, so* $NE(\sigma, \pi) = 1$*.*

Using this definition of normal embeddings computational testing indicates that, up to a sign, the Möbius function of $[\sigma, \pi]$ equals $\mathrm{NE}(\sigma, \pi)$ for 95% of intervals when $|\pi| < 9$. So in Part IV we use this definition of normal embeddings to develop a formula that says the Möbius function equals the number of normal embeddings, with sign given by the rank of the interval, plus an extra term. Furthermore, we conjecture that this extra term vanishes for a significant proportion of intervals. One such case where this second term vanishes is when two permutations have the same number of descents, which follows from Part III.

An interesting open question is how the definition of normal embeddings given here compares to those given in previous papers. Many of the previously considered definitions of normal have extra conditions to deal with decomposable permutations which have equal consecutive components. In Section 6.4 of Part IV we show that such equal consecutive components cause the extra term of our formula to be nonzero.

In terms of further work, the formula given in Part IV is not the final answer in the search to find an efficient formula for the Möbius function of $\mathcal{P}$. However, it gives a good basis for further investigation. If the intervals for which the extra term of the formula vanishes can be classified, we would get a polynomial time formula for a large class of permutations. Moreover, finding an efficient way to compute the extra term would give an efficient formula for the Möbius function of all intervals of the permutation poset.

Chapter 3

The Topology of the Permutation Poset

The other major results in Part III and Part IV are on the topology of the permutation poset. To study the topology of an interval of permutations we represent the interval as a simplicial complex. We can then study the topology of this complex in order to understand the structure of the underlying poset. First we introduce how we can represent these intervals as simplicial complexes.

3.1 Order Complexes

In order to study the topology of a poset we must represent the poset as a topological space, we use simplicial complexes for this purpose. A simplicial complex can be viewed as a collection of vertices, edges, triangles, tetrahedra and higher dimensional analogues, called *simplices*, that are glued together in certain ways and is formally defined as follows:

Definition 23. *An* abstract simplicial complex Δ *on the finite vertex set* V *is a nonempty collection of subsets of* V *such that:*

- $\{v\} \in \Delta$ *for all* $v \in V$,
- *if* $G \in \Delta$ *and* $F \subseteq G$ *then* $F \in \Delta$.

The elements of Δ *are called the* faces *and the maximal faces are called* facets. *The* dimension *of a face* F *is given by* $\dim F := |F| - 1$ *and the dimension of* Δ *is given by* $\dim \Delta := \max_{F \in \Delta} \dim F$.

Any abstract simplicial complex has a geometric representation obtained by embedding the complex in $\mathcal{R}^n$ to get a *geometric simplicial complex*. As such, from now on we simply refer to simplicial complexes or sometimes just complexes.

We can use simplicial complexes to represent a poset in the following way:

Definition 24. *A* chain *in a poset* P *is a totally ordered subset of* P. *The* order complex *of* P, *denoted* $\Delta(P)$, *contains the elements of* P *as vertices and any subset of vertices is a face of* $\Delta(P)$ *if the corresponding subset in* P *is a chain.*

See Figure 3.1 for an example of an order complex of a poset of permutations.

To allow us to easily apply topological methods to study posets, we refer to a poset and its order complex interchangeably. When considering a

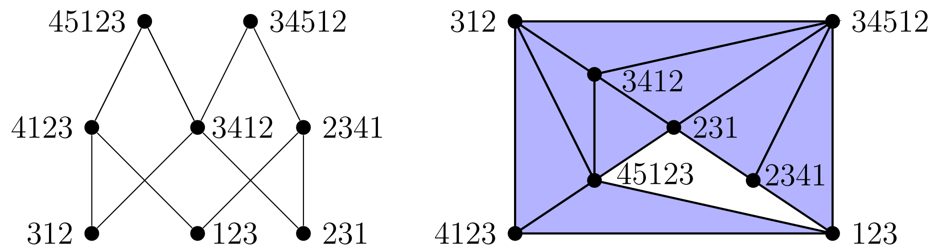


Figure 3.1: Left: Hasse diagram of $(12, 456123)$. Right: The order complex $\Delta(12, 456123)$.

poset it is often useful to have a top and bottom element. For example, this is necessary to consider the Möbius function. However, when considering the topology of a poset P we generally are only interested in the *interior*, that is, the poset without the top and bottom element which we denote P^o. Therefore, when considering the order complex of an interval $[\sigma, \pi]$ we actually consider the interior (σ, π) and use the notation $\Delta(\sigma, \pi)$. Moreover, when considering a poset P we use $\widehat{P}$ to denote the poset obtained by adding a top and bottom element to P.

Some properties of posets and their order complexes coincide nicely. For example, the *length* of a chain equals its number of elements minus one and the *rank* of a poset P, denoted $\mathrm{rk}(P)$, is the length of the longest chain. Therefore, the rank of a poset equals the dimension of the order complex of the poset. A simplicial complex is *pure* if all the facets have the same dimension and similarly a poset is *pure* if all the maximal chains have the same length. Therefore, it is clear that a pure poset has a pure order complex. We focus on order complexes of intervals of permutations which are always pure, so we assume throughout that a complex is pure unless otherwise stated. In many cases nonpure versions of the definitions and results exist, see [BW96] and [BW97].

Many other combinatorial objects can also be represented as simplicial complexes in a variety of ways. The most common objects to be represented in this way are graphs, for a thorough overview of this topic see [Jon08] and [Koz08, Chapter 9].

Now that we have a way to represent posets topologically we can study them with tools from topology. Homology theory is a very important field in the study of topology, however the topic lies outside of the scope of this thesis, so we refer the reader to [Hat02] for the necessary background. The most important topological invariants that we consider are the reduced Betti numbers and the reduced Euler characteristic which are defined thus:

Definition 25. *The* reduced Euler characteristic *of a simplicial complex* Δ *is defined as:*

$$\tilde{\chi}(\Delta) := \sum_{i=-1}^{\dim \Delta} (-1)^i \tilde{f}_i(\Delta), \tag{3.1}$$

where

$$\tilde{f}_i(\Delta) := \begin{cases} \text{the number of } i\text{-dimensional faces of } \Delta, & \text{if } i \neq 0 \\ 0, & \text{if } i = 0 \end{cases}.$$

The n*-th* reduced Betti number $\tilde{\beta}_n(\Delta)$ *is the rank of the* n*-th reduced homology group of* Δ*.*

The following useful result shows that the reduced Euler characteristic and the reduced Betti numbers are intrinsically linked:

Theorem 26. *(Euler-Poincaré formula)*

$$\tilde{\chi}(\Delta) = \sum_{i=-1}^{\dim \Delta} (-1)^i \tilde{\beta}_i(\Delta). \tag{3.2}$$

The field of poset topology is an extremely interesting field and, although we use many tools from it, we cannot fully cover the topic in this thesis and instead refer the reader to [Wac07] for a comprehensive overview.

3.2 The Topology of Order Complexes

The topology of a poset and its Möbius function are fundamentally linked, as can be seen in the following important proposition.

Proposition 27. *(Philip Hall Theorem) For any poset P,*

$$\mu(\widehat{P}) = \tilde{\chi}(\Delta(P)). \tag{3.3}$$

The Euler characteristic has been extensively studied and is well behaved with respect to many operations on simplicial complexes. Therefore, we can use tools from topology to compute the Möbius function of the underlying poset, for example:

Corollary 28. *If two posets P and Q have homotopy equivalent order complexes then $\mu(\widehat{P}) = \mu(\widehat{Q})$.*

If a complex satisfies certain conditions it can allow for easier computation of its topology. One such property is that of shellability, which was first implicitly consider in [Sch01] and formally introduced in [MB71].

Definition 29. *A simplicial complex* Δ *is* shellable *if the facets of* Δ *can be arranged in a linear order* $F_1, \ldots, F_t$ *such that the subcomplex*

$$\left(\bigcup_{i=1}^{k-1} \langle F_i \rangle\right) \cap \langle F_k \rangle,$$

where $\langle F \rangle := \{G : G \subseteq F\}$, *is pure and* $(\dim F_k - 1)$*-dimensional for all* $k = 2, \ldots, t$. *If an ordering of the facets satisfies this condition the ordering is called a* shelling *of* Δ.

See Figure 3.2 for an example of a shelling of a complex. Being shellable implies a complex has many nice properties, see [Wac07, Chapter 3]. One such property that is very useful for computing the topology is:

Theorem 30. *[Koz08, Theorem 12.3] A shellable simplicial complex* Δ *has the homotopy type of a wedge of top dimensional spheres.*

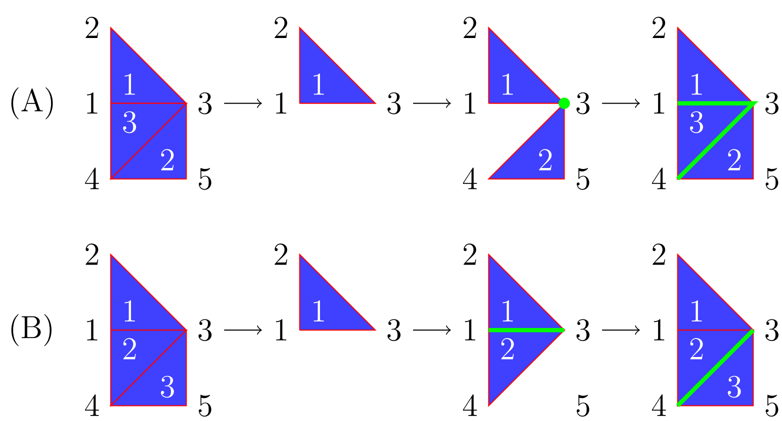


Figure 3.2: A complex with two orderings of its facets. Ordering (A) is not a shelling because the intersection of facets 1 and 2 does not have dimension 1. However, (B) is a shelling, therefore the complex is shellable.

When considering order complexes of posets we can combine Theorem 30 with Proposition 27 to get the following corollary:

Corollary 31. *For a poset P, if $\Delta(P)$ is shellable then it has the homotopy type of a wedge of $\mu(\widehat{P})$ spheres, all of dimension rk(P).*

3.3 Results on the Topology of the Permutation Poset

The first explicit results on the topology of the permutation poset appear in [MS15], where it is shown that almost all intervals are non-shellable. There are still large classes of intervals of permutations which are shellable. Furthermore, it is shown that any interval of layered permutations is shellable if and only if it has no disconnected subinterval of rank 3 or more.

In Part III we prove that an interval of permutations that all have the same number of descents is shellable. We then go on to conjecture that intervals $[1, \pi]$, where π has exactly one descent, are shellable if π avoids 456123 and 356124. We support this conjecture with a proof that such intervals do not contain any disconnected subintervals of rank at least 3, a known obstruction to shellability.

In Part IV we use topological tools to present a formula for the Möbius function of $[\sigma, \pi]$. To do this we construct a poset $A^{\sigma,\pi}$ which contains (σ, π) as a subposet. The poset $A^{\sigma,\pi}$ is constructed by taking the union of a collection of chains, so we can easily compute $\mu(A^{\sigma,\pi})$ using the inclusion-exclusion formula. We then retract $A^{\sigma,\pi}$ onto (σ, π) and record the change to the Möbius function caused by this retraction, thus obtaining a formula for $\mu(\sigma, \pi)$.

In terms of further work, many open questions remain in relation to the topology of the permutation poset. For example, intervals of certain classes of permutations are known to be shellable but it is unknown in general which intervals are shellable.

Part II

On the Möbius Function of Permutations With One Descent

Abstract

The set of all permutations, ordered by pattern containment, is a poset. We give a formula for the Möbius function of intervals $[1, \pi]$ in this poset, for any permutation π with at most one descent. We compute the Möbius function as a function of the number and positions of pairs of consecutive letters in π that are consecutive in value. As a result of this we show that the Möbius function is unbounded on the poset of all permutations. We show that the Möbius function is zero on any interval $[1, \pi]$ where π has a triple of consecutive letters whose values are consecutive and monotone. We also conjecture values of the Möbius function on some other intervals of permutations with at most one descent.

4.1 Introduction

Let σ and π be permutations of positive integers. We define an occurrence of σ as a *pattern* in π to be a subsequence of π with the same relative order of size as the letters in σ. For example, if $\sigma = 213$ and $\pi = 23514$ then there are two occurrences of σ in π as the subsequences 214 and 314. A permutation π is said to *avoid* a pattern σ if there are no occurrences of σ in π. The set of all permutations forms a poset $\mathcal{P}$, with a partial ordering defined as $\sigma \leq \pi$ if σ occurs as a pattern in π. An *interval* $[\sigma, \pi]$ in $\mathcal{P}$ is a subposet consisting of all permutations $z \in \mathcal{P}$ with $\sigma \leq z \leq \pi$. The Möbius function is defined recursively as follows: $\mu(\sigma, \lambda) = 0$ if $\sigma \not\leq \lambda$, $\mu(\sigma, \sigma) = 1$ for all σ and for $\sigma < \lambda$:

$$\mu(\sigma, \lambda) = -\sum_{\sigma \leq z < \lambda} \mu(\sigma, z).$$

We frequently use the term *Möbius value* of a permutation λ to refer to $\mu(1, \lambda)$ and we refer to permutations with a nonzero Möbius value as *nonzero* permutations. A *descent* in a permutation $\pi = \pi_1\pi_2 \ldots \pi_n$ is a decrease in the value of consecutive letters, that is, a descent at position i is where $\pi_i > \pi_{i+1}$.

Formulas for the Möbius function in this poset in certain special cases have been proved. Almost all such results so far are on permutations constructed using *direct sums*, where the direct sum of two permutations α and β, denoted $\alpha \oplus \beta$, is the concatenation of α with β', where β' is the permutation β with each letter increased in value by the length of α. For example, $213 \oplus 2413 = 2135746$. The first such result was by Sagan and

Vatter in [SV06], where they give a formula for the Möbius function on the poset of *layered* permutations, that is, permutations that can be written as the direct sum of a number of decreasing permutations. More general results are presented in [BJJS11] where a formula is given for the Möbius function of all *separable* permutations, that is, permutations avoiding both 3142 and 2413, along with many results for *decomposable* permutations, that is, permutations that can be written non-trivially as direct sums. It is also shown that the absolute value of the Möbius function has an upper bound in some of these cases.

In this paper we present some of the first results for the Möbius function on a substantial class of indecomposable permutations, the only other such result seems to be in [ST10], which gives certain cases in which the Möbius function is zero. As a result of this we show that $\mu(1, \pi)$ is unbounded, which does not seem to have been established before. Our main result is a formula for the Möbius function on the interval $[1, \pi]$ for any permutation π with at most one descent and that on such intervals the Möbius function is alternating in sign. Note that a permutation of length n with one descent is indecomposable unless it starts with 1 or ends with n, so our result applies to a substantial class of indecomposable permutations.

Define the subposet $\mathcal{P}_k \subseteq \mathcal{P}$ to be the poset containing permutations with exactly k descents. In this paper we mainly treat permutations from the subposets $\mathcal{P}_0$ and $\mathcal{P}_1$. We also use the notation $\mathcal{P}_k^n$ for the set of permutations of length n with exactly k descents and $\mathcal{P}^n$ for the set of all permutations of length n. As we often treat the Möbius function of the intervals $[1, \pi]$ we consider the Möbius function as a function of a single variable in the form of $\mu(\pi) := \mu(1, \pi)$. An *adjacency* in a permutation is two letters that are consecutive in position and have consecutive increasing values. For example, 24578136 has adjacencies 45 and 78, at positions 2 and 4. A permutation can also have a triple adjacency, or an adjacency of even greater length, as in the permutation 12456837 where there is the triple adjacency 456. The number of and positions of the adjacencies in a permutation will be key to our results. An important type of permutations from $\mathcal{P}_1$ are the permutations without adjacencies, which are the permutations where odd and even letters are separated from each other. We denote the even length permutations without adjacencies as $M_n = 246\ldots(2n)135\ldots(2n-1)$ and $W_n = 135\ldots(2n-1)246\ldots(2n)$ for $n > 1$.

In Section 4.2 we prove that $\mu(\pi) = 0$ for any permutation π containing a triple adjacency. In Section 4.3 we prove a result relating to permutations with no adjacencies that is useful in the proof in Section 4.4. In Section 4.4 we completely classify the Möbius function on the intervals $[1, \pi]$ where π has at most one descent. This proves the conjecture made in [Ste13] that $\mu(\pi) = \binom{n+1}{2}$ when π is of the form $246\ldots(2n)135\ldots(2n-1)$, which is the permutation without adjacencies M_n. This shows that $\mu(\pi)$ is unbounded in general, answering a question asked in [BJJS11], where it was shown that $|\mu(\pi)| \leq 1$ for all separable permutations π. In Section 4.5 we present additional conjectures we have not been able to prove on the Möbius function of permutations with at most one descent.

4.2 The Möbius Function on Permutations with a Triple Adjacency

In this section we present and prove a lemma stating that a permutation with a triple adjacency has a Möbius value of zero. While interesting in its own right, it is useful in proving the result in Section 4.4. But first we consider some notation and important points about adjacencies.

We defined an adjacency in the introduction as two letters that are consecutive in position and have consecutive increasing values. There is an analogous decreasing adjacency but we consider adjacencies to be increasing unless otherwise stated, because in $\mathcal{P}_1$ decreasing adjacencies are rare and do not play a role in our considerations. We denote the value of an adjacency by the value of its initial letter, so in the permutation 24578136 the adjacencies 45 and 78 have values 4 and 7. Notice that a triple adjacency consists of two adjacencies of two letters, for example we can split 456 into 45 and 56. When counting the adjacencies in a permutation we count adjacencies of two letters, therefore 12456837 has three adjacencies 12, 45 and 56.

Lemma 32. *If a permutation π contains a triple adjacency then $\mu(\pi) = 0$.*

Proof. We can easily check that $\mu(123) = 0$. Now assume that the claim holds for any permutation of length $m < n$ where $m \geq 3$. Given a permutation $\pi \in \mathcal{P}^n$ with a triple adjacency, removing any of the letters of the triple adjacency from π results in the same permutation, call this σ. Any permutation obtained from π by removing any of the letters not in the triple adjacency still has a triple adjacency hence by the inductive hypothesis has a zero Möbius value. Hence all nonzero permutations in $[1, \pi)$ occur in $[1, \sigma]$, implying:

$$\mu(\pi) = -\sum_{1 \leq z < \pi} \mu(1, z) = -\sum_{1 \leq z \leq \sigma} \mu(1, z) = 0.$$

□

The result in Lemma 32 also holds for the case of decreasing triple adjacencies, with an analogous proof. We can slightly generalise this result to give the following corollary, whose proof is analogous to the proof of Lemma 32 after suitably modifying the base case:

Corollary 33. *If a permutation π contains an adjacency (increasing or decreasing) of length $k \geq 3$, then $\mu(12 \ldots (k-2), \pi) = 0$.*

4.3 The Permutations with One Descent and No Adjacencies

We present a result on permutations with no adjacencies that is useful in proving the results in Section 4.4. Before stating the lemma, we introduce some notation and definitions along with a few remarks about the posets $\mathcal{P}_0$ and $\mathcal{P}_1$.

We say that two permutations are *related* if both or neither permutation begins with 1. For example the permutations 246135 and 2357146 are related as neither begins with 1 but 246135 and 135246 are not related.

Let the increasing permutation $12\dots k$ be denoted $\mathbf{k}$. Notice that the poset $\mathcal{P}_0$ forms a chain, as for any $k \geq 1$ the only length k permutation without a descent is the increasing permutation $\mathbf{k}$. As $\mathcal{P}_0$ is a chain it is easy to see that $\mu(\mathbf{k}) = 0$ for any $k > 2$.

It is also important to note that a permutation with k descents cannot contain, as a pattern, a permutation with more than k descents. Therefore, in any interval $[1, \pi]$, with $\pi \in \mathcal{P}_1$, any permutation $\lambda \in [\mathbf{1}, \pi]$ must be in $\mathcal{P}_0 \cup \mathcal{P}_1$. That is, the set $\mathcal{P}_0 \cup \mathcal{P}_1$ is an order ideal in $\mathcal{P}$, also called a *permutation class*. A permutation class can be uniquely determined by its *basis*, that is, the set of minimal permutations it avoids. The basis for $\mathcal{P}_0 \cup \mathcal{P}_1$ can be shown to be $\{321, 2143, 3142\}$. We remark that the poset of permutations with at most k descents, for any fixed k, is a permutation class, but the basis for the general case $k > 1$ is much more difficult to find. A formula is given in [BF13, Theorem 4.2] which can be used to calculate the size of such a basis but this formula is rather complicated.

Recall that the permutations without adjacencies are the permutations where the odd and even letters are separated from each other. The even length permutations without adjacencies are $M_n = 246\dots(2n)135\dots(2n-1)$ and $W_n = 135\dots(2n-1)246\dots(2n)$ for $n > 1$.

Lemma 34. *Let $\pi \in \mathcal{P}_1^n$ be a permutation with no adjacencies. Then π contains, as patterns, precisely all permutations in $\mathcal{P}_1$ of length less than n with at most two adjacencies except the following:*

1. *The permutations of length $n-1$ with two adjacencies.*

2. *The permutations of length $n-1$ with one adjacency that are not*

related to π.

3. *The permutations of length $n-2$ with two adjacencies that are not related to π.*

Proof. Let R and N be the subposets of $\mathcal{P}_1$ which contain the permutations, of length $m < n$, that are related and not related to π, respectively. Also denote the subposets of R and N with exactly k adjacencies as R^k and N^k, respectively. We need to prove that π contains all permutations $\sigma \in R^0 \cup N^0 \cup R^1$, all permutations $\sigma \in R^2 \cup N^1$ of length $m < n-1$ and all permutations $\sigma \in N^2$ of length $m < n-2$.

First consider the permutations in R. Note that R^0 is a chain and any permutation in R^0, of length $m < n$, can be obtained by removing the $n-m$ largest letters of π. To obtain a permutation $\sigma \in R^1$, of length $m < n$, where the adjacency has value i, it is necessary and sufficient to remove the letter $i+1$ from π and then to adjust to the correct length permutation by removing the $n-m-1$ largest letters. So to create any permutation in $R^0 \cup R^1$ there is only one letter that must be removed and thus all permutations in $R^0 \cup R^1$ of length $m \leq n-1$ can be obtained from π. Now consider a permutation $\tau \in R^2$, of length $m < n$, with adjacencies of value i and k. To create such a permutation, from π, we remove the letters of value $i+1$ and $k+1$ and then we adjust the length by removing the $n-m-2$ largest letters. So the permutations in R^2 require at least two letters to be removed and therefore all the permutations in R^2 of length $m \leq n-2$ can be obtained, but none of length $n-1$.

Now consider the permutations in N. Removing the letter 1 from π creates a unique length $n-1$ permutation λ which is in N^0. We can then apply the same argument as in the previous paragraph to λ instead of π. Hence we can obtain all permutations in $N^0 \cup N^1$ of length $m \leq n-2$ and all permutations in N^2 of length $m \leq n-3$. As λ is the only permutation of length $n-1$ in N^0 we can get all permutations in N^0 of length $m \leq n-1$.

Therefore we can obtain all permutations with at most two adjacencies in $R \cup N$ except for the following: The permutations in R^2 of length $n-1$, the permutations in N^2 of lengths $n-1$ and $n-2$ and the permutations in N^1 of length $n-1$. □

We provide an example of Lemma 34:

Example 35. *Consider the permutation* $\pi = 135246$. *By Lemma 34 we know the only permutations with at most two adjacencies not contained in* π *are:*

1. *The permutations of length* 5 *with two adjacencies, that is:* 12354, 41235, 12534, 34125, 12453, 31245, 15234, 23415, 14523, 23145, 13452 *and* 21345.
2. *The permutations of length* 5 *with one adjacency that are not related to* π, *that is:* 35124, 23514, 25134 *and* 24513.
3. *The permutations of length* 4 *with two adjacencies that are not related to* π, *that is:* 4123, 3412 *and* 2341.

4.4 The Möbius Function for Permutations with One Descent

In this section we present a formula for the Möbius function on the interval $[\mathbf{1}, \pi]$ where π is any permutation with at most one descent.

Theorem 36. *Given a permutation $\pi \in \mathcal{P}_0 \cup \mathcal{P}_1$, of length $n > 2$, the value of $\mu(\pi)$ can be computed from the number and positions of adjacencies in π, as follows:*

1. *If π begins with 12 or ends in $(n-1)n$ then $\mu(\pi) = 0$.*
2. *If π has a triple adjacency then $\mu(\pi) = 0$.*
3. *If π has more than two adjacencies then $\mu(\pi) = 0$.*
4. *If π has exactly two adjacencies then:*
   a) *If the first adjacency has greater value than the second then $\mu(\pi) = \pm 1$,*
   b) *If the first adjacency has lower value than the second then $\mu(\pi) = 0$.*
5. *If π has exactly one adjacency, at position $i \in \{1, \ldots, n-1\}$, and the descent is at position d, then: (see item 7 for calculating the sign)*
   a) *If $i < d$ and $\pi_1 \neq 1$ then $\mu(\pi) = \pm i$,*
   b) *If $i < d$ and $\pi_1 = 1$ then $\mu(\pi) = \pm(i-1)$,*
   c) *If $i > d$ and $\pi_n \neq n$ then $\mu(\pi) = \pm(n-i)$,*

d) If $i > d$ and $\pi_n = n$ then $\mu(\pi) = \pm(n - i - 1)$.

6. *If π has no adjacencies then:*

   a) If n is even and $\pi_1 = 1$, that is $\pi = W_{\frac{n}{2}}$, then $\mu(\pi) = -\binom{\frac{n}{2}}{2}$,

   b) If n is even and $\pi_1 = 2$, that is $\pi = M_{\frac{n}{2}}$, then $\mu(\pi) = -\binom{\frac{n}{2}+1}{2}$,

   c) If n is odd then $\mu(\pi) = \binom{\frac{n+1}{2}}{2}$.

7. *If $\mu(\pi) \neq 0$ then $\mu(\pi)$ is positive if and only if n is odd.*

Before proving Theorem 36 we make some remarks:

- Each permutation with one descent falls into at least one of the above classes.

- The above result agrees on permutations covered by more than one class. These cases are:

  - A permutation with one adjacency and beginning with 12 or ending with $(n-1)n$ has zero Möbius value by both part 1 and part 5.

  - A permutation with 12 at the beginning and $(n-1)n$ at the end has zero Möbius value by both part 1 and part 4b.

  - A triple adjacency can be treated as two consecutive adjacencies and the result states the Möbius value is zero by part 2 and part 4b.

– It is possible for a permutation to fall into all three of the first cases, such as 12354, and such a permutation has zero Möbius value according to all three cases.

For part 1 of Theorem 36, a permutation that begins with 12 or ends with $(n-1)n$ is decomposable so the proof follows directly from Corollary 3 in [BJJS11] and part 2 follows from Lemma 32.

We prove the remaining parts of Theorem 36 using an inductive argument throughout the following subsections. For a base case we need to consider all permutations of length $3 \leq n \leq 6$. We know certain permutations have zero Möbius value by the already proven parts 1 and 2 of Theorem 36. So we can leave such permutations. We now list the remaining permutations of length $3 \leq n \leq 6$ with one descent along with their calculated Möbius value and which case of Theorem 36 they fall into: $\mu(34125) = \mu(14523) = 1(4a)$, $\mu(3412) = \mu(145236) = \mu(256134) = \mu(346125) = \mu(356124) = -1(4a)$, $\mu(235614) = \mu(236145) = \mu(361245) = 0(4b)$, $\mu(231) = \mu(312) = \mu(13425) = \mu(14235) = \mu(23514) = \mu(25134) = 1(5)$, $\mu(1423) = \mu(3124) = \mu(1342) = \mu(2314) = \mu(134625) = \mu(136245) = \mu(235146) = \mu(251346) = -1(5)$, $\mu(24513) = \mu(35124) = 2(5)$, $\mu(245136) = \mu(351246) = \mu(146235) = \mu(135624) = -2(5)$, $\mu(132) = \mu(213) = 1(6)$, $\mu(1324) = \mu(2413) = -1(6)$, $\mu(13524) = \mu(24135) = 3(6)$, $\mu(135246) = -3(6a)$, $\mu(246135) = -6(6b)$.

It is straightforward to check that these results agree with Theorem 36.

The reason it is necessary to check the base case up to length $n = 6$ is so we can use Lemma 39 below to cancel out the Möbius values of sets of permutations in the intervals we consider.

From now on we assume that any permutation in $\mathcal{P}_0 \cup \mathcal{P}_1$, of length less than n, where $n > 6$, satisfies the claims in Theorem 36 and we prove that Theorem 36 then holds for permutations with at most one descent of length n and thus for any length. When referencing the induction hypothesis we add the part of Theorem 36 being referenced in brackets, for example (36.6a) for Theorem 36 part 6a.

By our inductive hypothesis (36.3) we can see that any nonzero permutation of length $m < n$ can have at most two adjacencies. If we combine this with Lemma 34 we see that a permutation of length $m < n$ with no adjacencies contains all nonzero permutations of length at most $m - 3$.

4.4.1 The Structure of the Proof

The remaining parts of the proof of Theorem 36 all follow a similar schema, which is outlined as follows:

1. Consider $\pi \in \mathcal{P}_1^n$.
2. Remove one letter from each adjacency in π or the largest letter if π has no adjacencies. This leaves a permutation λ with no adjacencies.
3. By the definition of the Möbius function, $\sum_{\sigma \in [\mathbf{1}, \lambda]} \mu(\sigma) = 0$.
4. Now we can compute $\mu(\pi)$ using $\mu(\pi) = -\sum_{\substack{\sigma < \pi \\ \sigma \not\leq \lambda}} \mu(\sigma)$.

We develop this schema in detail for the proof of Proposition 40 in the following subsection and then, as they are quite similar, the remaining parts of the proof are done in less detail.

We present two lemmas which we frequently reference throughout the proof:

Lemma 37. *Let $\sigma \in \mathcal{P}_1^m$, where $2 < m < n$, be a nonzero permutation satisfying either one of the following conditions:*

1. *Has exactly one adjacency, which is neither* 12 *nor* $(m-1)m$.
2. *Has exactly two adjacencies at least one of which is neither* 12 *nor* $(m-1)m$.

Then σ contains a length $m-1$ permutation λ with the same number of adjacencies as σ such that $\mu(\lambda) + \mu(\sigma) = 0$.

Proof. If σ has exactly one adjacency, at location i, then either this adjacency is before or after the descent. If the adjacency is before the descent, then by the induction hypothesis (36.5) the Möbius value of σ is a function of i. We know m must be to the right of i so removing m creates a length $m-1$ permutation λ with exactly one adjacency at location i, so by the induction hypothesis (36.5 and 36.7) $\mu(\sigma) = -\mu(\lambda)$. If the adjacency is after the descent then removing the letter 1 gives an analogous argument. This completes the first case.

If σ has exactly two adjacencies we can remove either the letter 1 or m which gives a length $m-1$ permutation λ which has two adjacencies of the same relative sizes as the adjacencies in σ. By the induction hypothesis (36.7) the sign of the Möbius function is alternating, therefore $\mu(\lambda) = -\mu(\sigma)$. □

Example 38. *1. Consider the permutation* 13425 *which is of the first form in Lemma 37. Removing the letter* 5 *gives the permutation* 1342*. We compute the Möbius values of these permutations as* $\mu(13425) = 1$ *and* $\mu(1342) = -1$.

2. Consider the permutation 24781356 *which is of the second form in Lemma 37. Removing the letter* 1 *gives the permutation* 1367245*. We compute the Möbius values of these permutations as* $\mu(24781356) = -1$ *and* $\mu(1367245) = 1$.

We can use Lemma 37 to show that the Möbius values of certain sets of permutations sum to zero.

Lemma 39. *Take a set* Δ^1 *of* k *nonzero permutations from* $\mathcal{P}_1^m$*, where* $4 < m < n$*, all with* $t > 0$ *adjacencies and where none of the adjacencies is* 12 *or* $(m-1)m$*. Then we can construct the following sets:*

- *A set* Δ^2 *of* $2k$ *permutations from* $\mathcal{P}_1^{m-1}$ *with exactly* t *adjacencies.*
- *A set* Δ^3 *of* k *permutations from* $\mathcal{P}_1^{m-2}$ *with exactly* t *adjacencies.*

Also the sum of the Möbius values of all the permutations in $\Delta^1 \cup \Delta^2 \cup \Delta^3$ *is zero.*

Proof. From each permutation in Δ^1 we have two options: We can remove the letter 1 or the letter m. Assume first that the removal of either of these letters from any of the permutations does not remove the descent from the permutation, then it is easy to see that this does not create or remove an adjacency. So to create Δ^2 we get two permutations for each permutation in Δ^1 by removing either 1 or m. To create Δ^3 we remove, from each permutation in Δ^1, both 1 and m. It is easy to see that, as we are only removing the letters 1 and m, the permutations in the union of Δ^2 and Δ^3 are distinct. This concludes the proof of the first part of the lemma.

To show that the Möbius values sum to zero we can apply Lemma 37. Recall that a permutation with $t > 2$ adjacencies has Möbius value zero by the induction hypothesis (36.3). As Δ^1 only contains nonzero permutations any permutation $\lambda \in \Delta^1$ must be of one of the forms in Lemma 37. First suppose it is of the first form, that is, it has one adjacency, and suppose this adjacency is before the descent. We can pair λ with the permutation obtained by removing the letter m from λ and their Möbius values sum to zero. Then, given the permutation $\lambda^1 \in \Delta^2$ obtained by removing the letter 1 from λ, we can pair this with the permutation $\lambda^{1,m} \in \Delta^3$ obtained by removing 1 and m from λ. By Lemma 37 we know the Möbius values of these two permutations sum to zero. We can do this for each permutation in Δ^1, which completes this case. An analogous argument applies to the case where λ has an adjacency after the descent or has two adjacencies.

If the removal of the letter 1 or m results in the removal of the descent from one of the permutations then we apply an analogous argument to entire proof above. In this argument we must account for the fact that for each permutation of this form there are two permutations that are increasing permutations of length greater than 2. As such, these permutations contain a triple adjacency and will have zero Möbius value and the Möbius value of the remaining permutations cancel as above. □

4.4.2 Theorem 36 Part 3

Recall that we are assuming Theorem 36 is true for any permutations of length $m < n$. We now consider part 3 of Theorem 36 for permutations of length n.

Proposition 40. *A permutation* $\pi \in \mathcal{P}_1^n$ *with more than two adjacencies has* $\mu(\pi) = 0$.

Proof. Suppose π has $k > 2$ adjacencies also suppose none of the adjacencies are 12 or $(n-1)n$ and there are no triple adjacencies. Then, by the inductive hypothesis (36.3), π contains no nonzero permutations of length greater than $n - k + 2$. There is a unique length $n - k$ permutation λ contained in π with no adjacencies. Let us ignore λ along with any other permutation in $[\mathbf{1}, \lambda]$, since their contributions to the Möbius value of π sum to zero. Then we can use Lemma 34 to consider the remaining permutations, that are possibly nonzero, occurring in π:

- Of length $n - k + 2$ there remain $s = \binom{k}{2}$ permutations with two adjacencies, call these $\Gamma^0 = \{\gamma_1^0, \ldots, \gamma_s^0\}$, where each of the γ_i^0's is obtained by removing a letter from all but two of the adjacencies in π.
- Of length $n - k + 1$ there remain:
  - k nonzero permutations with one adjacency, call these $\Delta^0 = \{\delta_1^0, \ldots, \delta_k^0\}$.

- $2\binom{k}{2}$ permutations with two adjacencies obtained by removing the letter 1 or the largest letter from each of the γ_i^0's, call these $\Gamma^1 = \{\gamma_1^1, \ldots, \gamma_{2s}^1\}$.

- Of length $n-k$ there remain:
  - All the permutations related to π that have two adjacencies, where at least one of the adjacencies is an original adjacency in π, call this set of permutations Ω^0.
  - $\binom{k}{2}$ permutations not related to π that have two adjacencies, both occurring in π, call these $\Gamma^2 = \{\gamma_1^2, \ldots, \gamma_s^2\}$.
  - $2k$ permutations with one adjacency obtained by removing the letter 1 or the largest letter from the δ_i^0's, call these $\Delta^1 = \{\delta_1^1, \ldots, \delta_{2k}^1\}$.

- Of length $n-k-1$ there remain:
  - All permutations with two adjacencies, where at least one of the adjacencies is an original adjacency in π, call this set of permutations Ω^1.
  - k permutations with one adjacency obtained by removing the letter 1 and the largest letter from each δ_i^0, call these $\Delta^2 = \{\delta_1^2, \ldots, \delta_k^2\}$.

- Of length $n-k-2$ all permutations not related to π with two adjacencies, where at least one of the adjacencies is an original adjacency in π, call this set of permutations Ω^2.

Note that by Lemma 39 the Möbius values in $\Delta^0 \cup \Delta^1 \cup \Delta^2$ sum to zero and the same is true of $\Gamma^0 \cup \Gamma^1 \cup \Gamma^2$ and $\Omega^0 \cup \Omega^1 \cup \Omega^2$. We know these sets satisfy the length conditions in Lemma 39 because the maximum number of adjacencies is $\frac{n-2}{2}$, this is because the letters 1 and n are not in adjacencies and there are no triple adjacencies, which implies $n - k \geq \frac{n+2}{2} \geq 4.5 > 4$. This implies $\mu(\pi) = 0$ and completes this case.

Now suppose one of the adjacencies in π is 12 or $(n-1)n$. If these adjacencies occur at the beginning or end, respectively, then this reduces to part 1 of Theorem 36. It is also possible that one of these adjacencies occurs directly after or before the descent in which case the proof follows from the proof above with minor modifications. These modifications arise from the fact that removing the letter 1 from π is equivalent to removing the adjacency 12 and likewise with the letter n and the adjacency $(n-1)n$. In certain cases, this may result in $n-k \not> 4$, and we must apply Lemma 37 to get the desired cancellation. □

4.4.3 Theorem 36 Part 4

Proposition 41. *Consider a permutation $\pi \in \mathcal{P}_1^n$ with exactly two adjacencies, at positions k and i. If the first adjacency has greater value than the second then $\mu(\pi) = \pm 1$, otherwise $\mu(\pi) = 0$.*

Proof. If π begins with 12 or ends with $(n-1)n$, then $\mu(\pi) = 0$ by part 1 of Theorem 36. Now consider the case π does not contain both the adjacencies 12 and $(n-1)n$.

Removing the letters π_i and π_k results in a permutation λ, of length $n-2$, with no adjacencies. As the Möbius values of all the permutations in $[\mathbf{1}, \lambda]$

sums to zero we can ignore any permutation in said interval. Now use Lemma 34 and consider the remaining permutations. By Lemma 39 the Möbius values of the remaining permutations with one adjacency sum to zero. Split the remaining permutations with two adjacencies into two sets A and B, where A are those obtained from π by removing the letters 1 or n (or both) and B are those obtained from π by removing a letter from an adjacency and then removing another letter to create a new adjacency that does not occur in π. As the largest permutations in B are of length $n-2 > 4$ we can apply Lemma 39 to see that the Möbius values of the permutations in B sum to zero.

This just leaves us to consider A. First assume π doesn't have the adjacencies 12 or $(n-1)n$ directly after or before the descent. Then A contains the following permutations with two adjacencies:

- A permutation δ of length $n-1$, obtained by removing the letter 1 from π.
- A permutation τ of length $n-1$, obtained by removing the letter n from π.
- A permutation σ of length $n-2$, obtained by removing letters 1 and n from π.

By the inductive hypothesis (36.4 and 36.7) it is clear that $\mu(\tau)+\mu(\sigma) = 0$. This means that $\mu(\pi) = -\mu(\delta)$. The relative values of the adjacencies in π are the same as in δ so, if the first adjacency has greater value than the second then $\mu(\pi) = -\mu(\delta) = \pm 1$, otherwise $\mu(\pi) = -\mu(\delta) = 0$. This completes the first case.

Now consider the case when π contains the adjacency 12 but not $(n-1)n$, then A only contains τ and $\mu(\pi) = -\mu(\tau)$. Similarly when π contains the adjacency $(n-1)n$ but not 12, then A only contains δ and $\mu(\pi) = -\mu(\delta)$. The result then follows by evaluating the value of δ or τ. This completes this case.

Finally consider the case π contains both adjacencies 12 and $(n-1)n$ and with $(n-1)n$ occurring before 12. In this case there are no permutations in the set denoted A above and not all the permutations with one adjacency cancel. So we repeat the argument above considering the permutations with one adjacency. □

Remark 42. *Note that the Möbius value in the above proof is computed as a negation of a permutation of length one less. Hence the Möbius value is alternating in the case of permutations with two adjacencies.*

4.4.4 Theorem 36 Part 5

Proposition 43. *Consider a permutation $\pi \in \mathcal{P}_1^n$ which has exactly one adjacency at position i and the descent at position d. Then:*

1. *If $i < d$ and $\pi_1 \neq 1$ then $\mu(\pi) = \pm i$,*
2. *If $i < d$ and $\pi_1 = 1$ then $\mu(\pi) = \pm(i-1)$,*
3. *If $i > d$ and $\pi_n \neq n$ then $\mu(\pi) = \pm(n-i)$,*
4. *If $i > d$ and $\pi_n = n$ then $\mu(\pi) = \pm(n-i-1)$.*

Proof. If π begins with 12 or ends with $(n-1)n$, then $\mu(\pi) = 0$ by part 1 of Theorem 36. Next, we consider the case where π doesn't have the adjacencies 12 or $(n-1)n$ directly before or after the descent.

Removing π_i from π creates a permutation λ with no adjacencies and we can ignore the interval $[\mathbf{1}, \lambda]$ as the Möbius values sum to zero by definition. We can apply Lemma 39 to the remaining permutations with two adjacencies to see that their Möbius values sum to zero. By Lemma 34 this leaves us to consider three permutations with one adjacency:

- Of length $n-1$ there remain two permutations with one adjacency, obtained by removing the letters 1 or n, call these σ_1 and σ_2 respectively.
- Of length $n-2$ there remains one permutation with one adjacency not related to π. This is obtained by removing the letters 1 and n from π, call this δ.

We consider the four cases in the statement of the proposition and obtain the Möbius value from the induction hypothesis (36.5):

- If $i < d$ then $\mu(\sigma_1) + \mu(\delta) = 0$. Hence $\mu(\pi) = -\mu(\sigma_2)$ which gives:
    1. If $\pi_1 \neq 1$ then $\mu(\pi) = -\mu(\sigma_2) = \pm i$.
    2. If $\pi_1 = 1$ then $\mu(\pi) = -\mu(\sigma_2) = \pm(i-1)$.
- If $i > d$ then $\mu(\sigma_2) + \mu(\delta) = 0$. Hence $\mu(\pi) = -\mu(\sigma_1)$ which gives:
    3. If $\pi_n \neq n$ then $\mu(\pi) = -\mu(\sigma_1) = \pm(n-i)$.
    4. If $\pi_n = n$ then $\mu(\pi) = -\mu(\sigma_1) = \pm(n-i-1)$.

We have completed this case of the proof.

If π contains the adjacency 12 or $(n-1)n$ then removing the letter n or 1, respectively, gives a permutation with one adjacency α. The Möbius values of all the other permutations sum to zero by Lemma 39, so $\mu(\pi) = -\mu(\alpha)$. Evaluating the four different cases of the proposition and using the inductive hypothesis (36.5) to get $\mu(\alpha)$ completes the proof. □

Remark 44. *Note that in the above proof for each case the Möbius value of π is a negation of a permutation of length one less. Therefore the sign of the Möbius value is alternating for all permutations with exactly one adjacency.*

4.4.5 Theorem 36 Part 6

Proposition 45. *Let π be a permutation in $\mathcal{P}_1^n$ with no adjacencies. Then:*

1. *If n is even and $\pi_1 = 1$, that is $\pi = W_{\frac{n}{2}}$, then $\mu(\pi) = -\binom{\frac{n}{2}}{2}$,*
2. *If n is even and $\pi_1 = 2$, that is $\pi = M_{\frac{n}{2}}$, then $\mu(\pi) = -\binom{\frac{n}{2}+1}{2}$,*
3. *If n is odd then $\mu(\pi) = \binom{\frac{n+1}{2}}{2}$.*

Proof. First note that π contains a permutation λ, with no adjacencies, of length $n-1$, obtained by removing the largest letter from π. As the Möbius values of all the permutations in $[\mathbf{1}, \lambda]$ sum to zero we can ignore any permutation in said interval. By Lemma 34 this leaves us to consider the following permutations which occur in π:

- Of length $n-1$ there remain:

- One permutation with no adjacencies obtained by removing the letter 1 from π.
- The permutations with one adjacency each obtained by removing a letter from π, excluding the letters 1 and n.

- Of length $n-2$ there remain:
  - The permutations not related to π with one adjacency. These are obtained by removing the letter 1 from each of the permutations with one adjacency of length $n-1$ listed above.
  - All permutations of length $n-2$ related to π with two adjacencies.
- Of length $n-3$ there remain the permutations not related to π with two adjacencies.

First consider the case n is even and $\pi_1 = 1$, that is when $\pi = 13\ldots(n-1)24\ldots n = W_{\frac{n}{2}}$. We will consider the permutations in $[1, W_{\frac{n}{2}}]$ based on number of adjacencies, and when needed, by the number removed to create an adjacency. We start with the nonzero permutations with two adjacencies. Note that all the length $n-3$ permutations with two adjacencies are obtained from the length $n-2$ permutations with two adjacencies by removing the letter 1. We can then apply Lemma 37 to see that the Möbius values of the permutations with two adjacencies sum to zero. We can repeat this argument with the permutations with one adjacency obtained from π by removing any of the letters $3, 4, \ldots, (n-1)$ to see that these cancel with the permutations of length $n-2$ with one adjacency. This leaves the permutations obtained from π by removing the letters 2 and 1, respectively. The first is of the form $124\ldots(n-2)3\ldots(n-1)$ and begins with 12 so has zero Möbius value by part 1 of Theorem 36. The second is $24\ldots(n-2)13\ldots(n-1)$ which by the induction hypothesis (36.6c) has Möbius value $\binom{\frac{n-1+1}{2}}{2}$, which implies $\mu(W_{\frac{n}{2}}) = -\binom{\frac{n}{2}}{2}$.

In the case n is odd and $\pi_1 = 1$, the argument is analogous. We find the permutation $24\ldots n13\ldots(n-1)$ has Möbius value $-\binom{\frac{n-1}{2}+1}{2}$, which implies $\mu(\pi) = \binom{n+1}{2}$.

Next consider the case where n is even and $\pi_1 = 2$, that is $\pi = 24\ldots n13...(n-1) = M_{\frac{n}{2}}$. First we consider permutations of length $n-1$ with one adjacency formed by removing one of the letters $3, 4, \ldots, (n-1)$. We can apply Lemma 37 to see that the Möbius value of all but one of these cancel with all but one of the length $n-2$ permutations with one adjacency. The only remaining length $n-2$ permutation is $124\ldots(n-2)35\ldots(n-3)$ which has zero Möbius value by part 1 of Theorem 36. The only remaining length $n-1$ permutation is $24\ldots(n-2)(n-1)13\ldots(n-3)$ which by the induction hypothesis (36.5a) has Möbius value $\frac{n}{2} - 1$.

Now consider the remaining permutations with two adjacencies. The permutation with the triple adjacency 123 contributes zero to the Möbius value by part 1 of Theorem 36. Removing the letter 2 and any letter $i > 3$ from π results in a permutation with adjacency 12 immediately after the descent. If i is even then the larger adjacency also appears after the descent so such a permutation contributes zero to the Möbius value by the induction hypothesis (36.4b). If i is odd then the adjacency appears before the descent. Since each such permutation has Möbius value -1 by the induction hypothesis (36.4a), and i is an odd number between 5 and n, the sum of the Möbius values of these permutations is $-\frac{n}{2}+2$. We can apply Lemma 37 to cancel all of the other permutations with two adjacencies in a similar way to the case $\pi_1 = 1$ above.

We must also consider the Möbius values of the permutations found by removing 2 or 1 from π. The permutation $35\ldots(n-1)124\ldots(n-2)$ has Möbius value $\frac{n}{2}-1$ by the induction hypothesis (36.5c). The permutation $13\ldots(n-1)24\ldots(n-2)$ has Möbius value $\binom{\frac{n}{2}}{2}$ by the induction hypothesis (36.6c). The Möbius value of π is given by the negation of the sum of the Möbius values of the permutations it contains, so we sum the above values and negate which gives:

$$\mu(\pi) = -\left(\binom{\frac{n}{2}}{2} + 2(\frac{n}{2}-1) - \frac{n}{2} + 2\right) = -\binom{\frac{n}{2}+1}{2}.$$

Finally we consider the case where n is odd and $\pi_1 = 2$, that is, $\pi = 24\ldots(n-1)13\ldots n$. The argument proceeds in an analogous manner to the previous case, except the sum of the Möbius values of the permutations with two adjacencies is $\frac{n-1}{2}-2$ and the Möbius values of the three permutations with one adjacency are $-\frac{n}{2}+1$, $-\frac{n}{2}+1$ and $\binom{\frac{n-1}{2}}{2}$, resulting in $\mu(\pi) = \binom{\frac{n+1}{2}}{2}$.

□

Remark:

- The nice form of the result in Proposition 45 raises the question of a direct combinatorial proof. We expect to present such a proof in the forthcoming paper [Smi14b] which analyses topological properties of some intervals in the poset $\mathcal{P}$.

- Notice that in the above cases the Möbius value is positive if and only if n is odd. Therefore the Möbius value is alternating.

4.4.6 Finishing the Proof of Theorem 36

Notice that the remarks after Propositions 41, 43 and 45 show that the Möbius value is alternating for all nonzero permutations, which implies the Möbius value is positive if and only if n is odd. This proves part 7 of Theorem 36. We have shown that if the classification of Theorem 36 holds for all permutations, of length less than n, with at most one descent, then it also holds for n. By induction, that completes the proof of Theorem 36. Parts 5 and 6 of Theorem 36 give us the following important corollary:

Corollary 46. *On the poset $\mathcal{P}$ the function $\mu(\pi)$ is unbounded.*

4.5 Conjectures on the Möbius Function for Intervals of Permutations with at Most One Descent

So far we have mainly concentrated on intervals of the form $[1, \pi]$. We now consider permutations where we allow the permutation 1 to change. We see that this change increases the complexity of computing the Möbius function quite drastically especially in the second conjecture we present, but also leads to some interesting results relating to the Möbius function being dependant on whether a permutation is separable.

4.5.1 The Möbius Function on the Intervals $[\sigma, M_n]$ and $[\sigma, W_n]$

In this subsection we examine intervals $[\sigma, \pi]$ where π is one of the two permutations of *even* length with no adjacencies and $\sigma \in \mathcal{P}_1$. Recall that these permutations with no adjacencies are denoted $M_n = 24\dots(2n)13\dots(2n-1)$ and $W_n = 13\dots(2n-1)24\dots(2n)$. This leads us to the following conjecture which has been checked by computer to hold for any pair (m, n) where $m < 12$ and $n < 7$:

Conjecture 47. *Given a permutation $\sigma \in \mathcal{P}_1^m$, let i be the number of adjacencies in σ. If $\sigma \leq \pi$ where $\pi \in \{M_n, W_n\}$ we have the following:*

- *If σ is separable, then:*
  - ◇ $\mu(\sigma, M_n) = \pm\binom{n+1}{m}$,
  - ◇ $\mu(\sigma, W_n) = \pm\binom{n+m-i-2}{m}$.
- *If σ is not separable, then:*

$$\mu(\sigma, \pi) = \pm\binom{n + \lfloor\frac{m-i-a}{2}\rfloor}{m}$$

$$\textit{where } a = \begin{cases} 0, & \textit{if } \sigma \textit{ and } \pi \textit{ are related} \\ 1, & \textit{otherwise} \end{cases}.$$

Also the Möbius value is positive if and only if m is even.

Recall that when considering an adjacency of length k we regard it as $k-1$ individual adjacencies.

Notice that Conjecture 47 only deals with intervals $[\sigma, \pi]$ where π is of even length. In Theorem 36 we can see that changing π between odd and even length has little effect on the Möbius function. In Conjecture 47, on the other hand, there is a substantial difference between the odd and even case.

4.5.2 The Möbius Function on the Interval $[M_m, \pi]$

We can reverse the idea in subsection 4.5.1 and consider intervals $[\sigma, \pi]$ where σ is a permutation without adjacencies and $\pi \in \mathcal{P}_0^n \cup \mathcal{P}_1^n$. In this subsection we conjecture a formula for the Möbius function on such intervals. This formula is somewhat complicated, but turns out to be computationally efficient, compared to the brute force method of computing from the recursive formula for the Möbius function. Before stating the result we define a few statistics on π:

- Let a be the number of adjacencies in π.
- Set $\hat{n} = \begin{cases} n-1, & \text{if } \pi_n = n \\ n, & \text{otherwise} \end{cases}$.
- Let the set $A = \{i_1, \ldots, i_a\}$ be the ordered sequence of the values of the adjacencies in π. Also add to A two phantom adjacencies i_0 and i_{a+1} which occur before and after the descent, respectively, with values:
$i_0 = \begin{cases} -1, & \text{if } \pi_1 \neq 1 \\ 0, & \text{otherwise} \end{cases}$ and $i_{a+1} = \hat{n} + 1$.

- A function:

$$\widehat{C}^{\alpha}_{\beta}(k,s) = \begin{cases} \binom{\alpha - 2k}{\beta}, & \text{if } 0 \leq k < \frac{s}{2} \\ \binom{\alpha - 2(s-k) + 1}{\beta}, & \text{if } \frac{s}{2} \leq k < s \end{cases}.$$

- A sequence $\hat{J} = \{\hat{j}_0, \ldots, \hat{j}_a\}$ where:

$$\hat{j}_k = \left\lfloor \frac{i_{k+1} - i_k - 2}{2} \right\rfloor.$$

- Split $\hat{J}$ into two sequences j^a and j^b in the following way:

$$\begin{cases} \hat{j}_k \in j^a, & \text{if } i_k \text{ and } i_{k+1} \text{ occur on the same side of the descent} \\ \hat{j}_k \in j^b, & \text{otherwise} \end{cases}.$$

- Set $s = \sum_{t=0}^{a} \hat{j}_t$.
- Trim j^a and j^b in the following way:

  1. If j^a is empty remove the largest element from j^b and set $\epsilon = 0$,
  2. If j^a is not empty let max_{j^a} be the largest element in j^a and remove it from j^a, then set $\epsilon = max_{j^a} - \sum_t j^a_t$,
  3. Then set $\alpha = |j^a|$ and $\beta = |j^b|$,
  4. Remove all zero elements from both sequences and if this results in j^b being empty set $\epsilon = 0$,
  5. Finally sort j^a into ascending order and j^b into decreasing order.

- Define the function $s_\theta(\kappa, \tau) = \sum_{t=\kappa}^{\tau} j^\theta_t$.

- Set $\lambda = \left\lceil \frac{\hat{n}}{2} \right\rceil + m - \left\lceil \frac{5a}{2} \right\rceil + \beta - t$ and $\sigma = 2m - 2a + \beta$ where $t = \begin{cases} 1, & \text{if } \pi_1 = 1 \text{ and } n \text{ is even and } \pi_n = n \\ 1, & \text{if } \pi_1 = 1 \text{ and } n \text{ is odd and } \pi_n \neq n \\ 0, & \text{otherwise} \end{cases}$.

For an example of these statistics see Example 49 below.

We can now state the conjecture which has been checked by computer tests to hold for all pairs (m, n) where $m < 6$ and $n < 12$:

Conjecture 48. *Consider the interval $[M_m, \pi]$ where $\pi \in \mathcal{P}_0^n \cup \mathcal{P}_1^n$ and λ, σ, j^a, j^b, ϵ, s, s_θ and $\widehat{C}$ are all as defined above, then:*

If π begins with 12, ends with $(n-1)n$ or contains a triple adjacency $\mu(M_m, \pi) = 0$, otherwise:

$$|\mu(M_m, \pi)| = \binom{\lambda}{\sigma} - \sum_{\tau=0}^{|j^b|-1} \sum_{\gamma=0}^{\tau} \sum_{\omega=\tau-\gamma}^{j^b_\gamma + s_b(\tau+1, |j^b|-1)-1} \widehat{C}^{\lambda-\tau-2}_{\sigma-\tau-1}(\omega, s) + \sum_{\tau=0}^{|j^a|-1} \left[\sum_{\gamma=1}^{j^a_\tau + s_a(0,\tau-1)} \widehat{C}^{\lambda-|j^b|-\tau}_{\sigma-|j^b|-\tau}(\gamma, s+1) + \sum_{\omega=1}^{\epsilon} \widehat{C}^{\lambda-|j^b|-\tau}_{\sigma-|j^b|-\tau}(\omega+1, s+1) \right].$$

Also the sign of μ is positive if and only if n is even.

Whilst Conjecture 48 is rather complicated it is significantly more computationally efficient than computing the Möbius function from its recursive definition. To see this consider the following example, for an interval of rank 20, whose computation from the recursive definition would take enormous time even on a fast computer:

Example 49. *Consider the interval*

$$I = [24681357, 2\,4\,6\,7\,9\,12\,14\,16\,18\,21\,23\,24\,26\,28\,1\,3\,5\,8\,10\,11\,13\,15\,17\,19\,20\,22\,25\,27].$$

We compute $\mu(I)$ using Conjecture 48, first extracting the following statistics from I:

- $a = 4$, $m = 4$ *and* $\hat{n} = 28$,
- $A = \{-1, 6, 10, 19, 23, 29\}$ *and* $\hat{J} = \{2, 1, 3, 1, 2\}$,
- *Before trimming:* $j^a = \{2, 3\}$ *and* $j^b = \{1, 1, 2\}$,
- *After trimming:* $j^a = \{2\}$, $j^b = \{2, 1, 1\}$, $\epsilon = 1$, $\alpha = 1$ *and* $\beta = 3$,
- $s = 9$, $\lambda = 11$ *and* $\sigma = 3$.

Putting this into the formula of Conjecture 48 we get:

$$\begin{aligned}
\mu(I) = \binom{11}{3} &- \sum_{\tau=0}^{2}\sum_{\gamma=0}^{\tau}\sum_{\omega=\tau-\gamma}^{j^b_\gamma+s_b(\tau+1,|j^b|-1)-1} \widehat{C}^{9-\tau}_{2-\tau}(\omega, 9) \\
&+ \sum_{\tau=0}^{0}\left[\sum_{\gamma=1}^{j^a_\tau+s_a(0,\tau-1)} \widehat{C}^{8-\tau}_{0-\tau}(\gamma, 10) + \sum_{\omega=1}^{1}\widehat{C}^{8-\tau}_{0-\tau}(\omega+1, 10)\right] \\
&= \binom{11}{3} - \sum_{\omega=0}^{4}\widehat{C}^{9}_{2}(\omega, 9) - \left[\sum_{\omega=1}^{2}\widehat{C}^{8}_{1}(\omega, 9) + \sum_{\omega=0}^{1}\widehat{C}^{8}_{1}(\omega, 9)\right] \\
&- \sum_{\omega=0}^{0}\widehat{C}^{7}_{0}(\omega, 9) + \sum_{\gamma=1}^{2}\widehat{C}^{8}_{0}(\gamma, 10) + \sum_{\omega=1}^{1}\widehat{C}^{8}_{0}(\omega+1, 10) \\
&= \binom{11}{3} - \binom{9}{2} - \binom{7}{2} - \binom{5}{2} - \binom{3}{2} \\
&- \binom{6}{1} - \binom{4}{1} - \binom{8}{1} - \binom{6}{1} - \binom{7}{0} + \binom{6}{0} + \binom{4}{0} + \binom{4}{0} \\
&= 165 - 36 - 21 - 10 - 3 - 6 - 4 - 8 - 6 - 1 + 1 + 1 + 1 = 73.
\end{aligned}$$

Whilst we cannot verify this is the correct value of the Möbius function on this interval, the example serves as a good indicator of the efficiency of the conjecture if it can be proved correct.

Part III

Intervals of Permutations with a Fixed Number of Descents are Shellable

Abstract

The set of all permutations, ordered by pattern containment, is a poset. We present an order isomorphism from the poset of permutations with a fixed number of descents to a certain poset of words with subword order. We use this bijection to show that intervals of permutations with a fixed number of descents are shellable, and we present a formula for the Möbius function of these intervals. We present an alternative proof for a result on the Möbius function of intervals $[1, \pi]$ such that π has exactly one descent. We prove that if π has exactly one descent and avoids 456123 and 356124, then the intervals $[1, \pi]$ have no nontrivial disconnected subintervals; we conjecture that these intervals are shellable.

5.1 Introduction and Preliminaries

A *permutation* of length n is an ordering of the integers $1, \ldots, n$, without repetitions. Given two permutations σ and π, we define an occurrence of σ as a *pattern* in π to be a subsequence of π with the same relative order of elements as in σ. For example, if $\sigma = 213$ and $\pi = 23514$ then there are two occurrences of σ in π, as the subsequences 214 and 314. The set of all permutations forms a poset $\mathcal{P}$, with a partial ordering defined by $\sigma \leq \pi$ if σ occurs as a pattern in π. An *interval* $[\sigma, \pi]$ in $\mathcal{P}$ is a subposet consisting of all permutations $z \in \mathcal{P}$ with $\sigma \leq z \leq \pi$. A *chain* in a poset P is a totally ordered subset $\{c_1 < \cdots < c_t\}$. For example, $21 < 2341 < 24513$ is a chain in $[1, 24513]$. The direct sum $\sigma \oplus \pi$ of two permutations σ and π is obtained by appending π to σ after adding the length of σ to each letter of π. For example, $213 \oplus 312 = 213645$. A *descent* occurs at i in a permutation $\pi_1 \ldots \pi_n$ if $\pi_i > \pi_{i+1}$. As an example, 23154 has descents at 2 and 4.

If $\sigma \leq \pi$, then $\mathrm{des}(\sigma) \leq \mathrm{des}(\pi)$. Therefore, if the permutations σ and π both have exactly k descents, then any permutation $\tau \in [\sigma, \pi]$ also has exactly k descents. We denote the induced subposet of all permutations with exactly k descents as $\mathcal{P}_k$. The Möbius function for a poset is defined recursively as follows: $\mu(a, b) = 0$ if $a \not\leq b$, $\mu(a, a) = 1$ for all a and for $a < b$:

$$\mu(a, b) = - \sum_{a \leq z < b} \mu(a, z).$$

One of the main goals of this paper is to study the Möbius function of P_k.

The *interior* of the interval $[\sigma, \pi]$, written (σ, π), is the set $[\sigma, \pi] - \{\sigma, \pi\}$.

The *order complex* of $[\sigma, \pi]$, written $\Delta(\sigma, \pi)$, is the simplicial complex whose faces are the chains contained in the interior (σ, π). When we attribute a topological property to an interval we mean the corresponding property of its order complex. We refer the reader to [Wac07] for extensive background on the subject of order complexes.

A simplicial complex is *pure* if all its maximal faces, which are called *facets*, have the same dimension. The order complex of an interval of permutations is always pure. A pure simplicial complex Δ is *shellable* if its facets can be arranged in linear order $F_1, \ldots, F_t$ in such a way that the subcomplex $\left(\cup_{i=1}^{k-1}\langle F_i\rangle\right) \cap \langle F_k\rangle$ is pure and $(\dim \Delta - 1)$-dimensional for $2 \leq k \leq t$, where $\langle F\rangle = \{G : G \subseteq F\}$, that is, $\langle F\rangle$ is the subcomplex generated by F. Again we refer the reader to [Wac07] for extensive background on the subject of shellability.

Let $\mathcal{A}$ be the poset of words on the alphabet of positive integers, with the partial order called *subword order* where $v \leq w$, with $w = w_1 \ldots w_n$, if there is a subsequence $w_{i_1} \ldots w_{i_m}$ in w such that $v = w_{i_1} \ldots w_{i_m}$. For example, $2132 \leq 212312$ but $2132 \not\leq 21233$. In [Bjö90] a formula was given for computing the Möbius function on intervals of $\mathcal{A}$ in polynomial time, and it is shown that all intervals in $\mathcal{A}$ are shellable. In this paper we present an order isomorphism, that is, an order-preserving bijection, from each interval in the permutation posets $\mathcal{P}_k$ to a corresponding interval in $\mathcal{A}$. This allows us to easily compute the Möbius function of intervals from the posets $\mathcal{P}_k$ and to show that they are shellable.

The *reduced Betti number* $\tilde{\beta}_k(X)$ of a simplicial complex X is the rank of the k-th reduced homology group of X (for background on the homology of simplicial complexes we refer the reader to [Koz08]). The Philip Hall Theorem and the Euler-Poincaré formula, which appear as Proposition 1.2.6 and Theorem 1.2.8 in [Wac07], combined state:

$$\mu(\sigma,\pi) = \tilde{\chi}(\Delta(\sigma,\pi)) = \sum_{i=-1}^{\dim \Delta(\sigma,\pi)} (-1)^i \tilde{\beta}_i(\Delta(\sigma,\pi)), \tag{5.4}$$

where $\tilde{\chi}(\Delta(\sigma,\pi))$ is the reduced Euler characteristic of the order complex of $[\sigma,\pi]$.

An important property of simplicial complexes is Cohen-Macaulayness, which has its origins in commutative algebra. A simplicial complex Δ is *Cohen-Macaulay* if $\operatorname{rank}(\tilde{H}_i(\ell k_\Delta F)) = 0$ for all $F \in \Delta$ and $i < \dim \ell k_\Delta F$, where $\ell k_\Delta F$ denotes the *link* of F and $\tilde{H}_i$ denotes the i'th reduced homology group. For a full explanation of this definition see [Wac07, Section 4]. A shellable simplicial complex is Cohen-Macaulay, as observed in [Sta96]. We use this property to compute the homology of intervals from the posets $\mathcal{P}_k$ for any $k \geq 0$.

There is a *generalised subword order*, defined in [SV06], where we take a poset P and let P^* denote the poset of finite words whose letters are elements of P. If u, $w \in P^*$ then $u \leq_{P^*} w$ if there is a subword $w_{i_1} \dots w_{i_{|u|}}$ such that $u_j \leq_P w_{i_j}$ for $1 \leq j \leq |u|$. If P is an antichain, then generalised subword order is precisely the subword order. In [SV06] a formula was presented for the Möbius function of words with generalised subword order when P is a chain. That paper also established an order isomorphism between posets of these words and posets of *layered permutations*, that is, permutations that can be expressed as a direct sum of decreasing permutations. For example, $1 \oplus 21 \oplus 321 \oplus 21 = 13265487$ is a layered permutation.

In [MS12] a formula was presented for the Möbius function of words with generalised subword order for any poset P, which covers both the words considered in the present paper and in [SV06]. In [Bjö80] it was shown that if an interval $\mathcal{I}$ contains a *nontrivial disconnected subinterval*, that is, a subinterval of rank at least 3 whose interior can split into two disjoint subposets, then $\mathcal{I}$ is not shellable. The first major result on the topology of intervals from the poset $\mathcal{P}$ appeared in [MS15], where it was shown that if P is a rooted forest, then any interval $[u, v]$ in P^* that does not contain a nontrivial disconnected subinterval is shellable. This result was then used to show that intervals of layered permutations that do not contain a nontrivial disconnected subinterval are shellable. Furthermore, it was conjectured that the same applies to the more general class of *separable permutations*, that is, the permutations that avoid 2413 and 3142.

In Section 5.2 we present a bijection from $\mathcal{P}$ to a subposet of $\mathcal{A}$. We show that when we restrict this bijection to $\mathcal{P}_k$ it is an order isomorphism. This allows us to draw on many useful results that have been proven for subword order, such as the shellability of intervals, and apply these results to permutations. In Section 5.3 we use this order isomorphism to present a formula for the Möbius function of intervals from the posets $\mathcal{P}_k$. We use this formula to prove a conjecture made in [Smi14a] and to present an alternative, simpler proof of [Smi14a, Theorem 5] on the Möbius function of intervals $[1, \pi]$ such that π has one descent. In Section 5.4 we show that if π has exactly one descent and avoids 456123 and 356124, then $[1, \pi]$ has no nontrivial disconnected subintervals and we conjecture that these intervals are shellable.

5.2 Bijection from Permutations to Words

In this section we present an order isomorphism from the poset P_k of permutations with exactly k descents to a subposet of $\mathcal{A}$. Let $\max(w)$ be the value of the largest letter in the word w. We now define the poset of words we consider:

Definition 50. *Let $\widehat{\mathcal{A}}$ denote the poset of words with subword order on the alphabet of all positive integers, with the additional conditions that for any $w \in \widehat{\mathcal{A}}$:*

AC1: *There is at least one occurrence of each letter $i \in \{1, \ldots, \max(w)\}$.*

AC2: *The rightmost occurrence of each letter $i \in \{1, \ldots, \max(w) - 1\}$ is preceded by an occurrence of $i + 1$.*

Let $\widehat{\mathcal{A}_k}$ denote the subposet of $\widehat{\mathcal{A}}$ of words w where $\max(w) = k$.

Example 51. *For example, $231423 \in \widehat{\mathcal{A}}$ but $1121343 \notin \widehat{\mathcal{A}}$ because the rightmost occurrence of 2 does not have a 3 to its left.*

The additional conditions in Definition 50 are very similar to the definition of a *restricted growth function*, which can be used to encode set partitions, see [Mil77]. To see the similarity we use the definition of a restricted growth function that appears in Question 106 in [Sta96, Chapter 1]. A *restricted growth function* is a sequence of the positive integers $1, \ldots, k$ with each letter occurring at least once and the first occurrence of i appearing before the first occurrence of $i+1$, for $1 \leq i \leq k-1$. If we consider AC2 reworded as beginning at the right end of the word, and travelling left, then the first occurrence of i must appear before the last occurrence of $i+1$. The key difference is that AC2 requires at least one occurrence of $i+1$ after the first i whereas a restricted growth function requires that all occurrences of $i+1$ are after the first i. As such, it is easy to see that $\widehat{\mathcal{A}}$ is a larger class than the class of restricted growth functions.

We know that the number of permutations of length n in $\mathcal{P}_k$ is the Eulerian number $A(n,k)$, see [Sta12]. We show that there is a length-preserving bijection from $\widehat{\mathcal{A}_k}$ to $\mathcal{P}_{k-1}$, which implies the number of words of length n in $\widehat{\mathcal{A}_k}$ is given by the Eulerian number $A(n,k-1)$.

When referring to both words and permutations we often use the notation α_i to refer to the letter at location i in α, and $|\alpha|$ to denote the length of α. Given a letter c of the permutation π, let $d_\pi(c)$ be the index of the *run* containing c, where a run is a maximal consecutive sequence of increasing letters. Therefore, $d_\pi(c)$ equals the number of descents preceding c in π, plus 1. For example, $d_{35241}(5) = 1$ and $d_{35241}(1) = 3$. Given a letter j of the word w, let $p_w(j)$ be the positions of the letter j in w, in increasing order. For example, $p_{21232}(2) = 135$. Now define the following functions:

$$f : \mathcal{P} \to \widehat{\mathcal{A}} \text{ by } \pi \mapsto d_\pi(1)d_\pi(2)\ldots d_\pi(|\pi|),$$

$$g : \widehat{\mathcal{A}} \to \mathcal{P} \text{ by } w \mapsto p_w(1), \ldots, p_w(\max(w)).$$

Now consider what these functions are doing. When applying f to π we first find the location of 1 in π and count the number of preceding descents, which gives the first letter, $d_\pi(1)$, of $f(\pi)$. We then repeat this for the letter 2 in π and continue up to n. To apply g to w we find the positions of each 1 in w and $g(w)$ begins with these positions in increasing order.

Then we find the positions of each 2 in w and we continue $g(w)$ with these positions in increasing order. We continue this up to $\max(w)$. For example, if $\pi = 263415$ then $f(\pi) = 312231$, and if $w = 214321$ then $g(w) = 261543$.

Before proceeding we define a term used for both permutations and words:

Definition 52. *Consider two elements $a \leq b$ of a poset of either words or permutations. An* embedding *of a in b is a sequence η of length $|b|$ such that the nonzero positions in η are the positions of an occurrence of a in b and removal of all the zeros from η results in a.*

Example 53. *The embeddings of the word* 2121 *in* 211221 *are* 210201*,* 210021*,* 201201 *and* 201021*. The embeddings of the permutation* 213 *in* 142356 *are* 021030, 020130, 021003 *and* 020103*.*

We now show that f and g are inverses of each other. First we show that f and g link the number of descents of permutations in $\mathcal{P}$ and the largest letter of words in $\widehat{\mathcal{A}}$.

Lemma 54. *Let f and g be defined as above.*

1. *If $\pi \in \mathcal{P}_k$, then $f(\pi) \in \widehat{\mathcal{A}_{k+1}}$.*
2. *If $w \in \widehat{\mathcal{A}_{k+1}}$, then $g(w) \in \mathcal{P}_k$.*

Proof. For (1), consider $\pi \in \mathcal{P}_k$ and let $w = f(\pi)$. It is clear that w is a word and that $d_\pi(\pi_n) = k + 1$. Also there must be an occurrence of all the letters $1, \ldots, k$ because for each i the letter at the location of the i-th descent maps to i. All that remains to be shown is that w satisfies AC2 in Definition 50. Let $w_t = i$ be the rightmost occurrence of the letter i. This implies the letter t at position j in π is the rightmost letter that is preceded by exactly i descents, and hence a descent occurs directly after π_j. Thus the letter π_{j+1} is mapped to $i+1$. Since $\pi_{j+1} < \pi_j$, the letter π_{j+1} is mapped to an earlier location in w than π_j. Therefore, $w_t = i$ is preceded by an occurrence of $i+1$. Since the argument holds for all i, this proves (1).

For (2), we need to show there are k descents in $g(w)$. By AC2 in Definition 50, the largest letter in $p_w(t)$ must have a greater value than the smallest letter in $p_w(t+1)$. Therefore, for each t there is a descent between $p_w(t)$ and $p_w(t+1)$ in $g(w)$. Since each $p_w(j)$ is increasing, these k are the only descents.

□

Lemma 55. *The map f is a bijection with inverse g.*

Proof. We prove this by showing that $fg = id_{\widehat{\mathcal{A}}}$ and $gf = id_{\mathcal{P}}$.

First consider $w \in \widehat{\mathcal{A}}$ and $v = f(g(w))$. If $w_i = t$ then $d_{g(w)}(i) = t$, since $i \in p_w(t)$ and thus in the t-th run of $g(w)$. Since we know that $w_i = t = d_{g(w)}(i) = v_i$ for all i, we conclude $w = v$.

Now consider $g(f(\pi))$ such that $\pi \in \mathcal{P}_k$, and let $\pi_t \dots \pi_{t+\lambda}$ be the j-th run of π for some $j \in \{1, \dots, k+1\}$. Each π_ℓ, where $\ell \in \{t, \dots, t+\lambda\}$, is mapped to the letter j in $f(\pi)$, and these are the only letters mapped to j. In turn only those letters are mapped into $p_{f(\pi)}(j)$. Since each segment is listed in increasing order, and this holds for all j, we have $g(f(\pi)) = \pi$. □

So f is a bijection from $\mathcal{P}$ to $\widehat{\mathcal{A}}$. Finally we need to see if this bijection is order-preserving. This is not true in general. For example, consider the permutations $132 \leq 2143$: Applying f yields $f(132) = 121 \not\leq 2132 = f(2143)$.

Consider the functions f_k obtained by restricting f to $\mathcal{P}_k$ and g_k obtained by restricting g to $\widehat{\mathcal{A}_k}$. We know by Lemma 54 that the image of f_k is $\widehat{\mathcal{A}_{k+1}}$ and the image of g_{k+1} is $\mathcal{P}_k$. Combining this with Lemma 55 implies f_k is a bijection. We now show that f_k and g_k are order-preserving:

Theorem 56. *The bijection f_k is an order isomorphism.*

Proof. Consider two permutations $\sigma, \pi \in \mathcal{P}_k$ with $\sigma \leq \pi$. Since σ and π have the same number of descents, thus the same number of runs, for any occurrence of σ in π the t-th run of σ must occur in the t-th run of π. If $\pi_{k_1} \dots \pi_{k_m}$ is an occurrence of σ in π, then $d_\pi(\pi_{k_1}) \dots d_\pi(\pi_{k_m}) = d_\sigma(\sigma_1) \dots d_\sigma(\sigma_m)$. Let $\pi_{t_1} \dots \pi_{t_m}$ be the reordering of $\pi_{k_1} \dots \pi_{k_m}$ in increasing order, then $d_\pi(\pi_{t_1}) \dots d_\pi(\pi_{t_m})$ occurs in $f_k(\pi)$ and is equal to $f_k(\sigma)$. Therefore, $f_k(\sigma) \leq f_k(\pi)$.

Now consider two words $v, w \in \widehat{\mathcal{A}_k}$ with $v \leq w$. Let η be an embedding of v in w, and let $\widehat{g_k}(\eta) = p_\eta(1) \dots p_\eta(k+1)$. It is easy to see that $p_\eta(t) \subseteq p_w(t)$, which implies $\widehat{g_k}(\eta) \leq g_k(w)$. Also $\widehat{g_k}(\eta)$ is an occurrence of $g_k(v)$, so $g_k(v) \leq g_k(w)$. □

Hence we have an order isomorphism between $\mathcal{P}_k$ and $\widehat{\mathcal{A}_{k+1}}$. One of our key results is the following corollary, which follows directly from [Bjö90, Theorem 3] and Theorem 56:

Corollary 57. *Any interval $[\sigma, \pi]$, where σ and π are permutations with the same number of descents, is shellable.*

Note that [Bjö90, Theorem 3] implies the stronger result that these intervals are dual CL-shellable which implies shellability. For a good survey of the implications of different types of shellability we refer the reader to [Wac07, Section 4.1].

We can also consider f as a map to the poset of words on generalised subword order, where the underlying poset is the chain of positive integers. In this case f is order-preserving, but g is not. For example, $211 \leq 212$ but $g(211) = 231 \not\leq 213 = g(212)$.

It is known that a shellable complex has the homotopy type of a wedge of spheres. Therefore, Corollary 57 gives the following result:

Corollary 58. *If σ and π are permutations with the same number of descents, then $\Delta(\sigma, \pi)$ is homotopy equivalent to a wedge of $|\mu(\sigma, \pi)|$ spheres of dimension* $\dim \Delta(\sigma, \pi) = |\pi| - |\sigma| - 1$.

5.3 Computing the Möbius Function

We can use Theorem 56 along with [Bjö90, Theorem 1], which also appears as [SV06, Theorem 2.1], to compute the Möbius function of any interval in $\mathcal{P}$ between permutations with the same number of descents. To do this we first need to define what a normal embedding is in the case of permutations. The definition we use is induced by the definition of a normal embedding in [SV06] after applying the bijection from Theorem 56:

Definition 59. *An* adjacency *in a permutation is a maximal sequence of consecutively valued letters in increasing consecutive order. The tail of an adjacency is all but the first letter of the adjacency. An embedding η of σ in π is normal if η_i is nonzero for each letter π_i in the tail of an adjacency. We* *use the notation from [Bjö90] and denote the number of normal embeddings of* σ *in* π *as* $\binom{\pi}{\sigma}_n$.

There is an analogous decreasing adjacency, but we are only interested in increasing adjacencies.

Example 60. *As in Example 53 consider* 213 *and* 142356*. The adjacencies in* 142356 *are* 23 *and* 56 *so the tails of the adjacencies are* 3 *and* 6*. Hence the only normal embedding is* 020103 *and therefore* $\binom{142356}{213}_n = 1$.

We use this definition to state the following result:

Proposition 61. *If* σ *and* π *are permutations with the same number of descents, then*

$$\mu(\sigma, \pi) = (-1)^{|\pi|-|\sigma|} \binom{\pi}{\sigma}_n .$$

Proof. This follows directly from Theorem 56 and [Bjö90, Theorem 1]. □

In [Bjö90] it was shown that $\binom{\pi}{\sigma}_n$ can be computed in polynomial time.

In Section 5.3.1 we use Proposition 61 to give a simpler proof of a result which appears in [Smi14a] and prove a conjecture from the same paper. First we present two corollaries:

Corollary 62. *Consider* $\sigma, \pi \in \mathcal{P}_k$*. Let* t *be the total number of letters in all the tails of all the adjacencies in* π*. If* $t > |\sigma|$*, then* $\mu(\sigma, \pi) = 0$.

This result doesn't hold if we remove the restriction on the number of descents. For example, consider $\sigma = 213$ and $\pi = 569341278$, which have one and two descents, respectively. The total number of letters in all the tails of 569341278 is $t = 4$ and $|\sigma| = 3$, but $\mu(312, 6745123) = 1 \neq 0$.

Corollary 62 is another part of the answer to a question posed in [BJJS11] asking when is $\mu(\sigma, \pi) = 0$. Whilst we cannot yet give a simple definitive answer to this question, there are results which present several classes of intervals with a zero Möbius function, such as results in [BJJS11], [Smi14a] and [ST10].

A result in [BJJS11] showed that if σ and π are separable permutations, then $|\mu(\sigma, \pi)|$ is at most the number of occurrences of σ in π. Proposition 61 implies this is also the case if we fix the number of descents, since an embedding corresponds to a unique occurrence.

Corollary 63. *If σ and π have the same number of descents, then $|\mu(\sigma, \pi)|$ is at most the number of occurrences of σ in π.*

5.3.1 Möbius Function of Permutations with at Most One Descent

Proposition 61 allows us to compute the Möbius function of an interval between two permutations with the same number of descents, but says nothing about intervals between permutations with different number of descents. Now we consider the intervals $[1, \pi]$, where $\pi \in \mathcal{P}_1$. In particular we present an alternative proof, which is both shorter and simpler than the original, of [Smi14a, Theorem 5]. We begin with a useful lemma which gives a formula for $\mu(1, \pi)$ for every permutation π with one descent.

Lemma 64. *If π has exactly one descent, then $\mu(1, \pi) = -\mu(21, \pi)$.*

Lemma 64 can be proved directly by considering the effect the removal of the increasing permutations has on the Möbius function. However, it also follows from Theorem 69, along with the Philip Hall Theorem and the Euler-Poincaré formula, so we omit the proof here.

We now present the alternative proof of [Smi14a, Theorem 5]. As in [Smi14a], we use the notation $\mu(\pi) := \mu(1, \pi)$. A *triple adjacency* indicates an adjacency of three letters, for example 234 in 52341. We use the notation *adjacency pair* to denote an adjacency of length 2. The value and position of an adjacency pair are given by the value and position of the first letter of the adjacency pair. We denote the two permutations of length n that have one descent and no adjacencies as $M_n = 246\dots135\dots$ and $W_n = 135\dots246\dots$. For example, $M_6 = 246135$ and $W_5 = 13524$.

As observed in [Smi14a], in Theorem 65 any overlap of cases agree in value. For example, if π contains the triple adjacency 234, then equivalently π contains the two adjacency pairs 23 and 34, the first of which has lower value; both cases imply $\mu(\pi) = 0$.

Theorem 65. *Given a permutation π of length $n > 2$, with exactly one descent, the value of $\mu(\pi)$ can be computed from the number and positions of adjacencies in π, as follows:*

1. *If $\mu(\pi) \neq 0$, then $\mu(\pi)$ is positive if and only if n is odd.*
2. *If π begins with 12 or ends in $(n-1)n$, then $\mu(\pi) = 0$.*
3. *If π has a triple adjacency, then $\mu(\pi) = 0$.*
4. *If π has more than two adjacency pairs, then $\mu(\pi) = 0$.*
5. *If π has exactly two adjacency pairs, then:*

a) If the first adjacency pair has greater value than the second, then $|\mu(\pi)| = 1$*,*

b) If the first adjacency pair has lower value than the second, then $\mu(\pi) = 0$*.*

6. *If* π *has exactly one adjacency pair, at position* $i \in \{1, \ldots, n-1\}$*, and the descent is at position* d*, then:*

   a) If $i < d$ *and* $\pi_1 \neq 1$*, then* $|\mu(\pi)| = i$*,*

   b) If $i < d$ *and* $\pi_1 = 1$*, then* $|\mu(\pi)| = i - 1$*,*

   c) If $i > d$ *and* $\pi_n \neq n$*, then* $|\mu(\pi)| = n - i$*,*

   d) If $i > d$ *and* $\pi_n = n$*, then* $|\mu(\pi)| = n - i - 1$*.*

7. *If* π *has no adjacencies, then:*

   a) If n *is even and* $\pi_1 = 1$*, so* $\pi = W_n$*, then* $\mu(\pi) = -\binom{\frac{n}{2}}{2}$*,*

   b) If n *is even and* $\pi_1 = 2$*, so* $\pi = M_n$*, then* $\mu(\pi) = -\binom{\frac{n}{2}+1}{2}$*,*

   c) If n *is odd, then* $\mu(\pi) = \binom{\frac{n+1}{2}}{2}$*.*

Proof. By Lemma 64, we know that $\mu(\pi) = -\mu(21, \pi)$. We can use Proposition 61 to compute $\mu(21, \pi)$, which implies the sign of $\mu(21, \pi)$ is given by $(-1)^{|\pi|-2}$. Therefore, $\mu(21, \pi)$ is positive if and only if n is even, combining this with $\mu(\pi) = -\mu(21, \pi)$ gives part 1.

We need to show that the absolute value of $\mu(21, \pi)$, which equals the number of normal embeddings, agrees with each of the cases in the theorem. We refer to the permutation 21 as σ, to avoid confusion between letters and permutations.

Case 2: If π begins with 12, then we must embed the 2 of σ as the 2 in π. However, there is no letter after the descent of value less than 2, so we cannot embed the 1 of σ anywhere. Similarly, if π ends in $(n-1)n$, then we must embed the 1 of σ as n in π. However, this leaves no valid position to embed the 2 of σ. Therefore, there are no normal embeddings of σ in π.

Case 3: If π has a triple adjacency at $\pi_i\pi_{i+1}\pi_{i+2}$, then any normal embedding of σ in π must be non-zero for $\pi_{i+1}\pi_{i+2}$. Therefore, σ must contain 12, which 21 does not. So there are no normal embeddings of σ in π.

Case 4 follows directly from Corollary 62.

Case 5: When there are two adjacency pairs, at locations k and j, there is only one embedding that might be normal, namely $\eta = \ldots 0\eta_{k+1}0 \ldots 0\eta_{j+1}0 \ldots$. If $\pi_k > \pi_j$, then we can set $\eta_{k+1} = 2$ and $\eta_{j+1} = 1$. Therefore, there is one normal embedding of σ in π. If $\pi_k < \pi_j$, then there is no way to make η an embedding of σ. Therefore, there are no normal embeddings of σ in π.

Case 6: In these cases we must embed one of the letters of 21 in the adjacency pair and can choose an appropriate place for the other letter. Denote the locations of the descent and adjacency pair as d and i, respectively. If $i < d$, then an embedding η of σ in π must have $\eta_{i+1} = 2$ and we can then embed the 1 from σ in any of the letters after the descent that have value less than π_i. Since the rest of π follows the same alternating pattern, because there are no more adjacencies, it is easy to see that this gives the desired results. The argument is analogous if $i > d$.

Case 7: Since there are no adjacencies in π, any embedding is normal.

Therefore, we need only count the number of embeddings. First consider the case when n is even and $\pi_1 = 1$. If we embed the letter 2 of σ in locations 1, 2, ..., $\frac{n}{2}$ and then count where we can embed the letter 1, then we get the following sequence $0, 1, 2, \ldots, \frac{n}{2} - 1$. Summing the sequence implies $\binom{\pi}{21}_n = \binom{\frac{n}{2}}{2}$. Repeating this for each case gives the desired results. □

We can also use Proposition 61 to prove one of the conjectures presented in [Smi14a]. In Proposition 67 we count the number of adjacency pairs, so a triple adjacency counts as two adjacency pairs and a length k adjacency counts as $k-1$ adjacency pairs. We say that two permutations with exactly one descent are *related* if they have the letter 1 on the same side of the descent. Let $\lfloor x \rfloor$ denote the *floor* of x, that is, the largest integer not greater than x.

Lemma 66. *Let σ be a permutation of length m with exactly one descent and i adjacency pairs. In σ the letter m occurs on the same side of the descent as the letter 1 if and only if $m - i$ is odd.*

Proof. If σ begins with the letter 1, then let $\tau = W_m$, otherwise let $\tau = M_m$. We can build σ from τ by going through each letter $k \in \{2, \ldots, m\}$ in τ. If k is not on the same side of the descent in τ as k is in σ, then move k to the opposing side of the descent, in the unique way that does not create a new descent.

We consider three cases that occur when moving a letter $k \in \{2, \ldots, m-1\}$. If k is not part of an adjacency pair, then moving it creates two new adjacency pairs $(k-1)k$ and $k(k+1)$. If k is part of one adjacency pair, then moving it destroys one adjacency pair but creates another. If k is part of two adjacency pairs, then moving it destroys both adjacency pairs. If $k = m$, then moving it either creates or destroys the adjacency pair $(m-1)m$. Therefore, each move of a letter k changes the number of adjacency pairs by -2, 0 or 2 for all $k \in \{2, \ldots, m-1\}$ and by 1 or -1 if $k = m$.

If m is odd, then 1 and m are on the same side of the descent in τ. If m is not moved whilst building σ from τ, then $m - i$ must be odd and m must be on the same side of the descent as 1 in σ. If m is moved, then it is on the opposite side of the descent and $m - i$ is even. The argument is analogous if m is even. $\square$

Proposition 67. *Given a permutation $\sigma \in \mathcal{P}_1$ of length m, let i be the number of adjacency pairs in σ. If $\sigma \leq \pi$ such that $\pi \in \{M_n, W_n\}$, then:*

$$\mu(\sigma, \pi) = (-1)^{n-m} \binom{\lfloor \frac{n+m-i-a}{2} \rfloor}{m},$$

where $a = \begin{cases} 0, & \text{if } \sigma \text{ and } \pi \text{ are related} \\ 1, & \text{otherwise} \end{cases}$.

Proof. Since both σ and π have exactly one descent, we can apply Proposition 61. The sign part of the result follows immediately. Since π has no adjacencies, any embedding of σ in π is normal, hence we need only count the number of embeddings. To do this we find it simpler to consider $f(\sigma)$ and $f(\pi)$, which are binary strings. Note that we consider an occurrence of a substring to occur in consecutive positions. For example, 101 has an occurrence of 10, but no occurrence of 11. We consider the different cases depending on whether n is odd or even, and whether σ and π are related.

First consider the case when σ and π are related and n is even. Suppose $\pi = M_n$, then $f(\pi) = 1010\dots$ and can be split into $n/2$ blocks, each consisting of a single 10. We can choose to embed a 10 from $f(\sigma)$ in either a single block of $f(\pi)$ or two separate blocks. For any other letter of $f(\sigma)$ we choose a single block of $f(\pi)$ in which to embed it. Thus, once we decide which 10s of $f(\sigma)$ to embed in single blocks of $f(\pi)$, all we need to do to determine an embedding is to pick a subset of blocks of $f(\pi)$.

Suppose we embed none of the 10s of $f(\sigma)$ in a single block of $f(\pi)$. We need to pick m of the $n/2$ blocks of $f(\pi)$, to embed one letter of $f(\sigma)$ in each of the selected blocks, which can be done in $\binom{\frac{n}{2}}{m}$ ways. Suppose we select r of the 10s in $f(\sigma)$ to embed in a single block. We need to choose $m - r$ of the 10s in $f(\pi)$ in which to embed the parts of $f(\sigma)$. Thus, we need to pick a total of m objects, some of them blocks of $f(\pi)$ to embed in and some of them 10s in $f(\sigma)$ to embed in a single block of $f(\pi)$. An occurrence of 10 in $f(\sigma)$ corresponds to a letter in σ that is after the descent and not the start of an adjacency pair, and there are $\lfloor \frac{m-i}{2} \rfloor$ such letters. Therefore, we have $\binom{\lfloor \frac{n}{2} \rfloor + \lfloor \frac{m-i}{2} \rfloor}{m}$ embeddings, and because n is even this gives the desired result. If $\pi = W_n$, n is even and σ and π are related, then the proof is analogous to when $\pi = M_n$, but considering substrings 01 instead of 10.

Now consider the case when n is odd and σ and π are related. By Lemma 66 we know that the largest letters in σ and π are on same sides of the descent if and only if $m - i$ is odd. Therefore, if $m - i$ is even, then we cannot embed anything in the final letter of π; thus this case is equivalent to when n is even and σ and π are related. If $m - i$ is odd, then we can embed a letter of σ in the largest letter of π; thus we have $\frac{n+1}{2}$ blocks of $f(\pi)$ to embed in. The remaining argument is analogous to when n is even, using the fact that as n and $m-i$ are odd $\lfloor\frac{n+1}{2}\rfloor + \lfloor\frac{m-i}{2}\rfloor = \lfloor\frac{n+m-i}{2}\rfloor$.

Finally consider the cases when σ and π are not related. In these cases we cannot embed anything in the first letter of π. Therefore, we can remove the first letter from π without changing the number of embeddings. So these cases are equivalent to when σ and π are related and π is of length $n-1$, the latter point accounting for the $-a$ in the equation. $\square$

5.4 Intervals of $[1,\pi]$ where π has One Descent

We have shown that intervals between two permutations with the same number of descents are shellable. Now we consider intervals of the form $[1,\pi]$ such that $\pi \in \mathcal{P}_1$. First we present a useful tool called the Quillen Fiber Lemma, which can be found as Theorem 15.28 in [Koz08]. Define the *upper ideal* as $Q_{\geq x} := \{y \in Q : y \geq x\}$.

Proposition 68. *(Quillen Fiber Lemma) Let $\phi : P \to Q$ be an order-preserving map between posets such that for any $x \in Q$ the complex $\Delta(\phi^{-1}(Q_{\geq x}))$ is contractible. Then the induced map between simplicial complexes $\Delta(\phi) : \Delta(P) \to \Delta(Q)$ is a homotopy equivalence.*

Note that the order complex of an upper ideal $Q_{\geq x}$ is always contractible to the point x. Now we consider the homology of the order complexes of intervals $[1,\pi]$ such that $\pi \in \mathcal{P}_1$.

Theorem 69. *If* $\pi \in \mathcal{P}_1$*, then the order complex* $\Delta(1,\pi)$ *is homotopy equivalent to a suspension of* $\Delta(21,\pi)$*. Therefore, the reduced Betti numbers of* $\Delta(1,\pi)$ *are* $\tilde{\beta}_n(\Delta(1,\pi)) = \tilde{\beta}_{n-1}(\Delta(21,\pi))$*, for* $n > 0$*, and* $\tilde{\beta}_0(\Delta(1,\pi)) = 0$.

Proof. Let $X = (1,\pi)$ and $A = X \setminus [123, \mathbf{k}]$, where $\mathbf{k} = 1\ldots k$ is the largest increasing permutation that occurs in π. The only permutations in A not in $(21,\pi)$ are 21 and 12. The permutations 21 and 12 occur as a pattern in every permutation in $(21,\pi)$. Therefore, in the order complex of A each of the points associated to 12 and 21 is the apex of a cone over $\Delta(21,\pi)$, so $\Delta(A)$ is a suspension of $\Delta(21,\pi)$.

We use the Quillen Lemma to show that $\Delta(X)$ is homotopically equivalent to $\Delta(A)$. Consider the map $f : X \to A$ defined by:

$$f(\sigma) = \begin{cases} 12, \text{ if } \sigma \in P_0 \\ \sigma, \text{ if } \sigma \in P_1 \end{cases} .$$

This map is order-preserving and $f^{-1}(A_{\geq a}) = X_{\geq a}$ which is an upper ideal, thus $\Delta(f^{-1}(A_{\geq a}))$ is contractible. Therefore, by the Quillen Fiber Lemma, f induces a homotopy equivalence between $\Delta(X)$ and $\Delta(A)$. Thus, $\Delta(X)$ is homotopically equivalent to a suspension of $\Delta(21,\pi)$. The result on the reduced Betti numbers then follows directly from the property of the suspension that $\tilde{H}_{n+1}(\operatorname{susp} X) = \tilde{H}_n(X)$. □

It is not true that all intervals $[1,\pi]$, $\pi \in P_1$, are shellable, as can be seen by the following example:

Example 70. *Consider the permutations* 456123 *and* 356124*. In the interval* $[1, 456123]$ *the subinterval* $[123, 456123]$ *is disconnected and of rank* 3*,* *which implies* $[1, 456123]$ *is not shellable. Similarly in* $[1, 356124]$ *the subinterval* $[123, 356124]$ *is disconnected and of rank* 3*. Consequently, if a permutation* $\pi \in \mathcal{P}_1$ *contains* 456123 *or* 356124 *the interval* $[1, \pi]$ *is not shellable.*

Whilst it is not true that the intervals $[1, \pi]$ are all shellable, we conjecture that containing 456123 or 356124 are the only obstructions to shellability for the intervals $[1, \pi]$ when $\pi \in \mathcal{P}_1$.

Conjecture 71. *If* $\pi \in P_1$ *and* π *avoids* 456123 *and* 356124*, then the interval* $[1, \pi]$ *is shellable.*

We have been unable to prove this conjecture, but we show that these intervals have no nontrivial disconnected subintervals. We prove this below, but first we need a result from [MS15] and the following definition:

Definition 72. *Let* η *be an embedding of* σ *in* π*. The* zero set *of* η*, which we denote* Z_η*, is the set* $\{i : \eta_i = 0\}$*. The zero set* Z_E *of a set of embeddings* E *is the union of the zero sets of all the embeddings in the set* E*.*

Example 73. *Let* $\sigma = 213$ *and* $\pi = 245136$*. Consider the following embeddings of* σ *in* π*:* $\eta_1 = 200130$*,* $\eta_2 = 200103$ *and* $\eta_3 = 020103$*. These embeddings have zero sets* $Z_{\eta_1} = \{2, 3, 6\}$*,* $Z_{\eta_2} = \{2, 3, 5\}$ *and* $Z_{\eta_3} = \{1, 3, 5\}$*, respectively. Therefore, the set* $\{\eta_1, \eta_2, \eta_3\}$ *has zero set* $\{1, 2, 3, 5, 6\}$*.*

Lemma 74. *(see [MS15, Proposition 5.3]) Consider two permutations* $\sigma < \pi$ *such that* $|\pi| - |\sigma| \geq 3$*. The interval* $[\sigma, \pi]$ *is not disconnected if the embeddings of* σ *in* π *cannot be partitioned into two non-empty sets* E_1 *and* E_2 *such that* $Z_{E_1} \cap Z_{E_2} = \emptyset$*.*

Proposition 75. *If $\pi \in \mathcal{P}_1$ and π avoids* 456123 *and* 356124*, then the interval $[1, \pi]$ has no disconnected subintervals of rank* 3 *or more.*

Proof. By Corollary 57 we know that intervals between two permutations in $\mathcal{P}_1$ are shellable, hence have no disconnected subintervals. All that remains is subintervals of the form $[\alpha, \beta]$, of rank 3 or more, with $\alpha \in \mathcal{P}_0$ (so α is an increasing permutation) and $\beta \in \mathcal{P}_1$. We show there is no way to split the embeddings of α in β into two sets with disjoint zero sets. To do this we separate the embeddings into three disjoint sets:

1. Embeddings with all of α embedded before the descent in β constitute the set E_1.

2. Embeddings with all of α embedded after the descent in β constitute the set E_2.

3. Embeddings with part of α embedded before the descent in β, and part after, constitute the set E_3.

Note that each embedding in E_1 has zeros in all positions after the descent. Similarly, all embeddings in E_2 have zeros in all positions before the descent. Therefore, it is not possible to split E_1 or E_2 into smaller sets that have disjoint zero sets. Moreover, E_3 cannot be split into smaller sets with disjoint zero sets. To see this note that, it is always possible to swap a nonzero letter with a zero letter directly to the right if after the descent, or directly to the left if before the descent. We can use this to build a sequence of embeddings between any two embeddings in E_3, where the elements in each adjacent pair in the sequence have only one letter differing in their zero sets. If the zero sets differ by only one element they cannot be disjoint. Since we can build such a sequence between any two embeddings in E_3, it is not possible to split E_3 into two sets with disjoint zero sets.

Suppose that all three sets are non-empty. Since both E_1 and E_2 are non-empty, it is not possible to make an embedding that uses all letters from one side of the descent and some letters from the other. This means that each embedding in E_3 must have a zero on both sides of the descent. So all embeddings in E_1 must be placed in the same set, all embeddings in E_3 must be placed in the same set as the embeddings in E_1 and all embeddings in E_2 must be placed in the same set as the embeddings in E_3. So we cannot split the embeddings into two sets with disjoint zero sets.

We now analyse three cases, depending on which of the three sets are empty.

First suppose E_1 is empty and that E_2 and E_3 are non-empty. Consider the embeddings in E_3. Unless an embedding embeds all its letters before the descent, and then some after, it has a zero before the descent, so must be put into the same set as E_2. Furthermore, as E_3 cannot be split into two sets with disjoint zero sets, the only way for E_2 and E_3 to have disjoint zero sets is if all the embeddings in E_3 have no zeros before the descent. We show that the only way such an embedding can exist is if $\beta = \beta_1\beta_2...\beta_d...\beta_i...\beta_n$ with $\beta_i > \beta_d$ and any letter strictly between β_d and β_i is less than β_d. Also the number of letters not between β_d and β_i must be exactly $|\alpha|$. Therefore, we can embed α as

$$\eta = \alpha_1...\alpha_d 0...0 \alpha_{d+1}...\alpha_a,$$

such that α_{d+1} is embedded in position i. To see this is the only possible embedding suppose there is another embedding $\hat{\eta} \neq \eta$. Since there cannot be a zero before the descent there must be a zero after $\hat{\eta}_i$. This implies it would also be possible to embed the sequence $\alpha_d ... \alpha_a$ after the descent, leaving a zero before the descent, contradicting our requirement for E_3.

If η is a valid embedding, then $\beta_{d-2}\beta_{d-1}\beta_d$ must be of one of two forms, either $c(c+1)(c+2)$ or $c(c+2)(c+3)$. Otherwise we could build valid embeddings of the form

$$\alpha_1 ... \alpha_{d-2} 00...0 \alpha_{d-1}\alpha_d\alpha_{d+1}...\alpha_a,$$

which has a zero before the descent, contradicting our requirement for E_3. We also know that there are $|\beta| - |\alpha| \geq 3$ letters smaller than β_d that occur after β_d. Therefore, the embedding η can only exist if there is an occurrence of either 456123 or 356124 in β. Since β avoids both these permutations η cannot be a valid embedding. So if E_1 is empty the embeddings cannot be split into disjoint zero sets.

An analogous argument shows that if E_2 is empty, then the embeddings cannot be split into disjoint zero sets.

Now suppose E_3 is empty but E_1 and E_2 are not. As E_3 is empty there can be no increasing sequence of length $|\alpha|$ spread across both sides of the descent. Using this we can repeat the same argument as above showing that $\beta_{d-2}\beta_{d-1}\beta_d$ must be of one of the forms $c(c+1)(c+2)$ or $c(c+2)(c+3)$. Therefore, if β avoids 456123 and 356124 this case cannot arise.

Therefore, if $\pi \in P_1$ and π avoids 456123 and 356124, then for any $1 \leq \alpha \leq \beta \leq \pi$ the embeddings of α in β cannot be split into two sets with disjoint zero sets. Thus, by Lemma 74, the interval $[\alpha, \beta]$ cannot be disconnected. Therefore, $[1, \pi]$ has no disconnected subintervals of rank 3 or more. $\square$

Part IV

A Formula for the Möbius Function of the Permutation Poset Based on a Topological Decomposition

Abstract

We present a two term formula for the Möbius function of intervals in the poset of all permutations, ordered by pattern containment. The first term in this formula is the number of so called normal occurrences of one permutation in another. Our definition of normal occurrences is similar to those that have appeared in several variations in the literature on the Möbius function of this and other posets, but simpler than most of them. The second term in the formula is complicated, but we conjecture that it equals zero for a significant proportion of intervals. We present some cases where the second term vanishes and others where it is nonzero. Computing the Möbius function recursively from its definition has exponential complexity, whereas the computation of the first term in our formula is polynomial and the exponential part is isolated to the second term, which seems to often vanish. We also present a new result on the Möbius function of posets connected by a poset fibration related to a result of Björner, Wachs and Welker.

6.1 Introduction

Let σ and π be permutations of positive integers. We define an *occurrence* of σ in π to be a subsequence of π with the same relative order of size as the letters in σ. For example, 132 occurs twice in 23541, as the subsequences 254 and 354. The *permutation poset* $\mathcal{P}$ consists of all permutations with the partial order $\sigma \leq \pi$ if there is an occurrence of σ in π. An *interval* $[\sigma, \pi]$ in $\mathcal{P}$ is the subposet $\{z \in \mathcal{P} \,|\, \sigma \leq z \leq \pi\}$. The *Möbius function* for a poset is defined recursively as: $\mu(a,b) = 0$ if $a \not\leq b$, $\mu(a,a) = 1$ for all a and, for $a < b$:

$$\mu(a,b) = -\sum_{a \leq z < b} \mu(a,z).$$

The first systematic study of the Möbius function of general posets appeared in [Rot64] and the first result pertaining to the Möbius function of intervals of $\mathcal{P}$ appeared in [SV06], where a formula for intervals of *layered* permutations was presented. A layered permutation is the direct sum of decreasing permutations, where the *direct sum* $\sigma \oplus \pi$ of two permutations σ and π is obtained by appending π to σ after adding the length of σ to each letter of π. For example, $312 \oplus 213 = 312546$. There is an analogous *skew sum* $\sigma \ominus \pi$ where π is appended to σ after the length of π is added to each element of σ. In [BJJS11] a formula for the Möbius function is presented for intervals of *decomposable* permutations, that is, permutations that can be written as the direct sum of two or more non-empty permutations. This formula, however, is recursive and bottoms out in intervals bounded by indecomposable permutations, for which there is no general formula for the Möbius function.

Furthermore, in [BJJS11] a formula is presented for intervals of *separable* permutations, that is, permutations that avoid 2413 and 3142, or equivalently, permutations that can be written using only direct sums, skew sums and the singleton permutation 1. A formula for the Möbius function of intervals of permutations with a fixed number of descents is given in [Smi14b], where a *descent* occurs at position i in a permutation $\pi = \pi_1 \dots \pi_n$ if $\pi_i > \pi_{i+1}$. Further results have been presented in [MS15, Smi14a, ST10]. However, the proportion of intervals $[\sigma, \pi]$ which satisfy any of these properties approaches zero as the length of π increases. There are indications that the formula we present here reduces the computation of the Möbius function to polynomial time for a significant proportion of intervals.

Many of the results on the Möbius function of intervals of $\mathcal{P}$, and also of some posets of words, are linked to the number of what have been termed *normal occurrences*, or *normal embeddings*, in the literature, see [Bjö90, Bjö93, BJJS11, SV06, Smi14b]. The first appearance of normal occurrences is in Björner's paper [Bjö90] where a formula for the Möbius function of intervals of words with subword order is presented. The definition of a normal occurrence has varied in these papers, but all follow a similar theme.

Our definition of normal occurrences, which is simpler than most previous ones, is based upon the *adjacencies* of a permutation, where an adjacency in a permutation is a maximal sequence of increasing or decreasing consecutively valued letters in consecutive positions and the *tail* of an adjacency is all but its first letter. A *normal occurrence* of σ in π, in our definition, is any occurrence that includes all the tails of all the adjacencies of π. This definition of normal occurrences based on adjacencies does not seem to have been considered previously, but in [Smi14b] we presented a slightly different version.

We present a formula, in Theorem 94, that shows the Möbius function of $[\sigma, \pi]$ is, up to a sign, equal to the number of normal occurrences of σ in π plus an extra term that seems to vanish for a significant proportion of intervals. For example, we know this extra term vanishes if σ and π have the same number of descents, which is a consequence of the result in [Smi14b]. Using *interval blocks*, which appear in [ST10], we prove that if for all permutations $\lambda \in [\sigma, \pi)$ there is a singleton interval block, that is, a letter of π which belongs to no occurrence of λ, the second term of the formula vanishes. The above mentioned cases are of zero proportion when the length of π goes to infinity, but computer tests indicate that for a substantial proportion of intervals the second term of our formula vanishes. Why that is the case is still a mystery, but this suggests that many more families of intervals than are now known may turn out to have a tractable Möbius function.

It is shown in [MS15] that if π is decomposable and has equal consecutive components then for any subpermutation σ obtained by removing $k > 1$ of the equal components, the interval $[\sigma, \pi]$ contains a disconnected subinterval. Many of the definitions of normal occurrences have an extra condition for the case when π has this property. We prove a result that indicates the second term of our formula for the Möbius function is often non-zero in this case. Exactly what the connection is between this second term and the topology of such intervals is another mystery.

Computing the Möbius function using the original recursive formula has exponential complexity, whereas our formula splits the computation into two parts. The first part, that is, computing the number of normal occurrences, can be done in polynomial time and the second part has exponential complexity in the general case, but computational evidence suggests that in a significant proportion of cases this second term vanishes. Our formula here is the first formula for arbitrary intervals of permutations that seems to have polynomial time complexity for a significant proportion of intervals.

In Section 6.2 we introduce some definitions, give a brief introduction to the topology of posets and present a poset fibration of $[\sigma, \pi]$ that we later use to compute $\mu(\sigma, \pi)$. In Section 6.3 we present and prove our main result, that the Möbius function of intervals of $\mathcal{P}$ equals the number of normal occurrences plus an extra term that we define. In Section 6.3.1 we present a result that links the Möbius function of two posets connected by a poset fibration satisfying a certain condition. This indicates there is possibly a more general condition for the main result of Björner, Wachs and Welker in [BWW05]. In Section 6.4 we apply our formula to show that the Möbius function of $[\sigma, \pi]$ equals the number of normal occurrences of σ in π if for each $\lambda \in [\sigma, \pi)$ there is at least one letter of π which is not in any occurrence of λ. We also show that the value of the second term of our formula for the Möbius function of $[\sigma, \pi]$ is often nonzero when π has a decomposition into a direct sum with consecutive equal components. Furthermore, we consider for which permutations all occurrences are normal.

6.2 Definitions and Preliminaries

In this section we introduce some definitions required to present our main result. We begin with an important property of permutations that is fundamental to our results:

Definition 76. *An* adjacency *in a permutation is a maximal sequence, of length* $\ell \geq 1$*, of increasing or decreasing consecutively valued letters in consecutive order. The* tail *of an adjacency of length at least* 2 *is all but the first letter of the adjacency. An adjacency of length* 1 *does not have a tail.*

Example 77. *The permutation* $\pi = 2314765$ *has adjacencies* 23*,* 1*,* 4 *and* 765 *and the tails are* 3 *and* 65*.*

Next we define embeddings and our version of normal embeddings. Embeddings are in one-to-one correspondence with occurrences, and we use embeddings instead of occurrences throughout the rest of the paper because they allow for easier presentation of the required definitions.

Definition 78. *Consider permutations* $\sigma \leq \pi$*. An* embedding η *of* σ *in* π *is a sequence of the same length as* π *such that the nonzero letters in* η *are the letters of an occurrence of* σ *in* π *and in the same positions in* η *as in* π*.*

An embedding η *of* σ *in* π *is* normal *if the positions of all the letters in all the tails of the adjacencies in* π *are nonzero in* η*. We denote the number of normal embeddings of* σ *in* π *as* $NE(\sigma, \pi)$*.*

Example 79. *For* $\sigma = 132$ *and* $\pi = 2314765$ *the sequence* 0300065 *is the only normal embedding of* σ *in* π*, so* $NE(\sigma, \pi) = 1$*.*

Proposition 80. *Computing* $NE(\sigma, \pi)$ *for a fixed* σ *can be done in time polynomial in the length of* π*.*

Proof. Counting the number of occurrences of σ in π, of lengths k and n, respectively, can be done in polynomial time $\mathcal{O}(n^k)$ by exhaustive search, and testing for normality is linear. □

We use the adjacencies of a permutation to break down the permutation and embeddings into smaller components.

Definition 81. *Consider permutations* $\sigma \leq \pi$ *and an embedding* η *of* σ *in* π*. Let* $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_t)$ *be the decomposition of* π *into its adjacencies, that is,* $\hat{\pi}_i$ *is a maximal increasing or decreasing permutation corresponding to the* i*-th adjacency of* π*.*

Define $\hat{\eta} := (\hat{\eta}_1, \ldots, \hat{\eta}_t)$ *where* $\hat{\eta}_i$ *is the permutation obtained from the nonzero letters that* η *embeds in the* i*'th adjacency of* π*. If* η *does not embed in any letters of the* i*'th adjacency then* $\hat{\eta}_i = \emptyset$*.*

Example 82. *If* $\sigma = 132$ *and* $\pi = 2314765$ *then* $\hat{\pi} = (12, 1, 1, 321)$ *and the embedding* $\eta = 0010760$ *gives* $\hat{\eta} = (\emptyset, 1, \emptyset, 21)$*.*

When considering embeddings the selection of letters within an adjacency is usually irrelevant. This is made formal by the following equivalence relation.

Definition 83. *Let* $E^{\sigma,\pi}$ *be the set of embeddings of* σ *in* π*. Define an equivalence relation on embeddings where* $\eta \sim \psi$ *if the only differences between* η *and* ψ *occur within adjacencies of* π*. Define* $\widehat{E}^{\sigma,\pi}$ *as the set containing the* *rightmost embedding, that is, the embedding where the nonzero letters are the furthest right, of each equivalence class of* $E^{\sigma,\pi}/\sim$*.*

Consider $\eta \in \widehat{E}^{\sigma,\pi}$ *and define the zero set of* η *as* $Z(\eta) = \{i \,|\, \eta_i = 0\}$*. Define* $EZ^{\sigma,\pi}$ *to be the set of sets of embeddings in* $\widehat{E}^{\sigma,\pi}$ *such that for each set* $S \in EZ^{\sigma,\pi}$ *we have* $\bigcap_{\eta\in S} Z(\eta) = \emptyset$*.*

When defining $\widehat{E}^{\sigma,\pi}$ we choose the rightmost embedding to ensure that all normal embeddings are in $\widehat{E}^{\sigma,\pi}$. Note that if $\eta \sim \psi$ then $\hat{\eta} = \hat{\psi}$, which can be used as an equivalent definition of the equivalence relation. The set $EZ^{\sigma,\pi}$ is upwards closed under containment because if we take any set $S \in EZ^{\sigma,\pi}$ adding a new embedding to S will result in a set that still has empty intersection of zero sets.

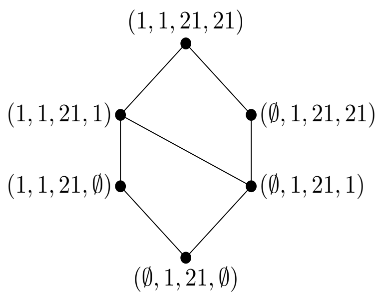
Example 84. *If* $\sigma = 132$ *and* $\pi = 413265$ *then the embedding* 013200 *has zero set* $Z(013200) = \{1, 5, 6\}$ *and*

$$\begin{aligned} E^{\sigma,\pi} =& \{013200,\ 400065,\ 010065,\ 003065,\ 000265\}, \\ \widehat{E}^{\sigma,\pi} =& \{013200,\ 400065,\ 010065,\ 000265\}, \\ EZ^{\sigma,\pi} =& \{\{013200,\ 400065\}, \{013200,\ 400065,\ 010065\}, \\ & \{013200,\ 400065,\ 000265\}, \{013200,\ 400065,\ 010065,\ 000265\}\}. \end{aligned}$$

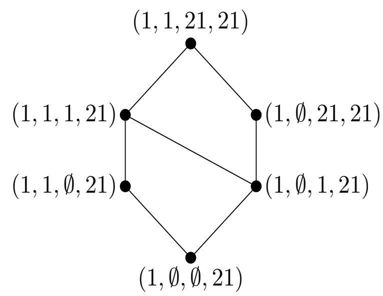
Using our decomposition we build posets from embeddings in the following way:

Definition 85. *Given an embedding* $\eta \in E^{\sigma,\pi}$ *define the poset* $P(\eta) := [\hat{\eta}_1, \hat{\pi}_1] \times \cdots \times [\hat{\eta}_t, \hat{\pi}_t]$ *and*

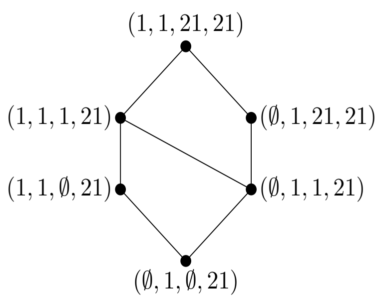
$$A^{\sigma,\pi} := \bigcup_{\eta\in\widehat{E}^{\sigma,\pi}} P(\eta)^o,$$



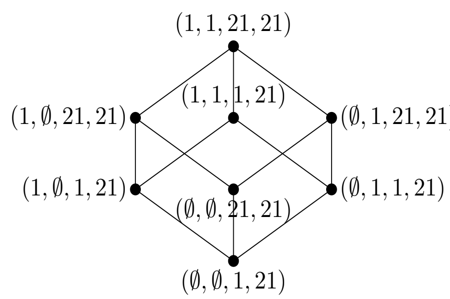
$P(013200) = [\emptyset, 1] \times [1, 1] \times [21, 21] \times [\emptyset, 21]$



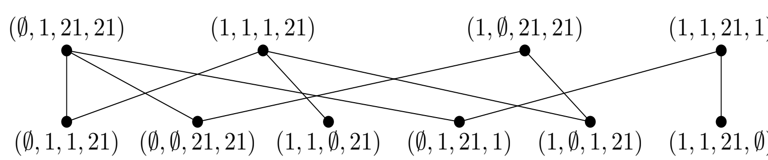
$P(400065) = [1, 1] \times [\emptyset, 1] \times [\emptyset, 21] \times [21, 21]$



$P(010065) = [\emptyset, 1] \times [1, 1] \times [\emptyset, 21] \times [21, 21]$



$P(000265) = [\emptyset, 1] \times [\emptyset, 1] \times [1, 21] \times [21, 21]$



$A^{132,413265}$

Figure 6.3: The posets of the embeddings of 132 in 413265 and the union $A^{132,413265}$ of their interiors.

where $P(\eta)^o$ denotes the interior of $P(\eta)$, that is, $P(\eta)$ with the top and bottom elements removed.

Example 86. *Consider $[132, 413265]$ and let η_1, η_2, η_3 and η_4 be the embeddings listed in $\widehat{E}^{\sigma,\pi}$ in Example 84. Then $\hat{\pi} = (1, 1, 21, 21)$ and $\hat{\eta}_1 = (\emptyset, 1, 21, \emptyset)$, $\hat{\eta}_2 = (1, \emptyset, \emptyset, 21)$, $\hat{\eta}_3 = (\emptyset, 1, \emptyset, 21)$ and $\hat{\eta}_4 = (\emptyset, \emptyset, 1, 21)$. See Figure 6.3 for $P(\eta_i)$ and $A^{132,413265}$.*

The poset $A^{\sigma,\pi}$ consists of the elements $\hat{\eta}$ for all $\eta \in \widehat{E}^{\lambda,\pi}$ and all $\lambda \in (\sigma, \pi)$. Therefore, we define a surjective poset map f from $A^{\sigma,\pi}$ to (σ, π) in the following way:

Definition 87. *Let $f : A^{\sigma,\pi} \to (\sigma, \pi)$ be the map which maps all elements $\hat{\eta}$, where $\eta \in \widehat{E}^{\lambda,\pi}$, to λ.*

Example 88. *If $[132, 413265]$ and $\hat{\eta} = (1, \emptyset, 1, 21)$ then $\eta = 400265 \in \widehat{E}^{2143,\pi}$, so $f(\hat{\eta}) = 2143$.*

6.2.1 The Topology of a Poset

We study the topology of a poset by constructing a simplicial complex from the poset in the following way:

Definition 89. *Let P be a poset. A* chain *in P is a totally ordered subset $\{z_1 < \cdots < z_t\}$. The* order complex *of P, denoted $\Delta(P)$, is the simplicial complex whose vertices are the elements of P and whose faces are the chains of P.*

When we refer to the order complex of an interval $[\sigma, \pi]$ we mean the order complex of the interior (σ, π), which we denote $\Delta(\sigma, \pi)$.

Example 90. *Consider the interval* $I = [123, 4567123]$*. An example of a chain in* $(123, 4567123)$ *is* $4123 < 456123$*. The order complex and Hasse diagram of* I *are given in Figure 6.4.*

We refer to a poset and its order complex interchangeably, so a topological property of a poset refers to that property of its order complex. For further background on order complexes and poset topology in general see [Wac07].

We can use the order complex of $[\sigma, \pi]$ to calculate $\mu(\sigma, \pi)$ due to the following formula, which is an application of the Philip Hall Theorem and the Euler-Poincaré formula for the reduced Euler characteristic, see [Wac07, Section 1.2]:

$$\mu(\sigma, \pi) = \tilde{\chi}(\Delta(\sigma, \pi)) = \sum_{i=-1}^{|\pi|-|\sigma|} (-1)^i \tilde{\beta}_i(\Delta(\sigma, \pi)), \tag{6.5}$$

where $\tilde{\chi}$ is the reduced Euler characteristic and $\tilde{\beta}_i$ is the i'th reduced Betti number, that is, the rank of the i'th reduced homology group. Therefore, by calculating the homology of $[\sigma, \pi]$ we can compute the Möbius function. For example, if we can show that $\Delta(\sigma, \pi)$ is contractible this implies $\mu(\sigma, \pi) = 0$, and if $\Delta(\sigma, \pi)$ and $\Delta(\alpha, \beta)$ are homotopically equivalent then $\mu(\sigma, \pi) = \mu(\alpha, \beta)$.

The first explicit results on the topology of intervals of permutations appear in [MS15] and [Smi14a].

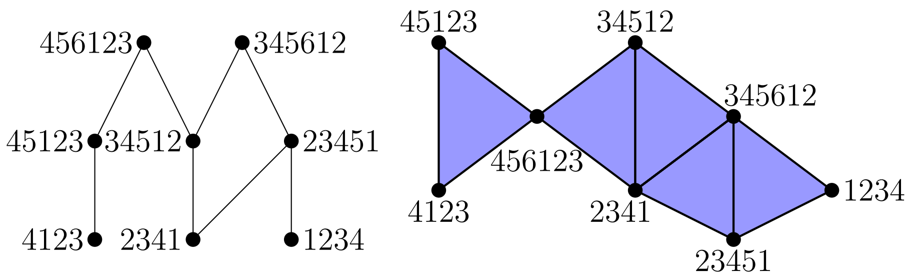


Figure 6.4: Left: Hasse diagram of $(123, 4567123)$. Right: The order complex $\Delta(123, 4567123)$.

6.3 The Main Result

We use the map f in Definition 87 to calculate the Möbius function of $[\sigma, \pi]$ by calculating $\mu(A^{\sigma,\pi})$ and the effect on the Möbius function when applying f. First we compute $\mu(A^{\sigma,\pi})$. Given a set A of posets the Möbius function of the union of A can be calculated using the following inclusion-exclusion formula, which can be seen as a consequence of the inclusion-exclusion formula for the Euler characteristic and Equation (6.5):

$$\mu\left(\bigcup_{a\in A} a\right) = \sum_{\substack{S\subseteq A\\ S\neq\emptyset}} (-1)^{|S|-1}\mu\left(\bigcap_{a\in S} a\right), \tag{6.6}$$

For more background on this see [Nar74]. Applying Equation (6.6) to $A^{\sigma,\pi}$ gives:

$$\mu(A^{\sigma,\pi}) = \sum_{\substack{S\subseteq \widehat{E}^{\sigma,\pi}\\ S\neq\emptyset}} (-1)^{|S|-1}\mu(\bigcap_{\eta\in S} P(\eta)^o). \tag{6.7}$$

To calculate this we need to know the Möbius function of the intersections $\cap_{\eta\in S}P(\eta)^o$. Note that when calculating the Möbius function of the interior (or intersection of interiors) we add the top and bottom elements back in. Therefore, a contractible intersection has Möbius function 0, an empty intersection has Möbius function -1 and $\mu(P(\eta)^o) = \mu(P(\eta))$.

Lemma 91. *If $S \subseteq \widehat{E}^{\sigma,\pi}$ and $|S| > 1$ then:*

$$\mu(\bigcap_{\eta \in S} P(\eta)^o) = \begin{cases} -1, & \textit{if } S \in EZ^{\sigma,\pi} \\ 0, & \textit{otherwise} \end{cases}.$$

Proof. Let $\hat{\pi} = (\hat{\pi}_1, \ldots, \hat{\pi}_t)$ and define the *join* of S to be $\vee S = (\max_{\eta \in S}(\hat{\eta}_1), \ldots, \max_{\eta \in S}(\hat{\eta}_t))$. The join is well defined because for each i the set $\{\eta_i \,|\, \eta \in S\}$ forms a chain, so there is a η_i that contains all others. The join of S is the smallest element contained in everything in S, so it is the bottom element of the intersection $I = \bigcap_{\eta \in S} P(\eta)^o$. Therefore, if $\vee S < \hat{\pi}$ then I is contractible and so has Möbius function 0, otherwise $\vee S = \hat{\pi}$ so I is empty and thus has Möbius function -1. If $\vee S = \hat{\pi}$ this implies that every letter of π is non-zero for some $\eta \in S$, that is, S has empty intersection of zero sets, so $S \in EZ^{\sigma,\pi}$. □

Example 92. *Consider our running example of* $[132, 413265]$. *If* $S^1 = \{013200, 010065\}$ *then* 013200 *decomposes to* $(\emptyset, 1, 21, \emptyset)$ *and* 010065 *decomposes to* $(\emptyset, 1, \emptyset, 21)$*, so the join is:*

$$\vee S^1 = (\max(\emptyset, \emptyset), \max(1, 1), \max(21, \emptyset), \max(\emptyset, 21)) = (\emptyset, 1, 21, 21).$$

Therefore, $\vee S^1 < \hat{\pi}$ *so the intersection is contractible. We can check this by looking at Figure 6.3 where we can see that the intersection* $P(013200) \cap P(010065)$ *is the single point* $(\emptyset, 1, 21, 21)$*, which is contractible.*

Now that we know the Möbius function of the intersections we can compute $\mu(A^{\sigma,\pi})$:

Lemma 93.

$$\mu(A^{\sigma,\pi}) = (-1)^{|\pi|-|\sigma|}\, NE(\sigma,\pi) + \sum_{S \in EZ^{\sigma,\pi}} (-1)^{|S|}.$$

Proof. We can split Equation (6.7) into two parts:

$$\mu(A^{\sigma,\pi}) = \sum_{\eta \in \widehat{E}^{\sigma,\pi}} \mu(P(\eta)^o) + \sum_{\substack{S \subseteq \widehat{E}^{\sigma,\pi} \\ |S|>1}} (-1)^{|S|-1} \mu(\bigcap_{\eta \in S} P(\eta)^o). \tag{6.8}$$

By Lemma 91 the second part of the right hand side of Equation (6.8) equals $\sum_{S \in EZ^{\sigma,\pi}} (-1)^{|S|}$.

By the definition of $P(\eta)$, and the identity $\mu(A \times B) = \mu(A)\mu(B)$, we know

$$\mu(P(\eta)) = \prod_{1 \le i \le t} \mu(\hat{\eta}_i, \hat{\pi}_i).$$

We know that $[\hat{\eta}_i, \hat{\pi}_i]$ is always a chain, so by the definition of normality if η is not normal there is some i such that $|\hat{\eta}_i| \le |\hat{\pi}_i| - 2$, so $\mu(\hat{\eta}_i, \hat{\pi}_i) = 0$ which implies $\mu(P(\eta)) = 0$. If η is normal then $|\hat{\pi}_i| - |\hat{\eta}_i| = 0$ or 1, so $\mu(\hat{\eta}_i, \hat{\pi}_i) = 1$ or -1, for all i. There are $|\pi| - |\sigma|$ parts $[\hat{\eta}_i, \hat{\pi}_i]$ with $\mu(\hat{\eta}_i, \hat{\pi}_i) = -1$, one for each zero in η, and the remaining have $\mu(\hat{\eta}_i, \hat{\pi}_i) = 1$. Therefore, $\mu(P(\eta)^o) = \mu(P(\eta)) = (-1)^{|\pi|-|\sigma|}$ for each normal embedding, so the first term in the right hand side of Equation (6.8) equals $(-1)^{|\pi|-|\sigma|}\, \mathrm{NE}(\sigma,\pi)$. □

We now present our formula for the Möbius function that applies to any interval of permutations:

Theorem 94. *For any permutations σ and π:*

$$\mu(\sigma,\pi) = (-1)^{|\pi|-|\sigma|}\, NE(\sigma,\pi) + \sum_{\lambda \in [\sigma,\pi)} \mu(\sigma,\lambda) \sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|}. \tag{6.9}$$

Proof. We take the poset $A^{\sigma,\pi}$ and for each $\lambda \in (\sigma, \pi)$ we retract $\widehat{E}^{\lambda,\pi}$ to a point we denote λ. This transforms $A^{\sigma,\pi}$ into the interval (σ, π). We need to know what effect this has on the Möbius function of $A^{\sigma,\pi}$.

We work our way from the bottom to the top so we can assume that all elements below the elements of $\widehat{E}^{\lambda,\pi}$ have already been retracted and all elements above have not. Define the poset $W(\lambda) := \{\tau \in A^{\sigma,\pi} \mid \tau \leq \eta$ or $\tau \geq \eta$ for some $\eta \in \widehat{E}^{\lambda,\pi}\}$. When we retract the elements of $\widehat{E}^{\lambda,\pi}$ to λ we retract $W(\lambda)$ onto a contractible poset, since in that poset the element λ is comparable to all other elements and thus represents a cone point in the corresponding order complex. This implies the change to the Möbius function is $-\mu(W(\lambda))$.

To compute $\mu(W(\lambda))$ we split $W(\lambda)$ into two disjoint parts

$$W(\lambda)^< := \{\tau \in W(\lambda) \mid \tau < \eta \text{ for some } \eta \in \widehat{E}^{\lambda,\pi}\},$$
$$W(\lambda)^\geq := \{\tau \in W(\lambda) \mid \tau \geq \eta \text{ for some } \eta \in \widehat{E}^{\lambda,\pi}\}.$$

The poset $W(\lambda)^<$ is isomorphic to (σ, λ) because all points below λ have already been retracted. The poset $W(\lambda)^\geq$ is equal to $\bigcup_{\eta \in \widehat{E}^{\lambda,\pi}} (P(\eta) \setminus \hat{\pi})$ which has Möbius function $-\sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|}$, by Lemma 91 and the inclusion-exclusion formula (this also follows from the Crosscut Theorem, see Proposition 104).

Because every element of $\widehat{E}^{\lambda,\pi}$ lies above every element of (σ, λ) this implies $W(\lambda) = W(\lambda)^< \star W(\lambda)^\geq$, where $\star$ denotes the topological join. Therefore,

$$-\mu(W(\lambda)) = -\mu(W(\lambda)^<)\mu(W(\lambda)^\geq) = \mu(\sigma, \lambda) \sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|}.$$

So we start with $\mu(A^{\sigma,\pi})$, given by Lemma 93, and then subtract $\mu(W(\lambda))$ for each $\lambda \in (\sigma, \pi)$, which gives the desired formula. □

Remark 95. *Computer tests indicate that about 95% of intervals $[\sigma, \pi]$, where*

$|\pi| < 9$, satisfy $\mu(\sigma, \pi) = (-1)^{|\pi|-|\sigma|} NE(\sigma, \pi)$. Thus, for these intervals the latter term in Equation (6.9) is zero.

Remark 96. *The complexity of counting the number of normal embeddings is polynomial so in the cases where we can show that the latter term of Equation (6.9) equals zero we have a polynomial time formula for the Möbius function. This is a dramatic improvement over the original recursive formula that has exponential complexity. However, computing the latter term of Equation (6.9) also has exponential complexity.*

Tests show that using Equation (6.9) is often much quicker than computing the Möbius function using the recursive formula. When computing the Möbius function of the rank 15 interval

$$[54123, 9\,7\,10\,4\,8\,1\,2\,6\,5\,3\,19\,17\,20\,14\,18\,11\,12\,16\,15\,13],$$

the formula in Equation (6.9) took 1.75 minutes and the recursive formula took 13.5 hours. Note that this interval has Möbius function -3 but no normal embeddings so the latter term of Equation (6.9) is nonzero in this case. Furthermore, using Equation (6.9) we were able to compute the Möbius function of a rank 16 interval in 1 hour and a rank 17 interval in 6 hours. However, if σ has a large number of occurrences in π then using Equation (6.9) can be quite slow. For example, if $\sigma = 2413$

and $\pi = 2\,4\,6\,8\,10\,1\,3\,5\,7\,9$ then there are 35 occurrences of σ in π and $\mu(\sigma, \pi)$ can be computed in 0.06 seconds using the recursive formula but takes 15.5 hours using Equation (6.9).

6.3.1 Poset Fibration

In this subsection we present a generalisation of the argument used to prove Theorem 94. We can view the pair $((\sigma, \pi), \{\widehat{E}^{\lambda,\pi}\}_{\lambda \in (\sigma,\pi)})$ as a poset fibration which makes f the projection map and $A^{\sigma,\pi}$ the total space. In [BWW05] various theorems are presented which relate two posets P and Q linked by a poset fibration f satisfying a certain condition, see Theorem 2.5 of [BWW05] for the most general form of this condition. However, our poset fibration does not always satisfy this condition, for example the condition is not true on the interval $[1, 456123]$. We present a new result with a different condition on the poset fibration that generalises the argument in the proof of Theorem 94. We let f^* and f^{-1} denote the image and preimage of f, respectively.

Proposition 97. *Let $f : P \to Q$ be a surjective poset map such that $f^*(P_{<p}) = Q_{<q}$, for any $q \in Q$ and $p \in f^{-1}(q)$. Then*

$$\mu(Q) = \mu(P) - \sum_{q \in Q} \mu(Q_{<q})\mu(f^{-1}(Q_{\geq q})).$$

Proof. We begin with P and for each $q \in Q$ we retract $f^{-1}(q)$ to a single point and observe the effect this retraction has on the Möbius function of P. We do this inductively from the bottom to the top, so when considering $q \in Q$ we assume all points in $P_{<p}$, for all $p \in f^{-1}(q)$, have been retracted.

To calculate the effect the retraction has on the Möbius function of the poset we consider

$$W(q) := \{p \in P \mid p < \lambda \text{ or } p \geq \lambda \text{ for some } \lambda \in f^{-1}(q)\}.$$

When we retract $f^{-1}(q)$ to a point we retract $W(q)$ to a contractible poset, which implies the change to the Möbius function is $-\mu(W(q))$. We can rewrite $W(q)$ in the following way:

$$\begin{aligned} W(q) &= \bigcup_{p \in f^{-1}(q)} P_{<p} \star P_{\geq p} = \bigcup_{p \in f^{-1}(q)} Q_{<q} \star P_{\geq p} \\ &= Q_{<q} \star \bigcup_{p \in f^{-1}(q)} P_{\geq p} = Q_{<q} \star f^{-1}(Q_{\geq q}). \end{aligned}$$

We can replace $P_{<p}$ with $Q_{<q}$ because our induction assumption is that $P_{<p}$ has been retracted and our condition of the proposition is $f^*(P_{<p}) = Q_{<q}$. Therefore, $-\mu(W(q)) = -\mu(Q_{<q})\mu(f^{-1}(Q_{\geq q}))$ and summing over all $q \in Q$ completes the proof. $\square$

Remark 98. *An interesting question is whether Proposition 97 can be generalised to show homotopy equivalence. Also, is there a more general condition that encompasses the conditions of both Proposition 97 and Theorem 2.5 of [BWW05]?*

6.4 Applications

By Theorem 94 we know that the Möbius function is linked to the number of normal embeddings, which depend on the adjacencies.

Lemma 99. *The average total number of letters in the tails of adjacencies in a permutation of length n is $2(\frac{n-1}{n})$. In particular, when n tends to infinity the average number of letters in the tails of adjacencies tends to 2.*

Proof. Note first that $k = 1$ cannot be in the tail of an increasing adjacency and n cannot be in the tail of a decreasing adjacencies. For $k > 1$ the number of permutations of length n in which k is in the tail of an increasing adjacency is $(n-1)!$, because these are precisely all permutations of the letters $1, 2, \ldots, n$ where $(k-1)k$ is regarded as a single letter. So the probability that a letter $k > 1$ is in the tail of an increasing adjacency is $(n-1)!/n! = 1/n$. Likewise, the probability that a letter $k < n$ is in the tail of a decreasing adjacency is $1/n$. Therefore, the probability that k is in the tail of an adjacency is

$$\begin{cases} \frac{1}{n}, & \text{if } k = 1 \text{ or } n \\ \frac{2}{n}, & \text{otherwise} \end{cases}.$$

Summing over all letters $k = 1, \ldots, n$ completes the proof. □

An embedding in a permutation π is likely to be normal if there is only a small proportion of letters in the tails of the adjacencies of π. Therefore, Lemma 99 indicates that the proportion of embeddings of λ in a random permutation π that are normal increases as the length of π increases. If a permutation has no adjacencies of size $\ell > 1$ then all embeddings will be normal, the proportion of such permutations tends to $1/e^2$ as the length of the permutations increase, see [Slo91].

By Remark 95 we suspect that the second part of Equation (6.9) vanishes for a significant proportion of intervals. The key to simplifying Equation (6.9) is answering the following question:

Question 100. *Given an interval* $[\sigma, \pi]$*, for which* $\lambda \in [\sigma, \pi)$ *is the following sum nonzero?:*

$$\sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|}. \tag{6.10}$$

A consequence of Proposition 3.3 in [Smi14b] is that if σ and π have the same number of descents then the sum in (6.10) equals zero for all $\lambda \in [\sigma, \pi)$. Another case where the sum in (6.10) equals zero is when $EZ^{\lambda,\pi}$ is always empty, which leads us to the following definition and proposition:

Definition 101. *We say an interval* $[\sigma, \pi]$ *has a* single block *if there exists some* i *such that* $\eta_i = 0$ *for any* $\eta \in \widehat{E}^{\sigma,\pi}$*. That is, there is a letter in* π *that is not contained in any of the occurrences in* $\widehat{E}^{\sigma,\pi}$*.*

We say an interval is single *if for all* $\lambda \in [\sigma, \pi)$ *the interval* $[\lambda, \pi]$ *has a single block.*

Our notation here follows from the idea of interval blocks in [ST10]. Computer tests show that 78.6% of intervals $[\sigma, \pi]$, where $|\pi| < 9$, are single and we found that about 39% of 48300 random rank 10 intervals, where $|\sigma| = 5$ and $|\pi| = 15$, are single. We suspect that the proportion of intervals that are single approaches zero as the rank tends to infinity, because the likelihood that there exists some $\lambda \in [\sigma, \pi]$ such that $[\lambda, \pi]$ does not have a single block increases as the rank of the interval increases.

Proposition 102. *If* $[\sigma, \pi]$ *is single then* $\mu(\sigma, \pi) = (-1)^{|\pi|-|\sigma|}\, NE(\sigma, \pi)$*.*

Proof. If $[\lambda, \pi]$ has a single block then $EZ^{\lambda,\pi}$ must be empty which implies $\sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|} = 0$. Therefore, if $[\sigma, \pi]$ is single then $\sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|} = 0$ for all $\lambda \in [\sigma, \pi)$, combining this with Equation (6.9) completes the proof. □

Intervals that contain a disconnected subinterval of rank at least 3 are non-shellable, as shown by Björner in [Bjö80], and thus not amenable to some of the elegant methods of topological combinatorics, see [MS15] for further background. In the rest of the paper we consider a particular type of interval that is known to be disconnected and show that the sum in (6.10) is nonzero for these intervals. Whether there is a topological "reason" for (6.10) being nonzero in these cases we don't know.

We consider decomposable permutations and write them in the form $\pi_1 \oplus \cdots \oplus \pi_n$ where each π_i, which we call a *component* of π, is indecomposable. Consider a permutation π, with a sequence of $\ell \geq 2$ equal consecutive components, and $\lambda \leq \pi$ obtained from π by removing k of the components from this sequence, where $\ell > k \geq 1$. The interval $[\lambda, \pi]$ is disconnected, which follows from results in [MS15], specifically Lemma 4.2 and Theorem 5.6. These intervals are the cause of the extra conditions in the formulas for the Möbius function that appear in [BJJS11] and [MS15].

The simplest permutations with equal consecutive components are the following:

Definition 103. *Given an indecomposable permutation λ let*

$$\lambda^n := \underbrace{\lambda \oplus \cdots \oplus \lambda}_{\times n}.$$

Before we continue let us introduce a very useful result for computing the Möbius function of lattices known as the Crosscut Theorem. We denote the join of $A \subseteq X$ as $\vee A := \min\{x \in X \mid x \geq a \text{ for all } a \in A\}$:

Proposition 104. *(Crosscut Theorem, see [Sta12, Corollary 3.9.4]) Let L be a finite lattice with top element $\hat{1}$ and bottom element $\hat{0}$. Let X be a subset of L such that $\hat{0} \notin X$ and for all $s \in L$, $s \neq \hat{0}$, there is some $t \in X$ such that $s \geq t$. Then*

$$\mu(\hat{0}, \hat{1}) = \sum_{\substack{A \subseteq X \\ \vee A = \hat{1}}} (-1)^{|A|}.$$

The Crosscut theorem is traditionally used to compute the Möbius function of a lattice, but we can use it in reverse if we can represent our problem as a lattice for which we already know the Möbius function.

Consider the interval $[\lambda^m, \lambda^n]$, for some indecomposable permutation λ, the embeddings of λ^m in λ^n can be considered as subsets of $[n] := \{1, \ldots, n\}$ of size m. So we can regard our problem as that of computing the Möbius function of a sublattice of the Boolean lattice:

Definition 105. *The* Boolean lattice B_n *is the poset of subsets of $[n]$ with the partial order given by inclusion.*

Define the truncated Boolean lattice $B_n^{\geq k}$ *as the subposet of B_n where all elements $a \in B_n$ such that $|a| \leq k$ are retracted to a single point $\hat{0}$. Similarly, define $B_n^{\leq k}$ as the subposet of B_n where all elements $a \in B_n$ such that $|a| \geq k$ are retracted to a single point $\hat{1}$.*

We take the notation for a truncated Boolean lattice from [Wac07, Section 3.2.1]. See Figure 6.5 for examples of truncated Boolean lattices.

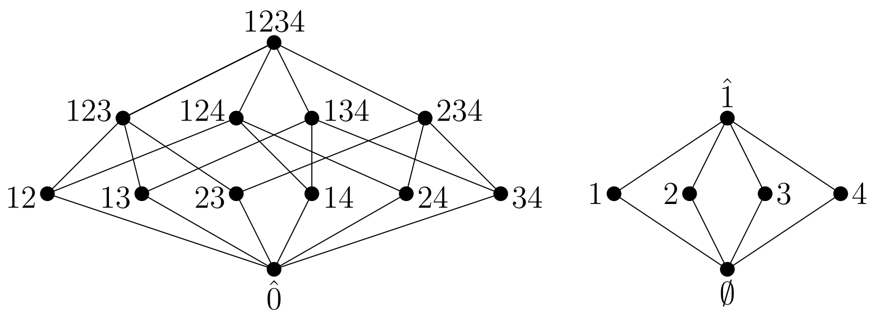


Figure 6.5: The truncated Boolean lattices $B_4^{\geq 2}$ and $B_4^{\leq 1}$.

The embeddings of λ^m in λ^n can be viewed as the atoms of the truncated Boolean lattice $B_n^{\geq m}$, so using the Crosscut theorem we can compute the sum in (6.10) for $[\lambda^m, \lambda^n]$ by computing $\mu(B_n^{\geq m})$. The Möbius function of a Boolean lattice is given by $\mu(B_n) = (-1)^n$, see [Rot64, Section 3]. We can use this to compute the Möbius function of the truncated Boolean lattice.

Lemma 106. *The Möbius function of a truncated Boolean lattice is given by:*

$$\mu(B_n^{\leq k}) = (-1)^{k-1}\binom{n-1}{k} \qquad \textit{and} \qquad \mu(B_n^{\geq k}) = (-1)^{n-k-1}\binom{n-1}{k-1}.$$

Proof. First consider $B_n^{\leq k}$. For each element $\lambda \in B_n^{\leq k}$, with $|\lambda| = \ell$, the interval $[\emptyset, \lambda]$ is isomorphic to the boolean lattice B_ℓ, therefore $\mu(\emptyset, \lambda) = (-1)^\ell$. There are $\binom{n}{\ell}$ elements in $B_n^{\leq k}$ with size ℓ, for $0 \leq \ell \leq k$. To compute $\mu(B_n^{\leq k})$ we need to sum all elements and negate, we do this by summing over ℓ. We can then apply an identity on the alternating sum of binomial coefficients, a proof of which can be found in Section 0 of [Kle63], this gives:

$$\mu(B_n^{\leq k}) = -\sum_{\ell=0}^{k}(-1)^\ell\binom{n}{\ell} = (-1)^k\binom{n-1}{k}.$$

Note that the lattice $B_n^{\geq k}$ is isomorphic to $(B_n^{\leq n-k})^*$, the dual of $B_n^{\leq n-k}$, that is, the lattice with the partial order reversed. Therefore, $\mu(B_n^{\geq k}) = \mu((B_n^{\leq n-k})^*) = \mu(B_n^{\leq n-k})$ which completes the proof. □

We can now present our result for the interval $[\lambda^m, \lambda^n]$:

Proposition 107. *Let λ be an indecomposable permutation, of length $\ell > 1$, and consider the interval $[\lambda^m, \lambda^n]$. Then:*

$$\sum_{S \in EZ^{\lambda^m, \lambda^n}} (-1)^{|S|} = (-1)^{n-m-1} \binom{n-1}{m-1}.$$

Proof. We can consider each embedding of λ^m in λ^n as a subset $a \subseteq \{1, \ldots, n\}$ with $|a| = m$. Therefore, the embeddings correspond to the atoms of the lattice $B_n^{\geq m}$. So we can apply the Crosscut theorem and Lemma 106 to complete the proof. □

We can generalise this result by adding in other components to the permutations to get the following proposition:

Proposition 108. *Consider a decomposable permutation $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ with a sequence of consecutive components $\pi_{i+1} = \cdots = \pi_{i+\alpha}$, with $\alpha > 1$. Let $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_m$ be the subpermutation of π obtained by reducing the sequence of equal components to length ℓ, for $0 \leq \ell \leq \alpha$. Then:*

$$\sum_{S \in EZ^{\lambda, \pi}} (-1)^{|S|} = (-1)^{\alpha-\ell-1} \binom{\alpha-1}{\ell-1}. \tag{6.11}$$

Proof. Let $\tilde{\pi} := \pi_{i+1} \oplus \cdots \oplus \pi_{i+\alpha}$ and $\tilde{\lambda} := \lambda_{i+1} \oplus \cdots \oplus \lambda_{i+\ell}$. Note that $\tilde{\pi} = \pi_{i+1}^{\alpha}$ and $\tilde{\lambda} = \pi_{i+1}^{\ell}$.

Given any set $S \in EZ^{\lambda,\pi}$ let $\tilde{S} \in EZ^{\tilde{\lambda},\tilde{\pi}}$ be the unique set which has the elements $\tilde{\eta} = \eta_{i+1} \oplus \cdots \oplus \eta_{i+\alpha}$, for each $\eta \in S$, with repetitions removed. Moreover, given any $A \in EZ^{\tilde{\lambda},\tilde{\pi}}$ consider the set $g^{\lambda,\pi}(A) := \{S \in EZ^{\lambda,\pi} \mid \tilde{S} = A\}$. We can rewrite the sum in Equation (6.11) as:

$$\sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|} = \sum_{A \in EZ^{\tilde{\lambda},\tilde{\pi}}} \sum_{S \in g^{\lambda,\pi}(A)} (-1)^{|S|}.$$

If we can show that $\sum_{S \in g^{\lambda,\pi}(A)} (-1)^{|S|} = (-1)^{|A|}$ for each $A \in EZ^{\tilde{\lambda},\tilde{\pi}}$ then the result follows from Proposition 107.

Claim 1. *Given any $A \in EZ^{\tilde{\lambda},\tilde{\pi}}$ we have* $\displaystyle\sum_{S \in g^{\lambda,\pi}(A)} (-1)^{|S|} = (-1)^{|A|}$.

Proof. We proceed by induction on n, the number of components of π. If $n = \alpha$, that is, $\tilde{\pi} = \pi$, then $g^{\lambda,\pi}(A) = \{A\}$ and the result follows trivially. Suppose that the claim is true if $n = \alpha + t - 1$ and consider the case where we prepend a component to π so $n = \alpha + t$, the case for appending a component is analogous. Let $\pi^{>1} := \pi_2 \oplus \cdots \oplus \pi_n$ and $\lambda^{>1} := \lambda_2 \oplus \cdots \oplus \lambda_m$.

For each set $S \in g^{\lambda,\pi}(A)$ we can remove η_1 from each $\eta \in S$ to get a unique set $S^{>1} \in g^{\lambda^{>1},\pi^{>1}}(A)$. Moreover, given a $B \in g^{\lambda^{>1},\pi^{>1}}(A)$ define the set $f(B) = \{S \in g^{\lambda,\pi}(A) \mid S^{>1} = B\}$. Then we can reformulate the sum in the claim as:

$$\sum_{S \in g^{\lambda,\pi}(A)} (-1)^{|S|} = \sum_{B \in g^{\lambda^{>1},\pi^{>1}}(A)} \sum_{S \in f(B)} (-1)^{|S|} \tag{6.12}$$

To compute the final sum in Equation (6.12) we need to consider what sets are in $f(B)$ given some set $B \in EZ^{\lambda^{>1},\pi^{>1}}$, with cardinality k. We can extend B to a set in $S \in EZ^{\lambda,\pi}$ by choosing where to embed λ_1 in π. For each embedding in $B = \{\eta^1, \ldots, \eta^k\}$ let ρ_j be the location of the leftmost component of $\pi^{>1}$ that η^j embeds in, so $2 \leq \rho_j$.

First we check that when extending B we get a set S in $EZ^{\lambda,\pi}$, that is, for all i there exists some $\psi \in S$ such that $\psi_i = \pi_i$. We know this is true for $i > 1$ because $B \in EZ^{\lambda^{>1},\pi^{>1}}$ and it is true for $i = 1$ because $\lambda_1 = \pi_1$ and there is some embedding $\eta^j \in B$ for which $\rho_j = 2$ which implies that to extend η^j we must embed λ_1 in π_1. Therefore, however we extend B it is in $EZ^{\lambda,\pi}$.

So we can extend each embedding $\eta_j \in B$ by choosing to embed λ_1 in one of $\pi_1, \ldots, \pi_{\rho_j}$ and we can take any number $t \geq 1$ of these embeddings to create a set in $f(B)$. This gives us the following equation:

$$\sum_{S \in f(B)} (-1)^{|S|} = \prod_{j=1}^{|B|} \sum_{\ell=1}^{\rho_j} (-1)^{|B|+\ell} \binom{\rho_j}{\ell} = (-1)^{|B|}. \tag{6.13}$$

So by Equation (6.13) we can replace the final sum in Equation (6.12) with $(-1)^{|B|}$ and we can use the inductive hypothesis to complete the proof.

□

□

Example 109. *Consider the permutation* $\pi = 2\,1\,5\,3\,4\,8\,6\,7\,11\,9\,10\,13\,12$ *which has the decomposition* $21 \oplus 312 \oplus 312 \oplus 312 \oplus 21$ *and so has a sequence of equal components* $\pi_2 = \pi_3 = \pi_4 = 312$ *of length* $\alpha = 3$. *If* $\lambda = 21 \oplus 312 \oplus 312 \oplus 21 = 2\,1\,5\,3\,4\,8\,6\,7\,10\,9$, *so* $\ell = 2$, *then Proposition 108 says*

$$\sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|} = (-1)^{3-2-1} \binom{3-1}{2-1} = 2.$$

We conjecture that we can generalise this further by removing elements from more than one sequence of equal components:

Conjecture 110. *Consider a decomposable permutation* $\pi = \pi_1 \oplus \cdots \oplus \pi_n$ *which has* t *sequences of equal components* $\pi_{i+1} = \cdots = \pi_{i+\alpha_i} \neq 1$ *of respective lengths* α_i*, for* $1 \leq i \leq t$*. Let* $\lambda = \lambda_1 \oplus \cdots \oplus \lambda_m$ *be the permutation obtained from* π *by, for each* i*, reducing the* i*-th sequence to length* ℓ_i*, with* $0 \leq \ell_i \leq \alpha_i$*. Then:*

$$\sum_{S \in EZ^{\lambda,\pi}} (-1)^{|S|} = (-1)^{\alpha-\ell-1} \prod_{i=1}^{t} \binom{\alpha_i - 1}{\ell_i - 1},$$

where $\alpha = \alpha_1 + \cdots + \alpha_t$ *and* $\ell = \ell_1 + \cdots + \ell_t$*.*

Example 111. *Consider the permutation* $\pi = 2\,1\,4\,3\,7\,5\,6\,10\,8\,9\,13\,11\,12$ *which has the decomposition* $21 \oplus 21 \oplus 312 \oplus 312 \oplus 312$ *so has* 2 *sequences of equal components* 21, 21 *and* 312, 312, 312 *of lengths* $\alpha_1 = 2$ *and* $\alpha_2 = 3$*. If* $\lambda = 21 \oplus 312 \oplus 312 = 21534867$*, so* $\ell_1 = 1$ *and* $\ell_2 = 2$*, the sum in* (6.10) *equals* -2*, this agrees with the formula in Conjecture 110:*

$$(-1)^{5-3-1} \binom{2-1}{1-1} \binom{3-1}{2-1} = -2.$$

Bibliography

[BF13] Mathilde Bouvel and Luca Ferrari. On the enumeration of d-minimal permutations. *Discrete Mathematics & Theoretical Computer Science*, 15(2):33–48, 2013. Cited on page 37.

[BJJS11] Alexander Burstein, Vít Jelínek, Eva Jelínková, and Einar Steingrímsson. The Möbius function of separable and decomposable permutations. *Journal of Combinatorial Theory. Series A*, 118(8):2346–2364, 2011. Cited on pages 18, 33, 34, 42, 77, 93, 94, and 112.

[Bjö80] Anders Björner. Shellable and Cohen-Macaulay partially ordered sets. *Transactions of the American Mathematical Society*, 260(1):159–183, 1980. Cited on pages 69 and 112.

[Bjö90] Anders Björner. The Möbius function of subword order. *Institute for Mathematics and its Applications*, 19:118, 1990. Cited on pages 9, 19, 20, 67, 74, 75, 76, and 94.

[Bjö93] Anders Björner. The Möbius function of factor order. *Theoretical Computer Science*, 117(1):91–98, 1993. Cited on page 94.

[BW96] Anders Björner and Michelle Wachs. Shellable nonpure complexes and posets I. *Transactions of the American Mathematical Society*, 348(4):1299–1327, 1996. Cited on page 25.

[BW97] Anders Björner and Michelle Wachs. Shellable nonpure complexes and posets II. *Transactions of the American Mathematical Society*, 349:3945–3976, 1997. Cited on page 25.

[BWW05] Anders Björner, Michelle Wachs, and Volkmar Welker. Poset fiber theorems. *Transactions of the American Mathematical Society*, 357(5):1877–1899, 2005. Cited on pages 96, 108, and 109.

[Eul68] Leonhard Euler. Remarques sur un beau rapport entre les séries des puissances tant directes que réciproques. *Memoires de l'academie des sciences de Berlin*, 17:83–106, 1768. Not cited.

[GAP14] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.4*, 2014. Not cited.

[Hat02] Allen Hatcher. *Algebraic Topology*. Cambridge University Press, 2002. Cited on page 25.

[Jon08] Jakob Jonsson. *Simplicial Complexes of Graphs*. Number 1928 in Lecture Notes in Mathematics. Springer, 2008. Cited on page 25.

[Kit11] Sergey Kitaev. *Patterns in Permutations and Words*. Monographs in Theoretical Computer Science. An EATCS Series. Springer, Heidelberg, 2011. Cited on pages 7 and 13.

[Kle63] Victor Klee. The Euler characteristic in combinatorial geometry. *American Mathematical Monthly*, pages 119–127, 1963. Cited on page 114.

[Knu68] Donald E Knuth. *The Art of Computer Programming, Vol. 1: Fundamental Algorithms*. Addison-Wesley, Reading, Mass, 1968. Cited on page 7.

[Knu70] Donald E. Knuth. Permutations, matrices, and generalized young tableaux. *Pacific Journal of Mathematics*, 34(3):709–727, 1970. Cited on page 7.

[Koz08] Dmitry Kozlov. *Combinatorial Algebraic Topology*. Algorithms and Computation in Mathematics. Springer, 2008. Cited on pages 25, 27, 68, and 84.

[MB71] P. Mani and H. Bruggesser. Shellable decompositions of cells and spheres. *Mathematica Scandinavica*, 29:197–205, 1971. Cited on page 26.

[Mil77] Stephen Milne. Restricted growth functions and incidence relations of the lattice of partitions of an n-set. *Advances in Mathematics*, 26(3):290–305, 1977. Cited on page 70.

[Möb32] August Ferdinand Möbius. Über eine besondere Art von Umkehrung der Reihen. *Journal für die Reine und Angewandte Mathematik*, 9:105–123, 1832. Cited on page 7.

[MS12] Peter R. W. McNamara and Bruce E. Sagan. The Möbius function of generalized subword order. *Advances in Mathematics*, 229(5):2741–2766, 2012. Cited on page 69.

[MS15] Peter R. W. McNamara and Einar Steingrímsson. On the topology of the permutation pattern poset. *Journal of Combinatorial Theory, Series A*, 134:1–35, 2015. Cited on pages 18, 28, 69, 86, 94, 95, 102, and 112.

[Nar74] Hiroshi Narushima. Principle of inclusion-exclusion on semilattices. *Journal of Combinatorial Theory, Series A*, 17(2):196–203, 1974. Cited on page 103.

[Rog78] D. G. Rogers. Ascending sequences in permutations. *Discrete Mathematics*, 22(1):35–40, 1978. Cited on page 7.

[Rot64] Gian-Carlo Rota. On the foundations of combinatorial theory I. Theory of Möbius functions. *Probability Theory and Related Fields*, 2(4):340–368, 1964. Cited on pages 7, 93, and 114.

[Rot81] Doron Rotem. Stack sortable permutations. *Discrete Mathematics*, 33(2):185–196, 1981. Cited on page 7.

[Sch01] Ludwig Schlafli. *Theorie der vielfachen Kontinuität*. Denkschriften der Schweizerischen Akademie der Naturwisschenschaften 38. Birkhäuser Basel, 1901. Cited on page 26.

[Slo91] Neil J. A. Sloane. The On-Line Encyclopedia of Integer Sequences. Published electronically at http://oeis.org, 1991. Sequence A002464. Cited on page 110.

[Smi14a] Jason P. Smith. On the Möbius function of permutations with one descent. *The Electronic Journal of Combinatorics*, 21:2.11, 2014. Cited on pages 1, 8, 69, 76, 77, 78, 81, 94, and 102.

[Smi14b] Jason P. Smith. Intervals of permutations with a fixed number of descents are shellable. *To appear in Discrete Mathematics*, 2014. arXiv:1405.2560v3. Cited on pages 1, 9, 56, 94, 95, and 111.

[Smi15] Jason P. Smith. A formula for the Möbius function of the permutation poset based on a topological decomposition. *arXiv preprint arXiv:1506.04406*, 2015. Cited on pages 1 and 9.

[SS85] Rodica Simion and Frank W. Schmidt. Restricted permutations. *European Journal of Combinatorics*, 6(4):383 – 406, 1985. Cited on page 7.

[ST10] Einar Steingrímsson and Bridget Eileen Tenner. The Möbius function of the permutation pattern poset. *Journal of Combinatorics*, 1(1):39–52, 2010. Cited on pages 33, 77, 94, 95, and 111.

[Sta96] Richard P. Stanley. *Combinatorics and Commutative Algebra*. Birkhčauser (Boston), 1996. Cited on pages 68 and 70.

[Sta12] Richard P. Stanley. *Enumerative Combinatorics. Vol. 1*. Cambridge Studies in Advanced Mathematics. Cambridge University Press, second edition, 2012. Cited on pages 7, 71, and 113.

[Ste13] Einar Steingrímsson. Some open problems on permutation patterns. In *Surveys in Combinatorics*, 409, pages 239–263. London Mathematical Society Lecture Note Series, 2013. Cited on pages 7 and 34.

[SV06] Bruce E. Sagan and Vincent Vatter. The Möbius function of a composition poset. *Journal of Algebraic Combinatorics*, 24(2):117–136, 2006. Cited on pages 17, 33, 68, 69, 75, 93, and 94.

[Wac07] Michelle L. Wachs. Poset topology: Tools and applications. In *Geometric Combinatorics*, volume 13 of *IAS/Park City Math. Ser.*, pages 497–615. Amer. Math. Soc., 2007. Cited on pages 7, 26, 27, 67, 68, 75, 102, and 113.