Discrete-State-Feedback Stabilisation of Highly Nonlinear Stochastic Hybrid Systems

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Abstract

A lot of practical systems, whose structure and parameters may change abruptly, can be modeled by the stochastic differential equations driven by continuous-time Markov chain (also known as hybrid SDEs). Among many interesting topics, stability has drawn intensive attention. In the case when a given hybrid SDE is unstable, it is a general practice to use state feedback control to achieve stabilisation. Theoretically, the design of feedback control is based on continuous-time state observations. However, in practice, it will be extremely costly and impossible to have continuous observations of the state for all time. So it is more realistic and costs less if the state is only observed at discrete times.

(Mao 2013) started the study of discrete-state-feedback stabilisation of hybrid SDEs. After several years' development, most results paid much attention to hybrid SDEs satisfying the classical linear growth condition, which could exclude many important real models. In this thesis, we will hence investigate this stabilisation problem without this restrictive condition, namely in highly nonlinear area.

We will firstly make some improvements on the existing results on this satbilisation problem of highly nonlinear hybrid SDEs. A new method will be given to estimate the difference between current-time state and discrete-time state, so that conditions imposed on the underlying systems will be less restrictive. To determine the upper bound of the observation duration, we will use optimisation method to avoid searching for free parameters and make the control rules be much more easily to verify.

Then, by taking different system structures (except for different system coefficients) in different Markovian modes into consideration, we will study the structured stabilisation of hybrid SDEs. The system structures are classified according to the view of Khasminskii-type condition. The control function will be designed

in a bounded state area, rather than every observable state, in order to reduce control cost. Further, we will extend the structured stabilisation problem to hybrid stochastic delay differential equations (SDDEs). The time delays will meet a weak condition, rather than the usually seen but restrictive differential assumption. In this case, more time delays such as piece-wise constant delay could be included. Moreover, time delays could influence our mode-structure classification scheme.

By making use of Lyapunov functional method and integral transform (for hybrid SDDEs), H_∞ stability, almost surely asymptotic stability, mean square exponential stability could be achieved. However, constructing an appropriate Lyapunov functional is always challenging, especially when integral transform method is invalid for some kinds of hybrid SDDEs, or the underlying systems are discontinuous. In this case, Razumikhin method, which is aimed to stability analysis for delay equations, will be powerful, since the discrete-time state feedback control itself is also a delay segment.

We will use Razumikhin idea to the stabilisation of hybrid SDDEs, where time delays will be relatively relaxed with little restriction. In other words, we are not able to use integral transform to deal with time delays now such as discrete-time delays. Next we will use this technique to study the stabilisation of hybrid SDEs by discrte-time state feedback control working intermittently and having rest time. Some important stability properties will be obtained in the sense of p-th moment exponential stability and almost surely exponential stability.

The successful applications to stochastic volatility model, neural networks, and oscillator systems demonstrate the practicability of our theory.

Contents

Notation

The general notations widely used in this thesis are list here. Other notations will be explained where they first appear.

$p := q$	:	p is defined by q or q is denoted by p.
A^{T}	:	the transpose of a vector of a matrix A.
$\mathrm{trace}(A)$	:	$= \sum_{i=1}^{d} a_{ii}$ for a square matrix $A = (a_{ij})_{d \times d}$.
$\lambda_{\min}(A)$	:	the smallest eigenvalue of a matrix A.
$\lambda_{\max}(A)$	:	the largest eigenvalue of a matrix A.
$\|A\|$	:	trace norm for a matrix A, i.e. $\|A\| = \sqrt{\mathrm{trace}(A^{\mathrm{T}}A)}$.
$\|\|A\|\|$	:	operator norm for a matrix A, i.e. $\|\|A\|\| = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}$.
$\emptyset$	:	the empty set.
A^C	:	the complement of A in S, i.e. $A^C = S - A$.
$A \subset B$	:	$A \cap B^C = \emptyset$.
$f : A \to B$	:	the mapping f from A to B.
$\mathbb{I}_A$	:	the indicator function of A, i.e. $\mathbb{I}_A(a) = 1$ if $a \in A$; otherwise 0.
$\mathbb{R} = \mathbb{R}^1$	:	the real line.
$\mathbb{R}_+$	:	the set of non-negative real numbers, i.e. $\mathbb{R}_+ = [0, \infty)$.
$\mathbb{N}$	:	the set of natural numbers, i.e. $\mathbb{N} = \{0, 1, 2, \cdots\}$.
$\mathbb{N}_+$	:	the set of positive natural numbers, i.e. $\mathbb{N}_+ = \mathbb{N} - \{0\}$.
$a \vee b$	:	the maximum of real numbers a and b.
$a \wedge b$	:	the minimum of real numbers a and b.
$[a]$	:	the integer part of real number a.
$\mathbb{R}^d$	:	the d-dimensional Euclidean space.
$\|x\|$	:	the Euclidean norm of $x \in \mathbb{R}^d$.
B_h	:	$= \{x \in \mathbb{R}^d : \|x\| \le h\}$.
$\sigma(\mathcal{C})$	:	the σ-algebra generated by $\mathcal{C}$.
$\mathcal{B} = \mathcal{B}^1$	:	the Borel-σ-algebra on $\mathbb{R}$.
$\mathcal{B}^d$	:	the Borel-σ-algebra on $\mathbb{R}^d$.
$C(D; \mathbb{R}^d)$	:	the family of continuous $\mathbb{R}^d$-valued functions defined on a domain D.
$C^m(D; \mathbb{R}^d)$	:	the family of continuously m-times differentiable $\mathbb{R}^d$-valued functions defined on D.

$C^{2,1}(D \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$	:	the family of real-valued functions defined on $D \times \mathbb{R}_+ \times \mathbb{S}$, which are continuously twice differentiable in $x \in D$ and once in $t \in \mathbb{R}_+$.
$C^2(D \times \mathbb{S}; \mathbb{R})$	:	the family of real-valued functions defined on $D \times \mathbb{S}$, which are continuously twice differentiable in $x \in D$.
V_t	:	$= \frac{\partial V}{\partial t}$.
V_x	:	$= \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d}\right)$.
V_{xx}	:	$= \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{d \times d}$.
$C([-\tau, 0]; \mathbb{R}^d)$	:	the family of continuous $\mathbb{R}^d$-valued functions ϕ defined on $[-\tau, 0]$ with norm $\|\|\phi\|\| = \sup_{-\tau \le \theta \le 0} \|\phi(\theta)\|$.
$L^p([a, b]; \mathbb{R}^d)$	:	the family of Borel-measurable functions $h : [a, b] \to \mathbb{R}^d$ such that $\int_a^b \|h(t)\|^p \mathrm{d}t < \infty$.
$(\Omega, \mathcal{F}, P)$	:	a complete probability space.
a.s.	:	almost surely, or with probability 1.
$E(\xi)$	:	the expectation of a random variable ξ.
$\|\|\xi\|\|_p$	:	$= (E\|\xi\|^p)^{\frac{1}{p}}$.
$L^p(\Omega; \mathbb{R}^d)$	:	the family of $\mathbb{R}^d$-valued random variables ξ such that $E\|\xi\|^p < \infty$.
$L^p_{\mathcal{F}_t}(\Omega; \mathbb{R}^d)$	:	the family of $\mathcal{F}_t$-measurable $\mathbb{R}^d$-valued random variables ξ such that $E\|\xi\|^p < \infty$.
$L^p_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^d)$	:	the family of $\mathcal{F}_t$-measurable $C([-\tau, 0]; \mathbb{R}^d)$-valued random variables φ such that $E\|\|\varphi\|\|^p < \infty$.
$C^b_{\mathcal{F}_t}([-\tau, 0]; \mathbb{R}^d)$	:	the family of $\mathcal{F}_t$-measurable bounded $C([-\tau, 0]; \mathbb{R}^d)$-valued random variables.
$\mathcal{L}^p_{\mathcal{F}_t}([a, b]; \mathbb{R}^d)$	:	the family of $\mathbb{R}^d$-valued $\mathcal{F}_t$-adapted stochastic process $\{f(t)\}_{a \le t \le b}$ such that $\int_a^b \|f(t)\|^p \mathrm{d}t < \infty$ a.s.
$\mathcal{L}^p(\mathbb{R}_+; \mathbb{R}^d)$	:	the family of $\mathbb{R}^d$-valued stochastic process $\{f(t)\}_{t \ge 0}$ such that $\int_0^T \|f(t)\|^p \mathrm{d}t < \infty$ a.s. for every $T > 0$.

Chapter 1

Introduction

1.1 Stabilisation problem

Stochastic differential equations (SDEs) appear as a useful modelling tool in many sciences, such as asset price movement in financial market (Lewis 2000, Vasicek 1977), population evolution process in crowded environment (Roberts & Saha 1999). However, due to the repairing work or environmental changes, a SDE may not keep staying in one mode for all the time. More generally, it will jump from one mode into another following some rules. For example, the famous Black-Scholes model (Black & Scholes 1973) is used to study the stock price. Usually, there are two modes in the overall stock market (e.g. (Yin & Zhou 2004)), bullish mode and bearish mode. In the bullish mode, the general market goes up, most stocks then increase. In the other mode, the market moves backward, most stocks so follows. While for investors, they need to make decisions during the continuous switches of these two modes.

Markovian switching, as one of the mode jumping rules, has drawn much attention. Because it can be used to describe one natural but important phenomenon, where the future mode only depends on the current mode and the information of past modes is of no use (see (Gillespie 1992, Anderson 1991, Norris 1998). In 1990's, Markovian switching process, was considered into the investigation of SDEs to modulate the time-varying structures or coefficients. Until now, there have been enormous publications in this area. For theory, we cite (Basak, Bisi & Ghosh 1996, Mao & Yuan 2006) as references. For applications, we recommend (Rong 2006, Li, Omotere, Qian & Dougherty 2017). Most of the existing results are concerned with SDE with Markovian switching in the following form

$$dx(t) = f(x(t), t, r(t))dt + g(x(t), t, r(t))dW(t),$$

where $x(t)$ is the system state taking values continuously, mode $r(t)$ is a Markov chain taking discrete values. Since there are two different states involved, we also call this kind of systems as hybrid SDEs. The detailed explanation of this equation will be given in Chapter 2.

One of the important topics in the investigation of hybrid SDEs is the stability analysis. Roughly speaking, stability depicts the ability of a system to resist disturbance caused by small changes in the initial condition or system parameters (see (Mao & Yuan 2006)). At most times, we want a hybrid SDE to remain stable and become less sensitive to these small changes. Therefore, when a given hybrid SDE is not stable, we need to impose exterior inputs into the system to derive it to a stable state. Such a process is the stabilisation and the inputs are often referred to controllers. In this thesis, we will focus on one important stabilisation problem: discrete-state-feedback stabilisation.

Given a hybrid SDE unstable, it is a general practice to use state feedback control $u(x, t, i)$ to achieve stabilisation. Traditionally, the control is designed based on continuous-time state observations, that is, $u(x(t), t, r(t))$. Stabilisation by continuous-time state feedback control for hybrid SDEs has been intensively studied (e.g. (Ji & Chizeck 1990, Shaikhet 1996, Mao, Yin & Yuan 2007, Deng, Luo & Mao 2012)). But to impose such a control, we need to continuously observe the underlying hybrid SDE to obtain the values of $x(t)$ for all the time. This is extremely costly, and sometimes seems a little impossible, such as pandemic intervention when the cases are collected day by day.

Therefore in practice, it is much wiser to implement the feedback control based on the state observations at discrete times, say by $0, \tau, 2\tau, \cdots$. This could be explained as follows: at the initial time 0, we observe the system and obtain the value of state $x(0)$, then input the control based on this information until we make the next observation at time τ. Our control hence becomes $u(x([t/\tau]\tau), t, r(t))$, which is obviously less costly. Although such a problem for deterministic systems has been widely investigated (see (Chammas & Leondes 1979, Hagiwara & Araki 1988, Allwright, Astolfi & Wong 2005)), it was (Mao 2013) that initialed the study of discrete-state-feedback stabilisation for stochastic systems. In this thesis, we will mainly pay attention to imposing discrete-time state feedback control into the drift coefficient. In other words, our controlled SDE becomes

$$\mathrm{d}x(t) = \Big(f(x(t), t, r(t)) + u(x([t/\tau]\tau), t, r(t))\Big)\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t).$$

1.2 Research methods

In 1892, Lyapunov created a very powerful tool in stability analysis known as Lyapunov second method. It transferred the problem from analysing system solutions directly into verifying system coefficients by an auxiliary function (so-called Lyapunov function). This method has been developed to deal with stability for stochastic systems by many authors, see (Arnold 1974, Kushner 1967, Mao & Yuan 2006, Khasminskii 2012). Thus the main method to study stability in this thesis is still the Lyapunov method.

If we let $\zeta(t) = t - [t/\tau]\tau$, we could find the controlled SDE is actually a delay equation with time delay $\zeta(t)$. The stability analysis of delay equations is absolutely more technical than non-delay ones, and we cite (Mohammed 1986, Kolmanovskii & Myshkis 1999, Mao & Yuan 2006) as references. But these classical results are not well-applicable to our control problem since $\zeta(t)$ behaves unusually with discontinuity. After several years' development, there are two useful methods to this problem.

The first one is the indirect technique. Construct an auxiliary SDE

$$\mathrm{d}y(t) = (f(y(t), t, r(t)) + u(y(t), t, r(t)))\mathrm{d}t + g(y(t), t, r(t))\mathrm{d}W(t),$$

which is proved to be stable in advance. Then estimate the state difference between $x(t)$ and $y(t)$, which is governed by the observation duration τ. The stability of $x(t)$ is obtained from the decomposition $x(t) = \big(x(t) - y(t)\big) + y(t)$, where the former is sufficiently small if letting τ work small enough, the latter is stable. This method is often called the comparison idea. It was very popular at the beginning stage of discrete-state-feedback stabilisation problem (e.g. (Mao 2013, Zhao, Zhang, Xu, Bai & Zhang 2017, Song, Zheng, Luo & Mao 2017)), since we only need pay attention to the stability analysis of the auxiliary system. This is a non-delay equation, and there is very rich literature in this problem (see, e.g. (Chen & Zhang 2004, Dragan, Morozan & Stoica 2004, Mao et al. 2007, Deng et al. 2012)). However, the value of τ derived by using this method is usually not very sharp. But even worse, this method only works well when the underlying hybrid SDE is globally Lipschitz continuous (see (Hu, Liu, Deng & Mao 2020)), which might exclude many important functions such as $x\sin(x)$ and x^2.

The other one is therefore becoming significant, which works directly to the controlled system. Now, to implement this idea, the technique of Lyapunov functional has received much attention. Lyapunov functional is an extension of Lyapunov function to the functional equations. One popular functional for the stabilisation problem by discrete-time state feedback control is given in the following type

$$\int_{-\tau}^{0}\int_{t+s}^{t}\big(\tau|f(x(v),v,r(v))+u(x([v/\tau]\tau),v,r(v))|^2+|g(x(v),v,r(v))|^2\big)\,\mathrm{d}v\mathrm{d}s.$$

By making use of this method, You et al. in (You, Liu, Lu, Mao & Qiu 2015) also obtained a better τ than that in (Mao 2013). Nevertheless, it should also be pointed out that this approach depends closely on the construction of Lyapunov functional. But as we all know, constructing Lyapunov functional effectively is sometimes really a challenge work. The functional we gave above is indeed suitable for some hybrid SDEs (see (You et al. 2015, Fei, Fei, Mao, Xia & Yan 2020, Mei, Fei, Fei & Mao 2020, Shi, Mao & Wu 2022)), but might also be useless for others (e.g. Chapters 6, 7). Then what should we do if an appropriate functional is out of reach?

Actually, such difficulty was overcame by Razumikhin in (Razumikhin 1956, Razumikhin 1960). He proposed a novel idea to check differential operator, and established theorems via Lyapunov function rather than Lyapunov functional, named as Razumikhin-type theorems. In the past two decades, Razumikhin-type theorems for stochastic stability have been developed (see (Mao 1996, Wu & Hu 2012, Cao & Zhu 2021)). But for our control problem, so far (Li, Lu, Kou, Mao & Pan 2018) was the only paper to use this approach.

1.3 Thesis outline

But it should be strengthened that most of results in discrete-state-feedback stabilisation problem mentioned above paid much attention to hybrid SDEs which satisfy the following linear growth condition

$$|f(x,t,i)|^2\vee|g(x,t,i)|^2\le K(1+|x|^2).$$

This might exclude many real models, such as the stochastic volatility model (see (Lewis 2000, Heston 1997))

$$\mathrm{d}x(t) = (\mu - \alpha x^3(t))\mathrm{d}t + \sigma x^{1.5}(t)\mathrm{d}W(t),$$

and stochastic population system (see (Kloeden & Platen 1992, Mao 2007))

$$\mathrm{d}x(t) = \mathrm{diag}(x_1(t), \cdots, x_d(t))((b + Ax(t) + Gx(t))\mathrm{d}t + Cx(t)\mathrm{d}W(t)).$$

Consequently, this thesis is devoted to the study of discrete-state-feedback stabilisation of highly nonlinear hybrid systems. Here, highly nonlinearity means the absence of linear growth condition.

Our aim is to make the underlying system more general: (i) weaken conditions imposed; (ii) study structured stabilisation; (iii) consider delay equations with general time delays, and to make the controller less costly: (i) make control rules easily checked; (ii) design bounded control; (iii) let control work intermittently. The main methods we will use are the Lyapunov functional technique and Razumikhin idea.

The layout of this thesis is organised as follows. Chapter 2 is set to prepare the fundamental theory of stochastic analysis. The elementary concepts in probability theory, stochastic process and Brownian motion are firstly given. Then after introducing Itô stochastic integral and Markov chains, the basic definitions and stability properties of hybrid SDEs are provided. It should be pointed out that Mao's books (Mao & Yuan 2006, Mao 2007) are the main sources of reference for this chapter. These results could also be found in many mathematical books or papers on stochastic analysis (see (Arnold 1974, Kloeden & Platen 1992, He, Wang & Yan 1992, Rong 2006)).

In Chapter 3, we make some improvements on current results about discrete-state-feedback stabilisation of highly nonlinear hybrid SDEs. The conditions imposed are weaken since we provide a new method to estimate the difference between current-time state and discrete-time state. The method to determine the value of observation duration is improved so that the control rules could be checked much easily. The stability types studied in this part are H_∞-stability and almost surely asymptotic stability.

Chapters 4 and 5 are devoted to structured stabilisation problem, where we consider there are different structures on different Markovian modes of underlying systems. The control function is designed in a bounded state area to reduce control cost. Except for stability types in Chapter 3, mean square exponential stability is our main interest. Chapter 4 continues going to hybrid SDEs, while Chapter 5 is concerned with hybrid SDDEs. Time delays are relaxed, at least the commonly used differential assumption being lifted.

In these three chapters, the method of Lyapunov functional is the main technique to study stability. But when it is difficult to construct an appropriate functional, Razumikhin method becomes very powerful.

In Chapter 6, we firstly provide a Razumikhin-type theorem for highly nonlinear SFDEs, then apply it to our stabilisation problem of hybrid SDDEs, where time delays are more general than before (in theory, the integral transform to eliminate delay effect is invalid). In Chapter 7, we use Razumikhin idea to the stabilisation of hybrid SDEs by discrete-time state feedback control working intermittently and having rest time. Since intermittent control is a piece-wise constant function, Razumikhin condition is generalised to time-inhomogeneous situation. The stability in these two parts includes p-th moment exponential stability and almost surely exponential stability.

At the end of each chapter, we also provide an application example from practical models.

Our main results in Chapters 3-7 could be found in the following publications:

- Xu, H. & Mao, X. (2023), 'Advances in discrete-state-feedback stabilization of highly nonlinear hybrid systems by Razumikhin technique', *IEEE Trans. Autom. Control* **68**(10), 6098-6113.
- Xu, H. & Mao, X. (2023), 'Improved delay-dependent stability of superlinear hybrid stochastic systems with general time-varying delays', *Nonlinear Anal.-Hybrid Syst.*, **50**, 101413.
- Xu, H. & Mao, X. (2024), 'Structured Stabilisation of Superlinear Delay Systems by Bounded Discrete-Time feedback control', *Automatica*, **159**, 111409.
- Xu, H. & Mao, X. (2024), 'Razumikhin technique to stabilisation of highly nonlinear hybrid systems by bounded discrete-time state feedback control working intermittently', *Numer. Algebr. Control Optim.*, Doi: 10.3934/naco.2024003.

Chapter 2

Stochastic Analysis

2.1 Probability theory

Measurable spaces

Probability theory is concerned with mathematical models of experiments involving randomness. The collection of all possible individual outcomes ω is called the sample space, denoted by Ω. A subset of Ω is refered to an event. But not every event is our interest, so we need put the interesting events together as a family, $\mathcal{F}$, which should be a *σ-algebra*

(i) $\Omega \in \mathcal{F}$;

(ii) if $A \in \mathcal{F}$, then the complement A^C should be in $\mathcal{F}$;

(iii) if the countable collection $\{A_n\}_{n \in \mathbb{N}_+} \subset \mathcal{F}$, then $\bigcup_{n=1}^{\infty} A_n \in \mathcal{F}$.

The pair $(\Omega, \mathcal{F})$ is then called a *measurable space*, and the elements of $\mathcal{F}$ is hence called $\mathcal{F}$-measurable sets instead of events.

If $\mathcal{C}$ is any family of subsets of Ω , it might not be a σ-algebra. But there is a smallest σ-algebra $\sigma(\mathcal{C})$ which covers $\mathcal{C}$. This $\sigma(\mathcal{C})$ is referred to the *σ-algebra generated by* $\mathcal{C}$. If $\Omega = \mathbb{R}^d$ and $\mathcal{C}$ is the family of all open sets in $\mathbb{R}^d$, then $\mathcal{B}^d := \sigma(\mathcal{C})$ is called the *Borel σ-algebra*, whose elements are called the *Borel sets*.

Random variables

A real-valued function $X : \Omega \to \mathbb{R}$ is said to be *$\mathcal{F}$-measurable* if for any $c \in \mathbb{R}$

$$\{\omega : X(\omega) \leq c\} \in \mathcal{F}.$$

In probability theory, the function X is also called a ($\mathcal{F}$-measurable) *random variable*. An $\mathbb{R}^d$-valued function $X(\omega) = (X_1(\omega), \cdots, X_d(\omega))^{\mathrm{T}}$ is said to be $\mathcal{F}$-*measurable* if all the elements $X_1, \cdots, X_d$ are $\mathcal{F}$-measurable. A $d \times m$-matrix-valued function $X(\omega) = (X_{ij}(\omega))_{d\times m}$ is $\mathcal{F}$-*measurable* if all X_{ij} are $\mathcal{F}$-measurable. If the measurable space is $(\mathbb{R}^d, \mathcal{B}^d)$, a $\mathcal{B}^d$-measurable function is usually called a *Borel measurable function*.

For any given function $X : \Omega \to \mathbb{R}^d$, we can define the *$\sigma$-algebra $\sigma(X)$ generated by X* as

$$\sigma(X) = \sigma\left(\left\{\{\omega : X(\omega) \in U\} : U \text{ is an open set of } \mathbb{R}^d\right\}\right).$$

Clearly, X is $\sigma(X)$-measurable. Moreover, if X is $\mathcal{F}$-measurable, then $\sigma(X) \subset \mathcal{F}$. For a collection of $\mathbb{R}^d$-valued functions $\{X_n\}_{n\in I}$, where I is an index set, the σ-algebra generated by this collection is given as

$$\sigma(\{X_n\}_{n\in I}) = \sigma\left(\bigcup_{n\in I} \sigma(X_n)\right).$$

Here, it should be pointed out that the union of two σ-algebras may not still be a σ-algebra.

Let A be a subset of Ω. Its *indicator function* $\mathbb{I}_A$ will be widely used, which is defined by

$$\mathbb{I}_A(\omega) = \begin{cases} 1, & \text{for } \omega \in A, \\ 0, & \text{for } \omega \notin A. \end{cases}$$

The indicator function $\mathbb{I}_A$ is $\mathcal{F}$-measurable if and only if $A \in \mathcal{F}$. The σ-algebra generated by $\mathbb{I}_A$ is easy to understand, i.e. $\sigma(\mathbb{I}_A) = \{\emptyset, \Omega, A, A^C\}$.

Probability measures

A *probability measure* P on a measurable space $(\Omega, \mathcal{F})$ is a function $P : \mathcal{F} \to [0, 1]$ such that

(i) for the entire space, $P(\Omega) = 1$;

(ii) for any disjoint sequence $\{A_n\}_{n\in\mathbb{N}_+} \subset \mathcal{F}$ (i.e. $A_n \cap A_k = \emptyset$ if $n \neq k$)

$$P\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n).$$

The triple $(\Omega, \mathcal{F}, P)$ is called a *probability space*. We can set $\bar{\mathcal{F}} = \{A \subset \Omega :$ there are $B, C \in \mathcal{F}$ such that $P(B) = P(C)$ with $B \subset A \subset C\}$. Then $\bar{\mathcal{F}}$ is a

σ-algebra, which is called the completion of $\mathcal{F}$. If $\mathcal{F} = \bar{\mathcal{F}}$, the probability space $(\Omega, \mathcal{F}, P)$ is said to be *complete*. In the sequel of this section, we will let $(\Omega, \mathcal{F}, P)$ be a complete probability space.

An 'event' $A \in \mathcal{F}$ is said to happen *almost surely* if $P(A) = 1$. In addition, the following properties of probability measure P are very useful

(1) **(complement rule)** $P(A^C) = 1 - P(A)$;

(2) **(inclusion rule)** $P(A \cup B) = P(A) + P(B) - P(A \cap B)$;

(3) **(subset rule)** if $A \subset B$, then $P(A) \leq P(B)$;

(4) **(almost sure rule)** if $P(A_n) = 1$ for all $n \in \mathbb{N}_+$, then $P\left(\bigcap_{n=1}^{\infty} A_n\right) = 1$;

(5) **(continuity rule)** if $\{A_n\}_{n\in\mathbb{N}_+}$ is increasing, $P\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} P(A_n)$.

Sometimes, given a sequence of sets $\{A_n\}_{n\in\mathbb{N}_+}$ in $\mathcal{F}$, we are more interested in the *upper limit of the sets*

$$\limsup_{n\to\infty} A_n = \{\omega : \omega \in A_n \text{ for infinitely many } n\} = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k.$$

In terms of its probability, we have the following well-known Borel-Cantelli Lemma, whose proof can be found in Section 2.7 in (Williams 1991).

Lemma 2.1. (First Borel-Cantelli lemma) *Let $\{A_n\}_{n\in\mathbb{N}_+}$ be a sequence of sets in $\mathcal{F}$ such that $\sum_{n=1}^{\infty} P(A_n) < \infty$. Then*

$$P\left(\limsup_{n\to\infty} A_n\right) = 0.$$

In other words, there is a set $\Omega_0 \in \mathcal{F}$ with $P(\Omega_0) = 1$ and a random integer n_0 such that for every $\omega \in \Omega_0$, we have $\omega \notin A_n$ whenever $n \geq n_0(\omega)$.

Independence

Let I be an index set. Sub-σ-algebras $\{\mathcal{F}_n\}_{n\in I} \subset \mathcal{F}$ are called *independent* if, whenever, $A_n \in \mathcal{F}_n$ $(n \in I)$, and any possible choice of indices $n_1, \cdots, n_k \in I$, then

$$P(A_{n_1} \cap \cdots \cap A_{n_k}) = P(A_{n_1}) \cdots P(A_{n_k}).$$

A collection of random variables $\{X_n\}_{n\in I}$ is said to be *independent* if the σ-algebras $\{\sigma(X_n)\}_{n\in I}$ generated by them are independent. A family of $\mathcal{F}$-measurable sets $\{A_n\}_{n\in I}$ is said to be *independent* if their indicator function $\{\mathbb{I}_{A_n}\}_{n\in I}$ are independent. In particular, two sets $A_1, A_2 \in \mathcal{F}$ are independent if $P(A_1 \cap A_2) =$

$P(A_1)P(A_2)$.

Now in the view of independence, we can give the second Borel-Cantelli Lemma (see Section 4.3 in (Williams 1991)).

Lemma 2.2. (Second Borel-Cantelli lemma) *If the sequence* $\{A_n\}_{n\in\mathbb{N}_+} \subset \mathcal{F}$ *is independent and* $\sum_{n=1}^{\infty} P(A_n) = \infty$*, then*

$$P\left(\limsup_{n\to\infty} A_n\right) = 1.$$

That is, there is a set $\Omega_0 \in \mathcal{F}$ *with* $P(\Omega_0) = 1$ *such that for every* $\omega \in \Omega_0$*, we can find a sub-sequence* $\{A_{n_k}\}_{k\in\mathbb{N}_+}$ *for* ω *belonging to all* A_{n_k}*.*

Expectation

If X is a real-valued random variable and is integrable with respect to P, the number

$$EX = \int_\Omega X(\omega)\mathrm{d}P(\omega)$$

is called the *expectation* of X. Let Y be another real-valued integrable random variable but independent from X. Then XY is also integrable and $E(XY) = EX \cdot EY$.

For an $\mathbb{R}^d$-valued random variable X, define $EX = (EX_1, \cdots, EX_d)^{\mathrm{T}}$. Let $p \in (0, \infty)$. The number $E|X|^p$ is said to be the *p-th moment* of X (here, $E|X|^p$ is always well-defined). The family of all $\mathbb{R}^d$-valued random variables X with $E|X|^p < \infty$ is denoted by $L^p = L^p\left(\Omega; \mathbb{R}^d\right)$. Clearly, L^p is a normed space with p-norm $||X||_p = (E|X|^p)^{\frac{1}{p}}$. In L^1, we have $|EX| \leq E|X|$. Moreover, the following probability inequalities will be widely used (see Section 6.13 in (Williams 1991))

(1) **(Hölder inequality)** if $p, q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $X \in L^p$, $Y \in L^q$

$$\left|E\left(X^{\mathrm{T}}Y\right)\right| \leq ||X||_p||Y||_q;$$

(2) **(Minkovski inequality)** if $p \geq 1$, $X, Y \in L^p$

$$||X + Y||_p \leq ||X||_p + ||Y||_p;$$

(3) **(Chebyshev inequality)** if $c > 0$, $p > 0$, $X \in L^p$

$$P(\{\omega : |X(\omega)| \geq c\}) \leq \frac{E|X|^p}{c^p}.$$

A simple appplication of Hölder inequality implies the monotonicity of p-norm, i.e. for $0 < r < p < \infty$, $X \in L^p$, $||X||_r \leq ||X||_p$.

Next, we introduce four important convergence modes. Let X and X_n $(n \in \mathbb{N}_+)$ be $\mathbb{R}^d$-valued random variables. The sequence $\{X_n\}_{n\in\mathbb{N}_+}$ is said to converge to X

(1) *with probability* 1 or *almost surely*: if $P(|X_n - X| \to 0) = 1$;

(2) *in probability*: if $P(\{\omega : |X_n(\omega) - X(\omega)| > \varepsilon\}) \to 0$ for every $\varepsilon > 0$;

(3) *in* L^p: if X_n, X belong to L^p and $E|X_n - X|^p \to 0$;

(4) *in distribution*: if $E(g(X_n)) \to E(g(X))$ for every continuous bounded function $g : \mathbb{R}^d \to \mathbb{R}$.

The following three integration convergence results are needed.

Theorem 2.1. (Monotonic convergence theorem) *If $\{X_n\}_{n\in\mathbb{N}_+}$ is a non-decreasing sequence of non-negative random variables, then*

$$\lim_{n\to\infty} EX_n = E\left(\lim_{n\to\infty} X_n\right).$$

Theorem 2.2. (Fatou lemma) *If $\{X_n\}_{n\in\mathbb{N}_+}$ is a sequence of non-negative random variables, then*

$$\liminf_{n\to\infty} EX_n \le E\left(\liminf_{n\to\infty} X_n\right).$$

Theorem 2.3. (Dominated convergence theorem) *Let $p \ge 1$, $\{X_n\}_{n\in\mathbb{N}_+} \subset L^p(\Omega;\mathbb{R}^d)$ and $Y \in L^p(\Omega;\mathbb{R}^d)$. Assume that $|X_n| \le Y$ a.s. and $\{X_n\}_{n\in\mathbb{N}_+}$ converges to X in probability. Then $X \in L^p(\Omega;\mathbb{R}^d)$, $\{X_n\}_{n\in\mathbb{N}_+}$ converges to X in L^p, and*

$$\lim_{n\to\infty} EX_n = E\left(\lim_{n\to\infty} X_n\right) = EX.$$

Their proof could be found in Sections 5.3, 5.4 and 13.7 in (Williams 1991), respectively.

Conditional expectation

Let $A, B \in \mathcal{F}$ with $P(B) > 0$. The *conditional probability* of A under condition B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}.$$

However, we frequently need the more general concept of condition expectation. Let X be a random variable in $L^1(\Omega;\mathbb{R})$ and $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. In general, X is not $\mathcal{G}$-measurable. Now we want to find an integrable $\mathcal{G}$-measurable random variable Y such that $E(\mathbb{I}_G Y) = E(\mathbb{I}_G X)$ for all $G \in \mathcal{G}$. By the Radon-Nikodym theorem (see Section 14.13 in (Williams 1991)), there indeed exists one such Y, almost surely unique. The random variable Y is called the *conditional expectation* of X under the condition $\mathcal{G}$, denoted by

$$Y = E(X|\mathcal{G}).$$

We list some important properties of conditional expectation as (all the equalities and inequalities shown hold almost surely)

(1) $E(E(X|\mathcal{G})) = EX$;

(2) $\mathcal{G} = \{\emptyset, \Omega\} \implies E(X|\mathcal{G}) = EX$;

(3) X is $\mathcal{G}$-measurable $\implies E(X|\mathcal{G}) = X$;

(4) X is a constant $c \implies E(X|\mathcal{G}) = c$;

(5) $X \geq 0 \implies E(X|\mathcal{G}) \geq 0$;

(6) $|E(X|\mathcal{G})| \leq E(|X||\mathcal{G})$;

(7) $E(aX + bY|\mathcal{G}) = aE(X|\mathcal{G}) + bE(Y|\mathcal{G})$ for constants a, b;

(8) X is $\mathcal{G}$-measurable $\implies E(XY|\mathcal{G}) = XE(Y|\mathcal{G})$;

(9) $\sigma(X)$, $\mathcal{G}$ are independent $\implies E(X|\mathcal{G}) = EX$;

(10) $\mathcal{G}_1 \subset \mathcal{G}_2 \subset \mathcal{F} \implies E\big(E(X|\mathcal{G}_2)\big|\mathcal{G}_1\big) = E(X|\mathcal{G}_1)$.

For an $\mathbb{R}^d$-valued random variable X, its *conditional expectation* under $\mathcal{G}$ is defined as $E(X|\mathcal{G}) = (E(X_1|\mathcal{G}), \cdots, E(X_d|\mathcal{G}))^{\mathrm{T}}$.

Finally, we recommend readers to (Williams 1991) for more information about probability theory.

2.2 Stochastic processes

Stochastic basis

A complete probability space $(\Omega, \mathcal{F}, P)$, with an increasing family of sub-σ-algebras of $\mathcal{F}$ (i.e. $\mathcal{F}_s \subset \mathcal{F}_t \subset \mathcal{F}$ for all $0 \leq s \leq t < \infty$), which satisfies the usual conditions

(i) *right-continuous*, i.e. $\mathcal{F}_t = \bigcap_{s>t} \mathcal{F}_s$ for all $t \geq 0$;

(ii) *complete*, i.e. $\mathcal{F}_0$ contains all P-null sets,

is called a *stochastic basis*. The family $\{\mathcal{F}_t\}_{t\geq 0}$ is often called a *filtration*. We also define $\mathcal{F}_\infty = \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$. Through this thesis, we will always let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis.

A family $\{X_t\}_{t\in I}$ of $\mathbb{R}^d$-valued random variables defined on $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ is said to be a *stochastic process* with index set I and state space $\mathbb{R}^d$. For each fixed $t \in I$, we have a $\mathbb{R}^d$-valued random variable $X_t(\omega)$. For each fixed $\omega \in \Omega$, we have a function of t, $X_t(\omega) : I \to \mathbb{R}^d$, which is called a *sample path* of the process. If I is identified, we often write $\{X_t\}_{t\in I}$ as $\{X_t\}$. Usually, $\{X_t\}_{t\geq 0}$ is denoted by X_t or $X(t)$ simply.

Let $\{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued process. It is said to be

(1) *continuous*: for almost all $\omega \in \Omega$, function $X_t(\omega)$ is continuous on $t \geq 0$;

(2) *cadlag*: if for almost all $\omega \in \Omega$, it is right-continuous and the left limit $\lim_{s\uparrow t} X_s(\omega)$ exists and is finite for all $t > 0$;

(3) *integrable*: if for every fixed $t \geq 0$, X_t is an integrable random variable;

(4) *square integrable*: if for every fixed $t \geq 0$, $E|X_t|^2 < \infty$;

(5) *adapted*: if for every fixed $t \geq 0$, random variable X_t is $\mathcal{F}_t$-measurable;

(6) *progressively measurable*: if for every fixed $T \geq 0$, $\{X_t\}_{0\leq t\leq T}$ regarded as a function of (t, ω) from $[0, T] \times \Omega$ is $\mathcal{B}([0, T]) \times \mathcal{F}_T$-measurable.

In particular, for a real-valued process $\{A_t\}_{t\geq 0}$, it is called *increasing* if for almost all $\omega \in \Omega$, function $A_t(\omega)$ is non-negative, increasing, right-continuous on $t \geq 0$.

Stopping times

A random variable $\tau : \Omega \to [0, \infty]$ is called a *stopping time* if for any $t \geq 0$

$$\{\omega : \tau(\omega) \leq t\} \in \mathcal{F}_t.$$

Since the technique of stopping times plays a crucial role in this thesis, we prepare two useful theorems.

Theorem 2.4. *If $\{X_t\}_{t\geq 0}$ is a progressively measurable process and τ is a stopping time, then $X_\tau \mathbb{I}_{\{\tau<\infty\}}$ is $\mathcal{F}_\tau$-measurable. In particular, if τ is finite, then X_τ is $\mathcal{F}_\tau$-measurable.*

Theorem 2.5. *Let $\{X_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued, cadlag, adapted process, and D an open subset of $\mathbb{R}^d$. Define*

$$\tau = \inf\{t \geq 0 : X_t \notin D\},$$

where we use the convention $\inf \emptyset = \infty$. Then τ is a stopping time, which is usually called the first exit time form D.

Martingales

An $\mathbb{R}^d$-valued, adapted, integrable process $\{M_t\}_{t\geq 0}$ is called a *martingale* (with respect to $\{\mathcal{F}_t\}_{t\geq 0}$) if for all $0 \leq s < t < \infty$

$$E(M_t|\mathcal{F}_s) = M_s \quad a.s.$$

If $X = \{X_t\}_{t\geq 0}$ is a progressively measurable process and τ is a stopping time, then $X^\tau = \{X_{t\wedge\tau}\}_{t\geq 0}$ is called a *stopped process* of X. The following is the well-known Doob stopping theorem, whose proof can be found in Theorem 2.35, p47, (He et al. 1992).

Theorem 2.6. (Doob stopping theorem) *Let $\{M_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued martingale, and θ, ρ two finite stopping times. Then*

$$E(M_\theta|\mathcal{F}_\rho) = M_{\theta\wedge\rho} \quad a.s.$$

In particular, if τ is a stopping time, the stopped process $M^\tau = \{M_{t\wedge\tau}\}$ is still a martingale.

We also prepare the following useful Doob martingale inequalities (see Theorem 1, p62 and Theorem 2, p64 in (Liptser & Shiryayev 1989)).

Theorem 2.7. (Doob martingale inequalities) *Let $\{M_t\}_{t\geq 0}$ be an $\mathbb{R}^d$-valued martingale, and $[a,b]$ be a bounded interval in $\mathbb{R}_+$.*

(1) If $p \geq 1$ and $M_t \in L^p(\Omega, \mathbb{R}^d)$, then for all $c \geq 0$

$$P\left(\left\{\omega : \sup_{a\leq t\leq b} |M_t(\omega)| \geq c|\right\}\right) \leq \frac{E|M_b|^p}{c^p}.$$

(2) If $p > 1$ and $M_t \in L^p(\Omega, \mathbb{R}^d)$, then

$$E\left(\sup_{a\leq t\leq b} |M_t|^p\right) \leq \left(\frac{1}{1-\frac{1}{p}}\right)^p E|M_b|^p.$$

If $M = \{M_t\}_{t\geq 0}$ is a real-valued, square-integrable, continuous martingale, then there is a unique continuous, integrable, adapted, increasing process, denoted by $\{\langle M, M\rangle\}$, such that $\{M_t^2 - \langle M, M\rangle\}$ is a continuous martingale vanishing at $t = 0$. The process $\{\langle M, M\rangle\}$ is called the *quadratic variation* of M. Particularly, for any finite stopping time τ, $EM_\tau^2 = E\langle M, M\rangle_\tau$.

A right-continuous, adapted process $M = \{M_t\}_{t\geq 0}$ is called a *local martingale* if there is an increasing sequence of stopping times $\{\tau_n\}_{n\in\mathbb{N}_+}$ with $\tau_n \uparrow \infty$ a.s. such that every $\{M_{t\wedge\tau_n} - M_0\}_{t\geq 0}$ is a martingale. By Theorem 2.6, every martingale is a local martingale. For a real-valued, continuous, local martingale $M = \{M_t\}_{t\geq 0}$, its *quadratic variation* $\{\langle M, M\rangle\}$ is the unique continuous, adapted process such that $\{M_t^2 - \langle M, M\rangle\}$ is a continuous, local martingale vanishing at $t = 0$.

The following result is the useful strong law of large numbers for martingales (see Theorem 10, p144 in (Liptser & Shiryayev 1989)).

Theorem 2.8. (Strong law of large numbers) *Let $M = \{M_t\}_{t\geq 0}$ be a real-valued, continuous, local martingale vanishing at $t = 0$. Let $\{A_t\}_{t\geq 0}$ be continuous, adapted, increasing process. If*

$$\lim_{t\to\infty} A_t = \infty \quad \textit{and} \quad \int_0^\infty \frac{\mathrm{d}\langle M, M\rangle_t}{(1 + A_t)^2} < \infty \quad \textit{a.s.},$$

then

$$\lim_{t\to\infty} \frac{M_t}{A_t} = 0 \quad \textit{a.s.}$$

To close this section, we provide a useful convergence theorem, which plays an important role in the stability analysis.

Theorem 2.9. (Semi-martingale convergence theorem) *Let $\{A_t\}_{t\geq 0}$ and $\{U_t\}_{t\geq 0}$ be two continuous, adapted, increasing processes with $A_0 = U_0 = 0$ a.s. Let $\{M_t\}_{t\geq 0}$ be a real-valued, continuous, local martingale with $M_0 = 0$ a.s. Let ξ be a non-negative, $\mathcal{F}_0$-measurable random variable. Define $X_t = \xi + A_t - U_t + M_t$ for all $t \geq 0$. If X_t is non-negative, then*

$$\left\{\lim_{t\to\infty} A_t < \infty\right\} \subset \left\{\lim_{t\to\infty} X_t \textit{ exists and is finite }\right\} \subset \left\{\lim_{t\to\infty} U_t < \infty\right\} \quad \textit{a.s.}$$

In particular, if $\lim_{t\to\infty} A_t < \infty$ a.s., then for almost all $\omega \in \Omega$,

$$\lim_{t\to\infty} X_t(\omega) \textit{ exists and is finite}, \quad \textit{and} \quad \lim_{t\to\infty} U_t(\omega) < \infty.$$

We cite (Liptser & Shiryayev 1989) (Theorem 7, p139) as a reference for this theorem.

Brownian motion

Brownian motion is the name given to the irregular movement of pollen grains suspended in water, observed by the Scottish botanist Robert Brown in 1928. Then it was used to explain by the random collisions with the molecules of water. In mathematics, it is natural to use a stochastic process $W_t(\omega)$ to describe the motion, explained as the position of the pollen grain ω at time t. The following is the mathematical definition of Brownian motion.

Definition 2.1. *Let* $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ *be a stochastic basis. A one-dimensional Brownian motion is a real-valued, continuous, adapted process* $\{W_t\}_{t\geq 0}$ *with the following properties*

(i) $W_0 = 0$ *a.s.;*

(ii) for $0 \leq s \leq t < \infty$*, the increment* $W_t - W_s$ *is normally distributed with expectation* 0 *and variance* $t - s$*;*

(iii) for $0 \leq s \leq t < \infty$*, the increment* $W_t - W_s$ *is independent of* $\mathcal{F}_s$*.*

In honor of American mathematician Norbert Wiener for his investigations on the mathematical properties of $W_t(\omega)$, we also call Brownian motion as *Wiener process*. Here we list some important properties

(1) let $c > 0$ and define $X_t = \frac{W_{ct}}{\sqrt{c}}$ for $t \geq 0$, then $\{X_t\}$ is a Brownian motion with respect to the filtration $\{\mathcal{F}_{ct}\}$;

(2) $\{W_t\}$ is a continuous, square-integrable martingale with quadratic variation $\langle W, W\rangle_t = t$ for all $t \geq 0$;

(3) applying the strong law of large numbers to $\{W_t\}$ yields that

$$\lim_{t\to\infty} \frac{W_t}{t} = 0 \quad \text{a.s.};$$

(4) for almost every $\omega \in \Omega$, the sample path $W_t(\omega)$ is nowhere differentiable.

Sometimes we shall speak of a Brownian motion $\{W_t\}_{0\leq t\leq T}$ on $[0, T]$, and the meaning is apparent. Next we define the d-dimensional Brownian motion.

Definition 2.2. *A* d*-dimensional process* $\left\{W_t = \left(W_t^1, \cdots, W_t^d\right)\right\}_{t\geq 0}$ *is called a* d*-dimensional motion if every* $\{W_t^i\}$ *is a one-dimensional Brownian motion, and* $\{W_t^1\}, \cdots, \{W_t^d\}$ *are independent.*

From this definition, the similar properties of one-dimensional Brownian motion hold for d-dimensional Brownian motion as well. Finally, we give a useful result.

Theorem 2.10. *Let* $M = \{M_t\}_{t\geq 0}$ *be a real-valued, continuous, local martingale such that* $M_0 = 0$ *and* $\langle M, M\rangle_t = t$ *a.s. Define the stopping time*

$$\tau_t = \inf\{s : \langle M, M\rangle_s > t\}.$$

Then $\{M_{\tau_t}\}_{t\geq 0}$ *is a one-dimensional Brownian motion with respect to the filtration* $\{\mathcal{F}_{\tau_t}\}_{t\geq 0}$*.*

We recommend readers to (Karatzas & Shreve 1991) for more knowledge of Brownian motion.

2.3 Itô integrals

In this section, we shall define the *Itô stochastic integral* $\int_0^t f(s)\mathrm{d}W(s)$ with respect to an m-dimensional Brownian motion for a class of $d \times m$-matrix-valued stochastic processes $\{f(t)\}_{t\geq 0}$, which was first proposed by K. Itô (see (Itô 1944)). Since the Brownian sample path is nowhere differentiable, the integral cannot be defined in the ordinary way. Thus we will give the definition of Itô stochastic integral step by step.

Fundamental ideas

We firstly focus on a basic situation. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis, and $W = \{W(t)\}_{t\geq 0}$ be a one-dimensional Brownian motion. The processes considered in this part should belong to the following space.

Definition 2.3. *Let* $0 \leq a < b < \infty$. *Denote by* $\mathcal{M}^2([a,b];\mathbb{R})$ *the space of all real-valued, measurable, adapted processes* $f = \{f(t)\}_{a\leq t\leq b}$ *such that*

$$||f||_{a,b}^2 = E\int_a^b |f(s)|^2 \mathrm{d}s < \infty. \tag{2.1}$$

We say f *and* $\bar{f}$ *are equivalent in* $\mathcal{M}^2([a,b];\mathbb{R})$ *if* $||f - \bar{f}||_{a,b}^2 = 0$.

Motivated by the definition of Lebesgue integral, we can show how to define the Itô integral for a process $f \in \mathcal{M}^2([a,b];\mathbb{R})$. The idea is natural: first define the integral $\int_0^t g(s)\mathrm{d}W(s)$ for a class of simple processes g; then use such simple processes g's to approximate any process f, and define the limit of $\int_0^t g(s)\mathrm{d}W(s)$ as the integral $\int_0^t f(s)\mathrm{d}W(s)$. Therefore, we need to know the concept of simple processes.

Definition 2.4. *A real-valued process* $g = \{g(t)\}_{a\leq t\leq b}$ *is said to be simple if there is a partition* $a = t_0 < t_1 < \cdots < t_n = b$ *of* $[a,b]$, *and bounded random variables* ξ_i, $0 \leq i \leq n-1$ *such that* ξ_i *is* $\mathcal{F}_{t_i}$*-measurable and*

$$g(t) = \xi_0 \mathbb{I}_{[t_0,t_1]}(t) + \sum_{i=1}^{n-1} \xi_i \mathbb{I}_{[t_i,t_{i+1}]}(t). \tag{2.2}$$

Denote by $\mathcal{M}^0([a,b];\mathbb{R})$ *the family of all such processes.*

Clearly, $\mathcal{M}^0([a,b];\mathbb{R}) \subset \mathcal{M}^2([a,b];\mathbb{R})$. We now introduce the Itô integral for such simple process.

Definition 2.5. (Itô integral of simple processes) *For a simple process g with the form of (2.2), define*

$$\int_a^b g(s)\mathrm{d}W(s) = \sum_{i=0}^{n-1} \xi_i (W(t_{i+1}) - W(t_i)) \tag{2.3}$$

and call it the Itô integral of g with respect to the Brownian motion W.

The integral $\int_a^b g(s)\mathrm{d}W(s)$ is $\mathcal{F}_b$-measurable. Further, it belongs to $L^2(\Omega;\mathbb{R})$.

Lemma 2.3. *For $g \in \mathcal{M}^0([a,b];\mathbb{R})$, we have*

$$E\int_a^b g(s)\mathrm{d}W(s) = 0 \qquad \text{and} \qquad E\left|\int_a^b g(s)\mathrm{d}W(s)\right|^2 = E\int_a^b |g(s)|^2\mathrm{d}s.$$

The latter is often called the *Itô isometry.* The linearity property is obvious.

Lemma 2.4. *Let $g_1, g_2 \in \mathcal{M}^0([a,b];\mathbb{R})$ and let c_1, c_2 be two real numbers. Then $c_1 g_1 + c_2 g_2 \in \mathcal{M}^0([a,b];\mathbb{R})$ and*

$$\int_a^b (c_1 g_1(s) + c_2 g_2(s))\mathrm{d}W(s) = c_1\int_a^b g_1(s)\mathrm{d}W(s) + c_2\int_a^b g_2(s)\mathrm{d}W(s).$$

For a general process in $\mathcal{M}^2([a,b];\mathbb{R})$, we will use the approximation result.

Lemma 2.5. *For any $f \in \mathcal{M}^2([a,b];\mathbb{R})$, there is a sequence $\{g_n\}_{n\in\mathbb{N}_+}$ of simple processes such that*

$$\lim_{n\to\infty} E\int_a^b |f(s) - g_n(s)|^2\mathrm{d}s = 0. \tag{2.4}$$

Lemmas 2.3 and 2.4 also tell us that $\left\{\int_a^b g_n(s)\mathrm{d}W(s)\right\}_{n\in\mathbb{N}_+}$ is a Cauchy sequence in $L^2(\Omega,\mathbb{R})$ since as $n, m \to \infty$

$$\begin{aligned} E\left|\int_a^b g_n(s)\mathrm{d}W(s) - \int_a^b g_m(s)\mathrm{d}W(s)\right|^2 =& E\left|\int_a^b (g_n(s) - g_m(s))\mathrm{d}W(s)\right|^2 \\ =& E\int_a^b |g_n(s) - g_m(s)|^2\mathrm{d}s \to 0. \end{aligned}$$

This leads to the following definition.

Definition 2.6. (Itô integral of $\mathcal{M}^2$-processes) *Let $f \in \mathcal{M}^2([a,b];\mathbb{R})$. The Itô integral of f with respect to W is defined by*

$$\int_a^b f(s)\mathrm{d}W(s) = \lim_{n\to\infty}\int_a^b g_n(s)\mathrm{d}W(s), \tag{2.5}$$

where $\{g_n\}_{n\in\mathbb{N}_+}$ is a sequence of simple processes such that (2.4) is satisfied.

After figuring out how to define the Itô integral for $\mathcal{M}^2$-processes, we give some nice properties. For $f, g \in \mathcal{M}^2([a,b];\mathbb{R})$,

(1) $\int_a^b f(s)\mathrm{d}W(s)$ is $\mathcal{F}_b$-measurable;

(2) $E\int_a^b f(s)\mathrm{d}W(s) = 0$;

(3) $E\left|\int_a^b f(s)\mathrm{d}W(s)\right|^2 = E\int_a^b |f(s)|^2\mathrm{d}s$;

(4) $\int_a^b (c_1 f(s) + c_2 g(s))\mathrm{d}W(s) = c_1\int_a^b f(s)\mathrm{d}W(s) + c_2\int_a^b g(s)\mathrm{d}W(s)$.

Generally, we are interested in that $[a,b] = [0,T]$ for some $T > 0$.

Definition 2.7. *Let $f \in \mathcal{M}^2([0,T];\mathbb{R})$. For $0 \le t \le T$, define*

$$I(t) = \int_0^t f(s))\mathrm{d}W(s),$$

where $I(0) = 0$ by definition. We call $\{I(t)\}_{0\le t\le T}$ the indefinite Itô integral of f.

It is easy to derive that $\{I(t)\}$ is an adapted, continuous, square-integrable martingale with quadratic variation given by

$$\langle I, I\rangle_t = \int_0^t |f(s)|^2\mathrm{d}s. \tag{2.6}$$

Generalised definitions

Let us next extend the Itô integral to more general cases. The first one is stochastic integrals with stopping times.

Definition 2.8. *Let $f \in \mathcal{M}^2([0,T];\mathbb{R})$ and τ be a stopping time such that $0 \le \tau \le T$. Then $\left\{f(t)\mathbb{I}_{[0,\tau]}(t)\right\}_{0\le t\le T} \in \mathcal{M}^2([0,T];\mathbb{R})$ clearly, and we define*

$$\int_0^\tau f(s))\mathrm{d}W(s) = \int_0^T f(s)\mathbb{I}_{[0,\tau]}(s)\mathrm{d}W(s).$$

Furthermore, if ρ is another stopping time with $0 \le \rho \le \tau$, we define

$$\int_\rho^\tau f(s))\mathrm{d}W(s) = \int_0^\tau f(s)\mathrm{d}W(s) - \int_0^\rho f(s)\mathrm{d}W(s) = \int_0^T f(s)\mathbb{I}_{[\rho,\tau]}(s)\mathrm{d}W(s).$$

We also have that

$$E\int_\rho^\tau f(s)\mathrm{d}W(s) = 0 \quad \text{and} \quad E\left|\int_\rho^\tau f(s)\mathrm{d}W(s)\right|^2 = E\int_\rho^\tau |f(s)|^2\mathrm{d}s.$$

Then let us pay attention to the multi-dimensional situation. Let $\{W(t)\}_{t\ge 0}$ be an m-dimensional Brownian motion, where $W(t) = (W_1(t), \cdots, W_m(t))^{\mathrm{T}}$, and

$\mathcal{M}^2([0,T];\mathbb{R}^{d\times m})$ denote the family of $d\times m$-matrix-valued, measurable, adapted process $f=\{(f_{ij}(t))_{d\times m}\}_{0\le t\le T}$ such that $E\int_0^T|f(s)|^2\mathrm{d}s<\infty$.

Definition 2.9. *Let $f\in\mathcal{M}^2([0,T];\mathbb{R}^{d\times m})$. Define the multi-dimensional indefinite Itô integral*

$$\int_0^t f(s)\mathrm{d}W(s)=\int_0^t \begin{pmatrix} f_{11}(s) & \cdots & f_{1m}(s)\\ \vdots & & \vdots\\ f_{d1}(s) & \cdots & f_{dm}(s)\end{pmatrix}\begin{pmatrix}\mathrm{d}W_1(s)\\ \vdots\\ \mathrm{d}W_m(s)\end{pmatrix}.$$

If ρ and τ are two stopping times $0\le\rho\le\tau\le T$, then

$$E\int_\rho^\tau f(s)\mathrm{d}W(s)=0 \quad \text{and} \quad E\left|\int_\rho^\tau f(s)\mathrm{d}W(s)\right|^2=E\int_\rho^\tau|f(s)|^2\mathrm{d}s.$$

Finally we shall extend Itô integral to a larger class of stochastic processes. Let $\mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$ denote the family of $d\times m$-matrix-valued, measurable, adapted process $f=\{f(t)\}_{t\ge0}$ such that

$$\int_0^T|f(s)|^2\mathrm{d}s<\infty \;\text{ a.s. for every }\; T>0.$$

It is easy to see that $\mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$ is a subspace of $\mathcal{L}^2\left(\mathbb{R}_+;\mathbb{R}^{d\times m}\right)$. Thus we will use the Itô integral of $\mathcal{M}^2$-processes to help define the Itô integral of $\mathcal{L}^2$-processes. Let $f\in\mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. For each $n\in\mathbb{N}_+$, define the stopping time

$$\tau_n=n\wedge\inf\left\{t\ge0:\int_0^t|f(s)|^2\mathrm{d}s\ge n\right\}.$$

Clearly $\tau_n\uparrow\infty$ a.s. More importantly, $\{f(t)\mathbb{I}_{[0,\tau_n]}(t)\}_{t\ge0}\in\mathcal{M}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$ so the integral $I_n(t)=\int_0^t f(t)\mathbb{I}_{[0,\tau_n]}(t)\mathrm{d}W(s)$ is well-defined. We could further derive that $I_k(t\wedge\tau_n)=I_n(t)$ for $1\le n\le k$ and $t\ge0$, which implies that

$$I_k(t)=I_n(t),\quad 0\le t\le\tau_n.$$

Definition 2.10. *Let $f\in\mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. The indefinite Itô integral of f with respect to W is the $\mathbb{R}^d$-valued process $\{I(t)\}_{t\ge0}$ defined by*

$$I(t)=I_n(t)\quad on\quad 0\le t\le\tau_n.$$

As before, we usually write $\int_0^t f(s)\mathrm{d}W(s)$ instead of $I(t)$.

To close this part, we present the well-known Burkholder-Davis-Gundy inequality to estimate the Itô integral for $\mathcal{L}^2$-processes.

Theorem 2.11. (BDG inequality) *Let $f\in\mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Define for any $t\ge0$*

$$x(t)=\int_0^t f(s)\mathrm{d}W(s) \quad \textit{and} \quad A(t)=\int_0^t f(s)\mathrm{d}s.$$

Then for every $p > 0$, we have

$$c_p E|A(t)|^{\frac{p}{2}} \le E\left(\sup_{0\le s\le t} |x(s)|^p\right) \le C_p E|A(t)|^{\frac{p}{2}},$$

where

$$\begin{aligned} c_p &= (p/2)^p\,, & C_p &= (32/p)^{p/2}\,, & &\textit{if } 0 < p < 2;\\ c_p &= 1, & C_p &= 4, & &\textit{if } p = 2;\\ c_p &= (2p)^{-p/2}, & C_p &= \left(p^{p+1}/2(p-1)^{p-1}\right)^{p/2}, & &\textit{if } p > 2. \end{aligned}$$

Itô formula

In this part, we will introduce the stochastic version of chain rule for the Itô integral, which is known as *Itô formula*, in the explicit calculations.

Let $W(t), t \ge 0$ be an m-dimensional Brownian motion. An d-dimensional *Itô process* is an $\mathbb{R}^d$-valued, continuous, adapted process $x(t), t \ge 0$, with $x(t) = (x_1(t), \cdots, x_d(t))^{\mathrm{T}}$, of the form

$$x(t) = x(0) + \int_0^t f(s)\mathrm{d}s + \int_0^t g(s)\mathrm{d}W(s),$$

where $f = (f_1, \cdots, f_d(t))^{\mathrm{T}} \in \mathcal{L}^1(\mathbb{R}_+;\mathbb{R}^d)$ and $g = (g_{ij})_{d\times m} \in \mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. We shall say that $x(t)$ has an *Itô differential* $\mathrm{d}x(t)$ on $t \ge 0$ given by

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}W(t).$$

Sometimes, we will speak of Itô process $x(t)$ and Itô differential $\mathrm{d}x(t)$ on $t \in [a, b]$, and the meaning is apparent.

Let $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R})$ denote the family of all real-valued functions $V(x,t)$ defined on $\mathbb{R}^d \times \mathbb{R}_+$ such that they are continuously twice differentiable in x and once in t with $V_t = \frac{\partial V}{\partial t}$, $V_x = \left(\frac{\partial V}{\partial x_1}, \cdots, \frac{\partial V}{\partial x_d}\right)$ and $V_{xx} = \left(\frac{\partial^2 V}{\partial x_i \partial x_j}\right)_{d\times d}$.

Theorem 2.12. (Itô formula) *Let $x(t)$ be a d-dimensional Itô process on $t \ge 0$ with the Itô differential*

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}W(t), \tag{2.7}$$

where $f \in \mathcal{L}^1(\mathbb{R}_+;\mathbb{R}^d)$ and $g \in \mathcal{L}^2(\mathbb{R}_+;\mathbb{R}^{d\times m})$. Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R})$. Then $V(x(t),t)$ is a real-valued Itô process with Itô differential

$$\mathrm{d}V(x(t),t) = \left(V_t(x(t),t) + V_x(x(t),t)f(t) + \frac{1}{2}\mathrm{trace}\left(g^{\mathrm{T}}(t)V_{xx}(x(t),t)g(t)\right)\right)\mathrm{d}t$$

$$+ V_x(x(t))g(t)\mathrm{d}W(t) \quad a.s.$$

Given $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R})$, define an operator $\mathcal{L}V : \mathbb{R}^d \times \mathbb{R}_+ \to \mathbb{R}$ by

$$\mathcal{L}V(x,t) = V_t(x,t) + V_x(x,t)f(t) + \frac{1}{2}\mathrm{trace}\left(g^{\mathrm{T}}(t)V_{xx}(x,t)g(t)\right), \qquad (2.8)$$

which is called the *diffusion operator* of the Itô process (2.7). If there is absence of t in the construction of a function $V(x,t)$, we simply write $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+;\mathbb{R})$ as $C^2(\mathbb{R}^d;\mathbb{R})$.

More detailed descriptions and proofs about this section can be found in (Mao & Yuan 2006) (Sections 1.5 and 1.6).

2.4 Markov chains

Markov chains

Definition 2.11. *A stochastic process $r(t), t \geq 0$ defined on a probability space $(\Omega, \mathcal{F}, P)$, which takes values in a countable set $\mathbb{S}$, is called a continuous-time Markov chain, if for all $t, h \geq 0$, all $i, j \in \mathbb{S}$, any times $0 \leq t_1 < \cdots < t_n < t$ and any $i_1, \cdots, i_n \in \mathbb{S}$*

$$\begin{aligned} &P(r(t+h) = j | r(t) = i, r(t_n) = i_n, \cdots, r(t_1) = i_1) \\ =&P(r(t+h) = j | r(t) = i). \end{aligned}$$

We are going to concentrate on homogeneous Markov chains: if for all $t, h \geq 0$ and $i, j \in \mathbb{S}$, the conditional probability $P(r(t+h) = j|r(t) = i)$ only depends on the time increment h, we say the Markov chain r is *homogeneous*. In this situation, $P(r(t+h) = j|r(t) = i) = P(r(h) = j|r(0) = i)$, and the function

$$p_{ij}(h) = P(r(h) = j|r(0) = i), \quad i, j \in \mathbb{S}, h \geq 0$$

is called the *transition function* or *transition probability* of the Markov chain r. The law of r is determined by the transition probability and its initial distribution $\lambda = (\lambda_i)_{i\in\mathbb{S}}$, where $\lambda_i = P(r(0) = i)$,

$$P(r(t) = j) = \sum_{i\in\mathbb{S}} \lambda_i p_{ij}(t).$$

In addition, we have that for each fixed t and i, $\sum_{j\in\mathbb{S}} p_{ij}(t) = 1$.

Next, we say the transition probability matrix $P(t) := (p_{ij}(t))_{i,j\in\mathbb{S}}$ is *standard* if for any $i \in \mathbb{S}$, $\lim_{t\downarrow 0} P_{ii}(t) = 1$.

Theorem 2.13. *Let $P(t)$ be a standard transition probability matrix. Then*

$$q_i = \lim_{t \to 0} \frac{1 - p_{ii}(t)}{t}$$

exists (but may be ∞) for all $i \in \mathbb{S}$.

A state $i \in \mathbb{S}$ is said to be stable if $q_i < \infty$.

Theorem 2.14. *Let $P(t)$ be a standard transition probability matrix and j be a stable state. Then $q_{ij} = p'_{ij}(0)$ exists and is finite for all $i \in \mathbb{S}$.*

From Theorem 2.14, for all $t \geq 0$, as $h \downarrow 0$,

$$P\left(r(t+h) = j | r(t) = i\right) = p_{ij}(h) = \begin{cases} 1+q_{ij}h + o(h), & \text{if } i = j, \\ q_{ij}h + o(h), & \text{if } i \neq j. \end{cases} \tag{2.9}$$

When $i \neq j$, we observe that $q_{ij} \geq 0$, which is called the *transition rate* from i to j. When $i = j$, since $\sum_{j \in \mathbb{S}} p_{ij}(t) = 1$, we have that $q_{ii} = -\sum_{j \neq i} q_{ij}$. The matrix $Q = (q_{ij})_{i,j \in \mathbb{S}}$ is called the *transition rate matrix* of r.

If the state space is finite, which we can take to be $\mathbb{S} = \{1, \cdots, S\}$, then the process is called a *continuous-time finite Markov chain.* Throughout this thesis, the underlying Markov chains are finite and all states are stable. For such a Markov chain, almost every sample path is a right-continuous step function.

It is useful to stress that a continuous-time Markov chain $r(t)$ with the transition rate matrix $Q = (q_{ij})_{S \times S}$ can be represented as a stochastic integral with respect to a Poisson random measure. Let $\Delta_{i,j}$ be consecutive, left-closed, right-open intervals of the real line with length q_{ij} such that $\Delta_{12} = [0, q_{12})$, $\Delta_{13} = [q_{12}, q_{12}+q_{13})$, $\cdots$, $\Delta_{1S} = \Big[\sum_{j=2}^{S-1} q_{1j}, \sum_{j=2}^{S} q_{1j}\Big)$, $\Delta_{21} = \Big[\sum_{j=2}^{S} q_{1j}, \sum_{j=2}^{S} q_{1j} + q_{21}\Big)$, $\cdots$, $\Delta_{2S} = \Big[\sum_{j=2}^{S} q_{1j} + \sum_{j=1, j\neq 2}^{S-1} q_{2j}, \sum_{j=2}^{S} q_{1j} + \sum_{j=1, j\neq 2}^{S} q_{2j}\Big)$ and so on. Define a function $\varphi : \mathbb{S} \times \mathbb{R} \to \mathbb{R}$ by

$$\varphi(i, y) = \begin{cases} j - i, & \text{if } y \in \Delta_{ij}, \\ 0, & \text{otherwise.} \end{cases} \tag{2.10}$$

Then

$$\mathrm{d}r(t) = \int_{\mathbb{R}} \varphi(r(t^-), y) Pois(\mathrm{d}t, \mathrm{d}y)$$

with initial data $r(0) = \lambda$, where $Pois(\mathrm{d}t, \mathrm{d}y)$ is a Poisson random measure with intensity $\mathrm{d}t \times Leb(\mathrm{d}y)$, in which Leb is the Lebesgue measure on the real line.

Generalised Itô formula

Form now on, unless otherwise specified, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ be a stochastic basis (i.e. $(\Omega, \mathcal{F}, P)$ is a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions). Let $W(t)$ be an m-dimensional Brownian motion defined on the probability space. Let $r(t)$ be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, \cdots, S\}$ with transition rate matrix $Q = (q_{ij})_{S\times S}$ given in (2.9). Assume that the Markov chain r and the Brownian motion W are independent under probability measure P.

Consider a d-dimensional Itô process on $t \geq 0$ with Itô differential

$$\mathrm{d}x(t) = f(t)\mathrm{d}t + g(t)\mathrm{d}W(t), \tag{2.11}$$

where $f \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$ and $g \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d\times m})$. The Itô formula established in Section 2.3 shows that a $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+; \mathbb{R})$-function V maps the Itô process $x(t)$ into another Itô process $V(x(t), t)$. But now, we shall consider the paired process $(x(t), r(t))$ and need to know how a function $V : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ will map $(x(t), r(t))$ into another process $V(x(t), t, r(t))$.

For this aim, let $C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$ denote the family of all real-valued functions $V(x, t, i)$ on $\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$ such that for each $i \in \mathbb{S}$, $V(x, t, i)$ are continuously twice differentiable in x and once in t. If $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$, define the diffusion operator $\mathcal{L}V : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ of the Itô process (2.11) by

$$\begin{aligned}\mathcal{L}V(x,t,i) =& V_t(x,t,i) + V_x(x,t,i)f(t) + \frac{1}{2}\mathrm{trace}\left(g^{\mathrm{T}}(t)V_{xx}(x,t,i)g(t)\right) \\ &+ \sum_{j\in\mathbb{S}} q_{ij}V(x,t,j).\end{aligned} \tag{2.12}$$

Theorem 2.15. (Generalised Itô formula) *If $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$, then for any $t \geq 0$*

$$\begin{aligned}&V(x(t), t, r(t)) \\ =& V(x(0), 0, r(0)) + \int_0^t \mathcal{L}V(x(s), s, r(s))\mathrm{d}s + \int_0^t V_x(x(s), s, r(s))g(s)\mathrm{d}W(s) \\ &+ \int_0^t \int_{\mathbb{R}} \Big(V(x(s), s, r(0) + \varphi(r(s), l)) - V(x(s), s, r(s))\Big)\bar{\mu}(\mathrm{d}s, \mathrm{d}l) \quad a.s.,\end{aligned}$$

where the function φ is defined by (2.10) and $\bar{\mu}(\mathrm{d}s, \mathrm{d}l) := \nu(\mathrm{d}s, \mathrm{d}l) - \mu(\mathrm{d}l)$ is a martingale measure. Here $\nu(\mathrm{d}s, \mathrm{d}l)$ is a Possion random measure with density $\mathrm{d}s \times \mu(\mathrm{d}l)$, in which μ is the Lebesgure measure on $\mathbb{R}$.

The proof can be found in Lemma 3, p104, (Skorokhod 1989). In particular, making use of the martingale property of Poisson measures and taking expectations on both sides of the last inequality gives a useful lemma.

Lemma 2.6. *Let $V \in C^{2,1}(\mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}; \mathbb{R})$ and τ_1, τ_2 be bounded stopping times such that $0 \le \tau_1 \le \tau_2$ a.s. If $V(x(t), t, r(t))$ and $\mathcal{L}V(x(t), t, r(t))$ are bounded on $t \in [\tau_1, \tau_2]$ with probability 1, then*

$$EV(x(\tau_2), \tau_2, r(\tau_2)) = EV(x(\tau_1), \tau_1, r(\tau_1)) + E\int_{\tau_1}^{\tau_2} \mathcal{L}V(x(s), s, r(s))\mathrm{d}s.$$

2.5 Hybrid stochastic differential equations

Let $f : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{d\times m}$ be Borel measurable functions. Then consider the *stochastic differential equations (SDEs) with Markovian switching* of the following form

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{2.13}$$

on $t \ge 0$ with the initial data

$$x(0) = \xi_0 \in \mathbb{R}^d, \qquad r(0) = i_0 \in \mathbb{S}. \tag{2.14}$$

Since there involve two different states, $x(t)$ taking values continuously and $r(t)$ taking discrete values, equation (2.13) is often referred to a *hybrid SDE*. $f(x, t, i)$ and $g(x, t, i)$ are called the *drift coefficient* and *diffusion coefficient*, respectively.

By the definition of Itô differential, the hybrid SDE (2.13) is equivalent to the following stochastic integration equation

$$x(t) = x(0) + \int_0^t f(x(s), s, r(s))\mathrm{d}s + \int_0^t g(x(s), s, r(s))\mathrm{d}W(s), \quad t \ge 0. \tag{2.15}$$

A $\mathbb{R}^d$-valued process $\{x(t)\}_{t\ge0}$ is a (global) solution of the hybrid SDE (2.13) if

(i) $x(t)$ is continuous and adapted;

(ii) $\{f(x(t), t, r(t))\}_{t\ge0} \in \mathcal{L}^1(\mathbb{R}_+; \mathbb{R}^d)$, $\{g(x(t), t, r(t))\}_{t\ge0} \in \mathcal{L}^2(\mathbb{R}_+; \mathbb{R}^{d\times m})$;

(iii) equation (2.13) is satisfied almost surely.

Moreover, a solution $x(t)$ is said to be *unique* if any other solution $\bar{x}(t)$ is such that $P(x(t) = \bar{x}(t) \text{ for all } t \ge 0) = 1$. Next we introduce the concept of local solution.

Definition 2.12. *Let σ_e be a stopping time such that $0 \le \sigma_e \le \infty$ a.s. An $\mathbb{R}^d$-valued, adapted, continuous process $\{x(t)\}_{0 \le t < \sigma_e}$ is called a local solution of the hybrid SDE (2.13) if there is an increasing sequence $\{\sigma_n\}_{n \in \mathbb{N}_+}$ of stopping times such that $0 \le \sigma_n \uparrow \sigma_e$ a.s. and*

$$x(t) = x(0) + \int_0^{t \wedge \sigma_n} f(x(s), s, r(s)) \mathrm{d}s + \int_0^{t \wedge \sigma_n} g(x(s), s, r(s)) \mathrm{d}W(s)$$

holds for any $t \ge 0$ and $n \in \mathbb{N}_+$ almost surely. If further, $\limsup_{t \to \sigma_e} |x(t)| = \infty$ whenever $\sigma_e < \infty$, then it is called a maximal local solution and σ_e is called the explosion time. A maximal local solution $\{x(t)\}_{0 \le t < \sigma_e}$ is said to be unique if any other maximal local solution $\{\bar{x}(t)\}_{0 \le t < \bar{\sigma}_e}$ is so that $\sigma_e = \bar{\sigma}_e$ and $x(t) = \bar{x}(t)$ for all $t \ge 0$ almost surely.

The following theorem is the classical result in existence of unique maximal local solution of the hybrid SDE (2.13).

Theorem 2.16. *Assume that for every $n \in \mathbb{N}_+$, there is a positive constant K_n such that for all $t \ge 0$, $i \in \mathbb{S}$ and those $x, y \in \mathbb{R}^d$ with $|x| \vee |y| \le k$*

$$|f(x,t,i) - f(y,t,i)|^2 \vee |g(x,t,i) - g(y,t,i)|^2 \le K_n |x-y|^2. \tag{2.16}$$

Then there is a unique maximal local solution of the hybrid SDE (2.13).

Condition (2.16) is the well-known Local Lipschitz condition, but it only guarantees the maximal local solution, which may explode at a finite time. To suppress the potential explosion and make sure the solution is global, the classical condition is the following linear growth condition.

Theorem 2.17. *Let the local Lipschitz condition (2.16) hold. Assume that* **(linear growth condition)** *there is a positive constant $K \ge 0$ such that*

$$|f(x,t,i)|^2 \vee |g(x,t,i)|^2 \le K(1 + |x|^2). \tag{2.17}$$

Then there is a unique global solution of the hybrid SDE (2.13), which belongs to $\mathcal{M}^2(\mathbb{R}_+; \mathbb{R}^d)$.

Among other topics of hybrid SDEs, stability is very important one. For a stable system, the trajectories which are close to each other at a specific instant should keep close to each other at the remaining instants. If we assume that

$$f(0,t,i) \equiv 0 \quad \text{and} \quad g(0,t,i) \equiv 0, \tag{2.18}$$

then the solution of the hybrid SDE (2.13) will remain to be zero if it starts from zero, namely $x(t;\xi_0,0,i_0) \equiv 0$. In other words, zero is an *equilibrium* state or a *trivial solution*.

Throughout this thesis, we will mainly focus on the stability of the hybrid SDE (2.13), in the sense of the following

(1) H_∞ *stability*: for all $(\xi_0, i_0) \in \mathbb{R}^d \times \mathbb{S}$

$$\int_0^\infty E|x(t;\xi_0,0,i_0)| < \infty;$$

(2) *p-th moment asymptotic stability*: for some $p > 0$ and all $(\xi_0, i_0) \in \mathbb{R}^d \times \mathbb{S}$

$$\lim_{t\to\infty} E|x(t;\xi_0,0,i_0)|^p = 0;$$

when $p = 2$, it is said to be *asymptotically stable in mean square*;

(3) *almost surely asymptotic stability*: for all $(\xi_0, i_0) \in \mathbb{R}^d \times \mathbb{S}$

$$P\left(\lim_{t\to\infty} |x(t;\xi_0,0,i_0)| = 0\right) = 1;$$

(4) *p-th moment exponential stability*: for some $p > 0$, there is a positive constant ε such that for all $(\xi_0, i_0) \in \mathbb{R}^d \times \mathbb{S}$

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(E|x(t;\xi_0,0,i_0)|^p\right) \le -\varepsilon;$$

when $p = 2$, it is said to be *exponentially stable in mean square*;

(5) *almost surely exponential stability*: there is a positive constant ε such that for all $(\xi_0, i_0) \in \mathbb{R}^d \times \mathbb{S}$

$$\limsup_{t\to\infty} \frac{1}{t} \log\left(|x(t;\xi_0,0,i_0)|\right) \le -\varepsilon \quad \text{a.s.}$$

More details about this section can be found in (Mao & Yuan 2006) (Chapters 3 and 5).

2.6 Useful inequalities

In the end of this chapter, we introduce several inequalities which are used frequently later on. We also recommend Chapter 2 in (Mao & Yuan 2006) as reference.

Arithmetic inequalities

Let us begin with the simplest inequality

$$2ab \le \varepsilon a^2 + \frac{1}{\varepsilon} b^2, \qquad \forall a, b \in \mathbb{R}, \quad \forall \varepsilon > 0. \tag{2.19}$$

Now we proceed to the Yong inequality, for any $a, b \in \mathbb{R}$ and $\varepsilon > 0$

$$|a|^p |b|^q \le \varepsilon |a|^{p+q} + \frac{q}{p+q} \left(\frac{p}{\varepsilon(p+q)} \right)^{\frac{p}{q}} |b|^{p+q}. \tag{2.20}$$

Another inequality frequently-used is the discrete Hölder inequality

$$\left| \sum_{i=1}^{n} a_i \right|^p \le H_p \sum_{i=1}^{n} |a_i|^p \tag{2.21}$$

for any $a_i \in \mathbb{R}$, $n \ge 2$, where $H_p = n^{p-1}$ if $p \ge 1$ and n^p if $p \in (0, 1)$. One more useful inequality is

$$|a^p - b^p| \le p(a - b)(a^{p-1} + b^{p-1}), \qquad \forall a, b \ge 0, \quad \forall p \ge 1. \tag{2.22}$$

Integral inequalities

When using the method of Lyapunov functional, the following inequality is very helpful. Let τ be a positive constant and $\phi : [-\tau, \infty) \to \mathbb{R}_+$ be an integrable function, then for any $t \ge 0$

$$\int_{-\tau}^{0} \int_{t+s}^{t} \phi(v) \mathrm{d}v \mathrm{d}s \le \tau \int_{t-\tau}^{t} \phi(v) \mathrm{d}v. \tag{2.23}$$

Then we give the well-known Gronwall inequality

Theorem 2.18. (Gronwall inequality) *Let $T > 0$ and $c \ge 0$. Let $u : [0, T] \to \mathbb{R}_+$ be a Borel-measurable bounded function, and $v : [0, T] \to \mathbb{R}_+$ be an integrable function. Then for any $t \in [0, T]$*

$$u(t) \le c + \int_0^t v(s) u(s) \mathrm{d}s \implies u(t) \le c \exp \left(\int_0^t v(s) \mathrm{d}s \right).$$

Matrix inequalities

For a matrix $A \in \mathbb{R}^{d \times m}$, its operator norm is given by $||A|| = \sup_{x \in \mathbb{R}^m, |x|=1} |Ax|$, while its trace norm is defined by $|A| = \sqrt{\text{trace}(A^{\mathrm{T}} A)}$. We always have

$$|Ax| \le ||A|| |x| \quad \text{and} \quad |Ax| \le |A| |x|, \quad \forall x \in \mathbb{R}^m.$$

For a symmetric matrix $A \in \mathbb{R}^{d\times d}$, denoted its largest and smallest eigenvalue by $\lambda_{\max}(A)$ and $\lambda_{\min}(A)$, respectively. Moreover,

$$\lambda_{\min}(A)|x|^2 \le x^{\mathrm{T}}Ax \le \lambda_{\max}(A)|x|^2, \quad \forall x \in \mathbb{R}^d.$$

Then the operator norm of a matrix $A \in \mathbb{R}^{d\times m}$ can be writen as

$$||A|| = \sqrt{\lambda_{\max}(A^{\mathrm{T}}A)}.$$

The theory of M-matrices will play an important role in the study stabilisation and control in this thesis. We present some conditions equivalent to non-singular M-matrix, and refer the reader to (Berman & Plemmons 1994) for more details.

Theorem 2.19. *If $A \in \mathbb{Z}^{d\times d} := \{A = (a_{ij})_{d\times d} : a_{ij} \le 0, i \ne j\}$, then the following statements are equivalent*

(1) A is a non-singular M-matrix;

(2) A^{-1} exists and all elements of A^{-1} are positive;

(3) There exists $x \gg 0$ in $\mathbb{R}^d$ such that $Ax \gg 0$.

Here $x \gg 0$ means all elements of x are positive.

Chapter 3

A note on stabilisation of hybrid SDEs by state feedback control observed at discrete times

3.1 Introduction

In Chapter 1, we have discussed the following stabilisation problem: given an unstable hybrid SDE

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{3.1}$$

on $t \geq 0$, compared with continuous-time state feedback control $u(x(t), t, r(t))$, it is more practical and less costly to use feedback control based on discrete-time state observations, say at times $0, \tau, 2\tau, \cdots$, to achieve the stabilisation of the controlled SDE

$$\mathrm{d}x(t) = \Big(f(x(t), t, r(t)) + u(x(t_\tau), t, r(t))\Big)\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t). \tag{3.2}$$

Here for convenience, we let $t_\tau = [t/\tau]\tau$. This stabilisation problem for stochastic systems was initially proposed by (Mao 2013). Traditionally, the coefficients f and g should satisfy the linear growth condition (see, e.g. (You et al. 2015, Li & Kou 2017)). But (Fei et al. 2020) eased this restriction and brought this stabilisation problem into highly nonlinear area. Although the theory developed therein has made great progress and more real models could be included, such as volatility model (Lewis 2000, Heston 1997), there are still two questions deserved our further discussion.

Firstly, we emphasise the key ingredient in (Fei et al. 2020) for stabilisation purpose, namely, the following condition: assume that there are non-negative constants χ_{i1}, $\hat{\chi}_{i1}$, positive constants χ_{i2}, $\hat{\chi}_{i2}$, and $p > 1$ such that for every $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$

$$\begin{cases} x^{\mathrm{T}} f(x,t,i) + \dfrac{1}{2}|g(x,t,i)|^2 \le \chi_{i1}|x|^2 - \chi_{i2}|x|^{p+1}, \\ x^{\mathrm{T}} f(x,t,i) + \dfrac{p}{2}|g(x,t,i)|^2 \le \bar{\chi}_{i1}|x|^2 - \bar{\chi}_{i2}|x|^{p+1}. \end{cases} \tag{3.3}$$

Condition (3.3) is indeed more advanced than the conventional linear growth condition. But the reader might wonder why we need to give two similar inequalities at the same time, particularly, the first one can be deduced from the other.

It is actually arisen from the effect of discrete-time state observations. To deal with this effect, we usually decompose the drift coefficient of the controlled SDE (3.2) as

$$\Big(f(x(t), t, r(t)) + u(x(t), t, r(t))\Big) + \Big(u(x(t_\tau), t, r(t)) - u(x(t), t, r(t))\Big) \tag{3.4}$$

and hope the second term (or $|x(t) - x(t_\tau)|$) could be small enough if the observation duration τ is sufficiently small. Currently, one popular method to estimate the second term is to compute $E|x(t) - x(t_\tau)|^2$. Then the estimation result (see, e.g. equation (47) in (Fei et al. 2020)) forces us to give two inequalities in condition (3.3) unavoidably. But is it possible for us to modify the estimation so that condition (3.3) could be relaxed? In this chapter, we will give a positive answer to this question (see Lemma 3.1 below). Owing to this modification, only the first inequality of condition (3.3) is required in the stability analysis.

Secondly, it should be highlighted that some conditions in (Fei et al. 2020) are not easily verified in practice. In particular, we need to find five free parameters $\hat{\chi}_j > 0$ $(j = 1, \cdots, 5)$ to let

$$L_1 U(x,t,i) + \hat{\chi}_1 |U_x(x,i)|^2 + \hat{\chi}_2 |f(x,t,i)|^2 + \hat{\chi}_3 |g(x,t,i)|^2 \le -\hat{\chi}_4 |x|^2 - \hat{\chi}_5 |x|^{2p}$$

hold for all $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$. Here, $U(x, i)$ is in the form of $|x|^2 + |x|^{p+1}$ and the definition of $L_1 U(x, t, i)$ can be found in (3.15). For more details, we refer to Condition 4.6 in (Fei et al. 2020). These free parameters all influence the bound of τ we obtain, and sometimes a bad choice may bring us a relatively small τ. Therefore, in this chapter, we will provide a new method to determine the value of τ, so that there is no need to find any free parameters in reality. In other words, conditions imposed on the original system and the control function can be verified much more easily than before.

Let us now start to develop these new techniques and establish our new theory.

3.2 Standing hypothesis

Suppose that our underlying system is described by the hybrid SDE (3.1) with the initial data

$$x(0) = \xi_0 \in \mathbb{R}^d, \qquad r(0) = i_0 \in \mathbb{S},$$

where $f : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous. Although the linear growth condition is not of our interest, we still do not want coefficients to grow very sharply. Hence the following *polynomial growth condition* is required.

Assumption 3.1. *Assume that there are non-negative constants H_1, H_2, $\tilde{H}_1$, $\tilde{H}_2$ and $p > 1$ such that for every $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,*

$$|f(x,t,i)| \le H_1|x| + H_2|x|^p \tag{3.5}$$

and

$$|g(x,t,i)|^2 \le \tilde{H}_1|x|^2 + \tilde{H}_2|x|^{p+1}. \tag{3.6}$$

But note that Assumption 3.1 cannot guarantee the hybrid SDE (3.1) has a unique global solution. For this purpose, the *Khasminskii-type condition* is always needed, which arises widely now in the study of highly nonlinear stochastic systems (see, e.g. (Mao & Yuan 2006, Fei et al. 2020, Shi et al. 2022)).

Assumption 3.2. *Assume that there exists a positive constant $\hat{\alpha}$ such that*

$$x^{\mathrm{T}} f(x,t,i) + \frac{p}{2}|g(x,t,i)|^2 \le \hat{\alpha}|x|^2 \tag{3.7}$$

for any $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$.

Under Assumptions 3.1 and 3.2, it is then easy to see from Theorem 3.19 in (Mao & Yuan 2006) that the hybrid SDE (3.1) has a unique global solution satisfying that for any $t > 0$, $\sup_{0\le s\le t} E|x(s)|^{p+1} < \infty$.

Before giving another assumption, let us make comments on condition (3.6).

Remark 3.1. *From conditions (3.5) and (3.7), compute that for any $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,*

$$|g(x,t,i)|^2 \le \frac{2}{p}\left(\hat{\alpha}|x|^2 + |x||f(x,t,i)|\right) \le \frac{2(\hat{\alpha}+H_1)}{p}|x|^2 + \frac{2H_2}{p}|x|^{p+1}.$$

This implies that condition (3.6) holds with $\tilde{H}_1 = \frac{2(\hat{\alpha}+H_1)}{p}$ *and* $\tilde{H}_2 = \frac{2H_2}{p}$*. Although condition (3.6) could be deduced from conditions (3.5) and (3.7), the values of* $\tilde{H}_1$ *and* $\tilde{H}_2$ *derived from this estimation are too rough. These two parameters could influence the upper bound of* τ*. The more accurate they are, the better* τ *we could obtain. As a result, we need to give condition (3.6) as a hypothesis seperately.*

Assumption 3.3. *For each* $i \in \mathbb{S}$*, assume that there are constants* $\gamma_i \geq 0$ *and* $\beta_i > 0$ *such that for every* $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$

$$x^{\mathrm{T}} f(x,t,i) + \frac{1}{2}|g(x,t,i)|^2 \leq \gamma_i |x|^2 - \beta_i |x|^{p+1}. \tag{3.8}$$

But the reader may find that Assumptions 3.2 and 3.3 look quite similarly. Is it necessary to give them at the same time? The following remark may be helpful.

Remark 3.2. *It should be pointed out that Assumptions 3.2 and 3.3 are both needed. At first, they play different roles. Assumption 3.2 is used to ensure the existence of global solution with certain moment properties, while Assumption 3.3 is for stabilisation and control design. Secondly, these two conditions are quite different, and any one cannot be deduced from the other. Let us use a scalar example operating just in one mode to explain this.*

Case 1: $f(x,t,1) = x - x^3 \sin^2(x)$, $g(x,t,1) = x$*. Then Assumption 3.2 is satisfied with* $\hat{\alpha} = 3.5$*. But we could not find a positive* β_1 *such that for all* $x \in \mathbb{R}$

$$x^{\mathrm{T}} f(x,t,1) + \frac{1}{2}|g(x,t,1)|^2 = 1.5|x|^2 - |x|^4 \sin^2(x) \leq 1.5|x|^2 - \beta_1 |x|^4.$$

Case 2: $f(x,t,1) = x - x^3$, $g(x,t,1) = x^2$*. It is easy to verify that Assumption 3.3 holds with* $\alpha_1 = 1$, $\beta_1 = 0.5$*. But Assumption 3.2 is not satisfied since*

$$x^{\mathrm{T}} f(x,t,1) + \frac{3}{2}|g(x,t,1)|^2 = |x|^2 + 0.5|x|^4.$$

3.3 Control design

Although the solution of the hybrid SDE (3.1) is in L^{p+1} under Assumptions 3.1 and 3.2, it might still be unstable. In this case, we need to design discrete-time state feedback control $u(x(t_\tau), t, r(t)$ to make the controlled SDE (3.2) stable. Here, the control function $u : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$ is supposed to be Borel-measurable. In this section, we shall propose some rules for the control $u(x(t_\tau), t, r(t))$ to meet the stabilisation aim.

Rule 3.1. *Design the control function* $u(x,t,i)$ *so that we can find a positive constant* K_M *to let*

$$|u(x,t,i) - u(y,t,i)| \le K_M|x-y| \tag{3.9}$$

hold for all $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$, *and moreover* $u(0,t,i) \equiv 0$.

Rule 3.1 tells us that the control function $u(x,t,i)$ should be globally Lipschitz continuous in x. This rule implies the following linear growth condition

$$|u(x,t,i)| \le K_M|x|, \quad \forall (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}. \tag{3.10}$$

This seems a little surprising since we would normally look for the control function which should be highly nonlinear given the high nonlinearity of the underlying hybrid SDE (3.1). But we will demonstrate that the globally Lipschitz continuous control function works well, which also makes the control design in practice and theoretical analysis much more easily, such as the following existence-and-uniqueness theorem.

Theorem 3.1. *Let Assumptions 3.1 and 3.2 hold, if the control function* $u(x,t,i)$ *meets Rule 3.1, then the controlled SDE (3.2) has a unique global solution* $x(t)$ *on* $t > 0$, *which satisfies that*

$$\sup_{0 \le s \le t} E|x(s)|^{p+1} < \infty. \tag{3.11}$$

This theorem can be shown in the same way as Theorem 7.13 in (Mao & Yuan 2006) so we omit the proof. Moreover, it should be pointed out that the controlled SDE (3.2) admits a trivial solution since $f(0,t,i) \equiv 0$ and $g(0,t,i) \equiv 0$ from Assumption 3.1, along with the requirement that $u(0,t,i) \equiv 0$ in Rule 3.1.

However, in order for the stability of the controlled SDE (3.2), Rule 3.1 is not enough, so more rules are needed. The following one is very critical.

Rule 3.2. *For each* $i \in \mathbb{S}$, *design the control function* $u(x,t,i)$ *such that there is a non-negative constant* κ_i *for*

$$x^{\mathrm{T}} u(x,t,i) \le -\kappa_i |x|^2, \qquad \forall (x,t) \in \mathbb{R}^d \times \mathbb{R}_+, \tag{3.12}$$

while for $A := -2\mathrm{diag}(\alpha_1, \cdots, \alpha_S) - Q$ *to be a non-singular* M*-matrix with* $\alpha_i = \gamma_i - \kappa_i$, *where* γ_i *is given in Assumption 3.3.*

But the reader may wonder if we can really find the control function $u(x,t,i)$ to make Rules 3.1 and 3.2 fulfilled. The following remark will deny this worry.

Remark 3.3. *In fact, there are a lot of control functions available under Assumption 3.3. For example, design the control function in the linear form* $u(x,t,i) = -\mathcal{A}_i x$ *with* $\mathcal{A}_i$ *being symmetric and positive-definite such that* $\lambda_{\min}(\mathcal{A}_i) \geq (\alpha+1)\gamma_i$ *with a sufficiently large* α. *It is then easy to see that Rule 3.1 holds with* $K_M = \max_{i\in\mathbb{S}} |\mathcal{A}_i|$. *Further, we have* $\kappa_i = (\alpha+1)\gamma_i$ *and* $A \approx 2\alpha \mathrm{diag}(\gamma_1, \cdots, \gamma_S)$ *when* α *is large enough. Thus, Rule 3.2 is satisfied.*

After knowing about the design and existence of control functions, we should pay attention to the observation duration τ in order to input the discrete-time state feedback control. It is certainly impossible to have a τ as large as we want since the information received from discrete-time state observations would be inadequate to achieve stabilisation. Therefore, the upper bound of τ is usually required.

Rule 3.3. *Let the observation duration* τ *be smaller than* τ^*, *which is determined by* $\tau^* = \max_{\varepsilon \in \left(0, \frac{1}{K_M \eta_M}\right)} \varphi(\varepsilon)$, *where*

$$\varphi(\varepsilon) = \frac{1}{K_M \eta_M} \left(\frac{1 - K_M \eta_M \varepsilon}{2H_1 + 2K_M + \frac{\tilde{H}_1}{\varepsilon} + \frac{1}{\eta_M}} \wedge \frac{2\beta_\eta}{2H_2 + \frac{\tilde{H}_2}{\varepsilon} + \frac{2\beta_\eta}{\eta_M}} \right),$$

in which $\eta_M = \max_{i\in\mathbb{S}} \eta_i$ *with* $(\eta_1, \cdots, \eta_S)^{\mathrm{T}} = A^{-1}(1, \cdots, 1)^{\mathrm{T}}$, $\beta_\eta = \min_{i\in\mathbb{S}}(\beta_i \eta_i)$.

We make some comments on this rule.

Remark 3.4. *At first, since* A *is a non-singular* M*-matrix, all* η_i $(i \in \mathbb{S})$ *are positive. As a result, the interval* $\left(0, \frac{1}{K_M \eta_M}\right)$ *is reasonable. Next, it is easy to find that* $\varphi(\varepsilon)$ *is a positive continuous function in* $\left(0, \frac{1}{K_M \eta_M}\right)$. *When* ε *tends to* 0 *or* $\frac{1}{K_M \eta_M}$, $\varphi(\varepsilon)$ *goes to zero. Therefore, there exists a number* $0 < \varepsilon^* < \frac{1}{K_M \eta_M}$ *such that* $\varphi(\varepsilon^*) = \max\left\{\varphi(\varepsilon) : 0 < \varepsilon < \frac{1}{K_M \eta_M}\right\}$. *In this case,* τ^* *is well-defined. It is also useful later that* $\tau^* < \frac{1}{K_M}$ *since* $\frac{1}{K_M}$ *is one bound of function* $\varphi(\varepsilon)$ *in* $\left(0, \frac{1}{K_M \eta_M}\right)$.

3.4 Lyapunov functional

The main method to study stability in this chapter is the technique of Lyapunov functional. Before that, we need some preparations. Let us firstly define a Lyapunov function $U(x,i) \in \mathbb{C}^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ by

$$U(x,i) = \eta_i |x|^2. \tag{3.13}$$

Define its corresponding operator $LU : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ with respect to the controlled SDE (3.2) by

$$\begin{aligned} LU(x, z, t, i) =& U_x(x, i)(f(x, t, i) + u(z, t, i)) \\ &+ \frac{1}{2}\text{trace}\left(g^{\mathrm{T}}(x, t, i)U_{xx}(x, i)g(x, t, i)\right) + \sum_{j=1}^{S} q_{ij}U(x, j). \end{aligned} \tag{3.14}$$

As mentioned before, to deal with the effect from discrete-time state observations, the basic idea is using decomposition (3.4). While from the perspective of stability theory, this means we should treat $LU(x, z, t, i)$ as $LU(x, z, t, i) = L_1U(x, t, i) + L_2U(x, z, t, i)$, where

$$\begin{aligned} L_1U(x, t, i) =& U_x(x, i)(f(x, t, i) + u(x, t, i)) \\ &+ \frac{1}{2}\text{trace}\left(g^{\mathrm{T}}(x, t, i)U_{xx}(x, i)g(x, t, i)\right) + \sum_{j=1}^{S} q_{ij}U(x, j) \end{aligned} \tag{3.15}$$

and

$$L_2U(x, z, t, i) = U_x(x, i)(u(z, t, i) - u(x, t, i)). \tag{3.16}$$

We then give the following remark to show the estimation of $L_1U(x, t, i)$ as well as the role of Assumption 3.3.

Remark 3.5. *From (3.8) and (3.12), we easily see that*

$$x^{\mathrm{T}}(f(x, t, i) + u(x, t, i)) + \frac{1}{2}|g(x, t, i)|^2 \le \alpha_i|x|^2 - \beta_i|x|^{p+1}$$

for any $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$. This implies that

$$\begin{aligned} L_1U(x, t, i) \le& 2\eta_i\left(x^{\mathrm{T}}(f(x, t, i) + u(x, t, i)) + \frac{1}{2}|g(x, t, i)|^2\right) + \sum_{j=1}^{S} q_{ij}\eta_j|x|^2 \\ \le& \left(2\eta_i\alpha_i + \sum_{j=1}^{S} q_{ij}\eta_j\right)|x|^2 - 2\eta_i\beta_i|x|^{p+1}. \end{aligned}$$

Noting that $2\eta_i + \sum_{j=1}^{S} q_{ij}\eta_j = -1$ and $\beta_\eta = \min_{i\in\mathbb{S}}(\beta_i\eta_i)$, we then have

$$L_1U(x, t, i) \le -|x|^2 - 2\beta_\eta|x|^{p+1}. \tag{3.17}$$

From the inequality above, we observe that Assumption 3.3 is actually used to make $L_1U(x, t, i)$ be negative, which is fundamental in Lyapunov stability analysis. The first term $\gamma_i|x|^2$ helps the design of control function, which could be found in Rule 3.2. The second term $\beta_i|x|^{p+1}$ is used to balance the high nonlinearity of f and g.

In view of mathematics, we have till seen that Assumption 3.1 depicted the high nonlinearity of our underlying SDE, Rule 3.1 (Assumption 3.2) aimed at global solution of the controlled SDE, Rule 3.2 (Assumption 3.3) was used to estimate $L_1U(x,t,i)$. But how about Rule 3.3? The following lemma gives the answer.

Lemma 3.1. *Under Assumptions 3.1, 3.2, 3.3, let the control function $u(x,t,i)$ satisfy Rules 3.1, 3.2, and the observation duration τ meets Rule 3.3. Then for any $t>0$, we have*

$$E|L_2U(x(t),x(t_\tau),t,r(t))| \le \phi_1 E|x(t)|^2 + \phi_2 E|x(t)|^{p+1} + \phi_3\int_{t-\tau}^t E|x(v)|^2\mathrm{d}v + \phi_4\int_{t-\tau}^t E|x(v)|^{p+1}\mathrm{d}v, \tag{3.18}$$

where $\phi_1 = \frac{K_M\eta_M}{1-K_M\tau}((H_1+2K_M)\tau+\varepsilon^*)$, $\phi_2 = \frac{K_M\eta_M}{1-K_M\tau}\frac{2H_2\tau}{p+1}$, $\phi_3 = \frac{K_M\eta_M}{1-K_M\tau}\left(H_1+\frac{\tilde{H}_1}{\varepsilon^*}\right)$, $\phi_4 = \frac{K_M\eta_M}{1-K_M\tau}\left(\frac{2pH_2}{p+1}+\frac{\tilde{H}_2}{\varepsilon^*}\right)$.

Proof. For any fixed $t>0$, there is an integer k such that $k\tau \le t < (k+1)\tau$. Hence $x(t_\tau)=x(k\tau)$, and also $x(v_\tau)=x(k\tau)$ for any $k\tau\le v\le t$. Then it is easy to see from (3.9) that

$$|L_2U(x(t),x(t_\tau),t,r(t))| \le 2K_M\eta_M|x(t)||x(t)-x(k\tau)|. \tag{3.19}$$

Letting $M_k=\int_{k\tau}^t g(x(v),v,r(v))\mathrm{d}W(v)$, we derive from (3.5) and (3.10) that

$$\begin{aligned}
&|x(t)||x(t)-x(k\tau)| \\
\le&|x(t)|\left(\int_{k\tau}^t(|f(x(v),v,r(v))|+|u(x(k\tau),v,r(v))|)\mathrm{d}v+|M_k|\right)\\
\le&\int_{k\tau}^t |x(t)|\left(H_1|x(v)|+H_2|x(v)|^p+K_M|x(k\tau)|\right)\mathrm{d}v+\frac{\varepsilon^*}{2}|x(t)|^2+\frac{1}{2\varepsilon^*}|M_k|^2\\
\le&\int_{k\tau}^t\left(\frac{H_1}{2}|x(t)|^2+\frac{H_1}{2}|x(v)|^2+\frac{H_2}{p+1}|x(t)|^{p+1}+\frac{pH_2}{p+1}|x(v)|^{p+1}\right)\mathrm{d}v\\
&+K_M\tau|x(t)||x(k\tau)|+\frac{\varepsilon^*}{2}|x(t)|^2+\frac{1}{2\varepsilon^*}|M_k|^2\\
\le&K_M\tau|x(t)||x(t)-x(k\tau)|+\left(\left(\frac{H_1}{2}+K_M\right)\tau+\frac{\varepsilon^*}{2}\right)|x(t)|^2+\frac{H_2\tau}{p+1}|x(t)|^{p+1}\\
&+\frac{H_1}{2}\int_{t-\tau}^t|x(v)|^2\mathrm{d}v+\frac{pH_2}{p+1}\int_{t-\tau}^t|x(v)|^{p+1}\mathrm{d}v+\frac{1}{2\varepsilon^*}|M_k|^2.
\end{aligned}$$

Recalling that $\tau < \tau^* < \frac{1}{K_M}$ in Remark 3.4, we further have

$$\begin{aligned}|x(t)||x(t)-x(k\tau)| \leq &\frac{1}{1-K_M\tau}\Bigg(\left(\left(\left(\frac{H_1}{2}+K_M\right)\tau+\frac{\varepsilon^*}{2}\right)|x(t)|^2+\frac{H_2\tau}{p+1}|x(t)|^{p+1}\right.\\ &\left.+\frac{H_1}{2}\int_{k\tau}^t|x(v)|^2\mathrm{d}v+\frac{pH_2}{p+1}\int_{k\tau}^t|x(v)|^{p+1}\mathrm{d}v+\frac{1}{2\varepsilon^*}|M_k|^2\right).\end{aligned}$$

Substituting this into (3.19), then taking expectations on both sides gives that

$$\begin{aligned}&E|L_2U(x(t),x(t_\tau),t,r(t))|\\ \leq&\phi_1E|x(t)|^2+\phi_2E|x(t)|^{p+1}+\frac{K_M\eta_M}{1-K_M\tau}\Bigg(H_1\int_{k\tau}^tE|x(v)|^2\mathrm{d}v\\ &+\frac{2pH_2}{p+1}\int_{k\tau}^tE|x(v)|^{p+1}\mathrm{d}v+\frac{1}{\varepsilon^*}\int_{k\tau}^tE|g(x(v),v,r(v))|^2\mathrm{d}v\Bigg)\\ \leq&\phi_1E|x(t)|^2+\phi_2E|x(t)|^{p+1}+\phi_3\int_{k\tau}^tE|x(v)|^2\mathrm{d}v+\phi_4\int_{k\tau}^tE|x(v)|^{p+1}\mathrm{d}v.\end{aligned}$$

Here, we have used (3.11) to deal with $E|M_k|^2$. The required assertion (3.18) follows since $t-\tau\leq k\tau$. The proof is complete. □

Clearly, Rule 3.3 is used to restrict the value of τ, which dominates the difference between $x(t)$ and $x(t_\tau)$. Let us say more about this estimation.

Remark 3.6. *In order to estimate $E|L_2U(x(t),x(t_\tau),t,r(t))|$, the traditional way is firstly using the Hölder inequality to split it as $E|U_x|^2$ and $E|x(t)-x(t_\tau)|^2$, then applying the classical method (namely, the Hölder inequality and the Itô isometry) to compute $E|x(t)-x(t_\tau)|^2$ such as (Fei et al. 2020, Shi et al. 2022). This makes us give restriction on $x^{\mathrm{T}}f(x,t,i)+\frac{p}{2}|g(x,t,i)|^2$ and use the Lyapunov function in the form of $|x|^2+|x|^{p+1}$. But in Lemma 3.1, we sharpen this estimation by treating $E|L_2U(x(t),x(t_\tau),t,r(t))|$ as a whole and making use of the constant property of t_τ between two observations. This improvement helps us weaken conditions and simplify the proof process since we only use Lyapunov function in the form of $|x|^2$.*

Now, we can give the Lyapunov functional used in this chapter by

$$V(x_t,t,r(t))=U(x(t),r(t))+\int_{-\tau}^0\int_{t+s}^t\left(\phi_3|x(v)|^2+\phi_4|x(v)|^{p+1}\right)\mathrm{d}v\mathrm{d}s \qquad (3.20)$$

on $t\geq 0$. Here, $x_t=\{x(t+\theta):-\tau\leq\theta\leq 0\}$. While for x_t to be well defined on $t\in[0,\tau]$, we set $x(\theta)=\xi_0$ for $\theta\in[-\tau,0)$.

3.5 Stabilisation results

In this part, we will show that the underlying unstable hybrid SDE (3.1) could indeed be stabilised by the discrete-time state feedback control $u(x(t_\tau), t, r(t))$ satisfying Rules 3.1, 3.2, 3.3. The first result is given in the sense of H_∞ stability.

Theorem 3.2. *Under the same conditions with Lemma 3.1, the solution of the controlled SDE (3.2) satisfies that*

$$\int_0^\infty E|x(t)|^{p+1}\mathrm{d}t < \infty \tag{3.21}$$

and

$$\sup_{0\le t<\infty} E|x(t)|^2 < \infty. \tag{3.22}$$

Proof. By the generalized Itô formula and the fundamental theory of calculus, it is easy to show that

$$V(x_t, t, r(t)) = V(\xi_0, 0, i_0) + \int_0^t \mathcal{L}V(x_s, s, r(s))\mathrm{d}t + M(t), \tag{3.23}$$

where

$$\begin{aligned}\mathcal{L}V(x_t, t, r(t)) =& LU(x(t), x(t_\tau), t, r(t)) + \phi_3\tau|x(t)|^2 + \phi_4\tau|x(t)|^{p+1}\\ &- \phi_3\int_{t-\tau}^t |x(v)|^2\mathrm{d}v - \phi_4\int_{t-\tau}^t |x(v)|^{p+1}\mathrm{d}v\end{aligned}$$

and $M(t)$ is a continuous local martingale vanishing at $t = 0$. The explicit form of $M(t)$ is of no use in this paper so we omit it here, but it can be found in Theorem 2.15. Taking expectations on both sides of (3.23), if necessary, using the procedure of stopping times, we could obtain that

$$EV(x_t, t, r(t)) \le V(\xi_0, 0, i_0) + \int_0^t E\mathcal{L}V(x_s, s, r(s))\mathrm{d}s. \tag{3.24}$$

This requires us to estimate $E\mathcal{L}V(x_s, s, r(s))$. Recalling the estimations of L_1U in (3.17) and L_2U in (3.18), we have

$$E\mathcal{L}V(x_s, s, r(s)) \le -(1-\phi_1-\phi_3\tau)E|x(s)|^2 - (2\beta_\eta - \phi_2 - \phi_4\tau)E|x(s)|^{p+1}.$$

Substituting this into (3.24) yields that

$$\begin{aligned}\eta_m E|x(t)|^2 \le& V(\xi_0, 0, i_0) - (1-\phi_1-\phi_3\tau)\int_0^t E|x(s)|^2\mathrm{d}s\\ &- (1-\phi_2-\phi_4\tau)\int_0^t E|x(s)|^{p+1}\mathrm{d}s,\end{aligned}$$

where $\eta_m = \min_{i\in\mathbb{S}} \eta_i$. Next, calculate

$$1-\phi_1-\phi_3\tau = \frac{1}{1-K_M\tau}\left(1-K_M\eta_M\varepsilon^* - K_M\eta_M\left(2H_1+2K_M+\frac{\tilde{H}_1}{\varepsilon^*}+\frac{1}{\eta_M}\right)\tau\right).$$

Since $\tau < \frac{1}{K_M}$ and

$$\tau < \tau^* = \varphi(\varepsilon^*) \le \frac{1}{K_M\eta_M}\frac{1-K_M\eta_M\varepsilon^*}{2H_1+2K_M+\frac{\tilde{H}_1}{\varepsilon^*}+\frac{1}{\eta_M}},$$

the number $1-\phi_1-\phi_3\tau$ is positive. Applying the similar analysis, we could also have that $1-\phi_2-\phi_4\tau > 0$. Therefore,

$$E|x(t)|^2 \le \frac{V(\xi_0,0,i_0)}{\eta_m}, \qquad \int_0^t E|x(s)|^{p+1}\mathrm{d}s \le \frac{V(\xi_0,0,i_0)}{1-\phi_2-\phi_4\tau}.$$

Since these hold for every $t > 0$, the assertions (3.21) and (3.22) follow immediately. The proof is complete. □

In Theorem 3.2, we have known that the controlled SDE (3.2) is H_∞ stable and also bounded in L^2. Then using the technique of stochastic LaSalle principle, we can conclude that the controlled SDE (3.2) is also almost surely asymptotically stable.

Theorem 3.3. *Let all the conditions in Lemma 3.1 hold. Then the solution of the controlled SDE (3.2) has the property that*

$$\lim_{t\to\infty} x(t) = 0 \quad a.s. \tag{3.25}$$

Proof. **Step 1.** From the proof above, we could also see that

$$\int_0^\infty E|x(t)|^2\mathrm{d}t < \infty,$$

which implies that $E\int_0^\infty |x(t)|^2\mathrm{d}t < \infty$ by the Fubini theorem. Therefore,

$$\int_0^\infty |x(t)|^2\mathrm{d}t < \infty \quad \text{a.s.},$$

and

$$\liminf_{t\to\infty} |x(t)| = 0 \quad \text{a.s.} \tag{3.26}$$

We claim that

$$\lim_{t\to\infty} |x(t)| = 0 \quad \text{a.s.} \tag{3.27}$$

If this is not true,

$$P\left(\limsup_{t\to\infty} |x(t)| > 0\right) > 0.$$

There is then a sufficiently small $\epsilon > 0$ such that

$$P(\Omega_1) \geq 3\epsilon, \quad \text{with} \quad \Omega_1 = \left\{ \limsup_{t\to\infty} |x(t)| \geq 2\epsilon \right\}. \tag{3.28}$$

Step 2. For any constant $h > |\xi_0|$, define the stopping time

$$\sigma_h = \inf\{t > 0 : |x(t)| \geq h\}.$$

By the Itô formula and conditions (3.8), (3.12), we obtain that

$$\begin{aligned} &E|x(t\wedge\sigma_h)|^2 \\ =&|\xi_0|^2 + E\int_0^{t\wedge\sigma_h} \left(2x(s)^{\mathrm{T}}F(s) + |G(s)|^2\right) \mathrm{d}s \\ \leq&|\xi_0|^2 + \int_0^{\infty} \left((2\gamma_M + K_M)E|x(s)|^2 + K_M E|x(s_\tau)|^2\right) \mathrm{d}s, \end{aligned}$$

where

$$F(t) = f(x(t), t, r(t)) + u(x(t_\tau), t, r(t)), \qquad G(t) = g(x(t), t, r(t)).$$

Hence, we can find a constant $C_1 > 0$ such that $E|x(t\wedge\sigma_h)|^2 \leq C_1$ for all the time t, which yields that $h^2 P(\sigma_h \leq t) \leq C_1$. Then we can choose h appropriately such that $\frac{C_1}{h^2} \leq \epsilon$. Letting $h \to \infty$, we have

$$P(\sigma_h < \infty) \leq \frac{C_1}{h^2} \leq \epsilon.$$

This tells us that

$$P(\Omega_2) \geq 1 - \epsilon, \quad \text{with} \quad \Omega_2 = \{|x(t)| \leq h, \ \forall\ 0 \leq t < \infty\}. \tag{3.29}$$

It follows from (3.28) and (3.29) that

$$P(\Omega_1 \cap \Omega_2) \geq 2\epsilon. \tag{3.30}$$

Step 3. Define a sequence of stopping times

$$\begin{aligned} a_1 =& \inf\{t > 0 : |x(t)|^2 \geq 2\epsilon\}, \\ a_{2n} =& \inf\{t > a_{2n-1} : |x(t)|^2 \leq \epsilon\}, \quad n = 1, 2, \cdots, \\ a_{2n+1} =& \inf\{t > a_{2n} : |x(t)|^2 \geq 2\epsilon\}, \quad n = 1, 2, \cdots. \end{aligned}$$

From (3.26) and the definitions of Ω_1, Ω_2, we observe that $a_{2n} < \infty$ whenever $a_{2n-1} < \infty$. Moreover, $\sigma_h = \infty$ and $a_n < \infty$ for all $n \geq 1$ in $\Omega_1 \cap \Omega_2$. Next calculate

$$\infty > E\int_0^{\infty} |x(t)|^2 \mathrm{d}t \geq \sum_{n=1}^{\infty} E\left(\mathbb{I}_{\{a_{2n-1}<\infty, \sigma_h=\infty\}} \int_{a_{2n-1}}^{a_{2n}} |x(t)|^2 \mathrm{d}t\right)$$

$$\geq \epsilon \sum_{n=1}^{\infty} E\left(\mathbb{I}_{\{a_{2n-1}<\infty,\sigma_h=\infty\}}(a_{2n}-a_{2n-1})\right). \tag{3.31}$$

Since functions f, g, u are all locally Lipschitz continuous, we can find a constant $\bar{K}_h$ such that

$$|F(t)|^2 \vee |G(t)|^2 \leq \bar{K}_h, \quad \text{whenever} \quad |x(t)| \vee |x(t_\tau)| \leq h.$$

By the Hölder inequality and the Doob martingale inequality, for any time $T > 0$

$$\begin{aligned}
& E\left(\mathbb{I}_{\{\sigma_h \wedge a_{2n-1}<\infty\}} \sup_{0\leq t\leq T} |x(\sigma_h \wedge (a_{2n-1}+t)) - x(\sigma_h \wedge a_{2n-1})|^2\right) \\
\leq & 2E\left(\mathbb{I}_{\{\sigma_h \wedge a_{2n-1}<\infty\}} \sup_{0\leq t\leq T} \left| \int_{\sigma_h\wedge a_{2n-1}}^{\sigma_h\wedge(a_{2n-1}+t)} F(s)\mathrm{d}s \right|^2\right) \\
& + 2E\left(\mathbb{I}_{\{\sigma_h \wedge a_{2n-1}<\infty\}} \sup_{0\leq t\leq T} \left| \int_{\sigma_h\wedge a_{2n-1}}^{\sigma_h\wedge(a_{2n-1}+t)} G(s)\mathrm{d}W(s) \right|^2\right) \\
\leq & 2TE\left(\mathbb{I}_{\{\sigma_h \wedge a_{2n-1}<\infty\}} \left| \int_{\sigma_h\wedge a_{2n-1}}^{\sigma_h\wedge(a_{2n-1}+T)} F(s)\mathrm{d}s \right|^2\right) \\
& + 8E\left(\mathbb{I}_{\{\sigma_h \wedge a_{2n-1}<\infty\}} \left| \int_{\sigma_h\wedge a_{2n-1}}^{\sigma_h\wedge(a_{2n-1}+T)} G(s)\mathrm{d}s \right|^2\right) \\
\leq & 2\bar{K}_h T(T+4).
\end{aligned} \tag{3.32}$$

Let $\theta = \frac{\epsilon}{2h}$, then $||x|^2 - |y|^2| \leq \epsilon$ whenever $|x-y| \leq \theta$ and $|x| \wedge |y| \leq h$. Choose T sufficiently small such that

$$\frac{2\bar{K}_h T(T+4)}{\theta^2} \leq \epsilon.$$

We could derive from (3.32) that

$$P\left(\{\sigma_h \wedge a_{2n-1} < \infty\} \cap \left\{\sup_{0\leq t\leq T} |x(\sigma_h \wedge (a_{2n-1}+t)) - x(\sigma_h \wedge a_{2n-1})| > \theta\right\}\right) < \epsilon.$$

Consequently,

$$\begin{aligned}
& P\left(\{a_{2n-1} < \infty, \sigma_h = \infty\} \cap \left\{\sup_{0\leq t\leq T} |x(a_{2n-1}+t) - x(a_{2n-1})| > \theta\right\}\right) \\
\leq & P\left(\{\sigma_h \wedge a_{2n-1} < \infty\} \cap \left\{\sup_{0\leq t\leq T} |x(\sigma_h \wedge (a_{2n-1}+t)) - x(\sigma_h \wedge a_{2n-1})| > \theta\right\}\right) \\
< & \epsilon.
\end{aligned}$$

This further implies that

$$
\begin{aligned}
& P\left(\{a_{2n-1}<\infty,\sigma_h=\infty\}\cap\left\{\sup_{0\le t\le T}|x(a_{2n-1}+t)-x(a_{2n-1})|\le\theta\right\}\right)\\
\ge & P\left(\{\sigma_h\wedge a_{2n-1}<\infty\}\right)\\
& -P\left(\{a_{2n-1}<\infty,\sigma_h=\infty\}\cap\left\{\sup_{0\le t\le T}|x(a_{2n-1}+t)-x(a_{2n-1})|>\theta\right\}\right)\\
\ge & 2\epsilon-\epsilon=\epsilon
\end{aligned}
$$

and

$$
\begin{aligned}
& P\left(\{a_{2n-1}<\infty,\sigma_h=\infty\}\cap\left\{\sup_{0\le t\le T}||x(a_{2n-1}+t)|^2-|x(a_{2n-1})|^2|^2\le\epsilon\right\}\right)\\
\ge & P\left(\{a_{2n-1}<\infty,\sigma_h=\infty\}\cap\left\{\sup_{0\le t\le T}|x(a_{2n-1}+t)-x(a_{2n-1})|\le\theta\right\}\right)\\
\ge & \epsilon. \qquad (3.33)
\end{aligned}
$$

Set

$$
\bar{\Omega}_n=\left\{\sup_{0\le t\le T}||x(a_{2n-1}+t)|^2-|x(a_{2n-1})|^2|^2\le\epsilon\right\}.
$$

Noting that $a_{2n}-a_{2n-1}\ge T$ in $\{a_{2n-1}<\infty,\sigma_h=\infty\}\cap\bar{\Omega}_n$, we derive form (3.31) and (3.33) that

$$
\begin{aligned}
\infty > & \epsilon\sum_{n=1}^{\infty}E\left(\mathbb{I}_{\{a_{2n-1}<\infty,\sigma_h=\infty\}}(a_{2n}-a_{2n-1})\right)\\
\ge & \epsilon\sum_{n=1}^{\infty}E\left(\mathbb{I}_{\{a_{2n-1}<\infty,\sigma_h=\infty\}\cap\bar{\Omega}_n}(a_{2n}-a_{2n-1})\right)\\
\ge & \epsilon T\sum_{n=1}^{\infty}P(\{a_{2n-1}<\infty,\sigma_h=\infty\}\cap\bar{\Omega}_n)\\
\ge & \epsilon T\sum_{n=1}^{\infty}\epsilon=\infty,
\end{aligned}
$$

which is a contradiction. Then (3.27) must hold. The proof is complete. □

3.6 Application to volatility model

Consider a scalar hybrid SDE in financial mathematics, which can be regarded as a generalisation of the well-known Heston stochastic volatility 1.5-model (see

(Lewis 2000, Heston 1997))

$$\mathrm{d}x(t) = x(t)\left(a_{r(t)} - b_{r(t)}|x(t)|\right)\mathrm{d}t + c_{r(t)}|x(t)|^{1.5}\mathrm{d}W(t) \tag{3.34}$$

on $t \geq 0$ with one-dimension Brownian motion $W(t)$. In order to avoid complicated calculations, we let $r(t)$ be a continuous Markov chain on the state space $\mathbb{S} = \{1, 2\}$ with transition rate matrix

$$Q = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}.$$

The parameters are given as

$$a_1 = 2, \quad a_2 = 0.2, \quad b_1 = 1, \quad b_2 = 0.4, \quad c_1 = 1, \quad c_2 = 0.5.$$

It is easy to verify that Assumptions 3.1 and 3.2 are satisfied with $H_1 = 4$, $H_2 = 2$, $\tilde{H}_1 = 0$, $\tilde{H}_2 = 1$, $p = 2$ and $\hat{\alpha} = 1.75$. Through computer simulation, we can find that the hybrid SDE (3.34) is unstable (see Fig. 3.1 middle).

Thus, we need to design the discrete-time state feedback control $u(x(t_\tau), r(t))$ to make the controlled SDE

$$\mathrm{d}x(t) = \Big(x(t)\left(a_{r(t)} - b_{r(t)}|x(t)|\right) + u(x(t_\tau), r(t))\Big)\mathrm{d}t + c_{r(t)}|x(t)|^{1.5}\mathrm{d}W(t) \tag{3.35}$$

become stable. Before the control design, compute

$$x\left(a_i x - b_i x|x|\right) + \frac{c_i^2}{2}|x|^3 \leq a_i|x|^2 - \left(b_i - \frac{c_i^2}{2}\right)|x|^3.$$

Then Assumption 3.3 holds with $\gamma_1 = 2$, $\gamma_2 = 0.2$, $\beta_1 = 0.5$, $\beta_2 = 0.275$. With this information, the control function can be given as

$$u(x, 1) = -4x, \qquad u(x, 2) = 0. \tag{3.36}$$

Remark 3.7. *Here we study an interesting phenomena that the state can be observed fully in mode 1 but in mode 2, it is not observable. Therefore, we can only design feedback control in mode 1, based on discrete-time state observations of course, but we cannot have feedback control in mode 2. For example, the financial market can be roughly divided as "bullish" mode and "bearish" mode. Sometimes, only "bearish" mode can cause investors' much attention, where the market can be observed easily and needed extra control.*

We can easily check that Rule 3.1 is satisfied with $K_M = 4$, Rule 3.2 holds with $\kappa_1 = 4$, $\kappa_2 = 0$, and

$$A = \begin{pmatrix} 5 & -1 \\ -6 & 5.6 \end{pmatrix},$$

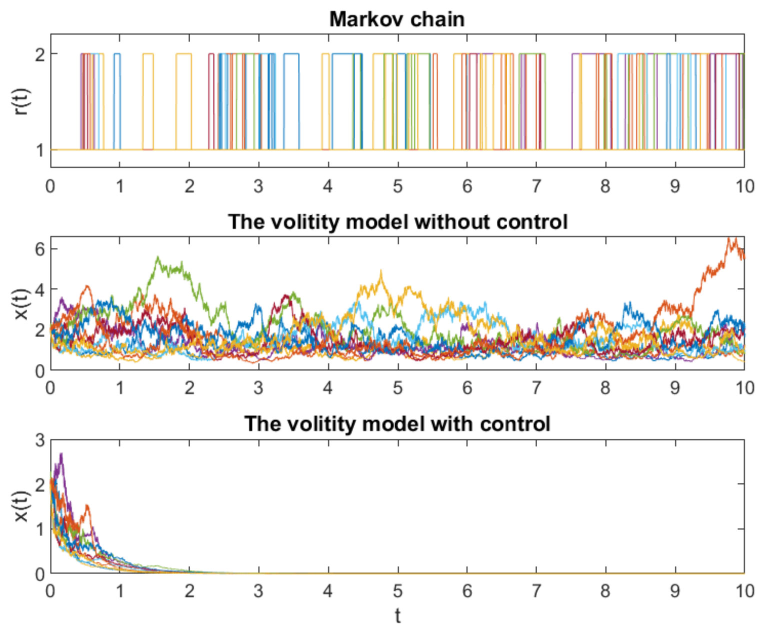

Figure 3.1: Ten sample paths of the Markov chain, the hybrid SDE (3.34) and the controlled SDE (3.35) with $\tau = 0.01$, using the truncated Euler-Maruyama method (see (Mao 2015)) with time step size 10^{-4}. For each path, the initial data is fixed given by $\xi_0 = 2$ and $i_0 = 1$.

which is clearly a non-singular M-matrix. We then obtain that $\eta_1 = 0.3$ and $\eta_2 = 0.5$. Using the method introduced in Rule 3.3, we get $\tau^* = 0.019643$. By Theorem 3.3, we can conclude that the controlled SDE (3.35) is almost surely asymptotically stable if $\tau < 0.019643$. We perform a computer simulation with $\tau = 0.01$ in Fig. 3.1 bottom, which supports our theoretical results clearly.

On the other hand, we see that

$$x\left(a_2 x - b_2 x|x|\right) + \frac{p}{2} c_2^2 |x|^3 \leq 2|x|^2,$$

which means that the second inequality of condition (3.3) is not satisfied. Thus the theory in (Fei et al. 2020) cannot be applied to this example, and we weaken the conditions in (Fei et al. 2020).

3.7 Summary

Compared with the existing papers on discrete-state-feedback stabilisation problem, this chapter presents a new method to estimate the difference between current-time state and discrete-time state. As a result, conditions imposed on the underlying system are less restrictive, at least replacing condition (3.3) by condition (3.8). An application example to the volatility model shows this clearly. The Lyapunov functional used in this chapter is also modified to adapt to this change. Moreover, a new method is given to determine the upper bound of the observation duration, so that there is no need to find any free parameters, and it will be much easier to verify conditions in reality.

Chapter 4

Stabilisation of hybrid SDEs with different structures by bounded discrete-time state feedback control

4.1 Introduction

In Chapter 3, we have made some improvements on the stabilisation problem by discrete-time state feedback control, including weaken and easily checked conditions. But there are two issues to be addressed in order to make our theory more useful and applicable.

At first, recalling condition (3.8), from the previous analysis, we see it plays a key role in stabilisation, which eliminates the effect from highly nonlinearity and makes the global Lipschitz continuous control design possible. It is certainly more advanced than condition (3.3). However, it is required for all modes, in particular, β_i should be strictly positive for any $i \in \mathbb{S}$. This seems a little restrictive in reality as this structure might be lost in some modes. For example, (Fei, Hu, Mao & Shen 2018) studied a population system described by

$$\begin{cases} \mathrm{d}x(t) = -2x(t)\mathrm{d}t + 0.8x(t)\mathrm{d}W(t), & \text{in mode 1: dry,} \\ \mathrm{d}x(t) = x(t)\left(1 - 2x^2(t)\right)\mathrm{d}t + 1.2x^2(t)\mathrm{d}W(t), & \text{in mode 2: rain.} \end{cases}$$

It is clear that condition (3.3) cannot be satisfied in mode 1 since $\beta_1 = 0$. This then tells us that the results in Chapter 3 are not well applicable to hybrid SDEs experiencing abrupt changes in their structures. Thus to deal with this situation, we need to consider structured stabilisation.

To the best of the authors' knowledge, although the structured stability has drawn many researchers' interest (e.g. (Fei et al. 2018, Shen, Mei & Deng 2019, Lu, Song & Zhu 2022)), there are few results on structured stabilisation. While recently, (Shi et al. 2022) made some efforts to this problem. They successfully designed a discrete-time state feedback control for hybrid SDEs with different structures in different modes. But it was still on the frame of original condition (3.3). Then in this chapter, we will consider the structured stabilisation problem based on our new settings.

The other one is concerned with control design. It should be underscored that in many papers studying discrete-state-feedback stabilisation problem such as (Fei et al. 2020, Shi et al. 2022), the control function $u(x,t,i)$ is usually designed on every observable discrete-time state, such as the linear form $\nu_i x([t/\tau]\tau)$ with $\nu_1 = -4$ and $\nu_2 = 0$ in Section 3.6. But this sometimes seems a little rough and would lead to some unnecessary cost. In general, the control cost is proportional to $|u(x(t_\tau),t,r(t))|$. Thus the control cost goes up as system state value $|x(t_\tau)|$ increases. Particularly, if the initial data is given large, the cost on the beginning stage will be relatively high. This then begs a question: is it really necessary to impose control on every discrete-time state? The answer at least in this paper is negative.

To see this clearly, let us go back to condition (3.8) again. For the i-th mode, we can rewrite the right-hand side of condition (3.8) by

$$\gamma_i|x|^2 - \beta_i|x|^{p+1} = -\gamma_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1} + \left(2\gamma_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1}\right).$$

Define a function $\phi : \mathbb{R}_+ \to \mathbb{R}$ by $\phi(z) = 2\gamma_i z^2 - \frac{\beta_i}{2}z^{p+1}$. When $z > R_i$, where $R_i = \left(\frac{4\gamma_i}{\beta_i}\right)^{\frac{1}{p-1}}$, we find that $\phi(z)$ is non-positive. This implies that when $|x| > R_i$,

$$x^{\mathrm{T}} f(x,t,i) + \frac{1}{2}|g(x,t,i)|^2 \le -\gamma_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1}.$$

There is hence no need to impose any control when $|x|$ exceeds R_i in the modes where condition (3.8) is true. In other words, the control is merely designed in a bounded state area $\left\{x \in \mathbb{R}^d : |x| \le R_i\right\}$. Roughly speaking, the control function in Section 3.6 could be modified by $u(x,t,1) = -4x$ if $|x| \le R_1$ and 0 otherwise. Such a control is much smaller than before so that the control cost could be reduced significantly, especially for large system states.

In conclusion, this chapter is devoted to the stabilisation of hybrid SDEs with different structures by bounded discrete-time state feedback control based on the theory in (Shi et al. 2022) and Chapter 3.

4.2 Problem statement

Consider a general hybrid SDE

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{4.1}$$

on $t \geq 0$ with the initial data $x(0) = \xi_0 \in \mathbb{R}^d$ and $r(0) = i_0 \in \mathbb{S}$, where $f(x, t, i)$ and $g(x, t, i)$ are locally Lipschitz continuous. Supposing it is unstable, we want to use discrete-time state feedback control to make the controlled system

$$\mathrm{d}x(t) = \Big(f(x(t), t, r(t)) + u(x(t_\tau), t, r(t))\Big)\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{4.2}$$

become stable.

4.2.1 Structures on original system

Since we are mostly interested in stabilisation, the structure classification always happens on the stability analysis and the corresponding conditions. Therefore, we still let Assumptions 3.1 and 3.2 hold to guarantee that there is a unique global solution of the hybrid SDE (4.1), which is in L^{p+1} for all the time. But the significant Assumption 3.3 will be modified as follows.

Assumption 4.1. *For simplicity, we divide the mode space* $\mathbb{S}$ *into two parts,* $\mathbb{S}_1 = \{1, \cdots, S_1\}$ *and* $\mathbb{S}_2 = \{S_1 + 1, \cdots, S\}$ *with* $1 \leq S_1 < S$.

For $i \in \mathbb{S}_1$, *assume that we can find constants* $\alpha_i, \hat{\alpha}_i \in \mathbb{R}$ *such that for every* $(x, t) \in \mathbb{R}^d \times \mathbb{R}_+$

$$\begin{cases} x^{\mathrm{T}} f(x, t, i) + \dfrac{1}{2}|g(x, t, i)|^2 \leq \alpha_i |x|^2, \\ x^{\mathrm{T}} f(x, t, i) + \dfrac{p}{2}|g(x, t, i)|^2 \leq \hat{\alpha}_i |x|^2. \end{cases} \tag{4.3}$$

These constants $\hat{\alpha}_i$ *should also make* $\hat{A} := -(p+1)\mathrm{diag}(\hat{\alpha}_1, \cdots, \hat{\alpha}_{S_1}) - Q$ *to be a non-singular* M*-matrix.*

For $i \in \mathbb{S}_2$, assume that there are constants $\gamma_i \geq 0$ and $\beta_i > 0$ such that

$$x^{\mathrm{T}} f(x,t,i) + \frac{1}{2}|g(x,t,i)|^2 \leq \gamma_i |x|^2 - \beta_i |x|^{p+1} \tag{4.4}$$

for any $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$.

This assumption reflects our idea of mode-structure classification.

Remark 4.1. *In view of Khasminskii-type condition, the structure in $\mathbb{S}_2$-modes remains the same as before, but in $\mathbb{S}_1$-modes, the structure differs since there is no requirement for positivity on β_i herein, namely, the serious presence of high-order term $|x|^{p+1}$. For example, the population system in the Introduction part satisfies Assumption 4.1 with $\mathbb{S}_1 = \{dry\}$ and $\mathbb{S}_2 = \{rain\}$. Therefore, our Assumption 3.3 before is indeed generalised. Moreover, since $|x|^{p+1}$ does not appear in Assumption 3.2, which is given for the existence-and-uniqueness theorem, there is no need to consider mode-structure classification in this assumption.*

Remark 4.2. *It is also worth mentioning that the classification scheme here is motivated by (Shi et al. 2022). But a little differently, for $\mathbb{S}_1$-modes, there was an additional linear growth condition requirement in (Shi et al. 2022). But in our thesis, we could not conclude from conditions (4.3) that the subsystems in $\mathbb{S}_1$-modes meet this condition as they are likely highly nonlinear, such as $f(x,t,i) = -x - 1.5x^3$, $g(x,t,i) = x^2$.*

4.2.2 Bounded-state-area feedback control

Next, we will explain how to design the bounded control function $u(x,t,i)$ according to mode-structure classification in Assumption 4.1. For convenience, we denote by $B_a = \{x \in \mathbb{R}^d : |x| \leq a\}$, $B_a^c = \{x \in \mathbb{R}^d : |x| > a\}$, $B_b - B_a = \{x \in \mathbb{R}^d : a < |x| \leq b\}$ for any $0 < a < b$.

Rule 4.1. *For $i \in \mathbb{S}_1$, let $u(x,t,i) = 0$ for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$.*

For $\mathbb{S}_2$-modes, firstly choose non-negative constants $\kappa_i (i \in \mathbb{S}_2)$ to let

$$A := -2\mathrm{diag}(\alpha_1, \cdots, \alpha_{S_1}, \gamma_{S_1+1} - \kappa_{S_1+1}, \cdots, \gamma_S - \kappa_S) - Q$$

be a non-singular M-matrix. Then for the i-th mode, set $R_i = \left(\frac{2\kappa_i}{\beta_i}\right)^{\frac{1}{p-1}}$ and

- *when $x \in B_{R_i}$, design $u(x,t,i)$ such that we can find a non-negative constant K_i such that for any $(x,y,t) \in B_{R_i} \times B_{R_i} \times \mathbb{R}_+$*

$$|u(x,t,i) - u(y,t,i)| \leq K_i |x-y|, \tag{4.5}$$

$$x^{\mathrm{T}}u(x,t,i) \le -\kappa_i|x|^2, \tag{4.6}$$

and moreover $u(0,t,i)=0$ *for all* $t\in\mathbb{R}_+$*;*

- *when* $x\in B_{2R_i}-B_{R_i}$*, let* $u(x,t,i)=u\left(\left(\frac{2R_i}{|x|}-1\right)x,t,i\right)$ *for all* $t\in\mathbb{R}_+$*;*
- *when* $x\in B^c_{2R_i}$*, let* $u(x,t,i)=0$ *for all* $t\in\mathbb{R}_+$*.*

Here for convenience, we let $K_i=\kappa_i=0$ for $i\in\mathbb{S}_1$. Let us now make some comments on this control rule.

Remark 4.3. *On the one hand, if we pay attention to the hybrid SDE (4.1) on* $\mathbb{S}_1$*, we find these subsystems might become stable. There is hence no need to impose any control when* $i\in\mathbb{S}_1$*. On the other hand, for* $\mathbb{S}_2$*-modes, the control function is designed in the bounded state area. But here, we have the following decomposition*

$$(\gamma_i-\kappa_i)|x|^2-\frac{1}{2}\beta_i|x|^{p+1}+\left(\kappa_i|x|^2-\frac{1}{2}\beta_i|x|^{p+1}\right), \tag{4.7}$$

where κ_i *can be chosen freely and is not required to be* $2\gamma_i$ *strictly as given in the Introduction part. This makes our decomposition more flexible to adapt to different actual needs.*

We also strengthen that although S_1*-subsystems have a certain stability property, this could not imply the stability of the whole system since we require* $\mathbb{S}_2\neq\emptyset$*. Therefore, the control design in* $\mathbb{S}_2$*-modes is still useful.*

Remark 4.4. *It should also be pointed out that we could in fact let*

$$u(x,t,i)=0, \qquad \forall x\notin B_{R_i}, (t,i)\in\mathbb{R}_+\times\mathbb{S}_2.$$

However, in our scheme, we set an additional connect area $B_{2R_i}-B_{R_i}$ *and require* $u(x,t,i)$ *to vanish when* $|x|\ge 2R_i$*. This is needed for the purpose of continuity of* $u(x,t,i)$ *in* x *to guarantee the existence of unique global solution of the controlled SDE (4.2). Certainty, we have many choices on how to construct the connect area and the control function in this area. Among others, in this paper, we select a relatively simple and easily implemented way, namely spherical symmetry. This can also guarantee the global Lipschitz continuity of* $u(x,t,i)$ *in* x *with the same Lipschitz coefficient assumed in* $x\in B_{R_i}$*, which is stated as Lemma 4.1.*

From the discussions above, after giving an appropriate κ_i, we see that the design of $u(x,t,i)$ for $\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{S}_1$ and $B^c_{R_i}\times\mathbb{R}_+\times\mathbb{S}_2$ is very clear. The remaining question is whether we could design $u(x,t,i)$ for $B_{R_i}\times\mathbb{R}_+\times\mathbb{S}_2$. Actually, there are lots of control functions available, such as the linear one $u(x,t,i)=-(1+\kappa_i)x$.

4.2.3 The upper bound of observation duration

By analogy with Rule 3.3, we need to determine an upper bound of τ. The basic idea is similar, but due to mode structures, we should be careful to some parameters. The following preparation work is hence wanted.

Let $(\eta_1, \cdots, \eta_S)^{\mathrm{T}} = A^{-1}(1, \cdots, 1)^{\mathrm{T}}$ and $(\hat{\eta}_1, \cdots, \hat{\eta}_{S_1})^{\mathrm{T}} = \hat{A}^{-1}(1, \cdots, 1)^{\mathrm{T}}$. Since A and $\hat{A}$ are both non-singular M-matrices, all $\eta_i (i \in \mathbb{S})$ and $\hat{\eta}_i (i \in \mathbb{S}_1)$ are positive. Denote by $\beta_{\eta_2} = \min_{i \in \mathbb{S}_2} (\beta_i \eta_i)$ and

$$\hat{\mu} = \frac{\beta_{\eta_2}}{1 + \max_{i \in \mathbb{S}_2} \left(\sum_{j=1}^{S_1} q_{ij} \hat{\eta}_j \right)}.$$

By the positivity of β_i assumed in Assumption 4.1 and properties of Q-matrix, it is easy to see that $\beta_{\eta_2} > 0$ and $\hat{\mu} > 0$.

Rule 4.2. *Let τ works smaller than $\tau^* := \max \left\{ \varphi(\varepsilon) : 0 < \varepsilon < \frac{1}{K_M \eta_{M_2}} \right\}$, where*

$$\varphi(\varepsilon) = \frac{1}{K_M \eta_{M_2}} \left(\frac{1 - K_M \eta_{M_2} \varepsilon}{2H_1 + 2K_M + \frac{\tilde{H}_1}{\varepsilon} + \frac{1}{\eta_{M_2}}} \wedge \frac{\hat{\mu}}{2H_2 + \frac{\tilde{H}_2}{\varepsilon} + \frac{\hat{\mu}}{\eta_{M_2}}} \right),$$

with $\eta_{M_2} = \max_{i \in \mathbb{S}_2} \eta_i$ and $K_M = \max_{i \in \mathbb{S}} K_i$.

Remark 4.5. *Compared with Rule 3.3, the key difference is the definition of φ, where $2\beta_\eta$ has been changed to $\hat{\mu}$, and η_M to η_{M_2}. If we do not consider mode-structure classification, then $\hat{\mu}$ will decay into $\min_{i \in \mathbb{S}} (\beta_i \eta_i)$, which is exactly Rule 3.3 (owing to bounded control scheme, here is β_η rather than $2\beta_\eta$). The detailed role of $\hat{\mu}$ will be explained in Remark 4.6.*

The analysis in Remark 3.4 also applies to τ^* here. That is, $\tau < \frac{1}{K_M}$, and there exists a $\varepsilon^* \in \left(0, \frac{1}{K_M \eta_{M_2}}\right)$ such that $\tau^* = \varphi(\varepsilon^*)$.

4.3 Main results

4.3.1 Control analysis

After giving our new control scheme, we need to discuss its theoretical properties. The first is for global solution, the other is about keeping the mode structures.

Global Lipschitz continuity

We give a lemma to see that $u(x,t,i)$ designed in Rule 4.1 could meet with Rule 3.1. From this point, our control is less costly than before.

Lemma 4.1. *Let Rule 4.1 hold. Then for all* $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,

$$|u(x,t,i) - u(y,t,i)| \le K_i |x-y|. \tag{4.8}$$

Proof. The result is clear for $i \in \mathbb{S}_1$. Then for any $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}_2$, we always assume that $|x| \le |y|$ without loss of generality. To show the desired assertion, let us consider the following six possible cases.

- For $x, y \in B_{R_i}$, it is just our condition (4.5).
- For $x, y \in B_{2R_i} - B_{R_i}$, we see that $\left(\frac{2R_i}{|x|} - 1\right) x$ and $\left(\frac{2R_i}{|y|} - 1\right) y$ are both in B_{R_i}. Thus by condition (4.5), we have
$$|u(x,t,i) - u(y,t,i)| \le K_i \left| \left(\frac{2R_i}{|x|} - 1\right) x - \left(\frac{2R_i}{|y|} - 1\right) y \right|.$$
Then compute
$$\begin{aligned}\left| \left(\frac{2R_i}{|x|} - 1\right) x - \left(\frac{2R_i}{|y|} - 1\right) y \right|^2 \le & \frac{4R_i}{|x||y|}(2R_i - |x| - |y|)(|x||y| - x^{\mathrm{T}} y) \\ & + |x-y|^2,\end{aligned}$$
which yields that $|u(x,t,i) - u(y,t,i)| \le K_i |x-y|$, since $|x|, |y| > R_i$.
- For $x, y \in B^c_{2R_i}$, the result is obvious.
- For $x \in B_{R_i}, y \in B_{2R_i} - B_{R_i}$, we have
$$\begin{aligned}|u(x,t,i) - u(y,t,i)| = & \left| u(x,t,i) - u\left(\left(\frac{2R_i}{|y|} - 1\right) y, t, i \right) \right| \\ \le & K_i \left| x - \left(\frac{2R_i}{|y|} - 1\right) y \right|.\end{aligned}$$
Because $|x| \le R_i$ and $R_i < |y| \le 2R_i$, we observe that
$$\left| x - \left(\frac{2R_i}{|y|} - 1\right) y \right|^2 - |x-y|^2 = 4(|y| - R_i)\left(\frac{x^{\mathrm{T}} y}{|y|} - R_i\right) \le 0.$$
The required assertion follows immediately.
- For $x \in B_{R_i}, y \in B^c_{2R_i}$, it is easy to derive that
$$|u(x,t,i) - u(y,t,i)| = |u(x,t,i)| \le K_i|x| \le K_i R_i \le K_i \big| |y| - |x| \big| \le K_i |x-y|.$$

- For $x \in B_{2R_i} - B_{R_i}, y \in B_{2R_i}^c$, we derive that

$$|u(x,t,i) - u(y,t,i)| = \left| u(x,t,i) - u\left(\frac{2R_i}{|y|}y,t,i\right)\right| \le K_i \left| x - \frac{2R_i}{|y|}y \right|.$$

Here we use the fact that $u(y,t,i) = u\left(\frac{2R_i}{|y|}y,t,i\right) = 0$ and the result in the second case. Next compute

$$\left| x - \frac{2R_i}{|y|}y \right|^2 - |x-y|^2 = (|y| - 2R_i)\left(\frac{2x^{\mathrm{T}}y}{|y|} - (2R_i + |y|)\right) \le 0.$$

The required assertion then follows.

The proof is therefore complete. □

From this lemma, we could see that $u(x,t,i)$ is global Lipschitz continuous in x, that is, Rule 3.1 is satisfied. Together with Assumptions 3.1 and 3.2, in view of Theorem 3.1, we can conclude that the controlled SDE (4.2) has a unique global solution $x(t)$, which satisfies that

$$\sup_{0\le s\le t} E|x(s)|^{p+1} < \infty, \qquad \forall t > 0.$$

Structures on controlled SDE

From Assumption 4.1 and Rule 4.1, we observe that the controlled SDE (4.2) also has different structures in different modes.

For $i \in \mathbb{S}_1$, since $u(x,t,i) \equiv 0$, we then derive that for every $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$,

$$\begin{cases} x^{\mathrm{T}}(f(x,y,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2 \le \alpha_i |x|^2, \\ x^{\mathrm{T}}(f(x,y,t,i) + u(x,t,i)) + \frac{p}{2}|g(x,t,i)|^2 \le \hat{\alpha}_i |x|^2. \end{cases} \tag{4.9}$$

For $\mathbb{S}_2$-modes, we have the following lemma.

Lemma 4.2. *Let Assumption 4.1 and Rule 4.1 hold. Then for $i \in \mathbb{S}_2$, we have that for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$*

$$x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2 \le \alpha_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1}, \tag{4.10}$$

where $\alpha_i = \gamma_i - \kappa_i$.

Proof. We divide the proof into two cases. Fix $(t,i) \in \mathbb{R}_+ \times \mathbb{S}_2$ arbitrarily. For $x \in B_{R_i}$, by conditions (4.4) and (4.6), it is easy to see that

$$x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2 \le (\gamma_i - \kappa_i)|x|^2 - \beta_i|x|^{p+1}$$

$$\leq \alpha_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1}.$$

On the other hand, for $x \notin B_{R_i}$, we see from the decomposition in Remark 4.3 that

$$x^{\mathrm{T}} f(x,t,i) + \frac{1}{2}|g(x,t,i)|^2 \leq (\gamma_i - \kappa_i)|x|^2 - \frac{\beta_i}{2}|x|^{p+1}$$

since $\kappa_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1} < 0$ when $|x| > R_i$. Then by the definition of $u(x,t,i)$, for $x \in B_{2R_i} - B_{R_i}$,

$$\begin{aligned} x^{\mathrm{T}} u(x,t,i) =& \frac{|x|}{2R_i - |x|}\left(\frac{2R_i}{|x|} - 1\right) x^{\mathrm{T}} u\left(\left(\frac{2R_i}{|x|} - 1\right) x, t, i\right) \\ \leq& -\kappa_i(2R_i - |x|)|x| \leq 0, \end{aligned}$$

and for $x \in B^c_{2R_i}$, $x^{\mathrm{T}} u(x,t,i) = 0$. Consequently, for $x \notin B_{R_i}$

$$x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2 \leq -\alpha_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1}.$$

Now we have shown the claim (4.10). □

4.3.2 Lyapunov functional

Since we have considered the structured stabilisation, the Lyapunov functional used in this chapter is a little different than before, which is given by

$$V(x_t, t, r(t)) = U(x(t), r(t)) + \int_{-\tau}^{0}\int_{t+s}^{t}\left(\varpi_1^*|x(v)|^2 + \varpi_2^*|x(v)|^{p+1}\right) \mathrm{d}v\mathrm{d}s. \quad (4.11)$$

Here, ϖ_1^*, ϖ_2^* are positive constants to be determined later, and the Lyapunov function $U(x,i) \in \mathbb{C}^2(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+)$ is defiend as

$$U(x,i) = \eta_i|x|^2 + \hat{\mu}\hat{\eta}_i|x|^{p+1}\mathbb{I}_{\{i\in\mathbb{S}_1\}}. \quad (4.12)$$

Recalling (3.14), we now need to estimate L_1U and L_2U, especially the first of which is not trivial.

Lemma 4.3. *Under Assumption 4.1 and Rule 4.1, for any* $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,

$$L_1U(x,t,i) \leq -|x|^2 - \hat{\mu}|x|^{p+1}. \quad (4.13)$$

Proof. For $i \in \mathbb{S}_1$, we have

$$\begin{aligned} L_1U(x,t,i) \leq& 2\eta_i\left(x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2\right) + \sum_{j=1}^{S} q_{ij}\eta_j|x|^2 \\ &+ (p+1)\hat{\mu}\hat{\eta}_i|x|^{p-1}\left(x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)) + \frac{p}{2}|g(x,t,i)|^2\right) \end{aligned}$$

$$+ \hat{\mu} \sum_{j=1}^{S_1} q_{ij} \hat{\eta}_j |x|^{p+1}.$$

It then follows from (4.9) that

$$L_1 U(x,t,i) \le \left(2\alpha_i \eta_i + \sum_{j=1}^{S} q_{ij}\eta_j \right) |x|^2 + \hat{\mu} \left((p+1)\hat{\alpha}_i \hat{\eta}_i + \sum_{j=1}^{S_1} q_{ij}\hat{\eta}_j \right) |x|^{p+1}.$$

Since $2\alpha_i \eta_i + \sum_{j=1}^{S} q_{ij}\eta_j = -1$ and $(p+1)\hat{\alpha}_i\hat{\eta}_i + \sum_{j=1}^{S_1} q_{ij}\hat{\eta}_j = -1$, we get

$$L_1 U(x,t,i) \le -|x|^2 - \hat{\mu}|x|^{p+1}.$$

For $i \in \mathbb{S}_2$, it is easy to derive that

$$\begin{aligned} L_1 U(x,t,i) \le & 2\eta_i \left(x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,t,i)|^2 \right) + \sum_{j=1}^{S} q_{ij}\eta_j |x|^2 \\ & + \hat{\mu} \sum_{j=1}^{S_1} q_{ij}\hat{\eta}_j |x|^{p+1}. \end{aligned}$$

Making use of (4.10), we have

$$L_1 U(x,t,i) \le \left(2\alpha_i \eta_i + \sum_{j=1}^{S} q_{ij}\eta_j \right) |x|^2 - \left(\beta_i \eta_i - \hat{\mu} \sum_{j=1}^{S_1} q_{ij}\hat{\eta}_j \right) |x|^{p+1}.$$

From the definition of $\hat{\mu}$, we deduce that for $i \in \mathbb{S}_2$,

$$\hat{\mu} + \hat{\mu} \sum_{j=1}^{S_1} q_{ij}\hat{\eta}_j \le \hat{\mu} \left(1 + \max_{i \in \mathbb{S}_2} \left(\sum_{j=1}^{S_1} q_{ij}\hat{\eta}_j \right) \right) = \beta_{\eta_2} \le \beta_i \eta_i.$$

This implies that

$$L_1 U(x,t,i) \le -|x|^2 - \hat{\mu}|x|^{p+1}.$$

The proof is therefore complete. □

Remark 4.6. *From the definition of our U-function, we could find that there is an additional term $\hat{\mu}\hat{\eta}_i|x|^{p+1}\mathbb{I}_{\{i\in\mathbb{S}_1\}}$ compared with (3.13). This is designed to let the estimation of $L_1U(x,t,i)$ for $\mathbb{S}_1$-modes behave similarly to $\mathbb{S}_2$-modes due to the absence of high-order term $|x|^{p+1}$ in this structure. Thus $\hat{\mu}$ is always refered to the mode balance parameter.*

The estimation of L_2U is an easy deduction from Lemma 3.1 since

$$\begin{aligned} |L_2 U(x,z,t,i)| = & \left| \left(2\eta_i x^{\mathrm{T}} + \hat{\mu}\hat{\eta}_i |x|^{p-1} x^{\mathrm{T}} \mathbb{I}_{\{i\in\mathbb{S}_1\}} \right) (u(z,t,i) - u(x,t,i)) \mathbb{I}_{\{i\in\mathbb{S}_2\}} \right| \\ \le & 2\eta_{M_2} |x| |u(z,t,i) - u(x,t,i)|. \end{aligned}$$

As a result, we only give the result and omit its proof.

Lemma 4.4. *Under Assumptions 3.1, 3.2, 4.1, let the control function $u(x, t, i)$ satisfy Rule 4.1, and τ meets Rule 4.2. Then for any $t > 0$, we have*

$$E|L_2U(x(t), x(t_\tau), t, r(t))| \leq \phi_1 E|x(t)|^2 + \phi_2 E|x(t)|^{p+1} + \phi_3 \int_{t-\tau}^{t} E|x(v)|^2 \mathrm{d}v + \phi_4 \int_{t-\tau}^{t} E|x(v)|^{p+1} \mathrm{d}v,$$

where $\phi_1 = \frac{K_M \eta_{M_2}}{1-K_M\tau}((H_1 + 2K_M)\tau + \varepsilon^*)$, $\phi_2 = \frac{K_M \eta_{M_2}}{1-K_M\tau}\frac{2H_2\tau}{p+1}$, $\phi_3 = \frac{K_M \eta_{M_2}}{1-K_M\tau}\left(H_1 + \frac{\tilde{H}_1}{\varepsilon^*}\right)$, $\phi_4 = \frac{K_M \eta_{M_2}}{1-K_M\tau}\left(\frac{2pH_2}{p+1} + \frac{\tilde{H}_2}{\varepsilon^*}\right)$.

4.3.3 Exponential stabilisation

If we use the same analysis in Theorems 3.2 and 3.3, we could obtain the H_∞ stability and asymptotic stability. But we should go further to discuss a more significant type of stability, i.e. exponential stability, which appears widely in our daily life such as the virus control.

Theorem 4.1. *Under the same conditions with Lemma 4.4, there exists a positive constant λ^* such that the solution of the controlled SDE (4.2) satisfies that*

$$\limsup_{t\to\infty} \frac{1}{t} \log(E|x(t)|^2) \leq -\lambda^*. \tag{4.14}$$

Proof. We divide the proof into two steps.

The first step is to guarantee the existence of λ^*. Define four functions on $\left[0, \frac{1}{\tau}\right)$ by $\varpi_1(\lambda) = \frac{\phi_3}{1-\lambda\tau}$, $\varpi_2(\lambda) = \frac{\phi_4}{1-\lambda\tau}$,

$$\Phi_1(\lambda) = 1 - \phi_1 - \varpi_1(\lambda)\tau - \eta_M\lambda, \qquad \Phi_2(\lambda) = \hat{\mu} - \phi_2 - \varpi_2(\lambda)\tau - \hat{\eta}_{M_2}\lambda,$$

where $\hat{\eta}_{M_2} = \max_{i\in\mathbb{S}_2} \hat{\eta}_i$ and $\eta_M = \max_{i\in\mathbb{S}} \eta_i$. It is easy to see that $\varpi_1(\cdot)$ is a positive increasing function and tends to infinity when $\lambda \to \frac{1}{\tau}$. This observation implies that $\Phi_1(\cdot)$ is decreasing and goes to negative infinity when λ approaches its right bound. Next, recalling the definition of ϕ_1 and ϕ_3, compute

$$\Phi_1(0) = 1 - \phi_1 - \phi_3\tau > 0.$$

Consequently, there exists a unique solution $\lambda_1^* \in \left(0, \frac{1}{\tau}\right)$ such that $\Phi_1(\lambda) = 0$. The same analysis applying to $\varpi_2(\cdot)$ and $\Phi_2(\cdot)$ yields that there is a unique solution $\lambda_2^* \in \left(0, \frac{1}{\tau}\right)$ so that $\Phi_2(\lambda) = 0$. Therefore, we let $\lambda^* = \lambda_1^* \wedge \lambda_2^*$.

The second step is to show the claim (4.14). We firstly choose the parameters set in the Lyapunov functional as

$$\varpi_1^* = \varpi_1(\lambda^*) = \frac{\phi_3}{1-\lambda^*\tau}, \qquad \varpi_2^* = \varpi_2(\lambda^*) = \frac{\phi_4}{1-\lambda^*\tau}, \tag{4.15}$$

which are all positive from the discussions above. Applying the generalised Itô formula to $e^{\lambda^* t}V(x_t, t, r(t))$, we obtain that for any $t \geq 0$ (if necessary, using the procedure of stopping times since $EV(x_t, t, r(t)) < \infty$ and $E|\mathcal{L}V(x_t, t, r(t))| < \infty$)

$$\begin{aligned} &e^{\lambda^* t}EV(x_t, t, r(t)) \\ \leq &V(\xi_0, 0, i_0) + \int_0^t e^{\lambda^* s}(\lambda^* EV(x_s, s, r(s)) + E\mathcal{L}V(x_s, s, r(s)))\mathrm{d}s. \end{aligned} \tag{4.16}$$

Recalling the estimations of L_1U in Lemma 4.3 and L_2U in Lemma 4.4, compute

$$\begin{aligned} E\mathcal{L}V(x_s, s, r(s)) \leq &-(1-\phi_1-\varpi_1^*\tau)E|x(s)|^2 - (\hat{\mu}-\phi_2-\varpi_2^*\tau)E|x(s)|^{p+1} \\ &-(\varpi_1^*-\phi_3)\int_{s-\tau}^s E|x(v)|^2\mathrm{d}v - (\varpi_2^*-\phi_4)\int_{s-\tau}^s E|x(v)|^{p+1}\mathrm{d}v. \end{aligned}$$

Recalling the definition of $V(x_s, s, r(s))$ and using inequality (2.23), we then have

$$\begin{aligned} &\lambda^* EV(x_s, s, r(s)) + E\mathcal{L}V(x_s, s, r(s)) \\ \leq &-\Phi_1(\lambda^*)E|x(s)|^2 - \Phi_2(\lambda^*)E|x(s)|^{p+1} \\ &-((1-\lambda^*\tau)\varpi_1^*-\phi_3)\int_{s-\tau}^s E|x(v)|^2\mathrm{d}v - ((1-\lambda^*\tau)\varpi_2^*-\phi_4)\int_{s-\tau}^s E|x(v)|^{p+1}\mathrm{d}v, \end{aligned}$$

where the definitions of $\Phi_1(\lambda^*)$ and $\Phi_2(\lambda^*)$ are given before, and we could see that they are both non-negative. Therefore, we obtain from (4.15) and (4.16) that

$$\eta_m e^{\lambda^* t}E|x(t)|^2 \leq e^{\lambda^* t}EV(x_t, t, r(t)) \leq V(\xi_0, 0, i_0),$$

where $\eta_m = \min_{i\in\mathbb{S}} \eta_i$. Letting $t \to \infty$ gives the desired assertion (4.14). The proof is hence complete. □

4.4 Application to neural networks

Consider a stochastic neural network with N ($N = 10$) neurons perturbed by a scalar Brownian motion $W(t)$, operating in two modes, busy and free. In free mode, it obeys the Hopfield model

$$\mathrm{d}x_j(t) = \left(-L_j x_j(t) + \sum_{k=1}^N \Pi_{jk}\vartheta_k(x_k(t))\right)\mathrm{d}t + \sigma x_j(t)\mathrm{d}W(t), \tag{4.17}$$

while in busy mode, it could be described by the Cohen-Grossberg neuron network

$$\mathrm{d}x_j(t) = -\Gamma x_j(t)\left(P\left(x_j^2(t)-\varrho\right) - \sum_{k=1}^{N}\tilde{\Pi}_{jk}\tilde{\vartheta}_k(x_k(t))\right)\mathrm{d}t + \tilde{\sigma}x_j^2(t)\mathrm{d}W(t). \quad (4.18)$$

Here $x_j(t)$ is the j-th neuron state, Γx_j represents the amplification function, $P\left(x_j^2-\varrho\right)$ is the behaved function, Π_{jk} and $\tilde{\Pi}_{jk}$ stand for the connection weight from neuron k to neuron j in free mode and busy mode, respectively, σ and $\tilde{\sigma}$ are the perturbation strength, $\vartheta_j(x_j) = \rho\frac{1-e^{-x_j}}{1+e^{-x_j}}$ and $\tilde{\vartheta}_j(x_j) = \tilde{\rho}\frac{e^{x_j}-e^{-x_j}}{e^{x_j}+e^{-x_j}}$ are the transfer functions, $L_j = \sum_{k=1}^{N}|\Pi_{jk}|$. For more information about neuron networks (4.17) and (4.18), we cite (Blythe, Mao & Liao 2001, Wang, Shu, Fang & Liu 2006, Ye, Michel & Wang 1995) for references.

This neuron network switches from one mode into the other according to a Markov chain $r(t)$ on the state space $\mathbb{S} = \{1, 2\}$ (1 for free mode, 2 for busy mode) with transition rate matrix

$$Q = \begin{pmatrix} -8 & 8 \\ 1 & -1 \end{pmatrix}.$$

The network parameters are given as $\varrho = 0.15$, $\rho = 0.3$, $\tilde{\rho} = 0.15$, $\Gamma = 3$, $P = 2.5$, $\sigma = 0.3$, $\tilde{\sigma} = 0.1$. The connection weight Π_{jk} and $\tilde{\Pi}_{jk}$ can be obtained from the network connection graphs with free mode in Fig. 4.1 and busy mode in Fig. 4.2. Take Fig. 4.1 as an example to explain the network connection graph: node j stands for the j-th neuron, directed edge (j, k) means the output of the k-th neuron is connected with the input of the j-th neuron, the number on the edge (j, k) is the value of Π_{jk}, if there is no edge between two nodes, these two neurons do not have direct interaction and the value of Π_{jk} is zero, such as $\Pi_{21} = 0.09$, $\Pi_{14} = 0$. Here, positive number represents the output-input connection is non-inverting, negative is inverting.

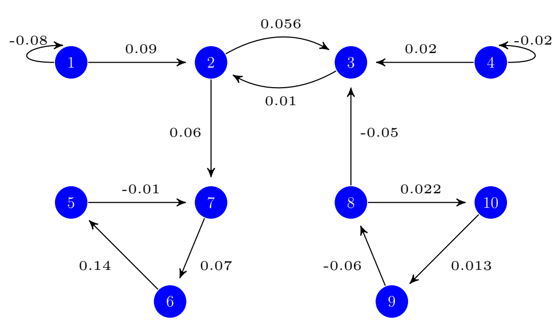

Figure 4.1: The neuron network connection at free mode.

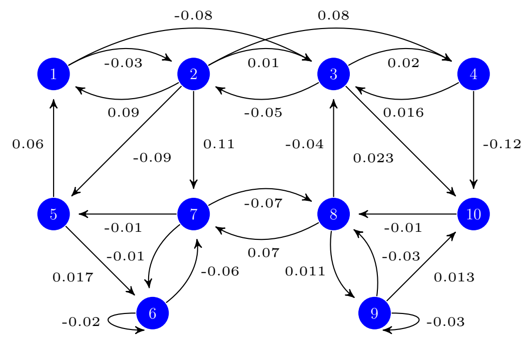

Figure 4.2: The neuron network connection at busy mode.

Let $x = (x_1, \cdots, x_N)^{\mathrm{T}}$, $x^2 = (x_1^2, \cdots, x_N^2)^{\mathrm{T}}$, $L = \mathrm{diag}(L_1, \cdots, L_N)$, $\vartheta(x) = (\vartheta_1(x_1), \cdots, \vartheta_N(x_N))^{\mathrm{T}}$, $\tilde{\vartheta}(x) = (\tilde{\vartheta}_1(x_1), \cdots, \tilde{\vartheta}_N(x_N))^{\mathrm{T}}$, $\mathcal{P} = (\varrho, \cdots, \varrho)^{\mathrm{T}}$, $\Pi = (\Pi_{jk})_{N\times N}$, $\tilde{\Pi} = (\tilde{\Pi}_{jk})_{N\times N}$. Then rewrite the network into a general form of the hybrid SDE as

$$\mathrm{d}x(t) = f(x(t), r(t))\mathrm{d}t + g(x(t), r(t))\mathrm{d}W(t). \tag{4.19}$$

Here $g(x, 1) = \sigma x$, $g(x, 2) = \tilde{\sigma}x^2$, and

$$f(x, 1) = -Lx + \Pi\vartheta(x), \qquad f(x, 2) = -\Gamma\mathrm{diag}(x)\Big(P\left(x^2 - \mathcal{P}\right) - \tilde{\Pi}\tilde{\vartheta}(x)\Big).$$

The simulation result in Fig. 4.3 shows the neural network (4.19) is unstable. We then want to design a state feedback control $u(x, i)$ based on discrete-time observations at $0, \tau, 2\tau, \cdots$ to stabilise it. The controlled network becomes

$$\mathrm{d}x(t) = (f(x(t), r(t)) + u(x(t_\tau), r(t)))\mathrm{d}t + g(x(t), r(t))\mathrm{d}W(t). \tag{4.20}$$

For each $1 \le j \le N$, we clearly have $|\vartheta_j(x_j)| \le \rho|x_j|$ and $|\tilde{\vartheta}_j(x_j)| \le \tilde{\rho}|x_j|$, which implies that

$$\begin{aligned}
|f(x, 1)| \le& (|L| + |\Pi|\rho)|x|, \\
|f(x, 2)| \le& \left(\Gamma P\varrho + \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{4}\right)|x| + \left(\Gamma P + \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{4}\right)|x|^3.
\end{aligned}$$

Moreover, it is easy to get

$$|g(x, 1)|^2 \le \sigma^2|x|^2, \qquad |g(x, 2)|^2 \le \tilde{\sigma}^2|x|^4.$$

Assumption 3.1 is satisfied with $H_1 = 1.1565$, $H_2 = 7.5315$, $\tilde{H}_1 = 0.09$, $\tilde{H}_2 = 0.01$,

$p = 3$. Next, since $|x|^4 \le N \sum_{j=1}^{N} |x_j|^4$, compute

$$x^{\mathrm{T}} f(x,1) \le -\left(\min_{1\le j\le N} L_j - |\Pi|\rho\right)|x|^2$$

and

$$x^{\mathrm{T}} f(x,2) \le \left(\Gamma P \varrho + \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{2}\right)|x|^2 - \frac{1}{N}\left(\Gamma P - \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{2}\right)|x|^4.$$

Therefore, Assumption 3.2 holds with $\hat{\alpha} = 1.188$. It is easy to see that $\mathbb{S}$ can be divided into two parts, $\mathbb{S}_1 = \{1\}$ and $\mathbb{S}_2 = \{2\}$ (Hopfield structure and Cohen-Grossberg structure, respectively). Making use of the estimation of $x^{\mathrm{T}} f(x,i)$ and $|g(x,i)|^2$ above, we obtain that for $i \in \mathbb{S}_1$, $\alpha_1 = 0.0662$, $\hat{\alpha}_1 = 0.1562$, and $\hat{A} = 7.375$, a non-singular M-matrix. While for $i \in \mathbb{S}_2$, we get $\gamma_2 = 1.188$, $\beta_2 = 0.7387$. As a result, Assumption 4.1 holds.

Then we choose $\kappa_2 = 2$ and design the control function as follows: for any $x \in \mathbb{R}^N$, $u(x,1) = 0$, and

$$u(x,2) = \begin{cases} -2x, & \text{if } |x| \le R_2, \\ -2\left(\frac{2R_2}{|x|} - 1\right)x, & \text{if } R_2 < |x| \le 2R_2, \\ 0, & \text{if } |x| > 2R_2, \end{cases} \tag{4.21}$$

with $R_2 = 2.327$. Consequently, Rule 4.1 is satisfied with $K_2 = 2$ and $A = \begin{pmatrix} 7.8675 & -8 \\ -1 & 2.624 \end{pmatrix}$ being a non-singular M-matrix. Compute $\hat{\eta}_1 = 0.1356$, $(\eta_1, \eta_2)^{\mathrm{T}} = (0.8402, 0.7013)^{\mathrm{T}}$, $\hat{\mu} = 0.0619$, and $\tau^* = 0.0173$. Up to now, we have verified all the conditions in Theorem 4.1. We conclude that the controlled network (4.20) is exponentially stable in mean square if $\tau < 0.0173$. We perform a simulation with $\tau = 0.01$ to support our theoretical results, which is demonstrated in Fig. 4.3.

4.5 Summary

Taking different system structures in different Markovian modes into consideration, this chapter studies the structured stabilisation of a class of hybrid SDEs by feedback control based on discrete-time state observations, in the sense of mean square exponential stability. The controller is designed in a bounded state area, rather than every observable state, to let the control less costly. An application to stochastic structured neural networks is given to demonstrate the practicability of the developed theory in multiple dimensional cases.

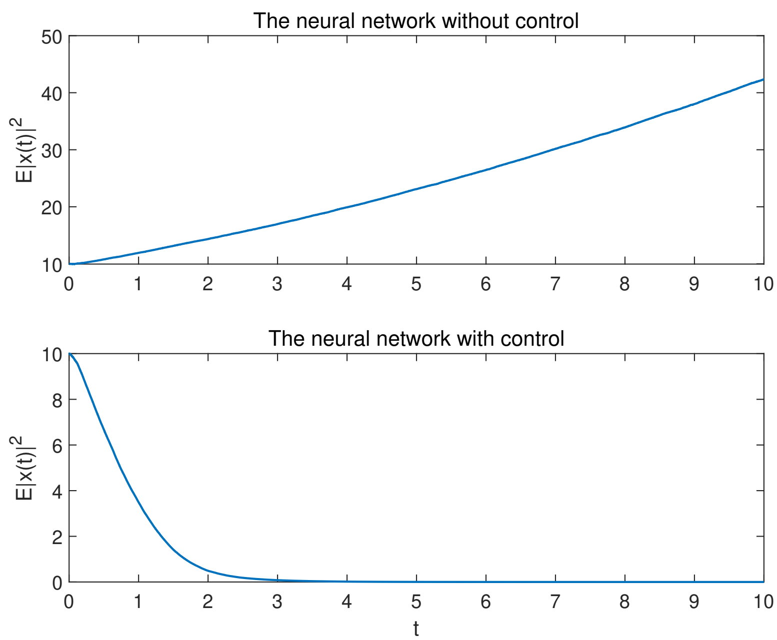

Figure 4.3: Computer simulations of $E|x(t)|^2$ of the neural network (4.19) (top), the controlled network (4.20) (bottom) using the truncated Euler-Maruyama method with step size 10^{-4} and sample size 200 as well as the fixed initial data for $\xi_0 = (\xi_{01}, \cdots, \xi_{0N})^{\mathrm{T}}$, where $\xi_{01} = \cdots = \xi_{0N} = 0.5$ and $i_0 = 1$ for all 200 samples.

Chapter 5

Structured stabilisation of hybrid delay systems by discrete-time state feedback control

5.1 Introduction

In the last two chapters, the discrete-state-feedback stabilisation problem we have been mainly concerned with is for non-delay systems. However, time delay is usually encountered unavoidably in our daily life. For example, in communication networks, a packet of data needs some time to travel through multiple devices, then be received at its destination and decoded (Fridman 2014). In terms of our interested SDEs, it means the system might not only be decided by the current state, but also depend on the past states. Then we could use a stochastic delay differential equation (SDDE) to describe such a system.

A general hybrid SDDE is given as

$$\mathrm{d}x(t) = f(x(t), x(t-\delta(t)), t, r(t))\mathrm{d}t + g(x(t), x(t-\delta(t)), t, r(t))\mathrm{d}W(t), \quad (5.1)$$

where $\delta(t)$ is the time-delay function. Using a discrete-time state feedback control to stabilise it (if unstable) has been studied for linear systems (Li & Kou 2017), highly nonlinear systems (Mei et al. 2020). But these results paid attention to the hybrid SDDE (5.1) with the same structure in every Markovian mode. Therefore, we will consider the structured stabilisation of hybrid SDDEs in this chapter.

But the reader might wonder if there is any need to further study this problem since we have investigated the structured stabilisation of hybrid SDEs in Chapter 4. Will it be an easy generalisation work? The answer is certainly not positive. In application, time delay could actually influence the mode-structure classification (see Example 5.1) and the control design (see Rule 5.2). Theoretically, the analysis of delay equations is much harder than non-delay ones since we need to tackle the effect arisen from the time delay (always negative). The conditions imposed and the Lyapunov functional constructed will become more complicated. As a result, the structured stabilisation of hybrid SDDEs deserves our investigation.

Secondly, it should be underscored that there is a widely imposed but restrictive condition on the time delay function $\delta(t)$ (e.g. (Hu, Mao & Shen 2013, Wang, Liu & Liu 2008, Min, Xu, Zhang & Ma 2019)), which is supposed to be a differentiable function and satisfies that

$$\frac{\mathrm{d}\delta(t)}{\mathrm{d}t} < 1, \qquad \forall t \geq 0. \tag{5.2}$$

This condition is just imposed owing to the mathematical need to deal with the time lag. However, many real-world time delays might miss this condition (e.g. (Li & Kou 2017, Zhang & Chen 2019, Dong & Mao 2022, Sun, Sun & Chen 2020, Qian & Zhao 2022, Gugat & Tucsnak 2011)).

For example, in the networked control systems, sawtooth delay appears frequently, such as $\delta(t) = \tau \sum_{k=0}^{\infty} \mathbb{I}_{[k,k+1)}(t)\,(t-k)$ (see, e.g. (Sun et al. 2020, Qian & Zhao 2022)). It was also found in (Gugat & Tucsnak 2011) that the energy of a vibrating system could decay exponentially with $2T$-periodic switching delay, namely, $\delta(t) = 4T$ in the first half of one period and $6T$ in the latter, where T represents the wave period. These delays are even discontinuous, let alone meeting condition (5.2). Therefore, it seems a little unreasonable to continue imposing this condition. And in this chapter, we will consider a weaker one (namely Assumption 5.1).

5.2 Model formulation

For delay systems, we need to prepare a few more notations. For some constant $\Delta > 0$, let $C\left([-\Delta, 0]; \mathbb{R}^d\right)$ represent the family of all continuous functions ξ from $[-\Delta, 0]$ to $\mathbb{R}^d$ and designate the norm of its element ξ by $||\xi|| = \sup_{-\Delta \leq \theta \leq 0} |\xi(\theta)|$.

Consider the hybrid SDDE (5.1) with the initial data

$$\{x(t) : -\Delta \le t \le 0\} = \xi \in C([-\Delta, 0]; \mathbb{R}^d), \quad r(0) = i_0 \in \mathbb{S}. \tag{5.3}$$

Here, drift coefficient $f : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$ and diffusion coefficient $g : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous in (x, y), $\delta : \mathbb{R}_+ \to [0, \Delta]$ is the system delay. The meaning of other notions appearing in the hybrid SDDE (5.1) stays unchanged. Also to avoid the abuse of notations, many of them will keep the same as given for SDEs if they are defined in the similar way.

5.2.1 General time delays

As mentioned before, the mathematical techniques used in many papers to tackle the delay effect, such as (Hu et al. 2013, Wang et al. 2008, Min et al. 2019), force authors to impose the differentiability condition (5.2) on the time delay $\delta(t)$, which is too restrictive in many real models. Consequently, in this chapter, we will consider a more general situation, by imposing the following assumption, which was firstly proposed by (Dong & Mao 2022).

Assumption 5.1. *Suppose that $\delta(t)$ is a Borel measurable function with the property that*

$$\Delta^* := \limsup_{\epsilon\to 0^+} \left(\sup_{s\ge -\Delta} \frac{\mu(I_{s,\epsilon})}{\epsilon} \right) < \infty, \tag{5.4}$$

where $\mu(\cdot)$ denotes the Lebesgue measure on the real line and

$$I_{s,\epsilon} = \{t \in \mathbb{R}_+ : t - \delta(t) \in [s, s+\epsilon)\}.$$

Although this assumption looks a little cumbersome, it is not so strong and can be met by many time-varying delay functions in practice.

Remark 5.1. *Let T be a positive constant, then the piecewise constant function*

$$\delta(t) = T \sum_{k=0}^{\infty} \mathbb{I}_{[(2k+1)T,(2k+2)T)}(t)$$

satisfies Assumption 5.1 with $\Delta^* = 2$. Moreover, if $\delta(t)$ is a Lipschitz continuous function with Lipschitz coefficient $\hat{h} \in [0, 1)$, namely

$$|\delta(t) - \delta(s)| \le \hat{h}|t - s|, \qquad \forall t, s \in \mathbb{R}_+,$$

then Assumption 5.1 is satisfied with $\Delta^* = \frac{1}{1-\hat{h}}$. In particular, if $\delta(t)$ is differentiable with derivative taking values in $[0, 1)$, then it satisfies Assumption 5.1.

These examples not only show that there is an abundant class of functions $\delta(t)$, but also condition (5.2) is a special case of Assumption 5.1. For more details, we refer the reader to (Dong & Mao 2022). But differently, the delay function $\delta(t)$ considered in this chapter is not needed to be bounded below by a positive constant. Of course we do not want to consider the case where $\delta(t) = 0$ for all $t \geq 0$ as the SDDE reduces to a SDE.

Next, we need to prepare a useful lemma, which plays a fundamental role when we discuss the properties of the hybrid SDDE (5.1).

Lemma 5.1. *Let Assumption 5.1 hold. Let $T > 0$ and $\varphi : [-\Delta, T] \to \mathbb{R}_+$ be a continuous function. Then*

$$\int_0^T \varphi(v - \delta(v))\mathrm{d}v \leq \Delta^* \int_{-\Delta}^T \varphi(v)\mathrm{d}v. \tag{5.5}$$

Proof. For any $\varepsilon > 0$, we derive from Assumption 5.1 that there is a positive constant $\bar{\epsilon}$ such that

$$\sup_{s \geq -\Delta} \frac{\mu(I_{s,\epsilon})}{\epsilon} \leq \Delta^* + \varepsilon, \qquad \forall \epsilon \in (0, \bar{\epsilon}).$$

Let n be a large integer so that $\frac{T+\Delta}{n} < \bar{\epsilon}$. Then we let $\epsilon = \frac{T+\Delta}{n}$ and $t_k = -\Delta + k\epsilon$ for $k = 0, 1, \cdots, n$. It is easy to see that

$$\mu(I_{t_k,\epsilon}) \leq \epsilon \sup_{s \geq -\Delta} \frac{\mu(I_{s,\epsilon})}{\epsilon} \leq (\Delta^* + \varepsilon)\epsilon.$$

By the definition of the Lebesgue integral, we have

$$\begin{aligned}\int_0^T \varphi(v - \delta(v))\mathrm{d}v &= \lim_{n\to\infty} \sum_{k=0}^{n-1} \varphi(t_k)\mu(I_{t_k,\epsilon}) \\ &\leq (\Delta^* + \varepsilon) \lim_{n\to\infty} \sum_{k=0}^{n-1} \varphi(t_k)\epsilon = (\Delta^* + \varepsilon) \int_{-\Delta}^T \varphi(v)\mathrm{d}v.\end{aligned}$$

Since ε is chosen arbitrarily, the required assertion (5.5) follows. The proof is complete. □

This lemma tells us how to tackle the effect of the time delay under our new Assumption 5.1. It should be pointed out that Δ^* given in Assumption 5.1 always satisfies that $\Delta^* \geq 1$. In fact, if we let $\varphi(t) \equiv 1$ for all $t \geq -\Delta$ in Lemma 5.1. Then this lemma tells us that $T \leq \Delta^*(T + \Delta)$ for any $T > 0$, which implies that $\Delta^* \geq \lim_{T\to\infty} \frac{T}{T+\Delta} = 1$.

5.2.2 Mode-structure classification

By analogy with Assumption 3.1, we still want to use polynomials to limit the growth of coefficients, which is given as follows when we consider hybrid SDDEs.

Assumption 5.2. *Assume that there are non-negative constants H_1, H_2, H_3, H_4, and $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{H}_3$, $\tilde{H}_4$, and $p > 1$ such that for every $(x, y, t, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,*

$$|f(x, y, t, i)| \leq H_1|x| + H_2|y| + H_3|x|^p + H_4|y|^p \tag{5.6}$$

and

$$|g(x, y, t, i)|^2 \leq \tilde{H}_1|x|^2 + \tilde{H}_2|y|^2 + \tilde{H}_3|x|^{p+1} + \tilde{H}_4|y|^{p+1}. \tag{5.7}$$

Recalling Remark 4.1, we do not need to conduct mode-structure classification in Assumption 3.2, owing to the absence of high-order term $|x|^{p+1}$. But for the delay equations, we have to do so since there exists a general Khasminskii-type condition with the appearance of $|x|^{p+1}$ and $|y|^{p+1}$, except for the classical one.

For convenience, we still divide $\mathbb{S}$ into two parts, $\mathbb{S}_1 = \{1, \cdots, S_1\}$ and $\mathbb{S}_2 = \{S_1 + 1, \cdots, S\}$ with $1 \leq S_1 < S$. The subsytems of the hybrid SDDE (5.1) in $\mathbb{S}_1$-modes and $\mathbb{S}_2$-modes satisfy the classical Khasminskii-type condition (condition (5.8)) and the generalized Khasminskii-type condition (condition (5.9)), respectively.

Assumption 5.3. *For $i \in \mathbb{S}_1$, suppose that there exist constants $\tilde{a}_i \in \mathbb{R}$ and $\tilde{b}_i \geq 0$ such that for all $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$*

$$x^{\mathrm{T}} f(x, y, t, i) + \frac{2p-1}{2}|g(x, y, t, i)|^2 \leq \tilde{a}_i|x|^2 + \tilde{b}_i|y|^2 \tag{5.8}$$

and for $\tilde{A} := -2p\mathrm{diag}(\tilde{a}_1, \cdots, \tilde{a}_{S_1}) - (q_{ij})_{i,j\in\mathbb{S}_1}$ to be a non-singular M-matrix.

For $i \in \mathbb{S}_2$, assume that there exist constants $\tilde{\gamma}_i, \tilde{b}_i, \tilde{d}_i \geq 0$ and $\tilde{c}_i > 0$ such that for any $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$x^{\mathrm{T}} f(x, y, t, i) + \frac{p}{2}|g(x, y, t, i)|^2 \leq \tilde{\gamma}_i|x|^2 + \tilde{b}_i|y|^2 - \tilde{c}_i|x|^{p+1} + \tilde{d}_i|y|^{p+1}. \tag{5.9}$$

From Assumption 5.3, it seems that the classification idea is quite similar to Chapter 4, and we just additionally consider this into the conditions for the existence-and-uniqueness theorem. However, we highlight that time delay could really influence our classification scheme. The existence of delay term or not in one mode will sometimes decide which class it will be put into. Let us give an example to explain this.

Example 5.1. *Consider a scalar example*

$$f(x,y,t,1) = -x - 3xy^2, \qquad f(x,y,t,2) = x - 4.5x^3, \qquad f(x,y,t,3) = -x^3,$$
$$g(x,y,t,1) = xy, \qquad g(x,y,t,2) = x^2, \qquad g(x,y,t,3) = y,$$

with transition rate matrix

$$Q = \begin{pmatrix} -2 & 1 & 1 \\ 9 & -18 & 9 \\ 5 & 5 & -10 \end{pmatrix}.$$

In this situation, time delay vanishes in mode 2. There are actually two classification schemes. Case 1: $\mathbb{S}_1 = \{1,2\}$*,* $\mathbb{S}_2 = \{3\}$*. It is easy to obtain that* $\tilde{A} = \begin{pmatrix} 8 & -1 \\ -9 & 6 \end{pmatrix}$*, which is a non-singular M-matrix and* $\tilde{c}_3 = 1 > 0$*. Case 2:* $\mathbb{S}_1 = \{1\}$*,* $\mathbb{S}_2 = \{2,3\}$*. We then have* $\tilde{A} = 8$*, a non-singular M-matrix, and* $\tilde{c}_2 = \tilde{c}_3 = 1 > 0$*.*

However, if we consider time delay into subsystem in mode 2 and let

$$g(x,y,t,2) = y^2,$$

then we only have one scheme, $\mathbb{S}_1 = \{1\}$*,* $\mathbb{S}_2 = \{2,3\}$*.*

5.2.3 Global solution

Let $(\tilde{\eta}_1, \cdots, \tilde{\eta}_{S_1})^{\mathrm{T}} = \tilde{A}^{-1}(1, \cdots, 1)^{\mathrm{T}}$. Since $\tilde{A}$ is a non-singular M-matrix, all $\tilde{\eta}_i$ are positive ($i \in \mathbb{S}_1$). Along with the properties of transition rate matrix and the fact that $\tilde{c}_i > 0$ for all $i \in \mathbb{S}_2$, the number $\tilde{\mu}$ is also positive, defined as

$$\tilde{\mu} = \frac{(p+1)\min_{i\in\mathbb{S}_2} \tilde{c}_i}{1 + \max_{i\in\mathbb{S}_2}\left(\sum_{j=1}^{S_1} q_{ij}\tilde{\eta}_j\right)}.$$

Now, we show that the hybrid SDDE (5.1) has a unique global solution.

Theorem 5.1. *Let Assumptions 5.1, 5.2, 5.3 hold. Further assume that*

$$\tilde{D} := 1 - (2p - 2 + 2\Delta^*)\max_{i\in\mathbb{S}_1}\left(\tilde{b}_i\tilde{\eta}_i\right) > 0$$

and

$$\tilde{\mu}\tilde{D} - \frac{(p+1)(p-1+(p+1)\Delta^*)}{2p}\max_{i\in\mathbb{S}_2} \tilde{d}_i \geq 0.$$

Then there is a unique global solution $x(t)$ of the hybrid SDDE (5.1) such that

$$\sup_{-\Delta\leq s\leq t} E\left(|x(s)|^{p+1} + |x(s)|^{2p}\mathbb{I}_{\{r(s)\in\mathbb{S}_1\}}\right) < \infty \tag{5.10}$$

for all $t > 0$.

Proof. We divide the whole proof into two steps.

Step 1. Set a function $\tilde{U} \in \mathbb{C}^2\left(\mathbb{R}^d \times \mathbb{S}; \mathbb{R}_+\right)$ by

$$\tilde{U}(x,i) = |x|^{p+1} + \tilde{\mu}\tilde{\eta}_i |x|^{2p}\mathbb{I}_{\{i\in\mathbb{S}_1\}},$$

while define a function $\mathbb{L}\tilde{U} : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ by

$$\begin{aligned}\mathbb{L}\tilde{U}(x,y,t,i) =& \tilde{U}_x(x,i)f(x,y,t,i) + \frac{1}{2}\text{trace}\left(g^{\mathrm{T}}(x,y,t,i)\tilde{U}_{xx}(x,i)g(x,y,t,i)\right)\\ &+ \sum_{j=1}^{S} q_{ij}\tilde{U}(x,j).\end{aligned}$$

We claim that

$$\mathbb{L}\tilde{U}(x,y,t,i) \le \zeta_1|x|^{p+1} + \zeta_2|y|^{p+1} - \zeta_3|x|^{2p} + \zeta_4|y|^{2p}, \tag{5.11}$$

where

$$\begin{aligned}\zeta_1 =& (p+1)\left(\max_{i\in\mathbb{S}_1}\tilde{a}_i \vee \max_{i\in\mathbb{S}_2}\tilde{\gamma}_i\right) + (p-1)\max_{i\in\mathbb{S}}\tilde{b}_i,\\ \zeta_2 =& 2\max_{i\in\mathbb{S}}\tilde{b}_i,\\ \zeta_3 =& \tilde{\mu} - (2p-2)\tilde{\mu}\max_{i\in\mathbb{S}_1}\left(\tilde{b}_i\tilde{\eta}_i\right) - \frac{(p+1)(p-1)}{2p}\max_{i\in\mathbb{S}_2}\tilde{d}_i,\\ \zeta_4 =& 2\tilde{\mu}\max_{i\in\mathbb{S}_1}\left(\tilde{b}_i\tilde{\eta}_i\right) + \frac{(p+1)^2}{2p}\max_{i\in\mathbb{S}_2}\tilde{d}_i.\end{aligned}$$

In fact, for $i \in \mathbb{S}_1$, it is easy to derive from (5.8) that

$$x^{\mathrm{T}}f(x,y,t,i) + \frac{p}{2}|g(x,y,t,i)|^2 \le \tilde{a}_i|x|^2 + \tilde{b}_i|y|^2.$$

This, together with (5.8) again, yields that

$$\begin{aligned}\mathbb{L}\tilde{U}(x,y,t,i) \le& (p+1)|x|^{p-1}\left(\tilde{a}_i|x|^2 + \tilde{b}_i|y|^2\right)\\ &+ \tilde{\mu}\left(2p\tilde{a}_i\tilde{\eta}_i + \sum_{j=1}^{S_1} q_{ij}\tilde{\eta}_j\right)|x|^{2p} + 2p\tilde{\mu}\tilde{b}_i\tilde{\eta}_i|x|^{2p-2}|y|^2.\end{aligned}$$

Since $2p\tilde{a}_i\tilde{\eta}_i + \sum_{j=1}^{S_1} q_{ij}\tilde{\eta}_j = -1$, by the Young inequality, we further have

$$\begin{aligned}\mathbb{L}\tilde{U}(x,y,t,i) =& \left((p+1)\tilde{a}_i + (p-1)\tilde{b}_i\right)|x|^{p+1} + 2\tilde{b}_i|y|^{p+1}\\ &- \tilde{\mu}|x|^{2p} + (2p-2)\tilde{\mu}\tilde{b}_i\tilde{\eta}_i|x|^{2p} + 2\tilde{\mu}\tilde{b}_i\tilde{\eta}_i|y|^{2p}.\end{aligned} \tag{5.12}$$

For $i \in \mathbb{S}_2$, making use of (5.9), we obtain that

$$\mathbb{L}\tilde{U}(x,y,t,i) \le (p+1)|x|^{p-1}\left(\tilde{\gamma}_i|x|^2 + \tilde{b}_i|y|^2 - \tilde{c}_i|x|^{p+1} + \tilde{d}_i|y|^{p+1}\right)$$

$$+\tilde{\mu}\sum_{j=1}^{S_1} q_{ij}\tilde{\eta}_j|x|^{2p}.$$

From the definition of $\tilde{\mu}$, we deduce that for $i \in \mathbb{S}_2$,

$$\tilde{\mu}+\tilde{\mu}\sum_{j=1}^{S_1} q_{ij}\tilde{\eta}_j \le \tilde{\mu}+\tilde{\mu}\max_{i\in\mathbb{S}_2}\left(\sum_{j=1}^{S_1} q_{ij}\tilde{\eta}_j\right) = (p+1)\min_{i\in\mathbb{S}_2}\tilde{c}_i \le (p+1)\tilde{c}_i.$$

This implies that

$$\begin{aligned}\mathbb{L}\tilde{U}(x,y,t,i) \le& \left((p+1)\tilde{\gamma}_i+(p-1)\tilde{b}_i\right)|x|^{p+1}+2\tilde{b}_i|y|^{p+1}\\ &-\tilde{\mu}|x|^{2p}+\frac{(p+1)(p-1)}{2p}\tilde{d}_i|x|^{2p}+\frac{(p+1)^2}{2p}\tilde{d}_i|y|^{2p}. \qquad (5.13)\end{aligned}$$

Combining with (5.12) and (5.13), the required (5.11) follows immediately.

Step 2. Since the system coefficients are locally Lipschitz continuous, we know that there is a unique maximal local solution $x(t)$ on $t \in [0,\sigma_e)$ by Theorem 7.12 in (Mao & Yuan 2006), where σ_e is the explosion time. Let $k_0 > 0$ be sufficiently large for $k_0 \ge ||\xi||$. For each integer $k \ge k_0$, define the stopping time

$$\sigma_k = \inf\left\{t \in [0,\sigma_e) : |x(t)| \ge k\right\}.$$

Clearly, σ_k is increasing as $k \to \infty$. Set $\sigma_\infty = \lim_{k\to\infty}\sigma_k$, whence $\sigma_\infty \le \sigma_e$ a.s. If we can show that $\sigma_\infty = \infty$ a.s., then $\sigma_e = \infty$ a.s., and the solution $x(t)$ is the global solution. Then, for any $k \ge k_0$ and $t \ge 0$, we derive from the generalised Itô formula and (5.11) that

$$\begin{aligned}E\tilde{U}(x(t\wedge\sigma_k), r(t\wedge\sigma_k)) \le& \tilde{U}(\xi(0),i_0)+E\int_0^{t\wedge\sigma_k}\mathbb{L}\tilde{U}(x(s),x(s-\delta(s)),s,r(s))\mathrm{d}s\\ \le& \tilde{U}(\xi(0),i_0)+E\int_0^{t\wedge\sigma_k}\Big(\zeta_1|x(s)|^{p+1}+\zeta_2|x(s-\delta(s))|^{p+1}\\ &-\zeta_3|x(s)|^{2p}+\zeta_4|x(s-\delta(s))|^{2p}\Big)\mathrm{d}s. \qquad (5.14)\end{aligned}$$

Making use of Lemma 5.1, we have

$$\begin{aligned}E\tilde{U}(x(t\wedge\sigma_k), r(t\wedge\sigma_k)) \le& C_1+E\int_0^{t\wedge\sigma_k}\Big(\zeta_1|x(s)|^{p+1}+\zeta_2|x(s-\delta(s))|^{p+1}\\ &-(\zeta_3-\zeta_4\Delta^*)|x(s)|^{2p}\Big)\mathrm{d}s,\end{aligned}$$

where $C_1 = \tilde{U}(\xi(0),i_0)+\zeta_4\Delta^*\Delta||\xi||^{2p}$. It is easy to compute $\zeta_3-\zeta_4\Delta^* \ge 0$ from the additional conditions in Theorem 5.1. Therefore,

$$\sup_{-\Delta\le s\le t} E\tilde{U}(x(s\wedge\sigma_k), r(s\wedge\sigma_k)) \le C_1+(\zeta_1+\zeta_2)\int_0^t \sup_{-\Delta\le v\le s} E|x(v\wedge\sigma_k)|^{p+1}\mathrm{d}s. \quad (5.15)$$

This implies that

$$\sup_{-\Delta\le s\le t} E|x(s\wedge\sigma_k)|^{p+1} \le \left(C_1+||\xi||^{p+1}\right)+(\zeta_1+\zeta_2)\int_0^t \sup_{-\Delta\le v\le s} E|x(v\wedge\sigma_k)|^{p+1}\mathrm{d}s.$$

Applying the Gronwall inequality gives that

$$\sup_{-\Delta\le s\le t} E|x(s\wedge\sigma_k)|^{p+1} \le \left(C_1+||\xi||^{p+1}\right)e^{(\zeta_1+\zeta_2)t}.$$

This particularly implies that

$$k^{p+1}P(\sigma_k\le t)\le E|x(t\wedge\sigma_k)|^{p+1}\le \left(C_1+||\xi||^{p+1}\right)e^{(\zeta_1+\zeta_2)t}<\infty.$$

We can hence let $k\to\infty$ to obtain that $P(\sigma_\infty\le t)=0$, namely, $P(\sigma_\infty>t)=1$. Since $t\ge 0$ is arbitrary, we must have that $P(\sigma_\infty=\infty)=1$ as required. Letting $k\to\infty$ in (5.15) gives that $\sup_{-\Delta\le s\le t}E\tilde{U}(x(s),r(s))<\infty$. Then the required assertion (5.10) follows since

$$\left(1\wedge\tilde{\mu}\min_{i\in\mathbb{S}_1}\tilde{\eta}_i\right)\left(|x|^{p+1}+|x|^{2p}\mathbb{I}_{\{i\in\mathbb{S}_1\}}\right)\le\tilde{U}(x,i).$$

The proof is therefore complete. □

Remark 5.2. *In the classical analysis of delay systems, time delay always plays a negative role. To eliminate this effect, the non-delay term should be strengthened, as a result of which we give two additional conditions in Theorem 5.1 except for Assumptions 5.1, 5.2, 5.3. In theory, one common method to cope with delay function is making use of the integral transform given in Lemma 5.1.*

From now on, since the subsequent stability analysis will be our main focus, we will not mention the conditions of Theorem 5.1 explicitly and assume they are true.

5.3 Control design

Suppose the hybrid SDDE (5.1) is unstable, we want to design a bounded feedback control $u(x,t,i)$, which is imposed at discrete times, say $0,\tau,2\tau,\cdots$, to make it become stable. Our controlled system is described as

$$\begin{aligned}\mathrm{d}x(t)=&\Big(f(x(t),x(t-\delta(t)),t,r(t))+u(x(t_\tau),t,r(t))\Big)\mathrm{d}t\\&+g(x(t),x(t-\delta(t)),t,r(t))\mathrm{d}W(t)\end{aligned}\tag{5.16}$$

on $t\ge 0$ with initial data ξ and i_0.

5.3.1 Additional assumption

As discussed in Assumption 3.3 or Assumption 4.1, in addition to all conditions in Theorem 5.1, we need to give another assumption on the hybrid SDDE (5.1) for the stabilisation aim.

Assumption 5.4. *For $i \in \mathbb{S}_1$, assume that there exist constants $a_i, \hat{a}_i \in \mathbb{R}$ and $b_i, \hat{b}_i \geq 0$ such that for any $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$,*

$$\begin{cases} x^{\mathrm{T}} f(x,y,t,i) + \frac{1}{2}|g(x,y,t,i)|^2 \leq a_i|x|^2 + b_i|y|^2, \\ x^{\mathrm{T}} f(x,y,t,i) + \frac{p}{2}|g(x,y,t,i)|^2 \leq \hat{a}_i|x|^2 + \hat{b}_i|y|^2, \end{cases} \tag{5.17}$$

while for the following two matrices

$$\begin{aligned} A_1 :=& -2\mathrm{diag}(a_1, \cdots, a_{S_1}) - (q_{ij})_{i,j\in\mathbb{S}_1}, \\ \hat{A} :=& -(p+1)\mathrm{diag}(\hat{a}_1, \cdots, \hat{a}_{S_1}) - (q_{ij})_{i,j\in\mathbb{S}_1} \end{aligned}$$

to be non-singular M-matrices. Moreover, assume that the constants

$$\begin{aligned} D_b :=& 1 - 2\Delta^* \max_{i\in\mathbb{S}_1}\left(b_i \sum_{j=1}^{S_1} (A_1^{-1})_{ij} \right), \\ \hat{D} :=& 1 - (p-1+2\Delta^*)\max_{i\in\mathbb{S}_1}(\hat{b}_i\hat{\eta}_i) \end{aligned}$$

are positive, where $(\hat{\eta}_1, \cdots, \hat{\eta}_{S_1})^{\mathrm{T}} = \hat{A}^{-1}(1, \cdots, 1)^{\mathrm{T}}$.

For $i \in \mathbb{S}_2$, there are non-negative constants γ_i, b_i, d_i, positive constant c_i so that for all $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$x^{\mathrm{T}} f(x,y,t,i) + \frac{1}{2}|g(x,y,t,i)|^2 \leq \gamma_i|x|^2 + b_i|y|^2 - c_i|x|^{p+1} + d_i|y|^{p+1}. \tag{5.18}$$

Further, letting $D_q = 1 + \max_{i\in\mathbb{S}_2}\left(\sum_{j=1}^{S_1} q_{ij}\hat{\eta}_j\right)$, assume that

$$D_d := \min_{i\in\mathbb{S}_2} c_i - \frac{D_q\Delta^*}{\hat{D}} \max_{i\in\mathbb{S}_2} d_i > 0.$$

Compared with Assumption 4.1, it seems more complicated in this one since we additionally need A_1 to be a non-singular M-matrix and the constants D_b, $\hat{D}$, D_d to be positive. These requirements are all blamed for the influence of the time delay, whose roles will be explained in Remark 5.3.

5.3.2 Control rules

Also due to the appearance of the time delay, the control should be designed more carefully than before. In this part, we will step by step give our control rules and explain the corresponding reasons.

Decomposition and bounded control

Recalling (4.7), we naturally come up with the decomposition scheme from condition (5.18) that

$$(\gamma_i - \kappa_i)|x|^2 + b_i|y|^2 - \frac{c_i}{2}|x|^{p+1} + d_i|y|^{p+1} + \left(\kappa_i|x|^2 - \frac{c_i}{2}|x|^{p+1}\right). \tag{5.19}$$

Rule 4.1 should hence be stayed. For the convenience of reading, we state it again.

Rule 5.1. *For $i \in \mathbb{S}_1$, let $u(x,t,i) = 0$ for all $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$.*

For $\mathbb{S}_2$-modes, firstly choose non-negative constants $\kappa_i (i \in \mathbb{S}_2)$ to let

$$A := -2\text{diag}(\alpha_1, \cdots, \alpha_{S_1}, \gamma_{S_1+1} - \kappa_{S_1+1}, \cdots, \gamma_S - \kappa_S) - Q$$

be a non-singular M-matrix. Then for the i-th mode, set $R_i = \left(\frac{2\kappa_i}{c_i}\right)^{\frac{1}{p-1}}$ and

- *when $x \in B_{R_i}$, design $u(x,t,i)$ such that we can find a non-negative constant K_i such that for any $(x,y,t) \in B_{R_i} \times B_{R_i} \times \mathbb{R}_+$*

$$|u(x,t,i) - u(y,t,i)| \le K_i|x-y|, \qquad x^{\mathrm{T}}u(x,t,i) \le -\kappa_i|x|^2,$$

  and moreover $u(0,t,i) = 0$ for all $t \in \mathbb{R}_+$;
- *when $x \in B_{2R_i} - B_{R_i}$, let $u(x,t,i) = u\left(\left(\frac{2R_i}{|x|} - 1\right)x, t, i\right)$ for all $t \in \mathbb{R}_+$;*
- *when $x \in B^c_{2R_i}$, let $u(x,t,i) = 0$ for all $t \in \mathbb{R}_+$.*

By Lemma 4.1, we can use the similar proof of Theorem 5.1 to show that there is a global solution of the controlled SDDE (5.16), which satisfies that

$$\sup_{-\Delta \le s \le t} E\left(|x(s)|^{p+1} + |x(s)|^{2p}\mathbb{I}_{\{r(s)\in\mathbb{S}_1\}}\right) < \infty, \qquad \forall t > 0.$$

Also from decomposition (5.19), it is easy to obtain the following lemma, that is, the controlled SDDE (5.16) also has different structures in different modes.

Lemma 5.2. *Let Assumption 4.1 and Rule 5.1 hold. Then for $i \in \mathbb{S}_1$, we derive* *that for every* $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$\begin{cases} x^{\mathrm{T}}(f(x,y,t,i)+u(x,t,i)) + \frac{1}{2}|g(x,y,t,i)|^2 \le a_i|x|^2 + b_i|y|^2, \\ x^{\mathrm{T}}(f(x,y,t,i)+u(x,t,i)) + \frac{p}{2}|g(x,y,t,i)|^2 \le \hat{a}_i|x|^2 + \hat{b}_i|y|^2. \end{cases} \tag{5.20}$$

For $i \in \mathbb{S}_2$, *we have that for all* $(x,y,t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$

$$\begin{aligned} &x^{\mathrm{T}}(f(x,y,t,i)+u(x,t,i)) + \frac{1}{2}|g(x,y,t,i)|^2 \\ \le& a_i|x|^2 + b_i|y|^2 - \frac{c_i}{2}|x|^{p+1} + d_i|y|^{p+1}, \end{aligned} \tag{5.21}$$

where $a_i = \gamma_i - \kappa_i$.

Lyapunov function and its estimations

The Lyapnov function $U : \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}_+$ used in this chapter will be similar to (4.12), given by $U(x,i) = \eta_i|x|^2 + \hat{\mu}\hat{\eta}_i|x|^{p+1}\mathbb{I}_{\{i\in\mathbb{S}_1\}}$ with mode balance parameter

$$\hat{\mu} = \frac{\min_{i\in\mathbb{S}_2}(c_i\eta_i)}{D_q},$$

where $(\eta_1,\cdots,\eta_S)^{\mathrm{T}} = A^{-1}(1,\cdots,1)^{\mathrm{T}}$, and $\hat{\eta}_i$, D_q have been given in Assumption 5.4. The corresponding operator $LU : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}_+$ with respect to the controlled SDDE (5.16) is defined as

$$\begin{aligned} LU(x,y,z,t,i) =& U_x(x,i)(f(x,y,t,i)+u(z,t,i)) \\ &+ \frac{1}{2}\mathrm{trace}\left(g^{\mathrm{T}}(x,y,t,i)U_{xx}(x,i)g(x,y,t,i)\right) + \sum_{j=1}^{S} q_{ij}U(x,j). \end{aligned}$$

It can be decomposed as $LU(x,y,z,t,i) = L_1U(x,y,t,i) + L_2U(x,z,t,i)$, where

$$\begin{aligned} L_1U(x,y,t,i) =& U_x(x,i)(f(x,y,t,i)+u(x,t,i)) \\ &+ \frac{1}{2}\mathrm{trace}\left(g^{\mathrm{T}}(x,y,t,i)U_{xx}(x,i)g(x,y,t,i)\right) + \sum_{j=1}^{S} q_{ij}U(x,j) \end{aligned}$$

and

$$L_2U(x,z,t,i) = U_x(x,i)(u(z,t,i) - u(x,t,i)).$$

By analogy with Lemma 4.3, one can use the same way of proofing (5.11) to show the estimation of L_1U as follows.

Lemma 5.3. *Let Assumption 5.4 and Rule 5.1 hold. Then for any* $(x,y,t,i) \in$

$\mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,

$$L_1U(x,y,t,i) \le -|x|^2 + 2\nu_1|y|^2 - (1-(p-1)\nu_2)\hat{\mu}|x|^{p+1} + (2\nu_2\hat{\mu} + 2\nu_3)|y|^{p+1}, \quad (5.22)$$

where $\nu_1 = \max_{i\in\mathbb{S}}(b_i\eta_i)$, $\nu_2 = \max_{i\in\mathbb{S}_1}(\hat{b}_i\hat{\eta}_i)$, $\nu_3 = \max_{i\in\mathbb{S}_2}(d_i\eta_i)$.

Dealing with time delays

Compared with Lemmas 4.3 and 5.3, we find there are delay terms $|y|^2$ and $|y|^{p+1}$ in the estimation of L_1U for SDDEs. In order to eliminate their effects, we should pick up $\kappa_i (i \in \mathbb{S}_2)$ to satisfy an additional rule.

Rule 5.2. *Ensure that κ_i chosen in Rule 5.1 makes the following numbers positive*

$$D_1 = 1 - 2\Delta^*\nu_1, \qquad D_2 = \hat{D}\hat{\mu} - 2\Delta^*\nu_3.$$

But the reader may wonder if we can find the appropriate κ_i to make Rules 5.1 and 5.2 fulfilled. The following remark will deny this worry.

Remark 5.3. *Since A_1 is a non-singular M-matrix required in Assumption 5.4, there is a constant κ large enough such that*

$$-2\mathrm{diag}(a_1, \cdots, a_{S_1}, \gamma_{S_1+1} - \kappa, \cdots, \gamma_S - \kappa) - Q$$

is a non-singular M-matrix. Therefore, we can choose $\kappa_i = \kappa$ for all $i \in \mathbb{S}_2$. Rule 5.1 hence holds. Then for sufficiently large κ,

$$\eta_i \approx \begin{cases} \sum_{j=1}^{S_1} (A_1^{-1})_{ij}, & \text{for } i \in \mathbb{S}_1, \\ \frac{1}{2\kappa}, & \text{for } i \in \mathbb{S}_2. \end{cases}$$

Therefore,

$$\begin{aligned} D_1 \approx & 1 - 2\Delta^* \max_{i\in\mathbb{S}_1}\left(b_i \sum_{j=1}^{S_1}(A_1^{-1})_{ij}\right) = D_b, \\ D_2 \approx & \frac{1}{\kappa}\left(\frac{\hat{D}}{D_q}\min_{i\in\mathbb{S}_2} c_i - \Delta^* \max_{i\in\mathbb{S}_2} d_i\right) = \frac{\hat{D}}{\kappa D_q} D_d. \end{aligned}$$

Since D_b, $\hat{D}$ and D_d are positive, Rule 5.2 could be satisfied. Therefore, we could see that the requirements on A_1, D_b, $\hat{D}$, D_d are used to guarantee the existence of κ_i. Certainly, in application, we need to make use of the special forms of f and g to take κ_i wisely.

Determination of observation duration

The upper bound of τ could be given in the similar way of Rule 4.2, but should take the time delay into account. Before that, let $\eta_{M_2} = \max_{i\in\mathbb{S}_2}\eta_i$ and $\mathcal{E} = \left(0, \frac{D_1}{K_M\eta_{M_2}}\right)$.

Rule 5.3. *Let τ works smaller than $\tau^* := \max_{\varepsilon\in\mathcal{E}}\varphi(\varepsilon)$, where*

$$\varphi(\varepsilon) = \frac{1}{K_M\eta_{M_2}}\left(\frac{D_1 - K_M\eta_{M_2}\varepsilon}{\varphi_1(\varepsilon)} \wedge \frac{D_2}{\varphi_2(\varepsilon)}\right)$$

with

$$\begin{aligned}\varphi_1(\varepsilon) =& \frac{D_1}{\eta_{M_2}} + 2K_M + 2H_1 + (1+\Delta^*)H_2 + \frac{\tilde{H}_1 + \Delta^*\tilde{H}_2}{\varepsilon},\\ \varphi_2(\varepsilon) =& \frac{D_2}{\eta_{M_2}} + 2H_3 + \frac{2(1+p\Delta^*)H_4}{p+1} + \frac{\tilde{H}_3 + \Delta^*\tilde{H}_4}{\varepsilon}.\end{aligned}$$

It is also easy to see that there is a $\varepsilon^* \in \left(0, \frac{1}{K_M\eta_{M_2}}\right)$ such that $\tau^* = \varphi(\varepsilon^*)$. To close this section, we give the estimation of L_2U, whose proof is quite similar to Lemma 4.4, so we omit it.

Lemma 5.4. *Under Assumption 5.4, let the control function $u(x,t,i)$ satisfy Rules 5.1 and 5.2, and τ meets Rule 5.3. Then for any $t > 0$, we have*

$$E|L_2U(x(t), x(t_\tau), t, r(t))| \le \phi_1 E|x(t)|^2 + \phi_2 E|x(t)|^{p+1} + \int_{t-\tau}^{t} E\Psi_1(v)\mathrm{d}v,$$

where

$$\Psi_1(t) = \phi_3|x(t)|^2 + \phi_4|x(t-\delta(t))|^2 + \phi_5|x(t)|^{p+1} + \phi_6|x(t-\delta(t))|^{p+1}$$

with

$$\begin{aligned}\phi_1 =& \frac{K_M\eta_{M_2}}{1-K_M\tau}((2K_M + H_1 + H_2)\tau + \varepsilon^*), & \phi_2 =& \frac{K_M\eta_{M_2}}{1-K_M\tau}\frac{2(H_3+H_4)\tau}{p+1},\\ \phi_3 =& \frac{K_M\eta_{M_2}}{1-K_M\tau}\left(H_1 + \frac{\tilde{H}_1}{\varepsilon^*}\right), & \phi_4 =& \frac{K_M\eta_{M_2}}{1-K_M\tau}\left(H_2 + \frac{\tilde{H}_2}{\varepsilon^*}\right),\\ \phi_5 =& \frac{K_M\eta_{M_2}}{1-K_M\tau}\left(\frac{2pH_3}{p+1} + \frac{\tilde{H}_3}{\varepsilon^*}\right), & \phi_6 =& \frac{K_M\eta_{M_2}}{1-K_M\tau}\left(\frac{2pH_4}{p+1} + \frac{\tilde{H}_4}{\varepsilon^*}\right).\end{aligned}$$

5.4 Stabilisation results

Before giving the Lyapunov functional used in this chapter, because of the time delay, we need to redefine x_t as $x_t = \{x(t+\theta)| -\tau-\Delta \le \theta \le 0\}$ for $t \ge 0$. For x_t

to be well defined for $t \in [0, \tau + \Delta]$, we set $x(\theta) = \xi(-\Delta)$ for $\theta \in [-\tau - \Delta, -\Delta)$. Then the Lyapunov functional will be of the form

$$V(x_t, t, r(t)) = U(x(t), r(t)) + \int_{-\tau}^{0} \int_{t+s}^{t} \Psi_2(v) \mathrm{d}v \mathrm{d}s$$

for any $t \geq 0$, where

$$\Psi_2(t) = \varpi_1^* |x(t)|^2 + \varpi_2^* |x(t - \delta(t))|^2 + \varpi_3^* |x(t)|^{p+1} + \varpi_4^* |x(t - \delta(t))|^{p+1}.$$

Here ϖ_1^*, ϖ_2^*, ϖ_3^*, ϖ_4^* are positive constants, which could be determined by the following remark.

Remark 5.4. *Let $\eta_M = \max_{i \in \mathbb{S}} \eta_i$ and $\hat{\eta}_{M_1} = \max_{i \in \mathbb{S}_1} \hat{\eta}_i$. Define six functions on $\left[0, \frac{1}{\tau}\right)$ as $\varpi_1(\lambda) = \frac{\phi_3}{1 - \lambda\tau}$, $\varpi_2(\lambda) = \frac{\phi_4}{1 - \lambda\tau}$, $\varpi_3(\lambda) = \frac{\phi_5}{1 - \lambda\tau}$, $\varpi_4(\lambda) = \frac{\phi_6}{1 - \lambda\tau}$,*

$$\Phi_1(\lambda) = 1 - \phi_1 - \varpi_1(\lambda)\tau - (2\nu_1 + \varpi_2(\lambda)\tau)\Delta^* e^{\lambda\tau} - \eta_M \lambda,$$
$$\Phi_2(\lambda) = (1 - (p-1)\nu_2)\hat{\mu} - \phi_2 - \varpi_3(\lambda)\tau - (2\nu_2\hat{\mu} + 2\nu_3 + \varpi_4(\lambda)\tau)\Delta^* e^{\lambda\tau} - \hat{\eta}_{M_1}\hat{\mu}\lambda.$$

Compared with Φ_1 and Φ_2 given in the first step of proofing Theorem 4.1, the most significant difference here is the appearance of $e^{\lambda\tau}$, which is caused by the integral transform of the time delay.

It is easy to see that all $\varpi_j(\cdot)$ are positive increasing functions and tend to infinity when $\lambda \to \frac{1}{\tau}$. Thus the decreasing function $\Phi_1(\cdot)$ goes to negative infinity when λ approaches its right bound. Next, compute

$$\begin{aligned}\Phi_1(0) =& D_1 - \phi_1 - (\phi_3 + \Delta^* \phi_4)\tau \\ =& \frac{1}{1 - K_M \tau} \left(D_1 - K_M \eta_{M_2} \varepsilon^* - K_M \eta_{M_2} \varphi_1(\varepsilon^*)\tau\right) > 0.\end{aligned}$$

Consequently, there exists a unique solution $\lambda_1^* < \frac{1}{\tau}$ such that $\Phi_1(\lambda) = 0$. Also we could find a unique solution $\lambda_2^* \in \left(0, \frac{1}{\tau}\right)$ so that $\Phi_2(\lambda) = 0$. Then $\Phi_1(\lambda)$ and $\Phi_2(\lambda)$ are non-negative for any $\lambda \in [0, \lambda^*]$, where $\lambda^* = \lambda_1^* \wedge \lambda_2^*$. Finally, we could let

$$\varpi_1^* = \varpi_1(\lambda^*), \qquad \varpi_2^* = \varpi_2(\lambda^*), \qquad \varpi_3^* = \varpi_3(\lambda^*), \qquad \varpi_4^* = \varpi_4(\lambda^*). \tag{5.23}$$

Theorem 5.2. *Under the same conditions with Lemma 5.4, there exists a positive constant λ^* such that the solution of the controlled SDDE (4.2) satisfies that*

$$\limsup_{t \to \infty} \frac{1}{t} \log(E|x(t)|^2) \leq -\lambda^*. \tag{5.24}$$

Proof. Applying the generalised Itô formula to $e^{\lambda^* t} V(x_t, t, r(t))$ yields that

$$e^{\lambda^* t} EV(x_t, t, r(t)) - V(x_0, 0, r_0)$$

$$\leq \int_0^t e^{\lambda^* s}(\lambda^* EV(x_s, s, r(s)) + E\mathcal{L}V(x_s, s, r(s)))\mathrm{d}s. \tag{5.25}$$

By Lemmas 5.3 and 5.4, we derive that

$$E\mathcal{L}V(x_s, s, r(s)) \leq EJ_1(s) - \int_{s-\tau}^{s} E(\Psi_2(v) - \Psi_1(v))\mathrm{d}v,$$

where

$$\begin{aligned} J_1(s) = & -(1 - \varpi_1^*\tau - \phi_1)|x(s)|^2 + (2\nu_1 + \varpi_2^*\tau)|x(s-\delta(s))|^2 \\ & - ((1-(p-1)\nu_2)\hat{\mu} - \varpi_3^*\tau - \phi_2)|x(s)|^{p+1} \\ & + (2\nu_2\hat{\mu} + 2\nu_3 + \varpi_4^*\tau)|x(s-\delta(s))|^{p+1}. \end{aligned}$$

Recalling the definition of $V(x_s, s, r(s))$, we then have

$$\begin{aligned} & \lambda^* EV(x_s, s, r(s)) + E\mathcal{L}V(x_s, s, r(s)) \\ \leq & EJ_2(s) - \int_{s-\tau}^{s} E((1-\lambda^*\tau)\Psi_2(v) - \Psi_1(v))\mathrm{d}v, \end{aligned} \tag{5.26}$$

where $J_2(s) = \lambda^* U(x(s), r(s)) + J_1(s)$. From (5.23), we see that $(1-\lambda^*\tau)\varpi_1^* - \phi_3 = 0$, $(1-\lambda^*\tau)\varpi_2^* - \phi_4 = 0$, $(1-\lambda^*\tau)\varpi_3^* - \phi_5 = 0$, $(1-\lambda^*\tau)\varpi_4^* - \phi_6 = 0$. Therefore,

$$(1-\lambda^*\tau)\Psi_2(v) = \Psi_1(v).$$

Substituting (5.26) into (5.25) shows that

$$EV(x_t, t, r(t)) \leq V(x_0, 0, r_0) + E\int_0^t e^{\lambda^* s} J_2(s)\mathrm{d}s. \tag{5.27}$$

Since $U(x,t) \leq \eta_M|x|^2 + \hat{\eta}_{M_1}\hat{\mu}|x|^{p+1}$, we have

$$\begin{aligned} \int_0^t e^{\lambda^* s} J_2(s)\mathrm{d}s \leq & -(1 - \varpi_1^*\tau - \phi_1 - \eta_M\lambda^*)\int_0^t e^{\lambda^* s}|x(s)|^2\mathrm{d}s \\ & + (2\nu_1 + \varpi_2^*\tau)\int_0^t e^{\lambda^* s}|x(s-\delta(s))|^2\mathrm{d}s \\ & - ((1-(p-1)\nu_2)\hat{\mu} - \varpi_3^*\tau - \phi_2 - \hat{\eta}_{M_1}\hat{\mu}\lambda^*)\int_0^t e^{\lambda^* s}|x(s)|^{p+1}\mathrm{d}s \\ & + (2\nu_2\hat{\mu} + 2\nu_3 + \varpi_4^*\tau)\int_0^t e^{\lambda^* s}|x(s-\delta(s))|^{p+1}\mathrm{d}s. \end{aligned}$$

Using Lemma 5.1 and $e^{\lambda^* s} \leq e^{\lambda^*\tau}e^{\lambda^*(s-\delta(s))}$ further gives that

$$\int_0^t e^{\lambda^* s} J_2(s)\mathrm{d}s \leq C_2 - \Phi_1(\lambda^*)E\int_0^t e^{\lambda^* s}|x(s)|^2\mathrm{d}s - \Phi_2(\lambda^*)E\int_0^t e^{\lambda^* s}|x(s)|^{p+1}\mathrm{d}s,$$

where $C_2 = e^{\lambda^*\tau}\Delta^*\tau\Big((2\nu_1+\varpi_2^*\tau)||\xi||^2 + (2\nu_2\hat{\mu}+2\nu_3+\varpi_4^*\tau)||\xi||^{p+1}\Big)$. From Remark 5.4, we see that $\Phi_1(\lambda^*)$ and $\Phi_2(\lambda^*)$ are non-negative. Therefore, we obtain from

(5.27) that

$$\left(\min_{i\in\mathbb{S}}\eta_i\right)e^{\lambda^* t}E|x(t)|^2 \le C_2 + V(x_0,0,r_0).$$

Letting $t\to\infty$ gives the desired assertion (5.24). The proof is hence complete. □

5.5 Application to neural networks

We continue to consider the neural network studied in Section 4.4. Due to the finite switching speed of amphfiers, time delay usually can not be avoided. The network model with delay can be described as for $j=1,\cdots,N$

$$\mathrm{d}x_j(t) = \begin{cases} \left(-L_j x_j(t) + \sum_{k=1}^N \Pi_{jk}\vartheta_k(x_k(t-\delta(t)))\right)\mathrm{d}t + \sigma x_j(t)\mathrm{d}W(t), \text{ in mode 1,} \\ -\Gamma x_j(t)\left(P\left(x_j^2(t)-\varrho\right) - \sum_{k=1}^N \tilde{\Pi}_{jk}\tilde{\vartheta}_k(x_k(t-\delta(t)))\right)\mathrm{d}t \\ +\tilde{\sigma}x_j^2(t)\mathrm{d}W(t), \text{ in mode 2.} \end{cases}$$

Using the same notations in Section 4.4, rewrite the neural network as a general hybrid SDDE

$$\mathrm{d}x(t) = f(x(t),x(t-\delta(t)),r(t))\mathrm{d}t + g(x(t),x(t-\delta(t)),r(t))\mathrm{d}W(t). \tag{5.28}$$

Here $g(x,1)=\sigma x$, $g(x,2)=\tilde{\sigma}x^2$, and

$$f(x,y,1) = -Lx + \Pi\vartheta(y), \qquad f(x,y,2) = -\Gamma\mathrm{diag}(x)\Big(P\left(x^2-\mathcal{P}\right) - \tilde{\Pi}\tilde{\vartheta}(y)\Big).$$

Time delay in a network sometimes is larger during business time than other time. For example, if we regard one second as time unit, then during the business period $\left[0,\frac{1}{3}\right)$, it can be 0.02 second, while in the non-business time $\left[\frac{1}{3},1\right)$, it will decrease to 0.01 second. Such a time delay can be described by the following piecewise-constant function

$$\delta(t) = \sum_{k=0}^{\infty}\left(0.02\mathbb{I}_{\left[k,k+\frac{1}{3}\right)}(t) + 0.01\mathbb{I}_{\left[k+\frac{1}{3},k+1\right)}(t)\right). \tag{5.29}$$

Obviously, this delay does not satisfy (5.2) but could meet with our Assumption 5.1 with $\Delta^*=2$ and $\Delta=0.02$.

Consider the same parameters given in Section 4.4 and time delay (5.29). It is easy to see that

$$|f(x,y,1)| \le |L||x| + |\Pi|\rho|y|$$

$$|f(x,y,2)| \leq \Gamma P\varrho|x| + \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{2}|x|^2 + \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{2}|y|^2 + \Gamma P|x|^3.$$

Assumption 5.2 is hence satisfied with $H_1 = 1.1565$, $H_2 = 0.0315$, $H_3 = 7.5315$, $H_4 = 0.0315$, $\tilde{H}_1 = 0.09$, $\tilde{H}_2 = 0$, $\tilde{H}_3 = 0.01$, $\tilde{H}_4 = 0$, $p = 3$. Next, compute

$$x^{\mathrm{T}} f(x,y,1) \leq -\left(\min_{1\leq j\leq N} L_j - \frac{\Pi|\rho|}{2}\right)|x|^2 + \frac{1}{2}|\Pi|\rho|y|^2,$$
$$x^{\mathrm{T}} f(x,y,2) \leq \Gamma P\varrho|x|^2 + \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{2}|y|^2 - \frac{1}{N}\left(\Gamma P - \frac{\Gamma|\tilde{\Pi}|\tilde{\rho}}{2}\right)|x|^4.$$

We derive that $\tilde{a}_1 = 0.2462$, $\tilde{A} = 6.5226$, $\tilde{b}_1 = 0.0099$, $\tilde{\gamma}_2 = 1.125$, $\tilde{b}_2 = 0.063$, $\tilde{c}_2 = 0.7287$, $\tilde{d}_2 = 0$. Then Assumption 5.3 is satisfied. Moreover, $\tilde{\eta}_1 = 0.1533$, $\tilde{\mu} = 2.5273$, $\tilde{D} = 0.9879$ and $\tilde{\mu}\tilde{D} - \frac{(p+1)(p-1+(p+1)\Delta^*)}{2p}\tilde{d}_M = 2.4967$. Until now, all the conditions in Theorem 5.1 are fulfilled. Thus, the neuron network (5.28) has a unique global solution. But it might still be unstable and the simulation results (see Fig. 5.1 top) also shows this clearly.

Therefore, we want to design a state feedback control $u(x,i)$ based on discrete-time observations at $0, \tau, 2\tau, \cdots$ to stabilise the neuron network (5.28). The controlled network then becomes

$$\begin{aligned} \mathrm{d}x(t) =& (f(x(t), x(t-\delta(t)), r(t)) + u(x(t_\tau), r(t)))\mathrm{d}t \\ &+ g(x(t), x(t-\delta(t)), r(t))\mathrm{d}W(t). \end{aligned} \tag{5.30}$$

Next, we obtain that for $i \in \mathbb{S}_1$, $a_1 = 0.0662$, $b_1 = 0.0099$, $\hat{a}_1 = 0.1562$, $\hat{b}_1 = 0.0099$, and $A_1 = 7.8675$ $\hat{A} = 7.375$, which are non-singular M-matrices. It is then easy to derive that $D_b = 0.995$, $\hat{D} = 0.992$. While for $i \in \mathbb{S}_2$, we get $\gamma_2 = 1.125$, $b_2 = 0.063$, $c_2 = 0.7387$, $d_2 = 0$, $D_d = 0.7387$. As a result, Assumption 5.4 holds.

Then we use the same control function given in (4.21). Rule 5.1 is true with $K_2 = 2$ and $A = \begin{pmatrix} 7.8675 & -8 \\ -1 & 2.75 \end{pmatrix}$ being a non-singular M-matrix. Compute $\hat{\mu} = 0.423$, $(\eta_1, \eta_2)^{\mathrm{T}} = (0.7884, 0.6503)^{\mathrm{T}}$, $\hat{\eta}_1 = 0.1356$. It is then easy to obtain that $D_1 = 0.8361$, $D_2 = 0.4196$. Rule 5.2 is hence fulfilled. Up to now, we have verified all the conditions in Theorem 5.2. Using the method introduced in Rule 5.3, we have $\tau^* = 0.0203$. By Theorem 5.2, we conclude that the controlled network (5.30) is exponentially stable in mean square if $\tau < 0.0203$. We perform a simulation with $\tau = 0.01$ to support our theoretical results, which is demonstrated in Fig. 5.1 bottom.

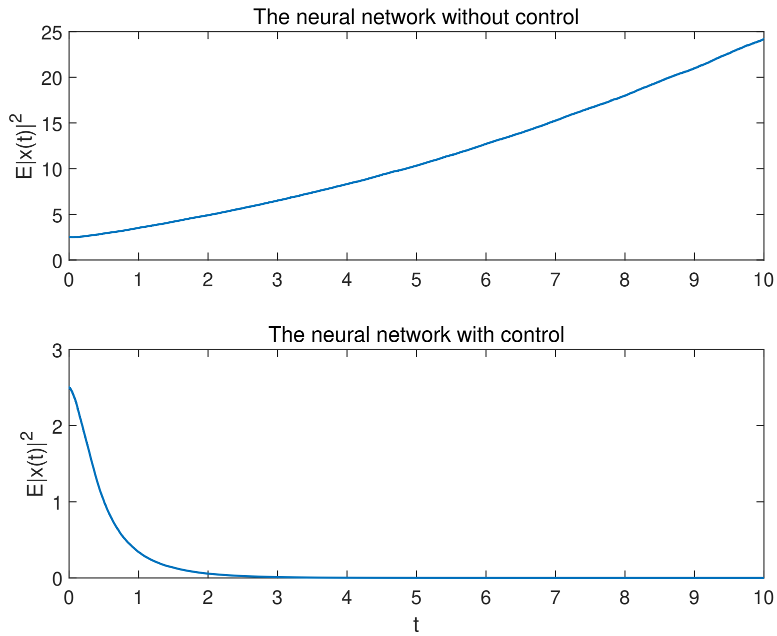

Figure 5.1: Computer simulations of $E|x(t)|^2$ of the neural network (5.28) (top), the controlled network (5.30) (bottom) using the truncated Euler-Maruyama method with step size 10^{-4} and sample size 200 as well as the fixed initial data for $\xi(t) = (\xi_1(t), \cdots, \xi_N(t))^{\mathrm{T}}$, where $\xi_1(t) = \cdots = \xi_N(t) = 0.5 + 0.5\cos(t)$, for $t \in [-0.02, 0]$ and $i_0 = 1$ for all 200 samples.

5.6 Summary

This chapter continues to consider the structured stabilisation problem by bounded discrete-time state feedback control but for SDDEs. In order to deal with the time delay, conditions imposed on the underlying system and rules on the control function become more complicated than non-delay ones. The analysis of global solution and stability is also not trivial. More importantly, it is found that delay could influence the mode-structure classification scheme. Meanwhile, our time delay is more general than the existing results, especially the commonly used differentiability assumption being relaxed.

Chapter 6

Razumikhin method to discrete-state-feedback stabilisation of hybrid systems with more general delays

6.1 Introduction

In Chapter 5, we have discussed the discrete-state-feedback stabilisation of hybrid SDDEs with more general time delays $\delta(t)$ satisfying Assumption 5.1, which include the commonly seen periodic switching delay (Gugat & Tucsnak 2011) and sawtooth delay (Sun et al. 2020). Although this improvement is much advanced, there are still some important delays not being covered such as discrete time delay. For example, in complex networks (see (Li, Shen, Wang, Huang & Luo 2019, Liu, Wang, Ma & Alsaadi 2019)), time delay usually behaves as $\delta(t) = t - [t/\Delta]\Delta$ with time unit Δ. Certainly the delay in our discrete-time state feedback control also belongs to such type of delay with $\Delta = \tau$. It is clear that discrete time delay cannot meet with Assumption 5.1 since

$$\Delta^* = \limsup_{\epsilon \to 0^+} \left(\sup_{s \geq -\Delta} \frac{\Delta}{\epsilon} \right) = \infty.$$

Theoretically speaking, the absence of this key assumption means we could not use Lemma 5.1, namely, the integral transform to handle the time delay. As a result, the Lyapunov functional method might not be valid. Then is it possible to study the stabilisation of hybrid SDDEs if we do not have Assumption 5.1 on the time delay, and what tools could we use?

Fortunately, we notice that in the study of the stability of delay systems, the Razumikhin technique has been proved as a very powerful tool. It is quite different from the classical Lyapunov idea, which requires us to check the negativity of the Lyapunov operator for every time like we have done before. While in Razumikhin method, we only need to do the verification work at some particular time (see Assumption 6.2). Due to this breakthrough idea, this method has been widely applied to stochastic functional differential equations (SFDEs). We cite (Mao & Yuan 2006, Janković, Randjelović & Jovanović 2009, Wu, Yin & Wang 2015, Zhu 2017, Cao & Zhu 2021) to readers for more details. Therefore, as a special case of SFDEs, the conditions given on time delays of SDDEs could be relatively relaxed.

So this begs a question naturally: can we use the Razumikhin method to investigate the stabilisation problem of discrete-time state feedback control? The answer is positive. In fact, (Li et al. 2018) successfully applied the Razumikhin method to the discrete-state-feedback stabilisation problem for a class of hybrid stochastic systems. And to our knowledge, so far (Li et al. 2018) has been the only paper to use the Razumikhin approach to investigate this kind of stabilisation problem. But unfortunately, they still required system coefficients to meet the linear growth condition.

Consequently, motivated by (Li et al. 2018), we will try to employ the Razumikhin technique to study this stabilisation problem of highly nonlinear hybrid SDDEs without Assumption 5.1 in this chapter. Since this work is full of challenge, to make the problem much easier, we will not consider the structured stabilisation here. Moreover, since time delay is quite general, the conditions imposed on the underlying systems will be a little different than before.

6.2 Razumikhin-type theorem

We first give our Razumikhin-type theorem of general SFDEs, which will be used later for discrete-state-feedback stabilisation problem. Except for stability property, this theorem will also be generalised to asymptotic boundedness of hybrid stochastic systems.

Let us consider a d-dimensional hybrid SFDE

$$\mathrm{d}x(t) = F(x_t, t, r(t))\mathrm{d}t + G(x_t, t, r(t))\mathrm{d}W(t) \tag{6.1}$$

on $t \geq 0$ with the initial data

$$\{x(\theta) : -h \leq \theta \leq 0\} = \hat{\xi} \in C([-h, 0]; \mathbb{R}^d), \quad r(0) = i_0 \in \mathbb{S}.$$

Here $x_t = \{x(t+\theta) : -h \leq \theta \leq 0\}$ is the past segment while $F : C([-h, 0]; \mathbb{R}^d) \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$, $G : C([-h, 0]; \mathbb{R}^d) \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous. For convenience, we extend $r(t)$ to $[-h, 0]$ by setting $r(\theta) = i_0$ for all $\theta \in [-h, 0]$.

Razumikhin-type theorem is usually given in terms of Lyapunov functions. Thus for any $V \in C^{2,1}(\mathbb{R}^d \times [-h, \infty) \times \mathbb{S}; \mathbb{R}_+)$, define an operator $\mathcal{L}V$ with respect to the hybrid SFDE (6.1) from $C([-h, 0]; \mathbb{R}^d) \times \mathbb{R}_+ \times \mathbb{S}$ to $\mathbb{R}$ by

$$\begin{aligned}\mathcal{L}V(\phi, t, i) =& V_t(\phi(0), t, i) + V_x(\phi(0), t, i)F(\phi, t, i) \\ &+ \frac{1}{2}\mathrm{trace}\left(G^{\mathrm{T}}(\phi, t, i)V_{xx}(\phi(0), t, i)G(\phi, t, i)\right) \\ &+ \sum_{j=1}^{S} q_{ij}V(\phi(0), t, j).\end{aligned} \tag{6.2}$$

Assume that for every initial data $\hat{\xi}$ and i_0, there exists a unique global solution $x(t)$ of the hybrid SFDE (6.1) satisfying the following conditions.

Assumption 6.1. *There is a $C^{2,1}$-function $V(x, t, i)$ such that for any $t \in \mathbb{R}_+$*

$$\sup_{0\leq s\leq t} EV(x(s), s, r(s)) < \infty, \qquad \sup_{0\leq s\leq t} E|\mathcal{L}V(x_s, s, r(s))| < \infty.$$

Moreover, $EV(x(t), t, r(t))$ and $E\mathcal{L}V(x_t, t, r(t))$ are right-continuous as functions of t.

Assumption 6.2. *For such $V(x, t, i)$ given in Assumption 6.1, assume that there exist constants $q > 1$, $\lambda_1 \geq 0$ and $\lambda_2 > 0$ such that*

$$E\mathcal{L}V(x_t, t, r(t)) \leq \lambda_1 - \lambda_2 EV(x_t(0), t, r(t)) \tag{6.3}$$

when x_t satisfies that for any $\theta \in [-h, 0]$

$$EV(x_t(\theta), t+\theta, r(t+\theta)) \leq qEV(x_t(0), t, r(t)). \tag{6.4}$$

Assumption 6.2 is the so-called Razumukhin-type condition. We only need to check $\mathcal{L}V$ at time t, when (6.4) happens. While for other time when $EV(x_t(\theta), t+\theta, r(t+\theta)) > qEV(x_t(0), t, r(t))$, it is naturally to obtain the negativity of $\mathcal{L}V$ since state in past time is larger than that in current time.

Theorem 6.1. *Let Assumptions 6.1, 6.2 hold. Then the solution of the hybrid SFDE (6.1) has the property that*

$$\limsup_{t\to\infty} EV(x(t),t,r(t)) \le \frac{\lambda_1}{\lambda},$$

where $\lambda = \min\left\{\lambda_2, \frac{\log(q)}{h}\right\}$*. In particular, if* $\lambda_1 = 0$,

$$\limsup_{t\to\infty} \frac{1}{t}\log EV(x(t),t,r(t)) \le -\lambda.$$

In the view of V, the hybrid SFDE (6.1) is moment asymptotically bounded and exponentially stable (when $\lambda_1 = 0$). Before diving into the proof, we make some comments about the right-continuity of EV and $E\mathcal{L}V$ required in Assumption 6.1.

Remark 6.1. *In subsequent Razumikhin analysis, we could find that it is crucial to require both* $EV(x(t),t,r(t))$ *and* $E\mathcal{L}V(x_t,t,r(t))$ *to be right-continuous. But to guarantee the right-continuity of these two functions is not trivial, especially for highly nonlinear systems. Moreover, even if we have the right-continuity of* $V(x(t),t,r(t))$ *and* $\mathcal{L}V(x_t,t,r(t))$*, we still cannot draw the conclusion that their expectations are right-continuous. Because in general, only the right-continuity of a process cannot guarantee its expectation remains right-continuous. The following example is very helpful to show this.*

Example 6.1. *Let* $W(t)$ *be a scalar Brownian motion. Define the stopping time*

$$T = \inf\{t \ge 0 : W(t) = 1\}.$$

It is easy to see that $T < \infty$ *a.s. from the recurrence of* $W(t)$*. Then for any* $t \ge 0$*, set* $Y(t) = W(t \wedge T)$*. By the Doob stopping theorem (Theorem 2.6),* $Y(t)$ *is actually a continuous martingale vanishing at* $t = 0$ *with the property that*

$$\lim_{t\to\infty} Y(t) = 1 \quad a.s.$$

Define a process $X(t)$ *by*

$$X(t) = \begin{cases} Y\left(\dfrac{1}{t-1}\right), & t > 1, \\ 1, & 0 \le t \le 1. \end{cases}$$

Since $\lim_{t\to 1^+} X(t) = \lim_{s\to\infty} Y(s) = 1$ *a.s., we observe that* $X(t)$ *is continuous (certainly right-continuous). However, we have that for* $0 \le t \le 1$, $EX(t) = 1$*, and for* $t > 1$

$$EX(t) = EY\left(\frac{1}{t-1}\right) = 0.$$

This means $EX(t)$ is not right-continuous at $t = 1$.

Now we give the proof of Theorem 6.1.

Proof. Fix any initial data $\hat{\xi}$ and i_0. Let $\eta \in (0, \lambda)$ be arbitrary and set $\bar{\lambda} = \lambda - \eta$. For $t \geq 0$, define

$$U(t) = \sup_{-h \leq \theta \leq 0} \left(e^{\bar{\lambda}(t+\theta)} EV(x(t+\theta), t+\theta, r(t+\theta)) \right).$$

From Assumption 6.1, we know that $U(t) < \infty$ for any $t \geq 0$, so $U(t)$ is well-defined. Letting $y_\eta(t) = \lambda_1 \int_0^t e^{\bar{\lambda}s} \mathrm{d}s$, we then claim that

$$D^+(U(t) - y_\eta(t)) \leq 0, \qquad t \geq 0. \tag{6.5}$$

If assertion (6.5) is true, we have

$$U(t) - y_\eta(t) \leq U(0) - y_\eta(0) \leq M, \qquad t \geq 0,$$

where $M = \sup_{-h \leq \theta \leq 0} V(\hat{\xi}(\theta), \theta, r(\theta))$. It then follows that for any $t \geq 0$,

$$e^{\bar{\lambda}t} EV(x(t), t, r(t)) \leq M + \lambda_1 \int_0^t e^{\bar{\lambda}s} \mathrm{d}s \leq M + \frac{\lambda_1}{\bar{\lambda}} e^{\bar{\lambda}t}.$$

Since η is arbitrary, we have

$$EV(x(t), t, r(t)) \leq Me^{-\lambda t} + \frac{\lambda_1}{\lambda}. \tag{6.6}$$

Finally, letting $t \to \infty$ gives

$$\limsup_{t \to \infty} EV(x(t), t, r(t)) \leq \frac{\lambda_1}{\lambda}.$$

If $\lambda_1 = 0$, we derive from (6.6) that

$$\limsup_{t \to \infty} \frac{1}{t} \log EV(x(t), t, r(t)) \leq -\lambda.$$

Now we show that assertion (6.5) is true. Fix $\hat{t} \geq 0$ arbitrarily. It is easy to observe that either

$$U(\hat{t}) > e^{\bar{\lambda}\hat{t}} EV(x(\hat{t}), \hat{t}, r(\hat{t}))$$

or

$$U(\hat{t}) = e^{\bar{\lambda}\hat{t}} EV(x(\hat{t}), \hat{t}, r(\hat{t})).$$

For the former case, we derive from the right-continuity of $EV(x(\cdot), \cdot, r(\cdot))$ that for all sufficiently small $\epsilon_1 \in (0, h)$,

$$U(\hat{t}) > e^{\bar{\lambda}t} EV(x(t), t, r(t)), \qquad \hat{t} \leq t \leq \hat{t} + \epsilon_1. \tag{6.7}$$

For $\hat{t}+\epsilon_1-h\le t<\hat{t}$, we naturally have

$$e^{\bar{\lambda}t}EV(x(t),t,r(t))\le \sup_{-h\le\theta\le 0}\Big(e^{\bar{\lambda}(\hat{t}+\theta)}EV(x(\hat{t}+\theta),\hat{t}+\theta,r(\hat{t}+\theta))\Big)=U(\hat{t}).$$

This together with (6.7) yields that $U(\hat{t}+\epsilon_1)\le U(\hat{t})$, and so

$$U(\hat{t}+\epsilon_1)<U(\hat{t})+\lambda_1\int_{\hat{t}}^{\hat{t}+\epsilon_1}e^{\bar{\lambda}s}\mathrm{d}s.$$

Then we have

$$U(\hat{t}+\epsilon_1)-y_\eta(\hat{t}+\epsilon_1)<U(\hat{t})-y_\eta(\hat{t}),$$

which indicates that

$$\begin{aligned}&D^+(U(\hat{t})-y_\eta(\hat{t}))\\ :=&\limsup_{\epsilon_1\to 0^+}\frac{\big(U(\hat{t}+\epsilon_1)-y_\eta(\hat{t}+\epsilon_1)\big)-\big(U(\hat{t})-y_\eta(\hat{t})\big)}{\epsilon_1}\le 0.\end{aligned}$$

On the other hand, if $U(\hat{t})=e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t}))$, we derive that for any $\theta\in[-h,0]$,

$$e^{\bar{\lambda}(\hat{t}+\theta)}EV(x(\hat{t}+\theta),\hat{t}+\theta,r(\hat{t}+\theta))\le e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t})).$$

Consequently,

$$\begin{aligned}EV(x(\hat{t}+\theta),\hat{t}+\theta,r(\hat{t}+\theta))\le& e^{-\bar{\lambda}\theta}EV(x(\hat{t}),\hat{t},r(\hat{t}))\\ \le& e^{\bar{\lambda}h}EV(x(\hat{t}),\hat{t},r(\hat{t}))\\ \le& qEV(x(\hat{t}),\hat{t},r(\hat{t})),\end{aligned}$$

where we have used the fact that $q\ge e^{\lambda h}$. Then by condition (6.3), we have

$$\begin{aligned}&E\mathcal{L}V(x_{\hat{t}},\hat{t},r(\hat{t}))+\bar{\lambda}EV(x(\hat{t}),\hat{t},r(\hat{t})\\ \le& E\mathcal{L}V(x_{\hat{t}},\hat{t},r(\hat{t}))+\lambda_2EV(x(\hat{t}),\hat{t},r(\hat{t}))\le\lambda_1<\lambda_1+\epsilon,\end{aligned}$$

where $\epsilon>0$ is an arbitrary constant. We therefore see from the right-continuity of $EV(x(t),t,r(t))$ and $E\mathcal{L}V(x_t,t,r(t))$ that for all $\epsilon_2\in(0,h)$ sufficiently small,

$$E\mathcal{L}V(x_t,t,r(t))+\bar{\lambda}EV(x(t),t,r(t))<\lambda_1+\epsilon \tag{6.8}$$

for any $\hat{t}\le t\le\hat{t}+\epsilon_2$. For each integer $k\ge 1$, define the stopping time

$$\sigma_k(\omega)=\inf\left\{t\ge\hat{t}:|x(t,\omega)|\ge k\right\},$$

which represents the first exiting time of sample path $x(t,\omega)$ leaving from the area $\{x\in\mathbb{R}^d:|x|<k\}$ after time $\hat{t}$. But this could be infinity since it is possible that for some ω, $x(t,\omega)$ would never go beyond that area. In this situation, the

time set $\{t \geq \hat{t} : |x(t,\omega)| \geq k\}$ is empty. For convenience, we denote $\sigma_k(\omega)$ by σ_k. Because the hybrid SFDE (6.1) admits a unique global solution, we observe that σ_k is increasing to infinity almost surely as $k \to \infty$. For each $k \geq 1$, by the generalized Itô formula, we have

$$\begin{aligned}&e^{\bar{\lambda}\hat{t}_k}V(x(\hat{t}_k),\hat{t}_k,r(\hat{t}_k)) - e^{\bar{\lambda}\hat{t}}V(x(\hat{t}),\hat{t},r(\hat{t}))\\ =&\int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda}s}\left(\mathcal{L}V(x_s,s,r(s)) + \bar{\lambda}V(x(s),s,r(s))\right)\mathrm{d}s + M_k,\end{aligned}$$

where $\hat{t}_k = (\hat{t}+\epsilon_2)\wedge\sigma_k$ and

$$M_k = \int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda}s}V_x(x(s),s,r(s))G(x_s,s,r(s))\mathrm{d}W(s).$$

Note that when $|x(\hat{t})| \geq k$ we have $\hat{t}_k = \hat{t}$; while when $|x(\hat{t})| < k$, since G is locally Lipschitz continuous, we see that for any $s \in [\hat{t},\hat{t}_k]$

$$e^{\bar{\lambda}s}V_x(x(s),s,r(s))G(x_s,s,r(s)) \leq e^{\bar{\lambda}(t+\epsilon_2)}L_k < \infty.$$

Therefore, we have $EM_k = 0$ and hence

$$\begin{aligned}&E\left(e^{\bar{\lambda}\hat{t}_k}V(x(\hat{t}_k),\hat{t}_k,r(\hat{t}_k))\right) - E\left(e^{\bar{\lambda}\hat{t}}V(x(\hat{t}),\hat{t},r(\hat{t}))\right)\\ =&E\int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda}s}\left(\mathcal{L}V(x_s,s,r(s)) + \bar{\lambda}V(x(s),s,r(s))\right)\mathrm{d}s.\end{aligned}$$

It is easy to see that for each $k \geq 1$,

$$\begin{aligned}&\left|\int_{\hat{t}}^{\hat{t}_k} e^{\bar{\lambda}s}\left(\mathcal{L}V(x_s,s,r(s)) + \bar{\lambda}V(x(s),s,r(s))\right)\mathrm{d}s\right|\\ \leq&\int_{\hat{t}}^{\hat{t}+\epsilon_2}\left|e^{\bar{\lambda}s}\left(\mathcal{L}V(x_s,s,r(s)) + \bar{\lambda}V(x(s),s,r(s))\right)\right|\mathrm{d}s.\end{aligned}$$

Since

$$\begin{aligned}&E\left|e^{\bar{\lambda}s}\left(\mathcal{L}V(x_s,s,r(s)) + \bar{\lambda}V(x(s),s,r(s))\right)\right|\\ \leq&e^{\bar{\lambda}s}\left(E|\mathcal{L}V(x_s,s,r(s))| + \bar{\lambda}EV(x(s),s,r(s))\right) < \infty\end{aligned}$$

holds for any $s \in [\hat{t},\hat{t}+\epsilon_2]$, by Fubini theorem, we have

$$\begin{aligned}&E\int_{\hat{t}}^{\hat{t}+\epsilon_2}\left|e^{\bar{\lambda}s}\left(\mathcal{L}V(x_s,s,r(s)) + \bar{\lambda}V(x(s),s,r(s))\right)\right|\mathrm{d}s\\ =&\int_{\hat{t}}^{\hat{t}+\epsilon_2}E\left|e^{\bar{\lambda}s}\left(\mathcal{L}V(x_s,s,r(s)) + \bar{\lambda}V(x(s),s,r(s))\right)\right|\mathrm{d}s < \infty.\end{aligned}$$

Letting $k \to \infty$ and using the Fatou lemma, the dominated convergence theorem, we obtain that

$$\begin{aligned}
&e^{\bar{\lambda}(\hat{t}+\epsilon_2)}EV(x(\hat{t}+\epsilon_2),\hat{t}+\epsilon_2,r(\hat{t}+\epsilon_2))\\
=&E\left(\liminf_{k\to\infty} e^{\bar{\lambda}t_k}V(x(t_k),t_k,r(t_k))\right)\\
\le&\liminf_{k\to\infty} E\left(e^{\bar{\lambda}t_k}V(x(t_k),t_k,r(t_k))\right)\\
=&e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t}))+E\int_{\hat{t}}^{\hat{t}+\epsilon_2} e^{\bar{\lambda}s}\Big(\mathcal{L}V(x_s,s,r(s))+\bar{\lambda}V(x(s),s,r(s))\Big)\mathrm{d}s.
\end{aligned}$$

Applying the Fubini theorem again as well as (6.8) yields that

$$\begin{aligned}
&e^{\bar{\lambda}(\hat{t}+\epsilon_2)}EV(x(\hat{t}+\epsilon_2),\hat{t}+\epsilon_2,r(\hat{t}+\epsilon_2))\\
<&e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t}))+\int_{\hat{t}}^{\hat{t}+\epsilon_2}(\lambda_1+\epsilon)e^{\bar{\lambda}s}\mathrm{d}s.
\end{aligned} \tag{6.9}$$

By analogy with (6.9), for any $\hat{t}\le t\le \hat{t}+\epsilon_2$, we have

$$\begin{aligned}
e^{\bar{\lambda}t}EV(x(t),t,r(t)) <&e^{\bar{\lambda}\hat{t}}EV(x(\hat{t}),\hat{t},r(\hat{t}))+(\lambda_1+\epsilon)\int_{\hat{t}}^{t} e^{\bar{\lambda}s}\mathrm{d}s\\
\le&U(\hat{t})+(\lambda_1+\epsilon)\int_{\hat{t}}^{\hat{t}+\epsilon_2} e^{\bar{\lambda}s}\mathrm{d}s.
\end{aligned}$$

For $\hat{t}+\epsilon_2-h\le t<\hat{t}$, it is also easy to see that

$$e^{\bar{\lambda}t}EV(x(t),t,r(t)) < U(\hat{t})+(\lambda_1+\epsilon)\int_{\hat{t}}^{\hat{t}+\epsilon_2} e^{\bar{\lambda}s}\mathrm{d}s$$

since $e^{\bar{\lambda}t}EV(x(t),t,r(t)) < U(\hat{t})$. Thus, we obtain that

$$U(\hat{t}+\epsilon_2)\le U(\hat{t})+\lambda_1\int_{\hat{t}}^{\hat{t}+\epsilon_2} e^{\bar{\lambda}s}\mathrm{d}s+\epsilon\int_{\hat{t}}^{\hat{t}+\epsilon_2} e^{\bar{\lambda}s}\mathrm{d}s.$$

Letting $\epsilon_2\to 0$ implies that $D^+(U(\hat{t})-y_\eta(\hat{t}))\le \epsilon e^{\bar{\lambda}\hat{t}}$. This holds for any $\epsilon>0$, so we must have $D^+(U(\hat{t})-y_\eta(\hat{t}))\le 0$. Since $\hat{t}$ is chosen arbitrarily, claim (6.5) is true. This therefore completes the proof. □

6.3 Stabilisation problem

6.3.1 Standing hypothesis

We will try to use the Razumikhin theory above to the stabilisation problem of the hybrid SDDE

$$\mathrm{d}x(t) = f(x(t),x(t-\delta(t)),t,r(t))\mathrm{d}t+g(x(t),x(t-\delta(t)),t,r(t))\mathrm{d}W(t), \tag{6.10}$$

with initial data

$$\{x(t) : -\Delta \le t \le 0\} = \xi \in C([-\Delta, 0]; \mathbb{R}^d), \quad r(0) = i_0 \in \mathbb{S}.$$

Here, time delay $\delta : \mathbb{R}_+ \to [0, \Delta]$ is Borel measurable, drift coefficient f and diffusion coefficient g are locally Lipschitz continuous. The polynomial growth condition is still required.

Assumption 6.3. *Assume that there are non-negative constants H_1, H_2, H_3, H_4 and $p > 1$ such that for every $(x, y, t, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,*

$$|f(x, y, t, i)| \le H_1|x| + H_2|y| + H_3|x|^p + H_4|y|^p. \tag{6.11}$$

Remark 6.2. *Here we do not give the condition on g since combining Assumption 6.3 with the following Assumption 6.4 could yield that*

$$|g(x, y, t, i)|^2 \le C\left(|x|^2 + |y|^2 + |x|^{p+1} + |y|^{p+1}\right) \tag{6.12}$$

using the method in Remark 3.1. Its detailed form is useless throughout this chapter as we will give a new method to estimate $|x(t) - x(t_\tau)|$.

In order to highlight the well-applicability of Razumikhin method, we will only consider that the structures on every mode keep the same. Since the delay integral transform is invalid, for the existence-and-uniqueness of global solution, we could only give the classical Khasminskii condition, rather than the generalised one.

Assumption 6.4. *Assume that there exist a pair of positive constants $\hat{a}$ and $\hat{b}$ such that*

$$x^{\mathrm{T}} f(x, y, t, i) + \frac{3p-2}{2}|g(x, y, t, i)|^2 \le \hat{a}|x|^2 + \hat{b}|y|^2 \tag{6.13}$$

for any $(x, t, i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$.

By Theorem 7.13 in (Mao & Yuan 2006), it is easy to see that the hybrid SDDE (6.10) has a unique global solution $x(t)$ such that for any $t > 0$

$$\sup_{-\Delta \le s \le t} E|x(s)|^{3p-1} < \infty. \tag{6.14}$$

For the purpose of control design, the following assumption is still wanted, which is also the classical Khasminskii condition owing to the absence of high-order delay term $|y|^{p+1}$.

Assumption 6.5. *For each $i \in \mathbb{S}$, assume that there are constants $\gamma_i \ge 0$, $\bar{\gamma}_i \ge 0$, $b_i \ge 0$, $\bar{b}_i \ge 0$, $c_i > 0$, $\bar{c}_i > 0$ such that for every $(x, y, t) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+$*

$$\begin{cases} x^{\mathrm{T}} f(x, y, t, i) + \dfrac{1}{2}|g(x, y, t, i)|^2 \le \gamma_i |x|^2 + b_i |y|^2 - c_i |x|^{p+1}, \\ x^{\mathrm{T}} f(x, y, t, i) + \dfrac{p}{2}|g(x, y, t, i)|^2 \le \bar{\gamma}_i |x|^2 + \bar{b}_i |y|^2 - \bar{c}_i |x|^{p+1}. \end{cases} \tag{6.15}$$

Before going to the control problem, let us make some comments on Assumptions 6.4 and 6.5.

Remark 6.3. *Recalling the corresponding parts in Chapter 3, readers might find that these two assumptions are stronger than Assumptions 3.2 and 3.3. In Assumption 6.4, the moment has been changed from $p+1$ to $3p-1$, while in Assumption 6.5, the traditional condition (3.3) has came back, namely, the second inequality of (6.15). We point out that they are both due to the application of Razumikhin-type Theorem 6.1. The former is needed for the right-continuity of $EV(x(t), t, r(t))$, $E\mathcal{L}V(x_t, t, r(t))$ in Assumption 6.1, and the latter is used to compare $E\mathcal{L}V(x_t, t, r(t))$ with $EV(x_t(0), t, r(t))$ in Assumption 6.2. The detailed theoretical explanation will be given in Theorem 6.2 and subsequent Remark 6.6.*

6.3.2 Control design

If the hybrid SDDE (5.1) is unstable, we want to design the discrete-time state feedback control to make our controlled system

$$\begin{aligned} \mathrm{d}x(t) = &\Big(f(x(t), x(t-\delta(t)), t, r(t)) + u(x(t_\tau), t, r(t)) \Big)\mathrm{d}t \\ &+ g(x(t), x(t-\delta(t)), t, r(t))\mathrm{d}W(t) \end{aligned} \tag{6.16}$$

become stable. We will give control rules one by one according to Assumption 6.5.

By condition (6.15), we have new decomposition scheme as

$$\begin{cases} (\gamma_i - \kappa_i)|x|^2 + b_i |y|^2 - \dfrac{c_i}{2}|x|^{p+1} + \left(\kappa_i |x|^2 - \dfrac{c_i}{2}|x|^{p+1} \right), \\ (\bar{\gamma}_i - \kappa_i)|x|^2 + \bar{b}_i |y|^2 - \dfrac{\bar{c}_i}{2}|x|^{p+1} + \left(\kappa_i |x|^2 - \dfrac{\bar{c}_i}{2}|x|^{p+1} \right). \end{cases} \tag{6.17}$$

As a result, the bounded control function should be modified as follows.

Rule 6.1. *Choose non-negative constants $\kappa_i (i \in \mathbb{S})$ to let*

$$A := -2\mathrm{diag}(\gamma_1 - \kappa_1, \cdots, \gamma_S - \kappa_S) - Q$$

be a non-singular M-matrix. Then for the i-th mode, set $R_i = \left(\frac{2\kappa_i}{c_i \wedge \bar{c}_i}\right)^{\frac{1}{p-1}}$ and

- *when $x \in B_{R_i}$, design $u(x,t,i)$ such that we can find a non-negative constant K_i such that for any $(x,y,t) \in B_{R_i} \times B_{R_i} \times \mathbb{R}_+$*
$$|u(x,t,i) - u(y,t,i)| \le K_i|x-y|, \qquad x^{\mathrm{T}}u(x,t,i) \le -\kappa_i |x|^2,$$
and moreover $u(0,t,i) = 0$ for all $t \in \mathbb{R}_+$;
- *when $x \in B_{2R_i} - B_{R_i}$, let $u(x,t,i) = u\left(\left(\frac{2R_i}{|x|} - 1\right)x, t, i\right)$ for all $t \in \mathbb{R}_+$;*
- *when $x \in B^c_{2R_i}$, let $u(x,t,i) = 0$ for all $t \in \mathbb{R}_+$.*

Under this rule, we could draw the similar conclusion for the controlled SDDE (6.16) from (6.14). But with a little more effort, we can have a better result.

Lemma 6.1. *Under Assumptions 6.3, 6.4, 6.5, let the control function $u(x,t,i)$ satisfy Rule 6.1, the solution of the controlled SDDE (6.16) has the property that*
$$E\left(\sup_{0\le s\le t} |x(s)|^{2p}\right) < \infty, \quad \forall t > 0. \tag{6.18}$$

Proof. It is easy to derive that for any $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$
$$|u(x,t,i) - u(y,t,i)| \le K_M|x-y| \tag{6.19}$$
with $K_M = \max_{i\in\mathbb{S}} K_i$, and naturally the linear growth condition
$$|u(x,t,i)| \le K_M|x|, \qquad \forall (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}. \tag{6.20}$$
Fix any time $t \ge 0$. Applying the Itô formula to $|x|^{2p}$, we see that for any $0 \le s \le t$
$$\begin{aligned}|x(s)|^{2p} =& |\xi(0)|^{2p} + \int_0^s 2p|x(v)|^{2p-2}x^{\mathrm{T}}(v)(f(x(v), x(v-\delta(v)), v, r(v)) \\ &+ u(x(v_\tau), v, r(v)))\mathrm{d}v + \int_0^s p|x(v)|^{2p-2}|g(x(v), x(v-\delta(v)), v, r(v))|^2\mathrm{d}v \\ &+ \int_0^s p(2p-2)|x(v)|^{2p-4}\left|x^{\mathrm{T}}(v)g(x(v), x(v-\delta(v)), v, r(v))\right|^2\mathrm{d}v \\ &+ \int_0^s 2p|x(v)|^{2p-2}x^{\mathrm{T}}(v)g(x(v), x(v-\delta(v)), v, r(v))\mathrm{d}W(v).\end{aligned}$$
From condition (6.13), for all $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$
$$x^{\mathrm{T}}f(x,y,t,i) + \frac{2p-1}{2}|g(x,y,t,i)|^2 \le \hat{a}|x|^2 + \hat{b}|y|^2.$$
Using this and (6.20) as well as the Young inequality, and then taking expectations on both sides, we get
$$E\left(\sup_{0\le s\le t}|x(s)|^{2p}\right)$$

$$
\begin{aligned}
&\leq |\xi(0)|^{2p} + E\int_0^t \left(2p\hat{\alpha} + (2p-2)\hat{b} + (2p-1)K_M\right)|x(v)|^{2p}\mathrm{d}v \\
&\quad + E\int_0^t \left(2\hat{b}|x(v-\delta(v))|^{2p} + K_M|x(v_\tau)|^{2p}\right)\mathrm{d}v + J(t),
\end{aligned} \tag{6.21}
$$

where

$$
J(t) = E\left(\sup_{0\leq s\leq t}\left|\int_0^s 2p|x(v)|^{2p-2}x^{\mathrm{T}}(v)g(x(v),x(v-\delta(v)),v,r(v))\mathrm{d}W(v)\right|\right).
$$

By the Burkholder-Davis-Gundy inequality (Theorem 2.11) , we compute

$$
\begin{aligned}
J(t) \leq& 3E\left(\int_0^t 4p^2|x(v)|^{4p-2}|g(x(v),x(v-\delta(v)),v,r(v))|^2\mathrm{d}v\right)^{\frac{1}{2}} \\
\leq& E\left(\sup_{0\leq s\leq t}|x(s)|^{2p}\int_0^t 36p^2|x(v)|^{2p-2}|g(x(v),x(v-\delta(v)),v,r(v))|^2\mathrm{d}v\right)^{\frac{1}{2}} \\
\leq& \frac{1}{2}E\left(\sup_{0\leq s\leq t}|x(s)|^{2p}\right) + 18p^2E\int_0^t |x(v)|^{2p-2}|g(x(v),x(v-\delta(v)),v,r(v))|^2\mathrm{d}v.
\end{aligned}
$$

Substituting this into (6.21) and using (6.12) gives that

$$
\begin{aligned}
E\left(\sup_{0\leq s\leq t}|x(s)|^{2p}\right) \leq& 2|\xi(0)|^{2p} + CE\int_0^t \Big(|x(v)|^{2p} + |x(v-\delta(v))|^{2p} + |x(v_\tau)|^{2p} \\
&+ |x(v)|^{3p-1} + |x(v-\delta(v))|^{3p-1}\Big)\mathrm{d}v,
\end{aligned} \tag{6.22}
$$

where C is a positive constant independent from t. Since $\sup_{0\leq s\leq t} E|x(s)|^{3p-1} < \infty$, we obtain that

$$
E\left(\sup_{0\leq s\leq t}|x(s)|^{2p}\right) \leq 2|\xi(0)|^{2p} + C\left(1 + \sup_{0\leq s\leq t} E|x(s)|^{3p-1}\right)t < \infty.
$$

This completes the proof. □

To deal with the time delay, the following rule is also required like Rule 5.2. Before that, let $(\eta_1, \cdots, \eta_S)^{\mathrm{T}} = A^{-1}(1, \cdots, 1)^{\mathrm{T}}$, and $a_i = \gamma_i - \kappa_i$, $\bar{a}_i = \bar{\gamma}_i - \kappa_i$. Denote by $\eta_M = \max_{i\in\mathbb{S}}\eta_i$, $\eta_m = \min_{i\in\mathbb{S}}\eta_i$, $a_M = \max_{i\in\mathbb{S}} a_i$, $\bar{a}_M = \max_{i\in\mathbb{S}}\bar{a}_i$, $b_M = \max_{i\in\mathbb{S}} b_i$, $\bar{b}_M = \max_{i\in\mathbb{S}}\bar{b}_i$, $\bar{c}_m = \min_{i\in\mathbb{S}}\bar{c}_i$, $b_\eta = \max_{i\in\mathbb{S}}(b_i\eta_i)$, $c_\eta = \min_{i\in\mathbb{S}}(c_i\eta_i)$.

Rule 6.2. *Ensure that κ_i chosen in Rule 6.1 also makes D_η positive, where*

$$
D_\eta = \frac{1}{\eta_M} - \frac{2b_\eta}{\eta_m} - 2\bar{b}_M.
$$

We give the following remark to show that we can also find the appropriate κ_i.

Remark 6.4. *Choose a sufficiently large number κ such that $\kappa > b_M + \bar{b}_M$ and $-2\mathrm{diag}(\gamma_1 - \kappa, \cdots, \gamma_S - \kappa) - Q$ is a non-singular M-matrix. Then let $\kappa_i = \kappa$ for* *each $i \in \mathbb{S}$. In this case, $\eta_i \approx \frac{1}{2\kappa}$, which implies that*

$$D_\eta \approx 2\kappa - 2b_M - 2\bar{b}_M > 0.$$

Here readers might find there is not any restriction on b_i and $\bar{b}_i$ unlike $D_b > 0$ required in Chapter 5, because we would use Razumikhin method to eliminate time delay rather than delay integral transform method.

Next we give a new method to determine the value of observation duration.

Rule 6.3. *Let $\underline{\varepsilon} = \frac{1}{c_\eta}\left((p+1)((\bar{a}_M \vee 0) + \bar{b}_M) + \frac{2b_\eta}{\eta_m}\right)$ and $\mathcal{E} = (\underline{\varepsilon}, \infty) \times (0, D_\eta)$. Then τ should work smaller than $\tau^* := \max\{\varphi(\varepsilon_1, \varepsilon_2) : (\varepsilon_1, \varepsilon_2) \in \mathcal{E}\}$, where*

$$\varphi(\varepsilon_1, \varepsilon_2) = \frac{1}{\varphi_2(\varepsilon_1, \varepsilon_2)}\big(c_\eta(\varepsilon_1 - \underline{\varepsilon}) \wedge (D_\eta - \varepsilon_2)\big)$$

with $\varphi_2(\varepsilon_1, \varepsilon_2) = \left(\frac{K_M^2 \eta_M \varepsilon_1}{\varepsilon_2} + \frac{(p+1)K_M^2}{2\bar{c}_m}\right)\left(\frac{2(a_M \vee 0) + 2b_M + 2H_1 + 2H_2 + 6K_M}{\eta_m \varepsilon_1} + 2H_3 + 2H_4\right)$.

It is easy to see that φ is a positive continuous function on $\mathcal{E}$. When $(\varepsilon_1, \varepsilon_2)$ tends to the boundary of $\mathcal{E}$, $\varphi(\varepsilon_1, \varepsilon_2)$ goes to zero. We are hence able to find $(\varepsilon_1^*, \varepsilon_2^*) \in \mathcal{E}$ such that $\tau^* = \varphi(\varepsilon_1^*, \varepsilon_2^*) = \max_{\mathcal{E}} \varphi(\varepsilon_1, \varepsilon_2)$. For convenience, denote by $\varphi_2^* = \varphi_2(\varepsilon_1^*, \varepsilon_2^*)$.

Remark 6.5. *It is worthy to point out that we set an additional parameter ε_1 here compared with the rules to determinate τ^* in previous chapters. In the subsequent stability analysis, we will see that ε_1^* is used to balance $\bar{a}_i$ and $\bar{b}_i$, since we do not require $-(p+1)diag(\bar{a}_1, \cdots, \bar{a}_S) - Q$ also to be a non-singular M-matrix as other papers (e.g. (Fei et al. 2020, Mei et al. 2020, Shi et al. 2022)).*

6.4 Stabilisation results

Let $h = \Delta + \tau$. Define, for $(\phi, t, i) \in C\left([-h, 0]; \mathbb{R}^d\right) \times \mathbb{R}_+ \times \mathbb{S}$,

$$F(\phi, t, i) = f(\phi(0), \phi(-\delta(t)), t, i) + u(\phi(-\zeta(t)), t, i),$$
$$G(\phi, t, i) = g(\phi(0), \phi(-\delta(t)), t, i),$$

where $\zeta(t) := t - t_\tau$ takes values in $[0, \tau]$. Then our controlled SDDE (6.16) becomesthe the hybrid SFDE (6.1) on $t \geq 0$ with initial data $\hat{\xi}(\theta) = \xi(\theta)$ for $\theta \in [-\Delta, 0]$ and $\hat{\xi}(\theta) = \xi(-\Delta)$ for $\theta \in [-\Delta - h, -\Delta]$.

6.4.1 Lyapunov function

The Lyapunov function $V(x,t,i)$ required in Theorem 6.1 will be of the form

$$V(x,t,i) = \varepsilon_1^* \eta_i |x|^2 + |x|^{p+1}. \tag{6.23}$$

Define two functions $L_1V, L_2V : \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}$ by

$$\begin{aligned} L_1V(x,y,t,i) =& 2\varepsilon_1^*\eta_i \left(x^{\mathrm{T}}(f(x,y,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,y,t,i)|^2 \right) \\ &+ (p+1)|x|^{p-1} \left(x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)) + \frac{p}{2}|g(x,y,t,i)|^2 \right) \\ &+ \sum_{j=1}^{S} q_{ij}\varepsilon_1^*\eta_j|x|^2 \end{aligned}$$

and

$$L_2V(x,z,t,i) = \left(2\varepsilon_1^*\eta_i x^{\mathrm{T}} + (p+1)|x|^{p-1}x^{\mathrm{T}}\right)(u(z,t,i) - u(x,t,i)).$$

Then the operator $\mathcal{L}V$ defined in (6.2) with respect to (6.1) could be rewritten as

$$\mathcal{L}V(\phi,t,i) = L_1V(\phi(0),\phi(-\delta(t)),t,i) + L_2V(\phi(0),\phi(-\zeta(t)),t,i).$$

Using decomposition (6.17), it is easy to derive that

$$\begin{cases} x^{\mathrm{T}}(f(x,y,t,i) + u(x,t,i)) + \frac{1}{2}|g(x,y,t,i)|^2 \le a_i|x|^2 + b_i|y|^2 - \frac{c_i}{2}|x|^{p+1}, \\ x^{\mathrm{T}}(f(x,y,t,i) + u(x,t,i)) + \frac{p}{2}|g(x,y,t,i)|^2 \le \bar{a}_i|x|^2 + \bar{b}_i|y|^2 - \frac{\bar{c}_i}{2}|x|^{p+1}, \end{cases} \tag{6.24}$$

which gives the following estimation of $L_1V(x,y,t,i)$.

Lemma 6.2. *Let all the conditions in Lemma 6.1 hold. Then for any* $(x,y,t,i) \in \mathbb{R}^d \times \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,

$$\begin{aligned} L_1V(x,y,t,i) \le & -\varepsilon_1^*|x|^2 + 2b_\eta\varepsilon_1^*|y|^2 - (c_\eta\varepsilon_1^* - (p+1)\bar{a}_M - (p-1)\bar{b}_M)|x|^{p+1} \\ &+ 2\bar{b}_M|y|^{p+1} - \frac{(p+1)\bar{c}_m}{2}|x|^{2p}. \end{aligned} \tag{6.25}$$

Since there is an additional term $|x|^{p+1}$ in our Lyapunov function than (3.13), we will give a new method to estimate $L_2V(x,z,t,i)$.

Lemma 6.3. *Under the same conditions in Lemma 6.1, we have*

$$\begin{aligned} EL_2V(x(t),x(t-\zeta(t)),t,r(t)) \le& \eta_M\varepsilon_1^*\varepsilon_2^*E|x(t)|^2 + \frac{(p+1)\bar{c}_m}{2}E|x(t)|^{2p} \\ &+ \varphi_2^*\tau \sup_{-h\le\theta\le 0} EV(x(t+\theta),t+\theta,r(t+\theta)). \end{aligned} \tag{6.26}$$

Proof. For any $t \geq 0$, there is some integer $k \geq 0$ such that $k\tau \leq t < (k+1)\tau$. Then it is easy to see that $v - \zeta(v) = k\tau$ for $v \in [k\tau, t]$. Using the Hölder inequality and (6.19) to split L_2V as

$$\begin{aligned}
&EL_2V(x(t), x(k\tau), t, r(t)) \\
\leq &\eta_M \varepsilon_1^* \left(\varepsilon_2^* E|x(t)|^2 + \frac{K_M^2}{\varepsilon_2^*} \right) E|x(t) - x(k\tau)|^2 \\
&+ \frac{(p+1)\bar{c}_m}{2} E|x(t)|^{2p} + \frac{(p+1)K_M^2}{2\bar{c}_m} E|x(t) - x(k\tau)|^2. \qquad (6.27)
\end{aligned}$$

It is then significant to estimate $|x(t) - x(k\tau)|^2$. Actually $x(s) - x(k\tau)$ is an Itô process on $[k\tau, t]$ with

$$\begin{aligned}
x(s) - x(k\tau) = &\int_{k\tau}^{s} (f(x(v), x(v - \delta(v), v, r(v)) + u(x(k\tau), v, r(v)))\mathrm{d}v \\
&+ \int_{k\tau}^{s} g(x(v), x(v - \delta(v), v, r(v))\mathrm{d}W(v).
\end{aligned}$$

Applying the Itô formula to the above process yields that

$$\begin{aligned}
&E|x(t) - x(k\tau)|^2 \\
=&E \int_{k\tau}^{t} \Big(2(x(v) - x(k\tau))^{\mathrm{T}} (f(x(v), x(v - \delta(v), v, r(v)) + u(x(k\tau), v, r(v))) \\
&+ |g(x(v), x(v - \delta(v), v, r(v))|^2 \Big) \mathrm{d}v \\
=&E \int_{k\tau}^{t} \Big(2x^{\mathrm{T}}(v)(f(x(v), x(v - \delta(v), v, r(v)) + u(x(v), v, r(v))) \\
&+ |g(x(v), x(v - \delta(v), v, r(v))|^2 \Big) \mathrm{d}v \\
&- E \int_{k\tau}^{t} 2x^{\mathrm{T}}(k\tau) f(x(v), x(v - \delta(v), v, r(v))\mathrm{d}v - E \int_{k\tau}^{t} 2x^{\mathrm{T}}(v) u(x(v), v, r(v))\mathrm{d}v \\
&+ E \int_{k\tau}^{t} 2x^{\mathrm{T}}(v) u(x(k\tau), v, r(v))\mathrm{d}v - E \int_{k\tau}^{t} 2x^{\mathrm{T}}(k\tau) u(x(k\tau), v, r(v))\mathrm{d}v.
\end{aligned}$$

Making use of (6.11), (6.20) and (6.24), we further derive that

$$\begin{aligned}
&E|x(t) - x(k\tau)|^2 \\
\leq &E \int_{k\tau}^{t} \left((2a_M + H_1 + 3K_M)|x(v)|^2 + \frac{2pH_3}{p+1} |x(v)|^{p+1} \right) \mathrm{d}v \\
&+ E \int_{k\tau}^{t} \left((2b_M + H_2)|x(v - \delta(v))|^2 + \frac{2pH_4}{p+1} |x(v - \delta(v))|^{p+1} \right) \mathrm{d}v \\
&+ E \int_{k\tau}^{t} \left((H_1 + H_2 + 3K_M)|x(k\tau)|^2 + \frac{2(H_3 + H_4)}{p+1} |x(k\tau)|^{p+1} \right) \mathrm{d}v
\end{aligned}$$

$$\leq \left(\frac{2(a_M \vee 0) + H_1 + 3K_M}{\varepsilon_1^* \eta_m} + \frac{2pH_3}{p+1} \right) \int_{k\tau}^t EV(x(v), v, r(v)) \mathrm{d}v$$
$$+ \left(\frac{2b_M + H_2}{\varepsilon_1^* \eta_m} + \frac{2pH_4}{p+1} \right) \int_{k\tau}^t EV(x(v-\delta(v)), v-\delta(v), r(v-\delta(v))) \mathrm{d}v$$
$$+ \left(\frac{H_1 + H_2 + 3K_M}{\varepsilon_1^* \eta_m} + \frac{2(H_3+H_4)}{p+1} \right) \int_{k\tau}^t EV(x(k\tau), k\tau, r(k\tau)) \mathrm{d}v$$
$$\leq \left(\frac{2(a_M \vee 0) + 2b_M + 2H_1 + 2H_2 + 6K_M}{\eta_m \varepsilon_1^*} + 2H_3 + 2H_4 \right) \tau$$
$$\times \sup_{-h \leq \theta \leq 0} EV(x(t+\theta), t+\theta, r(t+\theta)). \tag{6.28}$$

Substituting this into (6.27) gives that

$$EL_2 V(x(t), x(k\tau), t, r(t)) \leq \eta_M \varepsilon_1^* \varepsilon_2^* E|x(t)|^2 + \frac{(p+1)\bar{c}_m}{2} E|x(t)|^{2p}$$
$$+ \varphi_2^* \tau \sup_{-h \leq \theta \leq 0} EV(x(t+\theta), t+\theta, r(t+\theta)).$$

The proof is therefore complete. □

6.4.2 Exponential stabilisation

Next we could give our stabilisation results by Razumikhin-type theorem.

Theorem 6.2. *Under Assumptions 6.3, 6.4, 6.5, let the control function $u(x,t,i)$ satisfy Rules 6.1, 6.2, and the observation duration τ meet Rule 6.3. Then the solution of the controlled SDDE (6.16) obeys that*

$$\limsup_{t \to \infty} \frac{1}{t} \log E|x(t)|^{p+1} < 0. \tag{6.29}$$

Proof. To apply Theorem 6.1, we need to check Assumptions 6.1 and 6.2.

Step 1. For any $t \geq 0$, it is easy to see that

$$\sup_{0 \leq s \leq t} EV(x(s), s, r(s)) \leq \sup_{0 \leq s \leq t} E\left(\eta_M |x(t)|^2 + |x(t)|^{p+1} \right)$$
$$\leq C \left(1 + \sup_{0 \leq s \leq t} E|x(t)|^{p+1} \right) < \infty.$$

By (6.11), (6.12) and (6.20), compute

$$|\mathcal{L}V(x_t, t, r(t))| \leq \left(2\varepsilon_1^* \eta_M + (p+1)|x(t)|^{p-1} \right) \Big(|x(t)||f(x(t), x(t-\delta(t)), t, r(t))|$$
$$+ |x(t)||u(x(t-\zeta(t)), t, r(t))| + \frac{p}{2}|g(x(t), x(t-\delta(t)), t, r(t))|^2 \Big)$$
$$+ S\left(\max_{1 \leq i,j \leq S} |q_{ij}| \right) \eta_M |x(t)|^2$$

$$\leq C\Big(|x(t)|^2 + |x(t)|^{p+1} + |x(t)|^{2p} + |x(t-\delta(t))|^2 + |x(t-\delta(t))|^{p+1} + |x(t-\delta(t))|^{2p} + |x(t-\zeta(t))|^2\Big),$$

which implies that

$$\sup_{0\leq s\leq t} E|\mathcal{L}V(x_t,t,r(t))| \leq C \sup_{0\leq s\leq t} E\left(1 + |x(t)|^{2p} + |x(t-\delta(t))|^{2p} + |x(t-\zeta(t))|^{2p}\right)$$
$$\leq C\left(1 + \sup_{-h\leq s\leq t} E|x(t)|^{2p}\right) < \infty.$$

From the definition, it is clear that $V(x(t),t,r(t)$ and $\mathcal{L}V(x_t,t,r(t))$ are right-continuous on $t \geq 0$. For any sufficiently small $\epsilon > 0$, we have

$$\sup_{t\leq s\leq t+\epsilon} |V(x(s),s,r(s))| \leq C\left(1 + \sup_{0\leq s\leq t+\epsilon} |x(s)|^{p+1}\right)$$

and

$$\sup_{t\leq s\leq t+\epsilon} |\mathcal{L}V(x_s,s,r(s))| \leq C\left(1 + \sup_{0\leq s\leq t+\epsilon} |x(s)|^{2p}\right).$$

Since in Lemma 6.1, we have shown that

$$E\left(\sup_{0\leq s\leq t+\epsilon} |x(s)|^{2p}\right) < \infty,$$

using the Hölder inequality and the dominated convergence theorem shows that

$$\lim_{s\to t^+} EV(x(s),s,r(s)) = E\left(\lim_{s\to t^+} V(x(s),s,r(s))\right) = EV(x(t),t,r(t))$$

and

$$\lim_{s\to t^+} E\mathcal{L}V(x_s,s,r(s)) = E\left(\lim_{s\to t^+} \mathcal{L}V(x_s,s,r(s))\right) = E\mathcal{L}V(x_t,t,r(t)).$$

As a result, $EV(x(t),t,r(t)$ and $E\mathcal{L}V(x_t,t,r(t))$ are right-continuous. Up to now, all the conditions in Assumption 6.1 are fulfilled.

Step 2. Combing the estimations of L_1V in (6.25) and L_2V in (6.25), we derive that

$$\begin{aligned}
&E\mathcal{L}V(x_t,t,r(t))\\
\leq& -(1-\eta_M\varepsilon_2^*)\varepsilon_1^* E|x(t)|^2 + 2b_\eta\varepsilon_1^* E|x(t-\delta(t))|^2\\
&-(c_\eta\varepsilon_1^* - (p+1)\bar{a}_M - (p-1)\bar{b}_M)E|x(t)|^{p+1} + 2\bar{b}_M|x(t-\delta(t))|^{p+1}\\
&+\varphi_2^*\tau \sup_{-h\leq\theta\leq 0} EV(x(t+\theta),t+\theta,r(t+\theta))\\
\leq& -\left(\frac{1}{\eta_M} - \varepsilon_2^*\right) E\left(\varepsilon_1^*\eta_{r(t)}|x(t)|^2\right) - (c_\eta\varepsilon_1^* - (p+1)\bar{a}_M - (p-1)\bar{b}_M)E|x(t)|^{p+1}
\end{aligned}$$

$$+\left(\frac{2b_\eta}{\eta_m}+2\bar{b}_M+\varphi_2^*\tau\right)\sup_{-h\le\theta\le 0}EV(x(t+\theta),t+\theta,r(t+\theta)).$$

From Rule 6.3, we could find that

$$\frac{1}{\eta_M}-\varepsilon_2^*-\frac{2b_\eta}{\eta_m}-2\bar{b}_M=D_\eta-\varepsilon_2^*>\varphi_2^*\tau$$

and

$$c_\eta\varepsilon_1^*-(p+1)\bar{a}_M-(p-1)\bar{b}_M-\frac{2b_\eta}{\eta_m}-2\bar{b}_M=c_\eta(\varepsilon_1^*-\underline{\varepsilon})>\varphi_2^*\tau.$$

Therefore, there is a constant $q>1$ such that

$$\begin{cases}\dfrac{1}{\eta_M}-\varepsilon_2^*>q\left(\dfrac{2b_\eta}{\eta_m}+2\bar{b}_M+\varphi_2^*\tau\right),\\ c_\eta\varepsilon_1^*-(p+1)\bar{a}_M-(p-1)\bar{b}_M>q\left(\dfrac{2b_\eta}{\eta_m}+2\bar{b}_M+\varphi_2^*\tau\right).\end{cases}$$

If for some t, $\sup_{-h\le\theta\le 0}EV(x(t+\theta),t+\theta,r(t+\theta))\le qEV(x(t),t,r(t))$, then

$$\begin{aligned}&E\mathcal{L}V(x_t,t,r(t))\\ \le&-\left(\left(\frac{1}{\eta_M}-\varepsilon_2^*\right)\wedge\left(c_\eta\varepsilon_1^*-(p+1)\bar{a}_M-(p-1)\bar{b}_M\right)\right)EV(x(t),t,r(t))\\ &+q\left(\frac{2b_\eta}{\eta_m}+2\bar{b}_M+\varphi_2^*\tau\right)EV(x(t),t,r(t)).\end{aligned}$$

Consequently, condition (6.3) holds with $\lambda_1=0$ and

$$\lambda_2=\left(\left(\frac{1}{\eta_M}-\varepsilon_2^*\right)\wedge\left(c_\eta\varepsilon_1^*-(p+1)\bar{a}_M-(p-1)\bar{b}_M\right)\right)-q\left(\frac{2b_\eta}{\eta_m}+2\bar{b}_M+\varphi_2^*\tau\right).$$

Finally, the required assertion (6.29) follows from Theorem 6.1. □

Remark 6.6. *For hybrid SDDEs meeting the linear growth condition (e.g. (Mao & Yuan 2006)), it is very easy to prove the right-continuity of EV and $E\mathcal{L}V$ since we always have $E\left(\sup_{0\le s\le t}|x(s)|^r\right)<\infty$ for any $t\ge 0$ at any positive order r. However, in the highly nonlinear ones, the first step in the previous proof tells us that we need to impose extra assumptions to guarantee this, see Lemma 6.1. This is the main reason why we require $3p-1$ in Assumption 6.4.*

Theorem 6.2 is actually stronger than Theorem 5.2, where the moment is $p+1$ rather than mean square. This is the result that we give an additional requirement in Assumption 6.5, namely, $x^{\mathrm{T}}f(x,y,t,i)+\frac{p}{2}|g(x,y,t,i)|^2\le\bar{\gamma}_i|x|^2+\bar{b}_i|y|^2-\bar{c}_i|x|^{p+1}$. Owing to this condition, we could also obtain the almost surely exponential stability.

Theorem 6.3. *Let all the conditions in Theorem 6.2 hold. Then the controlled SDDE (6.16) has the property that*

$$\limsup_{t\to\infty}\frac{1}{t}\log|x(t)|<\infty \qquad a.s. \tag{6.30}$$

Proof. By Theorem 6.2, there is a constant λ such that

$$E|x(t)|^{p+1}\le Ce^{-\lambda t}, \qquad \forall t\ge h.$$

Let $k=2,3,\cdots$. For any $(k-1)h\le t\le kh$,

$$\begin{aligned}
&E\left(\sup_{(k-1)h\le t\le kh}|x(t)|\right)\\
\le&E|x((k-1)h)|+E\int_{(k-1)h}^{kh}|f(x(v),x(v-\delta(v)),v,r(v))|\mathrm{d}v\\
&+E\int_{(k-1)h}^{kh}|u(x(v-\zeta(v)),v,r(v))|\mathrm{d}v\\
&+E\left(\sup_{(k-1)h\le t\le kh}\left|\int_{(k-1)h}^{t}g(x(v),x(v-\delta(v)),v,r(v))\mathrm{d}W(v)\right|\right).
\end{aligned}$$

Using the similar way to show (6.22), we could get

$$\begin{aligned}
E\left(\sup_{(k-1)h\le t\le kh}|x(t)|\right)\le&E|x((k-1)h)|+C\int_{(k-1)h}^{kh}\Big(1+E|x(v)|^{p+1}\\
&+E|x(v-\delta(v))|^{p+1}+E|x(v-\zeta(v))|^{p+1}\Big)\mathrm{d}v\\
\le&Ce^{-\frac{\lambda}{p+1}(k-1)h}.
\end{aligned}$$

By the Chebyshev inequality,

$$\sum_{k=1}^{\infty}P\left(\sup_{(k-1)h\le t\le kh}|x(t)|>e^{-\frac{\lambda}{2(p+1)}(k-1)h}\right)\le\sum_{k=1}^{\infty}C_1e^{-\frac{\lambda}{2(p+1)}(k-1)h}<\infty.$$

The well-known Borel-Cantelli lemma then shows that for almost all $\omega\in\Omega$, there is positive integer $k_0=k_0(\omega)$ such that

$$\sup_{(k-1)h\le t\le kh}|x(t)|\le e^{-\frac{\lambda}{2(p+1)}(k-1)h}, \qquad k\ge k_0.$$

Hence for almost all ω,

$$\frac{1}{t}\log|x(t)|\le-\frac{\lambda}{2(p+1)}\frac{(k-1)h}{(k+1)h},$$

for $t \in [(k-1)h, kh]$, $k \geq k_0$. Letting $t \to \infty$ shows that

$$\limsup_{t\to\infty} \frac{1}{t}\log|x(t)| \leq -\frac{\lambda}{2(p+1)} < 0 \quad a.s.,$$

which is the required assertion (6.30). This completes the proof. □

6.5 Application to van der Pol–Duffing oscillator

Consider the modified stochastic van der Pol–Duffing oscillator studied in (Liu, Liu & Li 2021) described by

$$\begin{cases} \mathrm{d}x_1(t) = \Big(-(1+\hat{\lambda}_{r(t)})x_1(t) + \hat{B}_{r(t)}(x_2(t)-x_1(t))^3 + \hat{\lambda}_{r(t)}x_2(t) \\ \qquad\qquad - \hat{A}_{r(t)}x_1^3(t)\Big)\mathrm{d}t + \hat{\delta}_{r(t)}x_1(t-\delta(t))\mathrm{d}W_1(t) \\ \mathrm{d}x_2(t) = \Big(\hat{\lambda}_{r(t)}x_1(t) - \hat{\rho}_{r(t)}x_3(t) - \hat{B}_{r(t)}(x_2(t)-x_1(t))^3 - (\hat{\lambda}_{r(t)}+1)x_2(t) \\ \qquad\qquad - \hat{C}_{r(t)}x_2^3(t)\Big)\mathrm{d}t + \hat{\delta}_{r(t)}x_2(t-\delta(t))\mathrm{d}W_2(t) \\ \mathrm{d}x_3(t) = \Big(x_2(t) + \hat{\rho}_{r(t)}x_3(t) - \hat{D}_{r(t)}x_3^3(t)\Big)\,\mathrm{d}t + \hat{\delta}_{r(t)}x_3(t-\delta(t))\mathrm{d}W_3(t) \end{cases}$$

operating in two modes, following the Markov chain $r(t)$ with transition rate matrix $Q = \begin{pmatrix} -1 & 1 \\ 6 & -6 \end{pmatrix}$. Here $W_1(t)$, $W_2(t)$, $W_3(t)$ are independent scalar Brownian motions, $\delta(t)$ is the time delay occurring when the perturbation reaches the oscillator, which could be described by a discrete-time function

$$\delta(t) = t - [t/0.01]0.01.$$

Other parameters are given by

$$\begin{aligned} &\hat{\lambda}_1 = 0.5, \quad \hat{\rho}_1 = 0.2, \quad \hat{A}_1 = 1, \quad \hat{B}_1 = 0.2, \quad \hat{C}_1 = 0.8, \quad \hat{D}_1 = 0.8, \quad \hat{\delta}_1 = 0.3, \\ &\hat{\lambda}_2 = 0.3, \quad \hat{\rho}_2 = 0.1, \quad \hat{A}_2 = 0.8, \quad \hat{B}_2 = 0.4, \quad \hat{C}_2 = 1, \quad \hat{D}_2 = 1.2, \quad \hat{\delta}_2 = 0.2. \end{aligned}$$

Letting $x(t) = (x_1(t), x_2(t), x_3(t))^{\mathrm{T}}$ and $W(t) = (W_1(t), W_2(t), W_3(t))^{\mathrm{T}}$, we can rewrite the oscillator as

$$\mathrm{d}x(t) = f(x(t), r(t))\mathrm{d}t + g(x(t-\delta(t)), r(t))\mathrm{d}W(t), \tag{6.31}$$

where $g(y,i) = \hat{\delta}_i \mathrm{diag}(y_1, y_2, y_3)$ and

$$f(x,i) = \begin{pmatrix} -(1+\hat{\lambda}_i)x_1 + \hat{B}_i(x_2-x_1)^3 + \hat{\lambda}_i x_2 - \hat{A}_i x_1^3 \\ \hat{\lambda}_i x_1 - \hat{\rho}_i x_3 - \hat{B}_i(x_2-x_1)^3 - (\hat{\lambda}_i+1)x_2 - \hat{C}_i x_2^3 \\ x_2 + \hat{\rho}_i x_3 - \hat{D}_i x_3^3 \end{pmatrix}.$$

It is easy to check that Assumption 6.3 holds with $H_1 = 3.6056$, $H_2 = 0$, $H_3 = 3.7736$, $H_4 = 0$, $p = 3$. Then, compute

$$\begin{aligned} x^{\mathrm{T}} f(x,i) \leq & - x_2^2 - \hat{\rho}_i x_2 x_3 + \rho_i x_3^2 - \hat{A}_i x_1^4 - \hat{C}_i x_2^4 - \hat{D}_1 x_3^4 \\ \leq & \left(\frac{\hat{\rho}_i^2}{4} + \hat{\rho}_i \right) |x|^2 - \frac{1}{3}(\hat{A}_i \wedge \hat{C}_i \wedge \hat{D}_i)|x|^4 \end{aligned}$$

and $|g(y,i)|^2 \leq \hat{\delta}_i^2 |y|^2$. Hence, Assumption 6.4 holds with $\hat{a} = 0.21$ and $\hat{b} = 0.36$. Through computer simulation, we find that the oscillator (6.31) is unstable (see Fig. 6.1).

It is necessary to impose a discrete-time state feedback control $u(x(t_\tau), r(t))$ to make the controlled oscillator

$$\mathrm{d}x(t) = \Big(f(x(t), r(t)) + u(x(t_\tau), r(t)) \Big) \mathrm{d}t + g(x(t - \delta(t)), r(t)) \mathrm{d}W(t) \tag{6.32}$$

stable. Before the control design, we check that Assumption 6.5 is satisfied with $\gamma_1 = 0.21$, $\gamma_2 = 0.1025$, $b_1 = 0.045$, $b_2 = 0.02$, $c_1 = 0.2667$, $c_2 = 0.2667$, $\bar{\gamma}_1 = 0.21$, $\bar{\gamma}_2 = 0.1025$, $\bar{b}_1 = 0.135$, $\bar{b}_2 = 0.06$, $\bar{c}_1 = 0.2667$, $\bar{c}_2 = 0.2667$. Then we choose $\kappa_1 = 0.8$ and $\kappa_2 = 0.5$ to let $A = \begin{pmatrix} 2.18 & -1 \\ -6 & 6.795 \end{pmatrix}$ being a non-singular M-matrix. Design the control function as follows: for any $x \in \mathbb{R}^3$,

$$u(x,i) = \begin{cases} -\kappa_i x, & \text{if } |x| \leq R_i, \\ -\kappa_i \left(\dfrac{2R_2}{|x|} - 1 \right) x, & \text{if } R_i < |x| \leq 2R_i, \\ 0, & \text{if } |x| > 2R_i, \end{cases}$$

with $R_1 = 2.4495$, $R_2 = 1.9365$. Obviously, $K_1 = 0.8$, $K_2 = 0.5$. We then see that $(\eta_1, \eta_2) = (0.8845, 0.9282)$ and $D_\eta = 0.7174$, which implies that Rule 6.2 holds. Using the method given in Rule 6.3, we derive that $\tau^* = 0.002841$. Letting $\tau < 0.002841$, by Theorems 6.2 and 6.3, we conclude that the controlled oscillator (6.32) is exponentially stable in the sense of L^4 and almost surely. We perform a simulation with $\tau = 0.001$ to support our theoretical results in Fig. 6.1.

6.6 Summary

In this chapter, we firstly develop a Razumikhin-type theorem to study the asymptotic boundedness and moment exponential stability of hybrid SFDEs. Then we apply this generalised theory to our discrete-state-feedback stabilisation prob-

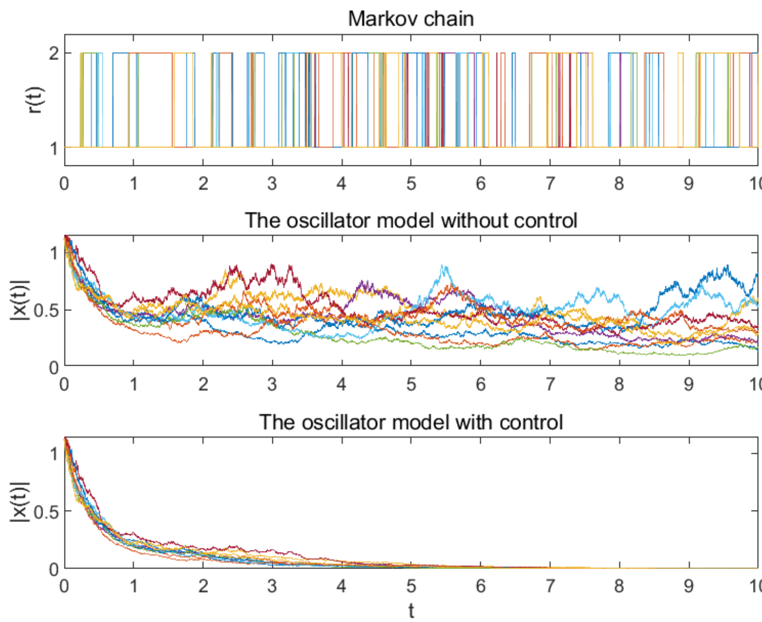

Figure 6.1: Ten sample paths of the Markov chain, the oscillator (6.31) and the controlled oscillator (6.32), using the truncated Euler-Maruyama method with time step size 10^{-4}. For each path, the initial data is fixed given by $\xi = (1 + \cos(t), 0.5 + \sin(t), 0)^{\mathrm{T}}$ for $t \in [-0.01, 0]$ and $i_0 = 1$.

lem of hybrid SDDEs with more general time delays in the sense of $(p+1)$-th moment exponential stability and almost sure exponential stability. Herein, there is only little restrictions on the time delay. Besides, compared with the Lyapunov functional method, Razumikhin technique could avoid the difficulty of constructing appropriate Lyapunov functionals and much complicated analysis.

Chapter 7

Intermittent control strategy to stabilisation of hybrid systems by generalised Razumikhin technique

7.1 Introduction

In Chapter 6, we have seen the effectiveness of Razumikhin method to the stabilisation problem of delay systems. Then in this part, we will present its another application, to the intermittent control strategy.

Currently, most of the discrete-state-feedback stabilisation results (see, e.g. (Li & Kou 2017, Fei et al. 2020, Li et al. 2018, Shi et al. 2022) and the precious chapters) are based on the controller imposed to the system for all the time without any rest. This undoubtedly will shorten the life of our controller. Therefore, a more practical technique is to let the controller working intermittently, where we divide the whole time periodically, and each period is consisted of working time and rest time. Then, the controller becomes

$$u(x(t_\tau), t, r(t)) \sum_{k=0}^{\infty} \mathbb{I}_{[kT, kT+\delta_\mu T)}(t),$$

where $T > 0$ is the control period, $\delta_\mu T$ is the working width with strength $\delta_\mu \in (0, 1)$. In theory, we do not need to know the value of T, which should be fixed according to practical needs. But in application, we always let $T = \tau$ to make the control design be easily implemented.

Due to its efficiency in reducing the control cost, intermittent control recently has drawn abundant interest (see, e.g. (Jiang, Hu, Lu, Mao & Mao 2021, Mao, You, Jiang & Mao 2023, Li, Feng & Liao 2007, Xia & Cao 2009, Zhang, Deng, Peng & Xie 2018, Wang, Hong & Su 2018, Liu et al. 2021, Liu & Wu 2020, Chen, Wang & Wu 2022, He, Ahn & Shi 2020)). But unfortunately, due to the difficulty in dealing with two discrete strategies, $x(t_\tau)$ and $I(t) := \sum_{k=0}^{\infty} \mathbb{I}_{[kT, kT+\delta_\mu T)}(t)$, at the same time, there are only a few results (e.g. (Jiang et al. 2021, Mao et al. 2023, Liu & Wu 2020, Chen et al. 2022, He et al. 2020)) considering them together. Until now, the comparison idea has been proven to be the most helpful method to study the intermittent discrete-time state feedback control. However, it only works well when the underlying hybrid SDEs are globally Lipschitz continuous (see Appendix in (Hu et al. 2020)). It is hence necessary to develop new techniques to deal with this stabilisation problem to cover more general models.

On the one hand, Lyapunov functional method might not be a good choice. Because it will be difficult to construct a continuous differential functional and apply the Itô formula for all time owing to the piece-wise constant property of $I(t)$. On the other hand, in the proof Theorem 6.1, only the right-derivative is used. As a result, Razumikhin method deserves our consideration.

But it should be pointed out Theorem 6.1 might not be used to the intermittent control problem directly. This is because condition (6.3) cannot be met. At first, $I(t)$ is discontinuous, which could not be considered into the construction of continuous Lyapunov function. Moreover, λ_2 is time-inhomogeneous, which would let the time-varying property of $I(t)$ be ignored. Therefore, it is wiser to establish the Razumikhin theory based on the function $\lambda_2(t)$ rather than the constant λ_2. This change will make our stability analysis more technical than before.

7.2 Control problem

To make the intermittent discrete-state-feedback stabilisation problem simple, we only try to consider the hybrid SDE

$$\mathrm{d}x(t) = f(x(t), t, r(t))\mathrm{d}t + g(x(t), t, r(t))\mathrm{d}W(t) \tag{7.1}$$

on $t \geq 0$ with initial data $x(0) = \xi_0 \in \mathbb{R}^d$ and $r(0) = i_0 \in \mathbb{S}$. Suppose that $f : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^d$ and $g : \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S} \to \mathbb{R}^{d\times m}$ are locally Lipschitz continuous. For the existence of a unique global solution, the polynomial growth condition and Khasminskii-type condition are still required.

Assumption 7.1. *Assume that there are non-negative constants H_1, H_2, and $p > 1$ such that for every $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$,*

$$|f(x,t,i)| \le H_1|x| + H_2|x|^p, \qquad \forall (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}. \tag{7.2}$$

Assumption 7.2. *Assume that there exists a positive constant $\hat{\alpha}$ such that*

$$x^{\mathrm{T}} f(x,t,i) + \frac{3p-1}{2}|g(x,t,i)|^2 \le \hat{\alpha}|x|^2, \qquad \forall (x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}.$$

For the aim of control design, the following assumption is needed, which is stronger than Assumption 3.3 since we will use Razumikhin method.

Assumption 7.3. *For each $i \in \mathbb{S}$, assume that there are non-negative constants γ_i, $\bar{\gamma}_i$, and positive constants β_i, $\bar{\beta}_i$ such that for any $(x,t) \in \mathbb{R}^d \times \mathbb{R}_+$*

$$\begin{cases} x^{\mathrm{T}} f(x,t,i) + \dfrac{1}{2}|g(x,t,i)|^2 \le \gamma_i|x|^2 - \beta_i|x|^{p+1}, \\ x^{\mathrm{T}} f(x,t,i) + \dfrac{p}{2}|g(x,t,i)|^2 \le \bar{\gamma}_i|x|^2 - \bar{\beta}_i|x|^{p+1}. \end{cases} \tag{7.3}$$

Given the hybrid SDE (7.1) is unstable, we would like to design a state feedback control based on discrete-time observations working intermittently to make the controlled SDE

$$\mathrm{d}x(t) = \Big(f(x(t),t,r(t)) + u(x(t_\tau),t,r(t))I(t)\Big)\mathrm{d}t + g(x(t),t,r(t))\mathrm{d}W(t) \tag{7.4}$$

become stable.

Based on Assumption 7.3, we first give the rule on the bounded control function.

Rule 7.1. *Choose non-negative constants $\kappa_i (i \in \mathbb{S})$ such that*

$$\begin{cases} A := -2\mathrm{diag}\,(\gamma_1 - \kappa_1, \cdots, \gamma_S - \kappa_S) - Q \\ \bar{A} := -(p+1)\mathrm{diag}\,(\bar{\gamma}_1 - \kappa_1, \cdots, \bar{\gamma}_S - \kappa_S) - Q \end{cases}$$

are non-singular M-matrices. Then for each $i \in \mathbb{S}$, set $R_i = \left(\frac{2\kappa_i}{\beta_i \wedge \bar{\beta}_i}\right)^{\frac{1}{p-1}}$. The control function can be designed as:

- *when $x \in B_{R_i}$, design $u(x,t,i)$ such that we can find a non-negative constant K_i to let for any $(x,y,t) \in B_{R_i} \times B_{R_i} \times \mathbb{R}_+$*

  $$|u(x,t,i) - u(y,t,i)| \le K_i|x-y|, \qquad x^{\mathrm{T}} u(x,t,i) \le -\kappa_i|x|^2,$$

  and moreover $u(0,t,i) = 0$ for all $t \in \mathbb{R}_+$;
- *when $x \in B_{2R_i} - B_{R_i}$, let $u(x,t,i) = u\left(\left(\frac{2R_i}{|x|} - 1\right)x, t, i\right)$ for all $t \in \mathbb{R}_+$;*

- *when* $x \in B_{2R_i}^c$, *let* $u(x,t,i) = 0$ *for all* $t \in \mathbb{R}_+$.

Then under Assumptions 7.1, 7.2, 7.3, Rule 7.1, the controlled SDE (7.4) admits a global solution $x(t)$, which satisfies that for any $t > 0$

$$\sup_{0\le s\le t} E|x(s)|^{3p-1} < \infty, \qquad E\left(\sup_{0\le s\le t} |x(s)|^{2p}\right) < \infty.$$

Rule 7.2. *Compared with Rule 6.1, we additionally require $\bar{A}$ to be a non-singular M-matrix since it is no longer appropriate to set a parameter to balance the positive term $|x|^{p+1}$ like ε_1^* in Chapter 6, as we will propose a totally different way to determine the value of τ.*

Let $(\eta_1, \cdots, \eta_S)^{\mathrm{T}} = A^{-1}(1,\cdots,1)^{\mathrm{T}}$ and $(\bar\eta_1, \cdots, \bar\eta_S)^{\mathrm{T}} = \bar{A}^{-1}(1,\cdots,1)^{\mathrm{T}}$. For convenience, denote by $\eta_M = \max_{i\in\mathbb{S}} \eta_i$, $\eta_m = \min_{i\in\mathbb{S}} \eta_i$, $\bar\eta_M = \max_{i\in\mathbb{S}} \bar\eta_i$, $\bar\eta_m = \min_{i\in\mathbb{S}} \bar\eta_i$, $\gamma_M = \max_{i\in\mathbb{S}} \gamma_i$, $\bar\beta_m = \max_{i\in\mathbb{S}} \bar\beta_i$, $K_M = \max_{i\in\mathbb{S}} K_i$. It is easy to see the constants

$$\begin{aligned}
\mu_1 :=& \frac{1}{\eta_M} \wedge \min_{i\in\mathbb{S}} \left(\frac{1+\eta_i\beta_i}{\bar\eta_i}\right), \\
\mu_2 :=& \max_{i\in\mathbb{S}} \left(\frac{1}{\eta_i}\left(2\eta_i\gamma_i + \sum_{j=1}^S q_{ij}\eta_j\right) \vee \frac{1}{\bar\eta_i}\left(-\eta_i\beta_i + (p+1)\bar\eta_i\bar\gamma_i + \sum_{j=1}^S q_{ij}\bar\eta_j\right)\right)
\end{aligned}$$

are positive.

Rule 7.3. *Choose $\delta_\mu \in \left(\frac{\mu_2}{\mu_1+\mu_2}, 1\right)$ and let τ^* be the unique root of*

$$\varphi(\tau) := \exp\left(\left(\frac{\mu_2}{\delta_\mu} - \mu_2\right)\tau\right) - \frac{1}{\varphi_1(\tau)}\left(\frac{\mu_1+\mu_2-\frac{\mu_2}{\delta_\mu}}{\sqrt{\tau}} - K_M\right), \qquad \tau \in (0, \hat\tau],$$

where $\hat\tau = \left(\frac{\mu_1+\mu_2-\frac{\mu_2}{\delta_\mu}}{K_M}\right)^2$ *and*

$$\varphi_1(\tau) = \left(\frac{2\gamma_M + 4K_M + 2H_1}{\eta_m} + \frac{2H_2}{\bar\eta_m}\right)\left(K_M\eta_M + \frac{(p+1)K_M^2\bar\eta_M}{2\bar\beta_m}\sqrt{\tau}\right).$$

Let τ works smaller than τ^*.

Remark 7.1. *Let $\delta_\mu \in \left(\frac{\mu_2}{\mu_1+\mu_2}, 1\right)$ be chosen. It is easy to see that $\varphi(\cdot)$ is an increasing continuous function on $(0, \hat\tau]$. Moreover, $\lim_{\tau\to 0^+}\varphi(\tau) = -\infty$, and $\varphi(\hat\tau) > 0$. Therefore, there is a unique root of φ, and so the definition of τ^* is clear.*

7.3 Stabilisation results

7.3.1 Useful lemmas

Define the Lyapunov function $U : \mathbb{R}^d \times \mathbb{S} \to \mathbb{R}_+$ by

$$U(x,i) = \eta_i|x|^2 + \bar{\eta}_i|x|^{p+1},$$

while define the operator $\mathcal{L}U(x,z,t,i)$ with respect to the controlled SDE (7.4) by

$$\mathcal{L}U(x,z,t,i) = L_1U(x,t,i) + L_2U(x,z,t,i)$$

where

$$\begin{aligned} L_1U(x,t,i) =& U_x(x,i)(f(x,t,i) + u(x,t,i)I(t)) \\ &+ \frac{1}{2}\text{trace}\left(g^{\mathrm{T}}(x,t,i)U_{xx}(x,i)g(x,t,i)\right) + \sum_{j=1}^{S} q_{ij}U(x,j) \end{aligned}$$

and

$$L_2(x,y,t,i) = U_x(x,i)(u(z,t,i) - u(x,t,i))I(t).$$

Owing to the intermittent control, we need to modify the estimations of L_1U and L_2U.

Lemma 7.1. *Let Assumptions 7.1, 7.2, 7.3, Rule 7.1 hold. Then*

$$L_1U(x,t,i) \le (-\mu_1 I(t) + \mu_2(1 - I(t)))U(x,i) - \frac{p+1}{2}\bar{\eta}_i\bar{\beta}_i|x|^{2p} \tag{7.5}$$

holds for any $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$.

Proof. At first, it easy to show that for any $(x,t,i) \in \mathbb{R}^d \times \mathbb{R}_+ \times \mathbb{S}$

$$\begin{cases} x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \dfrac{1}{2}|g(x,t,i)|^2 \le (\gamma_i - \kappa_i I(t))|x|^2 - \dfrac{\beta_i}{2}|x|^{p+1}, \\ x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \dfrac{p}{2}|g(x,t,i)|^2 \le (\bar{\gamma}_i - \kappa_i I(t))|x|^2 - \dfrac{\bar{\beta}_i}{2}|x|^{p+1}. \end{cases}$$

In fact, fix $(t,i) \in \mathbb{R}_+ \times \mathbb{S}$ arbitrarily. For $x \in B_{R_i}$, it is easy to see that

$$\begin{aligned} x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \frac{1}{2}|g(x,t,i)|^2 \le& (\gamma_i - \kappa_i I(t))|x|^2 - \beta_i|x|^{p+1} \\ \le& (\gamma_i - \kappa_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1}. \end{aligned}$$

On the other hand, we have that for $x \in B^c_{R_i}$, $x^{\mathrm{T}}u(x,t,i) \le 0$. Therefore,

$$x^{\mathrm{T}}(f(x,t,i) + u(x,t,i)I(t)) + \frac{1}{2}|g(x,t,i)|^2$$

$$
\begin{aligned}
&\le \gamma_i|x|^2 - \beta_i|x|^{p+1} \\
&= (\gamma_i - \kappa_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1} + \left(\kappa_i I(t)|x|^2 - \frac{\beta_i}{2}|x|^{p+1}\right) \\
&\le (\gamma_i - \kappa_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1}
\end{aligned}
$$

since $\kappa_i I(t)|x|^2 - \frac{\beta_i}{2}|x|^{p+1} \le \kappa_i|x|^2 - \frac{\beta_i}{2}|x|^{p+1} \le 0$ when $|x| > R_i$. Next, compute

$$
\begin{aligned}
&L_1U(x,t,i) \\
\le& 2\eta_i\left((\gamma_i - \kappa_i I(t))|x|^2 - \frac{\beta_i}{2}|x|^{p+1}\right) + \sum_{j=1}^{S} q_{ij}\eta_j|x|^2 \\
&+ (p+1)\bar{\eta}_i|x|^{p-1}\left((\bar{\gamma}_i - \kappa_i I(t))|x|^2 - \frac{\bar{\beta}_i}{2}|x|^{p+1}\right) + \sum_{j=1}^{S} q_{ij}\bar{\eta}_j|x|^{p+1} \\
=& \left(2\eta_i(\gamma_i - \kappa_i) + \sum_{j=1}^{S} q_{ij}\eta_j\right) I(t)|x|^2 + \left(2\eta_i\gamma_i + \sum_{j=1}^{S} q_{ij}\eta_j\right)(1 - I(t))|x|^2 \\
&+ \left(-\eta_i\beta_i + (p+1)\bar{\eta}_i(\bar{\gamma}_i - \kappa_i) + \sum_{j=1}^{S} q_{ij}\bar{\eta}_j\right) I(t)|x|^{p+1} \\
&+ \left(-\eta_i\beta_i + (p+1)\bar{\eta}_i\bar{\gamma}_i + \sum_{j=1}^{S} q_{ij}\bar{\eta}_j\right)(1 - I(t))|x|^{p+1} - \frac{p+1}{2}\bar{\eta}_i\bar{\beta}_i|x|^{2p} \\
=& - I(t)|x|^2 - (\eta_i\beta_i + 1)I(t)|x|^{p+1} + \left(2\eta_i\gamma_i + \sum_{j=1}^{S} q_{ij}\eta_j\right)(1 - I(t))|x|^2 \\
&+ \left(-\eta_i\beta_i + (p+1)\bar{\eta}_i\bar{\gamma}_i + \sum_{j=1}^{S} q_{ij}\bar{\eta}_j\right)(1 - I(t))|x|^{p+1} - \frac{p+1}{2}\bar{\eta}_i\bar{\beta}_i|x|^{2p} \\
\le& (-\mu_1 I(t) + \mu_2(1 - I(t)))U(x,i) - \frac{p+1}{2}\bar{\eta}_i\bar{\beta}_i|x|^{2p}.
\end{aligned}
$$

This completes the proof. □

Lemma 7.2. *Under the same conditions in Lemma 7.1, then the solution of the controlled SDE (7.4) satisfies that for any* $t \in \mathbb{R}_+$

$$
\begin{aligned}
&EL_2U(x(t), x(t_\tau), t, r(t)) \\
\le& \left(K_M EU(x(t), r(t)) + \varphi_1(\tau) \sup_{-\tau\le\theta\le 0} EU(x(t+\theta), r(t+\theta))\right)\sqrt{\tau}I(t) \\
&+ \frac{p+1}{2}E\left(\bar{\eta}_{r(t)}\bar{\beta}_{r(t)}|x(t)|^{2p}\right). \qquad (7.6)
\end{aligned}
$$

Proof. Let t be fixed. It is easy to derive from the global Lipschitz continuity of

$u(x,t,i)$ that

$$\begin{aligned}
&EL_2U(x(t),x(t_\tau),t,r(t))\\
\le&E\left(\left(2\eta_{r(t)}|x(t)|+(p+1)\bar{\eta}_{r(t)}|x(t)|^p\right)K_M|x(t)-x(t_\tau)|\right)I(t)\\
\le&E\Bigg(K_M\eta_{r(t)}\sqrt{\tau}|x(t)|^2+K_M\eta_{r(t)}\frac{1}{\sqrt{\tau}}|x(t)-x(t_\tau)|^2\\
&+\frac{p+1}{2}\bar{\eta}_{r(t)}\bar{\beta}_{r(t)}|x(t)|^{2p}+\frac{(p+1)K_M^2\bar{\eta}_{r(t)}}{2\bar{\beta}_{r(t)}}|x(t)-x(t_\tau)|^2\Bigg)I(t). \qquad (7.7)
\end{aligned}$$

We can find a non-negative integer n such that $n\tau\le t<(n+1)\tau$. Then we have $t_\tau=n\tau$, and $s_\tau=n\tau$ for any $s\in[n\tau,t]$. Applying the Itô formula yields that

$$\begin{aligned}
&E|x(t)-x(n\tau)|^2\\
\le&E\int_{n\tau}^t\left(2(x(s)-x(n\tau))^{\mathrm{T}}(f(x(s),s,r(s))+u(x(n\tau),s,r(s))I(s))\right.\\
&\left.+|g(x(s),s,r(s))|^2\right)\mathrm{d}s\\
\le&(2\gamma_M+K_M+H_1)\int_{n\tau}^t E|x(s)|^2\mathrm{d}s+\frac{2H_2p}{p+1}\int_{n\tau}^t E|x(s)|^{p+1}\mathrm{d}s\\
&+(3K_M+H_1)\int_{n\tau}^t E|x(n\tau)|^2\mathrm{d}s+\frac{2H_2}{p+1}\int_{n\tau}^t E|x(n\tau)|^{p+1}\mathrm{d}s\\
\le&\left(\frac{2\gamma_M+4K_M+2H_1}{\eta_m}+\frac{2H_2}{\bar{\eta}_m}\right)\tau\sup_{-\tau\le\theta\le0}EU(x(t+\theta),r(t+\theta)).
\end{aligned}$$

Substituting this into (7.7), we obtain the assertion (7.6). This ends the proof. □

If we use the same way as in the proof of Theorem 6.2, we can show that $EU(x(t),r(t))$ and $ELU(x(t),x(t_\tau),t,r(t))$ are right-continuous. But with a little more effort, we could see that $EU(x(t),r(t))$ is actually continuous.

Lemma 7.3. *Let all the conditions in Lemma 7.1 hold. Then as functions of t, $EU(x(t),r(t))$ is continuous, and $E\mathcal{L}U(x(t),x(t_\tau),t,r(t))$ is right-continuous.*

Proof. For any $t\in\mathbb{R}_+$, applying the generalized Itô formula to $U(x,i)$, we see that

$$\begin{aligned}
U(x(t),r(t))=&U(\xi_0,r_0)+\int_0^t\mathcal{L}U(x(s),x(s_\tau),s,r(s))\mathrm{d}s\\
&+\int_0^t U_x(x(s),r(s))g(x(s),s,r(s))\mathrm{d}W(s)+M(t), \qquad (7.8)
\end{aligned}$$

where $M(t)$ is a continuous martingale vanishing at $t=0$. It is easy to obtain that for any $(x,z,t,i)\in\mathbb{R}^d\times\mathbb{R}^d\times\mathbb{R}_+\times\mathbb{S}$

$$|\mathcal{L}U(x,z,t,i)|\le C\left(1+|x|^{2p}+|z|^{2p}\right)$$

and

$$|U_x(x,i)g(x,t,i)| \leq C(1+|x|^{2p}).$$

Since for any $s \in [0,t]$, $E|x(s)|^{2p} < \infty$,

$$E|\mathcal{L}U(x(s),x(s_\tau),s,r(s))| \leq C\left(1+E|x(s)|^{2p}+E|x(s_\tau)|^{2p}\right) < \infty$$

and

$$E|U_x(x(s),r(s))g(x(s),s,r(s))| \leq C\left(1+E|x(s)|^{2p}\right) < \infty.$$

Let n_0 be a sufficiently large integer for $n > |\xi_0|$. For each integer $n \geq n_0$, define the stopping time $\sigma_n = \inf\{t \in \mathbb{R}_+ : |x(t)| \geq n\}$. Clearly, $\sigma_n \uparrow \infty$ a.s. It then follows from (7.8) that

$$EU(x(t\wedge\sigma_n),r(t\wedge\sigma_n)) = U(x_0,r_0) + E\int_0^{t\wedge\sigma_n} \mathcal{L}U(x(s),x(s_\tau),s,r(s))\mathrm{d}s. \quad (7.9)$$

For each $n \geq n_0$, we have

$$\left|\int_0^{t\wedge\sigma_n} \mathcal{L}U(x(s),x(s_\tau),s,r(s))\mathrm{d}s\right| \leq \int_0^t |\mathcal{L}U(x(s),x(s_\tau),s,r(s))|\mathrm{d}s$$

and

$$E\int_0^t |\mathcal{L}U(x(s),x(s_\tau),s,r(s))|\,\mathrm{d}s = \int_0^t E\,|\mathcal{L}U(x(s),x(s_\tau),s,r(s))|\,\mathrm{d}s < \infty.$$

On the other hand,

$$U(x(t\wedge\sigma_n),r(t\wedge\sigma_n)) \leq \eta_M|x(t\wedge\sigma_n)|^2+\bar{\eta}_M|x(t\wedge\sigma_n)|^{p+1} \leq C\left(1+\sup_{0\leq s\leq t}|x(s)|^{p+1}\right).$$

Since $E\left(\sup_{0\leq s\leq t}|x(s)|^{p+1}\right) < \infty$, we can let $k \to \infty$ on both sides of (7.9) and use the dominated convergence theorem to get that

$$EU(x(t),r(t)) = U(x_0,r_0) + \int_0^t E\mathcal{L}U(x(s),x(s_\tau),s,r(s))\mathrm{d}s.$$

Clearly, $EU(x(t),r(t))$ is continuous at time t. □

7.3.2 Exponential stabilisation

It is challenging to give a time-inhomogeneous Razumukhin-type theorem for a general SFDE. Thus we only use the idea in Theorem 6.1 to show our exponential stability.

Theorem 7.1. *Let all the conditions in Lemma 7.1 hold and τ meet Rule 7.3.*

Then the solution of the controlled SDE (7.4) satisfies that

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(E|x(t)|^{p+1}\right)<0 \tag{7.10}$$

and

$$\limsup_{t\to\infty}\frac{1}{t}\log\left(|x(t)|\right)<0 \qquad a.s. \tag{7.11}$$

Proof. We divide the proof into three steps.

Step 1. For fixed τ, write $\varphi_1=\varphi_1(\tau)$ for simplicity. Since φ is increasing on $(0,\tau^*)$, we naturally have

$$\exp\left(\left(\frac{\mu_2}{\delta_\mu}-\mu_2\right)\tau\right)<\frac{1}{\varphi_1}\left(\frac{\mu_1+\mu_2-\frac{\mu_2}{\delta_\mu}}{\sqrt{\tau}}-K_M\right).$$

Therefore, we could choose a constant q such that

$$\exp\left(\left(\frac{\mu_2}{\delta_\mu}-\mu_2\right)\tau\right)<q<\frac{1}{\varphi_1}\left(\frac{\mu_1+\mu_2-\frac{\mu_2}{\delta_\mu}}{\sqrt{\tau}}-K_M\right). \tag{7.12}$$

If $\xi_0=0$, the result is obvious. Thus we always assume that $|\xi_0|>0$. If for some t, the solution satisfying that

$$\sup_{-\tau\le\theta\le 0}EU(x(t+\theta),r(t+\theta))\le qEU(x(t),r(t)), \tag{7.13}$$

we then derive from Lemmas 7.1 and 7.2 that

$$\begin{aligned}
&ELU(x(t),x(t_\tau),t,r(t))\\
\le&(-\mu_1 I(t)+\mu_2(1-I(t)))EU(x(t),r(t))-\frac{p+1}{2}E\left(\bar\eta_{r(t)}\bar\beta_r(t)|x(t)|^{2p}\right)\\
&+\left(K_M EU(x(t),r(t))+\varphi_1\sup_{-\tau\le\theta\le 0}EU(x(t+\theta),r(t+\theta))\right)\sqrt{\tau}I(t)\\
&+\frac{p+1}{2}E\left(\bar\eta_{r(t)}\bar\beta_{r(t)}|x(t)|^{2p}\right)\\
\le&-\Big((\mu_1-(K_M+\varphi_1 q)\sqrt{\tau})I(t)-\mu_2(1-I(t))\Big)EU(x(t),r(t)).
\end{aligned}$$

Letting $\varphi_2=\mu_1-(K_M+\varphi_1 q)\sqrt{\tau}$ and $\lambda_2(t)=\varphi_2 I(t)-\mu_2(1-I(t))$, we have

$$E\mathcal{L}U(x(t),x(t_\tau),t,r(t))\le-\lambda_2(t)EU(x(t),r(t)). \tag{7.14}$$

Step 2. For $t\ge 0$, define $\lambda(t)=\lambda_2(t)\wedge\frac{\log(q)}{\tau}$, and

$$V(t)=\exp\left(\int_0^t\lambda(s)\mathrm{d}s\right)EU(x(t),r(t)).$$

It is easy to see that $V(t)$ is well-defined, and further is a continuous function from

Lemma 7.3. Next, we claim that

$$V(t) \leq V(0), \qquad \forall t \geq 0, \tag{7.15}$$

where $V(0) = U(\xi_0, r_0)$ is a positive constant. If (7.15) is true, it then follows that

$$E|x(t)|^{p+1} \leq \frac{1}{\bar{\eta}_M} EU(x(t), r(t)) \leq \frac{V(0)}{\bar{\eta}_M} \exp\left(-\int_0^t \lambda(s)\mathrm{d}s\right).$$

Since $\delta_\mu < 1$, we can easily observe that

$$q < \frac{1}{\varphi_1}\left(\frac{\mu_1 + \mu_2 - \frac{\mu_2}{\delta_\mu}}{\sqrt{\tau}} - K_M\right) < \frac{1}{\varphi_1}\left(\frac{\mu_1}{\sqrt{\tau}} - K_M\right),$$

which implies that $\varphi_2 > 0$. As a consequence,

$$\begin{aligned}\int_0^t \lambda(s)\mathrm{d}s &= \left(\varphi_2 \wedge \frac{\log(q)}{\tau}\right)\int_0^t I(s)\mathrm{d}s - \mu_2\int_0^t (1 - I(s))\mathrm{d}s \\ &= -\mu_2 t + \left(\mu_2 + \varphi_2 \wedge \frac{\log(q)}{\tau}\right)\int_0^t I(s)\mathrm{d}s.\end{aligned}$$

For any fixed $t \geq 0$, we can find a non-negative integer k such that $kT \leq t < (k+1)T$. If $t \in [kT, kT + \delta_\mu T)$, we obtain that

$$\begin{aligned}\int_0^t \lambda(s)\mathrm{d}s &= -\mu_2 t + \left(\mu_2 + \varphi_2 \wedge \frac{\log(q)}{\tau}\right)(\delta_\mu kT + t - kT) \\ &= \left(-\mu_2 + \mu_2\delta_\mu + \varphi_2\delta_\mu \wedge \frac{\log(q)}{\tau}\delta_\mu\right)kT \\ &\quad + \left(-\mu_2 + \mu_2 + \varphi_2 \wedge \frac{\log(q)}{\tau}\right)(t - kT) \\ &\geq \left(-\mu_2 + \mu_2\delta_\mu + \varphi_2\delta_\mu \wedge \frac{\log(q)}{\tau}\delta_\mu\right)kT.\end{aligned}$$

From the first inequality of (7.12), we derive that $\frac{\log(q)}{\tau} > \frac{\mu_2}{\delta_\mu} - \mu_2$. The other side of (7.12) implies that $\varphi_2 > \frac{\mu_2}{\delta_\mu} - \mu_2$. In other words, $\bar{\lambda} := -\mu_2 + \mu_2\delta_\mu + \varphi_2\delta_\mu \wedge \frac{\log(q)}{\tau}\delta_\mu$ is positive, and so

$$\int_0^t \lambda(s)\mathrm{d}s \geq \bar{\lambda}(t - T).$$

If $t \in [kT + \delta_\mu T, (k+1)T)$, we have

$$\begin{aligned}\int_0^t \lambda(s)\mathrm{d}s &= -\mu_2 t + \left(\mu_2 + \varphi_2 \wedge \frac{\log(q)}{\tau}\right)\delta_\mu(k+1)T \\ &\geq -\mu_2 t + \left(\mu_2 + \varphi_2 \wedge \frac{\log(q)}{\tau}\right)\delta_\mu t \geq \bar{\lambda}(t - T).\end{aligned}$$

In conclusion, we have shown that for any $t \geq 0$

$$E|x(t)|^{p+1} \leq \frac{V(0)}{\bar{\eta}_M} \exp\left(-\int_0^t \lambda(s)\mathrm{d}s\right) \leq \frac{V(0)}{\bar{\eta}_M} \exp\left(\bar{\lambda}T\right) \exp\left(-\bar{\lambda}t\right),$$

which yields that

$$\frac{1}{t}\log\left(E|x(t)|^{p+1}\right) \leq \frac{1}{t}\log\left(\frac{V(0)}{\bar{\eta}_M}\exp\left(\bar{\lambda}T\right)\right) - \bar{\lambda}.$$

Letting $t \to \infty$ gives the assertion (7.10). After achieving the moment exponential stability, we can use the same analysis as in the proof of Theorem 6.3 to prove the assertion (7.11).

Step 3. The remaining work is to prove claim (7.15). Supposing not, there will be some $t > 0$ such that $V(t) > V(0)$. We can set $\hat{t} = \inf\{t > 0 : V(t) > V(0)\}$. Because of the continuity of $V(t)$, we see that for $0 \leq t < \hat{t}$, $V(t) \leq V(0)$; for $t = \hat{t}$, $V(\hat{t}) = V(0)$; and there is a sequence $\{t_n\}_{n\geq 1}$ such that $t_n > \hat{t}$, $t_n \downarrow \hat{t}$, and $V(t_n) > V(0)$.

On the other hand, for any $\theta \in [-\tau, 0]$, if $\hat{t} + \theta > 0$, we obtain that

$$EU(x(\hat{t}+\theta), r(\hat{t}+\theta)) \leq \exp\left(\int_{\hat{t}+\theta}^{\hat{t}} \lambda(s)\mathrm{d}s\right) EU(x(\hat{t}), r(\hat{t})) \leq qEU(x(\hat{t}), r(\hat{t}))$$

since $V(\hat{t}+\theta) \leq V(0) = V(\hat{t})$ and $\int_{\hat{t}+\theta}^{\hat{t}} \lambda(s)\mathrm{d}s \leq \frac{\log(q)}{\tau}(-\theta) \leq \log(q)$. Otherwise, as $\hat{t} \leq \tau$, we have

$$\begin{aligned} EU(x(\hat{t}+\theta), r(\hat{t}+\theta)) =& U(\xi_0, r_0) = V(\hat{t}) = \exp\left(\int_0^{\hat{t}} \lambda(s)\mathrm{d}s\right) EU(x(\hat{t}), r(\hat{t})) \\ \leq& qEU(x(\hat{t}), r(\hat{t})). \end{aligned}$$

In other words, we have shown that for any $-\tau \leq \theta \leq 0$,

$$EU(x(\hat{t}+\theta), r(\hat{t}+\theta)) \leq qEU(x(\hat{t}), r(\hat{t})),$$

which is exactly condition (7.13). Using the results in Step 1, we have

$$\begin{aligned} E\mathcal{L}U(x(\hat{t}), x(\hat{t}_\tau), \hat{t}, r(\hat{t})) \leq& -\lambda(\hat{t})EU(x(\hat{t}), r(\hat{t})) \\ \leq& -\lambda(\hat{t})EU(x(\hat{t}), r(\hat{t})) < -\lambda(\hat{t})EU(x(\hat{t}), r(\hat{t})) + \epsilon, \end{aligned}$$

where ϵ is an arbitrary positive constant. We can also find a non-negative integer K such that $KT \leq \hat{t} < (K+1)T$. Let $\delta_\mu > 0$ be small enough so that $\Delta_1 < (KT + \delta_\mu T - \hat{t})\mathbb{I}_{\{KT+\delta_\mu T > \hat{t}\}} + ((K+1)T - \hat{t})\mathbb{I}_{\{KT+\delta_\mu T \leq \hat{t}\}}$ and $\delta_\mu < \tau$. We then see from the right-continuity of $E\mathcal{L}U$ that

$$E\mathcal{L}U(x(t), x(t_\tau), t, r(t)) < -\lambda(t)EU(x(t), r(t)) + \epsilon, \qquad \forall t \in [\hat{t}, \hat{t} + \Delta_1].$$

The interval $[\hat{t}, \hat{t}+\Delta_1]$ is either in $[KT, KT+\delta_\mu T)$ or $[KT+\delta_\mu T, (K+1)T)$. Thus applying the generalised Itô formula to $\exp\left(\int_0^t \lambda(s)\mathrm{d}s\right) U(x(t), r(t))$ gives that

$$\begin{aligned}
& V(\hat{t} + \Delta_1) - V(\hat{t}) \\
=& \exp\left(\int_0^{\hat{t}+\Delta_1} \lambda(s)\mathrm{d}s\right) EU(x(\hat{t} + \Delta_1), r(\hat{t} + \Delta_1)) \\
& - \exp\left(\int_0^{\hat{t}} \lambda(s)\mathrm{d}s\right) EU(x(\hat{t}), r(\hat{t})) \\
=& \int_{\hat{t}}^{\hat{t}+\Delta_1} \exp\left(\int_0^s \lambda(v)\mathrm{d}v\right) \Big(E\mathcal{L}U(x(s), x(s_\tau), s, r(s)) + \lambda(s)EU(x(s), r(s))\Big)\mathrm{d}s \\
<& \epsilon \int_{\hat{t}}^{\hat{t}+\Delta_1} \exp\left(\int_0^s \lambda(v)\mathrm{d}v\right) \mathrm{d}s \le \epsilon \int_{\hat{t}}^{\hat{t}+\tau} \exp\left(\frac{\log(q)}{\tau} s\right) \mathrm{d}s.
\end{aligned}$$

Since $\int_{\hat{t}}^{\hat{t}+\tau} \exp\left(\frac{\log(q)}{\tau} s\right) \mathrm{d}s$ is a positive constant and ϵ is chosen arbitrarily, $V(\hat{t} + \Delta_1) - V(\hat{t}) \le 0$. For sufficiently large n with $t_n - \hat{t} \le \Delta_1$, we obtain that

$$V(t_n) \le V(\hat{t}) = V(0),$$

which is a contradiction with the fact that $V(t_n) > V(0)$ derived before. Therefore claim (7.15) must be true. The proof is therefore complete. □

7.4 Application to coupled oscillators

Consider the coupled Van der Pol–Duffing oscillator system, which is consisted of N oscillators and the n-th oscillator is described as

$$\begin{cases}
\mathrm{d}x_n(t) = \Big(- \big(a_n(r(t)) + b_n(r(t))\big)x_n(t) + B_n(r(t))(y_n(t) - x_n(t))^3 \\
\qquad\qquad + b_n(r(t))y_n(t) - A_n(r(t))x_n^3(t)\Big)\mathrm{d}t + \nu_n(r(t))x_n(t)\mathrm{d}w_n^{(1)}(t), \\
\mathrm{d}y_n(t) = \Big(b_n(r(t))x_n(t) - z_n(t) - B_n(r(t))(y_n(t) - x_n(t))^3 \\
\qquad\qquad - (b_n(r(t)) + 1)y_n(t) - C_n(r(t))y_n^3(t))\Big)\mathrm{d}t + \nu_n(r(t))y_n(t)\mathrm{d}w_n^{(2)}(t), \\
\mathrm{d}z_n(t) = \left(y_n(t) + \displaystyle\sum_{j=1}^{S} e_{nj}(r(t))\Pi_{nj}(z_n(t), z_j(t), r(t)) + P_n(z_n(t), r(t))\right) \mathrm{d}t \\
\qquad\qquad + \nu_n(r(t))z_n(t)\mathrm{d}w_n^{(3)}(t),
\end{cases}$$

where $x_n, y_n, z_n \in \mathbb{R}$, $a_n(i), b_n(i), A_n(i), B_n(i), C_n(i), \nu_n(i)$ are positive constants, $e_{nj}(i)$ stands for connection weight from oscillator j to oscillator n, $\Pi_{nj}(z_n, z_j, i)$ and $P_n(z_n, i)$ are locally Lipschitz continuous functions in the i-th mode. Here, we need to impose the following conditions on these functions.

Assumption 7.4. *For every $i \in \mathbb{S}$ and $n, j = 1, \cdots, N$, assume that there are positive constants $\Lambda_{nj}(i)$, $J_n^{(1)}(i)$, $J_n^{(2)}(i)$, $D_n^{(1)}(i)$, $D_n^{(2)}(i)$ so that for all $x, y \in \mathbb{R}$*

$$|\Pi_{n,j}(x, y, i)| \le \Lambda_{nj}(i)(|x| + |y|) \tag{7.16}$$

and

$$|P_n(x, i)| \le J_n^{(1)}(i)|x| + J_n^{(2)}(i)|x|^3, \qquad xP_n(x, i) \le D_n^{(1)}(i)|x|^2 - D_n^{(2)}(i)|x|^4. \tag{7.17}$$

Let $X_n = (x_n, y_n, z_n)^{\mathrm{T}}$, $X = \left(X_1^{\mathrm{T}}, \cdots, X_N^{\mathrm{T}}\right)^{\mathrm{T}}$, $W_n = \left(w_n^{(1)}, w_n^{(2)}, w_n^{(3)}\right)^{\mathrm{T}}$, $W = \left(W_1^{\mathrm{T}}, \cdots, W_N^{\mathrm{T}}\right)^{\mathrm{T}}$. Then the oscillator system can be written as

$$\mathrm{d}X(t) = F(X(t), r(t))\mathrm{d}t + G(X(t), r(t))\mathrm{d}W(t), \tag{7.18}$$

where

$$F(X, i) = \left(F_1^{\mathrm{T}}(X_1, i), \cdots, F_N^{\mathrm{T}}(X_N, i)\right)^{\mathrm{T}},$$

$$G(X, i) = \begin{pmatrix} G_1(X_1, i) & & \\ & \ddots & \\ & & G_N(X_N, i) \end{pmatrix},$$

with $G_n(X_n, i) = \nu_n(i)\mathrm{diag}(x_n, y_n, z_n)$ and

$$F_n(X_n, i) = \begin{pmatrix} -(a_n(i) + b_n(i))x_n + B_n(i)(y_n - x_n)^3 + b_n(i)y_n - A_n(i)x_n^3 \\ b_n(i)x_n - z_n - B_n(i)(y_n - x_n)^3 - (b_n(i) + 1)y_n - C_n(i)y_n^3 \\ y_n + \sum_{j=1}^S e_{nj}(i)\Pi_{nj}(z_n, z_j, i) + P_n(z_n, i) \end{pmatrix}.$$

With the detailed calculation, we derive that for each $i \in \mathbb{S}$, $|F(X, i)|^2 \le L_{1i}|X|^2 + L_{2i}|X|^6$, where

$$\begin{aligned} L_{1i} = &\max_{1\le n\le N} \left(4(a_n(i) + b_n(i))^2 + 5b_n^2(i)\right) \vee \max_{1\le n\le N} \left(4b_n^2(i) + 5(b_n(i) + 1)^2 + 3\right) \\ &\vee \left(\max_{1\le n\le N} \left(5 + 3(J_n^{(1)})^2 + 6N \sum_{j=1}^S (|e_{nj}(i)|\Lambda_{nj}(i))^2\right) \right. \\ &\left. + 6N \max_{1\le n,j\le N} (|e_{nj}(i)|\Lambda_{nj}(i))^2 \right) \end{aligned}$$

and

$$L_{2i} = \max_{1\le n\le N}\left(4A_n^2(i) + 288B_n^2(i)\right) \vee \max_{1\le n\le N}\left(5C_n^2(i) + 288B_n^2(i)\right) \vee \max_{1\le n\le N} 3(J_n^{(2)})^2.$$

Therefore, Assumption 7.1 is satisfied with $p = 3$, $H_1 = \max_{i\in\mathbb{S}}\sqrt{L_{1i}}$, $H_2 = \max_{i\in\mathbb{S}}\sqrt{L_{2i}}$. Next, compute

$$\begin{aligned} X^{\mathrm{T}}F(X,i) \le& \sum_{n=1}^{N}\Bigg(-a_n(i)x_n^2 - y_n^2 + \sum_{j=1}^{N}|e_{nj}(i)\Pi_{nj}(z_n,z_j,i)z_n| - A_n(i)x_n^4 \\ & - C_n(i)y_n^4 + z_nP_n(z_n,i)\Bigg) \\ \le& \sum_{n=1}^{N}\Bigg(\sum_{j=1}^{N}|e_{nj}(i)\Lambda_{nj}(i)(z_n^2 + |z_nz_j|)| + D_n^{(1)}(i)z_n^2 - A_n(i)x_n^4 \\ & - C_n(i)y_n^4 - D_n^{(2)}(i)z_n^4\Bigg). \end{aligned}$$

Since $|X|^4 \le 3N\sum_{n=1}^{N}(x_n^4 + y_n^4 + z_n^4)$, we further have

$$X^{\mathrm{T}}F(X,i) \le h_i|X|^2 - \frac{1}{3N}\min_{1\le n\le N}\left(A_n(i)\wedge C_n(i)\wedge D_n^{(2)}(i)\right)|X|^4,$$

where

$$h_i = \max_{1\le n\le N}\left(\frac{3}{2}\sum_{j=1}^{N}|e_{nj}(i)|\Lambda_{nj}(i) + D_n^{(1)}(i)\right) + \frac{1}{2}\max_{1\le n,j\le N}(|e_{nj}(i)|\Lambda_{nj}(i)).$$

It is easy to see that

$$|G(X,i)|^2 \le \sum_{n=1}^{N}\nu_n^2(i)(x_n^2 + y_n^2 + z_n^2) \le \max_{1\le n\le N}\nu_n^2(i)|X|^2.$$

As a result, Assumption 7.2 holds with $\hat{\alpha} = \max_{i\in\mathbb{S}}\left(h_i + 4\max_{1\le n\le N}\nu_n^2(i)\right)$. Assumption 7.3 is also satisfied with

$$\begin{aligned} \gamma_i =& h_i + \frac{1}{2}\max_{1\le n\le N}\nu_n^2(i), \qquad \bar{\gamma}_i = h_i + \max_{1\le n\le N}\nu_n^2(i), \\ \beta_i =& \bar{\beta}_i = \frac{1}{3N}\min_{1\le n\le N}\left(A_n(i)\wedge C_n(i)\wedge D_n^{(2)}(i)\right). \end{aligned}$$

But the oscillator system (7.18) might not be stable (see the simulation in Fig. 7.3). It is hence necessary to design controller according to the results above to achieve stabilisation. At first, the control function $\mathcal{U}(X,i)$ can be designed as follows.

Rule 7.4. *Choose non-negative constants $\kappa_i (i\in\mathbb{S})$ such that A and $\bar{A}$ are non-*

singular M-matrices. Then for each $i \in \mathbb{S}$, letting $R_i = \sqrt{\frac{2\kappa_i}{\beta_i}}$, we can design the control function as follows

$$\mathcal{U}(X,i) = \begin{cases} -\kappa_i X, & X \in B_{R_i}, \\ -\kappa_i \left(\dfrac{2R_i}{|X|} - 1\right) X, & X \in B_{2R_i} - B_{R_i}, \\ 0, & X \in B^c_{R_i}. \end{cases} \tag{7.19}$$

It is easy to verify that $\mathcal{U}(X,i)$ designed in Rule 7.4 meets Rule 7.1 with $K_i = \kappa_i$. Next, we let the feedback control $\mathcal{U}(X(t), r(t))$ working imminently with strength δ_μ and being observed at discrete times $0, \tau, 2\tau, \cdots$ In other words, the controlled oscillator system is given as

$$\mathrm{d}X(t) = (F(X(t), r(t)) + \mathcal{U}(X(t_\tau), r(t))I(t))\mathrm{d}t + G(X(t), r(t))\mathrm{d}W(t), \tag{7.20}$$

where $t_\tau = [t/\tau]\tau$ and $I(t) = \sum_{k=0}^{\infty} \mathbb{I}_{[kT, kT+\delta_\mu T)}(t)$ are the same as before. By Theorem 7.1, we can make the following assertion.

Theorem 7.2. *Let Assumption 7.4 hold and the control function $\mathcal{U}(X,i)$ be given in Rule 7.4. Using the method in Rule 7.3 to determine tha value of τ, then the controlled oscillator system (7.20) is exponential stable in the sense of L^4 and almost surely.*

For the sake of showing the viability of our results, a numerical example is provided below.

Example 7.1. *Let the Markov chain $r(t)$ taking values in $\mathbb{S} = \{1, 2\}$ with $Q = \begin{pmatrix} -10 & 10 \\ 10 & -10 \end{pmatrix}$. We consider the oscillator system (7.18) with 25 oscillators. The parameters are given as*

$$\begin{aligned} &a_n(1) = 0.2, \quad b_n(1) = 0.3, \quad A_n(1) = 1.6, \quad B_n(1) = 0.05, \quad C_n(1) = 1.7, \quad \nu_n(1) = 0.5, \\ &a_n(2) = 0.5, \quad b_n(2) = 0.4, \quad A_n(2) = 2, \quad\;\; B_n(2) = 0.03, \quad C_n(2) = 1.9, \quad \nu_n(2) = 0.8, \end{aligned}$$

and the functions are given as $\Pi_{n,j}(x,y,1) = 0.01(x-y)$, $\Pi_{n,j}(x,y,2) = 0.005(x-y)$, $P_n(x,1) = 0.5x - 1.5x^3$, $P_n(x,2) = 0.3x - 1.8x^3$ for all $n, j = 1, \cdots, 25$. The connection weight $(e_{n,j}(i))_{25\times 25}$ can be obtained from the connection graphs in Fig. 7.1 and Fig. 7.2. Here for both two modes, node n stands for the n-th oscillator, directed edge (n,j) means the output of the j-th oscillator is connected with the input of the n-th oscillator, the number on the edge (n,j) is the value of $e_{n,j}(i)$. It *is then easy to verify that Assumption 7.4 is satisfied with* $\Lambda_{n,j}(1) = 0.01$, $\Lambda_{n,j}(2) = 0.005$, $J_n^{(1)}(1) = D_n^{(1)}(1) = 0.5$, $J_n^{(2)}(1) = D_n^{(2)}(1) = 1.5$, $J_n^{(1)}(2) = D_n^{(1)}(2) = 0.3$, $J_n^{(2)}(2) = D_n^{(2)}(2) = 1.8$.

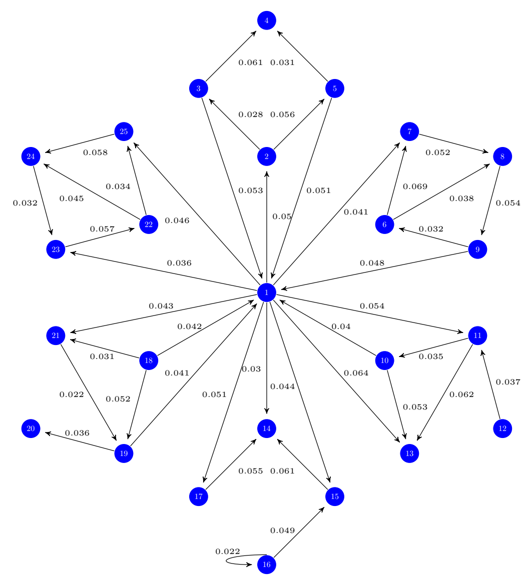

Figure 7.1: The oscillator connection graph at mode 1.

Through computer simulations (see Fig. 7.3), we find the oscillator system (7.18) is indeed unstable. Therefore, we want to use the controller $\mathcal{U}(X(t_\tau), r(t))I(t)$ *to realise stabilisation. Before that, we can easily get* $L_1(1) = 11.81$, $L_1(2) = 13.44$, $L_2(1) = 15.17$, $L_2(2) = 18.3092$, $h(1) = 0.5045$, $h(2) = 0.3012$, $\gamma(1) = 0.6295$, $\gamma(2) = 0.6212$, $\bar{\gamma}(1) = 0.7545$, $\bar{\gamma}(2) = 0.9412$, $\beta(1) = \bar{\beta}(1) = 0.02$, $\beta(2) = \bar{\beta}(2) = 0.024$. *We choose* $\kappa_1 = 6$ *and* $\kappa_2 = 5$, *as a result of which*

$$\mathcal{A} = \begin{pmatrix} 20.7411 & -10 \\ -10 & 18.7566 \end{pmatrix}, \quad \bar{\mathcal{A}} = \begin{pmatrix} 30.9821 & -10 \\ -10 & 26.2351 \end{pmatrix}$$

are non-singular M-matrices. The bounds of control area are given as $R_1 = 24.4949$ *and* $R_2 = 20.4124$. *Then Rule 7.4 is fulfilled. With detailed calculation, we derive that* $\mu_1 = 1.0391$, $\mu_2 = 4.2888$. *Thus we can take the control rate* $\delta_\mu = 0.9$ *to get the value of* τ^* *as* 2.17×10^{-6}. *By Theorem 7.2, we can conclude that the controlled oscillator system (7.20) is exponential stable in the sense of* L^4 *and almost surely if* $\delta_\mu = 0.9$ *and* $\tau < 2.17 \times 10^{-6}$. *The simulation results support our theory clearly (see Fig. 7.3).*

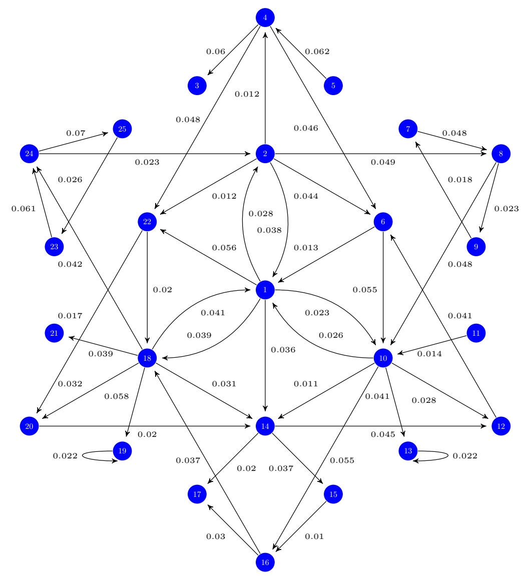

Figure 7.2: The oscillator connection graph at mode 2.

7.5 Summary

This chapter applies the Razumikhin idea to study the stabilisation of hybrid SDEs by discrete-time state feedback control, which works intermittently and is designed boundedly. Theoretically, the Razumikhin method is generalised in view of time-varying functions, rather than constants, where the time-inhomogeneous property of intermittent control could be fully made use of. In practice, the control cost could be reduced significantly since the controller is bounded, not observed continuously and having rest time. Moreover, there will be a wider range of applications especially for models that do not satisfy the linear growth condition. An example of the coupled Van der Pol–Duffing oscillator system is hence provided to show the practicability of the developed theory.

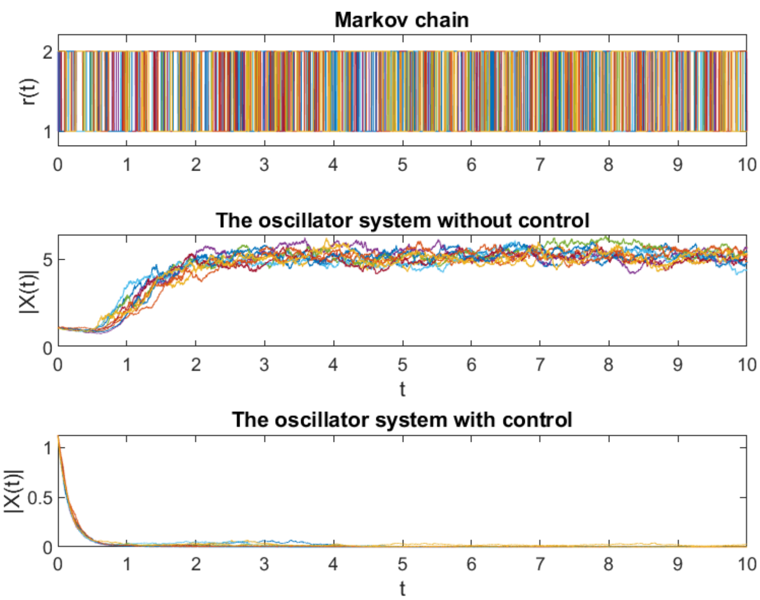

Figure 7.3: Ten sample paths of the Markov chain, the oscillator system (7.18), the controlled oscillator system (7.20) with $\delta_\mu = 0.9$ and $\tau = 1 \times 10^{-6}$. Here the initial data is fixed as $x_n(0) = 0.2$, $y_n(0) = 0.1$, $z_n(0) = 0$ for each $n = 1, \cdots, 25$.

Chapter 8

Conclusions and Future work

8.1 Conclusions

In this thesis, we have discussed the discrete-state-feedback stabilisation of highly nonlinear hybrid systems (namely, do not satisfy the classical linear growth condition).

In Chapter 3, we have shown that the unstable highly nonlinear hybrid SDEs could be stabilised by the globally Lipschitz continuous feedback controls based on discrete-time state observations, in the sense of H_∞ stability and almost surely asymptotic stability. By utilising the constant property of discrete-time states $x(t_\tau)$, we gave a new method in the estimation of difference between current-time state and discrete-time state. This improvement helped us to relax conditions imposed on the underlying systems and simplify the construction of Lyapunov functional as well as stability analysis. Moreover, we used the optimisation method to determine the value of τ, so that the inconvenience of finding free parameters was avoided.

But results developed in Chapter 3 only worked well for hybrid SDEs, where the structure in every Markovian mode was the same. The systems with changes between linear structures and highly nonlinear structures might not be included. Therefore, in Chapter 4, we considered the structured stabilisation of hybrid SDEs by discrete-time state feedback control. The mode-structure classification was made according to the Khasminskii-type condition, that is, the high-order term $|x|^{p+1}$ was strictly positive or vanished. Meanwhile, by spherical symmetry, we designed the feedback control in a bounded state area in order to reduce control cost. The stability studied was the every significant exponential stability.

Then we extended the structured stabilisation problem to hybrid SDDEs in Chapter 5. Condition on time delay was relaxed from differential assumption to a weak one, as a result of which the commonly seen sawtooth delay and piecewise constant delay could be covered. Due to the time delay effect, the conditions imposed became more complicated. Compared with the non-delay systems, time delay could influence the mode-structure classification scheme. Also for convenience, the mode space was divided into two sub-spaces, satisfying the classical Khasminskii-type condition and the generalised one, respectively.

The research technique in Chapters 3-5 was Lyapunov functional method. But constructing an appropriate functional was sometimes challenging. Even worse, it would less useful when integral transform failed or systems were discontinuous. In this case, Razumikhin method would be very helpful. Thus, in Chapters 6 and 7, we tried applying this method to our control problem. To highlight this idea, for simplicity, we did not consider the structured stabilisation. Also in these two parts, to use Razumikhin idea, we calculated the difference between current-time state and discrete-time state by Itô formula.

In Chapter 6, we firstly generalised the Razumikhin-type theorem to study the asymptotic boundedness and moment exponential stability of highly nonlinear functional equations. Then we used the developed theory to the stabilisation of hybrid SDDEs with more general time delays compared with Chapter 5. Since the time delay was relatively relaxed, we needed to give a little stronger conditions on the underlying systems. But stability properties became better, in the sense of $(p+1)$-th moment exponential stability and almost sure exponential stability. Chapter 7 was devoted to the stabilisation of hybrid SDEs by discrete-time state feedback control working intermittently, to let the controller have a rest time. Due to the discontinuity of intermittent control, the Razumikhin method was generalised to time-inhomogeneous functions, rather than constants.

In each chapter, we presented an example from real models. These applications to stochastic volatility model, neural networks, delayed nerual networks, van der Pol–Duffing oscillator and coupled oscillators showed the practicability of our theory.

8.2 Future work

After this thesis, there are still some problems deserved our further consideration.

In Chapters 4 and 5, due to mathematical techniques, we assumed that $\mathbb{S}_1$-subsystem had certain stability property. But it could be unstable, so this requirement seemed unreasonable in practice. On the other hand, we have noticed that there had been some results showing that we could only impose control on $\mathbb{S}_2$-subsystem to achieve stabilisation when $\mathbb{S}_1$-subsystem and $\mathbb{S}_2$-subsystem were both unstable, if their structures were the same. Therefore, in our future work, we will extend this theory to our stabilisation problem and get rid of the stability assumption on $\mathbb{S}_1$-subsystem.

In Chapters 6 and 7, because of the requirement to compare the past segment with current state in view of the same Lyapunov function, we gave two conditions in the assumption for control design purpose, rather than just one condition in the previous parts. Then in the future, we will continue to develop the Razumikhin technique to let this assumption relaxed. Also we will consider to use this method to structured stabilisation problem.

In Chapter 6, owing to the lack of integral transform method, the high-order term of delay $|y|^{p+1}$ vanished in the existence-and-uniqueness theorem unlike Chapter 5. We will try to consider the generalised Khasminskii-type condition for hybrid SDDEs with quite general time delays in the future.

While in Chapter 7, we only studied the intermittent control problem for hybrid SDEs. Then in the future work, we will consider it into delay systems.

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