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Empirical Bayesian Inference on Poisson Processes with a Clayton Prior Distribution

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Declaration

This thesis is the result of the author's original research. It has been composed by the author and has not been previously submitted for examination which has led to the award of a degree.

Signed:

Date:

Abstract

Dependency between rates of occurrence of events can exist for a variety of reasons. For example, management culture within organisations can have a similar impact on multiple outcomes. Modelling approaches that assume independence between event rates can be mathematically convenient, but they might also fail to account for all the information within the data since the existence of dependency means that data from one process can provide information about the rate of occurrence on similar processes. However, estimating correlated event rates is challenging. We address this challenge by developing an inference framework to account for such dependency using copulas in order to make full use of available data.

We develop an empirical Bayesian inference method based on a multivariate Poisson – Clayton with Gamma marginals probability model. The proposed model aims to capture both aleatory and epistemic uncertainties. We assume that events are generated from a homogeneous Poisson process capturing the pure inherent randomness in the observations, i.e. the aleatory uncertainty. Epistemic uncertainty is represented by the prior where the marginal distributions of event rates are Gamma, and the underlying correlation is captured by the Clayton copula. Of particular interest are situations where we might anticipate low rates of occurrence. The Clayton copula is appropriate for situations with left tail dependence, that is where low rates are considered relatively more correlated compared to high rates.

However, estimating copulas dependence parameter using count data can be challenging. Hence, we provide analytical expressions for estimating dependency of the Clayton copula as a function of the count data realised from Poisson processes. We examine the relative accuracy of the model and investigate the robustness of results under different parameter settings. To support comparison between the proposed model and existing theory, we consider the classic empirical Bayes method assuming independent Gamma priors. Findings are based on simulation experiments. We also evaluate our method when applied for supplier ranking using de - sensitised real data. We explicitly discuss the ranking problem from a Bayesian perspective, and we propose multiple ranking methods. We identify cases with different final rankings which further enhance the importance of not choosing to ignore dependency.

To my beloved grandfather Taki Dimitiou!

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Chapter 1

Introduction

1.1 Research Context and Motivation

Bayesian and empirical Bayesian models have been advocated within the context of risk and reliability. Kaplan (1983, 1985) introduced the two-stage or hierarchical Bayesian model for determining failure rates within a probabilistic risk analysis context. In particular, Kaplan (1983) described the two-stage Bayesian model as 'a simple procedure which operates on the data in such a way that the output of the first stage, i.e. the posterior distribution of the first stage, becomes the prior distribution of the second stage'. Since then, two - stage Bayesian models have been suggested for various purposes (see Iman & Hora, 1989; Hora & Iman, 1990; Bunea, Charitos, Cooke, & Becker, 2005; Vaurio, 2005).

In most applications of Bayesian inference for event rate processes, the event rates of each process are assumed to be statistically independent, given some parameters. If these unknown parameters are hyperparameters, then they can be fully specified through the prior distribution. However, in the presence of correlation, a multivariate distribution can be defined for capturing the dependency between all unknown parameters. Then by applying Bayes Theorem and so updating the prior on the observed data, the rates of events can be estimated. Thereby, the underlying dependence on the rates can be incorporated into the Bayesian modelling procedure, which ensures coherent and theoretically sound rates estimates, as well as, allows the rates estimates to be informed based on information for multiple events. While the assumption of independence is mathematically convenient, such approaches fail to account for all the information within the data, as the existence of dependency means that data from one process can provide information about the rate of occurrence on similar processes.

Estimating dependent event rates can be challenging. Dependency between rates of occurrence of events can exist for a variety of reasons. For example, management culture within organisations can have a similar impact on multiple outcomes. Inference of multiple event rates in the presence of dependency has been considered in supply chain risk problems. For example, Quigley, Wilson, Walls, and Bedford (2013) developed a Bayes linear Bayes model for correlated event rates within a Bayesian methodological framework. They consider that events generated from a homogeneous Poisson process and Gamma prior distributions; and, the correlation between the rates is modelled using a Hypergeometric function. Even if this approach performs significantly well, it has some limitations. The form of the multivariate prior in terms of its correlation and the marginal distributions could be considered restricted. The subjective character of the proposed method, which requires expert judgement for specifying the correlation coefficient on the rates could be considered challenging. Therefore, the need for creating an inference framework that considers dependency between multiple event rates and provides flexibility in terms of dependence structure motivated this study.

The underlying correlation between event rates can be complex. The dependence structure of event rates can be described as symmetric when low and high rates are similarly correlated, lower tail dependent when low rates are more correlated compared to high rates, and upper tail dependent when high rates appear more correlated compared to low rates. Copulas provide a means of describing and modelling such dependence structures (Nelsen, 2007). Therefore, developing an inference framework to account for dependency using copulas and make full use of empirical data available within a Bayesian methodological framework motivated this research.

This research investigates the impact of not accounting for dependency. Studies in supply chain discuss dependency within organisations. For example, Tseng (2010) investigate the relationship between organizational culture and knowledge conversion on corporate performance. Their results suggest that both organizational culture and knowledge conversion are positively correlated with the corporate performance. In particular, as they state 'an adhocracy culture enables knowledge conversion and enhances corporate performance more than clan and hierarchy cultures'. Considering three particular approaches, just in time, supply chain management and total quality management, Kannan and Tan (2005) examine the extend to which these approaches are correlated and further investigate how they may affect the overall business performance. According to their findings, understanding of supply chain dynamics and commitment to quality have the greatest effect on performance. Sezen (2008) investigate how supply chain integration, supply chain information sharing and supply chain design may affect the supply chain performance. Their findings suggest that integration and information sharing are correlated with performance measures, but their effect is lower than supply chain design. Green, Whitten, and Inman (2008) investigate the impact of logistics performance on organizational performance within the manufacturing sector. Their findings suggest that 'logistics performance is positively impacted by supply chain management strategy and that both logistics performance and supply chain management strategy positively impact marketing performance, which in turn positively impacts financial performance'.

From another angle, several studies investigate the supplier relations within supply chain and their impact on performance. Field and Meile (2008) investigate the relationship between supplier relations and satisfaction with the overall supplier performance. According to their analysis, cooperation and long-term commitment are significantly positively correlated with satisfaction with overall supplier performance. Kannan and Tan (2002) investigate the impact of supplier selection and assessment on business performance. They identify that assessment of a supplier' s willingness and ability to share information can have a significant impact on the organisation performance. Moreover, Carr, Kaynak, Hartley, and Ross (2008) suggest that supplier training and supplier involvement contribute significantly to the supplier' s operational performance. Terpend and Krause (2015) investigate the impact of incentives on supplier performance. They propose two categories: competitive, market - based incentives with which suppliers are rewarded based on their performance relatively to other suppliers, and cooperative incentives, where both buyer and supplier share benefits based on their joint performance. Their findings suggest that competitive incentives can be 'an effective approach to improving delivery, quality, innovation and flexibility, for purchases where the buyer - supplier relationship is characterized by balanced and moderate amounts of mutual dependence'. S. Li, Ragu-Nathan, Ragu-Nathan, and Rao (2006) explore the impact of supply chain management practices on competitive advantages and organizational performance. With focus on five SCM dimensions - strategic supplier partnership, customer relationship, level of information sharing, quality of information sharing, and postponement, their identified that higher levels of SCM practice lead to enhanced competitive advantage and improved organizational performance, and competitive advantage positive affects the organizational performance. Sánchez and Pérez (2005) explore the relationship between the dimensions of supply chain flexibility and organization performance by analysing a sample of 123 automotive suppliers. They identify a positive correlation between the flexibility and organization performance, however different flexibility dimensions have different impact on the organization performance. Key performance indicators (KPIs) can be defined as 'a set of metrics which reflect the operation performance' and consider critical for manufacturing operation management and continuous improvement. Kang, Zhao, Li, and Horst (2016) state that KPIs in a manufacturing system are not independent, and they may have intrinsic mutual relationships. Therefore, they introduce a multi - level hierarchical structure for identifying and analysing KPIs and their relationships in production systems.

Motivated by the challenges in supply chain, we propose methods for ranking correlated event rates. Hierarchical models have been proposed for quantitative comparisons between organisations (see Raudenbush & Bryk, 2002). Goldstein and Spiegelhalter (1996) discussed the statistical issues involved in providing comparisons between institutions in the area of health and education, and suggested rankings based on confidence intervals of the random effects associated with organisations. Hierarchical models have also been proposed in areas closely related to organizational comparisons. In the supply chain area, various methods and models have been proposed for supplier ranking (De Boer, Labro, & Morlacchi, 2001; Chai, Liu, & Ngai, 2013). Walls, Quigley, Parsa, and Comrie (2016) present a novel modelling suite using relevant historical data for supporting the analysis of risk in different stages of supplier life. They proposed an empirical Bayes method for ranking using Poisson process data with heterogeneous exposure to risk. However, their analysis stands only under the assumption of independence, which makes our proposed method distinct from theirs. As in many cases, when dealing with real world applications, the dependency between performance rates exists and therefore needs to be considered. Ranking under uncertainty can be a complicated process, and the final ranking result can be affected by various factors. We account for dependency on the rates, implying that it is driven by the operations within organisations rather than the occurrence of rare or extreme events. Of particular interest is where few data exist, resulting in low rates of occurrence. With emphasis on left tail dependence where low rates are considered relatively more correlated compared to high rates, Clayton copula is being investigated.

1.2 Research Aim and Objectives

The aim of this research is to develop an inference framework to account for dependency of event rates realised from Poisson process using copulas. This research associates the empirical Bayesian inference method with the Clayton copula, and makes full use of the empirical data available. The following objectives are to be achieved:

- Explicate the relationship between key statistics from the empirical data and the dependence parameter of the Clayton copula;
- Derive analytical expressions of the Clayton' s dependence parameter in relation to parameters 'known' from the empirical data;
- Evaluate the empirical Bayesian method using Clayton copula for capturing the underlying dependency on the rates of events;
- Investigate the impact of not accounting for dependency.

1.3 Thesis Structure

This section provides a structure of this thesis with a brief summary of the content in each chapter. Figure 1.3.1 illustrates this structure.

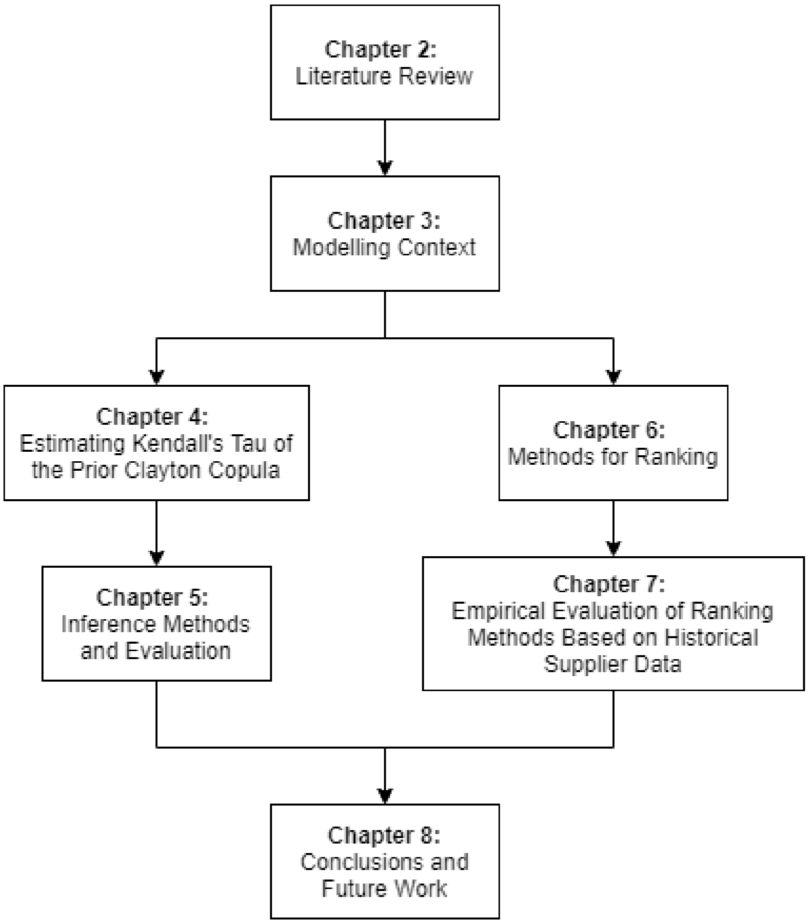


Figure 1.3.1: Diagram showing the structure of this thesis.

Firstly, the theoretical background to copulas including relevant dependence measures and description of bivariate copulas; and a literature review on Bayesian and empirical Bayesian approach in modelling along with applications in risk are presented in Chapter 2. We also examine the literature to discuss the gaps and hence to inform the research questions within the aim and objectives outlined above. In Chapter 3 we set up the modelling framework that is informed by the literature and will underpin all analysis reported in the thesis. We detail the empirical Bayesian considering dependency on the rates modelled with Clayton copula and the classic empirical methods assuming independence in Chapter 3. Chapter 4 describes the relationship between the empirical data realized from Poisson processes and the dependence parameter of the Clayton copula and presents the analytical expressions obtained. In Appendix A, we report the findings of the simulation study conducted. More detailed, we present all parameters chosen, and we discuss how different marginal parameter choices affect the correlation between Kendall's tau of the Prior and Poisson data. We also discuss how the proposed models perform using the theoretical settings and when the estimated prior parameters are used.

To examine the relative accuracy of the proposed empirical Bayesian model with Clayton copula, we conduct a simulation study. To support further comparison between the proposed model and existing theory, we consider the classic empirical Bayes method assuming independent Gamma priors. We investigate what happens if we choose to ignore dependency and perform the classic empirical Bayes method, and what are the consequences if there are any. A benchmarking study is conducted with which we identify and present cases where the consequences of our choice to ignore dependency are more significant than others. The simulation and benchmarking studies are presented and discussed in Chapter 5.

We investigate the impact of not accounting for dependency with an application to supplier ranking problem. Chapter 6 demonstrates our proposed methods for ranking. Particularly, our motivation, the purpose and the distinctiveness of the approach are explained in Sections 6.2, 6.3, 6.4, respectively. Relevant literature review on supplier selection and ranking process is also presented in Section C. The ranking problem from a Bayesian perspective and our methodological framework are explained in Section 6.5.

Ranking under uncertainty can be a complicated problem. Chapter 7 presents examples of analysis and model applications using de-sensitised real data from the prime manufacturer, considering multiple situations. Our approach is based on the empirical Bayes method considering the underlying dependence between the late delivery rate and the non - conformance rate. We consider three, mean rank, median rank and ranking by the cumulative distribution of ranks. The implications of each are also discussed. We further compare the proposed model with the classic empirical Bayes model and report our findings in Section 7.4. Finally, our conclusions and future work are presented in Chapter 8.

Chapter 2

Literature Review

2.1 Introduction

This chapter reviews the existing literature of Copulas along with the Bayesian and empirical Bayesian approach, with an emphasis on studies related to modelling dependency under uncertainty.

In particular, this chapter begins with a literature review of Copulas by discussing the most commonly used bivariate copula families and relevant dependence measures (Sections 2.2.3 and 2.2.2). A discussion about copula applications within the field of risk and reliability followed by a criticism of copulas are provided in Sections 2.2.4 and 2.2.5, respectively. The Bayesian and empirical Bayesian methodological frameworks are discussed in Section 2.3. Empirical Bayesian studies in risk are reviewed in Section 2.3.3. This chapter concludes by discussing the research gaps and defining the research questions of this research in Section 2.4.

2.2 Copulas

2.2.1 Definition

Copulas can describe the dependence between multiple event rates comprehensively and accurately. Copulas, from the Latin for "bond" or "tie", are distribution functions that bind multiple distribution functions under a specific dependence structure and allow to separate modelling of the dependency from modelling the marginal distributions (Nelsen, 2007). Nelsen (1999) characterised copula as "a function that joints multivariate distribution functions to their one – dimensional marginal distribution functions" or as "a distribution function whose one - dimensional margins are uniform on the interval $(0,1)$". The formal definition of Copulas and the Sklar's Theorem are presented in the following.

Definition. ***Copula*** *(Nelsen, 2007)*
An m - dimensional copula (or m - copula) is a function C from the unit m - cube $[0,1]^m$ *to the unit interval* $[0,1]$ *which satisfies the following conditions:*

1. $C(1,\cdots,1,a_n,1,\cdots,1) = a_n$ *for every* $n \leq m$ *and all* a_n *in [0, 1];*
2. $C(a_1,\cdots,a_m) = 0$ *if* $a_n = 0$ *for any* $n \leq m$*;*
3. *C is m - increasing.*

Theorem. ***Sklar' s Theorem*** *(Nelsen, 2007)*
Let H be a joint distribution function with margins F and G. Then there exists a copula C such that for all x, y *in* $\bar{R}$*,*

$$H(x,y) = C(F(x),G(y)).$$

If F and G are continuous, then C is unique; otherwise, C is uniquely determined on $RanF \times RanG$*. Conversely, if C is a copula and F and G are distribution functions,* *then the function H defined above is a joint distribution function with margins F and G.*

Copulas bind together univariate distributions,called marginal distributions or marginals, to form a multivariate distribution. Copulas capture the dependence structure between random variables. The dependence between any number of variables is characterized by the family of the copula C and the copula's dependence parameter θ. The family of the copula specifies the structure of the dependence, i.e. if there is a strong association in either of the tails (upper or lower tail dependence) or both (symmetric dependence), while the dependence parameter θ specifies the strength or even direction of the dependence. In the bivariate case, two random variables X and Y with cumulative distribution functions $F_X(x), F_Y(y)$ respectively, are' coupled' by copula C if their joint distribution function can be expressed as,

$$F_{X,Y}(x, y) = C(F_X(x), F_Y(y); \theta),$$

where θ represents the dependence parameter of the copula that measures the dependence between the marginals F_X and F_Y. Copulas, C, are cumulative distribution functions by definition, however the density function, if it exists, can be found by taking the partial derivatives of C with respect to each component. The bivariate density function can be expressed in terms of the copula density and the marginal densities as follows,

$$c(F_X(x), F_Y(y)) = \frac{f_{X,Y}(x, y)}{f_X(x) f_Y(y)} \qquad \Rightarrow \qquad f_{X,Y}(x, y) = c(F_X(x), F_Y(y)) f_X(x) f_Y(y)$$

where f_X, f_Y and $f_{X,Y}$ represent the marginal density functions and the non-zero density of variables X, Y. Following the Sklar's theorem, if F_X and F_Y are continuous, then copula C is unique; otherwise, copula C is uniquely determined on the Cartesian product of its marginals' ranges, $Ran\, F_X \times Ran\, F_Y$ (Kurowicka & Cooke, 2006; Nelsen, 2007).

2.2.2 Dependence Measures

Copulas are considered useful for modelling and describing the dependency between random variables (Nelsen, 1999). However, there are multiple ways one can measure dependence. In this section, we present the Pearson correlation coefficient and rank correlation measures as defined by Nelsen, 2007; Kurowicka & Cooke, 2006.

2.2.2.1 Correlation coefficient

The product moment correlation or Pearson linear correlation coefficient of two random variables (X, Y), is defined as

$$\rho_{X,Y} = \frac{cov(X,Y)}{\sigma_X \cdot \sigma_Y},$$

where $cov(X,Y) = E(XY) - E(X)E(Y)$, $\sigma_X, \sigma_Y > 0$, σ_X and σ_Y denote the standard deviations of X and Y, respectively.

The Pearson correlation coefficient, ρ_{XY}, (a) is a measure of linear correlation, (b) is symmetric, (c) the lower and upper bounds on the inequality $-1 \leq \rho_{XY} \leq 1$ measure perfect negative and positive linear correlation (known as normalization property), and (d) it is invariant with respect to linear transformations of the variables (Trivedi & Zimmer, 2007). However, it has some limitations which are listed below:

- The dependence structure of a multivariate distribution is not fully determined by the correlation matrix and it is not invariant under strictly increasing nonlinear transformations. For example, if $X \sim N(0,1)$ and $Y = X^2$, then $cov(X,Y) = 0$, but (X,Y) are clearly dependent. Therefore, having zero correlation, does not imply independence.

- It is not defined for some heavy-tailed distributions whose second moments do not exist, e.g., some members of the stable class and Student's t distribution with degrees of freedom equal to 2 or 1.

2.2.2.2 Rank correlation

Informally, a pair of random variables are concordant if "large" values of one tend to be associated with "large" values of the other, and "small" values of one with "small" values of the other. Let (x_i, y_i) and (x_j, y_j) denote two observations from a vector (X, Y) of continuous random variables. Then, (x_i, y_i) and (x_j, y_j) are concordant if $x_i < x_j$ and $y_i < y_j$ or if $x_i > x_j$ and $y_i > y_j$ (Nelsen, 2007). We define the "concordance function" Q which is the difference between the probability of concordance and discordance,between two vectors (X_1, Y_1) and (X_2, Y_2) of continuous random variables with (possibly) different joint distributions H_1 and H_2 , but with common margins F and G (Nelsen, 1999).

Theorem. ***Concordance Function*** *(Nelsen, 1999)*
Let (X_1, Y_1) and (X_2, Y_2) be independent vectors of continuous random variables with joint distribution functions H_1 and H_2, respectively, with common margins F (of X_1 and X_2)and G (of Y_1 and Y_2). Let C_1 and C_2 denote the copulas of (X_1, Y_1) and (X_2, Y_2), respectively, so that $H_I(x, y) = C_1(F(x), G(y))$ and $H_2(x, y) = C_2(F(x), G(y))$. Let Q denote the difference between the probabilities of concordance and discordance of (X_1, Y_1) and (X_2, Y_2), i.e., let

$$Q = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Then,

$$Q = Q(C_1, C_2) = 4 \iint_{I^2} C_2(u, v) \mathrm{d}C_1(u, v) - 1.$$

2.2.2.2.1 Spearman's rho

Consider two random variables X and Y with continuous distribution functions F_1 and F_2, respectively, and joint distribution function H. Then, Spearman's rank correlation or Spearman's rho is defined as,

$$\rho_{X,Y} = \rho(F_1(X), F_2(Y)).$$

Spearman's rho is the linear correlation between $F_1(X)$ and $F_2(Y)$, which are integral transforms of X and Y.

The population version of the rank correlation can be defined as proportional to the probability of concordance minus the probability of discordance for two vectors (X_1, Y_1) and (X_2, Y_2), where (X_1, Y_1) has distribution F_{XY} with marginal distribution functions F_X and F_Y and X_2, Y_2 are independent with distributions F_X and F_Y . Moreover (X_1, Y_1), (X_2, Y_2) are independent (Joe, 1997),

$$\rho = 3(P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0]).$$

2.2.2.2.2 Kendall's tau

Let (X_1, Y_1) and (X_2, Y_2) be independent and identically distributed random vectors, each with joint distribution function H. Then, Kendall's tau is defined as the probability of concordance minus the probability of discordance,

$$\tau = \tau_{X,Y} = P[(X_1 - X_2)(Y_1 - Y_2) > 0] - P[(X_1 - X_2)(Y_1 - Y_2) < 0].$$

Both measures are based on the concept of concordance, which refers to the property that large values of one random variable are associated with large values of another, whereas discordance refers to large values of one being associated with small values of the other. Both, Spearman's rho and Kendall's tau, can be expressed in terms of copulas (Nelsen, 1999).

If X and Y are continuous random variables with joint distribution function F and margins F_X and F_Y , let C denote the copula, that is, $F(x,y) = C(F_X(x), F_Y(y))$, then Spearman' s rho and Kendall' s tau for X and Y can be expressed as,

$$\rho_C = 12 \int\int_{I^2} [C(u,v) - uv] dudv$$
$$\tau_C = 4 \int\int_{I^2} C(u,v) dC(u,v) - 1.$$

2.2.2.2.3 Relationship between Kendall's tau and Spearman's rho

For continuous copulas, researchers convert the dependence parameter of the copula function to a measure such as Kendall's tau or Spearman's rho which are both bounded on the interval $[-1,1]$ and are independent of the functional forms of the marginal distributions.However, they are not simple functions of moments and hence computationally intensive.

While both Kendall's tau and Spearman's rho measure the probability of concordance between random variables with a given copula, the values of ρ and τ are often quite different. The relationship between ρ and τ varies considerably from family to family, but we summarise some universal inequalities for these measures (Balakrishnan & Lai, 2009; Kruskal, 1958).

1. $-1 < 3\tau - 2\rho \leq 1$
2. $\frac{1+\rho}{2} \geq \left(\frac{1+\tau}{2}\right)^2$
3. $\frac{1+\rho}{2} \geq \left(\frac{1-\tau}{2}\right)^2$
4. $\frac{3\tau-1}{2} \leq \rho \leq \frac{1+2\tau-\tau^2}{2}$, $\tau \geq 0$
5. $\frac{\tau^2+2\tau-1}{2} \leq \rho \leq \frac{1+3\tau}{2}$, $\tau \leq 0$.

2.2.2.3 Tail Dependence

The following tail dependence coefficients measure the dependence between the variables in the upper - right quadrant and the lower - left quadrant of $[0,1] \times [0,1]$ (Balakrishnan & Lai, 2009).

Definition. *Tail Dependence (Balakrishnan & Lai, 2009)*
The upper tail dependence coefficient (parameter) λ_U is the limit (if it exists) of the conditional probability that Y is greater than the $100ath$ percentile of G given that X is greater then the $100ath$ percentile F as a approaches 1,

$$\lambda_U = \lim_{a \to 1} Pr[Y > G^{-1}(a) \mid X > F^{-1}(a)].$$

If $\lambda_U > 0$, then X and Y are upper tail dependent and asymptotically independent otherwise.

Similarly, the lower tail dependence coefficient is defined as,

$$\lambda_L = \lim_{a \to 0} Pr[Y \leq G^{-1}(a) \mid X \leq F^{-1}(a)].$$

2.2.3 Bivariate Copulas

Copula functions are considered a powerful and useful tool for describing and modelling complex dependence structures (Nelsen, 2007). Multiple types of copulas exist, which describe different dependence structures. For example, Gumbel (1960) proposed a copula function that captures the upper tail dependence and Clayton (1978) presented a copula for capturing lower tail dependency. According to the literature, Gumbel copula is sensitive to upper tail dependence, Clayton copula is sensitive to lower tail dependence, and Frank copula is sensitive to symmetric dependence (Dodangeh, Shahedi, Shiau, & Mirakbari, 2017). We further discuss these copula families.

2.2.3.1 Product copula

The simplest copula, the product copula, has the form $C(u_1, u_2) = u_1 u_2$, where u_1 and u_2 take values in the unit interval of the real line. The product copula is important as a benchmark, as it corresponds to independence (Trivedi & Zimmer, 2007).

2.2.3.2 Farlie - Gumbel - Morgenstern copula

The Farlie - Gumbel - Morgenstern (FGM) copula has the following form,

$$C(u_1, u_2; \theta) = u_1 u_2 (1 + \theta(1 - u_1)(1 - u_2)).$$

The FGM copula was first proposed by Morgenstern (1956). The FGM copula is connected to the product copula. The FGM copula collapses to the product copula if the parameter θ takes the value zero, in the presence of independence. Notably, this copula family is not recommended in cases where strong dependence is under consideration (Trivedi & Zimmer, 2007).

2.2.3.3 Gaussian (Normal) copula

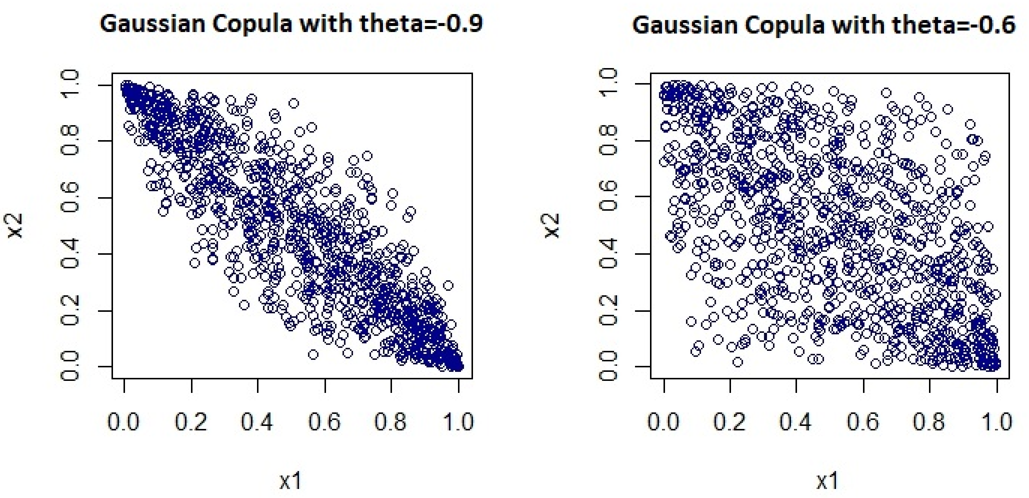
The Gaussian - Normal copula has the following form,

$$\begin{aligned} C(u_1, u_2; \theta) =& \Phi_G(\Phi^{-1}(u_1), \Phi^{-1}(u_2); \theta) \\ =& \int_{-\infty}^{\Phi^{-1}(u_1)} \int_{-\infty}^{\Phi^{-1}(u_2)} \frac{1}{2\pi(1-\theta^2)^{1/2}} \times \left(\frac{-(s^2 - 2\theta st + t^2)}{2(1-\theta^2)} \right) dsdt \end{aligned}$$

where Φ is the cumulative distribution function (CDF) of the standard normal distribution, and $\Phi_G(u_1, u_2)$ is the standard bivariate normal distribution with correlation parameter θ restricted to the interval $(-1, 1)$. Gaussian copula is considered flexible as it allows both positive and negative dependency. Figure 2.2.1 illustrates the bivariate Gaussian copula using different dependence parameters (from strong negative dependence to strong positive dependence), showing its symmetric property and how the dependence structure changes in different situations. Simulated samples of size 1000, generated using the following *R* commands,

normal.copula<- normalCopula(param = - , dim = 2)

x<- rCopula(n = 1000,normal.copula).



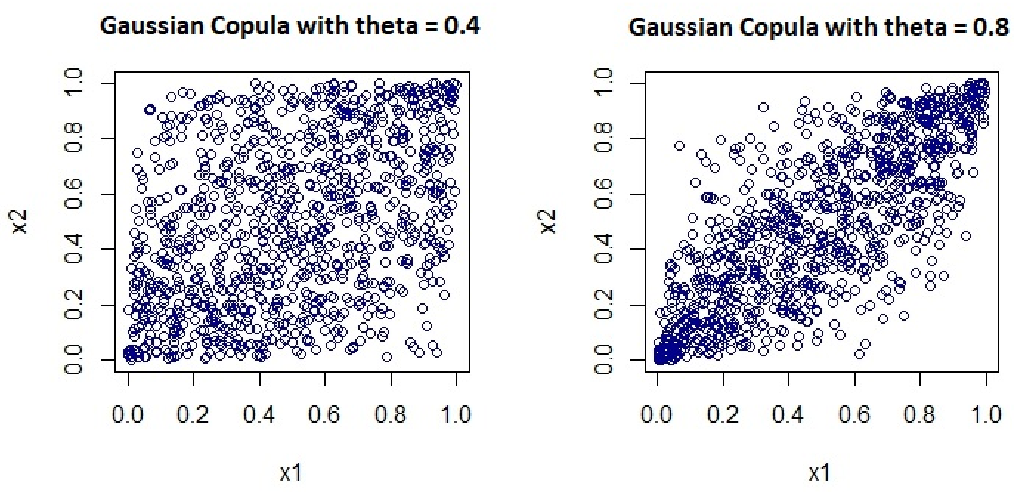


Figure 2.2.1: Graphical representation of the bivariate Gaussian copula showing the symmetric dependence structure in cases where the dependence parameter is -0.9, -0.6, 0.4 and 0.8.

2.2.3.4 Student t copula

The Student t Copula has the following form,

$$C^t(u_1, u_2; \theta_1, \theta_2) = \int_{-\infty}^{t_{\theta_1}^{-1}(u_1)} \int_{-\infty}^{t_{\theta_2}^{-1}(u_2)} \frac{1}{2\pi(1-\theta_2^2)^{1/2}} \times \left(\frac{-(s^2 - 2\theta_2 st + t^2)}{2(1-\theta_2^2)} \right)^{-(\theta_1+2)/2} dsdt$$

where $t_{\theta_1}(u_1)$ is the CDF of the standard univariate t - distribution with θ_1 degrees of freedom. The Student t copula has two parameters, θ_1, θ_2; parameter θ_1 refers to degrees of freedom and controls the heaviness of the tails, and θ_2 refers to the dependence parameter. Student t copula allows symmetric dependence in the tails and can be used for both positive and negative dependence.

Figures 2.2.2, 2.2.3, 2.2.4 provide graphical representation of the Student t copula in different situations. The degrees of freedom control the heaviness of the tails, where small values of degrees of freedom increase the tail dependency. However, careful consideration is required as relatively small numbers may lead to generate "wings" (see Figure 2.2.4). Worth also noticing that for large values of degrees of freedom, the Student t copula approximates the Gaussian Copula. Simulated samples of size 1000, generated using the following *R* commands,

student.cop<-tCopula(param = -, dim = 2, df= -)

s<-rCopula(1000, student.cop).

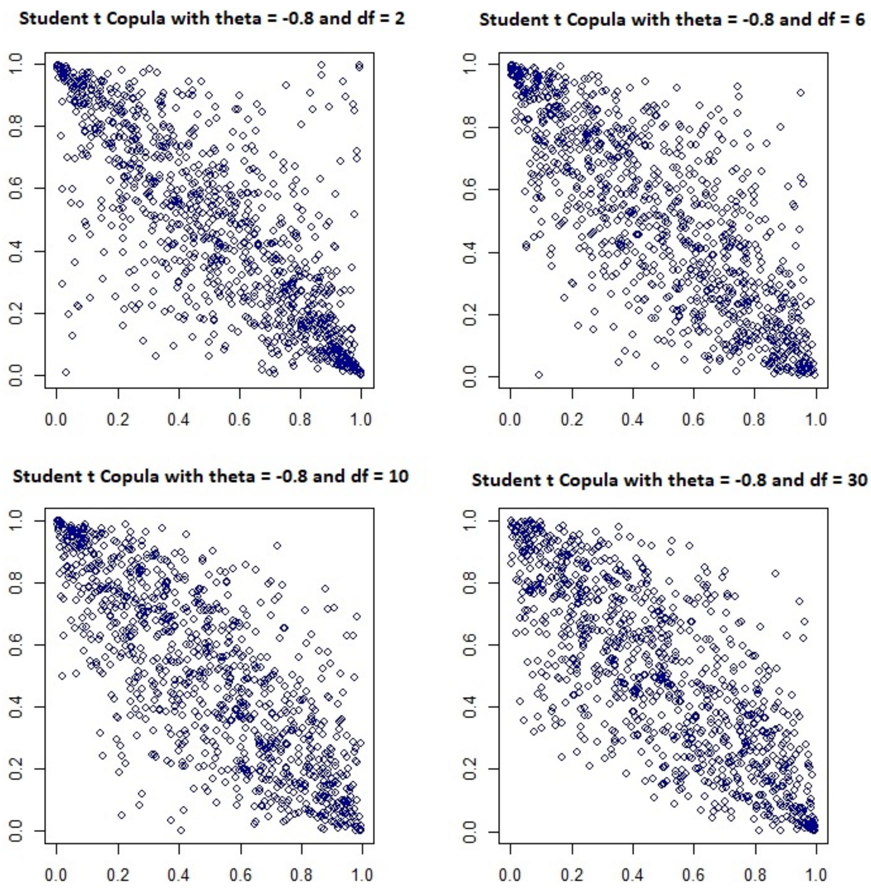


Figure 2.2.2: Graphical representation of the bivariate Student t copula showing the symmetric dependence structure in the tails when the dependence parameter θ is -0.8 with degrees of freedom equal to 2, 6, 10, 30. Showing that the number of degrees of freedom affect the heaviness of the tail dependency.

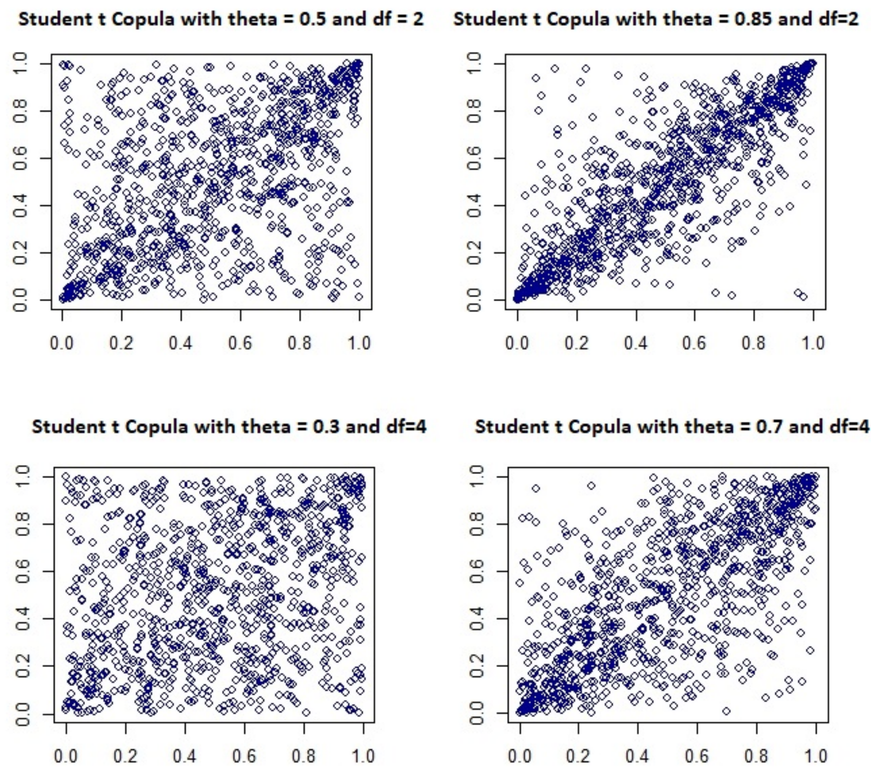


Figure 2.2.3: Graphical representation of the bivariate Student t copula showing the symmetric dependence structure in the tails in cases where the dependence parameter θ is 0.5 and 0.85, and the degrees of freedom is 2; and when θ is 0.3 and 0.7 with 4 degrees of freedom.

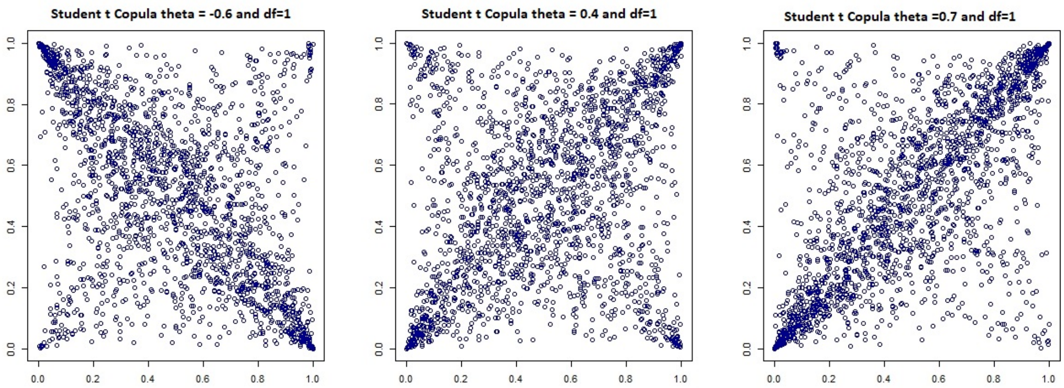


Figure 2.2.4: Graphical representation of the bivariate Student t copula showing that low degrees of freedom may generate "wings" effect. The dependence parameter θ is -0.6, 0.4 and 0.7 with 1 degree of freedom, respectively.

2.2.3.5 Clayton copula

The Clayton copula has the following form,

$$C(u_1, u_2; \theta) = (u_1^{-\theta} + u_2^{-\theta} - 1)^{-1/\theta}$$

with the dependence parameter θ restricted on the region $(0,\infty)$. The probability density function of the Clayton copula is given by,

$$c(u_1, u_2; \theta) = \frac{\partial^2 C(u_1, u_2; \theta)}{\partial u_1 \partial u_2} = (\theta + 1)(u_1 u_2)^{-(\theta+1)}(u_1^{-\theta} + u_2^{-\theta} - 1)^{-\frac{2\theta+1}{\theta}}.$$

The Clayton copula can only be used in cases where positive dependence is under investigation. Considered to be the most appropriate choice in cases where strong left tail dependence and relatively weak right tail dependence is observed. The low-tail dependence of the Clayton copula has the following form,

$$LT = \lim_{u_1 \to 0+} \frac{C(u_1, u_1)}{u_1} = \lim_{u_1 \to 0+} \frac{(2u_1^{-\theta} - 1)^{-1/\theta}}{u_1} = 2^{-1/\theta}.$$

There is also a closed-form formula describing Kendall's tau of Clayton copula as a function of the dependence parameter θ, as follows

$$\tau = \frac{\theta}{\theta + 2} \;\Rightarrow\; \theta = \frac{2\tau}{1-\tau}.$$

Figure 2.2.5 provides a graphical representation of the bivariate Clayton copula for dependence parameter θ 2, 5 and 8. We illustrate the effect of the dependence parameter on the left tail dependency. Simulated samples of size 1000, generated using the following *R* commands,

clayton.cop <- claytonCopula(param= -, dim = 2)

y <- rCopula(1000, clayton.cop).

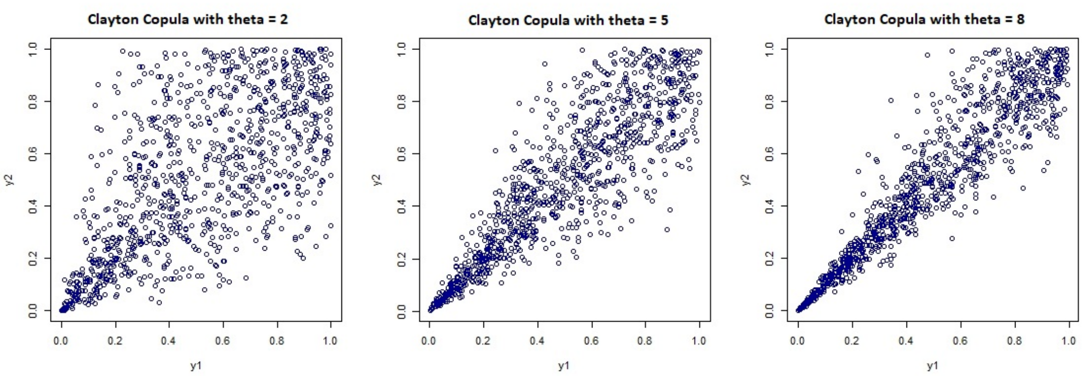


Figure 2.2.5: Graphical representation of the bivariate Clayton copula for dependence parameter θ 2, 5 and 8, showing the effect of the dependence parameter on the left tail.

2.2.3.6 Frank copula

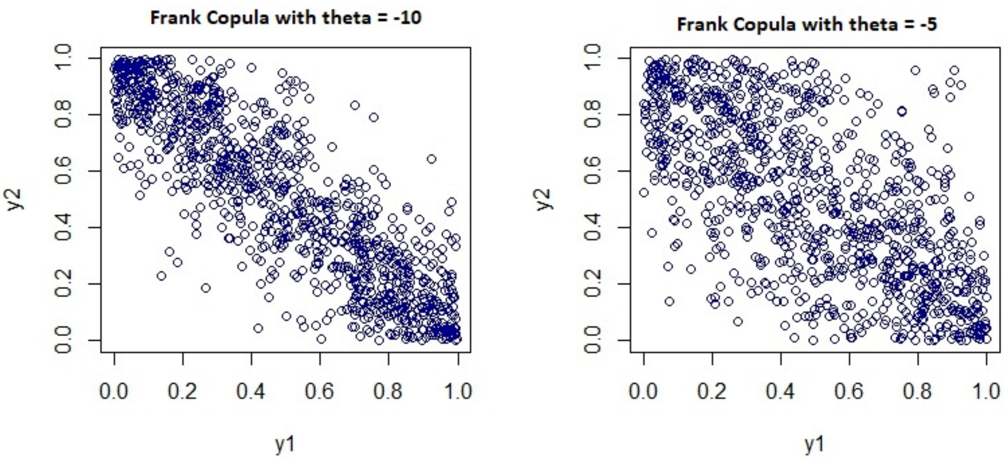
The Frank copula has the following form,

$$C(u_1, u_2; \theta) = -\theta^{-1} log \left\{ 1 + \frac{(e^{-\theta u_1} - 1)(e^{-\theta u_2} - 1)}{e^{-\theta} - 1} \right\}$$

with the dependence parameter $\theta \in (-\infty, +\infty)$. The Frank copula can only be used in situations where negative dependency is considered, and as the Gaussian and Student-t copulas, it allows symmetric dependency in the tails. Frank copula is considered the best choice in cases where we notice weak tail dependence and strong dependence centred in the middle of the distribution. Comparing Frank copula with Gaussian, simulation studies have shown that Franks' copula tail dependence is weaker than the Gaussian copula. Figure 2.2.6 shows a graphical representation of the bivariate Frank copula with dependence parameter θ -10, -5, 2 and 6. Simulated samples of size 1000, generated using the following *R* commands,

frank.cop <- frankCopula(param= - , dim = 2)

f <- rCopula(n=1000 , frank.cop).



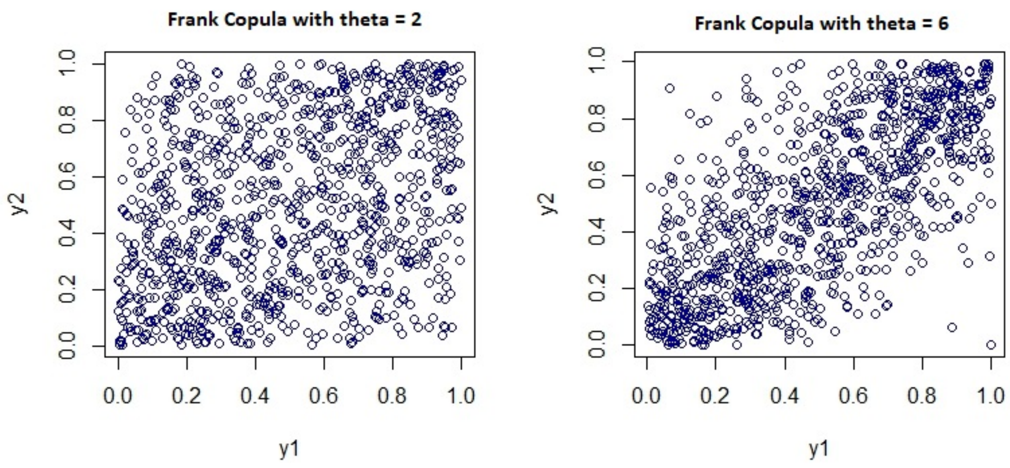


Figure 2.2.6: Graphical representation of the bivariate Frank copula with dependence parameter θ -10, -5, 2 and 6.

2.2.3.7 Gumbel copula

The Gumbel copula has the following form,

$$C(u_1,u_2;\theta) = \exp\left(-(\tilde{u}_1^{\theta}+\tilde{u}_2^{\theta})^{1/\theta}\right),$$

where $-\tilde{u}_j = log\ u_j$ and $\theta \in [1,\infty)$. Gumbel copula can only be used in cases where positive dependence is expected. Gumbel copula exhibits strong right tail dependence and relatively weak left dependence. Considered an appropriate copula choice in cases where we recognise strong correlation one high values and relatively weak correlation on low values. Figure 2.2.7 provide a graphical representation of the bivariate Gumbel copula for dependence parameter θ 2, 5 and 8. We illustrate the effect of the dependence parameter on the right tail dependency. Simulated samples of size 1000, generated using the following *R* commands,

gumbel.cop <- gumbelCopula(param = -, dim = 2)

g <- rCopula(1000, gumbel.cop).

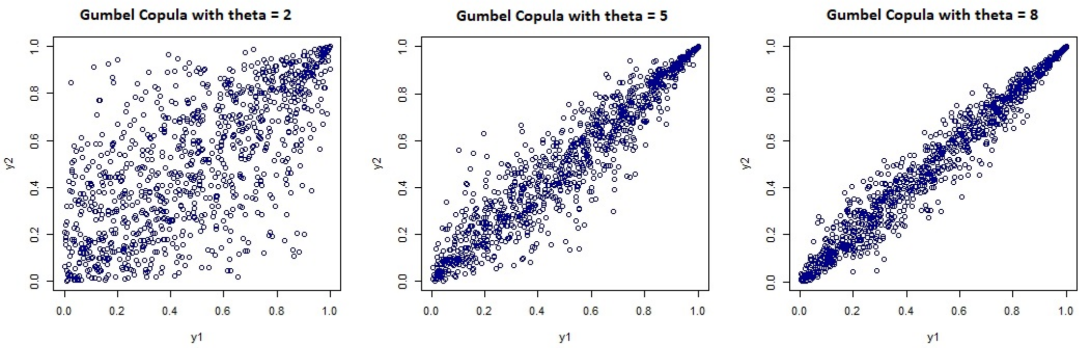


Figure 2.2.7: Graphical representation of the bivariate Gumbel copula for dependence parameter θ 2, 5 and 8, showing the effect of the dependence parameter on the right tail.

Having presented and discussed the most commonly used copula families, we now present their main properties in the summarizing table 2.2.1.

Copulas	Parameter Space	Properties
Gaussian	$\theta \in (-1,1)$	• Positive and negative dependence • Symmetric dependence in the tails • Linear correlation cases
Student t	$\theta_2 \in (-1,1)$, $\theta_1 \in \mathbb{R}^+$	• Positive and negative dependence • Symmetric dependence in the tails • Small θ_1 increases tail dependence • Large θ_1 approximates Gaussian
Clayton	$\theta \in (0,\infty)$	• Positive dependence • Asymmetric dependence • Strong left tail dependence
Frank	$\theta \in (-\infty,\infty)$	• Positive and negative dependence • Symmetric dependence in the tails • Weak tail dependence (Weaker than Gaussian) • Strong dependence centred in the middle
Gumbel	$\theta \in [1,\infty)$	• Positive dependence • Asymmetric dependence • Strong right tail dependence

Table 2.2.1: Summarizing the basic properties of commonly used copula families.

2.2.4 Copulas in Risk and Reliability

Apart from the unique dependence structure that copulas provide, copulas bind different marginal distributions under the same dependency with no restrictions in terms of the marginal choice (Kurowicka & Cooke, 2006). For example, one can choose between two Gamma marginal distributions or one Gamma and one Normal distribution; then, by using a copula of its choice with any of the two marginal combinations, a bivariate distribution will be constructed. Therefore, copula function is a powerful tool in modelling because primarily, it provides the flexibility of choosing among different dependence structures and secondly it provides flexibility in the marginal distributions. Both flexibilities are significantly necessary because, in real-world applications, not all dependencies between event rates are perfectly linear or well defined; and not in all situations, the marginal distributions can be chosen by only one class of distributions.

Copulas have been used in various systems for reliability modelling. Y. Wang and Pham (2011) use time-varying copulas to model the correlation between multiple degradation processes and random shocks within a competing risk model. Tang, Li, Zhou, Phoon, and Zhang (2013) use copulas for constructing bivariate distributions system reliability. Z.-w. An, Zhang, and Wang (2015) develop a reliability model for wind turbines based on failure parts correlation which is modelled using copulas. Z. An and Sun (2017) use copulas for modelling multiple dependent competing failure processes with shock loads above a certain level. D.-Q. Li et al. (2015) use copulas to construct the bivariate distribution of shear strength parameters and discuss its impact on Geotechnical System Reliability.Shen, Zhang, Zhuang, and Guo (2018) use mixed copulas to model the dependent function failure modes and evaluate the reliability of the Gear Door Lock System (GDLS).

One copula function can describe one specific dependence structure, but by mixing multiple copulas, one can capture multiple dependencies. For instance, Eryilmaz (2014) applied Clayton and Gumbel copulas for dynamic reliability modelling of dependent components of weighted-k-out-of-n systems. X. Wang, Wang, Chang, and Li (2020) use Clayton, Gumbel and Frank copulas to capture the correlation between contact fatigue failure and wear failure of bearings and establish a reliability model of rolling bearing based on multi correlation failure node. Morales-Nápoles, Paprotny, Worm, Abspoel-Bukman, and Courage (2017); Morales Nápoles, Worm, Abspoel-Bukman, Huibregtse, and Courage (2015) use one parameter bivariate copulas, Gaussian, Gumbel and Clayton, to analyse rain gauge data in the Netherlands and provide a better assessment of risks related to extreme rainfall events. Behrensdorf, Broggi, and Beer (2019) use multivariate copulas to represent realistic dependency structures between different networks.

2.2.5 Criticism of Copulas

Copulas have also been criticised. Particularly, to better inform and caution the audience about the problems associated with copulas Mikosch (2006) gave several criticisms. Some of these criticisms are listed below, as presented by Balakrishnan & Lai, 2009.

- "There are no particular advantages of using copulas when dealing with multivariate distributions. Instead, one can and should use any multivariate distribution which is suited to the problem at hand and which can be treated by statistical techniques."
- "The marginal distributions and the copula of a multivariate distribution are inextricably linked. The main selling point of the copula technology - separation of the copula (dependence function) from the marginal distributions - leads to a biased view of stochastic dependence, in particular when one fits a model to the data."
- "Various copula models (Archimedean, t-, Gaussian, elliptical, extreme value)

are mostly chosen because they are mathematically convenient; the rationale for their applications is murky."

- "Copulas are considered as an alternative to Gaussian models in a non- Gaussian world. Since copulas generate any distribution, the class is too big to be understood and to be useful."
- "There is little statistical theoretical theory for copulas. Sensitivity studies of estimation procedures and goodness-of-fit tests for copulas are unknown. It is unclear whether a good fit of the copula of the data yields a good fit to the distribution of the data."

We acknowledge that some of the concerns about how copulas should be used and which family should be chosen, i.e. not only because they are mathematically convenient, were reasonable, especially when there was a little statistical theoretical theory about copulas. If we consider this research study, where we investigate the dependence between unknown rates of events modelled using copulas, to our knowledge relevant studies have not yet been proposed in the literature.

On the other hand, we do not agree with the first criticism where it states that one could use any multivariate distribution instead of using copulas. In Bayesian methodology, we select a parametric distribution family and choose the appropriate parameters which will best represent our beliefs about the 'true' prior. However, it may not be possible to represent our beliefs about the 'true' prior with any of the standard parametric distributions, or any multivariate distribution; and even if we can, there are cases where two distributions are visually identical but in fact, are entirely different. Therefore, copulas can fill this gap and provide a more informative alternative that will best represent our beliefs.

Despite the criticism, copulas provide a number of flexibilities, e.g. can bind multiple distribution functions with no restrictions, and capture complex dependence structures by providing flexibility in terms of marginal choice, which are valuable in modelling, therefore they have piqued the interest of researchers and more and more studies contribute to knowledge. We aim to investigate copulas by incorporating them within an empirical Bayesian framework.

2.3 Bayesian and Empirical Bayesian Approach

Bayesian methods have been first introduced by Rev. Thomas Bayes, a minister and amateur mathematician, in 1763. In the 19th century, Laplace, Gauss and others showed some interest in the area, but other statisticians ignored and opposed the approach in the early 20th century. Fortunately, around 1950, the Bayesian methods have been actively advocated by statisticians such as L. J. Savage, Bruno de Finetti and many others (Carlin & Louis, 2008).

The Bayesian approach considered to be an increasingly effective and practical alternative to the classic, or frequentist, statistical philosophy for statistical analysis and design (Carlin & Louis, 2008). Different philosophical positions, conceptual justifications, arguments or mathematical proofs led to philosophical battles between the classical and Bayesian statisticians over the years. In the classical approach, as Carlin and Louis (2008) define, the procedures are evaluated based on imagining repeated sampling from the model, or the likelihood, which defines the probability distribution of the observed data conditional to some unknowns. Then, the properties of this procedure are evaluated within this sampling framework by assigning fixed values to those unknown parameters. On the other hand, the Bayesian approach requires not only a sampling model but also a prior distribution. The latter refers to our prior knowledge about all unknown parameters in the model. The conditional distribution of these unknowns given the observed data (posterior) is then obtained using the likelihood and the prior. In Bayesian analysis, the procedures can be evaluated through the repeated sampling of unknowns from the posterior distribution given the observed data. The empirical Bayesian (EB) approach differs from the classic Bayesian approach, as it occurs when we allow the observe data to play some role in determining the unknown parameters and consequently, the prior distribution.

Bayesian methods offer an alternative perspective, and often provide solutions to practical statistical problems. However, there were concerns regarding the practical issues arise from the application of such methods. While the methods were theoretically simple, required computing power and simulation techniques for evaluating complex integrals which at that time was challenging and discouraging. From a computational perspective, the issue has mainly been resolved (Carlin & Louis, 2008). We now present some of the advantages of using Bayesian methods, as presented in (Berger, 2013),

- ’Bayesian methods provide the user with the ability to incorporate prior information formally.’
- ’All Bayesian analyses follow directly from the posterior; no separate theories of estimation, testing, multiple comparisons, etc. are needed.’
- ’Bayes and EB procedures possess numerous optimality properties.’

Bayesians and classical statisticians have been criticising each other for multiple reasons over the years (see Carlin & Louis, 2008). Notably, Bayesian approaches have been criticised for not being able to deal with various examples, for over-relying on computationally convenient priors, and for being too fragile in their dependence on the priors; and classical approaches for not being able to incorporate relevant prior information, for not being efficient, flexible and coherent (i.e. ’a failure to process available information systematically, as a Bayesian approach would’ (Carlin & Louis, 2008). Classic statistical approaches do not dependent any prior beliefs, resulting in claims of ’objectivity’ which are often considered illusory by the Bayesians who believe that the underlying data mechanisms of such methods require ’myriads' assumptions. In contrast, Bayesians often remark that their only assumption is the prior family selection which should be explicitly declared and checked.

2.3.1 Bayesian Models

Bayesian models are considered parametric models in general, or stochastic models as relevant probability distribution functions are assigned to random variables or parameters involved. In contrast to the 'classical' parameter models, Bayesian models require a prior probability distribution or simply prior for the unknown parameter of the model. Within this context, the unknown parameter is considered a random variable rather than constant. Bayesian models consist of the following basic components,

- The (observed) data, denoted by y,
- The (unknown) model parameter, denoted by θ,
- The data – model distribution or likelihood, specified by $f(y|\theta)$ or $L(y|\theta)$,
- The prior distribution, specified by $f(\theta)$ or $\pi(\theta)$.

We now present the classic two-stage Bayesian model and address its main difference compared to the empirical Bayes model. We initially specify the likelihood $Y|\theta$ given the unknown parameter θ, as follows

$$Y|\theta \sim f(y|\theta).$$

In Bayesian methodology, the unknown parameter θ is not fixed but a random quantity or variable. The probability distribution of θ summarises and describes any prior knowledge or information about this quantity, and it is called prior distribution,

$$\theta \sim f(\theta) \text{ or } \pi(\theta).$$

Note that either Y or θ can be vectors, and the prior can be parametric or nonparametric. If the prior distribution has parameters, these are known as hyperparameters. The sequence of priors and parameters lead to hierarchical models in which the final step requires all remaining parameters to be known. In contrast to classic Bayesian model, the empirical Bayes methodology involves the estimation of the prior parameters by using the observed data available. Practically, the name 'empirical Bayes' arises from the fact that empirical data are used for estimating x and updating the prior distribution. The hyperparameters can be estimated using the maximum likelihood estimation method (MLE) or the method of moments (MME).

2.3.1.1 Posterior and predictive distribution

Inference about the unknown parameter θ is based on the posterior distribution, or estimated posterior in EB. The posterior distribution of θ is defined by

$$\pi(\theta|y) = \frac{f(y|\theta)\pi(\theta)}{f(y)},$$

where,

$$f(y) = \begin{cases} \int f(y|\theta)\pi(\theta)d\theta \text{ , if } \theta \text{ is continuous} \\ \\ \sum_{\theta} f(y|\theta)\pi(\theta) \text{ , if } \theta \text{ is discrete} \end{cases}$$

and is obtained following the Bayes' Theorem. Practically, the posterior distribution is the normalised product of the likelihood and the prior distribution. Since $f(y)$ is a constant quantity with respect to parameter θ, the posterior distribution may also be expressed as proportional to the product of the likelihood and the prior as follows,

$$f(\theta|y) \propto f(y|\theta) \cdot \pi(\theta).$$

Determining the posterior distribution requires the calculation of the normalisation constant $f(y)$, which sometimes can be challenging or event not analytically tractable. Considering now the predictive ability of Bayesian models, we define the predictive distribution for a future observation $y^\star$, which is independent of y given the parameter θ, as follows,

$$f(y^\star) = \int f(y^\star|\theta)\pi(\theta)\, d\theta.$$

As Puza (2015) stated, the predictive distribution summarises "the information concerning the likely value of a new observation, given the likelihood, the prior and the data we have observed so far."

2.3.1.2 Prior distribution

In the Bayesian approach as already mentioned, we assign probability distributions to describe not only the observed data but also the unknown model parameter θ. Mainly, our belief and any other relevant information we may have about the parameter θ are formed in a subjective probability, before even looking at the data y. Then this subjective probability combined with all information in the observed data, the posterior distribution from which our inference is derived emerges. However, determining the appropriate form of the prior distribution may be challenging. The prior distributions are elicited based on information from relevant historical studies or expert judgement, or both.

2.3.1.2.1 Elicited priors

We consider two different types of uncertainty in modelling, the aleatory and the epistemic. Aleatory uncertainty represents the pure inherent randomness in the observations, whilst epistemic uncertainty corresponds to the state of knowledge about the quantity of interest. The more data we observe, the more we learn, resulting in reduced epistemic uncertainty. Within the Bayesian methodological framework, the aleatory uncertainty is captured through the likelihood distribution function, while the prior distribution represents the epistemic uncertainty. The elicitation process requires experts expressing their beliefs about the unknown quantity in the model, and hence, the epistemic uncertainty in the form of a subjective probability distribution.

The elicitation process for constructing subjective probabilities can be a time - consuming and challenging process (Carlin & Louis, 2008). A simple approach describing the process is the following. Suppose that θ is univariate, we first consider a manageable number of "possible" θ values and then, we assign probability masses to each. The sum of the assigned probabilities needs to be 1, and their contribution needs to reflect on our beliefs as closely as possible. However, even if both conditions are satisfied, there are cases where some elicited priors provide more useful inference than others (Quigley, Bedford, & Walls, 2014). Another approach is to simply select a parametric distribution family and choose the appropriate parameters which will best represent our beliefs about the 'true' prior. Although the latter approach seems simple, we address two limitations. First, it may not be possible to represent our beliefs about the 'true' prior with any of the standard parametric distributions; and second, even if we can, there are cases where two distributions are visually identical but in fact, are entirely different. For example, the distributions Normal(0, 2.19) and Cauchy(0, 1) are visually similar, have identical percentiles (25th, 50th, 75th) but completely different properties (Carlin & Louis, 2008).

In addition, structured elicitation processes have also been developed for constructing subjective probability distributions (for further see Quigley et al., 2014; Carlin & Louis, 2008). Such processes usually require an 'expert' and an 'analyst' and aim to "minimise the impact of biases inherent in surfacing and capturing subjective expert judgement" (Quigley et al., 2014). The expert assesses the uncertainty about the quantity of interest, and the analyst collects relevant data and information from the expert and formulates the prior distribution.

Since the elicitation process relies on an expert's belief, it is almost impossible to identify a single correct subjective probability as each person interprets and forms their beliefs from different perspectives and through personal experience. However, calibration and information measures can be used to help develop proper scoring rules for creating priors via mathematical aggregation of multiple beliefs. Assessments of calibration can be obtained when the subjective probabilities can be compared to observed realisations or when observations of the quantity of interest are available or can be generated from the predictive distribution. The information measure can be assessed by calculating the mean squared error of the difference between the observed realisations and the elicited probabilities or based on information scores.

2.3.1.2.2 Conjugate priors

When we select the prior from a family distribution that is 'conjugate' with the distribution describing the data (likelihood), the posterior distribution belongs to the same family as the prior. Typically, such prior selections are considered more computationally convenient than others as they lead to well-defined posteriors and simplify calculations required. If the likelihood and the prior are not conjugate, the posterior cannot be defined by a well - known distribution function and therefore, we require numerical integration or Monte Carlo simulation methods for evaluating complex integrals.

2.3.1.2.3 Empirical prior elicitation methods

There are multiple other methods to construct prior distributions (for more details see Carlin & Louis, 2008). In contrast with all other methods mentioned previously, the empirical estimation method uses the observed data to determine the prior. Quigley and Walls (2018) developed a method for constructing subjective probability distributions by combining expert judgement and empirical data within an empirical Bayes framework.

The Empirical Bayes approach captures the epistemic uncertainty through experts identifying pool of analogous problems for which there is data and as such the data can be used to estimate the prior (Quigley & Walls, 2018). The empirical data are 'pooled' to estimate the unknown parameters of the prior, and the posterior estimate is the weighted average between the individual and the pool. The method of moments (MME) and the maximum likelihood estimation method (MME) can be used for estimating the prior parameters.

2.3.2 Bayesian Inference

2.3.2.1 Point estimation

Once the posterior distribution or density has been determined, the Bayesian point estimate of the unknown model parameter θ can be obtained. Mainly, there are three point estimates which are presented below (Puza, 2015).

- **Posterior mean**,

$$E(\theta|y) = \begin{cases} \int \theta f(\theta|y)\, d\theta \text{ , if } \theta \text{ is continuous,} \\ \sum_{\theta} \theta f(\theta|y) \text{ , if } \theta \text{ is discrete.} \end{cases}$$

- **Posterior mode**,

$$\begin{aligned} Mode(\theta|y) = \quad & \text{any values } m \in \Re \text{ which satisfies} \\ & f(\theta = m|y) = \underset{\theta}{max} f(\theta|y) \\ & \text{or } \lim_{\theta \to m} f(\theta|y) = sup\, f(\theta|y), \\ & \text{or the set of all such values.} \end{aligned}$$

- **Posterior median**,

$$Median(\theta|y) = \begin{array}{c} \text{any value } m \text{ of } \theta \text{ such that} \\ P(\theta \leq m|y) \geq 1/2 \text{ and } P(\theta \geq m|y) \geq 1/2, \\ \text{or the set of all such values.} \end{array}$$

2.3.2.2 Interval estimation

The statement "the probability that θ lies in C given the observed data y is at least $(1-\alpha)$" gives the 'informal' definition of the confidence interval. The formal definition of the 'Bayesian confidence interval' is given below.

Definition. ***Bayesian confidence interval***
A $100 \times (1-\alpha)\%$ *confidence interval for* θ *is a subset* C *of* Θ *such that,*

$$P(C|y) \geq 1-\alpha \;\Rightarrow\; \int_C p(\theta|y)\, d\theta \geq 1-\alpha,$$

where integration is replaced by summation if θ *is discrete.*

2.3.3 Empirical Bayes Applications in Risk

Bayesian and empirical Bayesian models have been widely used within the context of risk and reliability over the years. Notably, the two-stage or hierarchical Bayesian model was introduced by Kaplan (1983, 1985) for determining failure rates within a probabilistic risk analysis context. Kaplan (1983) described the two-stage Bayesian model as 'a simple procedure which operates on the data in such a way that the output of the first stage, i.e. the posterior distribution of the first stage, becomes the prior distribution of the second stage'. Iman and Hora (1989) used two-stage methodology for modelling recovery times with an application to the loss of off-site power at nuclear power plants. Hora and Iman (1990) used two-stage Bayesian modelling of initiating event frequencies at nuclear power plants. Bunea et al. (2005) performed different two-stage Bayesian models, comparing them to the one-stage model (Vaurio, 2005), for analysing reliability data for nuclear power facilities (called ZEDB project).

Moreover, estimating rare events or eliciting a prior when few or even zero occurrences observed can also be challenging. One way to overcome this challenge is following empirical Bayes inference methods. When using empirical Bayesian methods, the events of each process are pooled for estimating each event rate. Consequently, each rate is defined as the weighted average between the pooled event rate and the frequency of the event observed from data. Usually, Bayesian models suggest that the events are realised from Poisson distributions, which depend on the event rate and the exposure time for each event. There are many approaches which combine Poisson process data with empirical Bayes (EB) methods in risk analysis. For example, Ferdous, Uddin, and Pandey (1995) use EB to support inference for the Weibull distribution within a software reliability growth context and Vaurio (2002, 2005) discuss the application of EB for estimating the typical cause failure rates. Vaurio and Jänkälä (2006) use EB for failure rates and probabilities estimation methods within a Poisson modelling framework, Quigley, Bedford, and Walls (2007a) use EB for estimating the rate of occurrence of rare events within a Homogeneous Poisson process framework in a railway safety model, Quigley, Bedford, and Walls (2009) use EB for estimating the reliability development of one-shot device and Quigley and Walls (2011) mix Bayes and EB to anticipate the realisation of engineering concerns. Quigley, Hardman, Bedford, and Walls (2011) combine EB and expert judgement for estimating rare event frequency motivated by a Probabilistic Risk Assessment (PRA) project. They propose a novel and robust, although partially subjective, EB method in which a pool of events is used for estimating the frequency of rare events, and experts assess homogenisation factors under the assumption that events are generated from a Homogeneous Poisson Process.

In Bayesian inference, the event rates of each process are assumed to be statistically independent, given some parameters. If these unknown parameters are hyper-parameters, can then be fully specified through the prior distribution. In the presence of correlation, a multivariate distribution can be defined for capturing the dependency between all unknown parameters. Then by applying the Bayes Theorem and considering the observed data, the rates estimates can be updated. Thereby, the underlying dependence on the rates can be incorporated into the Bayesian modelling procedure, which primarily ensures coherent and theoretically sound rates estimates, and secondarily, allows the rates estimates to be informed based on information for multiple events. Considering dependency or not, the posterior distribution is not always analytically traceable. Therefore, numerical integration, Monte Carlo or Markov Chain Monte Carlo (MCMC) methods are of need, and hence can be computationally intensive (Gelman et al., 2013).

Considering dependency within the Bayesian methodological framework, Quigley et al. (2013) developed a Bayes linear Bayes model for correlated event rates. They consider that events generated from a homogeneous Poisson process and Gamma prior distributions; and, the correlation between the rates is modelled using a Hypergeometric function. Even if this approach performs significantly well compared to the full Bayesian model, it has some limitations. Mainly, the subjective character of the proposed method, which requires expert judgement for specifying the correlation coefficient on the rates could be considered challenging. Also, the choice of the marginal distributions and the dependence structure could be considered restricted and challenging in situations where other marginal distributions would be more appropriate or when the dependence structure is more complicated. Therefore, the need for creating a method that considers dependency between multiple event rates and provides flexibility in terms of marginal choice and dependence structure has motivated this study.

2.4 Research Gaps and Discussion

Estimating multiple event rates is challenging, especially when these unknown rates are correlated. There are problems in supply chain risk, safety risk analysis of a national railway network and reliability, where inference of multiple event rates in the presence of dependency is considered (see Quigley et al., 2013). Modelling approaches to these problems usually assume that the event rates of each process are conditionally statistical independent (Quigley et al., 2011). The underlying dependency between the event rates is assessed through subjective inference methods or require expert judgement; or is obtained empirically using the maximum likelihood estimation method or the method of moments (Quigley et al., 2013). In addition, the dependence structure between multiple event rates can be comparatively complex and therefore, multivariate distribution functions are of need to describe such structures. According to the literature, copula functions can be used for modelling and capturing such complex dependence structures (Nelsen, 2007). However, to our knowledge, empirical Bayesian methods combined with copulas within this context, i.e. prior copula, have not yet been explored or developed.

This research aims to fill this gap and contributes to the research field by developing an inference framework to account for dependency between multiple event rates realized from Poisson processes using copulas. We investigate the underlying dependence on the rates and not the correlation on the realisations of events (Poisson process data), implying that the underlying dependence on the rates is driven by the operations within organisations rather than the occurrence of rare or extreme events. Many studies in risk and supply chain area have been derived showing the existence of correlation between operations within organisations. For example, management culture within organisations can have a similar impact on multiple outcomes. Tseng (2010) investigate the relationship between organizational culture and knowledge conversion on corporate performance. Their results suggest that both organizational culture and knowledge conversion are positively correlated with the corporate performance. In particular, as they state 'an adhocracy culture enables knowledge conversion and enhances corporate performance more than clan and hierarchy cultures'. Of particular interest are situations where we anticipate low rates of occurrences which are relatively more correlated compared to the high. With emphasis on capturing the left tail dependency, an empirical Bayesian inference method combined with Clayton copula is proposed. We aim to evaluate the proposed method and answer to the following key research question,

R.Q. How good are the assessments of mean rates when using a moment - based inference approach within an Empirical Bayes method assuming dependency between the rates?

The empirical Bayes methodological framework involves the estimation of the prior parameters using only the empirical count data available. The maximum likelihood estimation method or the method of moments can be used for estimating the marginals prior parameters; however, estimating the copulas dependence parameter can be challenging. The multivariate prior distribution cannot always be analytically expressed, as are its derivatives or moments. To our knowledge, methods for describing the relationship between Poisson process data and prior dependency or estimating the copulas' dependence parameter within this context, i.e. prior copula, have not yet proposed in the literature. To bridge this gap, we initially investigate if there is a relationship between the empirical data and the prior dependence parameter and further how it can be explicated within this context. Therefore, the key research question to be answered is,

R.Q. How to estimate the dependence parameter of the Clayton copula with Gamma marginals using only empirical data realized from Poisson processes?

This research also investigates the impact of not accounting for dependency by providing an application of the proposed method. In particular, we propose methods for ranking. To our knowledge, there is a gap in the literature about methods for ranking correlated event rates, and therefore we shall focus on this area. Ranking under uncertainty can be a complicated process, and the final ranking result can be affected by various factors. Although much of our discussion is relevant to ranking on event rates, we concentrate on the area of supplier ranking. We believe that the methods we discuss are generally applicable whatever measures are chosen and for whatever purpose. Therefore, motivated by the challenges in supply chain, we propose a Bayesian method for supplier ranking considering the underlying dependence between the late delivery rate and the non – conformance rate. The key research question to be answered is,

R.Q. What is the impact of accounting for dependency in the context of ranking based on correlated event rates?

In the next chapter, we will specify the modelling context of this research, and define the empirical Bayesian model combined with Clayton copula. Methods for estimating the prior parameters and the posterior expectations will also be discussed in the following.

Chapter 3

Modelling Context

3.1 Introduction

Event rates of each process can be assumed to be statistically independent, given some parameters. If these unknown parameters are hyperparameters, can then be fully specified through the prior distribution (Carlin & Louis, 2008). In the presence of correlation, a multivariate distribution can be defined for capturing the dependency between all unknown parameters. Then by applying Bayes Theorem and considering the observed data, the rates estimates can be updated. Thereby, the underlying dependence on the rates can be incorporated into the Bayesian modelling procedure, which primarily ensures coherent and theoretically sound rates estimates, and secondly, allows the rates estimates to be informed based on information for multiple events.

We develop an empirical Bayes inference method based on a multivariate Poisson - Gamma probability model considering dependency on the prior captured by a Clayton copula. The proposed model aims to capture both aleatory and epistemic uncertainties. We assume that events are generated from a homogeneous Poisson process capturing the pure inherent randomness in the observations, i.e. the aleatory uncertainty. Epistemic uncertainty is represented by the prior where the marginal distributions of event rates are Gamma, and the underlying correlation is captured by the Clayton copula. Typical observable data are number of events, which lead to Poisson distribution, depending on the event rate and the exposure time for each event. The Clayton copula is chosen for modelling the correlation between event rates. The Clayton copula best describes left tail dependence structure, where lower rates are considered highly correlated compared to higher. If we now consider our motivation for this study, the copula choice is not unreasonable. We are investigating the underlying dependence on the rates and not the correlation on the realisations of events (Poisson process data), implying that the underlying dependence on the rates is driven by the operations within organisations rather than the occurrence of rare or extreme events. Our proposed empirical Bayes with Clayton copula model along with the classic empirical Bayes model in which the rates of events are considered independent, are defined in Sections 3.2 and 3.3, respectively.

3.2 Empirical Bayes with Copula Method

This method incorporates copulas within an empirical Bayes context, which aims to present and propose a new method for estimating multiple event rates assuming dependency on the rates. Figure 3.2.1 illustrates our methodological framework for the development of the empirical Bayes with Clayton copula model.

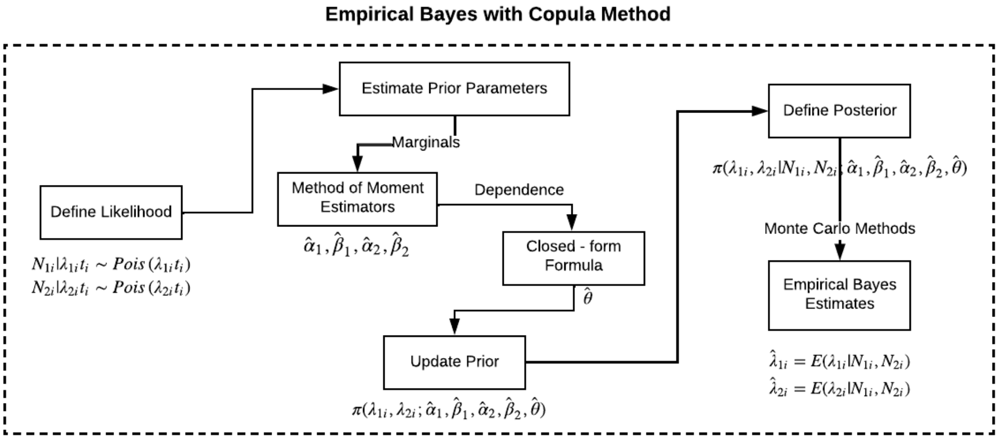


Figure 3.2.1: Diagram showing methodology followed for this study.

We consider conditionally independent bivariate Homogeneous Poisson Processes given their underlying rates. However, dependency in the epistemic uncertainty of these rates is allowed, as captured through a Clayton copula with Gamma marginal distributions for the prior distribution.

The prior distribution describes the variability of the rate of events within a pool, even before observing any data. The prior distribution is the probability density function measuring the Likelihood of an event, randomly chosen, having a rate of event of λ. We consider Gamma marginals as follows,

$$\Lambda_{ji} \sim Gamma(\alpha_j, \beta_j), \qquad j = 1,2, \; i = 1,2,\cdots,m$$

$$\pi(\lambda_{ji}) = \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_{ji}^{\alpha_j - 1} e^{-\beta_j \lambda_{ji}}, \qquad \lambda_{ji}, \; \alpha_j, \; \beta_j > 0.$$

Having the marginal distributions of the rate of events, we model their joint relationship by using a copula function. Clayton copula (bivariate case) with dependence parameter θ and Gamma marginal distributions was chosen. The prior has the following probability distribution function,

$$c(\lambda_1, \lambda_2) = c(F(\lambda_1), F(\lambda_2)) \cdot f(\lambda_1) \cdot f(\lambda_2)$$

where $F(\cdot)$ is the cumulative distribution function of the Gamma distribution, $f(\cdot)$ is the Gamma density probability function and $c(u, v)$ is the probability density function of the Clayton copula which has the following form,

$$c(u, v) = (\theta + 1)(uv)^{-(\theta+1)}(u^{-\theta} + v^{-\theta} - 1)^{-(2\theta+1)/\theta}, \ \theta \in (0, \infty).$$

The rate of events follows a constant rate over time, which implies a Homogeneous Poisson Process (HPP). The number of events that are realized over time interval t_i is denoted by the variable N_{ji}. The distribution of N_{ji} is Poisson and has the following probability density function,

$$N_{ji} \sim Poisson(\lambda_{ji}\ t_i)\ , \ j = 1, 2, \ i = 1, \cdots, m$$

$$P(N_{ji} = n_{ji} | \lambda_{ji}) = \frac{(\lambda_{ji}\ t_i)^{n_{ji}} e^{-\lambda_{ji}\ t_i}}{n_{ji}!}\ , \ \lambda_{ji} > 0, \ t_i > 0, \ n_{ji} = 0, 1, \cdots .$$

Following the empirical Bayesian methodology, the next step involves the prior specification, where all prior parameters $(\alpha_1, \alpha_2, \beta_1, \beta_2, \theta)$ should be estimated. The estimated marginal parameters $(\hat{\alpha}_j, \hat{\beta}_j)$ for $j = 1, 2$ are consistent estimators obtained using the Method of Moment (MME) presented in Section 3.4. Knowing that there is a closed-form formula for the Kendall's tau and the dependence parameter of a bivariate Clayton copula, we obtain the dependence parameter estimate $\hat{\theta}$ by obtaining the Kendall's tau estimation (see Chapter 4).

$$\tau = \frac{\theta}{\theta + 2} \qquad \Rightarrow \qquad \theta = \frac{2 \cdot \tau}{1 - \tau}.$$

After obtaining the prior estimates, the prior distribution is now updated. The final step of this method refers to the posterior distribution specification, which is not analytically traceable. Numerical methods need to be used to calculate the mean of the posterior distribution for the i^{th} event. Instead of numerical integration methods, a Monte Carlo - Simulation method is proposed for obtaining the expectations of the posterior distribution (see Section 3.5). Therefore, the empirical Bayes estimate of λ_{ji} is defined as the mean of the posterior distribution,

$$E(\lambda_{ji}|N_{1i},N_{2i}) = \frac{\int_0^\infty\int_0^\infty \lambda_{ji} \times c(\lambda_{1i},\lambda_{2i}) \times Pois(\lambda_{1i}) \times Pois(\lambda_{2i})\ d\lambda_{1i} d\lambda_{2i}}{\int_0^\infty\int_0^\infty c(\lambda_{1i},\lambda_{2i}) \times Pois(\lambda_{1i}) \times Pois(\lambda_{2i})\ d\lambda_{1i} d\lambda_{2i}},$$

$$\hat{\lambda}_{ji} = E(\lambda_{ji}|N_{1i},N_{2i})\,, \qquad j = 1,2\,,\ i = 1,2,\cdots,m.$$

3.3 Classic Empirical Bayes Model with Independent Event Rates

We now present the classic Gamma - Poisson empirical Bayesian model, where no underlying dependency on the rates is considered. The rate of events follows a constant rate over time, which implies Homogeneous Poisson Processes (HPP). The number of events that are realized over time interval t_i is denoted by the variable N_{ji}. The distribution of N_{ji} is Poisson and has the following probability density function,

$$N_{ji} \sim Poisson(\lambda_{ji}\ t_i)\,,\ j = 1,2,\ i = 1,\cdots,m$$

$$P(N_{ji} = n_{ji}|\lambda_{ji}) = \frac{(\lambda_{ji}\ t_i)^{n_{ji}} e^{-\lambda_{ji}\ t_i}}{n_{ji}!}\,,\ \ \lambda_{ji} > 0,\ \ t_i > 0,\ \ n_{ji} = 0,1,\cdots$$

We note that N_{1i}, N_{2i} are assumed conditionally independent given $\lambda_{1i}, \lambda_{2i}$. The realization from the HPP are assumed to be conditionally independent given the rates. The marginal prior distributions of the rates are chosen to be Gamma as follows,

$$\Lambda_{ji} \sim Gamma(\alpha_j, \beta_j), \qquad j = 1,2,\ i = 1,2,\cdots,m$$

$$\pi(\lambda_{ji}) = \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_{ji}^{\alpha_j - 1} e^{-\beta_j \lambda_{ji}}, \qquad \lambda_{ji}, \alpha_j, \beta_j > 0.$$

We then use the Method of Moments (see Section 3.4) to estimate the parameters of the prior distribution which are the Gamma parameters $(\alpha_1, \beta_1, \alpha_2, \beta_2)$. After obtaining the moment estimators, the prior distribution is updated. The final step refers to the calculation of the mean of the posterior distribution. In this case, the posterior distribution can be described by a well-known distribution function. Particularly, the posterior distribution is Gamma with the following density function,

$$\pi(\lambda_{ji}|N_{ji}, \alpha, \beta) = \frac{(\beta + t_i)^{\alpha + n_{ji}} \lambda_{ji}^{\alpha + n_{ji} - 1} e^{-(\beta + t_i)\lambda_{ji}}}{\Gamma(\alpha + n_{ji})}.$$

Finally, the empirical Bayes estimate of the rate of event, λ_{ji}, is the expectation of the posterior distribution and is defined as follows,

$$\begin{aligned} E(\lambda_{ji}|N_{ji}) &= \int_0^\infty \lambda_{ji} \pi(\lambda_{ji}|N_{ji}, \alpha, \beta)\ d\lambda_{ji} \\ &= \frac{\hat{\alpha} + n_{ji}}{\hat{\beta} + t_i} \\ &= \hat{\lambda}_{ji} \end{aligned}$$

3.4 Method of Moments Estimators

This is a method to estimate model parameters by matching the average moments observed in the data with the theoretical moments from the model and solving for the unknown parameters. For the use of the method of moments, the first two moment of N_{ji} are required,

$$E(N_{ji}) = E_{\lambda_{ji}}\left[E(N_{ji}|\lambda_{ji})\right] = \frac{\alpha_j}{\beta_j} t_i,$$

$$E(N_{ji}^2) = E_{\lambda_{ji}}\left[E(N_{ji}^2|\lambda_{ji})\right] = \frac{\alpha_j}{\beta_j} t_i + \frac{\alpha_j(\alpha_j+1)}{\beta_j^2} t_i^2.$$

Having the moments of N_{ji}, we obtain the moment estimates of the Gamma marginal parameter as follows,

$$\begin{matrix} \frac{\alpha_j}{\beta_j}\sum_{i=1}^{m} t_i = \sum_{i=1}^{m} N_{ji} \\ \\ \frac{\alpha_j}{\beta_j}\sum_{i=1}^{m} t_i + \frac{\alpha_j(\alpha_j+1)}{\beta_j^2}\sum_{i=1}^{m} t_i^2 = \sum_{i=1}^{m} N_{ji}^2 \end{matrix} \Rightarrow \begin{cases} \hat{\alpha}_j = \frac{U^2}{W-U^2} \\ \\ \hat{\beta}_j = \frac{U}{W-U^2} \end{cases}$$

where,

$$U = \frac{\sum_{i=1}^{m} N_{ji}}{\sum_{i=1}^{m} t_i} \quad , \qquad W = \frac{\sum_{i=1}^{m} N_{ji}^2 - \sum_{i=1}^{m} N_{ji}}{\sum_{i=1}^{m} t_i^2}, \quad \text{for } j = 1, 2.$$

Notably, the moment estimators are consistent, which means as the sample size increase towards infinity, the estimated parameters will converge with the true values, as presented by Quigley et al., 2013. Under the assumption that the Gamma marginal distributions are identically distributed meaning that they have equal shape and rate parameters, we consider the pool as one sample and calculate the moment estimators as follows,

$$\begin{matrix} \frac{\alpha}{\beta}\sum_{i=1}^{2m} t_i = \sum_{i=1}^{2m} N_i \\ \\ \frac{\alpha}{\beta}\sum_{i=1}^{2m} t_i + \frac{\alpha(\alpha+1)}{\beta^2}\sum_{i=1}^{2m} t_i^2 = \sum_{i=1}^{2m} N_i^2 \end{matrix} \Rightarrow \begin{cases} \hat{\alpha} = \frac{U^2}{W-U^2} \\ \\ \hat{\beta} = \frac{U}{W-U^2} \end{cases}$$

where,

$$U = \frac{\sum_{i=1}^{2m} N_i}{\sum_{i=1}^{2m} t_i} \quad , \qquad W = \frac{\sum_{i=1}^{2m} N_i^2 - \sum_{i=1}^{2m} N_i}{\sum_{i=1}^{2m} t_i^2}, \quad \text{for } j = 1, 2.$$

3.5 Monte Carlo Simulation – Summation Method

The posterior distribution is not always analytically traceable. Therefore, numerical integration methods are required for obtaining the empirical Bayes estimates by computing the posterior expectations. In numerical integration methods, we evaluate the integral over continuous function by computing the value of the function a finite number of points. We can obtain the desired accuracy by increasing the number of points, which in some situations can be computationally intensive. Numerical integration methods, in general, are either stochastic and simulation-based methods such as Monte Carlo, or deterministic such as many numerical integration methods like Simpson's Rule (Gelman et al., 2013).

We propose a simulation – summation method for calculating the posterior expectations, which is based on generated random samples from the prior distribution. We denote that desired accuracy of the simulation can be achieved by generating more samples. Description of the proposed simulation – summation method is presented below.

The empirical Bayes estimate of λ_{ji} is the mean of the posterior distribution,

$$\hat{\lambda}_{ji} = E(\lambda_{ji} | N_{1i}, N_{2i}), \qquad \text{for } j = 1, 2\ ,\ i = 1, 2, \cdots, m,$$

where

$$E(\lambda_{ji}|N_{1i},N_{2i}) = \frac{\int_0^\infty\int_0^\infty \lambda_{ji} \times c(\lambda_{1i},\lambda_{2i}) \times Pois(\lambda_{1i}) \times Pois(\lambda_{2i})\ d\lambda_{1i}d\lambda_{2i}}{\int_0^\infty\int_0^\infty c(\lambda_{1i},\lambda_{2i}) \times Pois(\lambda_{1i}) \times Pois(\lambda_{2i})\ d\lambda_{1i}d\lambda_{2i}}$$
$$= \frac{I_{ji}}{NC_i}.$$

The proposed simulation – summation method refers to the calculation of I_{ji} and NC_{ji} as follows,

$$I_{ji} = \int_0^\infty\int_0^\infty [\lambda_{ji} \times Pois(\lambda_{1i}) \times Pois(\lambda_{2i})] \times c(\lambda_{1i},\lambda_{2i})\ d\lambda_{1i}d\lambda_{2i}$$
$$= \frac{1}{k}\sum^k \left(\lambda_{jk} \times \exp^{-\lambda_{1k}} \cdot \frac{\lambda_{1k}^{n_{1i}}}{n_{1i}!} \times \exp^{-\lambda_{2k}} \cdot \frac{\lambda_{2k}^{n_{2i}}}{n_{2i}!}\right)$$
$$= \frac{1}{k}\sum^k (\lambda_{jk} \times Pois(\lambda_{1k}) \times Pois(\lambda_{2k}))$$

$$NC_i = \int_0^\infty\int_0^\infty [Pois(\lambda_{1i}) \times Pois(\lambda_{2i})] \times c(\lambda_{1i},\lambda_{2i})\ d\lambda_{1i}d\lambda_{2i}$$
$$= \frac{1}{k}\sum^k \left(\exp^{-\lambda_{1k}} \cdot \frac{\lambda_{1k}^{n_{1i}}}{n_{1i}!} \times \exp^{-\lambda_{2k}} \cdot \frac{\lambda_{2k}^{n_{2i}}}{n_{2i}!}\right)$$
$$= \frac{1}{k}\sum^k (Pois(\lambda_{1i}) \times Pois(\lambda_{2i}))$$

by simulating a large enough number of data (k) from the updated prior distribution.

Given that there is a pool of m bivariate Poisson Process data (n_{1i}, n_{2i}) and the empirical Bayes estimates of the λ_{1i} and λ_{2i} is I_{1i}/NC_i and I_{2i}/NC_i, respectively. The simulation algorithm of this method can be summarised by the following steps:

1. Generate k samples of $(\lambda_1, \lambda_2)^\star$ from updated prior.

2. For every pair (n_{1i}, n_{2i}) within the pool, calculate

$$\lambda_{ji} \times Pois(\lambda_{1i}) \times Pois(\lambda_{2i})$$

$$Pois(\lambda_{1i}) \times Pois(\lambda_{2i}).$$

for all k pairs $(\lambda_1, \lambda_2)^{\star}$.

3. Calculate the average,

$$I_{ji} = \frac{1}{k}\sum \left(\lambda_{jk} \times Pois(\lambda_{1k}) \times Pois(\lambda_{2k})\right)$$

$$NC_i = \frac{1}{k}\sum \left(Pois(\lambda_{1k}) \times Pois(\lambda_{2k})\right).$$

3.6 Summary

In this chapter, we specified the modelling context of this research. We defined our proposed empirical Bayesian with Clayton copula model and the classic empirical Bayesian model within the same context. We presented the method of Moment estimation (MME) which will be used for estimating the Gamma marginal parameters, (α_i, β_i). Moreover, we discussed how challenging is to obtain the posterior expectations analytically. Therefore, we proposed a Monte Carlo simulation - summation method to derive the posterior expectations. We also provided the simulation algorithm of the method.

Finally, we addressed that estimating the dependence parameter of the Clayton prior distribution can be challenging, as there is no relevant literature of methods or techniques that can be used within this context. Hence, we create an estimation method based on simulations for estimating Clayton's copula Kendall's tau. We present our proposed method in detail in the next chapter.

Chapter 4

Estimating Kendall's Tau of the Prior Clayton Copula

4.1 Introduction

In empirical Bayesian methodological framework, all prior parameters, i.e. the Gamma marginal parameters and the dependence parameter of the Clayton copula, need to be estimated by using the empirical data available. Well known methods can be used for estimating the Gamma marginal parameters, e.g. MME and MLE. In contrast, the estimation of the dependence parameter of the copula through count Poisson process can be challenging. In our case where Clayton copula is being investigated, the dependence parameter θ and Kendall's tau are associated by a closed-form formula (Nelsen, 2007), $\tau = \theta/(\theta+2)$, so by estimating Kendall's tau, the dependence parameter can be obtained.

Therefore, we develop models for predicting Kendall's tau of the prior Clayton by using the empirical data available. We also aim to provide closed - form expressions as a function of the count data realised from Poisson processes. Our findings are derived from a simulation study conducted considering multiple scenarios and relevant parameters, e.g. sample size, exposure time etc.

In this chapter, we explicitly discuss the parametric forms of the proposed models (see Section 4.2), and present the analytical proposed expression for obtaining an estimate of the Clayton Kendall's tau (see Section 4.4). Analytical discussion of our analysis of data and the simulation study is presented in Section 4.3. For further discussion see Appendix A.

4.2 Parametric Form of the Model

This method incorporates copulas within an empirical Bayes context, which aims to present and propose a new estimation method for the dependence parameter of the Clayton copula using Poisson process data. We consider conditionally independent bivariate Homogeneous Poisson Processes given their underlying rates but allow for dependency in the epistemic uncertainty of these rates as captured through a Clayton copula with Gamma marginal distributions for the prior distribution. This is discussed in Section 3.2.

Estimating the dependency between the underlying rates based on the observed count data is challenging. The relationship between the rates is obscured by the noise in the data introduced by the Poisson Process. Moreover, the dependency exists between continuously distributed random variables describing the value of the rates; however, the data, i.e. the number of events realised, are discrete and so can be a poor discriminator between low rates which they are likely to result in zero events with or without the presence of dependency.

There is a closed-form formula that describes the relationship between Kendall's tau and the dependence parameter of a bivariate Clayton copula (see Nelsen, 2007). Therefore, we can estimate θ by estimating Kendall's tau of the prior.

$$\tau = \frac{\theta}{\theta + 2} \qquad \Rightarrow \qquad \theta = \frac{2 \cdot \tau}{1 - \tau}.$$

The purpose of this study is to explicate the relationship between key statistics from the data and Kendall's tau from the Clayton copula. Under the assumption that the Gamma marginals are identically distributed ($E(\Lambda_1) = E(\Lambda_2)$ and $Var(\Lambda_1) = Var(\Lambda_2)$)based on findings derived empirically from the simulation study presented in Section 4.3, the relationship between Kendall's tau of Clayton copula and Poisson process data can be described as follows,

$$\tau_{copula} = f\left(\tau_{poisson}\right) + \epsilon \qquad \Rightarrow \qquad \tau_{copula} = A \times \tau_{poisson} + \epsilon \ , \qquad \epsilon \sim N(0, s)$$

where $A = A(\hat{\alpha}, \hat{\beta}, m, t)$ is a function of the marginal parameter estimates $(\hat{\alpha}, \hat{\beta})$, the size of the pool (m) and the exposure time (t). The proposed model considers all parameters involved, so once one knows these parameters can substitute them into the model and obtain an estimate of the dependence parameter of the Clayton copula.

According to our simulation study, function A has an affine relationship with prior mean and can be expressed as, $A(\hat{\alpha}, \hat{\beta}) = a_0 + b_0 \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}}\right)$ where slope (b_0) and intercept (a_0) can be expressed as (power) functions of prior variance,

$$b_0 = a_1 \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{b_1} \ , \qquad a_0 = 1 + a_2 \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{b_2} .$$

Considering that pool size (m) and exposure time (t) can be varied, we express the above coefficients as follows,

$$\begin{aligned} a_1 &= d_0 \cdot m^{d_1} \cdot t^{d_2}, \\ a_2 &= c_0 \cdot m^{c_1} \cdot t^{c_2}, \\ b_1 &= e_0 \cdot m^{e_1} \cdot t^{e_2}, \\ b_2 &= b_2 \quad \text{(not significant change)}. \end{aligned}$$

Summarizing, function A can be expressed as a function of the prior marginal estimates, size of the pool and exposure time, as follows,

$$\begin{aligned}
A(\hat{\alpha},\hat{\beta},m,t) =& a_0 + b_0 \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}}\right) \\
=& \left[1 + a_2 \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{b_2}\right] + \left[a_1 \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{b_1}\right] \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}}\right) \\
=& \left[1 + (c_0 \cdot m^{c_1} \cdot t^{c_2}) \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{b_2}\right] + \left[(d_0 \cdot m^{d_1} \cdot t^{d_2}) \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{e_0 \cdot m^{e_1} \cdot t^{e_2}}\right] \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}}\right)
\end{aligned}$$

The non-linear model that predicts the Kendall's tau of the Clayton copula considering the Kendall's tau of the Poisson data, the prior marginal estimates, the pool size and the exposure time is,

$$\tau_{copula} = \left\{ 1 + c_0 \cdot m^{c_1} \cdot t^{c_2} \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{b_2} + d_0 \cdot m^{d_1} \cdot t^{d_2} \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^2}\right)^{e_0 \cdot m^{e_1} \cdot t^{e_2}} \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}}\right) \right\} \times \tau_{poisson} + \epsilon$$

where error, $\epsilon \sim N(0,s)$, and $b_2, c_0, c_1, c_2, d_0, d_1, d_2, e_0, e_1, e_2$ are the unknown coefficients of the model.

The final step of this method involves the specification of the error, (ϵ). We define error/residual as the difference between "real" and "predicted" value of Kendall's tau of Clayton copula. The residuals are normally distributed with zero mean and standard deviation s. We define standard deviation, s, as the Root Mean Squared Error ($RMSE$)of residuals,

$$\begin{aligned}
RMSE =& \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\text{Residual}_i)^2} \\
=& \sqrt{\frac{1}{n}\sum_{i=1}^{n} (\text{Real}_i - \text{Predicted}_i)^2}
\end{aligned}$$

where n is the sample size. We aim to present a non-linear model that predicts $RMSE$ of residuals considering the prior marginal estimates, pool size and exposure time. $RMSE$ can be expressed as,

$$RMSE = a \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}}\right)^{e} \cdot \left(\frac{\hat{\alpha}}{\hat{\beta}^{2}}\right)^{d} \cdot m^{b} \cdot t^{c} + \epsilon_{R}$$

where $\epsilon_R \sim$ Normal distribution and it is defined as the difference between "real" and "estimated" $RMSE$ of the residuals.

4.3 Analysis of Data - Simulation Study for Estimating Kendall's Tau

4.3.1 Overview

In this section, we report the results of the simulation study conducted for estimating Kendall's tau of the Clayton copula. We aim to examine the relative accuracy of the estimates obtained using the proposed non-linear model. Notably, we compare the results obtained under different situations, considering all parameters of the model, with which we distinguish cases where relatively more accurate results are expected by identifying the best and worst-case scenarios. We initially examine the model performance by using the theoretical marginal parameters which imply the classic Bayes method; however, we then consider the empirical Bayes model by substituting the prior marginal estimates obtained by using the method of moment estimation.

The simulation process followed for this study can be summarised as follows. We initially define the prior distribution with fixed Gamma marginal parameters (under the assumption that both marginals are identically distributed) and a fixed dependency parameter. We then simulate data from the fully defined prior distribution, and, we simulate data from the Likelihood. The next step involves the calculation of the moment estimators for the Gamma marginals. Knowing that the estimators have to be positive since $\alpha, \beta \in (0, \infty)$, we need to check these values before we proceed. If they are positive, then the sample is accepted for further analysis. If not, the sample is rejected as it shows data underdispersion, where the variance is smaller than the mean. If the sample is accepted, we proceed to the final step, in which we calculate Kendall's tau of the prior and the Poisson data for further analysis. The methodological framework followed for this study is presented in this chapter. Figure 4.3.1 illustrates a flowchart of the simulation study and presents step by step all the stages followed throughout the process.

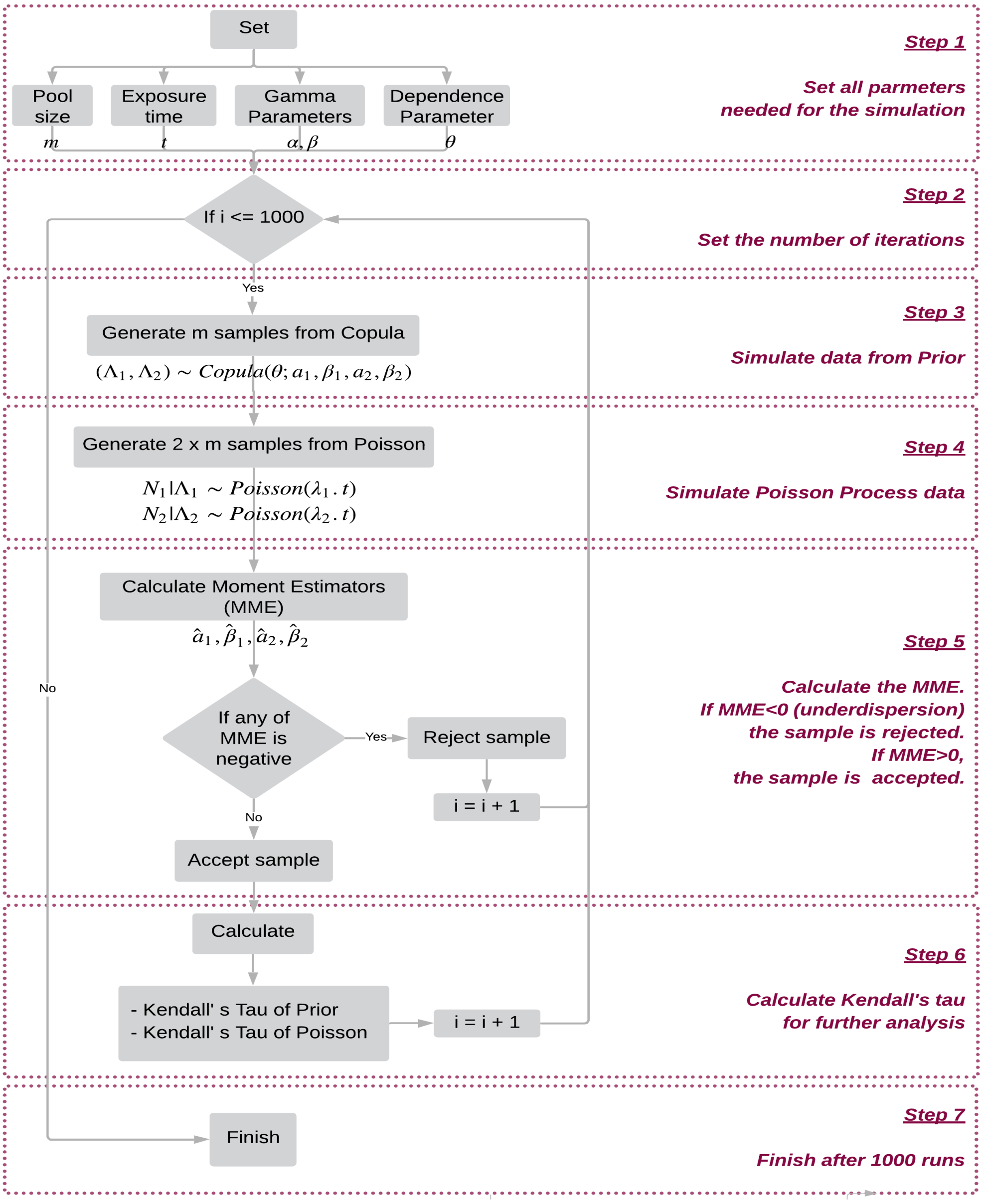


Figure 4.3.1: Showing methodology followed for the simulation study conducted for estimating the Kendall's tau of Clayton copula.

In the following, we present the parameters chosen for the study, and we discuss how different marginal choices may affect the relationship between Kendall's tau of the prior and the Poisson data. We also present the closed - form expressions derived after fitting the proposed model using both theoretical and estimated prior parameters.

4.3.2 Chosen Parameters

Sixteen (16) different sets of parameters are chosen for the Gamma marginal distributions. These parameters are chosen based on different mean and variance values. We are investigating how the model performs in situations where the marginals' mean is greater than its variance, the marginals' variance is greater than its mean, and both of the marginals have equal mean and variance. Also, we have six (6) different dependence parameters. Value 1 represents relatively weak dependency, and value 30 shows relatively strong dependency. Moreover, we set nine (9) different pool sizes and seven (7) different exposure times. We investigate the model performance when the size of the pool and the exposure time are relatively small (only 20 observations and exposure time equal to 5), and when it is large enough with larger exposure time (100 observations with an exposure time of 20). For every different combination of parameters, we run the simulation process multiple times (1000). All parameters chosen for this simulation study are presented in Table 4.3.1.

Runs:	1000
Pool size (m):	20, 30, 40, 50, $\cdots$, 100
Exposure time (t):	5, 7.5, 10, 12.5, $\cdots$, 20
Dependency (θ):	1, 2, 5, 10, 20, 30
Marginals:	
Mean	5, 10, 20, 40
Variance	2, 5, 10, 20
Total Number of datasets:	$16 \times 63 = 1008$ (for every θ)
Total Number of data:	$\simeq 6\,000\,000$

Table 4.3.1: Chosen parameters for this simulation study.

4.3.3 How Different Marginal Parameter Choices Affect the Relationship Between Kendall's Tau of Prior and Poisson Data

To investigate how different marginal parameter choices affect the relationship between Kendall's tau of Prior and Poisson data, we use over 6000 different parameter combinations. These combinations consist of different mean and variance values of the Gamma marginals, different pool sizes, exposure times and dependence parameters. Based on our simulation study, our findings show strong correlation between Kendall's tau of Prior and Poisson data when the variance is high, and the mean is relatively low. Notably, we observe that the best/more prominent and almost perfect positive linear correlation occurs when the variance is 20, and the mean is 5. We also note that this case is considered as the best-case scenario in all different combinations of pool sizes and exposure times.

Moreover, we expect that the model produces more accurate results when the marginal variance is high, and the marginal mean is low. We understand that as we increase the variability within a dataset, is becoming easier to observe the data and the individuals, and consequently to capture the dependency between the variables. On the other hand, as we have already addressed, Clayton copula focuses on left tail dependency. So, we are interested in the dependency of the lower rates, which occurs when the prior mean has relatively low values, i.e. low rates of occurrences. Figure 4.3.2 shows how different marginal parameters affect the correlation between Kendall's tau of Prior and Poisson data. There is a strong positive linear correlation for marginal mean 5 and marginal variance 20; however, when the prior variance decreases and the mean increases, the correlation is less clear and distinct. For the latter, the noise in the data introduced by the Poisson Process obscures the relationship between Kendall's tau of the prior and the data.

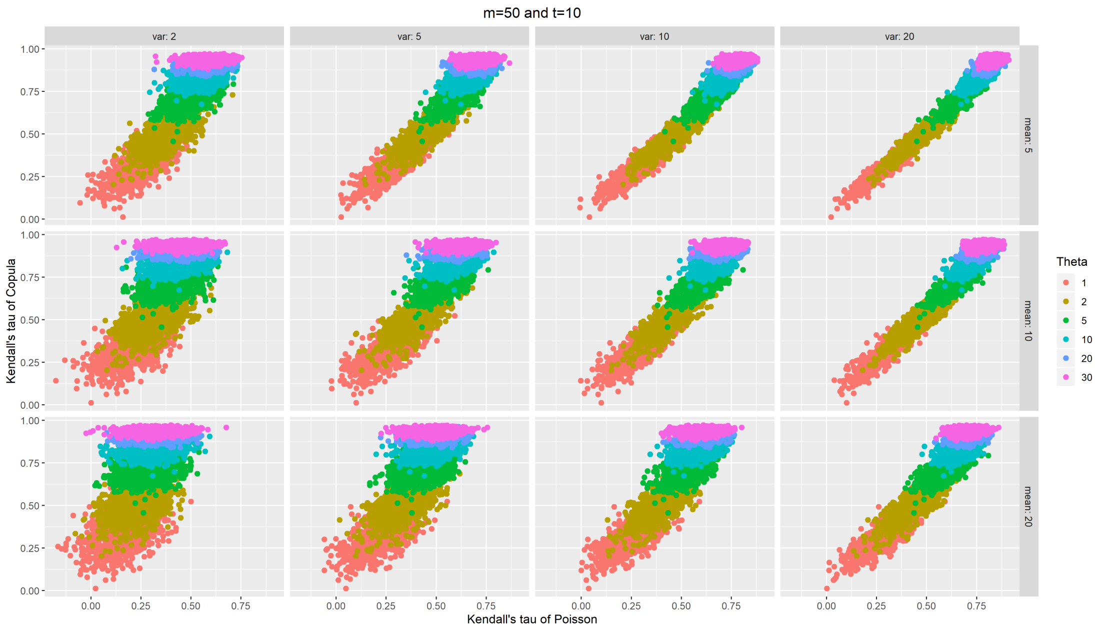


Figure 4.3.2: Showing the effect of different marginal choices on the correlation between Kendall's tau of Prior and Poisson data, with pool size of 50 and exposure time of 10. Different colors represent different dependence parameters.

According to the simulation results, the exposure time parameter also affects the relationship between the two measures. Notably, as the exposure time increases, the correlation between Kendall's tau of Prior and Poisson data is becoming more distinct, and less noise appears in the data. Figure 4.3.3 shows the best-case scenario where the prior mean is low (mean = 5), the prior variance is high (variance = 20), and the size of the pool is large (100 observations). Figure 4.3.4 presents the worst-case scenario where the prior mean is high (mean = 40), the prior variance is low (variance = 2), and the pool size is relatively small (20).

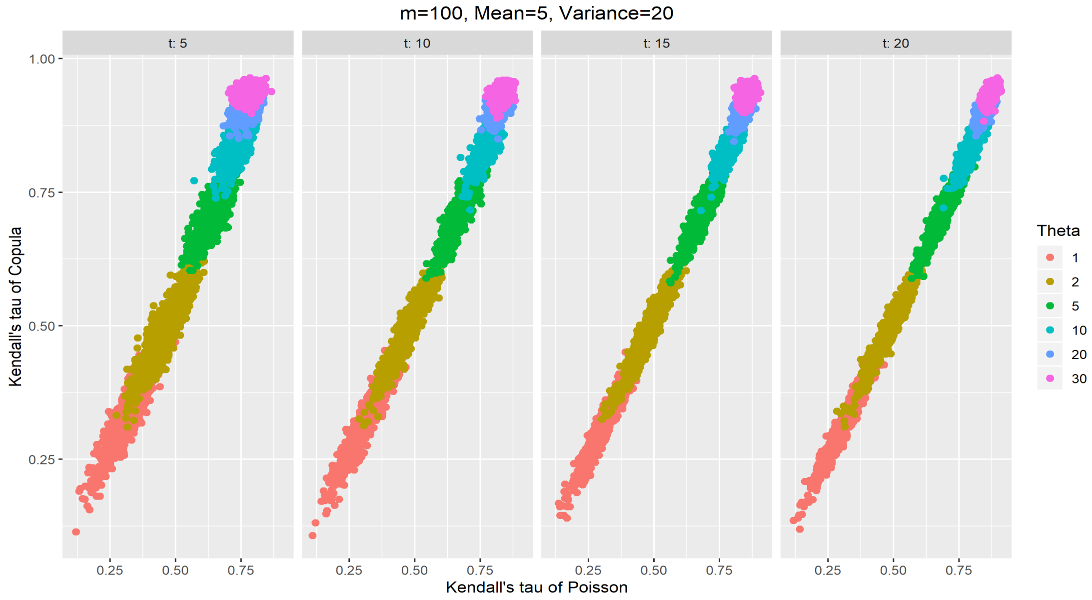


Figure 4.3.3: Showing how different values of exposure time t affect the correlation between Kendall's tau of Prior and Poisson data. We show the best-case scenario where mean=5, variance=20 and pool size=100. Different colors represent different dependence parameters.

Figure 4.3.4: Showing how different values of exposure time t affect the correlation between Kendall's tau of Prior and Poisson data. We show our worst-case scenario where mean=40, variance=2 and pool size=20. Different colors represent different dependence parameters.

Moreover, we denote the influence of the pool size on the correlation between Kendall's tau of Prior and Poisson data. As the size of the pool increases the linear correlation between Kendall's tau of Prior and Poisson data becomes more distinct, compared to smaller pool sizes. Figures 4.3.5, 4.3.6 show the effect of the pool sizes and present the best-case and worst-case scenario, respectively.

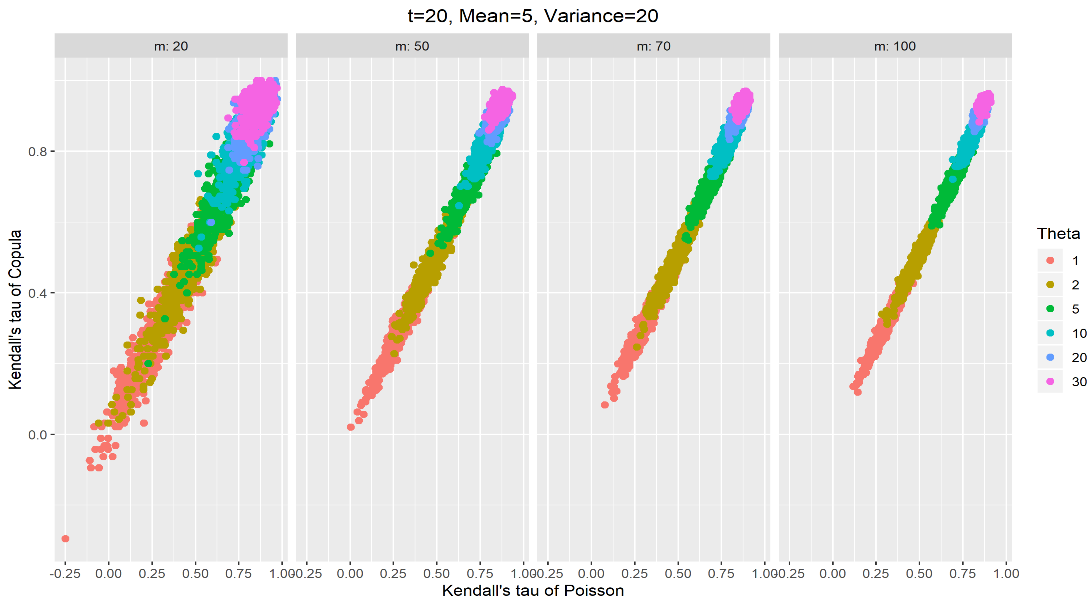


Figure 4.3.5: Showing how different pool sizes m affect the correlation between Kendall's tau of Prior and Poisson data. We show the best-case scenario where mean=5, variance=20 and exposure time=20. Different colors represent different dependence parameters.

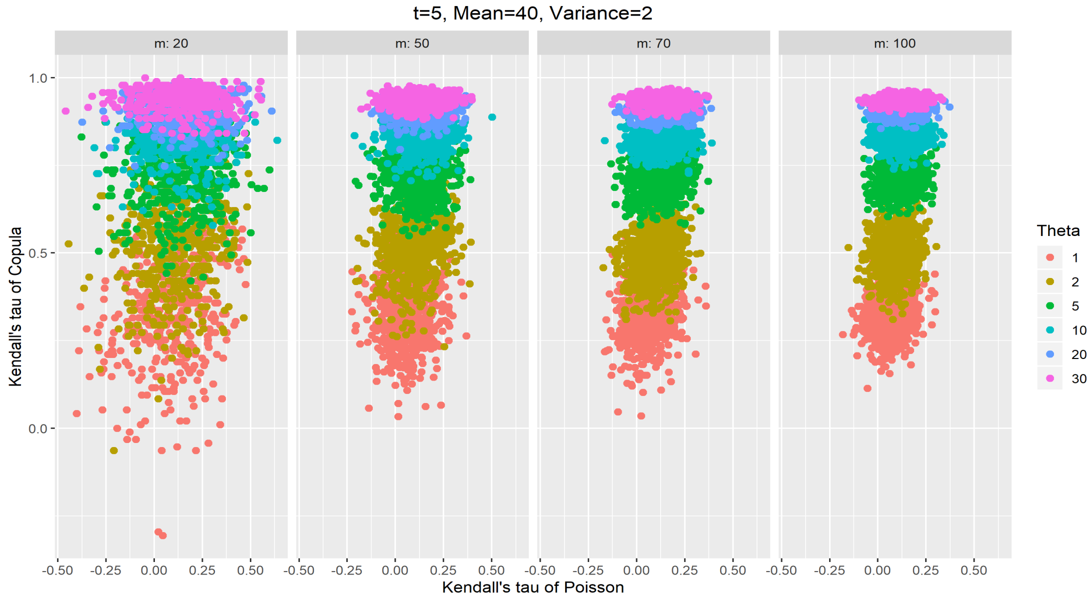


Figure 4.3.6: Showing how different pool sizes m affect the correlation between Kendall's tau of Prior and Poisson data. We show the worst-case scenario where mean=40, variance=2 and exposure time=5. Different colors represent different dependence parameters.

Summarizing, we have explored how different combinations of prior marginal mean and variance values, pool sizes and exposure times affect the relationship between Kendall's tau of prior and Poisson data. We now conclude that all models' parameters influence the relationship between the two measures differently. In particular, we expect more accurate results in cases where the prior variance, the pool size and the exposure time are large and the prior mean is small. In contrast, in cases where the prior variance is small and the prior mean is large, more noise is expected in the count data which will result in less accurate estimates of the prior Kendall's tau. Table 4.3.2 identifies best and worst-case scenarios across the study.

	Cases	
	Worst	**Best**
Mean	High ↗	Low ↙
Variance	Low ↙	High ↗
Pool Size	Low ↙	High ↗
Exposure Time	Low ↙	High ↗

Table 4.3.2: Showing best and worst case scenario considering all model parameters.

4.3.4 Model Fit

We now examine how the proposed non-linear model discussed in Section 4.2 performs in different situations. Firstly, we fit the model by using the theoretical marginal parameters and then using the estimates. By doing so, we imply the classic Bayesian model as opposed to the empirical Bayes model. The main difference between the two models, empirical and classic Bayes, is that for the latter we are not using any observed data for estimating and updating the prior distribution; instead, we are using the already known and fixed marginal parameters. Therefore, we compare both model fits. In the following, we present the model fit results and discuss the significance of the model coefficients, the overall error and the residuals.

Tables 4.3.3, 4.3.4 summarise the results of the model fit and present the estimated coefficients. We denote that the proposed model performs significantly well. The residual standard error is 0.1379, 24 iterations needed to convergence, and all coefficients are significant with a p-value less than $2e^{-16}$. It is also worth noting that coefficient $d0$ has a relatively small value, therefore ignoring it does not significantly affect the outcome of the model. However, we decide to keep all parameters in this study.

Formula:	$tau.cop \sim \left(1 + c0 \cdot m^{c1} \cdot t^{c2} \cdot \left(\frac{a}{b^2}\right)^{b2} + \right.$
	$\left. + d0 \cdot m^{d1} \cdot t^{d2} \cdot \left(\frac{a}{b^2}\right)^{e0 \cdot m^{e1} \cdot t^{e2}} \cdot \left(\frac{a}{b}\right)\right) \times$
	$\times\, tau.pois$
Residual standard error:	0.1379
Number of iterations to convergence:	24

Table 4.3.3: Model fit results.

	Estimate	Std. Error	t value	Pr(> |t|)
c0	4.53	0.06	77.62	0.00
c1	-0.13	0.00	-46.61	0.00
c2	-0.79	0.00	-258.11	0.00
b2	-0.79	0.00	-368.26	0.00
d0	0.06	0.00	129.15	0.00
d1	0.42	0.00	262.57	0.00
d2	-0.58	0.00	-367.50	0.00
e0	-0.27	0.00	-166.45	0.00
e1	0.18	0.00	149.60	0.00
e2	0.11	0.00	94.66	0.00

Table 4.3.4: Coefficients of model fit.

Figures 4.3.7, 4.3.8 show a comparison of Kendall's tau predictions against the actual prior Kendall's tau for the worst-case and best-case scenario, respectively. We acknowledge that there are predictions which exceed the upper and lower limit of Kendall's tau range, while $\tau \in (0,1)$. Therefore, we suggest all values exceeding this range to be considered as 0 and 1, respectively.

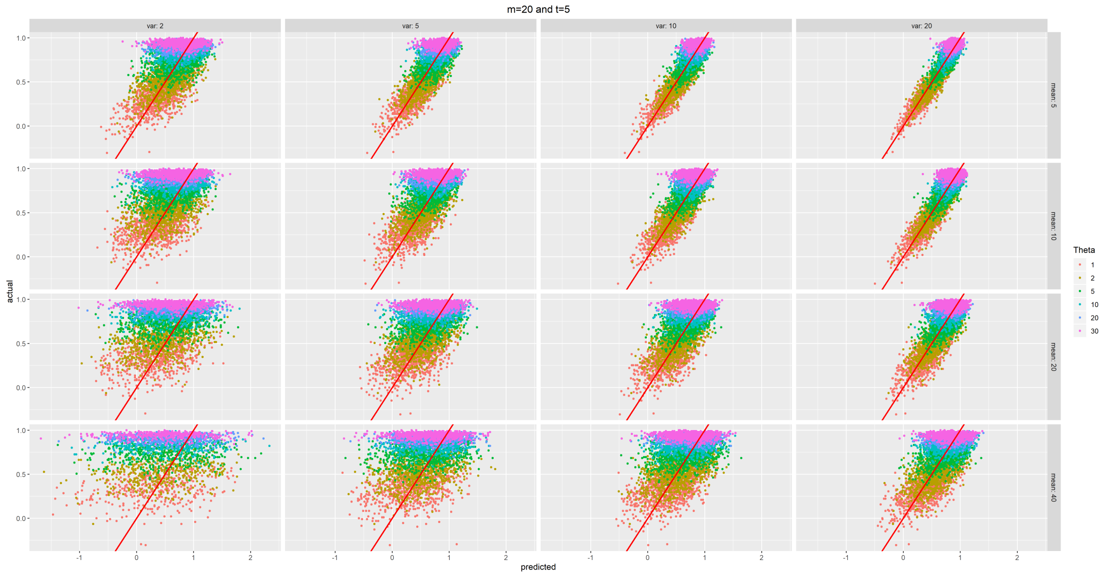


Figure 4.3.7: Showing the 'Actual' and 'Predicted' values of Kendall's tau in the worst-case scenario, where pool size is only 20 ($m = 20$) and exposure time $t = 5$. Different colours represent different values of the dependency.

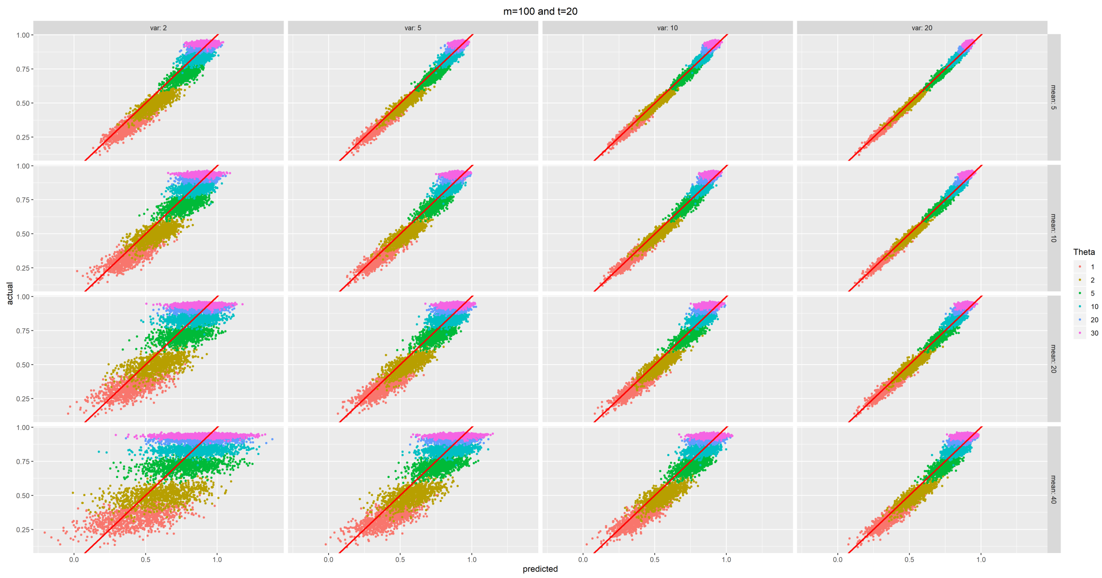


Figure 4.3.8: Showing the 'Actual' and 'Predicted' values of Kendall's tau in the best-case scenario, where pool size is 100 ($m = 100$) and exposure time $t = 20$. Different colours represent different values of the dependency.

Regarding the residuals, which are defined as the difference between real and estimated Kendall's tau values, we create QQ-plots against Normal distribution for all different sets of parameters chosen. Figures 4.3.9, 4.3.10 show that the model residuals are normally distributed even in the worst-case scenario (mean= 40, variance= 2, $m = 20$ and $t = 5$).

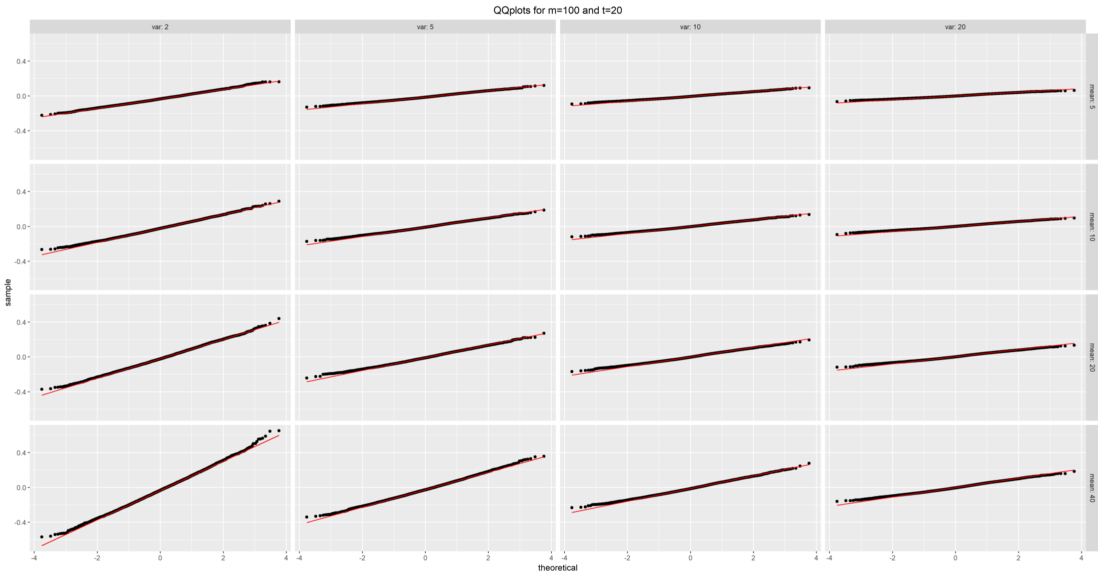


Figure 4.3.9: Showing that residuals are normally distributed, even in the worst-case scenario where mean= 40, variance= 2, $m = 20$ and $t = 5$.

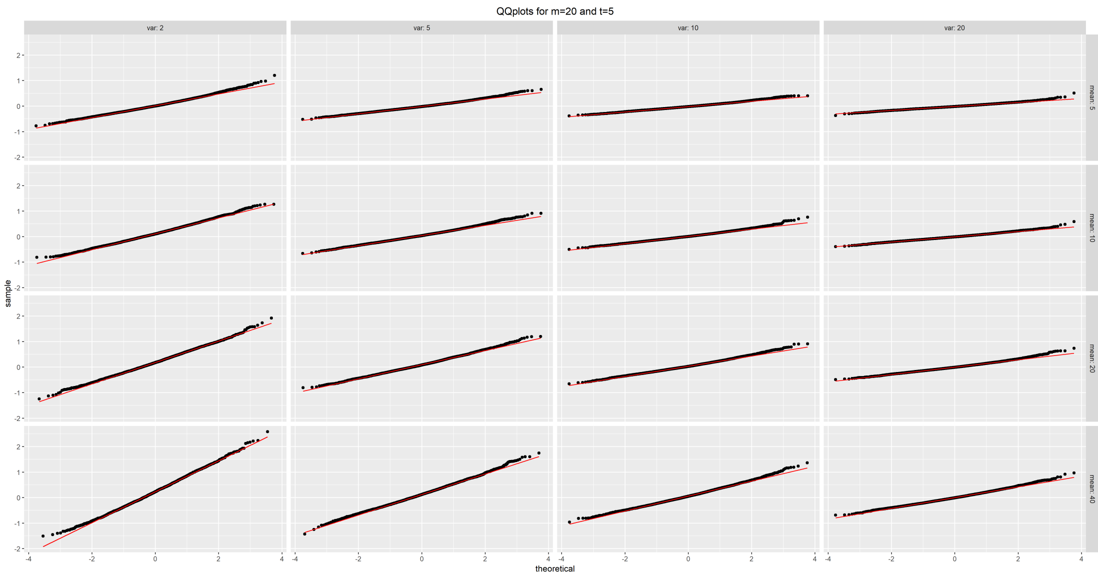


Figure 4.3.10: Showing that residuals are normally distributed. Best-case scenario where mean= 40, variance= 2, $m = 20$ and $t = 5$.

Additionally, we compare the distributions of the residuals between the best and worst case scenarios. Figures 7.3.5, 7.3.6 present the box-plots of residuals for the best and worst case of this study. In both cases, we notice that as the variance increases, the residuals' mean approximates zero, and the standard deviation of the estimate errors decreases, which indicates more certain estimates.

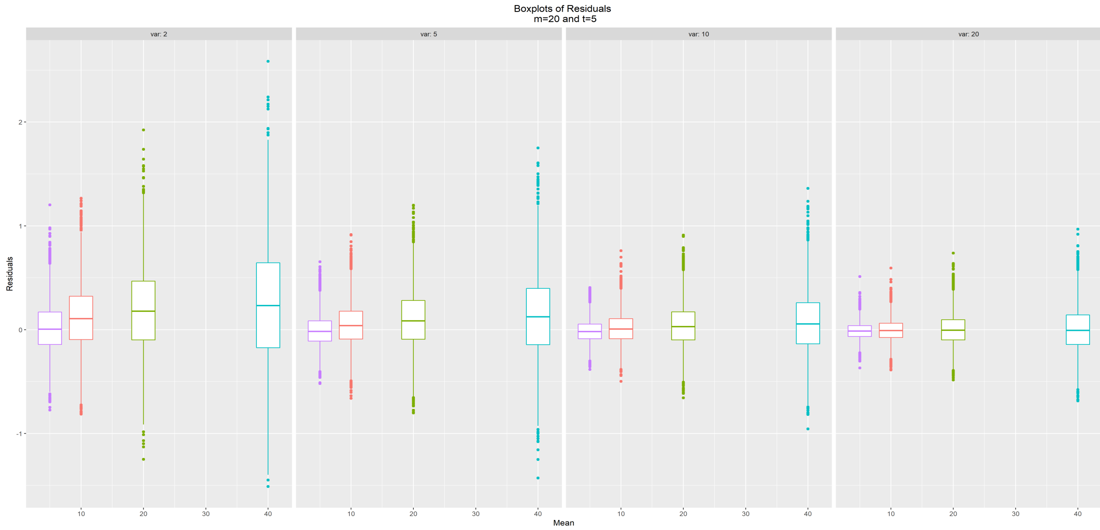


Figure 4.3.11: Multiple box-plots showing the distribution of residuals for the worst-case scenario, where mean= 40, variance= 2, $m = 20$ and $t = 5$.

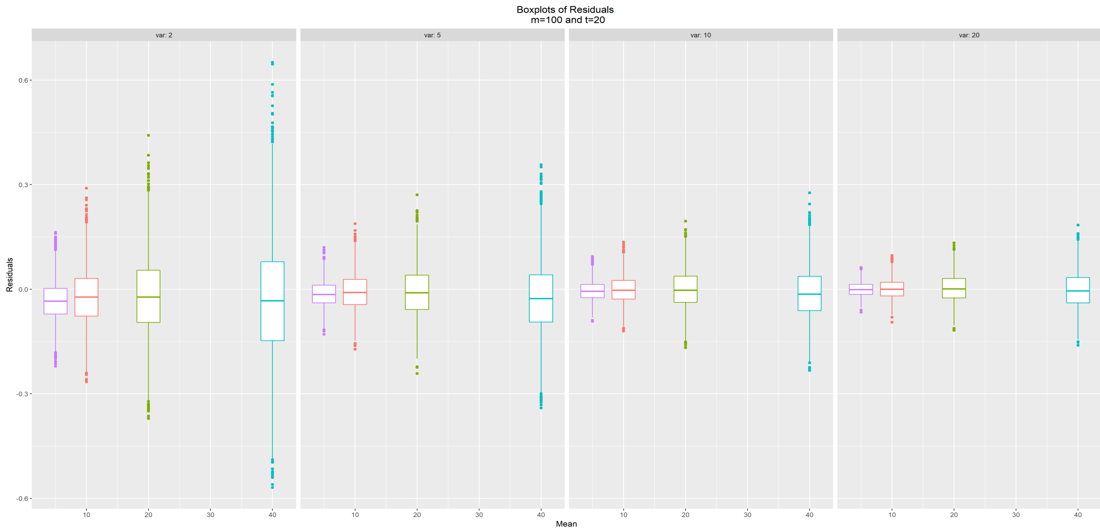


Figure 4.3.12: Multiple box-plots showing the distribution of residuals for the best-case scenario, where mean= 40, variance= 2, $m = 20$ and $t = 5$.

4.3.5 Comparison Between Classic and Empirical Bayes Model Fit

Within the empirical Bayesian methodology, we estimate and update the prior by using the observed Poisson process data; however, this is not the case for the classic Bayes method. We initially explore how the proposed model performs using the theoretical priors, which implies the full Bayesian model, and further investigate the empirical case in which the prior estimates replace the theoretical settings. This is particularly important, as making use of the Poisson process data available for estimating the copula dependence parameter is one of the motivations for this study.

To examine the performance of the proposed model using the estimated prior parameters of each sample obtained from the Method of Moments, we substitute the theoretical mean and variance values with the estimations. The overall residual standard error is 0.1384, the number of iterations needed to convergence is 27, and all coefficients are significant with p-value less than $e2^{-16}$, suggesting that the model performs significantly well. Table 4.3.5 summarizes information relevant to the model performance.

	Estimate	**Std. Error**	**t value**	**Pr(>|t|)**
c0	3.08	0.03	91.24	0.00
c1	-0.02	0.00	-7.53	0.00
c2	-0.82	0.00	-305.94	0.00
b2	-0.73	0.00	-470.67	0.00
d0	0.02	0.00	139.80	0.00
d1	0.49	0.00	333.18	0.00
d2	-0.41	0.00	-279.38	0.00
e0	-0.12	0.00	-153.24	0.00
e1	0.27	0.00	210.62	0.00
e2	0.24	0.00	207.82	0.00

Table 4.3.5: Non-linear model fit results using prior marginal estimated parameters.

We now compare the model fit results obtained from the classic and empirical Bayes. Table 4.3.6 compares both models by summarising the coefficient parameters, the residuals standard error and the number of iterations needed for convergence. We notice that both models produce similar results. Specifically, regarding the model coefficients, all are significant for both models; the residual standard errors and the number of iterations needed to convergence are also similar. Findings suggest that by making use of the empirical data available, we are able to obtain accurate estimates of the Kendall's tau for the Clayton copula formed as a prior distribution within an empirical Bayesian framework.

	Classic Bayes	**Empirical Bayes**
	Coef. Estimate	**Coef. Estimate**
c0	4.53	3.08
c1	-0.13	-0.02
c2	-0.79	-0.85
b2	-0.79	-0.73
d0	0.06	0.02
d1	0.42	0.49
d2	-0.58	-0.41
e0	-0.27	-0.12
e1	0.18	-0.27
e2	0.11	0.24
Residual Standard Error:	0.1379	0.1384
Iterations to convergence:	24	27

Table 4.3.6: Comparison between classic and empirical Bayes model fit.

4.4 Model Formula

In this section, we present the analytical expressions one can use to estimate Kendall's tau of the Clayton copula with Gamma marginals (bivariate case) when dealing with count data assuming that are generated from Poisson processes. All parameters needed for this formula can be easily obtained by using only the Poisson data at hand. Detailed discussion about the analysis of data, along with relevant results and conclusions can be found in Section 4.3 and Appendix A. Our proposed formula along with all parameters needed are as follows.

$$\tau_{\text{Clayton}} = \Bigg\{ 1 + 3.08 \cdot m^{-0.02} \cdot t^{-0.85} \cdot \left(\frac{\hat{a}}{\hat{b}^2}\right)^{-0.73} + \\ + 0.02 \cdot m^{0.49} \cdot t^{-0.41} \cdot \left(\frac{\hat{a}}{\hat{b}^2}\right)^{-0.12 \cdot m^{-0.27} \cdot t^{0.24}} \cdot \left(\frac{\hat{a}}{\hat{b}}\right) \Bigg\} \times \tau_{\text{poisson}} + \epsilon$$

where,

ϵ : Error, $\epsilon \sim N(0, s)$,

$$s = 1.17 \cdot m^{-0.41} \cdot t^{-0.53} \cdot \left(\frac{\hat{a}}{\hat{b}}\right)^{0.52} \cdot \left(\frac{\hat{a}}{\hat{b}^2}\right)^{-0.53},$$

m : pool size,

t : exposure time,

$\hat{a}$: shape parameter estimate of the Gamma marginals,

$\hat{b}$: rate parameter estimate of the Gamma marginals.

4.5 Summary

In this chapter, we defined our proposed non - linear regression model for predicting Kendall's tau of the prior Clayton. We also presented a non - linear regression model for predicting the Root Mean Squared Error of the predictions, ensuring that our model proposed is accurate even in the worst-case scenario. We discussed our methodology and presented the parametric form of the model addressing all parameters available, i.e. the exposure time, the pool size, and the Gamma marginal parameters. We discussed our analysis of data and presented relevant findings derived from a simulation study conducted considering multiple possible scenarios and parameters. Further discussion of the simulation study in relation to the model fit can be found in Appendix A.

Our findings suggest an affine relationship between Kendall's tau of the prior and the Poisson data. We observe a strong relationship in cases where the marginal variance, the pool size and the exposure time are large and the marginal mean is small. In contrast, a weak relationship occurs in cases where the marginal variance is small and the marginal mean is large, as more noise is introduced in the count data. From this relationship, expressions have been derived to enable the dependency measure of the Clayton copula on a prior to be estimated from Poisson process data. These expressions will be used in the next chapter for obtaining an estimate for the Clayton dependence parameter. Particularly, in Chapter 5, we will present our simulation and benchmarking studies for evaluating the empirical Bayesian with Clayton model.

Chapter 5

Inference Methods and Evaluation

5.1 Intoduction

An empirical Bayesian inference method combined with Clayton copula is developed for estimating multiple correlated event rates. To examine our proposed model's relative accuracy and investigate how good the assessments of mean event rates are when using a moment-based inference approach, we conduct a simulation study. Doing so allows us to assess the impact of different parameter choices, i.e. weak or strong dependency between rates, size of the pool and exposure time; and explore cases where relatively more accurate results are expected. Moreover, to support comparison between our proposed model and existing theory, we consider the classic empirical Bayes method assuming that the prior distributions of the rates are Gamma. We also investigate what happens if we choose to ignore dependency and perform a classic empirical Bayes method (Poisson - Gamma probability model); and what are the consequences, if there are any. We then identify cases where the consequences of our choice are more significant than others.

This chapter presents the simulation and benchmarking studies conducted for evaluating the empirical Bayes model proposed. The remainder of this chapter is structured as follows. Section 5.2 details the simulation study. In particular, Section

5.2.1 discusses the design of the study, Section 5.2.2 explains the algorithm, and Section 5.2.3 discusses the simulation results. Section 5.3 provides a detailed discussion of the benchmarking study. Section 5.4 summarizes this chapter.

5.2 Simulation Study

We design and conduct a simulation study to examine the relative accuracy of the model discussed in Section 3.2 and investigate how good the assessments of mean rates are when using a moment-based inference approach within an empirical Bayes method assuming dependency between the rates. The simulation study allows us to assess the impact of different parameter choices, i.e. weak or strong dependency between the rates, for a selection of different pool sizes and exposure times. Moreover, we simulate data from the empirical Bayes with Clayton copula model with known parameters. This will allow us to see how well the model approximates the real correlated rates in each case and more importantly, in which cases the model performs relatively better.

5.2.1 Simulation Design

Sixteen (16) different sets of parameters were chosen for the Gamma marginal distributions. These parameters were chosen based on different mean and variance values. We are interested to investigate how the model performs in situations where the marginal mean is greater than the variance, the marginal variance is higher than the mean, and both of the marginals have equal mean and variance. Also, we choose to have six (6) different values for the dependence parameter θ. Value 1 represents weak dependency, and value 30 shows strong dependency. Moreover, we set five (5) different pool size values and seven (7) different exposure time values. We are interested to see how our model performs when the size of the pool is relatively small (only 20 observations) and when it is large enough (100 observations). Finally, for every combination of parameters discussed above, we run the simulation process 1000 times. All parameters chosen for this simulation study are presented in tables 5.2.1 and 5.2.2.

Runs:	1000
Pool size (m):	20, 40, 60, 80, 100
Exposure time (t):	5, 7.5, 10, 12.5, $\cdots$, 20
Dependency (θ):	1, 2, 5, 10, 20, 30
Marginals:	
Mean	5, 10, 20, 40
Variance	2, 5, 10, 20
Total Number of sets:	$16 \times 35 = 560$ (for every θ)

Table 5.2.1: Chosen parameters for the simulation study.

		Prior Mean			
		5	**10**	**20**	**40**
Prior variance	**2**	(12.5, 2.5)	(50, 5)	(200, 10)	(800, 20)
	5	(5, 1)	(20,2)	(80, 4)	(320, 8)
	10	(2.5, 0.5)	(10, 1)	(40, 2)	(160, 4)
	20	(1.25, 0.25)	(5, 0.5)	(20, 1)	(80, 2)

Table 5.2.2: Chosen Gamma marginal parameters (shape, rate) for the simulation study.

The simulation process adopted for this study can be described as follows. Initially, we define the prior distribution with the chosen Gamma marginal parameters (under the assumption both marginals are identically distributed) and a fixed dependency parameter. We set the size of the pool and the exposure time. We then generate data from the fully defined prior distribution and the Likelihood (HPP). The next step involves the calculation of the moment estimators of the Gamma marginal parameters considering the observed data at hand; in our case, these are generated Poisson process data. Considering that the Gamma parameters, shape and rate, are defined to be positive, thus the moment estimators of these parameters need to be positive. If the moment estimators are positive, then the sample is considered accepted. However, if any of the estimated parameters is negative, the sample is considered rejected. In this situation, we are unable to use the moment estimators to update the prior and calculate the empirical Bayes estimates through the Posterior, as the generated sample shows data underdispersion, where the variance is smaller than the mean. Thus, our best guess for the rate of event is the average of the number of events realised for each ith event. Nevertheless, if the sample is accepted, we estimate the dependence parameter of the sample, and we proceed to the following step, which includes the prior and posterior specification. Finally, we calculate the empirical Bayes estimates of the rates of events through the posterior distribution using the simulation – summation method discussed in Section 3.5. Figure 5.2.1 illustrates the simulation process discussed.

We consider that the simulated data from the empirical Bayes with Clayton copula model along with the known parameters, which represent the 'real' event rates and the 'real' prior parameters, respectively. We are interested to see how well the empirical Bayes with Clayton copula model approximates the 'real' rates and how close our moment estimators of the Gamma marginals are to the 'real' parameters. As measures of accuracy, we consider the behaviour of the error and squared error. We know that the occurrence rate is λ_{ji} and the prediction is $\hat{\lambda}_{ji}$, so we are interested in the following quantities,

$$e_{ji} = \lambda_{ji} - \hat{\lambda}_{ji} \qquad \text{and} \qquad e_{ji}^2 = (\lambda_{ji} - \hat{\lambda}_{ji})^2.$$

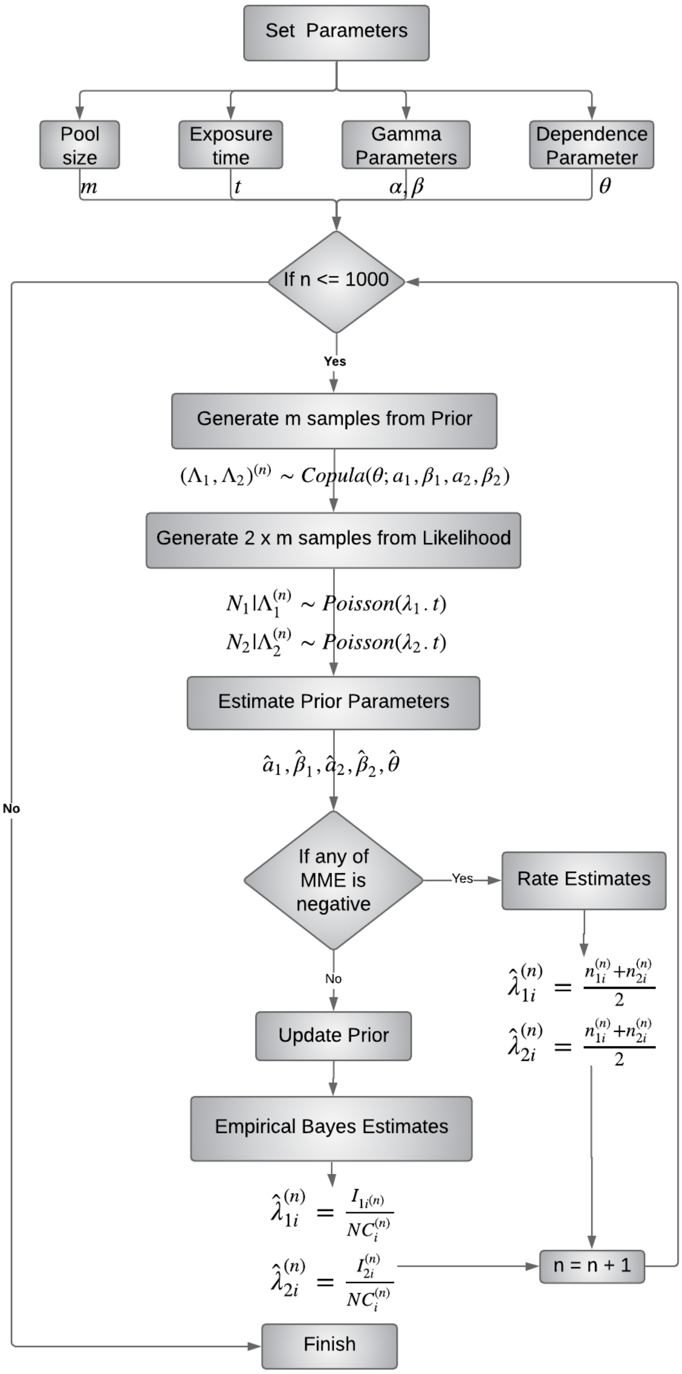


Figure 5.2.1: Diagram showing methodology followed for this simulation study.

5.2.2 Simulation Algorithm

In this section, we present the simulation algorithm adopted for this study. The simulation algorithm can be summarised in the following steps:

1. Set m (pool size), t (exposure time) and θ, α, β (prior parameters).

2. Generate $(\Lambda_1, \Lambda_2)_m$ from Clayton copula (Prior).

3. Generate $(N_1, N_2)_m$ from Poisson $(\Lambda_1 \cdot t, \Lambda_2 \cdot t)$ (Likelihood).

4. Calculate Moment estimators $\hat{\alpha}, \hat{\beta}$.

5. Calculate the dependence parameter estimator $\hat{\theta}$.

6. If $\hat{\alpha}, \hat{\beta} < 0$, calculate rate estimates $\hat{\lambda}_{ji} = (n_{1i} + n_{2i})/2$.

7. If $\hat{\alpha}, \hat{\beta} > 0$, calculate the Empirical Bayes estimates $(\hat{\Lambda}_1, \hat{\Lambda}_2)_m$.

8. Calculate e_{ji} and e^2_{ji}.

9. Repeat for specific number of realisations (n in 1 : 1000).

10. Calculate the following quantities (Bias and Mean Squared Error),

$$Bias = \frac{1}{n \times 2 \times m} \sum_{h=1}^{n} \sum_{j=1}^{2} \sum_{i=1}^{m} e_{ji}{}^{(h)},$$
$$MSE = \frac{1}{n \times 2 \times m} \sum_{h=1}^{n} \sum_{j=1}^{2} \sum_{i=1}^{m} e^2_{ji}{}^{(h)}.$$

5.2.3 Simulation Results

5.2.3.1 Overview

The simulation study is organised as 16 separate blocks. Each block corresponds to a different combination of Gamma marginal parameters (Set 1 to Set 16). Each one of these combinations is examined with different pool sizes, exposure times and dependence parameters as described previously.

In the following, we examine how the number of accepted samples is affected by all different parameter combinations, we evaluate the prior marginal expectations and the empirical Bayes with Clayton copula model errors. As measures of accuracy we consider the bias, the Mean Squared Error (MSE), the Root Mean squared Error (RMSE) and the Root Mean Squared Relative Error (RMSRE) of the errors.

5.2.3.2 Number of accepted samples

We repeat the simulation process over 1000 times for every different combination of parameters chosen. Notably, the number of iterations can be adjusted accordingly to the desired accuracy. In every iteration of the process, we check if the sample is accepted or rejected (due to underdispersion), so we are interested to see how many rejections occur over 1000 iterations and how different pool sizes and exposure times may affect the number of rejected samples. Figure 5.2.3 shows that when the true mean increases, we observe more rejected samples, and as the pool size and exposure time increase, fewer rejections are observed. Particularly, when the true variance is 2, we observe that as the true mean increases from 5 to 40, the number of rejected samples increases dramatically (see Figure 5.2.2). The smallest percentage of accepted samples occurs when the prior mean is relatively high and the prior variance is relatively low. This is not unreasonable. As already mentioned, the Gamma distribution is shifted towards or approximates the shape of the Normal distribution when its mean is relatively large. On the other hand, if the variance is relatively small, it is becoming much harder to identify the individuals within the pool. The individuals are close to each other, suggesting that all have similar rates. Consequently, capturing the dependence between them is becoming even more difficult resulting in the rejection of the sample.

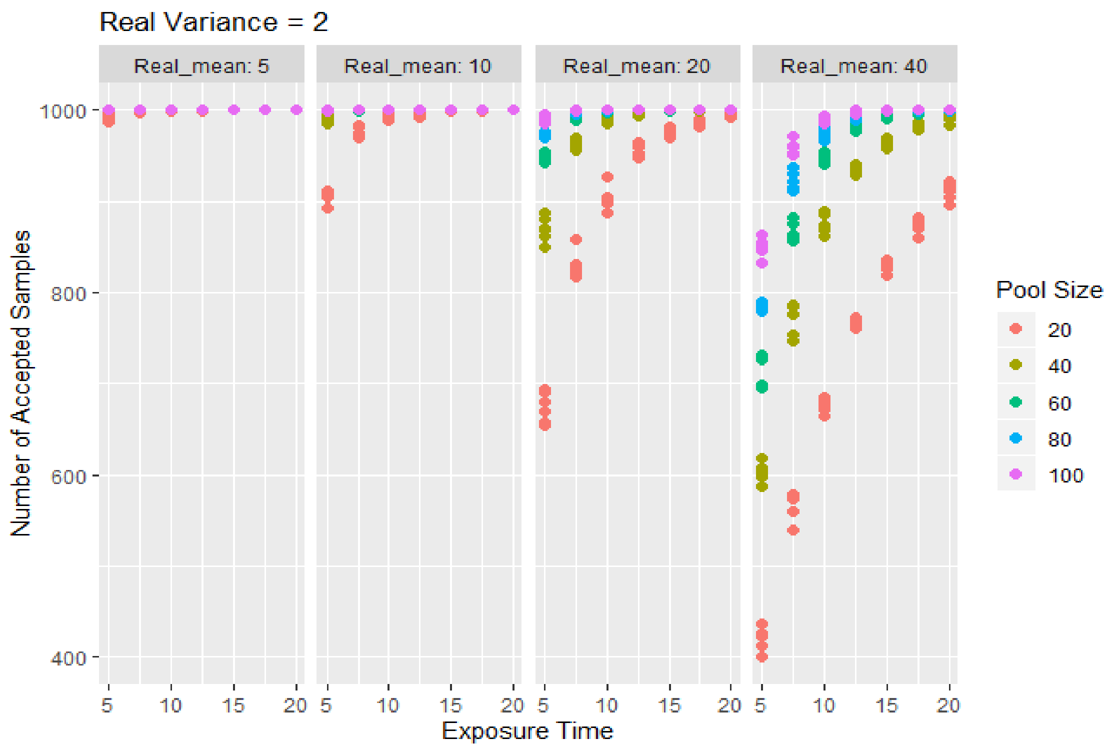


Figure 5.2.2: Showing how different pool sizes, exposure times and marginal mean choices affect the number of accepted samples, where true variance is 2.

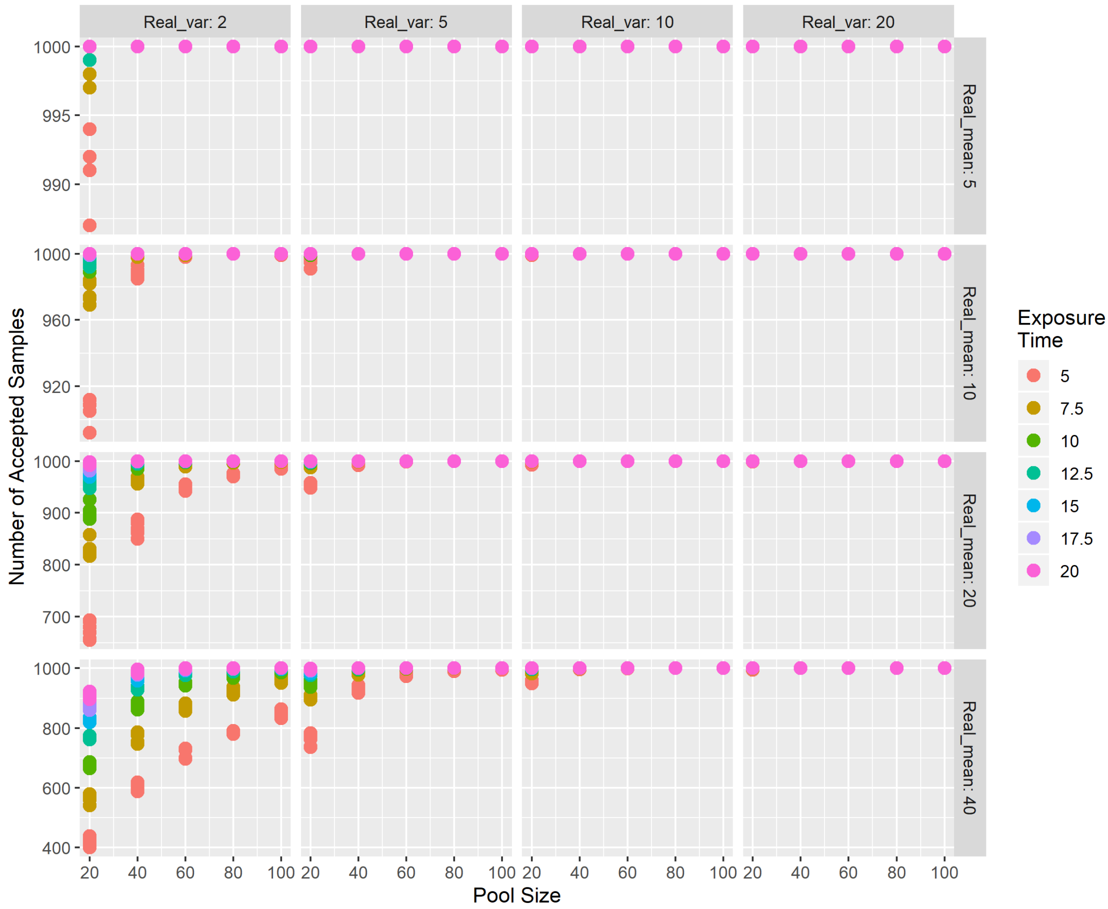


Figure 5.2.3: Showing how different pool sizes, exposure times, mean and variance choices affect the number of accepted samples.

5.2.3.3 Prior mean evaluation

Considering that identically distributed Gamma marginals with known parameters, we are investigating if the moment estimators of the prior mean is approximating the true mean. This is evaluated by calculating the errors between the 'true/real' mean and the estimation for every different combination of parameters. We consider the bias and the Mean Squared Error (MSE) of the mean errors as measures of accuracy. Figures 5.2.4, 5.2.5, 5.2.6, 5.2.7 show the Bias of prior mean errors of different pool size, exposure time, prior mean and dependence parameter combinations when the real variance is 2, 5, 10 or 20, respectively. Visuals show that bias in all cases is relatively small (range from -0.05 to 0.1) indicating the moment estimators are very close to the real values. Notably, bias approximates zero, especially when pool size and exposure time increase. We also notice that when the true variance is smaller (true variance = 2), the bias is relatively smaller rather than when the true variance is getting larger (true variance = 20).

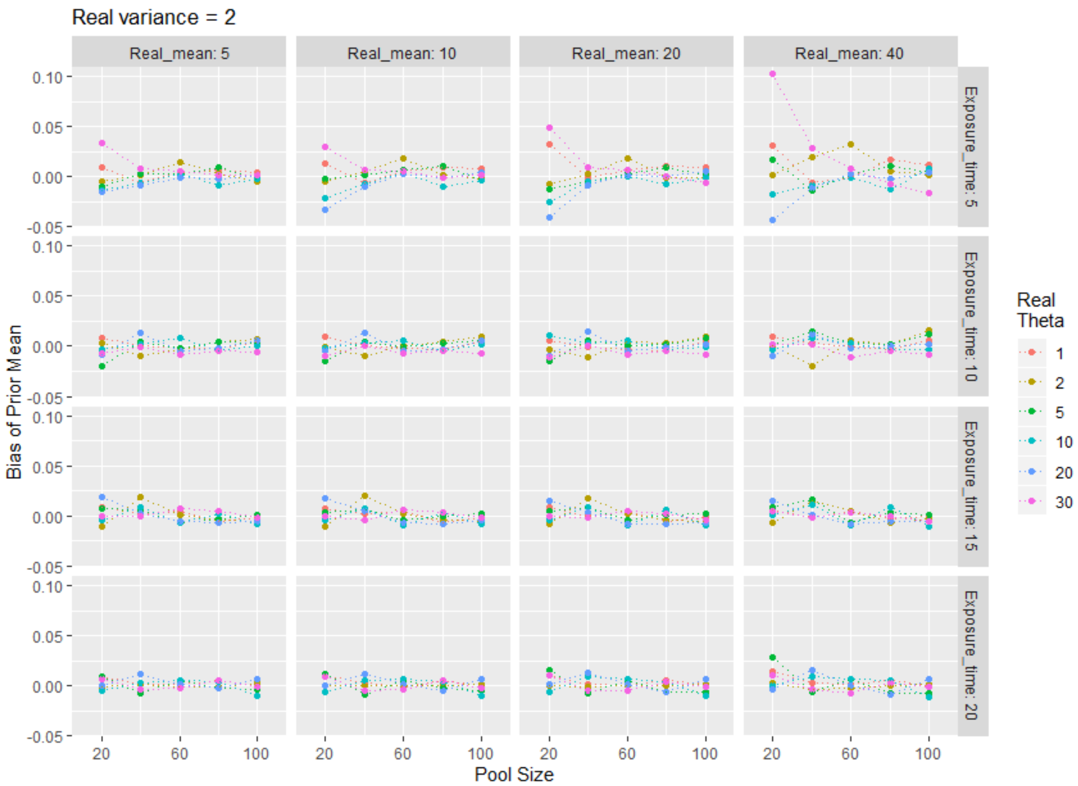


Figure 5.2.4: Showing bias results of the prior mean, when the true variance is 2.

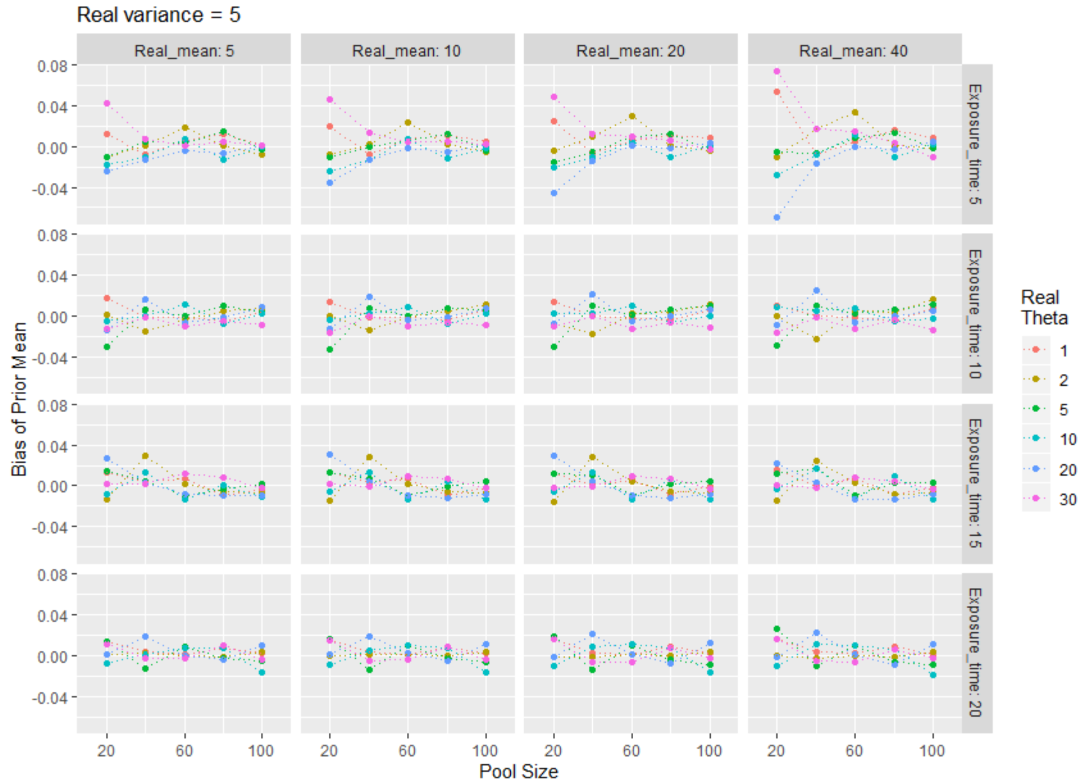


Figure 5.2.5: Showing bias results of the prior mean, when the true variance is 5.

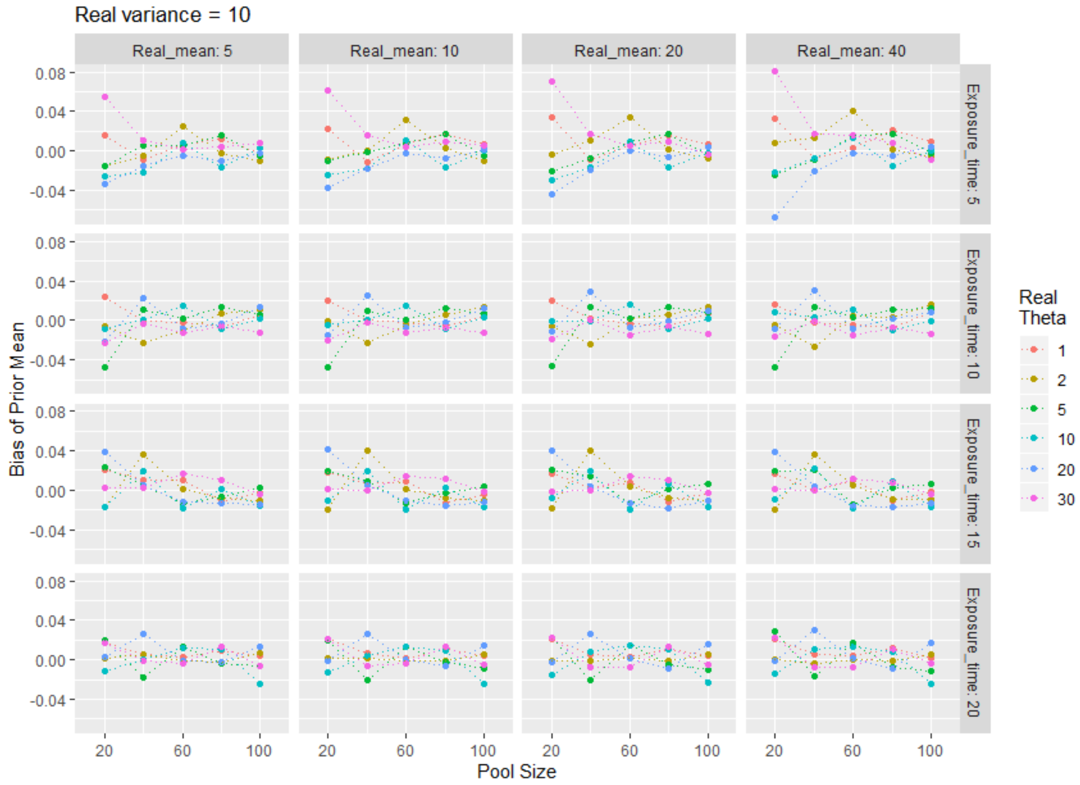


Figure 5.2.6: Showing bias results of the prior mean, when the true variance is 10.

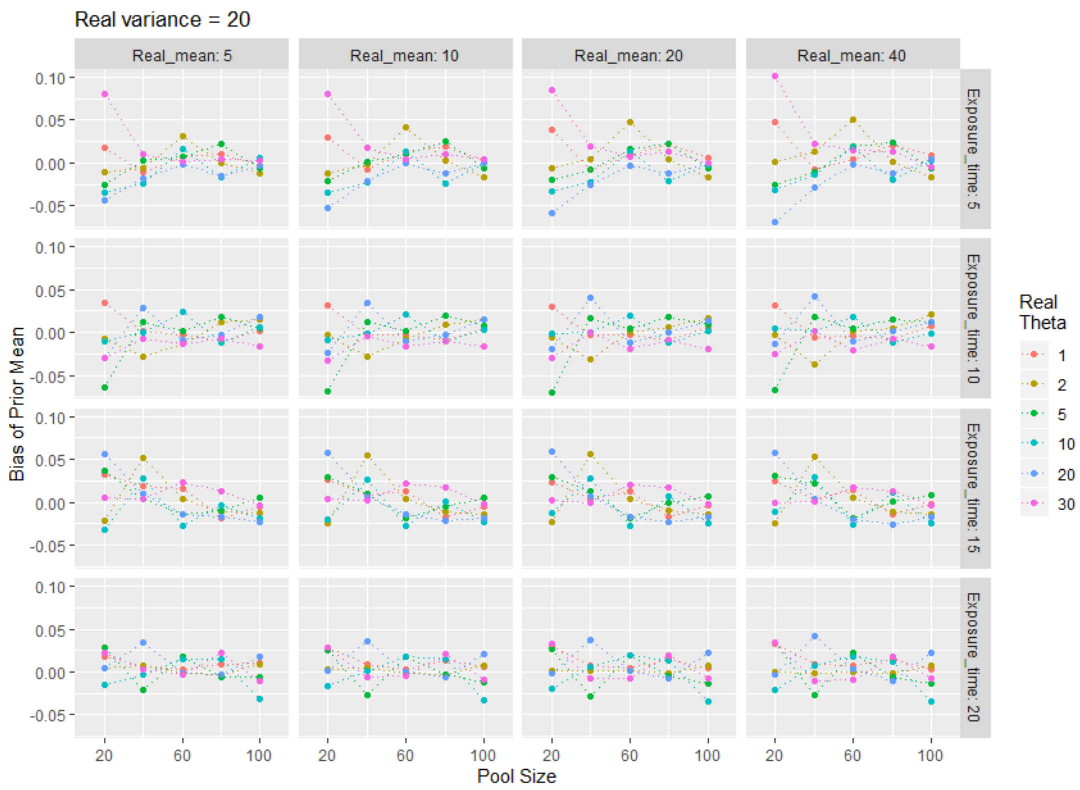


Figure 5.2.7: Showing bias results of the prior mean, when the true variance is 20.

Figures 5.2.8,5.2.9, 5.2.10, 5.2.11 show the MSE results of the prior mean errors. It is clear that MSE decreases as the pool size increases. We also observe that when the true variance is relatively small (true variance = 2), the MSE is significantly smaller than when the true variance is larger (true variance = 20). Moreover, we notice that in every block of four sets (i.e. when the true variance is 2 and true mean is 5, 10, 20, 40 and so on) the MSE slightly increases when the pool size and exposure time are relatively small. However, there is no significant difference between the sets within the blocks when the exposure time increases.

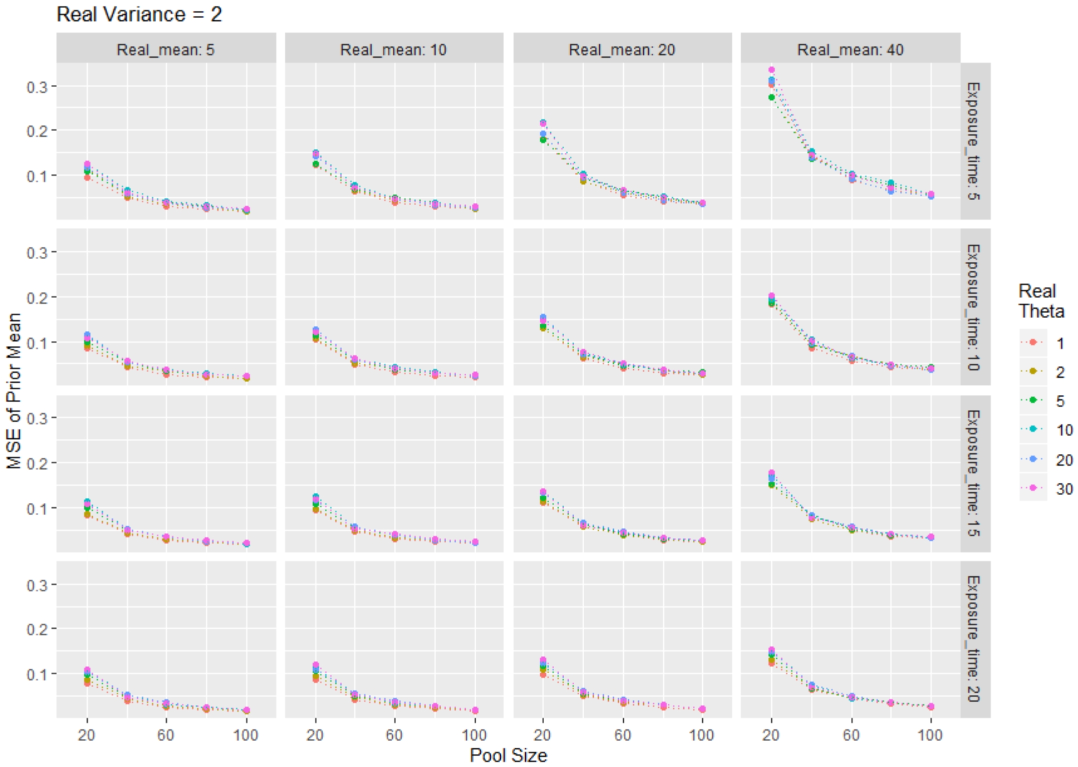


Figure 5.2.8: Showing MSE results of the prior mean, when the true variance is 2.

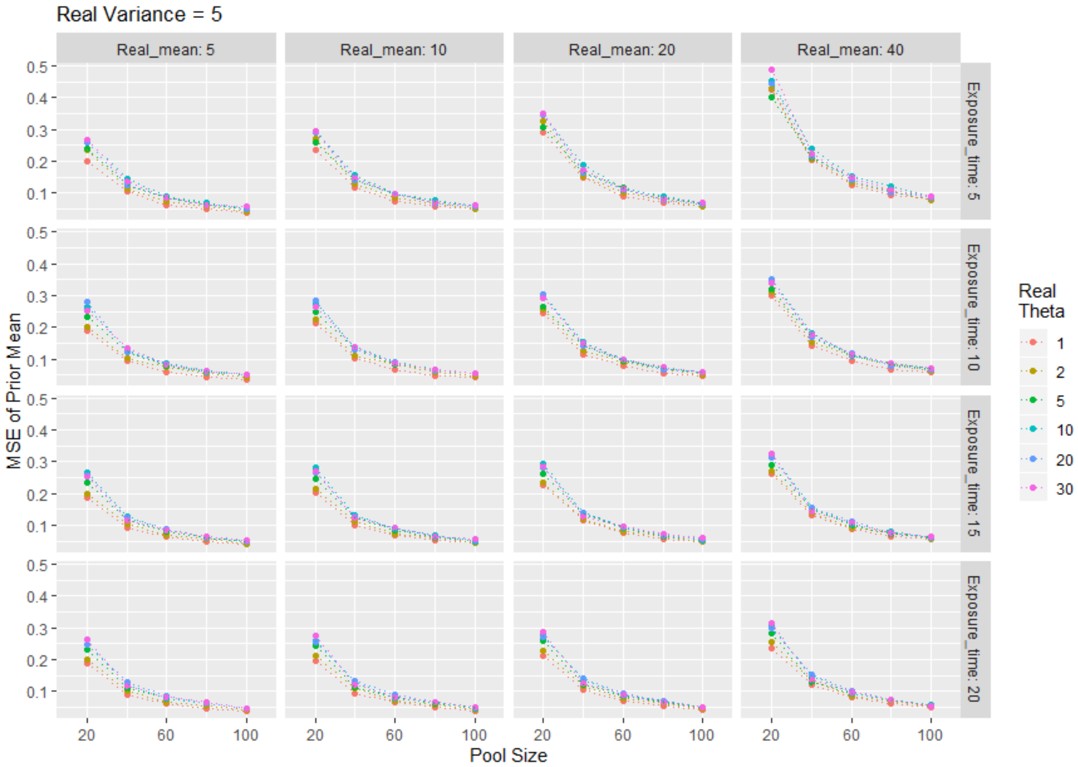


Figure 5.2.9: Showing MSE results of the prior mean, when the true variance is 5.

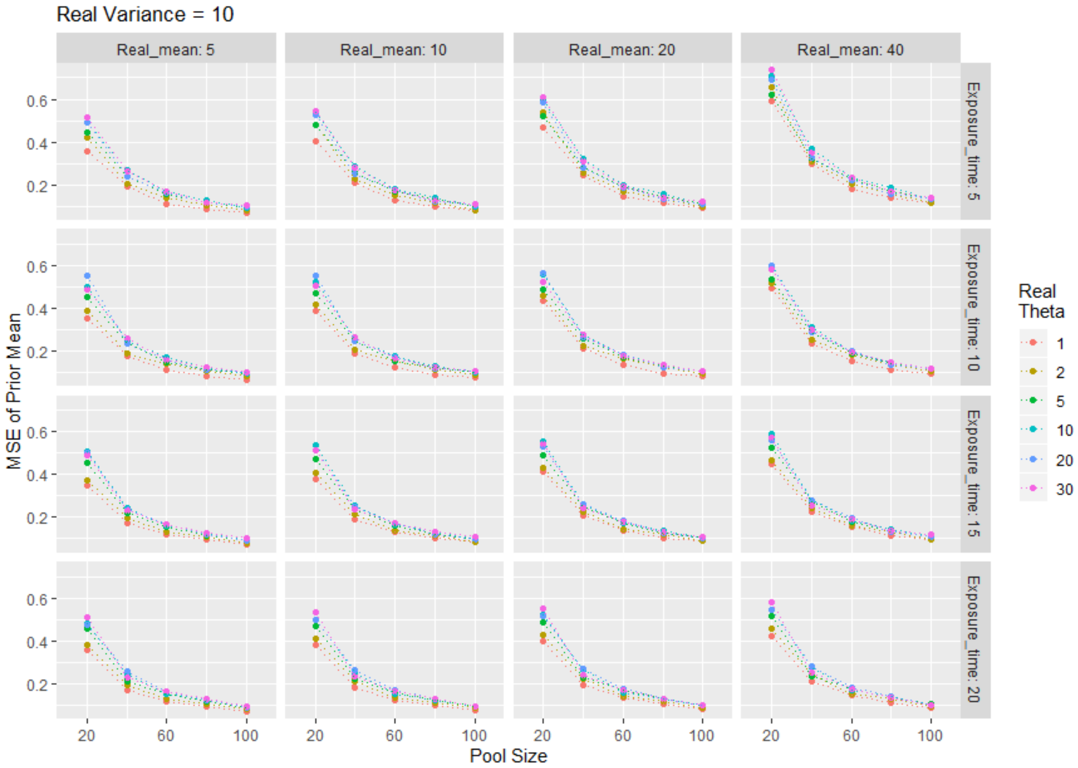


Figure 5.2.10: Showing MSE results of the prior mean, when the true variance is 10.

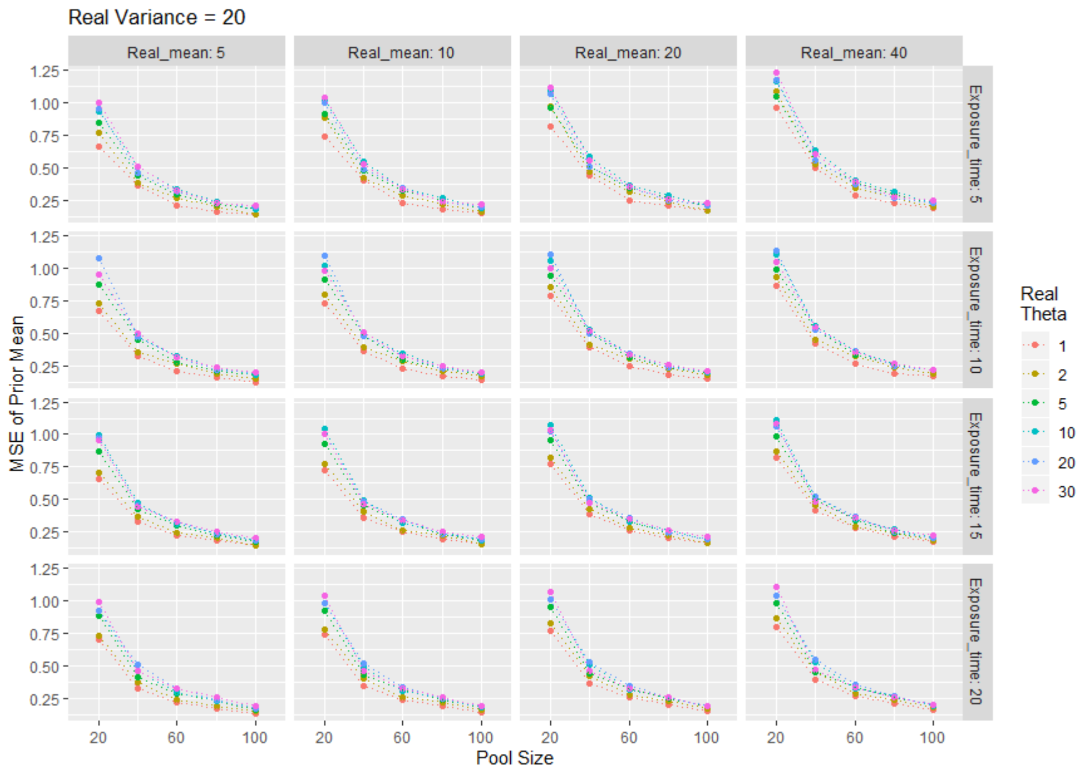


Figure 5.2.11: Showing MSE results of the prior mean, when the true variance is 20.

Figure 5.2.12 shows the MSE of the prior mean when the exposure time is set to be 5. In this particular case, we distinguish two different cases. Firstly, the smaller the prior variance, the smaller the MSE; and secondly the larger the pool size, the smaller the MSE. Moreover, we notice that when the prior mean increases, MSE also increases.

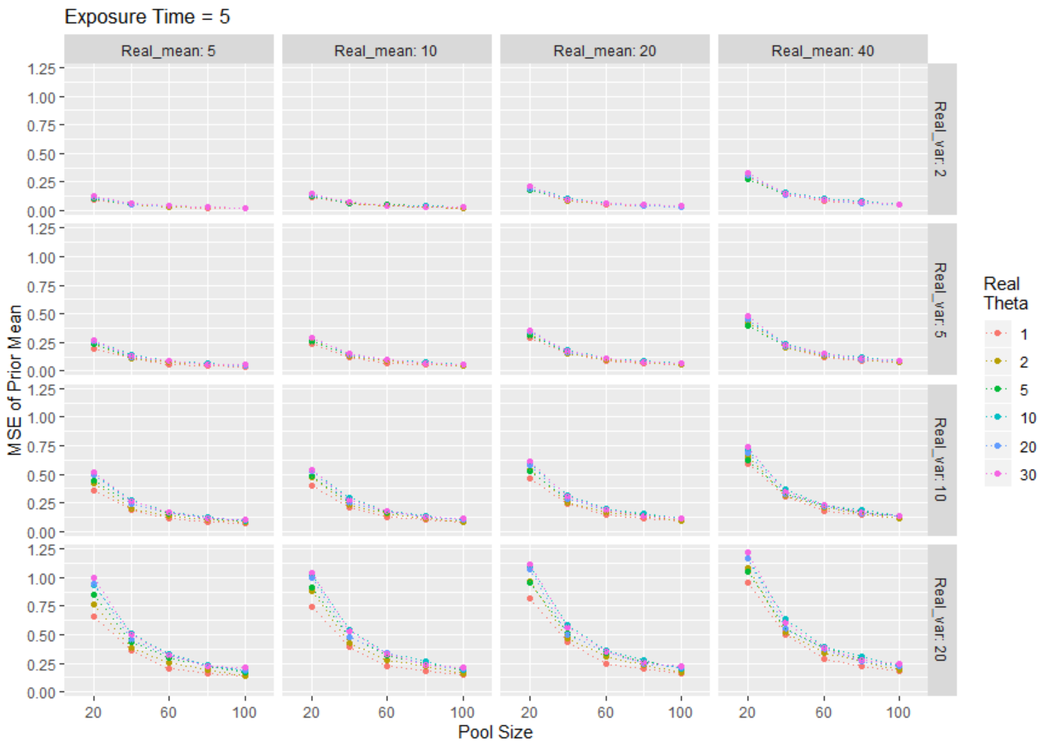


Figure 5.2.12: Showing the MSE of the prior mean when the exposure time is 5.

5.2.3.4 Empirical Bayes with Clayton prediction errors evaluation

In this section, we present the results from the simulation study conducted for evaluating the prediction rates errors. Errors are defined as the difference between the 'real' and the estimated event rates. Considering that the simulated data represent the 'real' event rate, we are investigating how well the proposed empirical Bayes with Clayton copula model approximates the 'real' rates. As measures of accuracy, we consider the Bias, the Mean Squared Error (MSE), the Root Mean Squared Error (RMSE) and the Root Mean Squared Relative Error (RMSRE) of the errors.

Figure 5.2.13 shows Bias, MSE, RMSE and RMSRE results of the errors when the true variance is 2, the true mean is 5 for all different combinations of pool sizes, exposure times and dependence parameter values. We observe that when the exposure time increases the MSE, RMSE and RMSRE of the errors decrease. Moreover, MSE, RMSE and RMSRE show a similar trend for all dependence parameter values. Particularly, the stronger the dependency between the rates, the lower the errors. However, we do not notice significant decrease when the pool size increases (MSE decreases from 0.65 to 0.6 for $\theta = 2$, pool size = 20 and exposure time = 5). Regarding the bias, we observe that when the pool size is smaller, the bias of errors are approximating zero as exposure time increases. However, when the pool size increases, we observe that bias is producing more negative values. Similar figures showing Bias, MSE, RMSE, RMSRE results for all different sets of parameters chosen can be found in Appendix B.1.1.

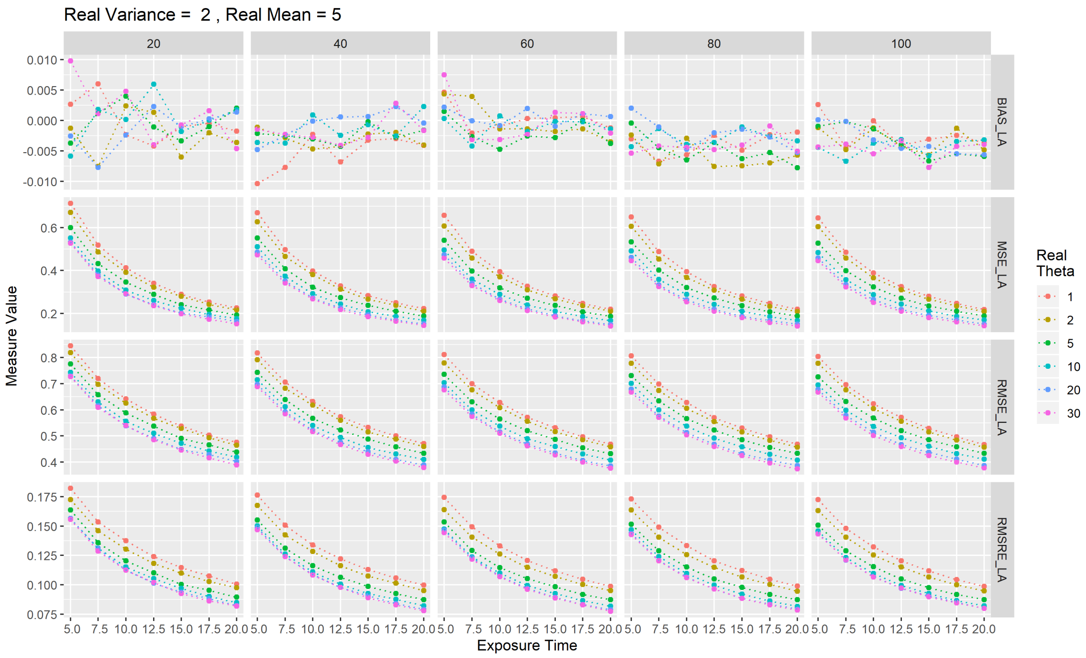


Figure 5.2.13: Showing how different pool sizes and exposure times affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is two and true mean is five.

Figure 5.2.14 shows Bias results of the prediction errors for all different parameter combinations when the size of the pool is 20. As the true variance and true mean increase, bias shows negative values, indicating that the predicted rates of events are overestimated. Considering this case where the pool size is 20, the worst-case scenarios are presented where the mean is relatively large (mean of 20 and 40), and the variance is also large (variance of 10 and 20). When using Bayesian models for estimating event rates in general, we prefer having large enough prior variance so that the rates can be distinct within the pool, but not too large as in this case the event rates are considered totally diffident to being part of the same pool. Moreover, we also notice that when are strongly dependent (with relatively larger real theta), results suggest that predicted event rates are overestimated (appears more strongly when the pool size increases, see figure 5.2.15). That is because of how sensitive the inverse Kendall's tau formula is, especially for rank correlation values close to 1. We remind that if the prior Kendall's tau tends to 1, then the dependence parameter tends to infinity.

$$\tau = \frac{\theta}{\theta + 2} \qquad \Rightarrow \qquad \theta = \frac{2 \cdot \tau}{1 - \tau} \qquad \text{therefore, as } \tau \to 1, \text{ then } \theta \to \infty$$

If there is indication of a strong rank correlation, e.g. $\tau = 0.95, 0.98 or 0.99$, then the dependence parameter θ will be 38, 98 and 198, respectively. Values of θ will range from 38 to 198, which shows how sensitive the inverse Kendall's tau formula is even for small decimal differences. To avoid this variability, an upper bound could be set for the dependence parameter θ, for rank correlation values over a certain value, e.g. θ could be considered 38, for all τ over 0.95. While this is reasonable in both cases, we recognise that may be an issue which needs careful consideration.

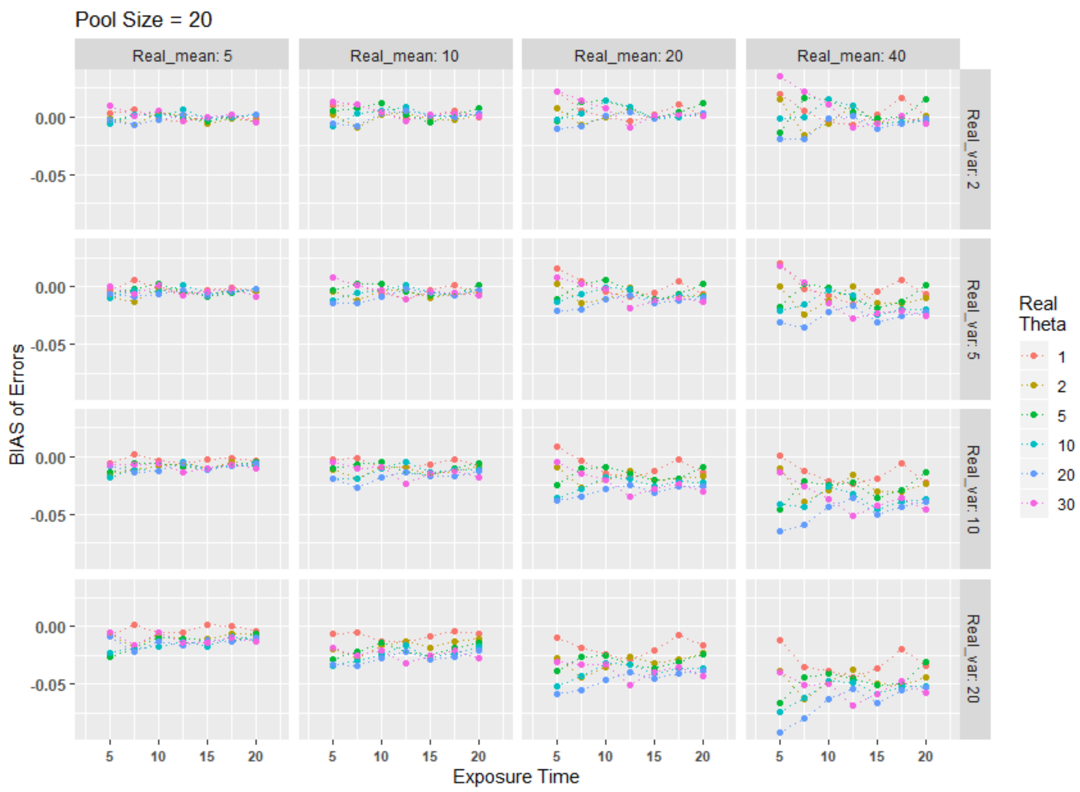


Figure 5.2.14: Showing how different exposure times, mean and variance choices affect bias of EB with Clayton prediction errors, while the pool size is 20.

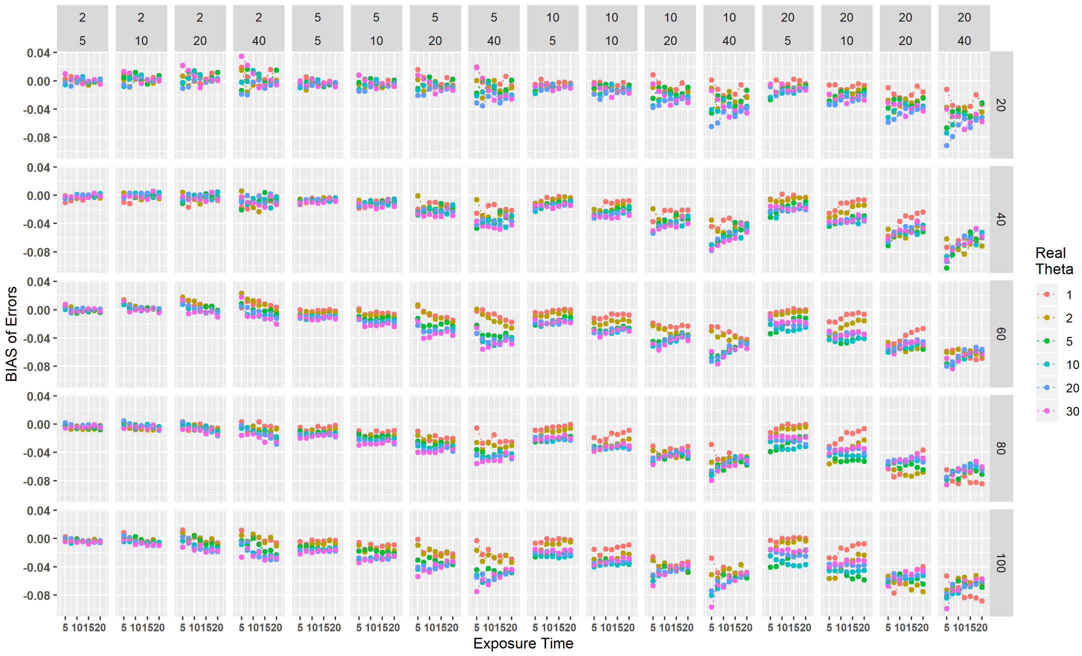


Figure 5.2.15: Showing how different pool sizes, exposure times, mean (5, 10, 20, 40) and variance (2, 5, 10, 20) choices affect bias of EB with Clayton prediction errors.

Figure 5.2.16 shows RMSE results of the prediction errors for all different parameter combinations of prior mean and variances, when the pool size is 20. It is clear that as the exposure time increases, RMSE decreases. We also notice that RMSE increases as the true mean increases. Similar decreasing trends are shown for all different variance values. Moreover, when the mean and variance are relatively large, RMSE results are shown more distinct in relation to the dependence parameter. The stronger the dependence, the lower the RMSE of the errors. On the other hand, we observe no significant difference between cases when the pool size increases. Figure 5.2.17 shows RMSE results of the prediction errors for all different parameter combinations considered for this study.

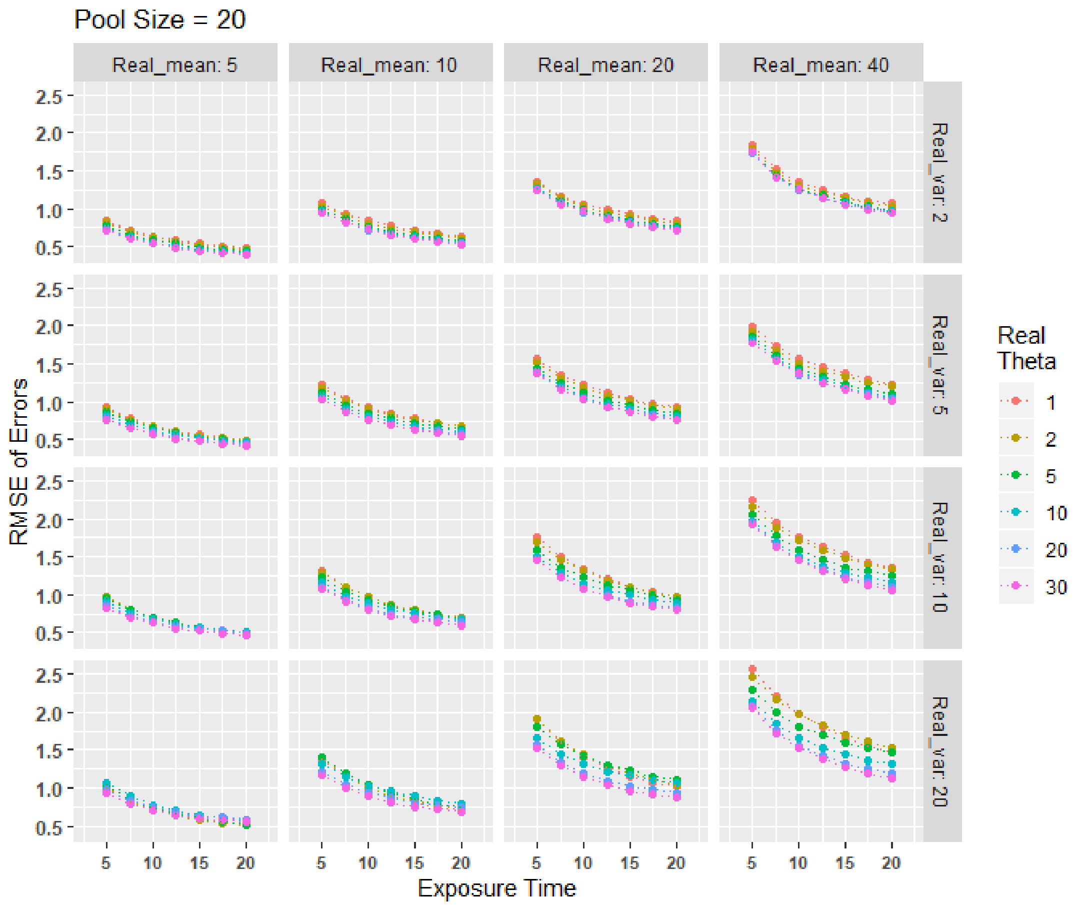


Figure 5.2.16: Showing RMSE results of the prediction errors for all different parameter combinations of prior mean and variances, when the pool size is 20.

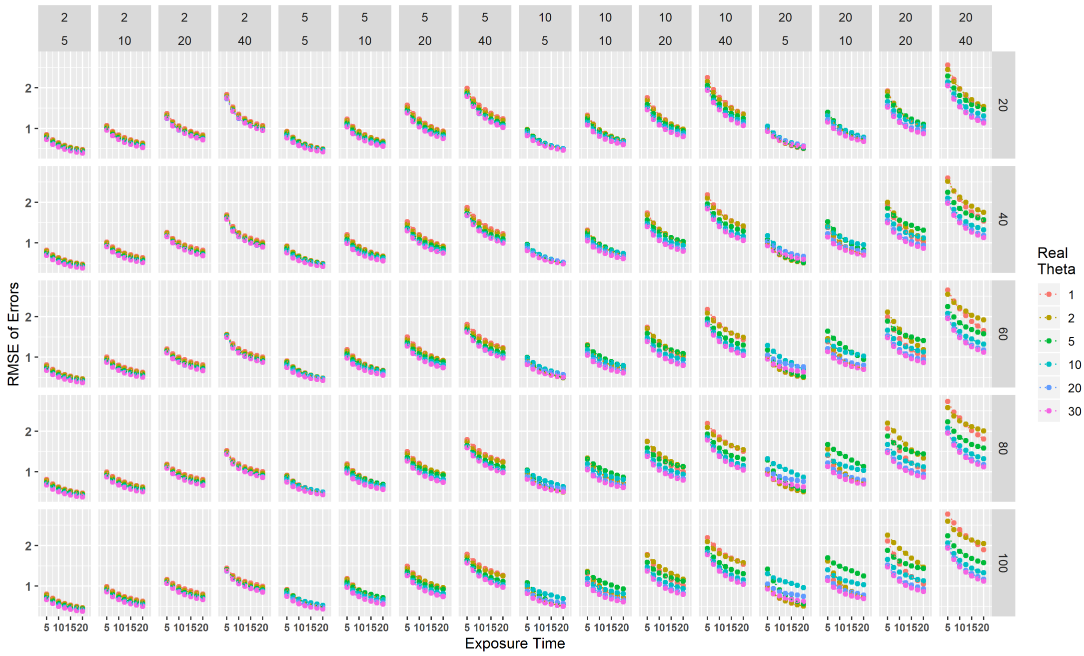


Figure 5.2.17: Showing how different pool sizes (20, 40, 60, 80, 100) (rows), exposure times, variance (2, 5, 10, 20) (first column label) and mean (5, 10, 20, 40) (second column label) choices affect RMSE of EB with Clayton prediction errors.

5.3 Benchmarking Study

5.3.1 Overview

To examine the relative accuracy of estimates obtained using the empirical Bayes with Clayton copula model (Model A) in Section 3.2 compared to the classic empirical Bayes model (Model B) and the Maximum Likelihood estimates (Model C) described in Section 3.3, respectively, we report the results from the simulation study conducted. We are investigating how the classic empirical Bayes model and Maximum Likelihood Estimates of the rates (without pooling) perform when the event rates are dependent. Practically, what happens if we choose to ignore the underlying dependency on the rates? Also, in which situations, if there are any, these models perform better compared to the empirical Bayes with Clayton copula model when having different selections of pool sizes, exposure times, dependence parameter and Gamma marginal parameters.

We simulate data from the empirical Bayes with Clayton copula model with known parameters, which represent the true rates of events. We choose to have the same parameters as presented in Section 5.2.1. Ignoring dependency between the rates, we then use Models B and C for estimating the rates of events. This will allow us to see how well we are recovering the true rates in each case and which of the three models used produces more accurate results which approximate the actual rates. As measures of accuracy, we consider the behaviour of the error and squared error. In table 5.3.1 we present the simulation process followed for this benchmarking study. Figure 5.3.1 illustrates our methodology and described Models A, B and C.

<table>
<tr><td colspan="3">
1. Set pool size m.
2. Set exposure time t.
3. Set Prior parameters θ, α_1, β_1, α_2, β_2.
4. Generate $(\Lambda_1, \Lambda_2)_m$ from Clayton copula.
5. Generate $(N_1, N_2)_m$ from Poisson$(\lambda_1 t, \lambda_2 t)$.

Using the same correlated data
</td></tr>
<tr><th>Model A
EB with Clayton</th><th>Model B
Classic EB</th><th>Model C
No pool</th></tr>
<tr>
<td>Consider dependency between the rates. Use Moment Estimators to update the prior and calculate the Empirical Bayes estimates.

The EB Estimates are,

$\hat{\lambda}_{ji} = E(\lambda_{ji}|N_{1i}, N_{2i}) = \frac{I_{ji}}{NC_i}$,
$j = 1, 2$, $i = 1, \cdots, m$</td>
<td>Ignore dependency between the rates and consider them as independent. Use Moment Estimators to update Gamma prior and calculate the Empirical Bayes estimates.

The EB Estimates are,

$\hat{\lambda}_{ji} = E(\lambda_{ji}|N_{ji}) = \frac{\hat{\alpha}+n_{ji}}{\hat{\beta}+t_i}$,
$j = 1, 2$, $i = 1, \cdots, m$</td>
<td>Use the generated samples without pooling. The estimates of the rates are $(N_1, N_2)_m$.

The estimates are,

$\hat{\lambda}_{ji} = n_{ji}$,
$j = 1, 2$, $i = 1, \cdots, m$</td>
</tr>
</table>

Table 5.3.1: Shows simulation process followed to compare Model A: Empirical Bayes with Clayton copula model with Model B: Classic Empirical Bayes and Model C: Rate Estimates without pooling.

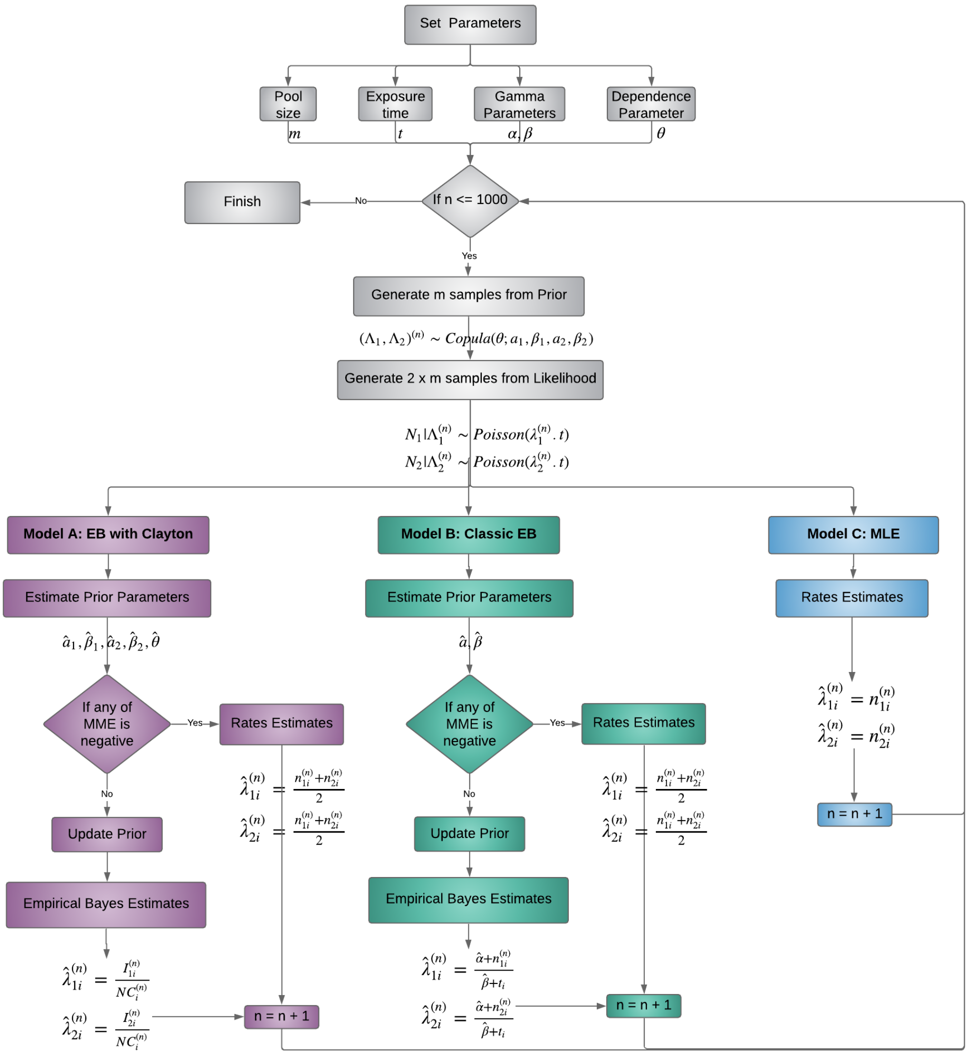


Figure 5.3.1: Diagram showing the methodology of the simulation process followed to compare Model A: Empirical Bayes with Clayton copula, Model B: Classic Empirical Bayes and Model C: Maximum Likelihood Estimators (without pooling).

5.3.2 Benchmarking Results

Considering that the occurrence rate is λ_{ji} and the prediction is $\hat{\lambda}_{ji}$, we are interested in the quantities, $e_{ji} = \lambda_{ji} - \hat{\lambda}_{ji}$ and $e^2_{ji} = (\lambda_{ji} - \hat{\lambda}_{ji})^2$ for every model. To measure the accuracy and compare Models A, B and C, we consider the Bias, the

Mean Squared Error (MSE) and the Root Mean Squared Error (RMSE) of the errors.

Figures 5.3.2, 5.3.3, 5.3.4, 5.3.5 show the Bias results of the errors for different combinations of exposure times and prior variance values for Models A, B and C when the pool size is 20 and the the real prior mean is 5, 10, 20 and 40, respectively. According to results presented in figure 5.3.2, we observe that Models A, B and C produce similar results, very close to zero, suggesting that all models perform significantly well. The only distinct difference occurs when the prior variance is relatively small (variance = 2), and the prior mean increases. In this case, Bias of Model B increases especially when the exposure time is relatively low, suggesting that Model B underestimates the rates of events. Particularly, we notice that Bias of Model B increases from 0.4 to 2.8, when the real mean increases from 10 to 20, and reaches Bias of 10 when the real variance increases to 40 (for real variance = 2 and exposure time = 5). Analytical visuals showing how the bias of Models A, B and C is affected by all parameters can be found in Appendix B.2.1.

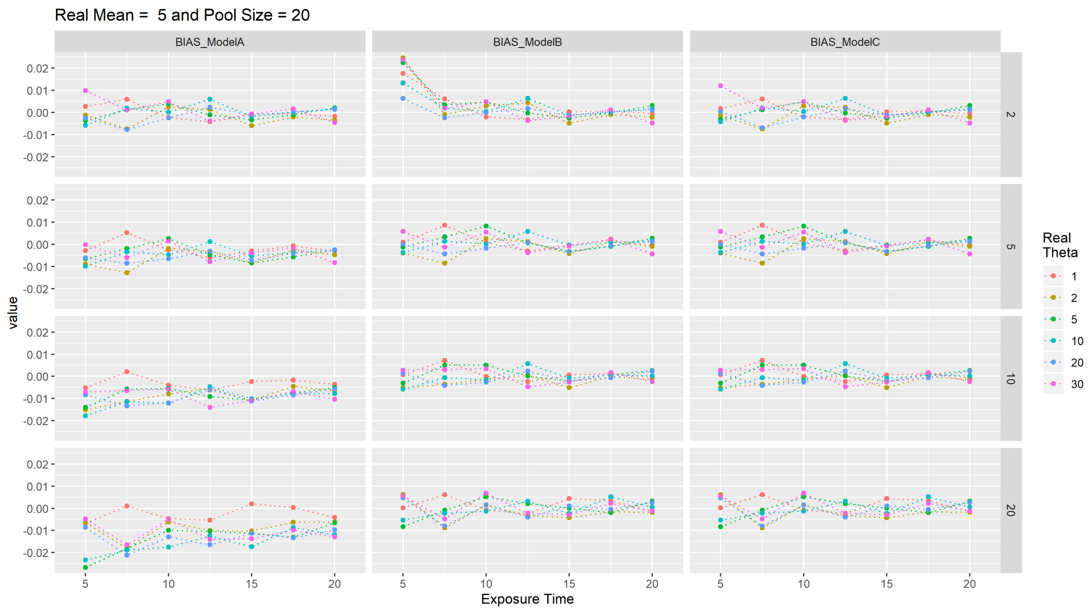


Figure 5.3.2: Showing how different pool sizes, exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect Bias for Models A, B and C, when the true prior mean is 5 and pool size is 20.

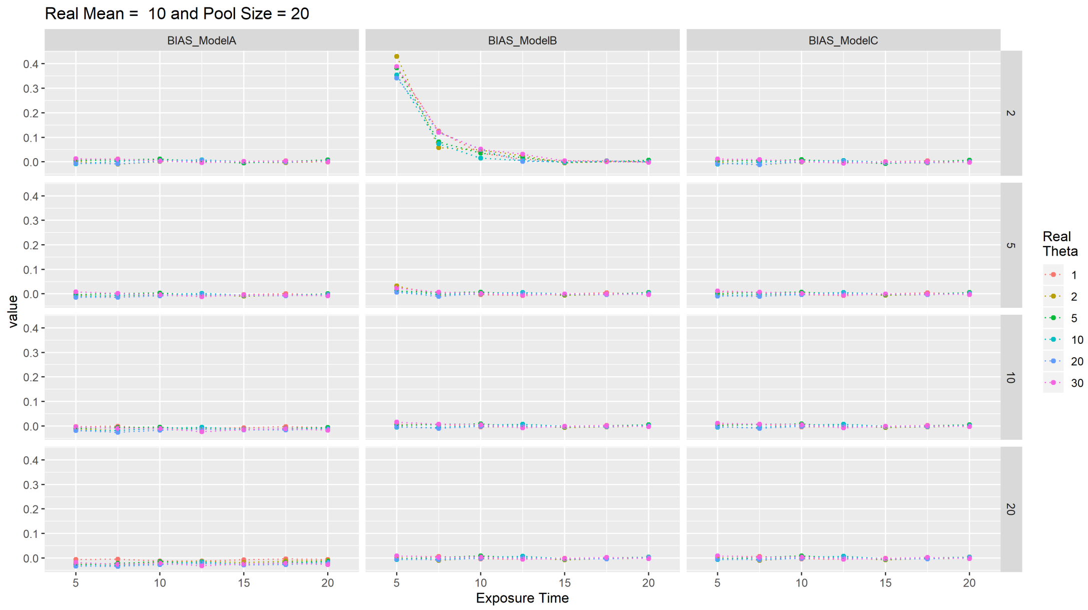


Figure 5.3.3: Showing how different pool sizes, exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect BIAS for Models A, B and C, when the true prior mean is 10 and pool size is 20.

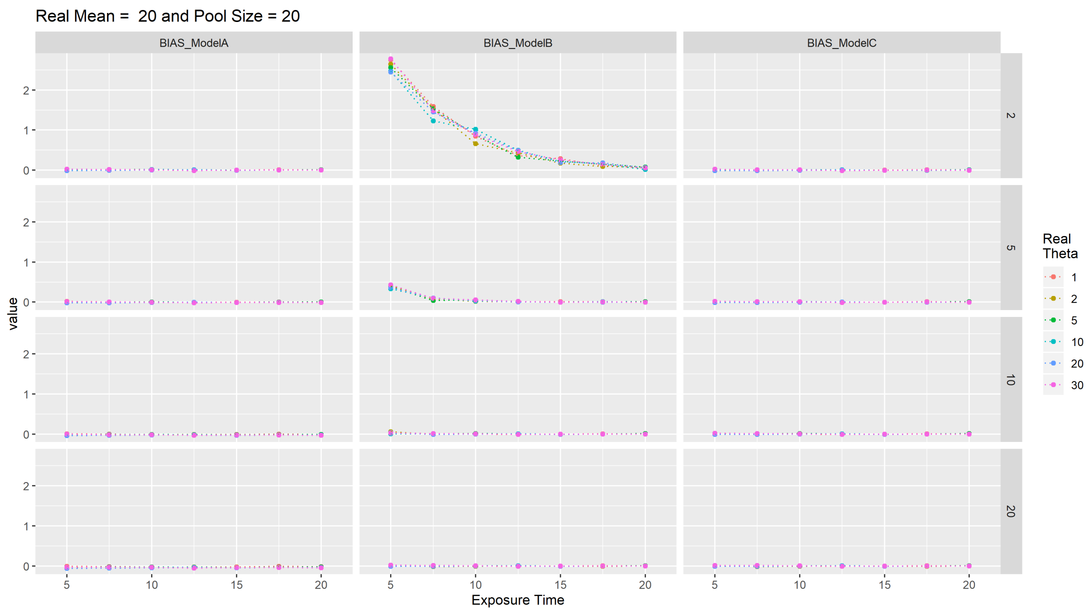


Figure 5.3.4: Showing how different pool sizes, exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect BIAS for Models A, B and C, when the true prior mean is 20 and pool size is 20.

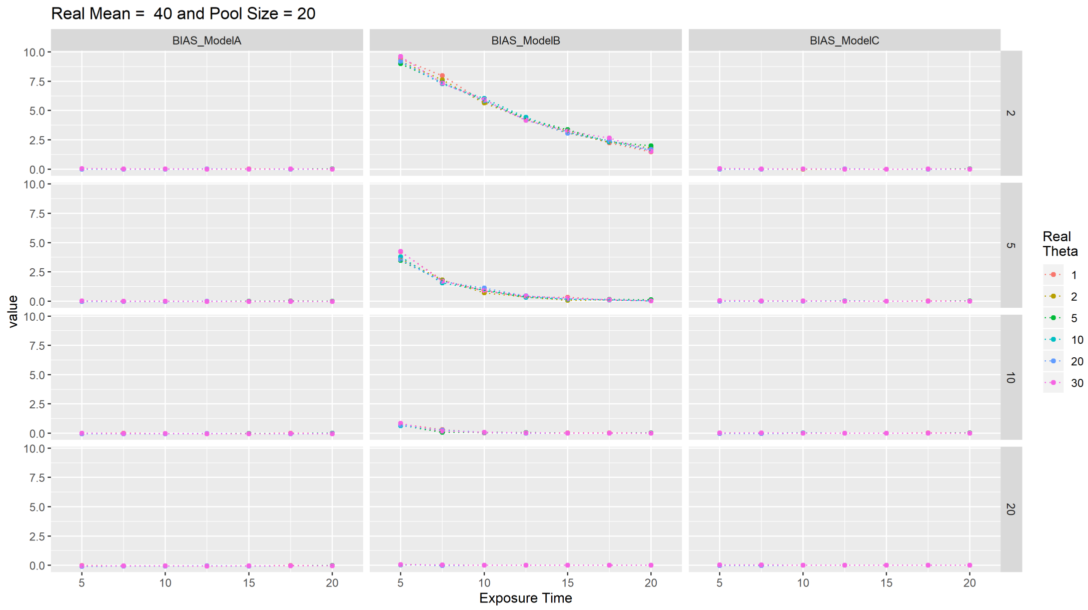


Figure 5.3.5: Showing how different pool sizes, exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect BIAS for Models A, B and C,when the true prior mean is 40 and pool size is 20.

Figures 5.3.6, 5.3.7, 5.3.8, 5.3.9 show the MSE results of the errors for different combinations of exposure times and prior variance values for Models A, B and C when the pool size is 20, and the real prior mean is 5, 10, 20 and 40, respectively. All visuals suggest that MSE decreases as exposure time increases. However, the larger the prior mean, the higher the MSE. Comparing Models A and B, figure 5.3.6 shows that both models have similar MSE when the dependence parameter is relatively small. However, when the dependence between the rates is becoming stronger, we observe that Model A performs better than Model B when the prior variance increases. Particularly when the exposure time is 5, and the dependence parameter is 30 (figure 5.3.9), the MSE of Model A is almost 33% less than the MSE of Model B, and almost 50% less than the MSE of Model C. Moreover, Model C is considered inferior compared to Models A and B. MSE of Model C increases significantly when the prior mean increases. Regarding Models A and B, we observe that when the prior variance increases the MSE of Model A is significantly smaller than the MSE of Model B, especially for large dependence parameters and low exposure times. Worth noticing that when the prior variance is large (prior variance = 20), the MSE of Model A is slightly larger than the MSE of Model B, for small dependence parameters indicating that may be worth ignoring the (weak) underlying dependence on the rates. Explicit visuals showing how different pool sizes, exposure times, dependence parameters and marginal parameters may affect the MSE of Models A, B, and C can be found in Appendix B.2.2.

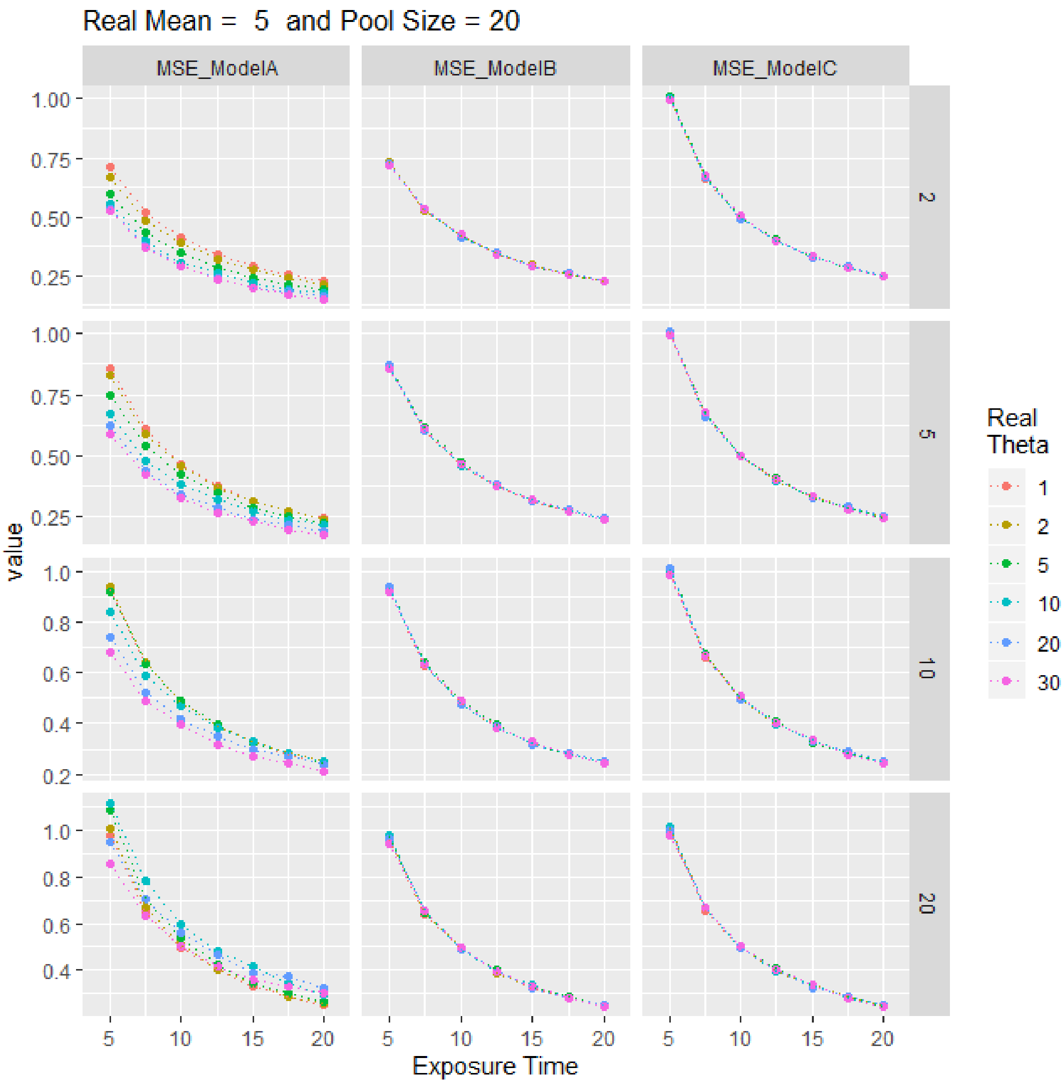


Figure 5.3.6: Showing how different exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect MSE for Models A, B and C, when the true prior mean is 5 and the pool size is 20.

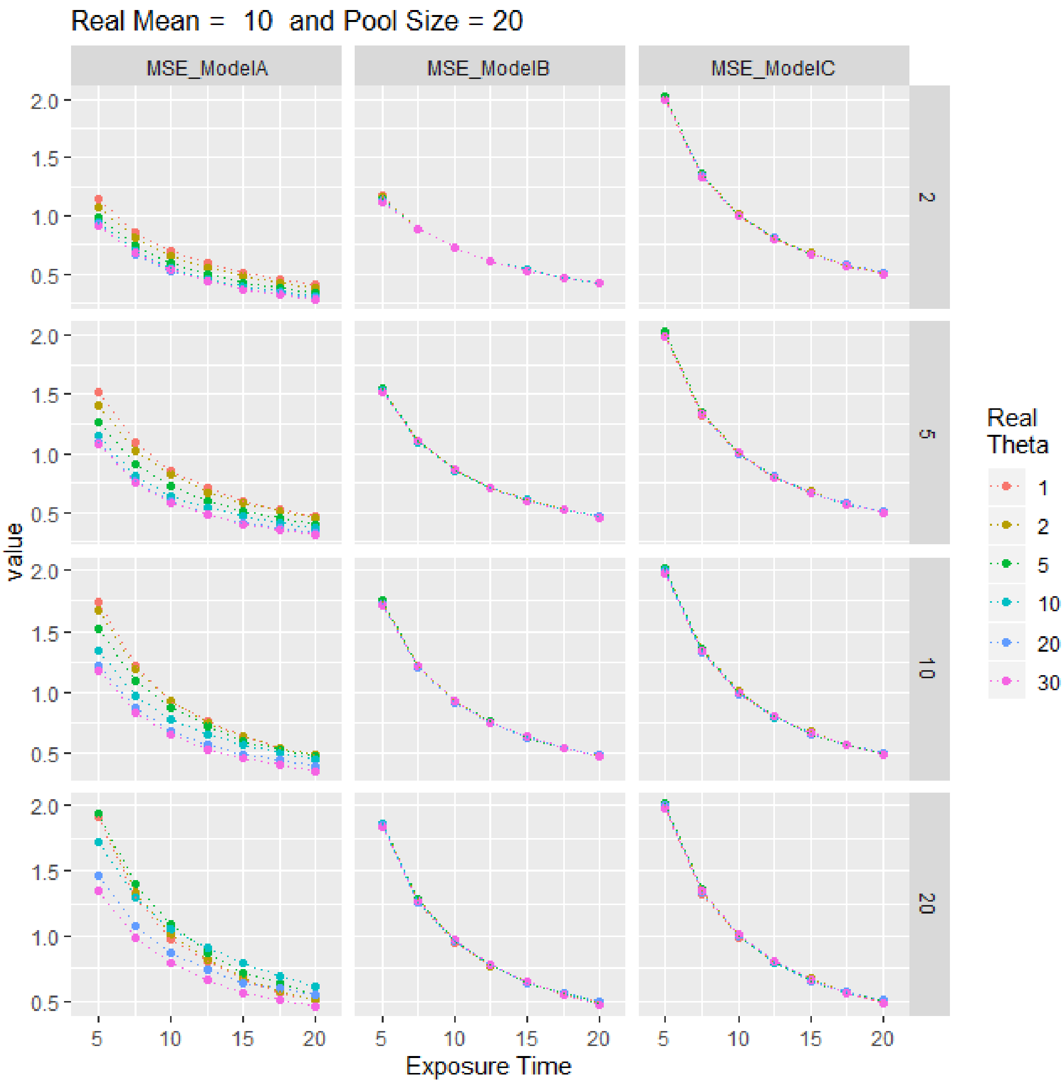


Figure 5.3.7: Showing how different exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect MSE for Models A, B and C, when the true prior mean is 10 and the pool size is 20.

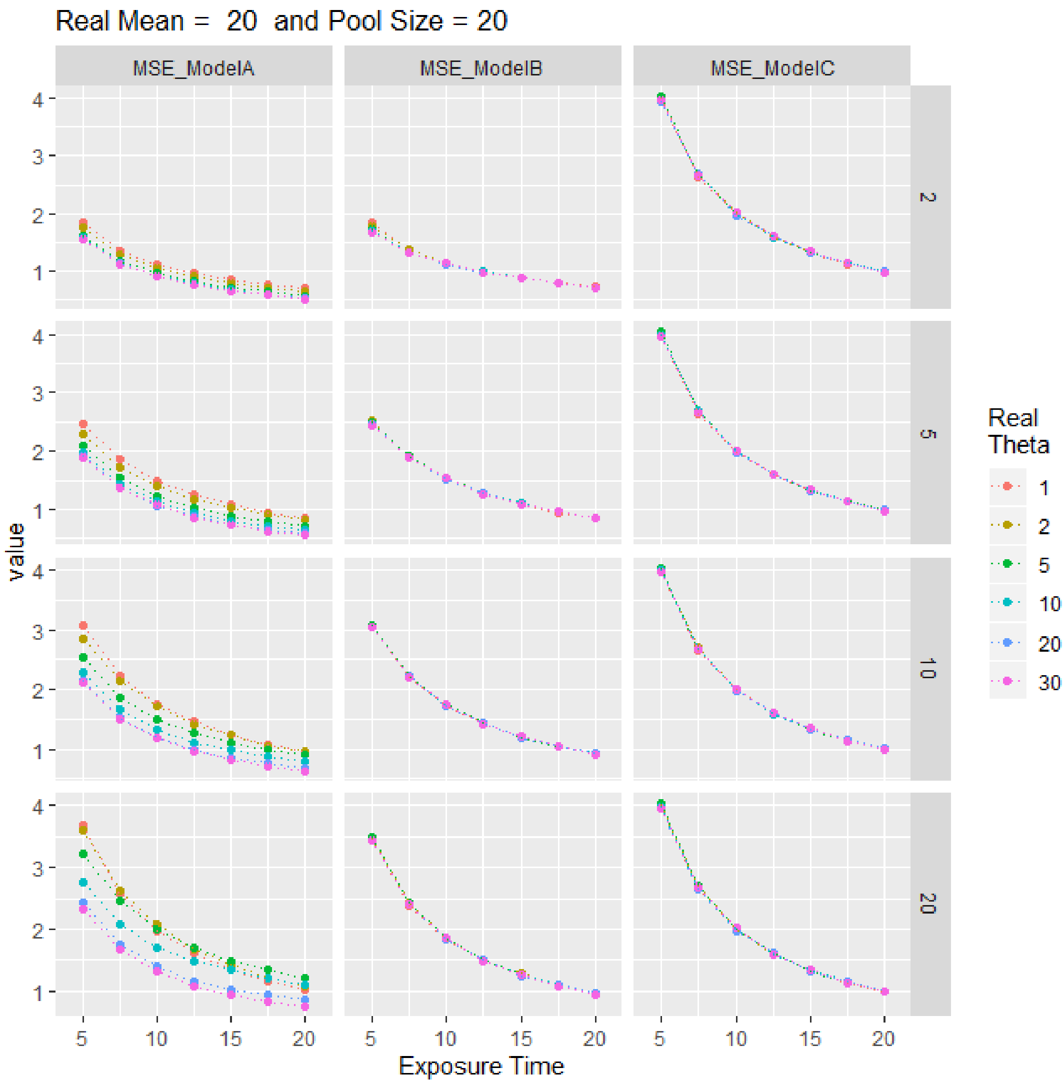


Figure 5.3.8: Showing how different exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect MSE for Models A, B and C, when the true prior mean is 20 and the pool size is also 20.

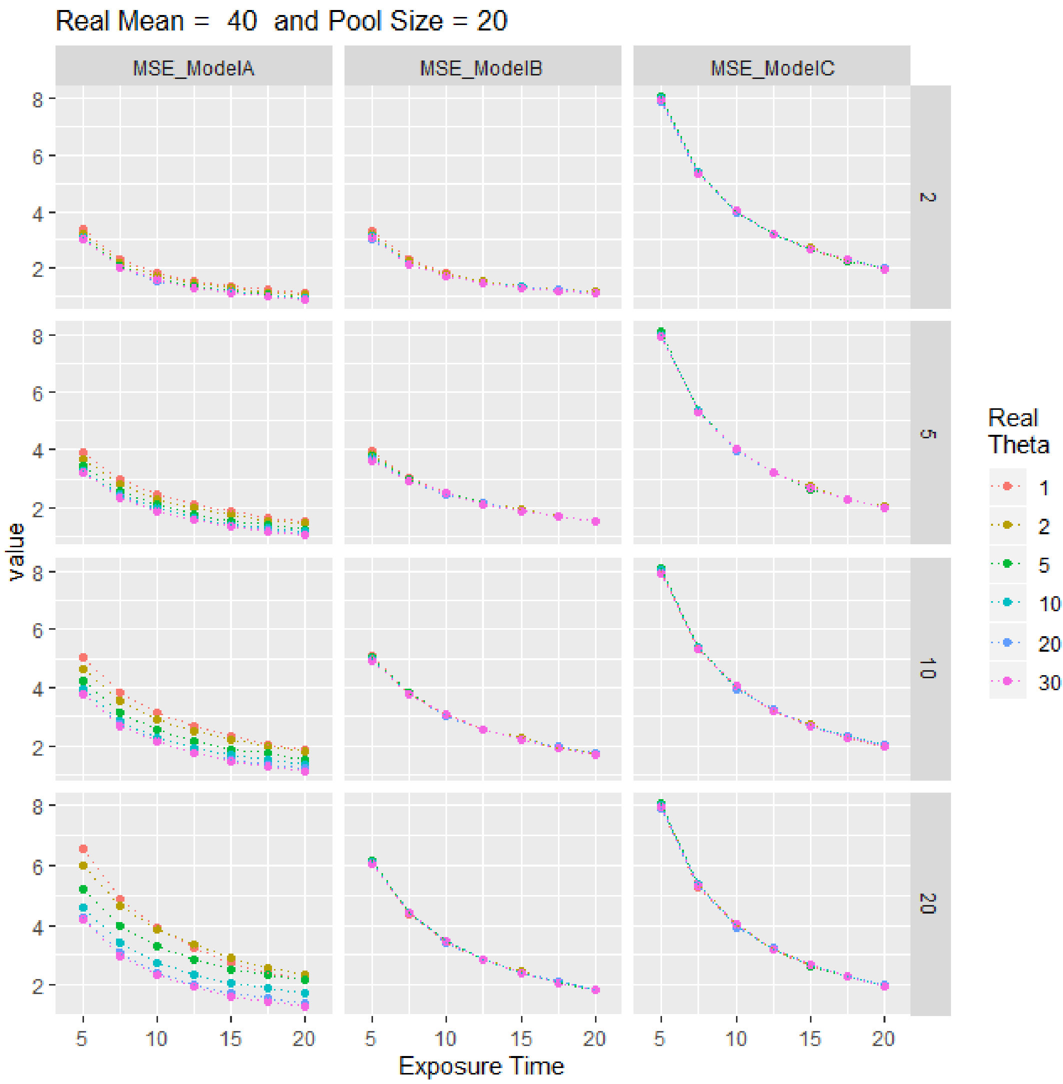


Figure 5.3.9: Showing how different exposure times, dependence parameters and prior variance values (2, 5, 10, 20) affect MSE for Models A, B and C, when the true prior mean is 40 and the pool size is 20.

Figure 5.3.10 shows the MSE results of the errors for different combinations of exposure times, pool sizes and prior mean values for Models A, B and C when the real prior variance is 20. According to this plot, we observe that the MSE increases as the prior mean increases. Model A performs better than Models B and C when the dependence parameter increases. However, for smaller dependence parameters, we notice that the MSE of Model A is larger than the MSE of Model B, especially when the pool size increases.

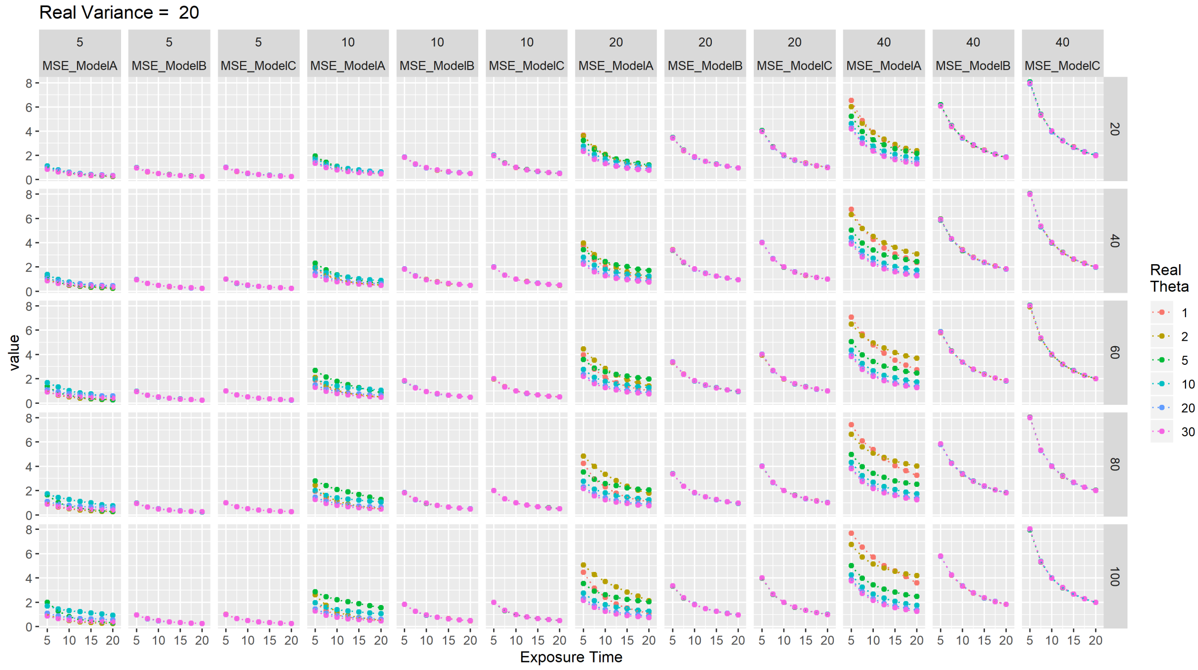


Figure 5.3.10: Showing how different pool sizes (20, 40, 60, 80, 100), exposure times, dependence parameters and prior mean values (5, 10, 20, 40) affect MSE for Models A, B and C, when the true prior variance is 20.

5.3.3 Comparison Between EB with Clayton Model and Classic EB Model

Comparing the empirical Bayes with Clayton copula model (Model A) to the classic empirical Bayes model (Model B) and the Maximum Likelihood Estimators method (Model C) (see Section 5.3.2), we conclude that there are cases where both Model A and B perform similarly well. However, there are cases where Model A outperforms and cases where class EB outperforms suggesting that may be worth ignoring dependency. We have also noticed that there are no cases where Model C outperforms Models A and B. Therefore, our goal is twofold. We initially provide further comparative analysis between the two models considering all parameters chosen for this study by identifying cases in which the proposed empirical Bayes with Clayton copula model is expected to outperform the classic Bayesian model; and, secondarily, to provide an answer to the following question: In which cases is it worth ignoring the underlying dependence on the event rates?

Doing so, we present analysis which aims at comparing proposed Model A with Model B. We consider cases across all different combinations of pool sizes, exposure times, dependence parameters and prior parameter combinations in which Models A and B are evaluated. We evaluate both models performance by comparing their MSE of errors. Particularly, we are interested in calculating the difference between the MSE of Model A and the MSE of Model B, as follows,

$$\text{Difference of MSE} = MSE_{ModelA} - MSE_{ModelB}.$$

If the difference of their MSE is close to zero, then we suggest that both models perform similarly. However, if the difference of greater than zero, then Model B performs relatively better compared to Model A, indicating that may be worth ignoring prior dependency. Lastly, if the difference between their MSE of errors is smaller than zero, we suggest that the proposed Model A performs relatively better than Model B. Summarising what is being discussed, Table 5.3.2 shows all expected cases.

Cases	**EB with Clayton (Model A)**	**Classic EB (Model B)**
$MSE_{ModelA} - MSE_{ModelB} = 0$	*Similar*	*Similar*
$MSE_{ModelA} - MSE_{ModelB} > 0$	*Worse*	*Better*
$MSE_{ModelA} - MSE_{ModelB} < 0$	*Better*	*Worse*

Table 5.3.2: Showing all possible cases.

Cases with a difference of MSE close to zero are defined with white colour, areas with a difference greater than zero with red and areas with difference lower than zero with green. Figures 5.3.11, 5.3.12, 5.3.13, 5.3.14, 5.3.15, 5.3.16 show cases with green, red and white gradient colour areas. White coloured areas suggest that both models perform similarly as they have similar MSE. Red coloured areas suggest that Model A has higher MSE than Model B, indicating that the classic EB model assuming independent event rates might be a better option. Lastly, green coloured areas suggest that Model A has lower MSE than Model B, indicating that the proposed Model A outperforms classic EB model. The cases are presented with respect to different prior mean and variance values, different pool sizes, exposure times and dependence parameters.

Concluding that the proposed empirical Bayes with Clayton copula model outperforms the classic empirical Bayes model in cases where the underlying dependence between event rates is moderate to strong, and the prior mean and variance are relatively large. In contrast, the classic EB model is suggested in cases where weak dependence between the rates occurs. Notably, in cases where the prior mean and variance are relatively large; and the size of the pool is large.

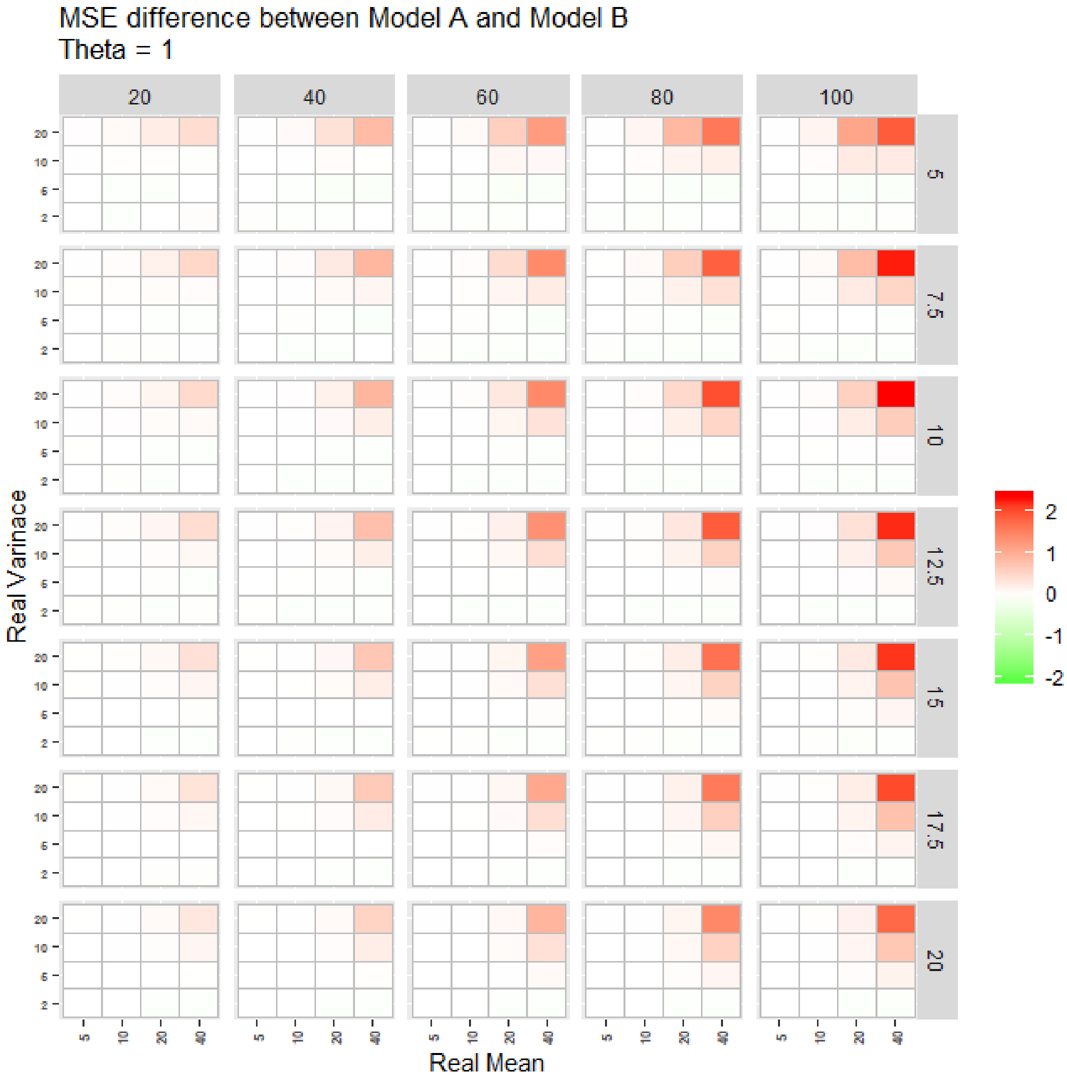


Figure 5.3.11: Compare Clayton (Model A) and Classical (Model B). Present cases of better and worse performance on 16 different combinations of prior mean and variance values. Sample size m = 20, 40, 60, 80, 100 (columns) and exposure time t = 5, 7.5, ..., 20 (rows). Green areas suggest the Clayton Model. Red areas suggest the classic EB model assuming independence.

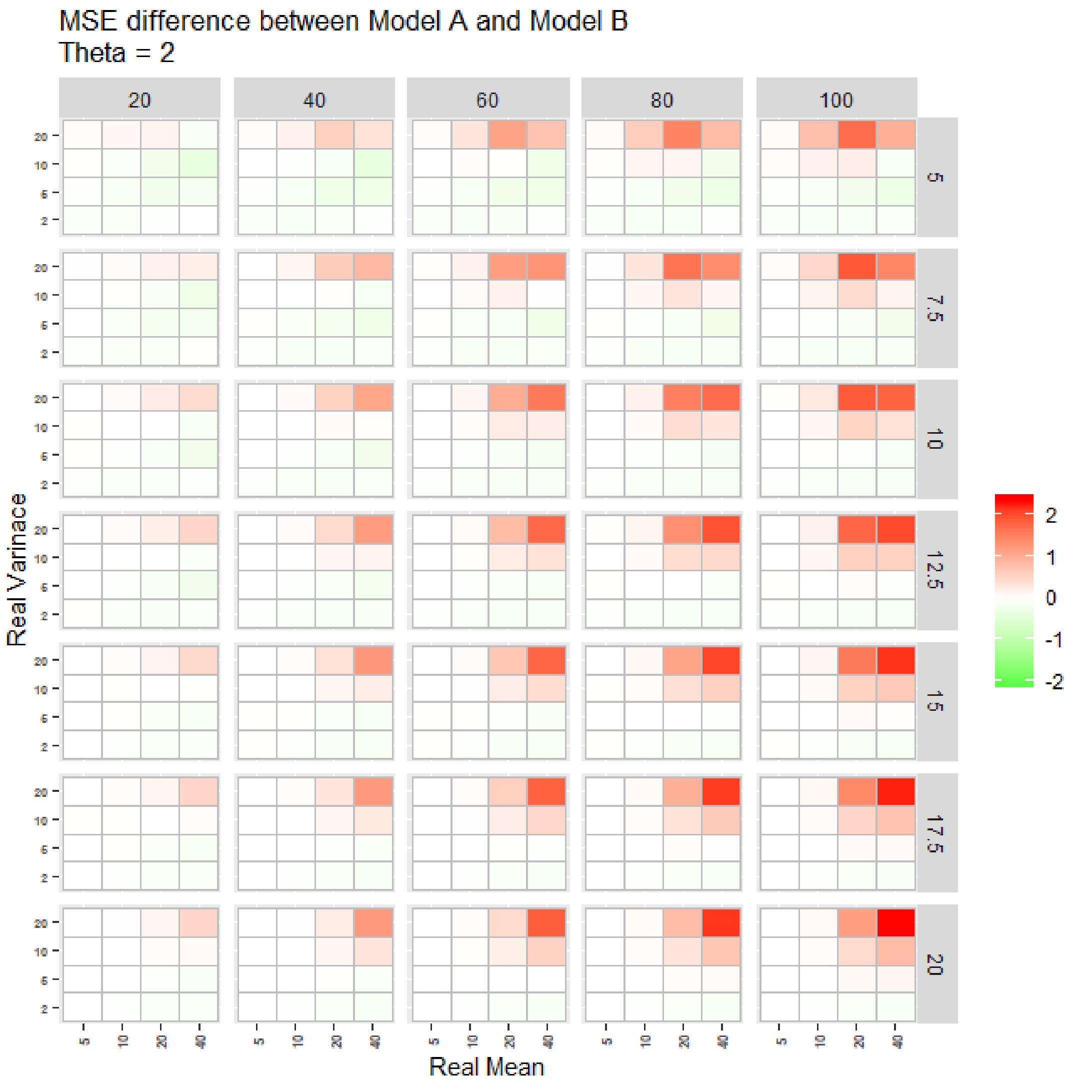


Figure 5.3.12: Compare Clayton (Model A) and Classical (Model B). Present cases of better and worse performance on 16 different combinations of prior mean and variance values. Sample size m = 20, 40, 60, 80, 100 (columns) and exposure time t = 5, 7.5, ..., 20 (rows). Green areas suggest the Clayton Model. Red areas suggest the classic EB model assuming independence.

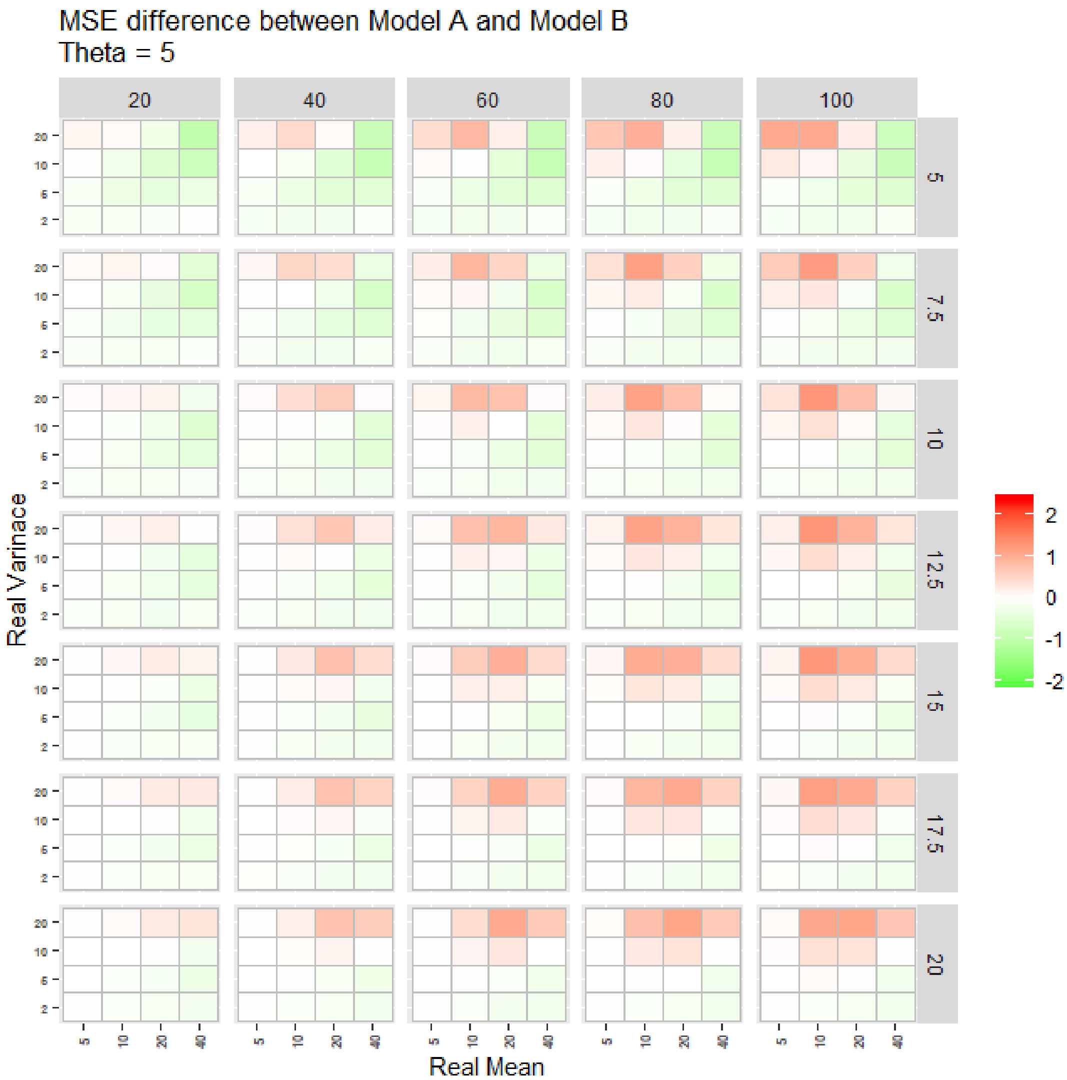


Figure 5.3.13: Compare Clayton (Model A) and Classical (Model B). Present cases of better and worse performance on 16 different combinations of prior mean and variance values. Sample size m = 20, 40, 60, 80, 100 (columns) and exposure time t = 5, 7.5, ..., 20 (rows). Green areas suggest the Clayton Model. Red areas suggest the classic EB model assuming independence.

Figure 5.3.14: Compare Clayton (Model A) and Classical (Model B). Present cases of better and worse performance on 16 different combinations of prior mean and variance values. Sample size m = 20, 40, 60, 80, 100 (columns) and exposure time t = 5, 7.5, ..., 20 (rows). Green areas suggest the Clayton Model. Red areas suggest the classic EB model assuming independence.

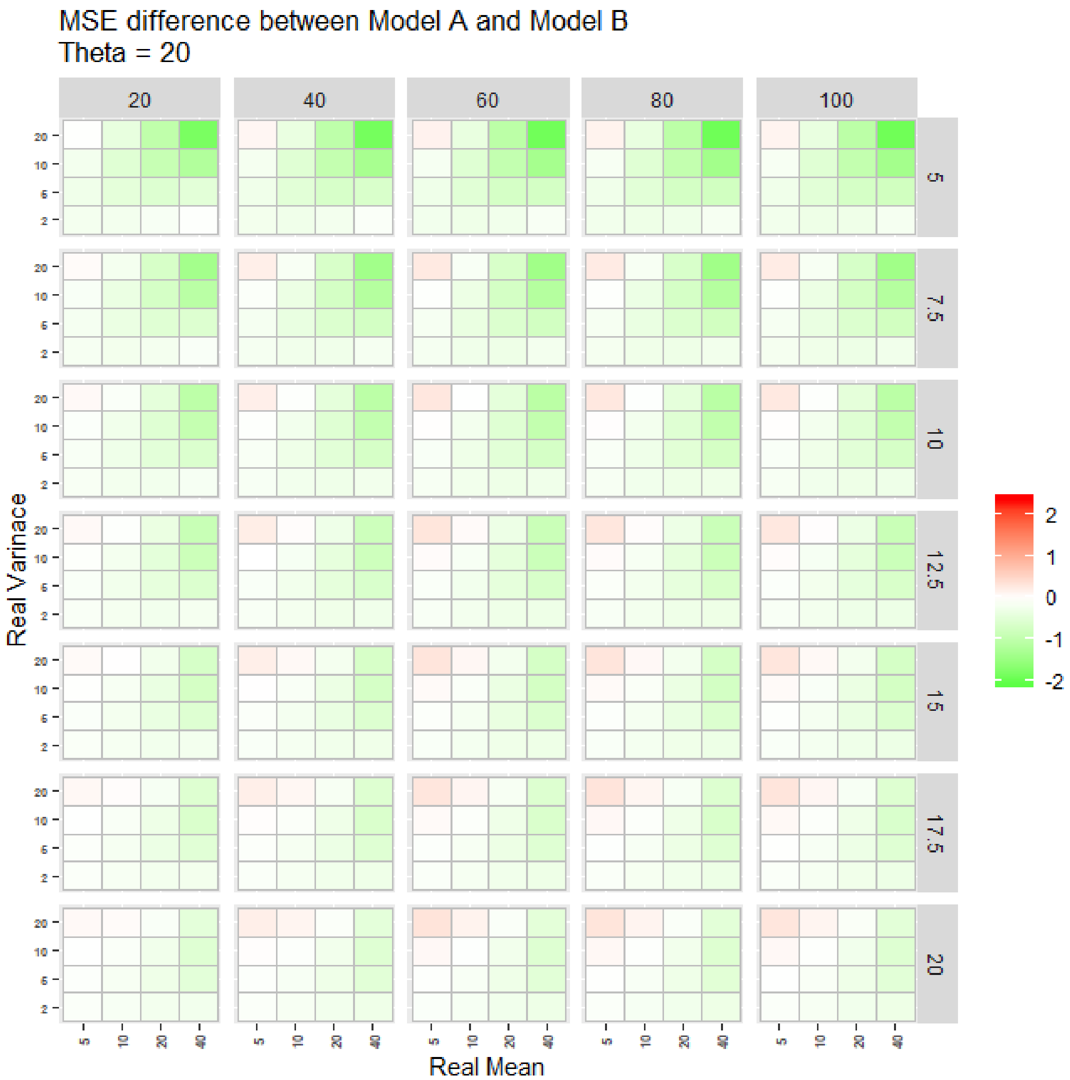


Figure 5.3.15: Compare Clayton (Model A) and Classical (Model B). Present cases of better and worse performance on 16 different combinations of prior mean and variance values. Sample size m = 20, 40, 60, 80, 100 (columns) and exposure time t = 5, 7.5, ..., 20 (rows). Green areas suggest the Clayton Model. Red areas suggest the classic EB model assuming independence.

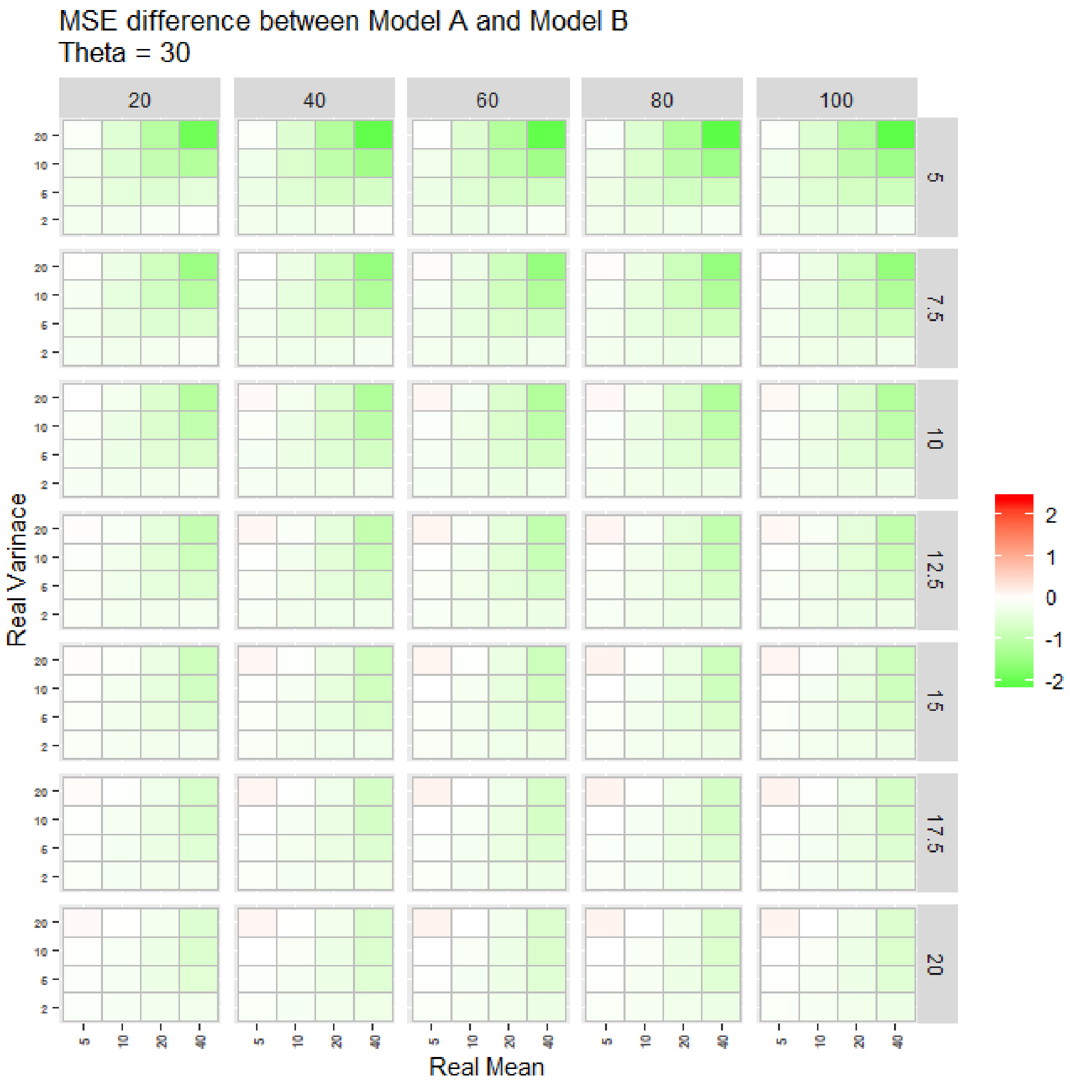


Figure 5.3.16: Compare Clayton (Model A) and Classical (Model B). Present cases of better and worse performance on 16 different combinations of prior mean and variance values. Sample size m = 20, 40, 60, 80, 100 (columns) and exposure time t = 5, 7.5, ..., 20 (rows). Green areas suggest the Clayton Model. Red areas suggest the classic EB model assuming independence.

5.4 Summary

In this chapter, we detailed the simulation and benchmarking studies conducted for evaluating the empirical Bayesian with Clayton copula model proposed. We presented the design of the study, the methodological framework followed and all relevant results. In particular, we discussed how the number of accepted samples was affected by all different parameter combinations. Our findings showed that when the true prior mean increases, more samples were rejected due to underdispersion, and as the pool size and exposure time increase, fewer rejections were observed. We also evaluated the prior marginal expectations and identified two different cases. Firstly, the smaller the prior variance, the smaller the MSE of the prior mean; and secondly the larger the pool size, the smaller the MSE of the prior mean.

Moreover, we evaluated the empirical Bayesian with Clayton copula model errors. Based on our simulation results, we observed that when the exposure time increases the MSE, RMSE and RMSRE of the errors decrease. We also identified a similar trend regarding the different dependence parameter values chosen. Particularly, the stronger the dependency between the rates, the lower the errors. However, we identified a less significant decrease when the pool size increases.

Finally, we reported the results of the benchmarking study conducted for comparing the empirical Bayesian with Clayton copula model (Model A) to the classic empirical Bayesian model (Model B) and the estimates of the rates obtained without pooling (Model C). Our findings showed that the MSE decreases as exposure time increases. However, the larger the prior mean, the higher the MSE of the rates. Comparing Models A and B, we observed similar results regarding the MSE when the dependence parameter is relatively small. However, when the dependence between the rates is becoming stronger, we observed that Model A performs significantly better than Model B when the prior variance increases. Lastly, Model C was considered inferior to Models A and B in most cases.

In the next chapter, we further investigate the impact of not accounting for dependency by providing an application of the proposed methods. Particularly, we will discuss methods for ranking in supply chain area with focus on methods for supplier ranking under uncertainty.

Chapter 6

Methods for Ranking

6.1 Introduction

The development of methods for evaluating the quality of organisations and providing quantitative comparative assessments between organisations has been a concern in various sectors, both public and private. Goldstein and Spiegelhalter (1996) discussed the need of establishing appropriate measures, known as ’performance indicators’, and interpreting with care and sensitivity apparent differences, and the need of considering model-based uncertainty when making quantitative comparisons. A performance indicator can be characterised as ’a summary statistical measurement on a system which is indented to be related to the ’quality’ of its functioning’ (Goldstein & Spiegelhalter, 1996). There may be multiple indicators that describe different aspects of the system and correspond to different objectives. If we consider a ranking problem from the supply chain context, a supplier performance indicator can be the number of on-time deliveries that indicates the delivery performance or the number of non-conforming items that indicates the quality performance.

Hierarchical models have been proposed for quantitative comparisons between organisations (see Raudenbush & Bryk, 2002). In particular, Goldstein and Spiegelhalter (1996) discussed the statistical issues involved in providing comparisons between institutions in the area of health and education, and suggested rankings based on confidence intervals of the random effects associated with organisations. However, as they state, 'an overinterpretation of a set of rankings where there are large uncertainty intervals can lead both to unfairness and inefficiency and unwarranted conclusions about changes in ranks' (Goldstein & Spiegelhalter, 1996). Therefore, careful consideration is required in situations where a single model is not sufficient and not able to capture all the variability. To this extent, Minotti and Vittadini (2010) proposed a two - step approach so - called the Local Multilevel (Hierarchical) model which aimed to capture the local behaviour by combining the Cluster - Weighted modelling with the Hierarchical modelling. They argued that non - homogeneity and non - linearity may appear in the individual - level relationships; and thus, choosing a single model which fails to capture such relationships will may cause a large loss of information. Therefore, with their two - step approach, firstly, the individual - level relationships (if exist or whenever they are) are captured and modelled, and secondly, the differences between organisations are locally identified.

Hierarchical models have also been proposed in areas closely related to organizational comparisons. In the supply chain area, various methods and models have been proposed for supplier ranking (De Boer et al., 2001; Chai et al., 2013). Walls et al. (2016) present a novel modelling suite using relevant historical data for supporting the analysis of risk in different stages of supplier life. They proposed an empirical Bayes method for ranking Poisson count data with heterogeneous exposure to risk. However, their comparative assessments stand only under the assumption of independence. To our knowledge, there is a gap in the literature about methods for ranking correlated event rates, and therefore we shall focus on this area. Although much of our discussion is relevant to ranking on event rates, we shall concentrate on the area of supplier ranking. We believe that the methods we discuss are generally applicable whatever measures (in our case, supplier key performance indicators) are chosen and for whatever purpose.

Supplier ranking models exist in the literature only as of the final phase of the supplier selection process (Chai & Ngai, 2020). More broadly, in the supplier selection process, multiple possible suppliers are of consideration with the ultimate goal of selecting one or more best-performing suppliers. All possible suppliers are initially evaluated and then ranked based on prespecified criteria. In contrast, the supplier ranking process that we intend to focus on is based on already existing suppliers within the organisation; and it requires the existence of relevant historical data with the goal of ranking already selected suppliers. Therefore, we choose to follow the supplier selection process, mainly focusing on the final phase, the supplier ranking. For further discussion about the supplier selection process in the supply chain management area see Appendix C.

Ranking under uncertainty can be a complicated process, and the final rankings may be affected by various factors. There are inevitable limitations that need to be considered when dealing with rankings. First, 'we should exert caution when applying statistical models to make comparisons between institutions, treating results as suggestive rather than definitive' (Goldstein & Spiegelhalter, 1996). Secondly, 'measurement of outcomes for research purposes is useful to help organisations to detect trends and spot extreme outliers' (Lilford, Mohammed, Spiegelhalter, & Thomson, 2004).

6.2 Motivation

This study has been motivated by the challenges supply chain managers face in several manufacturing companies. These challenges are all related to supplier evaluation and ranking problems. It is known that for several manufacturing companies, a high percentage of parts and subassemblies needed are outsourced from suppliers globally (Bag, 2018). Thus, maintaining close relationships with suppliers is essential, as their performance depends on theirs. Monitoring, developing, maintaining or exiting the relationship with the suppliers are several of the responsibilities that supply managers have. For the manufacturing companies, the set of suppliers frequently changes, as new suppliers are being added and/or existing ones are being exited. Additionally, the volume of work for each supplier is different and varies over time, depending on the company's workload and needs. Considering that manufacturers produce customised products along with the limited holding space and resource, it is impossible to keep a spare for every component type, so orders are placed as needed.

On-time delivery of parts to the required quality standard is necessary for the production line of the company. However, not on - time (early or late) deliveries of parts or non - conforming parts may cause disruption and production delays. Even if the supply managers theoretically understand the causes of supplier failure to deliver parts on - time to the expected quality which eventually affect their performance, they do not have analytical models to provide evidence about factors that drive supply risk. Such factors are directly connected with the supplier' performance and consequently shape and describe each supplier. Evaluating every supplier within the pool is becoming essential for the managers. Performance profiles and ranking systems of all suppliers under consideration are of need, to help the managers identify the position of each supplier within the pool and take action. Best performing suppliers will get all the credit, but for the poor performing ones further investigation will be of need. Therefore, analytical models that would aid better risk management and provide evidence about the suppliers' performance to support decision making are required.

6.3 Purpose of Risk Analysis

Empirical data, including supplier characteristics and performance records, can be available. This study shows how we can make use of such data to initially better understand and analyse each supplier, and further provide a robust comparative assessment of supplier performance and present various ranking systems, which are mostly focused on delivery and quality performance measures. Multiple supplier ranking systems have been considered for this study, which aim primarily to present a holistic view of different ranking approaches and perspectives, and further highlight how different modelling choices, from choosing the pool of suppliers to adopting a particular ranking method, may affect the rankings and produce different outcomes. To manage operations and schedule production, the manufacturing companies keep records of every supplier activity. We also show how this kind of data can be exploited to provide more insight into the supply process, and summarised so that they can be used in such applications.

We propose a method for ranking correlated event rates, which in this case are supplier performance rates. The exact value on which we intend to rank is unknown, but we have a prior distribution describing the uncertainty on the rates. However, how do we rank distributions? There are various ranking methods such as mean or median ranks, ranks based on the cumulative distribution function of the ranks etc. Every method may result in different conclusions about the position of each supplier within the pool showing that there is no absolute right or wrong answer, but multiple perspectives of ranks even under the same conditions/circumstances. Moreover, different modelling choices may lead to entirely different conclusions about the position of each supplier within the pool. For example, selecting the pool of suppliers is an essential factor in the ranking process. Considering this factor, more questions arise which make the process even more complicated. For example, should we use one pool with all suppliers or create smaller pools, how the size of the pool affects the position of the individuals within the pool, which is the most appropriate variable to be chosen as exposure time etc. Understanding that supplier ranking is a complicated process, and the final ranking result is affected by many factors, we aim to investigate several ranking methods and perspectives, and discuss possible similarities and/or dissimilarities.

Our goal is to develop analytical models that support analysis of risk for suppliers and provide a comparative performance analysis of the suppliers considering multiple ranking approaches/methods, using mainly two performance measures, the number of late deliveries and the number of non-conforming parts. Discussion about the differences among the methods proposed and relative results is also provided.

6.4 Distinctiveness of Approach

We ground our model in the theory and methods of stochastic processes, to develop sound ways of making use of historical data available in enterprise planning systems to support decision making on supplier ranking and selection issues. Our work relates to the broader literature of ranking on event rates. More particularly, our methodology relates to methods for analysing count data generated from Poisson processes, see e.g. Walls et al., 2016 and Quigley, Bedford, & Walls, 2007b. Walls et al. (2016) present a novel modelling suite using relevant historical data for supporting the analysis of risk in different stages of supplier life. They proposed an empirical Bayes method for ranking Poisson count data with heterogeneous exposure to risk, which focuses on late delivery rate and non – conformance rate. However, their analysis stands only under the assumption of independence, which makes our proposed model distinct from theirs. As in many cases, when dealing with real data applications, the dependency between performance rates exists and therefore needs to be considered.

Our model is based on the empirical Bayes method taking into consideration the underlying dependency between two key performance indicators, the late delivery rate and the non – conformance rate. The dependency is modelled using a Clayton copula function with Gamma marginal distributions. In the remainder, we discuss the methodological framework followed for the development of the proposed method for ranking based on supplier event rates and present the models developed. Even if we concentrate on ranking supplier event rates, we believe that the methods we propose are generally applicable whatever measures (event rates) are chosen and for whatever purpose.

6.5 Methodology for Ranking Supplier Event Rates

Our methodological framework is presented in Figure 6.5.1. It illustrates the relations between the empirical data (inputs), the modelling components and the ranking methods, as well as the related management decisions. Table 6.5.1 summarises the scientific method used and discusses more explicitly the modelling choices and management decisions associated.

Supplier Ranking System	
Method:	• Empirical Bayes inference
Modelling Choices:	• Select the pool of suppliers. • Select the appropriate variable which will best describe the exposure time. • Select a specific ranking method for ranking the suppliers.
Decision Support:	• Identify the position of a supplier within a pool of suppliers. • Identify the best and worst performing suppliers. • Compare suppliers based on Late delivery and Non - conformance rankings. • Compare the position of a supplier within the pool of suppliers between different ranking methods.

Table 6.5.1: Summary of the Supplier Ranking System developed and the related modelling choices and decisions.

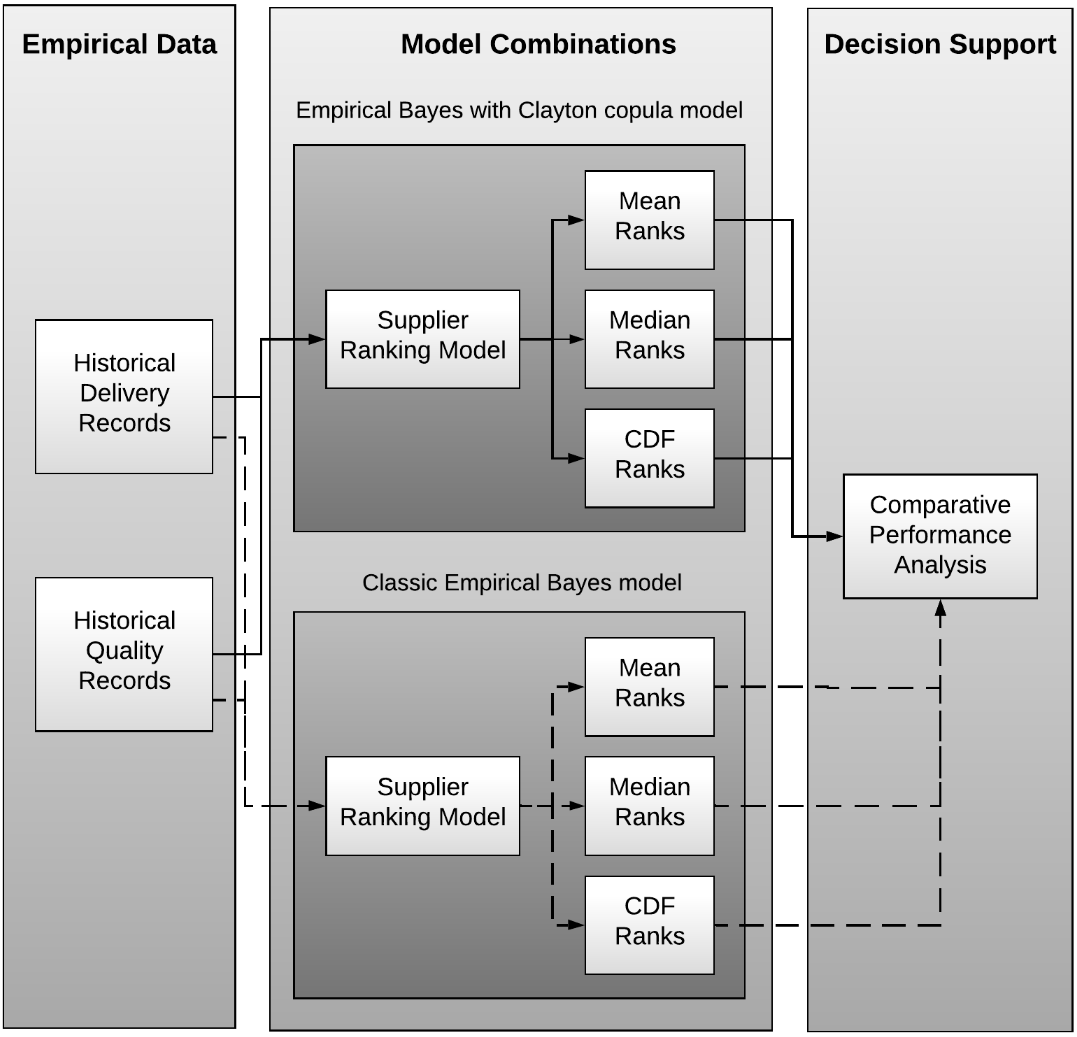


Figure 6.5.1: Methodological framework.

6.5.1 Supplier Ranking Criteria

According to the literature, several supplier selection criteria have been considered important over the years (see Appendix C.2). In particular, previous researches have shown that the most important criteria are the delivery performance and the price; however, recent studies show that quality has also reached a high ranking (Bharadwaj, 2004). For this particular study, two performance measures, delivery and quality, have been selected for supplier evaluation and ranking. Not on-time deliveries (early or late) directly affect the production line and might cause from a minor inconvenience, such as new production schedule and more workload (for late deliveries) or extra inventory cost (for early deliveries), to major late penalty charges from the end customer if the final product is not finished as expected. Notably, there are numerous factors, from both supplier and manufacturer parties or externally, that influence the delivery performance of a supplier. From the supplier perspective, for example, consider the situation where, during the production, there may occur machine failures that resulted in unanticipated rework or raw material problems. Also, during transportation from the supplier to the manufacturer, delays may occur because of the logistics company failure to pick up the parts on time or due to weather issues. Consider also the case where there may be miscommunication between the two parties due to data management systems problems, or where the manufacturer may be delayed in sending precise specifications of a newly designed part and so hold up progress at the supplier.

On the other hand, deliveries of non-conforming parts may also affect the production of the final product, and cause from minor inconvenience through to significant late penalty charges from the end customer if the final product is not finished at the expected time, but now due to the delay caused by the non-conforming parts delivered. Again, numerous factors, internal or external, may influence on the quality performance of a supplier. From the supplier perspective, for example, consider the situation where during the production, there may be machine failures resulting in unexpected non-conforming parts. Also, during transportation from the supplier to the manufacturer, quality issues may occur due to improper handling of the parts by the logistics company.

Therefore, evaluating the delivery and quality performance of a supplier is of key importance. We understand that the delivery and quality performances of a supplier represent a highly complex relationship between numerous internal and external factors. However, it is not always possible to consider every possible factor, and even if we wanted to, there is no sufficient historical data for all companies involved. The manufacturing companies keep delivery records with information relevant to the delivery and quality performance for every supplier. Each record represents one delivery and contains related information about the order, for example, supplier name (ID), part specification, ordered date, expected delivery date, order quantity, rejected quantity and so on. So, in this study, we focus on evaluating and ranking the suppliers under consideration using historical delivery and quality records available in every manufacturing company.

6.5.2 Bayes with Clayton Copula Method

The delivery and quality performance of a set of suppliers can be assessed based on simple statistical ratios. Knowing the number of orders of each supplier and the number of the late deliveries, the late delivery rate for each supplier can be obtained by calculating the ratio of the number of late deliveries over the total number of orders. Accordingly, the non – conformance can be obtained by calculating the ratio of the number of non-conforming parts over the total number of parts ordered. Then, rankings are given to each supplier based on their performance rates, and thus, all suppliers within the pool can be compared based on the rankings on both performance measures. However, by doing this, careful consideration is not being made about factors that as we have already mentioned are highly variable between suppliers. For example, the number and volume of the orders may vary for each supplier. Also, the two performance rates are considered as independent, which is not always the case, especially when dealing with real data. By ignoring the underlying dependency between the rates may mislead to incorrect conclusions about the position of a supplier within the pool which consequently can be financially costly for the manufacturing company. Therefore, the exposure of each supplier to risk needs to be considered when choosing a pool of suppliers for ranking purposes, as well as the underlying dependency between the two performance rates.

A Bayesian method for supplier ranking under uncertainty has been developed. We model the dependency between the two rates by using a Clayton copula, which describes cases with strong left-tail dependence. If we consider our motivation for this study, the Clayton copula seems appropriate for supplier ranking problems. As already mentioned, we consider the underlying dependence on the rates and not the correlation on the realisations of events (Poisson process data), implying that the underlying dependence on the rates is driven by the operations within organisations rather than the occurrence of rare or extreme events. Considering then this situation, we claim that when a supplier performs well on delivery, it usually performs well on quality as well; and when a supplier has a poor performance on one measure, it is uncertain how it would perform on the other, as other factors affect their operations within the organisation. Several studies in the literature agree with our assumption addressing that correlation exists between operations within organizations. Kang et al. (2016) define key performance indicators (KPIs) as 'a set of metrics which reflect the operation performance' and are considered critical for manufacturing operation management and continuous improvement. They also state that KPIs in a manufacturing system are not independent, and they may have intrinsic mutual relationships. Therefore, they introduce a multi - level hierarchical structure for identifying and analysing KPIs and their relationships in production systems. Note also that the exposure time is assumed to be the same for all suppliers within the pool.

We now discuss the methodology followed for this method. A more detailed description of the empirical Bayes with Clayton copula model can be found in Section 3.2. We denote the variable N_{ji} as the number of events that are realised over the time interval, t_i, for the jth performance measure and ith supplier. The distribution of N_{ji} is Poisson and has the following probability density function,

$$N_{ji} \sim Poisson(\lambda_{ji}\ t_i)\ ,\ \ j = 1,2,\ \ i = 1,\cdots,m$$

$$P(N_{ji} = n_{ji}|\lambda_{ji}) = \frac{(\lambda_{ji}\ t_i)^{n_{ji}} e^{-\lambda_{ji}\ t_i}}{n_{ji}!}\ ,\ \ \lambda_{ji} > 0,\ \ t_i > 0,\ \ \ n_{ji} = 0,1,\cdots,$$

where m represents the size of the pool. The prior distribution considering the underlying dependency of the rates represented by a Clayton copula (bivariate case) with Gamma marginal distributions is now presented. We denote the variable $\Lambda_{ji} \sim G(\alpha_j, \beta_j)$ as the rate of events for the ith supplier and jth performance rate. The prior distribution is as follows,

$$c(\lambda_1, \lambda_2) = c(F(\lambda_1), F(\lambda_2)) \cdot f(\lambda_1) \cdot f(\lambda_2)$$

where $F(\cdot)$ is the cumulative distribution function of the Gamma distribution, $f(\cdot)$ is the Gamma density probability function and $c(u, v)$ is the probability density function of the Clayton copula which has the following form,

$$c(u, v) = (\theta + 1)(uv)^{-(\theta+1)}(u^{-\theta} + v^{-\theta} - 1)^{-(2\theta+1)/\theta},\ \ \theta \in (0, \infty).$$

Following the Bayesian method, we define the predictive and posterior distributions, as follows,

$$P_{\text{Pred}}(\lambda_{ji}|N_{1i}, N_{2i}) = \int_0^\infty \int_0^\infty c(\lambda_{1i}, \lambda_{2i}) \cdot \text{Pois}(N_{1i} = n_{1i}|\lambda_{1i}) \cdot \text{Pois}(N_{2i} = n_{2i}|\lambda_{2i}) d\lambda_1 d\lambda_2$$

and,

$$P_{\text{Post}}(\lambda_{ji}|N_{1i}, N_{2i}) = \frac{c(\lambda_{1i}, \lambda_{2i}) \cdot \text{Pois}(N_{1i} = n_{1i}|\lambda_{1i}) \cdot \text{Pois}(N_{2i} = n_{2i}|\lambda_{2i})}{\int_0^\infty \int_0^\infty c(\lambda_{1i}, \lambda_{2i}) \cdot \text{Pois}(N_{1i} = n_{1i}|\lambda_{1i}) \cdot \text{Pois}(N_{2i} = n_{2i}|\lambda_{2i}) d\lambda_1 d\lambda_2}.$$

Since the prior distribution is Clayton copula with Gamma marginals, the posterior distribution cannot be obtained in a closed - form function or described as other well - known distribution functions. However, an empirical distribution of the rates for each supplier within the pool can be evaluated through simulations. The algorithm describing the simulation process for evaluating the posterior distribution is presented in Section 6.5.5.

6.5.3 Classic Bayes Method

In this section, we present the classic Bayesian model, where no underlying correlation on the rates is considered. The prior distributions of both rates are chosen to be Gamma as follows,

$$\Lambda_{1i} \sim Gamma(\alpha_1, \beta_1),$$
$$\Lambda_{2i} \sim Gamma(\alpha_2, \beta_2), \qquad i = 1, 2, \cdots, m \ ,$$

with the following probability distribution function,

$$\pi(\lambda_{ji}) = \frac{\beta_j^{\alpha_j}}{\Gamma(\alpha_j)} \lambda_{ji}^{\alpha_j - 1} e^{-\beta_j \lambda_{ji}} \ , \qquad \lambda_{ji}, \alpha_j, \beta_j > 0 \ , \ j = 1, 2.$$

The number of events that are realised over time interval t is denoted by the variables N_{1i}, N_{2i}. The distributions of N_{1i}, N_{2i} are Poisson and have the following probability density functions,

$$N_{1i} \sim Poisson(\lambda_{1i}\ t), \qquad N_{2i} \sim Poisson(\lambda_{2i}\ t)\ , \ i = 1, \cdots, m$$

$$P(N_{ji} = n_{ji} | \lambda_{ji}) = \frac{(\lambda_{ji}\ t)^{n_{ji}} e^{-\lambda_{ji}\ t}}{n_{ji}!} \ , \ \lambda_{ji} > 0, \ t > 0, \ n_{ji} = 0, 1, \cdots, j = 1, 2.$$

Then, the posterior distribution can be described by a well - known function, a

Gamma distribution, as follows,

$$\Lambda_{ji}|N_{ji} \sim Gamma(\hat{\alpha}_j + n_{ji}, \hat{\beta}_j + t),$$

$$\pi(\lambda_{ji}|N_{ji} = n_{ji}) = \frac{(\hat{\beta}_j + t)^{\hat{\alpha}_j + n_{ji}} \lambda_{ji}^{\hat{\alpha}_j + n_{ji} - 1} e^{-(\hat{\beta}_j + t)\lambda_{ji}}}{\Gamma(\hat{\alpha}_j + n_{ji})}.$$

6.5.4 Empirical Distribution of the Ranks

Summarising, we have structured the ranking problem from a Bayesian perspective. Mainly, we have a likelihood to explain the variability in observation given and inherent rate of interest, and a prior distribution to describe the uncertainty on the rates. We wish to rank on the inherent rate, but we have different amounts of uncertainty about each across the pool. As such, the rank, i.e. that which would be revealed if we knew the event rate for certain, is also uncertain.

Ranking under uncertainty is not a simple problem. We can express the probability distribution of the ranks for each rate, as follows,

$$P(R_i = k) = \sum_{S} \int_0^\infty \left[\prod_{x=1}^{k-1} \Pi_{s_x}(\lambda | \underset{\sim}{n}, \underset{\sim}{t}) \right] \left[\prod_{x=k+1}^{m} \left(1 - \Pi_{s_x}(\lambda | \underset{\sim}{n}, \underset{\sim}{t}) \right) \right] \cdot \pi_i(\lambda | \underset{\sim}{n}, \underset{\sim}{t}) \, d\lambda \ ,$$

$$k = 1, 2, \cdots, m,$$

where we denote,

- $\prod_{x=1}^{k-1} \Pi_{s_x}(\lambda | \underset{\sim}{n}, \underset{\sim}{t})$: $k-1$ supplier better than supplier i
- $\prod_{x=k+1}^{m} \left(1 - \Pi_{s_x}(\lambda | \underset{\sim}{n}, \underset{\sim}{t}) \right)$: $m-k$ suppliers worse than supplier i.

However, the probability distributions of the ranks have bi - modal or multi - modal shapes even for simple prior distributions. As such, we require Monte Carlo methods for obtaining the empirical distribution of the ranks, as described in Section 6.5.5.

Although we have obtained the distributions of the ranks, what would be an appropriate measure to rank on? There is no straightforward answer, as we are dealing with much uncertainty. Instead, we present three statistical criteria for ranking showing that there is no absolute right or wrong analysis but different perspectives and approaches, which eventually better inform decision making.

- Ranking by the mean of the distribution of ranks: $E(\lambda_{ji}|N_{1i}, N_{2i})$. This approach corresponds to a loss function the squared difference between the true and estimated rank.
- Ranking by the median of the distribution of ranks. With this approach, we consider a loss function the absolute difference between the actual and estimated ranks.
- Ranking by the cumulative distribution function (CDF) of the ranks: Ranking by the probability that i supplier belongs to the top k_1 or bottom k_2 suppliers, $P(R_i \leq k_1)$ and $P(R_i \geq k_2)$, respectively.

6.5.5 Simulation Algorithm

The posterior distribution cannot be expressed in closed - form, but can be evaluated through simulations. The simulation algorithm for obtaining the empirical posterior distribution of the rates of event can be summarised by the following steps:

1. Input $(N_{1i}, N_{2i})_m$: Number of late deliveries and number of non - conforming parts.
2. Input t_{1i}, t_{2i}: Exposure time (set to be the same for all suppliers within the pool)
3. Estimate Prior parameters.

4. Update Prior and define the posterior distribution.

5. Generate k pairs from the updated Prior, $(\lambda_{1k}, \lambda_{2k})_k$

6. For each supplier $i = 1, 2, \cdots, m$ within the pool, calculate:

   - The acceptance probability, $c^{(i)} = \text{Pois}(n_{1i}|\lambda_{1k}) \cdot \text{Pois}(n_{2i}|\lambda_{2k})$.
   - Normalize the acceptance probability, $c^{(i)}_{\text{norm}} = c^{(i)}_k / \sum_{h=1}^{k} c^{(i)}_h$.

7. Sample $(\lambda_1, \lambda_2)^{(i)}_s$, $s \leq k$ from the empirical distribution with the associated probabilities $c^{(i)}_{\text{norm}}$.

8. For every sample $s = 1, 2, \cdots, s$, transform rates into ranks $R^{(i)}_s$.

9. Calculate mean rank for each supplier, $R^{(i)} = \sum_{s=1}^{s} R^{(i)}_s / s$.

6.6 Summary

In this chapter, we discussed methods for ranking. We discussed the motivation and the distinctiveness of our approach. We also explained the methodological framework followed for the study. Although much of our discussion was relevant to ranking on event rates, we only concentrated on ranking supplier event rates. However, we believe that the methods we discussed are generally applicable when dealing with unknown event rates.

In particular, we developed a Bayesian method for ranking event rates under uncertainty. Our proposed model is based on the empirical Bayes method considering the underlying dependence between two performance measures, the late delivery rate and the non – conformance rate. We modelled the dependency between the two rates by using a Clayton copula, which describes cases with strong left-tail dependence.

Ranking under uncertainty can be challenging. Careful consideration needs to be made on multiple modelling choices. This will be investigated in the following chapter where we present examples of analysis and model applications using desensitised real data from the prime manufacturer considering multiple situations.

Chapter 7

Empirical Evaluation of Ranking Methods Based on Historical Supplier Data

7.1 Overview

In this chapter, we illustrate the application of the methods and models discussed in Chapter 6, and discuss the choices made throughout the modelling process. Note that we follow the empirical Bayes methodology, where the prior parameters need to be estimated before defined the predictive and posterior distributions using the observed data available. Specifically, the marginal Gamma parameters can be estimated through Moment estimation method; and the dependence parameter through the non-linear model (closed form expressions) proposed and discussed in Chapter 4. The analysis of data is also presented by describing the nature of data available and discussing the data preparation process. Note that the data used are 'real' de-sensitised data, and all analysis is aligned with an industry problem related to the supplier ranking. Discussion about the data available, data cleaning and preparation process is following.

7.2 Description of the Data Available

The data set under consideration originates from a prime manufacturer and includes records of purchase orders from multiple suppliers within five years. Every entry of the data set corresponds to a specific supplier delivery record of a particular type of product. Note that there are cases where multiple delivery entries are filed under the same purchase order; however, we consider every entry individually. These entries consist of relevant information about the delivery, including the supplier account, the date of the order was placed, the requested date, the delivery date, the specification of the type of part ordered; the ordered, delivered, rejected quantity and the relevant order price. The variables available are limited, considering the nature of our study; however, new variables can be defined using the existing ones. For example, covariates capturing the supplier delivery performance such as the number of days that an order is delivered early, on-time or late. We then consider the quality of the data available and proceed to the cleaning process. Records with missing values or invalid details (for example, negative lead time which indicates incorrect order or delivered date) in variables under consideration are removed from the data set.

Prime Manufacturer Data			
No of Entries: 27772			
No of Variables: 23			
No of Suppliers: 238			
Subsets	**No. Orders**	**No. Suppliers**	**No. Entries**
1	$[1,10]$	146	492
2	$(10,20]$	32	466
3	$(20,50]$	21	701
4	$(50,100]$	17	1189
5	$(100,500]$	10	1801
6	$(500,\infty)$	12	23123

Table 7.2.1: The prime manufacturer data available after the cleaning process.

After the data preparation and cleaning process, the data set consists of 27772 purchase orders and 23 variables of 238 suppliers in total. In table 7.2.1, we identify the total number of suppliers who have relatively low (less than 10) or relatively high (over 500) volume of orders within the pool of 238 suppliers. Table 7.2.2 shows standard summary statistics of key variables for randomly selected suppliers. Our intention is to provide a clear view of the raw data available (Table 7.2.3) and how they can be transformed and used in the ranking system.

Supplier ID	No of Orders	Mean Lead Time	Mean Late Del.	Mean Order Qty	Mean Rej. Qty
A13	502	25.66	2.34	15.12	0.08
A16	609	25	3	13.54	0.09
A18	178	26.04	1.96	19.88	0.16
A19	192	25.10	2.90	29.61	0.64
B13	286	5.70	1.30	68.21	0.55
H13	619	25.94	2.06	429.97	13.84
M13	279	26.59	1.41	21.47	0.24
R11	580	25.68	2.32	3.45	0.02
V10	681	26.20	1.80	6.12	0.10

Table 7.2.2: Showing standard summary statistics of selected suppliers.

	P. O.	Supplier Acct	Order Qty	Order Date	Delivery Date	Required Date	Rejected Qty
1 |	300101	B10	19	2011-09-09	2011-10-07	2011-10-07	0
2 |	300101	B10	7	2011-09-09	2011-10-07	2011-09-27	0
3 |	300101	B10	5	2011-09-09	2011-10-07	2011-10-07	0
4 |	300102	A16	100	2011-09-09	2011-10-07	2011-10-07	0
5 |	300102	A16	70	2011-09-09	2011-10-07	2011-10-02	0

Table 7.2.3: Extracted from raw data from prime manufacturer.

7.3 Supplier Ranking Applications

We present an illustration of the proposed supplier ranking method (see Section 6.5), applied on several selection sets of suppliers over different periods. The empirical data, discussed in Section 7.2, have been transformed into count numbers as the method requires. New variables consist of aggregated data that represent the total number of late deliveries and the total number of non - conforming parts (rejected quantity) have been defined. Note that the ranking method can be applied in any time horizon of interest and supplier rankings would correspond accordingly in each time frame (e.g. monthly, quarterly or yearly). However, we choose to have the same exposure time for all suppliers within the selected pool. For example, all suppliers within a selected pool have been tested by the manufacturer for the same time period (e.g. number of months) and had an approximately similar volume of orders and ordered quantity. Figure 7.3.1 shows the selected applications for this study. All four selected applications consist of different pools of suppliers within different periods (exposure times). Particularly sets 1 and 2 have the same exposure time, but the pool size differs. In set 2, the pool of suppliers under consideration is smaller than in set 1. Set 3 includes the largest pool across all applications; however, the exposure time is now set to 12 months. Lastly, set 4 has the same exposure time with set 3, but its pool size is significantly smaller than in set 3. All selected applications will be analysed and compared later on.

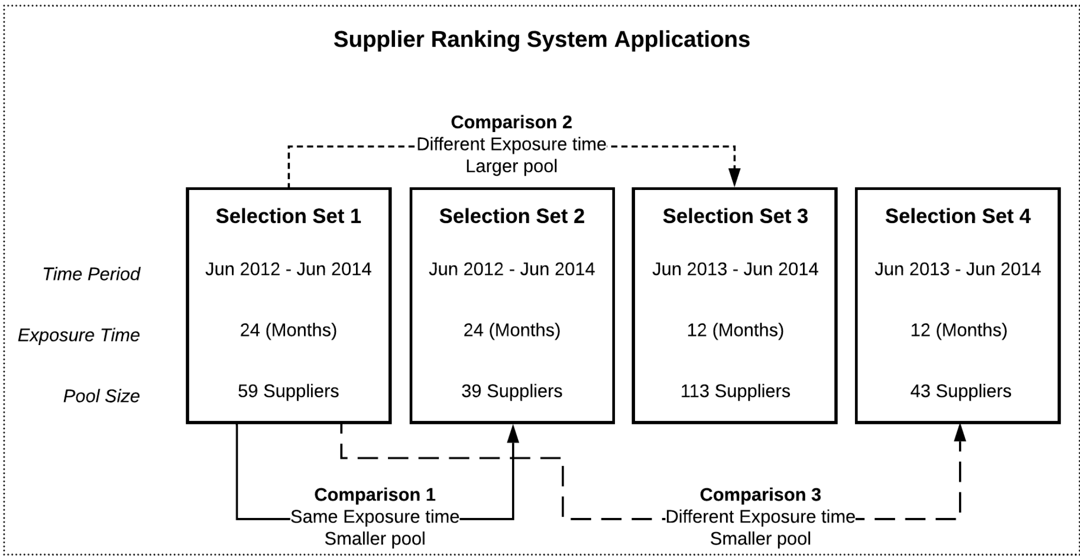


Figure 7.3.1: Selected supplier ranking applications.

As already mentioned, we follow the empirical Bayes method and thus we need to estimate the prior parameters. The marginal Gamma parameters are obtained by using the method of moments (see Quigley et al., 2007b). We also need to provide an estimate for the dependence parameter (see Chapter 4). The proposed formula takes as input the size of the pool (m), the exposure time (t), the estimated prior mean ($\hat{a}/\hat{b}$) and variance ($\hat{a}/\hat{b}^2$) assuming that both marginals are identically distributed, and the Kendall's tau between the observed data. However, when dealing with real data, this might not be the case, as the marginals could have different parameters. For this case, we propose to use uninformative prior for the prior Kendall's tau. We summarise the simulation process in the following steps.

Uninformative prior for obtaining an estimate of the prior Kendall's Tau, when having different marginals

Algorithm Steps

1. Set the Gamma marginal parameters (a, b), exposure time (t) and pool size (m).
2. Define uninformative Kendall's tau prior function $(\tau \in (0,1))$, $\pi(\tau) = 1 \sim U(0,1)$.
3. Generate τ from prior $\pi(\tau)$.
4. For $i = 1, 2, \cdots, N$.
5. Define the likelihood by using the function $f(\tau_f; \tau)$ (see Section 4.4).
6. Calculate $\tau_f^{(i)}$ from $f(\tau_f; \tau)$.
7. Go to Step 4 and repeat.
8. Go to Step 3 and repeat.

9. Update and define the posterior distribution,

$$\pi(\tau|\tau_f) = \frac{f(\tau_f;\tau)\cdot\pi(\tau)}{\int_0^1 f(\tau_f;\tau)\cdot\pi(\tau)\,d\tau_f}.$$

10. Given the data (and by using the function $f(\tau_f|\tau)$), we observe $\tau_f = \tau_{\text{obs}}$, so

$$\pi(\tau|\tau_f = \tau_{\text{obs}}) = \pi(\tau|\tau_f = \tau_{\text{obs}} \pm \epsilon).$$

11. An estimate of the prior Kendall's tau is the mean of the distribution,

$$\tau^\star = E\left[\pi(\tau|\tau_f = \tau_{\text{obs}})\right].$$

Moreover, we evaluate the suitability of the Clayton copula by using the parametric bootstrap method (Carlin & Louis, 2008), with which we create confidence intervals of Kendall's tau for the Clayton copula ensuring the existence of a rank correlation between the rates. The parametric bootstrap method simply mimics the simulation process of the empirical Bayes with Clayton model, by generating data. The process of the 'parametric' bootstrap is entirely generated from the prior rates, rather than by resampling (with or without replacement) from the observed data itself ('non-parametric bootstrap'). Mainly, we initially set the prior parameters, the exposure time and the sample size according to the observed data available, we then generate samples from the prior and the likelihood, estimate the prior dependence parameter and repeat.

In the following, we present the results of the applications of the proposed supplier ranking considering that the ranks are extracted from the empirical posterior distribution of the Late - delivery and Non - conformance rates. We show and compare rankings obtained using the mean, the median and the cumulative distribution function of the empirical distribution of the ranks. We also show that the position of each supplier may change if we choose a different pool or exposure time. We finally compare the suppliers' mean rankings obtained using the empirical Bayes with Clayton copula model to the Classic empirical Bayes method assuming no underlying dependency on the rates.

7.3.1 Transformed - Aggregated Real Data

Four (4) different selection sets have been chosen to illustrate the application of the proposed Supplier Ranking. We intend to provide a comparative analysis of supplier rankings across different settings, e.g. different exposure time and pool size. Selection sets 1 and 2 have the same exposure time, but different pool size. Selection set 3 has different exposure time and larger pool size compared to set 1; and lastly, the selection set 4 have different exposure time and smaller pool size compared to set 1. As already discussed, we use real de-sensitised prime manufacturer data which have been transformed into aggregate data, including the number of late deliveries and the number of non-conforming parts. Tables 7.3.1, 7.3.2, 7.3.3, 7.3.4 show the aggregate data used for the applications.

	ID	E.T.	L.D.	N.C.
1	A13	24	132	25
2	A15	24	24	32
3	A16	24	80	9
4	A18	24	35	3
5	A19	24	74	122
⋮				
55	S32	24	2	0
56	T13	24	1	0
57	U10	24	50	13
58	V10	24	54	23
59	W13	24	5	0

Table 7.3.1: Selection Set 1 - Transformed data. Showing the number of late deliveries and non - conformance parts of each supplier within a period of 2 years (24 months).

	ID	E.T.	L.D.	N.C.
1	A12	12	1	0
2	A13	12	77	0
3	A15	12	24	32
4	A16	12	50	9
5	A18	12	33	3
⋮				
109	T25	12	4	0
110	U10	12	35	13
111	V10	12	36	12
112	W10	12	2	0
113	W13	12	1	0

Table 7.3.3: Selection Set 3 - Transformed data. Showing the number of late deliveries and non - conformance parts of each supplier within a year (12 months).

	ID	E.T.	L.D.	N.C.
1	A13	24	132	25
2	A15	24	24	32
3	A16	24	80	9
4	A18	24	35	3
5	A19	24	74	122
⋮				
35	S10	24	573	443
36	S11	24	7	0
37	S20	24	24	14
38	U10	24	50	13
39	V10	24	54	23

Table 7.3.2: Selection Set 2 - Transformed data. Showing the number of late deliveries and non - conformance parts of each supplier within a period of 2 years (24 months).

	ID	E.T.	L.D.	N.C.
1	A13	12	77	0
2	A15	12	24	32
3	A16	12	50	9
4	A18	12	33	3
5	A19	12	36	1
⋮				
39	S11	12	6	0
40	S20	12	4	0
41	T13	12	1	0
42	U10	12	35	13
43	V10	12	36	12

Table 7.3.4: Selection Set 4 - Transformed data. Showing the number of late deliveries and non - conformance parts of each supplier within a year (12 months).

7.3.2 Model Set Up

After choosing the selection sets, we need to check and set up the model parameters before proceeding with the application of the proposed empirical Bayes with Clayton copula model. Considering the Gamma marginal distributions, the moment estimates of the prior parameters need to be estimated by using the observed count data and thus have to be positive. Table 7.3.5 shows that all Gamma prior parameter estimates by using the Moment Estimation method are positive. However, the Gamma priors for the selected sets do not have the same parameters. Hence, we follow the process described in Section 7.3 for obtaining an estimate of Clayton Kendall's tau. As starting points, we choose Kendall's tau values between 0 and 1 ($\tau \in \{0.1, 0.11, \cdots, 0.94, 0.95\}$). Figure 7.3.2 shows the distribution of Clayton Kendall's tau, given that the observed prior dependence is already known. Table 7.3.6 shows the observed Kendall's tau along with the updated tau and theta estimates obtained by using the uninformative prior.

	P.S.	E.T.	$\hat{a}_1$	$\hat{b}_1$	$\hat{a}_2$	$\hat{b}_2$	$\hat{a}_1/\hat{b}_1$	$\hat{a}_1/\hat{b}_1^2$	$\hat{a}_2/\hat{b}_2$	$\hat{a}_2/\hat{b}_2^2$
Selection Set 1	59	24	0.20	0.09	0.10	0.13	2.22	24.07	0.81	6.30
Selection Set 2	39	24	0.32	0.10	0.16	0.13	3.24	33.30	1.22	9.05
Selection Set 3	113	12	0.10	0.06	0.04	0.08	1.67	28.24	0.49	5.93
Selection Set 4	43	12	0.27	0.07	0.11	0.09	4.18	64.02	1.27	14.61

Table 7.3.5: Showing the model parameters; exposure time (E.T.), size of pool (P.S.), Gamma parameter estimates, estimated prior mean and variance for Late delivery rate and Non - conformance rate.

	Initial Predictions		Updated Predictions	
	τ_{init}	θ_{init}	τ_{pred}	θ_{pred}
Selection Set 1	0.62	3.26	0.67	3.98
Selection Set 2	0.54	2.31	0.53	2.25
Selection Set 3	0.53	2.24	0.69	4.35
Selection Set 4	0.50	1.99	0.51	2.07

Table 7.3.6: Showing the initial and updated (using an uninformative prior) Kendall's tau predictions.

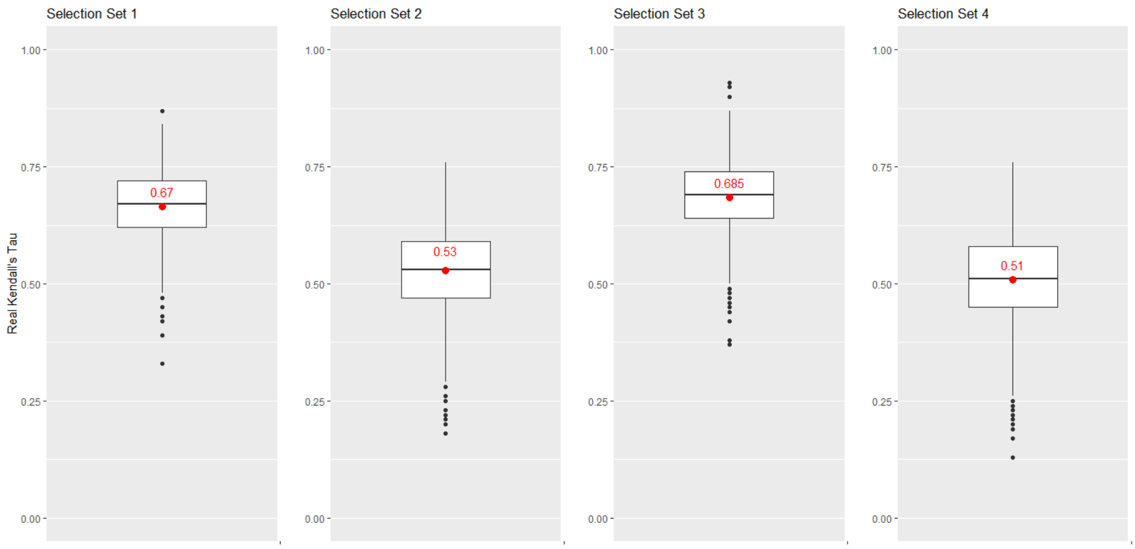


Figure 7.3.2: Kendall's tau distribution for all selection sets.

Since we have now fully defined all model parameters; the prior marginals and dependence parameter, the exposure time and the size of the pool, we demonstrate the suitability of the Clayton copula with the selected sets of data. As already addressed, we use the parametric bootstrap method to ensure the existence of rank correlation on the prior. The distributions of the prior Kendall's tau obtained from the bootstrap method are shown in figure 7.3.3. We observe that the rank correlation coefficient is not zero, which indicates dependence between the rates.

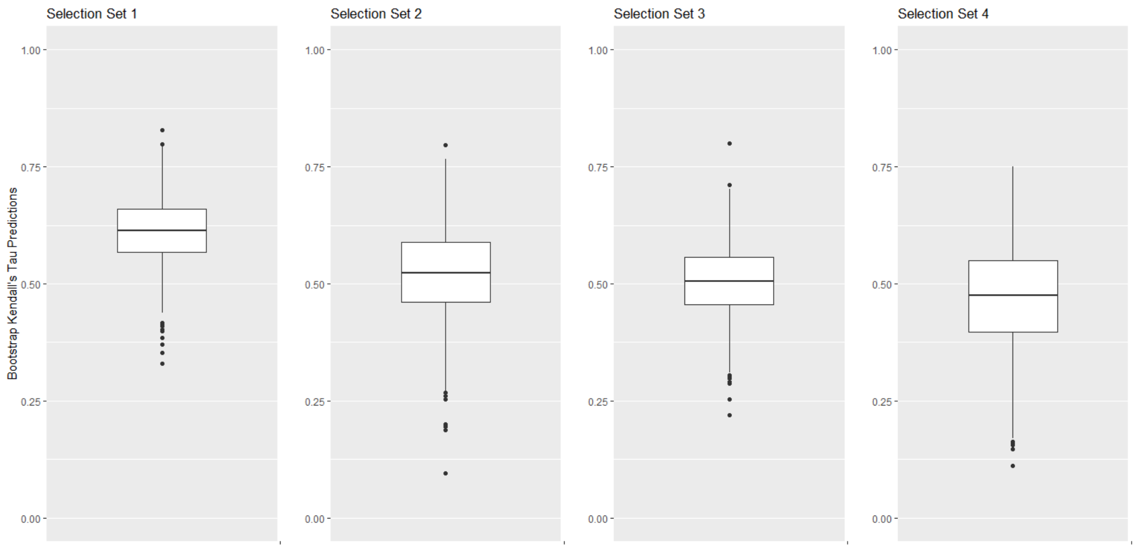


Figure 7.3.3: Bootstrap Kendall's tau predictions for all selection sets.

7.3.3 Mean and Median Ranking Results

We present different visualisations of the data analysis from the application of the proposed supplier ranking using the empirical Bayes model with Clayton copula on different data sets. We note that the ranks are based on the empirical distribution of the posterior distribution. Each supplier is presented by their code name (i.e. A13, A15, A16, etc.), as in the original data set. The distributions of the ranks in such applications cannot be expressed in closed-form formulas or well-known distribution functions, even if we choose the simplest priors. Hence, the distributions of the ranks for each supplier within the pool are expected to have shapes such as bi - modal or multi - modal. Figure 7.3.4 presents the probability distribution of the late delivery ranks for a selection of suppliers. We show how different the distributions of ranks are for each supplier, which make the ranking more interesting and challenging.

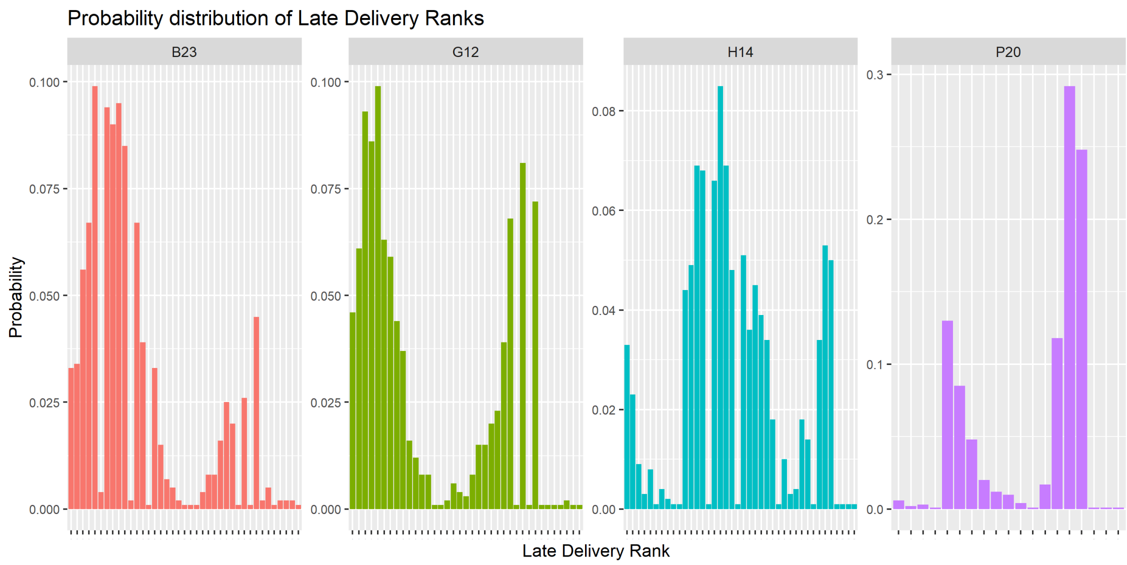


Figure 7.3.4: Probability distribution of late delivery ranks for Suppliers B23, G12, H14 and P20 from selection set 1.

Figures 7.3.5, 7.3.6, 7.3.7, 7.3.8 show the empirical distribution of the ranks for both rates and all four applications after sampling 1000 generated data out of 10000 generated data from the empirical posterior distribution of the rates. The distribution of the ranks is given for each one of the selected suppliers within each pool, where the red point denotes the mean rank. Information about the variation of the ranks for each supplier and the expected mean ranks are also shown through the boxplots. Figure 7.3.5 shows that the distribution of the ranks on both rates for some suppliers has less variation compared to others. We also observe that the distribution of the ranks on one rate has less variation compared to the other rate for some suppliers. For example, consider Suppliers A28 and A31 (set selection 1) whose distribution of the late delivery ranks has less variation compared to the non – conformance; this situation indicates that we are more confident about the rankings on late delivery rather than on non – conformance.

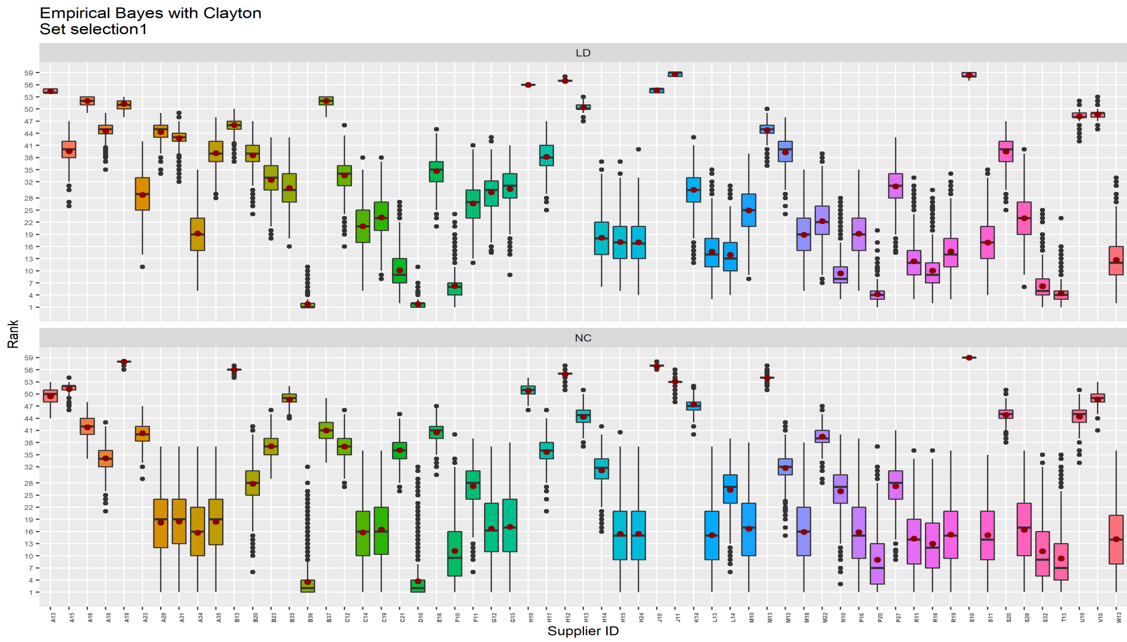


Figure 7.3.5: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 1.

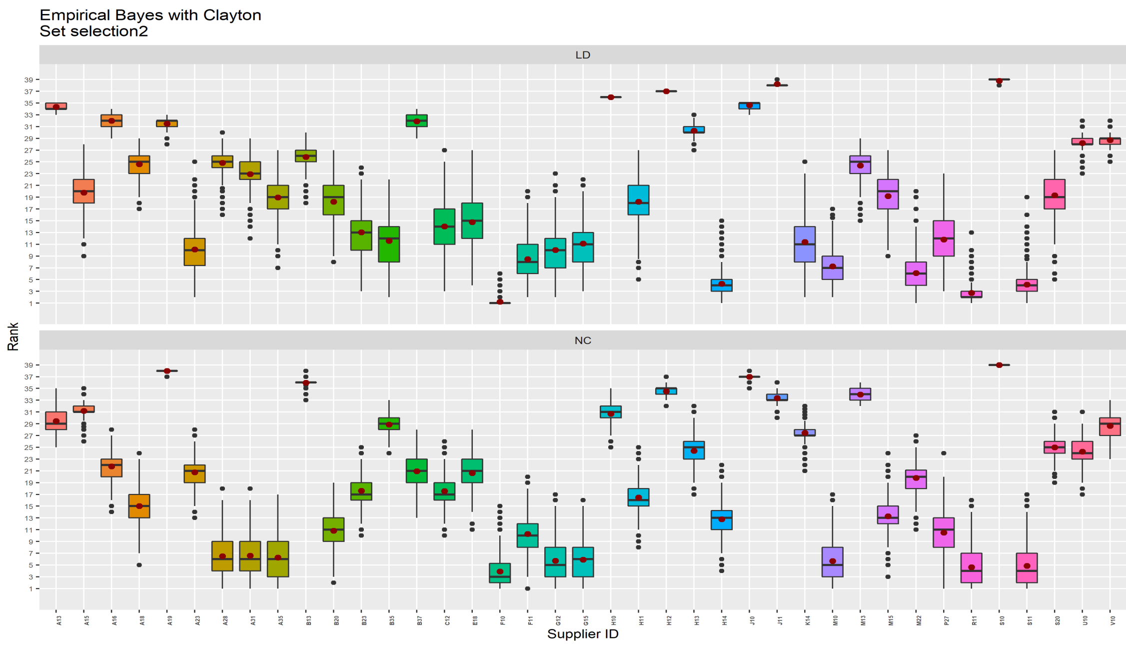


Figure 7.3.6: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 2.

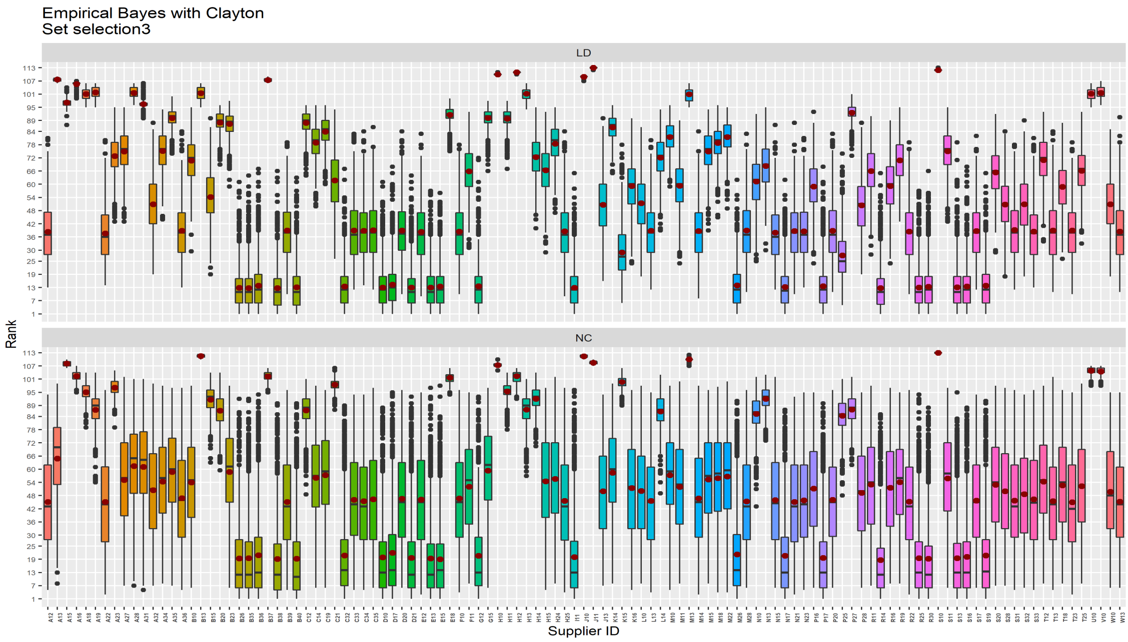


Figure 7.3.7: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 3.

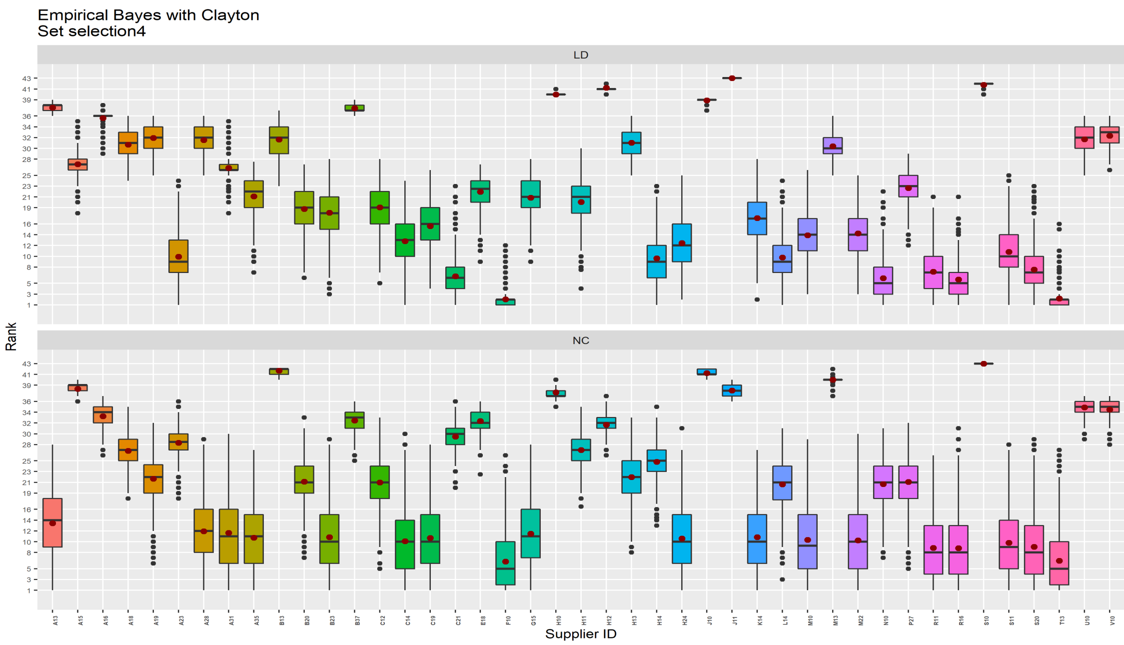


Figure 7.3.8: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 4.

We also show how the distribution of the ranks for individuals within different pool selection may change. For example, we compare the distribution of the ranks of Supplier A13 across all four applications. Figure 7.3.9 shows how the distribution of the ranks of Supplier A13 differs between measures and applications. We observe that non – conformance ranks have more variation compared to the late delivery ranks; and especially when the pool size increases, as in Set 3 where the pool consists of 113 suppliers.

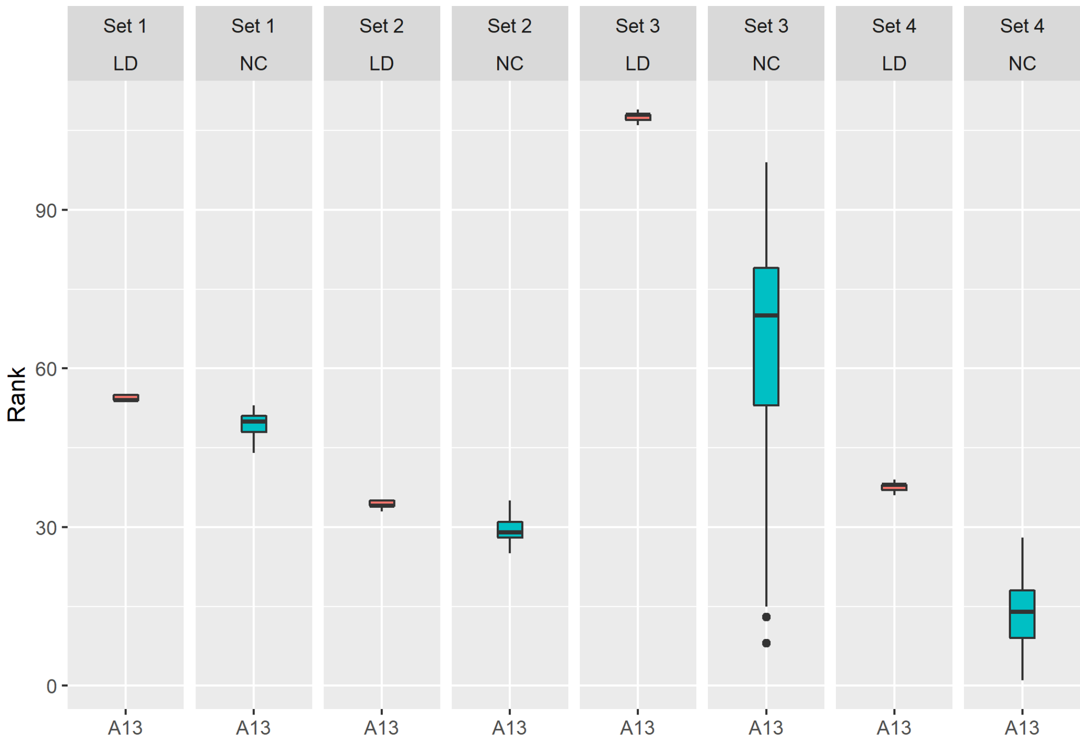


Figure 7.3.9: Empirical distribution of the ranks for both rates across applications of Supplier A13.

Figures 7.3.10, 7.3.11 show the mean ranks for all suppliers within the pool for both measures across all four applications. We compare the mean late delivery rank to the mean non - conformance rank of every supplier. According to the visuals, a relatively distinct left tail dependency between the non – conformance and late delivery mean ranks appears across all applications. We observe that the lower mean ranks appear more strongly dependent compared to the higher mean ranks, and the further we move to higher rankings, the more scattered the mean ranks appear. Clear discrimination of best and worst-performing suppliers across the pool is achieved, and thus we provide a comparative analysis of all suppliers within each pool. For example, we conclude that for the second application (set selection 2), Suppliers F10 and R11 consider as the best and second best-performing suppliers within the pool; Supplier S10 the worst and Suppliers A19, J10, H12 and J11 the second worst, according to the results presented in figure 7.3.10 (right part).

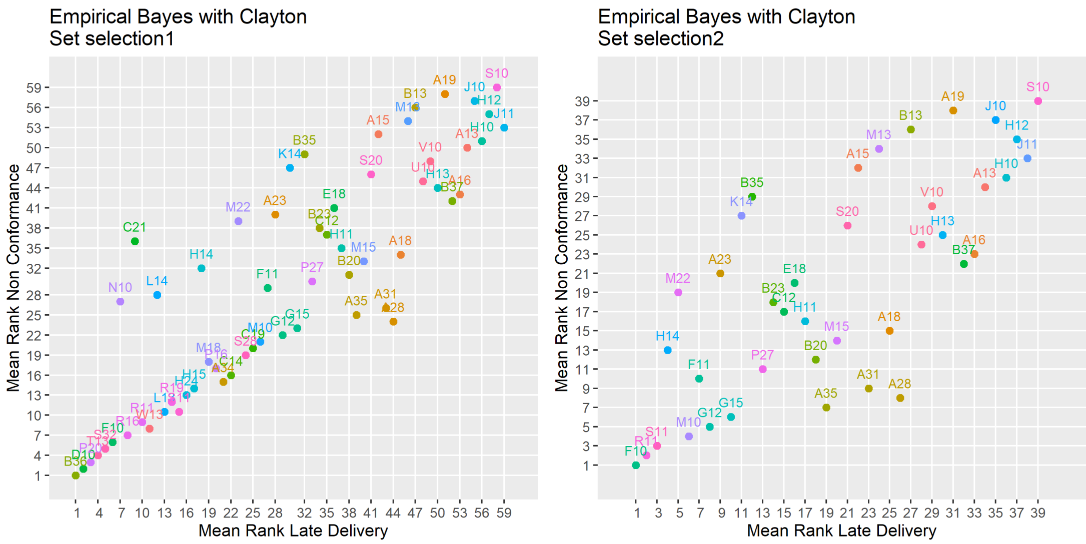


Figure 7.3.10: Showing the mean ranks of late delivery and non - conformance for all suppliers within the pool.

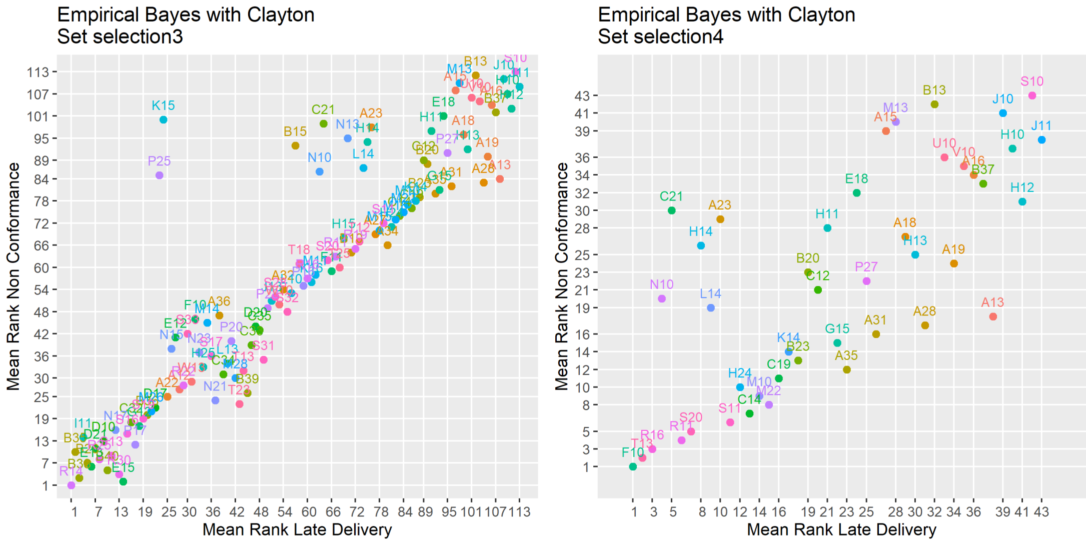


Figure 7.3.11: Showing the mean ranks of late delivery and non - conformance for all suppliers within the pool.

Apart from the mean ranks, we provide rankings obtained by the median of the distribution of the ranks for all suppliers within each selected pool. Figures 7.3.12 show that the median rankings are almost identical with the mean rankings for both late delivery and non – conformance measures across all applications. There are only a few exceptions where the median ranks are not the same as the mean ranks, but again they do not differ significantly, i.e. the late delivery mean and median ranks of selection set 3.

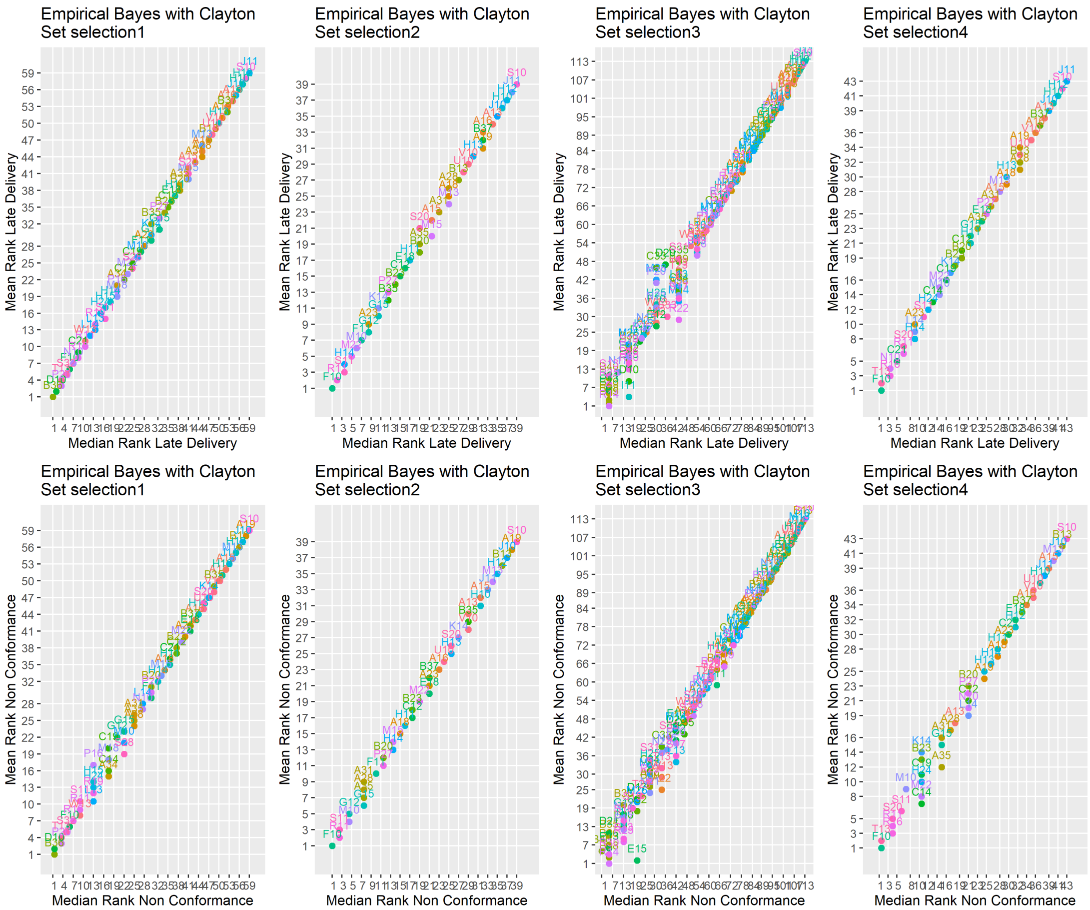


Figure 7.3.12: Comparison between mean and median rankings across applications.

7.3.4 Mean Ranks Comparison Across Applications

We have already chosen four (4) different selection sets/pools of suppliers with different exposure time and number of participants (as described in Section 7.3), and presented a comparative analysis of suppliers using the mean of the empirical distribution of the ranks (see Section 7.3.3). We now compare the mean rankings of selected suppliers across all four applications. We are interested in investigating how the position of a particular supplier may be affected by different pool size and/or different exposure time. For this particular comparison study, we identify and compare the position of 35 suppliers who participate in all four selection sets.

Figure 7.3.13 shows the late delivery and non – conformance mean ranks of all 35 suppliers across all four applications. We remind that the set 1 consists of 59 suppliers, set 2, set 3 and set 4 consist of 39, 113 and 43 suppliers, respectively. According to the visuals, we notice that all suppliers in sets 1, 2 and 4 show common trend and their rankings are similarly distributed within each pool. However, this is not the case in set 3; we observe that all selected 35 suppliers are shifted towards the higher rankings. This indicates that, even if some these suppliers have relatively good/low rankings in sets 1, 2 and 3, when choosing a different size pool (larger in this case) their ranking scores show that they do not perform as good as in the other pools. Notably, 34 out of 35 suppliers are considered to be in the worst 50% suppliers within the pool. We also notice that supplier F10 has been ranked 1st in sets 2 and 4, and 6th in set 1 for both measures, but in set 3 has achieved a mean rank of 32 on late delivery and 46 on non – conformance.

Figure 7.3.13: Showing the late delivery and non – conformance mean ranks of selected 35 suppliers across all four applications.

We also investigate which of the selected suppliers are ranked in the top or bottom 25% suppliers across all four selection sets. The top 25% suppliers are getting all the credit, but for the bottom 25% suppliers decisions need to be made about their fate in the company. Our intention is to provide multiple perspectives of the final mean rankings and show that different conclusions may be drawn under different situations and settings. Figure 7.3.14 shows the distinction of the selected 35 suppliers in top and bottom 25% suppliers within each set. As already addressed, none of the selected 35 suppliers is ranked in the to 25% suppliers in set 3, instead 28 suppliers are ranked in the bottom 25%. Suppliers F10, R11 and S11 are ranked in the top 25% suppliers in the sets 1, 2 and 4. On the other hand, we observe that supplier S10 considers the worst-performing supplier across all sets. Multiple comparisons can be made using such visuals which will eventually better inform the decision-maker by providing multiple perspectives of rankings across different applications.

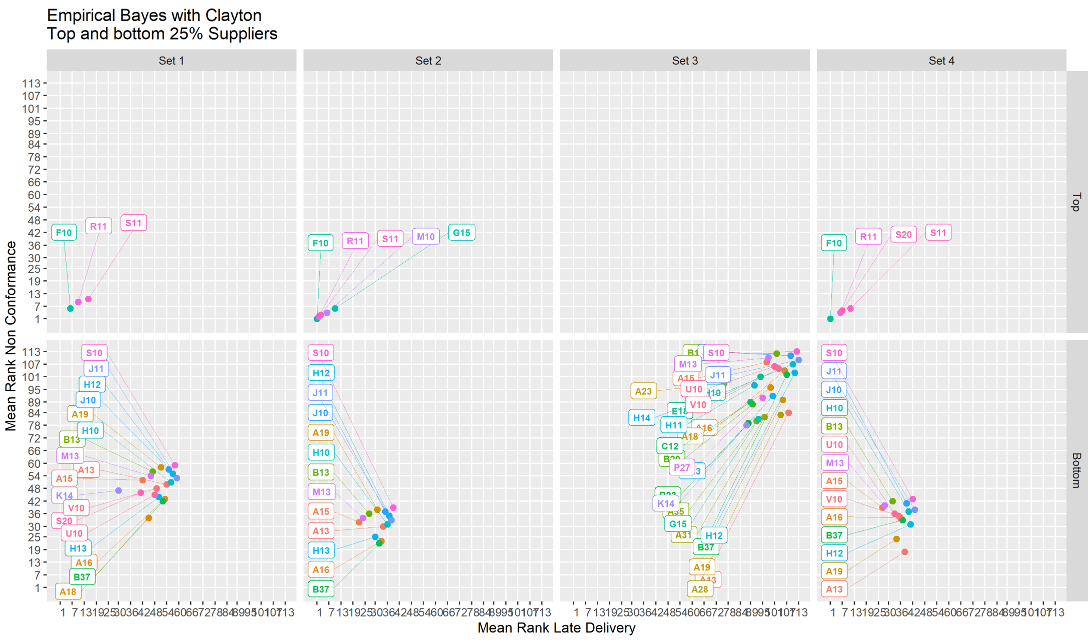


Figure 7.3.14: Showing the distinction of the selected 35 suppliers in top and bottom 25% suppliers within each set.

7.3.5 CDF Ranking Results

In this section, we propose another approach / method for supplier ranking by using the cumulative distribution function (CDF) of the ranks. For this method, an upper and a lower bound need to be set for obtaining the rankings at those specific bounds. For this particular study, we chose those bounds to represent the bottom and top 25% suppliers within each selection set. Figures 7.3.15 show the cumulative distribution function of both late delivery and non – conformance ranks for all four applications. Different coloured lines correspond to each supplier within the pool. According to the visuals, we notice different patterns across measures and applications. For instance, considering the non – conformance ranks of set 3, we notice distinct clusters of suppliers indicated by the similarity of their cumulative distribution functions. We also expect the suppliers who compose those clusters to have similar, if not the same, rankings. Worth noticing that by choosing different upper or lower bound, we also expect different ranking results. Figures 7.3.16 show the cumulative distribution ranks of late delivery and non – conformance ranks for the top and bottom 25% suppliers across all four applications. As expected, multiple suppliers across applications have similar rankings, appearing as overlapping points. That is reasonable considering the nature of the ranking method (ranking by the cumulative distribution function). For example, consider the case when two suppliers have similarly low ranks, and their empirical distributions of ranks have small variation, we expect that their cumulative distribution functions will quickly reach 1 with a similar trend. Thus, those two suppliers will eventually have the CDF ranking.

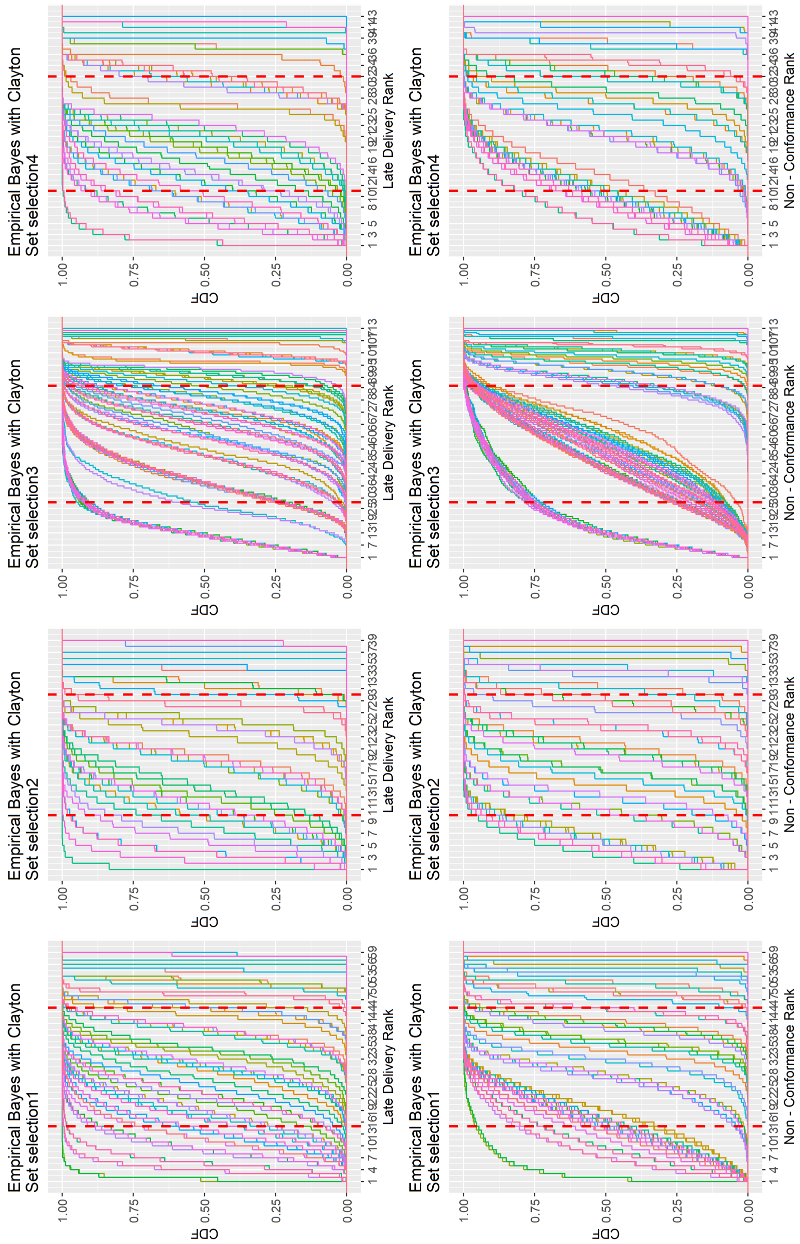


Figure 7.3.15: Showing the cumulative distribution function of late delivery and non - conformance ranks across applications. Vertical dashed lines represent the lower and upper bound.

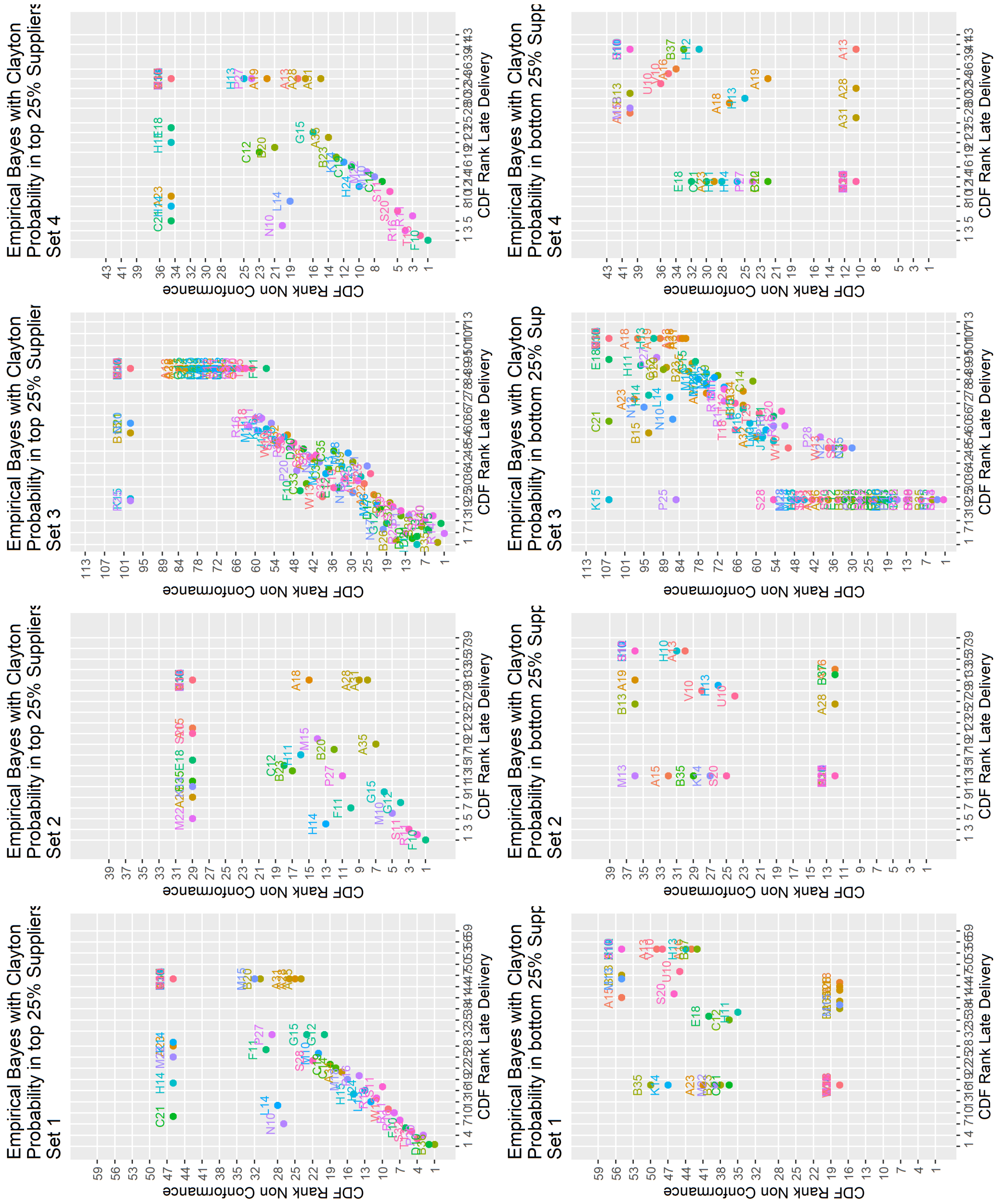


Figure 7.3.16: Showing the CDF ranks of late delivery and non - conformance for the top and bottom 25% suppliers across applications.

7.4 Empirical Bayes with Clayton Copula Vs Classic Empirical Bayes Rankings

In this section, we compare the empirical Bayes with Clayton copula model with the classic empirical Bayes model. Considering the classic empirical Bayes model, we assume that the late delivery and non – conformance rates are independent, and therefore we ignore any potential underlying dependency between them. However, how different the ranking results would be if we choose to ignore dependency. We intend to investigate the differences between the two models by analysing and comparing the position of individuals within the selected pools. We are mostly focusing on the distribution of the ranks and the mean rankings obtained using the classic empirical Bayes model (see Section 6.5.3).

We present different visuals aimed to compare the proposed supplier ranking using the empirical Bayes model with Clayton copula to the classic empirical Bayes model. Figure 7.4.1 shows the empirical distribution of the ranks for both rates and models. Again, the distribution of the ranks is given for all suppliers within each pool, where the red point denotes the mean rank. Comparing the distributions of the ranks of both models, we observe similar variations between rates and models, but there are differences, especially on the lower ranks. For example, consider the Supplier B36, its non – conformance mean rank is three (3) on the Clayton model, but around fourteen (14) on the classic empirical Bayes model, with more variation. Figure 7.4.2 identifies significant differences between methods and performance measures for suppliers B36 and D10. Similar comparative visuals for sets 2, 3 and 4 can be found in Figures D.1.1, D.1.2 and D.1.3.

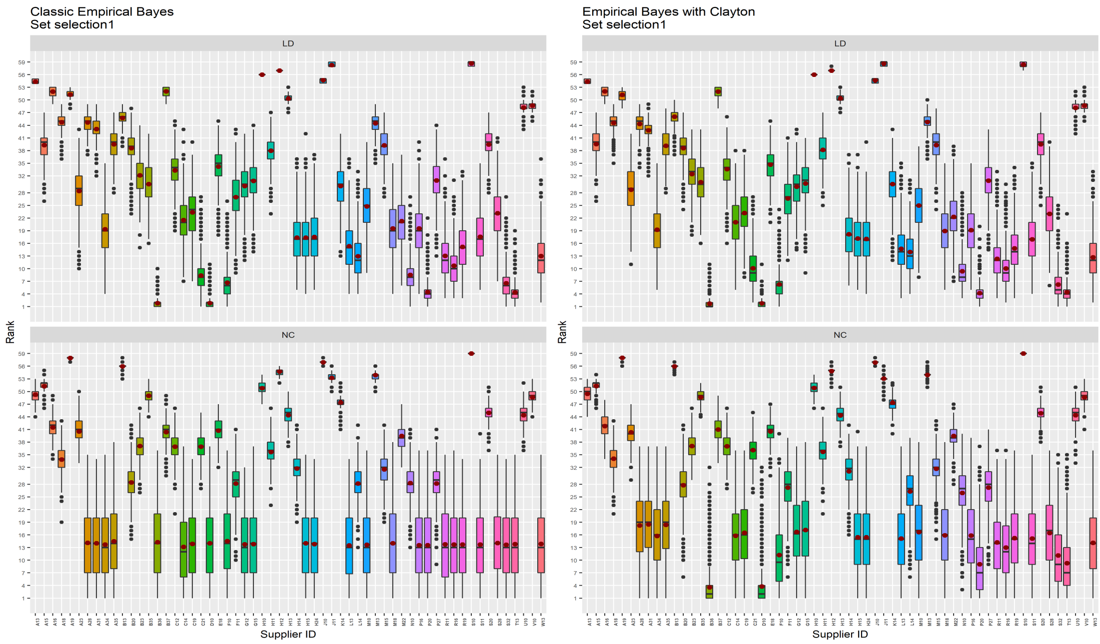


Figure 7.4.1: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 1. Comparison between classic empirical Bayes and empirical Bayes with Clayton copula.

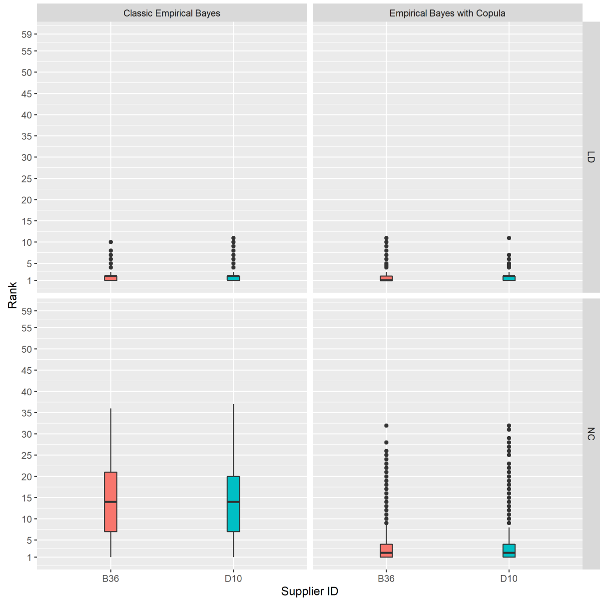


Figure 7.4.2: Showing the empirical distribution of the late delivery (LD) and non - conformance (NC) ranks for suppliers B38 and D10 within the pool of selection set 1. Comparison between classic empirical Bayes and empirical Bayes with Clayton copula.

Figure 7.4.3 show the mean ranks for all suppliers within the pool of set selection 1 for both models. We compare the mean late delivery rank to the mean non - conformance rank of every supplier. According to the visuals, a relatively distinct left tail dependency between the non – conformance and late delivery mean ranks appears on the Clayton model, whereas on the classic empirical Bayes model no definite trend or dependence is observed. We also observe that the lower mean ranks appear more strongly dependent compared to the higher mean ranks on the Clayton model; however, this is not the case for the classic empirical Bayes model. With the Clayton model, we achieve clear discrimination of best and worst-performing suppliers across the pool, but the distinction is not that clear for the classic model. We also notice differences between the mean ranks of the two model, especially on the lower ranks (best-performing suppliers). For example, for the Clayton model, Suppliers B36, D10 and P20 consider as the best, second and third best-performing suppliers within the pool. However, for the classic empirical Bayes model, Suppliers P20 and S32 consider as the best and second - best performing suppliers. The two models agree on the mean rankings of the worse performing supplier within the pool; specifically, both models consider Supplier S10 the worst and Suppliers A19, J10, H12 and J11 the second worst. Similar differences can be found across applications (see figure 7.4.4).

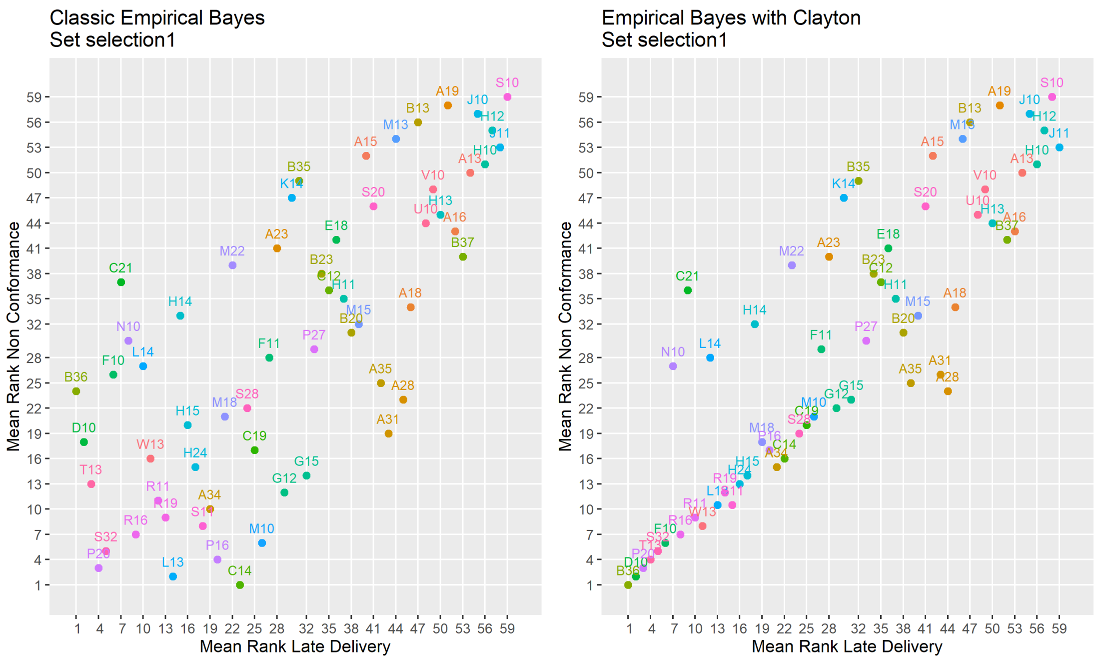


Figure 7.4.3: Comparing the mean ranks of late delivery and non - conformance for all suppliers within the pool of selection set 1 obtained by the classic empirical Bayes with the mean ranks obtained by the empirical Bayes with Clayton.

Figure 7.4.4: Comparing the mean ranks of late delivery and non - conformance for all suppliers within the pool of selection set 2, 3 and 4, obtained by the classic empirical Bayes with the mean ranks obtained by the empirical Bayes with Clayton.

Chapter 8

Conclusions and Future Work

8.1 Research Summary and Contribution

An inference method is developed for accounting for dependency between multiple unknown event rates using copulas. Our proposed method incorporates the underlying dependence on the rates, modelled using Clayton copula, into the Bayesian modelling framework, which primarily ensures coherent and theoretically sound rates estimates, and further, allows the rates estimates to be informed based on information for multiple events. Figure 8.1.1 illustrates the research aim, objectives and specific research questions answered in this research.

Our proposed empirical Bayes model captures both aleatory and epistemic uncertainties. Epistemic uncertainty is represented by the prior distribution. In particular, we used Clayton copula with Gamma marginals to describe the dependence between the rates, assuming that event rates realised from a homogeneous Poisson process capturing the pure inherent randomness in the observations, i.e. the aleatory uncertainty. Clayton copula specifies left tail dependence, where lower rates are considered highly correlated compared to higher. We capture the underlying dependence on the rates and not the correlation on the realisations of events (Poisson process data), implying that the underlying dependence on the rates is

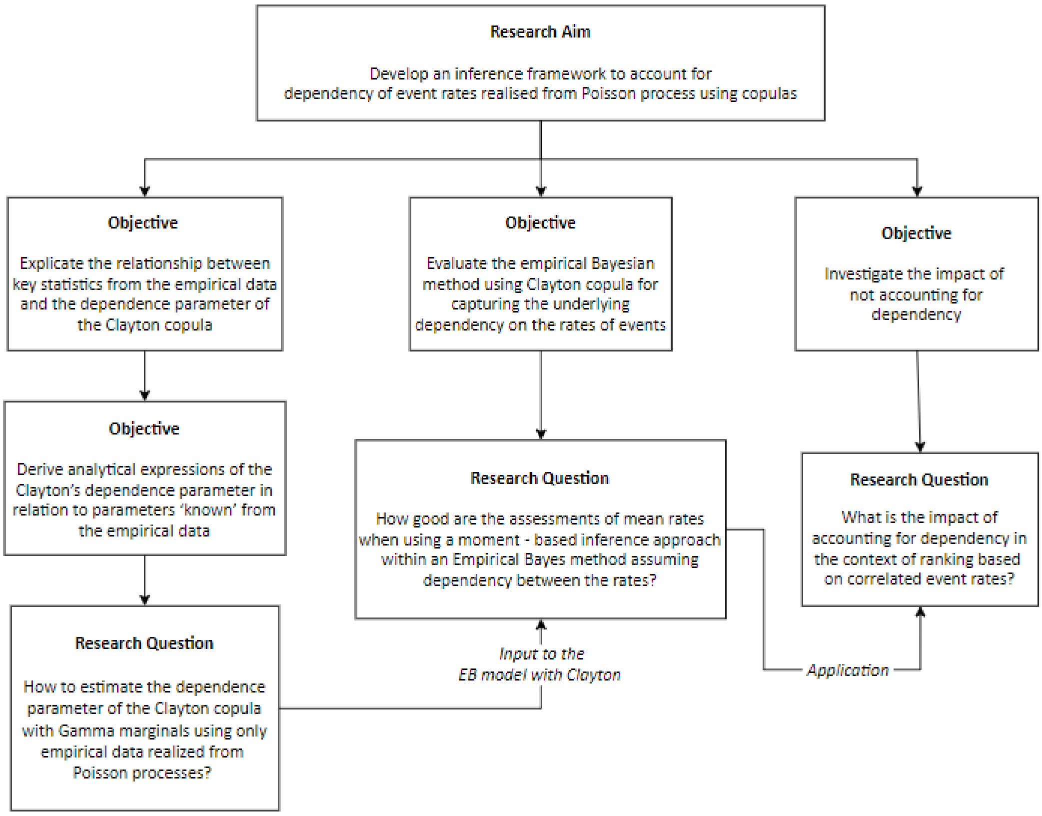


Figure 8.1.1: Diagram showing the research aim, objectives and research questions.

driven by the operations within organisations rather than the occurrence of rare or extreme events. Method of moment estimators was used for estimating the prior marginal distributions, and the analytical expression proposed in chapter 4 for estimating the dependence parameter θ of the Clayton copula. The posterior distribution is not analytically traceable in our case. Therefore, numerical integration methods are of need for obtaining the empirical Bayes estimates by computing the posterior expectations, which can be computationally intensive. Thereby, we proposed a simulation - summation method for calculating the posterior expectations, which is based on generated random samples from the updated prior distribution.

R.Q. How to estimate the dependence parameter of the Clayton copula with Gamma *marginals using only empirical data realized from Poisson processes?*

Following the empirical Bayesian methodological framework, all prior parameters including the dependence parameter of the copula need to be estimated. However, estimating the dependence parameter of the Clayton copula using only the empirical count data realized form Poisson processes can be challenging. Therefore, we conducted a simulation study in order to investigate, initially, if there is a relationship between the Poisson process data and the Clayton dependence parameter, and secondly, how such a relationship can analytically be explicated within this context.

According to our simulations, we identified a distinct relationship between the parameters available from the empirical data and the Kendall' s tau of the prior Clayton. knowing also that there is a closed - form formula that describes the dependence parameter θ in relation to the Kendall' s tau τ, of the Clayton copula (Nelsen, 2007), we focused on defining that relationship.

Therefore, we propose a new model for estimating the Kendall's tau of the Clayton copula with Gamma marginal distributions (bivariate case) using only the Poisson process data. Particularly, we developed a non - linear regression model for predicting Kendall's tau of the prior Clayton. Another non-linear regression model is also developed for predicting the Root Mean Squared Error of the predictions, ensuring that the model proposed is accurate even in the worst - case scenario.

A simulation study is conducted considering multiple possible situations and parameters, for example the exposure time, the sample size, and the marginal Gamma parameters. Our findings suggest that there is an affine relationship between the Kendall' s tau of the prior and the Poisson data, which in some cases, it is obscured by the noise in the data introduced by the Poisson Process. We conclude that all models' parameters affect the relationship between the two measures differently. Notably, we consider best - case scenarios cases where the exposure time, the pool size and the marginal variance are relatively high; and the marginal mean is low. To support further comparison, we compared the model fit results obtained from the application of full Bayesian model to the empirical Bayesian model. We discussed how the proposed models perform using the theoretical settings and how when the estimated prior parameters are used indicating empirical Bayes method. Our findings suggest that both models perform significantly well and produce similar results. At last, we provide analytical expressions for obtaining an estimate of the dependence parameter of the Clayton copula in relation to the empirical data available. The dependence estimate obtained from this model is necessary and is used as an input to the main empirical Bayesian model of this research.

R.Q. *How good are the assessments of mean rates when using a moment - based inference approach within an Empirical Bayes method assuming dependency between the rates?*

We conducted a simulation study to examine the relative accuracy of our proposed model, provide an answer to this research question, and further investigate what is the impact of choosing to ignore dependency. We considered multiple different parameters, including the size of the pool, the exposure time, the marginal Gamma parameters and levels of dependence (from relatively weak to strong). Our findings indicate that our proposed model provides accurate estimates of the prior rates. Notably, as the exposure time increases, the RMSE of the prediction errors decreases significantly. We also identified cases considering the Gamma marginal mean and variance, where the model is expected to perform relatively better compared to other situations. In particular, the smaller the marginal mean, the smaller the RMSE of the errors. We further evaluated the empirical Bayes with Clayton copula model in comparison with the classic empirical Bayes model by conducting a benchmarking study in which data were simulated from the Clayton prior and further from the Poisson distribution.

We compared the empirical Bayes estimates obtained using our proposed model to the estimates obtained from the classic empirical Bayes model by ignoring the underling dependence; and the estimates obtained without pooling. Our findings suggest that there are no cases where estimates obtained without pooling are more accurate compared to our proposed model and the classic empirical Bayes.

To provide further comparative analysis between the two models considering all parameters chosen for this study by identifying cases in which the proposed empirical Bayes with Clayton copula model is expected to outperform the classic Bayesian model; and, further,investigate in which cases, it is worth ignoring the underlying dependence on the event rates, we conducted a benchmarking study. Our findings indicate the proposed empirical Bayes with Clayton copula model outperforms the classic empirical Bayes model in cases where the underlying dependence between event rates is moderate to strong, and the prior mean and variance are relatively large. In contrast, the classic EB model is suggested in cases where weak dependence between the rates occurs. Particularly, in cases where the prior mean and variance are relatively large; and the size of the pool is large.

R.Q. What is the impact of accounting for dependency in the context of ranking based on correlated event rates?

We developed a method for ranking event rates under uncertainty. We structured the ranking problem from a Bayesian perspective as follows. We have a likelihood to explain the variability in observations given and inherent rate of interest, and a prior distribution to describe the uncertainty on the rates of events. We are interested in ranking on the inherent rate, but we have different amounts of uncertainty about each across the pool. As such, the rank, i.e. that which would be revealed if we knew the true rate for certain, is also uncertain. We were able to express the probability distribution of the ranks, but unusual bi-modal or even multi-modal distributions were created even for simple prior distributions. As such, we require Monte Carlo methods for obtaining the empirical distributions of the ranks.

Ranking under uncertainty can be challenging. Careful consideration needs to be made on multiple modelling choices. For example, how do we rank distributions? What would be an appropriate measure to rank on? There is no straightforward answer, as we are dealing with much uncertainty. Instead, we presented three statistical criteria for ranking; ranking by the mean, the median and the cumulative distribution of ranks, showing that there is no absolute right or wrong analysis but different perspectives and approaches, which eventually better inform decision making. Furthermore, what would be an appropriate pool to choose? Again, there is no straightforward answer.

To further investigate the impact of not accounting for dependency and provide an answer to the research question, we provide an application that focuses on supplier ranking. For this study, we used data for two supplier key performance indicators. In particular, our proposed model is based on the empirical Bayes method considering the underlying dependence between the late delivery rate and the non – conformance rate. We modelled the dependency between the two rates by using a Clayton copula, which describes cases with strong left-tail dependence. Considering our motivation for this study, the Clayton copula seems appropriate for supplier ranking problems. We consider the underlying dependence on the rates and not the correlation on the realisations of events (Poisson process data), implying that the underlying dependence on the rates is driven by the operations within organisations rather than the occurrence of rare or extreme events. Within this context, we can say that when a supplier performs well on delivery, it usually performs well on quality as well; and when a supplier has a poor performance on one measure, it is uncertain how it would perform on the other, as other factors affect their operations within the organisation.

We presented several examples of analysis and model applications using de - sensitised real data from the prime manufacturer considering multiple situations. For example, what happens if we select different size pools within the same time of exposure; or different exposure time and different pool size. We have also presented the analysis of data by describing the nature of data available and discussing the data preparation process. Furthermore, we have compared the proposed model with the classic empirical Bayes model, which assumes that the event rates are conditionally independent. Our findings suggest that mean and median rankings provide similar results. However, when comparing the proposed model with the classic EB model, the final rankings were not the same. A relatively distinct left tail dependency between the non – conformance and late delivery mean ranks appears on the Clayton model, whereas on the classic empirical Bayes model, no clear discrimination of the best - performing suppliers is observed.

With our proposed method considering dependency, we achieved a clear discrimination of best and worst - performing suppliers across the pool, but this is not the case for the classic empirical Bayesian model. Therefore, considering dependency can clarify the ranking as we provide more information into the analysis. On the other hand, if we choose to ignore the underlying dependency between the rates may mislead to incorrect conclusions about the position of a supplier within the pool which consequently can be financially costly for the company.

8.2 Research Limitations and Future Work

This research involves several limitations that can be turned into motivation for future work. Firstly, creating methods that consider the underlying dependence between multiple event rates and making use of the count data available have motivated this study. We understand that estimating copulas' dependence parameter using count data can be challenging. However, we have provided analytical expressions that can be used for obtaining an estimate of Kendall's tau of the Clayton copula with Gamma marginals when dealing with count data realized from Poisson processes. The proposed methods and conclusions are based on simulation studies conducted this research. Although we select a wide range of parameters, it is almost impossible to consider all possible scenarios. Therefore, an aspect for future work is to extend the study of simulations and analytically examine the asymptotic performance of the proposed method.

The need for creating methods that consider dependency between multiple event rates and provide flexibility in terms of marginal choice and dependence structure has also motivated this research. Although copulas are considered a powerful tool for modelling complex dependency, copulas combined with Poisson process data have not yet been explored within this context, i.e. prior copula. We account that the choice of the marginal distributions, the marginal parameters and the dependence structure could be considered restricted and challenging in situations where other marginal distributions would be more appropriate or when the dependence structure is different. However, copulas can provide such flexibilities. Therefore, exploring different copula families for identifying hidden relationships between prior copulas and Poisson process data and modelling complex dependence structures within Bayesian inference can be turned into motivation for future work. This research investigates the bivariate case of the Clayton copula where only two performance measures are under consideration. Another aspect for future work is to extend this method and explore higher - dimensional dependence modelling using vine copulas.

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Appendix A

Analysis of Data - Simulation Study for Estimating Kendall's Tau

A.1 Residual RMSE Model Fit

We are interested in the Root Mean Square Error (RMSE) of residuals. Figure A.1.1 shows the RMSE of residuals for all different sets of parameters, in which colours represent different marginal variance values and symbols represent different marginal mean values. We observe that RMSE is affected by the size of the pool, the exposure time and the marginal parameters. Particularly, as the pool size and exposure time increase, the RMSE always decreases. However, not in all cases, the decrease is equally significant. The marginal distributions contribute to the final form of the RMSE. For example, the highest RMSE appears when low variance, high mean, relatively small pool size and exposure time are set.

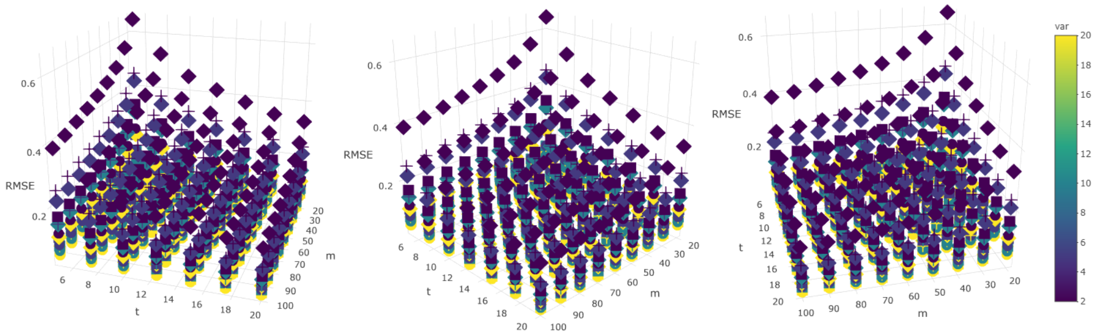

Figure A.1.1: Showing RMSE of residuals for all different sets of parameters. Colours represent different marginal variance values and symbols represent different marginal mean values.

Since RMSE is affected by all model parameters, we build a non-linear model that describes this relationship, as shown in Section 4.2. Tables A.1.1, A.1.2 summarise the results of the model fit. According to the results, the model performs relatively well. All coefficients are significant; the overall residual standard error is minimal (close to zero), and only 13 iterations needed to convergence.

Formula:	$rmse \sim a \cdot m^b \cdot t^c \cdot \left(\frac{a}{b^2}\right)^d \cdot \left(\frac{a}{b}\right)^e$
Residual standard error:	0.00537
Number of iterations to convergence:	13

Table A.1.1: RMSE model fit results.

	Estimate	**Std. Error**	**t value**	**Pr(> |*t*|)**
a	1.17	0.01	85.45	0.00
b	-0.41	0.00	-189.75	0.00
c	-0.53	0.00	-215.15	0.00
d	-0.53	0.00	-315.36	0.00
e	0.52	0.00	279.51	0.00

Table A.1.2: Coefficients of RMSE model fit.

Figure A.1.2 shows the comparison between predicted and actual RMSE values. Indicates that the predicted values of the model are very close to the actual RMSE values. Regarding the residuals of RMSE fit, we note that they follow Normal distribution (See figure A.1.3).

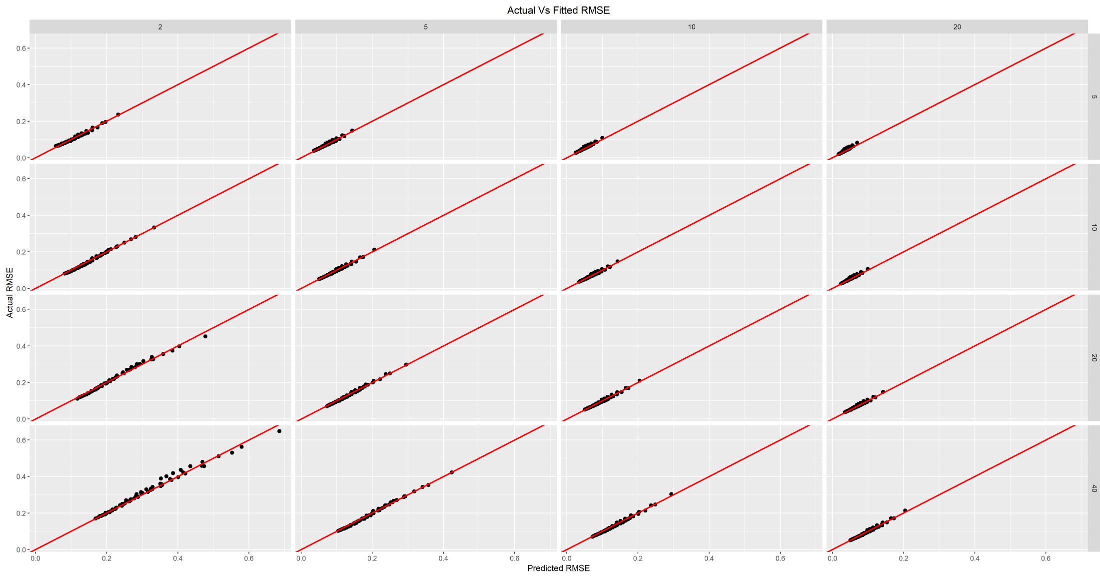

Figure A.1.2: Showing a comparison between the predicted RMSE values and the actual RMSE values.

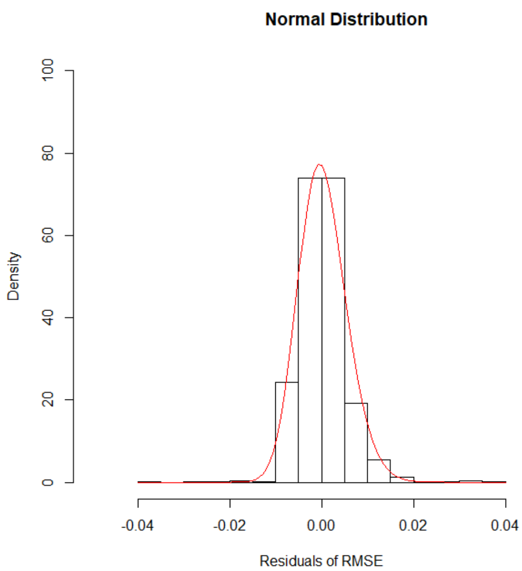

Figure A.1.3: Residuals of RMSE model follow Normal distribution.

Appendix B

Inference Methods and Evaluation Visuals

B.1 Simulation Study Visuals

B.1.1 Empirical Bayes with Clayton Model Prediction Errors

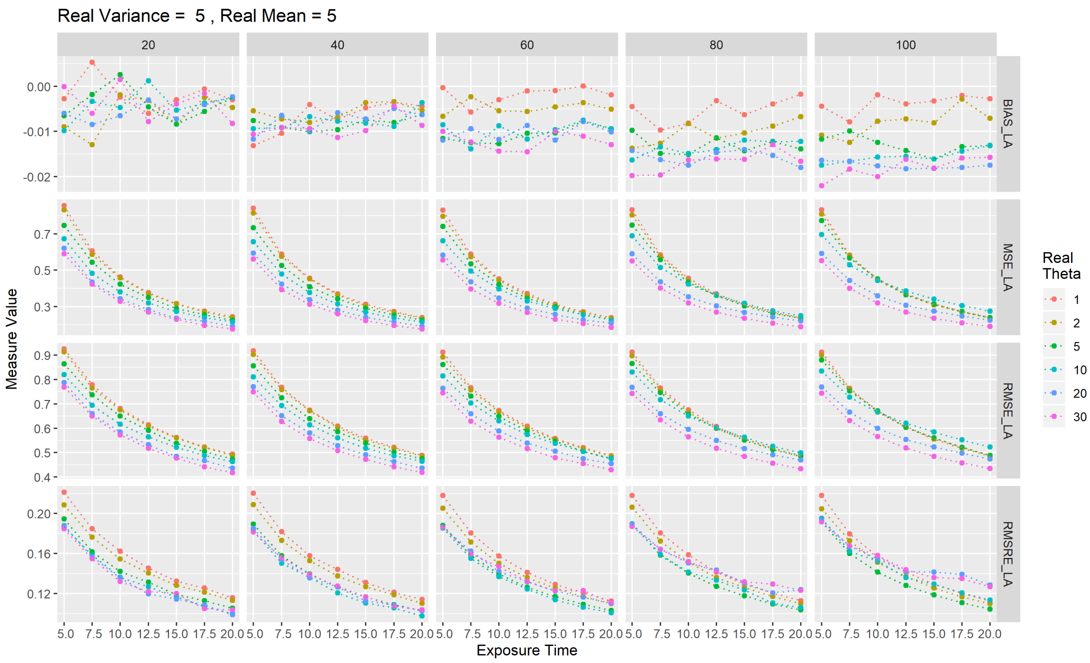

Figure B.1.1: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 5 and true mean is 5.

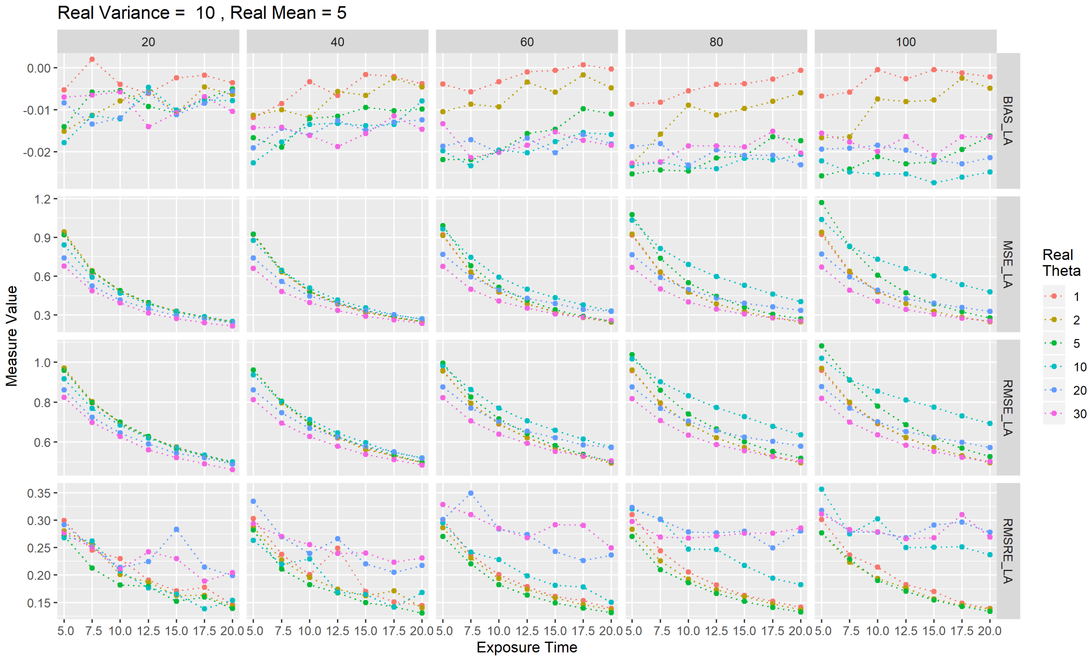

Figure B.1.2: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 10 and true mean is 5.

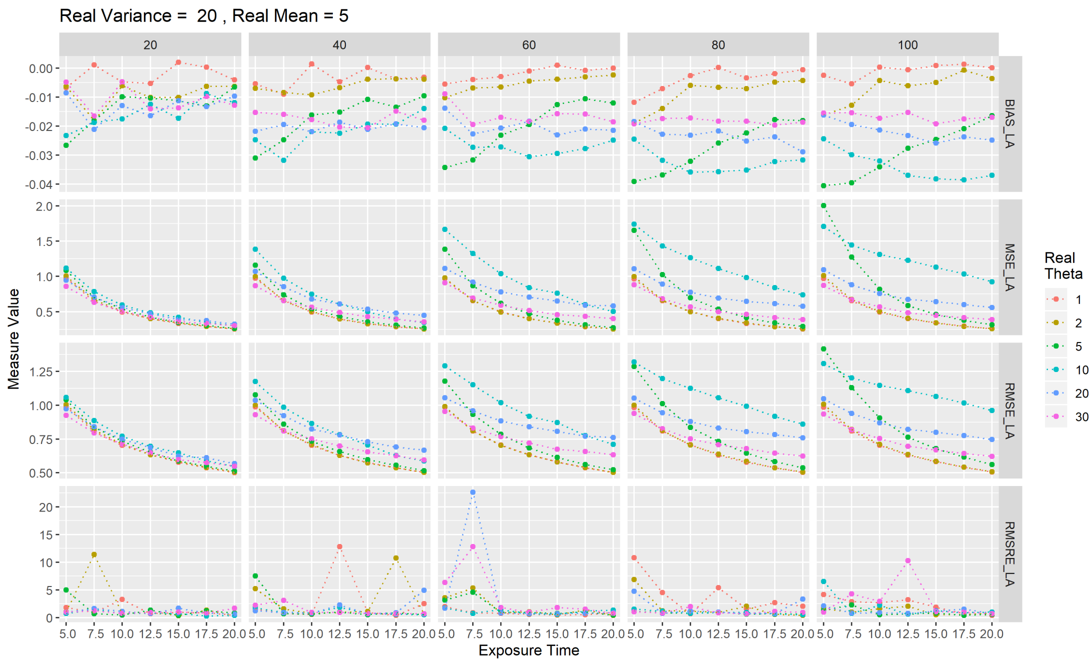

Figure B.1.3: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 20 and true mean is 5.

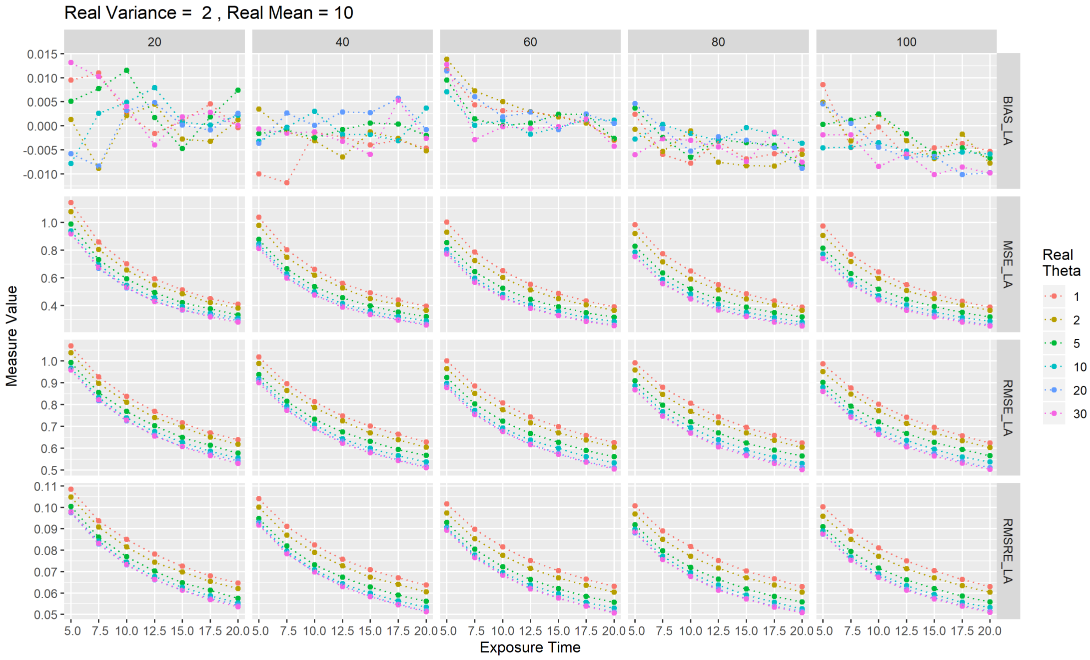

Figure B.1.4: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 2 and true mean is 10.

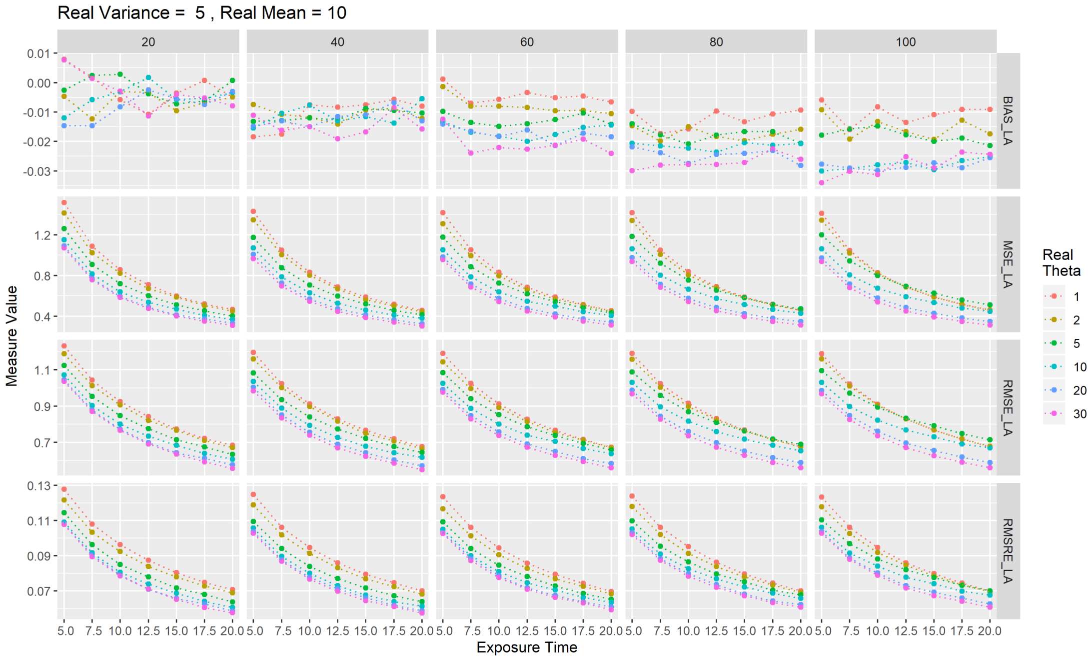

Figure B.1.5: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 5 and true mean is 10.

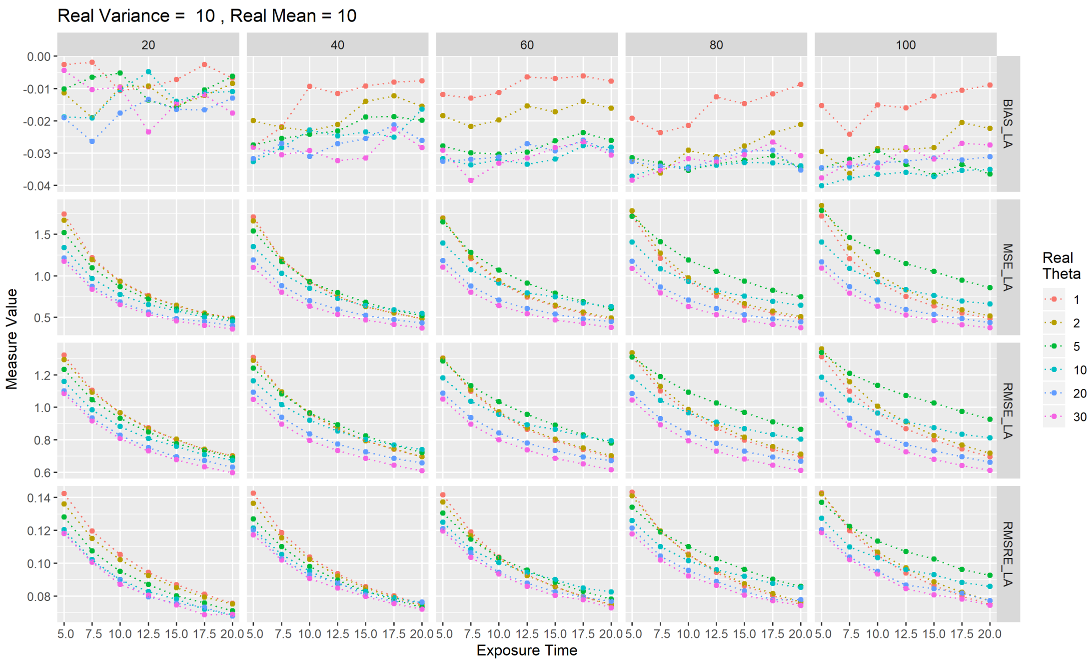

Figure B.1.6: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 10 and true mean is 10.

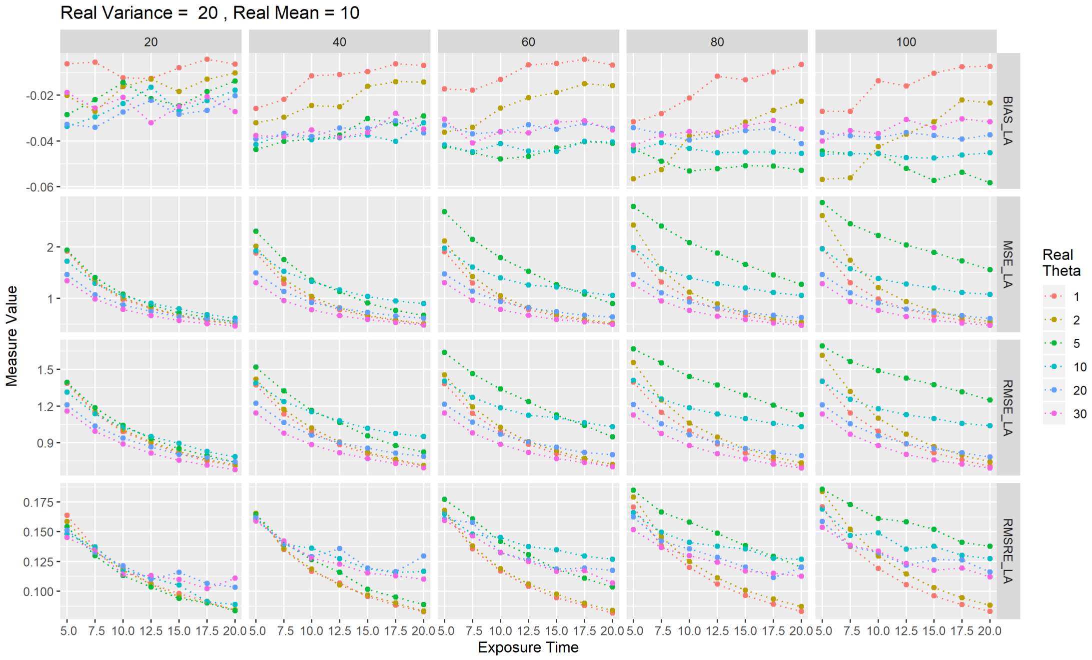

Figure B.1.7: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 20 and true mean is 10.

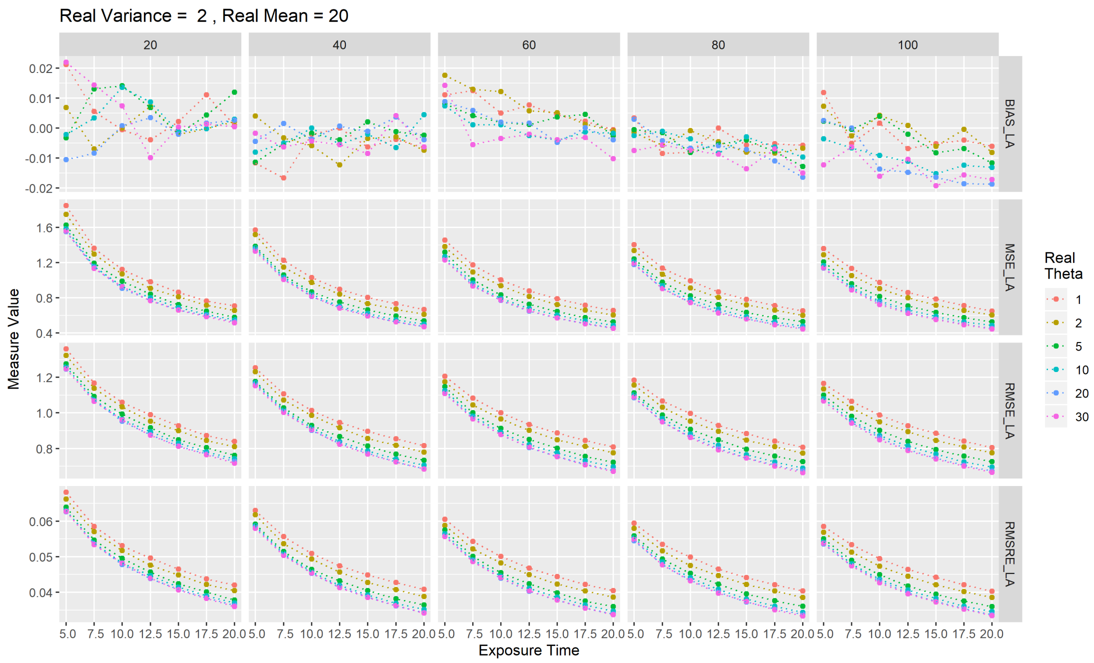

Figure B.1.8: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 2 and true mean is 20.

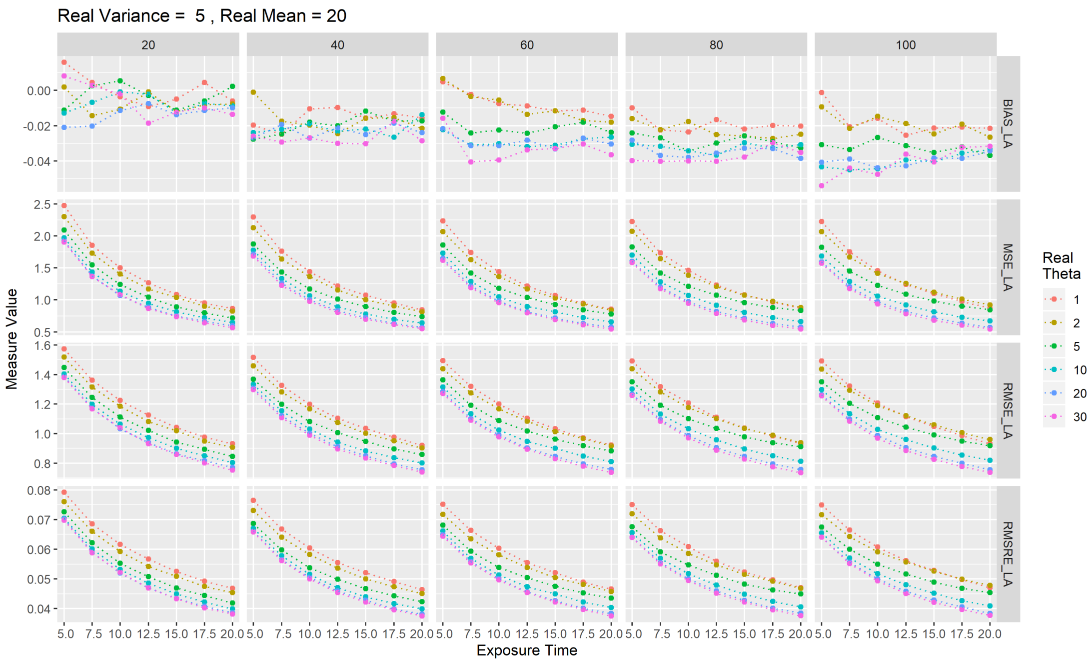

Figure B.1.9: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 5 and true mean is 20.

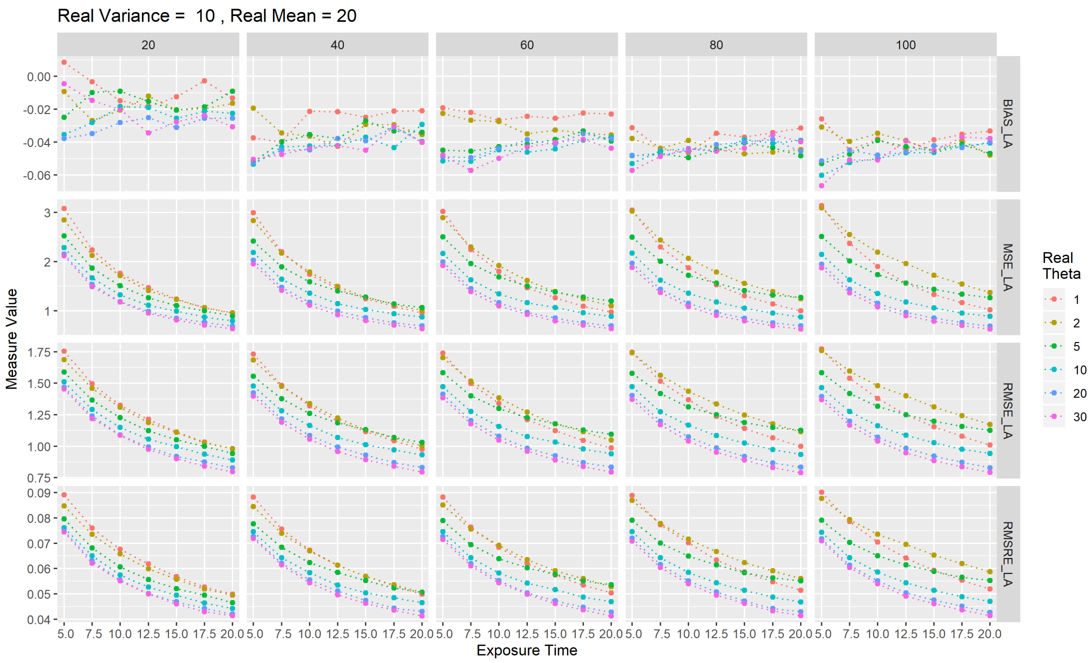

Figure B.1.10: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 10 and true mean is 20.

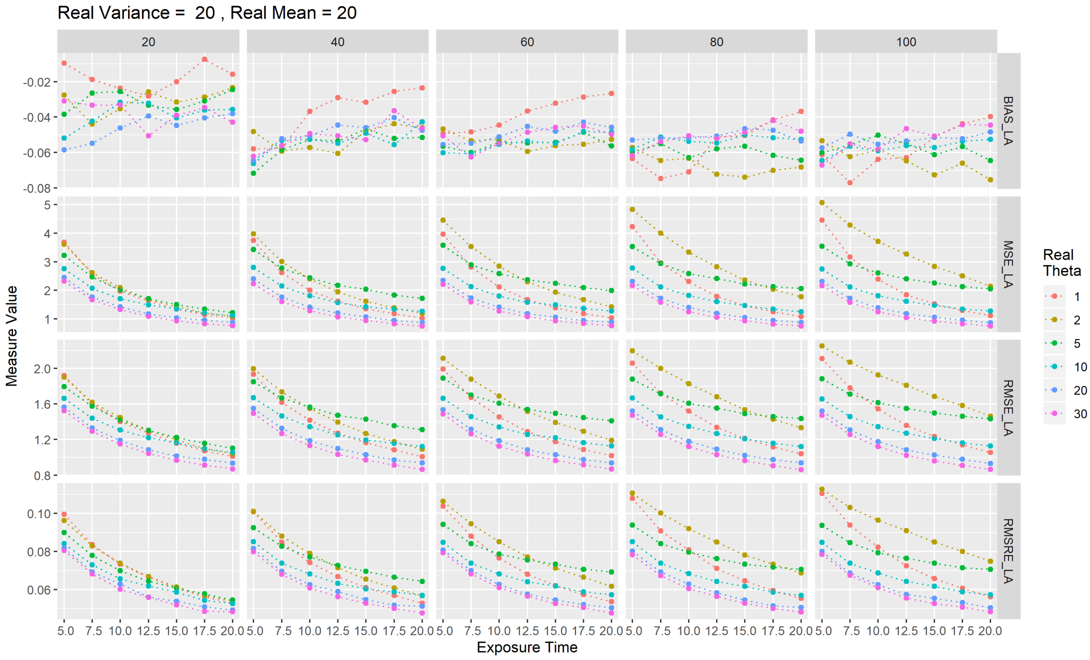

Figure B.1.11: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 20 and true mean is 20.

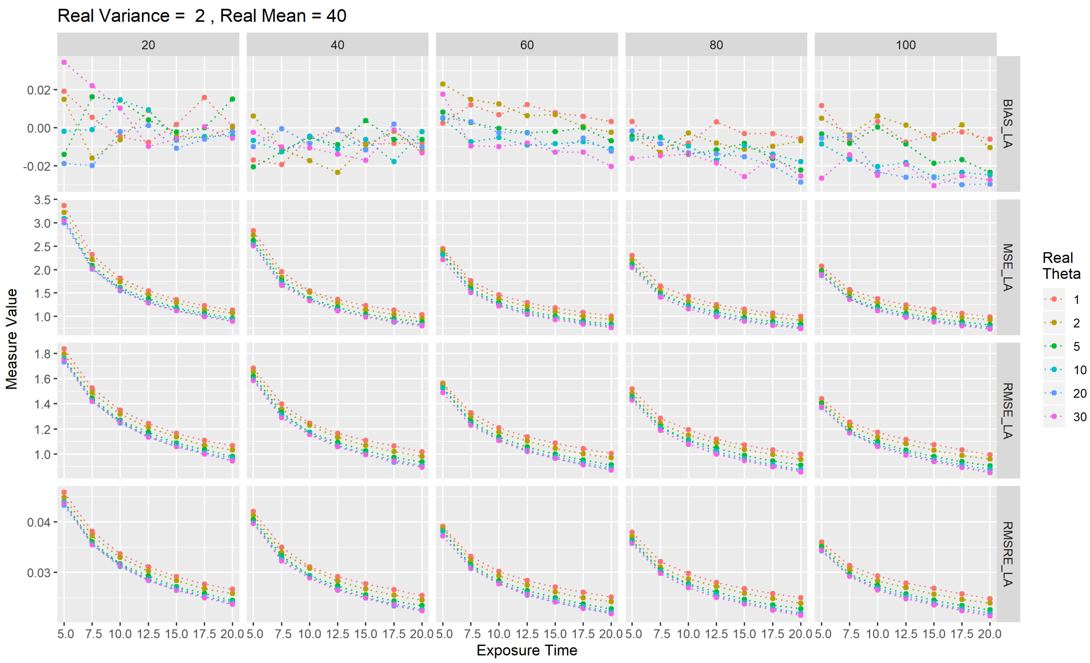

Figure B.1.12: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 2 and true mean is 40.

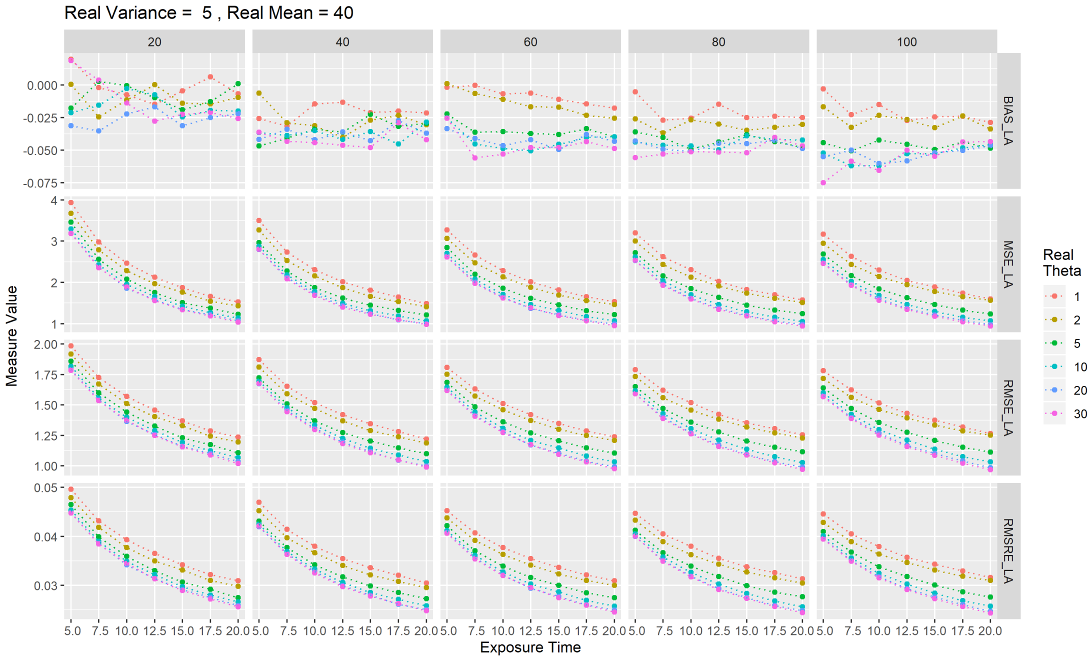

Figure B.1.13: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 5 and true mean is 40.

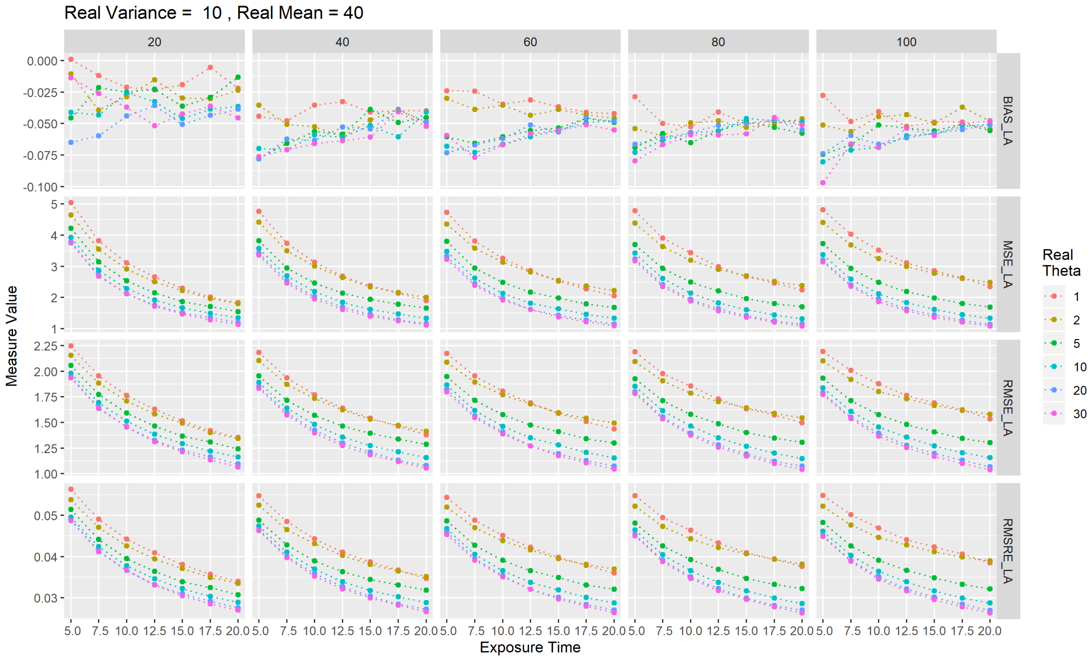

Figure B.1.14: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 10 and true mean is 40.

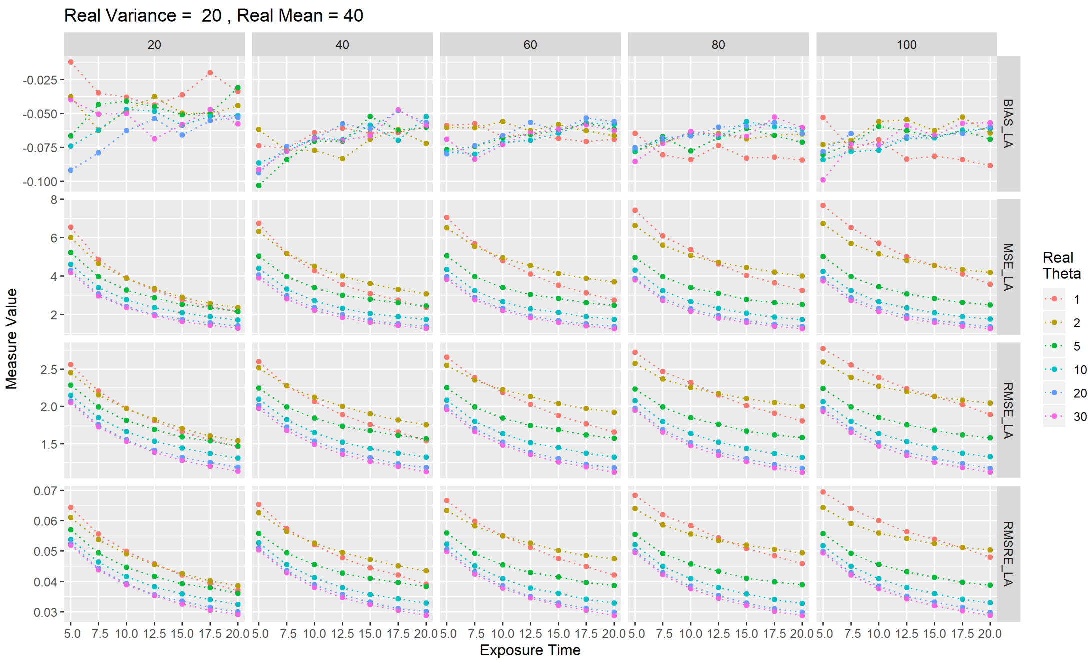

Figure B.1.15: Showing how different pool size, exposure time, mean and variance choices affect BIAS, MSE, RMSE and RMSRE of EB with Clayton prediction errors, when true variance is 20 and true mean is 40.

B.2 Benchmarking Study Visuals

B.2.1 Bias Results

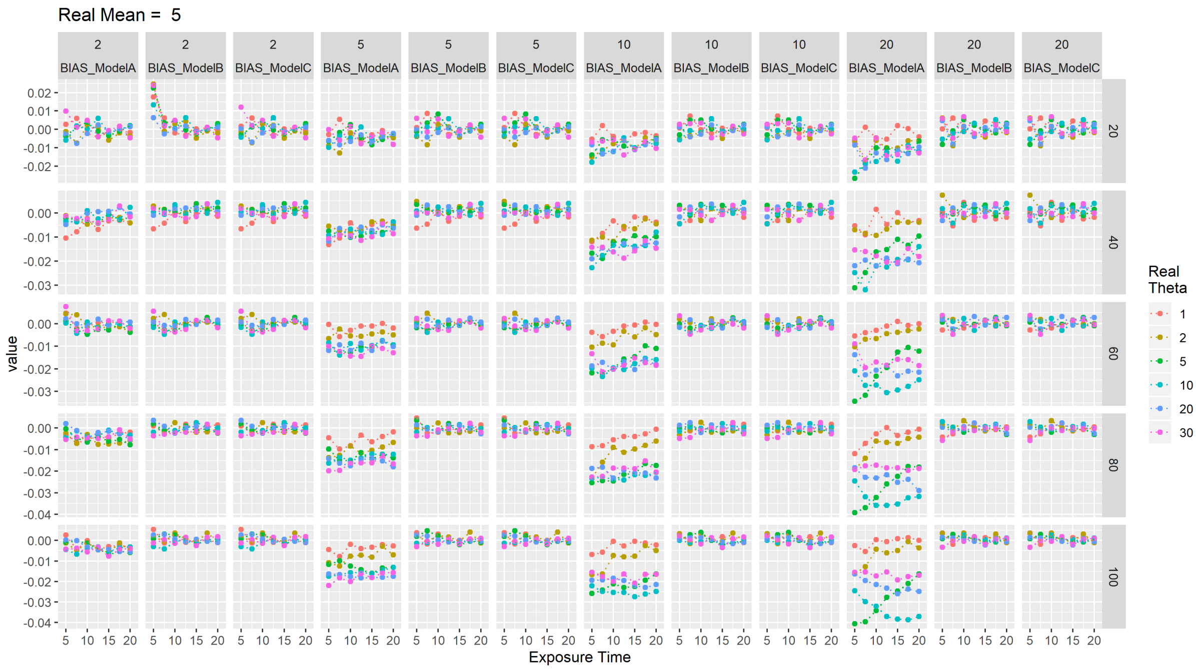

Figure B.2.1: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect BIAS for Models A, B and C, when the true prior mean is five.

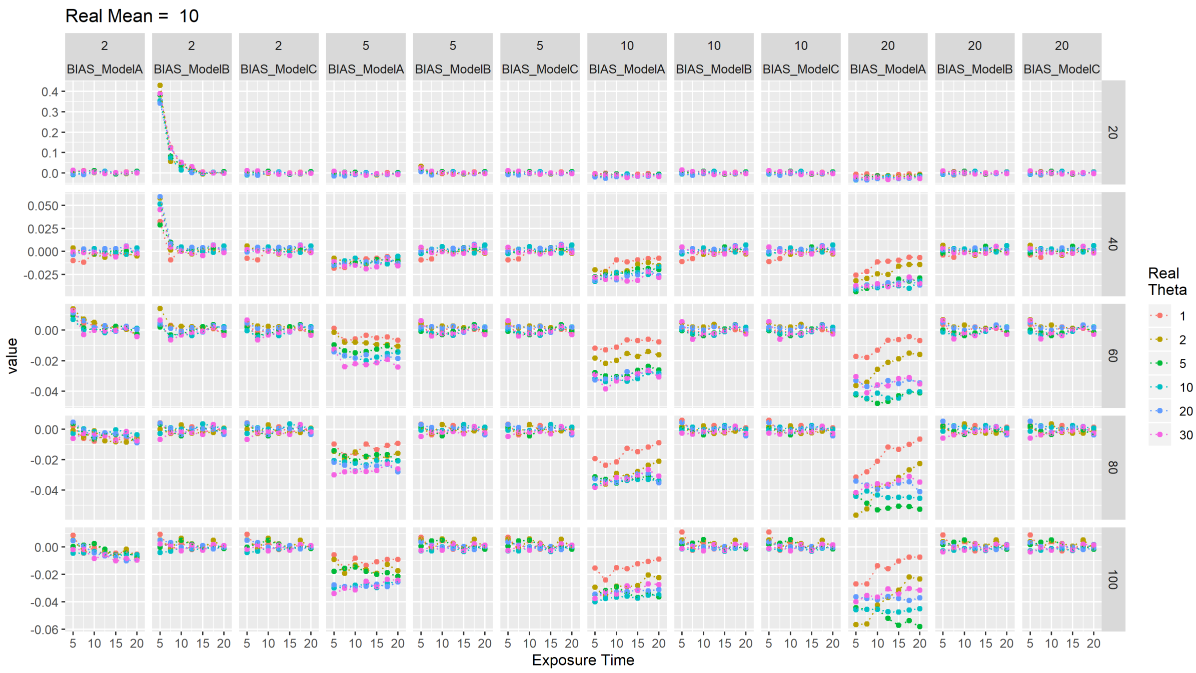

Figure B.2.2: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect BIAS for Models A, B and C,when the true prior mean is ten.

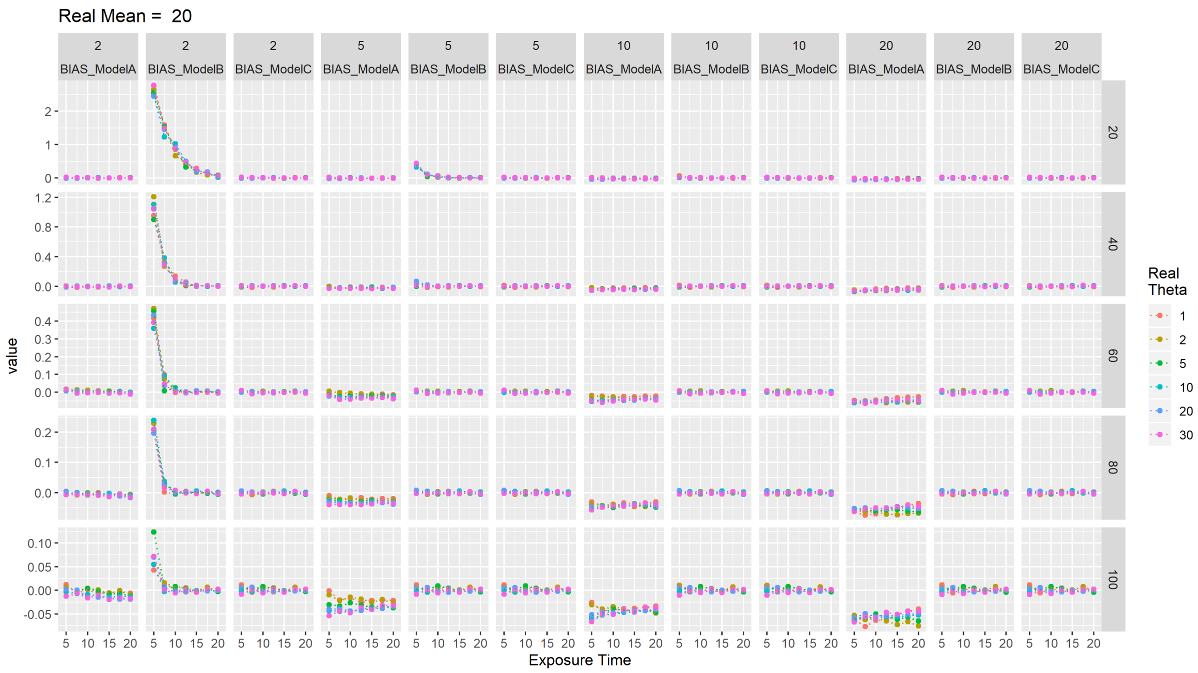

Figure B.2.3: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect BIAS for Models A, B and C,when the true prior mean is twenty.

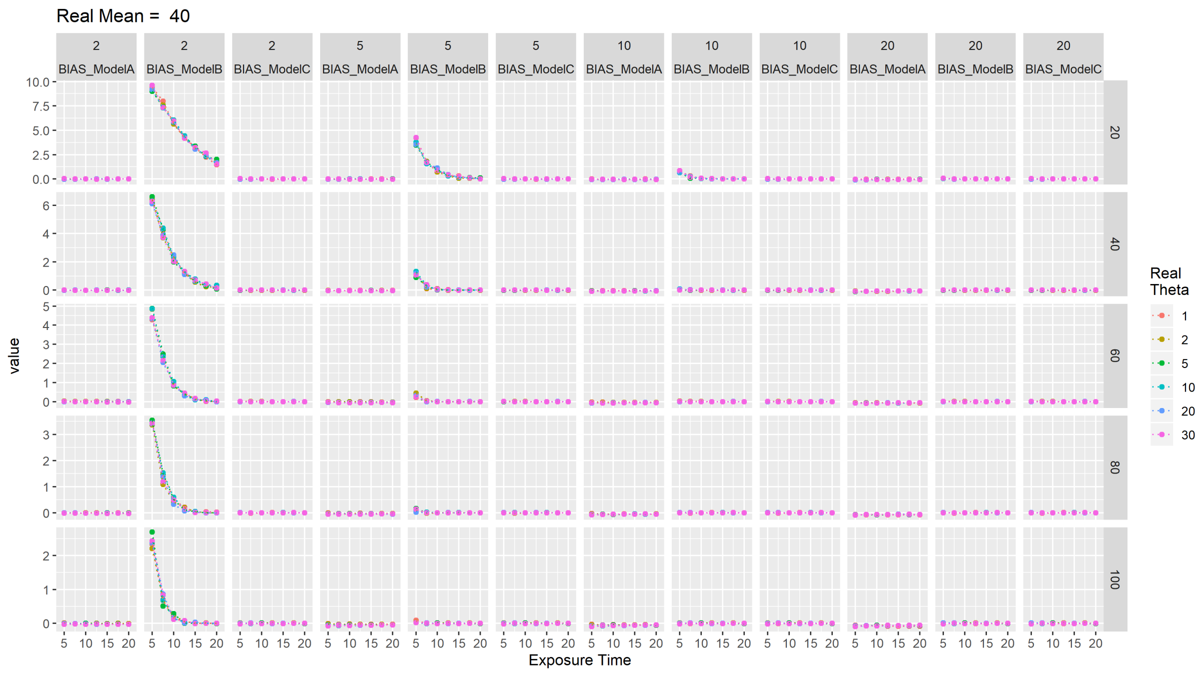

Figure B.2.4: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect BIAS for Models A, B and C,when the true prior mean is forty.

B.2.2 MSE Results

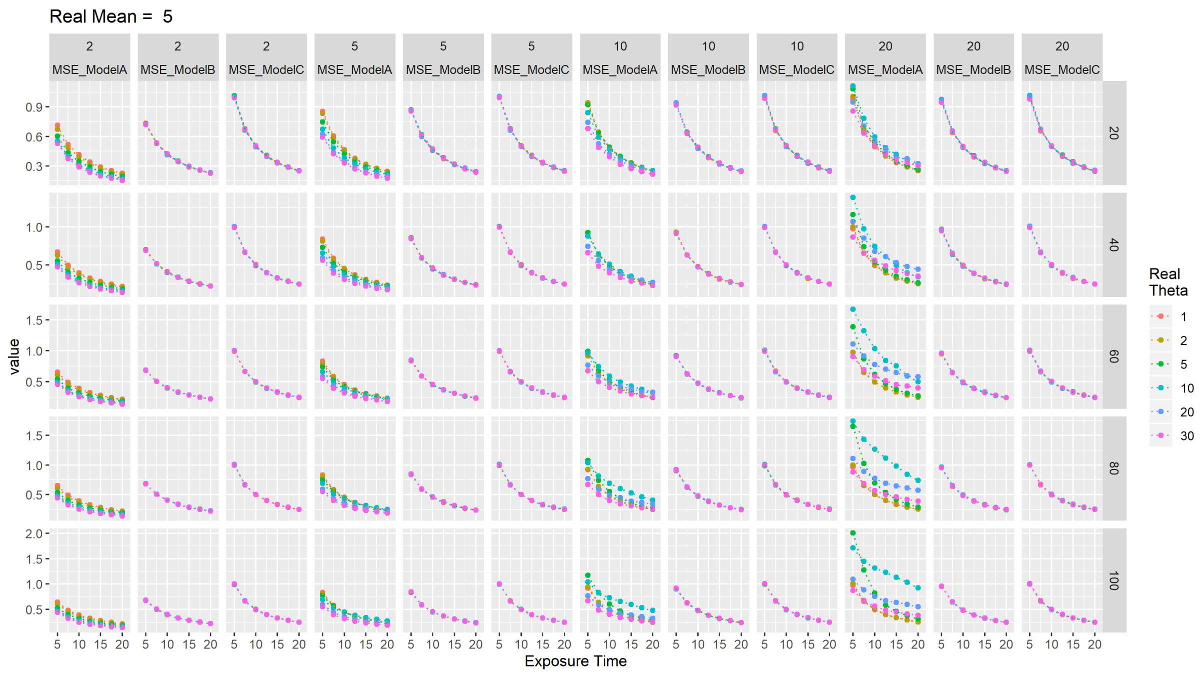

Figure B.2.5: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect MSE for Models A, B and C,when the true prior mean is five.

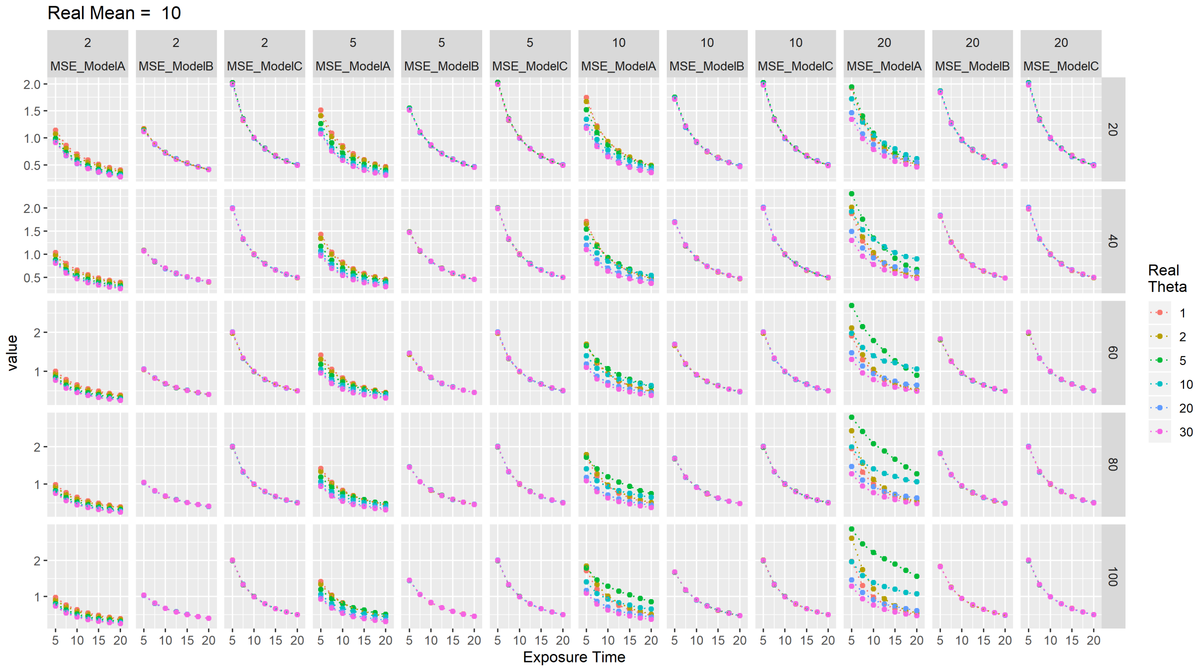

Figure B.2.6: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect MSE for Models A, B and C,when the true prior mean is ten.

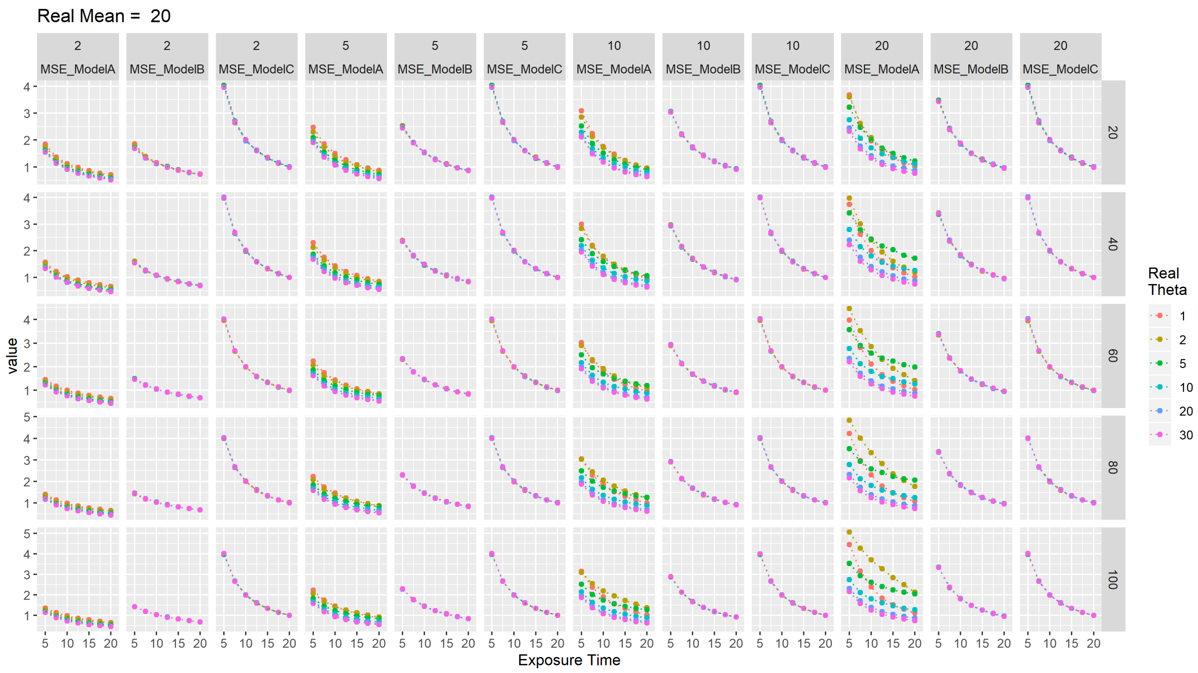

Figure B.2.7: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect MSE for Models A, B and C,when the true prior mean is twenty.

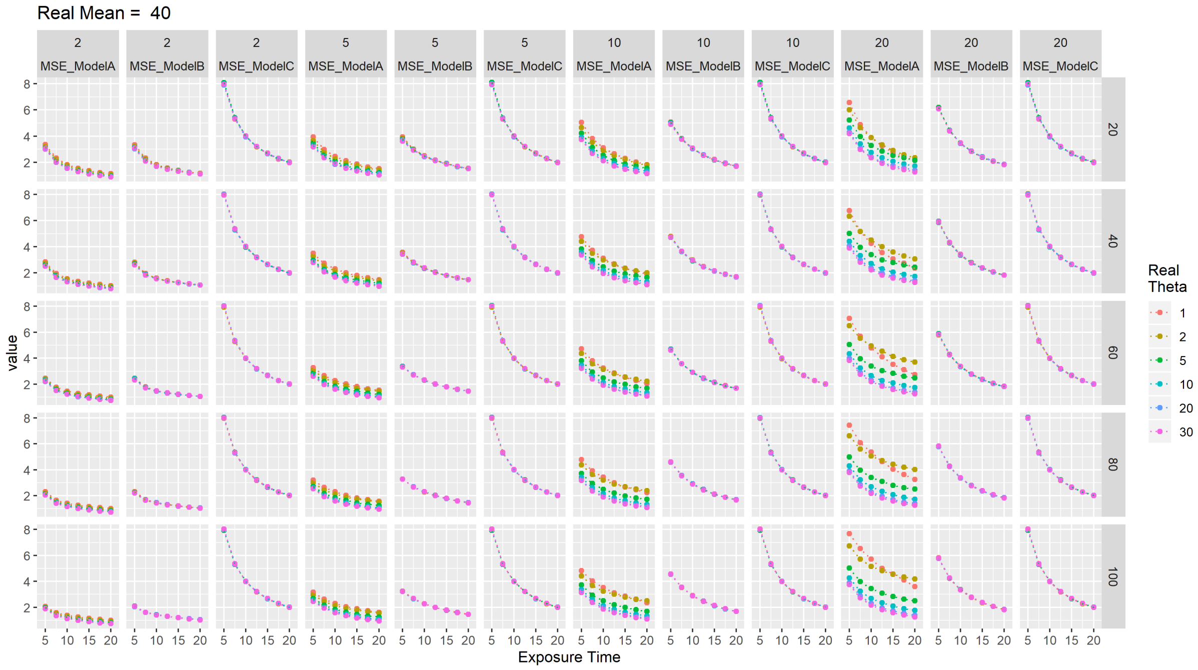

Figure B.2.8: Showing how different pool sizes, exposure times, dependence parameters and prior variance values affect MSE for Models A, B and C, when the true prior mean is forty.

Appendix C

Literature Review on Supplier Selection / Ranking Process

In this chapter, we present a literature review on the supplier selection process. Even if the supplier selection process seems distinct from supplier ranking, they are closely related. Mainly, the latter is part of the supplier selection process. In the supplier selection process, multiple possible suppliers are of consideration with the ultimate goal of selecting one or more best-performing suppliers. All possible suppliers are initially evaluated and then ranked based on prespecified criteria. In contrast, the supplier ranking process is based on already existing suppliers within the organisation; and it requires the existence of relevant historical data with the goal of ranking already selected suppliers. Therefore, we choose to follow the supplier selection process, mainly focusing on the final phase, the supplier ranking. We initially present different supplier selection frameworks along with relevant supplier selection criteria proposed in the past and then discuss the methods and models used for supplier selection and ranking in the supply chain management area. For further discussion see De Boer et al., 2001; Chai et al., 2013 and Chai & Ngai, 2020.

C.1 Supplier Selection Framework

The supplier selection process involves various phases; thus, it is essential to consider all these phases and not to concentrate only to the final step which refers to the supplier selection/ranking (De Boer et al., 2001). According to De Boer et al. (2001), the supplier selection process can be divided into four decision-making phases; the problem formulation, the formulation of criteria, the qualification and the final selection. We initially need to identify the purpose of the supplier selection process and then define the evaluation criteria. Considering the different purchasing situations is also essential, and as De Boer et al. (2001) propose, these categories could be (a) first time buys, (b) modified rebuys (leverage items), (c) straight rebuys (routine items) and (d) straight rebuys (strategic items). The authors argue that 'situational factors as the number of the suppliers available, the importance of the purchase and/or the supplier relationship and the amount and nature of the uncertainty present, are far more determinative for the suitability of a certain decision method in a particular situation' (De Boer et al., 2001). Consequently, they suggest only two determinant factors, importance and complexity.

Many authors in the past have examined complexity in the supplier selection process, for example Robinson et al. (1967) proposed three categories of purchasing situations a) the new task situation, b) the modified rebuy and c) the straight rebuy. A detailed description of the categorisation by Robinson et al. (1967) is presented in Table C.1.1. Kraljic presented another approach to the supplier selection area in 1983 (Kraljic, 1983). In his so-called Kraljic' s portfolio approach, importance and complexity are both examined based on two specific factors, the profit impact and the supply risk. Description of all characteristics of the approach is also presented in Table C.1.2.

New task situation	• Entirely new product/service; no previous experience • No (known) suppliers • High level of uncertainty with respect to the specification • Extensive problem solving; group decision - making
Modified rebuy	• New product/service to be purchased from known suppliers • Existing (modified) products to be purchased from new suppliers • Moderate level of uncertainty with respect to specification • Less extensive problem solving
Straight rebuy	• Perfect information concerning specification and supplier • Involves placing an order within existing contracts and agreements

Table C.1.1: Categorization of purchasing situations by Robinson et al. (1967).

	Low-supply risk	**High-supply risk**
Low-profit impact	***Routine items*** • Many suppliers • Rationalise purchasing procedures • Systems contracting • Automate/delegate	***Bottleneck items*** • Monopolistic supply market • Long-term contracts • Develop alternatives (internally) • Contingency planning
High-profit impact	***Leverage items*** • Many suppliers available • Competitive bidding • Short-term contracts • Active sourcing	***Strategic items*** • Few suppliers • Medium/long-term contracts • Supplier development/partnership • Continuous review

Table C.1.2: Kraljic (1983) portfolio approach

Following Kraljic (1983) portfolio approach and Robinson et al. (1967), De Boer et al. (2001) proposed a new supplier selection framework that aims to provide the purchaser with a clear distinction among the supplier selection situations, considering the four phases of the process discussed previously. The first distinction refers to the type of supplier selection, mainly, the distinction between the first time and repeated selection is considered. Summary of De Boer's supplier selection framework (De Boer, van der Wegen, & Telgen, 1998) is presented in Table C.1.3. We note that the proposed framework by De Boer et al. (1998) is closely related to the Kraljic's portfolio approach (Kraljic, 1983). Precisely, both approaches describe and distinguish the supplier selection situations based on leverage, routine and bottleneck/strategic items; and based on the impact of relationships between companies. In both approaches, the importance of the relationship between buyer and seller, which indicates that the final selection/choice may differ if two companies have close relationships, is also addressed.

	New task	**Modified rebuy** *(leverage items)*	**Straight rebuy** *(routine items)*	**Straight rebuy** (*strategic/bottleneck)*
Problem definition	Use a supplier or not?	Use more, fewer or other suppliers?	Replacing the current supplier?	How to deal with the supplier?
	Varying importance	Moderate/high importance	Low/moderate importance	High importance
	One-off decision	Repeating decision	Repeating decision	Repeating evaluation
Formulation of criteria	No historical data on suppliers available	Historical data on suppliers available	Historical data on suppliers available	Historical data on suppliers available, yet very few actual selections
	No previously used criteria available	Previously used criteria available	Previously used criteria available	Previously used criteria available
	Varying importance			
Qualification	Small initial set of suppliers	Large set of initial suppliers	Large set of initial suppliers	Very small set of suppliers

	Sorting rather than ranking	Sorting as well as ranking	Sorting rather than ranking	Sorting rather than ranking
	No historical records available	Historical data available	Historical data available	Historical data available
Choice	Small initial set of suppliers	Small to moderate set of initial suppliers	Small to moderate set of initial suppliers	Very small set of suppliers (often only one)
	Ranking rather than sorting	Ranking rather than sorting	Ranking rather than sorting	Historical data available
	Many criteria	Also: how to allocate volume?	Fewer criteria	Evaluation rather selection
	Much interaction	Fewer criteria	Less interaction	Sole sourcing
	No historical records available	Less interaction	Historical data available	
	Varying importance	Historical data available	Model used again	
	Model used once	Model used again	Single sourcing rather than multiple sourcing	

Table C.1.3: De Boer' s supplier selection framework (1998)

C.2 Supplier Selection Criteria

In this section, we discuss the initial phase of the supplier selection process, which involves the establishment of the ranking/evaluation criteria. Before we rank and finally select the suppliers, we need to define the purpose and determine the criteria on which the evaluation will be based. Over the years, the decision criteria vary due to the needs of every period. Previous researches have shown that the most important criteria are the delivery performance and the price; however, recent studies show that quality has also reached a high ranking (Bharadwaj, 2004).

Many studies focused on finding what and how many evaluation criteria are of need in the supplier selection process. For example, Stamm and Golhar (1993) identified thirteen (13) evaluation criteria, Ellram (1990) proposed eighteen (18) criteria and Rao and Kiser (1980) suggested sixty (60) criteria. Most of the evaluation criteria are quantitative, such as cost and delivery; however, Ellram (1990) proposed qualitative criteria in his supplier selection approach. The compatibility of management, the consistency of goal and the suppliers' strategy are some of his recommended qualitative evaluation criteria. Also, Dickson (1966) proposed a set of twenty-three (23) criteria suggesting that quality and delivery are two of the most critical factors. The performance history follows in third place in the ranking table. We also notice that price takes sixth place in the table, which indicates that it is not so relevant compared to the quality and delivery factors. Table C.2.1 shows the first ten criteria with the highest ranking, as mentioned by Dickson (1966).

Rank	Factor	Mean Rating
1	Quality	3.508
2	Delivery	3.417
3	Performance History	2.998
4	Warranties and Claim Policies	2.849
5	Production Facilities Capacity	2.775
6	Price	2.758
7	Technical Capability	2.545
8	Financial Position	2.514
9	Procedural Compliance	2.488
10	Communication System	2.426

Table C.2.1: Dicksons' supplier selection criteria (Weber et al. 1991)

Appendix D

Empirical Evaluation of Ranking Methods Based on Historical Supplier Data

D.1 Visual Comparison of Distribution of Ranks Between Classic EB and EB with Clayton Models.

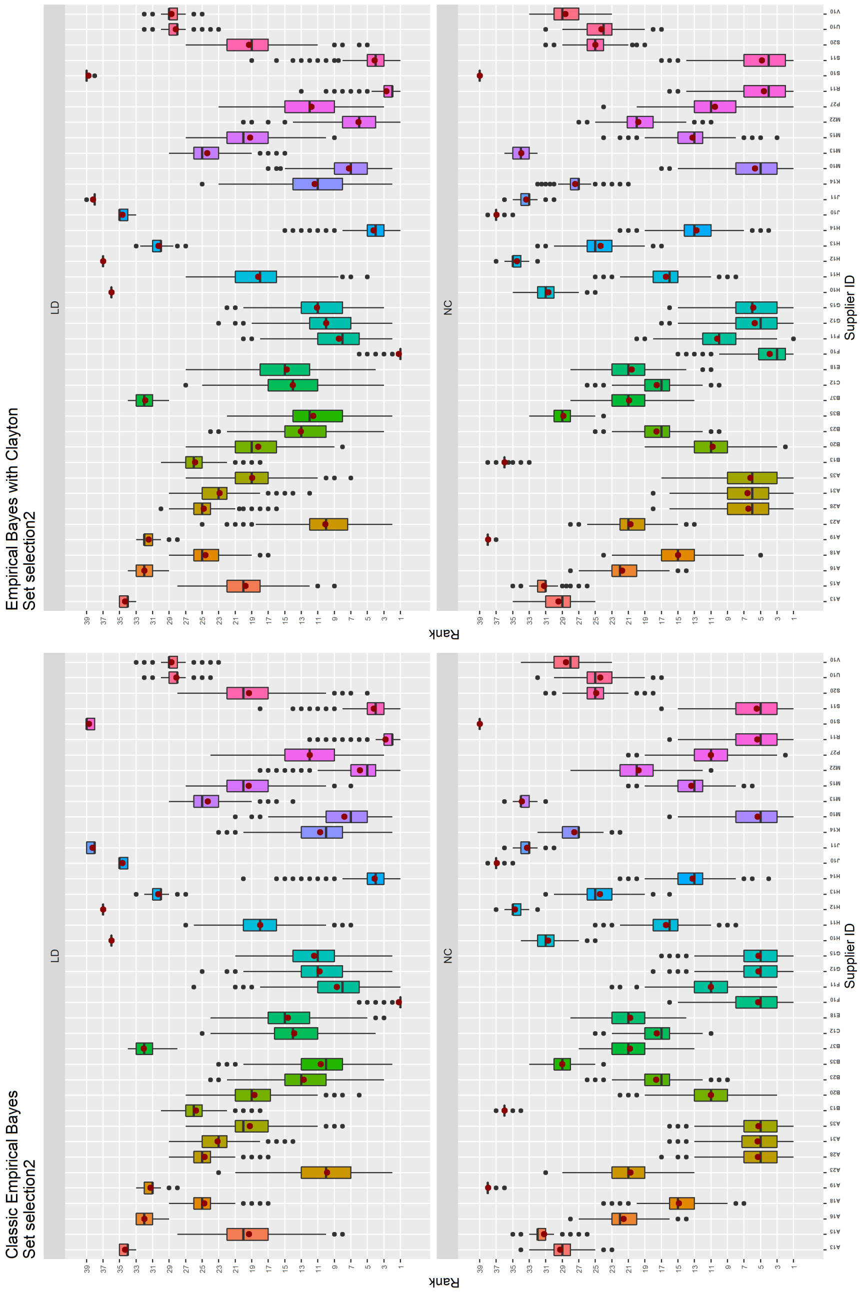

Figure D.1.1: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 2. Comparison between classic empirical Bayes and empirical Bayes with Clayton copula.

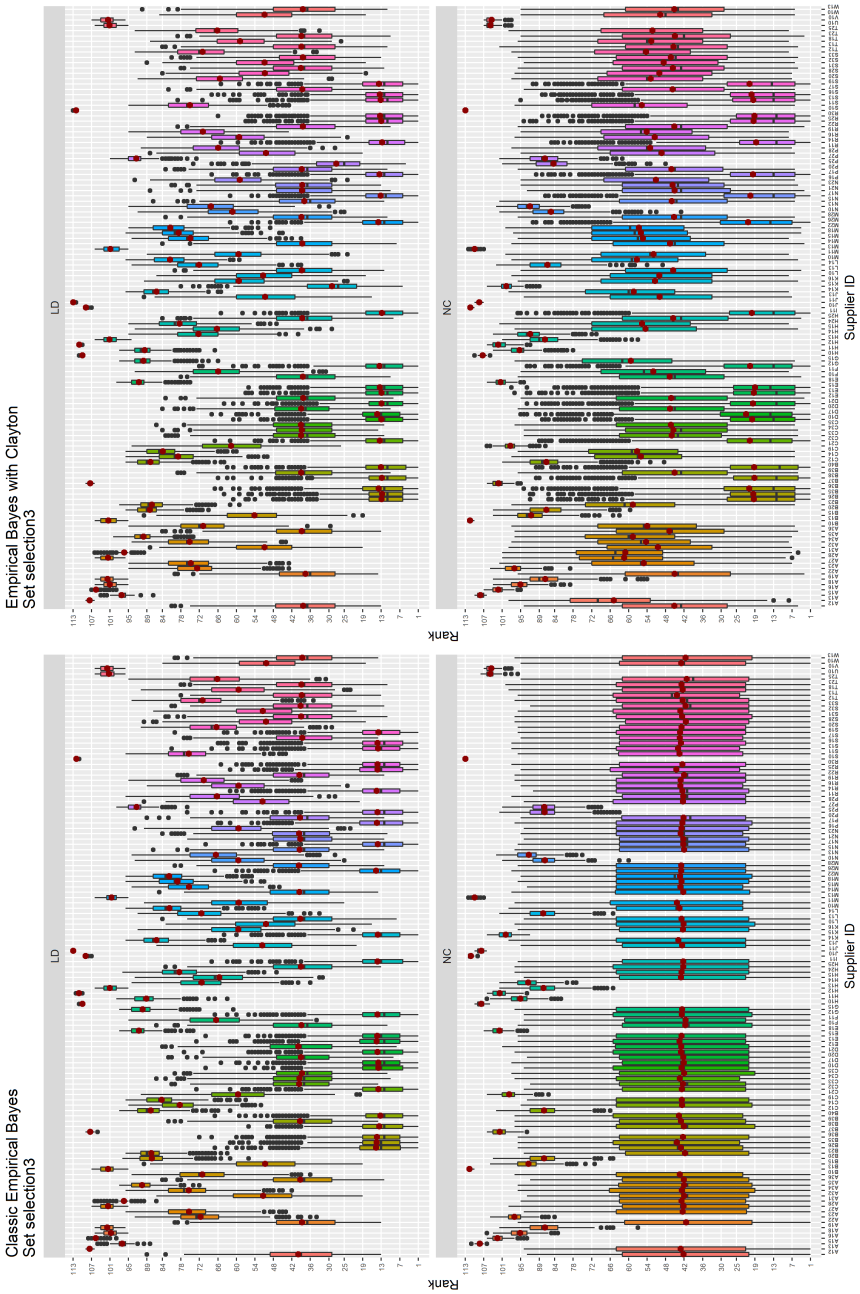

Figure D.1.2: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 3. Comparison between classic empirical Bayes and empirical Bayes with Clayton copula.

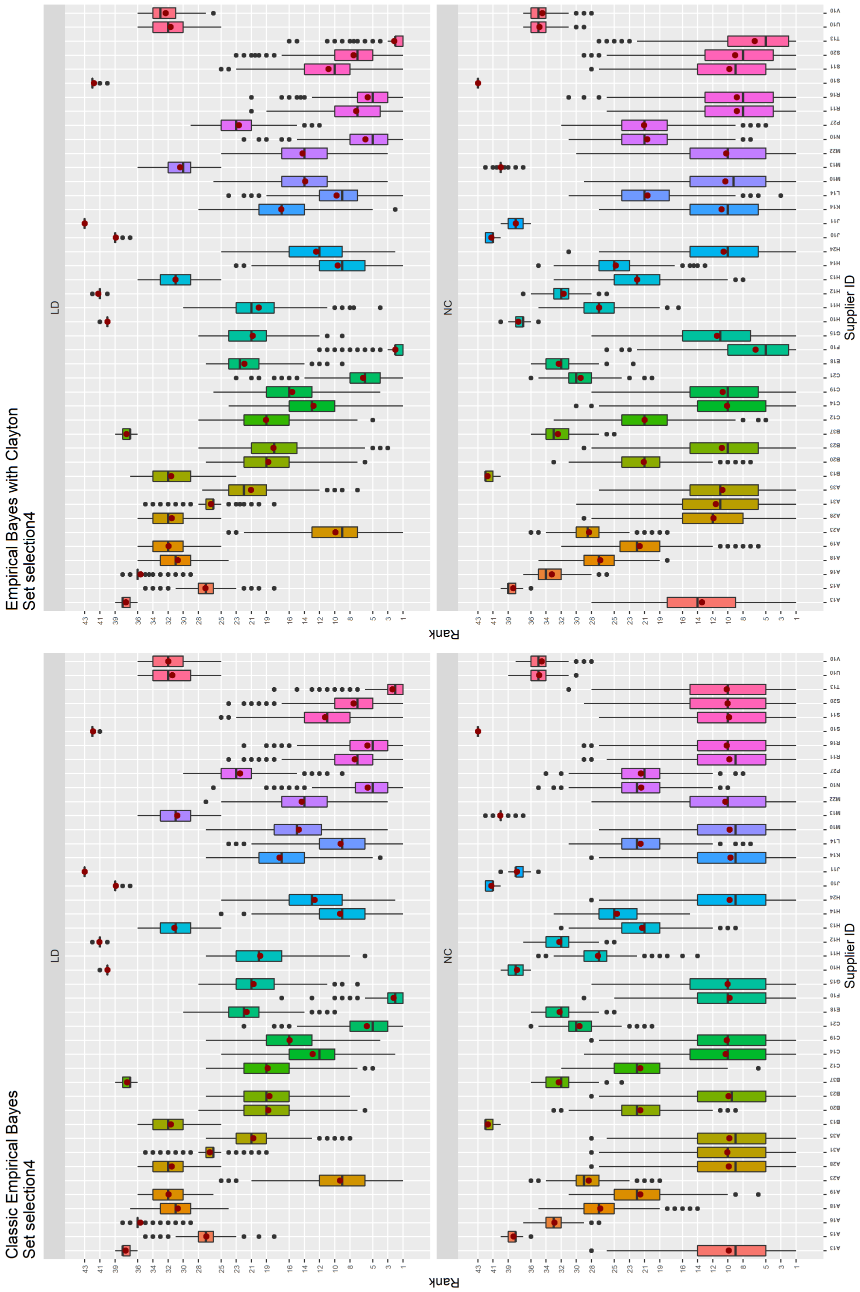

Figure D.1.3: Multiple boxplots showing the empirical distribution of the late delivery and non - conformance ranks for all suppliers within the pool of selection set 4. Comparison between classic empirical Bayes and empirical Bayes with Clayton copula.