A Classical View of the Quantum Vacuum

PhD Thesis

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This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

Abstract

In the coming years, a previously unexplored regime of quantum electrodynamics will be opened up to experimental study for the first time: the strong-field regime. Under the influence of strong electromagnetic fields, virtual particles in the quantum vacuum become polarised, and wave propagation in regions of strong field becomes nonlinear. This Thesis explores this regime using nonlinear vacuum electrodynamics.

The nonlinear nature of the vacuum imbues a region of strong field with an effective refractive index, such that wave propagation becomes analogous to propagation in a medium. This permits a novel view of an old problem concerning the energy-momentum tensor of light. In the context of light interacting with a medium two rival forms exist of the energy-momentum exist, each supposedly supported by theoretical and experimental evidence. By translating the problem to nonlinear electrodynamics, where the medium is replaced by a strong electromagnetic field, it is found that a much more precise statement can be made about which formulation should be adopted.

Maxwellian electrodynamics is known to be invariant under the conformal group, an extension of the usual Poincaré symmetry group. In general, nonlinear electrodynamics is invariant under Poincaré symmetries, and not the extended conformal group. The conformal group has been exploited in a wide range of areas of physics to simplify difficult problems. The possibility of using a conformally invariant, nonlinear theory of electrodynamics to describe strong-field physics is investigated. An entire class of conformally invariant nonlinear theories of electrodynamics is found, and their structure analysed. The role such theories may have in strong-field physics is then assessed, and it is found that in $(3 + 1)$ spacetime dimensions, the only physically meaningful conformally invariant theory of electrodynamics is Maxwell's theory.

A charged particle moving through a medium emits Cherenkov radiation when its velocity exceeds the phase velocity of light in that medium. Under the influence of a strong electromagnetic field the nonlinear nature of the vacuum allows for the possibility of high-energy particles to radiate via the Cherenkov process. The properties of this vacuum Cherenkov radiation are analysed from first principles, and applied to two physically relevant examples. It is found that this radiation process may be relevant to the excess signals of high-energy photons in astrophysical observations.

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Chapter 1

Introduction

Our current understanding of the laws of nature describes the world using four fundamental forces; the electromagnetic, weak, strong and gravitational forces. Of all of these, the one most relevant to everyday experience is the electromagnetic force, which describes the interaction between matter and light. The modern understanding of this force began with the revolutionary work of Faraday and Maxwell [1], the latter of which achieved one of the first instances of "unification" in physics, by demonstrating that electric and magnetic fields are manifestations of the same underlying electromagnetic force. The development of special relativity [2], and quantum mechanics led to a further paradigm shift in how we describe electromagnetism, through the work of Dirac[3, 4], and later Feynman [5–8], Schwinger [9–11], Tomonaga [12, 13], and Dyson [14, 15] with the development of quantum electrodynamics (QED).

QED represents the most precise and successful theory to ever be developed in physics, as evidenced by the extraordinary agreement between theoretical predictions [16, 17] and experimental data [18, 19], which give an accuracy to over 10 significant figures. The success of QED rests on the use of perturbation theory, where scattering processes are calculated by an expansion in powers of the QED-coupling, the fine-structure constant $\alpha = e^2/4\pi\epsilon_0 c\hbar \simeq 1/137$. Tests of QED have all but exhausted the high-energy frontier probed by the most powerful particle accelerators available, such as the Large Hadron Collider, and aside from some notable anomalous results (for example the "proton size puzzle", where the predicted and measured size of the proton appear to be in disagreement [20]), there has been almost no discrepancy in its predictions, or hints at new physics in this regime. QED is one part of a larger framework, the Standard Model of particle physics, which unites the electromagnetic, weak and strong forces under one common description: quantum field theory. So far, the electromagnetic and weak forces have been shown to be unified into the electroweak force, and a main goal of theoretical physics over the past 70 years has been to unify all three forces of the Standard Model, and ultimately gravity, under one framework. However, with no prediction of the Standard Model shown to be incorrect so far, how can we hope to make progress in pursuit of this goal? One thing which could be done is to look for unexplored parameter regimes within the Standard Model, to try and find where our descriptions break down, and progress can be made.

Recently, there has been a resurgence of interest in an essentially unexplored parameter regime, where it is hoped that the limits of QED can be tested. This is the so-called strong- or high-field regime, where instead of pushing to higher and higher energies, we consider the influence of high-intensity electromagnetic fields on the behaviour of the fundamental interactions of light and matter (for extensive reviews see [21–33]). Interest in the strong-field regime began early on in the history of QED, with significant contributions from Sauter [34], Heisenberg & Euler [35] and Schwinger [36]. It is well known that quantum field theories predict the existence of *virtual* particles, and that due to the uncertainty principle, particle-antiparticle pairs fluctuate in and out of existence in the quantum vacuum. In the standard, high-energy approach to QED, these particles play the role of mediating between incoming and outgoing particle states. Introducing a strong electromagnetic field changes the situation considerably. The electromagnetic field can interact with the virtual particles, which has the effect of changing the field equations from linear to nonlinear, thus opening up an entirely new regime in QED. The virtual particles are charged, and so interact with electromagnetic fields. With a strong enough field, they can become momentarily polarised, and real particles propagating through the region of strong field can interact strongly with the virtual particles, producing a vast range of effects which cannot be observed in vacuum. The work of Sauter [34] and Schwinger [36] described the scale at which these nonlinear effects become important, defining the critical field of QED, $E_S = m_e^2 c^3/|e|\hbar \simeq 1.32 \times 10^{18}$ V.m^{-1} (corresponding to an intensity $I_S \sim 10^{29}$ W.cm^{-2})[1]. This is the field strength required to perform work on the electron-positron pair equal to the rest mass energy of the particles over a Compton wavelength, which brings the particles on-shell.[2] In other words, *strong electromagnetic fields can create real particles from vacuum.* This is the so-called Schwinger mechanism for pair production [36]. Schwinger pair production is a fundamentally *nonperturbative* effect, meaning that the usual perturbation theory

[1]Here m_e is the electron mass, c is the speed of light, e is the electron charge, and $\hbar$ is the reduced Planck's constant.

[2]The terms "on-shell" and 'off-shell' refer to whether or not the particles satisfy the relativistic mass-shell condition. A particle with 4-momentum p^μ and mass m is said to be "on-shell" and real if it satisfies $p^2 = m^2$, or "off-shell" and virtual if $p^2 \neq m^2$.

which lies at the heart of the success of QED cannot be used.

Study of the effect of strong electromagnetic fields remained a purely theoretical pursuit until the 1960's due to the required field strengths being well beyond the reach of anything which could be experimentally produced. This changed with the invention of the laser, which initiated a new wave of theoretical interest in the strong-field regime (see for example [37–40]). Perhaps the most important development, which has made the study of these effects an experimental reality is the dramatic increases in power offered by the chirped-pulse amplification technique [24, 41], recently celebrated with the award of the Nobel prize in physics.[3] The advancement in laser power has steadily progressed since, with a number of petawatt class facilities currently online (e.g. [42–46]). These systems represent a current state-of-the-art, but still lie well below the critical field, E_S, in terms of the field strengths which can be achieved. Further strides towards the strong field limit will come over the next few decades, however, as a range of new facilities are either planned or under construction, which will begin a new dawn in high-intensity physics. These include upgrades to the VULCAN laser system at the Central Laser Facility in the UK [47, 48] and the APOLLON laser in France [49], both of which are envisaged to reach 10PW of power. Currently under construction is the Extreme Light Infrastructure (ELI), a European wide project aiming to deliver four facilities with a focus on the use and applications of high-intensity lasers. Of these, the Nuclear Physics program (ELI-NP) [50] is hoped to come online in the next few years,

[3] A major challenge in amplifying ultrashort laser pulses is that the high intensities involved can cause unwanted nonlinear effects in, or damage to, the gain medium. Chirped pulse amplification overcomes this by first temporally stretching and “chirping” (low frequency components move to the front) the pulse to create a longer pulse with lower peak power, before the interaction with the gain medium. The pulse is then amplified, compressed, and the “chirp” removed to create a short, high-intensity pulse without damaging the gain medium or generating unwanted nonlinear effects.

with projected intensities of $I \sim 10^{23}$—10^{24} W.cm^{-2} hoped to be achieved. Though still several orders of magnitude less than the critical intensity I_S, a vast array of effects are hoped to be observed [23, 25–28, 31–33]. These high-intensity laser systems, and others which have been planned, such as XCELS in Russia [51], are primarily responsible for the resurgence of interest in the strong-field regime.

One of the first effects to be investigated will be the unresolved problem of radiation reaction [52], the response of a charged particle to the radiation it emits in an electromagnetic field. Recently, the first data measuring the radiation reaction effect has been obtained by experimental groups using the Gemini laser at the Central Laser Facility in the UK [53, 54], which has prompted new questions about our theoretical understanding of the problem [54]. The problem may lie in the weakness of radiation reaction at the laser intensities used in the experiments, and so the increased field strengths provided at upcoming facilities will allow us to further probe this effect. Investigating pair production will also be a key goal at next generation facilities. As discussed above, nonperturbative Schwinger pair production requires a field strength of the order E_S, though various proposals have been made of configurations which could possibly bring down the pair-production threshold by several orders of magnitude (e.g. [55–66]). Aside from the Schwinger mechanism for pair production, there is also the trident process, where an electron emits a photon which subsequently decays into an electron-positron pair. Due to the interaction with a strong background field, this can occur by either the one-step (where the intermediate photon is off-shell) or two-step (where the intermediate photon is on-shell) processes [67]. The one-step process was measured in the landmark experiment E-144 at the Stanford Linear Accelerator Centre

(SLAC) [68, 69], twenty years ago, with no new experiments probing this pair production mechanism. The huge advances in field strength since this experiment will give a new opportunity to measure this effect. Related to the Schwinger pair-production mechanism is a range of processes related to to polarisation of the vacuum induced by strong fields, mentioned above, which will be discussed in more detail in the following Chapter but includes vacuum birefringence, photon splitting, and photon-photon scattering [23, 25, 27, 28, 31]. There are also several studies indicating that these future facilities may be able to investigate new physics beyond the Standard Model [26].

There are, however, also other sources of strong electromagnetic fields in the universe. In astrophysics, rapidly rotating neutron stars, or pulsars, are known to produce magnetic fields several orders of magnitude above the critical magnetic field of QED, $B_S = E_S/c = m_e^2 c^2/e\hbar \simeq 4.41 \times 10^9$ T (see for example [70–72]). Many of the processes described above have also been considered in such strong magnetic fields, such as Compton scattering [73] (which is closely related to radiation reaction) [74–76], photon splitting [77–79] and a range of other vacuum birefringence and vacuum polarisation effects, for example [80–88]. See also the reviews [89, 90]. Strong electromagnetic fields can also be generated in particle collisions using heavy nuclei, such as in the experiments conducted at the Relativistic Heavy Ion Collider (RHIC) [91], and the Large Hadron Collider (LHC) [92] (see for example [93]).

1.1 Outline of the thesis

To describe strong-field effects, two approaches are available. The first uses the full theory of QED, in the background of a strong electromagnetic field, and is typically referred to as strong-field QED. The second is based on the use of *effective nonlinear theories of electromagnetism*, which is the main topic of this thesis, and will be discussed in detail in Chapter 2. There are two prominent examples in the literature, the theories of Euler-Heisenberg, which is found as a low-energy limit of QED, and Born-Infeld, which in modern work is found as a low-energy limit of string theory. Each of these theories is built up from the Lorentz invariants of the electromagnetic field. It is possible to extend both of these theories to a class of generalised nonlinear theories, which can be studied in a very general sense based on the requirement of only depending on the two independent electromagnetic invariants of the field. Chapter 2 will discuss the effects which can be described by each, as well as give the technical details of generalised nonlinear theories of electromagnetism.

In the linear vacuum described by Maxwell's theory, there is no ambiguity in the energy-momentum of radiation. However, the nonlinearities induced in the vacuum by strong electromagnetic fields can be described as as analogous to a dielectric medium, imbuing the region of strong field with an effective refractive index. In the case of radiation interacting with a real dielectric medium, there is a well known issue surrounding the "correct" form of the energy-momentum of light as it propagates in the material, often referred to as the "Abraham-Minkowski controversy". We investigate this topic in an entirely novel way in Chapter 3, by analysing the energy-momentum of radiation moving though a strong electromagnetic background field. This gives an outline of the work contained within [94], as well as significantly extending the analysis.

As mentioned, the properties of the linear Maxwellian vacuum are well understood. Much of this can be attributed to the linearity of the theory, however, there is also an additional property of Maxwellian electrodynamics which lends to its simplicity. This is the fact that it obeys the conformal symmetry group, an extension of the usual Poincaré group which includes the additional symmetries of invariance under special conformal transformations and dilations (details to follow). In the quantisation procedure from Maxwell to QED, this conformal invariance is lost, and invariance is constrained to the Poincaré group. The main reason for this loss of conformal symmetry is the introduction of an explicit scale, namely the electron mass m_e in QED[4]. This constraint, however, is not *a priori* imposed on generalised nonlinear theories of electrodynamics. The possibility of obtaining a conformally invariant nonlinear theory, and whether such a theory could in some regime be used as a physically relevant description of strong-field physics is investigated in Chapter 4.

Each of the Chapters just described deal mainly with the formal aspects of nonlinear electrodynamics. The theories are rich in structure, and warrant the investigation of these properties, but perhaps of greater importance is their phenomenological implications. Many of these have been well studied, and an outline is presented in the next Chapter. An interesting effect, which has attracted less attention that its counterparts, is the possibility of particles propagating in the nonlinear vacuum emitting Cherenkov

[4] At a more fundamental level, the classical action of the Standard Model is *almost* conformally invariant with the only violations of the symmetry being due to the Higgs couplings. See discussion in [95] and references therein

radiation. Cherenkov radiation is usually considered in the context of particles propagating through a medium, and occurs when the particle velocity exceeds the phase velocity of light. In a strong electromagnetic field, wave propagation becomes nonlinear, reducing the phase velocity of light to less than the vacuum speed of light, c, providing the opportunity for relativistic particles moving through the region of strong field to emit radiation through the Cherenkov mechanism. This is the subject of Chapter 5, where we present the first fully quantitative, first principles analysis of the Cherenkov process in strong electromagnetic fields, and discuss the phenomenological implications in the context of both the high intensities provided by upcoming laser-matter experiments, and the strong magnetic fields found produced in astrophysical contexts. This work gives a more comprehensive view of the work presented in [96]. Finally, we summarise, provide some closing remarks, and discuss the outlook of the presented work in Chapter 6.

1.2 Units and conventions

Throughout this work a covariant notation of tensors and 4-vectors is used, where indices are raised and lowered using the metric tensor $g_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ with Greek indices running from $\mu = 0, ..., 3$. Thus, a vector $a^\mu = (a^0, \boldsymbol{a})$ contracted with the metric gives a covector $a_\mu = (a^0, -\boldsymbol{a})$. The Einstein summation convention for repeated tensor indices is assumed throughout, such that the Lorentz contraction of two 4-vectors is

$$(a.b) \equiv a^\mu b_\mu = a^0 b^0 - \boldsymbol{a}.\boldsymbol{b}. \tag{1.1}$$

We use bold-font to denote the spatial components of a 4-vector, and distinguish the contraction of 4-vectors (1.1) with the usual scalar product of 3-vectors $\boldsymbol{a}.\boldsymbol{b} = \sum_{i=1}^{3} a^i b^i$ using the bracketed notation above.

For a given rank-2 tensor, $A^{\mu\nu}$, the dual tensor will be denoted $\widetilde{A}^{\mu\nu} = \epsilon^{\mu\nu\alpha\beta} A_{\alpha\beta}/2$, where $\epsilon^{\mu\nu\alpha\beta}$ is the completely anti-symmetric Levi-Civita tensor. Where necessary, we will use the convention $\epsilon^{0123} = +1$, such that $\epsilon_{0123} = -1$.

We use natural units throughout, unless otherwise stated, such that $c = \hbar = \epsilon_0 = 1$, where c is the speed of light, $\hbar$ is the reduced Planck's constant, and ϵ_0 is the vacuum permittivity. The charge of the electron is $e = -|e| \simeq -1.602 \times 10^{-19}$ C, with mass $m_e \simeq 9.109 \times 10^{-31}$ kg. Where relevant, we will also use the proton mass $m_p \simeq 1.673 \times 10^{-27}$ kg.

Chapter 2

Nonlinear vacuum electrodynamics

The study of nonlinear theories of electrodynamics can be traced back to the early days of QED. In general, these theories aim to describe strong field processes, which are not captured by a Maxwellian description of electrodynamics, by using effective field theories which have nonlinear interaction terms between electromagnetic fields. The two most prominent examples are the theories due to Heisenberg & Euler [35], and Born & Infeld [97], which have very different origins and uses in the modern literature.

Euler-Heisenberg theory is obtained through integrating out the fermionic degrees of freedom in the path integral of quantum electrodynamics (QED), which then allows for a nonlinear interaction between real photons, as shown in Figure 2.1. The physical process responsible for this is the existence of fluctuations – in the form of virtual particle-antiparticle pairs – within the QED vacuum. In the presence of strong fields, these charged virtual pairs can become polarised and interact with photons passing through

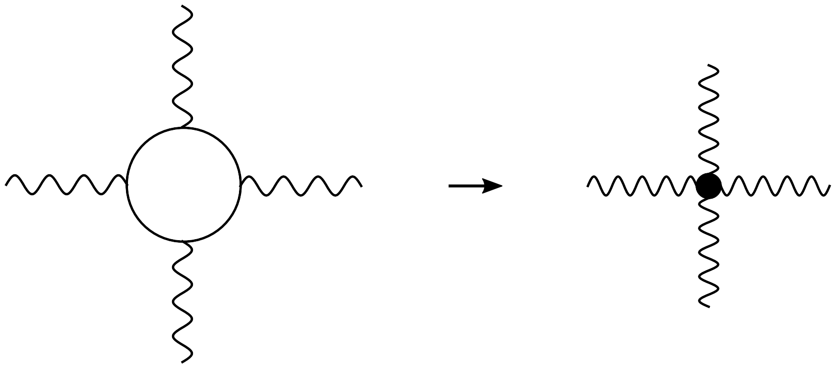

Figure 2.1: Euler-Heisenberg theory integrates out the fermionic degrees of freedom corresponding to virtual particles (solid lines; left), to give an effective theory where photons (wavy lines) interact directly (right).

high-field regions. As mentioned briefly in the introduction, the Euler-Heisenberg theory can be used to describe a wide range of effects related to strong-field QED (for an overview we direct the reader to the reviews [22, 23, 27, 98–103]). After integrating out the fermionic degrees of freedom, the theory is described by an effective action. The effects described by the Euler-Heisenberg theory can be defined as either dispersive, or absorptive processes, related to the real and imaginary parts of the effective action, respectively. Dispersive effects are those which describe photon-photon interactions. This is possible in a region of strong electromagnetic field, due to the fact that the vacuum behaves analogously to a dielectric material with well defined refractive index [22, 90]. This was recognised in the early work of Halpern [104], Euler & Kockel [105] and Weisskopf [106] as leading to the possibility of real photon-photon scattering (such as that shown in Figure 2.1), which was first calculated later by Karplus & Neuman [107], and is still an actively researched effect, with many different proposals for how it may be observed [23, 40, 108–116], and suggestions that it could be used as a probe of physics beyond the Standard Model [117]. There are also the related effects of photon splitting [77–79, 118–120] where an initial high energy photon interacts with the strong background field and "decays" into a pair (or more) of lower energy photons, and vacuum birefringence [39, 77, 121–127] which is completely analogous to birefringence in a medium, but induced by the strong electromagnetic background field. The latter of these, vacuum birefringence, is being investigated through the PVLAS experiment [128]. The absorptive processes, on the other hand, are those which take incoming photons to outgoing electron-positron pairs, i.e. pair production. This has become one of the most actively researched effects in the study of Euler-Heisenberg theory, and strong-field QED in general. The theory of Euler-Heisenberg has also been extended to non-abelian gauge fields [129–131], and had a significant impact on the study of quantum field theories in curved space-times [132–137].

Born-Infeld electrodynamics was originally proposed as a theory to describe the behaviour of the electron which solved the problem of the infinite self-energy which had plagued earlier calculations. Unlike the theory of Euler-Heisenberg, however, it is a fundamentally *classical* theory, having been derived essentially as a direct generalisation of Maxwellian electrodynamics. The rapid adoption of quantum physics into the description of particles and fields with the development of QED could in part be seen as a reason for why Born-Infeld theory was given less attention, though some interesting properties of the theory were recognised early on by Schrödinger. The theory admits an electric-magnetic duality [138], and allows for interesting exact solutions of solotonic electromagnetic waves [139, 140]. Much later, it was found that the Born-Infeld theory is the only known nonlinear generalisation of electrodynamics which preserves many of the key results of linear Maxwellian theory, specifically the lack of birefringence and shocks [141–143] (for a review of the general properties of the Born-Infeld theory, we refer the reader to [144]). Taking the theory as a fundamental description of electromagnetism and attempting to quantize the theory using the well established procedures of QED, however, led some, including Dirac [145], to believe that it was incompatible with quantum field theory. This would have consigned the theory to the realm of interesting, but unimportant, physical theories, had it not been for its rediscovery as a low-energy limit of certain string theories [146]. This has led to a resurrection of the theory, with numerous studies investigating this connection (for reviews see [147–149]). There are also several studies connecting General Relativity and Born-Infeld electrodynamics, such as modifications to General Relativity inspired by the Born-Infeld theory being found to exhibit some useful, and interesting properties [150].

While these two examples represent the most well known, and best studied, particular theories, all theories of nonlinear electrodynamics can be grouped into a generalised description. Analysing the behaviour of these generalised theories allows the fundamental properties shared among each class of theory to be defined, and also to show when an exceptional theory arises with different properties. There have been many studies on such general theories, for example [141–143, 151–165]. The general structure and properties of these theories is presented in the rest of this chapter.

2.1 General nonlinear vacuum electrodynamics

To begin, we first ask what requirements a nonlinear theory of electromagnetism must have, and how do we define them? Regardless of their origins, or uses in the literature, all theories of nonlinear electrodynamics are built up purely from electromagnetic fields, which must obey Lorentz invariance and gauge symmetry. As such the action integral S defining the theory must be a Lorentz scalar. Given an electromagnetic field $F_{\mu\nu}$ defined in terms of the electromagnetic gauge potential A_μ,

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \tag{2.1}$$

and its dual $\widetilde{F}_{\mu\nu} = \epsilon_{\mu\nu}{}^{\alpha\beta} F_{\alpha\beta}/2$ (where $\epsilon_{\mu\nu}{}^{\alpha\beta}$ is the Levi-Civita tensor), the only two Lorentz invariants[1] we can generate are (see e.g. [166, 167])

$$X = -\frac{1}{4} F^{\mu\nu} F_{\mu\nu} = \frac{1}{2}\left(\left|\boldsymbol{E}\right|^2 - \left|\boldsymbol{B}\right|^2\right) \tag{2.2}$$

$$Y = -\frac{1}{4} \widetilde{F}^{\mu\nu} F_{\mu\nu} = \boldsymbol{E}.\boldsymbol{B}. \tag{2.3}$$

The invariant X is a genuine Lorentz scalar, however the invariant Y is a pseudoscalar due to the fact that it is formed from the product of a vector (the electric field $\boldsymbol{E}$) and a pseudovector (the magnetic field $\boldsymbol{B}$). This means that under parity transformations $(t, \boldsymbol{x}) \to (t, -\boldsymbol{x})$ the pseudoscalar Y will change sign, $Y \to -Y$. Aside from the weak nuclear force [168], the fundamental forces are known to be invariant under parity transformations, and so any generalised nonlinear theory of electrodynamics must contain only even powers of the invariant Y. While it is possible to consider Lorentz scalars (or pseudoscalars) built up from higher order powers of the electromagnetic field tensor

[1] To be more precise, the invariants X and Y are the only independent invariants of the electromagnetic field in $(3+1)$ spacetime dimensions and in theories with no higher derivatives of the electromagnetic field. If higher derivative terms of the electromagnetic field $F^{\mu\nu}$ were included in the action of the theory, one could form further invariant quantities, e.g. $\partial_\mu F^{\mu\nu} \partial_\lambda F^{\lambda}{}_\nu$, $F^{\mu\nu} \partial_\lambda \partial^\lambda F_{\mu\nu}$, etc. Such theories are not considered in this work.

and its dual, these can be shown to reduce to functions of the two invariants X and Y, such that (2.2) and (2.3) represent the only *independent* invariants of the electromagnetic field [166, 167]. Appendix A gives some useful identities involving $F_{\mu\nu}$ and $\widetilde{F}_{\mu\nu}$, which will be used extensively throughout this thesis. With these considerations we can now define a general nonlinear theory of electrodynamics by using Lagrangian functions $\mathcal{L}(X, Y)$ which are arbitrary functions of the invariants (2.2) and (2.3), such that the action is,

$$S = \int d^4z\sqrt{-g}\mathcal{L}(X, Y), \tag{2.4}$$

where $g = \det g_{\mu\nu}$ is the determinant of the metric. Nonlinear theories of electromagnetism of this form are sometimes referred to as the Plebanski class of electrodynamics, owing to the works [142, 143] studying theories of this form (though Boillat also analysed this class independently [141]). The field equations are obtained in the usual way by varying the action with respect to the electromagnetic gauge potential, A^μ, which gives,

$$\partial_\mu H^{\mu\nu} = 0, \tag{2.5}$$

where we define the excitation tensor $H^{\mu\nu}$,

$$H_{\mu\nu} \equiv \frac{\partial\mathcal{L}}{\partial X}F_{\mu\nu} + \frac{\partial\mathcal{L}}{\partial Y}\widetilde{F}_{\mu\nu}. \tag{2.6}$$

This along with the usual Bianchi identity,

$$\partial_\mu \widetilde{F}^{\mu\nu} = 0, \tag{2.7}$$

describe the dynamics of the fields. The excitation tensor (2.6) can be interpreted as encoding the nonlinearities of the theory, and can be expressed in terms of the polarization and magnetization tensors of the strong-field vacuum[2]. The effect of this on radiation moving through the nonlinear vacuum is analogous to propagation through a dielectric medium, a fact that will be exploited later in this work.

Though the general class of nonlinear electrodynamics theories (2.4) will be the primary focus of this work, it will be useful also to consider the specific examples. First, in the notation presented above, the Lagrangian function of linear Maxwellian electrodynamics is simply,

$$\mathcal{L}_{\mathrm{M}} = X. \tag{2.8}$$

In this case, using (2.5), we recover the usual vacuum field equations of Maxwell's theory,

$$\partial_\mu F^{\mu\nu} = 0. \tag{2.9}$$

The success of Maxwellian electrodynamics in the weak-field limit also gives us a physical requirement for any nonlinear extension, in that it must reproduce the known

[2]An interesting corollary to this is that properties of the linear vacuum may be able to be derived from the underlying quantum theory, see for example [169, 170].

effects of linear electromagnetism in this limit. This fact will play a significant role in the discussion of conformally invariant nonlinear theories of electrodynamics in Chapter 4.

The Euler-Heisenberg Lagrangian function is derived from QED coupled to a strong background electromagnetic field, in the limit of the background fields being treated as constant.[3] This gives an effective nonlinear theory which describes the behaviour of strong fields interacting with virtual particles in the vacuum [35, 36],

$$\mathcal{L}_{\mathrm{EH}} = -\frac{1}{8\pi^2}\int_0^\infty \frac{d\eta}{\eta^3}\left\{\frac{e^2ab\eta^2}{\tanh(ea\eta)\tan(eb\eta)} - 1 - \frac{e^2\eta^2}{3}\left(a^2-b^2\right)\right\} e^{-\eta e E_S}, \tag{2.10}$$

where E_S is the critical field, and $\pm a$ and $\pm ib$ are the eigenvalues of the constant field tensor $F^{\mu\nu}$. These eigenvalues are related to the invariants (2.2) and (2.3) by

$$a = \sqrt{\sqrt{X^2+Y^2}+X}, \qquad b = \sqrt{\sqrt{X^2+Y^2}-X}. \tag{2.11}$$

The Lagrangian (2.10) is derived from the path integral representing the coupling of QED to a background gauge field A^μ,

$$Z[A] = \int D\psi D\bar{\psi} e^{iS_{\mathrm{QED}}[A,\psi,\bar{\psi}]}, \tag{2.12}$$

[3]Typically in strong-field QED, the particle energies and field strengths are strong enough that the formation length of a given process is typically much smaller than the scale of the structure of an inhomogeneous field, such that the field can be treated as at least locally constant [171].

where $S_{\text{QED}}[A,\psi,\bar{\psi}] = \int d^4z \mathcal{L}_{\text{QED}}[A,\psi,\bar{\psi}]$ is the QED action with Lagrangian function,

$$\mathcal{L}_{\text{QED}}[A,\psi,\bar{\psi}] = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,. \tag{2.13}$$

Here, γ^μ are the Dirac matrices, ψ and $\bar{\psi}$ are the fermion fields, $D_\mu = \partial_\mu + ieA_\mu$ is the covariant derivative, and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the electromagnetic field tensor. Under the approximation that the background field $F_{\mu\nu}$ can be treated as constant or slowly varying with respect to quantum fluctuations, the path integrals over ψ and $\bar{\psi}$ in (2.12) can be performed, giving the effective Lagrangian (2.10).

The theory of Born-Infeld, on the other hand, has completely different origins and uses in the modern literature. The original formulation [97], as discussed above, sought a theory of electrodynamics which self-consistently solved the problem of the infinite self-field of the electron. The theory is defined by the Lagrangian function,

$$\mathcal{L}_{\text{BI}}(X,Y) = \frac{1}{\kappa^2}\left(1 - \sqrt{1 - 2\kappa^2 X - \frac{\kappa^4}{4}Y^2}\right). \tag{2.14}$$

where the Born-Infeld parameter κ is an undefined constant of nature. In the original formulation of the theory, it essentially played the role of an absolute allowable electric field strength, with the correspondence $\kappa \propto 1/E_{\text{abs}}$, where the absolute field strength was estimated as $E_{\text{abs}} \simeq 2 \times 10^{20}$ V.m^{-1} [97]. This was to play a role analogous to the speed of light, c, in special relativity. Consider the Lagrangian L of a relativistic point particle of mass m and 3-velocity $\boldsymbol{v}$,

$$L = -mc^2\sqrt{1 - \frac{|\boldsymbol{v}|^2}{c^2}}. \tag{2.15}$$

Since if $|\boldsymbol{v}| > c$ the Lagrangian becomes complex, and no longer physically meaningful, the speed of light, c, plays the role of an upper limit on particle velocities. In complete analogy with this, Born & Infeld hoped to derive a Lagrangian function in which E_{abs} limited allowable electromagnetic field strengths, thus completely removing the infinite self-field problem, and arrived at (2.14). However, the advent of quantum mechanics, and subsequently quantum electrodynamics, gave a more universal approach to removing divergences through the process of renormalisation, which essentially ended the hopes for Born-Infeld as a fundamental theory of electromagnetism. In the modern literature, Lagrangian functions of the form (2.14) have been found as a low-energy limit of string theory [146], where the parameter κ is typically associated with the string tension.

The full forms of the Lagrangians of Euler-Heisenberg and Born-Infeld are valid for all applicable values of electromagnetic field strengths, and as such describe the wide range of effects detailed above. In many instances, however, a weak-field $E, B \ll E_S, B_S$ expansion of (2.10) and (2.14) can provide a description of many processes [90]. Using this approximation, we find that, both of the theories can be expressed in the form,

$$\mathcal{L} = X + \lambda_+ X^2 + \lambda_- Y^2, \tag{2.16}$$

to lowest non-trivial order, with the constants,

$$\text{Euler-Heisenberg:} \qquad \frac{1}{4}\lambda_+ = \frac{1}{7}\lambda_- = \frac{\alpha}{90\pi}\frac{1}{E_S^2} \tag{2.17}$$

$$\text{Born-Infeld:} \qquad \lambda_+ = \lambda_- = \frac{1}{2}\kappa^2. \tag{2.18}$$

As discussed above, the weak-field Lagrangian (2.16) is free of terms linear in the invariant Y, such as Y or XY in (2.16), as such terms lead to a violation of parity and are neglected on physical grounds. From (2.17) and (2.18) we can see that the two theories are fundamentally different. Euler-Heisenberg theory predicts the effect of vacuum birefringence, whereby the phase velocity of light propagating through a region of strong electromagnetic field depends on its polarisation. In the weak field limit, this can be seen as a direct consequence of the constants $\lambda_\pm$ defined in (2.17) not being equal.[4] On the other hand, the theory of Born-Infeld is defined in the weak-field limit by (2.18), where the equality of the two constants $\lambda_\pm$ means that the theory does not exhibit the birefringence effect, which along with the fact that it does not allow for shocks makes it unique in the class of nonlinear theories of electrodynamics [144]. Though less well motivated than Euler-Heisenberg theory as a physically meaningful effective description of strong-field physics, this (along with its discovery as a low-energy limit of string theory [146]) has been one of the main factors leading to its continued study. In the coming years, experiments hoping to measure vacuum birefringence such as the PVLAS experiment [128] will determine whether Born-Infeld theory is a physically meaningful

[4] This is shown explicitly in Chapter 5, where we derive the phase velocity of waves propagating through a strong electromagnetic field in the weak-field limit defined by Lagrangian functions of the form (2.16) (see equation (5.36)).

description of nature, or just an interesting but unphysical model of electrodynamics. Since Euler-Heisenberg theory is derived directly from QED, where observational consequences are sought in the work that follows we will use the simplified weak-field limit (2.16), with (2.17) defining the constants $\lambda_{\pm}$.

So far, we have presented the general theory of nonlinear electrodynamics, the resulting field equations, and highlighted the forms of the most commonly used theories. We are now in a position to turn our attention to a little discussed feature of light propagation in nonlinear electrodynamics, which will prove to be an interesting and novel take on an old problem.

Chapter 3

The energy-momentum of light

Since the beginning of the 20th century, a debate has been fought concerning the definition of the momentum of light in a medium. The behaviour of light in the linear vacuum has been well understood since the pioneering work of Maxwell. Due to the vacuum constitutive relations relating the displacement and electric fields $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E}$, and the magnetic field and magnetic induction $\boldsymbol{H} = \boldsymbol{B}/\mu_0$, the momentum density of a light wave, $\boldsymbol{g}$, can be equivalently described using either $\boldsymbol{g} = \boldsymbol{D} \times \boldsymbol{B}$ or $\boldsymbol{g} = \boldsymbol{E} \times \boldsymbol{H}/c^2$. However, if the wave is propagating through some material, the constitutive relations are changed to incorporate the electromagnetic properties of the medium, namely the polarisation, $\boldsymbol{P}$, and the magnetisation, $\boldsymbol{M}$, such that $\boldsymbol{D} = \varepsilon_0 \boldsymbol{E} + \boldsymbol{P}$ and $\boldsymbol{H} = \boldsymbol{B}/\mu_0 - \boldsymbol{M}$. The effect of this is that $\boldsymbol{D} \times \boldsymbol{B} \neq \boldsymbol{E} \times \boldsymbol{H}/c^2$, or in other words, we have two distinct ways of describing the momentum density of the light wave, which lead to different physical predictions of the total momentum carried by the wave in the medium. The debate surrounding which of these relations should be adopted to describe the momentum density is often referred to as the "Abraham-Minkowski controversy". Minkowski [172, 173] proposed that the momentum density should be $\boldsymbol{g}_M = \boldsymbol{D} \times \boldsymbol{E}$, corresponding to a wave with initial momentum $\boldsymbol{p}$ having momentum $\boldsymbol{p}_M = n\boldsymbol{p}$ inside a medium of refractive index n, and Abraham [174, 175] instead favoured the $\boldsymbol{g}_A = \boldsymbol{E} \times \boldsymbol{H}/c^2$ formulation, which gives a total momentum $\boldsymbol{p}_A = \boldsymbol{p}/n$.

Both $\boldsymbol{p}_M$ and $\boldsymbol{p}_A$ attributed to the rival forms of the momentum density can be derived from simple physical considerations [176]. To obtain the Minkowski form of the momentum, we consider an atom of mass m, with a transition frequency ω_0, which is moving away from a source of light at a velocity v. The source is producing light with a fixed frequency ω, and the entire system is immersed in a medium with a refractive index n [177], as shown in Figure 3.1. As it propagates, the atom will absorb photons from the field if the Doppler-shifter frequency of the light matches with the transition frequency, i.e. if

$$\omega_0 \simeq \omega \left(1 - \frac{nv}{c}\right), \tag{3.1}$$

with c the speed of light. After absorbing a photon, the atom will have a new velocity v', and by simple consideration of the conservation of energy and momentum we have,

$$\frac{1}{2}mv'^2 + \hbar\omega_0 = \frac{1}{2}mv^2 + \hbar\omega, \qquad mv' = mv + p_{\mathrm{photon}}, \tag{3.2}$$

with p_{photon} the momentum of the absorbed photon. Combining the two equations in

(3.2) we arrive at,

$$p_{\text{photon}} = \frac{\hbar\omega n}{c}\frac{2v}{v+v'} \simeq \frac{\hbar\omega n}{c}, \tag{3.3}$$

where we used the fact that the change in the velocity of the atom will be small, such that $v' \simeq v$. Since the photon in free space would have momentum $p = \hbar\omega/c$, we have arrived at the Minkowski form of the momentum $p_{\text{M}} = np$.

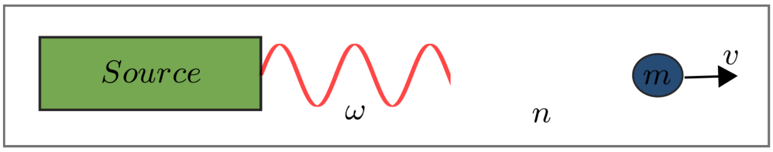

Figure 3.1: An atom and light source immersed in a medium with refractive index n. The atom has a phase transition frequency ω_0, and is moving with a velocity v away from the light source. The light is being produced at a frequency ω.

The derivation of the Abraham form of the momentum is also straightforward. In this case we instead consider a photon with an initial momentum $p = \hbar\omega/c$, which is travelling in the z-direction towards a transparent block with refractive index n [178] as shown in Figure 3.2. The block is initially at rest, and has a thickness L and mass M. As the photon passes through the block, its speed is reduced by the refractive index to c/n. By considering the uniform motion of the centre of mass of the system, we can determine the amount by which the block will be displaced from its initial position,

$$\Delta z = (n-1)L\frac{\hbar\omega}{Mc^2}. \tag{3.4}$$

The fact that the block has been displaced must mean that there has been a momentum transfer from the photon to the block. The momentum of the block can be determined as,

$$p_{\text{block}} = M\frac{\Delta z}{L(n/c)} = \left(1 - \frac{1}{n}\right)\frac{\hbar\omega}{c}. \tag{3.5}$$

Since the block was initially at rest, the total momentum of the block-photon system is simply the initial photon momentum $\hbar\omega/c$ Thus, by conservation of momentum, we find that,

$$p_{\text{photon}} = \frac{\hbar\omega}{cn}, \tag{3.6}$$

which is the Abraham form of the momentum of light $p_{\text{A}} = p/n$.

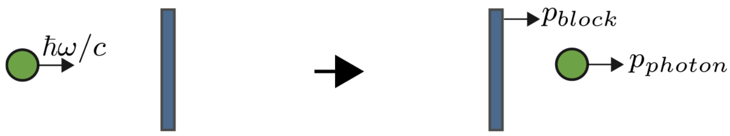

Figure 3.2: A photon with energy $\hbar\omega$ travels in the z-direction with speed c towards a transparent block at rest (mass M, thickness L, refractive index n).

These two simple examples highlight the conceptual difficulties underlying the Abraham-Minkowski debate. On the one hand, we arrived at the Minkowski form of the momentum by appealing to the conservation of energy and momentum. On the other hand, we arrived at a completely different form for the momentum by using the principles of the uniform motion of the centre of mass and the conservation of momentum. Throughout the latter half of the 20th century this conundrum fuelled many theoretical and experimental investigations, which found evidence supporting both Minkowski

[179–183] *and* Abraham [184–188]. More recently, the debate has been claimed as resolved in [189], where *both* forms of momentum are argued to be physically relevant, with the Minkowski form associated with the canonical momentum, and Abraham with the kinetic momentum. However some authors have highlighted the limitations of this analysis (see e.g. [190]).

Another postulated resolution to the problem has come from various authors (e.g. [191–193]), and lies instead in the importance of the material contribution to the total energy-momentum. It is suggested that by appropriately separating the total energy-momentum into wave and material parts that the approaches by Abraham and Minkowski are equivalent, with the discrepancies between them being related to which parts of the energy-momentum are attributed to the wave, and which are left to the material. While this provides a reasonable resolution to the problem, in practice accurately defining the energy-momentum of real materials is an almost impossible task, and it has been suggested that this difficulty has led to numerous studies claiming decisively one form of the momentum over the other. For a discussion of this, and a comprehensive review of the debate we refer the reader to [194].

The problem of defining the material contribution to the energy-momentum impacts the analysis of experiments, and so it is reasonable to ask whether the Abraham-Minkowski controversy can be viewed from an entirely different perspective which avoids the need for a material medium. Theories of electrodynamics which allow for the possibility of real photon-photon interactions, which we refer to collectively as nonlinear vacuum electrodynamics, offer the possibility of looking at the Abraham-Minkowski debate from a purely electromagnetic setup. As discussed above, these theories can be described with the analogy of the vacuum behaving as a medium under the influence of the nonlinear interactions of the fields. This gives us an entirely new way of approaching the Abraham-Minkowski debate, through analysing the energy-momentum of a purely electromagnetic set up, built up of fields which are characterised from the outset, and avoiding the difficulty in describing the energy-momentum of the medium.

The rest of this Chapter is structured as follows. In Section 3.1 we introduce the covariant Minkowski and Abraham energy-momentum tensors, and discuss some of the different features of these in the material context. The energy-momentum tensor is derived and analysed for the nonlinear theories of electrodynamics in 3.2, and we find some similarities appearing with the Minkowski form of the energy-momentum. An unexpected result from this consideration, however, is that we find that for an arbitrary electromagnetic field $F_{\mu\nu}$ there is a unification of the Abraham and Minkowski tensors. To be able to discriminate between the two approaches, we find that we must instead follow the analogy of light interacting with a medium more closely, by separating the full electromagnetic field $F_{\mu\nu}$ into a probe and background configuration. This is analysed in Section 3.3 where we show that the nonlinear energy-momentum of a probe field interacting with a strong background is naturally described by the Minkowski form of the energy-momentum. In Section 3.4 we highlight the difficulties in the interpretation of the Abraham energy-momentum in this context, and highlight the the evidence which seems to support the use of the Minkowski tensor. Finally, we summarise in Section 3.5.

3.1 Abraham and Minkowski energy-momentum tensors

Before considering the Minkowski and Abraham tensors in the context of nonlinear vacuum electrodynamics, we begin by defining each of these in the usual light-matter scenario using a covariant formalism, and highlight some of the key features of each. We consider an electromagnetic field defined by the anti-symmetric tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, where A_μ is the gauge field. The constitutive relations which describe how the electromagnetic fields of the medium interact with the incident field are encoded into the electromagnetic excitation tensor,

$$H_{\mu\nu} = \frac{1}{2} Z_{\mu\nu}{}^{\alpha\beta} F_{\alpha\beta} \,, \tag{3.7}$$

where the tensor $Z_{\mu\nu}{}^{\alpha\beta}$ describes the (possibly nonlinear) interaction between the electromagnetic fields, and encodes information about the properties of the medium.

With this notation, the Minkowski and Abraham energy-momentum tensors are, respectively [195, 196],

$$\Theta^\mu{}_\nu = H^{\mu\alpha} F_{\alpha\nu} + \frac{1}{4} \delta^\mu_\nu H^{\alpha\beta} F_{\alpha\beta} \,, \tag{3.8}$$

and,

$$\begin{aligned} \Omega^\mu{}_\nu = & \Theta^\mu{}_\nu + \frac{1}{2} \left(F^{\mu\alpha} H_{\alpha\nu} - H^{\mu\alpha} F_{\alpha\nu} \right) \\ & + \frac{1}{2} \left[u^\mu u_\alpha (F_{\nu\beta} H^{\alpha\beta} - H_{\nu\beta} F^{\alpha\beta}) + u_\nu u^\alpha (F^{\mu\beta} H_{\alpha\beta} - H^{\mu\beta} F_{\alpha\beta}) \right] . \end{aligned} \tag{3.9}$$

Here, the time-like ($u^2 = 1$) vector field u^μ which appears in the Abraham tensor has a physical interpretation as being the 4-velocity of the medium. The Abraham tensor (3.9) is also written here in terms of the Minkowski tensor (3.8). The main motivation of Abraham in deriving an alternative expression for the energy-momentum of the wave was the conservation of angular momentum. Since (3.8) is not (in general) symmetric, at the time it was believed this did not allow for angular momentum to be conserved, and so Abraham's formulation takes Minkowski's approach and introduces additional terms which make it manifestly symmetric. Non-symmetric energy-momentum tensors can, however, still conserve angular momentum [196], but there is still relevant motivation in wishing to obtain a symmetric tensor, which is necessary for example to couple to gravity in general relativity [197].

It is worthwhile here to highlight some basic features of (3.8) and (3.9). Firstly, taking the trace of each we find that,

$$\Theta^\mu{}_\mu = 0 = \Omega^\mu{}_\mu. \tag{3.10}$$

A traceless energy-momentum tensor indicates invariance under conformal transformations, which explicitly includes *scale invariance*. It could be argued that this is to be expected for Minkowski and Abraham, as each of these tensors aim to only describe the *wave* part of the energy-momentum, and since the wave is comprised of massless photons, there is an inherent scale invariance in this. However, the effect of a medium on a photon can be thought of, in some respects, as imbuing the photon with an effective mass, as the photon moves at speeds less than the speed of light in vacuum c, and so the traceless nature of (3.8) and (3.9) is somewhat surprising, if not counter-intuitive.

Secondly we can look at how each of these formulations conserve 4-momentum by taking the divergence. From the Minkowski tensor (3.8), the divergence can be expressed as a momentum-balance equation [196],

$$\partial_\mu \Theta^\mu{}_\nu + \mathcal{F}^{(m)}{}_\nu + \mathcal{F}^{(J)}{}_\nu = 0 \tag{3.11}$$

where the effective 4-force densities $\mathcal{F}^{(m)}{}_\nu$ and $\mathcal{F}^{(J)}{}_\nu$ are due to the macroscopic transfer of energy from the field to the medium and the presence of external currents and charges, respectively [196]. If we assume a closed system ($\mathcal{F}^{(J)}{}_\nu = 0$), and a simple (time-independent and homogeneous) medium ($\mathcal{F}^{(m)}{}_\nu = 0$), then 4-momentum is strictly conserved in the Minkowski formulation, despite (3.8) not being symmetric. In the case of Abraham, we find from (3.9) the appearance of an additional term, $\mathcal{F}^{(A)}{}_\nu$, in the momentum-balance equation,

$$\partial_\mu \Omega^\mu{}_\nu + \mathcal{F}^{(A)}{}_\nu = \partial_\mu \Theta^\mu{}_\nu, \tag{3.12}$$

which is generally called the Abraham 4-force density. Surprisingly, given that the Abraham formulation was derived on the basis of momentum conservation, even when $\partial_\mu \Theta^\mu{}_\nu = 0$, the Abraham 4-force term typically results in *non-conservation* of 4-momentum, $\partial_\mu \Omega^\mu{}_\nu \neq 0$. It should be noted, however, that numerous experimental investigations have claimed to have measured the Abraham force which results from $\mathcal{F}^{(A)}{}_\nu$ (e.g. [184, 185, 188]). Despite the fact that the Abraham 4-force density may be physical, the fact that the Minkowski formulation can preserve 4-momentum based purely on the symmetry conditions of the medium has been used to argue that (3.8) may be the more appropriate description of the energy-momentum of light in medium [196].

3.2 Energy-momentum in nonlinear electrodynamics

With the Minkowski and Abraham tensors now defined, and basic properties of each highlighted, we move on to the discussion of the energy-momentum tensor in the nonlinear theories of electrodynamics defined by the action (2.4). We obtain this in the usual way, varying (2.4) with respect to the metric tensor $g_{\mu\nu}$ which defines the energy-momentum tensor $T^{\mu\nu}$ as,

$$\delta S = -\frac{1}{2}\int d^4z\sqrt{-g}T^{\mu\nu}\delta g_{\mu\nu}. \tag{3.13}$$

The details of the calculation, and the useful identities which are used, are presented in Appendix B. We find that the energy-momentum tensor of nonlinear vacuum electrodynamics is given by,

$$T^{\mu}{}_{\nu} = H^{\mu\alpha}F_{\alpha\nu} - \delta^{\mu}_{\nu}\mathcal{L}, \tag{3.14}$$

where the excitation tensor $H^{\mu\nu}$ is defined in (2.6). What is immediately apparent from comparing this with (3.8) is that the only difference between Minkowski and the nonlinear energy-momentum appear in the strictly diagonal terms (those which are proportional to δ^μ_ν). Thus momentum density, which comes from the off-diagonal terms, are identical in the nonlinear and Minkowski formulations. As an aside, we also note that the energy-momentum tensors of Minkowski (3.8) and the nonlinear theory (3.14) can be expressed as,

$$T^\mu{}_\nu = \Theta^\mu{}_\nu - \delta^\mu_\nu \left(\mathcal{L} - X\frac{\partial \mathcal{L}}{\partial X} - Y\frac{\partial \mathcal{L}}{\partial Y} \right). \tag{3.15}$$

In this form the difference of the two energy-momentum tensors is proportional to the Legendre transform of the Lagrangian function $\mathcal{L} \equiv \mathcal{L}(X, Y)$, suggestive of the *Hamiltonian* of the nonlinear theory. This may hint at some interesting, deeper connection between the two, though we do not pursue this line of enquiry further here.

With a similar appearance, and identical off-diagonal terms, it is interesting to consider if there are differences between the properties of (3.14) and (3.8). Taking the trace of (3.14) we find,

$$T^\mu{}_\mu = -4\left(\mathcal{L} - X\frac{\partial \mathcal{L}}{\partial X} - Y\frac{\partial \mathcal{L}}{\partial Y} \right), \tag{3.16}$$

and since $\mathcal{L} \equiv \mathcal{L}(X, Y)$, in general, nonlinear extensions to Maxwellian electrodynamics give energy-momentum tensors which are not traceless.

A powerful feature of the energy-momentum tensor (3.14) is that regardless of the precise form of the Lagrangian function, the energy-momentum is always conserved for closed systems. This can be seen by taking the divergence of (3.14), and firstly expanding the derivative of the Lagrangian function in terms of derivatives of the

Lorentz invariants X and Y,

$$\partial_\mu T^\mu{}_\nu = (\partial_\mu H^{\mu\alpha}) F_{\nu\alpha} + H^{\mu\alpha}(\partial_\mu F_{\nu\alpha}) + \partial_\nu X \frac{\partial \mathcal{L}}{\partial X} + \partial_\nu Y \frac{\partial \mathcal{L}}{\partial Y}. \tag{3.17}$$

The first term here is zero, as a result of the field equations for nonlinear electromagnetism (2.5). Taking the derivatives of X and Y, we can express the divergence as,

$$\partial_\mu T^\mu{}_\nu = H^{\mu\alpha}(\partial_\mu F_{\nu\alpha}) - \frac{1}{2} H^{\alpha\beta} \partial_\nu F_{\alpha\beta}. \tag{3.18}$$

Finally, we can rewrite the last term using the alternative form of the Bianchi identity (2.7)

$$\partial_\mu F_{\nu\alpha} + \partial_\nu F_{\alpha\mu} + \partial_\alpha F_{\mu\nu} = 0, \tag{3.19}$$

to show finally that,

$$\partial_\mu T^\mu{}_\nu = 0. \tag{3.20}$$

In the case of Minkowski, which was also divergence free, we had to make further assumptions about the field and matter configurations. From the analysis presented here, there appears to be a much more natural interpretation of the Minkowski formulation in nonlinear electrodynamics, or rather, more obvious similarities between these two cases. This leads us to ask how we might interpret Abraham.

In the case of a real material medium, the tensor $H^{\mu\nu}$ will have some complicated structure, which ensures that the energy-momentum tensors of Abraham and Minkowski are distinct. In the nonlinear electrodynamics, however, with the excitation tensor defined in (2.6) we find that this leads to a unification of the Minkowski and Abraham approaches, such that,

$$\Theta^{\mu}{}_{\nu} = \Omega^{\mu}{}_{\nu}. \tag{3.21}$$

How can this be explained? We mentioned previously that one of the proposed resolutions to the Abraham-Minkowki debate is simply that the total energy-momentum of the system will be equivalent in both cases. The differences between the Abraham and Minkowski formulations then is attributed to how the total energy-momentum is separated into wave and material contributions[1]. The fact that the two approaches are unified in (3.21) could be viewed as the nonlinear electromagnetic statement of this same idea. In the analysis presented so far for the Minkowski and Abraham tensors in nonlinear electromagnetism, (3.8) and (3.9) respectively, no distinction has been made between the background field, and the probe field. Instead we have been working with the *full* electromagnetic field tensor $F_{\mu\nu}$. The tensors $\Theta^{\mu}{}_{\nu}$ and $\Omega^{\mu}{}_{\nu}$, defined with the

[1]This resolution to the Abraham-Minkowski debate would give some clarity on why theoretical studies have come out in favour of *both* formulations. Depending on how the problem is approached, and how the total energy-momentum of the system is separated, the approximations used to describe the energy-momentum contribution of the medium could lead to one or the other of the formulations to be obtained. However, it does not fully explain why there has been experimental work which purports to name one of the formulations correct, as experiments measure something real, which would be independent of how the theorists separate the energy-momentum. In response to this, Barnett [189] has suggested that *both* formulations are correct, but correspond to physically different forms of the momentum of light, with Minkowski associated with the canonical momentum, and Abraham with the kinetic momentum. Then, depending on how the experiment was set up, it would be sensitive to measuring one form of the momentum over the other.

full electromagnetic field $F_{\mu\nu}$, and the excitation tensor (2.6), can be thought of as the *total energy-momentum tensors of the system.* The question here is how do we build up a better picture, which could possibly lead to a clearer distinction between the two formulations? To do this, we will more closely follow the light-matter analogy by splitting the full electromagnetic field into a background and probe configuration, to mirror the separation into wave and material parts.

3.3 Perturbing around a strong background field

So far, the full electromagnetic tensor, $F^{\mu\nu}$, has been left completely arbitrary, and it is implicitly assumed that this accounts for *all* fields within the system, i.e. both the background field, and the probe field in which we are primarily interested. We choose to now separate the full electromagnetic tensor into its constituent parts, assuming that we can treat the probe field as a perturbation on the background field,

$$F^{\mu\nu} = \mathcal{F}^{\mu\nu} + f^{\mu\nu}, \tag{3.22}$$

where $\mathcal{F}^{\mu\nu}$ is the background, and $f^{\mu\nu}$ is the probe field. These conventions will be used throughout this Chapter, i.e. terms which depend on the background only ($\mathcal{O}(f^0)$) will use calligraphic font (e.g. $\mathcal{A}$), terms of linear order in the perturbation ($\mathcal{O}(f^1)$) will be lower-case (e.g. a) and terms second order ($\mathcal{O}(f^2)$) will use lower-case sans-serif font (e.g. a). For completeness, the background field is taken to be completely arbitrary and no assumptions are made about its spacetime dependence. The only condition which is placed on it is that the amplitude of the background field is the dominant contribution to the full electromagnetic field tensor, $|\mathcal{F}| \gg |f|$, and that the background is taken to be slowly varying with respect to the probe, such that terms linear in the probe f average to zero.

With (3.22) the Lorentz invariant parameters (2.2) and (2.3) become,

$$X = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}\mathcal{F}^{\mu\nu}f_{\mu\nu} - \frac{1}{4}f^{\mu\nu}f_{\mu\nu} = \mathcal{X} + 2x + \mathsf{x}, \tag{3.23}$$

$$Y = -\frac{1}{4}\widetilde{\mathcal{F}}^{\mu\nu}\mathcal{F}_{\mu\nu} - \frac{1}{2}\widetilde{\mathcal{F}}^{\mu\nu}f_{\mu\nu} - \frac{1}{4}\widetilde{f}^{\mu\nu}f_{\mu\nu} = \mathcal{Y} + 2y + \mathsf{y}, \tag{3.24}$$

where we have

$$\mathcal{X} = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} \qquad x = -\frac{1}{4}\mathcal{F}^{\mu\nu}f_{\mu\nu} \qquad \mathsf{x} = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu}\,. \tag{3.25}$$

and analogous definitions for Y.

Through using (3.23) and (3.24) (or equivalently (3.22)) we can perform Taylor expansions of the Lagrangian function and its derivatives, which are explicitly given in Appendix C. With these definitions, the excitation tensor (2.6) can be expressed order-by-order as,

$$H^{\mu\nu} = \mathcal{H}^{\mu\nu} + h^{\mu\nu} + \mathsf{h}^{\mu\nu} + \mathcal{O}(f^3), \tag{3.26}$$

where we omit higher order terms since these will be negligible provided the magnitude of the probe field is small compared to the background. We also only consider terms up to $\mathcal{O}(f^2)$, as the Minkowski (3.8) and Abraham (3.9) tensors are second order in the probe field.

3.3.1 Abraham and Minkowski forms of the probe energy-momentum

As mentioned above, the energy-momentum tensors of Minkowski (3.8) and Abraham (3.9) represent the *full* energy-momentum of the system, since the electromagnetic field F contains both the background and probe parts, and we assume a closed system with no free sources. In the case of light interacting with a real medium, the Minkowski and Abraham tensors describe only the energy-momentum of the light, and so it is necessary here to use the analogous description in terms of the probe f in (3.22) This is equivalent to simply making the substitutions $F^{\mu\nu} \to f^{\mu\nu}$ and $H^{\mu\nu} \to h^{\mu\nu}$. Thus we have,

$$\widetilde{\Theta}^{\mu}{}_{\nu} = h^{\mu\alpha} f_{\alpha\nu} + \frac{1}{4}\delta^{\mu}_{\nu} h^{\alpha\beta} f_{\alpha\beta}\,, \tag{3.27}$$

and,

$$\begin{aligned}\widetilde{\Omega}^{\mu}{}_{\nu} =& \widetilde{\Theta}^{\mu}{}_{\nu} + \frac{1}{2}\left(f^{\mu\alpha} h_{\alpha\nu} - h^{\mu\alpha} f_{\alpha\nu}\right) \\ &+ \frac{1}{2}\Big[u^{\mu} u_{\alpha}(f_{\nu\beta} h^{\alpha\beta} - h_{\nu\beta} f^{\alpha\beta}) + u_{\nu} u^{\alpha}(f^{\mu\beta} h_{\alpha\beta} - h^{\mu\beta} f_{\alpha\beta})\Big]\,. \end{aligned}\tag{3.28}$$

where we have used a tilde to distinguish these from the *full* tensors (3.8) and (3.9). Had we instead expanded the full tensors perturbatively, this would give both the background and probe contributions to the energy-momentum, not just the probe part which these tensors are meant to represent. Crucially, with the precise form of the first order excitation tensor (C.7) found in Appendix C, these two descriptions are *no longer* *equivalent*, i.e.,

$$\widetilde{\Theta}^{\mu}{}_{\nu} \neq \widetilde{\Omega}^{\mu}{}_{\nu}. \tag{3.29}$$

We now have two distinct descriptions of the energy-momentum of the probe field in the nonlinear interaction with the background field, giving a closer analogy to the light-matter interactions which are usually considered.

3.3.2 The nonlinear energy-momentum in the background/probe configuration

To proceed to calculate the perturbed nonlinear energy-momentum tensor, it is convenient to perform the expansion into powers of the probe field at the level of the action. The Lagrangian function can be Taylor expanded around the background, and expressed as (see Appendix C),

$$\mathcal{L}(X,Y) = \mathcal{L}(\mathcal{X},\mathcal{Y}) - \frac{1}{2}\mathcal{H}^{\mu\nu} f_{\mu\nu} - \frac{1}{4}h^{\mu\nu} f_{\mu\nu}. \tag{3.30}$$

The action is,

$$S = S^{(0)} + S^{(1)} + S^{(2)} + \mathcal{O}(f^3). \tag{3.31}$$

The first term

$$S^{(0)} = \int d^4z\sqrt{-g}\mathcal{L}(\mathcal{X},\mathcal{Y}) \tag{3.32}$$

depends only on the background, the second term

$$S^{(1)} = -\frac{1}{2}\int d^4z\sqrt{-g}\mathcal{H}^{\mu\nu}f_{\mu\nu} \tag{3.33}$$

is linear in the probe field, which would time-average to zero with an oscillating probe. The important term here is the last term which is second order in the probe,

$$S^{(2)} = -\frac{1}{4}\int d^4z\sqrt{-g}h^{\mu\nu}f_{\mu\nu}\,. \tag{3.34}$$

All orders of the excitation tensor (3.26) can be expressed in the form of a constitutive relation, analogous to (3.7), and we choose to express $h^{\mu\nu}$ in this way,

$$h_{\mu\nu} = \frac{1}{2}\chi_{\mu\nu}{}^{\alpha\beta}f_{\alpha\beta}\,. \tag{3.35}$$

In (3.7), the tensor $Z_{\mu\nu}{}^{\alpha\beta}$ defines the properties of the medium, and as such characterises the effect the medium will have on the passing probe wave. Analogously, the role $\chi_{\mu\nu}{}^{\alpha\beta}$ plays in (3.35) is to define the properties of the *nonlinear vacuum* under the influence of the strong electromagnetic field, encoding the nonlinear interaction which will be experienced by the probe field. This can be shown explicitly in the calculation of the energy-momentum tensor from (3.34). The term of interest is now,

$$S^{(2)} = -\frac{1}{16}\int d^4z\sqrt{-g}\left(g^{\mu\lambda}g^{\nu\rho} - g^{\mu\rho}g^{\nu\lambda}\right)\chi_{\lambda\rho}{}^{\alpha\beta}f_{\alpha\beta}f_{\mu\nu}\,, \tag{3.36}$$

where we explicitly insert (antisymmetric combination of) the metric tensors which raise the indices on $h^{\mu\nu}$ in (3.36).

To obtain the energy-momentum tensor, we again use the usual approach of varying (3.36) with respect to the metric $g_{\mu\nu}$,

$$\begin{aligned}\delta S^{(2)} =& -\frac{1}{16}\int d^4z\delta\left[\sqrt{-g}\left(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\rho}g^{\nu\lambda}\right)\chi_{\lambda\rho}{}^{\alpha\beta}\right]f_{\alpha\beta}f_{\mu\nu}\\ =&\frac{1}{2}\int d^4z\sqrt{-g}\mathsf{t}^{\mu\nu}\delta g_{\mu\nu},\end{aligned} \tag{3.37}$$

where we define $\mathsf{t}^{\mu\nu}$ as the $\mathcal{O}(f^2)$ contribution to the energy-momentum tensor. The only terms which may depend on the metric are those within the square brackets. There are two terms to consider here. The first comes from the variation of the terms explicitly involving the metric tensor,

$$-\frac{1}{8}\left[\delta\left(\sqrt{-g}\left(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\rho}g^{\nu\lambda}\right)\right)\right]\chi_{\lambda\rho}{}^{\alpha\beta}f_{\alpha\beta}f_{\mu\nu} =\frac{1}{2}\sqrt{-g}\left[h^{\mu\alpha}f_{\alpha}{}^{\nu}+\frac{1}{4}g^{\mu\nu}h^{\alpha\beta}f_{\alpha\beta}\right]\delta g_{\mu\nu}, \tag{3.38}$$

where we used the identities (B.2) and (B.6). We can immediately identify the term in the square brackets in (3.38) as the Minkowski form of the energy-momentum (3.27). This contribution to the total energy-momentum at $\mathcal{O}(f^2)$ has come from the explicit dependence on the metric. Another term remains, however, which comes from the variation of the constitutive tensor $\chi_{\mu\nu}{}^{\alpha\beta}$. Using the definition (C.7), and following an analogous procedure as presented in Appendix B for dealing with the metric dependence of terms appearing in $\chi_{\lambda\rho}{}^{\alpha\beta}$, we arrive at,

$$-\frac{1}{8}\left(\sqrt{-g}\left(g^{\mu\lambda}g^{\nu\rho}-g^{\mu\rho}g^{\nu\lambda}\right)\right)\delta\left[\chi_{\lambda\rho}{}^{\alpha\beta}\right]f_{\alpha\beta}f_{\mu\nu}=\frac{1}{2}\sqrt{-g}\mathsf{h}^{\mu\alpha}\mathcal{F}_{\alpha}{}^{\nu}\delta g_{\mu\nu}. \tag{3.39}$$

This is the term which is responsible for the non-zero trace of the energy-momentum tensor (at this order in the probe). In other words, the $\mathcal{O}(f^2)$ contribution to the energy-momentum tensor is,

$$\mathsf{t}^{\mu\nu}=\widetilde{\Theta}^{\mu\nu}+\mathsf{h}^{\mu\alpha}\mathcal{F}_{\alpha}{}^{\nu}, \tag{3.40}$$

where the second term comes entirely from the variation of the tensor χ. As mentioned previously, in the usual light-matter interaction (3.7), Z encodes the properties of the medium, and so the additional term alongside the Minkowski energy-momentum in (3.40) can have the interpretation of being the nonlinear vacuum analogies to the *material* contribution accompanying the wave part of the energy-momentum tensor.

With the Minkowski form appearing naturally, based on nothing but the usual variational procedure of deriving the energy-momentum, it is natural to ask the role, if any, Abraham may play in the nonlinear theories. It is always possible to arbitrarily separate the total energy-momentum into whatever parts we wish, and so there would be nothing stopping us, in principle, from separating out the energy-momentum into a description which involves the Abraham form, with some additional terms which we could say were the contribution from the strong background field. However, this would be very artificial, and in the next Section we discuss some of the difficulties with the

Abraham energy-momentum in nonlinear electrodynamics.

3.4 Minkowski or Abraham?

The purpose of this Chapter has been to look at the Abraham-Minkowski controversy from the point of view of nonlinear vacuum electrodynamics, to see what we can learn about the momentum of light in this particular situation. There are two key arguments which highlight the importance of the Minkowski energy-momentum tensor in nonlinear electrodynamics.

The first is that Minkowski appears naturally in the expansion of the full energy-momentum tensor in (3.40). It is always possible to arbitrarily separate the total energy-momentum (at a given order in the probe field), but doing that is rather artificial, and has no real physical motivation. We obtain the Minkowski form as a direct consequence of varying the explicit metric dependence in (3.36), with the additional terms coming from the variation of the constitutive tensor ${\chi_{\mu\nu}}^{\alpha\beta}$. This has a clear physical interpretation when compared with the analogy of light interacting with a real medium, where the constitutive tensor encodes the information about the material properties.

The second is related to the interpretation of the velocity field u^μ in the Abraham tensor describing the probe field (3.28). In the context of light-matter interactions, this has a reasonable and quantifiable interpretation as the 4-velocity of the material with which the probe field is interacting. In the case of the full tensor (3.9) in theories of nonlinear vacuum electrodynamics, terms proportional to u^μ vanish due to the form of the full excitation tensor (2.6), and there are therefore no problems presented with the appearance of the velocity field. For the Abraham energy-momentum tensor describing only the probe field (3.28) however, these terms do not vanish, and we are left with explicit dependence on the velocity field. Unlike in the usual context of light-matter interactions, we have no real medium present. Instead, the presence of fluctuations in the quantum vacuum and a strong background field lead to an incident probe field behaving analogously to how it behaves in a medium. Thus, there is no structure which can have a velocity field associated with it, and so u^μ has no real meaning, other than being a velocity field. As a result of this, and since the equation itself gives no indication of how to determine u^μ, any practical calculation would require a specific and arbitrary reference frame to be chosen, introducing a preferred frame of motion and thus violating Lorentz invariance. Unlike in the case of light-matter interactions, where the breaking of Lorentz invariance is a physical consequence of having a real medium, the nonlinear vacuum electrodynamics theories strictly retain this symmetry. While it could be argued that since the full energy-momentum tensor is Lorentz invariant and the symmetry breaking of the Abraham tensor would be compensated by another term, considering the Abraham form as an independent function which describes purely Lorentz invariant fields is troubling. Another argument which could be made is that there is a partial breaking of the Lorentz symmetry due to the background field itself. For example, a magnetic field introduces a preferred direction. However, performing a Lorentz transformation along the direction of the field, which leaves the form of the field invariant, changes the form of the Abraham energy-momentum, and so the Lorentz symmetry is violated. These problems are not found in the Minkowski formulation, giving a greater credibility to the use of the Minkowski tensor (3.27) as the correct description of the energy-momentum of light in nonlinear vacuum electrodynamics.

3.5 Summary

The correct way to describe the energy-momentum of light in a medium has had a long history of debate, with no decisive consensus being reached within the community as to which approach should be taken — Abraham or Minkowski. The nonlinear vacuum under the influence of strong fields can provide us with a purely electromagnetic way of looking at nonlinear light propagation, and we have considered the energy-momentum of light in theories of nonlinear vacuum electrodynamics. We have seen that for the full field, the nonlinear vacuum energy-momentum tensor (3.14) has obvious qualitative similarities to the Minkowski tensor, but that in fact, the form of the excitation tensor for the full theory (2.6) leads to a unification of the approaches of Abraham and Minkowski, possibly highlighting and giving evidence to the equality of the full energy-momentum of the system in these two different approaches. By separating the full electromagnetic field into a strong background and weak probe configuration, the Minkowski form appears naturally, with a clear physical interpretation of all the terms. Conversely, the Abraham energy-momentum plays no real role, and how to interpret it in this context appears to have no answer. These points, coupled with the clear violation of Lorentz invariance introduced by the Abraham form appears to promote Minkowski as the more correct description of the energy-momentum of light in nonlinear vacuum electrodynamics.

Chapter 4

Conformally invariant nonlinear electrodynamics

So far, the basic requirement of theories of nonlinear electrodynamics defined by the action (2.4) has been that they are built up from only Lorentz invariant contractions of the field, through the parameters X (2.2) and Y (2.3). The adherence to Lorentz invariance has been a foundational principle of modern theoretical physics since Einstein's revolutionary development of Special Relativity [2], leading ultimately to Quantum Field Theory. In the years that followed the development of the theory of Special Relativity, it was demonstrated by Bateman [198–200] and Cunningham [201] that Maxwell's theory of electromagnetism is invariant under a larger symmetry group than the Poincaré group which defines Lorentz invariance: the conformal group (see also the more recent work [202]). As well as the usual Lorentz symmetries, the conformal group includes invariance under scale, and special conformal, transformations. It is the largest extension of the Poincare symmetry group which leaves the lightcone invariant. The invariance of the lightcone (the path which defines the past and future propagation of a ray of light) is related to the definition of a conformal mapping as a transformation which preserves angles. We will discuss the technical features of the conformal group in the next Section.

Since the work of Bateman and Cunningham, conformal invariance in physical systems has had a varied and far reaching impact. For an excellent review on the historical development, we refer the reader to the review by Kastrup [203] (see also [204–206]). Arguably the biggest influence which conformal invariance has had on modern physics is the integral role it played in the development of gauge theory by Weyl. In an attempt to combine electromagnetism with Einstein's recently formulated theory of General Relativity, Weyl exploited a conformal rescaling of the metric tensor to arrive at a transformation which simultaneously transformed the metric and the electromagnetic potential [207, 208] (see also the review [209] and the discussion in the book by Wald [197]). It was then that the term "gauge transformation" first entered the lexicon of theoretical physics. While this original attempt was found to be problematic [203], it significantly influenced the direction of the early days of quantum mechanics, with Schrödinger adapting the gauge transformation [210], and Weyl establishing the usual gauge transformation we are familiar with in subsequent work, in the context of quantum mechanics [211, 212].

Aside from a few notable examples (e.g. [213–220]) there was little use of conformal invariance in the physics community for some time after the work of Weyl. Of these, the work of Pauli [219] provides an interesting interpretation of conformal invariance, with respect to the discussions presented in Chapter 3. He noted that conformal invariance of a theory can be determined by the trace of the energy-momentum tensor, such that for most theories, the signature of conformal invariance is the vanishing of the trace. We can see an example of this by considering the usual Maxwellian theory, defined by the Lagrangian function (2.8). In this case, the trace of the energy-momentum tensor vanishes. As discussed in the previous Chapter, the energy-momentum tensors of Minkowski (3.8) (or (3.27)) and Abraham (3.9) (or (3.28)) have the traceless property, but this is not observed in general nonlinear electromagnetic theories, where the energy-momentum tensor is defined by (3.14) (or (3.40)). We will return to this below.

After this work the use of conformal invariance in theoretical physics somewhat fell out of fashion until the work of Wess [221], who applied conformal invariance to quantum field theory, and investigated the resulting conservation laws and commutation relations for scalar, spinor and vector fields. This initiated further investigations which would lead to conformal field theory becoming a field within its own right. It was found that many of the interaction terms in the classical field theories of physics are conformally invariant in the work of Kastrup [222] (who also coined the term "special conformal transformation" [223]). This discovery led to the investigation of how conformal field theory may be useful in the description of the asymptotic behaviour of quantum field theories [224–227], and also gave a description of the special conformal transformation as a gauge transformation of Minkowski space [228], mirroring the initial use of the conformal transformation of Weyl discussed above. Conformal invariance then became an important topic in the theory of current algebras, where it was found to be partially conserved [229, 230], and using a procedure introduced by Wilson [231] was used as an asymptotic symmetry in lepton-hadron scattering [204, 232–235]. Following on from this, the energy-momentum tensor in QFT was revisited by several authors, and the conformal anomalies in the energy-momentum, as well as the divergence of the dilation and special conformal currents were found to be proportional to the β-function [236–246]. This demonstrated that theories for which the β-function vanishes, or has nontrivial fixed points, are conformally invariant. One of the most prominent theories for which this appears to be the case is the supersymmetric $\mathcal{N} = 4$ Yang-Mills theory (see e.g. [247]), which is used extensively in modern theoretical physics (discussed further below).

Perhaps the most significant advance in the use of conformal invariance was the development of the conformal bootstrap approach to quantum field theories [248–251], a nonperturbative technique which uses a non-Lagrangian formulation of quantum field theory to analyse the properties of correlation functions. This led to a revolution in the study of $2D$ conformal field theories, where it was found that the conformal group leads to an infinite set of transformations, which makes many models (such as the $2D$-Ising model) exactly solvable [252–256], and its use as a tool in statistical physics, studying critical phenomena was established.

The final use of conformal symmetry which will be mentioned is its use in the speculative area of string theory. The conjecture of Maldacena [257], known as the AdS/CFT (or gauge-gravity) conjecture, relates $D+1$ dimensional supergravity theories to D dimensional conformal field theories (made precise by Witten [258]). Since this work this conjectured duality has been one of the most active areas of research in theoretical physics, spawning thousands of papers on the various properties of quantum field theories (see reviews [259–266]), which would not be possible without the use of conformal invariance.

In light of the above, it is clear that exploiting conformal invariance in theoretical physics, whether in the sense of taking conformal symmetry to be fundamental or using the extended group to simplify more difficult problems, can lead to interesting new research directions and insights. The rest of this chapter is devoted to answering the question: can a conformally invariant, nonlinear theory of electromagnetism exist, and if so, can it be used as a model of strong field processes? The motivation behind this is the following. It is known that Maxwellian electrodynamics is a conformally invariant, linear theory of electromagnetism, but that a more complete description of electromagnetic phenomena is found by using the machinery of QED. In the quantization process, we lose the additional symmetries of the conformal group, obtaining instead a Poincaré invariant, nonlinear theory. So far in this thesis, we have shown how an effective field theoretic description, using classical nonlinear electromagnetism can be used to describe a wide range of quantum phenomena in strong electromagnetic fields, and it is natural to wonder if the extended symmetries offered by the conformal group could be exploited in the same way as its use in other areas of physics, as a means to find, or guide, solutions to difficult problems. It may also be possible that in some regime, such a theory could be used to model some aspects of Euler-Heisenberg theory, or other physics. A conformally invariant, nonlinear theory of electromagnetism would represent a true "middle ground" between the theories of Maxwell and QED, and may, like Euler-Heisenberg, be useful in the study of strong fields.

To approach this question, in Section 4.1 we outline the formal aspects, and technical details of the conformal group. We then proceed to apply conformal transformations to the nonlinear theories defined by the Lagrangian functions $\mathcal{L}(X,Y)$ in Section 4.2, as well as discuss the implications of the constrained Lagrangian functions, and we conclude by revisiting the energy-momentum tensor in the conformally invariant context.

4.1 The conformal group

The conformal group is an extension of the usual Poincaré group, to include scale/dilation transformations and special conformal transformations. The Poincaré group is the full symmetry group of special relativity, which includes invariance under translations, rotations and boosts. The associated algebra can be represented in terms of the 10 generators,

$$\text{Translations:} \qquad \mathcal{P}_\mu = i\partial_\mu, \tag{4.1}$$

$$\text{Lorentz transformations:} \qquad \mathcal{J}_{\mu\nu} = i\left(z_\mu\partial_\nu - z_\nu\partial_\mu\right), \tag{4.2}$$

which give the commutation relations,

$$[\mathcal{P}_\mu, \mathcal{P}_\nu] = 0, \tag{4.3}$$

$$[\mathcal{J}_{\mu\nu}, \mathcal{P}_\lambda] = -i\left(g_{\mu\lambda}\mathcal{P}_\nu - g_{\nu\lambda}\mathcal{P}_\mu\right), \tag{4.4}$$

$$[\mathcal{J}_{\mu\nu}, \mathcal{J}_{\lambda\rho}] = i\left(g_{\mu\rho}\mathcal{J}_{\nu\lambda} - g_{\mu\lambda}\mathcal{J}_{\nu\rho} + g_{\nu\lambda}\mathcal{J}_{\mu\rho} - g_{\nu\rho}\mathcal{J}_{\mu\lambda}\right), \tag{4.5}$$

where $g_{\mu\nu}$ is the Minkowski metric. Invariance under translations (4.1) corresponds to conservation of the associated energy-momentum tensor of the theory, $\partial_\mu T^{\mu\nu} = 0$, whereas invariance under Lorentz transformations (4.2) allows the energy-momentum tensor to be chosen to be symmetric $T^{\mu\nu} = T^{\nu\mu}$, due to the conserved current $J^{\mu\nu}_{\rho} = \frac{1}{2}\left(z^{\mu}T^{\nu}{}_{\rho} - z^{\nu}T^{\mu}{}_{\rho}\right)$.

To extend the Poincaré group to the conformal group, we introduce the dilation and special conformal transformations,

$$\text{Dilations:} \qquad \mathcal{D} = i\left(z^{\mu}\partial_{\mu}\right), \tag{4.6}$$

$$\text{Special conformal:} \qquad \mathcal{K}_{\mu} = 2iz_{\mu}z^{\lambda}\partial_{\lambda} - iz^{2}\partial_{\mu}, \tag{4.7}$$

corresponding to a further 5 generators, such that the conformal group is a 15 parameter symmetry group. These generators give us the further commutation relations,

$$[\mathcal{D}, \mathcal{K}_{\nu}] = i\mathcal{K}_{\nu}, \tag{4.8}$$

$$[\mathcal{D}, \mathcal{P}_{\nu}] = -i\mathcal{P}_{\nu}, \tag{4.9}$$

$$[\mathcal{K}_{\mu}, \mathcal{P}_{\nu}] = -2i\left(g_{\mu\nu}\mathcal{D} - \mathcal{J}_{\mu\nu}\right), \tag{4.10}$$

$$[\mathcal{K}_{\mu}, \mathcal{J}_{\nu\lambda}] = -i\left(g_{\mu\nu}\mathcal{K}_{\lambda} - g_{\mu\lambda}\mathcal{K}_{\nu}\right), \tag{4.11}$$

with all other commutators being zero. Dilation transformations require that the trace of the energy-momentum tensor can be expressed as the divergence of a current, $T^{\mu}{}_{\mu} = \partial^{\mu}J_{\mu}$, with J_{μ} often referred to as the the virial current [237]. The conserved current corresponding to this transformation is $D_{\mu} = z^{\lambda}T_{\mu\lambda} - J_{\mu}$. In this, the term $z^{\lambda}T_{\mu\lambda}$ causes the transformation, however on its own this is not enough to preserve scale invariance. This is the role that the virial current, J_{μ}, plays, by rescaling all fields such that scale invariance is ensured. The special conformal transformation requires that the trace of the energy-momentum tensor vanishes $T^{\mu}{}_{\mu} = 0$, with the associated conserved current $K_\mu = (2z^\lambda z^\nu - g^{\lambda\nu} z^2) T^{\mu}{}_{\lambda}$.

With the structure of the conformal group now established in terms of the generators, how do we view the transformation more generally? As mentioned previously, the conformal group is the symmetry group which leaves the light-cone invariant. There are two distinct, but related, uses of the term "conformal transformation" in the literature (for a good insight into the distinction see [203]). The first is interpreting a general conformal transformation as a coordinate transformation $z^\mu \to \tilde{z}^\mu$ which takes,

$$g_{\mu\nu} dz^\mu dz^\nu \to g_{\mu\nu} d\tilde{z}^\mu d\tilde{z}^\nu = \Omega^2(z) g_{\mu\nu} dz^\mu dz^\nu. \tag{4.12}$$

The function $\Omega(z)$ in this case depends on the particular transformation (see below). The view of conformal transformations as coordinate transformations, defined by (4.12), contains the full symmetries implied by the above generators. In this view, translations, Lorentz transformations, dilations and special conformal transformations correspond to the coordinate transforms,

$$\text{Translation:} \quad z^\mu \to \quad \tilde{z}^\mu = z^\mu + a^\mu, \qquad \Omega(z) = 1 \tag{4.13}$$

$$\text{Lorentz:} \quad z^\mu \to \quad \tilde{z}^\mu = \Lambda^{\mu}{}_{\nu} z^\nu, \qquad \Omega(z) = 1 \tag{4.14}$$

$$\text{Dilation:} \quad z^\mu \to \quad \tilde{z}^\mu = \lambda z^\mu, \qquad \Omega(z) = \lambda, \tag{4.15}$$

$$\text{Special:} \quad z^\mu \to \quad \tilde{z}^\mu = \frac{z^\mu - a^\mu z^2}{1 - 2(a.z) + a^2 z^2}, \qquad \Omega(z) = 1 - 2(a.z) + a^2 z^2, \tag{4.16}$$

with a^μ an arbitrary constant 4-vector, $\Lambda^{\mu}{}_{\nu}$ a Lorentz matrix, and λ a scaling parameter.

The special conformal transformation can also be considered as an inversion $z^\mu \to z^\mu/z^2$ followed by a translation, and another inversion.

The second interpretation, which is the method used by Weyl [207], is instead as a geometrical gauge transformation, where instead of a coordinate transformation we have a local rescaling of the metric,

$$\begin{aligned} g_{\mu\nu} \to \tilde{g}_{\mu\nu} =& \omega^2(z) g_{\mu\nu}, \\ g^{\mu\nu} \to \tilde{g}^{\mu\nu} =& \omega^{-2}(z) g^{\mu\nu} \end{aligned} \tag{4.17}$$

where $\omega(z)$ is now an *arbitrary* scalar function. This is the definition of conformal symmetry most commonly used in the physics literature. In this instance, the coordinates are kept fixed, and the transformation is a pure rescaling of the metric on unchanged coordinates. Thus, this transformation has the action of changing the metric to become curved, though still conformally flat. This interpretation of conformal invariance is more general than the coordinate transformation (4.12), in the sense that since it only directly accounts for dilation/scale transformations, and so is not as restrictive as the coordinate transform interpretation defined by (4.12). It is possible to have a scale invariant theory which is not conformally invariant [267], however the converse is not true (as scale transformations are a subset of the full conformal group). In many useful physical applications of conformal invariance, though, studying the scale transformations is sufficient for certain interesting phenomena to be discussed.

4.2 Conformally invariant nonlinear electrodynamics

As already discussed, the aim of this Chapter is the investigation of conformally invariant theories of nonlinear electrodynamics. Maxwell's theory is conformally invariant and linear, while QED is Poincaré invariant and nonlinear. Nonlinear theories of electrodynamics, such as Euler-Heisenberg, have been successfully utilised to study a range of physical processes in strong fields, but there are many instances of problems becoming intractable even in these simplified effective field theories. When a difficult problem is encountered in physics, the more symmetric we can make the problem, the more progress can usually be made. Often in the literature, conformal transformations can be used to turn a complicated problem in to a simpler one, and so it is natural to see if conformal symmetry could be exploited in nonlinear electrodynamics to produce an effective field theory of strong-field interactions in which problems can be tackled in a more tractable way.

In the following, we will adopt the definition of conformal invariance used by Weyl, as a rescaling of the metric tensor (4.17). Generally, this is not as strong a statement of conformal invariance, however, it is the most widely used definition in the physics literature, and also for the purposes investigated here gives identical results to using the coordinate transform definition (4.12).

4.2.1 Transformation of the action

Our starting point is again the nonlinear action (2.4). Since we are adopting the view that the coordinates remain the same, and it is instead the metric that transforms according to (4.17), then under the conformal transformation,

$$\sqrt{-g} \to \omega^4(z)\sqrt{-g}. \tag{4.18}$$

This can be shown directly by considering that for a general rank-2 tensor $A^{\mu}{}_{\nu}$, the determinant can be found by using the Levi-Civita tensor density $\tilde{\epsilon}^{\mu\nu\lambda\rho}$ (see eqn. (B.8)),

$$\det(A) = \frac{1}{4!}\tilde{\epsilon}^{\mu\nu\lambda\rho}\tilde{\epsilon}_{\alpha\beta\sigma\kappa}A_{\mu}{}^{\alpha}A_{\nu}{}^{\beta}A_{\lambda}{}^{\sigma}A_{\rho}{}^{\kappa}, \tag{4.19}$$

such that $g \equiv \det(g) \to \det(\tilde{g}) = \omega^8(z)g$. Thus, under the conformal transformation, for the action to be invariant, we also require that the Lagrangian function transforms as,

$$\mathcal{L}(X,Y) \to \omega^{-4}(z)\mathcal{L}(X,Y). \tag{4.20}$$

What is the effect then of the transformation on the invariants of the theory X and Y? The transformations are straightforward to calculate, though for the Y invariant the transformation is most clearly seen by using tensor density form of the Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta} = \tilde{\epsilon}^{\mu\nu\alpha\beta}/\sqrt{-g}$, (see (B.8) of Appendix B). Thus, we have that,

$$X \to \omega^{-4}(z)X, \qquad\qquad Y \to \omega^{-4}(z)Y, \tag{4.21}$$

As mentioned in the introduction to this Chapter, the function $\omega(x)$ is arbitrary, and so for convenience we simply set $\omega^{-4}(z) = \lambda$ in the following, but note that the results work for a generic spacetime dependent function. From (4.20) and (4.21) it is clear that under the action of the conformal transformation,

$$\lambda\mathcal{L}(X,Y) = \mathcal{L}(\lambda X, \lambda Y). \tag{4.22}$$

Differentiating both sides with respect to λ, the conformally invariant Lagrangian function takes on the simple form,

$$\mathcal{L}(X,Y) = X\frac{\partial\mathcal{L}(\lambda X,\lambda Y)}{\partial X} + Y\frac{\partial\mathcal{L}(\lambda X,\lambda Y)}{\partial Y}. \tag{4.23}$$

This is valid for arbitrary λ, and so we are free to set $\lambda = 1$,

$$\mathcal{L}(X,Y) = X\frac{\partial\mathcal{L}(X,Y)}{\partial X} + Y\frac{\partial\mathcal{L}(X,Y)}{\partial Y}. \tag{4.24}$$

We can arrive at a simpler form for the Lagrangian function by using “polar variables”,

$$X = R\cos\phi, \qquad Y = R\sin\phi, \tag{4.25}$$

with $R^2 = X^2 + Y^2$, such that,

$$R\frac{\partial}{\partial R} = X\frac{\partial}{\partial X} + Y\frac{\partial}{\partial Y}. \tag{4.26}$$

Using the new variables in (4.24), gives a differential equation in terms of R and ϕ

which can be integrated directly to give,

$$\mathcal{L}(R, \varphi) = RB(\varphi) \tag{4.27}$$

where $B(\varphi)$ is an arbitrary function of φ. Finally, returning to a description in terms of X and Y,

$$\mathcal{L}(X, Y) = XC\left(\frac{Y}{X}\right), \tag{4.28}$$

where $C(Y/X)$ is an arbitrary function of the *combination* Y/X (and is related to the function $B(\tan^{-1}(Y/X))$ in (4.27) by $C(Y/X) = \sqrt{1 + Y^2/X^2} B(\tan^{-1}(Y/X))$). This defines the full class of conformally invariant, nonlinear effective field theories of electromagnetism.

From this we can see that the Maxwellian Lagrangian function $\mathcal{L}_{\text{Max}} = X$ clearly falls within this more general class (in that case the function is simply $C = 1$). The aim of this work, as stated above, is to investigate this class of theories, and determine if they could be useful as a tool to model strong-field QED processes. Before discussing this, however, we will revisit the energy-momentum tensor of nonlinear electromagnetism, with the new added constraint of a conformally invariant theory, to check if the theories defined by (4.28) agree with our initial intuition about conformal invariance.

4.2.2 The energy-momentum tensor revisited

We first use the conformally invariant Lagrangian function (4.28) in the definition of the excitation tensor. Defining $\xi = Y/X$ we have,

$$H^{\mu\nu} = F^{\mu\nu} C(\xi) + \left(\widetilde{F}^{\mu\nu} - F^{\mu\nu}\xi\right) \frac{d}{d\xi} C(\xi). \tag{4.29}$$

Inserting this into the energy-momentum tensor (3.14), we find

$$T^{\mu}{}_{\nu} = \left(F^{\mu\alpha}F_{\alpha\nu} - \delta^{\mu}_{\nu}X\right) C(\xi) + \left(\widetilde{F}^{\mu\alpha}F_{\alpha\nu} - F^{\mu\alpha}F_{\alpha\nu}\,\xi\right) \frac{d}{d\xi} C(\xi). \tag{4.30}$$

As mentioned previously, the vanishing of the trace of the energy-momentum is typically taken to define invariance under conformal transformations. Checking this with (4.30), we find that this is the case, and for the Lagrangian functions (4.28),

$$T^{\mu}{}_{\mu} = 0. \tag{4.31}$$

So, as expected, the restriction to conformally invariant theories gets rid of the non-vanishing trace in the full energy-momentum tensor. In the previous Chapter, we noted that when the energy-momentum tensor is expanded in a background/probe configuration, the term appearing alongside the Minkowski tensor in (3.40) is responsible for the non-zero trace (since the Minkowski tensor is traceless). It can be easily shown by inserting (4.28) into the additional term that the trace now vanishes.

4.2.3 Physical implications?

We have demonstrated that it is possible to derive an entire class of Lagrangian functions which are nonlinear in the electromagnetic fields, and satisfy conformal invariance, however, can these theories be used to describe strong field processes in some limit? For an effective theory of electromagnetism to be physically meaningful, a basic requirement would be that it reproduces the weak field predictions of Maxwellian electrodynamics as the fields approach zero. If we write the arbitrary function $C(\xi)$ as,

$$C(\xi) = 1 + D(\xi), \tag{4.32}$$

then the weak field limit to obtain Maxwellian electrodynamics in vacuum corresponds to,

$$D(\xi)\Big|_{X,Y\to 0} \to 0. \tag{4.33}$$

However, since this function depends only on the *ratio* of the two invariants, the limit cannot be taken in a meaningful way. The only function which satisfies the above limit is $D(\xi) = 0$, i.e. *the only conformally invariant theory of electromagnetism is Maxwell's theory.* It has been known for some time that the conformal invariance of Maxwell's theory is found only in 4 space-time dimensions, but here we have a further hint at the special role Maxwell's theory plays, as being the only conformally invariant, and physically meaningful theory of electromagnetism in $4D$. Another, more physically intuitive explanation for the lack of a physically relevant, nonlinear conformal theory of electromagnetism is that a conformal transformation has the ability to change weak fields into strong fields, and vice versa, as highlighted by the transformation of the field invariants (4.21). In other words, without having a specific scale involved (such as the Schwinger critical field E_S), there is no real meaning to describing fields as either weak or strong, and there is no distinction between the two.

So to summarise this Chapter: It is possible to use the restriction to conformally invariant theories of electrodynamics, and develop an entire class of theories satisfying the larger conformal group. The Lagrangian functions of such theories take a very simple form, where the full class is defined by the family of functions $C(\xi)$, where $\xi = Y/X$. This dependence on the ratio of the two electromagnetic invariants, however, means that the only physically meaningful conformally invariant theory is Maxwellian electrodynamics.

Chapter 5

Cherenkov radiation from the nonlinear vacuum

So far in this thesis we have considered mainly technical aspects of nonlinear theories of electrodynamics. We now turn to a discussion of a phenomenological feature of such theories, which compared to other effects such as vacuum birefringence and pair production, has not had as much attention given to it in the literature.

In the case of a charged particle moving through a real material medium a well known effect occurs when the particle velocity exceeds the phase velocity of light in that medium — the phenomenon of Cherenkov (sometimes Vavilov-Cherenkov) radiation [268, 269]. The first theoretical work to explain these results was presented by Frank and Tamm [270] (though much earlier work by Heaviside [271] and Sommerfeld [272] considered similar effects). The critical mechanism behind this effect is essentially that due to the presence of a medium of refractive index n, the phase velocity of light becomes less than the speed of light, $v_p = c/n$. A particle travelling through the medium with

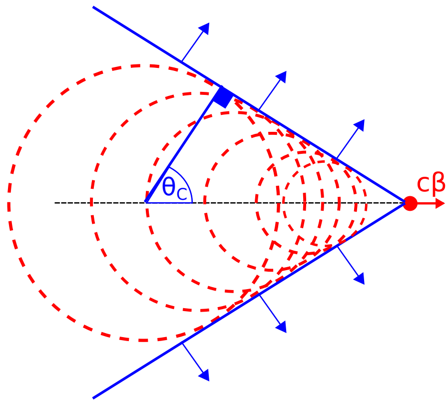

Figure 5.1: Cherenkov radiation occurs due to the build up of wavefronts (red dashed) with origin centred on the particle's orbit (black dashed). The constructive interference of these wavefronts produces Cherenkov radiation, which is emitted in a cone-like structure following the direction of the particle, which is moving with speed $c\beta$, with the cone defined by the Cherenkov angle θ_C.

velocity $c\beta > v_p$ will therefore outrun any electromagnetic waves it could emit. This can lead to the emission of real radiation due to the build up of wave fronts associated with the particle, producing the well known "Cherenkov cone" (analogous to the Mach cone of acoustics) of radiation behind the particle, as shown in Figure 5.1.

As discussed in detail above, the nonlinear interaction of electromagnetic fields in the vacuum leads to an effect on the propagation of light which is analogous to the behaviour of light in a material medium. Crucial to this is the possibility to observe what is sometimes referred to as "slow light" (see for example [164]), i.e. the phase velocity of the propagating wave can become less than c, the speed of light in vacuum. The presence of "slow light" in the nonlinear vacuum then should lend itself to the possibility of a particle with sufficient energy passing through a region of strong field to produce Cherenkov radiation, as well as the usual synchrotron radiation caused by acceleration in the field. The possibility of charged particles producing radiation by a Cherenkov-type mechanism in strong electromagnetic fields was first investigated by

Erber [273], who used insights gained from QED to obtain semi-quantitative estimates of Cherenkov radiation emitted by particles in a strong magnetic field (see also [274]). The Cherenkov effect was further studied in the work of Ritus [171, 275] in the context of radiative corrections and analysis of the photon propagator in strong field QED. This was primarily based on calculating the effective mass of a photon in a strong field, and using the real part of this to determine the Cherenkov rate in a constant crossed field. Ginzburg [276] considered Cherenkov radiation in the context of transition radiation, and determined that for a particle moving across the boundary from a region of no field to a region of strong magnetic field, normal transition radiation would always dominate over the possible Cherenkov radiation which may occur. More recently Cherenkov radiation obtained by a particle propagating through a photon gas was considered in [277–279], also in the context of the theory of Euler-Heisenberg. These studies each highlight the possibility of obtaining Cherenkov radiation, but do not completely develop the theory through a direct use of background field structures and analysis of propagation conditions of the radiation. In each case cited, the results are mainly qualitative, and rely on analysis based on the Euler-Heisenberg effective action. It is the aim of this Chapter to provide a more general look at Cherenkov radiation from the nonlinear vacuum. The analysis uses the most general form of the Lagrangian function $\mathcal{L} = \mathcal{L}(X, Y)$ such that it is valid for arbitrary nonlinear theories, and makes no assumptions about the background field structure, other than working in the limit where variations of the background can be neglected with respect to variations of the radiation field.

Cherenkov radiation has been studied quite extensively in a different (but related) context which is more in line with the approach presented here — in Lorentz violating extensions of (quantum) electrodynamics [280–287]. In these theories, instead of introducing nonlinearities between the fields (as in Euler-Heisenberg [35] or Born-Infeld [97] electrodynamics), the Cherenkov process is mediated by the inclusion of terms in the QED Lagrangian which allow for Lorentz violations at high energies, arising for example in some models of string theory [288, 289] and other "Beyond Standard Model" theories (e.g. [290–292]). While arising from different motivations, many of the results found in this line of enquiry will prove useful in developing a full, first-principles approach to Cherenkov radiation in nonlinear electrodynamics.

The rest of this Chapter is structured as follows. In Section 5.1 we present brief review of the key features of standard Cherenkov radiation found in the literature. In particular we derive the Cherenkov angle formula which can be used in any physical situation, whether it be in a real medium, or in an electromagnetic background, and discuss the power radiated per unit frequency in the case of an isotropic and homogeneous medium. Section 5.2 is devoted to reviewing wave propagation in nonlinear electrodynamics. Specifically, we show how the birefringent dispersion relation of the probe field is obtained by analysis of the propagating modes. The results from the preceding Section are then used to give the Cherenkov angle and power spectrum in nonlinear theories of electromagnetism in Section 5.3. Particular attention is given to how the expressions for the Cherenkov angle and power spectrum must be generalised to account for the background electromagnetic field, which acts analogously to an anisotropic medium. The expressions obtained here are, aside from a few reasonable assumptions, completely general, and can be used for arbitrary Lagrangians and background fields. One of the key features of Cherenkov radiation (in any context) is that the power spectrum has an explicit linear dependence on the emitted photon energy. In the case of particles interacting with real material media, a natural cut-off in the power spectrum occurs. We also discuss the cut-off in nonlinear theories of electromagnetism in Section 5.3. To gain some insight into the possible phenomenological applications of the theory, in Section 5.4 we specialise to the case of an energetic particle propagating in different constant background fields. For Cherenkov radiation to be observable, it must be non-negligible with comparison to the usual synchrotron radiation in electromagnetic fields. As such, we provide analysis which compares the synchrotron and Cherenkov spectra for the different field structures, to see if there are regimes in which Cherenkov may become comparable to, or dominate over, synchrotron radiation. Finally we summarise and conclude in Section 5.5.

5.1 Cherenkov radiation in a simple medium

Before discussing Cherenkov radiation in nonlinear theories of electrodynamics, it is useful to highlight some of the key features of the Cherenkov process in the case of a particle interacting with a simple medium. Consider a particle travelling through some medium with a velocity $\boldsymbol{\beta}$. The particle begins at the origin and as it propagates wave-fronts associated with the particle will be emitted. These wave-fronts will have their centre at the particle position and will propagate outwards with phase velocity $v_p(k)$. At some later time t, for a point $\boldsymbol{r}$ to lie on the wave-front emitted at the origin it will be constrained by the surface equation,

$$\Phi(\boldsymbol{r}, t) = \boldsymbol{r}^2 - v_p^2(k)t^2 = 0. \tag{5.1}$$

This makes no assumption about the background through which the particle is moving. Any anisotropies or inhomogeneities would enter into the constraint through the dependence of the phase velocity on the wave-vector.

An infinitesimal time δt after the first wavefront was emitted, another will be produced by the particle, centred at a new position. There will also be a constraint equation which defines points that lie on this new wave-front,

$$\begin{aligned}\Phi'(\boldsymbol{r}, t) =& (\boldsymbol{r} - \boldsymbol{\beta}\delta t)^2 - v_p^2(k)(t - \delta t)^2, \\ \simeq& \boldsymbol{r}^2 - v_p^2(k)t^2 - 2(\boldsymbol{r}.\boldsymbol{\beta} - v_p^2(k)t)\delta t = 0.\end{aligned} \tag{5.2}$$

Where we linearised in δt in the last line. As stated in the introduction, Cherenkov radiation is essentially due to the constructive interference of these wave-fronts. We therefore wish to find the points satisfying *both* of the constraint equations above, so we require,

$$2(\boldsymbol{r}.\boldsymbol{\beta} - v_p^2(k)t)\delta t = 0, \tag{5.3}$$

to be satisfied for all small times δt. Using $\boldsymbol{r}.\boldsymbol{\beta} = r\beta\cos\theta_C$, (with $r = |\boldsymbol{r}|$, $\beta = |\boldsymbol{\beta}|$) and substituting (5.1) into (5.3), we arrive at the definition of the Cherenkov angle,

$$\cos\theta_C = \frac{v_p(k)}{\beta}. \tag{5.4}$$

We can see immediately that this can only be satisfied when $\beta > v_p$, as stated in the introduction. When the phase velocity exceeds the particle velocity, the wavefronts never overlap and cannot interfere with each other, so no Cherenkov radiation is emitted. Essentially, Cherenkov radiation is generated due to the charge and current densities of particles. Since particles are located at a point, the charge and current densities involve delta-functions, and so contain Fourier components at all frequencies. When the Cherenkov condition (5.4) is satisfied, the radiation generated by these Fourier components can propagate, and real, measurable radiation is produced.

A derivation of the power spectrum can be found in standard textbooks (e.g. [293]). The essential features are that we begin from Maxwell's equations describing a particle travelling through a simple medium with a constant velocity, and Fourier transform to obtain expressions for the Fourier transformed fields. Then, the energy radiated as the particle travels is found by considering the energy flow through a tube surrounding the particle's trajectory. The end result is that the power radiated per unit frequency is given by,

$$\frac{dP}{d\omega} = \frac{e^2}{4\pi}\omega \sin^2\theta_C. \tag{5.5}$$

Cherenkov radiation in an isotropic medium is always linearly polarised in the plane spanned by $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{k}}$, i.e. the plane formed by the direction vectors of the particle velocity and the radiation. The polarisation will also be orthogonal to $\hat{\boldsymbol{k}}$. Orienting the coordinate system such that the particle velocity is in the $\hat{\boldsymbol{z}}$-direction, the Cherenkov angle then coincides with the polar angle of the emitted radiation, such that the spatial part of the (normalised) wavevector is

$$\hat{\boldsymbol{k}} = \sin\theta_C \cos\phi \hat{\boldsymbol{x}} + \sin\theta_C \sin\phi \hat{\boldsymbol{y}} + \cos\theta_C \hat{\boldsymbol{z}}. \tag{5.6}$$

Then, the unit normalised spatial part of the polarisation of ICR is

$$\hat{\boldsymbol{\epsilon}}_{\mathbf{0}} = \frac{\hat{\boldsymbol{\beta}} - \cos\theta_C \hat{\boldsymbol{k}}}{\sin\theta_C}. \tag{5.7}$$

In the usual discussions of Cherenkov radiation in the literature, the angle and power spectrum are calculated on the assumption of a homogeneous and isotropic medium, and so we refer to it as isotropic-medium Cherenkov radiation (ICR). Each of the expressions (5.4) and (5.5) are relatively simple in their structure, with (5.5) depending simply on the Cherenkov angle, which depends only on the ratio of the phase and particle velocities. As such, v_p and $\boldsymbol{\beta}$ are the key parameters. Above, we have made no assumptions on the medium (or background) in the derivation of the Cherenkov angle, and so it is clear that (5.4) should be valid for *any theory* in which there is a phase velocity less than the speed of light, such that highly energetic particles can emit Cherenkov radiation. This is what motivates us to consider this effect in nonlinear electrodynamics, where the nonlinear interaction between a strong background field and propagating radiation can lead to a reduced phase velocity.

5.2 Wave propagation in strong background fields

As just discussed, the above considerations and derivation of the Cherenkov angle gives us an opportunity to discuss Cherenkov radiation in the context of theories where we can have "slow-light", i.e. light moving at speeds less than the vacuum speed of light c. Thus, the first step to discuss this in the context of nonlinear electrodynamics is to look at how the nonlinearities modify the propagation of light moving through strong background fields. We adopt the procedure implemented in Chapter 3 of separating the electromagnetic field $F^{\mu\nu}$ into a strong, slowly varying background $\mathcal{F}^{\mu\nu}$ and a weak radiation field $f^{\mu\nu}$.

The background satisfies the field equations,

$$\partial_\mu \mathcal{H}^{\mu\nu} = 0, \qquad \partial_\mu \widetilde{\mathcal{F}}^{\mu\nu} = 0, \qquad \mathcal{H}^{\mu\nu} = \mathcal{L}_X \mathcal{F}^{\mu\nu} + \mathcal{L}_Y \widetilde{\mathcal{F}}^{\mu\nu}, \tag{5.8}$$

where the shorthand notation

$$\mathcal{L}_{Z_1 \dots Z_N} \equiv \left. \frac{\partial^N \mathcal{L}(X,Y)}{\partial Z_1 \dots \partial Z_N} \right|_{(X=\mathcal{X}, Y=\mathcal{Y})} \tag{5.9}$$

is used, and as before,

$$\mathcal{X} = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}, \qquad \mathcal{Y} = -\frac{1}{4}\widetilde{\mathcal{F}}^{\mu\nu}\mathcal{F}_{\mu\nu}. \tag{5.10}$$

Our concern is the dynamics of the probe field, $f_{\mu\nu}$. Since the background is much stronger than this, we can treat the probe field as a small perturbation, and so Taylor expand and linearise the field equations in the probe field. Using (5.8) and keeping only terms up to order $\mathcal{O}(f)$, we arrive at the field equation for the probe field,

$$\partial_\mu \left(\chi^{\mu\nu\alpha\beta} f_{\alpha\beta}\right) = 0, \tag{5.11}$$

where,

$$\begin{aligned} \chi^{\mu\nu\alpha\beta} \equiv & \mathcal{L}_X \left(g^{\mu\alpha} g^{\nu\beta} - g^{\mu\beta} g^{\nu\alpha}\right) - \mathcal{L}_{XY} \left(\mathcal{F}^{\mu\nu} \widetilde{\mathcal{F}}^{\alpha\beta} + \widetilde{\mathcal{F}}^{\mu\nu} \mathcal{F}^{\alpha\beta}\right) \\ & - \mathcal{L}_{XX} \mathcal{F}^{\mu\nu} \mathcal{F}^{\alpha\beta} - \mathcal{L}_{YY} \widetilde{\mathcal{F}}^{\mu\nu} \widetilde{\mathcal{F}}^{\alpha\beta} + \mathcal{L}_Y \epsilon^{\mu\nu\alpha\beta}. \end{aligned} \tag{5.12}$$

The probe field will also satisfy a Bianchi identity,

$$\partial_\mu \widetilde{f}^{\mu\nu} = 0. \tag{5.13}$$

The system of equations (5.11) and (5.13) has been extensively studied in the literature (e.g. [22, 118, 152]). We first write the probe field in terms of a single Fourier mode,

$$f^{\mu\nu} = \left(k^\mu a^\nu - k^\nu a^\mu\right) e^{ik.x}, \tag{5.14}$$

where k^μ is the wavevector of the radiation field, and a^μ is the gauge 4-potential. This solves the Bianchi identity (5.13). Then, by taking the background to be slowly varying with respect to the probe, such that we can neglect derivatives of the background, the remaining field equation (5.11) becomes simply an algebraic equation for the wavevector

k_μ of the radiation field,

$$\chi^{\mu\nu\alpha\beta} k_\nu k_\beta a_\alpha = 0. \tag{5.15}$$

This can then be solved by writing a_μ in terms of the 4-vectors,

$$a^+_\mu = \mathcal{F}^{\mu\nu} k_\nu, \qquad a^-_\mu = \widetilde{\mathcal{F}}^{\mu\nu} k_\nu, \qquad C_\mu = \mathcal{F}^{\mu\lambda}\mathcal{F}^{\nu\lambda} k_\nu, \tag{5.16}$$

which along with k_μ generically form a basis and are the only independent vectors which can be formed from the background field, its dual, and the wave-vector (any other contractions between these objects gives a vector proportional to one of the above). Thus we have,

$$a_\mu = c_+ a^+_\mu + c_- a^+_\mu + c_c C_\mu + c_k k_\mu, \tag{5.17}$$

which when inserted into (5.15) gives conditions on the parameters c_i ($i = \{+, -, c, k\}$), such that we find $c_c = 0$ and c_k is an arbitrary parameter, i.e. it is pure gauge, and we therefore set to zero, $c_k = 0$. The gauge field can then be explicitly given as,

$$a_\mu = c_+ a^+_\mu + c_- a^-_\mu, \tag{5.18}$$

where $a^\pm_\mu$ are the polarisation 4-vectors of the radiation field.

The coefficients c_+ and c_- can depend on scalar contractions involving the wave-vector (dependence is left implicit), and are related to each other through a system of equations which can be solved to determine the dispersion relation (see in particular [118, 294] for a treatment which has the most notational similarity to the work presented here). Defining the following parameters,

$$\mathcal{A} = \left[\mathcal{L}_X \left(\mathcal{L}_X + 2\mathcal{L}_{XY}\mathcal{Y} - 2\mathcal{L}_{YY}\mathcal{X}\right) + \left({\mathcal{L}_{XY}}^2 - \mathcal{L}_{XX}\mathcal{L}_{YY}\right)\mathcal{Y}^2\right], \tag{5.19}$$

$$\mathcal{B} = -\frac{1}{2}\left[\mathcal{L}_X \left(\mathcal{L}_{XX} + \mathcal{L}_{YY}\right) + 2\left({\mathcal{L}_{XY}}^2 - \mathcal{L}_{XX}\mathcal{L}_{YY}\right)\mathcal{X}\right], \tag{5.20}$$

$$\mathcal{C} = -\left[{\mathcal{L}_{XY}}^2 - \mathcal{L}_{XX}\mathcal{L}_{YY}\right], \tag{5.21}$$

where the shorthand notation (5.9) has been used again, we find that the dispersion relation can be expressed as,

$$k_\pm^2 = \Omega_\pm \mathcal{F}^\mu{}_\lambda \mathcal{F}_{\mu\rho} k_\pm^\lambda k_\pm^\rho, \qquad \Omega_\pm = \frac{-\mathcal{B} \pm \sqrt{\mathcal{B}^2 - \mathcal{A}\mathcal{C}}}{\mathcal{A}}, \tag{5.22}$$

with k_+^μ (k_-^μ) being the wavevector associated with the polarization 4-vector a_+^μ (a_-^μ). Thus, for a general theory of nonlinear vacuum electrodynamics we have *birefringence*, as evident by the two dispersion curves, labelled by the $\pm$ subscript, see figure 5.2. From (5.22), using $k^2 = (\omega^2 - |\boldsymbol{k}|^2) = (v_p^2 - 1)|\boldsymbol{k}|^2$, and defining the direction vector $\hat{k}^\mu = k^\mu/|\boldsymbol{k}|$, then the phase velocity of the probe is,

$$v_{p\pm}^2 = 1 + \Omega_\pm \mathcal{F}^\mu{}_\lambda \mathcal{F}_{\mu\rho} \hat{k}_\pm^\lambda \hat{k}_\pm^\rho. \tag{5.23}$$

Providing only that the background field can be treated as slowly varying with respect to the radiation field then this description of the phase velocity is entirely general. The symmetries of the background — e.g. anisotropy, homogeneity, etc. — are accounted

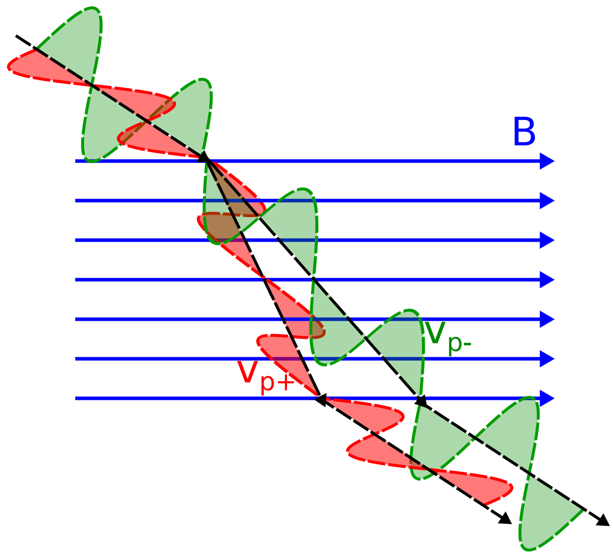

Figure 5.2: Demonstration of vacuum birefringence effect using the example of a magnetic field. Outside of the magnetic field the incident probe wave has polarisation aligned parallel (red), and perpendicular (green) to the magnetic field. The nonlinear interaction causes the parallel polarisation mode to acquire a phase velocity v_{p+}, and the perpendicular mode v_{p-}.

for through the contraction between the field (through the tensor $\mathcal{F}^{\mu}{}_{\lambda}\mathcal{F}_{\mu\rho}$) and the (normalised) wave-vectors $\hat{k}^{\mu}_{(\pm)}$. Any information about the specific theory used enters through the functions $\Omega_{\pm}$.

As an aside, we point out the interesting interpretation of the dispersion relation as being due to the presence of an *effective geometry* induced by the background field structure. This has been well discussed in the literature (see e.g. [153, 295] for more details), but rests essentially on the fact that we can derive the pair of effective metrics,

$$\mathcal{G}^{\mu\nu}_{\pm} = g^{\mu\nu} - \Omega_{\pm}\mathcal{F}^{\mu\lambda}\mathcal{F}^{\nu}{}_{\lambda}, \tag{5.24}$$

which we then use to define the light-cone structure underlying our theory,

$$\mathcal{G}_{\pm}^{\mu\nu} k_\mu k_\nu = 0, \tag{5.25}$$

instead of the usual $k^2 = g^{\mu\nu} k_\mu k_\nu = 0$ light-cone condition.

5.3 Cherenkov radiation in nonlinear electrodynamics

The gauge field solutions obtained above represent any probe field propagating on a strong field background, and so the task now is to specialise to the case where the probe field is radiation due to the Cherenkov effect. To describe Cherenkov radiation, we need to consider two things — the definition of the Cherenkov angle, θ_C, which is the emission angle of the radiation relative to the direction of the particle velocity $\boldsymbol{\beta}$; and the power radiated per unit frequency, $dP/d\omega$. For the case first considered by Frank & Tamm [270] of Cherenkov radiation from an isotropic and homogeneous medium (which we will refer to throughout as ICR), the quantities of interest are defined by (5.4) and (5.5). The question now becomes how can these definitions be generalised to the case of nonlinear vacuum electrodynamics?

5.3.1 Cherenkov angle in nonlinear electrodynamics

With respect to the Cherenkov angle definition, the generalisation is extremely straight forward. A background field will act analogously to an anisotropic medium, meaning that the phase velocity will no longer be a universal parameter, as is the case for an isotropic background, but will instead depend on the wavevector of the propagating radiation, i.e. $v_p \equiv v_p(k)$. In other words, the phase velocity depends on the propagation direction of the radiation relative to the direction of the background electromagnetic field. Thus, it is simply a case of taking this anisotropy in to account through including this dependence in (5.4), by using the phase velocity (5.23) such that,

$$\cos\theta_C = \frac{v_p(k)}{\beta} \qquad \Longrightarrow \qquad \cos^2\theta_C^{\pm} = \frac{1}{\beta^2}\left(1 + \Omega_\pm \mathcal{F}^{\mu}{}_{\lambda}\mathcal{F}_{\mu\rho}\hat{k}_\pm^{\lambda}\hat{k}_\pm^{\rho}\right), \tag{5.26}$$

which is valid in any (constant) background field and nonlinear theory. The anisotropic nature of the background enters through the contraction between the background field $\mathcal{F}^{\mu\nu}$ and the wavevector k^μ.

5.3.2 Power spectrum in nonlinear electrodynamics

To generalise the power radiated per unit angle, we need to consider what features of the radiation produced in the nonlinear theory are different due to the anisotropic background, when compared to ICR which is described by (5.5). Firstly, as indicated already, the phase velocity which the radiation experiences will depend on the direction in which it is travelling. This can be easily taken into account, as all it really means is that the power spectrum will depend on both the emitted frequency and on the azimuthal angle ϕ, such that we consider $d^2P/d\omega d\phi$, which is then integrated over the angular dependence to give the power radiated per unit frequency. Secondly, the polarisation of the radiation in the nonlinear theories will have a different orientation due to the anisotropic nature of the background. To take both of these differences in to account and generalise the power spectrum (5.5) we follow the approach of Altschul

[280], who considered Cherenkov radiation in the context of Lorentz violating extensions of electromagnetism. These types of theories are used as candidates for possible physics beyond the standard model, and characterise high energy effects by allowing for violations of Lorentz invariance at high energy. The nonlinear theories we are considering here strictly maintain this Lorentz symmetry, but the the field equations (5.13) and (5.11) are formally identical to the CPT-even Lorentz violating theories considered in [280], due to linearisation. The key observation in [280] is that Cherenkov modes corresponding to different wave vectors k^μ propagate independently, and hence behave as waves propagating in an isotropic medium with scalar refractive index $n = 1/v_p(\hat{k})$. Since these modes are independent, a distinct Cherenkov condition will be satisfied for each of the modes independently, and so we can treat them separately. So, to tackle the first difference — the dependence of the phase velocity on the wavevector — we recognise that although the phase velocity will depend on the relative angle between the radiation and the background, the phase velocity has a fixed value, the radiation will "see" what looks like a constant phase velocity as it propagates, as in the ICR case. So the real difference between the two types of radiation comes down to the differences in the polarisation, and if we can take this in to account in a formal way, then it should be possible to easily generalise (5.5) for nonlinear electrodynamics.

From the vector structure of the gauge field (5.18) we have two independent polarisation 4-vectors a_+^μ and a_-^μ, however the spatial parts of these will in general not coincide with $\hat{\boldsymbol{\epsilon}}_\mathbf{0}$. If we define the polarisation 3-vectors corresponding to a_+^μ and a_-^μ as $\hat{\boldsymbol{\epsilon}}_+$ and $\hat{\boldsymbol{\epsilon}}_-$ respectively, then only the projection of ICR along these directions will propagate. This leads to the result for the radiated power per unit frequency per unit azimuthal angle:

$$\frac{d^2P_\pm}{d\omega d\phi} = \frac{e^2}{8\pi^2}|\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}.\hat{\boldsymbol{\epsilon}}_\pm|^2\omega\sin^2\theta_C^\pm(\phi). \tag{5.27}$$

Here, $\hat{\boldsymbol{\epsilon}}_\pm$ are the (unit normalized) polarisation 3-vectors of the polarization modes $a^\mu_\pm$ (see below for details). The azimuthal dependence, i.e. the effect of anisotropy of the background, will in general arise through the overlap functions $|\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}.\hat{\boldsymbol{\epsilon}}_\pm|^2$ and the Cherenkov angle $\theta_C^\pm$. To obtain the total power per unit frequency, we simply integrate (5.27) with respect to ϕ, and take the sum of the contributions from each of the polarisation modes,

$$\frac{dP}{d\omega} = \frac{e^2}{8\pi^2}\sum_{+,-}\int_0^{2\pi} d\phi|\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}.\hat{\boldsymbol{\epsilon}}_\pm|^2\omega\sin^2\theta_C^\pm(\phi). \tag{5.28}$$

In generalising the power spectrum from the ICR case to nonlinear electrodynamics, the key detail is the overlap functions $|\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}.\hat{\boldsymbol{\epsilon}}_\pm|^2$ which give the projection of $\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}$, the ICR polarisation, along the polarisation 3-vectors $\hat{\boldsymbol{\epsilon}}_\pm$ which come from the nonlinear theory. The polarisation 3-vectors $\hat{\boldsymbol{\epsilon}}_\pm$ come from the 4-vectors $a^\mu_\pm$, however, they are not simply the spatial components $\boldsymbol{a}_\pm$ of $a^\mu_\pm$. To define $\hat{\boldsymbol{\epsilon}}_\pm$ as the polarisation 3-vectors from the nonlinear theory, we must work in a gauge where the temporal component of $a^\mu_\pm$ vanishes, i.e. the Weyl gauge $a^0_\pm = 0$. In general this will not be the case, as different background electromagnetic fields will give rise to different component structures of $a^\mu_\pm$. For example, a background electric field $\boldsymbol{\mathcal{E}}$ will introduce a nonzero a^0_+, and a background magnetic field $\boldsymbol{\mathcal{B}}$ will introduce a nonzero a^0_-,

$$\text{Electric:} \qquad a^0_+ = \boldsymbol{k}.\boldsymbol{\mathcal{E}}, \tag{5.29}$$

$$\text{Magnetic:} \qquad a^0_- = -\,\boldsymbol{k}.\boldsymbol{\mathcal{B}}. \tag{5.30}$$

To proceed, we make use of the fact that the decomposition of the electromagnetic field $f^{\mu\nu}$ in terms of the gauge 4-potential (5.14) is invariant under the gauge transformation,

$$a^\mu_\pm \to a'^\mu_\pm = a^\mu_\pm + \mathcal{C}_\pm k^\mu, \tag{5.31}$$

and choosing $\mathcal{C}_+ = -\boldsymbol{k}.\boldsymbol{\mathcal{E}}/\omega$ when we have a background electric field and $\mathcal{C}_- = \boldsymbol{k}.\boldsymbol{\mathcal{B}}/\omega$ for a background magnetic field will ensure the new gauge 4-potential $a'^\mu_\pm$ is in the Weyl gauge, and the spatial components will then be proportional to the unit normalised 3-vectors, $\boldsymbol{a}'_\pm \propto \hat{\boldsymbol{\epsilon}}_\pm$. It can then be deduced that the unit normalised polarisation 3-vectors are,

$$\hat{\boldsymbol{\epsilon}}_+ = \frac{\boldsymbol{\mathcal{E}} + v_p^{-1}\hat{\boldsymbol{k}} \times \boldsymbol{\mathcal{B}} - v_p^{-2}(\boldsymbol{\mathcal{E}}.\hat{\boldsymbol{k}})\hat{\boldsymbol{k}}}{|\boldsymbol{\mathcal{E}} + v_p^{-1}\hat{\boldsymbol{k}} \times \boldsymbol{\mathcal{B}} - v_p^{-2}(\boldsymbol{\mathcal{E}}.\hat{\boldsymbol{k}})\hat{\boldsymbol{k}}|}, \quad \hat{\boldsymbol{\epsilon}}_- = \frac{-\boldsymbol{\mathcal{B}} + v_p^{-1}\hat{\boldsymbol{k}} \times \boldsymbol{\mathcal{E}} + v_p^{-2}(\boldsymbol{\mathcal{B}}.\hat{\boldsymbol{k}})\hat{\boldsymbol{k}}}{|-\boldsymbol{\mathcal{B}} + v_p^{-1}\hat{\boldsymbol{k}} \times \boldsymbol{\mathcal{E}} + v_p^{-2}(\boldsymbol{\mathcal{B}}.\hat{\boldsymbol{k}})\hat{\boldsymbol{k}}|}, \tag{5.32}$$

where v_p is the phase velocity, which can either be obtained from the definition of the Cherenkov angle $v_p = \beta \cos\theta_C$ or from the dispersion relation (5.23).

5.3.3 Power spectrum cut-off

The power spectrum of both ICR (5.5) and nonlinear electrodynamics (5.28) are linear in the frequency ω, which would appear to suggest that as we go to higher and higher emitted photon energies $\hbar\omega$, the spectrum would suffer from divergences. In the case of ICR, or any Cherenkov radiation observed in a real medium, the cut-off in the spectrum is a natural phenomenon, which occurs because the dispersion relation has an explicit frequency dependence, and so the Cherenkov angle θ_C depends on ω. This results in a frequency dependent refractive index and for high frequencies the refractive index approaches unity. Since $n = 1/v_p$ no Cherenkov radiation can be generated, as $\cos\theta_C = 1/\beta n < 1$ cannot be satisfied at these high frequencies.

For the nonlinear theories, the dispersion relation (5.22) does not give a mechanism for a natural cut-off to emerge in an analogous way, and so we must find a suitable way of imposing a cut-off on the spectrum (5.28). This is a problem which has also been encountered in other contexts, such as the previously mentioned studies which considered Cherenkov radiation in Lorentz violating theories [280, 281]. So if Cherenkov radiation can really be observed from the nonlinear vacuum, we must assume the existence of some high-energy process which acts to provide a cut-off in the frequency spectrum in an analogous way.

We could simply impose a cut-off directly on the frequency ω, however frequency is not a Lorentz invariant quantity. The same statement holds for simply setting the cut-off as some energy, say the electron mass-energy $m_e c^2$. There should be a real, physical mechanism which conspires to create a cut-off in the spectrum, and so we should look towards a Lorentz and gauge invariant description for the cut-off. A useful parameter to consider then is the photon quantum nonlinearity parameter, χ_γ, which is defined by [171],

$$\chi_\gamma = \frac{|e|}{m_e^3}\sqrt{-\mathcal{F}^{\mu\lambda}\mathcal{F}^{\nu}{}_{\lambda}k_\mu k_\nu}, \tag{5.33}$$

for a photon of 4-momentum k_μ. A suitable cut-off for the theory could then be, for example,

$$\chi_\gamma^2 \lesssim 1, \tag{5.34}$$

It is not possible with the theory presented here to directly derive this cut-off limit, as it is beyond the scope of this theory, and we need to assume the existence of some new physics or other effects at high energies which would impose a natural cut-off. In the case of Euler-Heisenberg theory it is possible that including derivative corrections to the theory may provide a natural route to obtaining the cut-off. Since Euler-Heisenberg is derived on the basis of constant fields, when high frequencies are involved, these derivative corrections arise. This prospect is not covered in this work, though we hope to return to this in the future.

5.4 Weak-field Lagrangian and constant fields

We now know how to take into account the anisotropic background through (5.28), and can simply use the k-dependent phase velocity (5.23) to determine our Cherenkov angle. Our analysis so far has been completely general, with no assumptions made about the dynamics, other than the fact that we are treating the background field as slowly varying with respect to the radiation field. It will be more instructive now, to consider particular field configurations, and also work within a specified nonlinear theory. As such, we will now move to consider the quadratic weak-field Lagrangians defined in (2.16), which give the lowest order, parity preserving corrections to Maxwell's theory,

$$\mathcal{L} = X + \lambda_+ X^2 + \lambda_- Y^2.$$

With this simplification, the parameters $\Omega_\pm$, defined in (5.22) become, to leading order,

$$\Omega_\pm \simeq 2\lambda_\pm, \tag{5.35}$$

such that,

$$v_{p\pm}^2 \simeq 1 + 2\lambda_\pm \mathcal{F}_{\lambda\mu}\mathcal{F}^\lambda{}_\nu \hat{k}_\pm^\mu \hat{k}_\pm^\nu, \qquad \cos^2\theta_C^\pm \simeq \frac{1}{\beta^2}\left(1 + 2\lambda_\pm \mathcal{F}_{\lambda\mu}\mathcal{F}^\lambda{}_\nu \hat{k}_\pm^\mu \hat{k}_\pm^\nu\right). \tag{5.36}$$

From the definition of the phase velocity (and Cherenkov angle) above, and considering for example the values of the parameters $\lambda_\pm$ for Euler-Heisenberg (2.17) and the estimate for Born-Infeld (2.18) (see also text above this), we can see that the effect on the phase velocity is going to be a very small correction, which will be enhanced by having very energetic particles with $\gamma^2 = 1/(1-\beta^2) \gg 1$, and very strong electromagnetic fields. So, when considering what sort of background fields to look at, there are two candidates for physically meaningful sources of strong electromagnetic fields: lab-based high-powered lasers which can be approximated by a "constant crossed field", which is one that has equal and orthogonal electric and magnetic components, and astrophysical sources of magnetic fields such as pulsars, which can be approximated by a constant magnetic field.

If Cherenkov radiation due to nonlinear electromagnetic interactions is to be observed, it will have to be non-negligible with comparison to the usual synchrotron radiation which will occur due to the acceleration of the particle in the field. The synchrotron spectrum for a particle in a magnetic field of field strength B was first obtained by Schwinger [296]. For a particle of mass m and charge e moving perpendicularly to the magnetic field this is (reinstating factors of c),

$$\frac{dP_{\text{Synch}}}{d\omega} = \frac{\sqrt{3}}{2\pi}\frac{e^3 B}{mc\epsilon_0}\frac{\omega}{\omega_S}\int_{\omega/\omega_S}^{\infty} dx K_{5/3}(x), \tag{5.37}$$

where $K_\nu(x)$ is the order ν modified Bessel function of the second kind and

$$\omega_S = \frac{3}{2}\frac{eB}{m}\gamma^2. \tag{5.38}$$

In the case of a constant crossed field, with field strength E, we use (5.37) and (5.38) with the substitution $B \to 2E/c$ (where the factor of 2 comes from the fact that the constant crossed field has both electric and magnetic components).

In the above analysis, the particle is considered to be moving rectilinearly. It may be questioned whether this approximation can be used, as the strong field will deflect the particle. However, we will address this point below in Section 5.4.2, and highlight that for a wide range of parameter values which are considered, the approximation holds to a high degree of tolerance.

5.4.1 Constant crossed field

As discussed above, advances in laser technology have begun to open up the strong-field regime of quantum electrodynamics to experimental study. Doing QED calculations in strong background fields with realistic models of the laser field is an extremely challenging task, though many insights have been gained through using what is commonly referred to as a constant crossed field model of the laser. This approximation is considered to be valid in the ultrarelativistic regime, as for high particle energies and intense fields, even complicated pulse structures look approximately constant to the particle (this is the basis of what is known as the locally constant field approximation, used extensively in numerical simulations of strong-field QED processes e.g. [38, 171, 297, 298]).

To be more precise, a constant crossed field is a field with equal and perpendicular electric and magnetic components. This is essentially the zero-frequency (constant) limit of a plane-wave. In the coordinate basis (D.2) defined in Appendix D, a constant crossed field with magnitude E, with Poynting vector in the $\hat{\boldsymbol{z}}$-direction, and polarisation in the $\hat{\boldsymbol{y}}$-direction has an electromagnetic field tensor,

$$\mathcal{F}^{\mu\nu} = E\left[\left(\eta^{\mu} - \epsilon_3^{\mu}\right)\epsilon_2^{\nu} - \left(\eta^{\nu} - \epsilon_3^{\nu}\right)\epsilon_2^{\mu}\right]. \tag{5.39}$$

With this definition, the electromagnetic invariants of the background are,

$$\mathcal{X} = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu} = 0, \qquad \mathcal{Y} = -\frac{1}{4}\widetilde{\mathcal{F}}^{\mu\nu}\mathcal{F}_{\mu\nu} = 0. \tag{5.40}$$

We consider a charged particle to be counter-propagating with respect to the Poynting vector of the constant crossed field, as in this configuration the energy transfer between the field and the particle will be maximised, and the strongest effect observed. With this configuration, the Cherenkov angle θ_C coincides with the usual polar angle θ in spherical polar coordinates. We express the normalised wavevector $\hat{k}^\mu$ in terms of the coordinate basis (D.2),

$$\hat{k}_\nu = v_p\eta_\nu + \sin\theta_C\cos\phi\epsilon_{1\nu} + \sin\theta_C\sin\phi\epsilon_{2\nu} + \cos\theta_C\epsilon_{3\nu}. \tag{5.41}$$

Substituting (5.39) and (5.41) into the definition of the phase velocity (5.36) we find,

$$v_{p\pm}^2 = 1 - 2\lambda_\pm\left(v_{p\pm} + \cos\theta_C\right)^2. \tag{5.42}$$

Using $v_{p\pm} = \beta\cos\theta_C^\pm$, the Cherenkov angle takes on the simple form,

$$\cos^2\theta_C^\pm = \frac{1}{\left[\beta^2 + 2\left(1+\beta\right)^2\lambda_\pm E^2\right]}. \tag{5.43}$$

So for the case of the constant crossed field, the Cherenkov angle is independent of the azimuthal angle ϕ. As mentioned previously, Cherenkov radiation is only found when the condition $\cos^2\theta_C < 1$ is satisfied. Thus, we find that Cherenkov radiation can be emitted by a particle interacting with a constant crossed field when,

$$(\gamma+\sqrt{\gamma^2-1})^2E^2 > \frac{1}{2\lambda_\pm}, \tag{5.44}$$

where we have expressed things in terms of the Lorentz factor $\gamma^2 = 1/(1-\beta^2)$ of the particle. The inequality (5.44) highlights the need for extremely high field strengths and particle energies, when considering the magnitude of the parameters $\lambda_\pm$.

We now need to consider the polarisation 3-vectors $\hat{\boldsymbol{\epsilon}}_\pm$, which we previously defined through (5.32). We have the two polarization 4-vectors $a^+_\mu = \mathcal{F}^{\mu\nu}k_\nu$, and, $a^-_\mu = \widetilde{\mathcal{F}}^{\mu\nu}k_\nu$, which with the constant crossed field background (5.39) are,

$$a^\mu_+ = -\,E\left[(\eta^\mu-\epsilon^\mu_3)\,|\boldsymbol{k}|\sin\theta_C\sin\phi + (\omega+|\boldsymbol{k}|\cos\theta_C)\,\epsilon^\mu_2\right], \tag{5.45}$$

$$a^\mu_- = -\,E\left[(\eta^\mu-\epsilon^\mu_3)\,|\boldsymbol{k}|\sin\theta_C\cos\phi + (\omega+|\boldsymbol{k}|\cos\theta_C)\,\epsilon^\mu_1\right]. \tag{5.46}$$

As discussed above, to interpret the spatial components of these vectors as being proportional to the 3-polarisations, the temporal components must vanish. However, from (5.45) and (5.46), we find that,

$$a^0_+ = -\,E|\boldsymbol{k}|\sin\theta_C\sin\phi, \qquad a^0_- = -\,E|\boldsymbol{k}|\sin\theta_C\cos\phi, \tag{5.47}$$

and so we must work with the gauge transformed 4-vectors $a^\mu_\pm \to a'^\mu_\pm = a^\mu_\pm + \mathcal{C}_\pm k^\mu$ from (5.31), with the parameters,

$$\mathcal{C}_+ = v_p^{-1}E\sin\theta_C\sin\phi, \qquad \mathcal{C}_- = v_p^{-1}E\sin\theta_C\cos\phi. \tag{5.48}$$

Thus, taking the spatial components of the transformed gauge 4-vector a'^{μ}_{+} we have,

$$\boldsymbol{a}'_{+} = E|\boldsymbol{k}|v_p^{-1}\left\{\sin\theta_C\sin\phi\hat{\boldsymbol{k}} + v_p\sin\theta_C\sin\phi\hat{\boldsymbol{z}} - v_p\left[\cos\theta_C + v_p\right]\hat{\boldsymbol{y}}\right\}, \tag{5.49}$$

which has a normalisation of,

$$|\boldsymbol{a}'_{+}| = E|\boldsymbol{k}|v_p^{-1}\sqrt{v_p^2\left(\cos\theta_C + v_p\right)^2 - \left(v_p^2 - 1\right)\sin^2\theta_C\sin^2\phi}, \tag{5.50}$$

such that the polarisation 3-vector $\hat{\boldsymbol{\epsilon}}_{+}$ is given by,

$$\hat{\boldsymbol{\epsilon}}_{+} = \frac{\boldsymbol{a}'_{+}}{|\boldsymbol{a}'_{+}|} = \frac{\sin\theta_C\sin\phi\hat{\boldsymbol{k}} + v_p\sin\theta_C\sin\phi\hat{\boldsymbol{z}} - v_p\left[\cos\theta_C + v_p\right]\hat{\boldsymbol{y}}}{\sqrt{v_p^2\left(\cos\theta_C + v_p\right)^2 - \left(v_p^2 - 1\right)\sin^2\theta_C\sin^2\phi}}. \tag{5.51}$$

Similarly, the spatial components of the transformed gauge 4-vector a'^{μ}_{+} are,

$$\boldsymbol{a}'_{-} = E|\boldsymbol{k}|v_p^{-1}\left\{\sin\theta_C\cos\phi\hat{\boldsymbol{k}} + v_p\sin\theta_C\cos\phi\hat{\boldsymbol{z}} - v_p\left[\cos\theta_C + v_p\right]\hat{\boldsymbol{x}}\right\}, \tag{5.52}$$

which with the normalisation,

$$|\boldsymbol{a}'_{-}| = E|\boldsymbol{k}|v_p^{-1}\sqrt{v_p^4 + 2v_p^3\cos\theta_C + v_p^2\left(\cos^2\theta_C - \sin^2\theta_C\cos^2\phi\right) + \sin^2\theta_C\cos^2\phi} \tag{5.53}$$

gives the polarisation 3-vector

$$\hat{\boldsymbol{\epsilon}}_{-} = \frac{\boldsymbol{a}'_{-}}{|\boldsymbol{a}'_{-}|} = \frac{\sin\theta_C\cos\phi\hat{\boldsymbol{k}} + v_p\sin\theta_C\cos\phi\hat{\boldsymbol{z}} - v_p\left[\cos\theta_C + v_p\right]\hat{\boldsymbol{x}}}{\sqrt{v_p^4 + 2v_p^3\cos\theta_C + v_p^2\left(\cos^2\theta_C - \sin^2\theta_C\cos^2\phi\right) + \sin^2\theta_C\cos^2\phi}}. \tag{5.54}$$

Finally then, the squared overlap functions $|\hat{\boldsymbol{\epsilon}}_{\pm}.\hat{\boldsymbol{\epsilon}}_{\boldsymbol{0}}|^2$ are found by combining (5.51) and (5.54) with the polarisation 3-vector of ICR (5.7), giving respectively,

$$|\hat{\boldsymbol{\epsilon}}_{+}.\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}|^{2} = \frac{v_p^2 \sin^2\phi\,(v_p\cos\theta_C+1)^2}{v_p^4+2v_p^3\cos\theta_C-(v_p^2-1)\sin^2\theta_C\sin^2\phi+v_p^2\cos^2\theta_C}, \tag{5.55}$$

and,

$$|\hat{\boldsymbol{\epsilon}}_{-}.\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}|^{2} = \frac{v_p^2 \csc^2\theta_C\cos^2\phi\,(v_p\cos\theta_C+1)^2}{v_p^2\csc^2\theta_C(\cos\theta_C+v_p)^2-(v_p^2-1)\cos^2\phi}. \tag{5.56}$$

These have quite a complicated structure, and dependence on various parameters such as the Cherenkov angle θ_C and the phase velocity v_p. However, using the fact that $v_p \simeq 1+\mathcal{O}(\lambda)$, where $\lambda=\lambda_{\pm}$ is small we find that,

$$|\hat{\boldsymbol{\epsilon}}_{+}.\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}|^{2} \simeq \sin^2\phi+\mathcal{O}(\lambda), \tag{5.57}$$

and,

$$|\hat{\boldsymbol{\epsilon}}_{-}.\hat{\boldsymbol{\epsilon}}_{\mathbf{0}}|^{2} \simeq \cos^2\phi+\mathcal{O}(\lambda). \tag{5.58}$$

We find that to a within a high level of accuracy, these approximations give near identical results to the full expressions (5.55) and (5.56) when the rest of the structure of the power spectra (5.28) is taken into account.

The final thing we need before looking at the power spectrum is the energy cut-off which comes from the photon nonlinearity parameter χ_γ defined in (5.33), with the condition (5.34). Using the background field (5.39), we have,

$$\chi_\gamma^2 = \frac{e^2}{m_e^6} E^2 |\boldsymbol{k}|^2 \left(v_p + \cos\theta_C\right)^2 , \tag{5.59}$$

and using $v_p = \beta \cos\theta_C$,

$$\chi_\gamma^2 = \frac{e^2}{m_e^6} E^2 |\boldsymbol{k}|^2 \left(\beta + 1\right)^2 \cos^2\theta_C . \tag{5.60}$$

Now, using the definition of the Cherenkov angle in the constant crossed field (5.43), and inserting the resulting expression for χ_γ^2 into the inequality (5.34) we find the maximum emitted photon energy to be (reinstating factors of c and $\hbar$),

$$\hbar\omega_{\max} \simeq \frac{m_e^3 c^5}{2e\hbar E} \tag{5.61}$$

where we used the approximation $\omega^2 \simeq |\boldsymbol{k}|^2$. This approximation is valid here, as we only wish to have an approximate value for the energy cut-off. The cut-off here is really only a guide, based on the requirement that a physical cut-off will be an observable effect, and so must come from some gauge and Lorentz invariant mechanism, and so a first order approximation suffices. Alongside the Cherenkov angle (5.43) and polarisation overlap functions (5.57) and (5.58), we now have everything we need to compare the power spectrum (5.28) with the synchrotron spectrum (5.37) (with the substitution $B \to 2E/c$). We will also consider a specific theory, and since it is arguably the best motivated physically (as it is derived fom QED), we will work with Euler-Heisenberg, where the parameters $\lambda_\pm$ have the values (2.17).

The next generation of high-intensity laser facilities such as the Extreme Light Infrastructure [50] aim to achieve peak field strengths on the order of $E \sim E_S \times 10^{-3}$ (with E_S the critical field of QED). The high-energy particles which will predominantly be used in strong-field QED experiments at these facilities will be electrons generated through laser wakefield acceleration, with energies up to $\gamma \sim 10^5$ ($\simeq 50$ GeV). Thus, we consider this parameter regime in comparing the spectra from Cherenkov and synchrotron radiation in a constant crossed field. Figure 5.3 shows the calculated power per unit frequency due to each of the radiation processes as a function of the emitted photon energy $\hbar\omega$. The black dashed line represents the cut-off found from (5.61), $\hbar\omega_{\max} \sim (m_e^3 c^5)/(2e\hbar E) \simeq 0.25$ GeV. Below this limit, the synchrotron radiation is always the dominant process (by several orders of magnitude). Thus, observing the Cherenkov effect appears unlikely for even future laser facilities. This is primarily due to the limitation on the ability to produce high energy electrons in the lab. For $\gamma \gg 1$, the Cherenkov spectrum becomes roughly proportional to the square of the field strength, E^2, to leading order, and so for a fixed field strength, increasing the energy of the particles has very little effect on the Cherenkov spectrum. Conversely, the synchrotron spectrum becomes more and more suppressed as γ increases for fixed values of E. For the field strength considered here ($E \sim E_S \times 10^{-3}$) an electron Lorentz factor of $\gamma \sim 2.5 \times 10^6$, corresponding to an energy of 1.3 TeV would be required to have the contributions from Cherenkov and synchrotron processes approximately equal at the cut-off $\hbar\omega \sim 0.25$ GeV, as shown in Figure 5.4. There would also be the concern that to reach these high field strengths in a real experiment, strong focussing techniques need to be used to compress the laser pulse, and this brings in a significant range of other effects which would act to drown out the Cherenkov signal, or deplete the electron energy significantly to the point that by the time the electron reaches the peak intensity of the pulse, its energy has fallen below the Cherenkov threshold.

One may wonder whether considering heavier particles, such as protons, would offer a better opportunity to observe Cherenkov radiation, as the synchrotron spectrum will be suppressed. However, this turns out to not be the case, and we find instead that the Cherenkov spectrum is also suppressed. This is due to the fact that, although protons with much higher energies can be produced, for example in the LHC with energies of up to 7 TeV, the Cherenkov spectrum depends on the energy only through the Lorentz factor γ. Thus, even for the highest energy protons which can be produced in the lab, the suppression of γ by the proton mass leads to a similar situation as noted above for electrons. Thus, the possibility of observing Cherenkov radiation in the lab seems bleak.

Since the main obstacle for Cherenkov radiation to become comparable to the synchrotron radiation of a particle appears to be the availability of high energy particles, it is natural to turn our attention instead to astrophysics, where the only limit on the particle energy is the so-called GZK limit $\gamma \lesssim 10^{11}$ [299, 300], and so we consider a constant magnetic field to model the fields produced by pulsars.

5.4.2 Constant magnetic field

The availability of high energy particles and strong electromagnetic fields makes studying Cherenkov radiation in the context of astrophysics very natural. As mentioned previously, the only limit to the energy particles can have is the GZK limit $\gamma \lesssim 10^{11}$ [299,

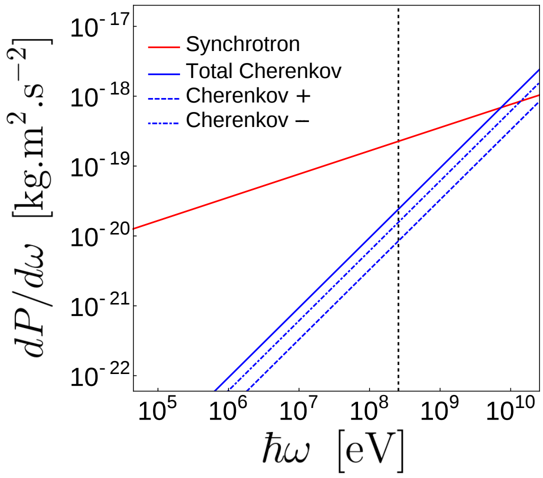

Figure 5.3: Radiated power per unit frequency from the interaction of an electron with $\gamma = 10^5$ ($\sim$ 50 GeV) and a crossed field with field strength $E = E_S \times 10^{-3}$, due to: Synchrotron radiation (red); total Cherenkov radiation (blue, solid); Cherenkov + mode (blue, dashed); Cherenkov − mode (blue, dot-dashed). The cut-off energy (black, dashed) is $\hbar\omega_{\max} \sim 0.25$ GeV.

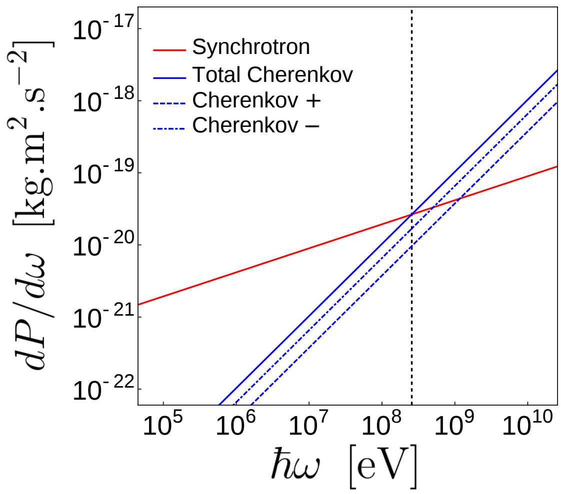

Figure 5.4: Radiated power per unit frequency from the interaction of an electron with $\gamma \sim 2.5 \times 10^6$ (1.3 TeV) and a crossed field with field strength $E = E_S \times 10^{-3}$, due to: Synchrotron radiation (red); total Cherenkov radiation (blue, solid); Cherenkov + mode (blue, dashed); Cherenkov − mode (blue, dot-dashed). The cut-off energy (black, dashed) is $\hbar\omega_{\max} \sim 0.25$ GeV.

300], and several pulsars have been observed to produce magnetic fields with strength up to and exceeding the the critical Schwinger magnetic field $B_S = E_S/c \simeq 4.4 \times 10^9$ T. We therefore turn our attention to consider Cherenkov radiation from particles moving in constant magnetic fields.

We consider a constant magnetic field of strength B, aligned in the $\hat{\boldsymbol{y}}$-direction, which in the same basis as before can be expressed as,

$$\mathcal{F}^{\lambda\rho} = B\left(\epsilon_1^\lambda \epsilon_3^\rho - \epsilon_3^\lambda \epsilon_1^\rho\right). \tag{5.62}$$

It can be shown that charged particles moving along the field lines cannot emit Cherenkov radiation, due to the fact that superluminal velocities $\beta > 1$ are required. As such, we consider particles moving perpendicular to the magnetic field, in the $\hat{\boldsymbol{z}}$-direction, which is also valid on the basis that any parallel component of the velocity could be removed by a Lorentz transformation along the field lines which leaves the background field invariant. With this configuration, as before, the Cherenkov angle θ_C corresponds to the usual polar angle θ in spherical polar coordinates, and so we use the normalised wavevector $\hat{k}_\mu$ from (5.41). Inserting the field structure (5.62) and (5.41) into the phase velocity definition (5.36) then gives us,

$$v_{p\pm}^2 \simeq 1 - 2\lambda_\pm B^2\left(1 - \sin^2\theta_C \sin^2\phi\right) \qquad \cos^2\theta_C^\pm \simeq \frac{1 - 2\lambda_\pm B^2 \cos^2\phi}{\beta^2 + 2\lambda_\pm B^2 \sin^2\phi}, \tag{5.63}$$

which gives the Cherenkov condition,

$$\gamma^2 B^2 > \frac{1}{2\lambda_\pm}. \tag{5.64}$$

We see here that unlike in the case of a constant crossed field, the Cherenkov angle now has an explicit dependence on the azimuthal angle ϕ.

Following the same procedure as before, we now need to consider the the polarization overlap functions which appear in the Cherenkov power spectrum (5.28). With the definition of the field (5.62) and the wavevector k^μ in spherical polar coordinates we find that,

$$a_+^\mu = -\, B|\boldsymbol{k}|\,(\epsilon_1^\mu \cos\theta_C - \epsilon_3^\mu \sin\theta_C \cos\phi) \tag{5.65}$$

$$a_-^\mu = -\, B\,(\eta^\mu |\boldsymbol{k}| \sin\theta_C \sin\phi + \epsilon_2^\mu \omega)\,. \tag{5.66}$$

Unlike for the crossed field, only the a_-^μ polarisation 4-vector has a nonzero temporal component, the a_+^μ 4-vector is already in the Weyl gauge $a_+^0 = 0$ and so we have,

$$\boldsymbol{a}_+ = -\, B|\boldsymbol{k}|\,(\hat{\boldsymbol{x}} \cos\theta_C - \hat{\boldsymbol{z}} \sin\theta_C \cos\phi)\,. \tag{5.67}$$

which gives,

$$|\boldsymbol{a}_+| = B|\boldsymbol{k}|\sqrt{1 - \sin^2\theta_C \sin^2\phi}. \tag{5.68}$$

The polarisation 3-vector $\hat{\boldsymbol{\epsilon}}_+$ is then found as before from,

$$\hat{\boldsymbol{\epsilon}}_+ = \frac{\boldsymbol{a}_+}{|\boldsymbol{a}_+|} = -\frac{(\hat{\boldsymbol{x}}\cos\theta_C - \hat{\boldsymbol{z}}\sin\theta_C\cos\phi)}{\sqrt{1-\sin^2\theta_C\sin^2\phi}}. \tag{5.69}$$

The other polarisation mode a_-^μ instead has a nonzero temporal component,

$$a_-^0 = -B|\boldsymbol{k}|\sin\theta_C\sin\phi, \tag{5.70}$$

and so we must use the gauge transformation $a_-^\mu \to a_-'^\mu = a_-^\mu + \mathcal{C}_- k^\mu$, with,

$$\mathcal{C}_- = v_p^{-1} B\sin\theta_C\sin\phi. \tag{5.71}$$

With the gauge transformation taken into account, we have,

$$\boldsymbol{a}'_- = B v_p^{-1}|\boldsymbol{k}|\sin\theta_C\sin\phi\hat{\boldsymbol{k}} - \omega B\hat{\boldsymbol{y}} \tag{5.72}$$

which with the normalisation,

$$|\boldsymbol{a}'_-| = B|\boldsymbol{k}|v_p^{-1}\sqrt{v_p^4 - \left(2v_p^2-1\right)\sin^2\theta_C\sin^2\phi} \tag{5.73}$$

gives the polarisation 3-vector,

$$\hat{\boldsymbol{\epsilon}}_- = \frac{\boldsymbol{a}_+}{|\boldsymbol{a}_+|} = \frac{\sin\phi\sin\theta_C\hat{\boldsymbol{k}} - v_p^2\hat{\boldsymbol{y}}}{\sqrt{v_p^4 - \left(2v_p^2-1\right)\sin^2\theta_C\sin^2\phi}}. \tag{5.74}$$

Combining together the polarisation 3-vectors (5.69) and (5.74) with the polarisation 3-vector of ICR (5.7), we find that the squared overlap functions,

$$|\hat{\boldsymbol{\epsilon}}_+ . \hat{\boldsymbol{\epsilon}}_{\mathbf{0}}|^2 = \frac{\cos^2\phi}{1 - \sin^2\theta_C \sin^2\phi}, \tag{5.75}$$

$$|\hat{\boldsymbol{\epsilon}}_- . \hat{\boldsymbol{\epsilon}}_{\mathbf{0}}|^2 = \frac{v_p^4 \cos^2\theta_C \sin^2\phi}{v_p^4 - (2v_p^2 - 1)\sin^2\theta_C \sin^2\phi} = \frac{\cos^2\theta_C \sin^2\phi}{1 - \sin^2\theta_C \sin^2\phi} + \mathcal{O}(\lambda_-^2), \tag{5.76}$$

where for $|\hat{\boldsymbol{\epsilon}}_- . \hat{\boldsymbol{\epsilon}}_{\mathbf{0}}|^2$ we used the fact that $v_p^2 = 1 + \mathcal{O}(\lambda_-)$.

The final thing to consider is the energy cut-off which comes from the photon nonlinearity parameter χ_γ defined in (5.33), with the condition (5.34). Using a similar procedure as for the constant crossed field case, we find that the highest emitted photons have an energy,

$$\hbar\omega_{\max} \sim \frac{m_e^3 c^4}{2e\hbar B}, \tag{5.77}$$

analogously to that of the constant crossed field. In fact, since we consider the cut-off to be an approximation, it would have sufficed to simply make the change $E \to Bc$.

We have everything we need to now consider the spectrum, and to compare it against the synchrotron emission in the constant magnetic field. We will choose to work with the Euler-Heisenberg theory again. We are considering high energy cosmic rays, which are predominantly protons, and so we consider the two radiation processes for these. This amounts to changing $m \to m_p$ in (5.37). Factors of m_e appearing in the Cherenkov spectrum (through the parameters $\lambda_\pm$) and the cut-off are not changed. The reason for this is that the nonlinear terms appearing in the Lagrangian (2.16) and the mass scale appearing in the cut-off (5.77) are (for Euler-Heisenberg) determined by the

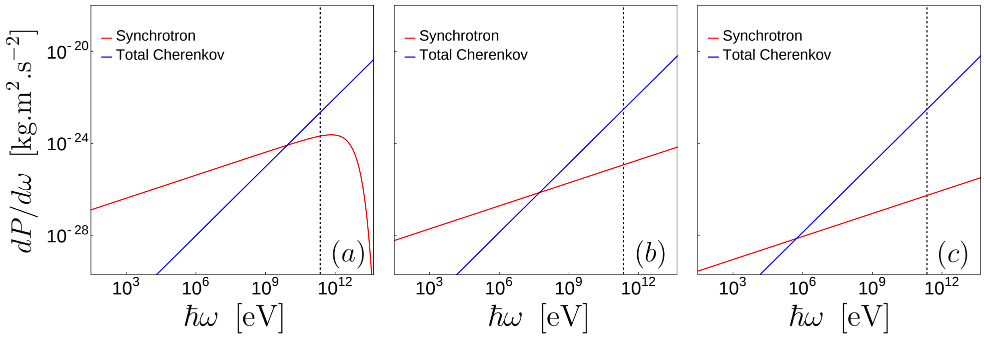

Figure 5.5: Radiated power per unit frequency from protons interacting with a constant magnetic field $B = 10^4$ T, with Lorentz factors: (a) $\gamma = 5 \times 10^7$, (b) $\gamma = 5 \times 10^9$, (c) $\gamma = 5 \times 10^{11}$. The cut-off energy is $\hbar\omega_{\text{max}} \sim 226$ GeV.

fluctuations of particle-antiparticle pairs in the vacuum, which will mainly be electrons and positrons. The total power radiated per unit frequency is again determined by (5.28).

In astrophysics, the strongest magnetic fields which have been observed are those produced by rapidly rotating pulsars. These objects have characteristic attributes of mass and radius, which with the rotational period determine the typical field strengths produced. There are two broad classes of pulsars, those with a relatively longer rotational period which have values of the magnetic field strength of around $B \sim 10^8$ T, and rapidly rotating "millisecond pulsars" which have typical values of $B < 5 \times 10^4$ T [301] (though there are many examples of pulsar magnetic fields with field strength $B > B_S$ [71]). We envisage the scenario where very high energy cosmic rays, in the form of protons, are propagating past, and interacting with the strong magnetic field of a pulsar. Figure 5.5 shows the spectra for Cherenkov and synchrotron radiation for a proton moving perpendicularly to the magnetic field of a millisecond pulsar, with $B = 10^4$ T, for different values of γ. For clarity we have only included the total

Cherenkov contribution. The cut-off energy found through (5.77) in this magnetic field is $\hbar\omega_{\max} \sim (m_e^3 c^4)/(e\hbar B) \simeq 226$ GeV. The particle energy γ is (a) $\gamma = 5 \times 10^7$, (b) 5×10^9 and (c) $\gamma = 5 \times 10^{11}$. Over this interval, the Cherenkov spectrum (as noted previously) remains almost constant, due to the leading order behaviour coming from the field strength B. For $\gamma = 10^7$, the intersection of the Cherenkov and synchrotron spectrum occurs at $\hbar\omega \simeq 8.5$ GeV, indicating that the highest energy emission will be predominantly due to the Cherenkov effect, rather than the usual synchrotron process. This intersection is roughly inversely proportional to the energy, *i.e.*, (b) $\hbar\omega \simeq 54$ MeV, (c) $\hbar\omega \simeq 0.54$ MeV. So for the highest energy proton cosmic rays, the highest energy radiation is completely dominated by the Cherenkov process.

Detections of high energy particles coming from astrophysical sources are predominantly cosmic rays which are mainly protons. However, the environment around neutron stars is also teaming with high energy electrons and positrons which have been generated through pair creation mechanisms. It is important to also consider these lighter particles, and how their lower mass affects the Cherenkov spectrum. Staying with the same field strength as above of $B = 10^4$ T, corresponding to millisecond pulsars, we have the same value for the cut-off energy in the spectrum. Figure 5.6 gives the corresponding spectra for electrons over the same values of γ as was considered for protons above. We see that in this case, synchrotron radiation is a much more dominant effect, with only electrons with the highest energies of $\gamma > 10^9$ emitting comparable amounts of Cherenkov radiation in the high energy end of the spectrum. This is to be expected, as noted previously, as the electrons are much lighter and so are more susceptible to oscillating in the magnetic field and producing synchrotron radiation.

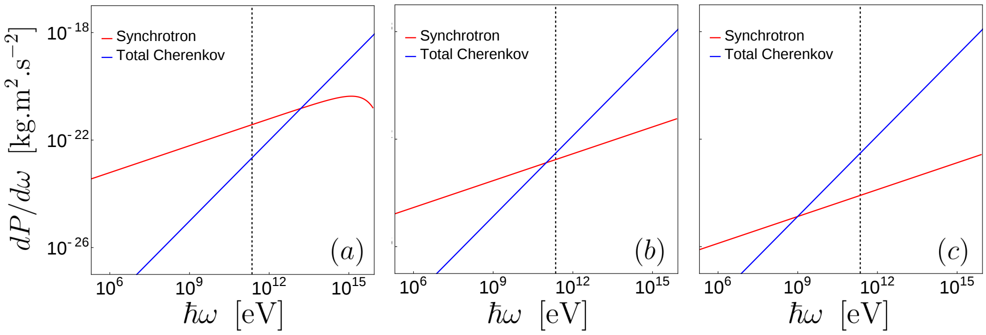

Figure 5.6: Radiated power per unit frequency from electrons interacting with a constant magnetic field $B = 10^4$ T, with Lorentz factors: (a) $\gamma = 5 \times 10^7$, (b) $\gamma = 5 \times 10^9$, (c) $\gamma = 5 \times 10^{11}$. The cut-off energy is $\hbar\omega_{\max} \sim 226$ GeV.

Millisecond pulsars represent the subsection of the pulsar population which produce the lowest magnetic fields. Much more common are pulsars with stronger magnetic fields surrounding them. For the vast majority of known objects of this class, a typical field strength is several orders of magnitude greater, at around a value of $B \sim 10^8$ T. Moving up to this field strength, the cut-off energy of the spectrum dramatically decreases due to the inverse relationship found in (5.77). However, for very high energy particles $\gamma \gg 1$, we find that the Cherenkov spectrum becomes essentially proportional to the field strength, meaning that although the spectrum is cut off at a lower energy, large amounts of Cherenkov radiation could still be produced in these fields. We therefore consider fields of $B = 10^8$ T for both protons (shown in Figure 5.7) and electrons (shown in Figure 5.8). The cut-off energy has now reduced by four orders of magnitude to $\hbar\omega_{\max} \sim 23$ MeV. As eluded to, although the cut-off energy has decreased, the dependence of the Cherenkov spectrum on the magnetic field strength has meant that even for the lowest energy electrons considered here, there is a regime at the highest end of the spectrum in which Cherenkov radiation is the dominant contributor over synchrotron radiation.

There is currently a debate within the astrophysics community concerning the origin of observed excesses of high energy photons found in recent data. For example observations of intense gamma rays from the Galactic Centre [302] have prompted a range of possible explanations, such as dark matter annihilation [303], and unresolved pulsar sources [304]. The Cherenkov process detailed in this Chapter provides a new, and so far unexplored, gamma ray production mechanism, which warrants further study in this context.

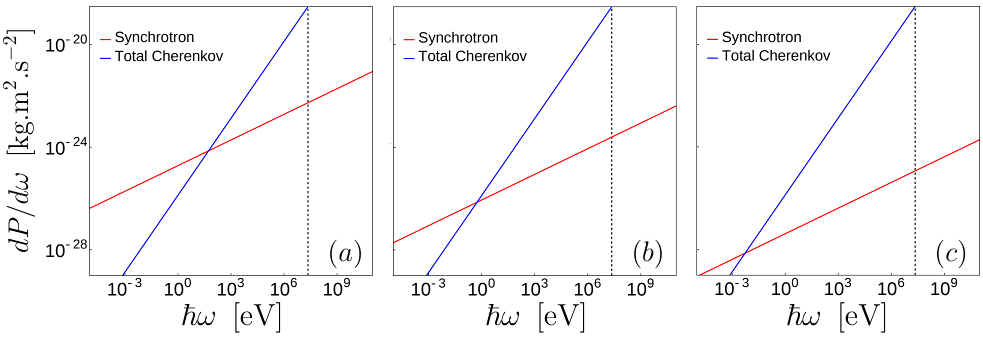

Figure 5.7: Radiated power per unit frequency from protons interacting with a constant magnetic field $B = 10^8$ T, with Lorentz factors: (a) $\gamma = 5 \times 10^7$, (b) $\gamma = 5 \times 10^9$, (c) $\gamma = 5 \times 10^{11}$. The cut-off energy is $\hbar\omega_{\text{max}} \sim 23$ MeV.

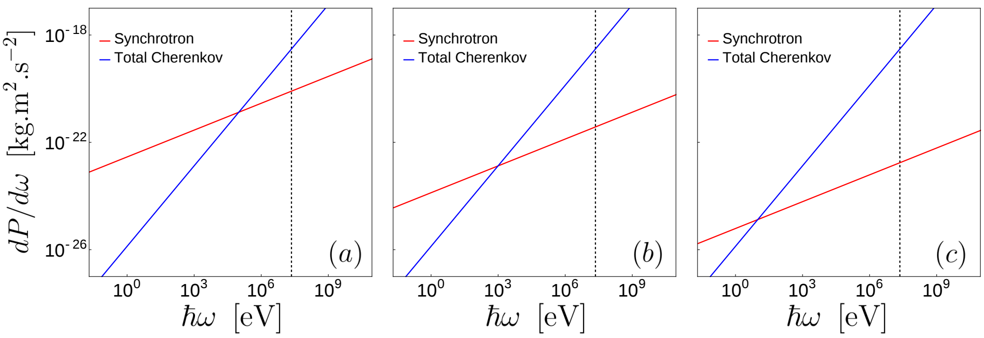

Figure 5.8: Radiated power per unit frequency from electrons interacting with a constant magnetic field $B = 10^8$ T, with Lorentz factors: (a) $\gamma = 5 \times 10^7$, (b) $\gamma = 5 \times 10^9$, (c) $\gamma = 5 \times 10^{11}$. The cut-off energy is $\hbar\omega_{\text{max}} \sim 23$ MeV.

Validity of the rectilinear motion approximation

In the above we have assumed that the particles are moving rectilinearly, i.e. in straight lines. In reality, no motion will ever be perfectly rectilinear, and so we must justify our use of the approximation here in the examples considered. We will therefore assume that a particle can turn through some angle $\theta_{\mathrm{max}} \ll 1$ and still be considered to move in a straight line during the emission process.

A particle with Lorentz factor $\gamma \gg 1$ and mass m in a magnetic field of strength B will undergo cyclotron oscillations with a characteristic radius,

$$R = \frac{\gamma m c}{eB}. \tag{5.78}$$

Since the particle is ultrarelativistic, we approximate its speed to be c. Now, in a magnetic field of strength B, we can approximate the phase velocity of emitted radiation as,

$$v_p \simeq c(1 - \lambda_{\pm} B^2). \tag{5.79}$$

Then, over the emission of a complete wavelength λ, the particle will travel a distance

$$d = \frac{\lambda}{\lambda_{\pm} B^2}, \tag{5.80}$$

meaning that during the emission the particle will deviate from rectilinear motion by an angle,

$$\theta = \frac{d}{R} = \frac{\lambda e}{\gamma m c \lambda_{\pm} B}. \tag{5.81}$$

It is valid to approximate the motion of the particle as rectilinear provided that it turns an angle $\theta < \theta_{\max}$ during the emission of a wavelength, for some $\theta_{\max} \ll 1$. With this condition it follows the results are reliable for wavelengths λ satisfying,

$$\lambda < \lambda_{\max} = \lambda_{+} B \frac{mc}{e} \gamma \theta_{\max}, \tag{5.82}$$

where the value λ_{+} has been chosen as, for Euler-Heisenberg theory, this is more restrictive than λ_{-}.

So, considering the value of the magnetic field used above of $B = 10^4$ T and using the proton mass $m = m_p$,

$$\lambda < \lambda_{\max} = \lambda_{+} B \frac{m_p c}{e} \gamma \theta_{\max} \simeq 1.7 \times 10^{-19} \gamma \theta_{\max} \text{ m}, \tag{5.83}$$

which corresponds to an emitted photon energy,

$$\hbar\omega > \hbar\omega_{\min} \simeq \frac{7.3 \times 10^{12}}{\gamma \theta_{\max}} \text{ eV}. \tag{5.84}$$

The lowest energy protons considered in this strength of magnetic field was $\gamma = 5 \times 10^7$, and assuming a tolerance of $\theta_{\max} = 10^{-2}$ rad, corresponds to $\hbar\omega_{\min} \simeq 15$ MeV.

The power spectrum has a linear dependence on the frequency ω, and in the range

$\omega_{\rm min} < \omega < \omega_{\rm max}$, where $\omega_{\rm max}$ is found from the cut-off condition (5.77), the ratio of the power radiated within this range to the total which would be radiated without the lower bound of $\omega_{\rm min}$ is,

$$\mathcal{R} = 1 - \frac{\omega_{\rm min}^2}{\omega_{\rm max}^2} \simeq 1 - (6.6 \times 10^{-5}). \tag{5.85}$$

As we increase the particle energy, the power radiated in this range increases significantly, and so we can say that over the vast majority of considered cases, the motion of the particle can be considered approximately rectilinear. Similar behaviour is seen for protons at the higher magnetic field strength considered above, and also for electrons, though for the lowest energy electrons the minimum energy $\hbar\omega_{\rm min}$ is much higher, since the electrons are more affected by the magnetic field in terms of their orbit. Despite this, the results presented above remain well approximated at the highest end of the spectrum by particles undergoing rectilinear motion.

5.5 Summary

To summarize, in this Chapter we have provided a comprehensive, quantitative study of the Cherenkov effect in nonlinear theories of vacuum electrodynamics. This effect — expected due to the reduced phase velocity of light predicted by these theories in regions of strong fields — may provide an alternative radiation mechanism for very high energy particles. We considered two specific examples of background field with relevance to future experimental or observational campaigns, and determined the possibility of observing Cherenkov radiation in each case. When the background field is taken to be a constant crossed field (approximating a laser pulse), the availability of high energy electrons appears to put observation of Cherenkov radiation out of reach. In contrast, astrophysics provides environments in which the vacuum Cherenkov effect may be observed, due to the presence of very high energy cosmic rays and strong magnetic fields. We have demonstrated that there are regimes in which radiation due to the nonlinear Cherenkov effect can become dominant over radiation produced through synchrotron emission, generating very high energy photons. A notable excess of gamma rays with energies in the GeV–TeV range has been observed in various astrophysical contexts, and the vacuum Cherenkov process could provide an alternative explanation for their origin, not previously considered in the literature.

Chapter 6

Summary and outlook

In the coming years, a number of high-power laser facilities will come online and allow unprecedented access to the strong-field regime of physics. The study of the processes which can be induced by such high-intensity fields represents an exciting opportunity for discovering new physics, and testing the limits of our already well established theories, such as QED. This Thesis presents a study of several aspects of strong-field physics, taking the approach of describing this regime using nonlinear effective theories of electrodynamics. Such theories represent a general framework within which to study possible extensions of Maxwellian electrodynamics into the strong-field regime, and analyse the general properties which they exhibit. One of the most powerful applications of nonlinear electrodynamics is the Euler-Heisenberg theory, which allows us to use a classical approach to studying the quantum nature of the vacuum. In Chapter 2, an overview of the use of nonlinear electrodynamics was given. How these theories are defined and the formalism which would be used throughout was also presented.

In Chapter 3, the structure of the energy-momentum tensor in generalised nonlinear theories of electrodynamics was considered, motivated in part by the historically controversial problem of Abraham & Minkowski in the context of light interacting with a material medium. In this case, a lot of the difficulty in coming to a definitive resolution of the Abraham-Minkowski problem has been attributed to not being able to fully characterise the part of the total energy-momentum associated with the medium. Without full knowledge of this, the arbitrary splitting of the full energy-momentum of the light-matter system appears to be the only way to circumvent the conceptual difficulties. This is not the case in theories of nonlinear electrodynamics, where the nonlinear interactions cause light to propagate analogously to its movement through a medium, but such theories give the added advantage of all the contributions to the total energy-momentum being known from the outset. The strict preservation of Lorentz invariance and the natural emergence of the Minkowski form of the energy-momentum tensor appears to support the use of Minkowski in these nonlinear theories when the total electromagnetic field is separated into a background and probe configuration. Taking a perturbative approach, it was shown that the component of the energy-momentum which is second order in the probe field is naturally expressed as the Minkowski tensor (describing the probe field) alongside an additional term which comes directly from the metric variation of the constitutive tensor. This gives a good indication that the extra term can be attributed to the strong-field vacuum in the interaction. This gives us a very intuitive physical interpretation. In future work it would be interesting to further study the more general role that the additional term plays. For example, the Minkowski energy-momentum is known to allow violation of the positive energy condition, which states that the energy-density $\rho = T^{\mu\nu}U_\mu U_\nu$ (with U_μ any time-like 4-vector) must be positive, $\rho \geq 0$. As such, the term which appears from the variation of the constitutive tensor may be responsible for ensuring that the total energy-density of the system is positive. This will be explored in future work.

Chapter 4 considered the possibility of exploiting conformal invariance in theories of nonlinear electrodynamics, with the aim of discovering some regime in which the extended symmetry group could be used to describe strong-field processes. Conformal symmetry, and conformal field theory, have been used extensively in the high-energy physics community over the last 50 years to tackle difficult, or intractable problems in order to help guide solutions by considering a simplified model. This Chapter explored whether conformal symmetry could be used in the same way in theories of nonlinear electrodynamics. It was found that a class of nonlinear conformally invariant theories of electrodynamics can be defined, where the Lagrangian function becomes restricted to a form which depends on an arbitrary function of a single variable. However, any theory which is used to describe strong-field processes must be able to reproduce the results of Maxwellian electrodynamics in the weak field limit. The structure of the conformally invariant theories does not allow for this, as the arbitrary function depends on the ratio of the electromagnetic field invariants Y/X. As such, there can only be a consistent weak field limit when the function $C(Y/X) = 1$. When this is the case, the conformally invariant theory is no longer nonlinear, but is in fact simply Maxwellian electrodynamics. In this way, it was shown that the only conformally invariant theory of electromagnetism in $(3 + 1)$ spacetime dimensions, which is compatible with known weak-field physics, is Maxwell's theory. It may be interesting to extend this study to the more abstract realm of higher-dimensional field theories. An interesting property of

Maxwellian electrodynamics is that it is only conformally invariant in $(3+1)$ spacetime dimensions (see for example [205, 305]), and this invariance is lost in higher dimensions. However, in string theory, conformal field theories are typically used and studied in higher dimensions, especially in the context of the AdS/CFT correspondence. Though maybe not of direct physical use, studying higher dimensional electrodynamics with conformal symmetry could provide an interesting insight into how these theories are structured more generally.

Finally, we turned our attention away from the more formal aspects of nonlinear electrodynamics and considered the effect of Cherenkov radiation in strong electromagnetic background fields in Chapter 5. This effect had been partially considered in much earlier work by several authors in the context of Euler-Heisenberg theory. In each case, however, much of the analysis was *ad hoc* and only semi-quantitative, relying on order of magnitude calculations. A complete, first principles approach to the effect was not to be found in the literature. Our approach provides a more comprehensive analysis of the Cherenkov effect in strong fields, giving a framework to quantitatively define the Cherenkov process for generalised nonlinear theories of electrodynamics and arbitrary slowly-varying background fields. The basis of this work is the dispersion relation of a radiation field propagating in a strong background, which can then be used to define the phase velocity of light—the key parameter needed to discuss Cherenkov radiation from ultrarelativistic particles. The contribution to the total emitted power from each of the polarisation modes of the radiation was determined, and the standard results of Cherenkov radiation in an isotropic medium were generalised to the case of nonlinear electrodynamics using the properties of these modes. Specialising to Euler-Heisenberg theory, and using two examples of strong background electromagnetic field, the physical implications for this radiation process were explored in the context of lab-based high-intensity laser experiments and astrophysics. In the case of laser-matter interactions, the Cherenkov process is unlikely to be observed with next generation laser facilities, primarily due to the current limit on the energies of particles which can be produced in the lab. Another aspect, briefly mentioned above, is that in a real experiment, although the constant crossed field gives a good first approximation to a laser pulse, in order to reach the field strengths required for the Cherenkov effect tightly focussed pulses are required. Since the field gradients in these realistic pulses will be very strong, it is expected that particles interacting with the pulse may radiate away too much of their energy before reaching the peak field strength, such that in practice the Cherenkov condition is never satisfied in these experiments. The example of a constant magnetic field was also considered, which gave some insight into the Cherenkov effect in astrophysical environments. In this scenario, the available particle energies are several orders of magnitude greater than what can be produced terrestrially, and the magnetic fields produced by pulsars can be even greater than the Scwhinger critical magnetic field B_S. Significant regimes in which the radiation due to the Cherenkov process can be dominant over the synchrotron background were found for both protons and electrons. For magnetic fields in the range investigated above, the Cherenkov mechanism gives a previously unidentified source of gamma rays with energies in the MeV to GeV regime. This means that the Cherenkov process may contribute to the excess signals of high energy gamma rays which have been observed in recent years. One thing which has not been considered in this work is how the Cherenkov signal in these astrophysical settings may compare to other known processes in the strong magnetic fields, for example the production of pions which then decay into photons. This will be investigated in future work in order to determine if the Cherenkov signal could still be distinguished. More generally, the approach taken in this work relied on the use of the rectilinear motion approximation. Though this is valid over a very large range of emitted photon energies, it would be interesting to investigate non-rectilinear motion, to develop a more robust theory. This could be achieved by using a Green's function approach to source the radiation from the accelerating particle, which would then give a more general understanding of the role non-rectilinear motion may play.

Nonlinear effective field theories of electrodynamics can be used as a powerful tool in the study of strong-field physics. There are, of course, other ways in which strong-field effects can be described, in particular using the full machinery of quantum electrodynamics. The Cherenkov process described above is related to ordinary nonlinear Compton scattering, where a charged particle absorbs photons from the background electromagnetic field and then subsequently radiates a photon. It would be interesting to investigate the loop corrections to nonlinear Compton scattering, to try and obtain a derivation of the Cherenkov process from the quantum field theory perspective.

Nonlinear effective field theories of the class (2.4) are not the only effective descriptions which can be derived from quantum electrodynamics. Much like Euler-Heisenberg theory is considered on the basis of constant electromagnetic fields, when rapidly oscillating (short wavelength) fields are present it is possible to obtain an effective theory which, in addition to containing nonlinearities, introduces higher-derivative terms to the theory [306]. As mentioned above, one of the challenges in describing Cherenkov radiation in nonlinear electrodynamics is that a cut-off has to be imposed, based on some fundamental assumptions. This is due to the fact that the dispersion relation does not contain any explicit dependence on the frequency ω. Higher derivative theories tend to alter the wave propagation in such a way that frequency dependent terms enter into the dispersion relation, and it may be possible that investigating this further could lead to a first principles derivation of the cut-off. This possibility will be investigated in future work.

Appendix A

Electromagnetic field identities

Throughout the work in the main body of the text, a number of different identities involving the electromagnetic field are used. Many of these are collected in this Appendix.

The Lorentz invariants of the electromagnetic field $F_{\mu\nu}$ are,

$$X = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}\,, \qquad\qquad Y = -\frac{1}{4}\widetilde{F}^{\mu\nu}F_{\mu\nu}\,. \tag{A.1}$$

In Chapters 3 and 5 the electromagnetic field is separated into a strong slowly varying background field $\mathcal{F}_{\mu\nu}$ and a weak probe/radiation field $f_{\mu\nu}$. We define the analogous invariants involving these fields as,

$$\mathcal{X} = -\frac{1}{4}\mathcal{F}^{\mu\nu}\mathcal{F}_{\mu\nu}\,, \qquad\qquad \mathcal{Y} = -\frac{1}{4}\widetilde{\mathcal{F}}^{\mu\nu}\mathcal{F}_{\mu\nu}\,, \tag{A.2}$$

$$x = -\frac{1}{4}\mathcal{F}^{\mu\nu}f_{\mu\nu}\,, \qquad\qquad y = -\frac{1}{4}\widetilde{\mathcal{F}}^{\mu\nu}f_{\mu\nu}\,, \tag{A.3}$$

$$\mathsf{x} = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu}\,, \qquad\qquad \mathsf{y} = -\frac{1}{4}f^{\mu\nu}f_{\mu\nu}\,. \tag{A.4}$$

As well as the invariant parameters above, which are Lorentz scalars (or pseudoscalars), there are also tensor identities involving the electromagnetic field. Again, for the full electromagnetic field tensor $F_{\mu\nu}$ we have,

$$\widetilde{F}^{\mu\lambda}F^{\nu}{}_{\lambda} = -\, g_{\mu\nu}Y, \tag{A.5}$$

which is found by using the symmetry properties of the Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$. We also have,

$$\widetilde{F}^{\mu\lambda}\widetilde{F}^{\nu}{}_{\lambda} - F^{\mu\lambda}F^{\nu}{}_{\lambda} = 2Xg^{\mu\nu}. \tag{A.6}$$

Again, these definitions can be extended to the case where we write $F_{\mu\nu} = \mathcal{F}_{\mu\nu} + f_{\mu\nu}$. The background ($\mathcal{O}(f^0)$) only identities are,

$$\widetilde{\mathcal{F}}^{\mu\lambda}\mathcal{F}^{\nu}{}_{\lambda} = -\, g_{\mu\nu}\mathcal{Y}, \qquad \widetilde{\mathcal{F}}^{\mu\lambda}\widetilde{\mathcal{F}}^{\nu}{}_{\lambda} - \mathcal{F}^{\mu\lambda}\mathcal{F}^{\nu}{}_{\lambda} = 2\mathcal{X}g^{\mu\nu}. \tag{A.7}$$

Identities first order ($\mathcal{O}(f)$) in the probe field are,

$$\widetilde{\mathcal{F}}_{\mu\alpha}f^{\alpha}{}_{\nu} + \widetilde{f}_{\mu\alpha}\mathcal{F}^{\alpha}{}_{\nu} = 2yg_{\mu\nu}, \tag{A.8}$$

$$\widetilde{\mathcal{F}}^{\mu\lambda}\widetilde{f}^{\nu}{}_{\lambda} + \widetilde{f}^{\mu\lambda}\widetilde{\mathcal{F}}^{\nu}{}_{\lambda} - \mathcal{F}^{\mu\lambda}f^{\nu}{}_{\lambda} - f^{\mu\lambda}\mathcal{F}^{\nu}{}_{\lambda} = 4xg^{\mu\nu}. \tag{A.9}$$

And finally, the terms second order ($\mathcal{O}(f^2)$) in the probe are,

$$\widetilde{f}^{\mu\lambda}f^{\nu}{}_{\lambda} = -\, g_{\mu\nu}\mathsf{y}, \qquad \widetilde{f}^{\mu\lambda}\widetilde{f}^{\nu}{}_{\lambda} - f^{\mu\lambda}f^{\nu}{}_{\lambda} = 2\mathsf{x}g^{\mu\nu}. \tag{A.10}$$

Appendix B

Calculation of the Energy-Momentum Tensor

In this appendix, we present the explicit calculation of the energy-momentum tensor in nonlinear electrodynamics. The energy-momentum is defined by (3.13)

$$\delta S = \int d^4 z \delta \left(\sqrt{-g}\mathcal{L}\right) = -\frac{1}{2}\int d^4 z \sqrt{-g} T^{\mu\nu} \delta g_{\mu\nu},$$

i.e. it is found by varying the action S with respect to the metric tensor $g_{\mu\nu}$. Expanding the variation, there are two terms,

$$\delta_g S = \int d^4 z \left[\mathcal{L}\left(\delta\sqrt{-g}\right) + \sqrt{-g}\left(\delta\mathcal{L}\right)\right], \tag{B.1}$$

In the first term of (B.1) we use the standard result,

$$\delta\sqrt{-g} = \frac{1}{2}\sqrt{-g}g^{\mu\nu}\delta g_{\mu\nu}. \tag{B.2}$$

For the second term we expand the variation of the Lagrangian function into variations of the field invariants X and Y,

$$\delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial X}\delta X + \frac{\partial\mathcal{L}}{\partial Y}\delta Y. \tag{B.3}$$

To obtain the variation of X we first express X as,

$$X = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{8}\left(g^{\mu\lambda}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\lambda}\right)F_{\mu\nu}F_{\lambda\sigma}\,. \tag{B.4}$$

Taking the variation, we have,

$$\begin{aligned}\delta X =& -\frac{1}{8}\left(g^{\nu\sigma}\delta g^{\mu\lambda} + g^{\mu\lambda}\delta g^{\nu\sigma} - g^{\nu\lambda}\delta g^{\mu\sigma} - g^{\mu\sigma}\delta g^{\nu\lambda}\right)F_{\mu\nu}F_{\lambda\sigma}\\ =& -\frac{1}{2}F_{\mu\lambda}F_{\nu}{}^{\lambda}\delta g^{\mu\nu}.\end{aligned} \tag{B.5}$$

We then use the known result,

$$\delta g^{\alpha\beta} = -g^{\alpha\mu}g^{\beta\nu}\delta g_{\mu\nu}, \tag{B.6}$$

such that,

$$\delta X = \frac{1}{2} F^{\mu\lambda} F^{\nu}{}_{\lambda}\, \delta g_{\mu\nu}. \tag{B.7}$$

To calculate the variation of Y, we express the dual tensor $\widetilde{F}^{\mu\nu}$ in a slightly different form to make the dependence on the metric more explicit. This enters through the Levi-Civita tensor $\epsilon^{\mu\nu\alpha\beta}$, which can be expressed in terms of a *tensor density* $\tilde{\epsilon}^{\mu\nu\alpha\beta}$ which takes on the values $(0, 1, -1)$ in any frame,

$$\epsilon^{\mu\nu\alpha\beta} = \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\mu\nu\alpha\beta}. \tag{B.8}$$

Thus the invariant Y can be expressed,

$$Y = -\frac{1}{8} \frac{1}{\sqrt{-g}} \tilde{\epsilon}^{\mu\nu\alpha\beta} F_{\mu\nu} F_{\alpha\beta}\,, \tag{B.9}$$

and so the variation is,

$$\delta_g Y = -\, \frac{1}{2} g^{\mu\nu} Y \delta g_{\mu\nu} = \frac{1}{2} \widetilde{F}^{\mu\lambda} F^{\nu}{}_{\lambda}\, \delta g_{\mu\nu}, \tag{B.10}$$

where we used the identity (A.5),

$$F^{\mu\alpha} F^{\nu}{}_{\alpha} = -\, g^{\mu\nu} Y$$

Hence, inserting (B.2), (B.7) and (B.10) into (B.1), and making use of the identity (A.5) we find that the energy-momentum tensor is given by (3.14) in the main text.

Appendix C

Perturbative expansions of Lagrangian and excitation tensor

In moving from the energy-momentum tensor in terms of the full electromagnetic field $F_{\mu\nu}$ to the background/probe configuration in Chapter 3, the clearest physical insight for the role each of the terms in the second order part of the energy-momentum (3.40) comes when we first use the perturbation $F_{\mu\nu} = \mathcal{F}_{\mu\nu} + f_{\mu\nu}$ at the level of the action. We begin with the the expanded invariants (3.23) and (3.24),

$$X = \mathcal{X} + 2x + \mathsf{x}, \qquad Y = \mathcal{Y} + 2y + \mathsf{y}, \tag{C.1}$$

which can be then used to perform a Taylor expansion of the Lagrangian function around the background values. To compare with Abraham and Minkowski, we consider terms up to $\mathcal{O}(f^2)$. We therefore have the perturbed Lagrangian function,

$$\begin{aligned}\mathcal{L}(X,Y) =& \mathcal{L}(\mathcal{X},\mathcal{Y}) + 2\left\{x\mathcal{L}_X + y\mathcal{L}_Y\right\} \\ &+ \left\{\mathsf{x}\mathcal{L}_X + \mathsf{y}\mathcal{L}_Y + 2x^2\mathcal{L}^2_{XX} + 4xy\mathcal{L}^2_{XY} + 2y^2\mathcal{L}^2_{YY}\right\} + \mathcal{O}(f^3) \qquad \text{(C.2)}\end{aligned}$$

and partial derivatives,

$$\begin{aligned}\frac{\partial\mathcal{L}(X,Y)}{\partial X} =& \mathcal{L}_X + 2\left\{x\mathcal{L}^2_{XX} + y\mathcal{L}^2_{XY}\right\} \\ &+ \left\{\mathsf{x}\mathcal{L}^2_{XX} + \mathsf{y}\mathcal{L}^2_{XY} + 2x^2\mathcal{L}^3_{XXX} + 4xy\mathcal{L}^3_{XXY} + 2y^2\mathcal{L}^3_{XYY}\right\} + \mathcal{O}(f^3),\end{aligned} \qquad \text{(C.3)}$$

$$\begin{aligned}\frac{\partial\mathcal{L}(X,Y)}{\partial Y} =& \mathcal{L}_Y + 2\left\{x\mathcal{L}^2_{XY} + y\mathcal{L}^2_{YY}\right\} \\ &+ \left\{\mathsf{x}\mathcal{L}^2_{XY} + \mathsf{y}\mathcal{L}^2_{YY} + 2x^2\mathcal{L}^3_{XXY} + 4xy\mathcal{L}^3_{XYY} + 2y^2\mathcal{L}^3_{YYY}\right\} + \mathcal{O}(f^3). \qquad \text{(C.4)}\end{aligned}$$

For clarity, the notation,

$$\mathcal{L}_X \equiv \left.\frac{\partial\mathcal{L}}{\partial X}\right|_{(\mathcal{X},\mathcal{Y})} \qquad \mathcal{L}^2_{XY} \equiv \left.\frac{\partial^2\mathcal{L}}{\partial X\partial Y}\right|_{(\mathcal{X},\mathcal{Y})} \qquad \text{(C.5)}$$

signifies that we first take the derivatives of $\mathcal{L}(X,Y)$, and then evaluate the result *on the background*.

We could go ahead and simply use (C.2) and proceed to perform the variation of the action with respect to the metric, however we can express C.2 in a much simpler form. Expanding the excitation tensor (2.6) and using (C.3), and (C.4) we have,

$$\mathcal{H}^{\mu\nu} = \mathcal{F}^{\mu\nu}\mathcal{L}_X + \widetilde{\mathcal{F}}^{\mu\nu}\mathcal{L}_Y, \tag{C.6}$$

$$h^{\mu\nu} = 2\mathcal{F}^{\mu\nu}\left\{x\mathcal{L}^2_{XX} + y\mathcal{L}^2_{XY}\right\} + f^{\mu\nu}\mathcal{L}_X + 2\widetilde{\mathcal{F}}^{\mu\nu}\left\{x\mathcal{L}^2_{XY} + y\mathcal{L}^2_{YY}\right\} + \widetilde{f}^{\mu\nu}\mathcal{L}_Y, \tag{C.7}$$

and,

$$\begin{aligned}\mathsf{h}^{\mu\nu} =& \mathcal{F}^{\mu\nu}\left\{\mathsf{x}\mathcal{L}^2_{XX} + \mathsf{y}\mathcal{L}^2_{XY} + 2x^2\mathcal{L}^3_{XXX} + 4xy\mathcal{L}^3_{XXY} + 2y^2\mathcal{L}^3_{XYY}\right\} \\ &+ \widetilde{\mathcal{F}}^{\mu\nu}\left\{\mathsf{x}\mathcal{L}^2_{XY} + \mathsf{y}\mathcal{L}^2_{YY} + 2x^2\mathcal{L}^3_{XXY} + 4xy\mathcal{L}^3_{XYY} + 2y^2\mathcal{L}^3_{YYY}\right\} \\ &+ 2f^{\mu\nu}\left\{x\mathcal{L}^2_{XX} + y\mathcal{L}^2_{XY}\right\} + 2\widetilde{f}^{\mu\nu}\left\{x\mathcal{L}^2_{XY} + y\mathcal{L}^2_{YY}\right\}.\end{aligned} \tag{C.8}$$

Making use of these definitions, we can rewrite the perturbed Lagrangian function (C.2) as,

$$\mathcal{L}(X,Y) = \mathcal{L}(\mathcal{X},\mathcal{Y}) - \frac{1}{2}\mathcal{H}^{\mu\nu}f_{\mu\nu} - \frac{1}{4}h^{\mu\nu}f_{\mu\nu}. \tag{C.9}$$

Appendix D

Coordinate basis

In Chapter 5 we consider two examples of background electromagnetic field: a constant crossed field, and a constant magnetic field. It is convenient to specify the form of each of these by defining a specific basis. We choose to work with the coordinate basis $\{\eta, \epsilon_i\}$ $(i = 1, 2, 3)$, where,

$$\eta^2 = 1 \qquad \epsilon_i.\epsilon_j = -\,\delta_{ij} \qquad \eta.\epsilon_i = 0. \tag{D.1}$$

Where necessary we will use the representation,

$$\eta_\mu = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon_{1\mu} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \qquad \epsilon_{2\mu} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \qquad \epsilon_{3\mu} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}. \tag{D.2}$$

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